NUMBER THEORY IN PHYSICS

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Several fields of mathematics have been closely associated to physics: this has always been the case for the theory of differential equations. In the early twentieth century, with the advent of general relativity and quantum mechanics, topics such as differential and Riemannian geometry, operator algebras and functional analysis, or group theory also developed a close relation to physics. In the past decade, mostly through the influence of string theory, algebraic geometry also began to play a major role in this interaction. Recent years have seen an increasing number of results suggesting that number theory also is beginning to play an essential part on the scene of contemporary theoretical and mathematical physics. Conversely, ideas from physics, mostly from quantum field theory and string theory, have started to influence work in number theory.

In describing significant occurrences of number theory in physics, we will, on the one hand, restrict our attention to quantum physics, while, on the other hand, we will assume a somewhat extensive definition of number theory, that will allow us to include arithmetic algebraic geometry. The territory is vast and an extensive treatment would go beyond the size limits imposed by the encyclopaedia. The choice of topics represented here inevitably reflects the limited knowledge, particular interests and bias of the author. Very useful references, collecting a lot of material on Number Theory and Physics, are the proceedings of the Les Houches conferences [1], [2], [3]. A "Number Theory and Physics" database is presently maintained online by Matthew R. Watkins.

In the following, we organized the material by topics in number theory that have so far made an appearance in physics and for each we briefly describe the relevant context and results. This singles out many themes. We shall first discuss occurrences in physics of a class of functions and their special values that are of great number theoretic importance. This includes the dilogarithm, the polylogarithms and multiple polylogarithms, and the multiple zeta values. We also discuss the most important symmetry groups of number theory, the Galois groups, and occurrences in physics of some forms of Galois theory. We then discuss how techniques from the arithmetic geometry of algebraic varieties, especially Arakelov geometry, play a role in string theory. Finally, we discuss briefly the theory of motives and outline its possible relation to quantum physics. From the physics point of view, it seems that the most promising directions in which number theoretic tools have come to play a crucial role are to be found mostly in the realm of rational conformal field theories and of noncommutative geometry, as well as in certain aspects of string theory.

Among the topics that are very relevant to this theme, but that I will not touch upon in this article, there are important subjects like the theory of "arithmetic quantum chaos", the use of methods of random matrix theory applied to the study of zeros of zeta functions, or mirror symmetry and its connection to modular forms. The interested reader can find such topics treated in other articles of this encyclopaedia and in the references mentioned above.

1. Dilogarithm, Multiple polylogarithms, multiple zeta values

The dilogarithm is defined as

$$Li_2(z) = \int_z^0 \frac{\log(1-t)}{t}\, dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

It satisfies the functional equation $Li_2(z) + Li_2(1-z) = Li_2(1) - \log(z)\log(1-z)$, where $Li_2(1) = \zeta(2)$, for $\zeta(s)$ the Riemann zeta function. A variant is given by the Rogers dilogarithm $L(x) = Li_2(x) + \frac{1}{2}\log(x)\log(1-x)$. For more details see Zagier's paper in [3].
The polylogarithms are similarly defined by the series $Li_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$. In quantum electrodynamics, there are corrections to the value of the gyromagnetic ratio, in powers of the fine structure constant. The correction terms that are known exactly involve special values of the zeta function like $\zeta(3)$, $\zeta(5)$ and values of polylogarithms like $Li_4(1/2)$. The series defining the polylogarithm function $Li_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}$ converges absolutely for all $s \in \mathbb{C}$ and $|z| < 1$ and has analytic continuation to $z \in \mathbb{C} \smallsetminus [1,\infty)$. The Fermi–Dirac and Bose–Einstein distributions are expressed in terms of the polylogarithm function as

$$\int_0^\infty \frac{x^s}{e^{x-\mu} \pm 1}\, dx = -\Gamma(s+1)\, Li_{1+s}(\pm e^{\mu}).$$

The multiple polylogarithms are functions defined by the expressions

$$\mathrm{Li}_{s_1,\ldots,s_r}(z_1, z_2, \ldots, z_r) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}. \tag{1.1}$$

By analytic continuation, the functions $\mathrm{Li}_{s_1,\ldots,s_r}(z_1, z_2, \ldots, z_r)$ are defined for all complex s_i and for z_i in the complement of the cut $[1,\infty)$ in the complex plane.
Multiple zeta values of weight k and depth r are given by the expressions

$$\zeta(k_1, \cdots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \tag{1.2}$$

with $k_i \in \mathbb{N}$ and $k_1 \geq 2$. These satisfy many combinatorial identities and nontrivial relations over $\mathbb{Q}$. For an informative overview on the subject see [7]. Notice that, in both (1.1) and (1.2), a different summation convention can also be found in the literature.

1.1. **Conformal field theories and the dilogarithm.**

There is a relation between the torsion elements in the algebraic K-theory group $K_3(\mathbb{C})$ and rational conformally invariant quantum field theories in two dimensions (see Werner Nahm's article [21]). There is, in fact, a map, given by the dilogarithm, from torsion elements in the Bloch group (closely related to the algebraic K-theory) to the central charges and scaling dimensions of the conformal field theories.
This correspondence arises by considering sums of the form

$$\sum_{m \in \mathbb{N}^r} \frac{q^{Q(m)}}{(q)_m}, \tag{1.3}$$

where $(q)_m = (q)_{m_1} \cdots (q)_{m_r}$, $(q)_{m_i} = (1-q)(1-q^2)\cdots(1-q^{m_i})$ and $Q(m) = m^t Am/2 + bm + h$ has rational coefficients. Such sums are naturally obtained from considerations involving the partition function of a bosonic rational CFT. In particular, (1.3) can define a modular function only if all the solutions of the equation

$$\sum_j A_{ij} \log(x_j) = \log(1 - x_i) \tag{1.4}$$

determine elements of finite order in an extension $\hat{B}(\mathbb{C})$ of the Bloch group, which accounts for the fact that the logarithm is multi-valued. The Rogers dilogarithm gives a natural group homomorphism $(2\pi i)^2 L : \hat{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$, which takes values in $\mathbb{Q}/\mathbb{Z}$ on the torsion elements. These values give the conformal dimensions of the fields in the theory.

1.2. **Feynman graphs.** Multiple zeta values appear in perturbative quantum field theory. Dirk Kreimer developed in [12] a connection between knot theory and a class of transcendental numbers, such as multiple zeta values, obtained by quantum field theoretic calculations as counterterms generated by corresponding Feynman graphs. Broadhurst and Kreimer [6] identified Feynman diagrams with up to 9 loops whose corresponding counterterms give multiple zeta values up to weight 15. Recently, Kreimer showed some deep analogies between residues of quantum fields and variations of mixed Hodge–Tate structures associated to polylogarithms.

Testing predictions about the standard model of elementary particles, in the hope of detecting new physics, requires developing effective computational methods handling the huge number of terms involved in any such calculation, *i.e.* efficient algorithms for the expansion of higher transcendental functions to a very high order. The interesting fact is that abstract number theoretic objects such as multiple zeta values and multiple polylogarithms, appear naturally in this context, *cf. e.g.* [19]. The explicit recursive algorithms are based on Hopf algebras and produce expansions of nested finite or infinite sums involving ratios of Gamma functions and Z–sums (Euler–Zagier sums), which naturally generalize multiple polylogarithms and multiple zeta values. Such sums typically arise in the calculation of multi-scale multi-loop integrals. The algorithms are designed to recursively reduce the Z-sums involved to simpler ones with lower weight or depth.

2. Galois Theory

Given a number field $\mathbb{K}$, which is an algebraic extension of $\mathbb{Q}$ of some degree $[\mathbb{K} : \mathbb{Q}] = n$, there is an associated fundamental symmetry group, given by the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$, where $\bar{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$. Even in the case of $\mathbb{Q}$, the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a very complicated object, far from being fully understood.

One can consider an easier symmetry group, which is the abelianization of the absolute Galois group. This corresponds to considering the field $\mathbb{K}^{ab}$, the *maximal abelian extension* of $\mathbb{K}$, which has the property that

$$\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})^{ab}.$$

The Kronecker–Weber theorem shows that for $\mathbb{K} = \mathbb{Q}$ the maximal abelian extension can be identified with the cyclotomic field (generated by all roots of unity), $\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$, and the Galois group is identified with $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*$, where

$\hat{\mathbb{Z}}^* = \mathbb{A}_f^*/\mathbb{Q}_+^*$. In general, for other number fields, one has the *class field theory isomorphism*

$$\theta : \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) \xrightarrow{\simeq} C_{\mathbb{K}}/D_{\mathbb{K}},$$

where $C_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$ is the group of idele classes and $D_{\mathbb{K}}$ the connected component of the identity in $C_{\mathbb{K}}$. In general, however, one does not have an explicit description of the generators of the maximal abelian extension $\mathbb{K}^{ab}$ and of the action of the Galois group. This is the content of the explicit class field theory problem, Hilbert 12th problem. Besides the Kronecker–Weber case, a complete answer is known in the case of imaginary quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, with $d > 1$ a positive integer. In this case generators are obtained by evaluating modular functions at a point τ in the upper half plane such that $\mathbb{K} = \mathbb{Q}(\tau)$ and the Galois action is described explicitly through the group of automorphisms of the modular field, through Shimura reciprocity. For a survey of the explicit class field theory problem and the case of imaginary quadratic fields see [24].

As we mentioned above, understanding the structure of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a fundamental question in number theory. Grothendieck described, in his famous proposal "Esquisse d'un programme", how to obtain an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on an essentially combinatorial object, the set of "dessins d'enfants". These are connected graphs (on a surface) such that the complement of the graph is a union of open cells and where vertices have two different markings, with the properties that adjacent vertices have opposite markings. Such objects arise by considering the projective line $\mathbb{P}^1$ minus three points. Any finite cover of $\mathbb{P}^1$ branched only over $\{0, 1, \infty\}$ gives an algebraic curve defined over $\bar{\mathbb{Q}}$. The dessin is the inverse image under the covering map of the segment $[0, 1]$ in $\mathbb{P}^1$. The absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the data of the curve and the covering map, hence on the set of dessins. A theorem of Bielyi shows that, in fact, all algebraic curves defined over $\bar{\mathbb{Q}}$ are obtained as coverings of the projective line ramified only over the points $\{0, 1, \infty\}$. This has the effect of realizing the absolute Galois group as a subgroup of outer automorphisms of the profinite fundamental group of the projective line minus three points. For a general reference on the subject, see [22]. For graphs on surfaces, including the theory of "dessins d'enfants" and many applications of interest to physicists, see also [13].

A different type of Galois symmetry of great arithmetic significance is *motivic* Galois theory. This will be discussed later in the section dedicated to motives, where we discuss a surprising occurrence in the context of perturbative quantum field theory and renormalization.

2.1. **Quantum Statistical Mechanics and Class Field Theory.** In quantum statistical mechanics, one considers an algebra of observables, which is a unital C^*-algebra $\mathcal{A}$ with a time evolution σ_t. States are given by linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(1) = 1$ and positivity $\varphi(x^* x) \geq 0$. Equilibrium states φ at inverse temperature β satisfy the KMS (Kubo–Martin–Schwinger) condition, namely, for all $x, y \in \mathcal{A}$ there exists a bounded holomorphic function $F_{x,y}(z)$ on the strip $0 < \Im(z) < \beta$, which extends continuously to the boundary, such that for all $t \in \mathbb{R}$

$$F_{x,y}(t) = \varphi(x\sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x). \tag{2.1}$$

Cases of number theoretic interest arise when one considers as algebra of observable the noncommutative space of commensurability classes of $\mathbb{Q}$-lattices up to scaling, with a natural time evolution determined by the covolume, [9]. A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ consists of a pair (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ together with a homomorphism of abelian groups $\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$. Two $\mathbb{Q}$-lattices are commensurable, $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$, iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$.

2.1.1. *The Bost–Connes system.* The quantum statistical mechanical system considered by Bost and Connes in [5] corresponds to the case of 1-dimensional $\mathbb{Q}$-lattices. The partition function of the system is the Riemann zeta function $\zeta(\beta)$. The system has spontaneous symmetry breaking at $\beta = 1$, with a single KMS state for all $0 < \beta \leq 1$. For $\beta > 1$ the extremal equilibrium states are parameterized by the embeddings of $\mathbb{Q}^{cycl}$ in $\mathbb{C}$ with a free transitive action of the idele class group $C_{\mathbb{Q}}/D_{\mathbb{Q}} = \hat{\mathbb{Z}}^*$. At zero temperature, the evaluation of KMS_∞ states on elements of a rational subalgebra intertwines the action of $\hat{\mathbb{Z}}^*$ by automorphisms of $(\mathcal{A}, \sigma_t)$ with the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the values of the states. This recovers the explicit class field theory of $\mathbb{Q}$ from a physical perspective.

2.1.2. *Noncommutative space of adele classes.* The algebra $\mathcal{A}$ of the Bost–Connes system is the noncommutative algebra of functions $f(r, \rho)$, for $\rho \in \hat{\mathbb{Z}}$ and $r \in \mathbb{Q}^*$ such that $r\rho \in \hat{\mathbb{Z}}$, with the convolution product

$$f_1 * f_2\,(r, \rho) = \sum_{s \in \mathbb{Q}^* : s\rho \in \hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s, \rho), \tag{2.2}$$

and the adjoint $f^*(r, \rho) = \overline{f(r^{-1}, r\rho)}$. According to the general philosophy of Connes style noncommutative geometry, it is the algebra of coordinates of the noncommutative space defined by the "bad quotient" $\mathrm{GL}_1(\mathbb{Q}) \backslash (\mathbb{A}_f \times \{\pm 1\})$ – a noncommutative version of the zero-dimensional Shimura variety $Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q}) \backslash (\mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\})$. Its "dual system" (in the sense of Connes's duality of type III and type II factors) is obtained by taking the crossed product by the time evolution. It gives the algebra of coordinates of the noncommutative space defined by the quotient $\mathbb{A}/\mathbb{Q}^*$. This is the noncommutative space of "adele classes" used by Connes in his spectral realization of the zeros of the Riemann zeta function.

2.1.3. *The* GL_2*-system.* A generalization of the Bost–Connes system was introduced by Connes and Marcolli in [9]. This corresponds to the case of 2-dimensional $\mathbb{Q}$-lattices. The partition function is the product $\zeta(\beta)\zeta(\beta - 1)$. The system in this case has two phase transitions, with no KMS states for $\beta \leq 1$. For $\beta > 2$, the extremal KMS states are parameterized by the invertible $\mathbb{Q}$-lattices, namely those for which ϕ is an isomorphism. The algebra $\mathcal{A}$ has an arithmetic structure given by a rational algebra of unbounded multipliers. This rational algebra contains modular functions and Hecke operators. At zero temperature, extremal KMS states can be evaluated on these multipliers. Symmetries of $(\mathcal{A}, \sigma_t)$ are realized in part by endomorphisms (as in the theory of superselection sectors) and the symmetry group acting on low temperature KMS states is the group of automorphisms of the modular field $\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^*$. For a generic set of extremal KMS_∞ states, evaluation at the rational algebra intertwines this action with the action on the values of an embedding of the modular field as a subfield of $\mathbb{C}$.

2.1.4. *The complex multiplication system.* In the case of an imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\tau)$, an analogous construction is possible. A 1-dimensional $\mathbb{K}$-lattice is a pair (Λ, ϕ) of a finitely generated $\mathcal{O}$-submodule Λ of $\mathbb{C}$, with $\Lambda\mathbb{K} = \mathbb{K}$, and a homomorphism of $\mathcal{O}$-modules $\phi : \mathbb{K}/\mathcal{O} \to \mathbb{K}\Lambda/\Lambda$. Two $\mathbb{K}$-lattices are commensurable iff $\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$. Connes, Marcolli, and Ramachandran [10] constructed a quantum statistical mechanical system describing the noncommutative space of commensurability classes of 1-dimensional $\mathbb{K}$-lattices up to scale. The partition function is the Dedekind zeta function $\zeta_{\mathbb{K}}(\beta)$. The system has a phase transition at $\beta = 1$ with a unique KMS state for higher temperatures and extremal KMS states parameterized by the invertible $\mathbb{K}$-lattices at lower temperatures. There is a rational subalgebra induced by the rational structure of the GL_2-system (1-dimensional $\mathbb{K}$-lattices are also 2-dimensional $\mathbb{Q}$-lattices with compatible notions of commensurability). The symmetries of the system are given by the idele class group $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$. The action is partly realized by endomorphisms corresponding to the possible presence of a non-trivial class group (for class number > 1). The values of extremal KMS_∞ states on the rational subalgebra intertwine the action of the idele class group with the Galois action on the values. This fully recovers the explicit class field theory for imaginary quadratic fields.

2.2. **Conformal Field Theory and the absolute Galois group.** Moore and Seiberg considered data associated to any rational conformal field theory, consisting of matrices, obtained as monodromies of some holomorphic multivalued functions on the relevant moduli spaces, satisfying polynomial equations. Under reasonable hypotheses, the coefficients of the Moore–Seiberg matrices are algebraic numbers. This allows for the presence of interesting arithmetic phenomena. Through the Chern–Simons/Wess–Zumino–Witten correspondence, it is possible to construct three-dimensional topological field theories from solutions to the Moore–Seiberg equations.

On the arithmetic side, Grothendieck proposed in his "Esquisse d'un programme" the existence of a Teichmüller tower given by the moduli spaces $M_{g,n}$ of Riemann surfaces of arbitrary genus g and number of marked points n, with maps defined by operations such as cutting and pasting of surfaces and forgetting marked points, all encoded in a family of fundamental groupoids. He conjectured that the whole tower can be reconstructed from the first two levels, providing, respectively, generators and relations. He called this a "game of Lego–Teichmüller". He also conjectured that the absolute Galois group acts by outer automorphisms on the profinite completion of the tower. The basic building blocks of the tower are provided by "pairs of pants", *i.e.* by projective lines minus three points.

This leads to a conjectural relation between the Moore–Seiberg equations and this Grothendieck–Teichmüller setting, *cf.* [11], according to which solutions of the Moore–Seiberg equations provide projective representations of the Teichmüller tower, and the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ corresponds to the action on the coefficients of the Moore–Seiberg matrices.

Rational conformal field theories are, in general, one of the most promising sources of interactions between number theory and physics, involving interesting Galois actions, modular forms, Brauer groups, complex multiplication. Some fundamental work in this direction was done by authors such as Borcherds, Gannon, etc.

3. Arithmetic algebraic geometry

In this section we describe occurrences in physics of various aspects of the arithmetic geometry of algebraic varieties.

3.1. **Arithmetic Calabi-Yau.** In the context of type II string theory, compactified on Calabi-Yau threefolds (see the relevant articles in this encyclopaedia for more information), Greg Moore [20] considered certain black hole solutions and a resulting dynamical system given by a differential equation in the corresponding moduli. The fixed points of these equations determine certain "black hole attractor varieties". In the case of varieties obtained from a product of elliptic curves or of a K3 surface and an elliptic curve, the attractor equation singles out an arithmetic property: the elliptic curves have complex multiplication. The class number of the corresponding imaginary quadratic field counts U-duality classes of black holes with the same area. Other results point to a relation between the arithmetic properties of Calabi-Yau threefolds and conformal field theory. For instance, it was shown by Schimmrigk that, in certain cases, the algebraic number field defined via the fusion rules of a conformal field theory as the field defined by the eigenvalues of the integer valued fusion matrices

$$\phi_i * \phi_j = (N_i)_j^k \, \phi_k$$

can be recovered from the Hasse–Weil L-function of the Calabi–Yau. An interesting case is provided by the Gepner model associated with the Fermat quintic Calabi-Yau threefold.

3.2. **Arakelov Geometry.** For $\mathbb{K}$ a number field and $O_{\mathbb{K}}$ its ring of integers, a smooth proper algebraic curve X over $\mathbb{K}$ determines a smooth minimal model $X_{O_{\mathbb{K}}}$, which defines an arithmetic surface $\mathcal{X}_{O_{\mathbb{K}}}$ over $\mathrm{Spec}(O_{\mathbb{K}})$. The closed fiber $X_\wp$ of $\mathcal{X}_{O_{\mathbb{K}}}$ over a prime $\wp \in O_{\mathbb{K}}$ is given by the reduction mod $\wp$.
When $\mathrm{Spec}(O_{\mathbb{K}})$ is "compactified" by adding the archimedean primes, one can correspondingly enlarge the group of divisors on the arithmetic surface by adding formal real linear combinations of irreducible "closed vertical fibers at infinity". Such fibers are only treated as formal objects. The main idea of Arakelov geometry is that it is sufficient to work with "infinitesimal neighborhood" $X_\alpha(\mathbb{C})$ of these fibers, given by the Riemann surfaces obtained from the equation defining X over $\mathbb{K}$ under the embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$ that constitute the archimedean primes. Arakelov developed a consistent intersection theory on arithmetic surfaces, by computing the contribution of the archimedean primes to the intersection indices using hermitian metrics on these Riemann surfaces and the Green function of the Laplacian.
A general introduction to the subject of Arakelov geometry can be found in [14].
Manin showed in [15] that these Green functions can be computed in terms of geodesics in a hyperbolic 3-manifold that has the Riemann surface $X_\alpha(\mathbb{C})$ as its conformal boundary at infinity.

3.2.1. *The Polyakov measure.* A first application to physics of methods of Arakelov geometry was an explicit formula obtained by Beilinson and Manin [4] for the Polyakov bosonic string measure in terms of Faltings's height function at algebraic points of the moduli space of curves.

The partition function for the closed bosonic string has a perturbative expansion $Z = \sum_{g \geq 0} Z_g$, with

$$Z_g = e^{\beta(2-2g)} \int_\Sigma e^{-S(x,\gamma)}\, Dx\, D\gamma, \tag{3.1}$$

written in terms of a compact Riemann surface Σ of genus g, maps $x : \Sigma \to \mathbb{R}^d$, and metrics γ on Σ. The classical action is of the form

$$S(x,\gamma) = \int_\Sigma d^2z \, \sqrt{|\gamma|}\, \gamma^{ab} \partial_a x^\mu \, \partial_b x^\mu. \tag{3.2}$$

Using the invariance of the classical action with respect to the semidirect product of diffeomorphisms of Σ and the conformal group, the integral is reduced (in the critical dimension $d = 26$ where the conformal anomaly cancels) to a zeta regularized determinant of the Laplacian for the metric on Σ and an integration over the moduli space M_g of genus g algebraic curves. Beilinson and Manin gave an explicit formula for the resulting Polyakov measure on M_g using results of Faltings on Arakelov geometry of arithmetic surfaces. In particular, their argument uses essentially the properties of the Faltings metrics on the invertible sheaves $d(L)$ given by the "multiplicative Euler characteristics" of sheaves L of relative 1-forms. For a suitable choice of bases $\{\phi_j\}$ and $\{w_j\}$ of differentials and quadratic differentials, the formula for the Polyakov measure is then of the form (up to a multiplicative constant)

$$d\pi_g = |\det B|^{-18} (\det \Im \tau)^{-13} \; W_1 \wedge \bar{W}_1 \wedge \cdots \wedge W_{3g-3} \wedge \bar{W}_{3g-3}, \tag{3.3}$$

with τ in the Siegel upper half space, $B_{ij} = \int_{a_i} \phi_j$, and the W_j given by the images of the basis w_j under the Kodaira–Spencer isomorphism.

3.2.2. *Holography.* In the case of the elliptic curve $X_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$, a formula of Alvarez-Gaume, Moore, and Vafa gives the operator product expansion of the path integral for bosonic field theory as

$$g(z,1) = \log\left(|q|^{B_2(\log|z|/\log|q|)/2} |1-z| \prod_{n=1}^{\infty} |1-q^n z| \, |1-q^n z^{-1}| \right), \tag{3.4}$$

where B_2 is the second Bernoulli polynomial. The expression (3.4) is in fact the Arakelov Green function on $X_q(\mathbb{C})$ (*cf.* [14]).
Using this and analogous results for higher genus Riemann surfaces, Manin and Marcolli showed in [18] that the result of [15] on Arakelov and hyperbolic geometry can be rephrased in terms of the AdS/CFT correspondence, or holography principle. The expression (3.4) can then be written as a combination of terms involving geodesic lenghts in the Euclidean BTZ black hole.
In the case of higher genus curves, the Arakelov Green function on a compact Riemann surface, which is related to the two point correlation function for bosonic field theory, can be expressed in terms of the semiclassical limit of gravity (the geodesic propagator) on the bulk space of Euclidean versions of asymptotically AdS_{2+1} black holes introduced by Kirill Krasnov.

4. Motives

There are several cohomology theories for algebraic varieties: de Rham, Betti, étale cohomology. de Rham and Betti are related by the periods isomorphism and comparison isomorphisms relate étale and Betti cohomology. In the smooth projective case, they have the expected properties of Poincaré duality, Künneth isomorphisms, etc. Moreover, étale cohomology provides interesting ℓ-adic representations of $\mathrm{Gal}(\bar{k}/k)$. In order to understand what type of information, such as maps or operations can be transferred from one to another cohomology, Grothendieck introduced the idea of the existence of a "universal cohomology theory" with realization functors to all the known cohomology theories for algebraic varieties. He called this the theory of *motives*. Properties that can be transferred between different cohomology theories are those that exist at the motivic level. A short introduction to motives can be found in [23].

The first constructions of a category of motives proposed by Grothendieck covers the case of smooth projective varieties. The corresponding motives form a $\mathbb{Q}$-linear abelian category of *pure motives*. Roughly, objects are varieties and morphisms are *correspondences* given by algebraic cycles in the product, modulo a suitable equivalence relation. The category also contains Tate objects generated by $\mathbb{Q}(1)$, which is the inverse of the pure motive $H^2(\mathbb{P}^1)$. Grothendieck's standard conjectures imply that the category of pure motives is equivalent to the category of representations Rep_G of a *motivic Galois group*, which in the case of pure motives is pro-reductive. The subcategory of pure Tate motives has motivic Galois group the multiplicative group $\mathbb{G}_m$. The situation is more complicated for *mixed motives*, for which constructions were only very recently proposed (for instance in the work of Voevodsky). These provide a universal cohomology theory for more general classes of algebraic varieties. Mixed Tate motives are the subcategory generated by the Tate objects. There is again a motivic Galois group. For mixed motives it is an extension of a pro-reductive group by a pro-unipotent group, with the pro-reductive part coming from pure motives and the pro-unipotent part from the presence of a weight filtration on mixed motives. The multiple zeta values appear as periods of mixed Tate motives.

4.1. **Renormalization and motivic Galois theory.** A manifestation of motivic Galois groups in physics arises in the context of the Connes–Kreimer theory of perturbative renormalization (for an introduction to this topic see the relevant article in this Encyclopaedia). In fact, according to the Connes–Kreimer theory, the BPHZ (Bogoliubov–Parasiuk–Hepp–Zimmerman) renormalization scheme with dimensional regularization and minimal subtraction can be formulated mathematically in terms of the Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z) \tag{4.1}$$

of loops in a pro-unipotent Lie group G, which is the group of characters of the Hopf algebra of Feynman graphs. Here the loop γ is defined on a small punctured disk around the critical dimension D, γ_+ is holomorphic in a neighborhood of D and γ_- is holomorphic in the complement of D in $\mathbb{P}^1(\mathbb{C})$. The renormalized value is given by $\gamma_+(D)$ and the counterterms by $\gamma_-(z)$.

In [8], Connes and Marcolli showed that the data of the Birkhoff factorization are equivalently described in terms of solutions to a certain class of differential systems with irregular singularities. This is obtained by writing the terms in the Birkhoff factorization as time ordered exponentials, and then using the fact that

$$\mathrm{T}e^{\int_a^b \alpha(t)\,dt} := 1 + \sum_{n=1}^{\infty} \int_{a \leq s_1 \leq \cdots \leq s_n \leq b} \alpha(s_1) \cdots \alpha(s_n)\; ds_1 \cdots ds_n$$

is the value $g(b)$ at b of the unique solution $g(t) \in G$ with value $g(a) = 1$ of the differential equation $dg(t) = g(t)\,\alpha(t)\,dt$.
The type of singularities are specified by physical conditions, such as the independence of the counterterms on the mass scale. These conditions are expressed geometrically through the notion of G-valued *equisingular connections* on a principal $\mathbb{C}^*$-bundle B over a disk Δ, where G is the pro-unipotent Lie group of characters of the Connes–Kreimer Hopf algebra of Feynman graphs. The *equisingularity* condition is the property that such a connection ω is $\mathbb{C}^*$-invariant and that its restrictions to sections of the principal bundle that agree at $0 \in \Delta$ are mutually equivalent, in the sense that they are related by a gauge transformation by a G-valued $\mathbb{C}^*$-invariant map regular in B, hence they have the same type of (irregular) singularity at the origin.
The classification of equivalence classes of these differential systems via the Riemann–Hilbert correspondence and differential Galois theory yields a Galois group $U^* = U \rtimes \mathbb{G}_m$, where U is pro-unipotent, with Lie algebra the free graded Lie algebra with one generator e_{-n} in each degree $n \in \mathbb{N}$. The group U^* is identified with the motivic Galois group of mixed Tate motives over the cyclotomic ring $\mathbb{Z}[e^{2\pi i/N}]$, for $N = 3$ or $N = 4$, localized at N.

4.2. **Speculations on arithmetical physics.** In a lecture written for the 25th Arbeitstagung in Bonn, Yuri Manin presented intriguing connections between arithmetic geometry (especially Arakelov geometry) and physics [16]. The theme is also discussed in his "reflections on arithmetical physics" [17]. These considerations are based on a philosophical viewpoint according to which fundamental physics might, like adeles, have archimedean (real or complex) as well as non-archimedean (p-adic) manifestations. Since adelic objects are more fundamental and often simpler than their archimedean components, one can hope to use this point of view in order to carry over some computation of physical relevance to the non-archimedean side where one can employ number theoretic methods.

4.2.1. *Adelic physics?* Some of the results mentioned in the previous sections seem to lend themselves well to this adelic interpretation. The quantum statistical mechanics of $\mathbb{Q}$-lattices relies fundamentally on adeles and it admits generalizations to systems associated to other algebraic varieties (Shimura varieties) that have an adelic description and adelic groups of symmetries. The result on the Polyakov measure also has an adelic flavor, in as it uses essentially the archimedean component of the Faltings height function. The latter is in fact a product of contributions from all the archimedean and non-archimedean places of the field of definition of algebraic points in the moduli space, so that one can expect that there would be an adelic Polyakov measure, of which one normally sees the archimedean side only. The Freund–Witten adelic product formula for the Veneziano string amplitude fits in the same context, with p-adic amplitudes

$$B_p(\alpha, \beta) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\, |1-x|_p^{\beta-1}\, dx$$

and $B_\infty(\alpha,\beta)^{-1} = \prod_p B_p(\alpha,\beta)$, *cf.* [25].

4.2.2. *Adelic physics and motives.* A similar adelic philosophy was taken up by other authors, who proposed ways of introducing non-archimedean and adelic geometries in quantum physics. A recent survey is given in [25]. For instance, Volovich [26] proposed space-time models based on cohomological realizations of motives, with étale topology "interpolating" between a proposed non-Archimedean geometry at the Planck scale and Euclidean geometry at the macroscopic scale. In this viewpoint, motivic L-functions appear as partition functions and actions of motivic Galois groups govern the dynamics.

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