Mathematics and Economics

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Mathematics and Economics: Big Successes in History

- Léon Walras,
  Éléments d'économie politique pure 1874
- Francis Edgeworth,
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- John von Neumann, Oskar Morgenstern,
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- Paul Samuelson,
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- Kenneth Arrow, Gérard Debreu,
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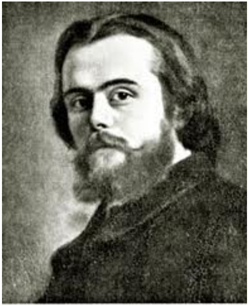

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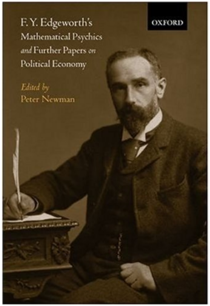

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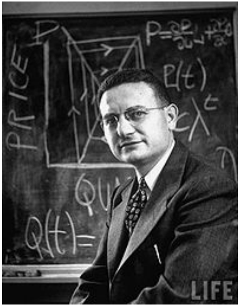

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Three Leading Questions

1. Rationality ?
   Isn't it simply wrong to impose heroic foresight and intellectual abilities to describe humans?
2. Egoism ?
   Humans show altruism, envy, passions etc.
3. Probability ?
   Doesn't the crisis show that mathematics is useless, even dangerous in markets?

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- These assumptions are frequently justified
- Aufklärung! . . . answers Kant's "Was soll ich tun?"
- design of institutions: good regulation must be robust against rational, egoistic agents – (Basel II was not, e.g.)
- Doubts remain . . .; Poincaré to Walras:

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- some blame mathematicians for financial crisis – nonsense, but
- does probability theory apply to single events like
- Ellsberg experiments

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based on Dufwenberg, Heidhues, Kirchsteiger, R., Sobel, *Review of Economic Studies 2010*

Empirical Evidence

- Humans react on their environment
- relative concerns, in particular with peers, are important
- especially in situations with few players
- not in anonymous situations

Fehr–Schmidt Other–regarding preferences matter in games, but not in markets

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General Equilibrium is the general theory of free, competitive markets with rational, self–interested agents

The Big Theorems

- Existence
- First Welfare Theorem: Equilibrium Allocations are efficient
- ... in the core, even
- Second Welfare Theorem: efficient allocations can be implemented via free markets and lump–sum transfers
- Core-Equivalence: in large economies, the outcome of rational cooperation (core) is close to market outcomes

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Simple, yet Famous Models

Other–Regarding Utility functions used to explain experimental data

- Fehr–Schmidt (Bolton–Ockenfels) introduce fairness and envy: $U_i = m_i - \frac{\alpha_i}{I-1}\sum_k \max\{(m_k - m_i), 0\} - \frac{\beta_i}{I-1}\sum_k \max\{(m_i - m_k), 0\}$
- Charness–Rabin: $m_i + \frac{\beta_i}{I-1}\left[\delta_i \min\{m_1, \ldots, m_I\} + (1-\delta_i)\sum_{j=1}^{I} m_j\right]$
- Edgeworth already has looked at $m^i + m^j$
- Shaked: such ad hoc models are not science (and Poincaré would agree)

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Mathematical Formulation

- in anonymous situations, an agent cannot debate prices or influence what others consume
- own consumption $x \in \mathbb{R}^L_+$, others' consumption $y \in \mathbb{R}^K$, prices $p \in \mathbb{R}^L_+$, income $w > 0$
- utility $u(x, y)$, strictly concave and smooth in x
- when is the solution $d(y, p, w)$ of

$$\text{maximize}\, u(x, y) \quad \text{subject to} \quad p \cdot x = w$$

independent of y ?

Definition

We say that agent i behaves as if selfish if her demand function $d_i\left(p, w, x^{-i}\right)$ does not depend on others' consumption plans x^{-i}.

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As–If Selfish Demand — Examples

- Clearly, standard "egoistic" utility functions $v_i(x_i) = v_i(x_{i1}, \ldots, v_{iL})$ lead to as-if selfish behavior
- Additive social preferences: let $U_i(x_i, x_j) = v_i(x_i) + v_j(x_j)$. Then marginal utilities are independent of x_j,
- Product Preferences: $U_i(x_i) = v_i(x_i)v_j(x_j) = v_i(x_{i1}, \ldots, v_{iL})v_j(x_{j1}, \ldots, v_{jL})$
  - marginal utilities do depend on others' consumption bundles but marginal rates of substitution do not! $\rightarrow$ as–if selfish behavior

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As–If Selfish Preferences

Theorem

Agent i behaves as if selfish if and only if her preferences can be represented by a separable utility function

$$V_i\left(m_i(x_i), x_{-i}\right)$$

where $m_i : X_i \to \mathbb{R}$ is the **internal utility function***, continuous, strictly monotone, strictly quasiconcave, and $V_i : D \subseteq \mathbb{R} \times \mathbb{R}_+^{(I-1)L} \to \mathbb{R}$ is an* **aggregator***, increasing in own utility m_i.*

Technical Assumption

Preferences are smooth enough such that demand is continuously differentiable. Needed for the “only if”.

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where $m_i : X_i \to \mathbb{R}$ is the **internal utility function**, *continuous, strictly monotone, strictly quasiconcave, and $V_i : D \subseteq \mathbb{R} \times \mathbb{R}_+^{(I-1)L} \to \mathbb{R}$ is an* **aggregator**, *increasing in own utility m_i.*

Technical Assumption

Preferences are smooth enough such that demand is continuously differentiable. Needed for the "only if".

General Consequences

- Free Markets are a good institution in the sense that they maximize *material* efficiency (in terms of $m_i(x_i)$)
- but not necessarily good as a social institution, i.e. in terms of real utility $u_i(x_i, x_{-i})$

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Inefficiency

Example

Take two agents, two commodities with the same internal utility and $U_i = m_1 + m_2$. Take as endowment an internally efficient allocation close to the edge of the box. Unique Walrasian equilibrium, but not efficient, as the rich agent would like to give endowment to the poor. Markets cannot make gifts!

Remark

Public goods are a way to make gifts. Heidhues/R. have an example in which the rich agent uses a public good to transfer utility to the poor agent (but still inefficient allocation).

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- in example: charity would lead to efficiency
- not true for more than 2 agents!
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Some (unplausible) preferences have to be ruled out:

Example

Hateful Society: $U_i = m_i - 2m_j$ for two agents $i \neq j$. No consumption is efficient.

Social Monotonicity

For $z \in \mathbb{R}^L_+ \setminus \{0\}$ and any allocation x, there exists a redistribution (z_i) with $\sum z_i = z$ such that

$$U_i(x + z) > U_i(x)$$

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Theorem

Under social monotonicity, the set of Pareto optima is included in the set of internal Pareto optima.

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Uncertainty Theory as new Probability Theory

As "P" is not exactly known, work with a whole class of probability measures $\mathscr{P}$, (Huber, 1982, Robust Statistics)

Knightian Decision Making

- Gilboa–Schmeidler: $U(X) = \min_{P \in \mathscr{P}^i} E^P u(x)$
- Föllmer–Schied, Maccheroni, Marinacci, Rustichini generalize to variational preferences

$$U(X) = \min_P E^P u(X) + c(P)$$

for a cost function c

- do not trust your model! be aware of sensitivities! do not believe in your EXCEL sheet!

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IMW Research on Optimal Stopping and Knightian Uncertainty

- Dynamic Coherent Risk Measures, *Stochastic Processes and Their Applications 2004*
- Optimal Stopping with Multiple Priors, *Econometrica, 2009*
- Optimal Stopping under Ambiguity in Continuous Time (with Xue Cheng), IMW Working Paper 2010
- The Best Choice Problem under Ambiguity (with Tatjana Chudjakow), IMW Working Paper 2009
- Chudjakow, Vorbrink, Exercise Strategies for American Exotic Options under Ambiguity, IMW Working Paper 2009
- Vorbrink, Financial Markets with Volatility Uncertainty, IMW Working Paper 2010
- Jan–Henrik Steg, Irreversible Investment in Oligopoly, Finance and Stochastics 2011

Optimal Stopping Problems: Classical Version

Let $\left(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t=0,1,2,\ldots}\right)$ be a filtered probability space.

- Given a sequence $X_0, X_1, \ldots, X_T$ of random variables
- adapted to the filtration $(\mathscr{F}_t)$
- choose a stopping time $\tau \leq T$
- that maximizes $\mathbb{E}X_\tau$.
- classic: Snell, Chow/Robbins/Siegmund: Great Expectations

Optimal Stopping Problems: Solution, Discrete Finite Time

based on R., Econometrica 2009

Solution

- Define the *Snell envelope* U via backward induction:

$$U_T = X_T$$
$$U_t = \max\{X_t, \mathbb{E}[U_{t+1}|\mathscr{F}_t]\} \qquad (t < T)$$

- U is the smallest supermartingale $\geq X$
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Two classics

The Parking Problem

- You drive along a road towards a theatre
- You want to park as close as possible to the theatre
- Parking spaces are free iid with probability $p > 0$
- When is the right time to stop? take the first free after 68%$1/p$ distance

Secretary Problem = When to Marry?

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Optimal Stopping with Multiple Priors: Discrete Time

We choose the following modeling approach

- Let $X_0, X_1, \ldots, X_T$ be a (finite) sequence of random variables
- adapted to a filtration $(\mathscr{F}_t)$
- on a measurable space $(\Omega, \mathscr{F})$,
- let $\mathscr{P}$ be a set of probability measures
- choose a stopping time $\tau \leq T$
- that maximizes

$$\inf_{P \in \mathscr{P}} \mathbb{E}^P X_\tau$$

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Assumptions

- (X_t) bounded by a $\mathscr{P}$–uniformly integrable random variable
- there exists a reference measure P^0: all $P \in \mathscr{P}$ are equivalent to P^0 (wlog, Tutsch, PhD 07)
- agent knows all null sets,
  Epstein/Marinacci 07
- $\mathscr{P}$ weakly compact in $L^1(\Omega, \mathscr{F}, P^0)$
- inf is always min,
  Föllmer/Schied 04, Chateauneuf, Maccheroni, Marinacci, Tallon 05

Extending the General Theory to Multiple Priors

Aims

- Work as close as possible along the classical lines
- Time Consistency
- Multiple Prior Martingale Theory
- Backward Induction

Time Consistency

- With general $\mathscr{P}$, one runs easily into inconsistencies in dynamic settings (Sarin/Wakker)
- Time consistency $\iff$ *law of iterated expectations*:

$$\min_{Q\in\mathscr{P}} \mathbb{E}^Q \left[\operatorname*{ess\,inf}_{P\in\mathscr{P}} \mathbb{E}^P \left[X \mid \mathscr{F}_t \right] \right] = \min_{P\in\mathscr{P}} \mathbb{E}^P X$$

- Literature on time consistency in decision theory /risk measure theory
  - Epstein/Schneider, R. , Artzner et al., Detlefsen/Scandolo, Peng, Chen/Epstein
  - time consistency is equivalent to *stability under pasting*:
    - let $P, Q \in \mathscr{P}$ and let $(p_t), (q_t)$ be the density processes
    - fix a stopping time τ
    - define a new measure R via setting

$$r_t = \begin{cases} p_t & \text{if } t \le \tau \\ p_\tau q_t / q_\tau & \text{else} \end{cases}$$

    - then $R \in \mathscr{P}$ as well

Multiple Prior Martingales

Definition

An adapted, bounded process (S_t) is called a multiple prior supermartingale iff

$$S_t \geq \operatorname*{ess\,inf}_{P \in \mathscr{P}} \mathbb{E}^P \left[S_{t+1} \mid \mathscr{F}_t \right]$$

holds true for all $t \geq 0$.
multiple prior martingale: $=$
multiple prior submartingale: $\leq$

Remark

- *Nonlinear notion of martingales.*
- *Different from $\mathscr{P}$–martingale (martingale for all $P \in \mathscr{P}$ simultaneously)*

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Characterization of Multiple Prior Martingales

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- *(S_t) is a multiple prior submartingale iff (S_t) is a $\mathscr{P}$–submartingale.*
- *(S_t) is a multiple prior supermartingale iff there exists a $P \in \mathscr{P}$ such that (S_t) is a P–supermartingale.*
- *(M_t) is a multiple prior martingale iff (M_t) is a $\mathscr{P}$–submartingale and for some $P \in \mathscr{P}$ a P–supermartingale.*

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For multiple prior supermartingales: $\Leftarrow$ holds always true.$\Rightarrow$ needs time–consistency.

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Let (S_t) be a multiple prior supermartingale.
Then there exists a multiple prior martingale M and a predictable, nondecreasing process A with $A_0 = 0$ such that $S = M - A$. Such a decomposition is unique.

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Let $(S_t)_{0\leq t\leq T}$ be a multiple prior supermartingale. Let $\sigma \leq \tau \leq T$ be stopping times. Then

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Not true without time consistency.

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With the concepts developed, one can proceed as in the classical case!

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$$U_t = \max\left\{X_t, \operatorname*{ess\,inf}_{P\in\mathscr{P}} \mathbb{E}^P\left[U_{t+1}|\mathscr{F}_t\right]\right\} \qquad (t < T)$$

- U is the smallest multiple prior supermartingale $\geq X$
- An optimal stopping time is given by $\tau^* = \inf\{t \geq 0 : X_t = U_t\}$.

Minimax Theorem

Question: what is the relation between the Snell envelopes U^P for fixed $P \in \mathscr{P}$ and the multiple prior Snell envelope U?

Theorem

$$U = \operatorname*{ess\,inf}_{P \in \mathscr{P}} U^P \,.$$

Corollary

Under our assumptions, there exists a measure $P^* \in \mathscr{P}$ such that $U = U^{P^*}$. The optimal stopping rule corresponds to the optimal stopping rule under P^*.

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Monotonicity and Stochastic Dominance

- Suppose that (Y_t) are iid under $P^* \in \mathscr{P}$ and
- for all $P \in \mathscr{P}$

$$P^*[Y_t \leq x] \geq P[Y_t \leq x] \qquad (x \in \mathbb{R})$$

- and suppose that the payoff $X_t = g(t, Y_t)$ for a function g that is isotone in y,
- then P^* is *for all optimal stopping problems* (X_t) the *worst–case measure*,
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Easy Examples

- Parking problem: choose the smallest p for open lots
- House sale: presume the least favorable distribution of bids in the sense of first–order stochastic dominance
- American Put: just presume the most positive possible drift

Diffusion Models

based on Cheng, R., IMW Working Paper 429

Framework now: Brownian motion W on a filtered probability space $(\Omega, \mathscr{F}, P_0, (\mathscr{F}_t))$ with the usual conditions

Typical Example: Ambiguous Drift $\mu_t(\omega) \in [-\kappa, \kappa]$

- $\mathscr{P} = \{P : W$ is Brownian motion with drift $\mu_t(\omega) \in [-\kappa, \kappa]\}$
- (for time–consistency: stochastic drift important!)
- worst case: either $+\kappa$ or $-\kappa$, depending on the state
- Let $\mathscr{E}_t X = \min_{P \in \mathscr{P}} E^P[X|\mathscr{F}_t]$
- we have the representation

$$-\mathscr{E}_t X = -\kappa Z_t dt + Z_t dW_t$$

  for some predictable process Z
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Time–consistent multiple priors are g–expectations

g–expectations

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Theorem (Delbaen, Peng, Rosazza Giannin)

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Optimal Stopping under g–expectations: Theory

Our Problem - recall

Let (X_t) be continuous, adapted, nonnegative process with $\sup_{t \le T} |X_t| \in L^2(P_0)$.
Let $g = g(\omega, t, z)$ be a standard concave driver (in particular, Lipschitz–continuous).
Find a stopping time $\tau \le T$ that maximizes

$$\mathcal{E}_0(X_\tau) .$$

Optimal Stopping under g–expectations: General Structure

Let

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathscr{E}_t(X_\tau)\,.$$

be the value function of our problem.

Theorem

1. *(V_t) is the smallest right–continuous g–supermartingale dominating (X_t);*
2. *$\tau^* = \inf\{t \geq 0 : V_t = X_t\}$ is an optimal stopping time;*
3. *the value function stopped at τ^*, $(V_{t\wedge\tau^*})$ is a g–martingale.*

Proof.

Our proof uses the properties of g–expectations like regularity, time–consistency, Fatou, etc. to mimic directly the classical proof (as, e.g., in Peskir,Shiryaev) with one additional topping: rightcontinous versions of g–supermartingales ∎

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Worst–Case Priors

Drift ambiguity

- V is a g–supermartingale
- from the Doob–Meyer–Peng decomposition

$$-dV_t = g(t, Z_t)dt - Z_t dW_t + dA_t$$

  for some increasing process A
- $= -\kappa|Z_t|dt - Z_t dW_t + dA_t$
- Girsanov: $= -Z_t dW_t^* + dA_t$ with kernel $\kappa \operatorname{sgn}(Z_t)$

Theorem (Duality for κ–ambiguity)

There exists a probability measure $P^* \in \mathscr{P}^\kappa$ such that $V_t = \operatorname{ess\,sup}_{\tau \geq t} \mathscr{E}_t(X_\tau) = \operatorname{ess\,sup}_{\tau \geq t} E^*[X_\tau | \mathscr{F}_t]$. In particular:

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Markov Models

- the state variable S solves a *forward SDE*, e.g.

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t, \quad S_0 = 1\,.$$

- Let

$$\mathscr{L} = \mu(x)\frac{\partial}{\partial x} + \sigma^2(x)\frac{\partial^2}{\partial x^2}$$

be the infinitesimal generator of S.

- By Itô's formula, $v(t, S_t)$ is a martingale if

$$v_t(t,x) + \mathscr{L}v(t,x) = 0 \tag{1}$$

- similarly, $v(t, S_t)$ is a *g–martingale* if

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PDE Approach: A Modified HJB Equation

Theorem (Verification Theorem)

Let v be a viscosity solution of the g–HJB equation

$$\max\left\{f(x) - v(t,x), v_t(t,x) + \mathscr{L}v(t,x) + \mathbf{g(t, v_x(t,x)\sigma(x))}\right\} = 0\,. \quad (3)$$

Then $V_t = v(t, S_t)$.

- nonlinearity only in the first–order term
- numeric analysis feasible
- ambiguity introduces an additional nonlinear drift term

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$$\max\left\{f(x) - v(t,x), v_t(t,x) + \mathscr{L} v(t,x) + \mathbf{g}(\mathbf{t}, \mathbf{v_x}(\mathbf{t}, \mathbf{x})\sigma(\mathbf{x}))\right\} = 0\,. \quad (3)$$

Then $V_t = v(t, S_t)$.

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More general problems

- With monotonicity and stochastic dominance, worst–case prior easy to identify
- In general, the worst–case prior is *path–dependent* even in iid settings
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- Shout Options
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based on Chudjakow, R., IMW Working Paper 413

- Call applicant j a *candidate* if she is better than all predecessors
- We are interested in $X_j = Prob[j \text{best} | j \text{candidate}]$
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Classic Secretary Problem

- only interested in the best applicant
- introduce $Y_n = 1$ if applicant n beats all previous applicants, else 0
- by uniform probability, the (Y_n) are independent and $P[Y_n = 1] = 1/n$.
- show that optimal rules must be simple, i.e. of the form

$$\tau_r = \inf \{k \geq r : Y_k = 1\}$$

- success of rule τ_r is

$$\frac{r-1}{N} \sum_{n=r}^{N} \frac{1}{n-1} \approx \frac{r-1}{N} \log \frac{N}{r-1}$$

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- we need a time–consistent multiple prior version of the model
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Reformulation in Adapted Payoffs

- problem: X is not adapted
- take $X_n = \min_{P \in \mathscr{P}} E^P \left[Z_n \middle| Y_1, \ldots, Y_n \right]$
- by the law of iterated expectations and the optional sampling theorem
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Reduction to a Monotone Problem

- $$X_n = Y_n \min_P P[Y_{n+1} = 0, \ldots, Y_N = 0]$$
  $$= Y_n \prod_{k=n+1}^{N} (1 - b_k)$$
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Solution

- the optimal stopping rule is simple
- the payoff of simple rule r is recursively given by
- $\phi(N) = a_N$
- $\phi(r) = a_r B_r + (1 - a_r)\phi(r+1)$
- explicit solution

$$\phi(r) = \sum_{n=r}^{N} \beta_n \prod_{k=r}^{n-1} \alpha_k$$

for

$$\beta_n = \frac{a_n}{1 - b_n}, \alpha_n = \frac{1 - a_n}{1 - b_n}$$

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American Straddle in the Bachelier Model for Drift Ambiguity

Suppose we want to stop $X_t = W_t$ under κ–ambiguity for an interest rate $r > 0$, i.e.

$$\max_{\tau} \mathcal{E}(|X_\tau| e^{-r\tau}) .$$

Claim: under the worst–case measure P^*, the process X has dynamics

$$dX_t = -\operatorname{sgn}(X_t)dt + dW_t^*$$

for the P^*–Brownian motion W^*.

American Straddle in the Bachelier Model for Drift Ambiguity

g–HJB equation: in the continuation set

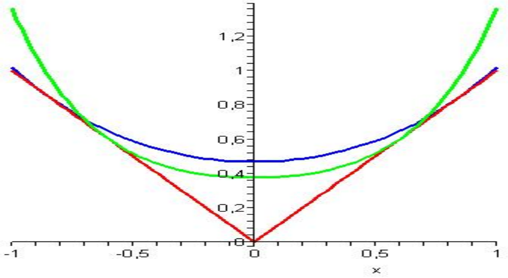
$$v_t + \frac{1}{2} v_{xx} - \kappa |v_x| = 0$$

Verification: solve the standard optimal stopping problem under P^*. There, the HJB equation reads

$$v_t + \frac{1}{2} v_{xx} - \kappa \, \mathrm{sgn}(x) v_x = 0$$

Show $\mathrm{sgn}(v_x) = \mathrm{sgn}(x)$, then this equation becomes the g–HJB equation and we are done.

American Straddle in the Bachelier Model for Drift Ambiguity



Evolution as Alternative to Rationality

- The other end of the scale: no rationality at all
- the forces of nature
  - overreproduction
  - selection
  - mutation
- as powerful as a basis for a theory as rationality
- Evolutionary Game Theory started with two biologists, Maynard Smith, Price, 1973
- Oechssler, R., *Journal of Economic Theory 2002*, Cressman, Hofbauer, R., *Journal of Theoretical Biology*, 2006 develop evolutionary game theory as dynamic systems on the Banach space of finite measures over metric spaces,

$$\frac{d}{dt}P_t(A) = \int_A \sigma\left(x, P_t\right) P_t(dx)$$

- Louge, R., Auctions, IMW Working Paper 2010

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Evolution of Languages

based on Jäger, Metzger, R., SFB 673 Project 6

Language

- sensation (Sinneseindruck) is complex: color, shape, size, location, temperature . . .
- only few words available

Job Market Signaling

Finance

Evolution of Languages

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Language

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Communication in large population between two randomly matched players

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Efficient Languages

Cooperative approach

- players use a meta–language to find the best language
- minimize $E\ l\left(\|s - i_{w(s)}\|\right) = \int_S l\left(\|s - i_{w(s)}\|\right) F(ds)$ over measurable functions $w : S \to W$ and $i : W \to S$

Theorem

Efficient languages exist.

Remark

Proof requires analysis of optimal signaling systems w given some interpretation i, then essentially compactness and continuity ...

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Efficient Languages: Optimal Signaling

Best Choice of Words

- Suppose the hearer interprets word w_j as point i_j
- suppose sensation s is given
- which word is the best?
- choose the word w_j such that the distance from interpretation i_j to sensation s is minimal
- $w^* = \arg\min \{||s - i_j|| : j = 1, \ldots, n\}$
- this leads to a **Voronoi tesselation**

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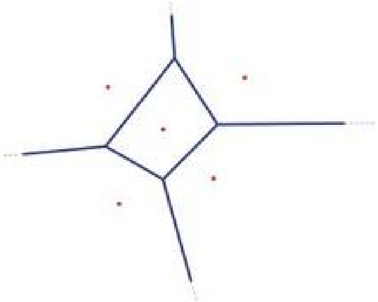
Voronoi Tesselations

Definition

Given distinct points $i_1, \ldots, i_n \in [0,1]^d$, the Voronoi tesselation assigns to (almost all) points $s \in [0,1]^d$ the unique closest point i_j to s. The convex set

$$C_j = \left\{ s \in [0,1]^d : \|s - i_j\| = \min_{k=1,\ldots,n} \|s - i_k\| \right\}$$

is called the Voronoi cell for i_j



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A Voronoi language consists of a Voronoi tesselation for the speaker and a best estimator interpretation for the hearer.

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Strict Nash equilibria are Voronoi languages with full vocabulary and vice versa.

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Case Study: $d = 1$, quadratic loss, Two Words

Speaker can say $w \in \{left, right\}$
uniform distribution

- Speaker chooses a threshold $\theta \in (0, 1)$
- says "left" if $s < \theta$, else "right", or vice versa
- Hearer interprets "left" as $i_1 = \theta/2$, "right" as $i_2 = (1 + \theta)/2$
- in equilibrium, i_1, i_2 must generate the Voronoi tesselation with boundary θ
- $(x_1 + x_2)/2 = \theta \Leftrightarrow \theta = 1/2$
- unique strict Nash equilibrium
- maximizes social welfare
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- Voronoi tesselations correspond to trapezoids
- there are only three (!) Voronoi languages (up to symmetry)
- only two with full vocabulary
  - left and right rectangle
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Two words in a rectangle with unequal sides

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