Mathematics for Economists

mainly optimization

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Unconstrained local necessary conditions

In the unconstrained case f is defined on an open set. Use the "variations" $g(t) \doteq f(x_0 + th)$ and chain rule.

Theorem

If x_0 is a local minimum then

1. $0 = f'_x(x_0) = (\partial f / \partial x_1 (x_0), \ldots, \partial f / \partial x_n (x_0))$.
2. $0 \leq h^T f''_{xx}(x_0) h = \sum_i \sum_j \partial^2 f / \partial x_i \partial x_j (x_0) h_i h_j$

Observe that one needs just twice differentiability at x_0.

Unconstrained local sufficient conditions

Theorem

If f is twice differentiable at x_0 and

$$f'_x(x_0) = 0$$

and the second derivative is

1. *positive definite then x_0 is a local minimum of f,*
2. *negative definite then x_0 is a local minimum of f.*

Observe that one needs just twice differentiability at x_0.

Unconstrained local sufficient conditions

Lemma

If $x^T Hx$ is a positive definite quadratic form on $\mathbb{R}^n$ then there is an $\alpha > 0$ such that

$$x^T Hx \geq \alpha \|x\|^2 .$$

Let $\alpha > 0$ be the minimum of $x^T Hx$ over the compact set $\|x\| = 1$. Obviously

$$\frac{x}{\|x\|}^T H \frac{x}{\|x\|} \geq \alpha > 0$$

which implies the lemma.

Unconstrained local sufficient conditions

As f is twice differentiable at x_0

$$f(x_0+h)-f(x_0)=\left\langle f'_x(x_0), h\right\rangle+\frac{1}{2}h^T f''_{xx}(x_0)\,h+o\left(\|h\|^2\right).$$

Let $H=f''_{xx}(x_0)$ and let $\alpha>0$ be the constant above. For any $\varepsilon<\alpha/2$ one has that

$$\left|o\left(\|h\|^2\right)\right|\leq\varepsilon\|h\|^2$$

for h small enough. Hence as $f'_x(x_0)=0$

$$f(x_0+h)-f(x_0)\geq\left(\frac{\alpha}{2}-\varepsilon\right)\|h\|^2\geq 0.$$

Constrained local necessary conditions

Theorem

Assume that φ_k $k = 0, 1, 2, \ldots, p$ *are differentiable and assume that* φ_0 *has a local minimum at* x_0 *on the set*

$$X \mathrel{\hat{=}} \{x \mid \varphi_k(x) = 0, k = 1, \ldots, p\}.$$

Then there are multipliers

$$l = (\lambda_0, \lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{p+1}$$

such that

$$\sum_{k=0}^{p} \lambda_k \varphi'_k(x_0) = 0.$$

If $\varphi'_k(x_0)$, $k = 1, \ldots, p$ *are linearly independent then* $\lambda_0 = 1$ *is possible.*

Constrained local necessary conditions

Theorem

Assume that we are in the regular case and φ_k, $k = 0, 1, \ldots, p$ are twice differentiable at x_0, where x_0 is a local minimum of the constrained optimization problem then

$$\left\langle \varphi_k'(x_0), h \right\rangle = 0, \qquad k = 1, 2, \ldots, p \Longrightarrow h^T L_{xx}''(x_0, \lambda) h \geq 0.$$

For local maximums one has

$$\left\langle \varphi_k'(x_0), h \right\rangle = 0, \qquad k = 1, 2, \ldots, p \Longrightarrow h^T L_{xx}''(x_0, \lambda) h \leq 0.$$

Recall that $L_{xx}''(x_0, \lambda)$ is the second derivative of the Lagrange function with respect to x at x_0.

Constrained local sufficient conditions

Theorem

Assume $L'_x(x_0, \lambda) = 0$, *that is let* x_0 *be a stationary point of the Lagrangian. Assume that we are in the regular case and* $\varphi_k, k = 0, 1, \ldots, p$ *are twice differentiable at* x_0. *If*

$$h^T L''_{xx}(x_0, \lambda) h > 0, \qquad h \neq 0$$

whenever $\langle \varphi'_k(x_0), h \rangle = 0$, $k = 1, 2, \ldots, p$ *then* x_0 *is a local minimum of the constrained optimization problem. If*

$$h^T L''_{xx}(x_0, \lambda) h < 0, \qquad h \neq 0$$

whenever $\langle \varphi'_k(x_0), h \rangle = 0$, $k = 1, 2, \ldots, p$ *then* x_0 *is a local maximum of the constrained optimization problem.*

Constrained local sufficient conditions

Let F be the vector function of the constraints and let $C \circeq \{x \mid F(x) = 0\}$. It is sufficient to show that there is a function K such that K has a local minimum on C at x_0 with U and $K(x_0) = \varphi_0(x_0)$ and $K(x) \leq \varphi_0(x), x \in U$. In this case if $x \in U \cap C$ then

$$\varphi_0(x) \geq K(x) \geq K(x_0) = \varphi_0(x_0)$$

so x_0 is a local minimum of φ_0 on C. Let

$$K(x) \circeq \varphi_0(x_0) - \langle \lambda, F(x) \rangle - \gamma \|F(x)\|^2$$

where γ is a large enough constant. Obviously if $x \in C$ then $F(x) = 0$ therefore $K(x_0) = \varphi_0(x_0)$ as $x_0 \in C$.

Constrained local sufficient conditions

One must show that

$$\varphi_0(x) \geq K(x) = \varphi_0(x_0) - \langle \lambda, F(x) \rangle - \gamma \|F(x)\|^2$$

That is

$$\begin{aligned} 0 &\geq \varphi_0(x_0) - (\varphi_0(x) + \langle \lambda, F(x) \rangle) - \gamma \|F(x)\|^2 = \\ &= \varphi_0(x_0) - L(x, \lambda) - \gamma \|F(x)\|^2 . \end{aligned}$$

Using the condition on the existence of the second derivative and the stationarity condition $L'_x(x_0, \lambda) = 0$ and that as x_0 is a feasible solution $L(x_0, \lambda) = \varphi_0(x_0)$ one must show that

$$\begin{aligned} 0 \geq & -(x - x_0)^T L''_{xx}(x_0, \lambda)(x - x_0) + o\left(\|x - x_0\|^2\right) - \\ & -\gamma \left\|F'(x_0)(x - x_0) + o(\|x - x_0\|)\right\|^2 . \end{aligned}$$

Obviously

$$\begin{gathered}\left\|F'\left(x_0\right)\left(x-x_0\right)+o\left(\left\|x-x_0\right\|\right)\right\|^2= \\ \left\|F'\left(x_0\right)\left(x-x_0\right)\right\|^2+ \\ +\left\|o\left(\left\|x-x_0\right\|\right)\right\|^2+ \\ +2\left\langle F'\left(x_0\right)\left(x-x_0\right), o\left(\left\|x-x_0\right\|\right)\right\rangle= \\ =\left\|F'\left(x_0\right)\left(x-x_0\right)\right\|^2+o\left(\left\|x-x_0\right\|^2\right).\end{gathered}$$

Constrained local sufficient conditions

By the conditions that

$$F'(x_0)\,h = 0 \Rightarrow h^T L''_{xx}(x_0, \lambda)\,h > 0$$

if γ is large enough and $h \neq 0$ the quadratic form

$$Q(h) \circeq h^T L''_{xx}(x_0, \lambda)\,h + \gamma \left\| F'(x_0)\,h \right\|^2 > 0.$$

Hence for some U around x_0

$$-Q(x - x_0) + o\left(\|x - x_0\|^2\right) \leq 0.$$

Constrained positive definite matrixes

Problem

Let $\mathbf{A}$ *be symmetric. When does* $x^T\mathbf{A}x \geq 0$ *for every* $\mathbf{B}x = 0$? *Or when does* $x^T\mathbf{A}x > 0$ *for every* $\mathbf{B}x = 0, x \neq 0$?

We can assume that $\mathbf{B}$ is fat and full rank. Solving the homogeneous equation $x_2 = -\mathbf{D}x_1$ where x_1 is the vector of free variables. If the number of the equations is p then the dimension of x_1 is $n-p$. Hence

$$\ker(\mathbf{B}) = \left\{x = \mathbf{P}y \mid y \in \mathbb{R}^{n-p}\right\} = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ -\mathbf{D} \end{pmatrix} y \mid y \in \mathbb{R}^{n-p}\right\}.$$

Hence one should study

$$x^T\mathbf{A}x = y^T\mathbf{P}^T\mathbf{A}\mathbf{P}y \overset{\circ}{=} y^T\mathbf{C}y, \qquad \dim\left(\mathbf{P}^T\mathbf{A}\mathbf{P}\right) = \dim(\mathbf{C}) = n-p$$

as $\operatorname{size}(\mathbf{P}) = n \times (n-p)$.

Constrained positive definite matrixes

One can use the Jacobi–Sylvester criteria on $\mathbf{C}$. To solve the sufficiency problem one should check the determinants of the $n-p$ leading principal minors of $\mathbf{C}$.

But to do this one should solve the equation $\mathbf{Bx} = \mathbf{0}$ and construct matrix $\mathbf{C}$. We want to avoid this and we will calculate the LAST $n-p$ leading principal determinant of the bordered matrix

$$\mathbf{H} \doteq \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{pmatrix}.$$

The size of $\mathbf{H}$ is $(n+p) \times (n+p)$. (It is not obvious why does it work. The proof is in the reader.)

Constrained positive definite matrixes

After some calculation one gets that:

1. Write the basis vectors of $\mathbf{B}$ first. $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2) = (\mathbf{B}_1, \mathbf{B}_1\mathbf{U})$, where $\mathbf{B}_1$ forms a basis for the column space of $\mathbf{B}$ and $\mathbf{U}$ are the coordinates of the other columns in $\mathbf{B}_2$.
2. Form the bordered Hessian

$$\begin{pmatrix} \underset{p \times p}{\mathbf{0}} & \underset{(p \times n)}{\mathbf{B}} \\ \underset{(n \times p)}{\mathbf{B}^T} & \underset{(n \times n)}{\mathbf{A}} \end{pmatrix}$$

3. Check the determinants of the last $n - p$ leading principal minors.
4. The sign of these determinants must be $(-1)^p$ in the constrained positive definite(minimum) case. In the negative definite (maximum) case the signs must alter staring with $(-1)^n$ for the largest minor.

Constrained positive definite matrixes

Example

Solve $x_1^2 + x_2^2 \to \min$, $x_1 + x_2 = 1$.

The Lagrangian is $L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 1)$.
The necessary condition

$$\begin{aligned}
2x_1 + \lambda &= 0, 2x_2 + \lambda = 0 \Rightarrow x_1 = x_2, \Rightarrow x_1 = x_2 = \frac{1}{2}. \\
\mathbf{A} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}
\end{aligned}$$

$\det(\mathbf{H}) = -4$. One should check $2 - 1$ determinants and for minimum the sign must be $-1 = (-1)^p$ as there is just one constraint, $p = 1$. Hence we have a local minimum.

Constrained positive definite matrixes

Example

Solve $x_1^2 + x_2^2 \to \min$, $x_1 + x_2 = 1, 2x_1 + 3x_2 = 4$.

The only solution is

$$x_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = -1, x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = 2.$$

But as one should check $n - p = 0$ determinants there is nothing to check. In this case the theorem formally does not applicable.

Constrained positive definite matrixes

Example

Solve $x_1 + x_2 \to \min, x_1^2 + x_2^2 = 1$.

The Lagrangian is

$$L(x_1, x_2, \lambda) = x_1 + x_2 + \lambda\left(x_1^2 + x_2^2 - 1\right).$$

The necessary conditions

$$1 + 2\lambda x_1 = 0, 1 + 2\lambda x_2 = 0, \Rightarrow x_1 = x_2, 2x_1^2 = 1, x_1 = \pm 1/\sqrt{2}.$$

Constrained positive definite matrixes

If $x_1 = x_2 = 1/\sqrt{2}$ then $\lambda = -1/\sqrt{2}$

$$\mathbf{A} = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \mathbf{B} = \left(2/\sqrt{2}, 2/\sqrt{2}\right), \mathbf{H} = \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \end{pmatrix}$$

$\det(\mathbf{H}) = 4\sqrt{2}$. As there are two variables this is the condition for local maximum, so $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a local maximum.

Constrained positive definite matrixes

If $x_1 = x_2 = -1/\sqrt{2}$ then $\lambda = 1/\sqrt{2}$

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \mathbf{B} = \left(-\sqrt{2}, -\sqrt{2}\right), \mathbf{H} = \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \end{pmatrix}$$

determinant: $-4\sqrt{2}$. As $p = 1$ this is the condition for the minimum so $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a local minimum.

Constrained positive definite matrixes

Example

Solve $x_1 x_2 \to \min, x_1^2 + x_2^2 = 1$.

The Lagrangian is

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda \left(x_1^2 + x_2^2 - 1 \right).$$

The necessary conditions

$$x_2 + 2\lambda x_1 = 0, x_1 + 2\lambda x_2 = 0.$$

If $x_1 = 0$ then $x_2 = 0$ which is not a solution. So $x_1 \neq 0, x_2 \neq 0$.

$$\frac{x_2}{x_1} = -2\lambda, \frac{x_1}{x_2} = -2\lambda, \frac{x_2}{x_1} = \frac{x_1}{x_2}, x_1^2 = x_2^2$$

Hence the four solutions are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Constrained positive definite matrixes

The Hessian of the Lagrangian is

$$\mathbf{A} = \begin{pmatrix} 2\lambda & 1 \\ 1 & 2\lambda \end{pmatrix}, \mathbf{B} = (2x_1, 2x_2).$$

For the first root $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \lambda = -1/2$

$$\mathbf{H} = \begin{pmatrix} 0 & 2/\sqrt{2} & 2/\sqrt{2} \\ 2/\sqrt{2} & -1 & 1 \\ 2/\sqrt{2} & 1 & -1 \end{pmatrix}$$

$\det(\mathbf{H}) = 8$ so as $n = 2$ the sign for maximum is $(-1)^n = 1$, hence $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a local maximum.

Constrained positive definite matrixes

For the third root $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \lambda = 1/2$

$$\mathbf{H} = \begin{pmatrix} 0 & -2/\sqrt{2} & 2/\sqrt{2} \\ -2/\sqrt{2} & 1 & 1 \\ 2/\sqrt{2} & 1 & 1 \end{pmatrix}$$

$\det(\mathbf{H}) = -8$. As $p = 1$ the sign for minimum must be $-1 = (-1)^p$, so $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a local minimum.

Constrained positive definite matrixes

Example

Solve the problem $x_1^2 + x_2^2 + x_3^2 \to \min, x_1 + x_2 + x_3 = 1$.

The Lagrangian is

$$L(x_1, x_2, x_3, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda(x_1 + x_2 + x_3 - 1).$$

Obviously $x_1 = x_2 = x_3 = 1/3$ is the solution.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

The determinant of $\mathbf{H}$ is -12.

Constrained positive definite matrixes

But we must calculate $3 - 1 = 2$ determinants: Deleting the last row and the last column

$$\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} = -4.$$

As both determinants are negative, $p = 1$, so the sign for the minimum is $(-1)^p = -1$ this is a local minimum.

Example

Solve the problem $x_1x_2x_3 \to \max, x_1^2 + x_2^2 + x_3^2 = 1$

The Lagrangian $x_1x_2x_3 + \lambda\left(x_1^2 + x_2^2 + x_3^2 - 1\right)$. The first order conditions

$$\frac{x_1x_2}{2x_3} = \frac{x_2x_3}{2x_1} = \frac{x_1x_3}{2x_2} = -\lambda,$$

Obviously the solutions of the first order conditions are

$$\begin{aligned} x_1^2 &= x_2^2 = x_3^2 = \frac{1}{3}. \\ \mathbf{A} &= \begin{pmatrix} 2\lambda & x_3 & x_2 \\ x_3 & 2\lambda & x_1 \\ x_2 & x_1 & 2\lambda \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_1 \end{pmatrix}. \end{aligned}$$

Constrained positive definite matrixes

A maximal solution is when $x_1 = x_2 = x_3 = 1/\sqrt{3}$ in this case

$$\lambda = -\frac{x_1 x_2}{2x_3} = -\frac{1}{2\sqrt{3}}$$

$$\mathbf{H} = \begin{pmatrix} 0 & 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

The determinant is $-\frac{16}{3}$. As there are three variables this is the right sign condition for the maximum as $-1 = (-1)^3$

Constrained positive definite matrixes

But we should also check another determinant as $n - p = 2$. Deleting the last row and column

$$\det \begin{pmatrix} 0 & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & .-1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} = \frac{16}{9}\sqrt{3}$$

Which is still good for the maximum as the sign is alternating.

Constrained positive definite matrixes

Now check the $x_1 = x_2 = x_3 = -1/\sqrt{3}$ which is the minimal solution.

$$\lambda = -\frac{x_1 x_2}{2x_3} = \frac{1}{2\sqrt{3}}$$

$$\mathbf{H} = \begin{pmatrix} 0 & -2/\sqrt{3} & -2/\sqrt{3} & -2/\sqrt{3} \\ -2/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ -2/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ -2/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}.$$

In this case the determinant is still $-\frac{16}{3}$, which is also compatible with the minimum rule as $p = 1$. (Observe that sign is compatible with maximum criteria as well.)

Constrained positive definite matrixes

But we should also check another determinant as $n - p = 2$. Deleting the last row and last column

$$\det\begin{pmatrix} 0 & -2/\sqrt{3} & -2/\sqrt{3} \\ -2/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ -2/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} = -\frac{16}{9}\sqrt{3}$$

Which is the right sign for the minimum condition as $p = 1$ and in the minimum case there is no alternation.

Constrained positive definite matrixes

Observe that $x_2 = x_3 = 0, x_1 = 1, \lambda = 0$ is also a solution of the equations

$$\begin{aligned} x_2 x_3 + 2\lambda x_1 &= 0 \\ x_1 x_3 + 2\lambda x_2 &= 0 \\ x_1 x_2 + 2\lambda x_3 &= 0 \end{aligned}$$

$$\mathbf{H} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The determinant is 4 which is not compatible with the max or the min rule.

Constrained positive definite matrixes

Example

Solve $x_1^2 - x_2^2 - x_3^2 \to \max, x_1^2 + x_2^2 + x_3^2 = 1.$

The Lagrangian

$$x_1^2 - x_2^2 - x_3^2 + \lambda \left(x_1^2 + x_2^2 + x_3^2 - 1 \right).$$

The necessary condition is

$$\begin{aligned} 2x_1 + \lambda 2x_1 &= 2x_1 (1 + \lambda) = 0 \\ -2x_2 + \lambda 2x_2 &= 2x_2 (-1 + \lambda) = 0 \\ -2x_3 + 2\lambda x_3 &= 2x_3 (-1 + \lambda) = 0. \end{aligned}$$

The relevant solution for the maximum is $x_1 = \pm 1, x_2 = x_3 = 0$.

Constrained positive definite matrixes

The Hessian

$$\mathbf{A}=\begin{pmatrix} 2+2\lambda & 0 & 0 \\ 0 & -2+2\lambda & 0 \\ 0 & 0 & -2+2\lambda \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \end{pmatrix}.$$

If $x_1 = 1, x_2 = 0, x_3 = 0$, then $\lambda = -1$

$$\mathbf{H}=\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

The determinant is -64. As there are three variables for maximum the sign must be $(-1)^3$.

Constrained positive definite matrixes

But we must also check

$$\begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 16$$

:Hence the determinants of the leading principal minors alternate, hence $(1,0,0)$ is a local maximum.

Constrained positive definite matrixes

Now let $x_1 = 0, x_2 = 1, x_3 = 0$ and $\lambda = 1$, which solves the equations.

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The determinant is 0. The problem comes from the fact that $(0, 1, 0)$ is not a isolated local minimum as $x_1 = 0, x_2^2 + x_3^2 = 1$ are all minimal solutions.

Constrained positive definite matrixes

Example

Solve $x_3^2 - x_1^2 - x_2^2 \to \max, x_1^2 + x_2^2 + x_3^2 = 1.$

The solution is $x_3 = 1$, $x_1 = x_2 = 0$. The (wrong) bordered Hessian is

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

The determinant is -64 which is fine. But $\begin{vmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 0$. (The problem is that the basis is at a wrong place.)

Example

Consumer saving problem:

1. The planning period is finite $t = 1, 2, \ldots, T$.
2. At every period t the consumer has wealth $w_t \geq 0$ and consumption $c_t \geq 0$. The utility function for the consumer is $u(c) = \sqrt{c}$.
3. At every t the set of feasible decisions are $\Phi(w_{t-1}) = [0, w_{t-1}]$.
4. There is a production function $f(w, c) = (w - c) \cdot (1 + r) \stackrel{\circ}{=} k \cdot (w - c)$ with $r \geq 0$.
5. The consumer is maximizing the aggregate utility $\sum_{t=0}^{T} u(c_t)$.

Dynamic programming

Problem

We want to maximize the aggregate reward

$$\sum_{t=1}^{T} r_t(s_t, a_t) \to \max,$$

under the conditions that

$$\begin{aligned} s_1 &\in S \\ s_t &= f_{t-1}(s_{t-1}, a_{t-1}), t = 2, \ldots, T \\ a_t &\in \Phi(s_t) \subseteq A, t = 1, 2, \ldots, T \end{aligned}$$

where a_t is the action one can choose and s_t is the state of the system. Obviously the objects in the problem are given before the optimization and are parameters of the problem.

Dynamic programming

Definition

1. S is called the state space.
2. A is called the action space.
3. $r_t(s, a)$ is the reward function at time period t.
4. $f_t(s, a)$ is the transition function at time period t.
5. $\Phi_t(s) \to 2^A$ is the correspondence of feasible actions at time period t.

Dynamic programming

Definition

1. The history h_t at time t is the sequence $h_t = (s_1, a_1, \ldots a_{t-1}, s_t)$. (Observe that there is no a_t but there is an s_t in the definition of the history.)
2. Strategy σ is a sequence of mappings σ_t.which gives the next action $a_t \in \Phi(s_t)$. The σ_t can depend on the whole actual history.h_t.
3. Σ denotes the set of strategies.
4. The value function is

$$V(s) \circeq \max_{\sigma \in \Sigma, \sigma_0 = s} W(\sigma) \circeq \max_{\sigma \in \Sigma, \sigma_0 = s} \sum_{t=1}^{T} r_t(\sigma_t).$$

   if the maximum is attained otherwise we write sup instead of max.
5. One can also define $V_t(s)$ as $V(s)$ above just we start the optimization at time period t.

Dynamic programming

Definition

A strategy is a Markovian strategy if the strategy is dependent only on the present state, that is independent of the history of path how we have got to the present state s.

Theorem (Bellman's principle)

Under some conditions on the model there is an optimal Markovian strategy and the sequence of value functions V_t satisfies the Bellman equation

$$V_t(s) = \max_{a \in \Phi_t(s)} \left\{ r_t(s, a) + V_{t+1}(f_t(s, a)) \right\}.$$

Dynamic programming

The conditions of the above theorem are quite reasonable. They are (just) guaranteeing the existence of the optimum.

1. $r_t(s, a)$ is continuous on $S \times A$ for every t.
2. $f_t(s, a)$ is continuous on $S \times A$ for every t.
3. Φ_t is a continuous, non-empty and compact valued correspondence.

Dynamic programming

Definition

A set-valued mapping, that is a correspondence, Φ is called

1. upper semi continuous if for every $x_n \to x, y_n \to y$ and $y_n \in \Phi(x_n)$ one gets that $y \in \Phi(x)$. This says that the

$$\text{Graph}(\Phi) \triangleq \{(x, y) \mid y \in \Phi(x)\}$$

   is a closed set;

2. lower semi continuous if for every $x_n \to x$ and $y \in \Phi(x)$ there is a sequence $y_n \in \Phi(x_n)$ such that $y_n \to y$;
3. it is continuous if both upper- and lower semi continuous.

Dynamic programming

The main advantage of the compact valued and continuous correspondences is that the parametric optimization problem

$$g(s) \circeq \max\left\{U(s,x) \mid x \in \Phi(s)\right\}$$

is a continuous function of the parameter s assuming that U and Φ are continuous and Φ is compact valued. In this case the correspondence

$$\Psi(s) \circeq \{x \mid x \in \Phi(s), U(s,x) = g(s)\}$$

is upper semi continuous.

1. First let us find the solution of the problem at $t = T$. In this case the state variable is the wealth w the correspondence

$$\Phi(w) = [0, w].$$

As $u(c) = \sqrt{c}$ the value function at time T is

$$V_T(w) = \max\{u(c) \mid c \in \Phi(T)\} = \sqrt{w}.$$

The optimal strategy is

$$\sigma_T(w) = w.$$

2. Now we move to time period $T-1$. By Bellman's principle one should solve the problem

$$\begin{aligned} V_{T-1}(w) &= \max_{c\in[0,w]}\left\{\sqrt{c}+V_T(k(w-c))\right\} = \\ &= \max_{c\in[0,w]}\left\{\sqrt{c}+\sqrt{k(w-c)}\right\}. \end{aligned}$$

Obviously the goal function is concave as $\sqrt{c}$ is concave, $k(w-c)$ is linear hence concave and the $x\mapsto\sqrt{x}$ is concave and increasing. So it is a convex KT-problem.

$$\frac{d}{dc}\left(\sqrt{c}+\sqrt{k(w-c)}\right)=\frac{1}{2\sqrt{c}}+\frac{1}{2\sqrt{k(w-c)}}(-k)=0$$

$$\frac{1}{\sqrt{c}}=\frac{\sqrt{k}}{\sqrt{w-c}}, ck=w-c$$

$$\sigma_{T-1}(w)=c=\frac{w}{1+k}.$$

In this case the value of the goal function is

$$\sqrt{\frac{w}{k+1}}+\sqrt{k\left(w-\frac{w}{k+1}\right)}=\sqrt{\frac{w}{k+1}}+\sqrt{k\left(\frac{w\left(k+1\right)-w}{k+1}\right)}=$$

$$=\sqrt{\frac{w}{k+1}}+\sqrt{\frac{k^2w}{k+1}}=\left(1+k\right)\sqrt{\frac{w}{k+1}}=\sqrt{1+k}\sqrt{w}.$$

But we should also check $c=0$ and $c=w$. If $c=0$ then

$$\sqrt{kw}<\sqrt{1+k}\sqrt{w},$$

if $c=w$ then

$$\sqrt{w}<\sqrt{1+k}\sqrt{w}$$

so

$$V_{T-1}\left(w\right)=\sqrt{1+k}\sqrt{w}$$

3. It gives us the induction hypothesis:

$$V_{T-t} = \sqrt{1+k+k^2+\ldots+k^t}\sqrt{w}$$
$$\sigma_{T-t} = \frac{w}{1+k+k^2+\ldots+k^t}$$

The only thing we should show is to prove this hypothesis:

$$V_{T-(t+1)}(w) = \max_{c\in[0,w]}\left\{\sqrt{c}+V_{T-t}(k(w-c))\right\} =$$
$$= \max_{c\in[0,w]}\left\{\sqrt{c}+\sqrt{1+k+k^2+\ldots+k^t}\sqrt{k(w-c)}\right\}.$$

Calculating the derivatives

$$
\begin{aligned}
\frac{1}{2\sqrt{c}} &= \sqrt{1+k+k^2+\ldots+k^t}\,\frac{k}{2\sqrt{k\left(w-c\right)}} \\
\sqrt{c} &= \frac{1}{\sqrt{1+k+k^2+\ldots+k^t}}\,\frac{\sqrt{w-c}}{\sqrt{k}} \\
kc &= \frac{1}{1+k+k^2+\ldots+k^t}\left(w-c\right) \\
c\left(k+\frac{1}{1+k+k^2+\ldots+k^t}\right) &= \frac{w}{1+k+k^2+\ldots+k^t} \\
c\,\frac{k+\ldots+k^{t+1}+1}{1+k+k^2+\ldots+k^t} &= \frac{w}{1+k+k^2+\ldots+k^t} \\
c &= \frac{w}{1+k+k^2+\ldots+k^t+k^{t+1}}.
\end{aligned}
$$

Dynamic programming

The value of the goal function is

$$V_{T-(t+1)}(w) = \sqrt{\frac{w}{\sum_{n=0}^{t+1} k^n}} + \sqrt{\sum_{n=0}^{t} k^n}\sqrt{k\left(w - \frac{w}{\sum_{n=0}^{t+1} k^n}\right)} =$$

$$= \sqrt{w}\left(\frac{1}{\sqrt{\sum_{n=0}^{t+1} k^n}} + \sqrt{\sum_{n=0}^{t} k^n}\sqrt{k\frac{\sum_{n=0}^{t+1} k^n - 1}{\sum_{n=0}^{t+1} k^n}}\right) =$$

$$= \sqrt{w}\left(\frac{1}{\sqrt{\sum_{n=0}^{t+1} k^n}} + \sqrt{\sum_{n=0}^{t} k^n}\sqrt{k\frac{\sum_{n=1}^{t+1} k^n}{\sum_{n=0}^{t+1} k^n}}\right) =$$

$$= \sqrt{w}\left(\frac{1}{\sqrt{\sum_{n=0}^{t+1} k^n}} + \sqrt{\sum_{n=1}^{t+1} k^n}\sqrt{\frac{\sum_{n=1}^{t+1} k^n}{\sum_{n=0}^{t+1} k^n}}\right) =$$

$$\sqrt{w}\frac{1}{\sqrt{\sum_{n=0}^{t+1} k^n}}\left(1 + \sum_{n=1}^{t+1} k^n\right) = \sqrt{w}\sqrt{\sum_{n=0}^{t+1} k^n}.$$

Cake eating problem

We have a cake of size w_1. We want to eat it in T periods. Our utility for the consumption plan $(c_t)_{t=1}^{T}$ is

$$\sum_{t=1}^{T} \beta^{t-1} u(c_t).$$

The constraints are

$$\sum_{t=1}^{T} c_t + w_{T+1} = w_1$$
$$w_{T+1} \geq 0, c_t \geq 0.$$

Cake eating problem

1. The utility function is twice continuously differentiable on $x > 0$.
2. $u'(c) > 0$. Increasing utility.
3. $u''(c) < 0$. Strictly concave utility.
4. $\lim_{c \searrow 0} u'(c) = \infty$. Excludes corner solutions. (Inada condition.)

Cake eating problem

As the set of feasible solutions is compact and the utility function is strictly concave there is a unique solution to the cake eating problem. We want to apply the cake eating problem for logarithmic type utility functions as well. Strictly speaking we cannot use this function as it is undefined at $x = 0$. One can observe that in this case the result still holds. One can use the definition $\ln 0 \overset{\circ}{=} -\infty$.

Cake eating problem

As the constraints are linear Slater's condition holds. The Lagrange function is

$$L = -\sum_{t=1}^{T} \beta^{t-1} u\left(c_t\right) + \lambda \left(w_1 - \sum_{t=1}^{T} c_t + w_{T+1} \right) - \sum_{t=1}^{T} \mu_t c_t - \varphi w_{T+1}.$$

Differentiating

$$\begin{aligned} \frac{\partial L}{\partial c_t} &= \lambda - \beta^{t-1} u'\left(c_t\right) - \mu_t = 0 \\ \frac{\partial L}{\partial w_{T+1}} &= \lambda - \varphi = 0. \end{aligned}$$

If $\varphi = 0$ then $\lambda = 0$, but in this case $\beta^{t-1} u'\left(c_t\right) = 0$ which is impossible as $u' > 0$. Hence $\varphi > 0$ which implies that $w_{T+1} = 0$. By the Inada condition $c_t > 0$ hence $\mu_t = 0$. So

$$\beta^{t_1 - 1} u'\left(c_{t_1}\right) = \lambda = \beta^{t_2 - 1} u'\left(c_{t_2}\right).$$

Cake eating problem

Definition

The relation

$$u'(c_t) = \beta u'(c_{t+1})$$

is called Euler equation.

Cake eating problem

Example

Solve the cake eating problem for the $u(x) \doteq \ln x$ function.

Observe that in this formulation one can use the logarithmic function as it is perfectly legitimate in the Kuhn-Tucker theory as in this case one has a convex open set U over which the whole story of optimization is considered. The Euler equation is

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}}, t = 1, 2, \ldots, T-1.$$

and we also have $\lambda > 0$ and $w_{T+1} = 0$ that is

$$\sum_{t=1}^{T} c_t = w_1.$$

Cake eating problem

It is easy to see that

$$\begin{aligned} c_1 &= \frac{w_1}{1+\beta+\ldots+\beta^{T-1}}, \\ c_2 &= \frac{\beta w_1}{1+\beta+\ldots+\beta^{T-1}}, \\ &\vdots \\ c_T &= \frac{\beta^{T-1} w_1}{1+\beta+\ldots+\beta^{T-1}} \end{aligned}$$

is a stationary point. As the problem is convex, it is the optimal solution.

Cake eating problem

We can solve the problem as a dynamic programming problem. With utility function $r_t(c_t) \circeq \beta^{t-1} u(c_t)$ and transition function $f(w, c) \circeq w - c$. The Bellman equation is

$$\begin{aligned} V_t(w) &= \max_{c \in [0,w]} \left(\beta^{t-1} u(c) + V_{t+1}(w-c)\right). \\ V_T(w) &= \beta^{T-1} u(w). \end{aligned}$$

Let $t = T - 1$. By the Inada condition we always have an interior solution so derivative of the function behind the maximum is zero:
$(c_T = w - c_{T-1})$

$$\beta^{T-2} u'(c) - V'_T(w-c) = 0.$$
$$\beta^{T-2} u'(c) - \beta^{T-1} u'(w-c) = 0$$
$$\beta^{T-2} u'(c_{T-1}) = \beta^{T-1} u'(w - c_{T-1}) = \beta^{T-1} u'(c_T)$$
$$u'(c_{T-1}) = \beta u'(c_T)$$

Cake eating problem

With backwards iteration for time $t = T - 2$

$$\begin{aligned} \beta^{T-3} u'(c_t) - V'_{T-1}(w_t - c_t) &= \beta^{T-3} u'(c_t) - V'_{T-1}(w_{T-1}) = 0 \\ \beta^{T-3} u'(c_{T-2}) &= V'_{T-1}(w_{T-1}). \end{aligned}$$

But how do we calculate

$$V'_{T-1}(w_{T-1})?$$

Cake eating problem

$$
\begin{aligned}
V_{T-1}(w) &= \max_{c\in[0,w]}\left(\beta^{T-2}u(c)+V_T(w-c)\right)= \\
&= \max_{c\in[0,w]}\left(\beta^{T-2}u(c)+\beta^{T-1}u(w-c)\right)= \\
&= \beta^{T-2}u(c(w))+\beta^{T-1}u(w-c(w)).
\end{aligned}
$$

By the envelope theorem if

$$f(p) \overset{\circ}{=} \max_x g(p,x) = g(p,x(p))$$

then

$$\frac{df(p)}{dp} = \frac{\partial g}{\partial p}(p,x(p)) + \frac{\partial g}{\partial x}(p,x(p))\frac{dx(p)}{dp} = \frac{\partial g}{\partial p}(p,x(p)),$$

Cake eating problem

Observe that the partial derivative is by the parameter of the goal function so with

$$\begin{aligned} g(p,x) &= \beta^{T-2}u(x)+\beta^{T-1}u(p-x) \\ \frac{\partial g}{\partial p}(p,x(p)) &= \beta^{T-1}u'(p-x(p)) \end{aligned}$$

that is

$$\frac{dV_{T-1}(w)}{dw}(w)=\beta^{T-1}u'(w-c(w)).$$

Which gives the Euler equation.

Cake eating problem

In the general case using the Fermat principle for the max

$$\beta^{T-t-1}u'\left(c_{T-t}\right)=V'_{T-t+1}\left(w_{T-t+1}\right)$$

By the Bellman equation

$$\begin{aligned}V_{T-t+1}\left(w\right) &= \max_{c\in[0,w]}\left(\beta^{T-t}u\left(c\right)+V_{T-t+2}\left(w-c\right)\right)= \\ &= \max_{u\in[0,w]}\left(\beta^{T-t}u\left(w-u\right)+V_{T-t+2}\left(u\right)\right).\end{aligned}$$

By the Envelope Theorem

$$V'_{T-t+1}\left(w\right)=\beta^{T-t}u'\left(w-u\left(w\right)\right)=\beta^{T-t}u'\left(c_{T-t+1}\right)$$

hence

$$\beta^{T-t-1}u'\left(c_{T-t}\right)=\beta^{T-t}u'\left(c_{T-t+1}\right)$$

Example

Solve the cake eating problem for the $u(x) \doteq \ln x$ function.

If $T = 2$ then the Euler equation is

$$\frac{1}{c_1} = \beta \frac{1}{c_2}.$$

By the constraint

$$c_1 + c_2 = w_1.$$

Solving the equation

$$c_1 = \frac{1}{1+\beta} w_1, \, c_2 = \frac{\beta}{1+\beta} w_1.$$

The value function is

$$\begin{aligned} V_1(w_1) &= \ln c_1 + \beta \ln c_2 = \ln \frac{1}{1+\beta} w_1 + \beta \ln \frac{\beta}{1+\beta} w_1 = \\ &= \ln \frac{1}{1+\beta} + \ln w_1 + \beta \ln w_1 + \beta \ln \frac{\beta}{1+\beta} \triangleq \\ &\triangleq A_2 + B_2 \ln w_1 . \end{aligned}$$

Cake eating problem

If $T = 3$ then by the Euler equations

$$\frac{1}{c_1} = \beta \frac{1}{c_2} = \beta^2 \frac{1}{c_3}$$

and the resource equation is

$$c_1 + c_2 + c_3 = w_1$$

This implies that

$$c_1 = \frac{w_1}{1 + \beta + \beta^2}, c_2 \frac{\beta w_1}{1 + \beta + \beta^2}, c_3 = \frac{\beta^2 w_1}{1 + \beta + \beta^2}$$

The value function is

$$V_1 = \ln c_1 + \beta \ln c_2 + \beta^2 \ln c_3 = A_3 + B_3 \ln w_1 .$$

Example

Solve the problem

$$\begin{aligned} \sum_{t=0}^{T} \beta^t U(c_t) &\rightarrow \max \\ c_t + k_{t+1} &= f(k_t). \end{aligned}$$

where k_0 is given.

Production saving model

The simplest way is to turn it to an unconstrained optimization problem that is

$$\sum_{t=0}^{T} \beta^t U\left(f\left(k_t\right) - k_{t+1}\right) \to \max$$

Production saving model

Consider the terms with k_{t+1} with $t < T$. It is in two terms of the sum

$$\beta^t U(f(k_t) - k_{t+1}) + \beta^{t+1} U(f(k_{t+1}) - k_{t+2})$$

Differentiating with respect to k_{t+1} and assuming that there is an internal solution

$$\beta^t U'(f(k_t) - k_{t+1})(-1) = \beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = 0$$

Definition

The second order difference equation

$$\begin{aligned} U'(f(k_t) - k_{t+1}) &= \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \\ k_0 &= a, k_{T+1} = 0 \end{aligned}$$

is the Euler equation for the production saving model.

Production saving model

We can get the same equation with dynamic programing. By the Bellman equation

$$\begin{aligned}
V_t(k_t) &= \max_{c_t\in[0,k_t]}\left(\beta^t U(c_t) + V_{t+1}(k_{t+1})\right) = \\
&= \max_{c_t\in[0,k_t]}\left(\beta^t U(c_t) + V_{t+1}(f(k_t) - c_t)\right) \\
V_{T+1} &= 0.
\end{aligned}$$

Assuming that no corner solution appears derivation by the control parameter c_t by Fermat's principle

$$\beta^t U'(c_t) + V'_{t+1}(f(k_t) - c_t)(-1) = 0,$$
$$\beta^t U'(c_t) = V'_{t+1}(f(k_t) - c_t) = V'_{t+1}(k_{t+1}).$$

To calculate $V'_{t+1}(k_{t+1})$ we want to use the envelope theorem. The parametric goal function is

$$\begin{aligned} g(k_{t+1}, c_{t+1}) &= \beta^{t+1} U(c_{t+1}) + V_{t+2}(k_{t+2}) = \\ &= \beta^{t+1} U(c_{t+1}) + V_{t+2}(f(k_{t+1}) - c_{t+1}). \end{aligned}$$

In the second case differentiating by the state parameter k_{t+1}

$$V'_{t+2}(f(k_{t+1}) - c_{t+1}) f'(k_{t+1})$$

which is again hopeless as it expresses V'_{t+1} with V'_{t+2}, in the first case k_{t+1} is not in the formula.

Production saving model

Again we rewrite the value function and introduce a new control parameter $k_{t+2} = f(k_{t+1}) - c_{t+1}$

$$\begin{aligned} V_{t+1}(k_{t+1}) &= \max_{c_{t+1}\in[0,k_{t+1}]} \left(\beta^{t+1} U(c_{t+1}) + V_{t+2}(k_{t+2})\right) = \\ &= \max_{k_{t+2}\in[0,f(k_{t+1})]} \left(\beta^{t+1} U(f(k_{t+1}) - k_{t+2}) + V_{t+2}(k_{t+2})\right) \end{aligned}$$

Differentiating the parametric goal function by the parameter k_{t+1} the second term's derivative is zero

$$V'_{t+1}(k_{t+1}) = \beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}).$$

Production saving model

Hence

$$\beta^t U'\left(f\left(k_t\right)-k_{t+1}\right)=\beta^t U'\left(c_t\right)=V'_{t+1}\left(k_{t+1}\right)=\beta^{t+1} U'\left(f\left(k_{t+1}\right)-k_{t+2}\right)$$

Hence

$$U'\left(f\left(k_t\right)-k_{t+1}\right)=\beta U'\left(f\left(k_{t+1}\right)-k_{t+2}\right) f'\left(k_{t+1}\right).$$

which is the same Euler equation for the production saving model.

Production saving model

We can also consider the problem as a Kuhn–Tucker problem

$$\begin{aligned} \sum_{t=0}^{T} \beta^t U(c_t) &\to \max \\ c_t + k_{t+1} - f(k_t) &\leq 0, \\ c_t, k_t &\geq 0 \end{aligned}$$

If we assume that if $x > 0$ then $f(x) > 0$ then Slater's condition holds
The Lagrange function is

$$\begin{aligned} L &= -\sum_{t=0}^{T} \beta^t U(c_t) + \sum_t \lambda_t (c_t + k_{t+1} - f(k_t)) - \\ &\quad - \sum_t \mu_t c_t - \sum_t \nu_t k_t. \end{aligned}$$

Production saving model

$$\frac{\partial L}{\partial c_t} = -\beta^t U'(c_t) + \lambda_t - \mu_t = 0.$$

By the Inada condition $\lambda_t > 0$. hence

$$c_t + k_{t+1} - f(k_t) = 0.$$

Production saving model

$$\frac{\partial L}{\partial k_t} = \lambda_{t-1} - \lambda_t f'(k_t) - \nu_t k_t = 0.$$

If we exclude corner solutions then $\nu_t = \mu_t = 0$ and then

$$\begin{aligned} \lambda_{t-1} &= \lambda_t f'(k_t) \\ \lambda_t &= \beta^t U'(c_t) \end{aligned}$$

Production saving model

$$\begin{aligned}
\beta^{t-1} U'(c_{t-1}) &= \beta^t U'(c_t) f'(k_t) \\
U'(c_{t-1}) &= \beta U'(c_t) f'(k_t) \\
U'(f(k_{t-1}) - k_t) &= \beta U'(f(k_t) - k_{t+1}) f'(k_t)
\end{aligned}$$

Production saving model

The Euler equation of the production saving model is

$$U'\left(c_t\right) = \beta U'\left(c_{t+1}\right) f'\left(k_{t+1}\right),$$

for the cake eating model the Euler equation is

$$U'\left(c_t\right) = \beta U'\left(c_{t+1}\right).$$

In the cake eating model there is no production so $f\left(k_t\right) = k_t$. From this it is clear that the production saving model is a generalization of the cake eating model. The cake eating model is a first order equation, but as the production saving model contains k_{t+1} it is a second order difference equation. with boundary conditions $k_0 = a, k_{T+1} = 0$ which is not easy to solve.

Production saving model

Example

Solve the production saving model with $U(c) = \ln c$, $f(k) = k^{\alpha}$.

As f should be concave hence $\alpha \leq 1$. Of course $\beta \leq 1$. The Euler equation is

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} f'(k_{t+1}) = \beta\alpha \frac{k_{t+1}^{\alpha-1}}{c_{t+1}}$$

Multiplying by k_{t+1}

$$\frac{k_{t+1}}{c_t} = \beta\alpha \frac{k_{t+1}^{\alpha}}{c_{t+1}} = \beta\alpha \frac{f(k_{t+1})}{c_{t+1}}.$$

Production saving model

Let $s_t \circeq k_{t+1} / k_t^{\alpha} = k_{t+1} / f(k_t)$ be the saving rate.

$$c_t = f(k_t) - k_{t+1}$$

hence

$$\frac{k_{t+1}}{c_t} = \frac{k_{t+1}}{f(k_t) - k_{t+1}} = \frac{k_{t+1}/f(k_t)}{1 - k_{t+1}/f(k_t)} = \frac{s_t}{1 - s_t}.$$

Also

$$s_{t+1} = \frac{k_{t+2}}{f(k_{t+1})} = \frac{f(k_{t+1}) - c_{t+1}}{f(k_{t+1})} = 1 - \frac{c_{t+1}}{f(k_{t+1})}$$

Production saving model

Hence the Euler equation is

$$\begin{aligned}
\frac{s_t}{1-s_t} &= \alpha\beta\frac{1}{1-s_{t+1}} \\
\frac{1-s_t}{s_t} &= \frac{1}{\alpha\beta}\left(1-s_{t+1}\right) \\
\alpha\beta\left(\frac{1}{s_t}-1\right) &= 1-s_{t+1} \\
s_{t+1} &= 1+\alpha\beta-\frac{\alpha\beta}{s_t}.
\end{aligned}$$

Production saving model

Or for backward iteration

$$
\begin{aligned}
\frac{1}{s_t} - 1 &= \frac{1 - s_{t+1}}{\alpha\beta} \\
\frac{1}{s_t} &= \frac{1 + \alpha\beta - s_{t+1}}{\alpha\beta} \\
s_t &= \frac{\alpha\beta}{1 + \alpha\beta - s_{t+1}}
\end{aligned}
$$

Production saving model

As $s_T = k_{T+1}/f(k_T) = 0$,

$$s_{T-1} = \frac{\alpha\beta}{1+\alpha\beta}$$

which implies with backward induction that

$$\begin{aligned}
s_{T-2} &= \frac{\alpha\beta}{1+\alpha\beta-\frac{\alpha\beta}{1+\alpha\beta}} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta+\alpha\beta(1+\alpha\beta)-\alpha\beta} = \\
&= \frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} \\
s_{T-t} &= \frac{\sum_{s=1}^{t}(\alpha\beta)^s}{1+\sum_{s=1}^{t}(\alpha\beta)^s}
\end{aligned}$$

Optimal stopping

Example

Optimal selling of a stock with independent offers.

Assume that we have $\xi_1, \xi_2, \ldots, \xi_T$ independent random offers for a stock. If we sell it in period t then we will get $\xi_t (1 + r)^{T-t}$ at the final period T. What is the optimal strategy?
One can think about this type of problems as a specific stochastic dynamic programing problem. At every time period we have just one possible action set: sell it or not. After we have sold the stock our set of strategies is already empty. Our goal is to maximize the expected payout.

Let $r = 0$ and let ξ uniformly distributed on $[0, 1]$ and let $T = 3$.
1. If you are in period $t = T = 3$, you have no choice, so your expected payout is $\mathbf{E}(\xi_3) = 1/2$.
2. If you are in period $t = T - 1 = 2$, then you should not take the variable if $\xi_2 < \mathbf{E}(\xi_3) = 1/2$ because if you wait one period more your expected payout is $1/2$, which is better. Your expected payout is

$$\begin{aligned}\mathbf{E}\left(\max\left(\frac{1}{2}, \xi_2\right)\right) &= \int_0^{1/2} \frac{1}{2} dx + \int_{1/2}^{1} x dx = \\ &= \frac{1}{4} + \left[\frac{x^2}{2}\right]_{1/2}^{1} = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8} = 0,625.\end{aligned}$$

Optimal stopping

3. If $\xi_1 < 5/8$ then you should not take ξ_1. Then you expected payout in this period

$$\begin{aligned}
\mathbf{E}\left(\max\left(\frac{5}{8}, \xi_1\right)\right) &= \int_0^{5/8} \frac{5}{8} dx + \int_{5/8}^1 x dx = \\
&= \frac{25}{64} + \left[\frac{x^2}{2}\right]_{5/8}^1 = \frac{25}{64} + \frac{1}{2} - \frac{1}{2}\frac{25}{64} = \\
&= \frac{1}{2} + \frac{1}{2}\frac{25}{64} = \frac{64+25}{128} = \frac{89}{128} = 0,695\,31.
\end{aligned}$$

Optimal stopping

The general iteration is $\alpha_T = 0$.

$$\begin{aligned}\alpha_t &= \mathbf{E}\left(\max\left(\alpha_{t+1}, \xi_t\right)\right) = \int_0^{\alpha_{t+1}} \alpha_{t+1} dx + \int_{\alpha_{t+1}}^1 x dx = \\ &= \alpha_{t+1}^2 + \frac{1}{2}\left(1 - \alpha_{t+1}^2\right) = \frac{1}{2}\left(1 + \alpha_{t+1}^2\right).\end{aligned}$$

Obviously from the construction $\alpha_t > \alpha_{t+1}$ as we are always increasing the integrand.
Is this the optimal strategy?

Optimal stopping

Now assume that $r = 100\%$ and let ξ uniformly distributed on $[0, 1]$ and let $T = 3$.

1. $t = T = 3$.you have no choice, so your expected payout is $\mathbf{E}(\xi_3) = 1/2$.
2. $t = T - 1 = 2$. If

$$\xi_2 < \frac{\mathbf{E}(\xi_3)}{1 + r} = \frac{1}{4}$$

you should continue as in this case your payout is $\xi_2 (1 + r) = 2\xi_2 < \mathbf{E}(\xi_3) = 1/2$.

The expected gain with this strategy

$$
\begin{aligned}
(1+r)\,\mathbf{E}\left(\max\left(\frac{1}{4},\xi_2\right)\right) &= 2\left(\int_0^{1/4}\frac{1}{4}dx+\int_{1/4}^{1}xdx\right)= \\
&= \frac{1}{8}+2\left[\frac{x^2}{2}\right]_{1/4}^{1}=\frac{1}{8}+1-\frac{1}{16}=\frac{17}{16}= \\
&= \frac{1}{4}\cdot\frac{1}{2}+\frac{3}{4}\cdot 2\cdot\frac{5}{8}= \\
&= \frac{1}{4}\cdot\mathbf{E}\left(\xi_3\right)+\frac{3}{4}\cdot(1+r)\cdot\frac{5}{8}.
\end{aligned}
$$

Where $5/8$ is the expected value of the uniform distribution on $[1/4, 1]$.

3. If $(1+r)^2 \xi_1 < 17/16$, that is if $\xi_1 < 17/64$ you should continue. Your expected gain is

$$\begin{aligned}
(1+r)^2 \mathbf{E}\left(\max\left(\frac{17}{64},\xi_1\right)\right) &= 4\left(\int_0^{17/64}\frac{17}{64}dx+\int_{17/64}^1 xdx\right) = \\
&= 4\left(\left(\frac{17}{64}\right)^2+\frac{1}{2}\left(1-\left(\frac{17}{64}\right)^2\right)\right) = \\
&= \frac{17}{64}\cdot\frac{17}{16}+2\left(1-\left(\frac{17}{64}\right)^2\right)=\frac{4385}{2048}
\end{aligned}$$

Optimal stopping

The optimal exercise boundary is

$$\alpha_3 = 0, \alpha_2 = \frac{1}{4}, \alpha_1 = \frac{17}{64}$$

The gains from these periods

$$\frac{1}{2}, \frac{17}{16} = 1,0625, \frac{4385}{2048} = 2,1411$$

Optimal stopping

Let ξ_t be exponential $\lambda = 1$. The backward iteration is with $\alpha_T = 0$

$$\begin{aligned}
\alpha_t &= \mathbf{E}\left(\max\left(\alpha_{t+1}, \xi_t\right)\right) = \int_0^{\alpha_{t+1}} \alpha_{t+1} \exp\left(-x\right) dx + \int_{\alpha_{t+1}}^{\infty} x \exp\left(-x\right) dx = \\
&= \alpha_{t+1} \int_0^{\alpha_{t+1}} \exp\left(-x\right) dx + \left[\frac{x \exp\left(-x\right)}{-1}\right]_{\alpha_{t+1}}^{\infty} + \int_{\alpha_{t+1}}^{\infty} \exp\left(-x\right) dx = \\
&= \alpha_{t+1}\left(1 - \exp\left(-\alpha_{t+1}\right)\right) + \alpha_{t+1} \exp\left(-\alpha_{t+1}\right) + \exp\left(-\alpha_{t+1}\right) = \\
&= \alpha_{t+1} + \exp\left(-\alpha_{t+1}\right).
\end{aligned}$$

Optimal stopping

$$
\begin{aligned}
\alpha_T &= 0. \\
\alpha_t &= \mathbf{E}\left(\max\left(\alpha_{t+1}, \xi_t\right)\right) = \int_0^{\alpha_{t+1}} \alpha_{t+1} dF(x) + \int_{\alpha_{t+1}}^{\infty} x dF(x) = \\
&= \alpha_{t+1} F\left(\alpha_{t+1}\right) + \int_{\alpha_{t+1}}^{\infty} x dF(x).
\end{aligned}
$$

Optimal stopping

Definition

Let $H_t \geq 0$ be a set of random payouts. The optimal stopping problem is to find

$$\sup_{\tau} \mathbf{E}\left(H\left(\tau\right)\right)$$

where τ is an arbitrary stopping time. (A discrete random variable τ is a stopping time if $\{\tau = t\} \in \mathcal{F}_t$ for every t where $\mathcal{F}_t$ is the set of observable events at time t. This means that at every moment of time t the condition of stopping is known at that moment. We want to exclude referring for the future in the conditions of stopping.)

Optimal stopping

With backward iteration one should formulate the variables

$$X_t \mathrel{\hat{=}} \max \left(H_t, \mathbf{E} \left(X_{t+1} \mid \mathcal{F}_t \right) \right),$$

where $\mathbf{E} \left(X_{t+1} \mid \mathcal{F}_t \right)$ is the conditional expectation of X_{t+1} given information available at time t.
It is not difficult to show that

$$X_t = \sup_{\tau \geq t} \mathbf{E} \left(H \left(\tau \right) \mid \mathcal{F}_t \right)$$

That is X_t is the best what one can get starting at time t. X_t is the same as the value function in DP.

Definition

X_t is called the Snell envelope.

Theorem

The optimal strategy is

$$\begin{aligned}\tau^* &= \min\left(t \mid H_t = X_t\right) = \min\left(t \mid H_t \geq X_t\right) = \\ &= \min\left(t \mid H_t \geq \mathbf{E}\left(X_{t+1} \mid \mathcal{F}_t\right)\right).\end{aligned}$$

The interpretation is quite simple. One must stop (in our case sell) when one sees that the present payout is better then the expected future payout given our knowledge at that moment.

In our case as the offers are independent so

$$\mathbf{E}\left(X_{t+1} \mid \mathcal{F}_t\right) = \mathbf{E}\left(X_{t+1}\right)$$

and the Snell envelope is

$$\begin{aligned} X_t &= \max\left(H_t, \mathbf{E}\left(X_{t+1}\right)\right) = \max\left(\xi_t\left(1+r\right)^{T-t}, \mathbf{E}\left(X_{t+1}\right)\right) = \\ &= \max\left(\xi_t\left(1+r\right)^{T-t}, \frac{\left(1+r\right)^{T-t}\mathbf{E}\left(X_{t+1}\right)}{\left(1+r\right)^{T-t}}\right) \doteq \\ &\doteq \left(1+r\right)^{T-t}\max\left(\xi_t, \alpha_t\right) \end{aligned}$$

$$\alpha_T \doteq 0, \alpha_t \doteq \frac{\mathbf{E}\left(X_{t+1}\right)}{\left(1+r\right)^{T-t}}.$$

$$\frac{X_t}{(1+r)^{T-t}} = \max(\xi_t, \alpha_t).$$

$$V_t(\xi_t) \triangleq \frac{X_t}{(1+r)^{T-t}} = \frac{(1+r)^{T-t}\max(\xi_t,\alpha_t)}{(1+r)^{T-t}} = \max(\xi_t,\alpha_t),$$

$$\alpha_t \triangleq \frac{\mathbf{E}(X_{t+1})}{(1+r)^{T-t}} = \frac{1}{1+r}\frac{\mathbf{E}(X_{t+1})}{(1+r)^{T-(t+1)}} = \frac{\mathbf{E}(V_{t+1}(\xi))}{1+r}.$$

Hence

$$V_t(\xi_t) = \max(\xi_t,\alpha_t) = \max\left(\xi_t, \frac{\mathbf{E}(V_{t+1}(\xi_t))}{1+r}\right).$$

Optimal stopping

Obviously

$$V_T(\xi) = \xi \leq \max\left(\xi, \frac{\mathbf{E}(V_{T-1}(\xi))}{1+r}\right) = V_{T-1}(\xi),$$

With induction

$$V_{t+1}(\xi) = \max\left(\xi, \frac{\mathbf{E}(V_{t+2}(\xi))}{1+r}\right) \leq \max\left(\xi, \frac{\mathbf{E}(V_{t+1}(\xi))}{1+r}\right) = V_t(\xi),$$

hence

$$\alpha_{t+1} = \frac{\mathbf{E}(V_{t+2}(\xi))}{1+r} \leq \frac{\mathbf{E}(V_{t+1}(\xi))}{1+r} = \alpha_t.$$

Definition

(α_n) is the optimal exercise boundary.

Optimal stopping

If $F(x)$ is the distribution function of (ξ_n) then the optimal exercise boundary is

$$
\begin{aligned}
\alpha_T &= 0, \\
\alpha_t &= \frac{\mathbf{E}\left(V_{k+1}\left(\xi_t\right)\right)}{1+r} = \frac{\mathbf{E}\left(\max\left(\xi_{t+1}, \alpha_{t+1}\right)\right)}{1+r} = \\
&= \frac{1}{1+r}\left(\int_0^{\alpha_{t+1}} \alpha_{k+1} dF + \int_{\alpha_{t+1}}^{\infty} x dF(x)\right) = \\
&= \frac{1}{1+r}\left(\alpha_{t+1} F\left(\alpha_{t+1}\right) + \int_{\alpha_{t+1}}^{\infty} x dF(x)\right).
\end{aligned}
$$

which is a backward induction for (α_t).

Optimal stopping

If (ξ_n) are uniform on $[0.1]$ then

$$\begin{aligned}
\alpha_T &= 0, \\
\alpha_n &= \frac{1}{1+r}\left(\alpha_{n+1}^2 + \left[\frac{x^2}{2}\right]_{\alpha_{n+1}}^1\right) = \frac{1}{1+r}\left(\alpha_{n+1}^2 + \frac{1}{2} - \frac{\alpha_{n+1}^2}{2}\right) = \\
&= \frac{1}{2(1+r)}\left(\alpha_{n+1}^2 + 1\right).
\end{aligned}$$

If $T = 3$ and $r = 100\%$ then

$$\alpha_3 = 0, \alpha_2 = \frac{1}{4}, \alpha_1 = \frac{1}{4}\left(\frac{1}{16} + 1\right) = \frac{17}{64}.$$

Example

Buying a stock with independent offers.

The Snell envelope is

$$\begin{aligned} X_T &= H_T \circeq \xi_T \\ X_n &= \min\left(\xi_n, \mathbf{E}\left(X_{n+1} \mid \mathcal{F}_n\right)\right) \end{aligned}$$

As the future is independent from the present $\mathbf{E}\left(X_{n+1} \mid \mathcal{F}_n\right) = \mathbf{E}\left(X_{n+1}\right)$.

$$X_n = \min\left(\xi_n, \mathbf{E}\left(X_{n+1}\right)\right),$$

hence if $\alpha_n \circeq \mathbf{E}\left(X_{n+1}\right)$ then the optimal strategy is

$$\tau^* = \min\left\{n \geq 0 \mid \xi_n \leq \alpha_n\right\} \wedge T.$$

At time T one must buy the stock.. At time $T-1$ one must buy it if ξ_{T-1} is smaller than the expected value. etc.

Optimal stopping

$$\begin{aligned}\alpha_T &= \mathbf{E}(\xi_T) \\ \alpha_n &= \alpha_{n+1}(1 - F(\alpha_{n+1})) + \int_0^{\alpha_{n+1}} x dF(x)\end{aligned}$$

If (ξ_n) is uniform on $[0, 1]$, then

$$\begin{aligned}\alpha_T &= \frac{1}{2} \\ \alpha_n &= \alpha_{n+1}(1 - \alpha_{n+1}) + \frac{\alpha_{n+1}^2}{2} = \alpha_{n+1} - \frac{\alpha_{n+1}^2}{2}.\end{aligned}$$

Example

Selling stock with recalling prices.

One should solve the problem

$$H_n \mathrel{\hat{=}} (1+r)^{T-n} \max_{k \le n} \xi_k$$

$$\begin{aligned} X_T &= H_T = \max_{n \le T} \xi_n = (1+r)^{T-T} \max_{n \le T} \xi_n, \\ X_n &= \max\left(H_n, \mathbf{E}\left(X_{n+1} \mid \mathcal{F}_n\right)\right) = \\ &= \max\left((1+r)^{T-n} \max_{k \le n} \xi_k, \mathbf{E}\left(X_{n+1} \mid \mathcal{F}_n\right)\right). \end{aligned}$$

Optimal stopping

Let

$$V_n \circeq \frac{X_n}{(1+r)^{T-n}}$$

then

$$\begin{aligned} V_T &= \max_{n \leq T} \xi_n, \\ V_n &= \max\left(\max_{k \leq n} \xi_k, \frac{\mathbf{E}\left(V_{n+1} \mid \mathcal{F}_n\right)}{1+r}\right). \end{aligned}$$

Optimal stopping

Theorem

If variables ξ and η are independent then one can use the relation

$$\mathbf{E}\left(f\left(\xi,\eta\right)\mid\xi=x\right)=\mathbf{E}\left(f\left(x,\eta\right)\right)$$

which is the same as

$$\mathbf{E}\left(f\left(\xi,\eta\right)\mid\xi\right)=\left.\mathbf{E}\left(f\left(x,\eta\right)\right)\right|_{x=\xi}.$$

Optimal stopping

As (ξ_k) are independent

$$\begin{aligned} V_{T-1} &= \max\left(\max_{k\leq T-1}\xi_k, \frac{\mathbf{E}\left(\max_{n\leq T}\xi_n \mid \mathcal{F}_{T-1}\right)}{1+r}\right) = \\ &= \max\left(\max_{k\leq T-1}\xi_k, \frac{\mathbf{E}\left(\max\left(\max_{n\leq T-1}\xi_n, \xi_T\right) \mid \mathcal{F}_{T-1}\right)}{1+r}\right) = \\ &= \max\left(\max_{k\leq T-1}\xi_k, h\left(\max_{k\leq T-1}\xi_k\right)\right) \end{aligned}$$

where

$$h(x) \circeq \frac{\mathbf{E}\left(\max\left(x, \xi_T\right)\right)}{1+r}.$$

Optimal stopping

Let

$$S \mathrel{\hat{=}} \{x \mid x \geq h(x)\} \mathrel{\hat{=}} \left\{x \mid x \geq \frac{\mathbf{E}(\max(x, \xi_T))}{1+r}\right\}$$

Obviously

$$\mathbf{E}(\max(x, \xi_T)) = xF(x) + \int_x^\infty w dF(w).$$

Optimal stopping

$$\begin{aligned} S &= \left\{ x \mid (1+r)\, x \geq xF(x) + \int_x^\infty w dF(w) \right\} = \\ &= \left\{ x \mid rx \geq x\left(F(x) - 1\right) + \int_x^\infty w dF(w) \right\} = \\ &= \left\{ x \mid rx \geq \int_x^\infty (w - x)\, dF(w) \right\}. \end{aligned}$$

The left side is increasing the right side decrasing so $S = \{x \geq a\}$, where a is the solution of the equation

$$(1 + r) a = aF(a) + \int_a^\infty x dF(x).$$

One should sell at $T - 1$ if

$$\left\{ \max_{n \leq T-1} \xi_n \in S \right\} = \left\{ \max_{n \leq T-1} \xi_n \geq a \right\},$$

otherwise one should continue. We show that

$$\tau^* = \min \left\{ n \mid \max_{k \leq n} \xi_k \in S \right\} \wedge T = \min \left\{ n \mid \max_{k \leq n} \xi_k \geq a \right\} \wedge T,$$

The reason is the one step ahead strategy. We work by induction on T. Let

$$S_n^{(T)} \doteq \left\{H_n^{(T)} \geq X_n^{(T)}\right\} \doteq \left\{H_n \geq X_n^{(T)}\right\}.$$

As $H_n^{(T)} = \max_{k \leq n} \xi_k$ is not changing.with T, but if we increase T then as $H_{T+1}^{T+1} \geq H_T^T$ therefore obviously $X_n^{(T)} \leq X_n^{(T+1)}$. Therefore

$$\begin{aligned} S_n^{(T+1)} &\doteq \left\{H_n^{(T+1)} \geq X_n^{(T+1)}\right\} = \\ &= \left\{H_n^{(T)} \geq X_n^{(T+1)}\right\} \subseteq \left\{H_n^{(T)} \geq X_n^{(T)}\right\} = S_n^{(T)} \end{aligned}$$

One must prove that $S_n^{(T)} \subseteq S_n^{(T+1)}$.

As we have already seen if $T=n+1$, then $S_n^{(T)}=\{\max_{k\leq n}\xi_k\geq a\}$. Now let $T=n+2$ that is two periods are still ahead. We must show that

$$S_n=S_n^{(T)}\subseteq S_n^{(T+1)}.$$

$$V_n\circeq\max\left(\max_{k\leq n}\xi_k,\frac{\mathbf{E}\left(V_{n+1}\mid\mathcal{F}_n\right)}{1+r}\right)=$$

$$=\max\left(\max_{k\leq n}\xi_k,\frac{\mathbf{E}\left(\max\left(\max_{k\leq n+1}\xi_k,\frac{\mathbf{E}(V_{n+2}|\mathcal{F}_{n+1})}{1+r}\right)\mid\mathcal{F}_n\right)}{1+r}\right).$$

As

$$\max_{k\leq n+1}\xi_k\geq\max_{k\leq n}\xi_k\geq a$$

using the induction hypothesis that if one periods left then the $\{x\mid x\geq a\}$ is the stopping region

$$\max\left(\max_{k\leq n+1}\xi_k,\frac{\mathbf{E}\left(V_{n+2}\mid\mathcal{F}_{n+1}\right)}{1+r}\right)=\max_{k\leq n+1}\xi_k.$$

Substituting back and recalling that

$$h(x) \circeq \frac{\mathbf{E}(\max(x,\xi_T))}{1+r} = \frac{\mathbf{E}(\max(\eta,\xi_T) \mid \eta = x)}{1+r}$$

On the set
$S_n = S_n^{(T)} = \{\max_{k\le n} \xi_k \ge a\} = \{\max_{k\le n} \xi_k \ge h(\max_{k\le n} \xi_k)\}$

$$V_n = \max\left(\max_{k\le n} \xi_k, \frac{\mathbf{E}(\max_{k\le n+1} \xi_k \mid \mathcal{F}_n)}{1+r}\right) =$$
$$= \max\left(\max_{k\le n} \xi_k, \frac{\mathbf{E}(\max(\max_{k\le n} \xi_k, \xi_{n+1}) \mid \mathcal{F}_n)}{1+r}\right) \circeq$$
$$\circeq \max\left(\max_{k\le n} \xi_k, h\left(\max_{k\le n} \xi_k\right)\right) = \max_{k\le n} \xi_k,$$

So $S_n^{(T)} \subseteq S_n^{(T+1)}$ hence the set $\{H_n \ge X_n\}$ is independent of the number of periods left.

Homework

1. Calculate the optimal strategy for stock selling if $T = 4$, $r = 0$ and the distribution of the price is the uniform distribution on $[0, 1]$.
2. Calculate the optimal strategy for stock selling if $T = 4$, $r = 0$ and the distribution of the price is the uniform distribution on $[0, 2]$.
3. Calculate the optimal strategy for stock selling if $T = 3$, $r = 1$ and the distribution of the price is the distribution of the dice rolling.
4. Calculate the optimal strategy for buying a stock if $T = 4$ and the distribution of the price is the uniform distribution on $[0, 1]$.

Stochastic dynamic programming

Definition

Stochastic dynamic programming problem on finite time horizon is

$$\mathbf{E}\left(g\left(x_T\right)+\sum_{k=0}^{T-1} u_k\left(x_k, u_k, \xi_k\right)\right) \rightarrow \max$$

$$x_{k+1}=f_k\left(x_k, u_k, \xi_k\right), u_k \in U_k\left(x_k\right), k=0,1, \ldots, T-1$$

$$x_0=a;$$

The value of the optimum is $J\left(x_0\right)$.

Stochastic dynamic programming

The dynamic programming algorithm is

$$\begin{aligned} J_N &= g(x_T) \\ J_k(x_k) &= \max_{u_k \in U_k} \mathbf{E}\left(u_k(x_k, u_k, \xi_k) + J_{k+1}(x_{k+1}) \mid \mathcal{F}_k\right) \\ &= \max_{u_k \in U_k} \mathbf{E}\left(u_k(x_k, u_k, \xi_k) + J_{k+1}(f(x_k, u_k, \xi_k)) \mid \mathcal{F}_k\right). \end{aligned}$$

This is a principle and not a general theorem. The usual interpretation of the conditional expectation is that we know the value of $\xi_0, \xi_1, \ldots, \xi_k$ and consider their values $s_0, s_1, \ldots, s_k$ as a parameter of the optimization in stage $k+1$. To simplify the problem one generally assumes that to solve the k-th problem it is sufficient to know x_k, including the determination of the distribution of ξ_{k+1}, that is the distribution of ξ_{k+1} depends only on x_k. This is the Markovian assumption about the random factor.

In the first optimal stopping problem above introduce a special state x^*

$$f_k(x_k, u_k, \xi_k) \triangleq \begin{cases} x^* & \text{if} \quad x_k = x^* \\ x^* & \text{if} \quad x_k \neq x^*, u_k = \text{stop} \\ \xi_k & \text{otherwise} \end{cases}$$

$$g(x_T) \triangleq \begin{cases} x_T & \text{if} \quad x_T \neq x^* \\ 0 & \text{otherwise} \end{cases}$$

$$u_k(x_k, u_k, \xi_k) \triangleq \begin{cases} \xi_k (1+r)^{T-k} & \text{if} \quad x_k \neq x^*, u_k = \text{stop} \\ 0 & \text{otherwise} \end{cases}$$

$$U(x_k) \triangleq \begin{cases} \emptyset & \text{if} \quad x_k = x^* \\ \{\text{stop}, \text{continue}\} & \text{otherwise} \end{cases}$$

In the optimal stopping problem

$$\begin{aligned} J_T &= H_T \\ J_{n-1} &= \max \begin{cases} H_{n-1} & \text{if} \quad u_k = \text{stop} \\ \mathbf{E}\left(J_n\left(f\left(x_k, u_k, \xi_k\right)\right) \mid \mathcal{F}_{n-1}\right) & \text{otherwise} \end{cases} . \end{aligned}$$

Liquidity modelling

There is a demand D for liquid resources, cash or euro in a bank or in a teller machine, with distribution function F. There are two costs: h is the holding cost and p is the penalty for not satisfying the demand. What is the level of optimal supply S^*? Our goal function is

$$J(S) = h \cdot \mathbf{E}\left((S-D)^+\right) + p \cdot \mathbf{E}\left((D-S)^+\right).$$

If there is a density function then

$$\mathbf{E}\left((D-S)^+\right) = \int_S^\infty (x-S) f(x)\, dx$$

$$\begin{aligned} \mathbf{E}\left((S-D)^+\right) &= \int_{\mathbb{R}} (S-x)^+ dF(x) = \int_{\mathbb{R}} (S-x)^+ f(x)\, dx = \\ &= \int_0^S (S-x) f(x)\, dx. \end{aligned}$$

Liquidity modelling

Hence

$$J(S) = h\int_0^S (S-x) f(x)\,dx + p\int_S^\infty (x-S) f(x)\,dx.$$

Using the formula

$$\begin{aligned}\frac{d}{dx}\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy &= f(x,\varphi_2(x))\,\varphi_2'(x) - \\ &\quad -f(x,\varphi_1(x))\,\varphi_1'(x) + \\ &\quad +\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial}{\partial x} f(x,y)\,dy\end{aligned}$$

$$\begin{aligned}\frac{dJ}{dS} &= h(S-S) + h\int_0^S 1\cdot f(x)\,dx + \\ &\quad -p(S-S) + p\int_S^\infty -1\cdot f(x)\,dx.\end{aligned}$$

Setting the derivative to zero

$$\begin{aligned} 0 &= h\int_0^S 1\cdot f(x)\,dx - p\int_S^\infty 1\cdot f(x)\,dx = \\ &= hF(S) - p(1 - F(S)). \end{aligned}$$

Solving it for S

$$(h+p)\,F(S) = p, \quad F(S^*) = \frac{p}{h+p}.$$

Liquidity modelling

Assume that there is some ordering cost c. In this case

$$J(S) = cS + h \cdot \mathbf{E}\left((S-D)^{+}\right) + p \cdot \mathbf{E}\left((D-S)^{+}\right).$$

Calculating the derivative

$$c + hF(S) - p(1 - F(S)) = 0.$$

Solving the equation

$$c - p + F(S)(h + p) = 0, \quad F(S^{*}) = \frac{p - c}{p + h}.$$

Liquidity modelling

If the distribution of D is discrete, $D = 0, 1, \ldots.$ then one must change the argument.

$$\begin{aligned} \mathbf{E}(D-S)^{+} &= \sum_{k=S}^{\infty} \mathbf{P}(D>k), \\ \mathbf{E}(S-D)^{+} &= \sum_{k=0}^{S-1} \mathbf{P}(D \leq k). \end{aligned}$$

Let

$$\begin{aligned} \Delta J(S) &= J(S+1) - J(S) = \\ &= h \sum_{k=0}^{S} \mathbf{P}(D \leq k) + p \sum_{k=S+1}^{\infty} \mathbf{P}(D > k) - \\ &\quad - h \sum_{k=0}^{S-1} \mathbf{P}(D \leq k) - p \sum_{k=S}^{\infty} \mathbf{P}(D > k). \end{aligned}$$

Liquidity modelling

$$\begin{aligned}\Delta J(S) &= h\mathbf{P}(D \leq S) - p\mathbf{P}(D > S) = \\ &= h(1 - \mathbf{P}(D > S)) - p\mathbf{P}(D > S) = \\ &= h - (h+p)\mathbf{P}(D > S).\end{aligned}$$

Obviously if $S = 0$ then $\Delta J(S) = h - (h+p)\mathbf{P}(D > 0)$. Also $\lim_{S \nearrow \infty} \Delta(J(S)) = h > 0$, and the expression is not decreasing when S is increasing. So the optimal, minimum, solution is

$$S^* = \min(S \mid \Delta J(S) \geq 0) = \min(S \mid J(S+1) \geq J(S)),$$

Liquidity modelling

$$\begin{aligned} S^* &= \min \{S \mid h-(h+p)\,\mathbf{P}(D>S) \geq 0\} = \\ &= \min \left\{S \mid \mathbf{P}(D>S) \leq \frac{h}{h+p}\right\} = \\ &= \min \left\{S \mid 1-\mathbf{P}(D \leq S) \leq \frac{h}{h+p}\right\} = \\ &= \min \left\{S \mid \mathbf{P}(D \leq S) \geq \frac{p}{h+p}\right\} = \\ &= \min \{S \mid \mathbf{P}(D \leq S) \geq \alpha\}. \end{aligned}$$

Observe that one must cumulate until we first time hit the level α.

Liquidity modelling

What does happen in a dynamic environment? The state equation is

$$y_{t+1} = (y_t + v - D_t)^+ , \quad y_0 = x,$$

where v is the control variable giving the amount of liquid resources ordered at time t and D_n is the random demand at time t and x is the starting value of the liquid resource.

$$\begin{aligned} I(x, v) &\stackrel{\circ}{=} cv + hx + p \cdot \mathbf{E}\left((D - (x+v))^+ \right) = \\ &= cv + hx + p \left(\int_{x+v}^{\infty} (z - (x+v))\, dF(z) \right) . \\ J(x, V) &= \sum_{n=0}^{\infty} \beta^n I(y_n, v_n) . \end{aligned}$$

Liquidity modelling

Theorem

If $p > c$, then there is an optimal strategy v^* such that

$$v^*(x) = \begin{cases} S^* - x & \text{if} \quad x \leq S^* \\ 0 & \text{if} \quad x > S^* \end{cases}.$$

The optimal value of S^* satisfies:

$$\overline{F}(S^*) = 1 - F(S^*) = \frac{c(1-\beta) + \beta h}{p - \beta c + \beta h}.$$

That is

$$F(S^*) = \frac{p - c}{p + \beta(h - c)}.$$

Compare with the static solution

$$F(S^*) = \frac{p - c}{p + h}.$$

Liquidity modelling

Example

Let the distribution of D exponential with $\lambda = 2$. Let the penalty $p = 3$ and the holding cost $h = 5$ and let $c = 1$. In the static model

$$\begin{aligned} F(S^*) &= 1 - \exp(-2S^*) = \frac{3-1}{3+5} = \frac{2}{8} = \frac{1}{4}. \\ \exp(-2S^*) &= \frac{3}{4}, S^* = -\frac{1}{2}\log\left(\frac{3}{4}\right) = 0{,}143\,84. \end{aligned}$$

Now let $\beta = 1/2$. In this case

$$\begin{aligned} F(S^*) &= 1 - \exp(-2S^*) = \frac{3-1}{3+\frac{1}{2}(5-1)} = \frac{2}{5}. \\ \exp(-2S^*) &= \frac{3}{5}, S^* = -\frac{1}{2}\log\left(\frac{3}{5}\right) = 0{,}255\,41 \end{aligned}$$

Problem

The problem

$$\sum_{t=1}^{\infty} \beta^{t-1} r(s_t, a_t) \to \max$$

$$s_{t+1} = f(s_t, a_t), t = 1, 2, \ldots$$

$$a_t \in \Phi(s_t), t = 1, 2, \ldots$$

is called stationary dynamic programming.

Infinite dynamic programming

1. $r(s, a)$ is bounded and continuous.
2. $f(s, a)$ is continuous.
3. Φ is continuous and compact valued.

Infinite dynamic programming

Theorem

Under the above conditions there is a stationary optimal strategy π. The value function solves the Bellman equation:

$$V(s) = \max_{a \in \Phi(s)} \left(r(s,a) + \beta V(f(s,a))\right).$$

As r is bounded the value function is also bounded and if a bounded function solves the equation then it is the value function of the problem.

Definition

A Markovian strategy is stationary if it is independent of the time parameter t.

Infinite dynamic programming

Sometimes the assumption that r is bounded too strong.

Theorem

If V is an optimal solution then it satisfies the Bellman equation. If V is a solution of the Bellman equation and for any feasible path $\beta^n V(x_n) \to 0$ then V is the optimal solution.

Infinite cake eating problem

The infinite cake eating problem is

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \to \max$$

$$w_{t+1} = w_t - c_t, t = 0, 1, \ldots$$

$$c_t \in [0, w_t], t = 0, 1, \ldots$$

w_0 is given.

Infinite cake eating problem

By the Bellman equation

$$\begin{aligned} V(w) &= \max_{c\in[0,w]} \left(u(c) + \beta V(w-c)\right) = \\ &= \max_{s\in[0,w]} \left(u(w-s) + \beta V(s)\right). \end{aligned}$$

The condition on optimality is

$$u'(c^*) = \beta V'(w-c^*)$$

or

$$u'(w-s^*) = \beta V'(s^*).$$

How we can calculate $V'(u)$?

Again we will use the Envelope Theorem. We use the second formulation

$$\begin{aligned} V'(w) &= u'(w - s^*) = u'(c(w)). \\ V'(w - c(w)) &= u'(c(w - c(w))), \end{aligned}$$

where $c(w)$ is the optimal consumption at w. Hence the Euler equation

$$u'(c(w)) = \beta u'(c(w - c(w))).$$

If π is the stationary policy function which gives the consumption at cake size w then

$$u'(\pi(w)) = \beta u'(\pi(w - \pi(w))).$$

Example

Solve the problem with $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$

We guess that $V(x) = \alpha \frac{x^{1-\sigma}}{1-\sigma}$ and the optimal policy is $\pi(x) = \lambda x$, with $0 < \lambda < 1$.

$$\alpha \frac{w^{1-\sigma}}{1-\sigma} = \max_{c \in [0,w]} \left(\frac{c^{1-\sigma}}{1-\sigma} + \frac{\beta\alpha}{1-\sigma} (w-c)^{1-\sigma} \right).$$

Differentiating

$$c^{-\sigma} + \beta\alpha (w-c)^{-\sigma} (-1) = 0, (\beta\alpha)^{-1/\sigma} (w-c) = c$$

$$(\beta\alpha)^{-1/\sigma} w = c \left(1 + (\beta\alpha)^{-1/\sigma}\right)$$

$$c = \frac{(\beta\alpha)^{-1/\sigma}}{1 + (\beta\alpha)^{-1/\sigma}} w = \frac{w}{1 + (\beta\alpha)^{1/\sigma}} = \lambda w$$

Infinite cake eating problem

Substituting back

$$\begin{aligned} \alpha\frac{w^{1-\sigma}}{1-\sigma} &= \frac{(\lambda w)^{1-\sigma}}{1-\sigma}+\frac{\beta\alpha}{1-\sigma}\left((1-\lambda)\,w\right)^{1-\sigma} \\ \alpha &= \lambda^{1-\sigma}+\beta\alpha\,(1-\lambda)^{1-\sigma} \end{aligned}$$

Infinite cake eating problem

If

$$\begin{aligned} \alpha &= \left(1-\beta^{1/\sigma}\right)^{-\sigma} \\ \lambda &= 1-\beta^{1/\sigma} \end{aligned}$$

then

$$\begin{aligned} \left(1-\beta^{1/\sigma}\right)^{-\sigma} &= \left(1-\beta^{1/\sigma}\right)^{(1-\sigma)} + \beta\left(1-\beta^{1/\sigma}\right)^{-\sigma}\beta^{(1-\sigma)/\sigma} = \\ &= \left(1-\beta^{1/\sigma}\right)^{(1-\sigma)} + \left(1-\beta^{1/\sigma}\right)^{-\sigma}\beta^{1/\sigma} = \\ &= \left(1-\beta^{1/\sigma}\right)^{-\sigma}\left(1-\beta^{1/\sigma}+\beta^{1/\sigma}\right) \end{aligned}$$

hence (α, λ) solves the equation.

Infinite cake eating problem

Obviously

$$\lim_{n\to\infty}\beta^{n}V\left(c_{n}\right)=0$$

so this is the optimal solution.

Example

Solve the problem if $u(c) = \ln c$.

In this case we guess that $V(w) = A + B \ln w$. Hence

$$A + B \ln w = \max_{s \in [0,w]} \left(\ln(w - s) + \beta \left(A + B \ln(s) \right) \right).$$

The first order condition

$$\frac{1}{w - s^*}(-1) + \beta B \frac{1}{s^*} = 0$$

$$\frac{-s^* + \beta B (w - s^*)}{(w - s^*) s^*} = 0$$

$$\beta B (w - s^*) = s^*$$

$$s^* = \frac{\beta B}{1 + B\beta} w, \Rightarrow c(w) = \frac{w}{1 + B\beta}$$

Infinite cake eating problem

$$
\begin{aligned}
A+B\ln w &= \ln\left(w-\frac{\beta B}{1+B\beta}w\right)+\beta\left(A+B\ln\frac{\beta B}{1+B\beta}w\right) \\
A+B\ln w &= \ln\left(\frac{w}{1+B\beta}\right)+\beta\left(A+B\ln\frac{\beta B}{1+B\beta}w\right) \\
A+B\ln w &= \ln w-\ln(1+B\beta)+\beta A+\beta B\ln\frac{\beta B}{1+B\beta}+\beta B\ln w \\
A+B\ln w &= -\ln(1+B\beta)+\beta A+\beta B\ln\frac{\beta B}{1+B\beta}+(1+\beta B)\ln w
\end{aligned}
$$

Infinite cake eating problem

As it holds for any w

$$\begin{aligned} B &= 1+\beta B, \quad \Rightarrow B = \frac{1}{1-\beta}. \\ \pi(w) &= c(w) = \frac{w}{1+\frac{\beta}{1-\beta}} = \frac{w(1-\beta)}{1-\beta+\beta} = w(1-\beta). \end{aligned}$$

For

$$A = -\ln(1+B\beta) + \beta A + \beta B \ln \frac{\beta B}{1+B\beta}$$

Infinite cake eating problem

$$\beta^n \ln(c_n) \leq \beta^n \ln(w) \to 0.$$

hence we can just prove that

$$\limsup_{n\to\infty} \beta^n V(c_n) = 0$$

Infinite cake eating problem

One can generalize the problem like we want to calculate

$$\sum_{t=0}^{T} \beta^t F\left(t, x_t, x_{t+1}\right) \to \max$$

where x_0 is given and $T \leq \infty$. If it is maximal and the optimum is at an interior point then using that the same variable x_{t+1} appears in two terms

$$\beta^t \frac{\partial F}{\partial x_{t+1}}\left(t, x_t, x_{t+1}\right) + \beta^{t+1} \frac{\partial F}{\partial x_t}\left(t, x_{t+1}, x_{t+2}\right) = 0.$$

This is a second order differential equation called Euler equation. If $T < \infty$ then we also have that

$$F_{x_{T+1}}\left(t, x_T, x_{T+1}\right) = 0.$$

In this case one can solve this equation with backward induction.

Infinite cake eating problem

In the cake eating problem with $u(x) = \ln x$

$$\sum_{t=0}^{T} \beta^t \ln\left(x_{t+1} - x_t\right) \to \max, x_0 = w.$$

Differentiating with respect to x_{t+1}

$$\beta^t \frac{1}{x_{t+1} - x_t} + \beta^{t+1} \frac{1}{x_{t+2} - x_{t+1}} (-1) = 0$$
$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}}.$$

Homework

1. Solve the consumption optimization problem with $u(c) = c^{\alpha}$, $\alpha > 0$. Consider the cases $\alpha < 1, \alpha = 1$, $\alpha > 1$.
2. Solve the following problem with constrained optimization and with dynamic programming where $s_t > 0$ and $c > 0$ are constants:
$$\sum_{t=1}^{T} s_t a_t^2 \to \max . \sum_{t=1}^{T} a_t = c, a_t \geq 0.$$
3. Solve the following problems with constrained optimization and with dynamic programming where $\delta_t > 0$ and $c > 0$ are constants:
$$\sum_{t=1}^{T} a_t^{\delta_t} \to \max, \sum_{t=1}^{T} a_t = c, a_t \geq 0.$$
4. Solve the following problems with constrained optimization and with dynamic programming
$$\prod_{t=1}^{T} a_t \to \max . \sum_{t=1}^{T} a_t = c, a_t \geq 0.$$

A typical example

$$\int_0^T \exp(-rt)\, U(C(t))\, dt \to \max$$

$$K'(t) = F(K(t)) - C(t) - bK(t)$$

$$K(0) = K_0,\ K(T) \geq 0,\ C(t) \geq 0$$

is an optimal control problem. First two lines are always there the third line, the boundary conditions, has many variations. The boundary conditions are very important!

A typical example

One should be careful as

$$\int_0^T \exp(-rt)\, U(C(t))\, dt \to \max$$
$$K'(t) = F(K(t)) - C(t) - bK(t)$$
$$K(0) = K_0,\ K(t) \geq 0,\ C(t) \geq 0.$$

is more difficult. It is an optimal control problem with path restrictions.

A typical example

One should be careful as

$$\int_0^T \exp(-rt)\, U(C(t))\, dt \to \max$$
$$K'(t) = F(K(t)) - C(t) - bK(t)$$
$$K(0) = K_0,\ C(t) \geq 0.$$

is much easier. This is the simplest optimal control problem.

Calculus of variations

If we drop $C(t) \geq 0$ and reformulate as

$$\int_0^T \exp(-rt)\, U\left(F(K(t)) - bK(t) - K'(t)\right) dt$$
$$K(0) = K_0$$

we get a calculus of variation problem.

Calculus of variations

$$\int_a^b F\left(t, x, \dot{x}\right) dt \to \begin{array}{c}\max \\ \min\end{array}$$

$$x(a) = x_a, x(b) = x_b$$

Let y be an arbitrary function, called variation, with $y(a) = y(b) = 0$. Let

$$\psi(\lambda) \mathrel{\hat=} \int_a^b F\left(t, x(t) + \lambda y(t), \dot{x}(t) + \lambda \dot{y}(t)\right) dt$$

If x is an optimal function then by Fermat's principle $\psi'(0) = 0$.

$$
\begin{aligned}
0 &= \psi'(0) \overset{\circ}{=} \frac{d}{d\lambda}\int_a^b F\left(t, x(t)+\lambda y(t), \dot{x}(t)+\lambda \dot{y}(t)\right) dt = \\
&= \int_a^b \frac{d}{d\lambda} F\left(t, x(t)+\lambda y(t), \dot{x}(t)+\lambda \dot{y}(t)\right) dt = \\
&= \int_a^b F'_x y + F'_{\dot{x}} \dot{y} dt \overset{\circ}{=} \\
&\overset{\circ}{=} \int_a^b q(t) y(t) + p(t) \dot{y}(t) dt.
\end{aligned}
$$

Integrating by parts and using that $y(a) = y(b) = 0$

$$\begin{aligned}
0 &= \int_a^b q(t)\,y(t) + p(t)\,y'(t)\,dt = \\
&= \int_a^b q(t)\,y(t)\,dt + \int_a^b p(t)\,y'(t)\,dt = \\
&= \int_a^b q(t)\,y(t)\,dt + [p(t)\,y(t)]_a^b - \int_a^b p'(t)\,y(t)\,dt = \\
&= \int_a^b q(t)\,y(t)\,dt - \int_a^b p'(t)\,y(t)\,dt = \\
&= \int_a^b \left(q(t) - p'(t)\right) y(t)\,dt.
\end{aligned}$$

As it is true for any variation y one has that

$$q(t) - p'(t) = 0,$$

that is

$$F'_x\left(t, x(t), \overset{\bullet}{x}(t)\right) - \frac{d}{dt}F'_{\overset{\bullet}{x}}\left(t, x(t), \overset{\bullet}{x}(t)\right) = 0,$$

where F'_x and $F'_{\overset{\bullet}{x}}$ are the partial derivative of F

Definition

This is the so-called the Euler–Lagrange differential equation.

$$\varphi_a(x(a)) + \varphi_b(x(b)) + \int_a^b F\left(t, x, \overset{\bullet}{x}\right) dt \to \begin{array}{c}\max\\ \min\end{array}$$

The argument is very similar. Now let y be an arbitrary (differentiable) function and let

$$\begin{aligned}\psi(\lambda) &\stackrel{\circ}{=} \varphi_a(x(a) + \lambda y(a)) + \varphi_b(x(b) + \lambda y(b)) + \\ &+ \int_a^b F\left(t, x(t) + \lambda y(t), \overset{\bullet}{x}(t) + \lambda \overset{\bullet}{y}(t)\right) dt.\end{aligned}$$

If the function x is optimal then by Fermat's principle $\psi'(0) = 0$.

Calculus of variations

$$0 = \psi'(0) = \varphi_a'(x(a))\, y(a) + \varphi_b'(x(b))\, y(b) + \int_a^b F_x' y + F_{\dot{x}}' \dot{y}\, dt.$$

Again integrating by parts

$$\int_a^b F_{\dot{x}}' \dot{y}\, dt = \left[F_{\dot{x}}' y \right]_a^b - \int_a^b \frac{d}{dt} F_{\dot{x}}' y\, dt =$$

$$= F_{\dot{x}}' y(b) - F_{\dot{x}}' y(a) - \int_a^b \frac{d}{dt} F_{\dot{x}}' y\, dt.$$

Hence

$$0 = \int_a^b \left(F_x' - \frac{d}{dt} F_{\dot{x}}' \right) y\, dt +$$

$$+ \left(\varphi_b'(x(b)) + F_{\dot{x}}' \left(b, x(b), \dot{x}(b) \right) \right) y(b) +$$

$$+ \left(\varphi_a'(x(a)) - F_{\dot{x}}' \left(a, x(a), \dot{x}(a) \right) \right) y(a).$$

Calculus of variations

As y is an arbitrary function, hence $y(b)$ and $y(a)$ is also arbitrary, so if x is an optimal solution then

$$\begin{aligned} F'_x\left(t, x(t), \dot{x}(t)\right) &= \frac{d}{dt} F'_{\dot{x}}\left(t, x(t), \dot{x}(t)\right) \\ \varphi'_b(x(b)) &= -F'_{\dot{x}}\left(b, x(b), \dot{x}(b)\right) \\ \varphi'_a(x(a)) &= F'_{\dot{x}}\left(a, x(a), \dot{x}(a)\right) \end{aligned}$$

Definition

The last two conditions are called the transversality conditions.

Example

Solve the problem

$$\int_0^2 \left(4 - 3x^2 - 16\dot{x} - 4\dot{x}^2\right)e^{-t}dt \to \max$$

$$x(0) = -8/3, x(2) = 1/3$$

In this case

$$F(t, x, \dot{x}) = \left(4 - 3x^2 - 16\dot{x} - 4\dot{x}^2\right)e^{-t}$$

$$F'_x = -6xe^{-t}, F'_{\dot{x}} = \left(-16 - 8\dot{x}\right)e^{-t}.$$

The Euler–Lagrange equation is

$$\begin{aligned} -6xe^{-t} &= \frac{d}{dt}\left(-16-8\dot{x}\right)e^{-t} = \\ &= e^{-t}\left(-1\right)\left(-16-8\dot{x}\right)+e^{-t}\left(-8\ddot{x}\right) \end{aligned}$$

Simplifying

$$\ddot{x}-\dot{x}-\frac{3}{4}x=2.$$

$$x\left(0\right)=-8/3, x\left(2\right)=1/3.$$

The characteristic equation is $\lambda^2 - \lambda - 3/4 = 0$. The roots are $\lambda_1 = -1/2, \lambda_2 = 3/2$. The $x \equiv -8/3$ is a particular solution so the general solution is

$$C_1 \exp\left(-\frac{1}{2}t\right) + C_2 \exp\left(\frac{3}{2}t\right) - \frac{8}{3}.$$

Using the boundary conditions

$$C_1 = -\frac{3}{e^3 - e^{-1}}, C_2 = \frac{3}{e^3 - e^{-1}}$$

Example

On what curve can the functional

$$\int_1^2 \left(\dot{x}\right)^2 - 2tx dt, \quad x\left(1\right) = 0, x\left(2\right) = -1$$

attain an extremum.

The kernel function is $F\left(t, x, \dot{x}\right) = \left(\dot{x}\right)^2 - 2tx$. The partial derivatives

$$F'_x = -2t, \quad F'_{\dot{x}} = 2\dot{x}$$

The Euler–Lagrange equation is

$$-2t = \frac{d}{dt}2\dot{x} = 2\ddot{x}, \quad \ddot{x} = -t$$

The solution is

$$\dot{x}\left(t\right) = -t^2/2 + c_1, x\left(t\right) = -\frac{t^3}{6} + c_1 t + c_2.$$

The boundary conditions are

$$\begin{aligned} 0 &= -\frac{1}{6} + c_1 + c_2 \\ -1 &= -\frac{8}{3} + 2c_1 + c_2 \end{aligned}$$

that is

$$c_1 + c_2 = \frac{1}{6}, \quad 2c_1 + c_2 = \frac{2}{6}$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1, \begin{vmatrix} 1/6 & 1 \\ 2/6 & 1 \end{vmatrix} = -1/6, \begin{vmatrix} 1 & 1/6 \\ 2 & 2/6 \end{vmatrix} = 0$$

hence $c_1 = 1/6, c_2 = 0$. The kernel is convex in $\left(x, \overset{\bullet}{x}\right)$ so it is a minimum.

Example

Find the extremal of the functional

$$\int_1^3 (3t - x)\, x dt, \quad x(1) = 1, x(3) = 4\frac{1}{2}.$$

The kernel function is

$$F\left(t, x, \overset{\bullet}{x}\right) = (3t - x)\, x.$$

$$F'_x = 3t - 2x, F'_{\overset{\bullet}{x}} = 0.$$

The Euler–Lagrange equation is $3t - 2x = 0$, so $x = 3/2t$. But at $t = 1$ $x(1) = 3/2$ the first boundary condition is not valid. Hence there is no extremal solution.

Calculus of variations

Example

Find the extremal solutions of

$$\int_0^1 \exp(x) + t\dot{x}\,dt,\ x(0) = 0,\ x(1) = \alpha.$$

The Euler–Lagrange equation is

$$\exp(x) = \frac{d}{dt}t = 1.$$

If $\alpha = 0$ then $x(t) = 0$ is a solution, otherwise there is no solution. As the kernel is a convex function, if there is an extremal solution then it is a minimum.

Example

Find the extremal of the functional

$$\int_0^{2\pi} \overset{\bullet}{x}^2(t) - x^2(t)\,dt, \quad x(0) = x(2\pi) = 1.$$

The kernel function is

$$F\left(t, x, \overset{\bullet}{x}\right) = \overset{\bullet}{x}^2 - x^2.$$
$$F'_x\left(t, x, \overset{\bullet}{x}\right) = -2x, F'_{\overset{\bullet}{x}}\left(t, x, \overset{\bullet}{x}\right) = 2\overset{\bullet}{x}.$$

The Euler–Lagrange equation is

$$-2x = \frac{d}{dt}2\overset{\bullet}{x}, \Rightarrow \overset{\bullet\bullet}{x} + x = 0.$$

which has the general solution $x(t) = c_1 \cos t + c_2 \sin t$. Putting the boundary conditions the extremal solutions are of the form

$$x(t) = \cos t + c \sin t.$$

Example

Find

$$\int_0^1 \sqrt{1+\left(\dot{x}\right)^2}\,dt \to \min$$

$$x(0)=0,\; x(1)=1.$$

The kernel function is

$$F\left(t,x,\dot{x}\right)=\sqrt{1+\left(\dot{x}\right)^2}$$

The Euler–Lagrange equation is

$$0=\frac{d}{dt}\frac{1}{2}\frac{1}{\sqrt{1+\left(\dot{x}\right)^2}}2\dot{x}$$

$$c = \frac{\overset{\bullet}{x}}{\sqrt{1 + \left(\overset{\bullet}{x}\right)^2}}$$

$$c^2 \left(1 + \left(\overset{\bullet}{x}\right)^2\right) = \left(\overset{\bullet}{x}\right)^2$$

$$\left(1 - c^2\right) \left(\overset{\bullet}{x}\right)^2 = c^2$$

$$\overset{\bullet}{x} = C \Rightarrow x = Ct + B.$$

Using the initial conditions $x(t) = t$.

We show that the kernel is convex.

$$\begin{aligned}
\frac{d}{dx}\sqrt{1+x^2} &= \frac{x}{\sqrt{x^2+1}} \\
\frac{d^2}{dx^2}\sqrt{1+x^2} &= \frac{\sqrt{x^2+1}-x\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{x^2+1-x^2}{\sqrt{x^2+1}}}{x^2+1} = \\
&= \frac{1}{\left(x^2+1\right)\sqrt{1+x^2}} > 0.
\end{aligned}$$

Hence the solution is a minimum.

Example

Find the extremal

$$\int_0^1 y\left(y'\right)^2 dx,\ y(0)=1,\ y(1)=\sqrt[3]{4}$$

The Euler-Lagrange equation is

$$\begin{aligned}\left(y'\right)^2 &= 2\frac{d}{dx}\left(yy'\right)=2\left(y'\right)^2+2yy'' \\ 2yy''+\left(y'\right)^2 &= 0.\end{aligned}$$

Calculus of variations

It is a second order non-linear equation. In this case one can try to substitute $u = y'$ and assume that y' depends on x just via y. Then

$$
\begin{aligned}
y''(x) &= \frac{d}{dx} y'(x) = \frac{d}{dx} u(y(x)) = \frac{d}{dy} u(y(x)) \frac{d}{dx} y(x) \\
&= \frac{d}{dy} u(y(x)) \, u(y(x)) .
\end{aligned}
$$

Using this

$$
\begin{aligned}
2yy'' + \left(y'\right)^2 &= 0 \\
2y\frac{du}{dx} + \left(y'\right)^2 &= 0 \\
2y(x)\frac{d}{dy}u(y(x))\frac{d}{dx}y(x) + u^2(x) &= 0 \\
2y(x)\frac{d}{dy}u(y(x))\,u(y(x)) + u^2(y(x)) &= 0.
\end{aligned}
$$

First we solve the equation

$$2yu(y)\frac{d}{dy}u(y)+u^2(y)=0.$$

If $u(y)=0$ then $y'(x)=0$ then $y(x)$ is a constant which is not a solution. We can simplify by u

$$\begin{aligned} 2y\frac{du}{dy}+u &= 0, \frac{1}{u}du=-\frac{1}{2y}dy \\ \ln u &= \frac{-1}{2}\ln y+c \\ u &= c\frac{1}{\sqrt{|y|}}, y'=c\frac{1}{\sqrt{|y|}} \end{aligned}$$

This is again an equation with separable variables

$$\begin{aligned} c\sqrt{|y|}dy &= dx \\ c_1 |y|^{3/2} &= x + c_2 \\ y &= c_3 \sqrt[3]{(x+c_2)^2}, y(0) = 1, y(1) = \sqrt[3]{4} \\ 1 &= c_3 \sqrt[3]{c_2^2}, \sqrt[3]{4} = c_3 \sqrt[3]{(1+c_2)^2} \end{aligned}$$

There are two solutions

$$\begin{aligned} c_3 &= c_2 = 1, y = \sqrt[3]{(x+1)^2} \\ c_3 &= \sqrt[3]{9}, c_2 = -\frac{1}{3}, y = \sqrt[3]{9}\sqrt[3]{\left(x-\frac{1}{3}\right)^2} = \sqrt[3]{(3x-1)^2}. \end{aligned}$$

Example

Find the extremal of

$$\int_1^e x\left(y'\right)^2 + yy'dx, y\left(1\right) = 0, y\left(e\right) = 1.$$

The Euler–Lagrange equation is

$$\begin{aligned} y' &= \frac{d}{dx}\left(x2y' + y\right) = 2y' + x2y'' + y' \\ 0 &= y' + xy''. \end{aligned}$$

Again introduce $u = y'$. (It is always working when y is missing.) Then

$$\begin{aligned} 0 &= u + xu'. \\ -\frac{1}{x}dx &= \frac{1}{u}du \\ -\ln|x| &= \ln|u| + c \\ \frac{c}{x} &= u = y' \\ y &= c_1 \ln x + c_2 \end{aligned}$$

Using the initial conditions the extremal solution is $y = \ln x$.

Calculus of variations

Example

Find the extremal solution of

$$\int_a^b y + \frac{(y')^3}{3} dx$$

The Euler–Lagrange equation is

$$\begin{aligned} 1 &= \frac{d}{dx}(y')^2, x + c = (y')^2, \\ y' &= \pm\sqrt{x+c}. \end{aligned}$$

As there are no boundary conditions

$$\begin{aligned} 0 &= -F'_{y'}(b, y(b), y'(b)) = (y')^2 = b + c \\ 0 &= F'_{y'}(a, y(a), y'(a)) = (y')^2 = a + c \end{aligned}$$

Which has no solution, hence there are no extremal solutions.

Calculus of variations

Example

$$2\pi \int_0^T x(t) \sqrt{1 + \left(\dot{x}\right)^2} dt \to \min$$
$$x(0) = a, x(T) = b$$

The kernel is

$$F\left(t, x, \dot{x}\right) = x\sqrt{1 + \left(\dot{x}\right)^2}$$

The Euler–Lagrange is

$$\sqrt{1 + \left(\dot{x}\right)^2} = \frac{d}{dt} \frac{x\dot{x}}{\sqrt{1 + \left(\dot{x}\right)^2}}$$

Calculus of variations

$$\sqrt{1+\left(\dot{x}\right)^2} = \frac{\sqrt{1+\left(\dot{x}\right)^2}\left(x\dot{x}\right)' - \left(\sqrt{1+\left(\dot{x}\right)^2}\right)' x\dot{x}}{1+\left(\dot{x}\right)^2}$$

$$\sqrt{1+\left(\dot{x}\right)^2} = \frac{\sqrt{1+\left(\dot{x}\right)^2}\left(\left(\dot{x}\right)^2 + x\ddot{x}\right) - \frac{2\dot{x}\ddot{x}}{2\sqrt{1+\left(\dot{x}\right)^2}} x\dot{x}}{1+\left(\dot{x}\right)^2}$$

$$1+\left(\dot{x}\right)^2 = \frac{\left(1+\left(\dot{x}\right)^2\right)\left(\left(\dot{x}\right)^2 + x\ddot{x}\right) - x\left(\dot{x}\right)^2\ddot{x}}{1+\left(\dot{x}\right)^2} =$$

$$1+\left(\dot{x}\right)^2 = \frac{\left(\dot{x}\right)^2 + x\ddot{x} + \left(\dot{x}\right)^4 + x\ddot{x}\left(\dot{x}\right)^2 - x\left(\dot{x}\right)^2\ddot{x}}{1+\left(\dot{x}\right)^2} =$$

$$1+\left(\dot{x}\right)^4 + 2\left(\dot{x}\right)^2 = \left(\dot{x}\right)^2 + x\ddot{x} + \left(\dot{x}\right)^4$$

Which simplifies to

$$1 = x\ddot{x} - \left(\dot{x}\right)^2$$

Calculus of variations

It is a non-linear equation. Assume that x' depends only on x and not on t. Let $u = x'$, $x'' = \frac{d}{dt}u' = du/dx \cdot dx/dt = du/dx \cdot u$

$$\begin{aligned}
1 &= xu\frac{du}{dx} - u^2 \\
\frac{1+u^2}{u} &= x\frac{du}{dx} \\
\frac{1}{2}\frac{2u}{u^2+1}du &= \frac{1}{x}dx \\
\ln\sqrt{1+u^2} &= \ln\left[x\right| + C, \sqrt{1+u^2} = Cx \\
\frac{x}{\sqrt{1+\overset{\bullet}{x}^2}} &= C
\end{aligned}$$

$$\left(\dot{x}\right)^2 = \frac{x^2}{C^2} - 1$$

Writing back to the Euler–Lagrange

$$x\ddot{x} - \left(\frac{x^2}{C^2} - 1\right) - 1 = 0$$

$$x\ddot{x} = \frac{x^2}{C^2}$$

$$\ddot{x} = \frac{x}{C^2}$$

The general solution is

$$x(t) = A\exp(-t/C) + B\exp(+t/C)$$

Writing back again to Euler-Lagrange

$$\begin{aligned} 4AB &= C^2 \\ x &= \alpha \cdot \mathrm{ch}\left(\frac{t-\beta}{\alpha}\right) \end{aligned}$$

where $\mathrm{ch}(t) = (\exp(t) + \exp(-t))/2$.

$$2\pi \int_0^T x(t) \sqrt{1 + \left(\dot{x}\right)^2} dt \to \min$$

$$x(0) = a$$

The solution of the Euler–Lagrange equation is still of the form

$$x(t) = \alpha \cdot \mathrm{ch}\left(\frac{t - \beta}{\alpha}\right)$$

but we should also have the condition $x(0) = a$ and the transversality

$$0 = \varphi_b'(x(b)) = -F_{\dot{x}}'\left(b, x(b), \dot{x}(b)\right) = -\frac{x(T)\dot{x}(T)}{\sqrt{1 + \left(\dot{x}(T)\right)^2}}.$$

That is $x(T)$ or $\dot{x}(T)$ is zero which is impossible for function $\mathrm{ch}(t)$.

Example

Solve the problem

$$\int_0^1 t\dot{x} + \left(\dot{x}\right)^2 dt \to \min, x\left(0\right) = 1.$$

The Euler–Lagrange is

$$0 = F'_x = \frac{d}{dt} F'_{\dot{x}} = \frac{d}{dt}\left(t + 2\dot{x}\right) = 1 + 2\ddot{x}.$$

$$C = t + 2 \cdot \dot{x}, \quad x\left(0\right) = 1$$

$$\dot{x} = \frac{C}{2} - \frac{t}{2}, \quad x\left(0\right) = 1$$

$$x\left(t\right) - 1 = \frac{C}{2} t - \frac{t^2}{4}.$$

$$x\left(t\right) = -\frac{t^2}{4} + \frac{C}{2} t + 1$$

The transversality condition is

$$
\begin{aligned}
0 &= -F'_{\dot{x}}\left(1, x(1), \dot{x}(1)\right) = -\left(t + 2\dot{x}\right)(1) = \\
&= -\left(1 + 2\left(-\frac{t}{2} + \frac{C}{2}\right)\right)(1) = -\left(1 - 2\left(\frac{1}{2} + \frac{C}{2}\right)\right)
\end{aligned}
$$

that is $C = 0$.

Theorem

If the kernel function $F\left(t, x, \dot{x}\right)$ is convex in $\left(x, \dot{x}\right)$ then every solution of the Euler–Lagrange equation is a global minimum.

Let x^* be a solution of the equation. Let $\Psi(y)$ be the value of the functional at y. For every y the variation

$$\begin{aligned}
\psi(\lambda) &= \Psi\left(x^* + \lambda\left(y - x^*\right)\right) = \Psi\left(\left(1-\lambda\right)x^* + \lambda y\right) = \\
&= \int_a^b L\left(t, (1-\lambda)x^* + \lambda y, (1-\lambda)\dot{x}^* + \lambda\dot{y}\right) \leq \\
&\leq (1-\lambda)\int_a^b L\left(t, x^*, \dot{x}^*\right) + \lambda\int_a^b L\left(t, y, \dot{y}\right) = \\
&= (1-\lambda)\psi(0) + \lambda\psi(1).
\end{aligned}$$

Calculus of variations

Hence ψ is convex. on the real line. As x^* satisfies the Euler-Lagrange equation $\psi'(0) = 0$ and as ψ is convex ψ has a minimum at $\lambda = 0$, so

$$\Psi(x^*) = \psi(0) \leq \psi(1) = \Psi(y)$$

for every y, which proves the theorem.

Example

Study the problem with kernel $F\left(t, x, \overset{\bullet}{x}\right) = \sqrt{1 + \left(\overset{\bullet}{x}\right)^2}$.

This is a convex function as

$$\begin{aligned} \frac{d}{du}\sqrt{1+u^2} &= \frac{u}{\sqrt{1+u^2}} \\ \frac{d^2}{du^2}\sqrt{1+u^2} &= \frac{1\sqrt{1+u^2} - u\frac{u}{\sqrt{1+u^2}}}{1+u^2} = \\ &= \frac{\frac{1+u^2-u^2}{\sqrt{1+u^2}}}{1+u^2} = \frac{1}{\left(1+u^2\right)^{3/2}} > 0. \end{aligned}$$

Example

Study the problem with kernel

$$F\left(t, x, \overset{\bullet}{x}\right) = x\sqrt{1 + \left(\overset{\bullet}{x}\right)^2}$$

This a product of two convex functions.

$$\det\left(H - \lambda I\right) = \begin{vmatrix} -\lambda & \overset{\bullet}{x} / \sqrt{1 + \left(\overset{\bullet}{x}\right)^2} \\ \overset{\bullet}{x} / \sqrt{1 + \left(\overset{\bullet}{x}\right)^2} & x / \left(1 + \left(\overset{\bullet}{x}\right)^2\right)^{3/2} - \lambda \end{vmatrix}$$

$$\lambda^2 - \frac{\lambda x}{\left(1+\left(\dot{x}\right)^2\right)^{3/2}} - \frac{\left(\dot{x}\right)^2}{1+\left(\dot{x}\right)^2}$$

$$\frac{x}{\left(1+\left(\dot{x}\right)^2\right)^{3/2}} \pm \sqrt{\frac{x^2}{\left(1+\left(\dot{x}\right)^2\right)^3} + 4\frac{\left(\dot{x}\right)^2}{1+\left(\dot{x}\right)^2}}$$

so the roots are positive and negative, so the function $F\left(t, x, \dot{x}\right)$ is not convex in $\left(x, \dot{x}\right)$.

Homework

- Analyze the problems

$$\int_0^1 x^2 + \left(\dot{x}\right)^2 dt \to \begin{matrix} \min \\ \max \end{matrix}, \quad x(0) = x(1) = 0.$$

- Analyze the problems

$$\int_0^T U\left(c - \dot{x}e^{rt}\right) dt, \quad x(0) = x_0, x(T) = 0$$

  where c and r and x_0 are positive constants, U is a continuously differentiable function.
- Solve the above problem if $U(z) = -e^{zv}/v$ where v is a positive constant.

Homework

- Solve the problem

$$\int_0^T e^{-t/4} \ln\left(2K - \dot{K}\right) dt \to \max, \quad K(0) = K_0, K(T) = K_T.$$

- Solve the problem

$$\int_0^T e^{-t/10} \left(\frac{1}{100} tK - \left(\dot{K}\right)^2\right) dt \to \max, \quad K(0) = 0, K(T) = S.$$

- Solve the problem

$$\int_0^1 1 - x^2 - \left(\dot{x}\right)^2 dt \to \max, \quad x(0) = 1.$$

- Solve the problem

$$\int_0^1 1 - x^2 - \left(\dot{x}\right)^2 dt \to \max, \quad x(1) = 1.$$

Homework

- Find the general solution of the Euler–Lagrange equation of the functional
$$\int_a^b \frac{\left(\dot{x}\right)^2}{t^3} dt$$
where $a > 0$.
- Find the general solution of the Euler–Lagrange equation of the functionals
$$\int_a^b \sqrt{t}\sqrt{1 + \left(\dot{x}\right)^2} dt$$
$$\int_a^b t\sqrt{1 + \left(\dot{x}\right)^2} dt$$
- Find the extremal solutions
$$\int_2^3 x^2 \left(1 - \dot{x}\right)^2 dx, \quad x(2) = 1, x(3) = \sqrt{3}.$$

Homework

- Find the extremal solutions

$$\int_a^b xy' + (y')^2 \, dx$$

- Find the extremal solutions

$$\int_0^\pi 4x\cos t + \dot{x}^2 - x^2 dt, \quad x(0) = x(\pi) = 0.$$

- Show that the linear functional

$$\int_a^b a(t)\dot{x}(t) + b(t)x(t) + c(t)\, dt,$$

where b and c are continuous and $a \neq 0$ is continuously differentiable has no extremal solution.

Optimal control

Definition

The problem

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \to \max$$

$$\overset{\bullet}{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0$$

$$u(t) \in U,$$

is called the optimal control problem. The variable $x(t_1)$ can be fixed or free. The set U is called the control region.

Example

The problem

$$\int_{t_0}^{t_1} f\left(t, x\left(t\right), u\left(t\right)\right) dt \rightarrow \max$$

$$\overset{\bullet}{x}\left(t\right) = u\left(t\right), \quad x\left(t_0\right) = x_0, x\left(t_1\right) = x_1, u\left(t\right) \in \left[0, 1\right]$$

is an optimal control problem. Observe that in this case there is a restriction on $\overset{\bullet}{x}$. The control region is $U = \left[0, 1\right]$.

Definition

The expression

$$H(t, x, u, p) = f(t, x, u) + p \cdot g(t, x, u)$$

is called the Hamiltonian. The variable p is called the adjoint or co-state variable. In some problems

$$H(t, x, u, p) = p_0 \cdot f(t, x, u) + p \cdot g(t, x, u)$$

and it is possible that $p_0 = 0$.

Optimal control

Theorem (Maximum principle)

If an $(x^*(t), u^*(t))$ *is an optimal solution of the problem then there is some adjoint function* $p(t)$ *such that* $u^*(t)$ *is maximizes the Hamiltonian at* $x^*(t)$

$$u \mapsto H(t, x^*(t), u, p(t)), \quad u \in U.$$

The adjoint function solves the differential equation

$$\frac{d}{dt}p(t) = -H'_x(t, x^*(t), u^*(t), p(t))$$

with some further conditions on the values of p *at the time* t_1 *depending on the boundary conditions on* $x(t_1)$. *This further conditions on* p *are called the transversality conditions:*

1. $x(t_1) = x_1$ *fixed then there is no restriction on* $p(t_1)$,
2. $x(t_1)$ *is free, in this case* $p(t_1) = 0$.

Optimal control

Theorem (Mangasarian)

If U is a convex set and $H(t, x, u, p(t))$ is concave in (x, u) then the solution of the Maximum Principle above $(x^*(t), u^*(t))$ is an optimal solution of the problem.

Theorem (Arrow)

Let

$$\widehat{H}(t, x, p) \stackrel{\circ}{=} \max \{H(t, x, u, p) \mid u \in U\}$$

assuming that the maximum is attained. If $\widehat{H}(t, x, p(t))$ is concave in x then if $(x^*(t), u^*(t))$ is a solution of the Maximum Principle above then $(x^*(t), u^*(t))$ solves the optimization problem.

Optimal control, regularity

Example

Solve the problem

$$\int_0^T u dt \longrightarrow \max$$

$$\overset{\bullet}{x} = u^2, \quad x(0) = x(T) = 0.$$

As $\overset{\bullet}{x} \geq 0$ and $x(0) = x(T) = 0$ the only feasible solution is $u = 0$, so there is an optimal solution. If $p_0 \neq 0$ then the Hamiltonian is $H = u + pu^2$. As there is no restriction on u at the maximum

$$0 = H_u = 1 + 2pu,$$

which is not satisfied by $u = 0$. If $H = p_0 \cdot u + p \cdot u^2$ then

$$H_u = p_0 + 2pu$$

is satisfied by $u = 0$ if $p_0 = 0$ and p is arbitrary.

Optimal control, regularity

Assume that $x(T)$ is free. In this case $p(T) = 0$ and if $p_0 = 0$ then

$$\begin{aligned}\frac{d}{dt}p(t) &= -H'_x(t, x^*(t), u^*(t), p(t)) = \\ &= 0 - p(t)\, g_x(t, x^*(t), u^*(t)).\end{aligned}$$

This is a linear differential equation with "initial condition" $p(T) = 0$. As $p(t) \equiv 0$ is a solution and as it is a linear equation so it has a unique solution $(p_0, p(t)) \equiv 0$ which is impossible.
We will ignore the regularity problem and we will always assume that $p_0 = 1$.

Optimal control, explanation, interpretation

One can interpret $p(t)$ as the shadow price for the state variable $x(t)$. At every moment of time the infinitesimal contribution is

$$\begin{aligned}\frac{dW(t)}{dt} &= f(t, x(t), u(t)) + (x(t)\,p(t))' = \\ &= f(t, x(t), u(t)) + x'(t)\,p(t) + x(t)\,p'(t) = \\ &= f(t, x(t), u(t)) + p(t)\,g(t, x(t), u(t)) + x(t)\,p'(t) = \\ &= H(t, x(t), u(t)) + x(t)\,p'(t).\end{aligned}$$

the direct inflow and the indirect change of the wealth. (Direct cash inflow and change of the value of the stocks.)

Optimal control, explanation, interpretation

If $(x(t), u(t))$ is optimal then they must be optimal at every moment of time. If not change the path accordingly. (May be not true as sometimes one must undertake some short term sacrifices. There are local and global criteria for optimality.) So if you are greedy then you must choose x as

$$\frac{\partial}{\partial x}\left(H(t, x, u, p) + xp'\right) = H_x + p' = 0 \Rightarrow p' = -H_x.$$

Observe that there are no constraints on the state variable. Now if $x(t)$ and therefore its price $p(t)$ is already given then $x(t)\, p\prime(t)$ is also given and one should directly maximize H with respect to u.(There are constraints on u so we cannot simply calculate the stationary points.)

Optimal control and calculus of variation

Let $F\left(t, x, \overset{\bullet}{x}\right)$ be a kernel for a calculus of variation problem. In this case

$$H = F(t, x, u) + pu.$$

By the optimality condition $H_u = 0 = F'_u + p$ that is

$$F'_u = F'_{\overset{\bullet}{x}} = -p, \quad \overset{\bullet}{p} = -H'_x = -F'_x$$

That is

$$\frac{d}{dt} F'_{\overset{\bullet}{x}} = -\frac{d}{dt} p = H'_x = F'_x.$$

Also by the condition on maximality by u at the optimal solution for every t

$$0 \geq H''_{u,u} = F''_{u,u} = F''_{\dot{x},\dot{u}}\left(t, x\left(t\right), \dot{x}\left(t\right)\right).$$

This is the so-called Legrende condition. One should remark that unfortunately the condition $F''_{\dot{x},\dot{u}} < 0$ is also just a necessary and not a sufficient condition.

Example

Solve the problem

$$\int_0^T 1 - tx(t) - u(t)^2 \, dt \to \max$$

$$\dot{x}(t) = u(t), \quad x(0) = x_0.$$

There is no boundary condition on $x(T)$ that is $x(T)$ is free. In this case the transversality condition is $p(T) = 0$. The Hamiltonian is

$$H(t, x, u, p) = 1 - tx - u^2 + pu.$$

Optimal control

If there is no restriction on $u \in U$ then to optimize the Hamiltonian with respect to u it is necessary that $H_u = 0$.

$$H'_u = -2u + p = 0, \quad \Rightarrow u^* = p/2.$$

The adjoint equation is

$$\frac{d}{dt}p(t) = -H'_x = t, \quad p(T) = 0.$$

From this

$$p(t) = \frac{1}{2}t^2 + C = \frac{1}{2}\left(t^2 - T^2\right).$$

This implies that

$$u^*(t) = \frac{1}{4}\left(t^2 - T^2\right).$$

Writing it back to the equation $\overset{\bullet}{x} = u$

$$x^{*}(t) - x(0) = \int_0^t u(s)\, ds = \int_0^t \frac{1}{4}\left(s^2 - T^2\right) ds = \frac{1}{12}t^3 - \frac{1}{4}T^2 t.$$

As the Hamiltonian is concave in (x, u) it is a real maximum.

Example

Solve the problem

$$\int_0^T x^2(t) + u^2(t)\,dt \to \min, \quad \overset{\bullet}{x}(t) = u(t)$$
$$x(0) = x_0.$$

The Hamiltonian is

$$\begin{aligned} H(t, x, u, p) &= -x^2 - u^2 + pu. \\ H'_x &= -2x, \quad H'_u = -2u + p. \end{aligned}$$

That is

$$\frac{dp}{dt} = 2x(t), \quad p(T) = 0, \quad u^*(t) = p(t)/2 \Rightarrow \frac{dx}{dt} = p(t)/2.$$

Optimal control

So

$$\frac{d^2p}{dt^2} = 2\frac{dx}{dt} = 2p(t)/2 = p(t).$$

It is a second order linear equation with characteristic polynomial $\lambda^2 - 1 = 0$. Using the general solution

$$p(t) = c_1 \exp(t) + c_2 \exp(-t), \quad p(T) = 0$$

On the other hand

$$\frac{dp}{dt}(0) = 2x(0) = 2x_0$$

$$\begin{aligned} c_1 &= 2x_0 e^{-T} / \left(e^T + e^{-T}\right), \quad c_2 = -2x_0 e^T / \left(e^T + e^{-T}\right) \\ x^*(t) &= \frac{1}{2}\frac{dp}{dt} = x_0 \frac{e^{T-t} + e^{t-T}}{e^T + e^{-T}} \end{aligned}$$

As there was no restriction on u this problem is in fact the same as the problem

$$\int_0^T x^2(t) + \left(\dot{x}\right)^2(t)\,dt \to \min, \quad x(0) = x_0.$$

The Euler–Lagrange equation is

$$2x = F'_x = \frac{d}{dt}F'_{\dot{x}} = \frac{d}{dt}2\dot{x} = 2\ddot{x}.$$

As $x(T)$ is free we need the transversality condition

$$0 = -F'_{\dot{x}}\left(t, x(T), \dot{x}(T)\right) = -2\dot{x}(T).$$

The characteristic polynomial of the second order linear equation is $\lambda^2 - 1 = 0$, so the general solution of the Euler–Lagrange equation is

$$x(t) = c_1 e^t + c_2 e^{-t}.$$

Using the initial condition and the transversality condition

$$\begin{aligned} x_0 &= c_1 + c_2, \\ 0 &= c_1 e^{T} - c_2 e^{-T}, c_1 = c_2 e^{-2T} \end{aligned}$$

$$x_0 = c_2 \left(1 + e^{-2T}\right), c_2 = x_0 / \left(1 + e^{-2T}\right) = \frac{x_0 e^{T}}{e^{T} + e^{-T}}$$

Hence

$$x(t) = \frac{x_0 e^{t-T}}{e^{T} + e^{-T}} + x_0 \frac{e^{T-t}}{e^{T} + e^{-T}}.$$

Example

Solve the problem

$$2\pi \int_0^T x\sqrt{1+u^2}dt \to \min, \quad \dot{x} = u, x(0) = x_0, x(T) = x_T.$$

The Hamiltonian is $H(t,x,u,p) = -x\sqrt{1+u^2} + pu$.

$$\frac{dp}{dt} = \sqrt{1+u^2}$$

Obviously $\dot{p} \neq 0$.

$$\begin{aligned} 0 &= H'_u = \frac{-xu}{\sqrt{1+u^2}} + p, \\ p &= \frac{xu}{\sqrt{1+u^2}} = \frac{xu}{\dot{p}} = \frac{x\dot{x}}{\dot{p}} \\ \dot{p}p &= \dot{x}x. \end{aligned}$$

Optimal control

$$\begin{aligned}
\frac{d}{dt}p^2 &= \frac{d}{dt}x^2, \quad x^2 = p^2 + C. \\
x^2 &= p^2 + C = \frac{(xu)^2}{1+u^2} + C \\
x^2 + x^2u^2 &= (xu)^2 + C\left(1+u^2\right) \\
x^2 &= C\left(1+u^2\right) = C\left(1+\left(\overset{\bullet}{x}\right)^2\right)
\end{aligned}$$

Optimal control

If $C = 0$ then $x \equiv 0$ which generally does not satisfy the boundary conditions. If $C \neq 0$ then $C > 0$ hence $x^2 > 0$

$$x^2 - C\left(\dot{x}\right)^2 = C$$

$$\begin{aligned} 2x\dot{x} &= 2C\dot{x}\ddot{x}, \\ \ddot{x} - \frac{x}{C} &= 0. \end{aligned}$$

From here the solution is the same as above.

Example

Let us consider the problem and try to apply Arrow''s condition.

$$2\pi \int_a^b x\sqrt{1+\left(\overset{\bullet}{x}\right)^2}dt \to \min .$$

The Hamiltonian is $H(t,x,u,p) = -x\sqrt{1+u^2}+pu$. This is a concave function in u. So for the maximization

$$0 = H_u' = p - x\frac{u}{\sqrt{1+u^2}}.$$

At $p(t)$ it is

$$p(t) = x\frac{u}{\sqrt{1+u^2}}$$

which is not solvable for every x, in u so the problem has no solution for every x so Arrow's condition is not applicable.

Optimal control, minimum problems

One can handle a minimum problem by changing the sign of the goal function. The Hamiltonian is

$$H = -f + p \cdot g, \quad \frac{dp}{dt} = -H'_x = -\left(-f_x + p \cdot g_x\right).$$

Multiplying by -1 and introducing $\widetilde{p} = -p$

$$-\frac{dp}{dt} = \left(-f_x + pg_x\right), \quad \frac{d\widetilde{p}}{dt} = -\left(f_x + \widetilde{p}g_x\right) = -\widetilde{H}'_x.$$

That is $\widetilde{p}$ is an solution of the adjoint equation of the maximum problem.

$$\int_{t_0}^{t_1} f dt \to \max, \quad \overset{\bullet}{x} = g.$$

$u^*(t)$ is maximizing $H = -f + p \cdot g = -\left(f + \widetilde{p} \cdot g\right)$ and therefore there is solution $\widetilde{p}$ of the adjoint equation for which $u^*(t)$ is minimizing the Hamiltonian $H = f + \widetilde{p}g$. In this case one can talk about a minimum principle.

Optimal control

Example

Solve the problem

$$\int_0^1 x(t)\,dt \to \max$$

$$\overset{\bullet}{x}(t) = x(t) + u(t), \quad x(0) = 0, \quad u(t) \in [-1, 1].$$

The Hamiltonian is

$$H(t, x, u, p) = x + p(x + u).$$

H is concave in (x, u) so every solution is a maximum. As there is no restriction on $x(1)$ the adjoint equation is

$$\overset{\bullet}{p} = -H'_x = -(1 + p), \quad p(1) = 0$$

Optimal control

The adjoint equation

$$\frac{d}{dt}p + p = -1$$

is linear

$$\begin{aligned} \left(e^t p\right)' &= -e^t \\ e^t p &= -e^t + C \\ p &= -1 + \frac{C}{e^t} \end{aligned}$$

$C = e$ as by the transversality condition $p(1) = 0$, hence

$$p(t) = e^{1-t} - 1.$$

Optimal control

As $p(t) > 0$ for every $t \in (0, 1)$ the max of the $H(t, x, u, p) = x + p(x + u)$ over $u \in [-1, 1]$ is $u^*(t) \equiv 1$. As

$$\overset{\bullet}{x}(t) = x(t) + u(t) = x(t) + 1.$$

It is again linear the general solution is

$$x(t) = Ce^t - 1.$$

Using the initial condition $x(0) = 0$ $C = 1$, so the optimal solution is

$$x(t) = \exp(t) - 1.$$

Example

Study the problem

$$\int_0^2 u^2 - xdt \longrightarrow \max, \quad \overset{\bullet}{x} = u, x(0) = 0, 0 \leq u \leq 1.$$

The Hamiltonian is $H(t, x, u, p) = u^2 - x + pu$. One cannot use the Mangasarian theorem as u^2 is convex and not concave.

$$\frac{dp}{dt} = -H'_x = 1. \quad \Rightarrow p(t) = t + C.$$

As $x(2)$ is free $p(2) = 0$ that is $2 + C = 0$ so $C = -2$,

$$p(t) = t - 2.$$

Optimal control

By the maximum principle $u^{*}(t)$ is maximizes

$$H(t, x, u, p(t)) = u^{2} - x + (t-2)u, u \in [0,1].$$

The optimization in u is independent of x. As H is convex in u the maximum is attained at the extremal points of $[0,1]$. Hence $u^{*}(t)$ is zero or one. If $u^{*}(t) = 0$ then $H_0 = -x$. If $u^{*}(t) = 1$ then $H_1 = 1 - x + (t-2)$ and in this case

$$H_0 \leq H_1 \Leftrightarrow 0 \leq 1 + t - 2 = t - 1.$$

So

$$\overset{\bullet}{x}(t) = u^{*}(t) = \begin{cases} 0 & \text{if} \quad t \in (0,1) \\ 1 & \text{if} \quad t \in (1,2) \end{cases}$$

Hence the optimal solution is

$$x^{*}(t) = \begin{cases} 0 & \text{if} \quad t \in [0,1] \\ t-1 & \text{if} \quad t \in (1,2] \end{cases}.$$

The partially optimized Hamiltonian is

$$\begin{aligned}\widehat{H}\left(t,x,p\left(t\right)\right) &= \max_{u\in[0,1]}\left(u^2-x+\left(t-2\right)u\right)= \\ &= \begin{cases} -x & \text{if} \quad t\in[0,1] \\ -x+t-1 & \text{if} \quad t\in(1,2] \end{cases}\end{aligned}$$

which is concave for every t. Hence we can use Arrow's theorem.

Optimal control

Example

Solve the problem

$$\int_0^1 2x(t) - x^2(t)\,dt \to \max$$

$$\overset{\bullet}{x}(t) = u(t), \quad x(0) = 0, x(1) = 0, u(t) \in [-1, 1].$$

The Hamiltonian is

$$\begin{aligned} H(t, x, u, p) &= 2x - x^2 + pu. \\ \frac{d}{dt}p &= -H'_x = 2x - 2. \end{aligned}$$

As there is a terminal condition on x there is no transversality condition on p.

Optimal control

The optimal u^* is maximizes the Hamiltonian $2x - x^2 + pu$ over $U = [-1, 1]$ so

$$u^*(t) = \begin{cases} 1 & \text{if} \quad p(t) > 0 \\ -1 & \text{if} \quad p(t) < 0 \end{cases}.$$

As $\dot{x} = u$ and $u \leq 1$ and $x(0) = 0$ $x(t) < 1$ for $t \in (0, 1)$. From the adjoint equation

$$\frac{d}{dt}p = -H'_x = 2x - 2 = 2(x - 1) < 0$$

so p is strictly decreasing.

Optimal control

- If $p(1) > 0$ then as p is decreasing $p(t) > 0$ for every $0 < t < 1$, which implies that $u^* \equiv 1$. In this case $\dot{x} = u^* = 1$. Using that $x(0) = 0$ $x(t) = t$, which is not feasible as the boundary condition $x(1) = 0$ is not valid. Hence $p(1) < 0$.
- If $p(0) < 0$ then as p is decreasing $p(t) < 0$ for every t. Hence one has that $u^* \equiv -1$ and therefore as $\dot{x} = u^* = -1$. Using that $x(0) = 0$ one has that $x(t) = -t$ and hence the boundary condition $x(1) = 0$ is not valid again. Hence $p(0) > 0$.

Optimal control

As p is continuous there is a point $t^* \in (0,1)$ where $p(t^*) = 0$. $p(t) > 0$ on $(0, t^*)$ hence on this interval $u^* = 1$ so as $x(0) = 0$ $x(t) = t$. On the interval $(t^*, 1)$ $p(t) < 0$ so $u^* = -1$ and $x(t) = -t + C$ on this interval. As x is continuous

$$x(t^*-) = t^* = -t^* + C = x(t^*+), \quad 2t^* = C$$

As $x(1) = 0$

$$C - t = 2t^* - t = 2t^* - 1 = 0, \quad t^* = 1/2.$$

Hence $x(t) = 1 - t$ if $t \geq 1/2$.

$$x(t) = \begin{cases} t & \text{if} \quad t < 1/2 \\ 1 - t & \text{if} \quad t \geq 1/2 \end{cases}$$

As

$$\frac{dp}{dt} = 2(x-1) = \begin{cases} 2(t-1) & \text{if} \quad t < 1/2 \\ 2(1-t-1) & \text{if} \quad t \geq 1/2 \end{cases}$$

$$\begin{aligned} p(t) &= \int_0^t 2(s-1)\,ds = t^2 - 2t + C, \quad t < 1/2 \\ 0 &= p\left(\frac{1}{2}\right) = \frac{1}{4} - 1 + C, \Rightarrow C = 3/4 \\ p(t) &= \int_{1/2}^t -2sds = -t^2 + \frac{1}{4}, \quad t \geq 1/2. \end{aligned}$$

$$\int_0^T f(t, x(t), u(t))\, dt + \phi(x(T)) \to \max$$

$$\overset{\bullet}{x}(t) = g(t, x(t), u(t)),\ x(0) = x_0$$

By the fundamental theorem of the calculus

$$\begin{aligned} \phi(x(T)) &= \phi(x(0)) + \int_0^T \phi'(x(t))\, \overset{\bullet}{x}(t)\, dt = \\ &= \phi(x_0) + \int_0^T \phi'(x(t))\, g(t, x(t), u(t))\, dt. \end{aligned}$$

As x_0 is fixed one can write the goal function as

$$\int_0^T f(t, x(t), u(t)) + \phi'(x(t))\, g(t, x(t), u(t))\, dt \to \max .$$

Hence

$$
\begin{aligned}
H = f\left(t, x\left(t\right), u\left(t\right)\right) + \phi'\left(x\left(t\right)\right) g\left(t, x\left(t\right), u\left(t\right)\right) + \\
+ p\left(t\right) g\left(t, x\left(t\right), u\left(t\right)\right) = \\
= f\left(t, x\left(t\right), u\left(t\right)\right) + \left(p\left(t\right) + \phi'\left(x\left(t\right)\right)\right) g\left(t, x\left(t\right), u\left(t\right)\right) = \\
= f\left(t, x\left(t\right), u\left(t\right)\right) + \widetilde{p}\left(t\right) g\left(t, x\left(t\right), u\left(t\right)\right).
\end{aligned}
$$

The adjoint equation with the transversality condition

$$\begin{aligned} \overset{\bullet}{p} &= -H'_x = -\left(f'_x + \phi'' g + \phi' g'_x + p g'_x\right) = \\ &= -\left(f'_x + \phi'' \overset{\bullet}{x} + \widetilde{p} g'_x\right), p\left(T\right) = 0 \end{aligned}$$

Optimal control, Bolza problem

$$\frac{d}{dt}\widetilde{p}(t) = \frac{d}{dt}p(t) + \phi''(x(t))\overset{\bullet}{x}(t).$$

Hence

$$\frac{d}{dt}\widetilde{p}(t) = -\left(f'_x + \widetilde{p}g'_x\right), p(T) = 0.$$

If $\widetilde{H} = f + \widetilde{p}g$ then

$$\frac{d}{dt}\widetilde{p}(t) = -\widetilde{H}'_x, \quad \widetilde{p} = \phi'(x(T)).$$

Optimal control, Bolza problem

Example

Solve

$$\int_0^1 \dot{x}^2 - xdt + x^2(1) \to \min, x(0) = 0$$

as an optimal control problem.

$$\begin{aligned} & \int_0^1 u^2 - xdt + x^2(1) \\ \dot{x} &= u, x(0) = 0. \end{aligned}$$

Optimal control, Bolza problem

$$
\begin{aligned}
H &= u^2 - x + pu, \overset{\bullet}{p} = -H_x = 1, p = t + C \\
p(1) &= 1 + C = 2x(1), C = 2x(1) - 1, p(t) = t + 2x(1) - 1 \\
0 &= H'_u = 2u + p = 2u + t + 2x(1) - 1, \\
u &= -x(1) + 1/2 - t/2.
\end{aligned}
$$

Optimal control, Bolza problem

$$\begin{aligned}
x(t) &= x(0) + \int_0^t -x(1) + 1/2 - t/2dt = \\
&= -x(1)\,t + 1/2t - t^2/4. \\
2x(1) &= \frac{1}{2}\cdot 1 - \frac{1}{4}\cdot 1^2 = \frac{1}{4}, x(1) = \frac{1}{8}, \\
x(t) &= \frac{3}{8}t - \frac{t^2}{4}.
\end{aligned}$$

Optimal control, Bolza problem

Example

Solve the problem

$$\int_0^1 u^2 dt + x^2(1) \to \min$$

$$\dot{x} = x + u, x(0) = 1$$

Optimal control, Bolza problem

$$H = u^2 + p(x + u),$$

$$\dot{p} = -H'_x = -p,$$

$$p = C \exp(-t)$$

$$p(1) = 2x(1) = \frac{C}{e},$$

$$C = 2e \cdot x(1), p = 2x(1) \exp(-t + 1),$$

Optimal control, Bolza problem

$$0 = H'_u = 2u + p = 2u + 2x(1)\exp(-t+1),$$
$$u = -x(1)\exp(-t+1)$$

$$\dot{x} = x - x(1)\exp(-t+1), x(0) = 1$$

$$\dot{x} - x = -x(1)\exp(-t+1), x(0) = 1,$$

$$\dot{x}e^{-t} - xe^{-t} = -x(1)e^{-2t+1}$$

$$xe^{-t} = -x(1)\int e^{-2t+1}dt + C =$$

$$= x(1)\frac{e^{-2t+1}}{2} + C,$$

$$x(t) = \frac{x(1)}{2}e^{-t+1} + Ce^{t}$$

Optimal control, Bolza problem

$$\begin{aligned} x(1) &= \frac{x(1)}{2} + Ce, 1 = x(0) = \frac{x(1)}{2}e + C, \\ \frac{x(1)}{2} &= Ce, C = 1 - \frac{x(1)}{2}e, C = 1 - Ce^2, C = \frac{1}{1+e^2} \\ x(1) &= \frac{2e}{1+e^2} \end{aligned}$$

$$x(t) = \frac{e^2}{1+e^2}e^{-t} + \frac{1}{1+e^2}e^t.$$

We can also write as

$$\int_0^1 \left(\dot{x} - x\right)^2 + 2x\dot{x}dt = \int_0^1 \dot{x}^2 + x^2 dt \to \min, x\left(0\right) = 1.$$

The Euler–Lagrange equation is

$$\frac{d}{dt} 2\dot{x} = 2\ddot{x} = 2x, \ddot{x} - x = 0.$$

The characteristic equation is $\lambda^2 - 1 = 0$ and the roots are ± 1. The general solution is

$$x\left(t\right) = C_1 \exp\left(-t\right) + C_2 \exp\left(t\right).$$

Optimal control, Bolza problem

As $x(0) = 1$ $C_1 + C_2 = 1$. By the transversality condition

$$0 = F'_{\dot{x}}(1) = 2\dot{x}(1),$$
$$\dot{x}(1) = 0,$$
$$0 = -C_1 \frac{1}{e} + C_2 e, C_1 = C_2 e^2$$

so $C_1 = e^2 / (1 + e^2)$ and $C_2 = 1 / (1 + e^2)$.

Homework

1. Solve the problems in calculus of variations with the method of optimal control.
2. Show with the method of optimal control that the problem $\int_0^1 \sqrt{1+\left(\dot{x}\right)^2}dt \to \max\ x(0)=0, x(1)=1$ has no solution.
3. Show with the method of optimal control that the problem $\int_0^1 t^{\alpha}\sqrt{1+\left(\dot{x}\right)^2}dt \to \min\ x(0)=0, x(1)=1$ has no solution if $\alpha > 0$.
4. Let U be a concave increasing utility function and let x be a continuously differentiable consumption strategy. Show that there is no such strategy x^* which maximizes the cumulated utility $\int_0^1 U(x(t))\,dt$ with $x(0)=0$ and $x(1)=1$. What can we say if we introduce the restriction $\left|\dot{x}\right| \leq 1$?

Stability

Definition

An equilibrium point x^* of a dynamic system X is called attractive if for any initial value x_0 the solution $X(t, x_0)$ is convergent to x^* that is

$$\lim_{t \to \infty} X(t, x_0) = x^*, \forall x_0.$$

If it is true only for some neighborhood of x^* then x^* is called locally attractive.

Definition

An equilibrium point x^* of a dynamic system is (Lyapunov) stable if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\|x_0 - x^*\| < \delta$ $\|X(t, x_0) - x^*\| < \varepsilon$. The system is (Lyapunov) asymptotically stable if it is stable and it is attractive. If it is true only for some neighborhood of x^* then x^* is called locally asymptotically stable.

Stability

Example

The solution $y^* = 0$ of the system

$$ay'' + by' + cy = 0$$

is asymptotically stable if the real part of the characteristic roots is negative. If the real part of the characteristic roots are non positive and characteristic roots are unique then the system is stable.

The only case worth considering is when $\lambda = 0$ is a double root. That is when $\lambda^2 = 0$, that is $y'' = 0$. In this case the general solution is $y(t) = c_1 + tc_2$ which is not stable. The general solution

$$x(t) = c_1 \cos \theta t + c_2 \sin \theta t$$

is stable but not asymptotically stable.

Example

The solution $y^* = 0$ of the system

$$y_{t+2} + ay_{t+1} + by_t = 0$$

is asymptotically stable if the absolute value of the characteristic roots is smaller than one. If the absolute value of the characteristic roots are not bigger than one and the characteristic roots are unique then the system is stable.

When $\lambda = \pm 1$ is a double root, then $y_{t+2} \pm 2y_{t+1} + y_t = 0$. In this case the general solution is $y_t = (\pm 1)^t (c_1 + tc_2)$ which is not stable. The general solution

$$y_t = c_1 \cos \theta t + c_2 \sin \theta t$$

is stable but not asymptotically stable.

Stability

Example

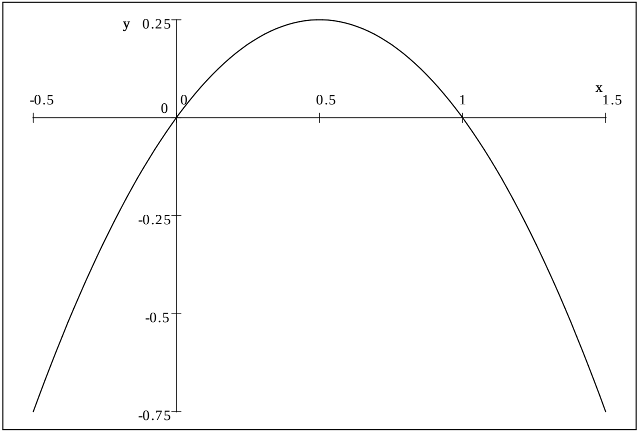
There is an example of a non-linear system which is attractive but not locally stable.

$$\begin{aligned} \dot{x}_1 &= \frac{x_1^2(x_2-x_1)+x_2^5}{(x_1^2+x_2^2)\left(1+(x_1^2+x_2^2)^2\right)} \\ \dot{x}_2 &= \frac{x_1^2(x_2-x_1)}{(x_1^2+x_2^2)\left(1+(x_1^2+x_2^2)^2\right)} \end{aligned}$$

Phase diagram in on dimension.

Let

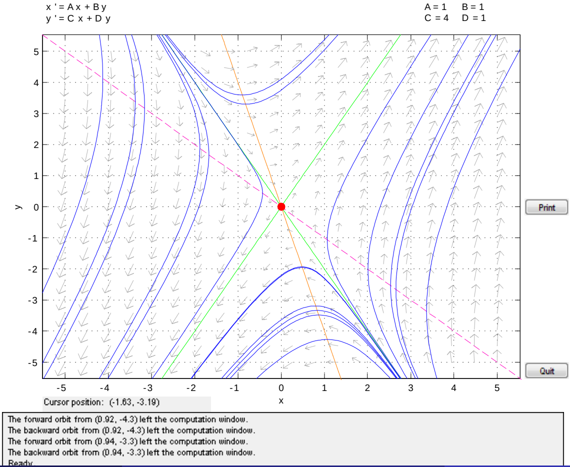
$$x' = x * (1 - x).$$



It has two equilibrium $x = 0$ and $x = 1$. $x = 0$ is instable, $x = 1$ is stable.

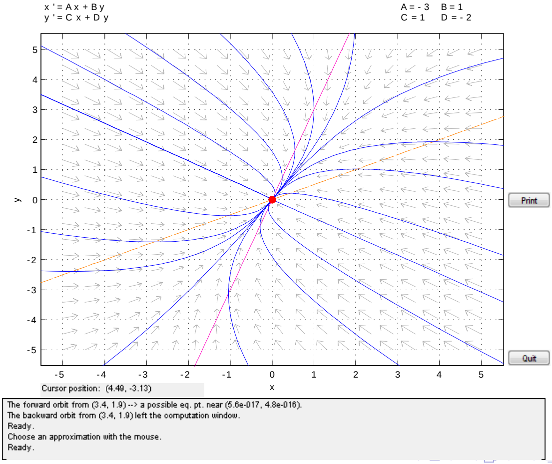
A saddle point.

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$



A node

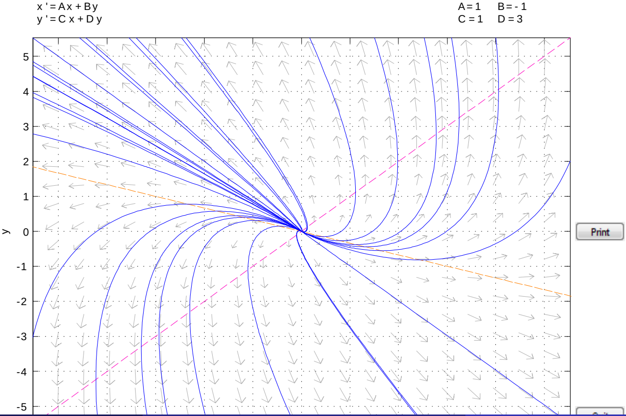
$$x' = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} x$$



Phase diagram

An improper node

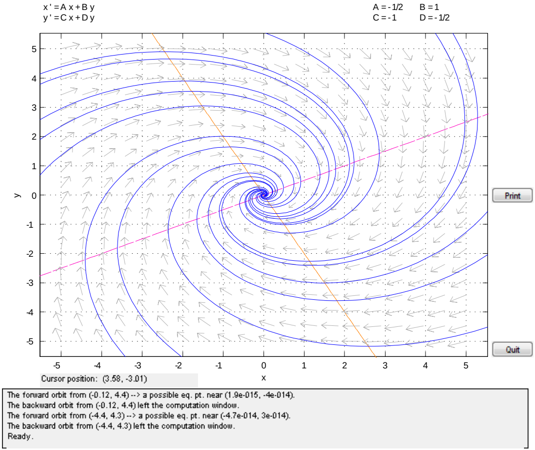
$$x' = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} x$$



Phase diagram

A focus

$$x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x$$



Phase diagram

1. Identify the four regions given by $x_1' = 0$ and $x_2' = 0$. This curves are called nullclines or demarcation lines.
2. Identify the directions of increase in all four regions.
3. We have a node when both eigenvalues are real and of the same sign. The node is stable when the eigenvalues are negative and unstable when they are positive.
4. When eigenvalues are real and of opposite signs we have a saddle point. The saddle is always unstable.
5. Focus (sometimes called spiral point) when eigenvalues are complex-conjugate. The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.
6. The center equilibrium occurs when a system has eigenvalues on the imaginary axis, namely, one pair of pure-imaginary eigenvalues. Centers in linear systems have concentric periodic orbits.