Acta Mathematicae Applicatae Sinica, English Series
Vol. 20, No. 4 (2004) 675–684

Acta Mathematicae Applicatae Sinica, English Series

A Capital Asset Pricing Model Under Stable Paretian Distributions in a Pure Exchange Economy

Xiao-hua Wang[1], **Zhi-xiong Wen**[2], **Zhuo Huang**[3]

[1,2]Department of Mathematics, Wuhan University, Wuhan 430072 China
[2](E-mail: zhxwen@whu.edu.cn)
[3]IAS, Wuhan University, Wuhan, 430072 China

Abstract In this paper, we established a Capital Asset Pricing Model (CAPM) subject to the assumption that the asset return rates obey symmetric stable Paretian distributions. This assumption seems to be closer to reality than the standard ones such as normality or finite variance. Conclusion similar to the original CAPM formula is drawn in this paper.

Keywords Stable Paretian distribution, CAPM, equilibrium, exchange economy
2000 MR Subject Classification

1 Introduction

The capital asset pricing model (CAPM) was the first attempt to explain the asset return behavior (with one factor) and has undergone considerable theoretical development in the last thirty years. In the context of CAPM a risky asset's beta with respect to the market portfolio is a sufficient statistic for its contribution to the riskiness of an individual's portfolio. Risky assets whose payoffs are positively correlated with those of the market portfolio have positive premiums. In such events, the higher the beta, the higher the premium is. That is, the equilibrium price of a risky asset price should be determined by its risk related to the market. In practice, financial experts can use regression to calculate the beta of each security and then work out its equilibrium price to guide their investments. In the classical versions of CAPM, the returns of risky assets are normally distributed, and the riskiness of a portfolio is measured by its variance. Merton[9] added a temporal dimension to CAPM by modeling asset returns by a diffusion process. Black[1] in his"zero-beta" version relaxed the assumptions on risk-free borrowing. Chamkerlain[2] showed that the hypothesis of normality can be replaced by the weaker one of finite variance. But neither the original CAPM nor its extensions seemed really satisfactory when empirically tested. All these papers assume square integrability or, more strongly, normality of asset returns. If this is not the case, the statistical test may suffer from inconsistency.

Perters[10] studied the frequency distributions of 5-Day and 90-Day Dow Jones Industrials returns from January 2, 1888 until December 31, 1991. He showed that both the return distributions were characterized by a high peak at the mean and fatter tails than the normal distribution. The two Dow distributions were virtually the same shape (that is, they were similar in terms of statistics). In fact, the results of empirical researches on another capital

Manuscript received January 20, 2003. Revised March 1, 2004.
Supported by the Special Funds for Major State Basic Research Projects of China

market returns indicated all their return rates do not obey normal distribution, but obey the distributions like the Dow distributions as above. Mandelbrot[7] proposed that the return rates in the capital market have stable Paretian distributions, which have peaks at the points of expectation, and fat tail. The distributions tend to have treads and discontinuity. Moreover it can adjust to skewness, mostly like the observed frequency distribution of securities.

Fama[3] established a CAPM for symmetric stable Paretian returns using the mathematics of portfolio frontier. In this paper, we also established a CAPM for symmetric stable Paretian returns within the framework of a pure exchange economy.

2 Stable Paretian Distributions and Their Properties

A random variable, denoted by R, is said to be stable Paretian if there exists for all positive a and b, and positive real numbers c and $d \in R$ such that

$$aR_1 + bR_2 \stackrel{d}{=} cR + d$$

where R_1 and R_2 are independent copies of R. Alternatively stated, R is Paretian stable if and only if its characteristic function is of the form

$$\Phi_R(\theta) = E(\exp iR\theta) = \begin{cases} \exp\Big\{ -\sigma^\alpha |\theta|^\alpha (1 - i\beta \text{sign}\theta \tan\dfrac{\pi\alpha}{2}) + i\mu\theta \Big\} & \text{if } \alpha \neq 1 \\ \exp\Big\{ -\sigma|\theta|(1 + i\beta\text{sign}\theta \ln|\theta|) + i\mu\theta \Big\} & \text{if } \alpha = 1 \end{cases}$$

where sign(x) is the sign function, $\alpha(0 < \alpha \leq 2)$ is the index of stability, $\beta(-1 \leq \beta \leq 1)$ the skewness parameter, $\sigma > 0$ the scale parameter, and $\mu \in R$ the location parameter. If $\alpha > 1$, the location parameter, μ, corresponds to the mean value of the distribution; and the scale parameter, σ, can be regarded as a generalized version of the standard deviation.

Follow Samorodnitaky and Taqqu[13], let $R = S_\alpha(\theta, \beta, \mu)$. For the symmetric case, i.e. $\beta = 0$, the characteristic function of the stable Paretian distribution reduces to

$$\Phi_R(\theta) = E(\exp iR\theta) = \exp\left(-\sigma^\alpha |\theta|^\alpha + i\mu\theta\right)$$

Similarly an n-dimensional vector of random variables, denoted by R, is stable Paretian, if there exists, for all positive a and b, a positive real number c and a vector $d \in R^n$, such that

$$R_1 + R_2 \stackrel{d}{=} cR + d$$

where R_1 and R_2 are independent copies of R. The characteristic function, $\Phi_R(\theta) = E(\exp(it\theta' R))$, of a stable Paretian vector is given by

$$\Phi_R(\theta) = \exp\Big\{ -\int_{S^n} |\theta' s|^\alpha \Big(1 - i\text{sign}(\theta' s)\tan\frac{\pi\alpha}{2}\Big)\Gamma(ds) + i\theta'\mu \Big\}, \qquad \text{if } \alpha \neq 1$$

and

$$\Phi_R(\theta) = \exp\Big\{ -\int_{S^n} |\theta' s|(1 + i\text{sign}(\theta' s)\ln|\theta' s|)\,\Gamma(ds) + i\theta'\mu \Big\}, \qquad \text{if } \alpha = 1$$

where $\theta = (\theta_1, \cdots, \theta_n)$; Γ represents a finite measure on the unit sphere $S^n \subset R^n$. The spectral measure Γ of a symmetric (about μ) stable vector R is also symmetric; and the characteristic function is of the form

$$\Phi_R(\theta) = \exp\Big\{ -\int_{s^n} |\theta' s|\,\Gamma(ds) + i\theta'\mu \Big\}.$$

Property 2.1. *If $R = S_\alpha(\sigma, \beta, \mu)$, then for any $a, b \in R$,*

$$aR + b = S_\alpha(|a|\sigma, \beta \text{sign} a, \mu + b).$$

Similar property holds for an n-dimensional stable Paretian vector.

Denote by $\sigma(\cdot)$ a function which assigns the scale parameter to a random variable, we can show that a linear combination of the elements of a stable Paretian vector satisfies

$$\sigma^\alpha(\theta' R) = \sigma^\alpha(\theta_1 R_1 + \cdots + \theta_n R_n) = \int_{S^n} |\theta' s|^\alpha \, \Gamma(ds).$$

If stable paretian distributions have the same index, skewness and location parameters, then they are first-order stochastically ordered according to their scale parameter values . i.e. for the distribution function $F(\cdot; \alpha, \beta, \sigma_i, \mu)$, which characteristic function is given by (2.1), we have

$$F(\cdot; \alpha, \beta, \sigma_1, \mu) \geq F(\cdot; \alpha, \beta, \sigma_2, \mu), \qquad \forall \; x$$

if and only if $\sigma_1 \leq \sigma_2$. Given a concave utility function, the asset associated with σ_1 will be preferred, if $\sigma_1 \leq \sigma_2$. Hence, the scale parameter is a suitable measure for risk.

Given the similarities between the variance in the Gaussion case, and quantity σ^α (the so-called variation) in the non-Gaussion stable Paretian case, as measures of risk, we next discuss the extension of the concept of covariance to that of covariation between two random variables.

Samorodnitsky and Taqqu[10] proposed the concept of covariation as a corresponding tool for symmetric stable Paretian laws.

The covariation between two symmetric stable Paretian random variables, say R_1 and R_2, with identical α's, denoted by $[R_1; R_2]_\alpha$, is defined by

$$[R_1; R_2]_\alpha = \int_{S^2} s_1 s_2{}^{<\alpha - 1>} \, \Gamma(ds)$$

where $x^{<k>} = |x|^k \text{sign}(x)$, let $0 < \alpha \leq 2$

In the Gaussion case, $\alpha = 2$

$$[R_1; R_2]_\alpha = \frac{1}{2} \text{cov}(R_1, R_2)$$

where $\text{cov}(R_1, R_2)$ is the covariance between R_1 and R_2.

Property 2.2. *If R_1, R_2, and R_3 are three symmetric stable Paretian random variables with identical α's, then*

a) $[\lambda R_1; R_2]_\alpha = \lambda [R_1; R_2]_\alpha$ *if* $\lambda \in R$ *and* $[R_1 + R_3; R_2]_\alpha = [R_1; R_2]_\alpha + [R_3; R_2]_\alpha$

b) $[R_1; \lambda R_2]_\alpha = \lambda^{<\alpha - 1>} [R_1; R_2]_\alpha$ *and* $[R_1; R_2 + R_3]_\alpha = [R_1; R_2]_\alpha + [R_1; R_3]_\alpha$

subject to the assumption that R_2 and R_3 are independent.

If R_1 and R_2 are independent, then their covariation is zero. However the converse is generally not true. Also, covariation is generally not symmetric, i.e., in general, $[R_1; R_2]\alpha \neq [R_2; R_1]_\alpha$.

Moreover, we have

$$[R_1; R_2]_\alpha \leq \sigma(R_1) \sigma^{\alpha - 1}(R_2)$$

and

$$\nu_\alpha(R) = [R; R]_\alpha = \sigma^\alpha(R)$$

where $\nu_\alpha(R) = [R; R]_\alpha$ is the so-called variation of α-stable Paretian R random variable.

Property 2.3. *If $R = (R_1, \cdots, R_n)$ is the basis of E_α which is a symmetric stable Paretian space. Let Ψ be a linear functional on E_α, then there exists a unique vector $\theta \in R^n$, such that*

$$\Psi(\cdot) \equiv [\cdot; \theta' R]_\alpha.$$

The property is based on the fact that if $R = (R_1, \cdots, R_n)$ is a linearly independent family of symmetric stable Paretian r.v. (with identical α's), then the function measuring the variation of a linear combination of these random variables

$$\gamma(\lambda) = \sigma^\alpha(\lambda' R) \qquad \lambda \in R^n$$

is strictly convex.

3 Asset-pricing Model Under Stable Paretian Distributions

Suppose the capital market consists of $K+1$ limited-liability assets, one of which is riskless asset and others are risky assets. That is , M =span$\{x_0, \cdots, x_K\}$ (represents linear space generated by x_j $j = 0, \cdots, K$), where x_j: $\Omega \to R$, $j = 0, \cdots, K$. $X = \{x_j\}_{j=1}^K$ is a linearly independent family of symmetric stable Paretian random variables (with identical α's), $x_0(\omega) = 1$ for any $\omega \in \Omega$.

As x_j is limited-liability security ($x_j \geq 0$), we have $p(x_j) \neq 0$. The return rate of x_j is defined as

$$r(x_j) = \frac{x_j - p(x_j)}{p(x_j)}, \qquad j = 0, \cdots, K.$$

Suppose investor $i = 1, \cdots, I$ each individual has a mean-risk utility function, i.e. $U^i\big(C_0, EC_1, \nu_\alpha(C_1)\big)$, where C_0, C_1 denote consumption at the initial and last periods respectively ,called consumption at period-0 and period-1.

Here

$$\frac{\partial U^i\big(C_0, EC_1, \nu_\alpha(C_1)\big)}{\partial EC_1} > 0,$$
$$\frac{\partial U^i\big(C_0, EC_1, \nu_\alpha(C_1)\big)}{\partial\big(\nu_\alpha(C_1)\big)} < 0.$$

Investor i has an initial endowment $\big(\overline{C_0^K}, \overline{C_1^K}(\omega)\big)$, $\omega \in \Omega$, $\overline{C_0^i} > 0$, $\overline{C_1^i}(\omega) = \overline{C_1^i}$, for any $\omega \in \Omega$, $\sum_{i=1}^I \overline{C_1^i} > 0$, and the initial endowment of assets $\{\overline{N_0^i}, \cdots, \overline{N_K^i}\}$, $\sum_{i=1}^I \overline{N_j^i} > 0$, $j = 0, \cdots, K$. In order to convert money into units of consumption, we suppose there exists an exogenous price level of consumption goods at period-0 $\rho_0 > 0$, and price level at period-1 $\rho_1(\omega) > 0$ for any $\omega \in \Omega$.

In this framework, investor i chooses $\{C_0, N_0, \cdots, N_K\}$ to maximize his utility function

$$\max U^i\big(C_0, EC_1, \nu_\alpha(C_1)\big) \tag{3.1}$$

subject to

$$\overline{C_0^i} + \sum_{j=0}^K \overline{N_j^i} p(x_j)/\rho_0 = C_0 + \sum_{j=0}^K N_j p(x_j)/\rho_0, \tag{3.2}$$

$$\overline{C_1^i} + \sum_{j=0}^K N_j x_j(\omega)/\rho_1(\omega) = C_1(\omega) \qquad \text{for any } \omega \in \Omega. \tag{3.3}$$

To simplify the calculations, we assume the consumption market has an exogenous inflation, that is, $\rho_1(\omega) = \rho_1$ for any $\omega \in \Omega$.

Using the constraint (3·3), we can rewrite the investor i's choice problem (3·1). Substituting the equations

$$\begin{aligned} EC_1 =& \overline{C_1^i} + \sum_{j=1}^{K} N_j E(x_j/\rho_1), \\ \nu_\alpha(C_1) =& \Big[\overline{C_1^i} + \sum_{j=0}^{K} N_j x_j/\rho_1; \overline{C_1^i} + \sum_{j=0}^{K} N_j x_j/\rho_1\Big]_\alpha, \\ =& \Big[\sum_{j=0}^{K} N_j x_j/\rho_1; \sum_{j=0}^{K} N_j x_j/\rho_1\Big]_\alpha = \nu_\alpha(\sum_{j=0}^{K} N_j x_j/\rho_1) \end{aligned}$$

into (3.1), we have

$$\max_{\{C_0,N_0,\cdots,N_K\}} U^i\Big(C_0, \overline{C_1^i} + \sum_{j=1}^{K} N_j E(x_j/\rho_1), \nu_\alpha\Big(\sum_{j=0}^{K} N_j x_j/\rho_1\Big)\Big) \tag{3.4}$$

subject to

$$\overline{C_0^i} + \sum_{j=0}^{K} \overline{N_j^i} p(x_j)\Big/\rho_0 = C_0 + \sum_{j=0}^{K} N_j p(x_j)/\rho_0. \tag{3.5}$$

Define the real return rate of asset j as

$$R(x_j) = \frac{x_j/\rho_1 - p(x_j)/\rho_0}{p(x_j)/\rho_0} \qquad \text{for any} \quad j = 0, \cdots, K$$

Specially, when $\rho_1 = \rho_0$, that is no inflation, the real return of assets equals nominal return rate.

Theorem 3.1 (Optimal Consumption and Choice of Assets).

$(C_0^i, C_1^i, N_0^i, \cdots, N_K^i)$ is investor i's optimal consumption and choice of assets if and only if it is the solution of the following equations.

$$E\big[R(x_j)\big] - R(x_0) + \alpha\Big(\frac{\partial U^i/\partial \nu_\alpha}{\partial U^i/\partial EC_1}\Big)\Big[R(x_j); \sum_{l=0}^{K} N_l^i x_l/\rho_1\Big]_\alpha = 0, \qquad j = 1, \cdots, K, \tag{3.6}$$

$$1 + R(x_0) = \frac{\partial U^i/\partial C_0}{\partial U^i/\partial EC_1}, \tag{3.7}$$

$$\overline{C_0^i} + \sum_{j=0}^{K} \overline{N_j^i} p(x_j)/\rho_0 = C_0^i + \sum_{j=0}^{K} N_j^i p(x_j)/\rho_0, \tag{3.8}$$

$$C_1^i(\omega) = \overline{C_1^i} + \sum_{j=0}^{K} N_j^i x_j(\omega)/\rho_1(\omega). \tag{3.9}$$

Proof. Consider the Lagrange Function

$$\begin{aligned} \pounds(c_0, n_0, \cdots, N_K, \lambda) =& U^i\Big(C_0, \overline{C_1^i} + \sum_{j=0}^{K} N_j E(x_j/\rho_1), \nu_\alpha\Big(\sum_{j=0}^{K} N_j x_j/\rho_1\Big)\Big) \\ &+ \lambda\Big(\overline{C_0^i} + \sum_{j=0}^{K} \overline{N_j^i} p(x_j)/\rho_0 - C_0 - \sum_{j=0}^{K} N_j p(x_j)/\rho_0\Big) \end{aligned}$$

subject to first order necessary and sufficient conditions stated as

$$\frac{\partial \pounds}{\partial C_0} = \frac{\partial U^i}{\partial C_0} - \lambda = 0, \tag{3.10}$$

$$\begin{aligned}\frac{\partial \pounds}{\partial N_j} =& \partial U^i/\partial(EC_1)E(x_j/\rho_1) + (\partial U^i/\partial \nu_\alpha)(\partial \nu_\alpha/\partial N_j) - \lambda p(x_j)/\rho_0 \\ =& \partial U^i/\partial(EC_1)E(x_j/\rho_1) + (\partial U^i/\partial \nu_\alpha)\Big(\alpha\Big[x_j/\rho_1; \sum_{l=0}^{K} N_l^i x_l/\rho_1\Big]_\alpha\Big) - \lambda p(x_j)/\rho_0 \\ =& 0 \qquad \text{for any} \quad j = 0, 1, \cdots, K \end{aligned} \tag{3.11}$$

$$\frac{\partial \pounds}{\partial \lambda} = \overline{C_0^i} + \sum_{j=0}^{K} \overline{N_j^i} p(x_j)/\rho_0 - C_0^i - \sum_{j=0}^{K} N_j^i p(x_j)/\rho_0 = 0 \tag{3.12}$$

when $j = 0$. Since $x_0(\omega) \equiv 1$ for any $\omega \in \Omega$. It also follows from (3.11) that

$$\frac{\partial U^i}{\partial EC_1} \frac{E(x_0/\rho_1)}{p(x_0)/\rho_0} = \lambda. \tag{3.13}$$

We substitute the equation $1 + R(x_0) = \frac{E(x_0/\rho_1)}{p(x_0)/\rho_0}$ into (3.10) and (3.13) and get

$$1 + R(x_0) = \frac{\partial U^i/\partial C_0}{\partial U^i/\partial EC_1}.$$

Substitute (3.13) into (3.11) we get

$$E[R(x_j)] - R(x_0) + \alpha\Big(\frac{\partial U^i/\partial \nu_\alpha}{\partial U^i/\partial EC_1}\Big)\Big[R(x_j); \sum_{l=0}^{K} N_l^i x_l/\rho_1\Big]_\alpha = 0, \qquad j = 1, \cdots, K.$$

Definition 3.1 (Competitive Equilibrium). *Competitive equilibrium is defined as the consumption and security prices* $\{C_0^i, C_1^i, N_0^i, \cdots, N_K^i\}_{i\in I}$ *and security prices* $\{p(x_0), \cdots, p(x_K)\}$ *satisfing:*

a) For any investor i, $(C_0^i, N_0^i, \cdots, N_K^i)$ *satisfy (3.1), (3.2) and (3.3) or equivalently*

b) (3.6) (3.7) (3.5) and (3.9)

$$\begin{aligned}&\sum_{i\in I} N_j^i = \sum_{i\in I} \overline{N_j^i}, \qquad j = 0, \cdots, K, \\ &\sum_{i\in I} C_0^i = \sum_{i\in I} \overline{C_0^i}, \\ &\sum_{i\in I} C_1^i = \sum_{i\in I} \overline{C_1^i} + \sum_{i\in I}\Big(\sum_{j=0}^{K} N_j^i x_j/\rho_0\Big).\end{aligned}$$

As for investor i, his optimal portfolio of risky assets $(\omega_1^i, \cdots, \omega_K^i)$ has a return $x_m^i \stackrel{\text{def}}{=}$

$\sum_{j=1}^{K} N_j^i x_j$. Thus the return rate can be written as

$$R(x_m^i) = \frac{\sum_{j=1}^{K} N_j^i x_j/\rho_1 - \sum_{j=1}^{K} N_j^i p(x_j)/\rho_0}{\sum_{j=1}^{K} N_j^i p(x_j)/\rho_0}$$

$$= \sum_{j=1}^{K} \frac{N_j^i p(x_j)/\rho_0}{\sum_{l=1}^{K} N_l^i p(x_l)/\rho_0} \cdot \frac{N_j^i x_j/\rho_1 - N_j^i p(x_j)/\rho_0}{N_j^i p(x_j)/\rho_0}$$

$$= \sum_{j=1}^{K} \omega_j^i R(x_j)$$

where $\omega_j^i = \frac{N_j^i p(x_j)/\rho_0}{\sum_{l=1}^{K} N_l^i p(x_l)/\rho_0}$, $\quad \sum_{j=1}^{K} \omega_j^i = 1$.

Theorem 3.2 (Investment Diversification). *If* $\{N_0^i, \cdots, N_K^i\}_{i\in I}$ *and* $\{p(x_0), \cdots, p(x_K)\}$ *are the equilibrium security allocation and prices, then* $N_j^i > 0,\ j = 1, \cdots, K$ *for any* $i \in I$.

Proof. Let

$$A^i \stackrel{def}{=} -\alpha\Big(\frac{\partial U^i/\partial \nu_\alpha}{\partial U^i/\partial EC_1}\Big),$$

obviously $A^i > 0$,

$$E[R(x_j)] - R(x_0) = A^i\Big[R(x_j); \sum_{l=1}^{K} N_l^i x_l/\rho_1\Big]_\alpha$$

$$= A^i\Big[R(x_j); \sum_{l=1}^{K} N_l^i p(x_l)/\rho_0\Big(\sum_{j=1}^{K} N_j^i x_j/\rho_1\Big)\Big/\Big(\sum_{l=1}^{K} N_l^i p(x_l)/\rho_0\Big)\Big]_\alpha$$

$$= A^i\Big[R(x_j); \sum_{l=1}^{K} N_l^i p(x_l)/\rho_0 R(x_m^i)\Big]_\alpha$$

$$E\big[R(x_j) - R(x_0)\big] = A^i\Big[R(x_j) - R(x_0); \sum_{l=1}^{K} N_l^i p(x_l)\Big/\rho_0 \sum_{s=1}^{K} \omega_s^i\big(R(x_s) - R(x_0)\big)\Big]_\alpha$$

$$= \Big[R(x_j) - R(x_0); \sum_{s=1}^{K} (A^i)^{1/(\alpha-1)} N_s^i p(x_s)/\rho_0\big(R(x_s) - R(x_0)\big)\Big]_\alpha.$$

Let $\Psi(x) = E\big[R(x_j) - R(x_0)\big]$ where $x = R(x_j) - R(x_0)$. Obviously, $\Psi(x)$ is a linear functional, according to Property (2.3), we can obtain

$$(A^i)^{1/(\alpha-1)} N_s^i p(x_s)/\rho_0 = (A^{i'})^{1/(\alpha-1)} N_s^{i'} p(x_s)/\rho_0, \qquad \forall i, i' \in I. \tag{3.14}$$

Therefore

$$\text{sign}(N_s^i) = \text{sign}(N_s^{i'}), \qquad s = 1, \cdots, K, \qquad \forall i \neq i', \quad i, i' \in I.$$

Finally, at the equilibrium point

$$\sum_{i\in I} N_j^i = \sum_{i\in I} \overline{N_j^i} > 0,$$

we have

$$N_j^i > 0, \qquad j = 1, \cdots, K, \qquad i \in I.$$

This theorem shows that every investor will hold a strictly positive amount of each kind of risky asset, whatever his utility function is and whatever initial amount of risky assets he holds.

Theorem 3.3 (CAPM). *If $\{N_0^i, \cdots, N_K^i\}_{i\in I}$ and $\{p(x_0), \cdots, p(x_K)\}$ are equilibrium security allocation and prices, then $\omega_j^i = \omega_j^{i'}$, $j = 1, \cdots, K$ for any $i \neq i'$, $i, i' \in I$ and we have the following equation*

$$E[R(x_j)] = R(x_0) + \frac{\big[R(x_j); R(x_m)\big]_\alpha}{\nu_\alpha(R(x_m))}\Big(E\big[R(x_m) - R(x_0)\big]\Big), \qquad j = 1, \cdots, K.$$

Proof. Sum equations (3.14) for s, we get

$$(A^i)^{(1/(\alpha-1))} \sum_{s=1}^K N_s^i p(x_s)/\rho_0 = (A^{i'})^{(1/(\alpha-1))} \sum_{s=1}^K N_s^{i'} p(x_s)/\rho_0 > 0, \qquad \forall i \neq i', \quad i, i' \in I,$$

therefore we get

$$\frac{N_j^i p(x_j)/\rho_0}{\sum\limits_{j=1}^K N_j^i p(x_j)/\rho_0} = \frac{N_j^{i'} p(x_j)/\rho_0}{\sum\limits_{s=1}^K N_j^{i'} p(x_j)/\rho_0}, \qquad j = 1, \cdots, K, \qquad \forall i \neq i', \quad i, i' \in I.$$

Thus, according to the definition of ω_j^i, we obtain

$$\omega_j^i = \omega_j^{i'}, \qquad j = 1, \cdots, K, \qquad \forall i \neq i', \quad i, i' \in I,$$

i.e.

$$R(x_m^i) = R(x_m^{i'}) \stackrel{\text{def}}{=} R(x_m), \qquad \forall\ i \neq i', \quad i, i' \in I.$$

On the other hand

$$\begin{aligned} E\big[R(x_j)\big] - R(x_0) =& A^i \Big[R(x_j); \sum_{l=1}^K N_l^i (p(x_l)/\rho_0) R(x_m)\Big]_\alpha \\ =& A^i \Big(\sum_{l=1}^K N_l^i p(x_l)/\rho_0\Big)^{\alpha-1} \big[R(x_j); R(x_m)\big]_\alpha, \end{aligned}$$

so we get

$$\begin{aligned} \sum_{j=1}^K \omega_j \big(E[R(x_j)] - R(x_0)\big) =& \sum_{j=1}^K \omega_j A^i \Big(\sum_{l=1}^K N_l^i p(x_l)/\rho_0\Big)^{\alpha-1} \big[R(x_j) - R(x_m)\big]_\alpha \\ =& A^i \Big(\sum_{l=1}^K N_l^i p(x_l)/\rho_0\Big)^{\alpha-1} \Big[\sum_{j=1}^K \omega_j R(x_j); R(x_m)\Big]_\alpha \\ =& A^i \Big(\sum_{l=1}^K N_l^i p(x_l)/\rho_0\Big)^{\alpha-1} \big[R(x_m); R(x_m)\big]_\alpha, \end{aligned}$$

i.e.,

$$E[R(x_m)] - R(x_0) = A^i \Big(\sum_{l=1}^K N_l^i p(x_l)/\rho_0\Big)^{\alpha-1} \nu_\alpha(x_m), \qquad \forall i \in I.$$

Since

$$(A^i)^{1/(\alpha-1)} N_s^i p(x_s)/\rho_0 = (A^{i'})^{1/(\alpha-1)} N_s^{i'} p(x_s)/\rho_0, \qquad \forall i, i' \in I,$$

i.e.

$$(A^i)\Big(\sum_{s=1}^{K} N_s^i p(x_s)/\rho_0\Big)^{\alpha-1} = (A^{i'})\Big(\sum_{s=1}^{K} N_s^{i'} p(x_s)/\rho_0\Big)^{\alpha-1}, \qquad \forall i, i' \in I,$$

we have

$$\frac{E\big[R(x_j)\big] - R(x_0)}{E\big[R(x_m)\big] - R(x_0)} = \frac{\big[R(x_j); R(x_m)\big]_\alpha}{\nu_\alpha\big(R(x_m)\big)},$$

i.e.,

$$E\big[R(x_j)\big] = R(x_0) + \beta_{im}\big(E\big[R(x_m)\big] - R(x_o)\big)$$

where

$$\beta_{im} = \frac{\big[R(x_j); R(x_m)\big]_\alpha}{\nu_\alpha\big(R(x_m)\big)}, \qquad j = 1, \cdots, K.$$

Here $R(x_0)$ is the risk free interest rate, $E[R(x_j)] - R(x_o)$ is security j's excess return beyond the market, $E[R(x_m)] - R(x_o)$ is the market risk premium and β_{im} is a similar measure of the security j's risk related to the market. Therefore, we reach a formula like the traditional CAPM formula, when the returns obey symmetric stable Paretian distributions. This result generalizes the CAPM theory and chare with it the same economic intuition, namely, there is no free lunch in the investment market; if you want to get excess returns, you must undertake excess risk related to the whole market.

4 Summary

When the asset return rates obey normal distributions, the CAPM formula given as

$$E\big(R(x_j)\big) = R(x_0) + \frac{\mathrm{cov}\big(R(x_j), R(x_m)\big)}{\mathrm{var}\big(R(x_m)\big)}\big(E(R(x_m)\big) - R(x_0)\big)$$

where $var\big(R(x_m)\big)$ is the variation of $R(x_m)$. In this paper, we established a similar CAPM formula which may be written as

$$E\big(R(x_j)\big) = R(x_0) + \frac{[R(x_j); R(x_m)]_\alpha}{\nu_\alpha\big(R(x_m)\big)}\big(E(R(x_m)) - R(x_0)\big)$$

when the asset return rates obey the symmetric stable Paretian distributions. However, we should also emphasize that this model assumes symmetric return distributions, which may not hold in a real economy. Moreover, we assume that returns have the same index of stability. Certainly returns are not Gaussion does,not of course, imply that they have the same index of stability. Thus, we need some weaker assumption than that a symmetric stable Paretian family share the same index of stability. As a result, unfortunately, however we can not expect to do the desirable properties better now. There is a long way to make our model more realistic. At the same time empirical tests should be carried out in order to test our assumptions and estimate parameters.

References

[1] Black, F. Captal market equilibrium with restricted borrowing. *Journal of Business*, 45: 444–454 (1972)

[2] Chamkerlain, G. A characterization of the distributions that imply mean-variance utility functions. *Journal of Economic Theory*, 29: 185–201 (1983)

[3] Fama, E. Stable models in testable asset pricing. *Journal of Political Economy*, 78: 30–55 (1970)

[4] Huang, C., D Litzenberger. Foundations for financial economics, North-Holland, New Yorkbdg, 1988

[5] Mandelbrot, B.B. Stable Paretian random functions and the mutliplicative raviation of income. *Econometrica*, 29: 517–543 (1961)

[6] Mandelbrot, B.B. Paretian distributions and income Maximization. *Quarterly Journal of Economics*, 76: 57–85 (1962)

[7] Mandelbrot, B.B. The stable Paretian income distribution when the apparent exponent is near two. *International Economic Revie*, 4: 111–116 (1963)

[8] Mandelbrot, B.B. Fractals and Scaling in Finance. Springer, 1997

[9] Merton, R. An intertemporal capital asset pricing model. *Econometrica*, 41: 867–887 (1973)

[10] Peters. Fractal market analysis. Wiley, New York, 1994

[11] Rachev, S.T., Mittnik, S. Stable Paretian models in finance. Wiley, New York, 2000.

[12] Ross, S. Mutual fund separation in financial theory: The separation distributions. *Journal of Economic Theory*, 17: 254–286 (1978)

[13] Samorodnitsky, G., Taqqu, M.S. Stable Non-Gaussian random process. Chapman & Hall, New York, 1994

[14] Schider, M. Some Structure theorems for symmetric stable laws. *Ann. Math. Statist*, 41: 412–421 (1970)

[15] Zolotavev, V.M. One-Dimensional Stable Distributions. Translation of Mathematical Monographs 65, American Mathematical Society, Providence, 1986