On 24 Forms of the Acoustic Wave Equation in Vortical Flows and Dissipative Media

L. M. B. C. Campos
Centro de Ciências e Tecnologias Aeronauticas e Espaciais (CCTAE),
Instituto Superior Técnico (IST),
1049-001 Lisbon, Portugal

*The 36 forms of the acoustic wave equation derived in an earlier review (Campos, L. M. B. C., 2007, "On 36 Forms of the Acoustic Wave Equation in Potential Flows and Inhomogeneous Media," Appl. Mech. Rev., **60**, pp. 149–171) were grouped in four classes, of which the last (Class IV) concerned sheared mean flows; another type of vortical flow is swirling flow, and thus the present review completes the preceding by starting with Class V of linear, nondissipative acoustic wave equations in axisymmetric swirling, and also sheared, mean flow. These include general swirl and, in particular, rigid body and potential vortex swirl, combined or not with shear, for axisymmetric or general nonaxisymmetric acoustic modes, in two types of media: (i) inhomogeneous isentropic and (ii) homogeneous homentropic. Besides the 14 acoustic wave equations in sheared and swirling mean flows, the remaining ten acoustic wave equations derived in the present review all concern waves in homogeneous and steady media at rest, with dissipation or nonlinear effects to second-order or a combination of these two opposing effects, viz., (i) Class VI of linear, nondissipative wave equations with weak or strong thermoviscous dissipation in a homogeneous medium at rest; (ii) Class VIIA nonlinear one-dimensional wave equations in steady, homogeneous medium at rest without dissipation, or with viscous or thermoviscous dissipation, also in the case of a duct of varying cross section; (iii) Class VIIB of weakly nonlinear, three-dimensional waves or beams with thermoviscous dissipation in a homogeneous steady medium at rest. The 24 forms of the acoustic wave equation derived in the present review add to the 36 forms in the preceding review to form the set of 60 acoustic wave equations, whose interconnections are indicated in a family tree at the end. Numerous examples of the applications of the wave equations to the physical world are given at the end of each written section.*
[DOI: 10.1115/1.2804329]

1 Introduction

The 36 forms of the acoustic wave equation (W1–W36) in the preceding review [1] ended with the case of a sheared mean flow. Another case of interaction of sound with vorticity is that of swirling mean flow, which may or may not be combined with shear. In both cases, there is no acoustic potential, so the acoustic pressure is used as wave variable again in the presence of swirl (W52). The simplest particular case is that of axisymmetric acoustic modes (W47), although it is simple to generalize it to nonaxisymmetric acoustic modes. In an axisymmetric mean flow with arbitrary swirl and shear, the acoustic wave equation is somewhat complex for arbitrary swirl (W50), although it simplifies somewhat for rigid body (W48) and potential vortex (W49) swirl. A significant simplification is to assume that the swirling component of the mean flow is of low Mach number, including a shear flow contribution; this leads to simpler wave equations in axisymmetric swirling and shear flow, either isentropic (W42) or homentropic (W41). Two particular cases of interest are as follows: (i) rigid body shear, when the angular velocity is constant in the isentropic (W44) or homentropic (W43) wave equation; (ii) the tangential mean flow velocity decaying as the inverse of distance, i.e. the potential vortex, also leads to a simplification of the wave equation in the isentropic (W46) or homentropic (W45) cases. A further simplification is to omit the shear flow component and consider only the swirl flow component, both for the isentropic (W40) and homentropic (W39) wave equations; conversely, no swirl and only axisymmetric sheared mean flow may be considered for nonaxisymmetric acoustic modes in isentropic (W38) and homentropic (W37) conditions.

All of the preceding 50 wave equations, of which 36 (viz., W1–W36) are in the earlier review [1], and 14 are the first (viz., W37–W50) in the present review are nondissipative. The dissipative acoustic wave equation (Sec. 3) is obtained most readily for linear waves in a medium at rest. The simplest case must take into account viscosity and thermal conduction, since the two effects are generally comparable; it is important to include shear as well as bulk viscosity, since the decoupling of the vorticity from the dilatation shows that the former is affected only by shear viscosity. Thus, it is possible to obtain a scalar dissipative acoustic wave equation for the dilatation, i.e., the divergence of the acoustic velocity. The linear dissipative wave equation in a homogeneous medium at rest (W52) involves products of diffusivities, in the case of strong dissipation. Since in many situations the viscous and thermal diffusivities are small, the neglect of their products leads to useful simplification of the dissipative wave equations, including linearly bulk and shear viscosities and thermal conduction (W51). In the case of weak dissipation of sinusoidal waves, the bulk and shear viscosities appear combined with the thermal conductivity, in a thermoviscous dissipation coefficient; the latter involves the specific heats at constant pressure and volume and appears also for weakly nonlinear and dissipative waves.

An interesting combination is nonlinear waves with linear dissipation, since it leads to a competition of two effects. Nonlinear unidirectional waves (W53) are specified by the conservation of Riemann invariants along characteristics (Sec. 4), implying that the propagation speed is larger in the compression phase of the wave than in the rarefaction phase, leading to a steepening of the wave form. This steepening can be countered by geometrical spreading, in the case of nonlinear waves in duct of nonuniform cross section (W54), e.g., spherical waves in a conical duct. For plane unidirectional nonlinear waves, linear dissipation is suffi-

Transmitted by Editorial Advisory Board Member E. H. Dowell.

Table 1 Fourteen linear, nondissipative wave equations in swirling and sheared mean flows

Medium entropy			Inhomogeneous isentropic	Homogeneous homentropic
Low Mach number swirl and shear	Shear only		W 38	W 37
	Swirl only		W 40	W39
	Both shear and swirl	General swirl	W 42	W 41
		Rigid body swirl	W 44	W 43
		Potential vortex swirl	W 46	W 45
Unrestricted swirl Mach number and shear	Axisymmetric modes		W 47	⋯
	Nonaxisymmetric modes	Rigid body swirl	W 48	⋯
		Potential vortex swirl	W 49	⋯
		Arbitrary swirl	W 50	⋯

cient to limit wave steepening, and leads to Burger's equation (W55), which can be generalized to ducts of nonuniform and unsteady cross section (W56), in the presence of shear and bulk viscosities. The method of characteristics can be checked, by deriving by elimination among the equations of motion, the wave equation for weakly nonlinear, linearly dissipative waves, including thermal conduction besides shear and bulk viscosities, both for one-dimensional free of waves (W57) and quasi-one-dimensional ducts of nonuniform cross section (W58). A final generalization concerns weakly nonlinear three-dimensional waves with thermoviscous dissipation (W59), including the case of beams (W60).

The last set of 10 acoustic wave equations include 2 nondissipative, 2 with viscous and 6 with thermoviscous dissipation; 6 one-dimensional and 4 in more than one dimension; 2 linear (in addition to the 21 in the first review and 14 in the present review) and 8 nonlinear (in addition to the 15 the first review). The total of 60 wave equations in (36 in the earlier review and 24 in the present one) are numbered sequentially from W1 to W60, and correspond to an approximate order of increasing complexity as indicated in Tables 1 and 2 of [1] and in Tables 1–4. A complete and detailed derivation is given of each wave equation, since it is crucial to establish its conditions of validity, as indicated in the list of wave equations in the Appendix. The notation of the present review is consistent with that of the earlier review, so only new symbols need be listed. The introduction and conclusion are specific to each review, and the common points are not restated. The family tree shows the hierarchy of the 60 wave equations and serves as final synthesis of the two reviews. The family tree may be used to find the most appropriate acoustic wave equation for a particular application; the list of wave equations then confirms the conditions of validity and specifies equivalent forms of that wave equation; the text then provides one or more proofs of the results.

The process of establishing 60 wave equations is made more efficient by using methods, which lead to a related set, so that there are only seven main derivations, with minor variants: (i) the acoustic variational principle applies (Class I) to linear, nondissipative acoustic waves in homogeneous or inhomogeneous, steady or unsteady, potential mean flows (W1–W9); (ii) the exact, nonlinear equation for a potential flow leads (Class II) to the corresponding nonlinear wave equations (W10–W15); (iii) the corresponding linear (W16–W24) and nonlinear (W25–W30) equations for nondissipative waves in a potential flow, in the case of quasi-one-dimensional propagation in a duct of varying cross section (Class III), are obtained by simple transformations; (iv) the linear, nondissipative wave equations in mean flows with (Class IV) shear (W31–W38) and/or with (Class V) swirl (W39 to W50) are obtained by elimination among the linearized equations of fluid, motion; (v) a different elimination, including viscous and thermal dissipation in a medium at rest, leads (Class VI) to the linear, dissipative acoustic wave equations (W51–W52); (vi) the conservation or decay of Riemann invariants along characteristics applies to (Class VIIA) unidirectional nonlinear waves (W53), also in a duct (W54), in the nondissipative case, and also in the presence of shear and bulk viscosities, both for free (W55) and ducted (W56) waves; (vii) elimination between the equations of motion adds thermal conduction to the viscous dissipation mechanisms for (Class VIIB) the free (W57) and ducted (W58) one-dimensional waves, and applies as well to three-dimensional weakly nonlinear waves with thermoviscous dissipation (W59), including beams (W60). The seven methods are not mutually exclusive, and can be used in some cases as cross-checks, allowing as many as three or four distinct derivations of the same wave equation.

Table 2 Two linear, dissipative wave equations in a homogeneous medium at rest

Dissipative	Weak	Strong
Thermoviscous	W 51	W 52

Table 3 Six nonlinear, one-dimensional wave equations in media at rest

Characteristics	Free	Ducted
Nondissipative	W53	W 54
Viscous	W55	W 56
Thermoviscous	W57	W 58

Table 4 Two weakly nonlinear, dissipative wave equations in a homogeneous medium at rest

Water waves	Three-dimensional	Beam
Thermoviscous	W59	W60

2 Acoustic Wave Equations in Swirling and/or Sheared Flows

The wave equation for axisymmetric acoustic modes in an axisymmetric mean flow (W36) can be extended to nonaxisymmetric modes (W38), and besides rotation or swirl of the mean flow around the axis can be superimposed on the mean shear flow (W42). The other particular case would be swirl without shear (W40). The wave equation for nonaxisymmentric acoustic modes in a mean axisymmetric sheared and swirling flow (W42) simplifies for two particular cases: (i) rigid body rotation [2,3] when the angular velocity is constant (W44); (ii) a potential vortex (W46), for which the tangential velocity decays [4–6] like the inverse of the radius, and thus the angular velocity decays like inverse square of the radius, and the rotation effect is included in the convected wave equation. These wave equations (W38, W40, W42, W44, W46) apply to an isentropic mean flow, which may be inhomogeneous, and simplify in homentropic or homogeneous mean flow reflectively to (W37, W39, W41, W43, W45). A mean flow with high Mach number swirl is always inhomogeneous, and leads to the more complex wave equations [3] with swirl plus shear (W50), including the cases of rigid body (W48) and potential vortex (W49) swirl, besides the case of axisymmetric acoustic modes (W47). The acoustic wave equations in shear flows describe the propagation of sound in shear layers and boundary layers. In the case of an axisymmetric duct, besides the shear layer near the wall, there may be swirl, e.g., due to turbomachinery. The impedance and rigid wall boundary conditions can be obtained also [3] for a sheared and swirling mean flow.

2.1 Equations of Fluid Mechanics in Cylindrical Coordinates. The fundamental equations of nondissipative fluids are the inviscid momentum equation:

$$\partial \mathbf{V}/\partial t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \Gamma^{-1}\nabla P = 0 \tag{1}$$

where $\mathbf{V}$ is the velocity, Γ the mass density, and P is the pressure, and the equation of continuity:

$$\partial \Gamma/\partial t + \mathbf{V}\cdot\nabla\Gamma + \Gamma\nabla\cdot\mathbf{V} = 0 \tag{2}$$

which may be combined with the adiabatic condition:

$$\partial P/\partial t + V\cdot\nabla P = C^2(\partial\Gamma/\partial t + \mathbf{V}\cdot\nabla\Gamma) \tag{3a}$$

$$C^2 \equiv (\partial p/\partial\Gamma)_s = \gamma P/\Gamma = \gamma RT \tag{3b}$$

where C is the adiabatic sound speed, and the last expression in Eq. (3b) applies to a perfect gas. Substitution of Eq. (2) in Eq. (3a) leads to

$$\partial P/\partial t + \mathbf{V}\cdot\nabla P + C^2\Gamma\nabla\cdot\mathbf{V} = 0 \tag{4}$$

Using cylindrical coordinates (r,θ,z), the last equation (4) becomes

$$\dot{P} + V_r\partial_r P + V_\theta r^{-1}\partial_\theta P + V_z\partial_z P + C^2[r^{-1}\partial_r(rV_r) + r^{-1}\partial_\theta V_\theta + \partial_z V_z] = 0 \tag{5}$$

and the momentum equation (1) has the cylindrical components:

$$\dot{V}_r + V_r\partial_r V_r + V_\theta r^{-1}\partial_\theta V_r + V_z\partial_z V_r - V_\theta^2 r^{-1} + \Gamma^{-1}\partial_r P = 0 \tag{6a}$$

$$\dot{V}_\theta + V_r\partial_r V_\theta + V_\theta r^{-1}\partial_\theta V_\theta + V_z\partial_z V_\theta + V_r V_\theta r^{-1} + \Gamma^{-1}\partial_\theta P = 0 \tag{6b}$$

$$\dot{V}_z + V_r\partial_r V_z + V_\theta r^{-1}\partial_\theta V_z + V_z\partial_z V_z + \Gamma^{-1}\partial_z P = 0 \tag{6c}$$

where dot denotes derivative with regard to time $\dot{P}\equiv\partial P/\partial t$ and $\partial_\xi\equiv\partial/\partial\xi$ partial derivative with regard to a cylindrical coordinate $\xi\equiv(r,\theta,z)$. The total state of the fluid is assumed to consist of

$$\{\mathbf{V},P,\Gamma,C\}(\mathbf{x},t) = \{\mathbf{V}_0,p_0,\rho_0,c_0\}(\mathbf{x}) + \{\vec{\mathbf{v}},p,\rho,c'\}(\mathbf{x},t) \tag{7}$$

of a steady, possibly inhomogeneous mean state, plus unsteady and inhomogeneous acoustic perturbations.

2.2 Acoustic Modes in an Axisymmetric Swirling and Sheared Flow. The mean flow is assumed to be axisymmetric, with a sheared axial velocity $U(r)$ and swirl specified by an angular velocity $\Omega(r)$, which may also be nonuniform, both depending on the radius alone:

$$\mathbf{v}_0 = \mathbf{e}_z U(r) + \vec{\mathbf{e}}_\theta r\Omega(r) \tag{8a}$$

and the corresponding linearized material derivative is

$$d/dt \equiv \partial/\partial t + \mathbf{v}_0\cdot\nabla = \partial/\partial t + U(r)\partial/\partial z + \Omega(r)\partial/\partial\theta \tag{8b}$$

The mean flow (8a) satisfies the momentum equation (6a)–(6c), where the pressure depends only on the radius, and its gradient is due to the centrifugal force:

$$\partial p_0/\partial\theta = 0 = \partial p_0/\partial z \quad dp_0/dr = \rho_0\Omega^2 r = (\gamma p_0/r)(\Omega r/c_0)^2 \tag{9}$$

where the sound speed (3b) for the mean state $c_0^2=\gamma p_0/\rho_0$ was introduced; it follows that for low Mach number swirl, the mean flow pressure may be taken as constant:

$$(\Omega r)^2 \ll [c_0(r)]^2 \quad p_0 = \text{const} \equiv p_* \tag{10a}$$

otherwise it is a function of the radius,

$$p_0(r) = p_* + \int_0^r \rho_0(r)[\Omega(r)]^2 r\,dr \tag{10b}$$

where $p_*=p_0(0)$ is the pressure on the axis. The adiabatic continuity equation (5) is linearized (7) as

$$c_0^{-2}[dp/dt + (dp_0/dr)v_r] + \rho_0[r^{-1}\partial_r(rv_r) + r^{-1}\partial_\theta v_\theta + \partial_z v_z] = 0 \tag{11}$$

using the linearized material derivative (8b). The axial component (6c) of the momentum equation is linearized:

$$dv_z/dt + v_r dU/dr + \rho_0^{-1}\partial_z p = 0 \tag{12}$$

and involves mean flow shear in the second term. The radial component of the momentum equation (6a) is linearized:

$$dv_r/dt - 2\Omega v_\theta - \Omega^2 r\rho_0^{-1}\rho + \rho_0^{-1}\partial_r p = 0 \tag{13}$$

where mean flow swirl appears in two terms, one coupling rotation to nonaxisymmetric modes $v_\theta\neq 0$ and to the other coupling rotation to the density perturbation $\rho\neq 0$. For nonaxisymmetric modes, the azimuthal component (6b) of the momentum equation is linearized:

$$dv_\theta/dt + (2\Omega + r d\Omega/dr)v_r + \rho_0^{-1}r^{-1}\partial_\theta p = 0 \tag{14}$$

where mean flow swirl appears the second term. Equations (11)–(14) coincide with those in Ref. [3], using the linearized material derivative (8b) for compact notation.

2.3 Wave Equation for Nonaxisymmetric Acoustic Modes. For axisymmetric acoustic modes, since $\partial_\theta\equiv\partial/\partial\theta=0$, the material derivative (86) omits the last term; since also $v_\theta=0$, it follows from Eq. (14) that axisymmetric acoustic modes can exist only if

$$v_\theta = 0 = \partial/\partial\theta \quad r d\Omega/dr + 2\Omega = 0 \tag{15a}$$

corresponding [4–6] to a potential flow vortex

$$\text{const} = \Omega r^2 = v_\theta r = \bar{\Gamma}/2\pi \tag{15b}$$

where $\bar{\Gamma}$ is the circulation. For nonaxisymmetric modes in a potential vortex, Eq. (14) simplifies to

$$\Omega r^2 = \text{const} \quad dv_\theta/dt + \rho_0^{-1} r^{-1} \partial_\theta p = 0 \tag{16}$$

which does not involve rotation; since for low Mach number swirl (13) simplifies to

$$(\Omega r)^2 \ll c_0^2 \quad dv_r/dt - 2\Omega v_\theta + \rho_0^{-1} \partial_r p = 0 \tag{17}$$

it follows that the rotation appears only in Eq. (17) and in the linearized material derivative (8*b*). In the case of rigid body rotation, the linearized material derivative also appears in Eq. (14), viz.,

$$\Omega = \text{const:l} \quad dv_\theta/dt + 2\Omega v_r + \rho_0^{-1} r^{-1} \partial_\theta p = 0 \tag{18}$$

this is a simplification of Eq. (14), distinct from the case (16) of a potential vortex.

Both for axisymmetric and nonaxisymmetric acoustic modes, the linearized adiabatic equation (3*a*):

$$dp/dt + v_r d p_0/dr = c_0^2 (d\rho/dt + v_r d\rho_0/dr) \tag{19}$$

leads on substitution of Eq. (9) to

$$d\rho/dt = c_0^{-2} dp/dt + r^{-1} v_r [\rho_0 (\Omega r/c_0)^2 - r d\rho_0/dr] \tag{20}$$

which simplifies for low Mach number swirl to

$$(\Omega r)^2 \ll c_0^2 \quad d\rho/dt = c_0^{-2} dp/dt - v_r d\rho_0/dr \tag{21}$$

The material derivative (8*b*) commutes with all components of the gradient:

$$\partial_\theta d/dt = d/dt \partial_\theta \tag{22a}$$

$$\partial_z d/dt = d/dt \partial_z \tag{22b}$$

except the radial

$$\partial_r d/dt - d/dt \partial_r = (dU/dr) \partial_z + (d\Omega/dr) \partial_\theta \tag{23}$$

The preceding results allow elimination of the acoustic wave equation in an axisymmetric unidirectional swirling and sheared mean flow, for nonaxisymmetric modes and low or high Mach number swirl.

2.4 Simplification for Low Mach Number Swirl and Shear. For the general case of nonaxisymmetric acoustic modes, in a low Mach number swirling flow, Eqs. (12) and (14) are unchanged, Eq. (13) simplifies to Eq. (17), and Eq. (11) simplifies to

$$c_0^{-2} dp/dt + \rho_0 (r^{-1} v_r + \partial_r v_r + r^{-1} \partial_\theta v_\theta + \partial_z v_z) = 0 \tag{24}$$

Applying d/dt to Eq. (24) and using Eqs. (22*a*), (22*b*), and (23) leads to

$$c_0^{-2} d^2 p/dt^2 + \rho_0 [r^{-1} dv_r/dt + \partial_r (dv_r/dt) + r^{-1} \partial_\theta (dv_\theta/dt) + \partial_z (dv_z/dt)] = \rho_0 [(dU/dr) \partial_z v_r + (d\Omega/dr) \partial_\theta v_r] \tag{25}$$

where Eqs. (17), (14), and (12) may be substituted

$$c_0^{-2} d^2 p/dt^2 - r^{-1} \partial_r p - \rho_0 \partial_r (\rho_0^{-1} \partial_r p) - r^{-2} \partial_{\theta\theta} p - \partial_{zz} p = 2\rho_0 (dU/dr) \partial_z v_r - 2\rho_0 \Omega \partial_r v_\theta + 2\rho_0 (d\Omega/dr + \Omega/r)(\partial_\theta v_r - v_\theta) \tag{26}$$

In the absence of shear and swirl (26) reduces to the inhomogeneous convected wave equation (see paper I [1] Eq. (178)), with an extra term relative to (209) in [1]:

$$dU/dr = 0 = \Omega \quad c_0^{-2} d^2 p/dt^2 - r^{-1} \partial_r p - \partial_{rr} p - r^{-2} \partial_{\theta\theta} p - \partial_{zz} p + \rho_0^{-1} (d\rho_0/dr) \partial_r p = 0 \tag{27}$$

because the acoustic modes are nonaxisymmetric $\partial_{\theta\theta} p \neq 0$.

In the presence of shear and/or swirl, the velocity perturbation is eliminated from Eq. (26) application of d/dt and use of Eqs. (22*a*), (22*b*), and (23), viz.,

$$\frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{\partial p}{\partial r}\right] = 2\rho_0 (dU/dr) \partial_z (dv_r/dt) - 2\rho_0 \Omega \partial_r (dv_\theta/dt) + 2\rho_0 \Omega [(dU/dr) \partial_z v_\theta + (d\Omega/dr) \partial_\theta v_\theta] + 2\rho_0 (d\Omega/dr + \Omega/r)[\partial_\theta (dv_r/dt) - dv_\theta/dt] \tag{28}$$

Substituting Eqs. (17) and (14) in Eq. (28), and using the condition (10*a*) of low Mach number swirl in the stronger form,

$$r^2 (\Omega + dU/dr)^2 \ll [c_0(r)]^2 \quad \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{\rho_0}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho_0}\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z \partial r} + 2\frac{d\Omega}{dr}\frac{\partial^2 p}{\partial \theta \partial r} - \frac{2\rho_0}{r}\left[\frac{d}{dr}\left(\frac{\Omega}{\rho_0}\right)\right]\frac{\partial p}{\partial \theta} = 0 \tag{29}$$

lead to the axisymmetric inhomogeneous convected shear and swirl acoustic wave equation valid for general, nonaxisymmetric acoustic modes in an axisymmetric sheared and swirling mean flow of low Mach number.

2.5 Acoustic Wave Equations for Isentropic Inhomogeneous Mean Flow. The inhomogeneous shear and swirl acoustic wave equation (29),

$$\text{W42:} \quad \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{\partial^2 p}{\partial r^2} - \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{dp}{\partial r}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z \partial r} + 2\frac{d\Omega}{dr}\frac{\partial^2 p}{\partial \theta \partial r} - \frac{2\rho_0}{r}\left[\frac{d}{dr}\left(\frac{\Omega}{\rho_0}\right)\right]\frac{\partial p}{\partial \theta} = 0 \tag{30}$$

in the absence of swirl $\Omega = 0$ simplifies to the axisymmetric inhomogeneous convected shear acoustic wave equation:

$$\text{W38:} \quad \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{\partial^2 p}{\partial r^2} - \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{\partial p}{\partial r}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z \partial r} = 0 \tag{31}$$

which extends (210) from Paper I [1] to nonaxisymmeytric acoustic modes for which $\partial^2 p/\partial \theta^2 \neq 0$; in the absence of shear $dU/dr = 0$ there is less simplification of Eq. (30) to the inhomogeneous swirl acoustic wave equation:

$$\text{W40:} \quad \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{\partial^2 p}{\partial r^2} - \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial t^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{\partial p}{\partial r}\right] + 2\frac{d\Omega}{dr}\frac{\partial^2 p}{\partial \theta \partial r} - 2\frac{\rho_0}{r}\left[\frac{d}{dr}\left(\frac{\Omega}{\rho_0}\right)\right]\frac{\partial p}{\partial \theta} = 0 \tag{32}$$

In the presence of shear and rigid body swirl $\Omega = \text{const}$, Eq. (30) simplifies to the shear and rigid body swirl acoustic wave equation:

$$\text{W44:} \quad \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2 p}{dt^2} - \frac{\partial^2 p}{\partial r^2} - \frac{1}{r}\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - \frac{\partial^2 p}{\partial z^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\frac{\partial p}{\partial r}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z \partial r} + 2\frac{\Omega}{\rho_0 r}\frac{d\rho_0}{dr}\frac{\partial p}{\partial \theta} = 0 \tag{33}$$

and in the case (15*a*) of potential vortex swirl, Eq. (30) simplifies to

$$\text{W46: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{\rho_0}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho_0}\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z\partial r} - 4\frac{\Omega}{r}\frac{\partial^2 p}{\partial\theta\partial r} + \frac{2\Omega}{r}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{2}{r}\right)\frac{\partial p}{\partial\theta} = 0 \tag{34}$$

which is the inhomogeneous shear and potential vortex swirl acoustic wave equation.

For low Mach number swirl, the mean flow pressure is constant (10*a*), and if the mean flow is homentropic, then it is homogeneous, allowing further simplification of the wave equations (30)–(34). The wave equation for general nonaxisymmetric acoustic modes in an axisymmetric sheared and swirling homogeneous mean flow with low Mach number is given by the following.

(i) In the presence of shear and swirl, by Eq. (29) the shear and swirl acoustic wave equation:

$$\text{W41: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z\partial r} + 2\frac{d\Omega}{dr}\frac{\partial^2 p}{\partial\theta\partial r} - \frac{2}{r}\frac{d\Omega}{dr}\frac{\partial p}{\partial\theta} = 0 \tag{35}$$

(ii) In the absence of swirl, by Eq. (31) the shear acoustic wave equation:

$$\text{W37: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z\partial r} = 0 \tag{36}$$

which generalizes (211) in paper I [1] to nonaxisymmetric acoustic modes, for which $\partial_{\theta\theta}p \neq 0$.

(iii) In the absence of shear, by Eq. (32) the swirl acoustic wave equation:

$$\text{W39: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{d\Omega}{dr}\frac{\partial^2 p}{\partial\theta\partial r} - \frac{2}{r}\frac{d\Omega}{dr}\frac{\partial p}{\partial\theta} = 0 \tag{37}$$

(iv) For rigid body swirl, by Eq. (33) the rigid body swirl acoustic wave equation:

$$\text{W43: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z\partial r} = 0 \tag{38}$$

which differs from Eq. (36) since Eq. (36) omits the last term in the linearized material derivative (8*b*).

(v) For potential vortex swirl, by Eq. (34) the potential vortex swirl acoustic wave equation:

$$\text{W45: } \frac{d}{dt}\left[\frac{1}{c_0^2}\frac{d^2p}{dt^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 p}{\partial\sigma_2} - \frac{\partial^2 p}{\partial z^2}\right] + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial z\partial r} - 4\frac{\Omega}{r}\frac{\partial^2 p}{\partial\theta\partial r} + \frac{4\Omega}{r^2}\frac{\partial p}{\partial\theta} = 0. \tag{39}$$

These homogeneous cases (35)–(39) can be compared with the inhomogeneous cases (30)–(34).

2.6 Acoustic Wave Equations for Homentropic Or Homogeneous Mean Flow.

For nonaxisymmetric acoustic modes, the acoustic pressure spectrum is introduced:

$$p(r,\theta,z,t) = \sum_{m=-\infty}^{+\infty} e^{im\theta}\int_{-}^{+}\int_{\infty}^{\infty} \tilde{p}(r;m,k,\omega)e^{i(kz-\omega t)}dk\,d\omega \tag{40}$$

where (i) the Fourier integrals involve the frequency ω and axial wave number k as for axisymmetric acoustic modes (212) in Paper I [1]; (ii) nonaxisymmetric acoustic modes introduce a Fourier series in the azimuthal direction θ with integral wave number m. The linearized material derivative (8*b*) leads to the Doppler shifted frequency:

$$d/dt \rightarrow -i\omega_4 \tag{41a}$$

$$\omega_4(r) \equiv \omega - kU(r) - m\Omega(r) \tag{41b}$$

which reduces to Eq. (41*c*) in the absence of swirl and to Eq. (41*d*) in the absence of shear:

$$\omega_3(r) \equiv \omega - kU(r) \tag{41c}$$

$$\omega_5(r) \equiv \omega - m\Omega(r) \tag{41d}$$

where Eq. (41*c*) coincides with (213b) in paper I [1]. Substitution in Eqs. (30)–(39) leads to the wave equations for the acoustic pressure spectrum of nonaxisymmetric modes in an axisymmetric sheared and swirling mean flow of low Mach number, in ten cases of increasing complexity from Eqs. (42)–(47), and particular cases (48)–(51), namely, (i) homogeneous mean flow without swirl (36):

$$\text{W37}^*: \quad d^2\tilde{p}/dr^2 + \lfloor r^{-1} + 2(k/\omega_3)dU/dr\rfloor d\tilde{p}/dr + [(\omega_3/c_0)^2 - k^2 - m^2/r^2]\tilde{p} = 0 \tag{42}$$

(ii) inhomogeneous mean flow without swirl (31):

$$\text{W38}^*: \quad d^2\tilde{p}/dr^2 + \lfloor r^{-1} - \rho_0^{-1}d\rho_0/dr + 2(k/\omega_3)dU/dr\rfloor d\tilde{p}/dr + [(\omega_3/c_0)^2 - k^2 - m^2/r^2]\tilde{p} = 0 \tag{43}$$

(iii) homogeneous mean flow without shear (37):

$$\text{W39}^*: \quad d^2\tilde{p}/dr^2 + \lfloor r^{-1} + 2(m/\omega_5)d\Omega/dr\rfloor d\tilde{p}/dr + [(\omega_5/c_0)^2 - k^2 - m^2/r^2 - 2(m/\omega_5)r^{-1}d\Omega/dr]\tilde{p} = 0 \tag{44}$$

(iv) inhomogeneous mean flow without shear (32):

$$\text{W40}^*: \quad d^2\tilde{p}/dr^2 + \lfloor r^{-1} - \rho_0^{-1}d\rho_0/dr + 2(m/\omega_5)d\Omega/dr\rfloor d\tilde{p}/dr + \{(\omega_5/c_0)^2 - k^2 - m^2/r^2 - 2(m/\omega_5)r^{-1}[d\Omega/dr - (\Omega/\rho_0)d\rho_0/dr]\}\tilde{p} = 0 \tag{45}$$

(v) homogeneous mean flow with swirl and shear (35):

$$\text{W41}^*: \quad d^2\tilde{p}/dt^2 + [r^{-1} + 2(kdU/dr + md\Omega/dr)/\omega_4]d\tilde{p}/dr + [(\omega_4/c_0)^2 - k^2 - m^2/r^2 - 2(m/\omega_4)r^{-1}d\Omega/dr]\tilde{p} = 0 \tag{46}$$

(vi) inhomogeneous mean flow with swirl and shear (30):

$$\text{W42}^*: \quad d^2\tilde{p}/dt^2 + [r^{-1} - \rho_0^{-1}d\rho_0/dr + 2(kdU/dr + md\Omega/dr)/\omega_4]d\tilde{p}/dr + \{[(\omega_4/c_0)^2 - k^2 - m^2/r^2 - 2(m/\omega_4)r^{-1}[d\Omega/dr - (\Omega/\rho_0)d\rho_0/dr]]\}\tilde{p} = 0 \tag{47}$$

(vii) homogeneous mean flow with rigid body swirl (38):

$$\text{W43}^*: \quad d^2\tilde{p}/dt^2 + [r^{-1} + 2(k/\omega_4)dU/dr]d\tilde{p}/dr + [(\omega_4/c_0)^2 - k^2 - m^2/r^2]\tilde{p} = 0 \tag{48}$$

(viii) inhomogeneous mean flow with rigid body swirl (39):

$$\text{W44}^*: \quad d^2\tilde{p}/dt^2 + [r^{-1} - \rho_0^{-1}d\rho_0/dr + 2(k/\omega_4)dU/dr]d\tilde{p}/dr + [(\omega_4/c_0)^2 - k^2 - m^2/r^2 + 2r^{-1}(m/\omega_4)(\Omega/\rho_0)d\rho_0/dr]\tilde{p} = 0 \tag{49}$$

(ix) homogeneous mean flow with potential vortex swirl:

$$\text{W45}^*: \quad d^2\tilde{p}/dr^2 + [r^{-1} + 2(kdU/dr - 2m\Omega/r)/\omega_4]d\tilde{p}/dr + [(\omega_4/c_0)^2 - k^2 - m^2/r^2 + 4(m/r^2)(\Omega/\omega_4)]\tilde{p} = 0 \tag{50}$$

(x) inhomogeneous mean flow with potential vortex swirl (34):

$$\text{W46}^*: \quad d^2\tilde{p}/dr^2 + [r^{-1} - \rho_0^{-1}d\rho_0/dr + 2(kdU/dr - 2m\Omega/r)/\omega_4]d\tilde{p}/dr + [(\omega_4/c_0)^2 - k^2 - m^2/r^2 + 2(m/\omega_4)(\Omega/r) \times (2/r + \rho_0^{-1}d\rho_0/dr)]\tilde{p} = 0 \tag{51}$$

The cases W37* / W38* in Eqs. (39) and (40) can be compared, respectively, with the cases W35* / W36* of axisymmetric acoustic modes in axisymmetric shear (214,215) in Paper I [1].

2.7 Impedance Wall Boundary Condition With Arbitrary Swirl.

In order to eliminate for the acoustic pressure among Eqs. (12)–(14) and Eq. (11), without the restriction on mean flow shear on swirl in Eq. (29), d/dt is applied to Eq. (13):

$$d^2v_r/dt^2 - 2\Omega dv_\theta/dt - \rho_0^{-1}\Omega^2 rd\rho/dt + \rho_0^{-1}d(\partial_r p)/dt = 0 \tag{52}$$

and (14) and (20) substituted to yield:

$$d^2v_r/dt^2 + \{2\Omega(2\Omega + rd\Omega/dr) - \Omega^2[(\Omega r/c_0)^2 - (r/\rho_0)d\rho_0/dr]\}v_r = -\rho_0^{-1}[d(\partial_r p)/dt + 2(\Omega/r)\partial_\theta p - (\Omega/c_0)^2 rdp/dt] \tag{53}$$

Introducing in addition to the pressure perturbation spectrum (40), the velocity perturbation spectrum $\tilde{v}_r$, leads to the relation

$$i\rho_0 X\tilde{v}_r = \omega_4 d\tilde{p}/dr - [2m\Omega + (\Omega r/c_0)^2\omega_4]\tilde{p}/r \tag{54}$$

where

$$X \equiv (\omega_4)^2 + \Omega^2[(\Omega r/c_0)^2 - (r/\rho_0)d\rho_0/dr] - 2\Omega(2\Omega + rd\Omega/dr) \tag{55}$$

involves the Doppler shifted frequency (41*b*).

The boundary condition at a cylindrical wall $r=r_0$ of impedance Z or specific impedance $\bar{Z} \equiv Z/\rho_0 c_0$ is

$$r = r_0 \quad \tilde{p}(r_0) = -Z\tilde{v}_r(r_0) = -\rho_0 c_0 \bar{Z}\tilde{v}_r(r_0) \tag{56}$$

In the general case of swirling flow (54), this implies the boundary condition

$$r = r_0 \quad d\tilde{p}/dr = \{r^{-1}[(\Omega r/c_0)^2 + 2m\Omega/\omega_4] - iX/(c_0\omega_4\bar{Z})\}\tilde{p} \tag{57}$$

For low Mach number swirl, this simplifies to

$$(\Omega r/c_0)^2 \ll 1 \quad d\tilde{p}/dr = [2(m/r)(\Omega/\omega_4) - iX_1/(c_0\omega_4\bar{Z})]\tilde{p} \tag{58}$$

where Eq. (55) reduces to

$$X_1 \equiv (\omega_4)^2 - \Omega^2(r/\rho_0)(d\rho_0/dr) - 2\Omega(2\Omega + rd\Omega/dr) \tag{59}$$

In the absence of swirl, there is the further simplification,

$$\Omega = 0 \quad d\tilde{p}/dr = -i\{\omega_4/(c_0\bar{Z})\}\tilde{p} \tag{60}$$

In the case of a rigid wall $Z=\infty$, the acoustic boundary condition (57) is

$$Z = \infty \quad d\tilde{p}/dr = [(\Omega r/c_0)^2 + 2m\Omega/\omega_4]\tilde{p}/r \tag{61}$$

for arbitrary swirl, and

$$(\Omega r/c_0)^2 \ll 1 \gg 1/\bar{Z} \quad d\tilde{p}/dr = 2(m/r)(\Omega/\omega_4)\tilde{p} \tag{62}$$

for low Mach number swirl, and

$$\Omega = 0 = 1/Z \quad d\tilde{p}/dr = 0 \tag{63}$$

in the absence of swirl.

2.8 Wave Equation With High Mach Number Swirl and Shear.

In order to obtain the acoustic wave equation for the pressure perturbation spectrum $\tilde{p}$ in Eq. (40), in the presence of arbitrary swirl and shear, it suffices [3] to eliminate $\tilde{v}_r$ between Eq. (54) and another independent relation between $\tilde{p}$ and $\tilde{v}_r$. The latter satisfy (11), viz.,

$$-i\omega_4 c_0^{-2}\rho_0^{-1}\tilde{p} + [1 + (\Omega r/c_0)^2]r^{-1}\tilde{v}_r + d\tilde{v}_r/dr + i(m/r)\tilde{v}_\theta + ik\tilde{v}_z = 0 \tag{64}$$

where Eq. (9) was used, and $\tilde{v}_\theta$ may be substituted from Eq. (14):

$$i\omega_4\tilde{v}_\theta = (2\Omega + rd\Omega/dr)\tilde{v}_r + i(m/r)\rho_0^{-1}\tilde{p} \tag{65}$$

and $\tilde{v}_z$ from Eq. (12):

$$i\omega_4\tilde{v}_z = (dU/dr)\tilde{v}_r + i(k/\rho_0)\tilde{p} \tag{66}$$

Substitution of Eqs. (65) and (66) in Eq. (64) yields

$$[(\omega_4/c_0)^2 - k^2 - m^2/r^2]\rho_0^{-1}\tilde{p} + i\omega_4 d\tilde{v}_r/dr + i\{(\omega_4/r)[1 + (\Omega r/c_0)^2] + m(2\Omega/r + d\Omega/dr) + k(dU/dr)\}\tilde{v}_r = 0 \tag{67}$$

Eliminating $\tilde{v}_r$ between (54) and (67) and leads to

$$\text{W50}^*: \quad \frac{d^2\tilde{p}}{dr^2} + \left[\frac{1}{r} + \frac{k}{\omega_4}\frac{dU}{dr} + \frac{m}{\omega_4}\frac{d\Omega}{dr} + \frac{\rho_0 X}{\omega_4}\frac{d}{dr}\left(\frac{\omega_4}{\rho_0 X}\right)\right]\frac{d\tilde{p}}{dr} + \left\{X\left(\frac{1}{c_0^2} - \frac{k^2 + m^2/r^2}{(\omega_4)^2}\right) - \frac{\rho_0 X}{\omega_4}\frac{d}{dr}\left\{\frac{\omega_4}{\rho_0 X r}\left[\frac{2m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\right\} - \frac{1}{r}\left[\frac{2m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\left\{\frac{1}{r}\left[1 + \left(\frac{\Omega r}{c_0}\right)^2\right] + \frac{m}{\omega_4}\left(2\frac{\Omega}{r} + \frac{d\Omega}{dr}\right) + \frac{k}{\omega_4}\frac{dU}{dr}\right\}\tilde{p} = 0 \tag{68}$$

which is the wave equation for the acoustic pressure spectrum of nonaxisymmetric modes in an axisymmetric isentropic mean flow with arbitrary shear and swirl involving the Doppler shifted frequency (41*b*) and the parameter (55). This equation has been derived before [3], using a different notation, with a difference in the coefficient of $d\tilde{p}/dr$.

The case of axisymmetric acoustic modes corresponds to $m=0$,

$$\text{W47}^*: \frac{d^2\tilde{p}}{dr^2} + \left[\frac{1}{r} + \frac{k}{\omega_3}\frac{dU}{dr} + \frac{\rho_0 X}{\omega_3}\frac{d}{dr}\left(\frac{\omega_3}{\rho_0 X}\right)\right]\frac{d\tilde{p}}{dr} + \left\{X\left[\frac{1}{c_0^2} - \left(\frac{k}{\omega_3}\right)^2\right] - \frac{\rho_0 X}{\omega_3}\frac{d}{dr}\left[\frac{\omega_3}{\rho_0 X r}\left(\frac{\Omega r}{c_0}\right)^2\right] - \frac{1}{r^2}\left(\frac{\Omega r}{c_0}\right)^2 \times \left[1 + \left(\frac{\Omega r}{c_0}\right)^2 + \frac{kr}{\omega_3}\frac{dU}{dr}\right]\right\}\tilde{p} = 0 \tag{69}$$

where the Doppler shifted frequency (41*b*) reduces to (41*c*) for $m=0$. It also simplifies in the case of low Mach number swirl, when Eq. (55) reduces to $X=(\omega_4)^2$ and the wave equation (68) reduces to

$$r^2\left(\Omega + \frac{dU}{dr}\right)^2 \ll c_0^2 \quad \frac{d^2\tilde{p}}{dr^2} + \left[\frac{1}{r} + \frac{k}{\omega_4}\frac{dU}{dr} + \frac{m}{\omega_4}\frac{d\Omega}{dr} + \rho_0\omega_4\frac{d}{dr}\left(\frac{1}{\rho_0\omega_4}\right)\right]\frac{d\tilde{p}}{dr} + \left[\left(\frac{\omega_4}{c_0}\right)^2 - k^2 - \frac{m^2}{r^2} - \frac{2m\rho_0\omega_4}{r}\frac{d}{dr}\left(\frac{\Omega}{\rho_0(\omega_4)^2}\right)\right]\tilde{p} = 0 \tag{70}$$

Noting that from Eq. (41*b*) follows

$$\omega_4 d(\omega_4^{-1})/dt = -\omega_4^{-1} d\omega_4/dr = (kdU/dr + md\Omega/dr)/\omega_4 \quad (71)$$

it is clear that Eq. (70) coincides with Eq. (47) in the low Mach swirl and shear approximation. This proves the consistency of the general wave equation (68) for unrestricted shear and swirl, with the particular cases of low Mach number shear and swirl. It would also be possible to derive the generalization of these from low Mach number swirl to unrestricted swirl and shear; since in the latter case the mean flow is inhomogeneous, this leads to five wave equations, of which Eq. (68) is one example. The number of wave equations would double from 15 to 30 by considering the particular cases of axisymmetric acoustic waves, of which Eq. (69) is an example. Of these 30 acoustic wave equations in axisymmetric sheared and/or swirling flow, only 16 are listed explicitly, viz., two more examples, in addition to the preceding cases (42)–(51) plus (68) and (69).

2.9 Rigid Body and Potential Vortex Swirl. For high Mach number shear (10*b*) the mean flow is always inhomogeneous. The simplest case is rigid body swirl:

$$\Omega = \text{const} \quad p_0(r) = p_* + \Omega^2 \int_0^r \rho_0(r) r dr \quad (72)$$

when Eq. (52) simplifies to

$$\Omega = \text{const} \quad X_2 \equiv (\omega_4)^2 - \Omega^2 [4 - (\Omega r/c_0)^2 + (r/\rho_0) d\rho_0/dr] \quad (73)$$

and the wave equation (68) becomes

$$\text{W48}^*: \quad \frac{d^2\tilde{p}}{dr^2} + \left[\frac{1}{r} + \frac{k}{\omega_4}\frac{dU}{dr} + \frac{\rho_0 X_2}{\omega_4}\frac{d}{dr}\left(\frac{\omega_4}{\rho_0 X_2}\right)\right]\frac{d\tilde{p}}{dr} + \left\{X_2\left[\frac{1}{c_0^2} - \frac{k^2 + m^2/r^2}{(\omega_4)^2}\right] - \frac{\rho_0 X_2}{\omega_4}\frac{d}{dr}\left\{\frac{\omega_4}{\rho_0 X_2 r}\left[\frac{2m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\right\} - \frac{1}{r^2}\left[\frac{2m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\left[1 + \left(\frac{\Omega r}{c_0}\right)^2 + \frac{2m\Omega}{\omega_4} + \frac{kr}{\omega_4}\frac{dU}{dr}\right]\right\}\tilde{p} = 0 \quad (74)$$

The next simplest case is potential vortex swirl (15*a*) and (15*b*), for which the mean flow pressure (10*b*) is given by

$$\Omega r^2 = \text{const} \quad p_0(r) = p_* + [r^2\Omega(r)]^2 \int_0^r \rho_0(r) r^{-3} dr \quad (75)$$

and Eq. (55) simplifies

$$r^2\Omega = \text{const} \quad X_3 \equiv (\omega_4)^2 + \Omega^2 [(\Omega r/c_0)^2 - (r/\rho_0) d\rho_0/dr] \quad (76)$$

and the wave equation (68) becomes

$$\text{W49}^*: \quad \frac{d^2\tilde{p}}{dr^2} + \left[\frac{1}{r} + \frac{k}{\omega_4}\frac{dU}{dr} - 2\frac{m}{\omega_4}\frac{\Omega}{r} + \frac{\rho_0 X_3}{\omega_4}\frac{d}{dr}\left(\frac{\omega_4}{\rho_\sigma X_3}\right)\right]\frac{d\tilde{p}}{dr} + \left\{X_3\left[\frac{1}{c_0^2} - \frac{k^2 + m^2/r^2}{(\omega_4)^2}\right] - \frac{\rho_0 X_3}{\omega_4}\frac{d}{dr}\left\{\frac{\omega_4}{\rho_0 X_3}\left[\frac{2m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\right\} - \frac{1}{r^2}\left[\frac{\bar{2}m\Omega}{\omega_4} + \left(\frac{\Omega r}{c_0}\right)^2\right]\left[1 + \left(\frac{\Omega r}{c_0}\right)^2 + \frac{kr}{\omega_4}\frac{dU}{dr}\right]\right\}\tilde{p} = 0 \quad (77)$$

For axisymmetric acoustic modes $m=0$, there is further simplification of Eqs. (77) and (74).

The most general wave equation with axisymmetric shear and swirl is Eq. (68), which applies to the acoustic pressure spectrum (40) of nonaxisymmetric modes (W50*) and simplifies to Eq. (69) for axisymmetric modes (W47*); for nonaxisymmetric modes, Eq. (68) simplifies to Eq. (74) for rigid body swirl (W48*) and Eq. (77) for potential vortex swirl (W49*). In the case of low Mach number, considering nonaxisymmetric acoustic modes, the wave equations may be written either (40) for the acoustic pressure p or for its spectrum $\tilde{p}$. The wave equation (30) for the acoustic pressure of nonaxisymmetric modes in low Mach number (29) mean axisymmetric shear and swirl (W42) simplifies to (33) for rigid body (W44) and (34) for potential vortex (W46) swirl; it also includes the cases (31) of shear only (W38) or (32) of swirl only (W40). In the case of low Mach number swirl, the mean flow may be homogeneous, in which case (W41) the wave equation for the acoustic pressure of nonaxisymmetric modes in a low Mach number homogeneous mean shear and swirl is Eq. (35). It simplifies to Eq. (36) for shear only (W37) and Eq. (37) for swirl only (W39). In the case of shear and swirl, it simplifies to Eq. (38) for rigid body (W43) and Eq. (39) for potential vortex (W45) swirl.

Taking the example of jet engines as ducted flows: (i) the noise radiation out of the inlet, upstream of the compressor is affected by shear flow [7–50] (these references also appear in Paper I [1]); (ii) the noise radiation out of the exhaust, downstream of the turbine is affected not only by shear but also by swirl [2–6,51–54]. Exact solutions of the acoustic wave equation in an axisymmetric swirling mean flow, with a uniform (not sheared) axial velocity, are known for two cases: (i) rigid body swirl [3], in terms of Bessel functions; (ii) potential flow swirl [6], using asymptotic expansions of extended Bessel functions [55,56]. The propagation of sound in sheared and swirling flows may lead to the appearance of critical layers, where the Doppler shifted frequency vanishes; these critical layers may also occur for other types of waves in fluids, e.g., water waves [57] and waves in gases [58]. The latter include gravity waves [59–64] associated with stratification, hydromagnetic waves in ionized fluids under external magnetic fields [65–68], inertial waves associated with rotation [69,70], and their couplings in magnetoacoustic-gravity-inertial waves [71,72].

Returning to the acoustics of ducts, the modal propagation is described by solving the appropriate wave equations (e.g., Classes I, II, IV, or V) with boundary conditions; this is simpler in the case of ducts of constant cross section as concerns modes [73–75], mean flow effects [76–78], acoustic momentum and energy balance [79,80], the effects of variable sound speed [81], changes in cross section [82], and radiation out of open ends [83–86]. The modal decompositions can be extended to ducts with nonuniform cross section and simple shapes, e.g., conical [87,88], otherwise approximate, and numerical and experimental methods are used [89–97]. There is no need to use boundary conditions in the case of the fundamental longitudinal mode, for quasi-one-dimensional propagation in ducts of nonuniform cross section, e.g., horns and nozzles [98–175] using wave equations of Class III. The simplest and most used [176–179] is Webster's horn equation, which has been extended to two dimensions [180], as for waveguides [181]. The acoustics of straight and bent tubes [182] is relevant to musical instruments [183–185] and to sound reproduction [186], either by electromechanical devices [150–154] or in concert halls [187–189].

The preceding applications of acoustic wave equations include several examples of sound in inhomogeneous [190,191] and moving [192–194] media. Nonuniform media may have a deterministic structure or have random inhomogeneities, i.e., the interface between two media may be a smooth surface (e.g., a plane) or may be randomly irregular. Flows may be laminar, i.e., with deterministic fluid variables, or turbulent with velocity, which is a random function of position and time. The convected wave equation applies to sound in turbulent media in the ray limit, when the wavelength is small relative to the length scale of the variation of the turbulent velocity; in this case the coefficients of the wave equation are random functions of position and time. There are theories of the interaction of sound with turbulence not restricted to high frequencies [195–197]. Turbulence is also one of the main mechanisms [198–201] of noise generation [202–221] together with fluid inhomogeneities [222–224]. Turbulence thus acts both as a source and a scatterer of sound, making more difficult the transposition between ground and in-flight noise measurements [225,226]. Acoustic measurements of sound propagating through turbulence [227–229] demonstrate random amplitudes and phases. The interference of phases is the common characteristic of all waves in random media [230–234] and leads to spectral and directional broadening of jet noise [26–28,235–240], and also to random pressures, relevant to (i) boundary-layer noise [241–249], and (ii) acoustic fatigue [250–252].

3 Wave Equations With Viscous and Thermal Dissipaton

All of the preceding 50 wave equations (W1–W50) are nondissipative. The dissipative acoustic wave equations are simplest for linear waves in a homogeneous medium at rest. In can be shown that the shear viscosity affects only the vorticity, and thus a scalar wave equation can be obtained for the dilatation. For linear acoustic waves in a homogeneous medium at rest, the dissipation of the dilatation by bulk and shear viscosity and thermal conduction are comparable. The linear, dissipative acoustic wave equation (W52) for the dilatation in a homogeneous medium at rest involves products of the bulk and shear viscosities by the thermal conductivity, which are relevant only in the case of strong dissipation. Since the diffusivities are often small, their products may be neglected, thus simplifying the linear dissipative acoustic wave equations in a homogeneous medium at rest, in the presence of weak shear and bulk viscosities and thermal conduction (W51). Sinusoidal waves are considered in both cases of strong and weak dissipation, the latter leading to the usual thermoviscous dissipation coefficient [253], which will reappear in connection with weakly nonlinear dissipative acoustic waves (Secs. 4.8 and 4.9).

3.1 Effects of Viscosities and Thermal Conductivity. For dissipative fluids, (i) the equation of continuity (2) is unchanged, and (ii) the viscous stresses are added to (1) the momentum equation:

$$D\mathbf{V}/Dt + \Gamma^{-1}\,\nabla\,P = \mu\nabla^2\mathbf{V} + (\upsilon + \mu/3)\,\nabla\,(\nabla\,.\,\mathbf{V}) \qquad (78)$$

where and D/Dt the exact material derivative,

$$D/Dt \equiv \partial/\partial t + \mathbf{V}\,.\,\nabla \qquad (79)$$

μ, υ denote, respectively, the shear and bulk kinematic viscosities or viscous diffusivities; (iii) the adiabatic equation (3a) is replaced by the energy equation in entropy production form:

$$\frac{1}{\gamma-1}\left(\frac{DP}{DT} - C^2\frac{D\Gamma}{DT}\right) = \zeta\nabla^2 T + \mu\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}(\nabla\,.\,\mathbf{V})\delta_{ij}\right]^2 + \upsilon(\nabla\,.\,\mathbf{V})^2 \qquad (80)$$

where ζ is the thermal conductivity and C the adiabatic sound speed (3b); (iv) the equation of state is used in the form $P(\Gamma,T)$.

The total state of the fluid is assumed to consist (compare with Eq. (7)) of a homogeneous mean state of rest plus a nonuniform unsteady perturbation:

$$\{\mathbf{v},p,P,T\}(\mathbf{x},t) = \{\mathbf{O},p_0,T_0\} + \{\mathbf{v},p,p,\Theta\}(\mathbf{x},t) \qquad (81)$$

linearizing with regard to the perturbations, the fundamental equations (1) and (78)–(80) lead to

$$\partial\rho/\partial t = -\,\rho_0\,\nabla\,\,.\,\mathbf{v} \qquad (82a)$$

$$\partial\mathbf{v}/\partial t - \mu\nabla^2\mathbf{v} - (\upsilon + \mu/3)\,\nabla\,(\nabla\,.\,\mathbf{v}) = -\,(1/\rho_0)\,\nabla\,p \qquad (82b)$$

$$\partial p/\partial t - c_0^2\partial\rho/\partial t = (\gamma-1)\zeta\nabla^2\Theta \qquad (82c)$$

where the adiabatic sound speed is now calculated $c_0^2=(\partial p_0/\partial p_0)$, from the pressure p_0 and mass density ρ_0 in the mean state. The linearized equation of state is

$$\rho' = (\partial p_0/\partial\rho_0)_T\rho + (\partial p_0/\partial T_0)_\rho\Theta \qquad (83)$$

where the thermodynamic derivatives are evaluated for the mean state.

3.2 Adiabatic and Isothermal Sound Speeds. The first coefficient of Eq. (83) is the isothermal sound speed (compare with the adiabatic sound speed (3b)):

$$\left(\frac{\partial P}{\partial\Gamma}\right)_T = \frac{\partial(P,T)}{\partial(P,S)}\frac{\partial(P,S)}{\partial(\Gamma,S)}\frac{\partial(\Gamma,S)}{\partial(\Gamma,T)} = C^2[(\partial S/\partial T)_\Gamma/(\partial S/\partial T)_P] \equiv C^2C_v/C_p = C^2/\gamma = RT \qquad (84)$$

where we used the specific heats at constant volume C_v and pressure C_p, and their ratio, which is the adiabatic exponent γ:

$$C_p \equiv T(\partial S/\partial T)_p \qquad (85a)$$

$$C_v \equiv T(\partial S/\partial T)_\rho \qquad (85b)$$

$$\gamma \equiv C_p/C_v \qquad (85c)$$

$$C_p - C_v = R \qquad (85d)$$

Note that the last equality in Eq. (84) applies to a perfect gas (3b) as well as (85d).

The second thermodynamic derivative in Eq. (83) follows the equation of state in the form $S(T,P(\Gamma,T))$, viz.,

$$\left(\frac{\partial S}{\partial T}\right)_\Gamma = \left(\frac{\partial S}{\partial T}\right)_P + \left(\frac{\partial S}{\partial P}\right)_T\left(\frac{\partial P}{\partial T}\right)_\Gamma \qquad (86)$$

leading to

$$\left(\frac{\partial P}{\partial T}\right)_\Gamma = \left[\left(\frac{\partial S}{\partial T}\right)_\Gamma - \left(\frac{\partial S}{\partial T}\right)_P\right]\Big/\left(\frac{\partial S}{\partial P}\right)_T = \frac{C_p - C_v}{\varepsilon T} = (\gamma-1)\frac{C_v}{\varepsilon T} \qquad (87)$$

where ε is the coefficient of thermal expansion,

$$\varepsilon \equiv -\,(\partial S/\partial P)_T = [\partial(1/\Gamma)/\partial T]_P \qquad (88)$$

Substituting Eqs. (84) and (87) in Eq. (83) yields

$$p = (c_0^2/\gamma)\rho + (\gamma-1)(C_v/T_0\varepsilon)\Theta \qquad (89)$$

as the linearized equation of state consistent with Eqs. (82a)–(82c) i.e., suitable for elimination of p, ρ, Θ, and leading to a single equation for the velocity perturbation $\mathbf{v}$, which is the linear dissipative acoustic wave equation.

3.3 Dissipative Wave Equation for the Velocity Perturbation. The process of elimination for $\mathbf{v}$ is started by taking the time derivative of Eq. (82b), and substituting Eqs. (89) and (82a) in succession:

$$\begin{aligned}\partial^2\mathbf{v}/\partial t^2 - \mu\nabla^2(\partial\mathbf{v}/\partial t) - (\upsilon+\mu/3)\,\nabla\,[\nabla\,.\,(\partial\mathbf{v}/\partial t)] &= -\,(1/\rho_0)\,\nabla\,(\partial p/\partial t)\\ &= -\,(\gamma-1)(C_v/T_0\varepsilon\rho_0)\,\nabla\,(\partial\Theta/\partial t) - (c_0^2/\gamma\rho_0)\,\nabla\,(\partial\rho/\partial t)\\ &= -\,(\gamma-1)(C_v/T_0\varepsilon\rho_0)\,\nabla\,(\partial\Theta/\partial t) + (c_0^2/\gamma)\,\nabla\,(\nabla\,.\,\mathbf{v})\end{aligned} \qquad (90)$$

another relation between the velocity $\mathbf{v}$ and temperature Θ perturbations is obtained from Eq. (82c), substituting Eqs. (89) and (82a):

$$\begin{aligned}\zeta\nabla^2\Theta &= (C_v/T_0\varepsilon)\partial\Theta/\partial t - \lfloor(c_0^2 - c_0^2/\gamma)/(\gamma-1)\rfloor\partial\rho/\partial t\\ &= (C_v/T_0\varepsilon)\partial\Theta/\partial t + (c_0^2/\gamma)\rho_0(\nabla\,.\,\mathbf{v})\end{aligned} \qquad (91)$$

The temperature perturbation can be eliminated between Eqs. (90) and (91),

$$\{\zeta\nabla^2 - (C_v/T_0\varepsilon)\partial/\partial t\}\{\partial^2/\partial t^2 - (c_0^2/\gamma)\nabla(\nabla . - \mu\nabla^2\ \partial/\partial t - (v + \mu/3)\nabla(\nabla . \partial/\partial t)\}\mathbf{v} = \{\zeta\nabla^2 - (C_v/T_0\varepsilon)\partial/\partial t\}[-(\gamma-1) \times(C_v/T_0\varepsilon\rho_0)\nabla(\partial\Theta/\partial t)] = \{-(\gamma-1)(C_v/T_0\varepsilon\rho_0)\nabla(\partial/\partial t)\} \times[(c_0^2/\gamma)\rho_0(\nabla . \mathbf{v})] = -c_0^2(1-1/\gamma)(C_v/T_0\varepsilon)\nabla[\nabla . (\partial\mathbf{v}/\partial t)] \quad (92)$$

leading to

$$\text{W52:}\quad \partial^3\mathbf{v}/\partial t^3 - c_0^2\nabla[\nabla . (\partial\mathbf{v}/\partial t)] = \mu\nabla^2(\partial^2\mathbf{v}/\partial t^2) + (v + \mu/3)\nabla[\nabla . (\partial^2\mathbf{v}/\partial t^2)] - (\alpha c_0^2/\gamma)\nabla^2[\nabla(\nabla . \mathbf{v})] + \alpha(\partial/\partial t) \times\{\nabla^2(\partial\mathbf{v}/\partial t) - (v+\mu/3)\nabla^2[\nabla(\nabla . \mathbf{v})] - \mu\nabla^4\mathbf{v}\} \quad (93)$$

for the velocity perturbation in a medium at rest, which is the linear dissipative acoustic wave equation; note that the parameter α in Eq. (94*a*)

$$\alpha \equiv T_0\varepsilon\zeta/C_v = \rho_0 T_0\varepsilon\bar{\alpha} \quad (94a)$$

$$\bar{\alpha} \equiv \zeta/\rho_0 C_v \quad (94b)$$

involves the thermal conductive diffusivity $\bar{\alpha}$ in Eq. (94*b*), multiplied by the dimensionless factor $\rho_0 T_0\varepsilon$; for a perfect gas it follows substituting the equation of state $P=\Gamma RT$ in Eq. (88) that $\varepsilon = R/P = 1/\Gamma T$, and thus the factor $\rho_0 T_0 \varepsilon = 1$ is unity, so that $\bar{\alpha} = \alpha$.

3.4 Dissipation of the Vorticity by Shear Viscosity. Taking the curl of the linearized momentum equation (82*b*) and denoting by dot a time derivative, e.g., $\mathbf{\Omega} \equiv \partial\mathbf{\Omega}/\partial t$, it follows that

$$\mathbf{\Omega} \equiv \nabla \wedge \mathbf{v} \quad \dot{\mathbf{\Omega}} = \mu\nabla^2\mathbf{\Omega} \quad (95)$$

the vorticity decouples and is dissipated only by shear viscosity. Thus, a plane wave solution for the unsteady vorticity,

$$\mathbf{\Omega}(\mathbf{x},t) = \mathbf{\Omega}_0 e^{i(\mathbf{k}.\mathbf{x}-\omega t)} \quad (96)$$

with wave vector $\mathbf{k}$ and "frequency" ω leads to the dispersion relation:

$$\omega = -ik^2\mu \quad (97)$$

which shows that there is only one "mode" with imaginary frequency, implying that a sinusoidal oscillation in space

$$\mathbf{\Omega}(\mathbf{x},t) = \mathbf{\Omega}_0 e^{i\mathbf{k}.\mathbf{x}} e^{-k^2\mu t} \quad (98)$$

decays exponentially in time, in proportion to the shear diffusity and to the square of the wave number.

3.5 Dissipative Wave Equation for the Dilatation. Taking the divergence of Eq. (93), it follows that the dilation Ψ satisfies a dissipative wave equation:

$$\text{W52}^*:\quad \Psi \equiv \nabla . \mathbf{v}$$

$$\dddot{\Psi} - c_0^2\nabla^2\dot{\Psi} = (4\mu/3 + v + \alpha)\nabla^2\ddot{\Psi} - c_0^2(\alpha/\gamma)\nabla^4\Psi - \alpha(4\mu/3 + v)\nabla^4\dot{\Psi} \quad (99)$$

which is generally of the third order in time and fourth order in position. The viscous and thermal diffusivities are generally small, and thus the last term in Eq. (21), involving their product

$$\alpha(4\mu/3 + \nu)\nabla^4\dot{\Psi} \ll (4\mu/3 + v)\nabla^2\ddot{\Psi} \quad (100)$$

can be neglected

$$\text{W51}^*:\quad \dddot{\Psi} - c_0^2\nabla^2\dot{\Psi} = (4\mu/3 + v + \alpha)\nabla^2\ddot{\Psi} - c_0^2(\alpha/\gamma)\nabla^4\Psi \quad (101)$$

for a plane wave,

$$\Psi(\mathbf{x},t) = \Psi_0 e^{i(\mathbf{k}.\mathbf{x}-\omega t)} \quad (102)$$

in the case of weak dissipation $(100) \equiv (103a)$,

$$\omega\nabla^2\Psi \sim \nabla^2\dot{\Psi} \gg \alpha\nabla^4\Psi \sim k^2\alpha\nabla^2\Psi \quad (103a)$$

$$\alpha \ll \omega/k^2 \quad (103b)$$

which is equivalent to (103*b*). The last two terms in the weakly dissipative acoustic wave equation (101) the viscous diffusivities are "interchangeable" and can be combined in a total viscous diffusivity (104*a*):

$$\beta \equiv \nu + 4\mu/3 \quad (104a)$$

$$\dddot{\Psi} - c_0^2\nabla^2\dot{\Psi} = (\beta+\alpha)\nabla^2\ddot{\Psi} - c_0^2(\alpha/\gamma)\nabla^4\Psi \quad (104b)$$

but the latter is not interchangeable with the thermal diffusivity in Eq. (26b). If the diffusivities are negligible

$$(\beta+\alpha)\nabla^2\ddot{\Psi} \ll \dddot{\Psi} \quad (105a)$$

$$\alpha\nabla^4\Psi \ll \nabla^2\dot{\Psi} \quad (105b)$$

then the classical wave equation W1 follows:

$$\beta,\alpha \ll \omega/k^2 \quad \ddot{\Psi} - c_0^2\nabla^2\Psi = 0 \quad (106)$$

in the agreement with Eq. (38) in Ref. [1]. Thus, there is a hierarchy of linear wave equations in a homogeneous medium at rest: (i) the classical wave equation W1 for negligible dissipation (106); (ii) the weakly dissipative classical wave equation W51 for small diffusivities (103*b*), which appear linearly in Eq. (101); (iii) the strongly dissipative classical wave equation (99):

$$\text{W52}*:\quad \dddot{\Psi} - c_0^2\nabla^2\dot{\Psi} = (\beta+\alpha)\nabla^2\ddot{\Psi} - (c_0^2\alpha/\gamma)\nabla^4\Psi - \alpha\beta\nabla^4\dot{\Psi} \quad (107)$$

which involves the product of viscous (104*a*) and thermal (94*a*) and (94*b*) diffusivities in Eq. (107).

3.6 Heat Equation as the Incompressible Limit. In the incompressible limit,

$$c_0 \to \infty \quad \dot{\Psi} = (\alpha/\gamma)\nabla^2\Psi = \alpha_1\nabla^2\Psi \quad (108)$$

the dissipative wave equation (99) reduces without approximation to a [254,255] heat equation with heat diffusivity:

$$\alpha_1 \equiv \alpha/\gamma = T_0\varepsilon\zeta/\gamma C_v = T_0\varepsilon\zeta/C_p = \rho_0 T_0\zeta\alpha_2 \quad (109a)$$

$$\alpha_2 \equiv \zeta/\rho_0 C_p \quad (109b)$$

which is similar to Eqs. (94*a*) and (94*b*) replacing the specific heat at constant volume by that at constant pressure (85*a*)–(85*c*). In conclusion, the dissipative acoustic wave equation W52* for the dilatation in a homogeneous medium at rest (99) includes limiting cases, the heat equation (108) and the W1 classical wave equation (106); in its general form (107), it involves the diffusivities (104*a*) and (94*a*) linearly in the case W51* of weak dissipation (103*b*) and second-order products (107) in the case W52* of strong dissipation.

The general linear, nondissipative acoustic wave equation in a homogeneous medium at rest (99) consists of eight terms: (i) the first two coincide with the classical wave equation (106) without dissipation; (ii) the next four are linear in the diffusivities, viz., shear μ and bulk ν viscous diffusivities and conductive α thermal diffusivitiy, the latter appearing also multiplying the isothermal sound speed c_0^2/γ in Eq. (84); (iii) the last two terms involve cross products of the diffusivities, viz., the viscous (shear and bulk) diffusivities times the thermal conductive diffusivity. As seen in Table 2, the most general linear, dissipative acoustic wave equation in a homogeneous medium at rest is (107)≡(99), which applies (W52*) in the presence of arbitrary shear and bulk viscosities and thermal conduction. In the case of weak diffusivities (W51*), it simplifies to Eq. (104*b*). In the incompressible limit $c_0 \to \infty$, all linear dissipative wave equations reduce to the heat equation (108), with heat condutivity as diffusivity (109*a*) and (109*b*). A vector heat equation (95) applies to the vorticity, with the shear viscosity as diffusivity. The equations for the vorticity (95) and dilatation (107)≡(99) follow from the linear, nondissipative acoustic wave equation W52 for the velocity perturbation in a homogeneous medium at rest, in the presence of shear and bulk viscosities and thermal conduction (93).

3.7 Sinusoidal Waves With Strong Dissipation. The shear and bulk viscosities appear combined (104*a*) in the exact dissipative wave equation for the dilatation (104*b*), which involves also the modified thermal diffusivity (94*a*) and (94*b*) and adiabatic (110*a*)≡(3*b*) and isothermal (110*b*)≡(84) sound speeds:

$$c_0^2 = (\partial p_0/\partial \rho_0)_{s_0} = \gamma p_0/\rho_0 = \gamma R T_0 \tag{110a}$$

$$(\partial p_0/\partial \rho_0)_{T_0} = c_0^2/\gamma = p_0/\rho_0 = RT_0 \equiv c_1^2 \tag{110b}$$

The exact dispersion relation for a plane wave (102) solution of Eq. (107) is of the third degree in frequency:

$$0 = \omega^3 + i(\alpha+\beta)k^2\omega^2 - k^2(c_0^2 + \alpha\beta k^2)\omega - ik^4c^2\alpha/\gamma \tag{111}$$

It has there roots, two corresponding to dissipative sound waves, and the third to a dissipative mode coupling thermal diffusion with viscosity and/or compressibility. Rewriting Eq. (111) in the form

$$0 = \omega^3 + i(\delta_1+\delta_2)\omega^2 - (\omega_0^2 + \delta_1\delta_2)\omega - i\delta_1\omega_1^2 = 0 \tag{112}$$

it involves (i) the frequencies for adiabatic (113*a*) and isothermal (113*b*) nondissipative sound waves:

$$\omega_0 \equiv c_0 k \tag{113a}$$

$$\omega_1 \equiv c_0 k/\sqrt{\gamma} \equiv \omega_0/\sqrt{\gamma} = c_1 k \tag{113b}$$

(ii) the damping rates (compare with Eqs. (97) and (98)) due to thermal (114*a*) and viscous (114*b*) dissipations,

$$\delta_1 \equiv \alpha k^2 \tag{114a}$$

$$\delta_2 \equiv \beta k^2 = (4\mu/3 + v)k^2 \tag{114b}$$

In order to solve Eq. (112) exactly, it is rewritten

$$\omega^3 + i\delta\omega^2 - \omega_2^2\omega - i\delta_1\omega_1^2 = 0 \tag{115}$$

where ω_2 is a strongly dissipative correction to Eq. (113*b*), the adiabatic sound frequency (116*a*):

$$\omega_2^2 \equiv \omega_0^2 + \delta_1\delta_2 = (c_0^2 + \alpha\beta)k^2 \tag{116a}$$

$$\delta \equiv \delta_1 + \delta_2 = (\alpha+\beta)k^2 = (\alpha + 4\mu/3 + v)k^2 \tag{116b}$$

and δ is the total damping rate (116*b*). The square term is eliminated from Eq. (115) by a simple substitution (117*a*), so that (115) simplifies to (117*b*):

$$\omega = \bar{\omega} - i\delta/3 \tag{117a}$$

$$\bar{\omega}^3 + a\bar{\omega} + b = 0 \tag{117b}$$

where

$$a \equiv \delta^2/3 - \omega_2^2 \tag{118a}$$

$$-b \equiv 2i\delta^3/27 - i\delta\omega_2^2/3 + i\delta_1\omega_1^2 \tag{118b}$$

The roots of Eq. (117*b*) are

$$\bar{\omega} = \sqrt[3]{-b/2+f} + \sqrt[3]{-b/2-f} \tag{119a}$$

$$f \equiv \sqrt{b^2/4 + a^3/27} \tag{119b}$$

Thus, the strongly dissipative acoustic wave equation (107) has three plane wave solutions (102), with arbitrary wave vector **k**, and three frequencies ω, which are given by Eqs. (117*a*) and (119*a*), where the constants (a,b,f) in Eqs. (118*a*), (118*b*), and (119*b*) are specified through Eqs. (116*a*) and (116*b*) by the frequencies for adiabatic (113*a*) and isothermal (113*b*) sound waves, and the thermal (114*a*) and viscous (114*b*) dampings. The frequencies (119*a*) are generally complex, so that

$$\exp\{i(\mathbf{k}\,.\,\mathbf{x} - \omega t)\} = \exp\{i[\mathbf{k}\,.\,\mathbf{x} - t\,\mathrm{Re}(\omega)]\}\exp[t\,\mathrm{Im}(\omega)] \tag{120}$$

so that the real part specifies the phase speed of wave fronts, and the imaginary part the decay in time, as in Eqs. (97) and (98). This will be illustrated next in the case of weak dissipation.

3.8 Dispersion Relation for Weak Dissipation. The case of weak dissipation (103*b*) is specified in terms of dampings (114*a*) and (114*b*) and sound frequencies (113*a*) and (113*b*) by

$$(\delta_1)^2, (\delta_2)^2, \delta_1\delta_2 \ll (\omega_0)^2, (\omega_1)^2, \omega_0\omega_1 \tag{121}$$

the dissipative terms (118*a*), (118*b*), and (119*b*) are approximated to lowest order $0(\delta)$ by

$$a = -\omega_0^2 \tag{122a}$$

$$-b = -i\delta\omega_0^2/3 + i\delta_1\omega_1^2 \tag{122b}$$

$$f \equiv i\omega_0^3/3\sqrt{3} \tag{122c}$$

and thus the three roots of Eqs. (117*a*) and (117*b*) are given by Eq. (119*a*), viz.,

$$\bar{\omega} = (\omega_0/\sqrt{3})\{\sqrt[3]{i}g_- + \sqrt[3]{-i}g_+\} \tag{123a}$$

where

$$g_\pm \equiv \sqrt[3]{1 \pm b/2f} = 1 \pm (\sqrt{3}/2\omega_0)(\delta/3 - \delta_1/\gamma) \tag{123b}$$

One pair of roots (123*a*) and (117*a*) is complex:

$$\omega_\pm = \pm\omega_0 - i\delta/2 + i\delta_1/2\gamma = \pm\omega_0 - i\delta_2/2 - i\delta_1(1-1/\gamma)/2 \tag{124}$$

and corresponds to sound waves propagating in opposite directions and specifies their damping to lowest order. The third root is pure imaginary ((114*a*) and (94*a*), (95)):

$$\omega_3 = i\delta_1/\gamma = i\alpha k^2/\gamma = -ik^2T_0\varepsilon\zeta/C_p \tag{125}$$

and corresponds to a purely damped motion, with the decay rate specified by thermal conduction at constant pressure:

$$\exp(i\omega_3 t) = \exp(-\alpha k^2 t/\gamma) = \exp(-T_0\varepsilon\zeta k^2 t/C_p) = \exp(-\alpha_1 k^2 t) \tag{126}$$

as in the classical heat equation (108) (compare with Eq. (98)). It could be expected that the imaginary part of Eq. (124) specifies the thermoviscous dissipation coefficient for acoustic waves. This will be confirmed next.

3.9 Acoustic Thermoviscous Dissipation Coefficient. The case of weak dissipation (103)*a*≡(127*a*) implies

$$\alpha\beta \ll \omega^2/k^4 \tag{127a}$$

$$\delta_1\delta_2 \ll \omega_0^2 \tag{127b}$$

for the diffusivities (114*a*) and (114*b*) and adiabatic sound frequency (113*a*), the condition (127*b*), which simplifies Eq. (112) to

$$0=\omega^3+i(\delta_1+\delta_2)\omega^2-\omega_0^2\omega-i\delta_1\omega_1^2 \quad (128)$$

corresponding to the weakly dissipative wave equation (104b), without the $\alpha\beta$ term in Eq. (107). In the absence of dissipation (106), sound waves have the adiabatic frequency (113a); thus one solution of Eq. (128) must be (113a) with a small correction, because dissipation is weak but non-negligible:

$$\omega=\pm\omega_0+\varepsilon \quad 0=\omega_0^2(2\varepsilon+i\delta_1+i\delta_2-i\delta_1/\gamma)+0(\varepsilon^2,\varepsilon\delta) \quad (129)$$

where the product of small quantities is also neglected. This may be solved for ε,

$$\varepsilon=-i[\delta_1(1-1/\gamma)+\delta_2]/2=\omega_\pm\mp\omega_0 \quad (130)$$

and agrees with the previous result (124). From (124)≡(130), it follows that sound waves have weak damping:

$$\exp(i\omega t)=\exp(\mp i\omega_0 t)\exp\{-[\delta_1(1-1/\gamma)+\delta_2]t/2\}=\exp(-i\omega_\pm t) \quad (131)$$

The latter can be set in the form:

$$\omega_\pm=\pm kc_0-i\vartheta/c_0 \quad (132)$$

so that the temporal damping rate is ϑ/c_0 and spatial damping rate ϑ is given by

$$\vartheta\equiv[\delta_2+\delta_1(1-1/\gamma)]/2c_0=[(4\mu/3+v)+\alpha(1-1/\gamma)]k^2/2c_0 \quad (133)$$

where Eqs. (114a) and (114b) were used; using Eq. (94a) leads, to the same order of approximation, to

$$\vartheta=(\omega^2/2c_0^2)[4\mu/3+v+(T_0\varepsilon\zeta)(1/C_v-1/C_p)] \quad (134)$$

which is known as the spatial damping coefficient [253] for linear sound waves; the acoustic thermoviscous dissipation coefficient (134) is restricted to weak damping as follows from restricting the analysis in Secs. 3.8 and 3.9 to it will reappear next in connection (Secs. 4.8 and 4.9) with weakly nonlinear dissipative acoustic waves in Eq. (200), bearing in mind that for a perfect gas $\rho_0T_0\varepsilon=1$ and $\alpha=\bar{\alpha}$ in Eqs. (94a) and (94b) and thus $T_0\varepsilon\zeta=\zeta/\rho_0$.

The thermoviscous dissipation coefficient for weak dissipation of sound can be deduced without recourse to wave equation, by [253] an energy balance; its derivation from the acoustic wave equation W51 with weak dissipation thus serves as an independent confirmation of the latter. Although the classical and convected wave equations appear in almost every textbook and monograph on acoustics, that does not seem to be the case for its extension to weak W51 or strong W52 thermoviscous dissipation. The thermoviscous damping of sound waves is usually treated in the weak dissipative case by means of the damping coefficient, e.g., in atmospheric acoustics [256–258]. The subject of outdoor sound propagation [259,260] concerns not only the aerial absorption of sound [261–267], but also the effect of ground impedance [268,269] and natural obstacles or man-made barriers to sound propagation. Viscothermal dissipation of sound also occurs near the walls of ducts [270,271]. Another mechanism of sound absorption is resonance in cavities [272–274], which may be nonlinear [275,276]. Sound absorption in ducts is enhanced by the use of acoustic liners, which may have uniform [35,36,40,277–282] or nonuniform [283–291] impedance. The sound absorption by acoustic liners is not mainly an effect of thermoviscous dissipation, but rather of vortex sheding from orifices in the liner, and the resultant acoustic-vortical interactions [292–295]. These sound scattering [296] and attenuation [297] effects fall under the general heading of fluid-structure interaction [298–303].

4 Nonlinear Dissipative and Inhomogeneous Wave Equations

The combination of nonlinear (Sec. 2) and dissipative (Sec. 3) waves is considered in the context of the theory of characteristics, first developed [304] for nonlinear isentropic one-dimensional waves (W53); there is an extension to nonuniform collapsible ducts both for isentropic characteristics (W54) and the weakly dissipative [305–307] case (W55). Whereas linear nondissipative waves in a homogeneous medium propagate an undeformed wave form, nonlinear sound propagates faster at the crests than at the troughs, leading to wave form steepening, and to shock formation; this process is opposed by dissipation. If the latter is weak, it leads to Burgers equation [308,309], which can be transformed to the heat equation [254,255] via the Cole–Hopf transformation [310,311], which thus provides a method of solution [312]. These one-dimensional wave equations with viscous dissipation can be extended to include thermal conduction as well (W57), allowing also for nonuniform and collapsible ducts (W58). The final extension to weakly nonlinear, weakly dissipative waves, is of interest (W59) to sonar pulses in water [313], including beams (W60), in the presence of viscous [314,315] or thermoviscous dissipation.

4.1 Conservation of Riemann Invariants Along Characteristics. Adding and subtracting the one-dimensional adiabatic equation of continuity (4) in the form:

$$\frac{1}{\Gamma C}\frac{\partial P}{\partial t}+\frac{V}{\Gamma C}\frac{\partial P}{\partial x}+C\frac{\partial V}{\partial x}=0 \quad (135)$$

to the one-dimensional inviscid momentum equation (1b):

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}+\frac{1}{\Gamma}\frac{\partial P}{\partial x}=0 \quad (136)$$

leads to

$$\left(\frac{\partial V}{\partial t}\pm\frac{1}{\Gamma C}\frac{\partial P}{\partial t}\right)+(V\pm C)\left(\frac{\partial V}{\partial x}\pm\frac{1}{\Gamma C}\frac{\partial P}{\partial x}\right)=0 \quad (137)$$

This can be written in the form

$$\text{W53:}\quad\left\{\frac{\partial}{\partial t}+(V\pm C)\frac{\partial}{\partial x}\right\} \quad (138a)$$

$$J_\pm(x,t)=0 \quad (138b)$$

showing that the Riemann invariants

$$J_\pm(x,t)\equiv V\pm\int(\Gamma C)^{-1}dP \quad (139)$$

are conserved along the characteristics:

$$(dx/dt)_\pm=V\pm C \quad (140)$$

which are the curves along which travel sound waves in opposite directions $\pm C$, at sound speed C superimposed on the velocity perturbation V, which can be comparable to C for a nonlinear wave. In adiabatic conditions,

$$\mathrm{S}=\text{const} \quad P\Gamma^\gamma=\text{const} \quad (141)$$

substituting the sound speed (3b) in the integral in Eqs. (139)

$$\int(\Gamma C)^{-1}dP=2C/(\gamma-1) \quad (142)$$

specifies

$$J_\pm=V\pm[2/(\gamma-1)]C \quad (143)$$

the Riemann invariants.

4.2 Wave Front Steepening for Nonlinear Waves. In the case of linear, nondissipative waves in a homogeneous medium at rest, the velocity perturbation v is small relative to the constant sound speed c_0, and (Eqs. (138) and (139)):

$$\left\{\frac{\partial}{\partial t}+(V\pm C)\frac{\partial}{\partial x}\right\}\left(V\pm\frac{2C}{\gamma-1}\right)=0 \tag{144}$$

simplify to:

$$v \ll c_0\text{const} \tag{145a}$$

$$\partial v/\partial t \pm c_0 \partial v/\partial x = 0 \tag{145b}$$

its solution involves arbitrary functions,

$$v_\pm(x,t)=\phi_\pm(x\pm c_0 t) \tag{146}$$

and shows that wave forms propagate without deformation in the positive and the negative x direction with constant phase velocity $\pm c_0$. Combining Eqs. (145a) and (145b) leads

$$\text{W1:}\quad 0=(\partial/\partial t+c_0\partial/\partial x)(\partial/\partial t-c_0\partial/\partial x)=\ddot{v}-c_0^2 v'' \tag{147}$$

to the classical wave equation (106), as should be expected for linear, nondissipative waves in a homogeneous medium at rest.

In the case of nonlinear, nondissipative waves (138), it follows that the Riemman invariants are given by

$$J_\pm(x,t)=\phi_\pm(x-(V\pm C)t) \tag{148}$$

Using Eq. (143) follows the coupled system,

$$V\pm 2C/(\gamma-1)=\phi_\pm(x-(V\pm C)t) \tag{149}$$

which confirms (140) that the propagation speed is the group velocity:

$$W_\pm \equiv V\pm C \tag{150}$$

Thus, the propagation speed is larger in the compression part $V>0$ of the wave than in the rarefaction part $V<0$, leading to wave form steepening and shock formation.

4.3 Conservation of the Self-Convected Group Velocity. For unidirectional waves, one of the Riemann invariants (143) is zero:

$$J_- = 0 \quad V=2C/(\gamma-1) \tag{151}$$

and the other can be expressed in terms of the velocity perturbation V or sound speed C alone:

$$J_+ = 2 \quad V=[4/(\gamma-1)]C \tag{152}$$

any other wave variable can be used, e.g., the group velocity,

$$W\equiv V+C=[(\gamma+1)/2]V=[(\gamma+1)/(\gamma-1)]C \tag{153}$$

The conservation of the Riemann invariant along the characteristic

$$\partial V/\partial t+(V+C)\partial V/\partial x=0 \tag{154}$$

applies to any wave variable, and in the case of the group velocity (153)

$$\text{W53}^*\text{:}\ \partial W/\partial t+W\partial W/\partial x=0 \tag{155}$$

shows that it is self-convected.

4.4 Weak Viscous Dissipation of Nonlinear Waves. The steepening of the wave front of nonlinear waves is opposed by viscosity, suggesting that the latter be added to Eq. (136), in the viscous momentum equation:

$$\partial V/\partial t+V\partial V/\partial x+\Gamma^{-1}\partial P/\partial x=\beta\partial^2 V/\partial x^2 \tag{156}$$

where β denotes the total viscosity (104a). Adding and subtracting Eq. (135) lead to

$$\text{W55:}\ \{\partial/\partial t+(V\pm C)\partial/\partial x\}J_\pm(x,t)=\beta\partial^2 V/\partial x^2 \tag{157}$$

which shows that the Riemann invariants (139)≡(143) are dissipated along the characteristics (140). For weak dissipation, it is still possible to consider unidirectional waves (151) and (152) and thus Eq. (157) simplifies to

$$\partial V/\partial t+(V+C)\partial V/\partial x=(\beta/2)\partial^2 V/\partial x^2 \tag{158}$$

Use of Eq. (153) shows that the group velocity satisfies Burger's equation,

$$\text{W55}^*\text{:}\ \partial W/\partial t+W\partial W/\partial x=(\beta/2)\partial^2 W/\partial x^2 \tag{159}$$

with half the total viscosity. This nonlinear convection plus diffusion equation was first introduced [308] as a one-dimensional viscous momentum equation without pressure gradient, in the hope of representing turbulence; in fact, "burgulence" is best seen as a competition between nonlinearity and diffusion [309].

4.5 Transformation of Burger's to a Heat Equation. The nonlinearity is accounted by noting that Eq. (159) in the form

$$-2\partial W/\partial t=\partial(W^2-\beta\partial W/\partial x)/\partial x \tag{160}$$

is satisfied if a function ψ exists such that

$$W=-\beta\partial(\log\psi)/\partial x \tag{161a}$$

$$W^2-\beta\partial W/\partial x=2\beta\partial(\log\psi)/\partial t \tag{161b}$$

From Eq. (161a) follows the Cole–Hopf [310,311] transformation,

$$\psi(x,t)=\exp\left\{-\beta^{-1}\int W(x,t)dx\right\} \tag{162}$$

and also by substitution into Eq. (161b), viz.,

$$\partial\psi/\partial t+(\beta/2)\partial^2\psi/\partial x^2=0 \tag{163}$$

the heat equation [254,255] such as Eqs. (95) and (108) with half the total viscosity $\beta/2$ as diffusivity.

4.6 Collapsible Duct of Varying Cross Section. In the case of propagation in a duct, whose cross-sectional area $A(x,t)$ can vary with position (nonuniform) and time (collapsible), the mass density per unit volume Γ is replaced by the mass density per unit length ΓA in the one-dimensional equation of continuity (2), viz.,

$$0=\frac{\partial P}{\partial t}(\Gamma A)+\frac{\partial}{\partial x}(\Gamma AV)=A\left(\frac{\partial\Gamma}{\partial t}+V\frac{\partial\Gamma}{\partial x}\right)+A\Gamma\frac{\partial V}{\partial x}+\Gamma\left(\frac{\partial A}{\partial t}+V\frac{\partial A}{\partial x}\right) \tag{164}$$

and the adiabatic condition (3a) can be used

$$0=\frac{A}{C^2}\left(\frac{\partial P}{\partial t}+V\frac{\partial P}{\partial x}\right)+A\Gamma\frac{\partial V}{\partial x}+\Gamma\frac{DA}{Dt} \tag{165}$$

Multiplying by $C/A\Gamma$,

$$\frac{1}{\Gamma C}\frac{\partial P}{\partial t}+\frac{V}{\Gamma C}\frac{\partial P}{\partial x}+C\frac{\partial V}{\partial x}=-\frac{C}{A}\frac{DA}{Dt} \tag{166}$$

which added or subtracted from Eq. (136) leads to

$$\text{W54:}\ \left\{\frac{\partial}{\partial t}+(V\pm C)\frac{\partial}{\partial x}\right\}J_\pm(x,t)=\mp\frac{C}{A}\left(\frac{\partial A}{\partial t}+V\frac{\partial A}{\partial x}\right) \tag{167}$$

showing that the Riemann invariants (139)≡(143) are not conserved along characteristics for a nonuniform $\partial A/\partial x\neq 0$ or collapsible $\partial A/\partial t\neq 0$ tube, due to wave reflections, which cancel in the sense that the sum of Eq. (167) implies

$$\left\{\frac{\partial}{\partial t}+(V+C)\frac{\partial}{\partial x}\right\}J_+ +\left\{\frac{\partial}{\partial t}+(V-C)\frac{\partial}{\partial x}\right\}J_- =0 \tag{168}$$

as a conservation law.

For gradual changes of cross section, unidirectional waves (151) and (152) may be considered in Eq. (167):

$$\partial V/\partial t + (V + C)\partial V/\partial x = -(C/2)D(\log A)/Dt \tag{169}$$

it follows that the group velocity (153) varies along characteristics

$$\text{W54}^*:\ \partial W/\partial t + W\partial W/\partial x = -[(\gamma + 1)/4]CD(\log A)/Dt \tag{170}$$

due to changes in cross section. The case of a rigid duct whose cross section $A \sim x^n$ varies like a power with exponent n of the axial coordinate:

$$A(x) = A(0)x^n \quad \partial W/\partial t + W\partial W/\partial x + (W/2)C(n/x) = 0 \tag{171}$$

includes, in particular, (i) for $n=0$ nonlinear plane waves in a uniform duct (155); (ii) for $n=2$ nonlinear spherical waves in a conical duct; (iii) the intermediate case $n=1$ corresponds to nonlinear cylindrical waves in duct with parabolic side profile, viz., $\sqrt{A} \sim \sqrt{x} \sim r_0$ in the case of circular cross section of radius r_0.

4.7 Nonlinear Dissipative Waves in a Variable Duct. Combining changes in cross section of the duct in the adiabatic continuity equation (166) with viscous dissipation in the momentum equation (156) implies that both effects cause the Riemann invariants (139) ≡ (143) to vary along characteristics:

$$\text{W56:}\ \left\{\frac{\partial}{\partial t} + (V \pm C)\frac{\partial}{\partial x}\right\}J_\pm = \beta\frac{\partial^2 V}{\partial x^2} \mp \frac{C}{A}\frac{DA}{Dt} \tag{172}$$

and although reflections cancel,

$$\left\{\frac{\partial}{\partial t} + (V \pm C)\frac{\partial}{\partial x}\right\}J_+ + \left\{\frac{\partial}{\partial t} + (V - C)\frac{\partial}{\partial x}\right\}J_- = 2\beta\frac{\partial^2 V}{\partial x^2} \tag{173}$$

dissipation breaks the conservation law (168). For weak viscous dissipation and gradual changes of cross section, a unidirectional wave (151) and (152) may be considered,

$$\partial V/\partial t + (V + C)\partial V/\partial x = (\beta/2)\partial^2 V/\partial x^2 - (C/2)(\log A)/Dt \tag{174}$$

use of Eq. (153) implies that the group velocity satisfies

$$\text{W56}^*:\ \partial W/\partial t + W\partial W/\partial x = (\beta/2)\partial^2 W/\partial x^2 - [(\gamma + 1)/4]CD(\log A)/dt \tag{175}$$

which is a Burgers equation (159) for half the viscosity, forced by duct area changes. In the case of rigid duct whose cross section increases with a power of the axial distance leads to

$$A(x) = A(0)x^n \quad \partial W/\partial t + W\partial W/\partial x = (\beta/2)\partial^2 W/\partial x^2 - (W/2)C(n/x) \tag{176}$$

which a linearly forced Burgers equation, generalizing Eq. (171) to include viscous dissipation. It includes the cases: (i) $n=0$ of nonlinear viscous plane waves (159); (ii) $n=2$ of nonlinear viscous spherical waves; (iii) $n=1$ of nonlinear viscous cylindrical waves. The method of characteristics has been applied to (W53) nonlinear isentropic waves (138) and (143); the extensions include (W54) variable ducts (167), viscous (W55) dissipation (157) and both (W56) together (172); the respective equations for the group velocity of unidirectional waves specify (W53*) free (155) and (W54*) forced (170) self-convection, and (W55*) free (159) and (W56*) forced (174) Burgers equations.

4.8 Thermoviscous Dissipation of Nonlinear Waves. The effects of viscosity and thermal conduction are comparable in the dissipation of linear acoustic waves in a homogeneous medium at rest (Sec. 3). Thus, it is desirable to extend the one-dimensional nonlinear wave equation (158) in the presence of viscosity (W55) to include thermal conduction (W57) as well. As an independent check of the method of characteristics (Secs. 4.1–4.4) used to derive W55, its generalization W57 will be obtained directly from the equations of motion. The inertia force, in the inviscid ((1)) or viscous (78) momentum equation, can be set into the form:

$$\Gamma DV_i/Dt = \Gamma(\partial V_i/\partial t + V_j\partial V_i/\partial x_j) = \partial(\Gamma V_i)/\partial t + \partial(\Gamma V_iV_j)/\partial x_j - V_i[\partial\Gamma/\partial t + \partial(\Gamma V_j)/\partial x_j] \tag{177}$$

where the term in square brackets vanishes, on account of the equation of continuity (2):

$$0 = \partial\Gamma/\partial t + V_i\partial\Gamma/\partial x_i + \Gamma\partial V_i/\partial x_i = \partial\Gamma/\partial t + \partial(\Gamma V_i)/\partial x_i \tag{178}$$

Substituting Eq. (173), the viscous momentum equation (78) takes the form

$$0 = \partial(\Gamma V_i)/\partial t + (\partial/\partial x_j)[\Gamma V_iV_j + P\delta_{ij} - \mu\Gamma\partial V_i/\partial x_j - (\nu + \mu/3)\Gamma\partial V_j/\partial x_i] \tag{179}$$

where $\mu\Gamma$ and $\nu\Gamma$ are, respectively, the static shear and bulk viscosities, since μ, ν denote the kinematic shear and bulk viscosities. Applying $\partial/\partial t$ to Eq. (178) and subtracting $\partial/\partial x_i$ applied to (183) lead to

$$\partial^2\Gamma/\partial t^2 = (\partial/\partial x_i\partial x_j)[\Gamma V_iV_j + P\delta_{ij} - \mu\Gamma\partial V_i/\partial x_j - (\nu + \mu/3)\Gamma\partial V_j/\partial x_i] \tag{180}$$

For a perturbation from a homogeneous mean state of rest (81), this may be approximated as

$$\partial^2\rho/\partial t^2 - c_0^2\nabla^2\rho = (\partial^2/\partial x_i\partial x_j)[(p - c_0^2\rho)\delta_{ij} + \rho_0 v_iv_j - \mu\rho_0\partial v_i/\partial x_j - (\nu + \mu/3)\rho_0\partial v_j/\partial x_i] \tag{181a}$$

where (i) the linear, nondissipative terms form the classical wave equation W1 on the left hand side (lhs) (106); (ii) the dissipative terms on the right hand side (rhs) are linearized, replacing total Γ by mean state ρ_0 density, for weak dissipation; (iii) the nonlinear, nondissipative terms are taken to second order, for weak nonlinearity. In the one-dimensional case, ((181*a*)) simplifies to

$$\partial^2\rho/\partial t^2 - c_0^2\partial^2\rho/\partial x^2 = (\partial^2/\partial x^2)(p - c_0^2\rho + \rho_0 v^2 - \rho_0\beta\partial v/\partial x) \tag{181b}$$

where β is the total kinematic viscosity (104*a*).

The pressure perturbation in Eq. (181*a*) and (181*b*) is specified by the equation of state $p(\rho, s)$, which in the weakly nonlinear, weakly dissipative approximation has the terms

$$p = (\partial p_0/\partial\rho_0)_s\rho + \tfrac{1}{2}(\partial^2 p_0/\partial\rho_0^2)_s\rho^2 + (\partial p_0/\partial s_0)_\rho s \tag{182}$$

viz., (i) the term linear on the density, which involves the adiabatic sound speed (110*a*), where the last equality is valid only for a perfect gas (3*b*); (ii) the weak nonlinearity is represented by the quadratic term on density, with coefficient,

$$(\partial^2 p_0/\partial\rho_0^2)_s = (\partial(c_0^2)/\partial\rho_0)_s = (\gamma/\rho_0)(\partial p_0/\partial\rho_0)_s - \gamma p_0/\rho_0^2 = (c_0^2/\rho_0)(\gamma - 1) \tag{183a}$$

where the ratio of specific heats γ was taken as a constant (85*c*); (iii) the weak dissipation is represented by taking only a linear term in the entropy (no cross term ρs or quadratic term s^2), whose coefficient

$$(\partial p_0/\partial s_0)_\rho = (\partial p_0/\partial T_0)_\rho(\partial T_0/\partial s_0)_\rho = (\gamma - 1)/\varepsilon \tag{183b}$$

is calculated from Eqs. (87) and (85*b*), and involves the thermal expansion coefficient (88). The equation of state ((182) and (110*a*); (183*a*) and (183*b*))

$$p = c_0^2\rho + (c_0^2/2\rho_0)(\gamma-1)\rho^2 + [(\gamma-1)/\varepsilon]s \quad (184)$$

can be substituted in the weakly nonlinear, weakly dissipative acoustic wave equation (181*b*):

$$\partial^2\rho/\partial t^2 - c_0^2\partial^2\rho/\partial x^2 = (\partial^2/\partial x^2)\{(c_0^2/2\rho_0)(\gamma-1)\rho^2 + \rho_0 v^2 + [(\gamma-1)/\varepsilon]s - \rho_0\beta\partial v/\partial x\} \quad (185)$$

In the quadratic nondissipative and linear dissipative terms on the rhs of Eq. (185), it is permissible to substitute linear nondissipative relations, because the error is the negligible (i.e., cubic nondissipative or quadratic dissipative terms). For example, the linear, one-dimensionnal nondissipative momentum equation,

$$\rho_0\partial v/\partial t = -\partial p/\partial x = -c_0^2\partial\rho/\partial x = c_0\partial\rho/\partial t \quad (186)$$

where Eq. (145*a*) was used with + sign, leads to the polarization relations,

$$v/c_0 = \rho/\rho_0 = p/(\rho_0 c_0^2) \quad (187)$$

which show that acoustic velocity perturbation ν is a fraction of the sound speed c_0, equal to the acoustic density perturbation ρ as a fraction of the mean state mass density ρ_0. Substituting Eq. (187) in the rhs of Eq. (185),

$$\partial^2\rho/\partial t^2 - c_0^2\partial^2\rho/\partial x^2 = (\partial^2/\partial x^2)\{(c_0^2/2\rho_0)(\gamma+1)\rho^2 - c_0\beta\partial\rho/\partial x + [(\gamma-1)/\varepsilon]s\} \quad (188)$$

only the density perturbation ρ appears, except in the last term, which is transformed next.

The entropy S satisfies the energy equation (79),

$$\Gamma TDS/DT = \zeta\nabla^2 T + \mu\left[\partial V_i/\partial x_j + \partial V_j/\partial x_i - \tfrac{2}{3}(\nabla\,.\,\mathbf{V})^2\delta_{ij}\right]^2 + \nu(\nabla\,.\,\mathbf{V})^2 \quad (189)$$

which can be linearized (81) in the weekly dissipative approximation:

$$\rho_0 T_0\partial s/\partial t = \zeta\nabla^2\Theta \quad (190)$$

This simplifies in the one-dimensional case to

$$(\zeta/\rho_0 T_0)\partial^2\Theta/\partial x^2 = \partial s/\partial t = -c_0\partial s/\partial x \quad (191)$$

which implies that the density perturbation is specified by

$$s = -(\zeta/\rho_0 T_0 c_0)\partial\Theta/\partial x \quad (192)$$

in terms of the temperature perturbation Θ. The latter is related to the density perturbation ρ by

$$\Theta/\rho = (\partial T_0/\partial\rho_0)_s = \partial(T_0,s_0)/\partial(\rho_0,s_0) = \frac{\partial(T_0,s_0)}{\partial(T_0,p_0)}\frac{\partial(T_0,p_0)}{\partial(s_0,\rho_0)}\frac{\partial(s_0,\rho_0)}{\partial(p_0,s_0)} = \varepsilon\frac{T_0}{C_p}c_0^2 \quad (193)$$

where the thermodynamic derivatives were evaluated in terms of the thermal expansion coefficient (88), the specific heat at constant pressure (85*a*) and the adiabatic sound speed (110*a*), all for the mean state. Substitution of Eq. (193) in Eq. (188) relates the entropy s and density ρ perturbations:

$$s = -(\zeta\varepsilon c_0/\rho_0 C_p)\partial\rho/\partial x \quad (194)$$

Substitution of Eq. (194) in Eq. (188) leads to the one-dimensional, weakly nonlinear, linearly dissipative acoustic wave equation:

$$\text{W57: } \partial^2\rho/\partial t^2 - c_0^2\partial^2\rho/\partial x^2 = (c_0^2/2\rho_0)(\gamma+1)\partial^2(\rho^2)/\partial x^2 - c_0\vec{\vartheta}\partial^3\rho/\partial x^3 \quad (195)$$

with the density perturbation ρ as variable, where the thermoviscous diffusivity

$$\bar{\vartheta} \equiv \beta + \zeta(\gamma-1)/\rho_o C_p = \nu + 4\mu/3 + (\zeta/\rho_0)(1/C_v - 1/C_p) = 2(c_0/\omega)^2\vartheta \quad (196)$$

which includes (i) the total kinematic viscosity (104*a*) and (ii) the effect of thermal conduction ζ divided by the mean state mass density ρ_0 to have the dimensions of diffusivity (L^2T^{-1}), taking into account the specific heats at constant volume C_v and pressure C_p. Note that the thermoviscous diffusivity (196) includes all the diffusivities appearing in the thermoviscous dissipation coefficient (134).

For linear, nondissipative waves (147), corresponding to the lhs of Eq. (195), the sound field consists of unidirectional waves (145*a*) and (145*b*) whose wave forms (146) do not change in a frame moving at sound speed, e.g.,

$$\xi \equiv x - c_0 t, \quad \partial/\partial t \to \partial/\partial t - c_0\partial/\partial\xi, \quad \partial/\partial x = \partial/\partial\xi \quad (197)$$

for a wave propagating in the positive x direction. The rhs of Eq. (195) implies that the wave form is deformed slowly in the scale of a wavelength, by two weak and opposing effects: (i) the quadratic nonlinearity, which causes wave form steepening (Sec. 4.2); (ii) the linear dissipation, which causes wave decay. Thus, changing from space-time $\rho(x,t)$ to coordinates (197) moving at sound speed $\rho(\xi,t)$, i.e., substituting (197) in Eq. (195), leads to

$$\partial^2\rho/\partial t^2 - 2c_0\partial^2\rho/\partial t\partial\xi - (c_0^2/2\rho_0)(\gamma+1)\partial^2(\rho^2)/\partial\xi^2 = -c_0\bar{\vartheta}\partial^3\rho/\partial\xi^3 \quad (198)$$

which may be simplified on account of the slow variation of the wave form:

$$\partial\rho/\partial t \ll 2c_0\partial\rho/\partial\xi \quad \partial\rho/\partial t + (c_0/2\rho_0)(\gamma+1)\rho\partial\rho/\partial\xi = (\bar{\vartheta}/2)\partial^2\rho/\partial\xi^2 \quad (199)$$

The linear relation (187) may be used in the quadratic nonlinear and linear dissipative terms in Eq. (199), and also in the slowly varying term $\partial\rho/\partial t$, leading to the one-dimensional, weakly nonlinear, weakly dissipative acoustic wave equation in terms of the velocity perturbation.

$$\text{W57}^*\text{: } \partial v/\partial t + [(\gamma+1)/2]v\partial v/\partial\xi = (\bar{\vartheta}/2)\partial^2 v/\partial\xi^2 \quad (200)$$

Introducing the group velocity for a unidirectional wave (153) leads to

$$\text{W57}^*\text{: } \partial W/\partial t + W\partial W/\partial\xi = (\bar{\vartheta}/2)\partial^2 W/\partial\xi^2 \quad (201)$$

which is the Burgers equation (159), with two differences: (i) the position x is replaced by the coordinate (197) moving at sound speed; (ii) the dissipation coefficient (197) includes viscous (104*a*) and thermal dampings as in the thermoviscous dissipation coefficient (134).

4.9 Weakly Nonlinear and Dissipative Three-Dimensional Beam. The preceding weakly nonlinear wave equation, with linear thermoviscous dissipation, was one-dimensional W57 in Eq. (195) and W57* in Eq. (204)≡(205), and will be extended to the three-dimensional case. Since there are important applications to sonars, i.e., nonlinear acoustic waves in water, it should be noted that the weakly nonlinear, weakly dissipative equation of state (182) applies not only to perfect gases ((110*a*); (183*a*) and (183*b*)) but also to liquids in the same form (184). For a liquid the equation of state of a perfect gas,

$$P = R\Gamma T \quad (202)$$

can be replaced by Tate's relation:

$$P = p_0 + B\lfloor(\Gamma/\rho_0)\gamma - 1\rfloor \quad (203)$$

which involves a constant B, in addition to the adiabatic exponent (85*c*) in Eq. (141). The adiabatic sound speed (3*b*) is given for liquid by

$$C^2 = (\partial P/\partial \Gamma)_s = (B\gamma/\rho_0)(\Gamma/\rho_0)^{\gamma-1} = (\gamma/\Gamma)(P - p_0 + B) \tag{204}$$

and its derivative with regard to the density by

$$(\partial^2 P/\partial \Gamma^2)_s = \lfloor \partial(C^2)/\partial \Gamma \rfloor_s = (\gamma/\Gamma)[-(P - p_0 + B)/\Gamma + (\partial P/\partial \Gamma)_s] = [\gamma(\gamma-1)/\Gamma^2](P - p_0 + B) = (\gamma - 1)C^2/\Gamma \tag{205}$$

which coincides with Eq. (183*a*) when applied to the mean state; since Eq. (183*b*) also holds for liquids, so does Eq. (184). Recalling the exact material derivative,

$$D\mathbf{V}/Dt = \partial \mathbf{V}/\partial t + \nabla(V^2/2) + (\nabla \wedge \mathbf{V}) \wedge \mathbf{V} \tag{206}$$

and assuming that weakly nonlinear and weakly dissipative sound waves remain irrotational

$$\nabla \wedge \mathbf{V} = 0 \quad D\mathbf{V}/Dt = \partial \mathbf{V}/\partial t + \nabla(V^2/2) \tag{207}$$

and thus the viscous momentum equation (78) simplifies to

$$\partial \mathbf{V}/\partial t + \nabla(V^2/2) + \Gamma^{-1}\nabla P = \beta \nabla^2 \mathbf{V} \tag{208}$$

where β is the total kinematic viscosity in Eq. (104*a*). The continuity equation (2) is used in the form

$$\Gamma(\nabla . \mathbf{V}) + \partial \Gamma/\partial t + \mathbf{V} . \nabla \Gamma = 0 \tag{209}$$

and the mean state is taken to be (81) homogeneous at rest.

When substituting Eq. (81) in Eqs. (208) and (209), the linear nondissipative terms are written on the lhs and on the rhs are retained only linear dissipative and nonlinear quadratic terms:

$$\rho_0 \partial \mathbf{v}/\partial t + \nabla p = -pc_0^{-2}\partial \mathbf{v}/\partial t - (\rho_0/2)\nabla(v^2) + \rho_0 \beta \nabla^2 \mathbf{v} \tag{210a}$$

$$\rho_0(\nabla . \mathbf{v}) + \partial \rho/\partial t = -\mathbf{v} . \nabla \rho - \rho \nabla . \mathbf{v} \tag{210b}$$

The equation of state (184) may be solved for ρ, and $p = c_0^2\rho$ replaced in the nonlinear and entropy terms:

$$\rho = c_0^{-2}p - [(\gamma-1)/2\rho_0 c_0^4]p^2 - [(\gamma-1)/c_0^2\varepsilon]s \tag{211}$$

substitution of Eq. (211) in Eq. (210*b*) leads to

$$\rho_0(\nabla . \mathbf{v}) + c_0^{-2}\partial p/\partial t = -c_0^2\mathbf{v} . \nabla p + \lfloor(\gamma-1)/2\rho_0 c_0^4\rfloor \partial(p^2)/\partial t - c_0^{-2}p(\nabla . \mathbf{v}) + [(\gamma-1)/c_0^2\varepsilon]\partial s/\partial t \tag{212}$$

The entropy perturbation is eliminated using Eqs. (190) and (193),

$$\partial s/\partial t = (\zeta/\rho_0 T_0)\nabla^2[(\varepsilon T_0 c_0^2/C_p)\rho] = (\varepsilon\zeta/\rho_0 C_p)\nabla^2 p \tag{213}$$

The velocity perturbation may be substituted from the linear relations (187) and

$$\rho_0 \partial \mathbf{v}/\partial t = -\nabla p \tag{214a}$$

$$\nabla . \mathbf{v} = -\rho_0^{-1}\partial \rho/\partial t = -\rho_0^{-1}c_0^{-2}\partial p/\partial t \tag{214b}$$

$$\nabla \wedge \mathbf{v} = 0 \nabla^2 \mathbf{v} = \nabla(\nabla . \mathbf{v}) = -\rho_0^{-1}c_0^{-2}\partial(\nabla p)/\partial t \tag{214d}$$

in the lhs of Eqs. (210) and (212),

$$\rho_0 \partial \mathbf{v}/\partial t + \nabla p = -c_0^{-2}\beta \partial(\nabla p)/\partial t \tag{215a}$$

$$\rho_0 \nabla . \mathbf{v} + c_0^{-2}\partial p/\partial t = (\gamma/2\rho_0 c_0^4)\partial(p^2)/\partial t - c_0^{-2}\mathbf{v} . \nabla p + [(\gamma-1)\zeta/c_0^2\rho_0 C_p]\nabla^2 p \tag{215b}$$

The velocity perturbation $\mathbf{v}$ may be eliminated between the lhs by applying the divergence to Eq. (215*a*) and subtracting $\partial/\partial t$ applied to Eq. (215*b*), leading to

$$\nabla^2 p - c_0^{-2}\partial^2 p/\partial t^2 + c_0^{-2}\bar{\vartheta}\nabla^2(\partial p/\partial t) = Y \tag{216a}$$

where the linear terms are the classical wave operator (106) with the thermoviscous diffusivity (196), and the nonlinear terms

$$\begin{aligned} Y + (\gamma/2\rho_0 c_0^4)\partial^2(p^2)/\partial t^2 &= c_0^{-2}[\mathbf{v} \cdot \nabla(\partial p/\partial t) + \nabla p \cdot \partial \mathbf{v}/\partial t] \\ &= -\rho_0 \mathbf{v} \cdot \nabla^2 \mathbf{v} - \rho_0^{-1}c_0^{-2}(\nabla p)^2 \\ &= -\rho_0^{-1}c_0^{-2}[p\nabla^2 p + (\nabla p)^2] \\ &= -\rho_0^{-1}c_0^{-4}[p\partial^2 p/\partial t^2 + (\partial p/\partial t)^2] \\ &= -(2\rho_0 c_0^4)^{-1}\partial^2(p^2)/\partial t^2 \end{aligned} \tag{216b}$$

were simplified using the linear, nondissipative relations (214*a*)–(214*c*), (187), and the classical wave equation (106) and relation $(\nabla p)^2 = c_0^{-2}(\partial p/\partial t)^2$ for plane waves.

Substituting (216*b*) in (216*a*) leads to the three-dimensional weakly nonlinear, linearly dissipative acoustic wave equation:

$$\text{W59: } \nabla^2 p - c_0^{-2}\partial^2 p/\partial t^2 + \bar{\vartheta}c_0^{-4}\partial^3 p/\partial t^3 = [(\gamma+1)/2\rho_0 c_0^4]\partial^2(p^2)/\partial t^2 \tag{217}$$

which consists of (i) the linear nondissipative terms (106) of the classical wave equation W1; (ii) linear dissipation specified by the thermoviscous diffusivity (196); (iii) quadratic, i.e., weakly nonlinear terms, which coincide with those in the Westerwelt [314] equation, which is valid in the nondissipative case:

$$\boldsymbol{\vartheta} = 0 \quad \nabla^2 p - c_0^{-2}\partial^2 p/\partial t^2 = [(\gamma+1)/2\rho_0 c_0^4]\partial^2(p^2)/\partial t^2 \tag{218}$$

Since the wave equation (217) represents a slow deformation of the wave form in coordinates moving at the sound speed (197), the retarded time (219*a*) is used:

$$\tau = t - x/c_0 \tag{219a}$$

$$\bar{x} = \chi x \tag{219b}$$

$$\bar{y} = \sqrt{\chi}y \tag{219c}$$

$$\bar{z} = \sqrt{\chi}z \tag{219d}$$

and a similarity transformation is used in Eq. (219*b*) in the propagation direction x with parameter χ, whereas the parameter $\sqrt{\chi}$ is used in the transverse directions (219*c*) and (219*d*) to represent a narrow beam:

$$\partial/\partial t = \partial/\partial \tau \quad \partial/\partial x = \chi \partial/\partial \bar{x} - c_0^{-1}\partial/\partial \tau \quad \{\partial/\partial y, \partial/\partial z\} = \sqrt{\chi}\{\partial/\partial \bar{y}, \partial/\partial \bar{z}\} \tag{220}$$

The substitution of (220) in Eq. (217) leads to

$$\text{W60: } (\partial/\partial \tau)\{\partial p/\partial \bar{x} + [(\gamma+1)/2\rho_0 c_0^3]p\partial p/\partial \tau - (\bar{\vartheta}/2c_0^3)\partial^2 p/\partial \tau^2\} = (c_0/2)[\partial^2 p/\partial \bar{x}^2 + \partial^2 p/\partial \bar{y}^2] \tag{221}$$

where the term in square brackets on the rhs is the two-dimensional Laplacian transverse to the direction of propagation. The weakly nonlinear wave equation for an acoustic beam (221) was first obtained without dissipation $\boldsymbol{\vartheta} = 0$ by Zabolotskaya and Khoklov [314]; dissipation by viscosity, i.e., $\bar{\vartheta} = \beta$ in Eq. (104*a*), was included by Kuznetsov [315], so that it is known as the KZK equation. In the present derivation, thermal conduction was included as well, since it is comparable to viscous dissipation in the total diffusivity (196). Using Eq. (187), the acoustic velocity perturbation appears as variable:

$$\text{W60: } (\partial/\partial \tau)\{\partial v/\partial \bar{x} + [(\gamma+1)/2c_0^2]v\partial v/\partial \tau - (\bar{\vartheta}/2c_0^3)\partial^2 v/\partial \tau^2\} = (c_0/2)(\partial^2 v/\partial \bar{y}^2 + \partial^2 v/\partial \bar{z}^2) \tag{222}$$

In the absence of transverse propagation, Eq. (222) reduces to the Burgers equation in curly brackets:

$$\partial/\partial \bar{y} = 0 = \partial/\partial \bar{z} \tag{23}$$

$$c_0\partial v/\partial \mathbf{x} + [(\gamma+1)/2](v/c_0)\partial v/\partial\tau = (\bar{\vartheta}/2c_0^2)\partial^2 v/\partial\tau^2$$

with nonlinearity parameter $(\gamma+1)/2$. Using Eq. (53), the group velocity leads to

$$\text{W60: } (\partial/\partial\tau)\{c_0\partial W/\partial\bar{x} + (W/c_0)\partial W/\partial\tau - (\bar{\vartheta}/2c_0^2)\partial^2 W/\partial\tau^2\} = c_0^2(\partial^2 W/\partial\bar{v}^2 + \partial^2 W/\partial\bar{z}^2) \quad (224)$$

In the absence of transverse propagation $\partial/\partial\bar{y}=0=\partial/\partial\bar{z}$, then Eq. (224) reduces to the terms in curly brackets, which coincide with Burgers equation (159), with time derivative $\partial/\partial t = c_0\partial/\partial\bar{x}$ and spatial derivative $\partial/\partial x = c_0^{-1}\partial/\partial\tau$.

The acoustic pressure perturbation p is related to the potential ϕ through (18,19) in Paper I:

$$-p/\rho_0 = d\phi/dt = \dot{\phi} + (\nabla\phi)^2 = \dot{\phi} + c_0^{-2}\phi'^2 \quad (225)$$

where the second expression is valid in the quadratic approximation. Substituting Eq. (225) in Eq. (217), the linear terms are unchanged and the nonlinear terms cancel on the lhs:

$$\text{W59: } \nabla^2\phi - c_0^{-2}\ddot{\phi} + \bar{\vartheta}c_0^{-4}\dddot{\phi} = [(\gamma+1)/2]c_0^{-4}\partial(\dot{\phi}^2)/\partial t \quad (226)$$

where on the rhs the acoustic pressure is replaced by the first term in Eq. (225), and in the quadratic approximation,

$$[(\gamma+1)/2]c_0^{-2}\partial(\dot{\phi}^2)/\partial t = (\gamma-1)\dot{\phi}c_0^{-2}\ddot{\phi} + \partial[\nabla\phi]^2/\partial t = (\gamma-1)\dot{\phi}\nabla^2\phi + 2\nabla\phi\cdot\nabla\dot{\phi} \quad (227)$$

Substituting Eq. (227) in Eq. (226) leads to the weakly nonlinear, linearly dissipative acoustic wave equation in a homogeneous medium at rest, in the form

$$\text{W59: } \ddot{\phi} - c_0^{-2}\nabla^2\phi + 2\nabla\phi\cdot\nabla\dot{\phi} + (\gamma-1)\dot{\phi}\nabla^2\phi = -\bar{\vartheta}c_0^{-2}\dddot{\phi} \quad (228)$$

which should be compared with the nonlinear classical wave equation W10, which applies ((76) in Paper I) to strongly nonlinear, nondissipative acoustic waves in a homogeneous medium at rest

$$\text{W10: } \ddot{\phi} - c_0^{-2}\nabla^2\phi + 2\nabla\phi\cdot\nabla\dot{\phi} + (\gamma-1)\dot{\phi}\nabla^2\phi + [(\gamma-1)/2] \times(\nabla\phi)^2\nabla^2\phi + \nabla\phi\cdot[(\nabla\phi\cdot\nabla)\nabla\phi] = 0 \quad (229)$$

viz., (i) the first four linear and quadratic nondissipative terms are the same in Eq. (229) and in Eq. (228); (ii) the last two cubic nonlinear terms in the exact nondissipative equation (229) are omitted in the weakly nonlinear approximation in Eq. (228); (iii) the linear dissipative term in Eq. (228) is absent in the nondissipative equation (229). An acoustic beam in three dimensions (224) differs from a one-dimensional wave in a duct of varying area because (i) there is no transverse propagation $\partial/\partial y=0=\partial/\partial z$; (ii) the remaining term in the Laplacian in Eq. (217) is replaced by the duct wave operator ((119) in Paper I):

$$\nabla^2\phi \to A^{-1}(A\phi')' = \phi'' + (A'/A)\phi' \quad (230)$$

leading to

$$\text{W58: } A^{-1}(Ap')' - c_0^{-2}\ddot{p} + \bar{\vartheta}c_0^{-4}\dddot{p} = [(\gamma+1)/2\rho_0c_0^4]\partial^2(p^2)/\partial t^2 \quad (231)$$

which is the one-dimensional, weakly nonlinear, linearly dissipative acoustic wave equation in a duct of varying cross section $A(x)$, containing a homogeneous fluid at rest. It can be expressed in terms of the acoustic potential ϕ using (225), viz.,

$$\text{W58: } \ddot{\phi} - c_0^2(\phi'' + \phi'A'/A) + 2\dot{\phi}'\phi' + (\gamma-1)\dot{\phi}(\phi'' + \phi'A'/A) = -\bar{\vartheta}c_0^{-2}\dddot{\phi} \quad (232)$$

which can be compared with W25 the nonlinear horn equation ((147) in Paper I), viz.,

$$\text{W25: } \ddot{\phi} - c_0^2(\phi'' + \phi'A'/A) + 2\dot{\phi}'\phi' + (\gamma-1)\dot{\phi}(\phi'' + \phi'A'/A) + [(\gamma-1)/2]\phi'^3A'/A + [(\gamma+1)/2]\phi'^2\phi'' = 0 \quad (233)$$

(i) the first six linear and quadratic terms coincide; (ii) the last two cubic terms in Eq. (233) are omitted in the weakly nonlinear approximation in Eq. (232); (iii) the linear dissipative term in Eq. (232) is omitted in the nondissipative equation (233).

Burgers equation has been derived by two approaches: (Sec. 4.4) the propagation W55 of unidirectional waves along characteristics $(158)\equiv(159)$ in the presence of weak viscous diffusivity (156); (Sec. 4.8) one-dimensional, weakly nonlinear W57 acoustic waves $(205)\equiv(199)$ with linear thermoviscous diffusivity (196). The Burgers equation simplifies (Sec. 4.3) to the conservation of the group velocity along characteristics $(154)\equiv(155)$ for nondissipative unidirectional waves W53; the generalization to one-dimensional nonlinear waves in nonuniform and collapsible tubes involves a forcing term in both W54 the nondissipative $(169)\equiv(170)$ and viscous W56 case $(176)\equiv(179)$. All these one-dimensional equations are particular cases of the three-dimensional weakly nonlinear acoustic wave equation W59 with thermoviscous dissipation (217), which leads to W60 for a narrow beam $(225)\equiv(226)\equiv(228)$, and also includes the case $(235)\equiv(236)$ of weakly nonlinear one-dimensional sound waves with thermoviscous dissipation in a duct varying cross-section W58.

Burgers and related equations [305–315] appear in connection with other weakly nonlinear, dissipative or dispersive waves [316–318], such as sound waves in a relaxing gas [319–321] or in a liquid with gas bubbles [322] or in a shock tube [323], and other cases not covered here. The preceding account is a typical example of the study of nonlinear waves [324–329]. The best known is the Korteweg–de Vries equation [330–333], which has soliton type solutions [334,335]; there are soliton-type solutions for other types of nonlinear wave equations. Nonlinear sound waves been studied and observed [336–338] in various situations, e.g., (i) resonances in cavities [276] and tubes [275]; (ii) in ducts [339,340] (iii) due to intense sound sources [341,342]; (iv) in air jets [343]; (v) in the presence of dissipation [344]; (vi) in ultrasonics [345]. Perhaps the most notorious of nonlinear waves is the sonic boom of aircraft [346]; shock cells [347] in supersonic jets [348,349] contribute to jet noise.

5 Discussion

The 60 forms of the acoustic wave equation obtained (36 in [1] and 24 in the present paper) include (i) linear and nonlinear waves; (ii) nondissipative waves and dissipation by shear and bulk viscosity and by thermal conduction; (iii) waves in homogeneous or inhomogeneous, and steady and unsteady mean states; (iv) media at rest and potential and vortical mean flows. Concerning (iv), the (a) potential flows were considered exhaustively, including cases of low Mach number and unrestricted Mach number; the (b) vortical mean flows were considered only for (b1) plane and (b2) spatial unidirectional shear and for (b3) axisymmetric shear and/or swirl. Altogether, these 60 acoustic wave equations represent a substantial extension and cover a much wider range of applications than the classical and convected wave equations, which are restricted to (i) linear (ii) nondissipative waves in (iii) homogeneous media (iv) at rest or in uniform motion.

The extensions of the classical and convected wave equations cover (I) each of the four aspects (i)–(iv) in isolation. A fair number of (II) double extensions were considered, e.g., (a) linear, nondissipative waves in inhomogeneous or unsteady flows (II1

=iii+iv); (b) nonlinear, nondissipative waves in homogeneous flows (II2=i+iv); (c) nonlinear, nondissipative waves in inhomogeneous medium at rest (II3=i+iii); (d) nonlinear, dissipative waves in homogeneous media at rest (II4=i+ii). A few cases of (III) triple extensions were considered, e.g., (a) nonlinear, nondissipative waves in inhomogeneous flows (IIIa=i+iii+iv) and (b) nonlinear, dissipative waves in inhomogeneous medium at rest (IIIb=i+ii+iii). The case (IV) of quadruple interactions is not tractable, so there is no single general wave equation from which all others could be derived. Between the 60 acoustic wave equations indicated and the ultimate nonlinear, dissipative wave equation in inhomogeneous moving media, there remains plenty of scope for further developments.

The extent to which the four combinations of (i) linear and nonlinear, (ii) dissipative and nondissipative waves, in (iii) homogeneous or inhomogeneous and steady or unsteady, media at (iv) rest or in motion have been covered is indicated in the Tables 1–4. Equivalently, the extent to which some combinations have been covered fairly thoroughly and others rather sparsely can be appreciated from the diagram of the family tree of acoustic wave equations (Fig. 1). The family tree is formed by applying successively the following criteria to the 60 wave equations: (i) dissipative (8) and nondissipative (52) waves; (ii) linear (40) or nonlinear (20) waves; (iii) ducted or one-dimensional (21) and free three-dimensional (39) waves; (iv) mean potential (30) or vortical (20) flows; (v) vortical mean flow with shear (8), swirl (2), or both (10); (vi) shear flow unidirectional (4) or axisymmetric (16); (vii) unidirectional shear flow plane (2) or spatial (2); (viii) axisymmetric flow (12) with axisymmetric (3) or nonaxisymmetric (9) acoustic modes; (ix) potential mean flow (30) of low Mach number (10), high-speed (10) or media at rest (10); (x) mean state homogeneous (27), inhomogeneous (27), or unsteady (6). The number of wave equations in each category can be cheked by following the family tree.

Of the 60 wave equations, only 8 are dissipative (see Tables 2–4), of which 6 nonlinear and 2 linear. Of the six nonlinear dissipative wave equations four are one-dimensional (3), viz., the Burgers equation (W55) for weak viscosity and its forced form (W56) in a quasi-one-dimensional duct plus the extensions to include thermal dissipation in the one-dimensional (W57) and quasi-one-dimensional (W58) cases. The three-dimensional extensions (4) include weakly nonlinear waves with thermoviscous dissipation (W59), and the particular case of beams (W60). The linear dissipative wave equations (Table 2) all apply to a homogeneous medium at rest and include shear and bulk viscosity and thermal conduction, with strong W52 or weak W51 diffusivities.

The 52 nondissipative wave equations consist of 38 linear and 14 nonlinear. Of the 14 nonlinear nondissipative wave equations, 2 concern one-dimensional characteristics (3) and the remaining 12 potential mean flows (Table 1). In each group, half the equations concern quasi-one-dimensional ducts: (i) the Riemann invariants along characteristics concern (Table 5) nonlinear nondissipative one-dimensional free W53 or ducted W54 waves; (iii) the 12 nonlinear, nondissipative acoustic wave equations in mean potential flows consist of the same six cases each for free three-dimensional (quasi-one-dimensional ducted) propagation. The six cases are (Table 1) nonlinear, nondissipative three-dimensional (quasi-one-dimensional) acoustic waves in a inhomogeneous high-speed W15 (W30), low Mach number W13 (W28) potential mean flow or inhomogeneous medium at rest W11 (W26), plus the six cases of nonlinear, nondissipative three-dimensional (quasi-one-dimensional) acoustic waves in a homogeneous high-speed W14 (W29) or low Mach number W12 (W27) mean flow or homogeneous medium at rest W10 (W25).

The 38 linear, nondissipative wave equations consist of 18 in potential flows (Table 1) and 20 in vortical flows (Tables 2 and 3). The 18 linear, nondissipative acoustic wave equations in potential mean flows divide (Table 1) equally in 9 for three-dimensional (quasi-one-dimensional) propagation in there cases: (i) high-speed mean potential flow, either homogeneous W7 (W22), or inhomogeneous W8 (W23), or unsteady W9 (W24) as well; (ii) low Mach number potential mean flow, either homogeneous W4 (W19), or inhomogeneneous W5 (W20), or unsteady W6 (W21) as well; (iii) medium at rest, either homogeneous W1 (W16), or inhomogeneous W2 (W17), or unsteady as well W3 (W18).

The 20 linear, nondissipative acoustic wave equations in vortical flows (Tables 2 and 3), consist of 8 with mean shear, 2 with swirl, and 10 with both shear and swirl. The 18 linear, nondissipative wave equations in sheared mean flows consist of 4 for unidirectional shear and 4 for axisimmetric shear, with half each for homogeneous or inhomogeneous medium. The 4 linear, nondissipative acoustic wave equations in homogeneous (inhomogeneous) unidirectional shear flows consist of the plane W31 (W33) and spatial W32 (W34) cases. The four linear nondissipative acoustic wave equations in homogeneous (inhomogeneous) axisymmetric shear flows concern axisymmetric W35 (W36) and nonaxisymmetric W37 (W38) acoustic modes.

The two linear, nondissipative acoustic wave equations in a swirling mean flow concern homogeneous W39 or inhomogeneous W40 fluid. Of the ten linear, nondissipative acoustic wave equations in axisymmetric sheared and swirling mean flow, six apply at low Mach number and four with unrestricted Mach number. The six linear, nondissipative acoustic wave equations in axisymmetric low Mach number sheared and swirling mean flows consist of homogeneous (inhomogeneous) fluids with arbitrary swirl W41 (W42), rigid body swirl W43 (W44), and potential vortex swirl W45 (W46). The four linear, nondissipative acoustic wave equations in axisymmetric sheared and swirling mean flows, with unrestricted Mach number, consist of one W47 for axisymmetric acoustic modes and three for nonaxisymmetric acoustic modes. The three linear, nondissipative wave equations for nonaxisymmetric acoustic modes in an axisymmetric mean sheared and swirling flow with unrestricted Mach number correspond to the general case of arbitrary swirl W50, and particular cases of rigid body W48 and potential vortex W49 swirl.

Of the 60 wave equations, 9 may be considered the core set, of which all others are particular cases: (i) the 9 (W1–W9) linear, nondissipative acoustic wave equations in a potential mean flow are all particular cases of the high-speed unsteady wave equation W9, consisting of ten terms (63) in [1] i.e., (I.63), which can be written in the compact form (I.45); (ii) the 9 (W16–W24) linear, nondissipative wave equations in a quasi-one-dimensional duct are all particular cases of the high-speed nozzle wave equation W24, which consist of 11 terms (I.131;) and can be written in the compact form (I.131); (iii) the 6 (W10–W15) nonlinear, nondissipative wave equations in a steady potential flow are all particular cases of the nonlinear high-speed wave equation W15, which consists of 16 terms (I.107) and can be written in the compact form (I.105), using the nonlinear (I.86) and self-convected (I.88a) material derivates; (iv) the 6 (W25–W30) nonlinear, nondissipative wave equations in a quasi-one-dimensional duct are all particular cases of the nonlinear high-speed nozzle wave equation W30, which consists of terms (I.166) and can be written in the compact form (I.164), using the nonlinear (I.151a) and self-convected (I.152a) material derivatives; (v) the 4 (W31–W34) linear, nondissipative wave equations in plane or spatial unidirectional shear mean flow are all particular cases of the spatial isentropic shear wave equation W34 for the acoustic pressure p in (I.195) or W34* for (I.200) the acoustic pressure spectrum $\tilde{p}$ defined by (I.197); (vi) the 12 linear, nondissipative wave equations for axisymmetric or nonaxisymmetric acoustic modes in an axisymmetric mean flow with low Mach number swirl and/or shear (W35–W46) are all particular cases of the linear, nondissipative low Mach number asisymmetric shear and swirl wave equation W46 for the acoustic pressure p in Eq. (39) of the present paper II, i.e., (II.39), or W46* for (II.51) the acoustic pressure spectrum $\tilde{p}$ in (II.40); (vii) the preceding (vi) plus the 4 linear, nondissipative acoustic wave

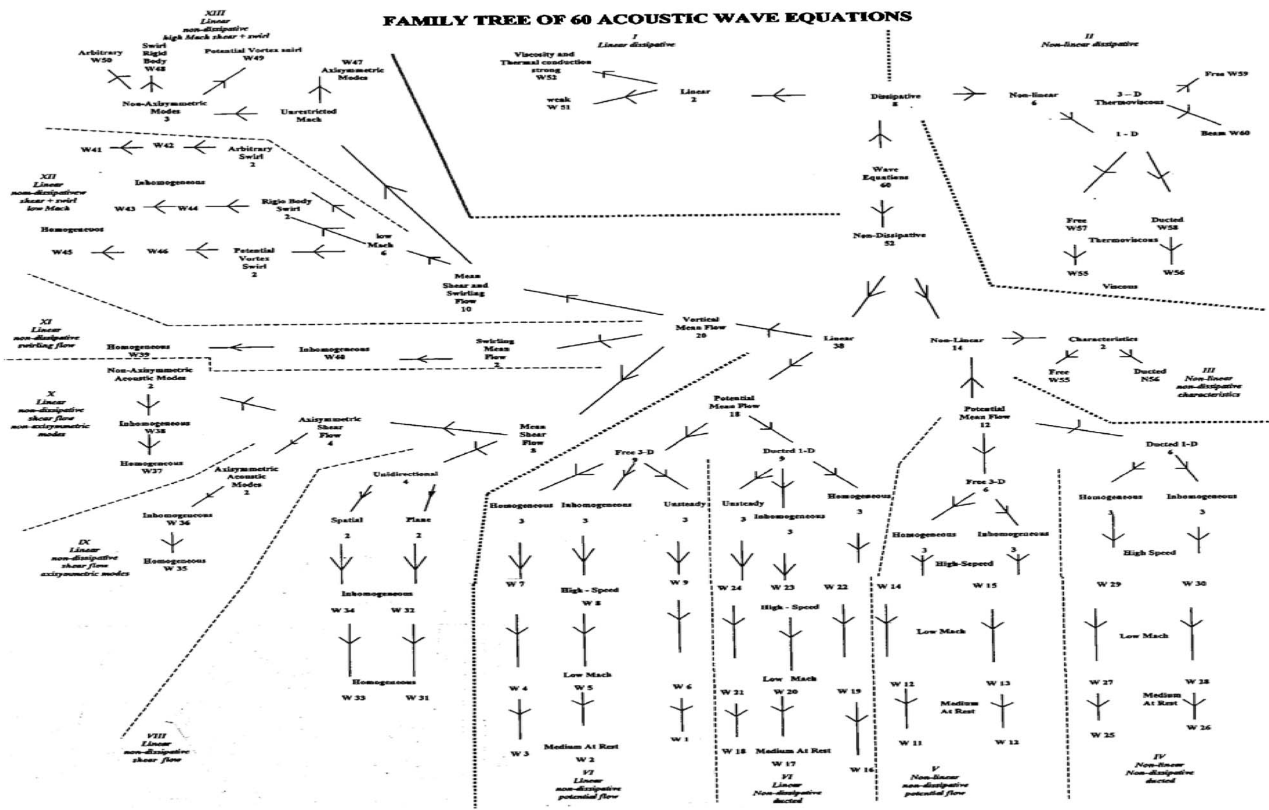

Fig. 1 Family tree of 60 acoustic wave equations, showing relations between them. It is divided in 13 regions I–XIII for different classes of wave equations. Includes all 60 wave equations, viz. W1–W36 listed in [1] in Tables 1 and 2, and W37–W60 listed in Tables 1–4 in the present paper.

equations for axisymmetric or nonaxisymmetric acoustic modes in axisymmetric mean flow with unrestricted shear an/or swirl (W37–W50) are all particular cases of the linear, nondissipative axisymmetric shear and swirl wave equation W50 for (II.68) the acoustic pressure spectrum $\tilde{p}$ in (II.40); (viii) the two linear, dissipative wave equations in a homogeneous medium at rest (W51–W52) are all particular cases of W52 including strong shear and bulk viscosities and conductive thermal diffusivity, using as variable the acoustic velocity perturbation $\mathbf{v}$ in (II.93), the vorticity perturbation $\mathbf{\Omega}$ in (II.95), and W52* the dilatation Ψ in (II.99); (ix) the weakly nonlinear one-dimensional or quasi-one-dimensional ducted acoustic waves without dissipation (W53 and W54), with viscous (W55 and W56) or thermoviscous (W57 and W58) dissipation, are all particular cases of the three-dimensional weakly nonlinear thermosviscous acoustic wave equation W59, including beams W60, and specified by the four terms in (II.221).

Although the nine core wave equations (W9, W15, W24, W30, W34, W46, W50, W54, W59) are sufficient to derive all 60 wave equations (W1–W60) as particular cases, the other 50 wave equations deserve explicit mention, because they have significant applications. Once the most appropriative acoustic wave equation for problem at hand has been identified either from the family tree or from Tables 1–6, the details of that wave equation can be found in the list in the Appendix, which contains the following information: (i) number of the wave equation (W1–W60) followed by name of the wave equation most commonly used, or otherwise a name thought appropriative; (ii) all the conditions of validity of the wave equation and the numbers of equations in the text, which equivalently describe the wave equation. This indicates the section of the text where the wave equation was proved; also the parameters in the wave equation can be traced to their defining equations through the list of symbols. The references attempt to indicate the first use of each wave equation, to the best of the author's limited historical knowledge. In 24 cases maybe the wave equation appears explicitly for the first time in the present paper, although it may be a particular case of a previously known wave equation; the latter situation can be verified from the family tree of wave equations, which shows the general and particular cases, and also from the text. The family tree is divided by dotted lines in 13 regions I–XIII containing distinct types of wave equations. With this understanding, it is possible that in the 60 wave equations, a maximum of 41 were derived by the author, 24 in this, and 19 in earlier papers.

The wave equations in the list were given technical rather than personal names since such nomenclature has proven to be historically quite inaccurate in some cases, e.g., the Webster [98] horn wave equation (I.117a,b), was derived four years earlier by Rayleigh [99], and can be traced more than one century back [350,106] to the research of Lagrange [101], Euler [102,351,104,105], Bernoulli [103], Poisson [352], and Green [123]. Also the same author can be the first to obtain several distinct wave equations, making a technical nomenclature the unambiguous choice. Of the nine core wave equations, seven were derived by the author, five in the present paper (W9, W24, W46, W54, W59), and two in earlier papers (W15, W30). The fact that possibly 41 of the 60 wave equations appear first in the present paper is a necessary consequence of trying to bridge the gaps in the literature, and presenting a unified and comprehensive account, within the limits of current knowledge. It has the natural consequence of providing many cross-checks, including multiple distinct derivations of several wave equations. This material should be a useful background when attempting to derive new wave equations valid in more general conditions than those presented here, for which there is plenty of scope left.

Nomenclature

a,b = parameters in the solution of acoustic thermoviscous dispersion relation (118*a*) and (118*b*)
c_0 = adiabatic sound speed of mean state (110*a*)
c_1 = isothermal sound speed of mean state (110*b*)
$f,g_\pm$ = parameters in the solution of acoustic thermoviscous dispersion relation (119*b*) and (123*b*)
k = axial wave number in Eq. (40) and Sec. 2 total wave number $|\mathbf{k}|$ in Eq. (111) and Sec. 3
$\mathbf{k}$ = wave vector (96)
m = azimuthal wave number (40)
r_0 = radius of cylindrical duct (56)
s = entropy perturbation (194)
v_r,v_θ,v_z = components of acoustic velocity perturbation in cylindrical coordinates (r,θ,z), e.g., in (112)–(114)
x,y,z = Cartesian coordinates
B = parameter in Tate's equation of state (207) for liquids
C_p,C_v = specific heats at constant pressure (85*a*) and volume (85*b*)
$J_\pm$ = Riemann invariants (139)
R = perfect gas constant (85*d*)
S = total entropy (193)
Y = set of terms in Eqs. (220)
$Z,\bar{Z}$ = wall impedance, specific impedance (56)
W = group velocity (153)
$\alpha,\bar{\alpha}$ = thermal conductive diffusivity (94*a*) and (94*b*)
α_1,α_2 = heat diffusivities (109*a*) and (109*b*)
β = total kinematic viscosity (104*a*)
γ = adiabatic exponent (85*c*)
δ_1,δ_2,δ = damping factors (114*a*), (114*b*), and (116*b*)
ε = coefficient of thermal expansion (88)
ψ = transformation variable from Burger's to heat equation (162)
ζ = thermal conductivity (80)
μ,ν = shear, bulk kinematic viscosities (78)
ω = wave frequencies (40)
ω_0,ω_1 = frequency of plane adiabatic (113*a*), isothermal (113*b*) sound waves
ω_2 = strongly dissipative correction to adiabatic frequency (116*a*)
$\bar{\omega}$ = transformation of wave frequency (117*a*)
τ = retarded time (223)
ϑ = acoustic thermoviscous dissipation coefficient (134)
$\bar{\vartheta}$ = thermoviscous diffusity (200)
χ = similarity parameter (223)
ξ = moving coordinate (201a)
$\bar{\Gamma}$ = circulation of a potential vortex (15*b*)
Ψ = acoustic dilatation (99)
Θ = temperature perturbation (81) and (82*c*)
$\mathbf{\Omega}$ = vorticity perturbation (95)

Appendix: List of Acoustic Wave Equations: W37 to W60

*W37-Acoustic shear wave equation. Linear, nondissipative, nonaxisymmetric acoustic modes in a unidirectional axisymmetric homentropic shear mean flow (36) ≡ (42).

*W38-Inhomogeneous acoustic shear wave equation. Idem isentropic (31) ≡ (43).

*W39-Swirl acoustic wave equation. Linear, nondissipative nonaxisymmetric waves in an axisymmetric swirling homentropic mean flow (37) ≡ (44).

*W40-Inhomogeneous swirl acoustic wave equation. Idem isentropic (32) ≡ (45).

*W41-Shear and swirl acoustic wave equation. Linear, nondissipative nonaxisymmetric waves in an axisymmetric homentropic mean flow with low Mach number shear and swirl (35) ≡ (46).

*W42-Inhomogeneous shear and swirl acoustic wave equation. Idem isentropic (30) ≡ (47).

*W43-Shear and rigid body swirl acoustic wave equation. Linear, nondissipative nonaxisymmetric waves in an axisymmetric homentropic mean flow with low Mach number shear and rigid body swirl (38) ≡ (48).

*W44-Inhomogeneous shear and rigid body swirl acoustic wave equation. Idem isentropic and rigid body swirl (33) ≡ (49).

*W45-Shear and potential vortex swirl acoustic wave equation. Idem homentropic and potential vortex swirl (39) ≡ (50).

*W46-Inhomogeneous shear and potential vortex swirl acoustic wave equation. Idem isentropic and potential vortex swirl (34) ≡ (51).

*W47-Axisymmetric high-speed shear and swirl acoustic wave equation. Linear, nondissipative, axisymmetric waves in an axisymmetric isentropic mean flow with shear and swirl of unrestricted Mach number (69).

*W48-High-speed shear and rigid body swirl acoustic wave equation. Linear, nondissipative nonaxisymmetric waves in an axisymmetric isentropic mean flow with shear and rigid body swirl of unrestricted Mach number (74).

*W49-High-speed shear and potential vortex swirl acoustic wave equation [3]. Idem, potential vortex swirl (77).

*W50-High-speed shear and swirl acoustic wave equation. Linear, nondissipative nonaxisymmetric waves in an axisymmetric isentropic mean flow with shear and swirl of unrestricted Mach number (68).

*W51-Dissipative acoustic wave equation. Linear, dissipative sound waves in a homogeneous medium at rest with weak shear and bulk kinematic viscosities and thermal conductive diffusivity (95) + (101).

*W52-Strongly dissipative acoustic wave equation. Idem with strong shear and bulk viscosities and and thermal conductive diffusivity (93) ≡ (95) + (99) ≡ (107).

*W53-Characteristic acoustic wave equation [304]. Nonlinear, nondissipative one-dimensional waves in an homogeneous medium at rest (138) ≡ (155).

*W54-Characteristic horn wave equation. Nonlinear, nondissipative quasi-one-dimensional waves in a collapsible duct of nonuniform cross section, containing a homogeneous fluid at rest (167) ≡ (170).

*W55-Viscous characteristic wave equation [305]. Nonlinear, one-dimensional waves in homogeneous viscous fluid at rest (157) ≡ (159).

*W56-Viscous characteristic horn wave equation. Nonlinear, quasi-one-dimensional waves in a collapsible duct of nonuniform cross section containing a homogeneous viscous fluid at rest (172) ≡ (175).

*W57-Nonlinear thermoviscous plane wave equation. Weakly nonlinear, weakly dissipative one-dimensional waves in a viscous and thermally conducting homogeneous steady fluid at rest (195) ≡ (200) ≡ (201).

*W58-Nonlinear, thermoviscous horn wave equation. Idem quasi-one-dimensional waves in a rigid duct of nonuniform cross section (231) ≡ (232).

*W59-Nonlinear, thermoviscous wave equation. Weakly nonlinear, weakly dissipative three-dimensional waves in a viscous and thermally conducting homogeneous steady fluid at rest (221).

*W60-Nonlinear, thermoviscous beam equation. Idem for a beam (225).

References

[1] Campos, L. M. B. C., 2006, "On 36 Forms of the Acoustic Wave Equations in Potential Flow and Inhomogeneous Media," Appl. Mech. Rev., **60**, pp. 149–171.
[2] Howe, M. S., 1977, "The Generation of Sound by Vorticity Waves in Swirling Flows," J. Fluid Mech., **81**, pp. 369–383.
[3] Tam, C. K. W., and Auriault, L., 1998, "The Wave Modes in Ducted Swirling Flows," J. Fluid Mech., **371**, pp. 1–20.
[4] Gobulev, V., and Atassi, H., 1996, "Sound Propagation in an Annular Duct Mean Potential Swirling Flow," J. Sound Vib., **198**, 601–606.
[5] Gobulev, V., and Atassi, H., 1998, "Acoustic-Vorticity Waves in Swirling Flows," J. Sound Vib., **209**, pp. 203–222.
[6] Campos, L. M. B. C, and Serrão, P. G. T. A., 2004, "On the Acoustics of Unbounded and Ducted Vortex Flows," SIAM J. Appl. Math., **65**, pp. 1353–1368.
[7] Mohring, W., 1972, "Energy Flux in Duct Flow," J. Sound Vib., **18**, pp. 101–109.
[8] Mohring, W., 1978, "Acoustic Energy Flux in Non-Homogeneous Ducts," J. Acoust. Soc. Am., **64**, pp. 1186–1189.
[9] Clebsch, A., 1857, "Ueber eine algemeine Transformation der hydrodynamische Gleichungen," Crelle, **64**, pp. 150–160.
[10] Campos, L. M. B. C., and Kobayashi, M. H., 2000, "On the Reflection and Transmission of Sound in a Thick Shear Layer," J. Fluid Mech., **420**, pp. 1–24.
[11] Haurwitz, W., 1932, "Zur Theorie der Wellenbewegungen in Luft und Wasser," Veroffentliche Geophysik Universiteit Leipzig, **6**, pp. 334–364.
[12] Kuchemann, D., 1938, "Störungbewegungen in einer Gasströmung mit Grenzschicht," Z. Angew. Math. Mech., **30**, pp. 79–84.
[13] Pridmore-Brown, D. C., 1959, "Sound Propagation in a Fluid Flowing Through an Attenuating Duct," J. Fluid Mech., **4**, pp. 393–406.
[14] Mohring, W., Muller, E. A., and Obermeier, F., 1983, "Problems in Flow Acoustics," Rev. Mod. Phys., **55**, pp. 707–723.
[15] Campos, L. M. B. C., Oliveira, J. M. G. S., and Kobayashi, M. H., 1999, "On Sound Propagation in Linear Shear Flow," J. Sound Vib., **95**, pp. 739–770.
[16] Mani, R., 1976, "The Influence of Jet Flow on Jet Noise. Part I. The Noise of Unheated Jets," J. Fluid Mech., **73**, pp. 753–778.
[17] Mani, R., 1976, "The Influence of Jet Flow on Jet Noise. Part II. The Noise of Heated Jets," J. Fluid Mech., **73**, pp. 779–793.
[18] Campos, L. M. B. C., and Serrão, P. G. T. A., 1999, "On the Acoustics of an Exponential Boundary Layer," Philos. Trans. R. Soc. London, Ser. A, **356**, pp. 2335–2378.
[19] Goldsteih, M., 1979, "Scattering and Distortion of Unsteady Motion an Transversely Sheared Mean Flows," J. Sound Vib., **91**, pp. 601–602.
[20] Goldstein, M., 1982, "High-Frequency Sound Emission From Multipole Sources Embedded in Arbitrary Transversely Sheared Mean Flows," J. Sound Vib., **80**, pp. 449–460.
[21] Campos, L. M. B. C., 1997, "On the Spectra of Aerodynamic Noise and Aeroacoustic Fatigue," Prog. Aerosp. Sci., **33**, pp. 353–389.
[22] Campos, L. M. B. C., 2006, "On Some Recent Advances in Aeroacoustics," J. Sound Vib., **11**, pp. 27–45.
[23] Graham, E. W., and Graham, B. B., 1969, "Effect of a Shear Layer on Plane Waves in a Fluid," J. Acoust. Soc. Am., **46**, pp. 169–175.
[24] Balsa, T. F., 1976, "The Farfield of High-Frequency Convected Singularities in Sheared Flow, With Application to Jet Noise Prediction," J. Fluid Mech., **74**, pp. 193–208.
[25] Balsa, T. F., 1976, "Refraction and Shielding of Sound From a Source in a Jet," J. Fluid Mech., **27**, pp. 513–529.
[26] Campos, L. M. B. C., 1978, "On the Spectral Broadening of Sound by Turbulent Shear Layers. Part I: Transmission of Sound Though Turbulent Shear Layers," J. Fluid Mech., **89**, pp. 723–749.
[27] Campos, L. M. B. C., 1978, "The Spectral Broadening of Sound by Turbulent Shear Layers. Part II: The Spectral Broadening of Sound and Aircraft Noise," J. Fluid Mech., **89**, pp. 750–783.
[28] Campos, L. M. B. C., 1983, "Sur la propagation du son dans les Écoulements non-uniformes et non-stationaires," Rev. Acoust., **67**, pp. 217–233.
[29] Guedel, A., 1978, "Scattering of an Acoustic Field by a Free Jet Shear Layer," J. Sound Vib., **100**, pp. 285–304.
[30] Myers, M. K., and Chuang, S. L., 1983, "Uniform Asymptotic Approximations for Duct Acoustic Modes in Thin Boundary-Layer Flow," AIAA J., **22**, pp. 1234–1241.
[31] Ffowcs-Williams, J. E., and Purshouse, M., 1981, "A Vortex Sheet Modeling of Boundary-Layer Noise," J. Fluid Mech., **113**, pp. 187–220.
[32] Hanson, D. B., 1984, "Shielding of Propfan Cabin Noise by the Fuselage Boundary Layer," J. Sound Vib., **92**, pp. 591–598.
[33] Tack, D. H., and Lambert, R. F., 1965, "Influence of Shear Flow on Sound Attenuation in Lined Ducts," J. Acoust. Soc. Am., **38**, pp. 655–666.
[34] Mugur, P., and Gladwell, G. M. L., 1969, "Acoustic Wave Propagation in a Sheared Flow Contained in a Duct," J. Sound Vib., **9**, pp. 28–48.
[35] Mariano, S., 1971, "Effect of Wall Shear Layers on the Sound Attenuation in Acoustically Lined Rectangular Ducts," J. Sound Vib., **19**, pp. 261–275.
[36] Eversman, W., 1971, "Effect of Boundary Layer on the Transmission and Attenuation of Sound in an Acoustically Treated Circular Duct," J. Acoust. Soc. Am., **39**, pp. 1372–1380.
[37] Shankar, P. N., 1971, "Acoustic Refraction by Duct Shear Layers," J. Fluid Mech., **47**, pp. 81–91.
[38] Eversman, W., and Bechenmeyer, R. J., 1972, "Transmission of Sound in Ducts With Thin Shear Layers: Convergence to the Uniform Flow Case," J. Acoust. Soc. Am., **52**, pp. 216–225.
[39] Shankar, P. N., 1972, "Sound Propagation in Duct Shear Layers," J. Sound

Vib., **22**, pp. 221–232.
[40] Ko, S. H., 1972, "Sound Attenuation in Acoustically Lined Circular Ducts in the Presence of Uniform Flow and Shear Flow," J. Sound Vib., **22**, pp. 193–210.
[41] Nayfeh, A. H., Kaiser, J. E., and Telionis, D. P., 1975, "Acoustics of Aircraft Engine-Duct Systems," AIAA J., **13**, pp. 130–153.
[42] Swinbanks, M. A., 1975, "Sound Field Generated by a Source Distribution in a Long Duct Carrying a Shear Flow," J. Sound Vib., **40**, pp. 51–76.
[43] Mani, R., 1980, "Sound Propagation in Parallel Sheared Flows in Ducts: The Mode Estimation Problem," Philos. Trans. R. Soc. London, Ser. A, **371**, pp. 393–412.
[44] Ishii, S., and Kakutani, T., 1987, "Acoustic Waves in Parallel Shear Flows in a Duct," J. Sound Vib., **113**, pp. 127–139.
[45] Goldstein, M., and Rice, E., 1979, "Effect of Shear on Duct Wall Impedance," J. Sound Vib., **30**, pp. 79–84.
[46] Jones, D. S., 1977, "The Scattering of Sound by a Simple Shear Layer," Philos. Trans. R. Soc. London, Ser. A, **284**, pp. 287–328.
[47] Jones, D. S., 1978, "Acoustic of a Splitter Plate," J. Inst. Math. Appl., **21**, pp. 197–209.
[48] Koutsoyannis, S. P., 1980, "Characterization of Acoustic Disturbances in Linearly Sheared Flows," AIAA J., **68**, pp. 187–202.
[49] Scott, J. N., 1979, "Propagation of Wave Through a Linear Shear Layer," AIAA J., **17**, pp. 237–245.
[50] Koutsoyannis, S. P., Katramcheti, K., and Galant, D. G., 1980, "Acoustic Resonances and Sound Scattering by a Shear Layer," AIAA J., **18**, pp. 1446–1450.
[51] Salant, R. F., 1968, "Symmetric Normal Modes in a Uniformly Rotating Fluid," J. Acoust. Soc. Am., **43**, pp. 1302–1303.
[52] Kerrebrock, J. L., 1977, "Small Disturbances in Turbomachine Annuli With Swirl," AIAA J., **15**, pp. 794–803.
[53] Greitzer, E. M., and Strand, T., 1978, "Axisymmetric Swirling Flows in Turbomachinery Annuli," ASME J. Eng. Power, **100**, pp. 618–629.
[54] Cooper, A., and Peake, N., 2000, "Trapped Acoustic Modes is Aeroengine Intakes With Swirling Flow," J. Fluid Mech., **419**, pp. 151–175.
[55] Campos, L. M. B. C., 2001, "On Some Solutions of the Extended Confluent Hypergeometric Differential Equation," J. Comput. Appl. Math., **137**, pp. 177–200.
[56] Campos, L. M. B. C., 2002, "On the Derivation of Asymptotic Expansions for Special Functions From the Corresponding Differential Equations," Integral Transforms Spec. Funct., **12**, pp. 227–236.
[57] Benjamim, T. B., 1967, "Internal Waves of Permanent Form in Fluids of Great Depth," J. Fluid Mech., **29**, pp. 559–592.
[58] Bretherton, F. P., 1968, "Propagation in Slowly Waveguides," Proc. R. Soc. London, Ser. A, **302**, pp. 555–576.
[59] Rayleigh, J. W. S., 1890, "On the Vibrations of an Atmosphere," Philos. Mag., **29**, pp. 173–180.
[60] Pedlosky, J., 1960, *Geophysical Fluid Dynamics*, 2nd ed., Springer, Berlin.
[61] Hines, C. O., 1974, *The Upper Atmosphere in Motion*, American Geophysical Union, Washington.
[62] Beer, T., 1974, *Atmospheric Waves*, Wiley, New York.
[63] Grossard, E. E., and Hooke, W., 1975, *Waves in the Atmosphere*, Elsevier, New York.
[64] Campos, L. M. B. C., 1983, "On Three-Dimensional Acoustic-Gravity Waves in Model Non-Isothermal Atmospheres," Wave Motion, **5**, pp. 1–14.
[65] Campos, L. M. B. C., 1983, "On Viscous and Resistive Dissipation of Hydrodynamic and Hydromagnetic Waves in Atmospheres," J. Mec. Theor. Appl., **2**, pp. 861–891.
[66] Campos, L. M. B. C., 1988, "On the Properties of Hydromagnetic Waves in the Vicinity of Critical Levels and Transition Layers," Geophys. Astrophys. Fluid Dyn., **40**, pp. 93–132.
[67] Campos, L. M. B. C., and Saldanha, R. S., 1991, "On Oblique Magnetohydrodynamic Waves in an Atmosphere Under a Magnetic Field of Arbitrary Direction," Geophys. Astrophys. Fluid Dyn., **56**, pp. 237–251.
[68] Campos, L. M. B. C., Isaeva, N. L., and Gil, P. J. S., 1999, "On the Reflection of Alfén Waves in the Inhomogeneous Solar Wind, Non-Linear Waves and Turbulence," Lect. Notes Phys., **52**, pp. 104–156.
[69] Greenspan, H. P., 1968, *Rotating Fluids*, Cambridge University Press, Cambridge.
[70] Lehnert, B., 1954, "Magnetohydrodynamic Waves Under the Action of Coriolis Forces," Astrophys. J., **119**, pp. 647–655.
[71] Campos, L. M. B. C., 1987, "On Waves in Gases. Part II: Interaction of Sound With Magnetic and Internal Modes," Rev. Mod. Phys., **59**, pp. 363–462.
[72] Campos, L. M. B. C., 1998, "On Hydromagnetic Waves in Atmospheres With Application to the Sun," Theor. Comput. Fluid Dyn., **10**, pp. 37–70.
[73] Barclay, D. W., Moodie, T. B., and Haddow, J. B., 1977, "Waves in Thin-Walled, Non-Uniform Perfectly Elastic Tubes Containing an Incompressible Inviscid Fluid," J. Acoust. Soc. Am., **62**, pp. 1–7.
[74] Cho, Y. C., and Ingard, K. U., 1983, "Mode Propagation in Non-Uniform Circular Ducts With Potential Flow," AIAA J., **21**, pp. 970–977.
[75] Keefe, D. H., 1984, "Acoustical Wave Propagation in Cylindrical Ducts: Transmission Line Parameter Approximations for Isothermal and Non-Isothermal Boundary Conditions," J. Acoust. Soc. Am., **75**, pp. 58–67.
[76] Morfey, C. L., 1971, "Sound Transmission and Generation in Ducts With Flow," J. Sound Vib., **14**, pp. 37–55.
[77] Leppington, F. G., and Levine, H., 1980, "Acoustics of a Blocked Duct With Flow," J. Sound Vib., **72**, p. 315.
[78] Davies, P. O. A. L., 1981, "Flow-Acoustic Coupling in Ducts," J. Sound Vib., **77**, pp. 191–209.
[79] Mohring, W., and Rahman, S., 1976, "The Influence of Perturbations of the Velocity and Speed of Sound on the Propagation of Sound Waves in Ducts," AIAA J., **76**, p. 493.
[80] Mohring, W., 1980, "Acoustic Momentum and Energy Theorems for Piecewise Uniform Ducts," J. Acoust. Soc. Am., **67**, pp. 1463–1471.
[81] Cole, J. E., 1979, "Sound Propagation in a Duct With Axial Sound Speed Variation: An Exact Solution," J. Sound Vib., **63**, pp. 237–246.
[82] Miles, J. W., 1944, "Reflection of Sound Due to a Change Cross Section of a Tube," J. Acoust. Soc. Am., **16**, pp. 14–19.
[83] Pinker, R. A., and Bryce, W. D., 1976, "The Radiation of Plane Wave Duct Noise from Jet Exhausts, Statically and in Flight," AIAA J., AIAA Paper No. 76–581.
[84] Cargill, A. M., 1982, "Low-Frequency Sound Radiation and Generation Due to the Interaction of Unsteady Flow With a Jet Pipe," J. Fluid Mech., **121**, pp. 59–105.
[85] Plumblee, H. E., and Dean, P. D., 1983, "Sound Measurements Within and in the Radiated Field of an Annular Duct With Flow," J. Sound Vib., **28**, pp. 715–735.
[86] Silcox, R. J., 1984, "Geometry and Static Flow Effects on Acoustic Radiation from Ducts," AIAA J., **22**, pp. 1087–1093.
[87] Barton, E. H., 1908, "On Spherical Radiation and Vibration in Conical Pipes," Philos. Mag., **15**, pp. 69–81.
[88] Hoersch, V. A., 1925, "Non-Radial Harmonic Vibrations Within a Conical Horn," Phys. Rev., **25**, pp. 218–229.
[89] Nayfeh, A. H., 1975, "Acoustic Waves in a Duct With Sinusoidaly Perturbed Walls and Mean Flow," J. Acoust. Soc. Am., **57**, pp. 1036–1039.
[90] Nayfeh, A. H., Kaiser, J. E., and Telionis, D. P., 1975, "Transmission of Sound Through Annular Ducts of Varying Cross Section," AIAA J., **13**, pp. 60–65.
[91] Kelly, J. J., Nayfeh, A. H., and Watson, L. T., 1982, "Acoustic Propagation in Partially-Choked Converging-Diverging Ducts," J. Sound Vib., **81**, pp. 519–534.
[92] Nayfeh, A. H., Shaker, B. S., and Kaiser, J. E., 1980, "Transmission of Sound Through Non-Uniform Circular Ducts With Compressible Mean Flows," AIAA J., **18**, pp. 515–525.
[93] Nayfeh, A. H., Kaiser, J. E., Marshall, R. L., and Hurst, C. J., 1980, "A Comparison of Experiment and Theory for Sound Propagation in Variable-Area Ducts," Sound Vib., **71**, pp. 241–259.
[94] Nayfeh, A. H., Kelly, J. J., and Watson, L. T., 1982, "Propagation of Spinning Acoustic Modes in Partially Choked Converging Ducts," J. Acoust. Soc. Am., **71**, pp. 796–802.
[95] Silcox, R. J., and Lester, H. C., 1982, "Sound Propagation Through a Variable-Area Duct: Experiment and Theory," AIAA J., **20**, pp. 1377–1384.
[96] Baumeister, K. J., Eversman, W., Astley, R. J., and White, J. W., 1984, "Acoustics in a Variable-Area Duct: Finite Element and Finite Difference Comparisons to Experiment," AIAA J., **21**, pp. 193–200.
[97] Uenishi, K., and Myers, M. K., 1984, "Two-Dimensional Acoustic Field in a Non-Uniform Duct Carrying Compressible Flow," AIAA J., **22**, pp. 1242–1248.
[98] Webster, A. G., 1919, "Acoustical Impedance and the Theory of Horns an the Phonograph," Proc. Natl. Acad. Sci. U.S.A., **5**, pp. 275–282.
[99] Rayleigh, J. W. S., 1916, "On the Propagation of Sound in Narrow Tubes of Variable Section," Philos. Mag., **31**, pp. 89–96.
[100] Weibel, W. S., 1955, "On Webster's Horn Equation," J. Acoust. Soc. Am., **75**, pp. 1705–1706.
[101] Lagrange, J. L., 1760, "Nouvelles Recherches Sur la Nature et Propagation du Son," Miscecania Turinesia, **2**, pp. 11–170.
[102] Euler, L., 1764, "De motu vibratorio cordarum inequalter crassarum," Commentari Academie Scientarum Petropolitana, **9**, pp. 264–304.
[103] Bernoulli, D., 1765, "Mémoire sur les vibrations des cordes d'une épaisseur Inégale," Mem. Acad. Sci. Berlin, **21**(1767), pp. 281–366.
[104] Euler, L., 1766, "Recherches sur le mouvement des cordes inegalement grosses," Miscecania Turinesia, **3**, pp. 27–59.
[105] Euler, L., 1771, "De motu aeris in tubis," Novi Comm. Acad. Scient. Petrop, **16**(1772), pp. 281–425.
[106] Truesdell, C. A., 1960, "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788," Commentari Academie Scientarum Petropolitana, **2**(16b), pp. 15–405.
[107] McLachlan, N. W., 1935, *Elements of Loudspeaker Practice*, Oxford University Press, London.
[108] McLachlan, N. W., 1936, *The New Acoustics*, Oxford University Press, London.
[109] Eisner, E., 1963, "Design of Sonic Amplitude Transformers for High Magnifications," J. Acoust. Soc. Am., **35**, pp. 1367–1377.
[110] Eisner, E., 1964, "The Design of Resonant Vibrators," *Physical Acoustics: Principles and Methods*, Mason, W. P., ed., Academic, New York, Vol. 1, pp. 353–368.
[111] Eisner, E., 1966, "Complete Solutions of Webster's Horn Equation," J. Acoust. Soc. Am., **41**, pp. 1126–1146.
[112] Campos, L. M. B. C., 1984, "Some General Properties of the Exact Acoustic Fields in Horns and Nozzles," J. Sound Vib., **95**, pp. 177–201.
[113] Campos, L. M. B. C., and Santos, A. J. P., 1988, "On the Propagation and Damping of Longitudinal Oscillations in Tapered Visco-Elastic Bars," J. Sound Vib., **126**, pp. 109–125.
[114] Shapiro, A. H., 1954, *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Ronald, New York.
[115] Eisenberg, N. A., and Kao, T. W., 1969, "Propagation of Sound Through a

Variable Area Duct With a Steady Mean Flow," J. Acoust. Soc. Am., **49**, pp. 169–175.
[116] Lumsdaine, E., and Ragab, S., 1977, "Effect of Flow in Quasi-One-Dimensional Acoustic Propagation in a Variable Duct of Finite Length," J. Sound Vib., **53**, pp. 47–51.
[117] Campos, L. M. B. C., 1984, "On the Propagation of Sound in Nozzles of Variable Cross-Section Containing a Low Mach Number Mean Flow," Z. Flugwiss. Weltraumforsch., **8**, pp. 97–109.
[118] Campos, L. M. B. C., and Lau, F. J. P., 1996, "On Sound in an Inverse Sinusoidal Nozzle With Low Mach Number Mean Flow," J. Acoust. Soc. Am., **100**, pp. 355–363.
[119] Campos, L. M. B. C., and Lau, F. J. P., 2001, "On the Convection of Sound in Inverse Catenoidal Nozzles," J. Sound Vib., **244**, pp. 195–209.
[120] Lau, F. J. P., and Campos, L. M. B. C., 2003, "On the Effect of Wall Undulations on the Acoustics of Ducts With Flow," J. Sound Vib., **270**, pp. 361–379.
[121] Poisson, S. D., 1817, "Sur le mouvement des fluid élastiques dans les tuyaux cylindrigues," Mem. Acad. Sci. Inst. Fr., **2**, pp. 305–402.
[122] Poisson, S. D., 1817, "Sur le mouvement des fluids élastiques dans les tuvaux cylindriques, et sur la théorie des instruments à vent," Mem. Acad. Sci. Inst. Fr., **2**, pp. 305–402.
[123] Green, G., 1837, "On the Motion of Waves in a Variable Canal of Small Depth and Width," Proc. Cambridge Philos. Soc., **6**, pp. 457–462.
[124] Heaviside, O., 1882, "Contributions to the Theory of the Propagation of Current in Wires," Electrical Papers, Chelsea Publ. New York, Vol. 1, pp. 141–179.
[125] Pyle, R. W., 1967, "Solid Torsional Horns," J. Acoust. Soc. Am., **41**, pp. 1147–1156.
[126] Campos, L. M. B. C., and Isaeva, N. L., 2001, "On Vertical Spinning Alfvén Waves in a Magnetic Flux Tube," J. Plasma Phys., **48**, pp. 415–434.
[127] Olson, H. F., and Wolff, I., 1930, "A Sound Concentrator for Microphones," J. Acoust. Soc. Am., **1**, pp. 410–417.
[128] Olson, H. F., 1938, "A Horn Consisting of Monifold Exponential Sections," Journal of the Society of Motion Picture Engineers, **30**, pp. 511–518.
[129] Salmon, V., 1946, "A New Family of Horns," J. Acoust. Soc. Am., **19**, pp. 212–218.
[130] Nagarkar, B. N., and Finch, R. D., 1971, "Sinusoidal Horns," J. Acoust. Soc. Am., **50**, pp. 23–31.
[131] Duhamel, J. M. C., 1939, "Sur les vibrations des gas dans les tuyaux cylindriques, coniques, etc," J. Math. Pures Appl., **14**, pp. 49–110.
[132] Barton, E. H., 1908, "On Spherical Radiation and Vibrations in Conical Pipes," Philos. Mag., **15**, pp. 69–81.
[133] Hoersch, V. A., 1925, "Non-Radial Harmonic Vibrations Within a Conical Horn," Philos. Mag., **25**, pp. 218–229.
[134] Stewart, G. W., 1920, "The Performance of Conical Horns," Phys. Rev., **16**, pp. 313–326.
[135] Ballantine, S., 1927, "On the Propagation Sound in the General Bessel Horn of Infinite Length," J. Franklin Inst., **203**, pp. 85–101.
[136] Bies, D. A., 1962, "Tapering Bar of Uniform Stress in Longitudinal Oscillation," J. Acoust. Soc. Am., **34**, pp. 1567–1569.
[137] Mawardi, O. K., 1949, "Generalized Solutions of Webster Horn Theory," J. Acoust. Soc. Am., **21**, pp. 323–330.
[138] Lambert, R. F., 1954, "Acoustical Studies of the Tractrix Horn," J. Acoust. Soc. Am., **26**, pp. 1024–1028.
[139] Pinkney, H. F. L., and Basso, G., 1963, "On the Classification of Families of Shapes for Rods of Axially Varying Cross-Section in Longitudinal Vibration," Nat. Res. Counc. Canada. Mech. Eng. Report No. MS-109.
[140] Molloy, C., 1975, "N-Parameter Ducts," J. Acoust. Soc. Am., **57**, pp. 1030–1036.
[141] Parodi, M., 1945, "Propagation sur une ligne elétrique sans pertes dont les parameters lineiques sont des functions exponentielles du carré de l'espace," J. Phys., **6**, pp. 331–332.
[142] Thiessen, G. J., 1950, "Resonance Characteristics of a Finite Catenoidal Horn," J. Acoust. Soc. Am., **22**, pp. 558–562.
[143] Goldstein, S., and MacLachlan, N. W., 1935, "Sound Waves of Finite Amplitude in an Exponential Horn," J. Acoust. Soc. Am., **6**, pp. 275–278.
[144] Merkulov, L. G., and Kharitonov, A. V., 1959, "Theory and Analysis of Sectional Concentrators," J. Acoust. Soc. Am., **5**, pp. 183–190.
[145] Merkulov, L. G., 1957, "Tapering Bar of Uniform Stress in Longitudinal Oscillation," J. Acoust. Soc. Am., **34**, pp. 1567–1569.
[146] Pochhammer, L., 1876, "Ueber Die Fortpflanzungschwindigkeiten der Schwingungen in einem unbegrentzen isotropen Kreiszylinder," Crelle, **81**, pp. 324–336.
[147] Salmon, V., 1946, "Generalized Plane Wave Horn Theory," J. Acoust. Soc. Am., **17**, pp. 199–211.
[148] Stevenson, A. F., 1951, "General Theory of Electro-Magnetic Horns," J. Appl. Phys., **22**, pp. 1447–1460.
[149] Schwartz, R. F., 1964, "Transformations in the Analysis of Non-Uniform Transmission Lines," J. Franklin Inst., **278**, pp. 163–278.
[150] Jordan, F. L., 1963, *Loudspeakers*, Focal, London.
[151] Olson, H. F., and Massa, F., 1939, *Applied Acoustics*, Blakiston, New York.
[152] Olson, H. F., 1940, *Elements of Acoustical Engineering*, Van Nostrand, New York.
[153] Moir, J., 1961, *High-Quality Sound Reproduction*, Chapman and Hall, London.
[154] Olson, H. F., 1972, *Modern Sound Reproduction*, Van Nostrand, New York.
[155] Benade, A. H., 1976, *Fundamentals of Musical Instruments*, Oxford University Press, London.
[156] Jeans, J., 1937, *Science and Music*, Cambridge University Press, Cambridge, England.
[157] Benade, A. H., 1980, *Horns, Strings and Harmony*, Doubleday, Garden City, NY.
[158] Berg, R. E., and Storck, D. G., 1982, *The Physics of Sound*, Prentice-Hall, Englewood Cliffs, NJ.
[159] Cabelli, A., 1980, "Acoustic Characteristics of Duct Bends," J. Sound Vib., **68**, pp. 369–388.
[160] Schroeder, M. R., 1967, "Determination of the Geometry of the Human Vocal Tract by Acoustic Measurements," J. Acoust. Soc. Am., **41**, pp. 1002–1010.
[161] Mermelstein, P., 1966, "Determination of the Vocal Tract Shape From Measured Formant Frequencies," J. Acoust. Soc. Am., **40**, pp. 1283–1294.
[162] Ishikawa, K., Matsudaira, M., and Kaneko, T., 1976, "Input Acoustic Impedance Measurement of the Subglottal System," J. Acoust. Soc. Am., **60**, pp. 160–197.
[163] Jackson, A. C., Butler, J. P., and Pyle, R. W., 1982, "Acoustic Input Impedance of Excised Dog Lungs," J. Acoust. Soc. Am., **64**, pp. 1020–1035.
[164] Dallos, P., 1973, *The Auditory Periphery*, Academic, New York.
[165] Lighthill, M. J., 1981, "Energy Flow in the Cochlea," J. Fluid Mech., **106**, pp. 149–160.
[166] Jackson, A. C., Butler, J. P., and Pyle, R. W., 1982, "Estimation of the Area Function of the Human Ear Canals by Sound Pressure Measurements," J. Acoust. Soc. Am., **73**, pp. 24–30.
[167] Stevin, G. O., 1984, "A Computational Model of the Ear Reflex," J. Acoust. Soc. Am., **55**, pp. 277–285.
[168] Pyle, R. W., 1965, "Duality Principle for Horns," J. Acoust. Soc. Am., **37**, p. 1178A.
[169] Campos, L. M. B. C., and Lau, F. J. P., 1996, "On the Acoustics of Low Mach Number Bulged, Throated and Baffled Nozzles," J. Sound Vib., **196**, pp. 611–633.
[170] Campos, L. M. B. C., 1996, "On Longitudinal Acoustic Propagation in Convergent and Divergent Nozzle Flows," J. Sound Vib., **117**, pp. 131–151.
[171] Myers, M. K., and Callegary, A. J., 1977, "On the Singular Behaviour of Linear Acoustic Theory in Near-Sonic Duct Flows," J. Sound Vib., **51**, pp. 517–531.
[172] Powel, A., 1959, "Propagation of a Pressure Pulse in a Compressible Mean Flow," J. Acoust. Soc. Am., **31**, pp. 1527–1535.
[173] Powel, A., 1960, "Theory of Sound Propagation Ducts Carrying High-Speed Mean Flows," J. Acoust. Soc. Am., **32**, pp. 1640–1646.
[174] Miles, J. H., 1981, "Acoustic Transmission Matrix for a Variable Area Duct or Nozzle Carrying Compressible Mean Flow," J. Acoust. Soc. Am., **69**, pp. 1577–1586.
[175] Nayfeh, A. H., Kelly, J. J., and Watson, L. T., 1982, "Acoustic Propagation in Waves by a Periodic Array," Wave Motion, **8**, pp. 225–234.
[176] Benade, A. H., and Jansson, E. V., 1974, "On Plane and Spherical Waves in Horns With Nonuniform Flare. I: Theory of Radiation, Resonance, Frequencies and Mode Conversion," Acustica, **31**, pp. 79–86.
[177] Jansson, E. V., and Benade, A. H., 1974, "On Plane and Spherical Waves in Horns With Non-Uniform Flare. II: Predictions and Measurements of Resonance Frequencies and Radiation Losses," Acustica, **31**, pp. 185–204.
[178] Zamorski, T., and Wyrzykowski, R., 1981, "Approximate Methods for the Solution of the Equation of Acoustic Wave Propagation in Horns," Arch. Acoust., **6**, pp. 237–285.
[179] Bostrom, A., 1983, "Acoustics Waves in a Duct With Periodically Varying Cross Section," Wave Motion, **5**, pp. 59–67.
[180] Yeow, K. W., 1974, "Webster Wave Equation in Two Dimensions," J. Acoust. Soc. Am., **56**, pp. 19–21.
[181] Hasegawa, A., 1983, "Propagation of Sound in Acoustic Slow Waveguides," Acustica, **52**, pp. 237–245.
[182] Brindley, G. S., 1973, "Speed of Sound in Bent Tubes and the Design of Wind Instruments," Nature (London), **246**, pp. 479–480.
[183] Cho, Y. C., 1980, "Rigorous Solution for Sound Radiation from Circular Ducts With Hyperbolic Horn Or Infinite Plane Baffle," J. Sound Vib., **69**, pp. 405–425.
[184] Caussé, R., Kergomard, J., and Lurton, X., 1984, "Input Impedance of Brass Musical Instruments-Comparison Between Experiment and Numerical Models," J. Acoust. Soc. Am., **75**, pp. 241–254.
[185] Fletcher, N. H., and Rossing, T. D., 1997, *The Physics of Musical Instruments*, Springer, New York.
[186] Meyer, J., 1972, *Akustik und Musikalishe Auffuhrungspraxis*, Verlag Erwin Bochinsky.
[187] Knudsen, V. O., and Harris, C. M., 1950, *Acoustical Designing in Architecture*, Acoustical Society of America, New York.
[188] Makrinenko, L. I., 1945, *Acoustics of Auditoriums in Public Buildings*, Acoustical Society of America, New York.
[189] Beranek, L. L., 1996, *Concert and Opera Halls*, Acoustical Society of America, New York.
[190] Brekhovskikh, L. M., 1960, *Waves in Layered Media*, 2nd ed., Academic, New York.
[191] Brekhovskikh, L. M., and Lysanov, Y. P., 1982, *Acoustic Propagation in the Ocean*, Springer.
[192] Blokhintsev, D. I., 1946, "The Propagation of Sound in an Inhomogeneous and Moving Medium," J. Acoust. Soc. Am., **18**, pp. 322–344.
[193] Franken, P. A., and Ingard, U., 1956, "Sound Propagation in a Moving Medium," J. Acoust. Soc. Am., **27**, pp. 1044–1050.

[194] Garrett, C. J. R., 1967, “The Adiabatic Invariant for Wave Propagation in a Non-Uniform Moving Medium,” Proc. R. Soc. London, Ser. A, **299**, pp. 26–27.
[195] Lighthill, M. J., 1953, “On the Energy Scattered From the Interaction of Turbulence With Sound and Shock Waves,” Proc. Cambridge Philos. Soc., **44**, pp. 531–551.
[196] Howe, M. S., 1973, “Kinetic Theory of Wave Propagation in Random Media,” Philos. Trans. R. Soc. London, Ser. A, **274**, pp. 523–549.
[197] Howe, M. S., 1973, “Multiple Scattering of Sound by Turbulence and Other Inhomogeneities,” J. Sound Vib., **27**, pp. 455–476.
[198] Lighthill, M. J., 1963, “Jet Noise,” AIAA J., **1**, pp. 1587–1617.
[199] Ffowcs-Williams, J. E., 1963, “The Noise From Turbulence Convected at High Speed,” Philos. Trans. R. Soc. London, **255**, pp. 471–503.
[200] Ffowcs-Williams, J. E., 1964, “Sound Production at the Edge of a Steady Flow,” J. Fluid Mech., **66**, pp. 791–816.
[201] Crighton, D. G., 1975, “Basic Principles of Aerodynamic Noise Generation,” Prog. Aerosp. Sci., **16**, pp. 31–96.
[202] Proudman, I., 1952, “The Generation of Sound by Isotropic Turbulence,” Proc. R. Soc. London, Ser. A, **214**, pp. 119–132.
[203] Lighthill, M. J., 1952, “On Sound Generated Aerodynamically, I. General Theory,” Proc. R. Soc. London, Ser. A, **211**, pp. 564–587.
[204] Lighthill, M. J., 1954, “On Sound Generated Aerodynamically. II. Turbulence as a Source of Sound,” Proc. R. Soc. London, Ser. A, **222**, pp. 1–32.
[205] Lighthill, M. J., 1961, “On Sound Generated Aerodynamically: Bakerian Lecture,” Proc. R. Soc. London, Ser. A, **267**, pp. 147–282.
[206] Powel, A., 1968, “Vortex Sound,” J. Acoust. Soc. Am., **36**, pp. 117–185.
[207] Campos, L. M. B. C., 1978, “On the Emission of Sound by an Ionized Inhomogeneity,” Proc. R. Soc. London, Ser. A, **359**, pp. 65–91.
[208] Howe, M. S., 1998, *Acoustics of Fluid-Structure Interaction*, Cambridge University Press, Cambridge, England.
[209] Lilley, G. M., 1974, “On Noise From Jets,” AGARD CP-131, Paper No. 13.1.
[210] Crighton, D. G., and Ffowcs-Williams, J. E., 1969, “Sound Generation by Turbulent Two-Phase Flow,” J. Fluid Mech., **36**, pp. 585–603.
[211] Colonius, T., Lele, S. K., and Moin, P., 1997, “Sound Generation in a Mixing Layer,” J. Fluid Mech., **330**, pp. 375–409.
[212] Goldstein, M. E., 2001, “An Exact form of Lilley’s Equation With a Velocity Quadrupole/Temperature Dipole Source Term,” J. Fluid Mech., **443**, pp. 231–236.
[213] Goldstein, M. E., 2003, “A Generalized Acoustic Analogy,” J. Fluid Mech., **488**, pp. 315–333.
[214] Musafr, R. E., 1993, “A Note on the Description of Jet Noise Source Terms,” Proceeding of the Institute of Acoustics, **15**, pp. 901–909.
[215] Curle, N., 1955, “On the Influence of Solid Boundaries Upon Aerodynamic Sound,” Proc. R. Soc. London, Ser. A, **231**, pp. 505–513.
[216] Goldstein, M. E., 1974, “Unified Approach to Aerodynamic Sound Generation in the Presence of Solid Boundaries,” J. Acoust. Soc. Am., **56**, pp. 497–506.
[217] Doak, P. E., 1960, “Acoustic Radiation From a Turbulent Fluid Containing Foreign Bodies,” Proc. R. Soc. London, Ser. A, **254**, pp. 129–145.
[218] Ffwocs-Williams, J. E., and Hawlings, D. L., 1968, “Sound Gemeration by Turbulence and Surfaces in Arbitrary Motion,” Philos. Trans. R. Soc. London, Ser. A, **264**, pp. 321–342.
[219] Lowson, M. V., 1965, “The Sound Field for Singularities in Motion,” Proc. R. Soc. London, Ser. A, **286**, pp. 559–572.
[220] Farassat, F., 1977, “Discontinuities in Aerodynamics: The Concept and Applications of Generalized Derivatives,” J. Sound Vib., **62**, pp. 165–153.
[221] Farassat, F., and Myers, M. K., 1988, “Extension of Kirchhoff’s Formula to Radiation From Moving Surfaces,” J. Sound Vib., **123**, pp. 451–460.
[222] Ffowcs-Williams, J. E., and Howe, M. S., 1975, “The Generation of Sound by Density Inhomogeneities in Low Mach Number Nozzle Flows,” J. Fluid Mech., **70**, pp. 605–622.
[223] Marble, F. E., and Candel, S. M., 1977, “Acoustic Disturbance From Gas Nonuniformities Convected Through a Nozzle,” J. Sound Vib., **55**, pp. 225–243.
[224] Jones, H., 1979, “The Generation of Sound by Flames,” Proc. R. Soc. London, Ser. A, **367**, pp. 291–309.
[225] Michalke, A., and Michel, U., 1979, “Prediction of Jet Noise Flight From Static Tests,” J. Sound Vib., **67**, pp. 341–367.
[226] McGowan, R. S., and Larson, R. S., 1984, “Relationship Between Static, Flight and Simulated Flight Jet Noise Measurements,” AIAA J., **22**, pp. 460–464.
[227] Schmidt, D. W., and Tillmann, P. M., 1970, “Experimental Study of Sound Phase Fluctuations Caused by Turbulent Wakes,” J. Acoust. Soc. Am., **47**, pp. 1310–1324.
[228] Ho, C. M., and Kovasznay, L. S. G., 1976, “Acoustical Shadowgraph,” Phys. Fluids, **19**, pp. 1118–1123.
[229] Ho, C. M., and Kovasznay, L. S. G., 1976, “Propagation of a Coherent Acoustic Wave Through a Turbulent Shear Flow,” J. Acoust. Soc. Am., **60**, pp. 40–45.
[230] Chernov, L. A., 1960, *Wave Propagation in a Random Medium*, McGraw-Hill, New York.
[231] Tatarski, V. I., 1961, *Wave Propagation in Turbulent Medium*, McGraw-Hill, New York.
[232] Uscinski, J. A., 1977, *Wave Propagation in Random Media*, McGraw-Hill, New York.
[233] Ishimaru, A., 1978, *Wave Propagation and Scattering in Random Media*, Academic P., 2 vols.
[234] Ogilvy, J. A., 1992, *Wave Scattering Random Rough Surfaces*, Institute of Physics, University of Reading, Berkshire.
[235] Bechert, D., and Pfizenmaier, E., 1975, “On the Amplification of Broadband Jet Noise by Pure Tone Excitation,” J. Sound Vib., **43**, pp. 581–587.
[236] Bechert, D. W., Michel, U., and Pfizenmaier, E., 1977, “Experiments on the Transmission of Sound Through Jets,” AIAA J., AIAA Paper No. 79–575.
[237] Blanc-Benon, P., and Juvé, D., 1981, “Effet d’un jet turbulent sur le niveau d’onde cohérente et sur I’intensité d’ un faisceau acoustique,” C. R. Seances Acad. Sci., Ser. 2, **293**, pp. 493–496.
[238] Blanc-Benon, P., and Juvé, D., 1981, “Elargissement d’un faisceau ultrasonore par transvergée d’un champ turbulent,” C. R. Acad. Sci., **292**, pp. 551–554.
[239] Blanc-Benon, P., and Juvé, D., 1982, “Coherence spatialle d’un faisceau ultrasone aprés tranversée d’une turbulence cinématique,” C. R. Acad. Sci., **294**, pp. 1221–1224.
[240] Blanc-Benon, P., and Juvé, D., 1982, “Corrélations spatiotemporelles d’ un faisceau aprés transversée d’ un turbulence cinématique,” C. R. Acad. Sci., **294**, pp. 1255–1258.
[241] Corcos, G. M., 1963, “The Resolution of Pressure,” J. Fluid Mech., **35**, pp. 192–199.
[242] Corcos, G. M., 1967, “The Resolution of in Turbulence at the Wall of a Boundary Layer,” J. Fluid Mech., **6**, pp. 59–70.
[243] Wilmarth, W. W., 1975, “The Structure of the Turbulent Pressure Field in Boundary Layer Flows,” Annu. Rev. Fluid Mech., **7**, pp. 187–210.
[244] Bull, M. K., 1996, “Wall Pressure Fluctuation in Turbulent Boundary Layers: Some Reflections in Forty Years of Search,” J. Sound Vib., **190**, pp. 299–315.
[245] Crow, S. C., 1967, “Visco Elastic Character Fine-Grained Isotropic Turbulence,” Phys. Fluids, **10**, pp. 1587–1589.
[246] Crow, S. C., 1968, “Visco Elastic Properties of Fine-Grained Incompressible Turbulence,” J. Fluid Mech., **33**, pp. 1–20.
[247] Chase, D. M., 1980, “Modelling the Wavevector-Frequency Spectrum of Turbulent Boundary Layer Wall Pressure,” J. Sound Vib., **70**, pp. 29–67.
[248] Chase, D. M., 1987, “The Character of the Turbulent Wall Pressure Spectrum at Sub Convective Wavenumbers and Suggested Comprehensive Model,” J. Sound Vib., **112**, pp. 125–147.
[249] Efimtsov, B. M., 1982, “Characteristics of the Field Turbulent Wall Pressure Fluctuations at Large Reynolds Numbers,” Sov. Phys. Acoust., **28**, pp. 289–292.
[250] Campos, L. M. B. C., 1992, “Effects on Acouctic Fatigue Loads of Multiple Reflections Between a Plate and Turbulent Wake,” Acustica, **76**, pp. 109–117.
[251] Campos, L. M. B. C., 1996, “On the Correlation of Acoustic Pressures Induced by a Turbulent Wake on a Nearby Wall,” Acust. Acta Acust., **82**, pp. 9–17
[252] Campos, L. M. B. C., Bourgine, A., and Bonomi, B., 1998, “Comparison of Theory and Experiment on Aeroacoustic Loads and Deflections,” J. Fluids Struct., **13**, pp. 3–35.
[253] Landau, L. D., and Lifshitz, E. F., *Fluid Mechanics*, Oxford University Press, London.
[254] Fourier, J., 1837, *Théorie analytique de la chaleur*, Dover, New York.
[255] Carslaw, H. S., and Jaeger, J. C., 1946, *Conduction of Heat in Solids*, 2nd ed., Oxford University Press, London.
[256] Brown, E. H., and Hall, F. F., 1978, “Advances in Atmospheric Acoustics,” Rev. Geophys. Space Phys., **16**, pp. 47–110.
[257] Bass, H. E., Sutherland, L. C., Zuckerwar, A. J., Blackstock, D. T., and Hest, D. M., 1995, “Atmospheric Absorption of Sound: Further Developments,” J. Acoust. Soc. Am., **97**, pp. 680–683.
[258] Sutherland, G. A., and Gaigle, G. A., 1998, “Atmospheric Sound Propagation,” *Handbook of Acoustics*, Wiley, New York, Vol. 28.
[259] Attenborough, S., and Taherzadeh, S., 1995, “Test Cases for Outdoor Sound Propagation,” J. Acoust. Soc. Am., **97**, pp. 173–191.
[260] Richards, T. L., and Attenborough, K., 1996, “Accurate FFT-based Hankel Transforms for Predictions of Outdoor Sound Propagation,” J. Sound Vib., **109**, pp. 157–167.
[261] Cerveneny, M. V., Popov, M., and Psensick, L., 1982, “Computation of Wave Fields in Inhomogeneous Media-Gaussian Beam Approach,” J. R. Astron. Soc. Can., **70**, pp. 109–126.
[262] Gilbert, K. E., and White, M. J., 1989, “Application of the Parabolic Equation to Sound Propagation in a Refracting Atmosphere,” J. Acoust. Soc. Am., **85**, pp. 630–637.
[263] Gilbert, K. E., Raspet, R., and Di, X., 1990, “Calculation of Turbulence Effects in an Upward-Refracting Atmosphere,” J. Acoust. Soc. Am., **87**, pp. 2428–2437.
[264] Esperance, A. L., Herzog, G. A., Daigle, G. A., and Nicolas, J., 1992, “Heuristic Model for Outdoor Sound Propagation Based on an Extension of the Geometrical Ray Theory in the Case of a Linear Sound Field Profile,” Appl. Acoust., **37**, pp. 111–139.
[265] Cabillet, Y., Schroeder, H., Daigle, G. A., and L’Esperance, A., 1993, “Aplication of the Gaussian Beam Approach to Sound Propagation in the Atmosphere,” J. Acoust. Soc. Am., **93**, pp. 3105–3116.
[266] Gilbert, K. E., and Di, X., 1993, “A Fast Green’s Function Method for One-Way Sound Propagation in the Atmosphere,” J. Acoust. Soc. Am., **94**, pp. 2343–2352.
[267] Raspet, R., Lee, S. W., Kuester, E., Chang, D. C., Richards, W. F., Gilbert, R.,

and Bong, N., 1985, "A Fast-Field Program for Sound Propagation in a Layered Atmosphere Above an Impedance Ground," J. Acoust. Soc. Am., **77**, pp. 345–352.
[268] Raspet, R., Baird, G., and Wu, W., 1992, "Normal Mode Solution for Low Frequency Sound Propagation in a Downward Refracting Atmosphere Above a Complex Impedance Plane," J. Acoust. Soc. Am., **91**, pp. 1341–1352.
[269] Li, K. M., 1994, "High-Frequency Approximation of Sound Propagation in a Stratified Moving Atmosphere Above a Porous Ground Surface," J. Acoust. Soc. Am., **95**, pp. 1840–1852.
[270] Page, N. W., and Mee, D. J., 1984, "Wall Effects on Sound Propagation in Tubes," J. Sound Vib., **93**, pp. 473–480.
[271] Kergomard, J., 1981, "Ondes Quasi-Stationnaires Dans Les Pavillons Avec Pertes Visco-Thermiques Aux Parois," Acustica, **48**, pp. 31–43.
[272] Cantrell, R. H., and Hart, R. W., 1964, "Interaction Between Sound and Flow in Acoustic Cavities: Mass, Momentrum and Energy Considerations," J. Acoust. Soc. Am., **36**, pp. 697–706.
[273] Cummings, A., 1984, "Acoustic Non-Linearities and Power Losses at Orfices," AIAA J., **22**, pp. 786–792.
[274] Leppington, F. G., and Levine, H., 1980, "Acoustics of a Blocked Duct With Flow," J. Sound Vib., **72**, pp. 303–315.
[275] Mortell, M. P., and Seymour, B. R., 1979, "Non-Linear Forced Oscillations in a Closed Tube: Continuous Solutions of a Functional Equation," Proc. R. Soc. London, Ser. A, **A367**, pp. 253–270.
[276] Keller, J. J., 1984, "Non-Linear Self-Excited Acoustic Oscillations in Cavities," J. Sound Vib., **94**, pp. 397–409.
[277] Koch, W., 1977, "Radiation of Sound from a Two-Dimensional Acoustically Lined Duct," J. Sound Vib., **55**, pp. 255–274.
[278] Koch, W., 1977, "Attenuation of Sound in Multi-Element Acoustically Lined Rectangular Ducts in the Absence of Mean Flow," J. Sound Vib., **52**, pp. 459–496.
[279] Ogimoto, K., and Johnston, G. W., 1979, "Modal Radiation Impedances for Semi-Infinite Unflanged Circular Ducts Including Flow Effects," J. Sound Vib., **62**, pp. 598–605.
[280] Koch, W., and Mohring, W., 1983, "Eigensolutions for Liners in Uniform Mean Flow Ducts," AIAA J., **21**, pp. 200–203.
[281] Rienstra, S. W., 1985, "Contributions to the Theory of Sound Propagation in Ducts With Bulk-Reacting Lining," J. Acoust. Soc. Am., **77**, pp. 1681–1685.
[282] Bies, D. A., Hansen, C. H., and Bridges, G. E., 1991, "Sound Propagation in Rectangular and Circular Cross-Section Ducts With Flow and Bulk-Reacting Liner," J. Sound Vib., **146**, pp. 47–80.
[283] Masterman, P. H., and Learricots, P. J. B., 1969, "Computer Method of Solving Waveguide-Iris Problems," Electron. Lett., **5**, pp. 23–25.
[284] Howe, M. S., 1983, "The Attenuation of Sound in a Randomly Lined Duct," J. Sound Vib., **87**, pp. 83–103.
[285] Watson, W. R., 1984, "An Acoustic Evaluation of Circumferentially Segmented Duct Liners," AIAA J., **22**, pp. 1229–1233.
[286] Fuller, C. R., 1984, "Propagation and Radiation of Sound from Flanged Circular Ducts With Circumferentially Varying Wall Admittances, I: Semi-Infinite Ducts," J. Sound Vib., **93**, pp. 321–340.
[287] Fuller, C. R., 1984, "Propagation and Radiation of Sound from Flanged Circular Ducts With Circumferentially Varying Wall Admittances, II: Finite Ducts With Sources," J. Sound Vib., **93**, pp. 341–351.
[288] Vaidya, P. G., 1985, "The Propagation of Sound in Ducts Lined with Circumferentially Non-Uniform Admittance of the Form $\eta_0 + \eta_q \exp(iq\theta)$," J. Sound Vib., **100**, pp. 463–475.
[289] Regan, B., and Eaton, J., 1999, "Modelling the Influence of Acoustic Liner Non-Uniformities on Duct Modes," J. Sound Vib., **219**, pp. 859–879.
[290] Campos, L. M. B. C., and Oliveira, J. M. G. S., 2004, "On the Acoustic Modes in a Cylindrical Nozzle with an Arbitrary Impedance Distribution," J. Acoust. Soc. Am., **116**, pp. 3336–3347.
[291] Campos, L. M. B. C., and Oliveira, J. M. G. S., 2004, "On the Optimization of Non-Uniform Acoustic Liners on Annular Nozzles," J. Sound Vib., **275**, pp. 557–576.
[292] Fabrikant, A. L., 1983, "Sound Scattering by Vortex Flows," Akust. Zh., **29**, p. 262; Sov. Phys. Acoust., **29**, pp. 152–155.
[293] Howe, M. S., 1983, "On the Scattering of Sound by a Vortex Ring," J. Sound Vib., **87**, pp. 567–571.
[294] Blevins, R. D., 1984, "Review of Sound Induced by Vortex Shedding from Cylinders," J. Sound Vib., **92**, pp. 455–470.
[295] Broadbent, E. G., and Moore, D. W., 1979, "Acoustic Destabilization of Vortices," Philos. Trans. R. Soc. London, **290**, pp. 353–371.
[296] Leppington, F. G., 1972, "Scattering of Quadrupole Sources Near the End of a Rigid Semi-Infinite Circular Pipe," Aeronautical Research Council, **1195**, pp. 5.
[297] Howe, M. S., 1995, "The Damping of Sound by Turbulent Wall Shear Layer," J. Acoust. Soc. Am., **98**, pp. 1723–1730.
[298] Blake, W. K., 1986, *Mechanics of Flow-Induced Sound and Vibration*, Academic, New York.
[299] Howe, M. S., 2002, *Theory of Vortex Sound*, Cambridge University Press, Cambridge.
[300] Nelson, P. A., and Elliott, S. J., 1992, *Active Control of Sound*, Academic Press.
[301] Junger, M. C., and Feit, D., 1972, *Sound, Structures and their Interaction*, 2nd ed., MIT, Cambridge, MA.
[302] Paidoussis, M. P., 1998, *Fluid-Structure Interactions*, Academic Press.
[303] Howe, M. S., 2002, *Acoustics of Fluid-Structure Interaction*, Cambridge University Press, Cambridge.
[304] Riemann., B., 1860, *Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Gesammelte Werke, Teubner, Berlin, pp. 156–173.
[305] Lighthill, M. J., 1956, "Viscosity Effects in Sound Waves of Finite Amplitude," *Surveys in Mechanics*, G. K. Batchelor and R. M. Davies, eds., Cambridge University Press, Cambridge, pp. 250–351.
[306] Campos, L. M. B. C., and Leitão, J. F. P. G., 1988, "On the Computation of Special Functions With Application to Non-Linear and Inhomogeneous Waves," Comput. Mech., **3**, pp. 343–360.
[307] Campos, L. M. B. C., 1984, "On a Theory of Solar Spicules and the Atmospheric Mass Balance," Mon. Not. R. Astron. Soc., **207**, pp. 547–574.
[308] Burgers, J. M., 1948, "A Mathematical Model Illustrating the Theory of Turbulence," Adv. Appl. Math., **10**, pp. 171–179.
[309] Burgers, J. M, *The Non-Linear Diffusion Equation*, Reidel, Dordrecht.
[310] Cole, J. P., 1950, "A Quasilinear Parabolic Equation Appearing in Aerodynamics," Q. J. Mech. Appl. Math., **9**, pp. 225–236.
[311] Hopf, E., 1951, "The Partial Differential Equation $\mu_t + \mu_x \mu = \mu_{xx}$," Commun. Pure Appl. Math., **3**, pp. 201–230.
[312] Compos, L. L. M. B. C., 1986, "On Waves in Gases, Part 1: Acoustics of Jests, Turbulence and Ducts," Rev. Mod. Phys., **58**, pp. 117–182.
[313] Westerwelt, P. J., 1963, "Parametric Acoustic Array," J. Acoust. Soc. Am., **35**, pp. 335–337.
[314] Zabolotskaya, E. A., and Khoklov, R. V., 1969, "Quasiplane Waves in Non-Linear Acoustics of Confined Beams," Akust. Zh., **25**, pp. 515–520, Sov. Phys. Acoust., **26**, pp. 217–220.
[315] Kuznetsov, V. P., 1970, "Equations of Non-Linear Acoustics," Akust. Zh., **16**, pp. 548–551, Sov. Phys. Acoust., **16**, pp. 467–470.
[316] Blackstock, D. T., 1972, "Non-Linear Acoustics," *American Institute of Physics Handbook*, D. E. Gray, ed., McGraw-Hill, New York.
[317] Beyer, R. T., 1974, *Non-Linear Acoustics*, Naval Sea Systems Command.
[318] Bjorno, L., 1976, "Non-Linear Acoustics," *In Acoustics and Vibration Progress*, R. W. B. Stephens and H. G. Leventhall, eds., Chapman & Hall, 2, pp. 101–198.
[319] Polyakova, A. L., Soluyan, S. I., and Kholov, R. V., 1962, "Propagation of Finite Disturbances in a Relaxing Medium," Sov. Phys. Acoust., **8**, pp. 78–82.
[320] Ockendon, H., and Spence, P. A., 1969, "Non-Linear Wave Propagation in a Relaxing Gas," J. Fluid Mech., **39**, pp. 329–345.
[321] Clarke, J. L., and Sinai, Y. L., 1977, "The Wave System Attached to a Slender Body in a Supersonic Relaxing Gas Stream. Basic Results: The Cone," J. Fluid Mech., **79**, pp. 499–524.
[322] Van Wijngarden, L., 1968, "On the Equations of Motion for Mixtures of Liquid and Gas Bubbles," J. Fluid Mech., **33**, pp. 465–474.
[323] Chester, W., 1964, "Resonant Oscillations in Closed Tubes," J. Fluid Mech., **18**, pp. 44–64.
[324] Friedlander, F. G., 1985, *Sound Pulses*, Cambridge University Press, Cambridge.
[325] Karpman, V. I., 1975, *Non-Linear Waves in Dispersive Media*, Pergamon, New York.
[326] Novikov, B. K., Rudenko, O. V., and Timoshenko, V. I., 1987, *Non-Linear Underwater Acoustics*, Acoustical Society of America, New York..
[327] Kelbert, M., and Sazonov, I., 1996, *Pulses and Other Wave Processes in Fluids*, Kluwer.
[328] Naugolnykh, K., and Ostrovsky, L., 1998, *Non-Linear Wave Processes in Acoustics*, Cambridge University Press, Cambridge.
[329] Drumheller, D. S., 1998, *Wave Propagation in Non-Linear Fluids and Solids*, Cambridge University Press, Cambridge.
[330] Miles, J. W., 1981, "The Korteweg–de Vries Equation: a Historical Essay," J. Fluid Mech., **106**, pp. 131–147.
[331] Gardner, C. S., Kruskal, J. M. D., and Miura, B. M., 1967, "Method for Solving the Korteweg de Vries Equation," Phys. Rev. Lett., **19**, pp. 1095–1097.
[332] Gardner, C. S., Greene, J. M., Kruskal, M. D., and Miura, R. M., 1974, "Korteweg–de Vries Equations and Generalizations. VI: Methods for Exact Solution," Commun. Pure Appl. Math., **27**, pp. 97–133.
[333] Freeman, N. C., 1979, "A Two-Dimensional Distributed Soliton Solution of the Korteweg–de Vries Equation," Proc. R. Soc. London, Ser. A, **366**, pp. 185–204.
[334] Boussinesq, J., 1871, "Théorie de I'intumencence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire," C. R. Hebd. Seances Acad. Sci., **72**, pp. 755–759.
[335] Ono, H., 1975, "Algebraic Solitary Waves in Stratified Fluids," J. Phys. Soc. Jpn., **39**, pp. 1082–1091.
[336] Bjomo, I., 1974, *Finite Amplitude Wave Effects in Fluids*, Science and Technology, Acoustical Society of America, New York.
[337] Blackstock, D. T., 1972, "Nonlinear Acoustics: Theoretical," *American Institute of Physics Handbook*, McGraw-Hill, New York, pp. 3–183.
[338] Beyer, R. T., 1972, "Noninear Acoustics: Experimental," *American Institute of Physics Handbook*, Mcgraw-Hill, New York, pp. 3–206.
[339] Peube, T. L., and Chasseriaux, J., 1973, "Non-Linear Acoustics in Ducts With Varying Cross Section," J. Sound Vib., **27**, pp. 533–546.
[340] Nayfeh, A. H., 1975, "Finite-Amplitude Plane Waves in Ducts With Varying Properties," J. Acoust. Soc. Am., **57**, pp. 1413–1415.
[341] Baxter, S. M., 1984, "The First-Order Non-Linear Sound Field of a Two-Frequency Spherical Source," J. Sound Vib., **94**, pp. 337–349.
[342] Scott, J. F., 1981, "Uniform Asymptotics for Spherical and Cylindrical Non-linear Acoustic Waves Generated by a Sinusoidal Source," Proc. R. Soc. London, Ser. A, **375**, pp. 211–230.

[343] Bjorno, I., and Larsen, P. N., 1984, "Noise of Air Jets From Rectangular Slits," Acustica, **54**, pp. 247–256.
[344] Blackstock, D. T., 1964, "Thermoviscous Attenuation of Plane, Periodic, Finite-Amplitude Sound Waves," J. Acoust. Soc. Am., **36**, p. 534.
[345] Rose, J. L., 1958, *Ultrasonic Waves in Solid Media*, Cambridge University Press, Cambridge.
[346] Hayes, W. D., 1973, "Sonic Boom," Annu. Rev. Fluid Mech., **3**, pp. 269–290.
[347] Seiner, J., and Yu, C., 1984, "Acoustic Near-Field Properties Associated Broadband Shock Noise," AIAA J., **22**, pp. 1207–1215.
[348] Prasad, P., 1973, "Non-Linear Wave Propagation on an Arbitrary Steady Transonic Flow," J. Fluid Mech., **57**, pp. 721–737.
[349] Scott, J. F., 1982, "The Non-Linear Propagation of Acoustic Noise," Proc. R. Soc. London, Ser. A, **55**, pp. 383–397.
[350] Truesdeep, C. A., 1955, "The Theory of Aerial Sound," Euleri Opera Omnia, **2**(13b), pp. XIX–LXX.
[351] Euler, L., 1759, "Supplement au et continuation des rechurches sur la propogation du son," Mem. Acad. Sci. Berlin, **15**, pp. 1–20 (1766).
[352] Poisson, S., 1807, "Memoire sur la théorie du son," J. Ec. Polytech. (Paris), **2**, pp. 367–430.