Particles and Fields

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Preface

Quantum field theory originally ment the theory of quantizing classical fields. Nowadays this notion is used for the general theory of quantum systems with **infinitly many degrees of freedom**. This is a vast subject, but we can cover only a few topics.

> **Warning:** This manuscript is just a condensed set of notes for the lecturer. Obviously, the explanations given here are insufficient for the non-expert. Maybe they will be extended in the future. For the time being these notes should be considered no more than an outline of the stuff to be elaborated in the lectures.

Purely mathematical proofs will mainly be skipped and replaced by suitable references.

Recommended Literature: (Araki, 1999; Baumgärtel, 1995; Bogolubov et al., 1990; Borchers, 1996; Buchholz, 1985; Fredenhagen, 1995; Haag, 1992; Horuzhy, 1990; Jost, 1965; Streater and Wightman, 1989)

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Chapter 1

General Quantum Theory

1.1 Basic Logical Structure

1.1.1 Classical Logic and General Notions[1]

Let $\mathcal{L}$ be a set of propositions fulfilling

$$E_1, E_2 \in \mathcal{L} \Longrightarrow E_1 \wedge E_2, \neg E_1 \in \mathcal{L}\,.$$

In common sense logic $E_1 \wedge E_2$ holds if and only if both E_1 and E_2 hold while $\neg E_1$ holds if and only if E_1 does not hold. Let us identify E_1 with E_2 whenever E_1 holds if and only if E_2 holds. Under these circumstances

$$E_1 \preccurlyeq E_2 \stackrel{\text{def}}{\Longleftrightarrow} E_1 \wedge E_2 = E_1 \tag{1.1}$$

defines a ***semi-ordering*** $\preccurlyeq$ on $\mathcal{L}$, i.e. the ordinary logical implication $\preccurlyeq$ is

reflexive:	$E \preccurlyeq E\,,$
transitive:	$E_1 \preccurlyeq E_2, E_2 \preccurlyeq E_3 \Longrightarrow E_1 \preccurlyeq E_3\,,$
anti-symmetric:	$E_1 \preccurlyeq E_2, E_2 \preccurlyeq E_1 \Longrightarrow E_1 = E_2\,.$

The semi-ordered set $(\mathcal{L}, \preccurlyeq)$ is even a ***lattice***, i.e. for any two elements $E_1, E_2 \in \mathcal{L}$ there is an infimum as well as a supremum, namely

$$E_1 \wedge E_2 = \inf_{\mathcal{L}}\{E_1, E_2\}\;, \quad E_1 \vee E_2 = \sup_{\mathcal{L}}\{E_1, E_2\}\,, \tag{1.2}$$

where $E_1 \vee E_2$ holds iff at least one of the propositions E_1 , E_2 holds.[2] This lattice has a ***universal lower bound***

$$\check{0} \stackrel{\text{def}}{=} \inf_{\mathcal{L}}\mathcal{L}$$

as well as a ***universal upper bound***

$$\check{1} \stackrel{\text{def}}{=} \sup_{\mathcal{L}}\mathcal{L}\,.$$

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[1]The definitions and notions introduced here are in agreement with those of (Birkhoff, 1967) and (Varadarajan, 2007).

[2]For lattices in general (1.2) serves as a definition for $\wedge$ and $\vee$, consistent with (1.1).

The negation $\neg$ is an ***orthocomplementation***, i.e. for all $E\,,E_1\,,\,E_2\in\mathcal{L}$ we have

$$\begin{aligned}(O_1):&\quad E\wedge\neg E=\check{0}\,,\\(O_2):&\quad E\vee\neg E=\check{1},\,,\\(O_3):&\quad \neg(\neg E)=E\,,\\(O_4):&\quad E_1\preccurlyeq E_2\Longrightarrow\neg E_2\preccurlyeq\neg E_1\,.\end{aligned}$$

The orthocomplemented lattice $(\mathcal{L},\preccurlyeq,\neg)$ is ***weakly modular***, i.e.:

$$E_1\preccurlyeq E_2\Longrightarrow E_2=(E_2\wedge E_1)\vee(E_2\wedge\neg E_1)\,.$$

A ***logic*** is defined to be a weakly modular orthocomplemented lattice $(\mathcal{L},\preccurlyeq,\neg)$ which is σ-complete; i.e. in which $\inf_{\mathcal{L}} A$ exists for every countable subset A of $\mathcal{L}$. A logic $(\mathcal{L},\preccurlyeq,\neg)$ is called ***classical*** if – as in the above example – it is ***distributive***; i.e.:[3]

$$\begin{aligned}(D_1):&\quad E_1\wedge(E_2\vee E_3)=(E_1\wedge E_2)\vee(E_1\wedge E_3)\,,\\(D_2):&\quad E_1\vee(E_2\wedge E_3)=(E_1\vee E_2)\wedge(E_1\vee E_3)\,.\end{aligned}$$

An **ordered** pair $(E_1,E_2)\in\mathcal{L}\times\mathcal{L}$ is called ***compatible*** if[4]

$$E_1=(E_1\wedge E_2)\vee(E_1\wedge\neg E_2)\ . \tag{1.3}$$

A ***probability measure*** on a logic $(\mathcal{L},\preccurlyeq,\neg)$ is defined to be a mapping w from $\mathcal{L}$ into the interval $[0,1]$ fulfilling the following two conditions:[5]

(W_1) : $w(1)=1\,.$
(W_2) : w is σ -***additive***; i.e.:
$w(\sup\{E_j:j=1,2,\dots\})=\sum_{j=1}^{\infty}w(E_j)$ if $E_j\preccurlyeq\neg E_k$ for $j\neq k\,.$

1.1.2 Quantum Logic

Every known concrete quantum theory is a *statistical theory* of the following type. It is affiliated with

1. A set $\mathcal{Q}$ of **macroscopic** prescriptions for preparing a 'state' of the system under consideration.

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[3]Thanks to orthocomplementation, (D_1) and (D_2) are equivalent.

[4]An orthocomplemented lattice $(\mathcal{L},\preccurlyeq,\neg)$ is weakly modular if and only if

$$(E_1,E_2)\text{ compatible }\Longleftrightarrow(E_2,E_1)\text{ compatible}$$

holds for all $E_1\,,\,E_2\in\mathcal{L}$ (Birkhoff, 1967, Theorem 21, p. 53).

[5]Actually, one should also make sure that $w(E_1)=w(E_2)=1\Longrightarrow w(E_1\wedge E_2)=1$ holds for all $E_1,E_2\in\mathcal{L}\,.$

2. A set $\mathcal{X}$ of **macroscopic** prescriptions for performing idealized ***simple tests*** (called ***questions*** by Piron) on the system under consideration[6] with only two possible outcomes referred to as 'yes' or 'no'.

3. A mapping[7]
$$w:\ \mathcal{Q}\times\mathcal{X}\longrightarrow[0,1]$$
with the following interpretation:[8]

> $w(S,T)$ is the probability[9] for the outcome 'yes' when performing a simple 'test' corresponding to T on the system in a 'state' corresponding to S.

Obviously, the 'tests' $T\in\mathcal{X}$ cannot separate elements $S_1,S_2\in\mathcal{Q}$, which are equivalent in the following sense:

$$S_1\sim S_2\stackrel{\text{def}}{\Longleftrightarrow}w(S_1,T)=w(S_2,T)\;\forall\,T\in\mathcal{X}\,.$$

Similarly the 'states' $S\in\mathcal{Q}$ cannot separate 'tests' $T_1,T_2\in\mathcal{X}$, which are equivalent in the sense that

$$T_1\sim T_2\stackrel{\text{def}}{\Longleftrightarrow}w(S,T_1)=w(S,T_2)\;\forall\,S\in\mathcal{Q}\,.$$

Therefore the appropriate mathematical formalism deals with the equivalence classes $[S]$ (also called ***states***) and $[T]$ (also called ***propositions*** or ***questions***) together with the (consistent) assignment

$$\omega\left(\hat{P}\right)\stackrel{\text{def}}{=}w(S,T)\quad\text{for }\omega=[S]\,,\ \hat{P}=[T]$$

rather than the specific prescriptions S,T and the mapping w.

> "What we call a state nowadays might turn out to be an equivalence class of states at later times. But this is only possible after having discovered new observables and new states at the same time because states and observables must be mutually separating." (Borchers, 1996, p. 2)

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[6] In relativistic quantum field theory we cannot assume that these tests can be performed within arbitrarily small time intervals. Therefore, as becomes evident by Theorem 1.2.2, we work in the so-called ***Heisenberg picture***, in which time evolution is attributed to the 'tests' rather than to the 'states'.

[7] Actually – as well known for open systems (Davies, 1976) – the probability for the outcome 'yes' or 'no' in a test performed before the 'state' state is prepared need not have any meaning. However for all known models of **closed** quantum systems the 'states' can be imagined as having been prepared as early as one likes. This is essential for standard scattering theory.

[8] We do **not** claim that S uniquely characterizes a **microscopic** state, nor do we claim that T fixes the microscopic details of a test!

[9] Compare, e.g. (Peres, 1995, p. 25).

$\mathcal{Q}$ and $\mathcal{X}$ are always (more or less implicitly) chosen such that the following three conditions are fulfilled:[10]

(I_1): For every $\hat{P} \in \mathcal{L} \stackrel{\text{def}}{=} \{[T] : T \in \mathcal{X}\}$ there is also an element $\neg\hat{P} \in \mathcal{L}$ fulfilling[11]

$$\omega(\neg\hat{P}) = 1 - \omega(\hat{P}) \quad \text{for all } \omega \in \mathcal{S} \stackrel{\text{def}}{=} \{[S] : S \in \mathcal{Q}\} \, .$$

(I_2): Let $\hat{P}_1, \hat{P}_2, \ldots \in \mathcal{L}$. Then there is an element $\hat{I} \in \mathcal{L}$ such that for all $\omega \in \mathcal{S}$

$$\omega(\hat{I}) = 1 \text{ if and only if } \omega(\hat{P}_j) = 1 \text{ for } j = 1, 2, \ldots$$

(I_3): Let $\hat{P}_1, \hat{P}_2, \ldots \in \mathcal{L}$ be such that

$$\omega(\hat{P}_j) = 1 \Longrightarrow \omega(\hat{P}_k) = 0 \quad \text{for all } \omega \in \mathcal{S} \quad \text{whenever } j < k \, .$$

Then there is an element $\hat{S} \in \mathcal{L}$ fulfilling

$$\omega(\hat{S}) = \omega(\hat{P}_1) + \omega(\hat{P}_2) + \ldots \quad \forall\, \omega \in \mathcal{S} \, .$$

Note that (I_1) defines a mapping $\neg : \mathcal{L} \longrightarrow \mathcal{L}$. Moreover, there is always a natural semi-ordering of the elements of $\mathcal{L}$ given by

$$P_1 \preccurlyeq P_2 \stackrel{\text{def}}{\Longleftrightarrow} \omega(P_1) \leq \omega(P_2) \; \forall\, \omega \in \mathcal{S} \, . \tag{1.4}$$

Theorem 1.1.1 (Structure Theorem) *If $\mathcal{L} \neq \emptyset$ and $\mathcal{S}$ fulfill conditions (I_1)—(I_3), then $(\mathcal{L}, \preccurlyeq, \neg)$, with $\preccurlyeq$ given by (1.4) and $\neg$ given by (I_1), is a **logic**, i.e. a σ-complete weakly modular lattice $(\mathcal{L}, \preccurlyeq)$. Moreover, under these conditions, every $\omega \in \mathcal{S}$ is a probability measure over $(\mathcal{L}, \preccurlyeq, \neg)$ fulfilling the **Jauch-Piron condition***

$$\left(\omega\left(\hat{P}_1\right) = 1 = \omega\left(\hat{P}_2\right) \Longrightarrow \omega\left(\hat{P}_1 \wedge \hat{P}_2\right) = 1\right) \quad \forall\, \hat{P}_1\,,\, \hat{P}_2 \in \mathcal{L} \, . \tag{1.5}$$

Proof: See (Doebner and Lücke, 1991, appendix) (see also (Maczyński, 1974) for related results). ∎

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[10] These conditions are designed to allow for classical reasoning as far as possible. Implicit in (I_3) and (I_1) is the following *standardization postulate*: For every $\hat{P} \in \mathcal{L} \setminus \{\hat{0}\}$ there exist a state $\omega \in \mathcal{S}$ with $\omega(\hat{P}) = 1$. Therefore semi-transparent windows, e.g., cannot be used for simple tests.

[11] A more general framework, allowing for nonlinear time evolution, was suggested in (Mielnik, 1974).

1.1.3 Quantum Reasoning

It seems natural to assign 'actual' properties $E_{\hat{P}}$ to the elements of $\mathcal{L}$ in the sense that:

A system in the state $\omega \in \mathcal{S}$ has property $E_{\hat{P}}$ **with certainty** if and only if[12] $\omega(P) = 1$. (1.6)

We are used giving names to these properties like 'spin up', 'positive energy' and so on. However, there is no evidence for the assumption that under all circumstances – independent of any test – the system has either property $E_{\hat{P}}$ or property $E_{\neg\hat{P}}$ – even though

$$\omega(\neg\hat{P}) = 1 - \omega(\hat{P}) \quad \forall\, \omega \in \mathcal{S}\,,\ \hat{P} \in \mathcal{L}$$

and even though tests corresponding to $\hat{P}$ and $\neg\hat{P}$ can typically be performed jointly.[13] This also becomes clear by the following lemma.

Lemma 1.1.2 (D. Pfeil) *For every finite set* $\hat{\mathcal{L}}$ *there is a* **classical** *logic* $(\mathcal{B}, \preccurlyeq_B, \neg_B)$ *and a mapping* $M : \hat{\mathcal{L}} \longrightarrow \mathcal{B}$ *for which the following holds:*

For every mapping $\omega : \hat{\mathcal{L}} \longrightarrow [0,1]$ *there is a probability measure* μ *on* $(\mathcal{B}, \preccurlyeq_B, \neg_B)$ *fulfilling*

$$\omega(\hat{P}) = \mu\Big(M(\hat{P})\Big) \quad \forall\, \hat{P} \in \hat{\mathcal{L}}$$

and

$$\hat{P}_1 \neq \hat{P}_2 \implies \mu\Big(M(\hat{P}_1) \cap M(\hat{P}_2)\Big) = \mu\Big(M(\hat{P}_1)\Big)\mu\Big(M(\hat{P}_2)\Big) \quad \forall\, \hat{P}_1\,,\, \hat{P}_2 \in \hat{\mathcal{L}}\,.$$

Proof: See (Lücke, 1996, Proof of Lemma 2.3). ∎

Now we should no longer be surprised[14] if, in orthodox quantum theory, we encounter quantum peculiarities such as[15]

$$\omega(\hat{P}) = 1 \not\Longrightarrow \Big(\omega(\hat{P} \wedge \hat{P}') = \omega(\hat{P}')\ \forall\, \hat{P}' \in \mathcal{L}\Big) \tag{1.7}$$

or[16]

$$\hat{P}_1 \wedge \hat{P}_2 = 0 \not\Longrightarrow \hat{P}_1 \preccurlyeq \neg\hat{P}_2\,. \tag{1.8}$$

[12]Recall Footnote 10.

[13]An example for the latter is given by the Stern-Gerlach experiment, where $\hat{P}$ corresponds to 'spin up' and $\neg\hat{P}$ to 'spin down'.

[14]According to Lemma 1.1.2, the (quantum logical) relations between (equivalence classes of) tests may be just a consequence of the (experimental) restrictions on the set of 'states'.

[15]See also (Szabó, 1996, Sect. 3.1).

[16]By (1.5), $\hat{P}_1 \wedge \hat{P}_2 = 0$ means that there is no preparable property guaranteeing both $E_{\hat{P}_1}$ and $E_{\hat{P}_2}$.

Nevertheless, ***simple quantum reasoning*** according to the following rules is consistent:

- Choose a **classical** sublogic $(\mathcal{L}_c, \preccurlyeq, \neg)$ of $(\mathcal{L}, \preccurlyeq, \neg)$ and forget about all the other elements of $\mathcal{L}$.
- Then imagine that every **individual** – in whatever situation – has either property $E_{\hat{P}}$ or $E_{\neg\hat{P}}$ **if** $\hat{P} \in \mathcal{L}_c$.
- For $\omega \in \mathcal{S}$, imagine that $\omega(\hat{P})$ is the relative number of individuals having property $E_{\hat{P}}$ in an ensemble corresponding to ω **if** $\hat{P} \in \mathcal{L}_c$.
- Imagine that

$$\begin{aligned}
&\preccurlyeq \text{ corresponds to common sense logical implication}\,,\\
&\neg \text{ corresponds to common sense logical negation,}\\
&\wedge \text{ corresponds to common sense logical `and'}\,,\\
&\vee \text{ corresponds to common sense logical `or'}\,.
\end{aligned}$$

This way all quantum peculiarities are avoided. For instance, in spite of (1.7), we may conclude

$$\left.\begin{array}{l} \omega(\hat{P}_1) = 1\,, \\ \hat{P}_1 \text{ compatible with } \hat{P}_2 \end{array}\right\} \Longrightarrow \omega(\hat{P}_1 \wedge \hat{P}_2) = \omega(\hat{P}_2)\ \forall\,\omega$$

or even

$$\begin{array}{l} \hat{P}_1 \text{ compatible}^{17} \text{ with } \hat{P}_2 \\ \Longrightarrow \omega(\hat{P}_1 \vee \hat{P}_2) = \omega(\hat{P}_1 \wedge \neg\hat{P}_2) + \omega(\neg\hat{P}_1 \wedge \hat{P}_2) + \omega(\hat{P}_1 \wedge \hat{P}_2)\ \forall\,\omega\,. \end{array}$$

Simple quantum reasoning naturally leads to the notion of *observable*:[18]

Definition 1.1.3 *An* ***observable*** *A of a physical system modeled by the logic $(\mathcal{L}, \preccurlyeq, \neg)$ is a σ-morphism $\hat{E}_A$ of the Borel ring on the real line*[19] *into $(\mathcal{L}, \preccurlyeq, \neg)$ which is unitary, i.e. $\hat{E}_A(\mathbb{R}) = \hat{1}$. It is called* ***bounded*** *iff $\hat{E}_A(\Delta) = 1$ for suitable compact $\Delta \in \mathbb{R}$.*

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[17]For compatible $\hat{P}_1, \hat{P}_2$:

$$\begin{aligned}
\hat{P}_1 \vee \hat{P}_2 = (\hat{P}_1 \vee \hat{P}_2) \wedge (\neg\hat{P}_2 \vee \hat{P}_2) &= (\hat{P}_1 \wedge \neg\hat{P}_2) \vee \hat{P}_2 \\
&= (\hat{P}_1 \wedge \neg\hat{P}_2) \vee (\neg\hat{P}_1 \wedge \hat{P}_2) \vee (\hat{P}_1 \wedge \hat{P}_2)
\end{aligned}$$

[18]The Borel ring on $\mathbb{R}^1$ could be replaced by an arbitrary classical logic; possibly associated with some physical dimension.

[19]The ***Borel sets*** on a locally compact space X form the smallest family of sets containing all compact subsets of X and being closed with respect to forming relative complements and countable unions.

The physical interpretation of $\hat{E}_A$ in the sense of quantum reasoning is as follows:

> Given $\omega \in \mathcal{S}$ and a Borel subset Δ of $\mathbb{R}^1$ then $\omega(\hat{E}_A(\Delta))$ can be imagined as the relative number of individuals, in an ensemble corresponding to ω, for which $A \in \Delta$.

Consequently, the expectation value[20] for A in an ensemble corresponding to ω is given by

$$\overline{A}(\omega) = \int \lambda \, \mathrm{d}\mu^A_\omega(\lambda)\,. \tag{1.9}$$

with Borel measure

$$\mu^A_\omega(B) \stackrel{\text{def}}{=} \omega\left(\hat{E}_A(B)\right) \quad \text{for Borel subsets } B \subset \mathbb{R}^1\,. \tag{1.10}$$

Simple quantum reasoning can be applied to a whole family observables $A_1, A_2, \ldots$ if and only if all the pairs

$$\left(\hat{E}_{A_j}(\Delta_j), \hat{E}_{A_k}(\Delta_k)\right)\;, \quad \Delta_j\,,\,\Delta_k \in \mathbb{R}$$

are compatible.[21]

In order to make predictions for multiple tests one has to know how states change as a result of a simple test. Here we assume[22]

> **Lüders' Postulate:** For every $\hat{P} \in \mathcal{L}$ there is at least one corresponding ***measurement of first kind***, i.e. a simple test T with $[T] = \hat{P}$ causing a transition[23] $\omega \mapsto \omega_{,\hat{P}}$ whenever the result is 'yes'. Here, if $\omega(\hat{P}) > 0$, $\omega_{,\hat{P}} \in \mathcal{S}$ is assumed to be uniquely characterized by the condition
>
> $$\omega_{,\hat{P}}(\hat{P}') = \omega(\hat{P}')/\omega(\hat{P}) \quad \forall\, \hat{P}' \preccurlyeq \hat{P}\,.$$

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[20] Of course, the expectation value may be infinite!

[21] In a logic $(\mathcal{L}, \preccurlyeq, \neg)$ the sublogic generated by $\hat{P}_1\,,\,\hat{P}_2\,,\ldots \in \mathcal{L}$ is classical if and only if all the pairs $(\hat{P}_j, \hat{P}_k)$ are compatible (Piron, 1976, §2-2).

[22] A first kind measurement corresponding to $\hat{P}$ does not destroy any of the properties $E_{\hat{P}'}$ with $\hat{P}'\,,\,\hat{P}$ compatible (Lücke, 1996). Usually, a test causes a much more drastic change of the state or even ends by absorbing the corresponding individual. A measurement of first kind, typically, would be approximately realized by means of a highly efficient filter.

[23] Naively interpreted, $\omega_{,\hat{P}}(\hat{P}')$ describes the *conditional probability* in the state ω for $E_{\hat{P}'}$ – defined by (1.6) – being true provided $E_{\hat{P}}$ is true. In ordinary quantum theory the new statistical operator $\hat{T}_{\omega_{,\hat{P}}}$ is given by $\hat{P}\hat{T}_\omega\hat{P}/\omega(\hat{P})$.

By Lüders' postulate,[24] given the initial state $\omega \in \mathcal{S}$, the probability for the ***homogeneous history*** $(\hat{P}_1, \ldots, \hat{P}_n)$ – i.e. for getting the answer 'yes' for all subsequent first kind measurements of a series corresponding to $\hat{P}_1, \ldots, \hat{P}_n \in \mathcal{L}$ – should be[25]

$$\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2)\cdots\omega_{,\hat{P}_1,\ldots,\hat{P}_{n-1}}(\hat{P}_n)\,.$$

Consistent quantum reasoning with respect to histories leads to the modern notion of ***decoherent histories***.

Given a history $(\hat{P}_1, \ldots, \hat{P}_n)$ not corresponding to a simple test, we can no longer be sure that there is an initial state for which $(\hat{P}_1, \ldots, \hat{P}_n)$ is certain, i.e., for which $\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2)\cdots\omega_{,\hat{P}_1,\ldots,\hat{P}_{n-1}}(\hat{P}_n) = 1$. Therefore the 'logic' of histories is weaker than that for simple tests and may provide a useful basis for generalizing quantum theory (Isham, 1995).

1.1.4 Symmetries and Dynamics

Just for simplicity we always use the following assumption, fulfilled in ordinary quantum theory:

$$\boxed{\mathcal{S} = \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)} \stackrel{\text{def}}{=} \text{set of \textbf{all} probability measures on } (\mathcal{L},\preccurlyeq,\neg)\,.} \tag{1.11}$$

Definition 1.1.4 *A **symmetry** of a physical system modeled*[26] *by the logic $(\mathcal{L},\preccurlyeq,\neg)$ is an automorphism of $(\mathcal{L},\preccurlyeq,\neg)$, i.e. a bijection of $\mathcal{L}$ onto itself preserving the least upper bound and the orthocomplementation. A **dynamical semi-group** for such a system is a family $\{\alpha_t\}_{t\in\mathbb{R}_+}$ of symmetries α_t fulfilling the following three conditions:*

(i) $\alpha_0(\hat{P}) = \hat{P} \quad \forall\, \hat{P} \in \mathcal{L}\,,$

(ii) $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1+t_2} \quad \forall\, t_1\,,\, t_2 \in \mathbb{R}_+\,,$

(iii) $t \longmapsto \alpha_t$ *is **weakly continuous**, i.e., for fixed $\hat{P} \in \mathcal{L}$ and $\omega \in \mathcal{S}$ the probability $\omega\left(\alpha_t(\hat{P})\right)$ is a* **continuous** *function of $t \in \mathbb{R}_+$.*

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[24] In the relativistic theory Lüders' postulate causes interesting problems (Schlieder, 1971) (see also (Mittelstaedt, 1983),(Mittelstaedt and Stachow, 1983)).

[25] Naively interpreted, $\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2)\cdots\omega_{,\hat{P}_1,\ldots,\hat{P}_{n-1}}(\hat{P}_n)$ is the probability for joint validity of the properties $E_{\hat{P}_1}, \ldots, E_{\hat{P}_1}$ in the state ω. Usually (see, e.g., (Omnès, 1994), (Griffith, 1995)), unfortunately, this is formulated in the Schrödinger picture, thus imposing unnecessary restrictions.

[26] If $\mathcal{S} \neq \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)}$ one should also require $\alpha^*(\mathcal{S}) = \mathcal{S}$ for the dual α^* of a symmetry α with respect to $\mathcal{S}$, defined by

$$(\alpha^*\omega)\,(\hat{P}) \stackrel{\text{def}}{=} \omega\left(\alpha(\hat{P})\right) \quad \forall\,\omega \in \mathcal{S}\,,\, \hat{P} \in \mathcal{L}\,.$$

Then the inverse of a symmetry need not be a symmetry.

The most important symmetries are the ***time-translations*** α_t, $t \in \mathbb{R}_+$:

> Let T be a macroscopic prescription for performing a simple test corresponding to $\hat{P} \in \mathcal{L}$. Then the prescription T_t to do everything prescribed by T just with time delay t characterizes a test corresponding to $\alpha_t(\hat{P})$.

The family $\{\alpha_t\}_{t\in\mathbb{R}_+}$ of time-translations, determining the ***dynamics*** of the system, is naturally assumed to be a dynamical semi-group,[27] **if** the system is homogeneous in time.

1.2 Orthodox Quantum Mechanics and Algebraic Formulation

1.2.1 Logic and Observables

In 'pioneer quantum mechanics' (Primas, 1981) (without superselection rules[28]) the logic $(\mathcal{L}, \preccurlyeq, \neg)$ described in Section 1.1.2 is realized as follows (***standard quantum logic***):

- $\mathcal{L}$ is given as the set of all projection operators[29] in some separable complex Hilbert space $\mathcal{H}$ of dimension ≥ 2.
- For arbitrary $\hat{P}_1, \hat{P}_2 \in \mathcal{L}$ we have

$$\begin{aligned} \hat{P}_1 \preccurlyeq \hat{P}_2 &\overset{\text{def}}{\iff} \hat{P}_1 \leq \hat{P}_2 \\ &\iff \left(\left\langle \Psi \mid \hat{P}_1\Psi\right\rangle \leq \left\langle \Psi \mid \hat{P}_1\Psi\right\rangle \ \forall\, \Psi \in \mathcal{H}\right). \end{aligned}$$

- For every $\hat{P} \in \mathcal{H}$ we have

$$\neg\hat{P} \overset{\text{def}}{=} \hat{1} - \hat{P}.$$

Then, if $\dim(\mathcal{H}) \geq 3$, Gleason's theorem (Gleason, 1957) tells us that for every $\omega \in \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)}$ there is a unique positive trace class operators[30] $\hat{T}_\omega \in \mathcal{L}(\mathcal{H})$ fulfilling

$$\omega(\hat{P}) = \mathrm{Tr}\left(\hat{T}_\omega \hat{P}\right) \quad \forall\, \hat{P} \in \mathcal{L}.$$

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[27] Note that the dual of a symmetry has always an inverse in $\mathcal{S} = \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)}$. In this sense evolution can always be extrapolated backwards in time, if (1.11) holds. Remember Footnote 7 in this connection.

[28] A system modeled by $(\mathcal{L}, \preccurlyeq, \neg)$ is said to possess ***superselection rules*** if the ***center***

$$\mathcal{C}_{(\mathcal{L},\preccurlyeq,\neg)} \overset{\text{def}}{=} \left\{\hat{P} \in \mathcal{L} : (\hat{P}, \hat{P}') \text{ compatible } \forall\, \hat{P}' \in \mathcal{L}\right\}$$

of $(\mathcal{L}, \preccurlyeq, \neg)$ is nontrivial ($\mathcal{L} \neq \mathcal{C} \neq \{\hat{0}, \hat{1}\}$). $(\mathcal{L}, \preccurlyeq, \neg)$ is called ***irreducible*** if $\mathcal{C} = \{\hat{0}, \hat{1}\}$.

[29] Their specific physical identification depends on the ***dynamics***, as discussed in (Mielnik, 1974) and (Lücke, 1995).

[30] Conversely, every trace class operator of trace 1 induces a probability measure on standard quantum logic.

Every positive trace class operator $\hat{T}$ of trace 1 can be written in the form

$$\hat{T} = \sum_{\nu=0}^{\infty} \underbrace{\lambda_\nu}_{\geq 0} \hat{P}_{\Psi_\nu} \,, \quad \sum_{\nu=0}^{\infty} \lambda_\nu = 1 \,, \; \mathcal{H} \ni \Psi_\nu \neq 0 \,\forall \nu \,,$$

Here we use the standard notation

$$\hat{P}_\Psi \Phi \stackrel{\text{def}}{=} \left\langle \frac{\Psi}{\|\Psi\|} \middle| \Phi \right\rangle \frac{\Psi}{\|\Psi\|} \quad \forall \Phi \in \mathcal{H} \,, \; \Psi \in \mathcal{H} \setminus \{0\} \,.$$

Hence

$$\omega_\Psi(\hat{P}) \stackrel{\text{def}}{=} \mathrm{Tr}\left(\hat{P}_\Psi \hat{P}\right) = \left\langle \frac{\Psi}{\|\Psi\|} \middle| \hat{P} \frac{\Psi}{\|\Psi\|} \right\rangle \quad \forall \Psi \in \mathcal{H} \setminus \{0\} \,, \; \hat{P} \in \mathcal{L} \,.$$

Now he have the following form of the **Lüders postulate:**

> For every $\hat{P} \in \mathcal{L}$ there is at least one corresponding measurement of first kind, i.e. a simple test T with $[T] = \hat{P}$ causing a transition $\omega \mapsto \omega_{,\hat{P}}$ whenever the result is 'yes', where
>
> $$\hat{T}_{\omega_{,\hat{P}}} = \frac{\hat{P}\hat{T}_\omega \hat{P}}{\mathrm{Tr}\left(\hat{P}\hat{T}_\omega \hat{P}\right)} \,.$$

Especially, if $\omega = \omega_\Psi$ for some $\Psi \in \mathcal{H} \setminus \{0\}$, a first kind measurement corresponding to $\hat{P}_\Phi$, $\Phi \in \mathcal{H} \setminus \{0\}$, causes the transition $\omega_\Psi \longrightarrow \omega_\Phi$ whenever the result is 'yes':

$$\omega_\Psi\left(\hat{P}_\Phi\right) > 0 \Longrightarrow \frac{\hat{P}_\Phi \hat{P}_\Psi \hat{P}_\Phi}{\mathrm{Tr}\left(\hat{P}_\Phi \hat{P}_\Psi \hat{P}_\Phi\right)} = \hat{P}_\Phi \,.$$

The corresponding ***transition probability*** is

$$\omega_\Psi\left(\hat{P}_\Phi\right) = \left| \left\langle \frac{\Phi}{\|\Phi\|} \middle| \frac{\Psi}{\|\Psi\|} \right\rangle \right|^2 \,.$$

Exercise 1 Prove the following:[31]

$$\hat{P}_1 \preccurlyeq \hat{P}_2 \iff \hat{P}_1\mathcal{H} \subset \hat{P}_2\mathcal{H} \,, \tag{1.12}$$
$$\hat{P}_1 \preccurlyeq \hat{P}_2 \iff \hat{P}_1 = \hat{P}_2\hat{P}_1 \,, \tag{1.13}$$

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[31] Assuming irreducibility, atomicity and (1.20) one may prove that – apart from some exceptional cases – $(\mathcal{L}, \preccurlyeq, \neg)$ is isomorphic to the logic of all projection operators on some *generalized Hilbert space* (Piron, 1976, Section 3–1). According to (1.21) the probability for the homogeneous history $(\hat{P}_1, \hat{P}_2)$ is $\omega_\Psi(\hat{P}_2\hat{P}_1\hat{P}_2)$, where $\hat{0} \leq \hat{P}_2\hat{P}_1\hat{P}_2 \leq \hat{1}$ but $\hat{P}_2\hat{P}_1\hat{P}_2 \stackrel{\text{i.g.}}{\notin} \mathcal{L}$. Note (Davies, 1976, Lemma 2.4), however.

$$\neg\hat{P}\mathcal{H} = \mathcal{H} \ominus \hat{P}\mathcal{H} \stackrel{\text{def}}{=} \{\Psi \in \mathcal{H} : \langle \Phi | \Psi \rangle = 0 \ \text{ for all } \Phi \in \hat{P}\mathcal{H}\}\,, \tag{1.14}$$

$$\hat{P}_1 \wedge \hat{P}_2 \wedge \ldots \stackrel{\text{def}}{=} \inf\{\hat{P}_1, \hat{P}_2, \ldots\} = \text{orthogonal projection onto } \bigcap_{j=1}^{\infty} \hat{P}_j\mathcal{H}\,, \tag{1.15}$$

$$\sup\{\hat{P}_1, \hat{P}_2, \ldots\} = \text{ orthogonal projection onto } \overline{\text{span}\,(\bigcup_{j=1}^{\infty} \hat{P}_j\mathcal{H})}\,, \tag{1.16}$$

$$\hat{P}_1 \wedge \hat{P}_2 = \text{ s-}\lim_{n\to\infty} (\hat{P}_1\hat{P}_2)^n \stackrel{\text{i.a.}}{\neq} \hat{P}_1\hat{P}_2\,, \tag{1.17}$$

$$\left(\hat{P}_1\,,\,\hat{P}_2\right) \text{ compatible } \iff \hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1\,, \tag{1.18}$$

$$\begin{aligned} \hat{P}_1 \text{ Atom } &\stackrel{\text{def}}{\iff} (\hat{P}_1 \neq 0 \text{ and } \hat{P} \preccurlyeq \hat{P}_1 \Longrightarrow \hat{P} \in \{0, \hat{P}_1\}) \\ &\iff \hat{P}_1\mathcal{H} \text{ 1-dimensional}\,, \end{aligned} \tag{1.19}$$

$$\hat{P}_1 \wedge \neg\hat{P}_2 = 0 \text{ and } \hat{P}_1 \text{ Atom } \stackrel{\text{“covering law”}}{\Longrightarrow} (\hat{P}_1 \vee \neg\hat{P}_2) \wedge \hat{P}_2 \text{ Atom}\,, \tag{1.20}$$

$$\omega_\Psi(\hat{P}) > 0 \Longrightarrow \frac{\hat{P}\hat{P}_\Psi\hat{P}}{\operatorname{Tr}\left(\hat{P}\hat{P}_\Psi\hat{P}\right)} = \frac{\hat{P}_{\hat{P}\Psi}}{\operatorname{Tr}\left(\hat{P}_{\hat{P}\Psi}\right)}\,. \tag{1.21}$$

Here, according to Definition 1.1.3, an observable A corresponds to a ***projection valued measure***, i.e. a mapping $\hat{E}_A$ from the ring of Borel-sets (over $\mathbb{R}^1$) into $\mathcal{L}$ such that:

(PVM$_1$) : $\hat{E}_A(\mathbb{R}^1) = \hat{1}\,,$ (***Normalization***)
(PVM$_1$) : $\hat{E}_A(\bigcup_{j=1}^{\infty} B_j) = s - \lim_{n\to\infty} \sum_{j=1}^{n} \hat{E}_A(B_j)\,,$
whenever the B_j are **mutually disjoint** Borel sets. (σ-***Additivity***)

$\hat{E}_A$ gives rise to a self-adjoint operator $\hat{A}$, uniquely characterized by[32]

$$\overline{A}(\omega_\Psi) = \left\langle \frac{\Psi}{\|\Psi\|} \middle| \hat{A}\frac{\Psi}{\|\Psi\|} \right\rangle \quad \forall\,\Psi \in D_{\hat{A}} \setminus \{0\} \tag{1.22}$$

(remember (1.9)/(1.10)), where

$$D_{\hat{A}} = \left\{ \Psi : \int_{\lambda\in\mathbb{R}^1} \lambda^2\, \omega_\Psi\left(\hat{E}_A(\mathrm{d}\lambda)\right) \right\} \tag{1.23}$$

is the domain of $\hat{A}$ (see e.g. (Achieser and Glasmann, 1965)). For this operator one usually writes

$$\hat{A} = \int_{\lambda\in\mathbb{R}^1} \lambda\, \hat{E}_A(\mathrm{d}\lambda)\,. \tag{1.24}$$

[32]This is very useful for heuristic physical identification of observables.

With this operator also have

$$\begin{array}{rcl} \overline{A}(\omega) & = & \operatorname{Tr}\left(\hat{T}_\omega \hat{A}\right) \quad \forall\, \omega \in \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)} \\ & & \textbf{if } \hat{A} \text{ is bounded} \end{array} \tag{1.25}$$

resp.

$$\begin{array}{rcl} \overline{A}(\omega) & = & \lim_{\Lambda_+\to+\infty} \operatorname{Tr}\left(\hat{T}_\omega \hat{A} \hat{E}_{\hat{A}}\left((0,\Lambda_+]\right)\right) \\ & & + \lim_{\Lambda_-\to-\infty} \operatorname{Tr}\left(\hat{T}_\omega \hat{A} \hat{E}_{\hat{A}}\left((0,\Lambda_-]\right)\right) \quad \forall\, \omega \in \mathcal{S}_{(\mathcal{L},\preccurlyeq,\neg)} \end{array}$$

if $\hat{A}$ is unbounded, where the l.h.s. is defined iff the r.h.s is.

According to the well known ***spectral theorem*** (see e.g. (Reed und Simon, 1972)), for every self-adjoint operator $\hat{A}$ there is a unique regular[33] projector-valued measure $\hat{E}_A$ fulfilling (1.22)/(1.23), called the ***spectral measure*** of $\hat{A}$. In this sense, according to Definition 1.1.3, the observables of orthodox quantum mechanics may be identified with the self-adjoint operators.[34]

Exercise 2 Determine the spectral measures for the following operators of elementary $L^2(\mathbb{R}^1, \mathrm{d}x)$-quantum mechanics:

(i) position operator,
(ii) linear momentum operator,
(iii) energy operator of the harmonic oscillator,
(iv) zero operator,
(v) identity operator.

Let $\hat{A}$ be a self-adjoint operator on the complex Hilbert space $\mathcal{H}$ with spectral measure $\hat{E}_A$ and let f be some complex-valued Borel-function[35] on $\mathbb{R}^1$. Then there is a unique operator $f(\hat{A})$, written

$$f(\hat{A}) = \int f(\lambda)\, \hat{E}_A(\mathrm{d}\lambda)\,, \tag{1.26}$$

with domain

$$D_{f(\hat{A})} \stackrel{\text{def}}{=} \left\{ \Psi \in \mathcal{H} : \int |f(\lambda)|^2\, \omega_\Psi\left(\hat{E}_A(\mathrm{d}\lambda)\right) < \infty \right\} \tag{1.27}$$

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[33] A projector-valued measure $\hat{E}_A$ is called ***regular*** if

$$\begin{array}{rcl} \hat{E}_A(B) & = & \sup\left\{\hat{E}_A(C) :\; B \supset C \text{ compact}\right\} \\ & = & \inf\left\{\hat{E}_A(\mathcal{O}) :\; B \subset \mathcal{O} \text{ open}\right\} \end{array}$$

holds for all Borel subsets B of $\mathbb{R}$ not only for $B = (-\infty, \lambda]\,,\ \lambda \in \mathbb{R}^1$.

[34] We skip physical dimensions to allow for addition of these operators.

[35] If B is a Borel subset of $\mathbb{C}$ then $f^{-1}(B)$ has to be a Borel subset of $\mathbb{R}^1$.

fulfilling

$$\left\langle \frac{\Psi}{\|\Psi\|} \middle| f(\hat{A}) \frac{\Psi}{\|\Psi\|} \right\rangle = \int f(\lambda)\, \omega_\Psi\left(\hat{E}_A(\mathrm{d}\lambda)\right) \quad \forall\, \Psi \in D_{f(\hat{A})} \setminus \{0\}\ .$$

If f is real-valued then $f(\hat{A})$ is self-adjoint on this domain and its spectral measure is characterized by

$$\hat{E}_{f(A)}(J) = \hat{E}_A\left(f^{-1}(J)\right) \qquad \text{for all intervals } J \subset \mathbb{R}\,. \tag{1.28}$$

If f is bounded then the operator $f(\hat{A})$ is bounded and

$$\left\| f(\hat{A}) \right\| \stackrel{\text{def}}{=} \sup_{\Psi \in \mathcal{H}\setminus\{0\}} \left\| \hat{A} \frac{\Psi}{\|\Psi\|} \right\| \leq \sup_{\lambda \in \mathbb{R}^1} |f(\lambda)|\ . \tag{1.29}$$

Exercise 3 Prove the following:

$$f(\lambda) = f_0 \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = f_0 \hat{1}\,, \tag{1.30}$$

$$f(\lambda) = g(\lambda) + h(\lambda) \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = \overline{g(\hat{A}) + h(\hat{A})}\,, \tag{1.31}$$

$$f(\lambda) = g(\lambda) h(\lambda) \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = \overline{g(\hat{A}) h(\hat{A})}\,, \tag{1.32}$$

$$g\left(f(\hat{A})\right) = h(\hat{A}) \ \textbf{ if } f \text{ is real-valued and } g\left(f(\hat{\lambda})\right) = h(\hat{\lambda}) \text{ for all } \lambda \in \mathbb{R}^1\,, \tag{1.33}$$

$$|f(\lambda)| = 1 \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) \text{ unitary}\,, \tag{1.34}$$

$$\hat{E}_A(B) = \chi_B(\hat{A}) \quad \text{for all Borel sets } B\,. \tag{1.35}$$

1.2.2 Symmetries and Dynamics

If $\mathcal{H}$ is a complex Hilbert space, let us denote by $(\mathcal{L}_\mathcal{H}, \preccurlyeq, \neg)$ the standard quantum logic described in 1.2.1.

Theorem 1.2.1 (Wigner) *Let $\mathcal{H}$ a be a complex Hilbert space of dimension ≥ 3. Then a map $\alpha : \mathcal{L}_\mathcal{H} \longrightarrow \mathcal{L}_\mathcal{H}$ is a symmetry of $(\mathcal{L}_\mathcal{H}, \preccurlyeq, \neg)$ iff there is either a unitary or an anti-unitary operator $\hat{U}$ with*[36]

$$\alpha(\hat{P}) = \hat{U} \hat{P} \hat{U}^* \quad \forall\, \hat{P} \in \mathcal{L}_\mathcal{H}\,.$$

Proof: See (Piron, 1976, §3–2). ∎

For the elements of a dynamical semi-group of $(\mathcal{L}_\mathcal{H}, \preccurlyeq, \neg)$ the choice of anti-unitary $\hat{U}$ is excluded:

[36]Especially, we have $\alpha(\hat{P}_\Phi) = \hat{P}_{\hat{U}\Phi}$.

Theorem 1.2.2 *Let $\mathcal{H}$ a be a complex Hilbert space of dimension ≥ 3 and let $\{\alpha_t\}_{t\in\mathbb{R}_+}$ be a dynamical semi-group of $(\mathcal{L}_\mathcal{H}, \preccurlyeq, \neg)$. Then there is a unique self-adjoint operator $\hat{H}$ fulfilling*

$$\alpha_t(\hat{P}) = e^{\frac{i}{\hbar}\hat{H}t}\hat{P}e^{-\frac{i}{\hbar}\hat{H}t} \quad \forall \hat{P} \in \mathcal{L}_\mathcal{H}\,,\, t \in \mathbb{R}_+\,.$$

Proof: See (Lücke, 1996, Sect. 3.3). ∎

1.2.3 Algebras of Bounded Observables

Unbounded observables $\hat{A}$ have the unpleasant feature that their domain is always smaller than $\mathcal{H}$, by the Hellinger-Toeplitz theorem (see (Reed und Simon, 1972, corollary to the Closed Graph Theorem II.12).) This causes lots of technical complications. Fortunately, from the principal point of view, it is sufficient to know the spectral operators $\hat{E}_A(J)$, which are always bounded, for all intervals $J \subset \mathbb{R}$. This allows for taking advantage of the powerful mathematical theory of algebras of bounded operators.

The set $\mathcal{L}(\mathcal{H})$ of **all** bounded operators $\hat{A}$ in $\mathcal{H}$ (with $D_{\hat{A}} = \mathcal{H}$) is a complex ***Banach algebra***, i.e. a complex Banach space[37] with associative[38] and distributive multiplication[39] fulfilling the so-called ***product inequality***

$$\|\hat{A}\hat{B}\| \leq \|\hat{A}\|\,\|\hat{B}\|\,.$$

Transition to the adjoint operator is a ***involution***, i.e. a mapping $\hat{A} \to \hat{A}^*$ fulfilling the following three conditions:

(I_1): $\left(\hat{A}^*\right)^* = \hat{A}$

(I_2): $(\hat{A}\hat{B})^* = \hat{B}^*\hat{A}^*$

(I_3): $(\alpha\hat{A} + \beta\hat{B})^* = \overline{\alpha}\hat{A}^* + \overline{\beta}\hat{B}^*$

$\mathcal{L}(\mathcal{H})$ is also a C^****-algebra***[40], i.e. a complex Banach algebra with involution *, obeying the condition

$$\|\hat{A}^*\hat{A}\| = \|\hat{A}\|^2$$

Exercise 4 Proof that $\|\hat{A}^*\| = \|\hat{A}\|$, hence also $\|\hat{A}\hat{A}^*\| = \|\hat{A}\|^2$, holds for every C^*-algebra.

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[37]In this case the set of all bounded operators with ordinary addition, multiplication by complex numbers, and operator norm.

[38]This means: $\hat{A}\left(\hat{B}\hat{C}\right) = \left(\hat{A}\hat{B}\right)\hat{C}$ for all elements $\hat{A}$, $\hat{B}$, $\hat{C}$ of the Algebra, and $(\alpha\beta)(\hat{A}\hat{B}) = (\alpha\hat{A})(\beta\hat{B})$ for all complex Numbers α, β and all elements $\hat{A}$, $\hat{B}$, $\hat{C}$ of the Algebra.

[39]In this case ordinary multiplication with operators and/or numbers.

[40]Called a *completely regular algebra* by Neumark (Neumark, 1959).

$\mathcal{L}(\mathcal{H})$ is even a ***von Neumann algebra***, i.e. a subalgebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$, $\mathcal{H}$ some complex Hilbert space, that is given by the ***commutant***

$$\mathcal{N}' \stackrel{\text{def}}{=} \left\{ \hat{A} \in \mathcal{L}(\mathcal{H}) : \, [\hat{A}, \hat{B}]_- = 0 \, \forall \, \hat{B} \in \mathcal{N} \right\}$$

of some $*$-invariant subset $\mathcal{N} \subset \mathcal{L}(\mathcal{H})$:

$$\mathcal{M} = (\mathcal{N} \cup \mathcal{N}^*)' \qquad \text{(and hence } \mathcal{M} = \mathcal{M}'') \, .$$

Exercise 5 Let $\mathcal{B}$ be a set and r a binary relation on $\mathcal{B}$. Show that

$$\mathcal{A}_1 \subset \mathcal{A}_2 \Longrightarrow \mathcal{A}_1^r \supset \mathcal{A}_2^r$$

holds for all subsets $\mathcal{A}_1$, $\mathcal{A}_2$ of $\mathcal{B}$, where

$$\mathcal{A}^r \stackrel{\text{def}}{=} \left\{ \hat{B} \in \mathcal{B} : \, r(\hat{B}, \hat{A}) \, \forall \, \hat{A} \in \mathcal{A} \right\} \quad \forall \mathcal{A} \subset \mathcal{B} \, .$$

Moreover, show for **symmetric** r that $\mathcal{A} \subset \mathcal{A}^{rr}$ and therefore also

$$\mathcal{A}^{rrr} = \mathcal{A}^r$$

holds for all $\mathcal{A} \subset \mathcal{B}$.

Another important Banach algebra with involution is the set $\mathcal{T}_1 \subset \mathcal{L}(\mathcal{H})$ of trace class operators with the norm

$$\|\hat{A}\|_{\text{Tr}} \stackrel{\text{def}}{=} \text{Tr}\sqrt{\hat{A}^* \hat{A}} \geq \|\hat{A}\| \, .$$

Identifying $\hat{A} \in \mathcal{L}(\mathcal{H})$ with the mapping

$$\hat{T} \to \text{Tr}\,(\hat{T}\hat{A})$$

we get

$$\mathcal{L}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})^* \stackrel{\text{def}}{=} \{ \text{ linear continuous mappings } \mathcal{T}_1(\mathcal{H}) \to \mathbf{C} \} \qquad (1.36)$$

(see (Bratteli and Robinson, 1979, Proposition 2.4.3)). Similarly,[41] we have¡

$$\mathcal{T}_1(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^* \qquad (1.37)$$

((Gaal, 1973, pp 100/101)) by this identification, where $\mathcal{C}_1(\mathcal{H})$ denotes the C^*-subalgebra[42] of $\mathcal{L}(\mathcal{H})$ consisting of all compact (=completely continuous) operators[43] in $\mathcal{H}$.

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[41]Note that (1.36) resp. (1.37) is analogous to $l^\infty = (l^1)^*$, resp. $l^1 = (c_0)^*$, where: $l^\infty \stackrel{\text{def}}{=} \{\text{bounded sequences}\}$, $c_0 \stackrel{\text{def}}{=} \{\text{null sequences}\}$, $l^1 \stackrel{\text{def}}{=} \{\text{absolutely convergent sequences}\}$

[42]See (Gaal, 1973, Theorem 3 and Proposition 4 of Section II.2)

[43]Compact operators are uniform limits of increasing countable sequences of finite rank operators (Gaal, 1973, Theorem 7 of Section II.2).

1.2.4 State Functionals

According to Gleason's theorem, if $\mathcal{H}$ is separable and of dimension > 2, every probability measure on the standard quantum logic is the restriction (to the projection operators) of a (unique) mapping $\omega : \mathcal{L}(\mathcal{H}) \longrightarrow \mathbb{C}$ of the form

$$\omega(\hat{A}) = \mathrm{Tr}(\hat{T}_\omega \hat{A}),\ \hat{T}_\omega \subset \mathcal{T}_1(\mathcal{H}),\ \mathrm{Tr}(\hat{T}_\omega) = 1,\ \hat{T}_\omega = \hat{T}_\omega^* \geq 0\,. \tag{1.38}$$

Exercise 6

(i) Given $\hat{P}_1, \ldots, \hat{P}_n \in \mathcal{L}_\mathcal{H}$ and $\Psi \in \mathcal{H} \setminus \{0\}$, show that the probability for the homogeneous history $\left(\hat{P}_1, \ldots, \hat{P}_n\right)$ in a state prepared according to ω_Ψ is

$$\frac{\left\|\hat{P}_n \cdots \hat{P}_1 \Psi\right\|^2}{\|\Psi\|^2} = \omega_\Psi \left(\hat{P}_1 \cdots \hat{P}_n \hat{P}_{n-1} \cdots \hat{P}_1\right)$$

and that

$$\hat{0} \leq \hat{P}_n \cdots \hat{P}_1 \hat{P}_{n-1} \cdots \hat{P}_1 \leq \hat{1}$$

even though $\hat{P}_n \cdots \hat{P}_1 \hat{P}_{n-1} \cdots \hat{P}_1 \overset{\text{i.g.}}{\notin} \mathcal{L}_\mathcal{H}$.

(ii) Show that, contrary to $\mathcal{L}_\mathcal{H}$, the set $\left\{\hat{F} \in \mathcal{L}(\mathcal{H}) : \hat{0} \leq \hat{F} \leq \hat{1}\right\}$ of all ***effects*** with its natural semi-ordering and 'orthocomplementation' is **not** a logic.[44]

The modern notion of state is as follows:[45] A ***state*** on a C^*-algebra $\mathcal{A}$ with unit is a mapping $\hat{A} \to \omega(\hat{A})$ of $\mathcal{A}$ into the **complex** numbers fulfilling the following three conditions for all $\hat{A}, \hat{B} \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

$$\begin{array}{lll} (S_1): & \omega(\hat{A} + \alpha\hat{B}) = \omega(\hat{A}) + \alpha\omega(\hat{B}) & (\textbf{\textit{linearity}}) \\ (S_2): & \omega(\hat{1}) = 1 & (\textbf{\textit{normalization}}) \\ (S_3): & \omega(\hat{A}^*\hat{A}) \geq 0 & (\textbf{\textit{positivity}}^{46}) \end{array}$$

Exercise 7 Show that the following three conditions are fulfilled for every state ω on a C^*-algebra with unit element:

$$\begin{array}{lll} (i) & |\omega(\hat{A}^*\hat{B})|^2 \leq \omega(\hat{A}^*\hat{A})\omega(\hat{B}^*\hat{B}) & (\textbf{\textit{Cauchy Schwarz inequality}}) \\ (ii) & \omega(\hat{A}^*) = \overline{\omega(\hat{A})} & (\textbf{\textit{hermiticity}}) \\ (iii) & \omega(\hat{A}_n) \to \omega(\hat{A}) \ \text{ if } \|\hat{A} - \hat{A}_n\| \to 0 & (\textbf{\textit{continuity}}) \end{array}$$

[44]**Hint:** Show that an effect $\hat{F}$ is a projection if and only if $\hat{0} = \inf_{\{\text{effects}\}} \left\{\hat{F}, (\hat{1} - \hat{F})\right\}$.

[45]This way explicit use of the Hilbert space becomes unnecessary, in principle (see Sect. 1.3.3 for details).

[46]It would be quite tedious to show in general for C^*-algebras that $\hat{A}^*\hat{A} = -\hat{B}^*\hat{B} \Longrightarrow \hat{A}^*\hat{A} = 0$.

However, due to $\mathcal{T}_1(\mathcal{H}) = \mathcal{C}(\mathcal{H})^*$, there are also so-called *singular* states,[47] for which $\omega(\hat{P}) = 0$ whenever $\hat{P}$ has finite rank. Of course, the restrictions of such functionals do not define probability measures on the standard quantum logic since they cannot be σ-additive. Therefore the state functionals need an additional characterization which relies on the following.

Lemma 1.2.3 *Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$, let I be some ordered index set, and let $\left\{\hat{A}_i\right\}_{i\in I} \subset \mathcal{M}$ be an increasing net of positive operators with $\sup_{\mathbb{R}}\left\{\left\|\hat{A}_i\right\| : i \in I\right\} < \infty$. Then $\sup_{\mathcal{L}(\mathcal{H})}\left\{\hat{A}_i : i \in I\right\}$ exists and is an element of the algebra $\mathcal{M}$.*

Proof: See (Bratteli and Robinson, 1979, Lemma 2.4.19). ▮

Definition 1.2.4 *A state ω on a von Neumann algebra $\mathcal{M}$ is said to be **normal**, iff*

$$\omega\left(\sup_{\mathcal{L}(\mathcal{H})}\left\{\hat{A}_i : i \in I\right\}\right) = \sup_{\mathbb{R}}\left\{\omega(\hat{A}_i) : i \in I\right\}$$

holds for every net fulfilling the requirements of Lemma 1.2.3.

Theorem 1.2.5 *Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$. Then the normal states ω on $\mathcal{M}$ are exactly those of the form (1.38).*

Proof: See (Bratteli and Robinson, 1979, Theorem 2.4.21). ▮

Exercise 8 Prove Theorem 1.2.5 for $\mathcal{M} = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is separable.[48]

According to Theorem 1.2.5 (and Gleason's theorem) the probability measures on the standard quantum logic correspond to restrictions of **normal** states on $\mathcal{L}(\mathcal{H})$ to the projection operators, if $dim(\mathcal{H}) > 2$.

Definition 1.2.6 *A state ω on a C^*-algebra $\mathcal{A}$ is said to be **mixed**, iff there are states $\omega_1 \neq \omega_2$ on $\mathcal{A}$ and a real number $\lambda \in (0,1)$ fulfilling*

$$\omega(\hat{A}) = \lambda\omega_1(\hat{A}) + (1-\lambda)\omega_2(\hat{A}) \qquad \text{for every } \hat{A} \in \mathcal{A}\,.$$

*Otherwise ω is said to be **pure**.*

[47]In general, a state ω on von Neumann algebra is called ***singular***, iff for every nonzero projection operator $\hat{P}$ there is another nonzero projection operator $\hat{P}'$ for which $\omega(\hat{P}') = 0$ and $\hat{P}' \preccurlyeq \hat{P}$.

[48]Hint: If $\left\{\hat{A}_i\right\}_{i\in I}$ is an increasing bounded net then so is $\left\{\sqrt{\hat{T}}\hat{A}_i\sqrt{\hat{T}}\right\}_{i\in I}$.

Exercise 9

(i) Show that the states ω_1, ω_2 of Definition 1.2.6 must both be **normal**, if ω is normal.

(ii) Prove that a normal state ω on $\mathcal{L}(\mathcal{H})$ is pure iff there is a normed vector $\Omega \in \mathcal{H}$ fulfilling

$$\omega(\hat{A}) = \langle\Omega|\,\hat{A}\Omega\big\rangle = \mathrm{Tr}(\hat{P}_\Omega \hat{A}) \quad \text{for all } \hat{A} \in \mathcal{L}(\mathcal{H})\,.$$

Exercise 10 Prove Theorem 1.3.6 for the case that $\pi(\mathcal{A}_1)$ is known to be a C^*-subalgebra of $\mathcal{A}_2$.

1.3 Algebraic Formulation of General Quantum Theory[49]

1.3.1 Partial States

If one is interested only in a certain subset[50] $\mathcal{A}$ of physical entities A, it is sufficient to know the ***partial state*** $\omega = \omega_{\text{total}}/\!\!\!\wedge\mathcal{M}$ of the proper normal state ω_{total} on $\mathcal{L}(\mathcal{H}_{\text{total}})$ with respect to the smallest von Neumann subalgebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H}_{\text{total}})$ that contains all the projection operators $\hat{E}_A\left((-\infty,\lambda]\right)$ with $A \in \mathcal{A},\ \lambda \in \mathbb{R}^1$.

An immediate consequence of Theorem 1.2.5 is the following.

Corollary 1.3.1 *Let $\mathcal{M}_2$ be a von Neumann subalgebra of the von Neumann algebra $\mathcal{M}_1$. Then the set of all normal states on $\mathcal{M}_2$ coincides with the set of all partial states of normal states on $\mathcal{M}_1$.*

We conclude that in orthodox quantum theory the relevant states are the **normal** states on the von Neumann subalgebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H}_{\text{total}})$ of interest.

Exercise 11 Show that even if ω_{total} is pure the partial state $\omega = \omega_{\text{total}}/\mathcal{M}$ may be mixed.

From now on we consider only quantum logics $(\mathcal{L}, \preccurlyeq, \neg)$ of the following type:[51]

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[49] See (Roberts and Roepstorff, 1969).

[50] A typical case would be the description of extended structures via their centers of mass.

[51] That the projection operators of a von Neumann algebra always form a sublogic $(\mathcal{L}_\mathcal{M}, \preccurlyeq, \neg)$ of the corresponding standard quantum logic follows from (1.17) (note $\hat{P}_1 \vee \hat{P}_2 = \neg\left(\neg\hat{P}_1 \wedge \neg\hat{P}_2\right)$) and Lemma 1.2.3. Necessary and sufficient conditions for a given quantum logic to be isomorphic to a sublogic of standard quantum logic are given in (Gudder, 1979).

There is a separable complex Hilbert space $\mathcal{H}$ and a **von Neumann subalgebra** $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ by which $(\mathcal{L}, \preccurlyeq, \neg)$ is realized in the following way:

- $\mathcal{L} = \mathcal{L}_{\mathcal{M}} \stackrel{\text{def}}{=} \left\{ \hat{P} \in \mathcal{M} : \hat{P}^* = \hat{P} = \hat{P}^2 \right\}$,
- $\hat{P}_1 \preccurlyeq \hat{P}_2 \stackrel{\text{def}}{\iff} \hat{P}_1 \le \hat{P}_2 \quad \forall \hat{P}_1, \hat{P}_2 \in \mathcal{L}$,
- $\neg \hat{P} \stackrel{\text{def}}{=} \hat{1} - \hat{P} \quad \forall \hat{P} \in \mathcal{L}$.

Exercise 12 For bounded self-adjoint operators it is known that they commute (in the naive sense) if and only if all their spectral projections commute[52] (Neumark, 1959, Theorem VII of §17.4) (or (Riesz and Sz.-Nagy, 1982, Theorem on Page 335)). Use this to show that

$$\hat{A} = \hat{A}^* \in \mathcal{M} \Longrightarrow \hat{E}_A(J) \in \mathcal{M}$$

holds for all $\hat{A} \in \mathcal{L}(\mathcal{H})$ and all intervals $J \subset \mathbb{R}$.

Theorem 1.3.2 (Generalized Gleason Theorem) *Let $\mathcal{M}$ be a von Neumann algebra with no type I_2 summand.*[53] *Every finitely additive probability measure ω on $\mathcal{L}_{\mathcal{M}}$ can be extended to a state on $\mathcal{M}$. This state is normal if and only if the corresponding probability measure is completely additive.*

Proof: See (Maeda, 1989). ∎

We conclude[54] that the physically relevant states will **always** be the **normal** states on the corresponding von Neumann algebra $\mathcal{M}$.

Definition 1.3.3 *A* $*$-***morphism*** *of a C^*-algebra $\mathcal{A}_1$ into a C^*-algebra $\mathcal{A}_2$ is a mapping γ of $\mathcal{A}_1$ into $\mathcal{A}_2$ respecting linearity, multiplication, and involution:*

$$\begin{aligned} (M_1): &\quad \gamma(\hat{A} + \beta\hat{B}) = \gamma(\hat{A}) + \beta\gamma(\hat{B}) \\ (M_2): &\quad \gamma(\hat{A}\hat{B}) = \gamma(\hat{A})\gamma(\hat{B}) \\ (M_3): &\quad \gamma(\hat{A}^*) = \gamma(\hat{A})^* \end{aligned}$$

A $*$-***morphism*** *of a C^*-algebra $\mathcal{A}_1$ into a C^*-algebra $\mathcal{A}_2$ is called a* $*$-***isomorphism*** *if it is a bijection (one-one and onto). A* $*$-***automorphism*** *of a C^*-algebra $\mathcal{A}$ is a $*$-isomorphism of $\mathcal{A}$ onto itself.*

[52] For unbounded self-adjoint operators the latter is the appropriate definition for commutativity (Reed und Simon, 1972, Sect. VIII.5).

[53] The latter means, if $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$, that there is no projection $\hat{P} \in \mathcal{L}_{\mathcal{H}}$ onto a 2-dimensional subspace of $\mathcal{H}$ for which $\mathcal{M} = \hat{P}\mathcal{L}(\mathcal{H})\hat{P} + (\hat{1} - \hat{P})\mathcal{M}(\hat{1} - \hat{P})$ (see (Neumark, 1959, §38 Nr. 3 Theorem 2)).

[54] Remember our assumption (1.11).

Now Theorem 1.3.2 has the following consequence (remember Definition 1.1.4):

Corollary 1.3.4 *Let $\mathcal{M}$ be a von Neumann algebra with no type I_2 summand and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a dynamical semi-group for some system modeled by $(\mathcal{L}_{\mathcal{M}}, \preccurlyeq, \neg)$. Then $\{\alpha_t\}_{t\geq 0}$ is the restriction to $\mathcal{L}_{\mathcal{M}}$ of a weakly* continuous*[55] *1-parameter semi-group of $*$-automorphisms of $\mathcal{M}$.*

Proof: See (Lücke, 1996, Appendix A). ∎

In the Haag-Doplicher-Roberts theory (see (Haag, 1992)) the relevant Neumann algebra $\mathcal{M}$ is to be constructed by weak closure of a suitable representation of some C^*-algebra. Therefore we have to discuss the latter concept.

Definition 1.3.5 *A **representation of a C^*-algebra** $\mathcal{A}$ in the complex Hilbert space $\mathcal{H}$ is a $*$-morphism π of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$. The representation is said to be **faithful** iff π is an injection.*

Theorem 1.3.6 *Let γ be a $*$-morphism of the C^*-algebra $\mathcal{A}_1$ into the C^*-algebra $\mathcal{A}_2$. Then $\gamma(\mathcal{A}_1)$ is a C^*-subalgebra of $\mathcal{A}_2$ and:*[56]

$$\left\|\gamma(\hat{A})\right\|_{\mathcal{A}_2} \leq \left\|\hat{A}\right\|_{\mathcal{A}_1} \quad \forall\, \hat{A} \in \mathcal{A}_1 .$$

Proof: See (Bratteli and Robinson, 1979, Lemma 2.3.1) ((Dixmier, 1969, sections 1.3.7 and 1.8.3)). ∎

An immediate consequence of Theorem 1.3.6 is the following

Corollary 1.3.7 *Let π be a representation of the C^*-algebra $\mathcal{A}$ in $\mathcal{H}$. Then $\pi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. If the representation is faithful, then:*

$$\left\|\pi(\hat{\mathcal{A}})\right\|_{\mathcal{L}(\mathcal{H})} = \left\|\hat{A}\right\|_{\mathcal{A}} \quad \forall\, \hat{A} \in \mathcal{A} .$$

Warning: Even if $\mathcal{A}$ is a von Neumann algebra and π is **faithful** it may happen[57] that $\sup_{\mathcal{L}(\mathcal{H})}\left\{\pi(\hat{A}_i) : i \in I\right\} \notin \pi(\mathcal{A})$ for some net $\{\hat{A}^i\}_{i\in I} \subset \mathcal{A}$ of the type considered in Lemma 1.2.3. Then $\pi(\mathcal{A})$ is not a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ and π will not map $(\mathcal{L}_{\mathcal{A}}, \preccurlyeq, \neg)$ onto a sublogic of $(\mathcal{L}, \preccurlyeq, \neg)$.

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[55] $\{\alpha_t\}_{t\geq 0}$ is ***weakly* continuous*** iff $\omega\left(\alpha_t(\hat{A})\right)$ is continuous in t for all normal states ω and all $\hat{A} \in \mathcal{M}$ (Bratteli and Robinson, 1979, Propositon 2.4.3).

[56] Thus application to the special case π =identity shows that the norm of a C^*-algebra is uniquely fixed by the algebraic structure.

1.3.2 GNS-Representation

Theorem 1.3.8 *Let $\mathcal{A}$ be a C^*-algebra with $\hat{1}$ and let ω be a state on $\mathcal{A}$. Then the set of equivalence classes*

$$[\hat{A}_1] \stackrel{\text{def}}{=} \{\hat{A}_2 \in \mathcal{A},\ \omega\left((\hat{A}_1 - \hat{A}_2)^*(\hat{A}_1 - \hat{A}_2)\right) = 0\}\ ,\ \hat{A}_1 \in \mathcal{A}$$

with linear structure

$$\alpha[\hat{A}] + \beta[\hat{B}] \stackrel{\text{def}}{=} [\alpha\hat{A} + \beta\hat{B}]$$

and inner product

$$\left\langle [\hat{A}] \middle| \, [\hat{B}] \right\rangle \stackrel{\text{def}}{=} \omega(\hat{A}^*\hat{B})$$

is a complex pre-Hilbert space. Moreover, continuous extension of the operators

$$\pi_\omega(\hat{A})[\hat{B}] \stackrel{\text{def}}{=} [\hat{A}\hat{B}]$$

onto the completion $\mathcal{H}_\omega$ of this pre-Hilbert space[58] *yields a representation π_ω of $\mathcal{A}$ in $\mathcal{H}_\omega$, the so-called* ***GNS-representation*** *of $\mathcal{A}$ given by ω. With $\Omega_\omega \stackrel{\text{def}}{=} [\hat{1}]$ we have*

$$\omega(\hat{A}) = \langle \Omega_\omega | \, \pi_\omega(\hat{A})\Omega_\omega \Big\rangle \qquad \forall\, \hat{A} \in \mathcal{A}$$

and the vector Ω_ω is ***cyclic*** *with respect to $\pi_\omega(\mathcal{A})$; i.e. $\overline{\pi_\omega(\mathcal{A})\Omega_\omega} = \mathcal{H}_\omega$.*

Exercise 13 Prove Theorem 1.3.8.[59]

Theorem 1.3.9 *Let ω be a* **normal** *state on the von Neumann algebra $\mathcal{M}$. Then the GNS-representation π_ω is* ***normal****, i.e.*

$$\sup_{\mathcal{L}(\mathcal{H})} \left\{\pi_\omega(\hat{A}_i) : i \in I\right\} \;=\; \pi_\omega\left(\sup_{\mathcal{M}}\left\{\hat{A}_i : i \in I\right\}\right)$$

holds[60] *for every increasing uniformly bounded net $\{\hat{A}_i\}_{i\in I} \subset \mathcal{M}$. Moreover, $\pi_\omega(\mathcal{M})$ is a von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_\omega)$.*

Proof: See (Bratteli and Robinson, 1979, Theorem 2.4.24). ■

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[57]Just consider the GNS representation π_ω (Theorem 1.3.8) for the mixture ω of a separating state ω on $\mathcal{L}(\hat{\mathcal{H}})$ (see (Bratteli and Robinson, 1979, Prop. 2.5.6)) with a non-normal state on $\mathcal{L}(\hat{\mathcal{H}})$ (recall Statement (i) of Exercise 9).

[58]For general *-algebras this extension should be skipped; see also (Antoine and Ôta, 1989).

[59]**Hint:** First of all show, using Schwarz's inequality (statement (iii) of Exercise 7), that [0] is a ***left ideal*** of $\mathcal{A}$, i.e. that [0] is a linear subspace of $\mathcal{A}$ and that $\hat{A}[0] \subset [0] \quad \forall \hat{A} \in \mathcal{A}$ (see (Bratteli and Robinson, 1979, Sect. 2.2.3)).

[60]This means, if ω' is a normal state on $\pi_\omega(\mathcal{M})$ then $\hat{\omega}(\hat{A}) \stackrel{\text{def}}{=} \omega'(\pi_\omega(\hat{A})$ is a normal state on $\mathcal{M}$.

Definition 1.3.10 *Let $\mathcal{A}$ be a C^*-algebra and let π_1, π_2 be representations of $\mathcal{A}$ in $\mathcal{H}_1$ resp. $\mathcal{H}_2$. Then π_2 is said to be* ***unitarily equivalent*** *to π_1 iff there is a unitary mapping $\hat{U}$ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ fulfilling*

$$\hat{U}\pi_1(\hat{A}) = \pi_2(\hat{A})\hat{U} \quad \forall \hat{A} \in \mathcal{A}\,.$$

Corollary 1.3.11 *Let ω be a state on the C^*-algebra $\mathcal{A}$ with $\hat{1}$, let π be a representation of $\mathcal{A}$ in $\mathcal{H}$, and let Ω be a cyclic Vector (with respect to $\pi(\mathcal{A})$) fulfilling*

$$\omega(\hat{A}) = \langle \Omega |\, \hat{A}\Omega \rangle \quad \forall \hat{A} \in \mathcal{A}$$

Then, according to Theorem 1.3.8, π is unitarily equivalent to the GNS-representation π_ω.

Exercise 14 Prove Corollary 1.3.11 and show[61] that – contrary to what Bratteli an Robinson claim (Bratteli and Robinson, 1979, beginning of Section 2.4.4) – equality of of the sets of vector states belonging to the representations π_1, π_2 of $\mathcal{A}$ does **not** imply unitary equivalence of π_1 and π_2, in general.[62]

Definition 1.3.12 *Let π be a representation in $\mathcal{H}$ of the C^*-algebra $\mathcal{A}$. Then π is said to be (topologically)* ***irreducible*** *iff $\mathcal{H}$ and $\{0\}$ are the only closed subspaces of $\mathcal{H}$ that are mapped into themselves by all $\hat{A} \in \mathcal{A}$.*[63] *Otherwise π is said to be* ***reducible****.*

Exercise 15 Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces and let ω be a pure normal state on the von Neumann subalgebra

$$\mathcal{M} \stackrel{\text{def}}{=} \left\{ \hat{A}_1 \otimes \hat{1} \quad \hat{A}_1 \in \mathcal{L}(\mathcal{H}_1) \right\}$$

of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Show that the GNS-representation π_ω of $\mathcal{M}$ is irreducible and unitarily equivalent to the representation

$$\pi(\hat{A}_1 \otimes \hat{1}) \stackrel{\text{def}}{=} \hat{A}_1 \quad \forall \hat{A}_1 \in \mathcal{L}(\mathcal{H}_1)$$

of $\mathcal{M}$ in $\mathcal{H}_1$.

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[61] Inspect the direct sums $\pi_1 \stackrel{\text{def}}{=} \oplus_{t \in \mathbf{R}^1} \pi$ and $\pi_2 \stackrel{\text{def}}{=} \oplus_{n=1}^{\infty} \pi$ of one and the same representation π.

[62] In case both π_1 and π_2 are irreducible Bratteli's and Robinson's statement is a consequence of Corollary 1.3.11 and Theorem 1.3.13, below.

[63] That cyclicity does **not** imply irreducibility, in general, can be clearly seen by inspecting the algebra of all functions of the position observable in the Schrödinger representation (compare Section 1.3.3).

Theorem 1.3.13 *Let ω be a state on the C^*-algebra $\mathcal{A}$ with $\hat{1}$ and let π_ω be the corresponding GNS-representation of $\mathcal{A}$ in $\mathcal{H}_\omega$. Then the following four statements are equivalent:*

1. *π_ω is irreducible.*
2. *ω is pure.*
3. **Every** *$\Psi \in \mathcal{H}_\omega \setminus \{0\}$ is cyclic with respect to $\pi_\omega(\mathcal{A})$.*
4. *$(\pi_\omega(\mathcal{A}))' = \left\{\alpha\hat{1} : \alpha \in \mathbb{C}\right\}$.*

Proof: See (Bratteli and Robinson, 1979, Prop. 2.3.8 and Theorem 2.3.19.) ∎

Exercise 16 Let $\mathcal{A}$ be a C^*-algebra with $\hat{1}$. Prove the following three statements:[64]

(i) $\mathcal{A}$ is the linear span of all its unitary[65] elements $\hat{U}$.

(ii) Every representation of $\mathcal{A}$ is unitarily equivalent to a suitable direct sum of either cyclic or trivial representations of $\mathcal{A}$.

(iii) A cyclic representation π of $\mathcal{A}$ is unitarily equivalent to a suitable GNS-representation iff it is nontrivial, i.e. iff $\pi(\hat{A}) \neq 0$ for at least one $\hat{A} \in \mathcal{A}$.

Concluding remark: Let $\mathcal{A}$ be a C^*-algebra (with $\hat{1}$). Then one may prove (Bratteli and Robinson, 1979, Lemma 2.3.23) that for every $\hat{A} \in \mathcal{A} \setminus \{0\}$ there is a state ω on $\mathcal{A}$ for which $\omega(\hat{A}) \neq 0$. Hence, if $E_\mathcal{A}$ denotes the set of all states on $\mathcal{A}$,

$$\pi = \bigoplus_{\omega \in E_\mathcal{A}} \pi_\omega$$

is a **faithful** representation of $\mathcal{A}$. This shows that every C^*-algebra is *-isomorphic to a suitable C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, for suitable $\mathcal{H}$!

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[64] **Hint for (i):** First show that the formal Taylor expansion of $\hat{B} \pm i\sqrt{\hat{1} - \hat{B}^2}$ with respect to the 'variable' $\hat{B}$ converges in $\mathcal{A}$ for self-adjoint $\hat{B} \in \mathcal{A}$ with $\|\hat{B}\| < 1$ and the limit is unitary.
Hint for (ii): Use (i) and Zorn's lemma.

[65] Here, unitarity of $\hat{U}$ means $\hat{U}^* = \hat{U}^{-1}$.

1.3.3 Canonical Quantization

In elementary quantum mechanics of n '1-dimensional' distinguishable particles without inner degrees of freedom and without (further) constraints one uses the state space

$$\mathcal{H}_n \stackrel{\text{def}}{=} L^2(\mathbb{R}^n, dx_1, \ldots, dx_n)$$

and the time zero position operators

$$(\hat{x}_\nu \Psi)(x_1, \ldots, x_n) \stackrel{\text{def}}{=} x_\nu \Psi(x_1, \ldots, x_n) \tag{1.39}$$

with obvious domains.[66]

Exercise 17 Let $\mathcal{M}_{\text{pos}}$ denote the smallest von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_n)$ containing all bounded functions of every $\hat{x}_\nu$. Prove the following four statements:

(i) $\mathcal{M}_{\text{pos}}$ is ***maximally abelian***, i.e. $\mathcal{M}_{\text{pos}}$ coincides with its commutant[67]

$$\mathcal{M}'_{\text{pos}} = \left\{ \hat{B} \in \mathcal{L}(\mathcal{H}_n) \ : \ [\hat{B}, \hat{A}] = 0 \ \forall \hat{A} \in \mathcal{M}_{\text{pos}} \right\}$$

(ii) $\mathcal{H}_n$ contains a dense set of vectors which are all cyclic with respect to the identical representation of $\mathcal{M}_{\text{pos}}$.

(iii) The identical representation of $\mathcal{M}_{\text{pos}}$ is reducible.

(iv) The von Neumann logic of $\mathcal{M}_{\text{pos}}$ does not contain any atom.

Translation of particle ν at time zero by a_ν corresponds to a symmetry of the time zero standard logic. The corresponding $*$-automorphism α_{a_ν} is implemented by the unitary operator $\hat{U}_\nu(-a_\nu)$ defined by

$$[\hat{U}_\nu(-a_\nu)\psi](x_1, \ldots, x_n) \stackrel{\text{def}}{=} \psi(x_1, \ldots, x_\nu - a_\nu, \ldots, x_n)\,, \tag{1.40}$$

i.e.:

$$\alpha_{a_\nu}(\hat{A}) = \hat{U}_\nu(-a_\nu)\hat{A}\hat{U}_\nu(-a_\nu)^{-1}\,.$$

By Equation (1.39), the operators $\hat{U}_\nu(\tau)$ for fixed ν fulfill the relation

$$\hat{U}_\nu(\tau_1)\hat{U}_\nu(\tau_2) = \hat{U}_\nu(\tau_1 + \tau_2)$$

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[66] Compare Exercise 2, Statement (i), and Equation (1.23).

[67] First show that

$$\begin{aligned}&\int \overline{\phi(x_1, \ldots, x_n)}(\hat{T}\psi)(x_1, \ldots, x_n)dx_1 \ldots dx_n \\ &= \int \overline{\phi(x_1, \ldots, x_n)}(\hat{T}\chi_M)(x_1, \ldots, x_n)\psi(x_1, \ldots, x_n)dx_1 \ldots dx_n\end{aligned}$$

holds for $\hat{T} \in \mathcal{M}'_{\text{pos}}$ and $\phi, \psi \in \mathcal{H}_n$ whenever $\text{supp}\phi \subset M$.

and depend continuously on the parameter τ, i.e. they form a ***continuous 1-parameter unitary group***. Hence, by Stone's theorem, there exist unique self-adjoint operators $\hat{p}_1, \dots, \hat{p}_n$ with

$$\hat{U}_\nu(\tau) = e^{+\frac{i}{\hbar}\hat{p}_\nu \tau} \qquad \text{for } \nu = 1, \dots, n\,. \tag{1.41}$$

$\hat{p}_\nu$ is interpreted as time zero momentum operator for particle ν. If ψ is sufficiently regular we have the Taylor expansion

$$\psi(x_1, \dots, x_\nu + \tau, \dots, x_n) = e^{\tau \partial_{x_\nu}} \psi(x_1, \dots, x_n)$$

which, by (1.40) and (1.41), gives

$$(\hat{p}_\nu \psi)(x_1, \dots, x_n) = \frac{\hbar}{i} \partial_{x_\nu} \psi(x_1, \dots, x_n)\,. \tag{1.42}$$

This, finally, yields the ***canonical commutation relations***

$$\left.\begin{array}{l} [\hat{p}_\nu, \hat{x}_\mu] = \frac{\hbar}{i}\delta_{\nu\mu}\hat{1} \\ [\hat{x}_\nu, \hat{x}_\mu] = [\hat{p}_\nu, \hat{p}_\mu] \end{array}\right\} \text{ on a suitable domain.} \tag{1.43}$$

Exercise 18 Explain why Heisenberg's uncertainty relations are not valid for the angular momentum L_3 and its corresponding angular variable $\varphi \in [0, 2\pi)$ of an ordinary quantum mechanical particle (in $\mathbb{R}^3$) even though their observables obey the canonical commutation relations

$$[\hat{L}_3, \hat{\varphi}] = \frac{\hbar}{i}\hat{1} \qquad \text{on some invariant dense domain.}$$

In order to avoid domain problems one replaces the fundamental relations (1.43) by the corresponding ones obeyed by the continuous 1-parameter groups (1.41) and

$$\hat{V}_\mu(s) \stackrel{\text{def}}{=} e^{i\hat{x}_\mu s} \tag{1.44}$$

i.e. by the so-called ***Weyl relations***:

$$\hat{U}_\nu(\tau)\hat{V}_\mu(s) = e^{i\tau s \delta_{\nu\mu}} \hat{V}_\mu(s)\hat{U}_\nu(\tau)\,, \tag{1.45}$$

$$\hat{U}_\nu(\tau_1)\hat{U}_\nu(\tau_2) = \hat{U}_\nu(\tau_1 + \tau_1)\,, \qquad \hat{V}_\mu(s_1)\hat{V}_\mu(s_2) = \hat{V}_\mu(s_1 + s_1)\,,$$
$$\hat{U}_\nu(-\tau) = \hat{U}_\nu(\tau)^* \neq 0 \neq \hat{V}_\mu(-s) = \hat{V}_\mu(s)^*\,.$$

Remark: (1.45) shows that

$$(p_1, \dots, p_n; x_1, \dots, x_n; t) \longmapsto e^{2\pi i h t} e^{\pi i h \sum_{\nu=1}^n p_\nu x_\nu} \prod_{\mu=1}^n \hat{V}_\mu(2\pi x_\mu)\hat{U}_\mu(h p_\mu)$$

is a representation of the ***Heisenberg group*** $\mathbf{H}_n$, i.e. of $\mathbb{R}^{2n+1}$ with multiplication

$$\begin{array}{l} (p_1, \dots, p_n; x_1, \dots, x_n; t)(p'_1, \dots, p'_n; x'_1, \dots, x'_n; t') \\ \stackrel{\text{def}}{=} \left(p_1 + p'_1, \dots, p_n + p'_n; x_1 + x'_1, \dots, x_n + x'_n; t + t' + \frac{1}{2}\sum_{\nu=1}^n (p_\nu x'_\nu - p'_\nu x_\nu)\right) \end{array}$$

(compare (Folland, 1989, Sect. 1.2)).

Theorem 1.3.14 *There is a C^*-algebra $\mathcal{A}_n^{\mathrm{B}}$, unique up to *-isometry, that is generated by elements $\hat{U}_\nu(\tau)$, $\hat{V}_\mu(s)$ $(\nu,\mu \in \{1,\dots,n\}\ ;\ \tau, s \in \mathbb{R})$ fulfilling the Weyl relations.*

Proof: See (Bratteli and Robinson, 1981, Theorem 5.2.8). ∎

The representation of the so-called **CCR-*algebra*** $\mathcal{A}_n^{\mathrm{B}}$ given by the C^*-subalgebra of $\mathcal{L}(\mathcal{H}_n)$ generated by all $\hat{U}_\nu(\tau)$ and $\hat{V}_\mu(s)$ defined above is called the **Schrödinger representation**. From now on, let us identify $\mathcal{A}_n^{\mathrm{B}}$ with its Schrödinger representation.

Exercise 19 Prove[68] that the Schrödinger of $\mathcal{A}_n^{\mathrm{B}}$ representation is **irreducible**.

Every state ω on $\mathcal{A}_n^{\mathrm{B}}$ is uniquely fixed by[69] its ***generating functional***

$$E_\omega(z_1,\dots,z_n) \stackrel{\mathrm{def}}{=} \omega\left(\hat{W}_1(z_1)\dots\hat{W}_n(z_n)\right), \tag{1.46}$$

$$\text{where:}\qquad \hat{W}_\nu(s+i\tau) \stackrel{\mathrm{def}}{=} \hat{U}_\nu(\frac{\tau}{2})\hat{V}_\nu(s)\hat{U}_\nu(\frac{\tau}{2})\,.$$

If one takes for ω the so-called ***Fock ground state***[70]

$$\omega_{\mathrm{F}}(\hat{A}) \stackrel{\mathrm{def}}{=} \left\langle \Omega_{\mathrm{F}} \middle| \hat{A}\Omega_{\mathrm{F}} \right\rangle, \quad \text{where:}\ \ \Omega_{\mathrm{F}}(x_1,\dots,x_n) = \pi^{-\frac{n}{4}} e^{-\frac{1}{2}(x_1^2+\dots x_n^2)},$$

one gets the so-called ***Fock functional***

$$E_{\mathrm{F}}(s_1,\tau_1;\dots) \stackrel{\mathrm{def}}{=} E_{\omega_{\mathrm{F}}}(s_1+i\tau_1;\dots)\,,$$

which uniquely characterizes the so-called ***Fock representation***, i.e. the GNS-representation $\pi_{\mathrm{F}} \stackrel{\mathrm{def}}{=} \pi_{\omega_{\mathrm{F}}}$. According to Exercise 19 and Corollary 1.3.11 the Fock representation is unitary equivalent to the Schrödinger representation.

Exercise 20 Show that[71]

$$E_{\mathrm{F}}(s_1,\tau_1;\dots;s_n,\tau_n) = \exp\left(-\frac{1}{4}(s_1^2+\tau_1^2+\dots s_n^2+\tau_n^2)\right).$$

[68] Recall Statement (i) of Exercise 17 and Theorem 1.3.13.

[69] Recall (1.45) and Exercise 13.

[70] The ground state of n independent harmonic oscillators with mass and circular frequency equal to 1.

[71] Recall (1.40) and note that $\int_{-\infty}^{+\infty} dx\, e^{-(z+z_0)^2} = \sqrt{\pi}$ for arbitrary complex z_0.

Definition 1.3.15 *A representation π of $\mathcal{A}_n^{\mathrm{B}}$ is called **regular** if the $\pi\left(\hat{U}_\nu(\tau)\right)$ $\pi\left(\hat{V}_\mu(\tau)\right)$ depend strongly continuously on the parameter τ and coincide with the unit operator for $\tau = 0$.*

Obviously, the Fock representation of $\mathcal{A}_n^{\mathrm{B}}$ is regular.

Theorem 1.3.16 (Stone - von Neumann) *Let π be a* **regular**[72] *representation of $\mathcal{A}_n^{\mathrm{B}}$ (with* **finite** *n) in $\mathcal{H}$. Then π is unitarily equivalent to some direct sum of Fock representations (resp. Schrödinger representations) of $\mathcal{A}_n^{\mathrm{B}}$.*

Proof: See (Bratteli and Robinson, 1981, Corollary 5.2.15) or (Folland, 1989, pp 35–36). ∎

Exercise 21 Using the results of Exercise 20, prove Theorem 1.3.16.[73]

Let $\mathcal{M}$ denote the von Neumann subalgebra of $\mathcal{L}(\mathcal{H}_{\mathrm{tot}})$ generated by those **non-relativistic** n-particle observables which refer only to the motion of **identifiable** particles with respect to one space dimension but not to inner degrees of freedom. If – as usual – we assume $\mathcal{M}$ to be $*$-isomorphic to $\mathcal{L}\left(L^2(\mathbb{R}^{3n}, dx_1, \dots, dx_n)\right)$, interpreting (1.41) and (1.44) in the standard way (compare Exercise 15), we see from Theorem 1.3.16:

> The partial states of physically realizable ensembles of the kind described above correspond to ***regular*** states[74] ω on the C^*-algebra $\mathcal{A}_n^{\mathrm{B}}$ characterized by the canonical commutation relations in Weyl form. Here the self-adjoint elements of $\mathcal{A}_n^{\mathrm{B}}$ may be interpreted as time-zero observables ('quantum kinematics') in accord with 1.41, 1.44, and 1.45. Time evolution in the sense of Section 1.1.4 has to be unitary, according to Theorem 1.1.4.

In this sense replacement of the complex numbers (c-numbers) p_ν, q_ν by operators $\hat{p}_\nu$, $\hat{q}_\nu$ (q-numbers) fulfilling the commutation Relations (1.43) (as a consequence of (1.45)) is called a ***quantization*** of the system described above.

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[72]For the classification of strongly measurable (not necessarily regular) representations in non-separable Hilbert spaces see (Cavallaro et al., 1998).

[73]Show, first of all, that

$$\Omega \stackrel{\text{def}}{=} \lim_{\epsilon\longrightarrow 0} \pi\left(\prod_{\nu=1}^{n}\left(\int ds\, e^{-\frac{s^2}{2}}\hat{V}_\nu(s)\right)\left(\int d\tau\, e^{-(\epsilon\tau)^2}\hat{U}_\nu(\tau)\right)\right)\psi$$

is normalized for suitable ψ and that the generating functional E_ω, corresponding to the vector state

$$\omega(\hat{A}) \stackrel{\text{def}}{=} \left\langle \Omega \right| \pi(\hat{A})\Omega \Big\rangle \qquad \forall \hat{A} \in \mathcal{A}_n^{\mathrm{B}},$$

coincides with the Fock functional.

[74]I.e. π_ω is regular.

For infinitely many degrees of freedom ($n \to \infty$) the situation is much more complicated:

Let $\{\Psi_j\}_{j\in\mathbb{Z}_+}$ be complete orthonormal system of $\mathcal{H}_1 = L^2(\mathbb{R}^1, dx)$ with $\Psi_0 = \Omega_{\mathrm{F}}(x_1)$ and consider

$$\{(\Psi_{j_1}, \Psi_{j_2}, \ldots, \Psi_0, \Psi_0, \ldots)\}_{j_1,j_2,\ldots\in\mathbb{Z}_+}$$

as a complete orthonormal system of a complex Hilbert space $\left(\bigotimes_{\nu=1}^{\infty}\mathcal{H}_1\right)^{\Omega_{\mathrm{F}}^\infty}$, where $\Omega_{\mathrm{F}}^\infty \stackrel{\text{def}}{=} (\Psi_0, \Psi_0, \ldots)$, called an ***infinite tensor product*** of $\mathcal{H}_1$ with itself. Then, given n, the definition

$$\pi_n(\underbrace{\hat{W}_\nu(z)}_{\in\mathcal{A}_n^{\mathrm{B}}})(\Psi_{j_1}, \ldots, \Psi_{j_\nu}, \ldots) \stackrel{\text{def}}{=} (\Psi_{j_1}, \ldots, \hat{W}_1(z)\Psi_{j_\nu}, \ldots)\,,$$

for $\nu = 1, \ldots, n$ and complex z (compare (1.46)), fixes a representation π_n of $\mathcal{A}_n^{\mathrm{B}}$ that is unitary equivalent to the corresponding Schrödinger representation.

The C^*-subalgebra $\mathcal{A}_\infty^{\mathrm{B}}$ of $\mathcal{A}_\infty^{\mathrm{B}}$ of $\mathcal{L}\left(\left(\bigotimes_{n=1}^{\infty}\mathcal{H}_1\right)^{\Omega_{\mathrm{F}}^\infty}\right)$ generated by $\bigcup_{n=1}^{\infty}\pi_n(\mathcal{A}_n^{\mathrm{B}})$ is called the ***CCR-algebra***. It is considered to be the algebra of time zero observables of a Bose system with 'infinitely many degrees of freedom' (see also (Yurtsever, 1993) and (Borchers, 1996, Sect. I.2)). Because of

$$\pi_\nu(\mathcal{A}_\nu^{\mathrm{B}}) \subset \pi_{\nu+1}(\mathcal{A}_{\nu+1}^{\mathrm{B}})$$

the Weyl relations 1.45 hold also for

$$\hat{U}_\nu^\infty(\tau) = \pi_\nu\left(\hat{U}_\nu(\tau)\right) \qquad \hat{V}_\nu^\infty(s) = \pi_\nu\left(\hat{V}_\nu(s)\right)\,,$$

$$\text{where} \qquad \hat{U}_\nu(\tau), \hat{V}_\nu(s) \in \mathcal{A}_\nu^{\mathrm{B}}\,,$$

instead of $\hat{U}_\nu(\tau), \hat{V}_\nu(s)$ for arbitrary $\nu, \mu \in \mathbb{N}$.

The **regular** state ω_{F} on the CCR-algebra with generating functional

$$E_{\mathrm{F}}(z_1, \ldots, z_n, 0, 0, \ldots) \stackrel{\text{def}}{=} \left\langle \Omega_{\mathrm{F}} \middle| \hat{W}_1^\infty(z_1), \ldots, \hat{W}_n^\infty(z_n)\Omega_{\mathrm{F}}^\infty \right\rangle$$

(compare (1.46)) is called the ***Fock vacuum***. The corresponding GNS-representation $\pi_{\mathrm{F}} = \pi_{\omega_{\mathrm{F}}}$ is called the ***Fock representation*** of $\mathcal{A}_\infty^{\mathrm{B}}$.

Exercise 22 Show that the identical representation of $\mathcal{A}_\infty^{\mathrm{B}}$ is unitary equivalent to the Fock representation and **irreducible**.

1.3.4 Spontaneously Broken Symmetries

Let $\hat{U}_1, \hat{U}_2, \ldots$ be unitary operators in $\mathcal{H}_1$ and define

$$\Omega_N^\infty \stackrel{\text{def}}{=} (\hat{U}_1\Psi_0, \ldots, \hat{U}_N\Psi_0, \Psi_0, \Psi_0, \ldots) \qquad \text{for } N = 1, 2, \ldots .$$

Then, for finite N, the GNS-representation π_{ω_N} of $\mathcal{A}_\infty^{\text{B}}$ for the vector state

$$\omega_N(\hat{A}) \stackrel{\text{def}}{=} \left\langle \Omega_N^\infty \middle| \, \hat{A}\Omega_N^\infty \right\rangle$$

is unitary equivalent to the Fock representation. In the limit $N \to \infty$, however, we get a state ω_∞ the GNS-representation π_{ω_∞} of which is **regular**, irreducible, and faithful[75] but, in general,[76] **not unitary equivalent** to the Fock representation.

Exercise 23 Show, by Theorem 1.3.13, that the von Neumann completion of $\pi_{\omega_{N_1}}(\mathcal{A}_\infty^{\text{B}})$ in $\mathcal{H}_{\omega_{N_1}}$ is isomorphic to the von Neumann completion of $\pi_{\text{F}}(\mathcal{A}_\infty^{\text{B}})$ in $\mathcal{H}_{\omega_\infty}$. Explain why, nevertheless, $\pi_{\omega_{N_1}}$ may be unitary **in**equivalent to π_{F}.

The above construction shows us that Theorem 1.3.16 does not hold for $n = \infty$ but that there is a myriad of – up to know unclassified – physically relevant, regular, irreducible representations of $\mathcal{A}_\infty^{\text{B}}$ which are inequivalent to the Fock representation !

Problem: Which is the von Neumann algebra $\mathcal{M}$ corresponding to $\mathcal{A}_\infty^{\text{B}}$ in the sense of Section 1.3.1 and **how** is $\mathcal{A}_\infty^{\text{B}}$ embedded into $\mathcal{M}$?

As already pointed out, one does not always know the von Neumann algebra $\mathcal{M}$ of the considered partial theory in the sense of Section 1.3.1, but only – up to C^*-algebra isometry – a C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$ generating $\mathcal{M}$.

This is the reason for using also C^*-algebras which are not von Neumann algebras, in quantum statistical mechanics and relativistic quantum field theory.

Regarding the physical relevance of states[77] we then need suitable criteria (such as regularity of states over $\mathcal{A}_N^{\text{B}}$).

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[75] Limit states of a similar kind play an important role in statistical quantum mechanics (Araki and Woods, 1963) and constructive quantum field theory (Wightman, 1967).

[76] One may prove that

$$(\Phi_1, \Phi_2, \ldots) \in \left(\bigotimes_{\nu=1}^{\infty} \mathcal{H}_1 \right)^{\Omega_{\text{F}}^\infty} \iff \sum_{\nu=1}^{\infty} |1 - \langle \Psi_0 | \, \Phi_\nu \rangle| < \infty .$$

[77] The main problem is to characterize those those partial states on $\mathcal{A}$ which are restrictions of **normal** states on $\mathcal{M}$.

Definition 1.3.17 *A state ω on the C^*-algebra $\mathcal{A}$ is called* ***normal***[78] ***with respect to the representation*** *π on $\mathcal{A}$ in $\mathcal{H}$, if there is an operator $\hat{T}_\omega \in \mathcal{T}_1(\mathcal{H})$ with*

$$\omega(\hat{A}) = \mathrm{Tr}\left(\hat{T}_\omega \pi(\hat{A})\right) \quad \forall \hat{A} \in \mathcal{A}\,.$$

The set of all states which are normal with respect to π is called the ***folium*** *$\mathcal{S}_\pi$ corresponding to π .*

Definition 1.3.18 *Two representations π_1, π_2 of a C^*-algebra $\mathcal{A}$ are called* ***quasi equivalent*** *if their folia coincide, i.e. if $\mathcal{S}_{\pi_1} = \mathcal{S}_{\pi_2}$.*

Theorem 1.3.19 *Let π_1, π_2 be* ***non-degenerate*** *(i.e. $\pi_j(\mathcal{A})\Psi = \{0\} \Longrightarrow \Psi = 0$) representations of the C^*-algebra $\mathcal{A}$. Then π_1, π_2 are quasi equivalent if and only if a suitable direct sum of π_1 is unitary equivalent to a suitable sum of π_2 .*

Proof: See (Bratteli and Robinson, 1981, Theorem 2.4.26). ∎

Definition 1.3.20 *A physical symmetry corresponding to the $*$-automorphism (or $*$-anti-automorphism) φ of the C^*-algebra $\mathcal{A}$ is said to be* ***spontaneously broken*** *by the state ω on $\mathcal{A}$ if the GNS-representations corresponding to ω and $\varphi_*\omega$ are* **not** *quasi-equivalent, i.e. if $\mathcal{S}_{\pi_\omega} \neq \mathcal{S}_{\pi_{\varphi_*\omega}}$.*

Exercise 24 Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ and let $\hat{U}$ be an anti-unitary operator on $\mathcal{H}$.

(i) Show that

$$\gamma(\hat{A}) \stackrel{\text{def}}{=} \hat{U}\hat{A}^*\hat{U}^* \quad \text{for } \hat{A} \in \mathcal{A}$$

defines a $*$-***antiautomorphism***, i.e. for all $\hat{A}$, $\hat{B} \in \mathcal{A}$ and all $z \in \mathbb{C}$:

$$\begin{aligned} (A_1): \quad \gamma(\hat{A} + z\hat{B}) &= \gamma(\hat{A}) + z\,\gamma(\hat{B})\,, \\ (A_2): \quad \gamma(\hat{A}\hat{B}) &= \gamma(\hat{B})\gamma(\hat{A})\,, \\ (A_3): \quad \gamma(\hat{A}^*) &= \gamma(\hat{A})^*\,. \end{aligned}$$

(ii) Show that γ , as defined above, is the only $*$-antiautomorphism of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ with

$$\gamma(\hat{P}) = \hat{U}\hat{P}\hat{U}^* \quad \forall\, \hat{P} \in \mathcal{L}_\mathcal{H}\,.$$

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[78] In view of Theorem 1.2.5

Exercise 25 Show that symmetries may be broken by regular states on $\mathcal{A}^{\mathrm{B}}_{\infty}$ but not by regular states on $\mathcal{A}^{\mathrm{B}}_{N}$, $N < \infty$.

Remarks:

(i) In physically relevant theories (like QED) the symmetries corresponding to homogeneous Lorentz transformations are spontaneously broken (see e.g. (Buchholz, 1986)).

(ii) For interesting speculations regarding spontaneous breaking of time translation symmetry see (Rieckers,).

(iii) Spontaneous breaking of gauge symmetries plays a decisive role in the Weinberg-Salam theory of electroweak interactions (see e.g. (Mohapatra, 1986) and (Watkins, 1986)).

Exercise 26 Let ω be a state and φ a $*$-automorphism of the C^*-algebra $\mathcal{A}$. Show[79] that the representation π of $\mathcal{A}$, defined by

$$\pi(\hat{A}) \stackrel{\text{def}}{=} \pi_\omega\left(\varphi^{-1}(\hat{A})\right) \qquad \text{for } \hat{A} \in \mathcal{A}\,,$$

is unitary equivalent to $\pi_{\varphi_*\omega}$ but not necessarily to π_ω .

[79] **Hint:** First, show that

$$\hat{U}\pi_{\varphi_*\omega}(\hat{A})\Omega_{\varphi_*\omega} \stackrel{\text{def}}{=} \pi_\omega\left(\varphi^{-1}(\hat{A})\right)\Omega_\omega \qquad \text{for } \hat{A} \in \mathcal{A}$$

(consistently!) defines a unitary mapping $\hat{U}$ from $\mathcal{H}_{\varphi_*\omega}$ onto $\mathcal{H}_\omega$.

Chapter 2

Massive Scalar Fields

2.1 Free Neutral Scalar Fields

We are going to describe systems of noninteracting, indistinguishable, relativistic point 'particles' on Minkowski space[1] with rest mass $m > 0$ having no internal degrees of freedom and no charge. We use natural units throughout, especially

$$\boxed{c = \hbar = 1\,.}$$

2.1.1 1-Particle Space

Momentum Space Representation

The ***three-momentum*** of a classical relativistic point particle is

$$\mathbf{p} = m_{\mathbf{v}}\,\mathbf{v}\,,$$

where $\mathbf{v}$ is its velocity and

$$m_{\mathbf{v}} = \frac{m}{\sqrt{1 - |\mathbf{v}|^2}}$$

its ***inertial mass*** coinciding[2] – thanks to natural units – with its ***energy*** (divided by c=1)

$$p^0 = \omega_{\mathbf{p}} \stackrel{\text{def}}{=} \sqrt{m^2 + |\mathbf{p}|^2} > 0\,.$$

Therefore,[3]

$$\mathbf{v} = \frac{\mathbf{p}}{\omega_{\mathbf{p}}}\,.$$

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[1] For generalization to curved space-time see, e.g. (Verch, 1997) and references given there.
[2] Just compare the squares of $m_{\mathbf{v}}$ and p^0.
[3] This, obviously, is consistent with $\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathbf{p}(t)} = \mathbf{v}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t)\,.$

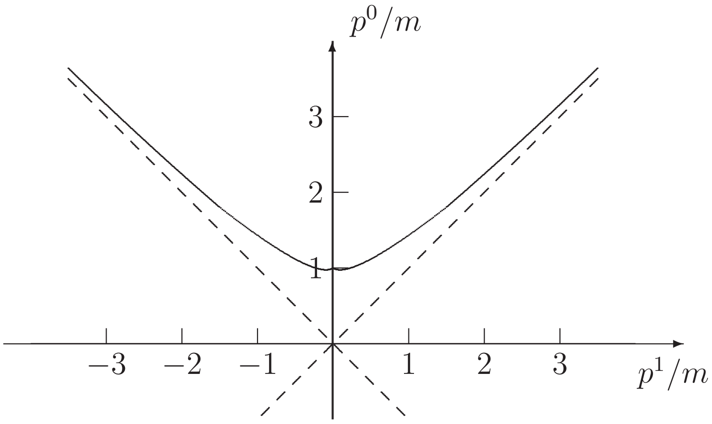

Figure 2.1: 1-particle mass-shell, restricted to the p^0-p^1-plane

Moreover, we have

$$\boxed{p \cdot p \stackrel{\text{def}}{=} p^0 p^0 - \mathbf{p} \cdot \mathbf{p} = m^2 \,, \quad p^0 > 0} \tag{2.1}$$

for its ***four-momentum***

$$p = (p^0, p^1, p^2, p^3) = (p^0, \mathbf{p}) \,.$$

When changing the inertial system (but not the origin of the resp. system) p has to be transformed by the same Lorentz matrix as $x = (x^0, \mathbf{x})$:

$$x, p \stackrel{\text{change of ref.syst.}}{\longrightarrow} x' = \Lambda x, p' = \Lambda p \tag{2.2}$$

Exercise 27 Let $\left\{\underline{p}(s)\right\}_{s \in \mathbb{R}}$ be some (sufficiently well behaved) curve on the ***1-particle mass shell*** $M_m \stackrel{\text{def}}{=} \left\{p \in \mathbb{R}^4 : p^0 = \omega_{\mathbf{p}}\right\}$. Let $(p^0(s), p^1(s), p^2(s), p^3(s))$ resp. $(p'^0(s), p'^1(s), p'^2(s), p'^3(s))$ be the coordinates of $\underline{p}(s)$ in the inertial System L resp. L', related to each other by a special Lorentz transformation:

$$p'^0 = \frac{p^0 - u\, p^1}{\sqrt{1 - u^2}} \,, \quad p'^1 = \frac{p^1 - u\, p^0}{\sqrt{1 - u^2}} \,, \quad p'^2 = p^2 \,, \quad p'^3 = p^3$$

(u fixed). Show that

$$\frac{\frac{\mathrm{d}p'^1}{\mathrm{d}s}}{\omega_{\mathbf{p}'}} = \frac{\frac{\mathrm{d}p^1}{\mathrm{d}s}}{\omega_{\mathbf{p}}} \,.$$

Quantum mechanical 1-particle state space:

The pure 1-particle states are given – in the sense of orthodox quantum mechanics – by the vectors of the separable complex Hilbert space

$$\mathcal{H}_0^{(1)} \stackrel{\text{def}}{=} \mathrm{L}^2\left(\mathbb{R}^3, \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\right)$$

with inner product

$$\langle \check{f} | \check{g} \rangle \stackrel{\text{def}}{=} \int \overline{\check{f}(\mathbf{p})}\, \check{g}(\mathbf{p})\, \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \quad \forall \check{f}, \check{g} \in \mathcal{H}_0^{(1)} . \tag{2.3}$$

Obviously,

$$\left(\hat{U}_0(a,\Lambda)\check{f}\right)(\mathbf{p}) \stackrel{\text{def}}{=} \left(e^{ipa}\,\check{f}\left(\overrightarrow{\Lambda^{-1}p}\right)\right)_{|p^0=\omega_{\mathbf{p}}} \tag{2.4}$$

defines a representation of $\mathcal{P}_+^\uparrow$ ('restricted' Poincaré group), i.e.

$$\hat{U}_0(a_2,\Lambda_2)\,\hat{U}_0(a_1,\Lambda_1) = \hat{U}_0(\Lambda_2 a_1 + a_2, \Lambda_2\Lambda_1) \quad \forall\,(a_2,\Lambda_2)\,,\,(a_1,\Lambda_1) \in \mathcal{P}_+^\uparrow\,,$$

which is (strongly) continuous. By Exercise 27 the measure $\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}$ is Lorentz invariant. Hence the representation (2.4) is **unitary**, i.e.:

$$\left\langle \hat{U}_0(a,\Lambda)\check{f} \middle| \hat{U}_0(a,\Lambda)\check{g} \right\rangle = \langle \check{f} | \check{g} \rangle \quad \forall \check{f}, \check{g} \in \mathcal{H}_0^{(1)} . \tag{2.5}$$

It is to be interpreted as follows:

$\hat{U}_0(a,\Lambda)\check{f}$ corresponds to an ensemble that, with respect to the coordinates $x' \stackrel{\text{def}}{=} \Lambda^{-1}(x-a)$, is to be described in exactly the same way as an ensemble corresponding to $\check{f}$ is to be described with respect to the coordinates x.	(2.6)

According to (2.4), with the projection-valued measure

$$\hat{E}_0(J)\,\check{f}(\mathbf{p}) \stackrel{\text{def}}{=} \begin{cases} \check{f}(\mathbf{p}) & \text{if } (\omega_{\mathbf{p}}, \mathbf{p}) \in J \\ 0 & \text{otherwise} \end{cases} \quad \text{for Borel sets } J \subset \mathbb{R}^4 \tag{2.7}$$

we have[4]

$$\begin{aligned} \hat{U}_0(a) &\stackrel{\text{def}}{=} \hat{U}_0(a, \mathbb{1}_4) = \exp\left(i\overline{\hat{P}_0\,a}\right) = \int e^{ipa}\,\hat{E}_0(dp)\,, \\ \text{where} \quad \hat{P}_0 &\stackrel{\text{def}}{=} \int p\,\hat{E}_0(dp)\,. \end{aligned} \tag{2.8}$$

[4] As usual, $\overline{\hat{P}_0\,a}$ denotes the closure of the (essentially self-adjoint) operator $\hat{P}_0 a$.

$\hat{E}_0$ is to be interpreted as the spectral measure for 4-momentum, i.e. for $\left\|\check{f}\right\| = 1$:

$$\begin{aligned}
\left\langle \check{f} \mid \hat{E}_0(J)\check{f} \right\rangle \underset{(2.7)}{=} & \int_{(\omega_{\mathbf{p}},\mathbf{p})\in J} \left|\check{f}(\mathbf{p})\right|^2 \frac{\mathrm{d}\mathbf{p}}{2\omega_{\mathbf{p}}} \\
\underset{\text{Interpret.}}{=} & \text{ probability for: } p \in J \text{ in a state } \widehat{=} \check{f} \\
= & \text{ probability for: } p \in J \cap M_m \text{ in a state } \widehat{=} \check{f}\,.
\end{aligned} \tag{2.9}$$

This is equivalent to interpreting $\hat{P}_0$ as energy-momentum operator (= observable of 4-momentum):

$$\begin{aligned}
\left\langle \check{f} \mid \hat{P}_0^\mu\, \check{f} \right\rangle &= \int_{p^0=\omega_{\mathbf{p}}} p^\mu \left|\check{f}(\mathbf{p})\right|^2 \frac{\mathrm{d}\mathbf{p}}{2p^0} \\
&= \left\{ \begin{array}{l} \text{expectation value for the } \mu\text{-component of the} \\ \text{(time-independent) 4-momentum in a state } \widehat{=} \check{f} \end{array} \right.
\end{aligned} \tag{2.10}$$

(if $\|\check{f}\| = 1$). This interpretation is also suggested by the relations

$$\hat{U}_0(a,\Lambda)^{-1}\,\hat{P}_0^\mu\,\hat{U}_0(a,\Lambda) = \Lambda^\mu{}_\nu \hat{P}_0^\nu \tag{2.11}$$

and

$$\hat{P}_0^0 \geq m\hat{1}\,, \quad \hat{P}_0 \cdot \hat{P}_0 = m^2\hat{1}\,. \tag{2.12}$$

Space-Time Representation

Instead of the $\check{f}(\mathbf{p})$ one may also use the corresponding wave functions[5]

$$f^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \check{f}(\mathbf{p})\, e^{-ipx}\, \frac{\mathrm{d}\mathbf{p}}{2p^0}\,, \tag{2.13}$$

which uniquely characterize the $\check{f}(\mathbf{p})$ due to

$$\check{f}(\mathbf{p}) = (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} e^{ipx}\, i\overleftrightarrow{\partial}_0\, f^+(x)\, \mathrm{d}\mathbf{x} \quad \forall\, x^0 \in \mathbb{R}\,, \tag{2.14}$$

where

$$g(x)\overleftrightarrow{\partial}_0 f(x) \stackrel{\text{def}}{=} g(x)\frac{\partial}{\partial x^0}f(x) - \left(\frac{\partial}{\partial x^0}g(x)\right) f(x)\,. \tag{2.15}$$

Exercise 28 Show, for sufficiently well behaved $\check{f}(\mathbf{p})$, that

$$\jmath_\mu(x) \stackrel{\text{def}}{=} \overline{f^+(x)}\, i\overleftrightarrow{\partial}_\mu f^+(x)$$

[5]Assume the $\check{f}(\mathbf{p})$ to be sufficiently well behaved, for the moment.

is a conserved Lorentz vector field[6] with

$$\int j^0(x)\,\mathrm{d}\mathbf{x} = \left\|\check{f}\right\|^2 \quad \forall\, x^0 \in \mathbb{R}\,,$$

but that $j^0(x)$ is not nonnegative,[7] in general.

The $f^+(x)$ transform according to

$$\check{g}(\mathbf{p}) = \hat{U}_0(a,\Lambda)\check{f}(\mathbf{p}) \underset{(2.4)}{\Longrightarrow} g^+(x) = f^+\left(\Lambda^{-1}(x-a)\right) \tag{2.16}$$

and are solutions of the ***Klein-Gordon equation***

$$\left(\Box + m^2\right) f^+(x) = 0 \tag{2.17}$$

($\Box \stackrel{\text{def}}{=} (\partial_0)^2 - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$). The idea is that $|f^+(x)|^2$ describes at least roughly the localization in space-time. There are fundamental obstructions[8] for defining a position operator in the sense of Sect. 1.2.2 (Hegerfeldt, 1974),(Hegerfeldt and Ruijsenaars, 1980). The most natural definition[9] would be the one given by Newton and Wigner, according to which – in full analogy to nonrelativistic quantum mechanics – the 3-dimensional Fourier transform of the time-dependent ***momentum amplitude***

$$e^{-i\omega_{\mathbf{p}}x^0}\,\frac{\check{f}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}$$

(compare (2.9)) is interpreted as ***position amplitude***; i.e.

$$\int_V |f_{\text{N.W.}}(x)|^2\,\mathrm{d}\mathbf{x} \;=\; \text{probability for "}\mathbf{x}\in V \text{ at time } x^0 \text{ in a state} \mathrel{\widehat{=}} \check{f}\text{"} \tag{2.18}$$

(if $\|f\| = 1$), where[10]

$$\begin{aligned} \check{f}_{\text{N.W.}}(\mathbf{p}) &= \frac{\check{f}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}\,,\\ f_{\text{N.W.}}(x) &\underset{(2.13)}{=} (2\pi)^{-3/2}\int_{p^0=\omega_{\mathbf{p}}} \check{f}_{\text{N.W.}}(\mathbf{p})e^{-ipx}\,\mathrm{d}\mathbf{p}\,. \end{aligned} \tag{2.19}$$

Here one easily realizes the following problem:[11]

[6] I.e.: $\partial_\mu j^\mu = 0\,,\ \overline{f^+\left(\Lambda^{-1}(x-a)\right)}\, i\overleftrightarrow{\partial}_\mu f^+\left(\Lambda^{-1}(x-a)\right) = \Lambda^\mu{}_\nu j^\nu\left(\Lambda^{-1}(x-a)\right).$

[7] Check $j^0(o)$ for

$$\check{f}(\mathbf{p}) \sim \omega_{\mathbf{p}}\left(\delta_\epsilon(\mathbf{p}-\mathbf{p}_1) - \frac{1}{2}\delta_\epsilon(\mathbf{p}-\mathbf{p}_2)\right)$$

where δ_ϵ is sufficiently close to the delta function and $\mathbf{p}_1$, $\mathbf{p}_2$ are fixed momenta with $|\mathbf{p}_1| < |\mathbf{p}_2|$.

[8] From the relativistic point of view this is quite satisfactory (see (Crewther, 1995, Sect. 1)).

[9] See (Wightman, 1962) for a very detailed discussion.

[10] Note that $\|\check{f}\| = \int |f_{\text{N.W.}}(x)|^2\,\mathrm{d}\mathbf{x}$.

[11] See Exercise 29, below. A consistent definition of strict localization for relativistic quantum field theory was given in (Knight, 1961) – not on the 1-particle level, of course.

Even though – in the relativistic theory – the velocity is bounded by $|\mathbf{v}| \leq 1$, there can exist at most one instant of time at which the 'particle' is localized within a bounded space region in the sense of Newton and Wigner!

As usual, let us denote, for $n \in \mathbb{N}$, by $\mathcal{S}(\mathbb{R}^n)$ the linear topological space of all complex-valued C^∞ functions $\varphi(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ for which all the norms

$$\|\varphi\|_N \stackrel{\text{def}}{=} \sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n} \left|1 + \sum_{\nu=1}^n x_\nu^2\right|^N \sup_{\substack{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n \\ \alpha_1+\ldots+\alpha_n<N}} \left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \varphi(x_1, \ldots, x_n)\right|$$

$(N \in \mathbb{N})$ are finite; the sets

$$U_{N,\epsilon} \stackrel{\text{def}}{=} \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_N < \epsilon\}$$

$(N \in \mathbb{N}, \epsilon > 0)$ forming a basis of open neighborhoods of 0, by definition.

A function $f^+(x)$ is called a ***positive frequency smooth Klein-Gordon solution*** iff it is of the form (2.13) with $\check{f} \in \mathcal{S}(\mathbb{R}^3)$. The function $f^-(x)$ is called a ***negative frequency smooth Klein-Gordon solution*** iff it is the complex conjugate of a positive frequency Klein-Gordon solution: $f^-(x) = \overline{f^+(x)}$. Finally, the function $f(x)$ is called a ***smooth Klein-Gordon solution*** iff it is of the form

$$f(x) = f^+(x) + f^-(x)$$

with $f^+(x)$ resp. $f^-(x)$ a positive resp. negative frequency smooth Klein-Gordon solution.[12]

Exercise 29 Let f^+ be a positive frequency smooth Klein-Gordon solution. Using the easy part of the Paley-Wiener theorem (see, e.g., (Gårding and Lions, 1959, Theorem 7·1.5.)), saying that the Fourier transform of a (generalized) function with compact support is an entire analytic function, prove the following statements:

(i) There is at most one instant of time x^0 for which $f^+(x)$, considered as a function of $\mathbf{x}$, vanishes outside some bounded subset of $\mathbb{R}^3$.

(ii) There is no instant of time x^0 for which both $f^+(x)$ and $\frac{\partial}{\partial x^0} f^+(x)$ vanish outside some bounded subset of $\mathbb{R}^3$.

As mentioned above, in spite of the obstructions for defining a fully satisfactory position operator, the transformation rule (2.16) suggests that $|f^+(x)|^2$ describes at least roughly the localization in space-time for a particle with momentum amplitude $\check{f}(\mathbf{p})$. This expectation is confirmed by the following Lemma.[13]

[12] Note that this decomposition of $f(x)$ is unique.

[13] A weaker version of Lemma 2.1.1 was originally proved in (Ruelle, 1962).

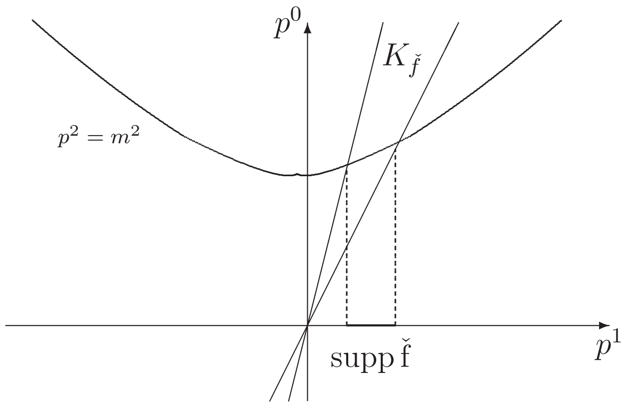

Figure 2.2: Velocity cone $K_{\check{f}}$ ($=$asymptotic localization region for $f(x)$), restricted to the p^0-p^1-plane

Lemma 2.1.1 (Ruelle) *Let $f^+(x)$ be a positive frequency smooth Klein-Gordon solution. Then for every $N \in \mathbb{Z}_+$ there is a constant C for which*

$$\begin{aligned} &\|(t, \mathbf{v}t)\|^N \left|f^+(t - x^0, \mathbf{v}t - \mathbf{x})\right| \\ &\le (1 + \|x\|)^N C \quad \forall x \in \mathbb{R}^4\,,\ t \in \mathbb{R}^1,\ \mathbf{v} \in \mathbb{R}^3 \setminus \left\{\tfrac{\mathbf{p}}{\omega_\mathbf{p}} : \mathbf{p} \in \operatorname{supp}\check{f}\right\}. \end{aligned}$$

Proof: See (Lücke, 1974b, Appendix 2). ∎

Exercise 30 Proof Lemma 2.1.1 for the special case

$$\mathbf{v} = (v, 0, 0)\ , v \notin \left\{\frac{p^1}{\omega_\mathbf{p}} : \mathbf{p} \in \operatorname{supp}\check{f}\right\}\ , \quad x = 0$$

by substitution of variables

$$p^1 \longrightarrow \xi \stackrel{\text{def}}{=} \omega_\mathbf{p} - \mathbf{p}\cdot\mathbf{v}$$

and N-fold partial integration[14] with respect to ξ.

2.1.2 Fock Space

Free n-Particle System (Momentum Representation)

As in nonrelativistic quantum mechanics, n-particle states are described by functions of n-times as many variables as in the 1-particle case. Since the particles cannot

[14]Note that $dp^1 = \frac{\omega_{\mathbf{p}(\xi)}}{p^1(\xi) - v\omega_{\mathbf{p}(\xi)}}\, d\xi$ and rewrite differentiation with respect to ξ as differentiation with respect to p^1 after partial integration.

be distinguished and have spin 0, we require these functions to be symmetric with respect to exchange of 3-momenta:[15]

$$\mathcal{H}_0^{(n)} = \left\{ \check{f}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathrm{L}^2\left(\mathbb{R}^{3n}, \tfrac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}}\right) : \check{f}_n(\mathbf{p}_{\pi 1}, \ldots, \mathbf{p}_{\pi n}) = \check{f}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \ \forall \pi \in \mathrm{S}_n \right\},$$

$$\langle \check{f}_n \mid \check{g}_n \rangle \stackrel{\text{def}}{=} \int \overline{\check{f}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n)} \check{g}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \frac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}}. \tag{2.20}$$

Again, the corresponding representation

$$\left(\hat{U}_0(a, \Lambda)\check{f}_n\right)(\mathbf{p}_1, \ldots, \mathbf{p}_n) \stackrel{\text{def}}{=} \left(e^{i(p_1 + \ldots + p_n)a} \check{f}_n\left(\overrightarrow{\Lambda^{-1}p_1}, \ldots, \overrightarrow{\Lambda^{-1}p_n}\right)\right)_{|p_j^0 = \omega_{\mathbf{p}_j}} \tag{2.21}$$

of $\mathcal{P}_+^\uparrow$ fulfills (2.5) (unitarity), (2.8) (4-dim. spectral representation), and (2.11) (transformation behaviour of $\hat{P}_0$).

Exercise 31 Determine the spectral measure $\hat{E}_0$ of $\hat{P}_0$ on $\mathbb{R}^4$ (generalization of (2.7)).

Exercise 32 Show that for every $n \in \mathbb{N}$ and every function w on $\{0,1\}^n$ the equation

$$\sum_{\nu=1}^{n} \nu \sum_{(b_1, \ldots, b_n) \in M_\nu} w(b_1, \ldots, b_n) = \sum_{\mu=1}^{n} \sum_{\substack{(b_1, \ldots, b_n) \in \{0,1\}^n \\ b_\mu = 1}} w(b_1, \ldots, b_n)$$

holds where

$$M_\nu \stackrel{\text{def}}{=} \left\{ (b_1, \ldots, b_n) \in \{0,1\}^n : \sum_{\mu=1}^{n} b_\mu = \nu \right\} \quad \text{for } \nu = 1, \ldots, n.$$

If the particles could be distinguished we had

$$\int_{\mathbf{p}_\nu \in B_3} \left|\check{f}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n)\right|^2 \frac{d\mathbf{p}_1 \cdots \mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}} = \left\{ \begin{array}{l} \text{probability for particle } \nu \text{ having a} \\ \text{three-momentum } \mathbf{p}_\nu \in B_3 \text{ in a state } \widehat{=} \check{f}_n \end{array} \right.$$

under obvious conditions. Therefore, according to Exercise 32, (2.9) becomes

$$\int_{B_3} \left\|\hat{a}_0(\mathbf{p})\check{f}_n\right\|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} = \left\{ \begin{array}{l} \text{expectation value for the} \\ \text{number of particles with } \mathbf{p} \in B_3 \end{array} \right. \tag{2.22}$$

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[15] As usual, we denote by S_n the group of all permutation of n elements. We do not consider the – much more complicated – possibility of para-Bose statistics (see e.g. (Ohnuki and Kamefuchi, 1982) and references given there; especially (Stolt and Taylor, 1970),(Hartle and Taylor, 1969)).

for normalized $\check{f}_n \in \mathcal{S}\left(\mathbb{R}^{3n}\right) \subset \mathcal{H}_0^{(n)}$, $n > 0$, if $\hat{a}_0(\mathbf{p})$ denotes the linear mapping

$$\left(\hat{a}_0(\mathbf{p})\check{f}_n\right)\underbrace{\left(\mathbf{p}_1,\ldots,\mathbf{p}_{n-1}\right)}_{\text{absent for n}\leq 1} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } n = 0 \\ \sqrt{n}\check{f}_n\left(\mathbf{p}\underbrace{,\mathbf{p}_1,\ldots,\mathbf{p}_{n-1}}_{\text{absent for n=1}}\right) \in \mathcal{H}_0^{(n-1)} & \text{for } n > 0 \end{cases} \tag{2.23}$$

from $\mathcal{S}\left(\mathbb{R}^{3n}\right) \subset \mathcal{H}_0^{(n)}$ into $\mathcal{S}\left(\mathbb{R}^{3(n-1)}\right) \subset \mathcal{H}_0^{(n-1)}$; where

$$\mathcal{S}\left(\mathbb{R}^0\right) \stackrel{\text{def}}{=} \mathcal{H}_0^{(0)} \stackrel{\text{def}}{=} \mathbb{C}\,, \quad \left\langle \check{f}_0 \mid \check{g}_0 \right\rangle \stackrel{\text{def}}{=} \overline{\check{f}_0}\,\check{g}_0\,.$$

(2.10) becomes

$$\begin{aligned} \left\langle \check{f}_n \mid \hat{P}_0^\mu \check{f}_n \right\rangle &= \int_{p^0=\omega_{\mathbf{p}}} p^\mu \left\|\hat{a}_0(\mathbf{p})\check{f}_n\right\|^2 \frac{d\mathbf{p}}{2p^0} \\ &= \int_{p_j^0=\omega_{\mathbf{p}_j}} \left(p_1^\mu + \ldots + p_n^\mu\right) \left|\check{f}_n\left(\mathbf{p}_1,\ldots,\mathbf{p}_n\right)\right|^2 \frac{d\mathbf{p}_1\cdots d\mathbf{p}_n}{2p_1^0\cdots 2p_n^0} \\ &= \begin{cases} \text{expectation value for the } \mu\text{-component of the} \\ \text{total 4-momentum in a state} \mathrel{\widehat{=}} \check{f}_n \end{cases} \end{aligned} \tag{2.24}$$

for normalized $\check{f}_n \in \mathcal{S}\left(\mathbb{R}^{3n}\right) \cap \mathcal{H}_0^{(n)}$, but instead of (2.12) we have

$$\hat{P}_0^0 \geq nm\hat{1}\,, \quad \hat{P}_0 \cdot \hat{P}_0 \geq (nm)^2\hat{1}\,. \tag{2.25}$$

Total State Space

If one does not want – or even cannot – fix the particle number, it is convenient to identify $\mathcal{H}_0$ with the so-called ***Fock-space***:

$$\mathcal{H}_0 = \bigoplus_{n=0}^{\infty} \mathcal{H}_0^{(n)}\,.$$

Here the elements of $\mathcal{H}_0$ are sequences

$$\underline{\check{f}} \stackrel{\text{def}}{=} \left\{\check{f}_0, \check{f}_1, \check{f}_2, \ldots\right\}$$

with $\check{f}_n \in \mathcal{H}_0^{(n)}$ for $n = 0, 1, \ldots$ and $\left\|\underline{\check{f}}\right\| < \infty$, where[16]

$$\left\langle \underline{\check{f}} \mid \underline{\check{g}} \right\rangle \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left\langle \check{f}_n \mid \check{g}_n \right\rangle\,.$$

[16]To be absolutely precise, one should use different symbols for the various inner products.

The corresponding unitary representation $\hat{U}_0(a,\Lambda)$ of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}_0$ is given by[17]

$$\hat{U}_0(a,\Lambda)\underline{\check{f}} \stackrel{\text{def}}{=} \left\{\check{f}_0, \hat{U}_0(a,\Lambda)\check{f}_1, \hat{U}_0(a,\Lambda)\check{f}_2, \ldots\right\}.$$

Again, (2.8) and

$$\hat{P}_0^0 \geq 0\,, \quad \hat{P}_0 \cdot \hat{P}_0 \geq 0\,.$$

hold, while (2.22) becomes

$$\int_{\mathcal{G}} \left\|\hat{a}_0(\mathbf{p})\underline{\check{f}}\right\|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} = \left\{ \begin{array}{l} \text{expectation value for the} \\ \text{number of particles with } \mathbf{p} \in \mathcal{G} \end{array} \right. \tag{2.26}$$

for **normalized** $\underline{\check{f}} \in D_0$. Here, the domain D_0 is defined by

$$D_0 \stackrel{\text{def}}{=} \left\{\underline{\check{f}} \in \mathcal{H}_0 : \ \check{f}_n \in \mathcal{S}\left(\mathbb{R}^{3n}\right) \ \forall n\,,\ \check{f}_n = 0\ \forall\, n > n_0\left(\underline{\check{f}}\right)\right\} \tag{2.27}$$

and the ***annihilation operator*** (field) $\hat{a}_0(\mathbf{p})$ by

$$\hat{a}_0(\mathbf{p})\underline{\check{f}} \stackrel{\text{def}}{=} \left\{\hat{a}_0(\mathbf{p})\check{f}_1, \hat{a}_0(\mathbf{p})\check{f}_2, \ldots\right\} \quad \forall \underline{\check{f}} \in D_0\,. \tag{2.28}$$

As a consequence of (2.23) and (2.4) we thus have

$$\begin{array}{rcl} \hat{U}_0(a,\Lambda)^{-1}\hat{a}_0(\mathbf{p})\hat{U}_0(a,\Lambda) & = & e^{+ipa}\hat{a}_0\left(\overrightarrow{\Lambda^{-1}p}\right)_{|_{p^0=\omega_{\mathbf{p}}}}\,, \\ \hat{U}_0(a,\Lambda)\hat{a}_0(\mathbf{p})\hat{U}_0(a,\Lambda)^{-1} & = & e^{-i\Lambda pa}\hat{a}_0\left(\overrightarrow{\Lambda p}\right)_{|_{p^0=\omega_{\mathbf{p}}}}\,. \end{array} \tag{2.29}$$

According to (2.28), $\hat{a}_0(\mathbf{p})$ annihilates the so-called ***vacuum vector*** $\Omega_0 \stackrel{\text{def}}{=} \{1, 0, \ldots\}$ just as the energy-momentum operator does:

$$\hat{a}_0(\mathbf{p})\Omega_0 = \left(\hat{1} - \hat{U}_0(a,\Lambda)\right)\Omega_0 = \hat{P}_0^\mu\Omega_0 = 0\,. \tag{2.30}$$

2.1.3 The Free Field

Creation Operators in Momentum Space

From (2.24)/(2.28) we conclude **formally**:

$$\hat{P}_0^\mu = \int_{p^0=\omega_{\mathbf{p}}} \hat{a}_0(\mathbf{p})^* p^\mu \hat{a}(\mathbf{p})\,\frac{d\mathbf{p}}{2p^0}\,. \tag{2.31}$$

[17] Similarly, to be precise, different symbols should be used for the various subrepresentations.

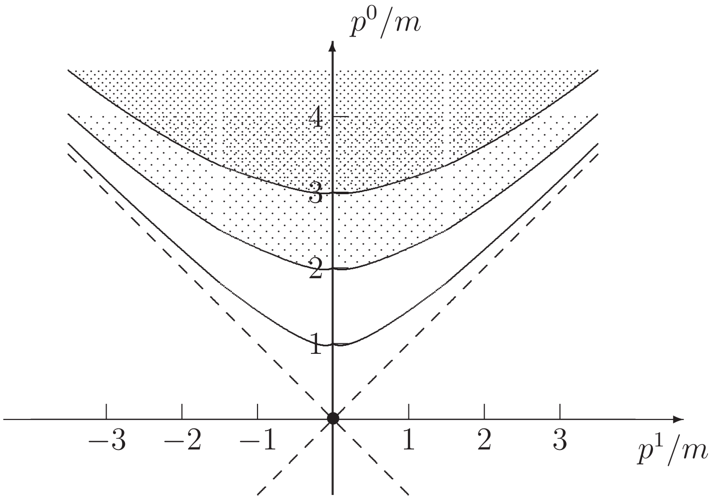

Figure 2.3: Spectrum of $\hat{P}_0$ (= support of $\hat{E}_0$), restricted to the p^0-p^1-plane

Unfortunately, however, the adjoint $\hat{a}_0(\mathbf{p})^*$ of $\hat{a}_0(\mathbf{p})$ does **not** exist for fixed $\mathbf{p}$:

The restriction to D_0 of the adjoint $\hat{a}_0(\check{\chi})^*$ of

$$\hat{a}_0(\check{\chi}) \stackrel{\text{def}}{=} \int \hat{a}_0(\mathbf{p})\check{\chi}(\mathbf{p})\, d\mathbf{p} \quad \text{for } \check{\chi} \in \mathcal{S}(\mathbb{R}^3)$$

is given by[18]

$$\left(\hat{a}_0(\check{\chi})^* \underline{\check{f}}\right)_0 = 0\,,$$
$$\left(\hat{a}_0(\check{\chi})^* \underline{\check{f}}\right)_1 (\mathbf{p}_1) = 2\omega_{\mathbf{p}_1} \overline{\check{\chi}(\mathbf{p}_1)}\, f_0\,,$$
$$\left(\hat{a}_0(\check{\chi})^* \underline{\check{f}}\right)_{n+1} (\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{\nu=1}^{n+1} 2\omega_{\mathbf{p}_\nu} \overline{\check{\chi}(\mathbf{p}_\nu)} \check{f}_n (\mathbf{p}_1, \ldots, \not{\mathbf{p}}_\nu, \ldots, \mathbf{p}_{n+1})\,,$$

for $\underline{\check{f}} \in D_0$; hence, formally, by

$$\left(\hat{a}_0(\mathbf{p})^* \underline{\check{f}}\right)_{n+1} (\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{\nu=1}^{n+1} 2\omega_{\mathbf{p}}\, \delta(\mathbf{p} - \mathbf{p}_\nu) \check{f}_n (\mathbf{p}_1, \ldots, \not{\mathbf{p}}_\nu, \ldots, \mathbf{p}_{n+1}) \tag{2.32}$$

for $\underline{\check{f}} \in D_0$. This means,

$$\hat{a}_0^*(\check{\chi}) \stackrel{\text{def}}{=} \int \hat{a}_0^*(\mathbf{p})\check{\chi}(\mathbf{p})\, d\mathbf{p} \stackrel{\text{def}}{=} \left(\int \hat{a}_0(\mathbf{p})\overline{\check{\chi}(\mathbf{p})}\, d\mathbf{p}\right)^* \restriction D_0 \tag{2.33}$$

creates a particle with momentum space wave function $2\omega_{\mathbf{p}}\, \check{\chi}(\mathbf{p})$. As a simple consequence of (2.23) we get the canonical commutation relations

$$\boxed{\begin{array}{c} [\hat{a}_0(\mathbf{p}), \hat{a}_0(\mathbf{p}')]_- = [\hat{a}_0^*(\mathbf{p}), \hat{a}_0^*(\mathbf{p}')]_- = 0\,, \\ [\hat{a}_0(\mathbf{p}), \hat{a}_0^*(\mathbf{p}')]_- = 2\omega_{\mathbf{p}}\, \delta(\mathbf{p} - \mathbf{p}')\,. \end{array}} \tag{2.34}$$

[18] We write $\not{\mathbf{p}}_\nu$ when $\mathbf{p}_\nu$ has to be skipped.

The precise meaning of (2.34) is[19]

$$\left.\begin{array}{l}[\hat{A}_{\check{f}},\hat{A}_{\check{g}}]_-=[\hat{A}^*_{\check{f}},\hat{A}^*_{\check{g}}]_-=0\\ [\hat{A}_{\check{f}},\hat{A}^*_{\check{g}}]_-=\langle\check{f}\mid\check{g}\rangle_{\mathcal{H}_0^{(1)}}\,\hat{1}\end{array}\right\}\quad\forall\,\check{f}\,,\,\check{g}\in\mathcal{S}(\mathbb{R}^3)$$

on D_0 , where

$$\hat{A}_{\check{f}}\stackrel{\text{def}}{=}\int\hat{a}_0(\mathbf{p})\,\frac{\overline{\check{f}(\mathbf{p})}}{2\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}\,,\quad\text{hence}\quad\hat{A}^*_{\check{f}}\restriction D_0=\int\hat{a}^*_0(\mathbf{p})\,\frac{\check{f}(\mathbf{p})}{2\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}\,.$$

The present representation of the canonical commutation relations has the ***Fock property***

$$\hat{A}_{\check{f}}\Omega=0\quad\forall\,\check{f}\in\mathcal{S}(\mathbb{R}^3)\,.$$

Moreover, according to (2.29) and (2.33), we have

$$\begin{aligned}\hat{U}_0(a,\Lambda)^{-1}\hat{a}^*_0(\mathbf{p})\hat{U}_0(a,\Lambda)&=e^{-ipa}\hat{a}^*_0\left(\overrightarrow{\Lambda^{-1}p}\right)_{|_{p^0=\omega_{\mathbf{p}}}}\,,\\ \hat{U}_0(a,\Lambda)\hat{a}^*_0(\mathbf{p})\hat{U}_0(a,\Lambda)^{-1}&=e^{+i\Lambda pa}\hat{a}^*_0\left(\overrightarrow{\Lambda p}\right)_{|_{p^0=\omega_{\mathbf{p}}}}\,.\end{aligned}\tag{2.35}$$

Expressions like

$$\langle\Psi_1\mid\hat{a}^*_0(\mathbf{p})\Psi_2\rangle\;,\quad\Psi_1\in\mathcal{H}\,,\;\Psi_2\in D_0\,,$$

being well-defined only when *smeared* by some test function $\check{\chi}$,

$$\int\langle\Psi_1\mid\hat{a}^*_0(\mathbf{p})\Psi_2\rangle\;\check{\chi}(\mathbf{p})\,\mathrm{d}\mathbf{p}\;\stackrel{\text{def}}{=}\;\langle\Psi_1\mid\hat{a}^*_0(\check{\chi})\Psi_2\rangle\;,$$

($\check{\chi}\in\mathcal{S}(\mathbb{R}^3)$, here), are called ***generalized functions*** (if linearly and continuously depending on the test function from a suitable topological *test space*). This name indicates that many operations, defined for ordinary functions, may be generalized to these functionals:

Let $\hat{K},\hat{K}'$ be 'sufficiently well-behaved' linear operators fulfilling

$$\int\left(\hat{K}\check{\chi}_1\right)(\mathbf{p})\check{\chi}_2(\mathbf{p})\,\mathrm{d}\mathbf{p}=\int\check{\chi}_1(\mathbf{p})\hat{K}'\check{\chi}_2(\mathbf{p})\,\mathrm{d}\mathbf{p}\tag{2.36}$$

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[19]Note that, if $\{\check{f}_\nu\}_{\nu\in\mathbb{N}}$ is a complete orthonormal system in $\mathcal{H}_0^{(1)}$, the operators

$$\hat{p}_\nu\stackrel{\text{def}}{=}\frac{\hat{A}^*_{\check{f}_\nu}\restriction D_0+\hat{A}_{\check{f}_\nu}}{\sqrt{2}}\;,\quad\hat{x}_\nu\stackrel{\text{def}}{=}\frac{\hat{A}^*_{\check{f}_\nu}\restriction D_0-\hat{A}_{\check{f}_\nu}}{i\sqrt{2}}$$

fulfill the relations (1.43) with $\hbar=1$ and that

$$\hat{P}_0\restriction D_0=\sum_{\nu=1}^{\infty}\left\langle f_\nu\,\middle|\,\hat{P}_0f_\nu\right\rangle\hat{A}^*_{\check{f}_\nu}\hat{A}_{\check{f}_\nu}\,.$$

for all test functions $\check{\chi}_1, \check{\chi}_2$. then it is natural to define, e.g.,

$$\int \left(\hat{K} F\right)(\mathbf{p})\, \check{\chi}(\mathbf{p})\, \mathrm{d}\mathbf{p} \stackrel{\text{def}}{=} \int F(\mathbf{p})\, \hat{K}' \check{\chi}(\mathbf{p})\, \mathrm{d}\mathbf{p} \quad \forall\, \check{\chi} \in \mathcal{S}(\mathbb{R}^3) \tag{2.37}$$

for continuous linear functionals F on $\mathcal{S}(\mathbb{R}^3)$. Correspondingly, we then define

$$\int \left(\hat{K} \hat{a}_0^*\right)(\mathbf{p}) \check{\chi}(\mathbf{p})\, \mathrm{d}\mathbf{p} = \int \hat{a}_0^*(\mathbf{p}) \hat{K}' \check{\chi}(\mathbf{p})\, \mathrm{d}\mathbf{p} \tag{2.38}$$

for test functions $\check{\chi}$. Generalizations of these prescriptions are obvious and will be used without special explanation.

Exercise 33 Determine the following operations on $\hat{a}_0^*(\mathbf{p})$:

(i) partial differentiation
(ii) multiplication by suitable functions
(iii) transformation of variables (e.g. Poincaré transformations)
(iv) Fourier transformation

Field Operators in Minkowski Space

Similarly to the wave functions $f(x)$ (see (2.13)) one defines the ***field operators***[20]

$$\hat{\Phi}_0^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \hat{a}_0(\mathbf{p})\, e^{-ipx} \frac{\mathrm{d}\mathbf{p}}{2\omega_{\mathbf{p}}}\,, \tag{2.39}$$

(***positive frequency part*** or ***creation part***) and

$$\hat{\Phi}_0^-(x) \stackrel{\text{def}}{=} \left(\hat{\Phi}_0^+(x)\right)^* \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=-\omega_{\mathbf{p}}} \hat{a}_0^*(-\mathbf{p})\, e^{-ipx} \frac{\mathrm{d}\mathbf{p}}{2\omega_{\mathbf{p}}}\,, \tag{2.40}$$

(***negative frequency part*** or ***annihilation part***) on D_0 , which are both solutions of the Klein-Gordon equation[21]

$$\left(\Box + m^2\right) \hat{\Phi}_0^\pm(x) = 0 \tag{2.41}$$

(in the sense of generalized functions) and, thanks to (2.29), transform according to

$$\hat{U}_0(a,\Lambda)\hat{\Phi}_0^\pm(x)\hat{U}_0(a,\Lambda)^{-1} = \hat{\Phi}_0^\pm(\Lambda x + a)\,. \tag{2.42}$$

[20]The exact definition is by multiplication with $\exp(-i\omega_{\mathbf{p}} x^0)/2\omega_{\mathbf{p}}$ and subsequent 3-Fourier transform w.r.t. $\mathbf{p}$, both in the sense of generalized functions.

[21]Note that, for $\underline{\check{f}} \in D_0$, $\left\langle \underline{\check{f}} \,\middle|\, \hat{\Phi}_0^+(x) \right\rangle$ resp. $\left\langle \underline{\check{f}} \,\middle|\, \hat{\Phi}_0^-(x) \right\rangle$ is a positive resp. negative frequency smooth Klein Gordon solution.

It it convenient to add both parts to get a hermitian field operator

$$\hat{\Phi}_0(x) \stackrel{\text{def}}{=} \hat{\Phi}_0^+(x) + \hat{\Phi}_0^-(x)\,. \tag{2.43}$$

Note that it it is sufficient to smear $\hat{\Phi}_0(x)$ in the space variables, i.e.

$$\hat{\Phi}_\psi(x^0) \stackrel{\text{def}}{=} \int \hat{\Phi}_0(x)\,\psi(\mathbf{x})\,d\mathbf{x}$$

is well defined for $x^0 \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$.

Exercise 34 Show for arbitrary $x^0 \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$:

(i) $\psi = \overline{\psi} \implies \left(\hat{\Phi}_\psi(x^0)\right)^* \restriction D_0 = \hat{\Phi}_\psi(x^0)$.

(ii) Every $\underline{\check{f}} \in D_0$ is an entire analytic vector for $\hat{\Phi}_\psi(x^0)$, i.e.: $\sum_{\nu=0}^\infty \frac{1}{\nu!} \left\| \left(\lambda \hat{\Phi}_\psi(x^0)\right)^\nu \underline{\check{f}} \right\| < \infty \quad \forall\, \lambda > 0$.

(iii) $\hat{\Phi}_\psi(x^0) D_0 \subset D_0$.

According to Exercise 34 and a well-known theorem by Nelson (Reed und Simon, 1972, Sect. X.4) $\hat{\Phi}_\psi(x^0)$ has a unique self-adjoint extension if ψ is real-valued. By (2.42), $\hat{\Phi}(x)$ transforms like the observable of the field strength of some Lorentz invariant scalar field:

$$\hat{U}_0(a,\Lambda)\hat{\Phi}_0(x)\hat{U}_0(a,\Lambda)^{-1} \underset{(2.42)}{=} \hat{\Phi}_0(\Lambda x + a)\,.$$

This suggests the interpretation

$$\hat{\Phi}_\psi(x^0) = \left\{ \begin{array}{l} \text{observable of the mean value } \int \Phi_0(x)\,\psi(\mathbf{x})\,d\mathbf{x} \\ \text{of the classical field}^{22}\ \Phi_0(x) \text{ at time zero}\,. \end{array} \right. \tag{2.44}$$

Since, for real-valued ψ, $\hat{\Phi}_\psi(x_0)$ should correspond to some measurement performable within $\operatorname{supp}\psi$ at time x^0 the condition of ***local commutativity***[23]

$$\left[\hat{\Phi}_0(x), \hat{\Phi}_0(y)\right]_- = 0 \quad \text{for } x \times y\,. \tag{2.45}$$

(also called ***microcausality*** for observable fields) should be fulfilled (compare Footnote 29). Indeed, (2.45) is a consequence of

$$\left[\hat{\Phi}_0(x), \hat{\Phi}_0(y)\right]_- = i\Delta_m(x-y) \stackrel{\text{def}}{=} \left\langle \Omega_0 \mid \left[\hat{\Phi}_0(x), \hat{\Phi}_0(y)\right]_- \Omega_0 \right\rangle \tag{2.46}$$

[22] As we will learn from the Haag-Ruelle scattering theory in Section 2.3, it is not that important to give a concrete physical interpretation for $\Phi_0(x)$.

[23] We use $x \times y$ as a shorthand notation for $(x-y)^2 < 0$; i.e. for x being space-like to y.

and the fact that

$$\Delta_m(x) = -i(2\pi)^{-3} \int \frac{p^0}{|p^0|} \delta(p^2 - m^2) e^{-ipx} \, \mathrm{d}p \,, \tag{2.47}$$

being an odd Lorentz invariant distribution, vanishes for $x \times 0$ (Güttinger and Rieckers, 1968) (see also Footnote 60).

Exercise 35 Show, for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^4)$, that

$$\int \hat{\Phi}_0(x) \, \varphi(x) \, \mathrm{d}x = \sqrt{2\pi} \, \left(\hat{a}_0(\check{\varphi}_-) + \hat{a}_0^*(\check{\varphi}_+) \right) \,,$$

where

$$\check{\varphi}_\pm(p) \stackrel{\text{def}}{=} \left(\frac{\widetilde{\varphi}(\pm p)}{2p^0} \right)_{|p^0 = \omega_{\mathbf{p}}}$$

and

$$\widetilde{\varphi}(p) \stackrel{\text{def}}{=} (2\pi)^{-1} \int \varphi(x) \, e^{+ipx} \, \mathrm{d}x \qquad \text{(Fourier transform)} \,.$$

By Exercise 35 we easily see that Ω_0 is a ***cyclic vector*** for the algebra $\mathcal{P}_0$ generated by $\hat{1} \restriction D_0$ and the smeared field field operators $\hat{\Phi}_0(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$, i.e.:

$$\mathcal{P}_0 \, \Omega \text{ is dense in } \mathcal{H}_0 \,.$$

Obviously, the common domain D_0 has the following invariance properties:[24]

$$\hat{\Phi}_0(\varphi) D_0 \subset D_0 \supset \hat{U}_0(a, \Lambda) D_0 \tag{2.48}$$

(hence $\mathcal{P}_0 \Omega \subset D_0$).

Exercise 36

(i) Determine the observable for the particle density according to Newton and Wigner.

(ii) What changes will arise for $\hat{\Phi}_0(x)$, if one defines $\hat{a}_0(\mathbf{p})$ by (2.23)/(2.28) using **anti**-symmetric $\check{f}_n(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ (and $\hat{a}_0^*(\mathbf{p})$ by (2.33), again)?

(iii) Determine the norm[25] of $\overline{\hat{\Phi}_0(\varphi)} \restriction \mathcal{H}_0^{(n)}$ as a function of $n \in \mathbb{Z}_+$.

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[24] We even have $D_0 = \hat{U}_0(a, \Lambda) D_0$.

[25] **Hint:** Recall Exercise 35 and study the operators considered in Footnote 19 first.

2.2 Wightman Theory for Neutral Scalar Fields

2.2.1 Wightman Axioms

A ***Wightman theory*** of a single neutral scalar field $\hat{\Phi}(x)$ is characterized by the following assumptions (*Wightman axioms*):

0. Assumptions of Relativistic Quantum Theory:

The pure states are given – in the sense of orthodox quantum theory – by the vectors Ψ of some separable,[26] complex Hilbert space $\mathcal{H}$ on which a (strongly) continuous, unitary representation $\hat{U}(a,\Lambda)$ of $\mathcal{P}_+^\uparrow$ acts according to the following interpretation:

> An ensemble corresponding to $\hat{U}(a,\Lambda)\Psi$ is to be described with respect to the coordinates $x' = \Lambda^{-1}(x-a)$ in exactly the same way as an ensemble corresponding to $\Psi \in \mathcal{H}$ is to be described with respect to the coordinates x .

This representation fulfills the following ***spectrum condition:***

For the spectral measure $\hat{E}$ of the observable

$$\hat{P} = \int p\,\hat{E}(\mathrm{d}p)$$

of 4-momentum, uniquely characterized by[27]

$$\left\langle \Psi \mid \hat{U}(a,\mathbb{1}_4)\Psi \right\rangle = \int e^{ipa} \left\langle \Psi \mid \hat{E}(\mathrm{d}p)\Psi \right\rangle \quad \forall\, \Psi \in \mathcal{H}\,,$$

we have

$$\hat{E}(B) = 0 \text{ for all Borel } B \subset \mathbb{R}^4 \setminus \overline{V_+}\,,$$

where

$$V_\pm \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^4 : xx > 0\,,\ \pm x^0 > 0 \right\}\,.$$

There is a normed vector Ω , unique up to a phase factor, that is invariant under the representation of $\mathcal{P}_+^\uparrow$:

$$\hat{U}(a,\Lambda)\Omega = \Omega \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,.$$

This vector describes the ***vacuum state*** of the theory.

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[26] Actually, separability is a consequence of the separability of the test function space and cyclicity of the vacuum state.

[27] That such a projection-valued measure $\hat{E}$ on $\mathbb{R}^4$ exists is guaranteed by the so-called ***SNAG theorem*** (see (Streater and Wightman, 1989, p. 92) and references given there).

I. Assumptions about the Domain and Continuity of the Field:

The field $\hat{\Phi}(x)$ is a ***hermitian operator-valued, tempered generalized function with invariant domain*** $D \subset \mathcal{H}$; i.e. a **linear** mapping[28]

$$\begin{aligned} \hat{\Phi} \,:\, \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D,D) \\ \varphi &\longmapsto \hat{\Phi}(\varphi) = \underbrace{\int \hat{\Phi}(x)\varphi(x)\,\mathrm{d}x}_{\text{formal}} \end{aligned}$$

for which all the

$$\int \left\langle \Psi \mid \hat{\Phi}(x)\Psi \right\rangle \varphi(x)\,\mathrm{d}x \stackrel{\text{def}}{=} \left\langle \Psi \mid \hat{\Phi}(\varphi)\Psi \right\rangle \,, \Psi \in D\,,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a,\Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \subset D\,, \quad \hat{U}(a,\Lambda)D \subset D\,, \quad \hat{\Phi}(\varphi)D \subset D\,, \quad \hat{\Phi}(\overline{\varphi}) = \hat{\Phi}(\varphi)^* {\restriction} D\,.$$

II. Transformation Law of the Field:

The field transforms according to

$$\hat{U}(a,\Lambda)\hat{\Phi}(x)\hat{U}(a,\Lambda)^{-1} = \hat{\Phi}(\Lambda x + a) \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,.$$

III. Local Commutativity (Microscopic Causality):

The smeared fields $\hat{\Phi}(\varphi_1)$, $\hat{\Phi}(\varphi_2)$ commute whenever the supports of the test functions φ_1 , $\varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other.[29] Formally:

$$x \times y \implies [\hat{\Phi}(x), \hat{\Phi}(y)]_- = 0\,.$$

Finally, the vacuum vector Ω is required to be **cyclic** with respect to the algebra $\mathcal{F}_0$ generated by $\hat{1}{\restriction}D$ and the smeared field operators $\hat{\Phi}(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$. This means:

$$\mathcal{F}_0\,\Omega \text{ is dense in } \mathcal{H}\,.$$

Obviously, all the Wightman axioms are fulfilled for the free field $\hat{\Phi}_0(x)$.

[28] As usual, if X and Y are linear spaces, we denote by $L(X,Y)$ the set of all linear mappings from X into Y. We do no longer assume smearing of the field in the space variables to be sufficient. Actually it can be shown that smearing in x^0 would be sufficient (Borchers, 1964).

[29] That this condition has to be fulfilled in order to avoid acausal effects even at the microscopic scale, if the field $\hat{\Phi}(x)$ is fully observable, is shown in (Schlieder, 1971).

2.2.2 Remarks on the Choice of the Space of Test Functions

Originally (Wightman, 1956), Wightman used the Schwartz space

$$\mathcal{D}(\mathbb{R}^4) \stackrel{\text{def}}{=} \left\{\varphi \in \mathcal{S}(\mathbb{R}^4) : \text{supp}\varphi \text{ compact}\right\}$$

with the usual topology:

$\varphi_\nu \longrightarrow \varphi$ in $\mathcal{D}(\mathbb{R}^4)$ if and only if[30]

$$D_x^\alpha \varphi_\nu(x) \longrightarrow D_x^\alpha \varphi(x) \text{ uniformly in } x \in \mathbb{R}^4 \quad \forall\, \alpha \in \mathbb{Z}_+^4$$

and if there is a compact subset $K \subset \mathbb{R}^4$ outside which all the φ_ν vanish.

Since the Fourier transform $\widetilde{\varphi}$ of a test function $\varphi \in \mathcal{D}(\mathbb{R}^4)$ is always an entire analytic function, the generalized functions on $\mathcal{D}(\mathbb{R}^4)$ are ***non-localizable***, in general, i.e. there is no notion of support in the ordinary sense for this class of Distributions. In his original program (Wightman, 1956), Wightman indicated corresponding problems by the remark:

"We shall assume that $F^{(n)}$ has a Fourier transform."

Now, the Fourier transform $\widetilde{F}^{(n)}$ of a generalized function $F^{(n)}$ on $\mathcal{D}(\mathbb{R}^4)$ is always well defined on the Fourier dual $\widetilde{\mathcal{D}}(\mathbb{R}^4)$ of $\mathcal{D}(\mathbb{R}^4)$. What Wightman ment was that his use of the Laplace transform of $\widetilde{F}^{(n)}$ should be justified. Using the notion of quasi-support for nonlocalizable generalized functions, introduced in (Bümmerstede and Lücke, 1974) (via *local continuity*, as explained in 3.3.3), and the corresponding definition of Laplace transform, introduced in (Lücke, 1984, Sect. 4), there is no problem at all. Another method to justify Wightman's results without additional assumptions was presented earlier in (Borchers, 1964, Sect. 5).

The choice of test function space may well be crucial (Wightman, 1981), since:

Nobody could construct a Wightman field (for the test space $\mathcal{S}(\mathbb{R}^4)$) with nontrivial interaction.

If physical space-time $\mathbb{R}^1 \times \mathbb{R}^3$ is replaced by a toy space-time $\mathbb{R}^1 \times \mathbb{R}^d$ with $d < 3$ then the test space $\mathcal{S}(\mathbb{R}^{d+1})$ is known to be suitable.[31]

E.P. Osipov (hep-th/9608115) claims to be able to construct a field $\hat{\Phi}(x)$ on the physical space-time with nontrivial S-matrix fulfilling all the Wightman axioms with $\mathcal{S}(\mathbb{R}^4)$ **replaced** by some ***Jaffe space***[32] $\mathcal{J}(\mathbb{R}^4)$, where

$$\mathcal{D}(\mathbb{R}^4) \cap \mathcal{J}(\mathbb{R}^4) \quad \text{is dense in} \quad \mathcal{J}(\mathbb{R}^4) \subset \mathcal{S}(\mathbb{R}^4)\,.$$

[30] We use standard multi-index notation: $D_x^\alpha \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x^0}\right)^{\alpha_0} \cdots \left(\frac{\partial}{\partial x^3}\right)^{\alpha_3}$.

[31] See (Streater and Wightman, 1989, Appendix) for a neat review of the construction of corresponding models. For more details see (Glimm and Jaffe, 1981).

[32] See (Jaffe, 1967) for Jaffe's class of test spaces for **localizable** fields.

Examples of nonlocalizable fields are carefully studied in (Rieckers, 1971). How the condition of microcausality has to be modified for these examples is shown in (Lücke, 1974a).

2.2.3 Mathematical Tools

Theorem 2.2.1 (Schwartz's Nuclear Theorem) *Let $n_1, n_2 \in \mathbb{N}$ and let*

$$\begin{array}{rcl} F \, : \, \mathcal{S}(\mathbb{R}^{n_1}) \times \mathcal{S}(\mathbb{R}^{n_2}) & \longrightarrow & \mathbb{C} \\ (\varphi_1, \varphi_2) & \longmapsto & F(\varphi_1, \varphi_2) \end{array}$$

be linear and continuous in each argument separately. Then there is a one and only one generalized function $F(x_1, x_2)$ on $\mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with

$$\int F(x_1, x_2)\, \varphi_1(x_1)\varphi_2(x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 = F(\varphi_1, \varphi_2) \quad \forall\, (\varphi_1, \varphi_2) \in \mathcal{S}(\mathbb{R}^{n_1}) \times \mathcal{S}(\mathbb{R}^{n_2})\,.$$

Proof:[33] See (Gelfand and Wilenkin, 1964, Chapter I §1 Nr. 2). ∎

Theorem 2.2.1 (together with the Hahn-Banach theorem) implies that[34]

$$\mathcal{S}(\mathbb{R}^{n_1}) \otimes \mathcal{S}(\mathbb{R}^{n_2}) \text{ is dense in } \mathcal{S}(\mathbb{R}^{n_1+n_1}) \quad \forall\, n_1\,,\, n_2 \in \mathbb{N}\,. \tag{2.49}$$

Therefore the following Lemma allows iteration (Corollary 2.2.3, below) of the nuclear theorem.

Lemma 2.2.2 *Let $n \in \mathbb{N}$ and let $\{F_\nu\}_{\nu \in \mathbb{Z}_+} \subset \mathcal{S}(\mathbb{R}^n)'$ be such that $\lim_{\nu \to \infty} F_\nu(\varphi)$ exists for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$F(\varphi) \stackrel{\text{def}}{=} \lim_{\nu \to \infty} F_\nu(\varphi) \quad \textit{for } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

defines a tempered generalized function on $\mathcal{S}(\mathbb{R}^n)$; i.e. $F \in \mathcal{S}(\mathbb{R}^n)'$.

Proof: See (Gelfand and Schilow, 1962, Ch. I §5 No. 6). ∎

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[33] See also (Bümmerstede and Lücke, 1974, Appendix).

[34] As usual, we denote by $\mathcal{S}(\mathbb{R}^{n_1}) \otimes \mathcal{S}(\mathbb{R}^{n_2})$ the algebraic tensor product of $\mathcal{S}(\mathbb{R}^{n_1})$ and $\mathcal{S}(\mathbb{R}^{n_2})$ realized as the linear span of the set of all $\varphi \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ of the form

$$\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)\,, \quad \varphi_j \in \mathcal{S}(\mathbb{R}^{n_j}) \text{ for } j = 1, 2\,.$$

Corollary 2.2.3 *Let $k, n_1, \ldots, n_k \in \mathbb{N}$ and let*

$$\begin{array}{rcl} F \,:\, \mathcal{S}(\mathbb{R}^{n_1}) \times \ldots \times \mathcal{S}(\mathbb{R}^{n_k}) & \longrightarrow & \mathbb{C} \\ (\varphi_1, \ldots, \varphi_k) & \longmapsto & F(\varphi_1, \ldots, \varphi_k) \end{array}$$

be linear and continuous in each argument separately. Then there is a one and only one generalized function $F \in \mathcal{S}(\mathbb{R}^{n_1+\ldots+n_k})'$ with

$$\int F(x_1, \ldots, x_k)\, \varphi_1(x_1) \cdots \varphi_k(x_k)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k = F(\varphi_1, \ldots, \varphi_k)$$

for all $(\varphi_1, \ldots, \varphi_k) \in \mathcal{S}(\mathbb{R}^{n_1}) \times \ldots \times \mathcal{S}(\mathbb{R}^{n_k})$.

Given $n \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$, let us define

$$\left(\hat{K}_\Delta \varphi\right) \left(\frac{x_1 + \ldots + x_n}{n}, x_1 - x_2, \ldots, x_{n-1} - x_n\right) \stackrel{\text{def}}{=} \varphi(x_1, \ldots, x_n)$$

for $x_1, \ldots, x_n \in \mathbb{R}^4$. Then $\hat{K}_\Delta$ and its inverse $\hat{K}_\Delta^{-1}$ are linear continuous mappings of $\mathcal{S}(\mathbb{R}^{4n})$ into itself with

$$\begin{array}{l} \int \left(\hat{K}_\Delta \varphi\right) (X, \xi_1, \ldots, \xi_{n-1})\, \psi(X, \xi_1, \ldots, \xi_{n-1})\, \mathrm{d}X \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{n-1}) \\ = \int \varphi(x_1, \ldots, x_n) \left(\hat{K}_\Delta^{-1} \psi\right) (x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \quad \forall\, \varphi\,,\, \psi \in \mathcal{S}(\mathbb{R}^{4n})\,. \end{array}$$

Therefore, as explained in 2.1.3, $\hat{K}_\Delta F$ is well defined for tempered **generalized** functions F on $\mathbb{R}^{4n}$.

Lemma 2.2.4 *Let $n \in \mathbb{R}^4$ and let $W \in \mathcal{S}(\mathbb{R}^{4n})'$ fulfill*

$$W(x_1 + a, \ldots, x_n + a) = W(x_1, \ldots, x_n) \quad \forall\, a \in \mathbb{R}^4\,.$$

Then there is a unique generalized function $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$ with

$$W(x_1, \ldots, x_n) = \mathcal{W}(x_1 - x_2, \ldots, x_{n-1} - x_n)\,,$$

i.e.:

$$\begin{array}{l} \int W(x_1, \ldots, x_n)\, \varphi(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ = \int \mathcal{W}(\xi_1, \ldots, \xi_{n-1}) \left(\int \left(\hat{K}_\Delta\, \varphi\right) (a, \xi_1, \ldots, \xi_{n-1})\, \mathrm{d}a\right) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \end{array}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

Sketch of proof:[35] Let us choose some $\chi \in \mathcal{S}(\mathbb{R}^4)$ with

$$\int \chi(a)\,\mathrm{d}a = 1$$

and use the short-hand notation

$$\begin{aligned} \check{x} &\stackrel{\text{def}}{=} (x_1,\dots,x_n)\,, & \check{\xi} &\stackrel{\text{def}}{=} (\xi_1,\dots,\xi_{n-1})\,,\\ \mathrm{d}\check{x} &\stackrel{\text{def}}{=} \mathrm{d}x_1\cdots\mathrm{d}x_n\,, & \mathrm{d}\check{\xi} &\stackrel{\text{def}}{=} \mathrm{d}\xi_1\cdots\mathrm{d}\xi_{n-1}\,. \end{aligned}$$

Then

$$\begin{aligned} &\int W(\check{x})\,\varphi(\check{x})\,\mathrm{d}\check{x}\\ &= \int W(\check{x})\,\varphi(x_1-a,\dots,x_n-a)\,\mathrm{d}\check{x}\\ &= \int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\left(\hat{K}_\Delta\varphi\right)(X-a,\check{\xi})\mathrm{d}X\mathrm{d}\check{\xi}\\ &= \int \left(\int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\left(\hat{K}_\Delta\varphi\right)(X-a,\check{\xi})\mathrm{d}X\mathrm{d}\check{\xi}\right)\chi(a)\,\mathrm{d}a\\ &= \int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\left(\int \left(\hat{K}_\Delta\varphi\right)(X-a,\check{\xi})\,\chi(a)\,\mathrm{d}a\right)\mathrm{d}X\mathrm{d}\check{\xi}\\ &= \int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\left(\int \left(\hat{K}_\Delta\varphi\right)(a,\check{\xi})\,\chi(X-a)\,\mathrm{d}a\right)\mathrm{d}X\mathrm{d}\check{\xi}\\ &= \int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\left(\int \left(\hat{K}_\Delta\varphi\right)(a,\check{\xi})\,\chi(X)\,\mathrm{d}a\right)\mathrm{d}X\mathrm{d}\check{\xi}\\ &= \int \left(\hat{K}_\Delta W\right)(X,\check{\xi})\chi(X)\left(\int \left(\hat{K}_\Delta\varphi\right)(a,\check{\xi})\,\mathrm{d}a\right)\mathrm{d}X\mathrm{d}\check{\xi} \end{aligned}$$

and hence

$$\mathcal{W}(\xi_1,\dots,\xi_{n-1}) = \int \left(\hat{K}_\Delta\,W\right)(X,\xi_1,\dots,\xi_{n-1})\,\chi(X)\,\mathrm{d}X\,. \qquad \blacksquare$$

Theorem 2.2.5 (Bochner-Schwartz) *Let $\mathcal{W} \in \mathcal{D}(\mathbb{R}^4)$ be* ***positive semi definite****, i.e. fulfill*

$$\int \mathcal{W}(x-y)\,\overline{\varphi(x)}\varphi(y)\,\mathrm{d}x\mathrm{d}y \geq 0 \quad \forall\,\varphi \in \mathcal{D}(\mathbb{R}^4)\,.$$

Then there are a (unique) positive Borel measure μ on $\mathbb{R}^4$ and some $k \in \mathbb{Z}_+$ with

$$\int (1+\|p\|)^{-k}\,\mu(\mathrm{d}p) < \infty$$

and[36]

$$\int \widetilde{\mathcal{W}}(q)\,\psi(q)\,\mathrm{d}q = \int \psi(q)\,\mu(\mathrm{d}q) \quad \forall\,\psi \in \widetilde{\mathcal{D}}(\mathbb{R}^4)\,.$$

Hence $\mathcal{W}$ is the restriction to $D(\mathbb{R}^4)$ of a tempered generalized function.

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[35] The interchanges of (ordinary and/or formal) integration may be easily justified by intermediate regularization of the involved generalized functions. An alternative, more indirect, proof can be found in (Streater and Wightman, 1989, p. 39/40).

[36] Of course, $\widetilde{\mathcal{D}}(\mathbb{R}^4)$ means the Fourier dual of $\mathcal{D}(\mathbb{R}^4)$: $\widetilde{\mathcal{D}}(\mathbb{R}^4) \stackrel{\text{def}}{=} \{\widetilde{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^4)\}$.

Proof: See (Gelfand and Wilenkin, 1964, Ch. II §3). ∎

Lemma 2.2.6 *Let $n \in \mathbb{R}^4$ and let $W \in \mathcal{S}(\mathbb{R}^{4n})'$ fulfill*

$$W(x_1 + a, \ldots, x_n + a) = W(x_1, \ldots, x_n) \quad \forall\, a \in \mathbb{R}^4$$

as well as

$$\mathrm{supp}\widetilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V_+} \implies \int W(x_1, \ldots, x_\nu + a, \ldots, x_n + a)\, \chi(a)\, \mathrm{d}a = 0$$

for all $\chi \in \mathcal{S}(\mathbb{R}^4)$ and $\nu = 2, \ldots, n$. Then for the Fourier transform

$$\widetilde{\mathcal{W}}(q_1, \ldots, q_{n-1}) \stackrel{\text{def}}{=} (2\pi)^{-2(n-1)} \int \mathcal{W}(\xi_1, \ldots, \xi_{n-1})\, e^{i(\xi_1 q_1 + \ldots + \xi_{n-1} q_{n-1})}\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1}$$

of the generalized function

$$\mathcal{W}(\xi_1, \ldots, \xi_{n-1}) = W(\xi_1 + \ldots + \xi_{n-1}, \xi_2 + \ldots + \xi_{n-1}, \ldots, \xi_{n-1}, 0)$$

given by Lemma 2.2.4 we have:

$$\mathrm{supp}\widetilde{\mathcal{W}} \subset \overline{V_+} \times \cdots \times \overline{V_+}\,.$$

Sketch of proof: By Lemma 2.2.4 we have for all $\nu \in \{2, \ldots, n\}$

$$W(x_1, \ldots, x_{\nu+1} + a, \ldots, x_n + a) = \mathcal{W}(\xi_1, \ldots, \xi_\nu - a, \ldots, \xi_{\nu-1}) \quad \forall\, a \in \mathbb{R}^4$$

and hence for all $\chi \in \mathcal{S}(\mathbb{R}^4)$ with $\mathrm{supp}\widetilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V_+}$

$$\begin{aligned} 0 &= \int \left(\int \mathcal{W}(\xi_1, \ldots, \xi_\nu - a, \ldots, \xi_{n-1})\, e^{i(\xi_1 q_1 + \ldots + \xi_{n-1} q_{n-1})}\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \right) \chi(a)\, \mathrm{d}a \\ &= \left(\int \mathcal{W}(\xi_1, \ldots, \xi_{n-1})\, e^{i(\xi_1 q_1 + \ldots + \xi_{n-1} q_{n-1})}\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \right) \int \chi(a) e^{i q_\nu a}\, \mathrm{d}a \\ &= (2\pi)^{2(n-1)}\, \widetilde{\mathcal{W}}(q_1, \ldots, q_{n-1})\, \widetilde{\chi}(q_\nu)\,. \end{aligned}$$ ∎

Lemma 2.2.7 *Let $1 < n \in \mathbb{N}$ and let $\mathcal{W}(\xi_1, \ldots, \xi_{n-1}) \in \mathcal{S}(\mathbb{R}^{4(n-1)})$ be such that*

$$\mathrm{supp}\widetilde{\mathcal{W}} \subset \overline{V_+} \times \cdots \times \overline{V_+}\,.$$

Then there is one and only one holomorphic function $\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(z_1, \ldots, z_{n-1})$ on

$$\mathcal{T}_{n-1} \stackrel{\text{def}}{=} \left\{ (z_1, \ldots, z_{n-1}) \in \mathbb{C} : \Im(z_\nu) \in V_- \text{ for } \nu = 1, \ldots, n-1 \right\},$$

fulfilling the condition[37]

$$\begin{aligned} &\int \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1})\, \widetilde{\varphi}(\xi_1, \ldots, \xi_{n-1})\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \\ &= \int \widetilde{\mathcal{W}}(q_1, \ldots, q_{n-1}) \left(e^{q_1 \eta_1 + \ldots + q_{n-1}\eta_{n-1}} \varphi(q_1, \ldots, q_{n-1}) \right) \mathrm{d}q_1 \cdots \mathrm{d}q_{n-1}\,. \end{aligned}$$

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[37] Note that

$$\int F(q_1, \ldots, q_{n-1})\, \widetilde{\varphi}(q_1, \ldots, q_{n-1})\, \mathrm{d}q_1 \cdots \mathrm{d}q_{n-1} = \int \widetilde{F}(q_1, \ldots, q_{n-1})\, \varphi(q_1, \ldots, q_{n-1})\, \mathrm{d}q_1 \cdots \mathrm{d}q_{n-1}$$

for all $F \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$ and $\varphi \in \mathcal{S}(\mathbb{R}^{4(n-1)})$.

*This so-called **Laplace transform***

$$\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1}) \\ = \underbrace{\int \widetilde{\mathcal{W}}(q_1, \ldots, q_{n-1}) e^{-i(q_1(\xi_1+i\eta_1)+\ldots+q_{n-1}(\xi_{n-1}+i\eta_{n-1}))}\, \mathrm{d}q_1 \cdots \mathrm{d}q_{n-1}}_{\text{formal}}$$

of $\widetilde{\mathcal{W}}$ fulfills[38]

$$\lim_{V_- \ni \eta_1, \ldots, \eta_{n-1} \to 0} \int \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1})\, \psi(\xi_1, \ldots, \xi_{n-1})\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \\ = \int \mathcal{W}(\xi_1, \ldots, \xi_{n-1})\, \psi(\xi_1, \ldots, \xi_{n-1})\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-1} \quad \forall\, \psi \in \mathcal{S}(\mathbb{R}^{4(n-1)})\,.$$

Sketch of proof: Choose some $\chi \in C^\infty(\mathbb{R}^{4(n-1)})$ with

$$\chi(\check{\xi}) = \begin{cases} 1 & \text{if } \left\|\check{\xi} - \check{\xi}'\right\| < 1 \text{ for all } \check{\xi}' \in V_+ \times \cdots \times V_+ \\ 0 & \text{if } \left\|\check{\xi} - \check{\xi}'\right\| > 2 \text{ for some } \check{\xi}' \in V_+ \times \cdots \times V_+ \end{cases}$$

and define

$$\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1}) \\ \stackrel{\text{def}}{=} \int \widetilde{\mathcal{W}}(q_1, \ldots, q_{n-1}) \Big(\underbrace{e^{-i(q_1(\xi_1+i\eta_1)+\ldots+q_{n-1}(\xi_{n-1}+i\eta_{n-1}))}\chi(q_1, \ldots, q_{n-1})}_{\in \mathcal{S}(\mathbb{R}^{4(n-1)})}\Big)\, \mathrm{d}q_1 \cdots \mathrm{d}q_{n-1}$$

for $\eta_1, \ldots, \eta_{n-1} \in V_-$. ■

The following theorem shows that $\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1})$ is already fixed by its (distributional) boundary values on any open subset $\mathcal{O}$ of $\mathbb{R}^{n-1}$.

Theorem 2.2.8 (Edge-of-the-Wedge) *Let $n \in \mathbb{N}$, let $\check{\mathcal{O}}$ be an open subset of $\mathbb{C}^n$ for which $\mathcal{O} \stackrel{\text{def}}{=} \check{\mathcal{O}} \cap \mathbb{R}^n \neq 0$, let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^n$ with apex at 0, and let L be a holomorphic function on*

$$\mathcal{B} \stackrel{\text{def}}{=} (\mathbb{R}^n + i\,\mathcal{C}) \cap \check{\mathcal{O}}$$

such that

$$\lim_{\mathcal{C} \ni y \to 0} \int L(x + iy)\, \varphi(x)\, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathcal{D}(\mathbb{R}^n) \textit{ with } \operatorname{supp}\varphi \subset \mathcal{O}\,.$$

Then

$$L(x + iy) = 0 \quad \forall\, x + iy \in \mathcal{B}\,.$$

[38] Compare (Lücke, 1984, Theorem 4).

Proof: See (Streater and Wightman, 1989, Theorem 2-17). ∎

Corollary 2.2.9 *Let* $1 < n \in \mathbb{N}$ *and* $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$. *If*

$$\mathrm{supp}\widetilde{\mathcal{W}} \subset \overline{V_+} \times \ldots \times \overline{V_+}$$

then either $\mathcal{W} = 0$ *or* $\mathrm{supp}\mathcal{W} = \mathbb{R}^{4(n-1)}$.

Proof: Direct consequence of Lemma 2.2.7 and Theorem 2.2.8. ∎

2.2.4 Some Standard Results

By Corollary 2.2.3, for every $n \in \mathbb{N}$ and every $\Psi \in D$ there is a unique generalized function

$$\left\langle \Psi \mid \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\Psi \right\rangle \in \mathcal{S}(\mathbb{R}^{4n})'$$

with

$$\int \left\langle \Psi \mid \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\Psi \right\rangle \varphi_1(x_1) \cdots \varphi_n(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \left\langle \Psi \mid \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n)\Psi \right\rangle$$

for all $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$. Thus, especially, the so-called ***n-point functions***

$$W(x_1, \ldots, x_n) = \left\langle \Omega \mid \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\Omega \right\rangle , \tag{2.50}$$

are well defined as generalized functions on $\mathcal{S}(\mathbb{R}^{4n})$. The relativistic transformation law for $\hat{\Phi}(x)$ and the invariance of the vacuum imply

$$\int W(\Lambda x_1 + a, \ldots, \Lambda x_n + a)\varphi_1(x_1) \cdots \varphi_n(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \left\langle \Omega \mid \hat{U}(a,\Lambda)\hat{\Phi}(\varphi_1) \underbrace{\hat{U}(a,\Lambda)^{-1}\hat{U}(a,\Lambda)}_{=\hat{1}} \cdots \hat{U}(a,\Lambda)\hat{\Phi}(\varphi_n) \underbrace{\hat{U}(a,\Lambda)^{-1}\Omega}_{=\Omega} \right\rangle$$

and, by the same reasoning,

$$\int W(x_1, \ldots, x_\nu + a, \ldots, x_n + a)\varphi_1(x_1) \cdots \varphi_n(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \left\langle \Omega \mid \hat{\Phi}(\varphi_1) \cdots \hat{U}(a)\hat{\Phi}(\varphi_2) \cdots \hat{\Phi}(\varphi_n)\Omega \right\rangle$$

for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$, and $\nu = 2, \ldots, n$. By (2.49), this implies

$$W(x_1, \ldots, x_n) \stackrel{\text{def}}{=} W(\Lambda x_1 + a, \ldots, \Lambda x_n + a) \quad \forall\, n \in \mathbb{N}\,,\ (a, \Lambda) \in \mathcal{P}_+^\uparrow \tag{2.51}$$

and, thanks to the spectrum condition,[39]

$$\operatorname{supp}\widetilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V_+} \implies \int W(x_1, \ldots, x_\nu + a, \ldots, x_n + a)\, \chi(a)\, \mathrm{d}a = 0\,.$$

Therefore, according to Lemma 2.2.4 and Lemma 2.2.6, for every natural number $n > 1$ there is a generalized function $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})$ with

$$W(x_1, \ldots, x_n) = \mathcal{W}(x_1 - x_2, \ldots, x_{n-1} - x_n) \tag{2.52}$$

and

$$\operatorname{supp}\widetilde{\mathcal{W}} \subset \overline{V_+} \times \cdots \times \overline{V_+}\,. \tag{2.53}$$

Exercise 37 Show the following:

(i) For every $\varphi \in \mathcal{S}(\mathbb{R}^4)$ there is a unique[40] operator

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\, \varphi(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \in L(D, \mathcal{H})$$

with

$$\left\langle \Psi \,\middle|\, \left(\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\, \varphi(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right) \Psi \right\rangle$$
$$= \int \left\langle \Psi \mid \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Psi \right\rangle \varphi(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \quad \forall\, \Psi \in D\,.$$

(ii) For every $\Psi \in D$

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\, \varphi(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \Psi$$

depends strongly continuously on $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

(iii) For all $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n)\, \varphi_1 \cdots \varphi_n\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n)\,.$$

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[39] Note that $\int \hat{U}(a)\, \chi(a)\, \mathrm{d}a = \int \widetilde{\chi}(p)\, \hat{E}(\mathrm{d}p)$.

[40] Recall that $\left\langle \Phi \mid \hat{A}\Psi \right\rangle = \frac{1}{4} \sum_{k=0}^{3} \left\langle \Phi + i^k \Psi \mid \hat{A}\,(\Phi + i^k \Psi) \right\rangle \quad \forall\, \Phi, \Psi \in D\,,\ \hat{A} \in L(D, \mathcal{H})$.

By Exercise 37, without any restriction of generality, we may assume[41]

$$\int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)\,\varphi(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n D \subset D$$

and

$$\begin{aligned}\left(\int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)\,\varphi(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right)^* \restriction D \\ = \int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)\,\overline{\varphi(x_n,\ldots,x_1)}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\end{aligned} \tag{2.54}$$

for all $n \in \mathbb{N}$ and all $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

Obviously, by the GNS technique (recall Footnote 58 of Chapter 1), a Wightman theory with $D = \mathcal{F}_0\Omega$ can be reconstructed from its n-point functions, up to unitary equivalence.[42]

Lemma 2.2.10 *In a field theory of the type described in 2.2.1, with* $\dim\mathcal{H} > 1$, *the field operator* $\hat{\Phi}(x)$ *cannot be defined pointwise for* $x \in \mathbb{R}^4$.

Proof:[43] Obviously, the generalized function $\mathcal{W}(\xi) \in \mathcal{S}(\mathbb{R}^4)'$ associated with the 2-point function is positive semi-definite. Hence, by Theorem 2.2.5, there is a polynomially bounded positive Borel measure μ on $\mathbb{R}^4$ with

$$W(x_1,x_2) = \mathcal{W}(x_1 - x_2) = \int e^{ip(x_1-x_2)}\mu(\mathrm{d}p)\,.$$

By (2.51), μ has to be Lorentz covariant:[44]

$$\mu(B) = \mu(\Lambda B) \quad \forall\,\text{Borel } B \in \mathbb{R}^4\,,\ \Lambda \in L_+^{\uparrow}\,.$$

Let us assume that $\hat{\Phi}(x)$ is well defined for every $x \in \mathbb{R}^4$. Then $\mu(\mathbb{R}^4) = \mathcal{W}(0)$ must be finite. Hence there is a number $\mu_0 \geq 0$ with

$$\mu(B) = \begin{cases} \mu_0 & \text{if } 0 \in B \\ 0 & \text{else} \end{cases} \qquad \forall\,\text{Borel } B \in \mathbb{R}^4\,.$$

This implies

$$\left\langle \hat{\Phi}(x_1)\Omega \mid \hat{\Phi}(x_2)\Omega \right\rangle = W(x_1,x_2) = \mathcal{W}(x_1 - x_2) = \mu_0 \quad \forall\, x_1\,,\, x_2 \in \mathbb{R}^4$$

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[41] Here we use

$$\begin{aligned}\left(\int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_{n_1})\,\varphi_1(x_1,\ldots,x_{n_1})\,\mathrm{d}x_1\cdots\mathrm{d}x_{n_1}\right)\left(\int \hat{\Phi}(y_1)\cdots\hat{\Phi}(y_{n_2})\,\varphi_2(y_1,\ldots,y_{n_2})\,\mathrm{d}y_1\cdots\mathrm{d}y_{n_2}\right) \\ = \int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_{n_1})\hat{\Phi}(y_1)\cdots\hat{\Phi}(y_{n_2})\,\varphi_1(x_1,\ldots,x_{n_1})\varphi_2(y_1,\ldots,y_{n_2})\,\mathrm{d}x_1\cdots\mathrm{d}x_{n_1}\mathrm{d}y_1\cdots\mathrm{d}y_{n_2}\end{aligned}$$

of course.

[42] See (Streater and Wightman, 1989, Sect. 3-4) for full details.

[43] The main part of this proof, which uses neither microcausality nor the spectrum condition, was presented in (Jaffe, 1967, Introduction).

[44] As usual we denote by $L_+^{\uparrow}$ the set of all Lorentz transformations Λ with $(0,\Lambda) \in \mathcal{P}_+^{\uparrow}$.

even though

$$\left\|\hat{\Phi}(x)\Omega\right\| = \left\|\hat{U}(x)\hat{\Phi}(0)\Omega\right\| = \left\|\hat{\Phi}(0)\Omega\right\| \quad \forall\, x \in \mathbb{R}^4 \,.$$

This, in turn, implies

$$\hat{\Phi}(x)\Omega = \hat{\Phi}(0)\Omega \quad \forall\, x \in \mathbb{R}^4$$

and hence

$$\hat{U}(a,\Lambda)\hat{\Phi}(x)\Omega = \hat{\Phi}(x)\Omega \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,,\; x \in \mathbb{R}^4 \,.$$

Thanks to uniqueness of the vacuum state, therefore,

$$\hat{\Phi}(x)\Omega \sim \Omega \quad \forall\, x \in \mathbb{R}^4 \,.$$

By cyclicity of the Ω, finally, this gives $\dim \mathcal{H} = 1$. Since this contradicts the assumptions of the lemma, $\hat{\Phi}(x)$ cannot be well defined for every $x \in \mathbb{R}^4$. ∎

For the theory of a single neutral scalar field the theorem on the connection between **spin and statistics** becomes especially simple:[45]

Theorem 2.2.11 *There is no quantum field $\hat{\Phi}(x) \neq 0$, not necessarily fulfilling microcausality but all the other assumptions of 2.2.1, for which*

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0 \,.$$

Sketch of proof: Thanks to (2.51) and Lemma 2.2.4 there are $L_+^\uparrow$-invariant generalized functions $F(\xi)\,,\, G(\xi) \in \mathcal{S}(\mathbb{R}^4)'$ with

$$F(x-y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle - \left\langle \Omega \mid \hat{\Phi}(y)\hat{\Phi}(x) \right\rangle$$

and

$$G(x-y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle + \left\langle \Omega \mid \hat{\Phi}(y)\hat{\Phi}(x) \right\rangle \,.$$

Since F is also odd, we have[46]

$$\xi \times 0 \implies F(\xi) = 0 \,.$$

Now, assume local anticommutativity:

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0 \,.$$

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[45] Actually, **strict** local anti-commutativity is not necessary (Lücke, 1979). See also (Guido and Longo, 1995) and (Davidson, 1995) for a purely algebraic version using strict locality.

[46] Recall the reasoning for (2.45). For general fields one has to exploit the BHW theorem (Theorem 2.2.19, below).

Then, because of

$$\mathcal{W}(x-y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle = \frac{1}{2}\left(F(x-y) + G(x-y)\right) ,$$

we have

$$\xi \times 0 \implies \mathcal{W}(\xi) = 0 .$$

By (2.53) and Corollary 2.2.9, therefore, $\mathcal{W} = 0$ and hence $\hat{\Phi}(\varphi)\Omega = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$. By cyclicity of Ω, this implies $\hat{\Phi}(x) = 0$. ∎

Theorem 2.2.12 (Reeh-Schlieder) *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 with possible exception of microcausality*[47] *and let $\mathcal{O} \neq \emptyset$ be an open subset of $\mathbb{R}^4$. Then the vacuum state Ω is cyclic with respect to the algebra $\mathcal{F}_0(\mathcal{O}) \subset L(D,D)$ generated by $\hat{1}\restriction D$ and all $\hat{\Phi}(\varphi)$ with $\varphi \in \mathcal{S}(\mathbb{R}^4)$.*

Sketch of proof: By cyclicity of Ω with respect to $\mathcal{F}_0 = \mathcal{F}_0(\mathbb{R}^4)$ it is sufficient to prove[48]

$$\begin{aligned} &\left\langle \Psi \mid \hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \right\rangle = 0 \quad \forall\, \varphi_1,\ldots,\varphi_n \in \mathcal{S}(\mathcal{O}) \\ \implies &\left\langle \Psi \mid \hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \right\rangle = 0 \quad \forall\, \varphi_1,\ldots,\varphi_n \in \mathcal{S}(\mathbb{R}^4) \end{aligned} \tag{2.55}$$

for all $\Psi \in D$ and $n \in \mathbb{N}$:

By appropriate transformation of coordinates we get a generalized function $L(\xi_1,\ldots,\xi_n) \in \mathcal{S}(\mathbb{R}^{4n})'$ with

$$L(-x_1, x_1 - x_2, \ldots, x_{n-1} - x_n) = \left\langle \Psi \mid \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)\Omega \right\rangle$$

and

$$\mathrm{supp} L \neq \mathbb{R}^{4n} ,$$

thanks to the assumption in (2.55). On the other hand, we have

$$\mathrm{supp}\widetilde{L} \subset \overline{V_+} \times \ldots \times \overline{V_+} ,$$

by essentially the same reasoning as for (2.53). Therefore[49] Corollary 2.2.9 implies $L = 0$, hence the r.h.s. of (2.55). ∎

> Theorem 2.2.12 "can be interpreted as meaning that it is difficult to isolate a system described by fields from outside effects."
> (Streater and Wightman, 1989, p. 139)

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[47] Strict localizability is essential, however.

[48] As usual, $\mathcal{S}(\mathcal{O})$ denotes the (topological) subspace of all $\varphi \in \mathcal{S}(\mathbb{R}^4)$ with $\mathrm{supp}\varphi \subset \mathcal{O}$.

[49] Note that translation of the Fourier transform corresponds to multiplication of the original function by some function with constant modulus one.

Corollary 2.2.13 *Let all the assumptions of Theorem 2.2.12 be fulfilled. Moreover, let* $\hat{A} \in L(D, \mathcal{H})$ *commute with* $\mathcal{F}(\mathcal{O})$ *in the sense that*[50]

$$\left\langle \Psi \mid \hat{A}\hat{C}\Psi \right\rangle = \left\langle \hat{C}^*\Psi \mid \hat{A}\Psi \right\rangle \quad \forall\, \Psi \in D\,,\, \hat{C} \in \mathcal{F}(\mathcal{O})$$

and assume $D \subset D_{\hat{A}^*}$ *. Then*

$$\hat{A}\Omega = 0 \implies \hat{A} = 0\,.$$

Proof: If $\hat{A}\Omega = 0$ then we have

$$\left\langle \Psi \mid \hat{A}\hat{C}\Omega \right\rangle = \left\langle \hat{C}^*\Psi \mid \hat{A}\Omega \right\rangle = 0\,,$$

for all $\Psi \in D$ and $\hat{C} \in \mathcal{F}(\mathcal{O})$. By the Reeh-Schlieder theorem and thanks to $D \in D_{\hat{A}^*}$ this implies $\hat{A} = 0$. ∎

Corollary 2.2.14 *Let all the assumptions of Corollary 2.2.13 be fulfilled and let* $\hat{A}$ *be positive in the sense that*

$$\left\langle \Psi \mid \hat{A}\Psi \right\rangle \geq 0 \quad \forall\, \Psi \in D\,.$$

Then

$$\left\langle \Omega \mid \hat{A}\Omega \right\rangle = 0 \implies \hat{A} = 0\,.$$

Sketch of proof: Assume $\left\langle \Omega \mid \hat{A}\Omega \right\rangle = 0$. Since $\hat{A}$ is positive, it has a positive self-adjoint extension $\hat{\hat{A}}$ (***Friedrichs' theorem***, see e.g. (Yosida, 1971, Ch. XI §7)). With this extension we have

$$\left\langle \sqrt{\hat{\hat{A}}}\Omega \mid \sqrt{\hat{\hat{A}}}\Omega \right\rangle = \left\langle \Omega \mid \hat{\hat{A}}\Omega \right\rangle = 0$$

and hence

$$\hat{A}\Omega = \sqrt{\hat{\hat{A}}}\left(\sqrt{\hat{\hat{A}}}\Omega\right) = 0\,.$$

By Corollary 2.2.13, this implies $\hat{A} = 0$. ∎

> **Remark:** Corollary 2.2.14 shows that there cannot be any local positive energy density or positive 0-component of a local current density with vanishing vacuum expectation value in a field theory of the type described in 2.2.1.

[50] This condition is obviously fulfilled for $\hat{A} \in \mathcal{F}_0(\underline{\mathcal{O}'})$ with $\underline{\mathcal{O}'} \times \mathcal{O}$.

Theorem 2.2.15 (Borchers) *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 and let $\Psi \in \mathcal{H}$. Then*[51]

$$\left(\hat{U}(a)\Psi = \Psi \quad \forall\, a \in \mathbb{R}^4\right) \implies \Psi \sim \Omega\,.$$

Proof: See (Borchers, 1962, Theorem 3). ∎

Corollary 2.2.16 *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1. Then $\mathcal{F}_0$ is* ***irreducible*** *in the sense that every $\hat{C} \in \mathcal{L}(\mathcal{H})$ with*

$$\left\langle \Psi \mid \hat{C}\hat{\Phi}(\varphi)\Psi \right\rangle = \left\langle \hat{\Phi}(\varphi)^*\Psi \mid \hat{C}\Psi \right\rangle \quad \forall\, \Psi \in D\,,\, \varphi \in \mathcal{S}(\mathbb{R}^4) \tag{2.56}$$

must be a multiple of $\hat{1}$.

Sketch of proof: Let $\hat{C} \in \mathcal{L}(\mathcal{H})$ fulfill (2.56). Then it is sufficient to prove

$$\hat{C}\Omega = c\Omega \tag{2.57}$$

for some $c \in \mathbb{C}$ since the latter implies

$$\begin{aligned} \left\langle \Psi \mid \hat{C}\hat{A}\Omega \right\rangle &\underset{(2.56)}{=} \left\langle \hat{A}^*\Psi \mid \hat{C}\Omega \right\rangle \\ &\underset{(2.57)}{=} c\left\langle \Psi \mid \hat{A}\Omega \right\rangle \quad \forall \Psi \in D\,,\, \hat{A} \in \mathcal{F}_0 \end{aligned}$$

and hence $\hat{C} = c\hat{1}$, thanks to cyclicity of Ω. In order to prove (2.57) it is sufficient, by Theorem 2.2.15, to show

$$\hat{U}(a)\hat{C}\Omega = \hat{C}\Omega \quad \forall\, a \in \mathbb{R}^4$$

which, by cyclicity and translation invariance[52] of Ω, is equivalent to

$$\left\langle \hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \mid \hat{\hat{U}}(a)\hat{C}\Omega \right\rangle$$

being independent of $a \in \mathbb{R}^4$ for all $n \in \mathbb{N}$ and $\varphi_1 \ldots \varphi_n \in \mathcal{S}(\mathbb{R}^4)$. The latter,

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[51] Let $\mathcal{T}$ denote the subspace of all translational invariant vectors. Obviously, then, $\mathcal{T}$ is invariant under $\mathcal{P}_+^\uparrow$. Therefore, if we already new $\mathcal{T}$ to be finite dimensional the statement of Theorem 2.2.15 were a simple consequence of the fact that there are no non-trivial unitary representations of $L_+^\uparrow$.

[52] Translation invariance of Ω ensures that also $\left\langle \hat{1}\Omega \mid \hat{U}(a)\Omega \right\rangle$ is a-independent.

however, is an easy consequence of (2.56) and the Wightman axioms:

$$
\begin{aligned}
&\int_{\overline{V_+}} e^{+iap} \left\langle \hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \mid \hat{E}(\mathrm{d}p)\hat{C}\Omega \right\rangle \\
&= \left\langle \hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \mid \hat{U}(a)\hat{C}\Omega \right\rangle \\
&= \left\langle \left(\hat{U}(-a)\hat{\Phi}(\varphi_1)\hat{U}(a)\right)\cdots\left(\hat{U}(-a)\hat{\Phi}(\varphi_n)\hat{U}(a)\right)\Omega \mid \hat{C}\Omega \right\rangle \\
&= \left\langle \Omega \mid \hat{C}\left(\hat{U}(-a)\hat{\Phi}(\varphi_n)\hat{U}(a)\right)^*\cdots\left(\hat{U}(-a)\hat{\Phi}(\varphi_1)\hat{U}(a)\right)^*\Omega \right\rangle \\
&= \left\langle \hat{C}^*\Omega \mid \hat{U}(-a)\left(\hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\right)^*\Omega \right\rangle \\
&= \int_{\overline{V_+}} e^{-iap} \left\langle \hat{C}^*\Omega \mid \hat{E}(\mathrm{d}p)\left(\hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\right)^*\Omega \right\rangle . \quad \blacksquare
\end{aligned}
$$

In the following we shall denote by $\mathcal{F}(\mathcal{O})$, $\mathcal{O}$ any open subset of $\mathbb{R}^4$, the algebra generated by $\hat{1}\!\restriction\! D$ and all operators $\hat{A}$ of the form

$$\hat{A} = \int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n)\,\varphi(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

(recall Exercise 37) with $n \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathcal{O}\times\cdots\times\mathcal{O})$.

Exercise 38 Show for the free field $\hat{\Phi}(x) = \hat{\Phi}_0(x)$, described in 2.1, that

$$\left.\begin{array}{l}\hat{A}\in\mathcal{F}(\mathcal{O})\\ [\hat{A},\mathcal{F}(\mathcal{O})]_- = 0\end{array}\right\} \Longrightarrow \hat{A}\sim\hat{1}\!\restriction\! D$$

holds for every bounded open set $\mathcal{O}\subset\mathbb{R}^4$.

Lemma 2.2.17 *Let all the assumption of 2.2.1 be fulfilled.*[53] *Then*

$$\hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \in \hat{E}\left(\overline{V_+}\cap(\mathrm{supp}\widetilde{\varphi}_1+\ldots\mathrm{supp}\widetilde{\varphi}_n)\right)\mathcal{H}$$

holds for all $n\in\mathbb{N}$ and $\varphi_1,\ldots,\varphi_n\in\mathcal{S}(\mathbb{R}^4)$.

Sketch of proof: By

$$\hat{U}(a) = \int e^{ipa}\,\hat{E}(\mathrm{d}p) \tag{2.58}$$

and the basic relation

$$\delta(p-p') = (2\pi)^{-4}\int e^{i(p-p')}\,\mathrm{d}a \tag{2.59}$$

[53] Actually, microcausality will not be used in the proof of Lemma 2.2.17.

of Fourier calculus we have

$$\hat{E}(B) = (2\pi)^{-4} \int_B \left(\int \hat{U}(a)\, e^{-ipa}\, \mathrm{d}a \right) \mathrm{d}p \quad \forall\, \text{Borel}\, B \subset \mathbb{R}^4 . \tag{2.60}$$

By the transformation behaviour of $\hat{\Phi}(x)$ and translation invariance of Ω this implies

$$\begin{aligned} &\hat{E}(B)\hat{\Phi}(\varphi_1)\cdots\hat{\Phi}(\varphi_n)\Omega \\ &= (2\pi)^{-4} \int \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_n) \left(\int_B \left(\int \varphi_1(x_1 - a)\cdots\varphi_1(x_n - a)\, e^{-ipa}\, \mathrm{d}a \right) \mathrm{d} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \Omega \\ &\qquad\qquad \forall\, \text{Borel}\, B \subset \mathbb{R}^4 . \end{aligned}$$

Since $\int \varphi_1(x_1 - a)\cdots\varphi_1(x_n - a)\, e^{-ipa}\, \mathrm{d}a$ can only be nonzero for $p \in \mathrm{supp}\widetilde{\varphi}_1 + \ldots + \mathrm{supp}\widetilde{\varphi}_n$, this together with the spectrum condition implies the statement of the lemma. ∎

Theorem 2.2.18 (Jost-Schroer) *Let $\mathcal{H}$ $\hat{U}(a,\Lambda)$, Ω, D and $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 with $\dim \mathcal{H} > 1$ and let $\mathcal{H}_0$ $\hat{U}_0(a,\Lambda)$, Ω_0, D_0, $\hat{\Phi}_0(x)$ be the corresponding objects of the free field theory described in 2.1. If also $\hat{\Phi}(x)$ fulfills the free Klein-Gordon equation*

$$\left(\Box + m^2\right) \hat{\Phi}(x) = 0$$

and if $D = \mathcal{F}(\mathbb{R}^4)\Omega$ then there are a unitary mapping $\hat{V} : \mathcal{H}_0 \longrightarrow \mathcal{H}$ and a constant $\lambda > 0$ with:

$$\begin{aligned} &D = \hat{V} D_0 , \quad \Omega = \hat{V}\Omega \\ &\hat{\Phi}(x) = \lambda\, \hat{V} \hat{\Phi}_0(x) \hat{V}^{-1} , \\ &\hat{U}(a,\Lambda) = \hat{V}\hat{U}_0(a,\Lambda)\hat{V}^{-1} \quad \forall\, (a,\Lambda) \in \mathcal{P}_+^\uparrow . \end{aligned}$$

Sketch of proof: Assume that $\hat{\Phi}(x)$ fulfills the Klein-Gordon equation. This implies $(\Box_\xi + m^2)\, \mathcal{W}(\xi) = 0$ and hence, by (2.53),

$$\mathrm{supp}\widetilde{W} \subset M_m = \left\{ P \in \mathbb{R}^4 : p^0 = \omega_{\mathbf{p}} \right\} . \tag{2.61}$$

From the proof of Lemma 2.2.10 we also know that there is a positive $L_+^\uparrow$-invariant measure μ with

$$\left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y)\Omega \right\rangle = \mathcal{W}(x-y) = \int e^{ip(x-y)} \mu(\mathrm{d}p)$$

Therefore, there must exist a $\lambda^2 \geq 0$ with

$$\mu(\mathrm{d}p) = \frac{\lambda^2}{(2\pi)^3} \theta(p^0)\delta(p^2 - m^2)\, \mathrm{d}p .$$

Without restriction of generality we may assume $\lambda = 1$. Then we have[54]

$$\left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y)\Omega \right\rangle = i\Delta_m^{(+)}(x-y) \stackrel{\text{def}}{=} (2\pi)^{-3} \int \theta(p^0)\delta(p^2 - m^2) e^{-ip(x-y)}\, \mathrm{d}p . \tag{2.62}$$

[54]Vice versa, (2.62) implies the Klein-Gordon equation, by Corollary 2.2.13. Therefore, assumption (2.61) would have been sufficient. In this sense the results of (Baumann, 1986) are much stronger than the Jost-Schroer theorem. For generalization in a different direction see (Steinmann, 1982).

The Klein-Gordon equation allows us to define

$$\hat{\Phi}^{(\pm)}(x) \stackrel{\text{def}}{=} (2\pi)^{-2} \int_{\pm p^0 \geq 0} \tilde{\hat{\Phi}}(p)\, e^{-ipx}\, dp\,. \tag{2.63}$$

With this definition we have

$$\hat{\Phi}(x) = \hat{\Phi}^{(+)}(x) + \hat{\Phi}^{(-)}(x)\,, \tag{2.64}$$

$$\hat{\Phi}^{(+)}(x)\Omega = 0\,, \tag{2.65}$$

and hence

$$\left\langle \Omega \mid \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(+)}(y)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(+)}(y)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(-)}(y)\Omega \right\rangle = 0\,, \tag{2.66}$$

the latter because of $\hat{\Phi}^{(-)}(x)^* = \hat{\Phi}^{(+)}(x)$. Moreover, by (2.62), it is clear that

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_- = \left\langle \Omega \mid [\hat{\Phi}(x), \hat{\Phi}(y)]_-\Omega \right\rangle \hat{1}\!\!\bigwedge D \tag{2.67}$$

would imply

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_- = i\Delta_m(x-y)\hat{1}\!\!\bigwedge D \tag{2.68}$$

(recall (2.47)). Since statements corresponding to (2.63), (2.65), and (2.68) hold for the free theory, this shows that (2.67) would imply that the n-point functions of $\hat{\Phi}(x)$ are the same[55] as those of $\hat{\Phi}_0(x)$. In view of the GNS representation, described by Theorem 1.3.8 (recall Footnote 58 of Chapter 1), therefore, it is sufficient to prove (2.67). By Corollary 2.2.13, (2.67) follows from

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_-\Omega \sim \Omega\,. \tag{2.69}$$

To prove (2.69), let us consider states of the form

$$\check{\Psi} = \hat{\Phi}^{(+)}(\varphi_+)\hat{\Phi}^{(-)}(\varphi_-)\Omega\,, \quad \varphi_\pm \in \mathcal{S}(\mathbb{R}^4)\,.$$

For such $\check{\Psi}$ Lemma 2.2.17 implies

$$\check{\Psi} = \hat{E}\left(\overline{V_+} \cap \left(-\mathrm{supp}\hat{\Phi}^{(+)} - \mathrm{supp}\hat{\Phi}^{(-)}\right)\right)\check{\Psi}\,.$$

Since[56]

$$\overline{V_+} \cap \left(-\mathrm{supp}\hat{\Phi}^{(+)} - \mathrm{supp}\hat{\Phi}^{(-)}\right) = \{0\}\,,$$

[55]By (2.64) it is sufficient to check the vacuum expectation values $\left\langle \Omega \mid \hat{\Phi}^{(\sigma_1)}(x_1)\cdots\hat{\Phi}^{(\sigma_n)}(x_n)\Omega \right\rangle$ for arbitrary $n \in \mathbb{N}$ and $\sigma_\nu \in \{+,-\}$. By (2.68), then, it is sufficient to check the expectation values of the form $\left\langle \Omega \mid \hat{\Phi}^{(+)}(x_1)\cdots\hat{\Phi}^{(+)}(x_{n_+})\hat{\Phi}^{(-)}(x_{n_++1})\cdots\hat{\Phi}^{(-)}(x_{n_++n_-})\Omega \right\rangle$. By (2.65), however, all of them vanish, as for the free field.

[56]By the Klein-Gordon equation we have $\mathrm{supp}\tilde{\hat{\Phi}} \subset M_m \cup (-M_m)$ and hence $\mathrm{supp}\widetilde{\hat{\Phi}^{(\pm)}} \subset \pm M_m$, by (2.63).

this means $\check{\Psi} = \hat{E}(\{0\})\check{\Psi}$. Thus, by (2.58), $\check{\Psi}$ is translational invariant and thus a multiple of Ω, by Borchers' theorem. Therefore:

$$\begin{aligned}
\hat{\Phi}^{(+)}(x)\hat{\Phi}^{(-)}(y)\Omega &= \left\langle \Omega \mid \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(-)}(y)\Omega \right\rangle \Omega \\
&\underset{(2.64),(2.66)}{=} \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y)\Omega \right\rangle \\
&\underset{(2.62)}{=} i\Delta_m^{(+)}(x-y)\Omega\,.
\end{aligned}$$

By (2.65), (2.64), and

$$\Delta_m(x-y) = \Delta_m^{(+)}(x-y) - \Delta_m^{(+)}(y-x)$$

this implies

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_-\Omega = i\Delta_m(x-y)\Omega + [\hat{\Phi}^{(-)}(x), \hat{\Phi}^{(-)}(y)]_-\Omega\,.$$

Therefore, it is sufficient to prove

$$F_\Psi(x,y) \stackrel{\text{def}}{=} \left\langle \Psi \mid [\hat{\Phi}^{(-)}(x), \hat{\Phi}^{(-)}(y)]_-\Omega \right\rangle = 0$$

for all $\Psi \in D$. This, however, is guaranteed by Corollary 2.2.9 since, obviously, $\operatorname{supp} F_\Psi \neq \mathbb{R}^8$ and $\operatorname{supp} \widetilde{F_\Psi} \subset \overline{V_+} \times \overline{V_+}$ (recall Footnote 56). ∎

2.2.5 PCT Theorem

The defining representation of the ***Lorentz group*** $L = L(\mathbb{R})$ is well known to be given by the real 4×4-matrices Λ fulfilling

$$\Lambda^{\mathrm{T}}\,\eta\,\Lambda = \eta \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.70}$$

Similarly, the defining representation of the ***complex Lorentz group*** $L = L(\mathbb{C})$ is given by the **complex** 4×4-matrices Λ fulfilling (2.70). Its subgroup

$$L_+(\mathbb{C}) \stackrel{\text{def}}{=} \{\Lambda \in L(\mathbb{C}) : \det\Lambda = +1\}$$

is called the ***proper complex Lorentz group***. Obviously:

$$L_+^\uparrow \subset L_+(\mathbb{C}) \ni +\mathbb{1}_4\,, -\mathbb{1}_4\,. \tag{2.71}$$

Theorem 2.2.19 (Bargmann-Hall-Wightman) *Let $n, N \in \mathbb{N}$, $n > 1$, let $\Lambda \longrightarrow S(\Lambda)$ be an irreducible $N \times N$-matrix representation of $L_+^\uparrow$, and let $\mathcal{W}_1, \ldots, \mathcal{W}_N$ be holomorphic functions on $\mathcal{T}'_{n-1}$ with*

$$\mathcal{W}_\mu(z_1, \ldots, z_{n-1}) = \sum_{\nu=1}^{N} S_{\mu\nu}(\Lambda^{-1}) \mathcal{W}_\nu(\Lambda z_1, \ldots, \Lambda z_{n-1}) \quad \forall \mu \in \{1, \ldots, N\} \tag{2.72}$$

for all $(z_1, \ldots, z_{n-1}) \in \mathcal{T}_{n-1}$ and all $\Lambda \in L_+^\uparrow$. Then the $\mathcal{W}_\nu$ have unique single-valued analytic continuations onto the ***extended tube***

$$\mathcal{T}'_{n-1} \stackrel{\text{def}}{=} \{(\Lambda z_1, \ldots, \Lambda z_{n-1}) : \Lambda \in L_+(\mathbb{C}), (z_1, \ldots, z_{n-1}) \in \mathcal{T}_{n-1}\}$$

fulfilling (2.72) for all $(z_1, \ldots, z_{n-1}) \in \mathcal{T}'_{n-1}$ and all $\Lambda \in L_+(\mathbb{C})$, where $S(\Lambda)$ is to be extended to the corresponding irreducible representation of $L_+(\mathbb{C})$.

Proof: See (Bogush and Fedorov, 1977, Sect. 3.4). ∎

Corollary 2.2.20 *Let all the assumptions of 2.2.1, with possible exception of microcausality,*[57] *be fulfilled and let $1 < n \in \mathbb{N}$. Then the Laplace transform $\left(\mathcal{L}\widetilde{\mathcal{W}}\right)$ of $\widetilde{\mathcal{W}}$ has a single valued analytic continuation onto the extended tube $\mathcal{T}'_{n-1}$ fulfilling the conditions*

$$\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\Lambda z_1, \ldots, \Lambda z_{n-1}) = \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(z_1, \ldots, z_{n-1}) \qquad \forall \Lambda \in L_+(\mathbb{C}),\ (z_1, \ldots, z_{n-1}) \in \mathcal{T}'_{n-1} \tag{2.73}$$

and[58]

$$\mathcal{W}(\psi) = \int \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1, \ldots, \xi_{n-1})\, \psi(\xi_1, \ldots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1} \qquad \forall \psi \in \mathcal{S}(\mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}). \tag{2.74}$$

Sketch of proof: Note, first of all, that (2.52) and (2.51) imply

$$\mathcal{W}(\Lambda\xi_1, \ldots, \Lambda\xi_{n-1}) = \mathcal{W}(\xi_1, \ldots, \xi_{n-1}) \quad \forall \Lambda \in L_+^\uparrow.$$

This together with (2.53) and Lemma 2.2.7 gives

$$\left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1}) = \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\Lambda\xi_1 + i\Lambda\eta_1, \ldots, \Lambda\xi_{n-1} + i\Lambda\eta_{n-1}) \quad \forall \Lambda \in L_+^\uparrow$$

[57] Microcausality would allow further holomorphic continuation when $n > 2$ (see, e.g., (Tomozawa, 1963)).

[58] Note that $\mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}$ is an open subset of $\mathbb{R}^{4(n-1)}$, since $\mathcal{T}'_{n-1}$ is open in $\mathbb{C}^{4(n-1)}$.

as an equation for holomorphic functions on $\mathcal{T}_{n-1}$. Therefore, by Theorem 2.2.19, $\left(\mathcal{L}\widetilde{\mathcal{W}}\right)$ has a single valued analytic continuation onto the extended tube $\mathcal{T}'_{n-1}$ fulfilling (2.73). From Lemma 2.2.7 we also know that

$$\mathcal{W}(\psi) = \lim_{V_-\ni\eta_1,\ldots,\eta_{n-1}\to 0} \int \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\xi_1+i\eta_1,\ldots,\xi_{n-1}+i\eta_{n-1})\,\psi(\xi_1,\ldots,\xi_{n-1})\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{n-1}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^{4(n-1)})$. Since every Jost point[59]

$$\check{\underline{\xi}} \stackrel{\text{def}}{=} (\underline{\xi}_1,\ldots,\underline{\xi}_{n-1}) \in \mathcal{J}_{n-1} \stackrel{\text{def}}{=} \mathcal{T}'_{n-1}\cap\mathbb{R}^{4(n-1)}$$

has a complex open neighborhood $\mathcal{O}_{\check{\underline{\xi}}} \subset \mathcal{J}_{n-1}$, this implies

$$\mathcal{W}(\psi) = \int \left(\mathcal{L}\widetilde{\mathcal{W}}\right)(\check{\xi})\,\psi(\check{\xi})\,\mathrm{d}\check{\xi} \quad \forall\,\psi\in\mathcal{S}\left(\mathcal{O}_{\check{\underline{\xi}}}\cap\mathbb{R}^{4(n-1)}\right).$$

From this (2.74) follows by standard distribution theoretical techniques (choice of a suitable *partition of unity*). ∎

Theorem 2.2.21 (Jost) *Let* $1 < n \in \mathbb{N}$ *and* $(\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{4(n-1)}$. *Then*

$$\begin{aligned}&(\xi_1,\ldots,\xi_{n-1}) \in \mathcal{T}'_{n-1}\\ \iff &\left(\sum_{\nu=1}^{n-1}\lambda_\nu\xi_\nu\right)\left(\sum_{\nu=1}^{n-1}\lambda_\nu\xi_\nu\right) < 0 \quad \textit{for all } \lambda_1,\ldots,\lambda_{n-1}\geq 0 \textit{ with } \sum_{\nu=1}^{n-1}\lambda_\nu > 0\,.\end{aligned}$$

Proof: See (Streater and Wightman, 1989, Theorem 2-12). ∎

Corollary 2.2.22 (PCT Theorem) *Let all the assumptions of 2.2.1, with possible exception of microcausality, be fulfilled and let* $1 < n \in \mathbb{N}$. *Then the PCT condition*

$$W(x_1,\ldots,x_n) = W(-x_n,\ldots,-x_1) \tag{2.75}$$

is equivalent to the condition of ***weak local commutativity***

$$W(x_1,\ldots,x_n) = W(x_n,\ldots,x_1) \quad \textit{for } (x_1-x_2,\ldots,x_{n-1}-x_n)\in\mathcal{J}_{n-1}\,. \tag{2.76}$$

Proof: By Corollary 2.2.20, since $-\mathbb{1}_4 \in L_+(\mathbb{C})$, (2.76) is equivalent to[60]

$$W(x_1,\ldots,x_n) = W(-x_n,\ldots,-x_1) \quad \text{for } (x_1-x_2,\ldots,x_{n-1}-x_n)\in\mathcal{J}_{n-1}\,,$$

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[59] The elements of $\mathcal{T}'_{n-1}\cap\mathbb{R}^{4(n-1)}$ are usually called ***Jost points***.

[60] Since $\mathcal{J}_1 = \left\{\xi\in\mathbb{R}^4 : \xi \times 0\right\}$, this shows once more that

$$\left\langle\Omega \mid [\hat{\Phi}(x),\hat{\Phi}(y)]_-\Omega\right\rangle = 0 \quad \text{for } x \times y$$

holds even if microcausality is not assumed, as a consequence of the other Wightman axioms.

i.e. to

$$F(\check{\xi}) \stackrel{\text{def}}{=} \mathcal{W}(\xi_1, \dots, \xi_{n-1}) - \mathcal{W}(\xi_{n-1}, \dots, \xi_1) = 0 \quad \forall \check{\xi} \in \mathcal{J}_{n-1} \,. \tag{2.77}$$

By Corollary 2.2.9, since (2.53) implies

$$\operatorname{supp} F \subset \overline{V_+} \times \dots \times \overline{V_+} \,,$$

(2.77) and hence (2.76) is equivalent to $F = 0$, i.e. to (2.75). ∎

Exercise 39 Show that (2.75) is equivalent to existence of an anti-unitary Operator $\hat{\theta}$ fulfilling the conditions

$$\hat{\theta}\hat{\theta} = \hat{1} \,, \quad \hat{\theta}\Omega = \Omega \,,$$

$$\hat{\theta}\,\hat{\Phi}(\varphi)\,\hat{\theta} = \left(\int \hat{\Phi}(-x)\,\varphi(x)\,\mathrm{d}x\right)^* \restriction D \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4) \,,$$

and

$$\hat{\theta}\,\hat{U}(a)\,\hat{\theta} = \hat{U}(-a) \quad \forall a \in \mathbb{R}^4 \,.$$

2.3 S-Matrix for Self-interacting Neutral Scalar Fields

2.3.1 General Scattering Theory

The main concern of scattering theory is asymptotic (for $t \to \pm\infty$) identification of an interacting system (IS) with a suitable "free" system (FS):

$$\begin{aligned} \check{\mathcal{H}} &\stackrel{\text{def}}{=} \text{set of all (pure) states of the IS} \,, \\ \check{\mathcal{H}}_0 &\stackrel{\text{def}}{=} \text{set of all (pure) states of the FS} \,. \end{aligned}$$

Here, we use the ***Heisenberg picture***, i.e. the states describe the corresponding system for all times until some "measurement" is taking place. Whether the considered systems are classical or quantum does not matter, so far. Thus, given $\check{\Psi} \in \check{\mathcal{H}}$, the basic problem is to find the (hopefully unique) "free" states $\check{\Psi}_\pm \in \check{\mathcal{H}}_0$ such that

$$\text{for } t \to \pm\infty \ \check{\Psi} \text{ "looks like" } \check{\Psi}_\pm \,.$$

The precise meaning of the latter has to be specified by some ***asymptotic condition*** (AC) as sketched in Fig. 2.4. For example, in **potential scattering of classical particles**, sketched in Fig. 2.5, the Heisenberg states are given by the solutions $\mathbf{x}(t)$ of the classical equations of motion and $\mathbf{x}_\pm(t)$ being *free* means:

$$\mathbf{x}_\pm(t) = \mathbf{x}_\pm + \mathbf{v}_\pm t \,.$$

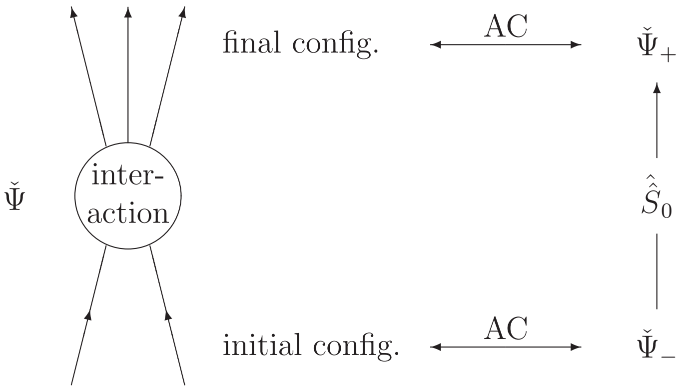

Figure 2.4: Asymptotic identification of the IS (left) with the FS (right) via the AC

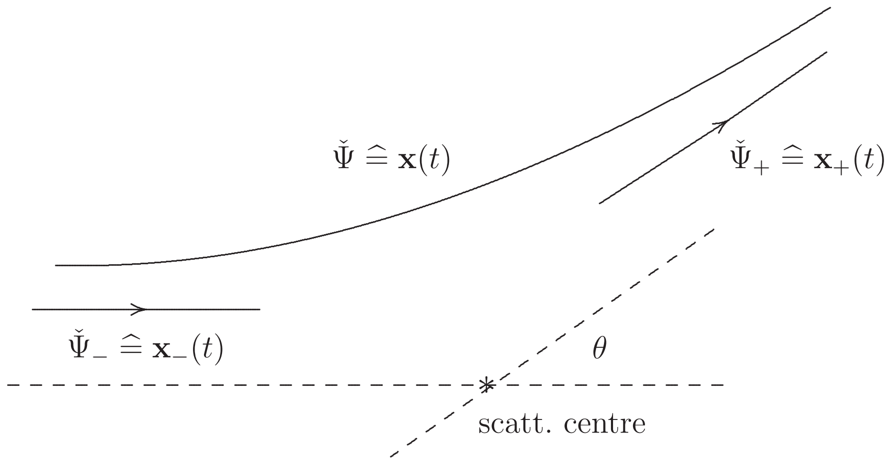

Figure 2.5: Asymptotics for a classical particle

For potentials of sufficiently short range forces the ***asymptotic condition*** is:[61]

$$\mathbf{v}_\pm = \lim_{t\to\pm\infty} \dot{\mathbf{x}}(t) \ , \quad \mathbf{x}_\pm = \lim_{t\to\pm\infty} \mathbf{x}(t) - \mathbf{v}_\pm t \ .$$

$\check{\Psi} \in \check{\mathcal{H}}$ is called a ***scattering state***, if there are $\check{\Psi}_- \in \check{\mathcal{H}}_0$ **and** $\check{\Psi}_+ \in \check{\mathcal{H}}_0$ fulfilling the AS for $\check{\Psi}$. Obviously, this condition need not always be fulfilled (***bounded states***, particle *capture* etc.).

FS and AC have to meet the following **requirement:**

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[61]See (Reed und Simon, 1972, Vol. III, Thm. XI.1). For the Coulomb potential the "free states" have to be taken of the form

$$\mathbf{x}_\pm(t) = \mathbf{x}_\pm + \mathbf{v}_\pm t + \mathbf{d}_\pm \ln t$$

(see (Reed und Simon, 1972, Vol. III, Sect. 9)).

For every $\check{\Phi}_0 \in \check{\mathcal{H}}_0$ there is exactly one $\check{\Psi} \in \check{\mathcal{H}}$ with $\check{\Psi}_- = \check{\Phi}_0$; and similarly for '+' instead of '-'.

Then we can define the following ***generalized wave operators***:

$$\begin{aligned} \hat{\hat{V}}_{\text{in}}\check{\Phi}_0 &\stackrel{\text{def}}{=} \text{the state } \check{\Psi} \in \check{\mathcal{H}} \text{ for which } \check{\Psi}_- = \check{\Phi}_0 \,, \\ \hat{\hat{V}}_{\text{out}}\check{\Phi}_0 &\stackrel{\text{def}}{=} \text{the state } \check{\Psi}' \in \check{\mathcal{H}} \text{ for which } \check{\Psi}'_+ = \check{\Phi}_0 \,. \end{aligned} \tag{2.78}$$

Now, the subject of scattering theory is to study the relation between $\check{\Psi}_- = \hat{\hat{V}}_{\text{in}}^{-1}\check{\Psi}$ and $\check{\Psi}_+ = \hat{\hat{V}}_{\text{out}}^{-1}\check{\Psi}$ for arbitrary scattering states $\check{\Psi}$.

The ***scattering operator in the Heisenberg picture*** is

$$\hat{\hat{S}} \stackrel{\text{def}}{=} \hat{\hat{V}}_{\text{in}}\hat{\hat{V}}_{\text{out}}^{-1} \tag{2.79}$$

and maps $\check{\mathcal{H}}_{\text{out}} \stackrel{\text{def}}{=} \hat{\hat{V}}_{\text{out}}\check{\mathcal{H}}_0$ one-to-one onto $\check{\mathcal{H}}_{\text{in}} \stackrel{\text{def}}{=} \hat{\hat{V}}_{\text{in}}\check{\mathcal{H}}_0$:

$$\hat{\hat{S}} \quad \underbrace{\hat{\hat{V}}_{\text{out}}\check{\Phi}_0}_{\text{looks like } \check{\Phi}_0 \text{ for } t\to+\infty} \quad = \quad \underbrace{\hat{\hat{V}}_{\text{in}}\check{\Phi}_0}_{\text{looks like } \check{\Phi}_0 \text{ for } t\to-\infty} \quad . \tag{2.80}$$

Definition (2.79) has the advantage of being applicable even in case $\check{\mathcal{H}}_{\text{in}} \not\subset \check{\mathcal{H}}_{\text{out}}$, contrary to definition (2.82) below, and of being independent of the special choice for the realization of the free system. Its drawback is that $\hat{\hat{S}}$ describes the relation between $\check{\Psi}_-$ and $\check{\Psi}_+$ only indirectly (via $\hat{\hat{V}}_{\text{out}}$ or $\hat{\hat{V}}_{\text{in}}$):

$$\left.\begin{aligned} \hat{\hat{S}}\hat{\hat{V}}_{\text{out}}\check{\Psi}_- \quad &\left(\underset{(2.79)}{=} \hat{\hat{V}}_{\text{in}}\check{\Psi}_- = \check{\Psi} \right) \quad = \hat{\hat{V}}_{\text{out}}\check{\Psi}_+ \\ \hat{\hat{S}}\hat{\hat{V}}_{\text{in}}\check{\Psi}_- \quad &\left(= \hat{\hat{S}}\check{\Psi} = \hat{\hat{S}}\hat{\hat{V}}_{\text{out}}\check{\Psi}_+ \right) \quad = \hat{\hat{V}}_{\text{in}}\check{\Psi}_+ \end{aligned}\right\} \text{ for scattering states } \check{\Psi} \,. \tag{2.81}$$

If $\check{\mathcal{H}}_{\text{in}} \subset \check{\mathcal{H}}_{\text{out}}$, this relation may be described directly via the ***scattering operator in the interaction picture***

$$\hat{\hat{S}}_0 \stackrel{\text{def}}{=} \hat{\hat{V}}_{\text{out}}^{-1}\hat{\hat{V}}_{\text{in}} \,, \tag{2.82}$$

namely:

$$\hat{\hat{S}}_0\check{\Psi}_- = \check{\Psi}_+ \,. \tag{2.83}$$

In this case, i.e. when $\check{\mathcal{H}}_{\text{in}}$ contains only scattering states (no capture),

$$\hat{\hat{S}} = \hat{\hat{V}}_{\text{out}}\hat{\hat{S}}_0\hat{\hat{V}}_{\text{out}}^{-1} \,. \tag{2.84}$$

In case of ***weak asymptotic completeness***, i.e. if $\check{\mathcal{H}}_{\text{in}} = \check{\mathcal{H}}_{\text{out}}$, we also have

$$\hat{\hat{S}} = \hat{\hat{V}}_{\text{in}}\hat{\hat{S}}_0\hat{\hat{V}}_{\text{in}}^{-1} \,, \tag{2.85}$$

since then $D_{\hat{\hat{V}}_{\text{in}}^{-1}} = D_{\hat{\hat{V}}_{\text{out}}^{-1}}$. For potential scattering of classical particles the action of $\hat{\hat{S}}$ and $\hat{\hat{S}}_0$ is sketched in Fig. 2.6.

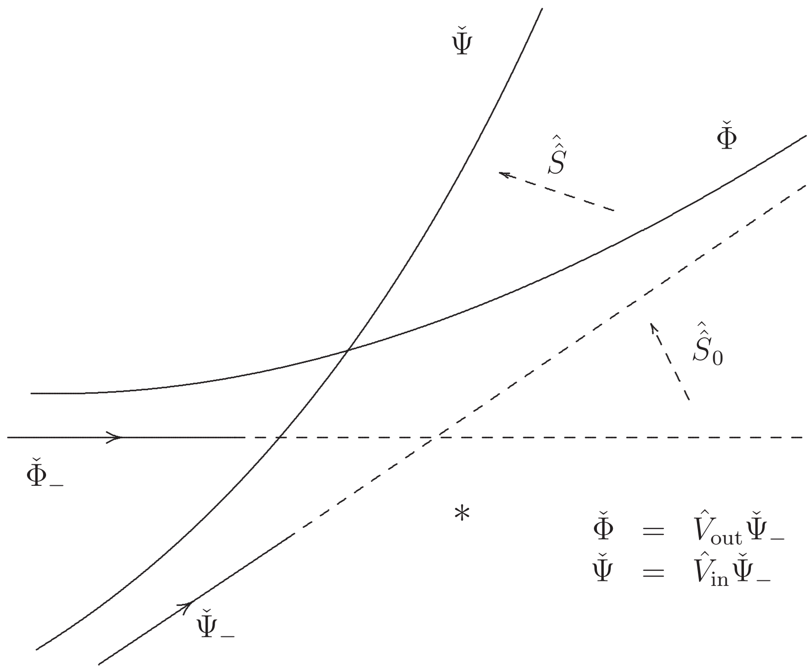

Figure 2.6: Two versions of the S-matrix

2.3.2 Asymptotic condition for massive neutral scalar particles

In quantum mechanics without superselection rules, the pure states of the WS resp. FS are given by the 1-dimensional subspaces $\check{\Psi}$ resp. $\check{\Psi}_0$ of some complex Hilbert space $\mathcal{H}$ resp. $\mathcal{H}_0$:

$$\check{\mathcal{H}}_{(0)} = \left\{\check{\Psi}_{(0)} = \left\{\lambda\Psi_{(0)} : \lambda \in \mathbb{C}\right\} : \Psi_{(0)} \in \mathcal{H}_{(0)}\right\} .$$

The generalized wave operators $\hat{\hat{V}}_{\substack{\text{out}\\(\text{in})}}$ are given by isometric (linear) mappings

$$\hat{V}_{\substack{\text{out}\\(\text{in})}} : \mathcal{H}_0 \longrightarrow \mathcal{H}_{\substack{\text{out}\\(\text{in})}} \stackrel{\text{def}}{=} \hat{V}_{\substack{\text{out}\\(\text{in})}}\mathcal{H}_0 \subset \mathcal{H}$$

via:

$$\hat{\hat{V}}_{\substack{\text{out}\\(\text{in})}}\check{\Psi}_0 \stackrel{\text{def}}{=} \left\{\lambda\hat{V}_{\substack{\text{out}\\(\text{in})}}\Psi_0 : \lambda \in \mathbb{C}\right\} \quad \text{for } \Psi_0 \in \mathcal{H}_0 .$$

Let us consider the theory of a neutral scalar field as specified in 2.2.1, describing the IS, and assume:

1. The restriction of the representation $\hat{U}(a, \Lambda)$ of $\mathcal{P}_+^\uparrow$ to $\hat{E}(M_m)\mathcal{H}$ is irreducible.

2. The free field theory described in 2.1.3 represents a FS suitable for asymptotic description of the IS.

3. The corresponding scattering theory fulfills the condition of ***asymptotic completeness***:
$$\mathcal{H}_{\rm in} = \mathcal{H}_{\rm out} = \mathcal{H}\,.$$

By asymptotic completeness, then, the scattering isometries $\hat V_{\substack{\rm out\\(\rm in)}}$ are even unitary mappings from $\mathcal{H}_0$ onto $\mathcal{H}$. The chosen FS can only be suitable if
$$\hat U(a,\Lambda)\,\hat V_{\substack{\rm out\\(\rm in)}} = \hat V_{\substack{\rm out\\(\rm in)}}\,\hat U_0(a,\Lambda) \quad \forall\,(a,\Lambda)\in\mathcal{P}_+^\uparrow\,. \tag{2.86}$$

This implies
$$\mathrm{supp}\hat E = \mathrm{supp}\hat E_0 = \{0\}\cup M_m\cup\left\{p\in\mathbb{R}^4 : p^2\geq(2m)^2\,,\,p^0>0\right\} \tag{2.87}$$

and
$$\hat V_{\substack{\rm out\\(\rm in)}}\mathcal{H}_0^{(0)} = \hat E\left(\{0\}\right)\mathcal{H}\,,\quad \hat V_{\substack{\rm out\\(\rm in)}}\mathcal{H}_0^{(1)} = \hat E(M_m)\mathcal{H}\,.$$

Since $\hat U(a,\Lambda)\restriction\hat E\,(M_m)\,\mathcal{H}$ is irreducible and since the vacuum state is uniquely characterized by $\mathcal{P}_+^\uparrow$-invariance, we may assume without loss of generality:
$$\mathcal{H}_0^{(0)}\oplus\mathcal{H}_0^{(1)} = \hat E\left(\{0\}\cup M_m\right)\mathcal{H}\,,\quad \hat V_{\substack{\rm out\\(\rm in)}}\restriction\left(\mathcal{H}_0^{(0)}\oplus\mathcal{H}_0^{(1)}\right) = \hat 1\restriction\left(\mathcal{H}_0^{(0)}\oplus\mathcal{H}_0^{(1)}\right)\,. \tag{2.88}$$

By unitarity of $\hat V_{\substack{\rm out\\(\rm in)}}$, then, it is sufficient to determine the vectors
$$\Psi_{\substack{\rm out\\(\rm in)}}(\check\chi_1,\ldots,\check\chi_n) \stackrel{\rm def}{=} \hat V_{\substack{\rm out\\(\rm in)}}\left(\hat a_0^*(\check\chi_1)\cdots\hat a_0^*(\check\chi_n)\Omega\right) \tag{2.89}$$

(recall (2.33)) for all $n>1$ and functions $\check\chi_1,\ldots,\check\chi_n\in\mathcal{D}(\mathbb{R}^3)$ that are ***nonoverlapping***, i.e.:
$$\nu\neq\mu \Longrightarrow \mathrm{supp}\,\check\chi_\nu\cap\mathrm{supp}\,\check\chi_\mu = \emptyset\,.$$

We want to characterize the states corresponding to vectors of the form (2.89) by their expectation values for localized measurements corresponding to **bounded** observables.[62] The use of the field $\hat\phi(x)$ for this is just to associate with every open region $\mathcal{O}\subset\mathbb{R}^4$ the corresponding ***algebra of local observables***[63] $\mathcal{A}(\mathcal{O})$, i.e. the von Neumann algebra generated be all **bounded** observables corresponding to measurements performable within $\mathcal{O}$. Once the local algebras $\mathcal{A}(\mathcal{O})$ are specified, we may forget about the field $\hat\phi(x)$ as far as the S matrix is concerned.

In order to be able to interpret the closed smeared field operators $\overline{\int dx\,\hat\Phi(x)\varphi(x)}$, where $\varphi=\overline{\varphi}\in\mathcal{S}(\mathbb{R}^4)$, as observables of the φ-weighted 'field strength' $\int dx\,\Phi(x)\varphi(x)$, let us assume that before closure the operators $\hat\Phi(\varphi)\stackrel{\rm def}{=}\int dx\,\hat\Phi(x)\varphi(x)$ on D are essentially self-adjoint and that in case $\mathrm{supp}\,\varphi_1 \times \mathrm{supp}\,\varphi_2$ also the spectral projection

[62] A more general formalism, suitable also for nonlocalizable fields, was developed in (Lücke, 1983).

[63] See (Thomas and Wichmann, 1998) and references given there for further details.

operators of $\hat{\Phi}(\varphi_1)$ commute[64] with those of $\hat{\Phi}(\varphi_2)$. Then,[65] given an open region $\mathcal{O} \subset \mathbb{R}^4$, $\mathcal{A}(\mathcal{O})$ should be identified with the von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ generated by all spectral operators of the selfadjoint operators $\overline{\hat{\Phi}(\varphi)}$, $\varphi = \overline{\varphi} \in \mathcal{S}(\mathcal{O})$. This also ensures that the following four conditions are fulfilled:[66]

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)\,, \tag{2.90}$$

$$\mathcal{O}_1 \times \mathcal{O}_2 \implies \Big[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2\Big]_- = \{\hat{0}\}\,, \tag{2.91}$$

$$\hat{U}(a,\Lambda)\mathcal{A}(\mathcal{O}))\hat{U}(a,\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O} + a)\,, \tag{2.92}$$

$$\Big(\bigcup_{R>0} \mathcal{A}\left(U_R(0)\right)\Big)' = \left\{\lambda\hat{1} : \lambda \in \mathbb{C}\right\}\,. \tag{2.93}$$

By Ruelle's lemma (Lemma 2.1.1) the influence of an asymptotic particle on measurements outside its velocity cone should fall off very rapidly. This statement has to be made precise in order to get a suitable asymptotic condition:

Asymptotic Condition:[67] Let $\check{\chi}_1, \dots \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ be **nonoverlapping** and let the vectors $\hat{a}_0^*(\chi_1)\Omega_0\,, \dots\,, \hat{a}_0^*(\chi_n)\Omega_0 \in \mathcal{H}_0^{(1)}$ be normalized. Moreover, let $\check{K} = -\check{K}$ be a closed cone for which

$$\check{K} \cap K_{\check{\chi}_n} = \{0\}\,, \quad K_{\check{\chi}_n} \stackrel{\text{def}}{=} \left\{\left(t, \frac{\mathbf{p}}{\omega_{\mathbf{p}}} t\right) : t \in \mathbb{R}\,, \mathbf{p} \in \operatorname{supp}\chi_n\right\}\,.$$

Then, for sufficiently small[68] $\epsilon > 0$ there exists a sequence of positive numbers $C_1, C_2, \dots$ for which

$$\begin{aligned} |t|^N \Big| \Big\langle \Psi_{\substack{\text{out}\\(\text{in})}}(\check{\chi}_1, \dots \check{\chi}_n) \Big| \hat{A}_t \Psi_{\substack{\text{out}\\(\text{in})}}(\check{\chi}_1, \dots \check{\chi}_n) \Big\rangle \\ - \Big\langle \Psi_{\substack{\text{out}\\(\text{in})}}(\check{\chi}_1, \dots \check{\chi}_{n-1}) \Big| \hat{A}_t \Psi_{\substack{\text{out}\\(\text{in})}}(\check{\chi}_1, \dots \check{\chi}_{n-1}) \Big\rangle \Big| \\ \le \|\hat{A}_t\|\, C_N \quad \forall\, t \underset{(<)}{>} 0\,,\ N \in \mathbb{N}\,,\ \hat{A}_t \in \mathcal{A}\left(\check{K} \cap U_{\epsilon|t|}(\Sigma_t)\right)\,, \end{aligned}$$

where:

$$\Sigma_t \stackrel{\text{def}}{=} \left\{X \in \mathbb{R}^4 : x^0 = t\right\}\,.$$

Definition 2.3.1 ——— *Draft, November 9, 2007* ———

[64]This is a stronger version of the Wightman axiom (v); for an interesting sufficient condition see (Borchers and Zimmermann, 1964), again.

[65]For more general considerations concerning the connection between local algebras of bounded

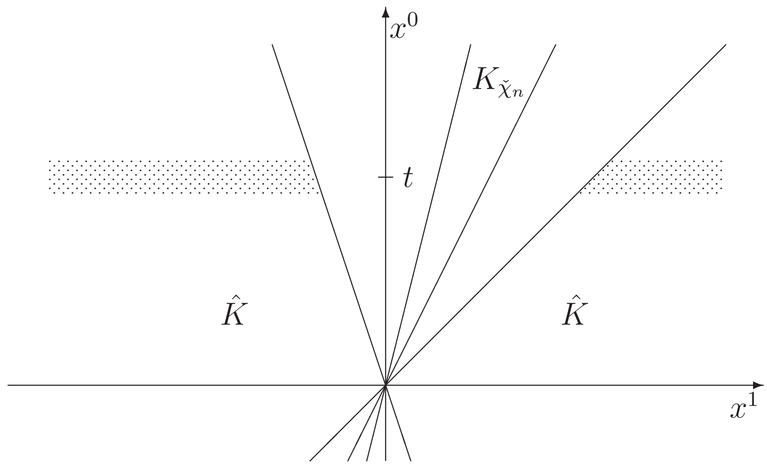

Figure 2.7: The regions (dotted) of measurement $\check{K} \cap U_{\epsilon t}(\Sigma_t)$, $t > 0$

2.3.3 Evaluation of the Asymptotic Condition

The evaluation of the asymptotic condition depends crucially on the following representation of 1-particle states:

Given $\check{\chi} \in \mathcal{D}(\mathbb{R}^3)$, *a sequence* $\left\{\hat{B}_t\right\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ *is called a* ***Haag-Ruelle-Kastler sequence*** *(HRK sequence) for*

$$\Psi = \hat{a}_0^*(\check{\chi})\Omega_0 \in \mathcal{H}_0^{(1)} \subset \mathcal{H}$$

if the following three conditions are fulfilled:

(i)
$$\lim_{t\to\pm\infty} |t|^N \left\|\hat{B}_t\Omega_0 - \Psi\right\| = 0 \quad \forall\, N \in \mathbb{N}\,.$$

(ii) *For every* $N \in \mathbb{N}$ *there is a sequence of local operators*

$$\hat{A}_t \in \mathcal{A}\left(U_{\frac{|t|}{N}}\left(K_{\check{\chi}} \cap \Sigma_t\right)\right)$$

with

$$\lim_{t\to\pm\infty} |t|^N \left\|\hat{A}_t - \hat{B}_t\right\| = 0\,.$$

operators and local Wightman fields see (Wollenberg, 1985) and (Driessler et al., 1986b).

[66]Statement (2.93) is a simple consequence of Corollary 2.2.16 and the spectral theorem.

[67]The factor $|t|^N$ with **arbitrary** N is appropriate for short-range forces, only. For long-range forces there are only very limited results (Buchholz, 1977).

[68]The ***causal completion*** $\mathcal{O}_t' \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^4 : x \times \mathcal{O}_t\right\}$ of $\mathcal{O}_t = \check{K} \cap U_{\epsilon|t|}(\Sigma_t)$ must not intersect $K_{\check{\chi}_n}$.

(iii)
$$\lim_{t\to\pm\infty} |t|^{-N} \left\| \hat{B}_t \right\| = 0 \quad \textit{for sufficiently large } N \in \mathbb{N}\,.$$

In order to prove existence of HRK sequences with the additional properties formulated in Lemma 2.3.3, below, we need the following variant of Lemma 2.2.17:

Lemma 2.3.2 *Let $\hat{A} \in \mathcal{L}(\mathcal{H})$ and let $\varphi \in \mathcal{S}(\mathbb{R}^4)$. Then for every Borel $B \subset \mathbb{R}^4$:*
$$\Psi \in \hat{E}(B)\mathcal{H} \implies \int \hat{A}(x)\,\varphi(x)\,\mathrm{d}x\,\Psi \in \hat{E}\left(\overline{V_+} \cap (B + \operatorname{supp}\widetilde{\varphi})\right)\mathcal{H}\,,$$
where
$$\hat{A}(x) \stackrel{\text{def}}{=} \hat{U}(x)\,\hat{A}\,\hat{U}(x)^{-1} \quad \textit{for } x \in \mathbb{R}^4\,.$$

Sketch of proof: Let B and $\hat{B}$ be arbitrary Borel subsets of $\mathbb{R}^4$ and let $\Psi \in \hat{E}(B)\mathcal{H}$. Then, since (2.60) implies
$$\hat{E}(B) = (2\pi)^{-2} \int \hat{U}(a)\,\widetilde{\chi}_B(-a)\,\mathrm{d}a\,,$$
where χ_B denotes the characteristic function of B :
$$\begin{aligned}
&(2\pi)^4 \hat{E}(\hat{B}) \int \hat{A}(x)\,\varphi(x)\,\mathrm{d}x\,\Psi \\
&= \int \hat{U}(\hat{a})\,\hat{A}(x)\,\hat{U}(a)\,\widetilde{\chi}_{\hat{B}}(-\hat{a})\,\widetilde{\chi}_B(-a)\varphi(x)\,\mathrm{d}\hat{a}\mathrm{d}a\mathrm{d}x\,\Psi \\
&= \int \hat{A}(x+\hat{a})\,\hat{U}(\hat{a}+a)\,\widetilde{\chi}_{\hat{B}}(-\hat{a})\,\widetilde{\chi}_B(-a)\varphi(x)\,\mathrm{d}\hat{a}\mathrm{d}a\mathrm{d}x\,\Psi \\
&= \int \hat{A}(x)\,\hat{U}(a)\,\widetilde{\chi}_{\hat{B}}(-\hat{a})\,\widetilde{\chi}_B(\hat{a}-a)\varphi(x-\hat{a})\,\mathrm{d}\hat{a}\mathrm{d}a\mathrm{d}x\,\Psi\,.
\end{aligned}$$
Since
$$\int \widetilde{\chi}_{\hat{B}}(-\hat{a})\,\widetilde{\chi}_B(\hat{a}-a)\varphi(x-\hat{a})\,\mathrm{d}\hat{a} = 0 \quad \text{for } \hat{B} \cap (B + \operatorname{supp}\widetilde{\varphi}) = \emptyset\,,$$
this implies
$$\int \hat{A}(x)\,\varphi(x)\,\mathrm{d}x\,\Psi \in \hat{E}\left(B + \operatorname{supp}\widetilde{\varphi}\right)\mathcal{H}$$
and hence, by the spectrum condition, the statement of the lemma. ∎

Lemma 2.3.3 *For every $\check{\chi} \in \mathcal{D}(\mathbb{R}^3)$ and $\epsilon > 0$ there is a HKR sequence $\left\{\hat{B}_t\right\}_{t\in\mathbb{R}}$ for $\hat{a}_0^*(\check{\chi})\Omega_0$ fulfilling the following two conditions:*[69]

(i)
$$\lim_{t\to\pm\infty} |t|^N \left\| \hat{B}_t^* \hat{B}_t \Omega_0 - \left\langle \Omega_0 \mid \hat{B}_t^* \hat{B}_t \Omega_0 \right\rangle \Omega_0 \right\| = 0 \quad \forall\, N \in \mathbb{N}\,.$$

[69] Actually, by Lemma 2.3.2 and Borchers' theorem, (i) is a consequence of (ii) for sufficiently small $\epsilon > 0$.

(ii)
$$\hat{B}_t^* \hat{E}(B)\mathcal{H} \subset \hat{E}\Big(B - U_\epsilon\left(M_{\check{\chi}}\right)\Big)\mathcal{H} \quad \forall\, t \in \mathbb{R}\,,\ Borel\, B \subset \mathbb{R}^4\,.$$
where
$$M_{\check{\chi}} \stackrel{\text{def}}{=} \{(\omega_{\mathbf{p}}, \mathbf{p}) :\ \mathbf{p} \in \operatorname{supp}\check{\chi}\}\ .$$

Sketch of proof: Let $\check{\chi} \in \mathcal{D}(\mathbb{R}^4)$ and $\epsilon > 0$. The essential point is to show that there is an ***almost local*** operator $\hat{A}$, i.e. a bounded operator $\hat{A}$ with
$$\lim_{0<R\to\infty} R^N \inf\left\{\left\|\hat{A} - \hat{B}\right\| :\ \hat{B} \in \mathcal{A}\left(U_R(0)\right)\right\} = 0 \quad \forall\, N \in \mathbb{N}\,,$$
for which
$$\check{f}_{\hat{A}}(\mathbf{p}) = 1 \quad \forall\, \mathbf{p} \in \operatorname{supp}\check{\chi}\,, \tag{2.94}$$
where
$$\check{f}_{\hat{A}} \stackrel{\text{def}}{=} \left(\hat{E}(M_m)\hat{A}\Omega_0\right)_1\ .$$

By (2.93), $\{0\}$ and $\mathcal{H}$ itself are the only closed subspaces of $\mathcal{H}$ which are invariant with respect to[70]
$$\mathcal{A}_{\mathrm{loc}} \stackrel{\text{def}}{=} \bigcap_{R<0} \mathcal{A}\left(U_R(0)\right)\ .$$
Therefore
$$\overline{\mathcal{A}_{\mathrm{loc}}\Omega_0} = \mathcal{H}$$
and consequently
$$\check{f}_{\hat{A}} \neq 0 \quad \text{for some } \hat{A} \in \mathcal{A}_{\mathrm{loc}}\,.$$
Moreover, from
$$\begin{aligned}\hat{E}(M_m)\hat{U}(0,\Lambda)\hat{A}\hat{U}(0,\Lambda)^{-1}\Omega_0 &= \hat{U}(0,\Lambda)\hat{E}(M_m)\hat{A}\Omega_0\\ &\underset{(2.86),(2.88)}{=} \hat{U}_0(0,\Lambda)\hat{E}(M_m)\hat{A}\Omega_0\end{aligned}$$
and (2.4) we conclude
$$\check{f}_{\hat{U}(0,\Lambda)\hat{A}\hat{U}(0,\Lambda)^{-1}}(\mathbf{p}) = \check{f}_{\hat{A}}\left(\overrightarrow{\Lambda^{-1}p}\right)_{|p^0=\omega_{\mathbf{p}}} \quad \forall\, \hat{A} \in \mathcal{A}_{\mathrm{loc}}\,,\ \mathbf{p} \in \mathbb{R}^3\,. \tag{2.95}$$
Therefore, choosing some Haar measure μ on $L_+^\uparrow$, we get
$$\check{f}_{\hat{A}_{\check{\delta}}}(\mathbf{p}) = \int \check{f}_{\hat{A}}\left(\overrightarrow{\Lambda^{-1}p}\right)_{|p^0=\omega_{\mathbf{p}}} \check{\delta}(\Lambda)\,\mu\left(d\Lambda\right)\,,$$
for sufficiently well-behaved $\check{\delta}$, where
$$\hat{A}_{\check{\delta}} \stackrel{\text{def}}{=} \int \hat{U}(0,\Lambda)\hat{A}\hat{U}(0,\Lambda)^{-1}\check{\delta}(\Lambda)\,\mu\left(d\Lambda\right)\ .$$

[70]The orthogonal projection onto a nontrivial invariant subspace would commute with all elements of $\mathcal{A}_{\mathrm{loc}}$, in contradiction to (2.93).

Now, for suitable $\check{\delta}$

$$\begin{aligned} g(\Lambda') &\stackrel{\text{def}}{=} \check{f}_{\hat{A}_{\check{\delta}}}\left(\overrightarrow{\Lambda'(m,0)}\right) \\ &= \int \check{f}_{\hat{A}}\left(\overrightarrow{\Lambda^{-1}(m,0)}\right)\check{\delta}(\Lambda'\Lambda)\,\mu\,(\mathrm{d}\Lambda) \\ &\in C^\infty(\mathbb{R}^3)\setminus\{0\} \end{aligned}$$

and therefore

$$\check{f}_{\hat{A}} \in C^\infty(\mathbb{R}^3)\setminus\{0\} \qquad \text{for suitable } \hat{A} \in \mathcal{A}_{\mathrm{loc}}\,.$$

By (2.4), we also have

$$\check{f}_{\hat{U}(x)\hat{A}\hat{U}(x)^{-1}}(\mathbf{p}) = \left(e^{ipx}\check{f}_{\hat{A}}(\mathbf{p})\right)_{|_{p^0=\omega_\mathbf{p}}}$$

and hence

$$\check{f}_{\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi(x)\,\mathrm{d}x}(\mathbf{p}) = (2\pi)^2\check{f}_{\hat{A}}(\mathbf{p})\widetilde{\varphi}(\omega_\mathbf{p},\mathbf{p})\,. \tag{2.96}$$

Therefore

$$0 \le \check{f}_{\hat{A}} \in C^\infty(\mathbb{R}^3)\setminus\{0\} \qquad \text{for suitable almost local } \hat{A}\,.$$

From this, using first (2.95) and then (2.96) again, we easily get (2.94) for some almost local $\hat{A}$.

Note that for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and

$$f^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2}\int_{p^0=\omega_\mathbf{p}} \check{\chi}(\mathbf{p})\,e^{-ipx}\,\mathrm{d}\mathbf{p} \tag{2.97}$$

we have

$$\begin{aligned} \varphi_t(x) &\stackrel{\text{def}}{=} \int_{y^0=t} \varphi(x-y)\,f^+(y)\,\mathrm{d}\mathbf{y} \\ &= (2\pi)^{-2}\int\left((2\pi)^3\widetilde{\varphi}(p)\,e^{i(\omega_\mathbf{p}-p^0)t}\check{\chi}(\mathbf{p})\right)e^{-ipx}\,\mathrm{d}p\,. \end{aligned} \tag{2.98}$$

By (2.96) this implies

$$\begin{aligned} \check{f}_{\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x)\,\mathrm{d}x}(\mathbf{p}) &= 2\omega_\mathbf{p}\,\check{\chi}(\mathbf{p}) \\ &= \left(\hat{a}_0^*(\check{\chi})\Omega_0\right)_1(\mathbf{p}) \end{aligned}$$

for all $t \in \mathbb{R}$ **if**

$$\check{f}_{\hat{A}}(\mathbf{p}) = 1 \quad \forall\,\mathbf{p} \in \operatorname{supp}\check{\chi}$$

and

$$\widetilde{\varphi}(\omega_\mathbf{p},\mathbf{p}) = (2\pi)^{-7/2}2\omega_\mathbf{p} \quad \forall\,\mathbf{p} \in \operatorname{supp}\check{\chi}\,. \tag{2.99}$$

In other words:

> There is an almost local operator $\hat{A}$ with
>
> $$(2.97)\text{–}(2.99) \implies \hat{E}(M_m)\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x)\,\mathrm{d}x\Omega_0 = \hat{a}_0^*(\check{\chi})\Omega_0\,.$$

Choosing φ such that

$$\mathrm{supp}\widetilde{\varphi} \cap \mathrm{supp}\hat{E} \subset M_m \tag{2.100}$$

we even have

$$\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x)\,\mathrm{d}x\Omega_0 = \hat{a}_0^*(\check{\chi})\Omega_0 \quad \forall\, t \in \mathbb{R}\,,$$

by Lemma 2.3.2. Since, obviously,

$$\sup_{x\in\mathbb{R}^4} \left|f^+(x)\right| < \infty$$

this together with Ruelle's Lemma (Lemma 2.1.1) shows that (2.97) – (2.100) guarantee $\left\{\hat{B}_t \stackrel{\mathrm{def}}{=} \int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x)\,\mathrm{d}x\right\}_{t\in\mathbb{R}}$ to be a HRK sequence for $\hat{a}_0^*(\check{\chi})\Omega_0$. In order to fulfill also conditions (i) and (ii) of the lemma, it is sufficient – thanks to Lemma 2.3.2, spectrum condition and Borchers' theorem – to choose φ such that also

$$\mathrm{supp}\widetilde{\varphi} \subset U_\delta\left(M_{\check{\chi}}\right)$$

holds with $\delta > 0$ sufficiently small to ensure

$$U_\delta\left(M_{\check{\chi}}\right) \cap U_\delta\left(-M_{\check{\chi}}\right) \subset U_m(0)\,. \qquad \blacksquare$$

Exercise 40 Show that the HRK sequence of Lemma 2.3.3 may be constructed in the form

$$\hat{B}_t = \int_{x^0=t} \hat{B}(x)\, i\overleftrightarrow{\partial}_0\, f^+(x)\,\mathrm{d}\mathbf{x}\,, \quad \hat{B}(x) \stackrel{\mathrm{def}}{=} \hat{U}(x)\,\hat{B}\,\hat{U}(x)^{-1}\,,$$

with f^+ defined by (2.97), where $\hat{B}$ is an almost local operator fulfilling

$$(\Box_x + m^2)\hat{B}(x)\Omega_0 = 0$$

and hence

$$\hat{B}_\Sigma\,\Omega_0 = \hat{a}_0^*(\check{\chi})\Omega_0\,, \quad \hat{B}_\Sigma \stackrel{\mathrm{def}}{=} \int_\Sigma \hat{B}(x)\, i\overleftrightarrow{\partial}_\mu\, \chi^+(x)\,\mathrm{d}\sigma_\mu\,,$$

for every (sufficiently well-behaved) spacelike hypersurface Σ (without finite boundary points). Show that $\hat{B}$ may be chosen such that also

$$\hat{B}_\Sigma^*\hat{B}_\Sigma\,\Omega_0 \sim \Omega_0$$

holds for all these surfaces.

Theorem 2.3.4 *Let $\check{\chi}_1, \ldots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ be nonoverlapping and let*

$$\|\hat{a}_0^*(\check{\chi}_\nu)\Omega_0\| = 1 \quad \forall \, \nu \in \{1, \ldots, n\} \ .$$

Then the assumptions made for the considered theory and the AB *imply existence of real numbers $\varphi_\pm$ with*

$$\lim_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_n) - \exp\left(i \, \varphi_{\underset{(-)}{+}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n,t} \Omega_0 \right\| = 0 \tag{2.101}$$

for all $N \in \mathbb{N}$ and HRK *sequences $\left\{\hat{A}_{\nu,t}\right\}_{t \in \mathbb{R}}$ for the $\hat{a}_0^*(\check{\chi}_\nu)\Omega_0$.*

Sketch of proof: Obviously, the (2.101) holds for $n = 1$ with $\varphi_\pm = 1$. Therefore, it is sufficient to prove the theorem for $n = n'$ assuming it to be valid for $n < n'$:

Without restriction of generality we may assume existence of some $\epsilon >=$ with

$$M_{\check{\chi}_1} + \ldots + M_{\check{\chi}_{n'}} - U_\epsilon\left(M_{\check{\chi}_1}\right) \ldots - U_\epsilon\left(M_{\check{\chi}_{n'}}\right) \subset U_m(0) \,. \tag{2.102}$$

Let us first consider HRK sequences of the type given by Lemma 2.3.3 and prove the the lemma by induction w.r.t. n . Exploiting the AB and the definition of HRK sequences we easily see that

$$\begin{aligned}
\lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \Bigg(& \left\| \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'}) \right\|^2 \\
& - \left\| \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\|^2 \Bigg) = 0
\end{aligned}$$

holds for all $N \in \mathbb{N}$. Since we already know that there are real $\varphi'_+ \, , , \, \varphi'_-$ with

$$\lim_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \exp\left(i \, \varphi'_{\underset{(-)}{+}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\| = 0$$

for all $N \in \mathbb{N}$, we conclude:

$$\begin{aligned}
& \lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'}) \right\| \\
&= \lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\| \\
&= \lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \exp\left(i \, \varphi'_{\underset{(-)}{+}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\| \\
&= \lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \exp\left(i \, \varphi'_{\underset{(-)}{+}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \hat{A}^*_{n'-1,t} \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\| \\
&= \lim\nolimits_{t \to \underset{(-)}{+} \infty} |t|^N \left\| \exp\left(i \, \varphi'_{\underset{(-)}{+}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1, \ldots, \check{\chi}_{n'-1}) \right\| \\
&= 0 \qquad \forall \, N \in \mathbb{N} \,.
\end{aligned}$$

Since, obviously,

$$\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) = \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_{\pi 1},\ldots,\check{\chi}_{\pi n'}) \quad \forall\, \pi \in \mathrm{S}'_n$$

this implies

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \left\| \hat{A}_{\nu,t}\hat{A}^*_{\nu,t}\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \right\| = 0$$

for all $\nu \in \{1,\ldots,n'\}$, $N \in \mathbb{N}$ and hence, by iteration,

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \left\| \hat{A}_{1,t}\hat{A}^*_{1,t}\cdots\hat{A}_{n',t}\hat{A}^*_{n',t}\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \right\| = 0$$

for all $N \in \mathbb{N}$. Exploiting local commutativity, once again, this gives

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \left\| \hat{A}_{1,t}\cdots\hat{A}_{n',t}\hat{A}^*_{1,t}\cdots\hat{A}^*_{n',t}\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \right\| = 0 \tag{2.103}$$

for all $N \in \mathbb{N}$. Now, for the HRK sequences of the type specified in Lemma 2.3.3 we have

$$\hat{A}^*_{1,t}\cdots\hat{A}^*_{n',t}\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \in \hat{E}\left(M_{\check{\chi}_1}+\ldots+M_{\check{\chi}_{n'}} - U_\epsilon\left(M_{\check{\chi}_1}\right)\ldots - U_\epsilon\left(M_{\check{\chi}_{n'}}\right)\right)\mathcal{H}\,,$$

since

$$\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \in \hat{E}\left(M_{\check{\chi}_1}+\ldots+M_{\check{\chi}_{n'}}\right)\mathcal{H}\,.$$

By (2.102), the spectrum condition, and Borchers' theorem, therefore,

$$\hat{A}^*_{1,t}\cdots\hat{A}^*_{n',t}\Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \sim \Omega \sim \Omega_0\,.$$

This together with (2.103) shows that there is a complex-valued function $\rho(t)$ with

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \left\| \rho(t)\hat{A}_{1,t}\cdots\hat{A}_{n',t}\Omega_0 - \Psi_{\underset{(\mathrm{in})}{\mathrm{out}}}(\check{\chi}_1,\ldots,\check{\chi}_{n'}) \right\| = 0 \quad \forall\, n \in \mathbb{N}\,.$$

Since, obviously,

$$\lim_{t\to \underset{(-)}{+}\infty} \left\| \rho(t)\hat{A}_{1,t}\cdots\hat{A}_{n',t}\Omega_0 \right\| = 1 \lim_{t\to \underset{(-)}{+}\infty} \left\| \hat{A}_{1,t}\cdots\hat{A}_{n',t}\Omega_0 \right\|\,,$$

we are left to prove

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \sup_{s>t} \left\| \hat{A}_{1,t}\cdots\hat{A}_{n',t}\Omega_0 - \hat{A}_{1,s}\cdots\hat{A}_{n',s}\Omega_0 \right\| = 0 \quad \forall\, n \in \mathbb{N}\,,$$

as far as the special HRK sequences are concerned. This however, is a simple consequence of

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \sup_{s,s'\in(t,t+1)} \left\| \hat{A}_{1,s}\cdots\hat{A}_{\nu,s'}\cdots\hat{A}_{n',s}\Omega_0 - \hat{A}_{1,s}\cdots\hat{A}_{n',s}\Omega_0 \right\| = 0 \quad \forall\, n\in\mathbb{N}\,.$$

Finally, it is an easy consequence of local commutativity and Definition 2.3.1 that

$$\lim_{t\to \underset{(-)}{+}\infty} |t|^N \left\| \hat{A}_{1,t}\cdots\hat{A}_{n,t}\Omega_0 - \hat{A}_{1,t}\cdots\hat{A}'_{\nu,t}\cdots\hat{A}_{n,t}\Omega_0, t \right\|$$

holds for all $N\in\mathbb{N}$ and $\nu\in\{1,\ldots,n\}$, if $\left\{\hat{A}'_{\nu,t}\right\}_{t\in\mathbb{R}}$ is any other HRK sequence for $\hat{a}_0^*(\check{\chi}_\nu)\Omega_0$; i.e.:
(2.101) does not dependent on the special choice for the HRK sequences. ∎

Exercise 41 Show the following:

(i) Unitary mappings $\hat{V}_{\underset{(\mathrm{in})}{\mathrm{out}}}$ from $\mathcal{H}_0$ onto $\mathcal{H}$ fulfilling (2.86), (2.88), and the asymptotic condition do exist.

(ii) The numbers $\varphi_\pm$ may[71] depend on n but not on the functions $\check{\chi}_\nu$.

Exercise 42 Assuming that the local algebras are given by a neutral scalar Wightman field as described above, show the following for the PCT operator $\hat{\theta}$ considered in Exercise 39 and the corresponding operator $\hat{\theta}_0$ of the FS:[72]

(i)
$$\hat{\theta}\hat{U}(a,\Lambda) = \hat{U}(-a,\Lambda)\hat{\theta} \quad \forall\,(a,\Lambda)\in\mathcal{P}_+^\uparrow\,,$$

(ii)
$$\hat{\theta}\hat{V}_{\mathrm{out}} = \hat{V}_{\mathrm{in}}\hat{\theta}_0\,, \quad \hat{\theta}\hat{V}_{\mathrm{in}} = \hat{V}_{\mathrm{out}}\hat{\theta}_0\,,$$

(iii) The described scattering theory is PCT-invariant in the sense that[73]

$$\hat{\theta}\hat{\hat{S}} = \hat{\hat{S}}^{-1}\hat{\theta}\,.$$

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[71]Suitable extension of the asymptotic condition implies $\varphi_\pm = 1$ for all n (see (Lücke, 1983)).
[72]Note that (i) implies $\hat{\theta}\mathcal{A}(\mathcal{O}) = \mathcal{A}(-\mathcal{O})\hat{\theta}$ for all open $\mathcal{O}\subset\mathbb{R}^4$.
[73]For a (complicated) proof not using this assumption see (Epstein, 1967).

2.3.4 Cluster Properties of the S-Matrix

Definition 2.3.5 *Let* $\{M_t\}_{t\in\mathbb{R}} \subset \mathbb{R}^4$ *and* $\left\{\hat{B}_t\right\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{H})$. *Then* $\hat{B}_t$ *is called* ***asymptotically localized*** *in* M_t *if the following two conditions are fulfilled:*

(i) *For every* $N \in \mathbb{N}$ *there is a sequence of local operators*

$$\hat{A}_t \in \mathcal{A}\left(U_{\frac{|t|}{N}}(M_t)\right)$$

with

$$\lim_{t\to\pm\infty} |t|^N \left\|\hat{A}_t - \hat{B}_t\right\| = 0\,.$$

(ii)

$$\lim_{t\to\pm\infty} |t|^{-N} \left\|\hat{B}_t\right\| = 0 \quad \textit{for sufficiently large } N \in \mathbb{N}\,.$$

An immediate consequence of Definition 2.3.5 is the following

Corollary 2.3.6 *For* $\nu = 1, 2$ *let* $\hat{B}_{j,t}$ *be asymptotically localized in* $M_{j,t}$. *Then* $\hat{B}_t \stackrel{\text{def}}{=} \hat{B}_{1,t}\hat{B}_{2,t}$ *is asymptotically localized in* $M_t \stackrel{\text{def}}{=} M_{1,t} \cup M_{2,t}$.

Definition 2.3.7 *Let* $\{M_t\}_{t\in\mathbb{R}} \subset \mathbb{R}^4$ *and* $\Psi \in \mathcal{H}$. *Then* $\left\{\hat{B}_t\right\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ *is called a* M_t***-sequence*** *for* Ψ *if the following two conditions are fulfilled:*[74]

(i) $\lim_{t\to\pm\infty} |t|^N \left\|\hat{B}_t\Omega_0 - \Psi\right\| = 0 \quad \forall\, N \in \mathbb{N}$.

(ii) $\hat{B}_t$ *is asymptotically localized in* M_t.

Theorem 2.3.8 *Let* $\check{\chi}_1, \ldots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ *be non-overlapping and let* Σ *be a smooth spacelike hypersurface (without finite boundary points) above resp. below*[75] *all the sets*

$$(K_{\check{\chi}_\nu} + a_\nu) \cap \left(K_{\check{\chi}_\mu} + a_\mu\right)\ , \quad \nu \neq \mu\,,$$

for given $a_1, \ldots, a_n \in \mathbb{R}^4 \setminus \Sigma$. *Then there are* $(t\Sigma - ta_\nu)$*-sequences* $\left\{\hat{B}_{\nu,t}\right\}_{t\in\mathbb{R}}$ *for the 1-particle states* $\hat{a}_0^*(\check{\chi}_\nu)\Omega_0$ *and a real number* φ_{out} *resp.* φ_{in} *with*[76]

$$\lim_{\lambda\to+\infty} |t|^N \Big\|\Psi_{\text{ex}}\left(\hat{U}_0(\lambda a_1)\check{\chi}_1, \ldots, \hat{U}_0(\lambda a_n)\check{\chi}_n\right) - e^{i\varphi_{\text{ex}}}\hat{B}_{1,\lambda}(\lambda a_1)\cdots\hat{B}_{n,\lambda}(\lambda a_n)\Omega_0\Big\| = 0 \quad \forall\, N \in \mathbb{N}$$

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[74] Note that for $\Psi = \hat{a}_0^*(\check{\chi})\Omega_0$ and $M_t = K_{\check{\chi}} \cap \Sigma_t$ the M_t-sequences are just HRK-sequences.

[75] We say Σ is above (below) a set $M \subset \mathbb{R}^4$ if $M \subset \left\{(x^0 \underset{(+)}{-} t, \mathbf{x}) : x \in \Sigma\,,\, t > 0\right\}$.

[76] The special choice for the $(t\Sigma - ta_\nu)$-sequences is not essential. As usual, we use the notation $\hat{B}(x) = \hat{U}(x)\hat{B}\,\hat{U}(x)^{-1}$ for $\hat{B} \in \mathcal{L}(\mathcal{H})$, $x \in \mathbb{R}^4$.

where ex=out resp. ex=in .

Proof: Straightforward application of the described standard techniques. ∎

Theorem 2.3.9 (Fredenhagen) *Let $\tau > 0$ and let $\hat{A}$, $\hat{B} \in \mathcal{L}(\mathcal{H})$ fulfill the condition*[77]

$$[\hat{U}(t)\hat{A}\,\hat{U}(t)^{-1}, \hat{B}]_- = 0 \quad \forall\, t \in [-\tau, +\tau]\,.$$

Then

$$\left|\left\langle \Omega_0 \mid \hat{A}\hat{B}\Omega_0 \right\rangle - \left\langle \Omega_0 \mid \hat{A}\Omega_0 \right\rangle \left\langle \Omega_0 \mid \hat{B}\Omega_0 \right\rangle\right| \le e^{-m\tau} \sqrt{\left\|\hat{A}^*\Omega_0\right\| \left\|\hat{B}\Omega_0\right\| \left\|\hat{A}\Omega_0\right\| \left\|\hat{B}^*\Omega_0\right\|}\,.$$

Proof: See (Fredenhagen, 1985). ∎

All we need for the derivation of cluster properties of the S-matrix is the following immediate consequence of Theorem 2.3.9.

Corollary 2.3.10 *For $\nu = 1, 2$ let $\hat{B}_{j,t}$ be asymptotically localized in $M_{j,t}$. If there is some ϵ for which*

$$\lambda > \frac{1}{\epsilon} \implies M_{1,\lambda} \times U_{\epsilon\lambda}(M_{2,\lambda})\,,$$

then

$$\lim_{\lambda\to+\infty} \lambda^N \left\|\left\langle \Omega_0 \mid \hat{B}_{1,\lambda}\hat{B}_{2,\lambda}\Omega_0 \right\rangle - \left\langle \Omega_0 \mid \hat{B}_{1,\lambda}\Omega_0 \right\rangle \left\langle \Omega_0 \mid \hat{B}_{2,\lambda}\Omega_0 \right\rangle\right\| = 0 \quad \forall\, N \in \mathbb{N}\,.$$

Now the cluster properties of the S-matrix are an immediate consequence:

Corollary 2.3.11 *Let $\check{\chi}_1, \ldots, \check{\chi}_n\, a_1, \ldots, a_n$ and Σ be given as in Theorem 2.3.8 for 'above'. Moreover, let $\check{\chi}'_1, \ldots, \check{\chi}'_{n'}\, a'_1, \ldots, a'_n$ and Σ' be given as in Theorem 2.3.8 for 'below'. Finally, let $I \subset \{1, \ldots, n\}$ and $I' \subset \{1, \ldots, n'\}$ be such that*

$$\left(\left(K_{\check{\chi}_{\nu_1}} + a_{\nu_1}\right) \cap \Sigma\right) \cup \left(\left(K_{\check{\chi}'_{\nu'_1}} + a'_{\nu'_1}\right) \cap \Sigma'\right)$$

is spacelike relative to

$$\left(\left(K_{\check{\chi}_{\nu_2}} + a_{\nu_2}\right) \cap \Sigma\right) \cup \left(\left(K_{\check{\chi}'_{\nu'_2}} + a'_{\nu'_2}\right) \cap \Sigma'\right)$$

for all $\nu_1 \in I$, $\nu'_2 \in I'$ and $\nu_2 \in \{1, \cdots, n\} \setminus I$, $\nu'_2 \in \{1, \ldots, n'\} \setminus I'$.

[77] As usual, we write $\hat{U}(t)$ for $\hat{U}\left((t, 0, 0, 0), \mathbb{1}_4\right)$.

Then there is a real number ρ with

$$\lim_{\lambda\to+\infty} \lambda^N \left| e^{i\rho} \langle \Psi_{\mathrm{out}}(\lambda) \mid \Psi_{\mathrm{in}}(\lambda)\rangle - \langle \Psi_{1,\mathrm{out}}(\lambda) \mid \Psi_{1,\mathrm{in}}(\lambda)\rangle \langle \Psi_{2,\mathrm{out}}(\lambda) \mid \Psi_{2,\mathrm{in}}(\lambda)\rangle \right| = 0$$

for all $N \in \mathbb{N}$, where

$$\begin{aligned}
\Psi_{\mathrm{out}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{out}} \left(\prod_{\nu'\in\{1,\dots,n'\}} \left(\hat U(\lambda a'_{\nu'}) \hat a_0^*(\check\chi'_{\nu'}) \left(\hat U(\lambda a'_{\nu'})\right)^{-1}\right) \Omega_0 \right), \\
\Psi_{1,\mathrm{out}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{out}} \left(\prod_{\nu'\in I'} \left(\hat U(\lambda a'_{\nu'}) \hat a_0^*(\check\chi'_{\nu'}) \left(\hat U(\lambda a'_{\nu'})\right)^{-1}\right) \Omega_0 \right), \\
\Psi_{2,\mathrm{out}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{out}} \left(\prod_{\nu'\in\{1,\dots,n'\}\setminus I'} \left(\hat U(\lambda a'_{\nu'}) \hat a_0^*(\check\chi'_{\nu'}) \left(\hat U(\lambda a'_{\nu'})\right)^{-1}\right) \Omega_0 \right), \\
\Psi_{\mathrm{in}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{in}} \left(\prod_{\nu\in\{1,\dots,n\}} \left(\hat U(\lambda a_{\nu}) \hat a_0^*(\check\chi_{\nu}) \left(\hat U(\lambda a_{\nu})\right)^{-1}\right) \Omega_0 \right), \\
\Psi_{1,\mathrm{in}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{in}} \left(\prod_{\nu\in I} \left(\hat U(\lambda a_{\nu}) \hat a_0^*(\check\chi_{\nu}) \left(\hat U(\lambda a_{\nu})\right)^{-1}\right) \Omega_0 \right), \\
\Psi_{2,\mathrm{in}}(\lambda) &\stackrel{\mathrm{def}}{=} \hat V_{\mathrm{in}} \left(\prod_{\nu\in\{1,\dots,n\}\setminus I} \left(\hat U(\lambda a_{\nu}) \hat a_0^*(\check\chi_{\nu}) \left(\hat U(\lambda a_{\nu})\right)^{-1}\right) \Omega_0 \right).
\end{aligned}$$

Typical consequences of Corollary 2.3.11 are illustrated by Figures 2.9 and 2.8 (see also (Lücke, 8384)).[78] Roughly speaking, Corollary 2.3.11 shows that, from the macroscopic point of view, the S-matrix does not violate the causality principle.

It should have become clear from the evaluation of the asymptotic condition, that the field $\hat\Phi(x)$ itself is not necessary to determine the S-matrix once the set of almost local operators is determined. Note that the representation $\hat U(a,\Lambda)$ of $\mathcal{P}_+^\uparrow$ is already fixed, up to unitary equivalence, by the FS. Of course, one cannot expect that the Hamiltonian $\hat P^0$ itself determines the physical picture of the dynamics unless the physical interpretation of the other observables is sufficiently well established. The above considerations show that it is sufficient to specify the macroscopically localized observables consistently, in order to select the S-matrix.

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[78]Figure 2.8 illustrates the case $n = 4$, $n' = 6$, $I = I' = \{1,2\}$; using the notation $K_\nu \stackrel{\mathrm{def}}{=} K_{\check\chi_\nu} + \lambda a_\nu$, $K'_\nu \stackrel{\mathrm{def}}{=} K_{\check\chi'_\nu} + \lambda a'_\nu$

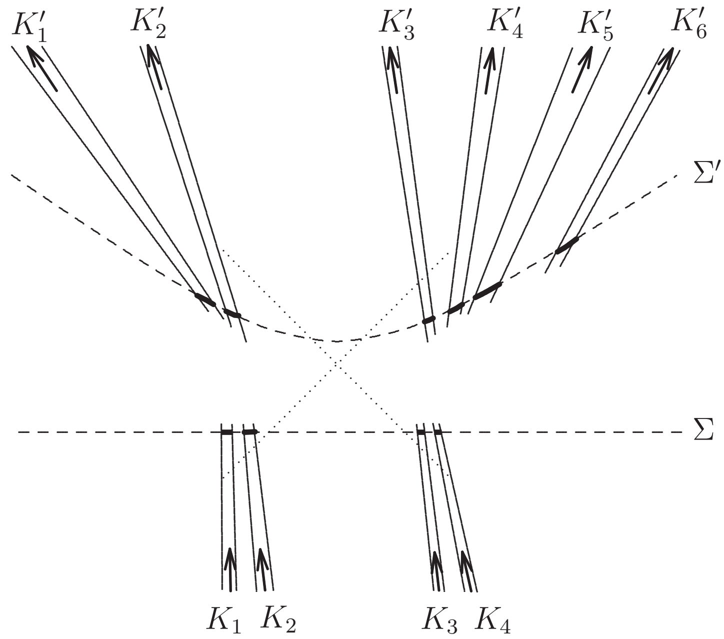

Figure 2.8: Macroscopically independent: '1+2 ⟶ 1+2' and '3+4 ⟶ 3+...+6'

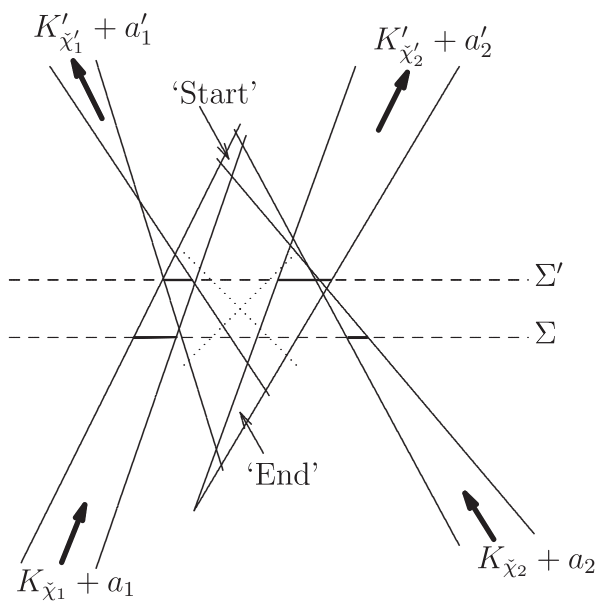

Figure 2.9: Macroscopically forbidden: 'output before input'

2.4 Charged Scalar Fields

2.4.1 Free Charged Scalar Fields

Fields Operators

It is known, **nowadays**, that for every charged particle there is an *antiparticle* with opposite charge. Let (2.39) describe such a particle in the sense of 2.1.3 and, similarly,

$$\breve{\hat{\Phi}}_0^{(+)}(x) \stackrel{\mathrm{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \breve{\hat{a}}_0(\mathbf{p})\, e^{-ipx} \frac{\mathrm{d}\mathbf{p}}{2p^0}$$

the corresponding antiparticle on its Fock space $\breve{\mathcal{H}}_0$ with domain $\breve{D}_0$ and representation $\breve{\hat{U}}_0(a,\Lambda)$ of $\mathcal{P}_+^\uparrow$. Then both particles may be described simultaneously by the ***charged scalar field***

$$\hat{\Phi}_q(x) \stackrel{\mathrm{def}}{=} \hat{\Phi}_0^{(+)}(x) \otimes \mathbb{1} + \left(\mathbb{1} \otimes \breve{\hat{\Phi}}_0^{(+)}(x)\right)^* \tag{2.104}$$

on

$$D_q \stackrel{\mathrm{def}}{=} D_0 \otimes \breve{D}_0 \subset \mathcal{H}_q \stackrel{\mathrm{def}}{=} \mathcal{H}_0 \otimes \breve{\mathcal{H}}_0 \,,$$

generalizing (2.43). As for the neutral scalar field, the relations[79]

$$\left(\Box + m^2\right) \hat{\Phi}_q(x) = 0 \,, \tag{2.105}$$

$$\hat{U}_q(a,\Lambda)\hat{\Phi}_q(x)\hat{U}_q(a,\Lambda)^{-1} = \hat{\Phi}_q(\Lambda x + a)\,, \tag{2.106}$$

$$\left[\hat{\Phi}_q(x), \hat{\Phi}_q(y)\right]_- = 0 \;\text{ for } x \times y\,, \tag{2.107}$$

are fulfilled for the charged field, where

$$\hat{U}_q(a,\Lambda) \stackrel{\mathrm{def}}{=} \hat{U}_0(a,\Lambda) \otimes \breve{\hat{U}}_0(a,\Lambda)\,.$$

Nevertheless, **the charged field cannot be interpreted as observable of whatever field strength** since, according to (2.104), it is not hermitian. This is the price to be paid for the important commutation relation

$$\left[\hat{Q}, \hat{\Phi}_q(x)\right]_- = -q\, \hat{\Phi}_q(x)\,, \quad \text{where:} \tag{2.108}$$

$\hat{Q}$ = observable of the **additive** charge,
q = charge of the particle,
$-q$ = charge of the antiparticle,

valid on a suitable domain. The main purpose of the **charged** scalar field is to create a dense set of well-interpreted states (for scattering theory) out of the vacuum.

Now it is important to supplement (2.107) by[80]

$$\left[\hat{\Phi}_q(x), \hat{\Phi}_q^*(y)\right]_- = 0 \;\text{ for } x \times y\,. \tag{2.109}$$

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[79] (2.107) holds for all x, y.

[80] More precisely, (2.46) becomes $\left[\hat{\Phi}_q(x), \hat{\Phi}_q^*(y)\right]_- = i\Delta(x-y)$.

Local Gauge Transformations

Since the charged scalar field $\hat{\Phi}_q(x)$ is non-hermitian, anyway, it may well be replaced by

$$\hat{\Phi}_q^\lambda(x) \stackrel{\text{def}}{=} e^{i\hat{Q}\lambda(x)}\hat{\Phi}_q(x)e^{-i\hat{Q}\lambda(x)} \underset{(2.108)}{=} e^{-iq\lambda(x)}\hat{\Phi}_q(x) \tag{2.110}$$

if, simultaneously, the Klein-Gordon equation (2.105) is replaced by

$$\left((\partial_\mu + iq(\partial_\mu\lambda)(x))\,(\partial^\mu + iq(\partial^\mu\lambda)(x)) + m^2\right)\hat{\Phi}_q^\lambda(x) = 0\,, \tag{2.111}$$

where $\lambda(x)$ denotes an arbitrary, but sufficiently smooth, real-valued function on $\mathbb{R}^4$. As a direct generalization this leads to the Klein-Gordon equation

$$\left((\partial_\mu + iqA_\mu(x))\,(\partial^\mu + iqA^\mu(x)) + m^2\right)\hat{\Phi}^\lambda(x) = 0 \tag{2.112}$$

for the quantized scalar field $\hat{\Phi}(x)$ interacting with the *external* (classical) electromagnetic field $A^\mu(x)$, being invariant under ***gauge transformations of second kind***

$$\hat{\Phi}(x) \longrightarrow e^{-iq\lambda(x)}\hat{\Phi}\,, \quad A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\lambda(x)\,. \tag{2.113}$$

Consistency Considerations

Exchanging the roles of particles and antiparticles results, according to (2.104), in the transition

$$\hat{\Phi}_{(q)}(x) \longrightarrow \hat{\Phi}^*_{(q)}(x)\,, \quad q \longrightarrow -q\,.$$

Obviously, (2.112) is invariant under this transformation (since $A^\mu(x)$ is real).

For sufficiently well-behaved external fields $A^\mu(x)$ there exist solutions $\hat{\Phi}$ of (2.112) respecting the interaction picture (Seiler, 1978).

Interpreting classical solutions of the Klein-Gordon equation as expectation values of the quantized Klein-Gordon field solves the well-known problem raised by creation of negative frequency contributions in certain scattering problems (see e.g. (Baym, 1969, Chapt. 22)).

Replacing the classical electromagnetic potential $A^\mu(x)$ in (2.112) by the corresponding quantum field $\hat{A}^\mu(x)$ leads to the difficult problem of defining products of quantized fields (basic problems of *renormalization theory*).

2.4.2 Wightman Theory for Charged Scalar Fields

Wightman Axioms

A ***Wightman Theory*** of a single **charged** scalar field $\hat{\Phi}(x)$ is characterized by the following assumptions (*Wightman axioms*):

0. Assumptions of Relativistic Quantum Theory:

Exactly the same as those for the Wightman Theory of a single **neutral** scalar field, as formulated in Section 2.2.1.

I. Assumptions about the Domain and Continuity of the Field:

There are two fields $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ defined as operator-valued, tempered, generalized functions with invariant domain $D \subset \mathcal{H}$; i.e. **linear** mappings

$$\begin{aligned} \hat{\Phi} \,:\, \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D,D) \\ \varphi &\longmapsto \hat{\Phi}(\varphi) = \underbrace{\int \hat{\Phi}(x)\varphi(x)\,\mathrm{d}x}_{\text{formal}} \end{aligned}$$

and

$$\begin{aligned} \hat{\Phi}^* \,:\, \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D,D) \\ \varphi &\longmapsto \hat{\Phi}^*(\varphi) = \underbrace{\int \hat{\Phi}^*(x)\varphi(x)\,\mathrm{d}x}_{\text{formal}} \end{aligned}$$

for which all the

$$\int \left\langle \Psi \mid \hat{\Phi}(x)\Psi \right\rangle \varphi(x)\,\mathrm{d}x \stackrel{\text{def}}{=} \left\langle \Psi \mid \hat{\Phi}(\varphi)\Psi \right\rangle \,, \Psi \in D\,,$$

and

$$\int \left\langle \Psi \mid \hat{\Phi}^*(x)\Psi \right\rangle \varphi(x)\,\mathrm{d}x \stackrel{\text{def}}{=} \left\langle \Psi \mid \hat{\Phi}^*(\varphi)\Psi \right\rangle \,, \Psi \in D\,,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a,\Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \subset D\,, \quad \hat{U}(a,\Lambda)D \subset D\,, \quad \hat{\Phi}(\varphi)D \subset D \supset \Phi^*(\varphi)D\,.$$

The fields $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ are related by

$$\left\langle \Psi \mid \hat{\Phi}^*(\varphi)\Psi \right\rangle = \left\langle \Psi \mid \left(\hat{\Phi}(\overline{\varphi})\right)^* \Psi \right\rangle \quad \forall \Psi \in D\,,\, \varphi \in \mathcal{S}(\mathbb{R}^4)\,. \tag{2.114}$$

II. Transformation Law of the Field:

The field operators $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ transform according to

$$\hat{U}(a,\Lambda)\hat{\Phi}(x)\hat{U}(a,\Lambda)^{-1} = \hat{\Phi}(\Lambda x + a) \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow$$

and[81]

$$\hat{U}(a,\Lambda)\hat{\Phi}^*(x)\hat{U}(a,\Lambda)^{-1} = \hat{\Phi}^*(\Lambda x + a) \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,.$$

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[81]By (2.114) the transformation law of $\hat{\Phi}(x)$ implies that of $\hat{\Phi}^*(x)$ and vice versa.

III. Local Commutativity:

The smeared fields $\hat{\Phi}(\varphi_1)$ and $\hat{\Phi}(\varphi_2)$ resp. $\hat{\Phi}^*(\varphi_2)$ commute whenever the supports of the test functions φ_1 , $\varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other.[82] Formally:

$$x \times y \implies [\hat{\Phi}(x), \hat{\Phi}(y)]_- = [\hat{\Phi}(x), \hat{\Phi}^*(y)]_- = 0\,.$$

Finally, the vacuum vector Ω is required to be **cyclic** with respect to the algebra $\mathcal{F}_0$ generated by $\hat{1}\restriction D$ and the smeared field field operators $\hat{\Phi}(\varphi)$ and $\hat{\Phi}^*(\varphi)$ with $\varphi \in \mathcal{S}(\mathbb{R}^4)$:

$$D_0 \stackrel{\text{def}}{=} \mathcal{F}_0\,\Omega \text{ is dense in } \mathcal{H}\,.$$

Obviously, all these axioms are fulfilled for the free **charged** field $\hat{\Phi}(x) = \hat{\Phi}_q(x)$, if we define $D \stackrel{\text{def}}{=} D_q$ and

$$\hat{\Phi}^*(\varphi) \stackrel{\text{def}}{=} \left(\hat{\Phi}(\overline{\varphi})\right)^* \restriction D \qquad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^4)\,.$$

PCT and Spin-Statistics Theorem

The 'connection between spin and statistics' for the theory of a single charged field is given by the following two theorems.

Theorem 2.4.1 *There is no charged field $\hat{\Phi}(x) \neq 0$, fulfilling all the Wightman axioms with the possible exception of local commutativity, for which*

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) + \hat{\Phi}^*(y)\hat{\Phi}(x) = 0\,.$$

Sketch of proof: The techniques used in Section 2.2.4 show that the expectation values of products of field operators exist as generalized functions and that there are $L_+^\uparrow$-invariant generalized functions $W, \check{W} \in \mathcal{S}(\mathbb{R}^4)'$ with

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}^*(y)\Omega \right\rangle &= W(x-y)\,, \\ \left\langle \Omega \mid \hat{\Phi}^*(x)\hat{\Phi}(y)\Omega \right\rangle &= \check{W}(x-y)\,, \end{aligned}$$

and

$$\operatorname{supp}\widetilde{W} \subset \overline{V_+} \supset \operatorname{supp}\widetilde{\check{W}}\,. \tag{2.115}$$

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[82]Note that that this condition can be shown to be necessary to avoid acausal effects.

Now, assume

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) + \hat{\Phi}^*(y)\hat{\Phi}(x) = 0\,,$$

i.e.

$$W(\xi) + \check{W}(-\xi) = 0 \quad \text{for } \xi \times 0\,.$$

Since $\check{W}(\xi) - \check{W}(-\xi)$, being an odd $L_+^\uparrow$-invariant distribution, vanishes for spacelike ξ this implies

$$F(\xi) \stackrel{\text{def}}{=} W(\xi) + \check{W}(\xi) = 0 \quad \text{for } \xi \times 0\,.$$

Since, by (2.115),

$$\text{supp}\widetilde{F} \subset \overline{V_+} \supset \text{supp}\widetilde{\check{W}}$$

Corollary 2.2.9 tells us that $F = 0$, i.e:

$$\left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}^*(y)\Omega \right\rangle + \left\langle \Omega \mid \hat{\Phi}^*(-y)\hat{\Phi}(-x)\Omega \right\rangle = 0\,.$$

Therefore, we have

$$\begin{aligned} 0 &= \int \left(\left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}^*(y)\Omega \right\rangle + \left\langle \Omega \mid \hat{\Phi}^*(-y)\hat{\Phi}(-x)\Omega \right\rangle \right) \varphi(x)\overline{\varphi(y)}\,\mathrm{d}x\mathrm{d}y \\ &= \left\| \left(\hat{\Phi}(\varphi)\right)^* \Omega \right\|^2 + \left\| \int \hat{\Phi}(x)\varphi(-x)\,\mathrm{d}x\,\Omega \right\|^2 \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and hence

$$\hat{\Phi}^*(x)\Omega = \hat{\Phi}(x)\Omega = 0\,.$$

By cyclicity of Ω, however, this would imply $\hat{\Phi}(x) = 0$. ∎

Theorem 2.4.2 *There is no charged field $\hat{\Phi}(x) \neq 0$, fulfilling all the Wightman axioms with the possible exception of local commutativity, for which the conditions*

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) - \hat{\Phi}^*(y)\hat{\Phi}(x) = 0$$

and

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0$$

hold.

Sketch of Proof:[83] Assume

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) - \hat{\Phi}^*(y)\hat{\Phi}(x) = 0$$

and

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0\,.$$

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[83]This proof may be applied to a much more general situation (see (Streater and Wightman, 1989, Theorem 4.8)).

Then for all $\varphi\,,\,\psi \in \mathcal{D}(\mathbb{R}^4)$ with

$$\mathrm{supp}\varphi \times \mathrm{supp}\psi$$

we have

$$\begin{aligned} 0 &\leq \left\| \hat{\Phi}^*(\varphi)\hat{\Phi}(\psi)\Omega \right\|^2 \\ &= \left\langle \Omega \mid \left(\hat{\Phi}(\psi)\right)^* \left(\hat{\Phi}^*(\varphi)\right)^* \hat{\Phi}^*(\varphi)\hat{\Phi}(\psi)\Omega \right\rangle \\ &= -\left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi)\right)^* \hat{\Phi}^*(\varphi) \left(\hat{\Phi}(\psi)\right)^* \hat{\Phi}(\psi)\Omega \right\rangle . \end{aligned}$$

Since

$$\hat{U}(a)\hat{\Phi}(\psi)\hat{U}(a)^{-1} = \int \hat{\Phi}(x)\psi(x-a)\,\mathrm{d}x \quad \forall\, a \in \mathbb{R}^4\,,$$

this implies

$$\left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi)\right)^* \hat{\Phi}^*(\varphi)\hat{U}(\lambda a) \left(\hat{\Phi}(\psi)\right)^* \hat{\Phi}(\psi)\Omega \right\rangle \leq 0$$

for spacelike a and sufficiently large $\lambda = \lambda(a, \varphi, \psi)$. On the other hand, however, one may prove[84] that

$$\begin{aligned} &\lim_{\lambda\to\infty} \left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi)\right)^* \hat{\Phi}^*(\varphi)\hat{U}(\lambda a) \left(\hat{\Phi}(\psi)\right)^* \hat{\Phi}(\psi)\Omega \right\rangle \\ &= \left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi)\right)^* \hat{\Phi}^*(\varphi)\Omega \right\rangle \left\langle \Omega \mid \left(\hat{\Phi}(\psi)\right)^* \hat{\Phi}(\psi)\Omega \right\rangle \\ &= \left\| \hat{\Phi}^*(\varphi)\Omega \right\|^2 \left\| \hat{\Phi}(\psi)\Omega \right\|^2 . \end{aligned}$$

Therefore

$$\hat{\Phi}^*(x)\Omega = \hat{\Phi}(x)\Omega = 0\,,$$

which, by cyclicity of Ω, would imply $\hat{\Phi}(x) = 0$. ∎

Now, of course, the PCT theorem has to involve both $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$:

Theorem 2.4.3 *Let $n \in \mathbb{N}$ and let $\hat{\Phi}(x)$ be a charged scalar field fulfilling all the Wightman axioms with the possible exception of local commutativity. Then the PCT condition*

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}_1(x_1)\cdots\hat{\Phi}_n(x_n)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}_n(-x_n)\cdots\hat{\Phi}_1(-x_1)\Omega \right\rangle \\ \forall\, x \in \mathbb{R}^{4n}\,,\ \hat{\Phi}_\nu \in \left\{\hat{\Phi}, \hat{\Phi}^*\right\} \end{aligned} \tag{2.116}$$

is equivalent to the condition of ***weak local commutativity***

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}_1(x_1)\cdots\hat{\Phi}_n(x_n)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}_n(x_n)\cdots\hat{\Phi}_1(x_1)\Omega \right\rangle \\ \text{for } (x_1 - x_2, \ldots, x_{n-1} - x_n) \in \mathcal{J}_{n-1}\,,\ \hat{\Phi}_\nu \in \left\{\hat{\Phi}, \hat{\Phi}^*\right\} . \end{aligned} \tag{2.117}$$

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[84]See (Araki et al., 1962, Theorem 3) for a proof of this ***cluster property*** not depending on 0 being an isolated point of the energy-momentum spectrum. See also (Maison, 1968) and, for a C^* algebraic version, (Baumgärtel, 1995, Theorem 1.2.5).

Proof: Analogous to that of Corollary 2.2.22

Show that (2.116) is equivalent to existence of an anti-unitary Operator $\hat\theta$ fulfilling the conditions

$$\hat\theta\hat\theta = \hat 1\,, \quad \hat\theta\Omega = \Omega\,,$$

$$\hat\theta\,\hat{\hat\Phi}(\varphi)\,\hat\theta = \left(\int \hat{\hat\Phi}(-x)\,\varphi(x)\,\mathrm{d}x\right)^* \bigwedge D \quad \forall\,\varphi \in \mathcal{S}(\mathbb{R}^4)\,,\; \hat{\hat\Phi} \in \left\{\hat\Phi, \hat\Phi^*\right\}$$

and

$$\hat\theta\,\hat U(a,\Lambda)\,\hat\theta = \hat U(-a,\Lambda) \quad \forall\,(a,\Lambda)\in\mathcal{P}_+^\uparrow\,.$$

2.4.3 Scattering Theory

Now assume that the IS is described by a charged scalar field $\hat\Phi(x)$ fulfilling all the Wightman axioms and that the corresponding FS may be described by the free charged scalar field $\hat\Phi_q(x)$.

Since the 1-particle states are charged they can no longer be approximated by states of the form $\hat B\Omega$, $\hat B \in \mathcal{A}_{\mathrm{loc}}$. Here the local observable algebras have to be replaced by some net of ***local field algebras*** $\mathcal{F}_{\mathrm{b}}(\mathcal{O})$. It is not evident how to define these algebras of **bounded** operators.[85] This problem can be be avoided by working with unbounded operators (Lücke, 1983). For simplicity, however, let us assume that the $\mathcal{F}_{\mathrm{b}}(\mathcal{O})$ are specified and fulfill the conditions of ***isotony***

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathcal{F}_{\mathrm{b}}(\mathcal{O}_1) \subset \mathcal{F}_{\mathrm{b}}(\mathcal{O}_2)\,,$$

local commutativity

$$\mathcal{O}_1 \times \mathcal{O}_2 \Longrightarrow [\mathcal{F}_{\mathrm{b}}(\mathcal{O}_1), \mathcal{F}_{\mathrm{b}}(\mathcal{O}_2)]_- = \{\hat 0\}\,,$$

Poincaré covariance

$$\hat U(a,\Lambda)\mathcal{F}_{\mathrm{b}}(\mathcal{O})\hat U(a,\Lambda)^{-1} = \mathcal{F}_{\mathrm{b}}(\Lambda\mathcal{O}+a)\,,$$

and ***irreducibility***

$$\left(\bigcup_{R>0}\mathcal{F}_{\mathrm{b}}(U_R(o))\right)' = \{\lambda\hat 1 : \lambda\in\mathbb{C}\}$$

(compare (2.90)–(2.93)). Then the corresponding net of local observable algebras is given by

$$\mathcal{A}(\mathcal{O}) \stackrel{\mathrm{def}}{=} \left\{\hat A \in \mathcal{F}_{\mathrm{b}}(\mathcal{O}) : [\hat A, \hat Q]_- = 0\right\}\,,$$

where $\hat Q$ is the ***charge operator*** (uniquely) defined by

$$\left.\begin{array}{rcl} [\hat Q, \hat\Phi(\varphi)]_- &=& -q\hat\Phi(\varphi) \\ [\hat Q, \hat\Phi^*(\varphi)]_- &=& +q\hat\Phi^*(\varphi) \end{array}\right\} \quad \forall\varphi\in\mathcal{S}(\mathbb{R}^4)\,, \quad \hat Q\Omega = 0\,, \quad \hat Q^* = \hat Q\,,$$

[85] See (Driessler et al., 1986a) for a detailed discussion of this point.

and the scattering theory described for the neutral field can be applied to the charged field with the following modifications:

1. Everywhere, except in the asymptotic condition, the field algebras $\mathcal{F}_{\mathrm{b}}(\mathcal{O})$ have to be used instead of the observable algebras.

2. Asymptotic states with arbitrary number of particles **and** antiparticles have to be considered.

3. The HRK sequences have to be defined for both particle and antiparticle states.

Now, of course, the resulting PCT invariance of the transition probabilities also involves interchange of particles and antiparticles.

Chapter 3

$\lambda\Phi_4^4$ Perturbation Theory

"Renormalization theory has a history of egregious errors by distinguished savants. It has a justified reputation for perversity; a method that works up to 13$^{\text{th}}$ *order in the perturbation series fails in the 14*$^{\text{th}}$ *order. Arguments that sound plausible often dissolve into mush when examined closely. The worst that can happen often happens. The prudent student would do well to distinguish sharply between what has been proved and what has been made plausible, and in general he should watch out!""*

A. S. Wightman (Velo and Wightman, 1976, p. 16)

3.1 General Aspects

3.1.1 Interaction Picture

General Definition

Let us **formally** assume that the 'same' **instant** measurements can be performed on the IS as well as on the FS and that the expectation values for all such measurements performable at a fixed time t determine the corresponding state uniquely. Then we say

"$\check{\Psi} \in \check{\mathcal{H}}$ resp. $\check{\Psi}_0 \in \check{\mathcal{H}}_0$ looks like $\check{\Phi}_0 \in \check{\mathcal{H}}_0$ at time t"

if all the expectation values for identical measurements to be performed at time t predicted by $\check{\Psi}$ resp. $\check{\Psi}_0$ are the same as those predicted by $\check{\Phi}_0$.

Moreover, let us assume that for every state $\check{\Psi}$ and for every instant of time t there is a state $\check{\Psi}_{\text{I}}(t)$ of the FS such that $\check{\Psi} \in \check{\mathcal{H}}$ looks like $\check{\Psi}_{\text{I}}(t) \in \check{\mathcal{H}}_0$ at time t . Then we call $\check{\Psi}_{\text{I}}(t)$ the ***instantaneous state*** at time t of the IS ***in the interaction picture***, if the IS is in the actual state $\check{\Psi}$. See Fig. 3.1 for the example of a classical particle moving in an external potential.

In general, if the interaction picture exists, the AC should be of the form

$$\check{\Psi}_{\text{I}}(t) \underset{t\to\pm\infty}{\longrightarrow} \check{\Psi}_{\pm} \tag{3.1}$$

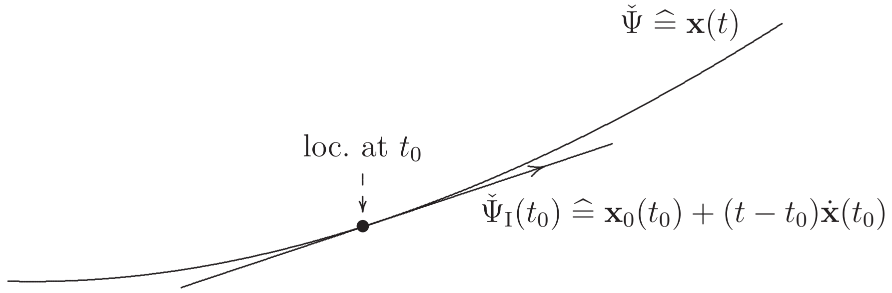

Figure 3.1: Interaction picture for a classical particle

for suitable specification of the type of convergence.

Formalization (in view of quantum theory)

Assuming that the interaction picture exists, in the sense described above, let us introduce the following notation:

$$
\begin{array}{lcl}
\check{\mathcal{A}}_0 & \stackrel{\text{def}}{=} & \text{set of all instantaneous measurement performable at time } 0\,,\\
\check{\alpha}_{\Delta t}(A) & \stackrel{\text{def}}{=} & \text{measurement } A \in \check{\mathcal{A}}_0 \text{ time-shifted by } \Delta t\,,\\
\check{\mathcal{A}} & \stackrel{\text{def}}{=} & \bigcup_{t\in\mathbb{R}} \left\{\check{\alpha}_t(A) :\ A \in \check{\mathcal{A}}_0\right\}\,,\\
E\left(A, \check{\Psi}\right) & \stackrel{\text{def}}{=} & \text{expectation value of } A \in \check{\mathcal{A}} \text{ for the IS in state } \check{\Psi}\,,\\
E_0\left(A, \check{\Psi}_0\right) & \stackrel{\text{def}}{=} & \text{expectation value of } A \in \check{\mathcal{A}} \text{ for the FS in state } \check{\Psi}_0\,.
\end{array}
$$

While the set of all instantaneous measurement procedures is the same for the IS and the FS, the instantaneous states develop differently in time:[1]

$$
\begin{array}{lcl}
\hat{\hat{U}}(t)\check{\Psi} & \stackrel{\text{def}}{=} & \text{state of the IS, at time } t \text{ looking like } \check{\Psi} \text{ at time } 0\,,\\
\hat{\hat{U}}_0(t)\check{\Psi}_0 & \stackrel{\text{def}}{=} & \text{state of the FS, at time } t \text{ looking like } \check{\Psi}_0 \text{ at time } 0\,.
\end{array}
\tag{3.2}
$$

Consistency requires

$$
E_{(0)}\left(\check{\alpha}_t(A), \hat{\hat{U}}_{(0)}(t)\check{\Psi}_{(0)}\right) = E_{(0)}\left(A, \check{\Psi}_{(0)}\right) \quad \forall\, A \in \check{\mathcal{A}}_0\,. \tag{3.3}
$$

The instantaneous state $\check{\Psi}_{\mathrm{I}}(t)$ in the interaction picture at time t of the IS in the actual state $\check{\Psi}$ is determined by

$$
E_0\left(\check{\alpha}_t(A), \check{\Psi}_{\mathrm{I}}(t)\right) = E\left(\check{\alpha}_t(A), \check{\Psi}\right) \quad \forall\, A \in \check{\mathcal{A}}_0\,. \tag{3.4}
$$

Defining

$$
\hat{\hat{W}}(t)\check{\Psi} \stackrel{\text{def}}{=} \check{\Psi}_{\mathrm{I}}(t)\,,
$$

[1] **Warning:** In general, $\hat{\hat{U}}(t)$ and $\hat{\hat{U}}_0(t)$ depend on the choice for the origin of the time-scale.

by (3.1) we formally get

$$\hat{\hat{W}}(t)\check{\Psi} \underset{t\to\underset{(-)}{+}\infty}{\longrightarrow} \check{\Psi}_{\underset{(-)}{+}} \underset{(2.78)}{=} \hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}}^{-1}\check{\Psi} \quad \forall\, \check{\Psi} \in \check{\mathcal{H}}_{\underset{(\mathrm{in})}{\mathrm{out}}}\,,$$

i.e.:

$$\hat{\hat{W}}(t)^{-1} \underset{t\to\underset{(-)}{+}\infty}{\longrightarrow} \hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}}\,. \tag{3.5}$$

By (3.3) and (3.4), assuming (without restriction of generality)

$$\hat{\hat{W}}(0) = \text{identity mapping}\,,$$

we have

$$\hat{\hat{W}}(t) = \hat{\hat{U}}_0(t)\hat{\hat{U}}(t)^{-1}\,, \quad \hat{\hat{W}}(t)^{-1} = \hat{\hat{U}}(t)\hat{\hat{U}}_0(t)^{-1} \tag{3.6}$$

and therefore, by (2.82), (3.5) implies

$$\boxed{\begin{aligned}&\hat{\hat{S}}_0 = \lim_{t_\pm\to\pm\infty} \hat{\hat{\Omega}}(t_+,t_-)\,, \text{ where:}\\ &\hat{\hat{\Omega}}(t_+,t_-) \stackrel{\text{def}}{=} \hat{\hat{U}}_0(t_+)\hat{\hat{U}}(t_+)^{-1}\hat{\hat{U}}(t_-)\hat{\hat{U}}_0(t_-)^{-1}\,.\end{aligned}} \tag{3.7}$$

The type of limit, depending on the model, has to be suitably defined, of course. If, in addition, we also have homogeneity in time, i.e.[2]

$$\hat{\hat{U}}_{(0)}(t_1)\hat{\hat{U}}_{(0)}(t_2) = \hat{\hat{U}}_{(0)}(t_1+t_2)\,,$$

(3.5) and (3.6) imply

$$\hat{\hat{U}}(t)\hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}} = \hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}}\hat{\hat{U}}_0(t)\,, \quad \hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}}^{-1}\hat{\hat{U}}(t) = \hat{\hat{U}}_0(t)\hat{\hat{V}}_{\underset{(\mathrm{in})}{\mathrm{out}}}^{-1}\,. \tag{3.8}$$

and, by (2.79)/(2.82), therefore:

$$[\hat{\hat{S}}_{(0)}, \hat{\hat{U}}_{(0)}(t)]_- = 0\,.$$

Application to Quantum Theory

In quantum theory the states

$$\check{\Psi}_{(0)} = \omega_{\Psi_{(0)}}$$

are given by state vectors $\Psi_{(0)}$ from the corresponding Hilbert space $\mathcal{H}_{(0)}$ and the time translations $\hat{\hat{U}}_{(0)}(t)$ are given by unitary operators $\hat{U}_{(0)}(t)$ in $\mathcal{H}_{(0)}$,

$$\hat{\hat{U}}_{(0)}(t)\,\check{\Psi}_{(0)} = \omega_{\hat{U}_{(0)}(t)\Psi_{(0)}}\,,$$

[2]In this case $\hat{\hat{U}}(t)$ and $\hat{\hat{U}}_0(t)$ are independent of the choice for the origin of the time-scale.

depending strongly continuously on t. Therefore, according to Stone's theorem, we may define the Hamiltonian $\hat{H}_{(0)}(t)$ by[3]

$$\hat{H}_{(0)}(t) \stackrel{\text{def}}{=} \left(i\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{U}_{(0)}(t)^{-1}\right)\right)\hat{U}_{(0)}(t) = -\hat{U}_{(0)}(t)^{-1}\, i\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{(0)}(t)\,,$$

even if we do not have homogeneity in time. This implies

$$\begin{aligned} i\partial_t\hat{\Omega}(t,t_-) &= \hat{H}_{\mathrm{I}}(t)\hat{\Omega}(t,t_-)\,, \text{ where:} \\ \hat{H}_{\mathrm{I}}(t) &\stackrel{\text{def}}{=} \hat{U}_0(t)\left(\hat{H}(t)-\hat{H}_0(t)\right)\hat{U}_0(t)^{-1}\,, \end{aligned} \tag{3.9}$$

which, formally, is equivalent to[4]

$$\hat{\Omega}(t,t_-) = \hat{1} - i\int_{t_-}^{t}\hat{H}_{\mathrm{I}}(t_+)\hat{\Omega}(t_+,t_-)\,\mathrm{d}t_+\,, \tag{3.10}$$

since $\hat{\Omega}(t_-,t_-) = \hat{1}$, by (3.7). As usual, this integral equation may be formally solved by iteration giving the so-called ***Dyson series***.

3.1.2 Canonical Field Quantization

Field Equations

Hoping to convert the free Klein-Gordon theory into a model with nontrivial S-matrix, one adds a ***local self-interaction*** term of the form[5] $\lambda_{\mathrm{b}}\vdots F(\hat{\Phi}(x))\vdots$ with ***coupling constant*** $\lambda_{\mathrm{b}} > 0$ as ***perturbation*** to the Klein-Gordon equation:[6]

$$\left(\Box + m_{\mathrm{b}}^2\right)\hat{\Phi}(x) = -\lambda_{\mathrm{b}}\vdots F(\hat{\Phi}(x))\vdots\,. \tag{3.11}$$

Best studied is the so-called $\lambda\,(\Phi^4)_4$-theory given formally by $F(\Phi) = 2\Phi^3$. Nevertheless, nobody succeeded up to now in giving this theory a precise meaning by rigorous construction. This is due to tremendous technical difficulties connected with 4-dimensionality of physical space-time.[7] In 2- or 3-dimensional model space-time these difficulties are much less severe and have already been overcome (See

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[3]Note that $\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}^{-1}\right)\hat{U} + \hat{U}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\hat{U} = 0$.

[4]In naive quantum electrodynamics (before renormalization) one has:

$$\hat{H}_{\mathrm{I}}(t) = \int_{x^0=t} g_{\mu\nu}\hat{\jmath}^{\mu}_{\mathrm{free}}(x)\hat{A}^{\nu}_{\mathrm{free}}(x)\,\mathrm{d}\mathbf{x}\,.$$

[5]Locality, i.e. dependence of the interaction term on the field values at the space-time point x, corresponds to the **point particle** picture.

[6]For the corresponding **classical** field theory see (Reed, 1976). For the problem of defining the operator function $F(\hat{\Phi}(x))$ via $\vdots\;\vdots$ (in the sense of (3.20)) see (Segal, 1962; Segal, 1983). Negative values of m_{b}^2 lead to *spontaneous symmetry breaking*.

[7]Actually there are even indications that such a construction is not possible; see (Bég and Furlong, 1985) and references given there.

e.g. (Glimm and Jaffe, 1981; Constantinescu, 1980) and (Streater and Wightman, 1989, Appendix).)

Time-Zero Fields

Let us assume that there is a rigorous construction for the interacting theory formally described above. Moreover assume – in spite of all knowledge to the contrary – that the interaction picture, as described in Sect. 3.1.1, is applicable to this theory with the corresponding "free" system being described by the neutral scalar field $\hat{\Phi}_0(x)$ with *physical mass*[8] m :

$$\begin{aligned} \hat{\Phi}(0,\mathbf{x}) &= \hat{\Phi}_0(0,\mathbf{x})\,, \\ \hat{\Pi}(0,\mathbf{x}) \stackrel{\text{def}}{=} \partial_0 \hat{\Phi}(x)_{|_{x^0=0}} &= \hat{\Pi}_0(0,\mathbf{x}) \stackrel{\text{def}}{=} \partial_0 \hat{\Phi}_0(x)_{|_{x^0=0}}\,. \end{aligned} \tag{3.12}$$

Under these conditions

$$\vdots G(\hat{\Phi}(x),\hat{\Pi}(x))\vdots = \hat{U}(x^0){:}G\left(\hat{\Phi}_0(0,\mathbf{x}),\hat{\Pi}_0(0,\mathbf{x})\right){:}\hat{U}(x^0)^{-1} \tag{3.13}$$

might be a good definition[9] for well-behaved functionals G of $\Phi(x)$ and $\Pi(x)$. Here $::$ denotes normal ordering, i.e. the factors of monomials have to be interchanged – as if they were commuting – such that no creation operator is on the right of any annihilation operator.

In any case, by (2.39), (2.40), and (2.43), we have

$$\hat{\Phi}_0(x) = (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \frac{d\mathbf{p}}{2p^0} \left(\hat{a}(\mathbf{p})e^{-ip^0x^0} + \hat{a}^*(-\mathbf{p})e^{+ip^0x^0}\right) e^{+i\mathbf{p}\mathbf{x}} \tag{3.14}$$

and hence **for real-valued** $\varphi \in \mathcal{S}(\mathbb{R}^3)$:

$$\begin{gathered} \int d\mathbf{x}\, \hat{\Phi}_0(0,\mathbf{x})\varphi(\mathbf{x}) = \hat{A}_{\widetilde{\varphi}} + \hat{A}^*_{\widetilde{\varphi}}\,, \quad \int d\mathbf{x}\, \hat{\Pi}_0(0,\mathbf{x})\varphi(\mathbf{x}) = -i\hat{A}_{2\omega\widetilde{\varphi}} + i\hat{A}^*_{2\omega\widetilde{\varphi}}\,, \\ \text{where:} \quad \widetilde{\varphi}(\mathbf{p}) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int d\mathbf{x}\, \varphi(\mathbf{x})e^{-i\mathbf{p}\mathbf{x}}\,, \quad \hat{A}_{\check{f}} \stackrel{\text{def}}{=} \int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}\, \hat{a}(\mathbf{p})\overline{\check{f}(\mathbf{p})}\,. \end{gathered} \tag{3.15}$$

By (3.14) and (2.34), (3.12) implies the ***canonical commutation relations***

$$\boxed{\begin{gathered} \left[\hat{\Phi}(0,\mathbf{x}),\hat{\Pi}(0,\mathbf{y})\right]_- = i\delta(\mathbf{x}-\mathbf{y})\,, \\ \left[\hat{\Phi}(0,\mathbf{x}),\hat{\Phi}(0,\mathbf{y})\right]_- = \left[\hat{\Pi}(0,\mathbf{x}),\hat{\Pi}(0,\mathbf{y})\right]_- = 0 \end{gathered}} \tag{3.16}$$

on D_0 , as defined by (2.27).

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[8]In case m coincided with the *bare mass* m_b there would be no chance for $e^{i\hat{H}t} = \hat{V}_{\text{out}} e^{i\hat{H}_0 t} \hat{V}_{\text{out}}^{-1}$ to hold together with (3.21), since the latter implies coincidence of the spectra of $\hat{H}$ and $\hat{H}_0$.

[9]Note, however, that $:G\left(\hat{\Phi}_0(0,\mathbf{x}),\hat{\Pi}_0(0,\mathbf{x})\right):$ is well defined as quadratic form on D_0 (see, e.g., (Reed und Simon, 1972, Sect. VIII.6) for the definition of quadratic forms) but not necessarily as $L(D_0,\mathcal{H})$-valued generalized function of $\mathbf{x}$.

Let $\left\{\check{f}_\mu\right\}_\mu$ be a complete orthonormal system of scalar 1-particle wave functions fulfilling

$$\check{f}_\mu(\mathbf{p}) = \overline{\check{f}_\mu(-\mathbf{p})} \tag{3.17}$$

and define

$$\varphi_\mu(\mathbf{x}) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int \mathrm{d}\mathbf{p}\, \check{f}_\mu(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}}\,, \quad \psi_\mu(\mathbf{x}) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int \mathrm{d}\mathbf{p}\, \frac{\check{f}_\mu(\mathbf{p})}{4\omega_{\mathbf{p}}} e^{i\mathbf{p}\mathbf{x}}\,. \tag{3.18}$$

Then, by (3.12)/(3.15) and in agreement with (3.16),

$$\begin{aligned} \pi\left(\hat{U}_\nu(\tau)\right) &\stackrel{\text{def}}{=} \exp\left(\overline{i\int \mathrm{d}\mathbf{x}\hat{\Pi}(0,\mathbf{x})\tau\psi_\nu(\mathbf{x})}\right)\,, \\ \pi\left(\hat{V}_\mu(\tau)\right) &\stackrel{\text{def}}{=} \exp\left(\overline{i\int \mathrm{d}\mathbf{x}\hat{\Phi}(0,\mathbf{x})\tau\varphi_\mu(\mathbf{x})}\right)\,, \end{aligned} \tag{3.19}$$

defines a regular representation[10] of the Weyl commutation relations (1.45):

$$\hat{U}_\nu(\tau)\hat{V}_\mu(s) = e^{i\tau s\delta_{\nu\mu}}\hat{V}_\mu(s)\hat{U}_\nu(\tau) \quad \text{etc.}$$

Even if (3.12) is not fulfilled, the field theory is called ***canonical***, whenever (3.19) is a representation of the Weyl commutation relations. Then π has a unique extension to a true representation of the CCR algebra.

Exercise 44 Show that, provided (3.12) holds, the representation of the CCR algebra resulting from (3.19) is equivalent to the Fock representation (discussed in Sect. 1.3.3).[11]

Exercise 45 Using (3.15), show the following:

(i) The (identical representation of the) C^*-algebra (in $\mathcal{H}_0$) generated by the bounded functions of smeared time-zero fields $\hat{\Phi}_0(0,\mathbf{x})$, $\hat{\Pi}_0(0,\mathbf{x})$ is **irreducible**.

(ii) Ω_0 is **cyclic** w.r.t. (the restriction of this representation to) the abelian subalgebra generated by bounded functions of the smeared field $\hat{\Phi}_0(0,\mathbf{x})$.

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[10] Note the nontrivial dependence on the mass value m.

[11] For comparison with the theory of ordinary independent quantum oscillators, note that

$$\pi(\hat{a}_\nu) \stackrel{\text{def}}{=} \frac{1}{2}\pi(\hat{x}_\nu) + i\pi(\hat{p}_\nu) = \frac{\sqrt{mD}\pi(\hat{x}_\nu) + i\hat{p}_\nu}{\sqrt{2m}} \quad \text{for } m = D = \frac{1}{2}\,.$$

Use of the Canonical Commutation Relations

If the canonical commutation relations hold and if $\vdots\,\vdots$ is defined such that[12]

$$
\begin{aligned}
&[\vdots\hat{A}(\mathbf{x}')^N\vdots, \hat{B}(\mathbf{x})]_- \\
&= \begin{cases} \vdots N\hat{A}(\mathbf{x}')^{N-1}[\hat{A}(\mathbf{x}'),\hat{B}(\mathbf{x})]_-\vdots & \text{if } \{\hat{A}(.),\hat{B}(.)\} = \{\hat{\Phi}(0,.),\hat{\Pi}(0,.)\} \\ & \text{or } \{\hat{A}(.),\hat{B}(.)\} = \{\partial_j\hat{\Phi}(0,.),\hat{\Pi}(0,.)\} \\ 0 & \text{if } \hat{A}(.) \in \{\hat{\Phi}(0,.),\partial_j\hat{\Phi}(0,.)\} \ni \hat{B}(.) \\ & \text{or } \hat{A}(.) = \hat{B}(.) = \hat{\Pi}(0,.)\,, \end{cases}
\end{aligned}
\tag{3.20}
$$

holds for $j \in \{1,2,3\}$ we get[13] a formal solution of the field equation (3.11) in the form

$$
\boxed{
\begin{aligned}
&\hat{\Phi}(t,\mathbf{x}) = e^{i\hat{H}t}\hat{\Phi}(0,\mathbf{x})e^{-i\hat{H}t}\,, \text{ where:} \\
&\hat{H} \stackrel{\text{def}}{=} \frac{1}{2}\int \vdots\left(\left(\hat{\Pi}(0,\mathbf{x})\right)^2 + \left(\boldsymbol{\nabla}\hat{\Phi}(0,\mathbf{x})\right)^2 + m_\text{b}^2\left(\hat{\Phi}(0,\mathbf{x})\right)^2 + 2\lambda_\text{b}U\left(\hat{\Phi}(0,\mathbf{x})\right)\right)\vdots\,\mathrm{d}\mathbf{x}\,, \\
&\qquad + \text{const.} \\
&U \stackrel{\text{def}}{=} \text{indefinite integral of } F\,,
\end{aligned}
}
\tag{3.21}
$$

which – in a cutoff version[14] – was the original starting point for *constructive field theory*.

Reminder: In **classical** Lagrange field theory the Hamiltonian

$$
H(\Phi_t,\Pi_t) = \frac{1}{2}\int\left(\Pi_t(\mathbf{x})^2 + (\boldsymbol{\nabla}\Phi_t(\mathbf{x}))^2 + m_\text{b}^2\Phi_t(\mathbf{x})^2 + 2\lambda_\text{b}U(\Phi_t(\mathbf{x}))\right)\mathrm{d}\mathbf{x}\,,
$$

corresponds to the Lagrangian

$$
L\left(\Phi_t,\dot{\Phi}_t\right) = \frac{1}{2}\int\left(\left(\dot{\Phi}_t(\mathbf{x})\right)^2 - (\boldsymbol{\nabla}\Phi_t(\mathbf{x}))^2 - m_\text{b}^2\Phi_t(\mathbf{x})^2 - 2\lambda_\text{b}U(\phi_t(\mathbf{x}))\right)\mathrm{d}\mathbf{x}
$$

with

$$
\Pi_t(\mathbf{x}) \stackrel{\text{def}}{=} \frac{\delta L}{\delta\dot{\Phi}_t(\mathbf{x})} = \dot{\Phi}_t(\mathbf{x})\,.
$$

The Euler-Lagrange equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta\dot{\Phi}_t(\mathbf{x})} - \frac{\delta L}{\delta\Phi_t(\mathbf{x})} = 0
$$

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[12]Note that, for bounded operators, $[\hat{A}\hat{B},\hat{C}]_- = \hat{A}[\hat{B},\hat{C}]_- + [\hat{A},\hat{C}]_-\hat{B}$ and, consequently:

$$
[\hat{A},\hat{B}]_- \sim \hat{1} \Longrightarrow [\hat{A}^N,\hat{B}]_- = N\hat{A}^{N-1}[\hat{A},\hat{B}]_-\,.
$$

For (3.12)/(3.13), (3.20) is (formally) a consequence of Wick's theorem (Theorem 3.2.1).

[13]Thanks to

$$
\begin{aligned}
[\vdots\hat{A}(\mathbf{x}')^N\vdots,\hat{\Pi}(0,\mathbf{x})]_- &= i\delta(\mathbf{x}-\mathbf{x}')\vdots\frac{\partial}{\partial\hat{\Phi}(0,\mathbf{x}')}\hat{A}(\mathbf{x}')^N\vdots\,, \\
[\vdots\hat{A}(\mathbf{x}')^N\vdots,\hat{\Phi}(0,\mathbf{x})]_- &= -i\delta(\mathbf{x}-\mathbf{x}')\vdots\frac{\partial}{\partial\hat{\Pi}(0,\mathbf{x}')}\hat{A}(\mathbf{x}')^N\vdots\,.
\end{aligned}
$$

[14]Note that, e.g., even $:\hat{\Phi}_0(0,\mathbf{x})^4:$ is not well-defined (except on 2-dimensional space-time).

(more precisely: family of equations, parameterized by $\mathbf{x}$) for this Lagrangian is the field equation

$$\left(\Box + m_{\mathrm{b}}^2\right)\Phi(x) = -\lambda_{\mathrm{b}} F\left(\Phi(x)\right)\ , \quad \Phi(t,\mathbf{x}) \stackrel{\mathrm{def}}{=} \Phi_t(\mathbf{x})\,.$$

Note that the Euler-Lagrange equation is equivalent to the Hamilton equations

$$\begin{aligned}\dot{\Phi}_t(\mathbf{x}) &= -\left\{H\left(\Phi_t,\Pi_t\right),\Phi_t(\mathbf{x})\right\} = \frac{\delta H}{\delta \Pi_t(\mathbf{x})}\,,\\ \dot{\Pi}_t(\mathbf{x}) &= -\left\{H\left(\Phi_t,\Pi_t\right),\Pi_t(\mathbf{x})\right\} = -\frac{\delta H}{\delta \Phi_t(\mathbf{x})}\,,\end{aligned}$$

where the ***Poisson bracket*** $\{,\}$ is defined by

$$\{F,G\} \stackrel{\mathrm{def}}{=} \int \left(\frac{\delta F}{\delta\Phi_t(\mathbf{x}')}\frac{\delta G}{\delta\Pi_t(\mathbf{x}')} - \frac{\delta F}{\delta\Pi_t(\mathbf{x}')}\frac{\delta G}{\delta\Phi_t(\mathbf{x}')}\right) d\mathbf{x}'$$

for functionals F, G of Φ_t, Π_t.

Indeed,[15]

$$\begin{aligned}\dot{\hat{\Phi}}(t,\mathbf{x}) &= e^{i\hat{H}t}\,[i\hat{H},\hat{\Phi}(0,\mathbf{x})]_-\,e^{-i\hat{H}t}\\ &\underset{(3.21)}{=} e^{i\hat{H}t}\,\frac{i}{2}\int [{:}\Pi(0,\mathbf{x}')^2{:},\hat{\Phi}(0,\mathbf{x})]_-\,d\mathbf{x}'\,e^{-i\hat{H}t}\\ &\underset{(3.13),(3.16)}{=} e^{i\hat{H}t}\,i\int {:}\Pi(0,\mathbf{x}')[\Pi(0,\mathbf{x}'),\hat{\Phi}(0,\mathbf{x})]_-{:}\,d\mathbf{x}'\,e^{-i\hat{H}t}\\ &\underset{(3.20)}{=} e^{i\hat{H}t}\,\Pi(0,\mathbf{x})\,e^{-i\hat{H}t}\\ &\underset{(3.16)}{} \end{aligned} \tag{3.22}$$

and a similar **formal** calculation verifies

$$\ddot{\hat{\Phi}}(t,\mathbf{x}) \underset{(3.22)}{=} e^{i\hat{H}t}[i\hat{H},\Pi(0,\mathbf{x})]_- e^{-i\hat{H}t}$$

to be in agreement with the field equation (3.11).

However, locality and relativistic covariance of the formal solution (3.21) are not so easy to establish.[16]

Exercise 46 Show for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^3)$ that

$$\int \frac{\left|\widetilde{\varphi}(\mathbf{p}_1+\mathbf{p}_2)\right|^2}{\left(\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}\right)^3}\,d\mathbf{p}_1 d\mathbf{p}_2 < \infty$$

and therefore

$$\int {:}\left(\hat{\Pi}(0,\mathbf{x})\right)^2 + \left(\boldsymbol{\nabla}\hat{\Phi}(0,\mathbf{x})\right)^2 + m_{\mathrm{b}}^2\left(\hat{\Phi}(0,\mathbf{x})\right)^2{:}\,\varphi(\mathbf{x})\,d\mathbf{x} \;\in\; L(D_0,\mathcal{H})\,.$$

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[15] Note that we do not care about the domain of definition for $\hat{H}$. For ${:}F(\hat{\Phi}_0(0,\mathbf{x})){:}\Omega_0 \neq 0$, the case of interest, there are obvious difficulties with the Fock representation.

[16] Note that $[\hat{A},\hat{B}]_- = 0 \overset{\text{i.g.}}{\not\Longrightarrow} [e^{i\hat{A}},e^{i\hat{B}}]_- = 0$ (see e.g. (Reed und Simon, 1972, Vol. I, Sect. VIII.5) and (Fröhlich, 1977)).

Inapplicability of the Fock Representation

Let us assume (3.12) to be valid. Then

$$\begin{aligned}\hat U(0,\mathbf{a})\Phi_0(0,\mathbf{x})\hat U(0,\mathbf{a})^{-1} &= \hat U(0,\mathbf{a})\Phi(0,\mathbf{x})\hat U(0,\mathbf{a})^{-1}\\ &= \Phi(0,\mathbf{x}+\mathbf{a})\\ &= \Phi_0(0,\mathbf{x}+\mathbf{a})\\ &= \hat U_0(0,\mathbf{a})\Phi_0(0,\mathbf{x})\hat U_0(0,\mathbf{a})^{-1}\end{aligned}$$

and the corresponding conclusion for $\Pi_0(0,\mathbf{a})$ imply (compare Exercise 45(i)):

$$\hat U(0,\mathbf{a}) = e^{i\varphi(\mathbf{a})}\hat U_0(0,\mathbf{a})$$

for suitable real-valued $\varphi(\mathbf{a})$. There is only one 1-dimensional subspace which is invariant under all $\hat U_0(0,\mathbf{a})$, and this contains the Fock vacuum Ω_0, characterized up to a factor by $\hat{\mathbf{P}}_0\Omega_0 = 0$. For the ***physical vacuum*** state vector Ω, characterized up to a factor by

$$\hat U(a)\,\Omega = \Omega \quad \forall\, a \in \mathbb{R}^4\,,$$

we have

$$\hat{\hat U}_0(0,\mathbf{a})\Omega = e^{-i\varphi(\mathbf{a})}\hat{\hat U}(0,\mathbf{a})\Omega = e^{-i\varphi(\mathbf{a})}\Omega\,.$$

Therefore also Ω is an element of the invariant 1-dimensional subspace, i.e.:

$$\Omega = e^{i\alpha}\Omega_0\,.$$

This means that both $\hat U(0,\mathbf{a})$ and $\hat U_0(0,\mathbf{a})$ leave Ω_0 (and Ω) invariant and thus have to coincide:

$$\hat U_0(0,\mathbf{a}) = \hat U(0,\mathbf{a}) \quad \forall\, \mathbf{a} \in \mathbb{R}^3\,.$$

Since, for obvious physical reasons, (3.8) should be supplemented by

$$\hat U(0,\mathbf{a}) = \hat V_{\rm out}\hat U_0(0,\mathbf{a})\hat V_{\rm out}^{-1} \quad \forall\, \mathbf{a} \in \mathbb{R}^3\,,$$

we get commutativity of $\hat V_{\rm out}$ with all $\hat U_0(0,\mathbf{a})$. Now, since $\hat P^0_0$ is a function of $\hat{\mathbf{P}}_0$, commutativity with all $\hat U_0(0,\mathbf{a})$ implies commutativity with all $\hat U_0(t)$. Thus, by (3.8) we have

$$\hat U_0(t,0) = \hat U(t,0) \quad \forall\, t \in \mathbb{R}$$

and hence

$$\hat\Phi_0(x) = \hat\Phi(x) \quad \forall\, x \in \mathbb{R}^4\,.$$

Haag's theorem[17] says that this conclusion is correct even without (3.8) and its generalization and without specification of $\hat H$, if the theory fulfills the Wightman

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[17] A rigorous proof is given in (Streater and Wightman, 1989, Sect. 4.5). In a first step, relativistic covariance is used to show that

$$\left\langle \Omega \mid \hat\Phi(x)\hat\Phi(y)\Omega\right\rangle = \left\langle \Omega_0 \mid \hat\Phi_0(x)\hat\Phi_0(y)\Omega_0\right\rangle \qquad (*)$$

holds for $x \times y$, since the equations holds for $x^0 = y^0 = 0$. This together with the spectrum condition implies that $(*)$ holds for all $x, y \in \mathbb{R}^4$, as can be shown by standard techniques of axiomatic field theory (for a stronger result see (Lücke, 1979, Corollary)). Then the Jost-Schroer theorem (Theorem 2.2.18) says that $(*)$ can only hold for all $x, y \in \mathbb{R}^4$ if $\hat\Phi_0(x)$ and $\hat\Phi(x)$ are unitarily equivalent.

axioms formulated in 2.2.1.

So, unfortunately, it is not possible to define the Hamiltonian on a suitable domain to make the formal solution (3.21) of (3.11) a true one – as long as one insists in (3.12) for **all** $\mathbf{x} \in \mathbb{R}^3$.

In spite of *Haag*'s theorem there is still hope (compare (Baumann,)) that a nontrivial **canonical** $\lambda\,(\Phi^4)_4$-theory might exist.[18] For such a quantum field theory, of course, the representation π given by (3.19) must be **in**equivalent to the Fock representation. All this was illustrated in *constructive field theory* by several models living on space-time of dimension < 4.

For the free field, of course, nothing is wrong with the ***canonical quantization*** procedure (3.21) (for $\lambda_\mathrm{b} = 0$).

3.2 Canonical Perturbation Theory

3.2.1 Dyson Series and Wick's Theorem

Let us consider an IS for which the interaction picture works as described in Section 3.1.1:

$$\boxed{\begin{array}{rcl} \hat{S}_0 & \underset{(3.7)}{=} & \lim\limits_{t_\pm \to \pm\infty} \hat{\Omega}(t_+, t_-)\,, \\ \hat{\Omega}(t_+, t_-) & \underset{(3.10)}{=} & \hat{1} - i \displaystyle\int_{t_-}^{t_+} \hat{H}_\mathrm{I}(t)\hat{\Omega}(t, t_-)\,\mathrm{d}t\,, \\ \hat{H}_\mathrm{I}(t) & \underset{(3.9)}{=} & \hat{U}_0(t)\left(\hat{H}(t) - \hat{H}_0(t)\right)\hat{U}_0(t)^{-1}\,. \end{array}} \tag{3.23}$$

Let us assume

$$\hat{H}_\mathrm{I}(t) = \lambda \hat{\hat{H}}_\mathrm{I}(t)$$

and that – on a suitable domain – $\hat{\Omega}(t, t_-)$ depends sufficiently smoothly on λ. Then the Leibniz rule gives

$$\partial_\lambda^n \hat{\Omega}(t, t_-) = -i \int_{t_-}^{t} \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\partial_\lambda^\nu \hat{H}_\mathrm{I}(t')\right) \partial_\lambda^{n-\nu} \hat{\Omega}(t', t_-)\,\mathrm{d}t' \quad \text{for } n > 0$$

and, because of

$$\left(\partial_\lambda^\nu \hat{H}_\mathrm{I}(t)\right)_{|\lambda=0} = \begin{cases} \hat{\hat{H}}_\mathrm{I}(t) & \text{for } \nu = 1\,, \\ 0 & \text{else} \quad, \end{cases}$$

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[18] Note that existence of a unitary operator $\hat{U}_R$ fulfilling the conditions

$$\hat{\Phi}(0, \mathbf{x}) = \hat{U}_R \hat{\Phi}_0(0, \mathbf{x}) \hat{U}_R^{-1}\,, \quad \hat{\Phi}(0, \mathbf{x}) = \hat{U}_R \hat{\Pi}_0(0, \mathbf{x}) \hat{U}_R^{-1}\,,$$
$$\vdots F\left(\hat{\Phi}(x)\right)\vdots = \hat{U}(x^0)\hat{U}_R {:} F\left(\hat{\Phi}_0(0, \mathbf{x})\right){:} \hat{U}_R^{-1} \hat{U}(x^0)^{-1}$$

for $|x| < R$ would have been sufficient for the formal proof of (3.11) in this region.

the iteration formula

$$\partial_\lambda^n \hat{\Omega}(t,t_-)_{|\lambda=0} = -i \int_{t_-}^{t} n\hat{\hat{H}}_{\mathrm{I}}(t')\left(\partial_\lambda^{n-1}\hat{\Omega}(t',t_-)\right)_{|\lambda=0} \mathrm{d}t' \,.$$

Since,

$$\hat{\Omega}(t,t_-)_{|\lambda=0} = \hat{1}\,,$$

this gives the (not necessarily converging) Taylor expansion

$$\begin{aligned}\hat{\Omega}(t_+,t_-) &= \hat{1} + \sum_{n=1}^{\infty}(-i)^n \underbrace{\int\limits_{t_-<t_1<\ldots t_n<t_+}\!\!\!\cdots\int \hat{H}_{\mathrm{I}}(t_n)\cdots\hat{H}_{\mathrm{I}}(t_1)\,\mathrm{d}t_1\cdots\mathrm{d}t_n}_{\frac{1}{n!}\sum_{\pi\in S_n}\int\limits_{t_-<t_{\pi1}<\ldots<t_{\pi n}<t_+}\!\!\!\cdots\int T\left(\hat{H}_{\mathrm{I}}(t_n)\cdots\hat{H}_{\mathrm{I}}(t_1)\right)\mathrm{d}t_1\cdots\mathrm{d}t_n} \\ &= \hat{1} + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_-}^{t_+}\cdots\int_{t_-}^{t_+} T\left(\hat{H}_{\mathrm{I}}(t_n)\cdots\hat{H}_{\mathrm{I}}(t_1)\right)\mathrm{d}t_1\cdots\mathrm{d}t_n\,,\end{aligned}$$

where[19]

$$T\left(\hat{H}_{\mathrm{I}}(t_n)\cdots\hat{H}_{\mathrm{I}}(t_1)\right) \stackrel{\text{def}}{=} \hat{H}_{\mathrm{I}}(t_{\pi n})\cdots\hat{H}_{\mathrm{I}}(t_{\pi 1}) \quad \text{for } \pi\in S_n \text{ with } t_{\pi 1}<\ldots<t_{\pi n}$$

is the so-called ***chronological product*** of $\hat{H}_{\mathrm{I}}(t_n),\ldots,\hat{H}_{\mathrm{I}}(t_1)$. The usual shorthand notation for the resulting formal perturbation expansion of the S-matrix is:

$$\boxed{\hat{S}_0 = T\,\exp\left(-i\int_{-\infty}^{+\infty}\hat{H}_{\mathrm{I}}(t)\mathrm{d}t\right)\,.} \tag{3.24}$$

In view of $\lambda\Phi_4^4$-theory let us formally assume

$$\boxed{\hat{H}_{\mathrm{I}}(x^0) = i\int g(x)\hat{S}_1(x)\,\mathrm{d}\mathbf{x}\,,} \tag{3.25}$$

where $\hat{S}_1(x)$ is a normal ordered function of the **free** field $\hat{\Phi}_0(x)$ and its derivatives at the space-time point x. Then (3.24) becomes:

$$\boxed{\begin{aligned}&\hat{S}_0 = \hat{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\int\cdots\int \hat{S}_n(x_1,\ldots,x_n)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,,\\ &\text{where }\ \hat{S}_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right)\,.\end{aligned}} \tag{3.26}$$

In order to facilitate evaluation of the S-matrix elements the chronological products should be expressed by normal ordered products. Formally this may be done by applying Wick's theorem.

[19]This is the definition for Bose fields, only. However, even if the theory contained Fermi fields the Hamiltonian ought to be of Bose type.

Theorem 3.2.1 (Wick's theorem) *Let*[20]

$$\hat{\chi}_\nu(x) \in \left\{ \hat{\Phi}_0(x), \partial_0 \hat{\Phi}_0(x), \ldots, \partial_3 \hat{\Phi}_0(x) \right\} \quad \textit{for } \nu = 1, \ldots, j_N$$

and let $j_1 < j_2 < \ldots < j_N$ *. Then*

$$\begin{aligned} &:\hat{\chi}_1(x_1)\cdots\hat{\chi}_{j_1}(x_{j_1})::\hat{\chi}_{j_1+1}(x_{j_1+1})\cdots\hat{\chi}_{j_2}(x_{j_2}):\cdots:\hat{\chi}_{j_{N-1}+1}(x_{j_{N-1}+1})\cdots\hat{\chi}_{j_N}(x_{j_N}): \\ &=:\exp\Big(\sum_{\substack{\nu \le j_k < \mu \\ \text{for suit. } k}} \int\int dx dy \, \langle \Omega_0 \mid \hat{\chi}_\nu(x)\hat{\chi}_\mu(y)\Omega_0\rangle \frac{\delta}{\delta\hat{\chi}_\nu(x)} \frac{\delta}{\delta\hat{\chi}_\mu(y)} \Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \hat{\chi}_1(x_1)\cdots\hat{\chi}_{j_N}(x_{j_N}):, \end{aligned}$$

if – formally – the fields $\hat{\chi}_\nu(x)$ *are considered as independent functional variables.*

Proof: From the simple chain of equations

$$\begin{aligned} &\hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_n)\hat{\Phi}_0^-(x_{n+1}) \\ &= \hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_{n-1})\hat{\Phi}_0^-(x_{n+1})\hat{\Phi}_0^+(x_n) \\ &\quad + \hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_{n-1})[\hat{\Phi}_0^+(x_n), \hat{\Phi}_0^-(x_{n+1})]_- \\ &= \hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_{n-2})\hat{\Phi}_0^-(x_{n+1})\hat{\Phi}_0^+(x_{n-1})\hat{\Phi}_0^+(x_n) \\ &\quad + \hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_{n-2})[\hat{\Phi}_0^+(x_{n-1}), \hat{\Phi}_0^-(x_{n+1})]_-\hat{\Phi}_0^+(x_n) \\ &\quad + \hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_{n-1})[\hat{\Phi}_0^+(x_n), \hat{\Phi}_0^-(x_{n+1})]_- \\ &\vdots \\ &= \hat{\Phi}_0^-(x_{n+1})\hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_n) + \sum_{\nu=k}^n [\hat{\Phi}_0^+(x_\nu), \hat{\Phi}_0^-(x_{n+1})]_- \prod_{\substack{\mu=k \\ \mu\neq\nu}}^n \hat{\Phi}_0^+(x_\mu) \end{aligned}$$

we easily conclude that

$$\begin{aligned} &\hat{\Phi}_0^-(x_1)\cdots\hat{\Phi}_0^-(x_{k-1})\hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_n)\hat{\chi}_{n+1}(x_{n+1}) \\ &= :\hat{\Phi}_0^-(x_1)\cdots\hat{\Phi}_0^-(x_{k-1})\hat{\Phi}_0^+(x_k)\cdots\hat{\Phi}_0^+(x_n)\hat{\chi}_{n+1}(x_{n+1}): \\ &\quad + \sum_{\nu=k}^n [\hat{\Phi}_0^+(x_\nu), \hat{\chi}_{n+1}(x_{n+1})]_- \hat{\Phi}_0^-(x_1)\cdots\hat{\Phi}_0^-(x_{k-1}) \prod_{\substack{\mu=k \\ \mu\neq\nu}}^n \hat{\Phi}_0^+(x_\mu) \end{aligned}$$

holds for $\hat{\chi}_{n+1}(x) \in \left\{ \hat{\Phi}_0^-(x), \hat{\Phi}_0^+(x) \right\}$. This implies

$$\begin{aligned} &:\hat{\chi}_1(x_1)\cdots\hat{\chi}_n(x_n):\hat{\chi}_{n+1}(x_{n+1}) \\ &= :\hat{\chi}_1(x_1)\cdots\hat{\chi}_{n+1}(x_{n+1}): + \sum_{\nu=1}^n \langle\Omega_0 \mid \hat{\chi}_\nu(x_\nu)\hat{\chi}_{n+1}(x_{n+1})\Omega_0\rangle : \prod_{\substack{\mu=1 \\ \mu\neq\nu}}^n \hat{\chi}_\mu(x_\mu): \end{aligned}$$

or, written in a suggestive formal way,

$$\begin{aligned} &:\hat{\chi}_1(x_1)\cdots\hat{\chi}_n(x_n):\hat{\chi}_{n+1}(x_{n+1}) \\ &=:\exp\left(\sum_{\nu=1}^n \int\int dx dy \, \langle\Omega_0 \mid \hat{\chi}_\nu(x)\hat{\chi}_{n+1}(y)\Omega_0\rangle \frac{\delta}{\delta\hat{\chi}_\nu(x)} \frac{\delta}{\delta\hat{\chi}_{n+1}(y)} \right) \\ &\qquad\qquad\qquad\qquad \hat{\chi}_1(x_1)\cdots\hat{\chi}_{n+1}(x_{n+1}): \end{aligned}$$

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[20]Here, the $\hat{\chi}_\nu$ could also be partial derivatives of various Bose fields. For the more general case, where some of the $\hat{\chi}_\nu(x)$ are Fermi fields, see Chapter 4.

for

$$\hat\chi_1(x_1),\ldots,\hat\chi_{n+1}(x_{n+1})\in\left\{\hat\Phi_0^+(x),\hat\Phi_0^-(x)\right\}. \qquad (*)$$

Induction w.r.t. n, finally, gives

$$\begin{aligned}&\hat\chi_1(x_1)\cdots\hat\chi_n(x_n)\hat\chi_{n+1}(x_{n+1})\\ &=:\exp\left(\sum_{1\le\nu<\mu\le n+1}\int\!\!\int \mathrm{d}x\mathrm{d}y\,\langle\Omega_0\mid\hat\chi_\nu(x)\hat\chi_\mu(y)\Omega_0\rangle\frac{\delta}{\delta\hat\chi_\nu(x)}\frac{\delta}{\delta\hat\chi_\mu(y)}\right)\\ &\qquad\qquad\hat\chi_1(x_1)\cdots\hat\chi_{n+1}(x_{n+1}):,\end{aligned}$$

provided $(*)$ holds. From this the statement of the theorem follows easily for

$$\hat\chi_1(x)=\ldots=\hat\chi_{n+1}(x)=\hat\Phi_0(x)$$

and then, by just forming derivatives, the full statement. ∎

Corollary 3.2.2 *Let $P_1,\ldots,P_n$ be polynomials. Then*[21]

$$\begin{aligned}&:P_1(\hat\Phi_0(x_1)):\cdots:P_n(\hat\Phi_0(x_n)):\\ &=:\prod_{\nu<\mu}\exp\left(\left\langle\Omega_0\mid\hat\Phi_0(x_\nu)\hat\Phi_0(x_\mu)\Omega_0\right\rangle\frac{\partial}{\partial\hat\Phi_0(x_\nu)}\frac{\partial}{\partial\hat\Phi_0(x_\mu)}\right)P_1(\hat\Phi_0(x_1))\cdots P_n(\hat\Phi_0(x_n)):\end{aligned}$$

if – formally – $\hat\Phi_0(x_1),\ldots,\hat\Phi_0(x_n)$ are considered as independent scalar variables.

Exercise 47 Use Corollary 3.2.2 to give a formal proof of[22]

$$:e^{\hat\Phi_0(x)}::e^{\hat\Phi_0(y)}:=e^{\left\langle\Omega_0|\hat\Phi_0(x)\hat\Phi_0(y)\Omega_0\right\rangle}:e^{\hat\Phi_0(x)+\hat\Phi_0(y)}:$$

and the corresponding formula for the time-ordered product of the two (normal ordered) exponentials.

For simplicity let us assume

$$\hat S_1=:P\left(\hat\Phi_0(x)\right):$$

for some polynomial P. Then Corollary 3.2.2 implies **formally**

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[21] For the definition of powers of 2-point functions see Equation (3.32) and Exercise 49. Actually, a rigorous proof of Corollary 3.2.2 is not straightforward.

[22] For a rigorous definition of expressions of the form $\sum_{n=0}^\infty c_n{:}\hat\Phi_0(x)^n{:}$ with arbitrary $c_n\in\mathbb{R}$ see (Rieckers, 1971).

$$T\left(:\hat{\Phi}_0(x)^4::\hat{\Phi}_0(y)^4:\right) \mathrel{\widehat{=}}$$

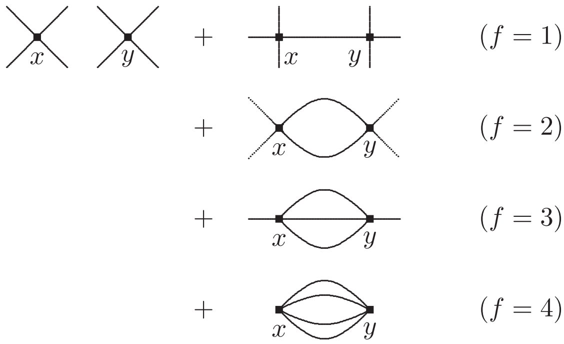

Figure 3.2: Wick's theorem interpreted in terms of diagrams

$$\begin{aligned}
&T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right)\\
&= :P\left(\hat{\Phi}_0(x_1)\right)\cdots P\left(\hat{\Phi}_0(x_n)\right):\\
&\quad + \sum_{\nu<\mu}\sum_{f_{\nu\mu}=1}^{\deg P} :P\left(\hat{\Phi}_0(x_1)\right)\cdots \underbrace{P\left(\hat{\Phi}_0(x_\nu)\right)\cdots P\left(\hat{\Phi}_0(x_\mu)\right)}_{f_{\nu\mu}}\cdots P\left(\hat{\Phi}_0(x_n)\right):\\
&\quad + \frac{1}{2}\sum_{\nu_1<\mu_1}\sum_{\substack{\nu_2<\mu_2\\(\nu_2,\mu_2)\neq(\nu_1,\mu_1)}}\sum_{f_{\nu_1\mu_1},f_{\nu_2\mu_2}=1}^{\deg P} :P\left(\hat{\Phi}_0(x_1)\right)\cdots\\
&\quad \cdots \underbrace{P\left(\hat{\Phi}_0(x_{\nu_1})\right)\cdots P\left(\hat{\Phi}_0(x_{\nu_2})\right)\cdots P\left(\hat{\Phi}_0(x_{\mu_1})\right)}_{f_{\nu_1\mu_1}}\cdots P\left(\hat{\Phi}_0(x_{\mu_2})\right)\cdots P\left(\hat{\Phi}_0(x_n)\right):\\
&\qquad (\text{contraction } f_{\nu_2\mu_2} \text{ between } x_{\nu_2} \text{ and } x_{\mu_2})\\
&\quad + \quad \ldots
\end{aligned} \tag{3.27}$$

where

$$\underbrace{\qquad}_{f_{\nu\mu}} \mathrel{\widehat{=}} \frac{1}{f_{\nu\mu}!}\left(i\Delta_{\mathrm{F}}(x_\nu - x_\mu)\frac{\partial}{\partial\hat{\Phi}_0(x_\nu)}\frac{\partial}{\partial\hat{\Phi}_0(x_\mu)}\right)^{f_{\nu\mu}}$$

and

$$\begin{aligned}
\Delta_{\mathrm{F}}(x-y) &\stackrel{\text{def}}{=} \lim_{\epsilon\to+0}(2\pi)^{-4}\int\frac{1}{p^2-m^2+i\epsilon}e^{-ip(x-y)}\,\mathrm{d}p\\
&= -i\left\langle\Omega_0 \mid T\left(\hat{\Phi}_0(x)\hat{\Phi}_0(y)\right)\Omega_0\right\rangle \quad \text{for } x\neq y\,.
\end{aligned} \tag{3.28}$$

This sum over all ***contraction*** schemes with f-fold contraction lines may be easily identified with a corresponding sum of diagrams.[23]

In Fig. 3.2 this is sketched for the special case $P(\xi) = \xi^4$, $n = 2$. Here the

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[23]Subdiagrams of the type  (***tadpoles) do never occur – thanks to Wick ordering of*** $\hat{S}_1(x)$.

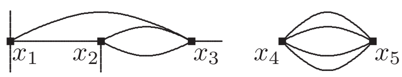
concrete meaning of the sum of diagrams is

$$\begin{aligned} T\left(:\hat\Phi_0(x)^4::\hat\Phi_0(y)^4:\right) &= \;:\hat\Phi_0(x)^4\hat\Phi_0(y)^4: \\ &\quad +:\left(i\Delta_{\mathrm F}(x-y)\frac{\partial}{\partial\hat\Phi_0(x)}\frac{\partial}{\partial\hat\Phi_0(y)}\right)\hat\Phi_0(x)^4\hat\Phi_0(y)^4: \\ &\quad +\tfrac{1}{2}:\left(i\Delta_{\mathrm F}(x-y)\frac{\partial}{\partial\hat\Phi_0(x)}\frac{\partial}{\partial\hat\Phi_0(y)}\right)^2\hat\Phi_0(x)^4\hat\Phi_0(y)^4: \\ &\quad +\tfrac{1}{3!}:\left(i\Delta_{\mathrm F}(x-y)\frac{\partial}{\partial\hat\Phi_0(x)}\frac{\partial}{\partial\hat\Phi_0(y)}\right)^3\hat\Phi_0(x)^4\hat\Phi_0(y)^4: \\ &\quad +\tfrac{1}{4!}:\left(i\Delta_{\mathrm F}(x-y)\frac{\partial}{\partial\hat\Phi_0(x)}\frac{\partial}{\partial\hat\Phi_0(y)}\right)^4\hat\Phi_0(x)^4\hat\Phi_0(y)^4:\,, \end{aligned}$$

giving the **formal** result

$$\begin{aligned} T\left(:\hat\Phi_0(x)^4::\hat\Phi_0(y)^4:\right) &= \;:\hat\Phi_0(x)^4\hat\Phi_0(y)^4: \\ &\quad +16i\Delta_{\mathrm F}(x-y):\hat\Phi_0(x)^3\hat\Phi_0(y)^3: \\ &\quad -72\Delta_{\mathrm F}(x-y)^2:\hat\Phi_0(x)^2\hat\Phi_0(y)^2: \\ &\quad -96i\Delta_{\mathrm F}(x-y)^3:\hat\Phi_0(x)\hat\Phi_0(y): \\ &\quad +24\Delta_{\mathrm F}(x-y)^4\,. \end{aligned} \tag{3.29}$$

Exercise 48 Evaluate



considered as a single diagram (not a sum of diagrams).

Recall that, according to (3.28),

$$\Delta_{\mathrm F}(x) = \begin{cases} +\Delta_+(x) & \text{for } x\in\mathbb{R}^4\setminus\overline{V_-}\,,\\ -\Delta_-(x) & \text{for } x\in\mathbb{R}^4\setminus\overline{V_+}\,,\end{cases}$$

where[24]

$$\begin{aligned} \Delta_+(x-y) &\stackrel{\text{def}}{=} -i\left\langle\Omega_0 \mid \hat\Phi_0(x)\hat\Phi_0(y)\Omega_0\right\rangle \\ &= -i(2\pi)^{-3}\int \mathrm{d}p\,\theta(p^0)\delta(p^2-m^2)e^{-ip(x-y)} \end{aligned} \tag{3.30}$$

and

$$\Delta_-(x) \stackrel{\text{def}}{=} -\Delta_+(-x)\,. \tag{3.31}$$

Since $\operatorname{supp}\widetilde{\Delta_\pm}(p)\subset\overline{V_\pm}$, we may recursively define

$$\widetilde{\Delta^{n+1}_\pm}(p) \stackrel{\text{def}}{=} (2\pi)^{-2}\left(\widetilde{\Delta^n_\pm} * \widetilde{\Delta_\pm}\right)(p)\,, \tag{3.32}$$

(see Exercise 49) in spite of the singularities[25] of $\Delta_\pm(x)$ on the light cone. Actually, this definition has to be used in Corollary 3.2.2 and to fix $\Delta^n_{\mathrm F}(x)$ for $x\neq 0$:

$$\Delta^n_{\mathrm F}(x) \stackrel{\text{def}}{=} \begin{cases} \Delta^n_+(x) & \text{for } x\in\mathbb{R}^4\setminus\overline{V_-}\,,\\ (-1)^n\Delta^n_-(x) & \text{for } x\in\mathbb{R}^4\setminus\overline{V_+}\,.\end{cases} \tag{3.33}$$

[24] In (2.62) we wrote $\Delta^{(+)}_m$ instead of Δ_+ .

[25] See (Bogoliubov and Shirkov, 1959, Sect. 15.2).

Supplemented by (3.33), (3.27) becomes a rigorous equation on $\{(x_1,\ldots,x_n)\in\mathbb{R}^{4n}: x_\nu \neq x_\mu \text{ for } \nu\neq\mu\}$.

Exercise 49 Let $M=\overline{M}\subset\mathbb{R}^4$ and let $F,G\in\mathcal{S}(\mathbb{R}^4)'$ fulfill the conditions

$$\operatorname{supp}\tilde F\subset\overline{V_+}\supset\operatorname{supp}\tilde G$$

and

$$\operatorname{supp}F\subset M\,.$$

Show that convolution of $\tilde F$ with $\tilde G$ is well-defined and that the Fourier transform FG of $(2\pi)^{-2}\tilde F*\tilde G$ fulfills the conditions

$$\operatorname{supp}(FG)\subset M\,,\quad \operatorname{supp}\widetilde{FG}\subset\overline{V_+}$$

and, if both F and G are sufficiently regular:

$$(FG)(x)=F(x)G(x)\quad\text{pointwise}\,.$$

The nontrivial problem is extension of (3.33) to all of $\mathbb{R}^4$ for $n>1$. No doubt, (3.33) may be extended to a Lorentz invariant tempered distribution on all of $\mathbb{R}^4$ (see Sect. 3.2.2). This extension is unique up to addition of a Lorentz invariant distribution with point-like support at the origin. However, without further restrictions there is no hope to extract physically relevant information.

3.2.2 Counter Terms and Renormalization

The guiding heuristic principle for minimizing the arbitrariness in the definition of $\Delta_{\mathrm F}(x)^n$ is to make it no more singular at the origin than necessary.[26] For $n=1$ this means to take (3.28). For $n>1$ the allowed (tempered) solutions may be constructed as follows:

One introduces a suitable covariant regularization $\Delta_{\mathrm F,M}$ of $\Delta_{\mathrm F}$ depending on a parameter M such that for finite M the naive definition

$$\widetilde{\Delta^n_{\mathrm F,M}}(p)\stackrel{\text{def}}{=}(2\pi)^{-2(n-1)}\left(\widetilde{\Delta_{\mathrm F,M}}*\cdots*\widetilde{\Delta_{\mathrm F,M}}\right)(p)\tag{3.34}$$

works and:

$$\int\Delta_{\mathrm F,M}(x)\varphi(x)\,\mathrm dx \underset{M\to\infty}{\longrightarrow} \int\Delta_{\mathrm F}(x)\varphi(x)\,\mathrm dx\quad\forall\,\varphi\in\mathcal S(\mathbb R^4)\,,$$

$$\int\Delta^n_{\mathrm F,M}(x)\varphi(x)\,\mathrm dx \underset{M\to\infty}{\longrightarrow} \begin{cases}\int\Delta^n_+(x)\varphi(x)\,\mathrm dx & \forall\,\varphi\in\mathcal S\left(\mathbb R^4\setminus\overline{V_-}\right)\,,\\ (-1)^n\int\Delta^n_-(x)\varphi(x)\,\mathrm dx & \forall\,\varphi\in\mathcal S\left(\mathbb R^4\setminus\overline{V_+}\right)\,.\end{cases}$$

For given n this regularization has to fulfill the requirement that

$$\Delta^n_{\mathrm F,M}(x)-\sum_{\nu=1}^N A_{\nu,M}\,\Box_x^{\nu-1}\delta(x)$$

[26] See also (Epstein and Glaser, 1973, Sect. 5).

has a ($M \to \infty$)-limit $\Delta^n_{\mathrm{F,reg}}(x)$ in $\mathcal{S}(\mathbb{R}^4)'$ for a minimal number $N = N(n)$ of suitable sequences $A_{1,M}, \ldots, A_{N,M}$. This way $\Delta^n_{\mathrm{F,reg}}(x)$ is fixed up to addition of a distribution of the form

$$\sum_{\nu=1}^{N} c_\nu \,\Box_x^{\nu-1}\delta(x)$$

with **finite** coefficients $c_1, \ldots, c_N$. A suitable regularization is for example ***Pauli-Villars regularization***

$$\begin{aligned}\Delta_{\mathrm{F},M}(x) &\stackrel{\mathrm{def}}{=} (2\pi)^{-4}\int\left(\frac{1}{p^2-m^2+i\epsilon}-\frac{1}{p^2-M^2+i\epsilon}\right)e^{-ipx}\,\mathrm{d}p\\ &= (2\pi)^{-4}\int\frac{m^2-M^2}{(p^2-m^2+i\epsilon)(p^2-M^2+i\epsilon)}e^{-ipx}\,\mathrm{d}p\,.\end{aligned}$$

This gives[27]

$$N(2) = 1\,,\quad N(3) = 2\,;$$

i.e.:

$$\begin{aligned}\Delta^2_{\mathrm{F,reg}}(x) &= \lim\nolimits_{M\to\infty}\left(\Delta^2_{\mathrm{F},M}(x) - A_M\delta(x)\right),\\ \Delta^3_{\mathrm{F,reg}}(x) &= \lim\nolimits_{M\to\infty}\left(\Delta^3_{\mathrm{F},M}(x) - B_M\delta(x) - C_M\,\Box_x\delta(x)\right)\end{aligned}$$

in the topology of $\mathcal{S}(\mathbb{R}^4)'$ for suitable A_M, B_M, C_M.

Of course, these sequences have to diverge for $M \to \infty$ in order to compensate the so-called ***ultraviolet divergences*** appearing in (3.34) for $M \to \infty$. This is what is meant by the usual saying:

> The ultraviolet infinities introduced by formal use of
>
> $$\begin{aligned}\Delta^n_{\mathrm{F}}(x) &= \Delta^n_{\mathrm{F},\infty}(x)\\ &= (2\pi)^{-4(n+1)}\int\frac{1}{p_1^2-m^2+i\epsilon}\,\frac{1}{(p_2-p_1)^2-m^2+i\epsilon}\cdots\\ &\qquad\cdots\frac{1}{(p_n-p_{n-1})^2-m^2+i\epsilon}e^{-ip_nx}\,\mathrm{d}p_n\cdots\mathrm{d}p_1\end{aligned}$$
>
> can be removed by **infinite** *counter terms*, e.g. $A_\infty\,\delta(x)$ for $n = 2$ resp. $B_\infty\,\delta(x) + C_\infty\,\Box_x\delta(x)$ for $n = 3$.

The essential result of the above considerations is the following:

For (3.26) in the limit[28] $g \to 1$ with $\hat{S}_1(x) = \lambda\!:\!\hat{\Phi}_0(x)^4\!:$ any deviation of the working definition for Δ^n_{F}, used in (3.29), from the physically correct one (if such exists at all) may be compensated – at least up to second order in λ – by adding suitable ***counter terms*** C_ν of **higher order** in λ:

$$\begin{aligned}\lambda\!:\!\hat{\Phi}_0(x)^4\!: \;\longrightarrow\; &\lambda\!:\!\hat{\Phi}_0(x)^4\!: + C_1(\lambda)\!:\!\hat{\Phi}_0(x)^4\!: + C_2(\lambda)\!:\!\hat{\Phi}_0(x)^2\!:\\ &+ C_3(\lambda)\!:\!\hat{\Phi}_0(x)\,\Box_x\hat{\Phi}_0(x)\!: + C_4(\lambda,x)\hat{1}\,.\end{aligned}\qquad(3.35)$$

[27] See (Bogoljubov and Širkov, 1984, Sects. 23.2 and 25.2).

[28] We ignore the subtleties indicated by Exercise 52. See Section 3.3.2 in this connection.

The highly nontrivial[29] result of renormalization theory is that this compensation works for all orders of canonical perturbation theory for $\lambda{:}\hat{\Phi}_0(x)^4{:}$.

Let us now indicate why such counter terms may be unavoidable in the construction of solutions to interacting field equations (see e.g. (Feldman and Raczka, 1977)).

We had already seen in Sect. 3.1.2 that the **formal** success of canonical quantization does not depend on the choice of the 'physical' mass m. It does not even depend on the normalization of the interacting field. More precisely, let $\hat{\Phi}(x)$ be a solution of

$$(\Box + \check{m}_{\mathrm{b}})\,\hat{\Phi}(x) = -4\check{\lambda}{:}\hat{\Phi}(x)^3{:} \tag{3.36}$$

for $\check{m} = m_{\mathrm{b}}$ and $\check{\lambda} = Z\lambda_{\mathrm{b}}$ fulfilling the canonical commutation relations (3.16), the Hamiltonian being

$$\hat{H} = \frac{1}{2}\int_{x^0=0} \vdots\left(\dot{\hat{\Phi}}(x)^2 + \left(\boldsymbol{\nabla}\hat{\Phi}(x)\right)^2 + m_{\mathrm{b}}^2\,\hat{\Phi}(x)^2 + 2Z\,\lambda_{\mathrm{b}}\hat{\Phi}(x)^4\right)\vdots\,d\mathbf{x} \;+\; \delta E \tag{3.37}$$

(compare Sect. 3.1.2). Then

$$\hat{\Phi}_Z(x) \stackrel{\text{def}}{=} \sqrt{Z}\hat{\Phi}(x)$$

fulfills (3.36) for $\check{m} = m_{\mathrm{b}}$ and $\check{\lambda} = \lambda_{\mathrm{b}}$. The Hamiltonian for $\hat{\Phi}_Z(x)$ is the same as for $\hat{\Phi}(x)$, of course, but in terms of $\hat{\Phi}_Z(x)$ it is given by

$$\hat{H} = \frac{1}{2Z}\int_{x^0=0} \vdots\left(\dot{\hat{\Phi}}_Z(x)^2 + \left(\boldsymbol{\nabla}\hat{\Phi}_Z(x)\right)^2 + m_{\mathrm{b}}^2\,\hat{\Phi}_Z(x)^2 + 2\lambda_{\mathrm{b}}\,\hat{\Phi}_Z(x)^4\right)\vdots\,d\mathbf{x} + \text{const}\,. \tag{3.38}$$

Exercise 50 Show that (3.38) and

$$\hat{\Pi}_Z(x) \stackrel{\text{def}}{=} \frac{1}{Z}\dot{\hat{\Phi}}_Z(x) = \frac{1}{\sqrt{Z}}\dot{\hat{\Phi}}(x)$$

correspond to the canonical formalism for the (classical) Lagrangian

$$L(\Phi_Z, \dot{\Phi}_Z) = \frac{1}{2Z}\int_{x^0=0}\left(\dot{\Phi}_Z(x)^2 - (\boldsymbol{\nabla}\Phi_Z(x))^2 - m_{\mathrm{b}}^2\,\Phi_Z(x)^2 - 2\lambda_{\mathrm{b}}\,\Phi_Z(x)^4\right)d\mathbf{x}\,.$$

Therefore, given a solution $\hat{\Phi}_Z(x)$ of (3.36) for $\check{m} = m_{\mathrm{b}}$ and $\check{\lambda} = \lambda_{\mathrm{b}}$ with asymptotic 'free' time evolution governed by[30]

$$\hat{H}_0 = \frac{1}{2}\int_{x^0=0} {:}\left(\dot{\hat{\Phi}}_0(x)^2 + \left(\boldsymbol{\nabla}\hat{\Phi}_0(x)\right)^2 + m^2\,\hat{\Phi}_0(x)^2\right){:}\,d\mathbf{x}$$

[29] A rough idea of how to proceed (in the Bogoliubov-Parasiuk-Hepp sense) may be extracted from Sect. 3.3.2. Considerable complication is caused by so-called *overlapping divergences*.

[30] In principle, of course, $\hat{\Phi}_Z(x)$ could be associated with several (asymptotic) particles of different masses.

(compare (3.21)), it may be necessary to scale $\hat{\Phi}_Z(x)$ to $\hat{\Phi}(x)$ in order to have (3.37) and

$$\hat{\Phi}(0,\mathbf{x}) = \hat{\Phi}_0(0,\mathbf{x}) \,, \dot{\hat{\Phi}}(0,\mathbf{x}) = \dot{\hat{\Phi}}_0(0,\mathbf{x}) \,,$$

(at least up to local equivalence, if possible at all) where $\hat{\Phi}_0(\mathbf{x})$ denotes the free field with mass m (not m_b). Then in (3.23), **formally** assuming (3.13), we get

$$\boxed{\begin{aligned} \hat{H}_\mathrm{I}(t) &= \int_{x^0=t} :\left(-\frac{1}{2}\delta m^2\,\hat{\Phi}_0(x)^2 + Z\lambda_\mathrm{b}\,\hat{\Phi}_0(x)^4\right):\mathrm{d}\mathbf{x} \;+\; \delta E \\ &= \int_{x^0=t} :\left(-\frac{1}{2}\delta m^2\,\hat{\Phi}_0(x)^2 + \lambda\,\hat{\Phi}_0(x)^4 + \delta\lambda\,\hat{\Phi}_0(x)^4\right):\mathrm{d}\mathbf{x} \;+\; \delta E \,, \end{aligned}} \tag{3.39}$$

where

$$\delta m^2 \stackrel{\mathrm{def}}{=} m^2 - m_\mathrm{b}^2 \,, \quad \delta\lambda \stackrel{\mathrm{def}}{=} Z\lambda_\mathrm{b} - \lambda \,,$$

λ denoting the *physical* (i.e. the *renormalized*) *coupling constant* to be fixed by some convention. Introduction of the counter terms[31]

$$-\frac{1}{2}\delta m^2 :\hat{\Phi}_0(x)^2: \,, \quad \delta\lambda :\hat{\Phi}_0(x)^4: \,, \quad \delta E$$

in (3.39) corresponds[32] to (3.35). Correct choice of the counter terms is called ***renormalization***.

In perturbation theory the coefficients $\delta m^2, \delta\lambda, \delta E$ are considered as power series in λ with coefficients depending on m. Their choice has to be adapted to the working definition of time ordering to meet physical requirements, especially stability of the vacuum and 1-particle states and, formally:[33]

$$\delta m^2{}_{|\lambda=0} = \delta\lambda_{|\lambda=0} = \delta E_{|\lambda=0} = 0 \,. \tag{3.40}$$

Exercise 51 For $s > 0$ show[34] that also

$$\hat{\hat{\Phi}}(x) \stackrel{\mathrm{def}}{=} s\,\hat{\Phi}_1(sx)$$

fulfills the canonical commutation relations (3.16) with $\hat{\Pi}(x) = \partial_0\hat{\Phi}(x)$ and the field equation (3.36) for $\hat{m} = sm_\mathrm{b}$, $\check{\lambda} = \lambda_\mathrm{b}$. Moreover, show that the Hamiltonian for

[31] Thanks to the counter terms there is now some chance that the r.h.s. of (3.37) may be rigorously defined by suitable limiting procedures as in lower-dimensional constructive field theory. However, the counter terms cannot all be finite, because of Haag's theorem.

[32] Recall that $:\hat{\Phi}_0(x)\,\Box\,\hat{\Phi}_0(x): \,=\, -m^2 :\hat{\Phi}_0(x)^2:$. Therefore an arbitrary term proportional to $:\hat{\Phi}_0(x)\,\Box\,\hat{\Phi}_0(x):$ can be extracted from $\delta m^2 :\hat{\Phi}_0(x)^2:$ and compensated by change of δm^2 .

[33] Actually, as pointed out for Equation (3.35), δm^2 and $\delta\lambda$ are assumed to be of higher order in λ .

[34] This exercise indicates that – as far as perturbative calculations of S elements are concerned – taking a mass different from the physical mass m for the free field could be balanced by suitable change of the counter terms.

$\hat{\hat{\Phi}}(x)$ is

$$\hat{\hat{H}} = \frac{1}{2}\int_{x^0=0} :\left(\sum_{\mu=0}^{3}\left(\partial_\mu \hat{\hat{\Phi}}(x)\right)^2 + (s m_{\rm b})^2\,\hat{\hat{\Phi}}(x)^2 + 2\lambda_{\rm b}\,\hat{\hat{\Phi}}(x)^4\right): d\mathbf{x} \;+\; \text{const}\,.$$

Exercise 52 Show that $\int :\hat{\Phi}_0(x)^4: dx$ exists as a quadratic form but not as an operator on D_0. Moreover, show that $\int :\hat{\Phi}_0(x)^2: dx$ does not even exist as a quadratic form on D_0.

3.2.3 Feynman Rules

The perturbative expansion (3.24) for (3.39) – evaluated by Coroll. 3.2.2 – may be represented by diagrams. For these diagrams we need three types of vertices which we will draw as •, ○, and ⦿. Each vertex of type • or ⦿ is connected to exactly four solid lines, each vertex of type ○ to exactly two solid lines. These lines may either have free ends (***external lines***) or connect to another vertex (***internal lines***). Finally, the vertices of such a diagram G have to be indexed from 1 to V_G, where

$$V_G \stackrel{\rm def}{=} \text{number of vertices of } G\,.$$

diagrams of this kind will be called ***admitted*** (for $\lambda\hat{\Phi}^4_4$-perturbation theory in x-space).

To write down the formal operator $\hat{A}_G$ represented by an admitted diagram G one has to apply the following ***Feynman rules***:

1. Write down a factor

$$\begin{array}{rll} -i\lambda & \text{for each vertex} & \bullet\,, \\ \frac{i}{2}\delta m & \text{for each vertex} & \circ\,, \\ -i\delta\lambda & \text{for each vertex} & \odot\,. \end{array}$$

2. For every pair of vertices with indices ν and μ, if directly connected by at least one internal line, write down a factor

$$i^{f_{\nu\mu}}\,\Delta_{\rm F}^{f_{\nu\mu}}(x_\nu - x_\mu)\,,$$

where $f_{\nu\mu}$ is the number of internal lines directly connecting these vertices.

3. Multiply by the symmetry factor[35]

$$\prod_{1\le\nu<\mu\le V_G}\frac{1}{f_{\nu\mu}!}\prod_{1\le\alpha\le V_G}\frac{l_\alpha!}{\left(l_\alpha - \sum_{\beta=1}^{V_G} f_{\alpha\beta}\right)!}\,,$$

[35] Recall the evaluation of Fig. 3.2.

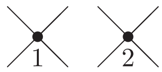
where

$$l_\alpha \stackrel{\text{def}}{=} \begin{cases} \text{number of lines} \\ \text{attached to vertex } \alpha\,. \end{cases}$$

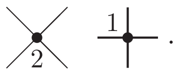
4. For every external line write down a field operator $\hat{\Phi}_0(x_\nu)$, where ν is the index of the vertex to which the line is attached.

5. Normal order the resulting monomial and integrate all field variables over $\mathbb{R}^4$.

Any two diagrams have to be considered as ***equal*** if they differ only by their diagramatical realization.[36] For example,



is to be considered as equal to

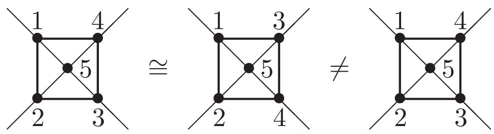


Then, for suitable[37] φ, we have[38]

$$e^{i\varphi}\hat{S}_0 = \hat{1} + \sum_{G \text{ admitted}} \frac{1}{V_G!}\hat{A}_G \tag{3.41}$$

as an equation for formal power series in λ of quadratic forms on D_0.

Any two diagrams G_1, G_2 are called ***equivalent*** ($G_1 \cong G_2$) if they differ only by the distribution of their vertex indices; e.g.:



Then the number of elements in the equivalence class $[G]$ of a diagram G is $\frac{V_G!}{I_G}$, where

$$I_G \stackrel{\text{def}}{=} \begin{cases} \text{number of permutations of the vertex indices} \\ \text{that do not change the diagram } G\,. \end{cases}$$

Therefore[39] (3.41) is equivalent to

$$e^{i\varphi}\hat{S}_0 = \hat{1} + \sum_{[G]} \frac{1}{I_G}\hat{A}_G\,. \tag{3.42}$$

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[36]Consider the lines as elastic strings. Then all elastic deformations leave the diagrams unchanged.

[37]Actually, everything should be defined with suitable cutoffs first. When removing these limits φ becomes infinite and compensates the infinite contributions of the ***vacuum diagrams***, i.e. those diagrams G for which $\hat{A}_G \sim \hat{1}$. This will be used in (3.44).

[38]Thanks to (3.40), only a finite number of diagrams contributes to each order in λ on the r.h.s. of (3.41).

[39]Note that $G_1 \cong G_2 \Longrightarrow \hat{A}_{G_1} = \hat{A}_{G_2}$, thanks to integration over all field variables, even though $G_1 \cong G_2$ does not imply $G_1 = G_2$, in general.

Denote by $G_1 \cdots G_N$ the diagram consisting of the disjoint subdiagrams $G_1, \ldots, G_N$ with natural renumbering of their vertices. Then

$$\hat{A}_{G_1 \cdots G_N} = :\hat{A}_{G_1} \cdots \hat{A}_{G_N}:$$

and

$$I_{G_1 \cdots G_N} = I_{G_1} \cdots I_{G_N} E_{G_1 \cdots G_N} \,,$$

where

$$E_{G_1 \cdots G_N} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{number of permutations } \pi \in S_N \text{ with} \\ ([G_{\pi 1}], \ldots, [G_{\pi N}]) = ([G_1], \ldots [G_N]) \,. \end{array} \right.$$

Thus we have, formally,

$$\frac{1}{I_{G_1 \cdots G_N}} \hat{A}_{G_1 \cdots G_N} = \frac{1}{E_{G_1 \cdots G_N}} : \frac{1}{I_{G_1}} \hat{A}_{G_1} \cdots \frac{1}{I_{G_N}} \hat{A}_{G_N} : \tag{3.43}$$

for every set of diagrams $\{G_1, \ldots, G_N\}$.

An admitted diagram is called ***connected*** if any two vertices are connected by a chain of internal lines. Obviously, for every diagram G there is a unique N-tuple of connected diagrams $(G_1, \ldots, G_N)$ with $G = G_1 \cdots G_N$. Consequently, by (3.43), (3.42) may be written as

$$e^{i\varphi} \hat{S}_0 = :\exp\Bigg(\sum_{\substack{[G] \\ G \text{ connected}}} \frac{1}{I_G} \hat{A}_G \Bigg) : .$$

With the physically natural requirement[40]

$$\left\langle \Omega_0 \mid \hat{S}_0 \Omega_0 \right\rangle = 1$$

(***stability of the vacuum***) this implies:

$$\boxed{\begin{array}{l} \hat{S}_0 = :\exp\Big(\sum\limits_{[G] \in \mathcal{G}} \frac{1}{I_G} \hat{A}_G \Big) : = :\exp\Big(\sum\limits_{\substack{G \text{ admitted} \\ [G] \in \mathcal{G}}} \frac{1}{V_G !} \hat{A}_G \Big) : , \\ \text{where:} \\ \qquad \mathcal{G} \stackrel{\text{def}}{=} \left\{ [G] : G \text{ connected}, \; \hat{A}_g \not\propto \hat{1} \right\} . \end{array}} \tag{3.44}$$

In order to evaluate (3.44) one has to fix, first of all, δm as a power series in λ depending on m and $\delta\lambda$. For this the physical requirement

$$\left\langle \Psi \mid \hat{S}_0 \Psi \right\rangle = \langle \Psi \mid \Psi \rangle \quad \forall \, \Psi \in \mathcal{H}_0^{(1)}$$

(***stability of 1-particle states***) is sufficient. Finally, one has to fix $\delta\lambda$ as a power series in λ depending only on m. For this some convention concerning the 2-2-scattering amplitude – depending on the preferred technical interpretation of the coupling constant – is necessary.

[40] Actually, only $\left\langle \Omega_0 \mid \hat{S}_0 \Omega_0 \right\rangle = \hat{1}$ has to be required.

Exercise 53 Calculate the total cross section[41]

$$\sigma(p_1,p_2) = \frac{(2\pi)^2}{4\sqrt{(p_1p_2)^2-m^4}}\int_{q_j^0=\omega_{\mathbf{q}_j}}\left|\frac{\left\langle \hat a_0^*(\mathbf{q}_1)\hat a_0^*(\mathbf{q}_2)\Omega_0 \mid (\hat S_0-\hat 1)\hat a_0^*(\mathbf{p}_1)\hat a_0^*(\mathbf{p}_2)\Omega_0\right\rangle}{\delta(q_1+q_2-p_1-p_2)}\right|^2 \times\delta(q_1+q_2-p_1-p_2)\,\frac{\mathrm{d}\mathbf{q}_1\mathrm{d}\mathbf{q}_2}{2q_1^0 2q_2^0}$$

of elastic scattering of two particles with initial 4-momenta p_1, p_2 to first order in the renormalized coupling constant λ.

> **Remark:** The derivation of (3.44) also shows, that one may set $\varphi = 0$ in (3.41) **if** summation is restricted to those diagrams which do not have disjoint parts without any external line.

3.3 Bogoliubov-Shirkov Theory

3.3.1 Basic Assumptions

Bogoliubov and Shirkov (Bogoliubov and Shirkov, 1959) assume that there is a whole family of (interaction picture) S-matrices $\hat S_0(g)$ depending sufficiently smoothly on $g \in \mathcal{S}(\mathbb{R}^4,\mathbb{R})$, where $\mathcal{S}(\mathbb{R}^4,\mathbb{R})$ denotes the subspace of real-valued elements of $\mathcal{S}(\mathbb{R}^4)$. If

$$0 \le g(x) \le 1 \quad \forall x \in \mathbb{R}^4 \tag{3.45}$$

then $g(x)$ is interpreted as a degree to which the interaction is 'switched on' at x (formally by replacing the renormalized coupling constant λ in (3.39) by $g(x)\lambda$). $\hat S_0(g)$ is assumed to fulfill the following conditions for all $g \in \mathcal{S}(\mathbb{R}^4)$ fulfilling (3.45):

1. $\hat S_0(g)$ is unitary.
2. $\hat S_0(0) = \hat 1$.
3. ***Relativistic Covariance:*** [42]

   $$\hat U_0(a,\Lambda)\hat S_0(g)\hat U_0(a,\Lambda)^{-1} = \hat S_0\left(\{a,\Lambda\}g\right) \quad \forall (a,\Lambda) \in \mathcal{P}_+^\uparrow,$$

   where

   $$\left(\{a,\Lambda\}g\right)(x) \stackrel{\text{def}}{=} g\left(\Lambda^{-1}(x-a)\right).$$

[41] For a derivation of this formula see e.g. (Itzykson and Zuber, 1980a, Sect. 5-1-1). For its evaluation see also 4.3.2.

[42] For theories with fermions a representation of iSL$(2,\mathbb{C})$ has to be used; see 4.2.2.

4. ***Bogoliubov-Shirkov causality***:

$$\left(x^0 < y^0\ \forall (x,y) \in \operatorname{supp} g_1 \times \operatorname{supp} g_2\right) \Longrightarrow \hat S_0(g_1+g_2) = \hat S_0(g_2)\hat S_0(g_1)\,.$$

Moreover, the functional derivatives

$$\hat S_n(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \left(\frac{\delta}{\delta g(x_1)}\cdots\frac{\delta}{\delta g(x_n)}\hat S_0(g)\right)_{|g=0}$$

are assumed to exist as operator-valued distributions on $\mathcal{S}(\mathbb{R}^{4n})$ with invariant dense domain D_0 (defined by (2.27)) for the 'smeared' $\hat S_n(x_1,\ldots,x_n)$ and their adjoints. With this definition we have the following formal Taylor expansion:[43]

$$\begin{aligned}\hat S_0(g) &= \left(\exp\left(\int \mathrm{d}x\, g(x)\frac{\delta}{\delta \breve g(x)}\right)\hat S_0(\breve g)\right)_{|\breve g=0}\\ &= \underbrace{\hat 1}_{=\hat S_0(0)} + \sum_{n=1}^{\infty}\frac{1}{n!}\int \hat S_n(x_1,\ldots,x_n)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,.\end{aligned}$$

The $\hat S_n(x_1,\ldots,x_n)$ are the central objects of the Bogoliubov-Shirkov theory.[44] Its advantage is that it does not depend on the full-fledged interaction picture.[45]

An immediate consequence of their definition is the permutation symmetry of the $\hat S_n$:

$$\boxed{\hat S_n(x_1,\ldots,x_n) = \hat S_n(x_{\pi 1},\ldots,x_{\pi n})\ \forall \pi \in S_n\,.} \tag{3.46}$$

Therefore

$$\hat S(M) \stackrel{\text{def}}{=} \begin{cases}\hat S_n(\xi_1,\ldots,\xi_n) & \text{for } M = \{\xi_1,\ldots,\xi_n\} \neq \emptyset\\ \hat 1 & \text{for } M = \emptyset\end{cases}$$

is a consistent definition (if the M are considered as sets of $\mathbb{R}^4$-variables).

Exercise 54 Given the formal power series

$$\begin{aligned}S(g) &= S(\emptyset) + \sum_{n=1}^{\infty}\frac{1}{n!}\int S\left(\{x_1,\ldots,x_n\}\right)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,,\\ R(g) &= R(\emptyset) + \sum_{n=1}^{\infty}\frac{1}{n!}\int R\left(\{x_1,\ldots,x_n\}\right)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\end{aligned}$$

in g , show that

$$S(g)\,R(g) = (S*R)(\emptyset) + \sum_{n=1}^{\infty}\frac{1}{n!}\int (S*R)\left(\{x_1,\ldots,x_n\}\right)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

[43] Many properties of the $\hat S_n$ may be easily read off from this formal power series.
[44] See (Stora, 1971; Epstein and Glaser, 1973) for a more elaborated version.
[45] This does not mean, however, that it is physically better motivated.

holds (in the sense of formal power series) with the convolution product[46]

$$(S * R)(X) \stackrel{\text{def}}{=} \sum_{M \subset X} S(M)R(X \setminus M) \quad \text{for } X \subset \{x_1, x_2, x_3, \ldots\} \ .$$

Relativistic covariance implies ***formal covariance***:

$$\boxed{\hat{U}_0(a,\Lambda)\hat{S}_n(x_1,\ldots,x_n)\hat{U}_0(a,\Lambda)^{-1} = \hat{S}_n\left(\Lambda x_1 + a, \ldots, \Lambda x_n + a\right) .} \tag{3.47}$$

Unitarity of $\hat{S}_0(g)$ implies that all functional derivatives of $\hat{V}(g) = \hat{S}_0(g)\hat{S}_0(g)^*$ vanish at g fulfilling (3.45). For $g = 0$ this gives

$$\begin{aligned} 0 &= \left(\frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \left(\hat{S}_0(g)\hat{S}_0(g)^*\right)\right)_{|g=0} \\ &= \sum_{M \subset X_n} \underbrace{\left(\left(\prod_{x \in M} \frac{\delta}{\delta g(x)}\right)\hat{S}_0(g)\right)_{|g=0}}_{=\hat{1} \text{ for } M=\emptyset} \left(\left(\prod_{x' \in X_n \setminus M} \frac{\delta}{\delta g(x')}\right)\hat{S}_0(g)^*\right)_{|g=0} , \end{aligned} \tag{3.48}$$

$$X_n \stackrel{\text{def}}{=} \{x_1, \ldots, x_n\} \ .$$

Therefore, since

$$\left(\frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \hat{S}(g)^*\right)_{|g=0} = \hat{S}_\nu(x_1, \ldots, x_n)^* ,$$

we have ***formal unitarity***:[47]

$$\boxed{0 = \sum_{M \subset X_n} \hat{S}(M)\hat{S}(X_n \setminus M)^* \quad \text{for } X_n = \{x_1, \ldots, x_n\} \ , \ n > 0 \, .} \tag{3.49}$$

Let $g_1, g_2 \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$ fulfill the conditions

$$\begin{aligned} g_1(x), g_2(x) &\in [0,1] \quad \forall x \in \mathbb{R}^4 , \\ x^0 < y^0 &\quad \forall (x,y) \in \operatorname{supp} g_1 \times \operatorname{supp} g_2 \, . \end{aligned} \tag{3.50}$$

Then Bogoliubov-Shirkov causality and unitarity imply

$$\hat{S}_0(g_1 + g_2)\hat{S}_0(g_1)^* = \hat{S}_0(g_2) \tag{3.51}$$

and therefore

$$(3.50) \Longrightarrow \frac{\delta}{\delta g_1(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g_1)^*\right)_{|g=g_1+g_2} = 0 \text{ for } (x,y) \in \operatorname{supp} g_1 \times \operatorname{supp} g_2 \, . \tag{3.52}$$

[46]This convolution product has many useful applications; see e.g. (Stora, 1971; Borchers, 1972; Doebner and Lücke, 1977; Hegerfeldt, 1985).

[47]Since unitarity means $\hat{S}_0(g)\hat{S}_0(g)^* = \hat{1} = \hat{S}_0(g)^*\hat{S}_0(g)$, also $0 = \sum_{M \subset X_n} \hat{S}(M)^*\hat{S}(X_n \setminus M)$ holds.

Remark: Conversely, (3.52) implies

$$\partial_{\lambda_1}\partial_{\lambda_2}\left(\hat{S}_0(\lambda_1 g_1+\lambda_2 g_2)\hat{S}_0(\lambda_1 g_1)^*\right)=0 \quad \forall\,\lambda_1,\lambda_2\in[0,1]$$

hence constancy of

$$\partial_{\lambda_2}\left(\hat{S}_0(\lambda_1 g_1+\lambda_2 g_2)\hat{S}_0(\lambda_1 g_1)^*\right)$$

in λ_1 and thus[48]

$$\hat{S}_0(g_1+g_2)\hat{S}_0(g_1)^*=\hat{S}_0(g_2)\underbrace{\hat{S}_0(0)^*}_{=\hat{1}}$$

under the assumption (3.50). In other words: (3.52), thanks to unitarity and $\hat{S}_0(0)=\hat{1}$, implies Bogoliubov-Shirkov causality.

Applying Bogoliubov-Shirkov causality once more we get from (3.52)

$$(3.50)\Longrightarrow\left(\frac{\delta}{\delta g(x)}\left(\frac{\delta\hat{S}_0(g)}{\delta g(y)}\hat{S}_0(g)^*\right)\right)_{|g=g_1+g_2}=0 \text{ for } (x,y)\in\operatorname{supp}g_1\times\operatorname{supp}g_2\,.$$

Evaluating this — or directly (3.52) — at $g=0$ gives

$$\hat{K}(y;X_n)=0 \quad \text{for } \max\left\{y^0-x_1^0,\ldots,y^0-x_n^0\right\}>0\,,$$

where

$$\hat{K}(y;X_n)\stackrel{\text{def}}{=}\frac{\delta}{\delta g(x_1)}\cdots\frac{\delta}{\delta g(x_n)}\left(\frac{\delta\hat{S}_0(g)}{\delta g(y)}\hat{S}_0(g)^*\right)_{|g=0}\,,$$

i.e.

$$\hat{K}(y;X_n)=\sum_{M\subset X_n}\hat{S}(\{y\}\cup M)\hat{S}(X_n\setminus M)^*\,.$$

This way, by Bogoliubov-Shirkov causality and relativistic covariance, we get ***formal causality***:

$$\boxed{\sum_{M\subset X_n}\hat{S}(\{y\}\cup M)\hat{S}(X_n\setminus M)^*=0 \quad \textbf{if } y-x\in\mathbb{R}^4\setminus\overline{V_-} \text{ for some } x\in X_n\,,\ n>0\,.} \tag{3.53}$$

Exercise 55 Show that (3.53) implies

$$\boxed{\frac{\delta}{\delta g(x)}\left(\frac{\delta\hat{S}_0(g)}{\delta g(y)}\hat{S}_0(g)^*\right)=0 \text{ for } y-x\in\mathbb{R}^4\setminus\overline{V_-}}$$

in the sense of formal power series in g.

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[48] Since then $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\hat{S}_0(g_1+\lambda g_2)\hat{S}_0(g_1)^*-\hat{S}_0(\lambda g_2)\hat{S}_0(0)^*\right)=0\,.$

3.3.2 General Solution

For $y = x_{n+1}$ (3.53) implies

$$\begin{aligned} \hat{S}(X_{n+1}) &= -\sum_{\substack{M \subset X_n \\ M \neq X_n}} \hat{S}\left(\{x_{n+1}\} \cup M\right) \hat{S}(X_n \setminus M)^* \\ &\text{if } x_{n+1} - x \in \mathbb{R}^4 \setminus \overline{V_-} \text{ for some } x \in X_n\,. \end{aligned} \tag{3.54}$$

This shows that the $\hat{S}_n(x_1, \ldots, x_n)$ are fixed on

$$\mathbb{R}^{4n}_{\neq} \stackrel{\text{def}}{=} \left\{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} : x_\nu \neq x_\mu \text{ for } \nu \neq \mu\right\} \tag{3.55}$$

for $n = 2, 3, \ldots$ once

$$\boxed{\hat{L}_{\mathrm{I}}(x) \stackrel{\text{def}}{=} -i\hat{S}_1(x)} \tag{3.56}$$

is given. Of course, $\hat{L}_{\mathrm{I}}(x)$ has to meet certain requirements. (3.49) for $n = 1$ means

$$\boxed{\hat{L}_{\mathrm{I}}(x) = \hat{L}_{\mathrm{I}}(x)^*\,.} \tag{3.57}$$

Therefore evaluation of (3.54) for $n = 1$ gives

$$\hat{S}(\{x_1, x_2\}) = \hat{S}_1(x_2)\hat{S}_1(x_1) = -\hat{L}_{\mathrm{I}}(x_2)\hat{L}_{\mathrm{I}}(x_1) \quad \text{for } x_2 - x_1 \in \mathbb{R}^4 \setminus \overline{V_-}\,. \tag{3.58}$$

Since this does not depend on the choice of indices, we conclude that[49]

$$\boxed{[\hat{L}_{\mathrm{I}}(x), \hat{L}_{\mathrm{I}}(y)]_- = 0 \quad \text{for } x \times y\,.} \tag{3.59}$$

By (3.47), finally, we have

$$\boxed{\hat{U}(a,\Lambda)\hat{L}_{\mathrm{I}}(x)\hat{U}(a,\Lambda)^{-1} = \hat{L}_{\mathrm{I}}(\Lambda x + a)\,.} \tag{3.60}$$

Summing up:

$\hat{L}_{\mathrm{I}}(x)$ must be a hermitian, scalar, local operator field.

Conversely, these properties of $\hat{L}_{\mathrm{I}}(x)$ guarantee that (3.46), (3.47), (3.49), and (3.53) are fulfilled on $\mathbb{R}^{4n}_{\neq}$ by[50]

$$\hat{S}^{\mathrm{T}}_n(x_1, \ldots, x_\nu) \stackrel{\text{def}}{=} T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_\nu)\right) \quad \text{for } \nu = 2, 3, \ldots \tag{3.61}$$

[49] Note that, $x \times y \implies x_2 - x_1 \in \mathbb{R}^4 \setminus \overline{V_-} \ni x_1 - x_2\,.$

[50] (3.49), for example, is obvious for $n = 1$ and therefore follows for $n = 2, 3, \ldots$ – if restricted to $\mathbb{R}^{4n}_{\neq}$ – because

$$\begin{aligned} &\sum_{M' \subset X_{n+1}} \hat{S}^T(M')\hat{S}^T\left(X_{n+1} \setminus M'\right)^* \\ &= \hat{S}_1(x_{n+1}) \sum_{M \subset X_n} \hat{S}^T(M)\hat{S}^T(X_n \setminus M)^* + \sum_{M \subset X_n} \hat{S}^T(M)\hat{S}^T(X_n \setminus M)^* \hat{S}_1(x_{n+1})^* \end{aligned}$$

holds on the subregion of $\mathbb{R}^{4(n+1)}_{\neq}$ characterized by: $x^0_{n+1} > x^0_j \quad$ for $j = 1, \ldots, n$. For (3.53) see also (Bogoliubov and Shirkov, 1959, Sect. 18.5).

if T is a (linear) covariant time ordering operation, i.e. fulfills the conditions

$$T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_\nu)\right) = \hat{S}_1(x_{\pi 1})\cdots\hat{S}_1(x_{\pi\nu}) \quad \text{for } \pi \in S_\nu \text{ with } x^0_{\pi 1} > \ldots > x^0_{\pi\nu} \tag{3.62}$$

and

$$\hat{U}_0(a,A)T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_\nu)\right)\hat{U}_0(a,A)^{-1} = T\left(\hat{S}_1(\Lambda x_1 + a)\cdots\hat{S}_1(\Lambda x_\nu + a)\right). \tag{3.63}$$

(on all of $\mathbb{R}^{4\nu}$ for $\nu = 2,3,\ldots$). Therefore:[51]

> On the restricted region $\mathbb{R}^{4n}_{\neq}$ there is no other choice for $\hat{S}_n(x_1,\ldots,x_n)$ than (3.61).

The difficult problem is physically correct extension of $\hat{S}_n(x_1,\ldots,x_n)$ to all of $\mathbb{R}^{4n}$ $(n = 2,3,\ldots)$.

If the $\hat{S}_\nu(x_1,\ldots,x_\nu)$ are known for $\nu \le n$ then $\hat{S}_{n+1}(x_1,\ldots,x_{n+1})$ is fixed by (3.54) even on the complement of

$$\mathbb{R}^{4(n+1)}_{=} \stackrel{\text{def}}{=} \left\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{4(n+1)} : x_1 = x_2 = \ldots = x_{n+1}\right\},$$

since there must be a pair $x_\nu, x_\mu \in \{x_1,\ldots,x_{n+1}\}$ for which $x_\nu - x_\mu \in \mathbb{R}^4 \setminus \overline{V_-}$ unless $x_1 = x_2 = \ldots = x_n$. In other words:

> If $\hat{S}_1(x),\ldots,\hat{S}_n(x_1,\ldots,x_n)$ are fixed then $\hat{S}_{n+1}(x_1,\ldots,x_{n+1})$ is unique up to addition of an operator field $\hat{A}_{n+1}(x_1,\ldots,x_{n+1})$ which is ***quasilocal***, i.e.
> $$\operatorname{supp}\hat{A}_{n+1}(x_1,\ldots,x_{n+1}) \subset \mathbb{R}^{4(n+1)}_{=}.$$

Further details: Similarly to (3.52), starting from

$$\hat{S}_0(g_2)^*\hat{S}_0(g_1+g_2) = \hat{S}_0(g_1)$$

instead of (3.51), we get

$$(3.50) \Longrightarrow \frac{\delta}{\delta g_1(x)}\left(\hat{S}_0(g_2)^*\frac{\delta\hat{S}_0(g)}{\delta g(x)}\right)_{|g=g_1+g_2} = 0 \text{ for } (x,y) \in \operatorname{supp} g_1 \times \operatorname{supp} g_2$$

and hence

$$\sum_{M\subset X_n}\hat{S}(X_n\setminus M)^*\hat{S}(\{y\}\cup M) = 0 \quad \textbf{if } y - x \in \mathbb{R}^4\setminus\overline{V_-} \text{ for some } x \in X_n,\ n > 0.$$

Therefore, the difference

$$D(X_n;y) \stackrel{\text{def}}{=} R(X_n;y) - A(X_n;y)$$

[51] Recall that (3.53) implies (3.54).

of the *retarded* function

$$R(X_n; y) \stackrel{\text{def}}{=} \sum_{M \subset X_n} \hat{S}(X_n \setminus M)^* \hat{S}(\{y\} \cup M)$$

and the *advanced* function

$$A(X_n; y) \stackrel{\text{def}}{=} \sum_{M \subset X_n} \hat{S}(\{y\} \cup M) \hat{S}(X_n \setminus M)^*$$

vanishes whenever one of the arguments x_ν is spacelike w.r.t. y. If all the $S(X_\nu)$ are known for all $\nu \leq n$ then also $D(X_n; x_{n+1})$ and $\hat{S}(X_{n+1}) - A(X_n; x_{n+1})$ are known and determination of $\hat{S}(X_{n+1})$ is equivalent to physically correct splitting of $D(X_n; x_{n+1})$ into an advanced function $A(X_n; x_{n+1})$ and a retarded function $A(X_n; x_{n+1})$. The main difficulty of such a splitting is to get $A(X_n; x_{n+1})$ and $R(X_n; x_{n+1})$ Lorentz covariant (see (Steinmann, 1963; Epstein, 1966)).

Note that one may work with $\hat{S}_0(x)^{-1}$ instead of $\hat{S}_0(x)^*$ and use (3.57) instead of formal unitarity (see (Epstein and Glaser, 1973, Sections 1.2 and 2)).

Let us use this to analyze a given formal power series $\hat{S}_0(g)$ that fulfills all the Bogoliubov-Shirkov requirements.

Let $\hat{A}_2, \ldots, \hat{A}_n$ be quasilocal operators and let T be a (linear) time ordering operation, defined for all direct products $\hat{C}_j$ composed of elements from $\left\{\hat{S}_1, \hat{A}_2, \ldots, \hat{A}_n\right\}$, fulfilling the following requirements:

$$\begin{array}{l} T\Big(\hat{C}_1(x_1, \ldots, x_j)\, \hat{C}_2(y_1, \ldots, y_k)\Big) = T\Big(\hat{C}_1(x_1, \ldots, x_j)\Big)\Big(\hat{C}_2(y_1, \ldots, y_k)\Big) \\ \textbf{if } \sup\left\{x_1^0, \ldots, x_j^0\right\} > \sup\left\{y_1^0, \ldots, y_k^0\right\}, \end{array} \tag{3.64}$$

$$\begin{aligned} T\Big(\hat{C}_1(x_1, \ldots, x_j)\, T\Big(\hat{C}_2(y_1, \ldots, y_k)\Big)\Big) &= T\Big(T\Big(\hat{C}_1(x_1, \ldots, x_j)\Big)\hat{C}_2(y_1, \ldots, y_k)\Big) \\ &= T\Big(\hat{C}_1(x_1, \ldots, x_j)\, \hat{C}_2(y_1, \ldots, y_k)\Big) \\ &= T\Big(\hat{C}_2(y_1, \ldots, y_k)\, \hat{C}_1(x_1, \ldots, x_j)\Big), \end{aligned} \tag{3.65}$$

$$\hat{U}_0(a, A) T\Big(\hat{C}(x_1, \ldots, x_\nu)\Big)\hat{U}_0(a, A)^{-1} = T\Big(\hat{C}(\Lambda x_1 + a, \ldots, \Lambda x_\nu + a)\Big). \tag{3.66}$$

Exercise 56 Show the following:

1. The $\hat{A}_\nu$ are local relative to $\hat{S}_1$ in the sense that

$$x \times y_1 \implies \Big[\hat{S}_1(x), \hat{A}_\nu(y_{\pi 1}, \ldots, y_{\pi\nu})\Big]_- = 0 \quad \forall \pi \in S_\nu .$$

2. The $\hat{A}_\nu$ are local relative w.r.t. each other in the sense that

$$x_1 \times y_1 \implies \Big[\hat{A}_\mu(x_1, \ldots, x_\mu), \hat{A}_\nu(y_{\pi 1}, \ldots, y_{\pi\nu})\Big]_- = 0 \quad \forall \pi \in S_\nu .$$

3. The $\hat{A}_\nu$ transform according to

$$\hat{U}(a,\Lambda)\,\hat{A}_\nu(x_1,\dots,x_\nu)\,\hat{U}(a,\Lambda)^{-1} = \hat{A}_\nu\left(\Lambda x_1 + a,\dots,\Lambda x_\nu + a\right) \quad \forall\,(a,\Lambda)\in\mathcal{P}_+^\uparrow\,.$$

Then we define

$$\hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(\{x_1,\dots,x_\nu\}) \stackrel{\text{def}}{=} \frac{\delta}{\delta g(x_1)}\cdots\frac{\delta}{\delta g(x_\nu)}\hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(g)_{|g=0}\,,$$

where

$$\hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(g) \stackrel{\text{def}}{=} T\left(\exp\left(\int \hat{S}_1(x)g(x)\,\mathrm{d}x + \sum_{\nu=2}^{n}\int \hat{A}_\nu(x_1,\dots,x_\nu)g(x_1)\cdots g_\nu(x_\nu)\,\mathrm{d}x_1\cdots\mathrm{d}x_\nu\right)\right)$$

(in the sense of formal power series).

Exercise 57 Show that $\hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(g)$ fulfills all the Bogoliubov-Shirkov requirements if also

$$\hat{A}_\nu(x_{\pi 1},\dots,x_{\pi\nu}) = \hat{A}_\nu(x_1,\dots,x_\nu) = -\hat{A}_\nu(x_1,\dots,x_\nu)^* \qquad \forall\,\pi\in S_\nu\,,\ \nu\le n \tag{3.67}$$

holds.

Now assume

$$\hat{S}(\{x_1,\dots,x_\nu\}) = \hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(\{x_1,\dots,x_\nu\}) \quad \text{for } \nu\le n\,. \tag{3.68}$$

Then, according to the above considerations,

$$\hat{A}_{n+1}(x_1,\dots,x_{n+1}) \stackrel{\text{def}}{=} \hat{S}(\{x_1,\dots,x_{n+1}\}) - \hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}(\{x_1,\dots,x_{n+1}\}) \tag{3.69}$$

must be a quasilocal operator field, since (3.54) holds for both $\hat{S}$ and $\hat{S}^T_{\hat{A}_2,\dots,\hat{A}_n}$.

Exercise 58 Show that, with $\hat{A}_{n+1}$ defined by (3.69), the statements of Exercise 56 and (3.67) hold also for n replaced by $n+1$.

If we can extend[52] the T-operation to all direct products $\hat{C}_j$ composed of elements from $\left\{\hat{S}_1,\hat{A}_2,\dots,\hat{A}_{n+1}\right\}$ in a way respecting (3.64)–(3.66) we get (3.68) for n replaced by $n+1$. This way we are lead to the following ***Bogoliubov-Shirkov***

[52]In general, it is not at all obvious that such an extension is possible. Up to now, nobody could provide a proof for existence of the **covariant** T-products for interacting Wightman fields (compare (Steinmann, 1963; Epstein, 1966) to see the difficulties).

conjecture:

> Let $\hat{S}_0(g)$ be a formal power series fulfilling all the Bogoliubov-Shirkov requirements and let T be a (linear) covariant time ordering operation fulfilling (3.64)–(3.66) for arbitrary muliple direct products $\hat{C}_j$ of $\hat{S}_1$ with itself. Then there is a sequence of quasilocal operators $\hat{A}_2, \hat{A}_3, \ldots$ and a suitable extension of T for which
>
> $$\hat{S}_0(g) = T\Big(\exp\Big(\int \mathrm{d}x \hat{S}_1(x) g(x) \; + \; \hat{A}(g)\Big)\Big) \,, \quad \text{where}$$
> $$\hat{A}(g) \stackrel{\text{def}}{=} \sum_{\nu=2}^{\infty} \int \hat{A}_\nu(x_1, \ldots, x_\nu) g(x_1) \cdots g(x_\nu) \, \mathrm{d}x_1 \cdots \mathrm{d}x_\nu \,,$$
>
> holds in the sense of formal power series in g.

The transition

$$\int \hat{S}_1(x) g(x) \, \mathrm{d}x \longrightarrow \int \hat{S}_1(x) g(x) \, \mathrm{d}x + \hat{A}(g)$$

is to be considered as ***renormalization*** from the Bogoliubov-Shirkov point of view. Without correct choice of the ***counter terms*** $\hat{A}(g)$ transition to the ***adiabatic limit*** ($g(x) \longrightarrow 1$) will typically not be possible (recall Exercise 52).

3.3.3 Generalization to Nonlocalizable Test Spaces

As already pointed out in 2.2.2, there is no physical justification for the technical requirement that $\hat{\Phi}_0(g)$ be defined for all **tempered** g. Therefore other test spaces should be taken under consideration. Convenient families of test spaces, parameterized by $s \geq 0$, are the following:[53]

$$\begin{aligned} S^s(\mathbb{R}^{4n}) &\stackrel{\text{def}}{=} \bigcup_{A>0} S^{s,A} && \text{(inductive limit)}\,, \\ J^s(\mathbb{R}^{4n}) &\stackrel{\text{def}}{=} \bigcap_{A>0} S^{s,A} && \text{(projective limit)}\,, \end{aligned}$$

where

$$S^{s,A}(\mathbb{R}^{4n}) \stackrel{\text{def}}{=} \left\{ \varphi \in \mathcal{S}(\mathbb{R}^{4n}) : \|\varphi\|^s_{A,N} < \infty \; \forall \, N \in \mathbb{N} \right\} \,,$$

the topology of $S^{s,A}(\mathbb{R}^{4n})$ being given by the family of norms[54]

$$\|\varphi\|^s_{A,N} \stackrel{\text{def}}{=} \sup_{\check{x} \in \mathbb{R}^{4n}} (1 + \|\check{x}\|)^N \sup_{\check{\alpha} \in \mathbb{Z}^{4n}_+} (A + \frac{1}{N})^{-|\check{\alpha}|} \check{\alpha}^{-s\check{\alpha}} \left|\varphi^{(\check{\alpha})}(\check{x})\right| \,.$$

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[53] Efimov (see (Efimov, 1968) and references given there) used slightly different test spaces.

[54] We write

$$\check{x} = (x_1, \ldots, x_n) \,, \quad \|\check{x}\| = \sqrt{\sum_{\nu=1}^{n} \sum_{\mu=0}^{3} (x^\mu_\nu)^2} \,, \quad \check{p}\check{x} = \sum_{\nu=1}^{n} \sum_{\mu=0}^{3} p^\mu_\nu x^\mu_\nu$$

Note that

$$s_1 < s_2 \implies S^{s_1} \preccurlyeq S^{s_2}\,,$$

where $S^{s_1} \preccurlyeq S^{s_2}$ means that S^{s_1} is contained in S^{s_2} as a set and that the topology of S^{s_1} is finer than that induced by S^{s_2}.

The elements of $S^s(\mathbb{R}^{4n})$ resp. $J^s(\mathbb{R}^{4n})$ may be characterized by their Fourier transforms

$$\tilde\varphi(\check p) \stackrel{\text{def}}{=} (2\pi)^{-2n} \int \varphi(\check x)\, e^{+i\check p\check x}\, \mathrm{d}\check x$$

as follows:

$$\begin{aligned}\varphi \in S^s(\mathbb{R}^{4n}) &\iff \left(\exists\, A>0:\ \|\tilde\varphi\|_s^{A,N} < \infty\ \forall\, N\in\mathbb{N}\right),\\ \varphi \in J^s(\mathbb{R}^{4n}) &\iff \left(\forall\, A>0:\ \|\tilde\varphi\|_s^{A,N} < \infty\ \forall\, N\in\mathbb{N}\right).\end{aligned}$$

Here

$$\|\tilde\varphi\|_s^{A,N} \stackrel{\text{def}}{=} \sup_{\check p\in\mathbb{R}^{4n}}\ \sup_{\substack{\check\alpha\in\mathbb{Z}_+^{4n}\\ |\check\alpha|\le N}} g_s\left((A+\frac{1}{N})^{-1}\,\|\check p\|\right)\left|\tilde\varphi^{(\check\alpha)}(\check p)\right|,$$

where

$$g_s(t) \stackrel{\text{def}}{=} \sup_{\mu\in\mathbb{Z}_+} \mu^{-s\mu}\,|t|^\mu \quad \text{for } t\in\mathbb{R}^1\,.$$

Note that

$$e^{-\frac{se}{2}}\, e^{\frac{s}{e}|t|^{1/s}} \le g_s(|t|) \le e^{\frac{s}{e}|t|^{1/s}} \quad \forall\, t\in\mathbb{R}^1\,,$$

if $s>0$, while

$$g_0(t) = \begin{cases} 1 & \text{for } |t|\le 1\,.\\ \infty & \text{else}\,.\end{cases}$$

This implies that $S^s(\mathbb{R}^{4n})$ contains only entire analytic functions if $s<0$ and that

$$\varphi\in S^0(\mathbb{R}^{4n}) \iff \tilde\varphi\in\mathcal{D}(\mathbb{R}^{4n}) \quad , \qquad J^0(\mathbb{R}^4) = \emptyset\,.$$

Moreover, the elements φ of $S^1(\mathbb{R}^{4n})$ are seen to be analytic in a complex neighborhood of $\mathbb{R}^{4n}$ (depending on φ). Finally, as shown by Roumieu (Roumieu, 1960; Roumieu, 1963) (see also (Gelfand and Schilow, 1962, Kapitel IV)):

$$s>1 \implies S^s(\mathbb{R}^{4n})\cap\mathcal{D}(\mathbb{R}^{4n}) \text{ dense in } S^s(\mathbb{R}^{4n})\,.$$

and use standard multi-index notation:

$$|\check\alpha| = \sum_{\nu=1}^{n}\sum_{\mu=0}^{3}\alpha_\nu^\mu\,,\quad \check\alpha^{\check\beta} = \prod_{\nu=1}^{n}\prod_{\mu=0}^{3}(\alpha_\nu^\mu)^{\beta_\nu^\mu}\,,\quad \check\alpha^{-s\check\beta} = \left(\check\alpha^{\check\beta}\right)^{-s},$$

$$\varphi^{(\check\alpha)}(\check x) = D_{\check x}^{\check\alpha}\varphi(\check x)\,,\quad D_{\check x}^{\check\alpha} = \prod_{\nu=1}^{n}\prod_{\mu=0}^{3}\left(\frac{\partial}{\partial x_\nu^\mu}\right)^{\alpha_\nu^\mu}.$$

Therefore the standard definition of *support* may be applied to generalized functions on $S^s(\mathbb{R}^{4n})$ if and only if $s > 1$. For the spaces $J^s(\mathbb{R}^{4n})$ the situation is quite similar.

For $s < 1$, if one wants to test by functions $\varphi \in S^s(\mathbb{R}^{4n})$ whether a given $F \in S^s(\mathbb{R}^{4n})'$ is 'localized' within a closed subset M of $\mathbb{R}^{4n}$, there is essentially only one possibility:

Check whether

$$F(\varphi_\nu) \underset{\nu\to\infty}{\longrightarrow} 0$$

holds for every sequence $\varphi_1, \varphi_2, \ldots$ with

$$\varphi_\nu \underset{\nu\to\infty}{\overset{\text{suitably}}{\longrightarrow}} 0 \quad \text{on } M\,.$$

The questions is just how to specify what 'suitably' should mean, here.

Let us interpret 'suitably' as 'S^s-like' in the sense that all the φ_ν are elements of $S^s(\mathbb{R}^{4n})$ and[55]

$$\left(\varphi_\nu \underset{\nu\to\infty}{\overset{S^s\text{-like}}{\longrightarrow}} 0 \quad \text{on } M\right) \overset{\text{def}}{\iff} \left(\exists A > 0 : \|\varphi_\nu\|^{s,M}_{A,N} \underset{\nu\to\infty}{\longrightarrow} 0 \;\forall\, N \in \mathbb{N}\right),$$

where[56]

$$\|\varphi\|^{s,M}_{A,N} \overset{\text{def}}{=} \sup_{\check{x}\in M} (1 + \|\check{x}\|)^N \sup_{\check{\alpha}\in\mathbb{Z}^{4n}_+} (A + \frac{1}{N})^{-|\check{\alpha}|} \check{\alpha}^{-s\check{\alpha}} \left|\varphi^{(\check{\alpha})}(\check{x})\right|\,.$$

Let $F \in S^s(\mathbb{R}^{4n})'$ and $M = \overline{M} \subset \mathbb{R}^{4n}$. Then M is called a ***quasi-support*** *of F with respect to* $S^s(\mathbb{R}^{4n})$ if and only if

$$\left(\varphi_\nu \underset{\nu\to\infty}{\overset{S^s\text{-like}}{\longrightarrow}} 0 \quad \text{on } M\right) \implies F(\varphi_\nu) \underset{\nu\to\infty}{\longrightarrow} 0$$

holds for arbitrary $\varphi_\nu \in S^s(\mathbb{R}^{4n})$. A neutral scalar quantum field $\hat{\Phi}(x)$ with domain D is called ***essentially local*** *with respect to* $S^s(\mathbb{R}^{4n})$ if and only if $\{(x, y) \in \mathbb{R}^4\times\mathbb{R}^4 : x - y \in \overline{V}\}$ is a quasi-support of $\left\langle \Psi_1 \mid [\hat{\Phi}(x), \hat{\Phi}(y)]_- \Psi_2 \right\rangle$ with respect to $S^s(\mathbb{R}^{4n})$ for all $\Psi_1, \Psi_2 \in D$.

For $s > 1$ every quasi-support w.r.t. S^s contains the (ordinary) support as a subset, whereas for $s < 1$:

$$\left.\begin{array}{l} M_1 \text{ quasi-support of } F \\ \exists \epsilon > 0 : \; M_1 \subset U_\epsilon(M_2) \end{array}\right\} \implies M_2 \text{ quasi-support of } F\,.$$

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[55] Efimov required the $\varphi_1, \varphi_2, \ldots$ to be what he called *projecting sequences with support M* (see (Alebastrov and Efimov, 1974) and references given there). This especially means that the analytic continuations $\varphi_\nu(\check{z})$ of the $\varphi_\nu(\check{x})$ have to converge uniformly to zero in the region $M + i\mathbb{R}^{4n}$ and uniformly to one in every region of the form $K + i\mathbb{R}^{4n}$ with K a compact subset of $\mathbb{R}^{4n} \setminus M$.

[56] Note that $\|\varphi\|^{s,\mathbb{R}^{4n}}_{A,N} = \|\varphi\|^s_{A,N}$, so that every element of $S^s(\mathbb{R}^{4n})'$ is localized in $\mathbb{R}^{4n}$, at least.

This is why we speak about 'quasi'-supports. For instance, $\{0\}$ is a quasi-support of $F(x) = e^{-\|x\|^2}$ with respect to $S^{1/4}(\mathbb{R}^4)$. Thus, obviously, a quasi-support is not a domain of **strict** localization, in general. Nevertheless the *PCT* theorem and the spin-statistics theorem could be proved for fields which are only essentially local[57] with respect to $S^0(\mathbb{R}^4)$ (see (Lücke, 1984; Lücke, 1986)).

Now it is evident how to generalize the Bogoliubov-Shirkov theory:

Formulate everything – except Bogoliubov-Shirkov causality – with S^s instead of $\mathcal{S}$ and replace Bogoliubov-Shirkov causality by the requirement of ***generalized Bogoliubov-Shirkov causality***:[58]

$$\boxed{\begin{array}{l}\{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x - y \in \overline{V_+}\} \text{ is a quasi-support} \\ \text{of } \dfrac{\delta}{\delta g(x)} \left(\dfrac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right)_{|g=0} \quad \text{with respect to } S^s(\mathbb{R}^8)\,.\end{array}}$$

Naturally, we call $M = \overline{M}$ a quasi-support of $F \in J^s(\mathbb{R}^{4n})'$ with respect to $J^s(\mathbb{R}^{4n})$ if and only if

$$\left(\varphi_\nu \xrightarrow[\nu \to \infty]{J^s\text{-like}} 0 \quad \text{on } M \right) \implies F(\varphi_\nu) \xrightarrow[\nu \to \infty]{} 0\,,$$

where

$$\left(\varphi_\nu \xrightarrow[\nu \to \infty]{J^s\text{-like}} 0 \quad \text{on } M \right) \stackrel{\text{def}}{\iff} \left(\forall\, A, N > 0 : \|\varphi_\nu\|^{s,M}_{A,N} \xrightarrow[\nu \to \infty]{} 0 \right)\,.$$

Now, one would like to have a convenient criterion for $\overline{V_+}$ being a quasi-support. For tempered distributions we have the following.

Theorem 3.3.1 *Let $\mathcal{L}(p + iq)$ be an analytic function on the* tube $\mathbb{R}^4 + iV_+$ *having the following property:*

For every $\eta \in V_+$ there is a polynomial P_η for fulfilling

$$|\mathcal{L}\,(p + i(\eta + a))| \leq |P_\eta(p + ia)| \quad \forall\, p \in \mathbb{R}^4\,,\; a \in V_+\,.$$

Then there is a tempered distribution $F(x)$ with

$$F(\varphi) = \lim_{\epsilon \to +0} \int \mathcal{L}\,(p + i(\epsilon, 0, 0, 0))\, \tilde{\varphi}(-p)\, \mathrm{d}p \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^4)\,.$$

For this distribution

$$\operatorname{supp} F \subset \overline{V_+}\,.$$

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[57] In spite of the – generally misinterpreted – result of (Borchers and Pohlmeyer, 1968) there are examples of **nonlocal** tempered fields which are essentially local w.r.t. $S^0(\mathbb{R}^4)$, as shown in (Bümmerstede and Lücke, 1975, Sect. 5).

[58] We choose the spaces J^s since we are primarily interested in the case $s = 1$, here. For the general study of axiomatic field theory the case $s = 0$ was the real challenge.

Conversely, for every tempered distribution with such support there is a unique function $\mathcal{L}$ of the type specified above, namely the Laplace transform

$$\mathcal{L}(p+iq) = (2\pi)^{-2} \int F(x)\, e^{+i(p+iq)x}\, \mathrm{d}x\,.$$

Proof: See (Streater and Wightman, 1989, Theorem 2.8). ∎

An immediate consequence of Theorem 3.3.1 is

$$\operatorname{supp} \Delta_{\mathrm{ret}}(x) \subset \overline{V_+}$$

for the ***retarded commutator***

$$\Delta_{\mathrm{ret}}(x) \stackrel{\mathrm{def}}{=} \lim_{\epsilon\to+0} (2\pi)^{-4} \int \frac{e^{-ipx}}{(p^0+i\epsilon)^2 - \mathbf{p}^2 - m^2}\, \mathrm{d}p\,.$$

Note that, by Cauchy's integral theorem,

$$\Delta_{\mathrm{F}}(x) = \Delta_{\mathrm{ret}}(x) - \Delta_-(x)$$

for $\Delta_{\mathrm{F}}(x)$ as defined in (3.28) and $\Delta_-(x)$ as defined in (3.30)/(3.31).

Exercise 59 Let $\{\mu_\alpha\}_{\alpha\in\mathbb{Z}^4_+}$ be a family of complex-valued Borel measures μ_α on $\mathbb{R}^4$ and assume

$$\sup_{\alpha\in\mathbb{Z}^4_+} A^{-|\alpha|}\alpha^{+s\alpha} \int (1+\|x\|)^{-A}\, |\mu_\alpha|(\mathrm{d}x) < \infty \quad \text{for } A>0 \text{ large enough}\,,$$

where $|\mu|$ denotes the ***variation*** of the complex-valued measure μ :

$$|\mu|(E) \stackrel{\mathrm{def}}{=} \sup\left\{ \left| \int_E f\, \mathrm{d}\mu \right| \,:\, f\,:\, E\to\mathbb{C} \text{ measurable and } |f|\le 1 \right\} \tag{3.70}$$

(compare (Halmos, 1950, p. 124)). Show[59] that

$$F(x) = \sum_{\alpha\in\mathbb{Z}^4_+} D^\alpha_x \mu_\alpha(x)$$

converges in the weak topology of $J^s(\mathbb{R}^4)'$ and that

$$M = \bigcap_{\alpha\in\mathbb{Z}^4_+} \operatorname{supp} \mu_\alpha$$

is a quasi-support of F with respect to $J^s(\mathbb{R}^4)$.

[59]See (Lücke, 1984) for a proof of the converse of the corresponding statement for $S^s(\mathbb{R}^4)$.

In the nonlocalizable case, a useful substitute of Theorem 3.3.1 is the following.

Theorem 3.3.2 *Let $s>0$, let $\mathcal{L}(p+iq)$ be an analytic function on $\mathbb{R}^4+iV_+$, and assume that*

$$\sup_{p+iq\in\mathbb{R}^4+iV_+} \left(\|q\|\,\mathrm{dist}\,(q,\partial V_+)\right)^A e^{-\left(\frac{\|p\|+\|q\|}{A}\right)^{1/s}} |\mathcal{L}(p+iq)| < \infty$$

holds for sufficiently large $A>0$. Then

$$\tilde{F}(p) = \lim_{\epsilon\to+0} \mathcal{L}(p+iq)$$

converges in the weak topology of $\widetilde{J^s}(\mathbb{R}^4)'$. Moreover, every closed cone with apex at $x=0$ and containing $\overline{V_+}\setminus\{0\}$ in its interior is a quasi-support[60] *of $F(x)$ with respect to $J^s(\mathbb{R}^4)$.*

Proof: See (Fainberg and Soloviev, 1992, Theorem 4). ∎

Moreover the following result is useful:

Theorem 3.3.3 *Let*[61] *$s>0$ and let $\{x\in\mathbb{R}^4 : x^0\geq 0\}$ be a quasi-support (w.r.t. $J^s(\mathbb{R}^4)$) of $F(x)\in J^s(\mathbb{R}^4)'$. If $F(x)$ is Lorentz invariant then also $\overline{V_+}$ is a quasi-support of $F(x)$.*

Proof: See (Bümmerstede, 1976, Theorem 4.7). ∎

Let $\mathcal{V}(t)$ be an entire analytic function of ***order*** $\frac{1}{2s}$, i.e.:

$$\exists\rho>0:\ \sup_{z\in\mathbb{C}^1} e^{-\rho|z|^{\frac{1}{2s}}} |\mathcal{V}(z)| < \infty\,. \tag{3.71}$$

Then, by Theorems 3.3.2 and 3.3.3,

$$\Delta^{\mathcal{V}}_{\mathrm{ret}}(x) \stackrel{\mathrm{def}}{=} \lim_{\epsilon\to+0}(2\pi)^{-4}\int \frac{\mathcal{V}\left(\frac{p^2}{m^2}\right)e^{-ipx}}{(p^0+i\epsilon)^2-\mathbf{p}^{\,2}-m^2}\,\mathrm{d}p \tag{3.72}$$

is an element of $J^s(\mathbb{R}^4)$ with quasi-support $\overline{V_+}$. If, in addition,

$$\mathcal{V}(1)=1 \tag{3.73}$$

[60] Presumably, $\overline{V_+}$ itself is a quasi-support of $F(x)$ — as obvious for $s>1$. A generalization of Theorem 3.3.2 for $s=0$ was proved in (Soloviev, 1997).

[61] Actually, Theorem 3.3.3 is also valid for $s=0$.

then also

$$\begin{aligned}\Delta_{\mathrm{F}}^{\mathcal{V}}(x) &\stackrel{\text{def}}{=} \Delta_{\text{ret}}^{\mathcal{V}}(x) - \Delta_{-}(x) \\ &= \lim_{\epsilon\longrightarrow +0} (2\pi)^{-4} \int \frac{\mathcal{V}\left(\frac{p^2}{m^2}\right) e^{-ipx}}{p^2 - m^2 - i\epsilon}\, \mathrm{d}p\end{aligned} \tag{3.74}$$

is a well defined element of $J^s(\mathbb{R}^4)$. If $\mathcal{V}(t)$ is even of sufficiently fast decrease, e.g.

$$\sup_{t\in\mathbb{R}^1} t^2 \left|\mathcal{V}(t)\right| < \infty\,, \tag{3.75}$$

then there is a 'canonical' definition for powers of the modified Feynman propagator $\Delta_{\mathrm{F}}^{\mathcal{V}}(x)$ using ***Feynman parameterization***

$$\begin{aligned}\frac{1}{c_1\cdots c_n} &= (n-1)! \int_0^1 \mathrm{d}\xi_1 \cdots \int_0^1 \mathrm{d}\xi_n \frac{\delta(1-\xi_1-\ldots-\xi_n)}{(c_1\xi_1+\ldots c_n\xi_n)^n} \\ &\quad \text{if } 1 + c_1\xi_1 + \ldots + c_n\xi_n \neq 0\ \forall \xi_1,\ldots,\xi_n \geq 0\end{aligned} \tag{3.76}$$

(see (Alebastrov and Efimov, 1973, Sect. 4.3)).

Proof of (3.76):

$$\begin{aligned}\frac{1}{n!c_1\cdots c_n} &= \int_0^\infty \mathrm{d}\xi_1 \cdots \int_0^\infty \mathrm{d}\xi_n \frac{1}{(1+c_1\xi_1+\ldots+c_n\xi_n)^{n+1}} \\ &= \int_0^\infty \mathrm{d}\xi_1 \cdots \int_0^\infty \mathrm{d}\xi_n \int_0^\infty \mathrm{d}\lambda \frac{\delta(\lambda-\xi_1-\ldots-\xi_n)}{(1+c_1\xi_1+\ldots+c_n\xi_n)^{n+1}} \\ &= \int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\xi_1' \cdots \int_0^\infty \mathrm{d}\xi_n' \frac{\delta\left((1-\xi_1'-\ldots-\xi_n')\lambda\right)}{(1+(c_1\xi_1'+\ldots+c_n\xi_n')\lambda)^{n+1}}\lambda^n \\ &= \int_0^\infty \mathrm{d}\xi_1' \cdots \int_0^\infty \mathrm{d}\xi_n' \int_0^\infty \mathrm{d}\lambda \frac{\delta(1-\xi_1'-\ldots-\xi_n')}{(\lambda^{-1}+c_1\xi_1'+\ldots+c_n\xi_n')^{n+1}}\lambda^{-2} \\ &= \int_0^\infty \mathrm{d}\xi_1' \cdots \int_0^\infty \mathrm{d}\xi_n' \int_0^\infty \mathrm{d}\lambda' \frac{\delta(1-\xi_1'-\ldots-\xi_n')}{(\lambda'+c_1\xi_1'+\ldots+c_n\xi_n')^{n+1}} \\ &= \frac{1}{n}\int_0^1 \mathrm{d}\xi_1' \cdots \int_0^1 \mathrm{d}\xi_n' \frac{\delta(1-\xi_1'-\ldots-\xi_n')}{(c_1\xi_1'+\ldots+c_n\xi_n')^n} \qquad \blacksquare\end{aligned}$$

Therefore, if we denote by $T_{\mathcal{V}}$ the time-ordering resulting by replacing – after formal application of Wick's theorem – the products of the ordinary Feynman propagator $\Delta_{\mathrm{F}}(x)$ by the 'canonical' products of the modified Feynman propagator $\Delta_{\mathrm{F}}^{\mathcal{V}}(x)$,

$$\hat{S}_0^{\mathcal{V}}(g) \stackrel{\text{def}}{=} T^{\mathcal{V}}\left(\exp\left(\int \hat{S}_1(x)\,g(x)\,\mathrm{d}x\right)\right) \quad \text{for } g \in J^s(\mathbb{R}^4) \tag{3.77}$$

fulfills all the requirements of the generalized Bogoliubov-Shirkov theory.[62]

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[62]Generalized Bogoliubov-Shirkov causality can be proved by Theorem 3.3.2 and the Alebastrov-Efimov analysis of $\frac{\delta}{\delta g(x)}\left(\frac{\delta \hat{S}_0(g)}{\delta g(y)}\hat{S}_0(g)^*\right)_{|g=0}$ in (Alebastrov and Efimov, 1974, Sect. 7). Of course, counterterms have still to be added if the adiabatic limit $g \longrightarrow 1$ is to exist.

Unfortunately, as a consequence of the Phragmén-Lindelöf theorem (see, e.g., Theorem 4.1.1 of (Lücke, ftm)), (3.71) and (3.75) are incompatible with each other for $s \geq 1/2$. For $s < 1/2$ there are plenty of entire functions $\mathcal{V}(t)$ fulfilling both (3.71) and (3.75) (see (Gelfand and Schilow, 1962, Kap. IV §8)). This is why Efimov (see (König, 1993) and references given there) suggested to use test spaces of entire functions in order to 'regularize' the perturbative expansion of the S-matrix.

For $s = 1$ we may still have

$$\sup_{t \in \mathbb{R}_+} t^2 \mathcal{V}(-t) < \infty \tag{3.78}$$

in addition to (3.73). Take

$$\mathcal{V}(t) = \left(\frac{\sin \sqrt{-t}}{\sqrt{-t}} \right)^4 ,$$

for instance. Now, the expressions resulting from the 'canonical' definition of powers of $\Delta_{\mathrm{F}}^{\mathcal{V}}(x)$ – thanks to transition to Euclidean momenta – are well defined, if only (3.78) holds. Efimov used this for the **definition** of 'regularized' solutions of the generalized Bogoliubov-Shirkov theory[63] for $J^1(\mathbb{R}^4)$ – motivated by suitable quantization of formally nonlocal field theories (Efimov, 1974).

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[63]For generalization to gauge theories see, e.g., (Moffat, 1990; Cornish, 1992) and references given there.

Chapter 4

Quantum Electrodynamics

4.1 The Free Electromagnetic Field Operators

4.1.1 Wightman Theory

Axioms

The ***Wightman axioms*** for the free electromagnetic field $\hat{F}^{\mu\nu}(x)$ follow from those for the neutral scalar field by straightforward adaption:

0. Assumptions of Relativistic Quantum Theory:

The same as in 2.2.1.

I. Assumptions about the Domain and Continuity of the Fields:

The field operators $\hat{F}^{\mu\nu}(x)$ are ***hermitian operator-valued, tempered generalized functions with invariant domain*** $D \subset \mathcal{H}$; i.e. **linear** mappings

$$\begin{array}{rcl} \hat{F}^{\mu\nu} \,:\, \mathcal{S}(\mathbb{R}^4) & \longrightarrow & L(D,D) \\ \varphi & \longmapsto & \hat{F}^{\mu\nu}(\varphi) = \underbrace{\int \hat{F}^{\mu\nu}(x)\varphi(x)\,\mathrm{d}x}_{\text{formal}} \end{array}$$

for which all the

$$\int \left\langle \Psi \mid \hat{F}^{\mu\nu}(x)\Psi \right\rangle \varphi(x)\,\mathrm{d}x \stackrel{\text{def}}{=} \left\langle \Psi \mid \hat{F}^{\mu\nu}(\varphi)\Psi \right\rangle \,,\Psi \in D\,,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a,\Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \in D\,, \quad \hat{U}(a,\Lambda)D \subset D\,, \quad \hat{F}^{\mu\nu}(\varphi)D \subset D\,, \quad \hat{F}^{\mu\nu}(\overline{\varphi}) = \hat{F}^{\mu\nu}(\varphi)^* \restriction D\,.$$

II. Transformation Law of the Fields:

The fields transform according to

$$\hat{U}(a,\Lambda)^{-1}\hat{F}^{\mu\nu}(x)\hat{U}(a,\Lambda) = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\hat{F}^{\alpha\beta}\left(\Lambda^{-1}(x-a)\right) \quad \forall\,(a,\Lambda)\in\mathcal{P}^{\uparrow}_{+}\,.$$

Remark: Recall that $\hat{U}(a,\Lambda)\Psi$ is to be described w.r.t. the coordinates $x' = \Lambda^{-1}(x-y)$ in exactly the same way as Ψ is to be described w.r.t. the coordinates x and that the field expectation values should transform like the classical fields (see, e.g., Eq. (2.9) of (Lücke, edyn). Therefore:

$$\left\langle \hat{U}(a,\Lambda)\Psi \mid \hat{F}^{\mu\nu}\left(\Lambda x + a\right)\hat{U}(a,\Lambda)\right\rangle = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\left\langle \Psi \mid \hat{F}^{\alpha\beta}(x)\,\Psi\right\rangle\,.$$

III. Local Commutativity (Microscopic Causality):

The smeared fields $\hat{F}^{\mu\nu}(\varphi_1)$, $\hat{F}^{\alpha\beta}(\varphi_2)$ commute whenever the supports of the test functions φ_1, $\varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other. Formally:

$$x \times y \implies [\hat{F}^{\mu\nu}(x), \hat{F}^{\alpha\beta}(y)]_- = 0\,.$$

Again, the vacuum vector Ω is required to be **cyclic** with respect to the algebra $\mathcal{F}_0$ generated by $\hat{1}\bigwedge D$ and the smeared field operators $\hat{F}^{\mu\nu}(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$, $\mu,\nu \in \{0,1,2,3\}$. Finally, the field operators have to fulfill the free Maxwell equations:[1]

$$\partial_\nu \hat{F}^{\mu\nu}(x) = 0\,, \tag{4.1}$$

$$\partial^\nu \epsilon_{\mu\nu\alpha\beta}\hat{F}^{\alpha\beta}(x) = 0\,. \tag{4.2}$$

Essential Uniqueness of the Wightman Theory

Using the results of (Oksak and Todorov, 1969) and (Pohlmeyer, 1969) one man prove[2] the following variant of the Jost-Schroer theorem (Theorem 2.2.18):

Theorem 4.1.1 *The **Wightman theory of the free electromagnetic field**, as described above*[3] *is unique up to unitary equivalence and up to some common constant factor of the field operators, if D is chosen to be the smallest linear subspace of $\mathcal{H}$ containing Ω and being invariant under all the smeared field operators.*

A realization of the Wightman theory of the free electromagnetic field will be given in 4.1.3.

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[1] As usual, we define

$$\epsilon_{\mu\nu\alpha\beta} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is an even permutation of } (0,1,2,3)\,, \\ -1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is an odd permutation of } (0,1,2,3)\,, \\ 0 & \text{else}\,. \end{cases}$$

[2] See also (Fredenhagen, 2001, III.4).

[3] As pointed out by D. Buchholz (private communication) it is sufficient to require the validity of the wave equation for every component of the field tensor rather than the full set of Maxwell equations.

Vacuum Fluctuations

Although we will see that

$$\left\langle \Omega \mid \hat{F}^{\mu\nu}(x)\Omega \right\rangle = 0\,,$$

as to be expected, cyclicity of Ω implies that the 2-point functions

$$\left\langle \Omega \mid \hat{F}^{\mu\nu}(x)\hat{F}^{\mu\nu}(y)\Omega \right\rangle$$

cannot all vanish.[4] Therefore the variance of the field strength is nonzero already in the vacuum state. This is due to ***vacuum fluctuations***, which also cause *spontaneous emission* of photons from atoms in excited states (see, e.g. (Baym, 1969, S. 278 ff.)) and may be experimentally verified by the so-called *Casimir effect* (see, e.g. (Itzykson and Zuber, 1980a, Section 3-2-4)).

The permanent presence of perturbations like those connected with vacuum fluctuations is one of the main problems of quantum field theory.

4.1.2 Problems With the Quantized Potentials

Desirable Properties

Just as in classical electrodynamics it is convenient – and for coupling to the charged matter fields also necessary – to introduce potentials $\hat{A}^\mu(x)$ which we should like to fulfill the following requirements:[5]

(i) The $\hat{\mathbf{A}}(\mathbf{x}, t)$ are operator-valued tempered generalized functions with invariant dense domain $D_A \subset \mathcal{H}$.

(ii) There is a nondegenerate[6] continuous sesquilinear form $(.\mid .)$ on $\mathcal{H}$ w.r.t. which

[4]This fact implies interesting restrictions on the joint measurability of different field components (Bohr and Rosenfeld, 1933) — a necessary supplement to the Heisenberg uncertainty relations of ordinary quantum mechanics.

[5]See also (Strocchi, 1977) and (Lücke, edyn, Sect. 2.3.2). Of course, it would be nice to have $D_A = D_F = D$ and $(.\,|) = \langle .\mid .\rangle$. By Strocchi's Theorem (Theorem 4.1.2), however, this is not possible. While it is easy to see that quantized potentials can always be constructed in some bigger space with indefinite metric – even in the noninteracting case (Bongaarts, 1977) – there are almost no physical hints on which properties can be expected for such auxiliary field operators.

[6]In general a quadratic form $(.\mid .)$ on $\mathcal{H}\times\mathcal{H}$ is called ***nondegenerate*** if the mapping

$$\mathcal{H} \ni \Psi \longmapsto (\Psi \mid .) \in \mathcal{H}'$$

is a bijection. For Hilbert spaces $\mathcal{H}$ this is equivalent to:

$$(\Phi \mid \Phi') = 0\;\forall\,\Phi' \in \mathcal{H} \quad\Longrightarrow\quad \Phi = 0\,.$$

the $\hat{\mathbf{A}}(\mathbf{x},t)$ are hermitian:[7]

$$\left(\Phi_1 \mid \hat{\mathbf{A}}(\mathbf{x},t)\Phi_2\right) = \left(\hat{\mathbf{A}}(\mathbf{x},t)\Phi_1 \mid \Phi_2\right) \ \forall \Phi_1,\Phi_2 \in D_A\,. \tag{4.3}$$

(iii) There is a representation $\hat{V}(a,\Lambda)$ of $\mathcal{P}_+^\uparrow$ with

$$D_{\hat{V}(a,\Lambda)} = R_{\hat{V}(a,\Lambda)} = D_A \tag{4.4}$$

which is strongly continuous w.r.t. $\langle . \mid . \rangle$, unitary w.r.t. $(. \mid .)$, i.e.

$$\left(\Phi_1 \mid \hat{V}(a,\Lambda)\Phi_2\right) = \left(\hat{V}(a,\Lambda)^{-1}\Phi_1 \mid \Phi_2\right) \quad \forall\, \Phi_1,\Phi_2 \in D_A\,, \tag{4.5}$$

and transforming the $\hat{\mathbf{A}}(\mathbf{x},t)$ according to

$$\hat{V}(a,\Lambda)^{-1}\hat{\mathbf{A}}(\mathbf{x},t)\hat{V}(a,\Lambda) = (\Lambda_A)^\mu{}_\nu\, \hat{A}^\nu\left(\Lambda_A^{-1}(x-a)\right)\,. \tag{4.6}$$

(iv) There is a vector $\Omega \in D_A$, **unique** up to a phase factor, fulfilling

$$(\Omega \mid \Omega) = 1 \quad \text{and} \quad \hat{V}(a,\Lambda)\Omega = \Omega \quad \forall\,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,. \tag{4.7}$$

(v)

$$x \times y \implies \left[\hat{\mathbf{A}}(\mathbf{x},t), \hat{A}^\nu(y)\right]_- = 0\,. \tag{4.8}$$

(vi)

$$\Box\, \hat{A}^\nu(x) = 0 \quad \forall\,\nu \in \{0.\ldots,3\}\,. \tag{4.9}$$

(vii) There is a linear subspace D_F of D_A fulfilling the conditions

$$\left(\Phi_1 \mid \partial_\mu \hat{\mathbf{A}}(\mathbf{x},t)\Phi_2\right) = 0 \quad \forall\,\Phi_1,\Phi_2 \in D_F\,, \tag{4.10}$$

$$\hat{F}^{\mu\nu}_{\hat{A}}(x) D_F \subset D_F \supset \hat{V}(a,\Lambda) D_F \quad \forall\,\mu,\nu \in \{0,\ldots,3\}\,,(a,\Lambda) \in \mathcal{P}_+^\uparrow\,, \tag{4.11}$$

and

$$\Omega \in D_F \quad (\subset D_A \subset \mathcal{H})\,. \tag{4.12}$$

where

$$\hat{F}^{\mu\nu}_{\hat{A}}(x) \stackrel{\text{def}}{=} \partial^\mu \hat{A}^\nu(x) - \partial^\nu \hat{\mathbf{A}}(\mathbf{x},t)\,, \tag{4.13}$$

(viii)

$$\int \left(\Phi_1 \mid \hat{V}(a,\mathbb{1}_4)\Phi_2\right) e^{-ip^\mu a_\mu} \mathrm{d}a = 0 \quad \forall\, p \in \mathbb{R}^4 \setminus \overline{V_+}\,,\ \Phi_1,\Phi_2 \in D_A\,. \tag{4.14}$$

[7] Of course, we should like to take $(. \mid .) = \langle . \mid . \rangle$. However, as will be shown in Corollary 4.1.4, this would be in contradiction to the other assumptions made below.

(ix)
$$\overline{\mathcal{Z}_A}^{\langle.|.\rangle} = \mathcal{H}\,,$$
where $\mathcal{Z}_A$ denotes the smallest linear subspace of D_A containing Ω and being invariant under all $\hat{\mathbf{A}}(\mathbf{x},t)$.

(x) For every $\Phi \in D_F$ and every $\epsilon > 0$ there is a $\Phi' \in \mathcal{Z}_F$ with
$$(\Phi - \Phi' \mid \Phi - \Phi') < \epsilon\,,$$
where $\mathcal{Z}_F$ denotes the smallest linear subspace of D_F containing Ω and being invariant under all $\hat{F}^{\mu\nu}_{\hat{A}}(x)$.

If $(.\mid.)$ were positive definite,
$$\mathcal{H}_F \stackrel{\text{def}}{=} \overline{D_F}^{(.|.)}\,,\ \overline{\hat{V}(a,\Lambda)}^{(.|.)}\,,\ \Omega \text{ and } \hat{F}^{\mu\nu}$$
would give a Wightman theory of the free electromagnetic field.

No-Go Theorem

Theorem 4.1.2 (Strocchi) *Unless all $\hat{F}^{\mu\nu}_{\hat{A}}(x)$ vanish, conditions (i)–(x) imply*
$$\boxed{\partial_\mu \hat{F}^{\mu\nu}_{\hat{A}}(x)\Omega \neq 0\,.} \tag{4.15}$$

Proof: See (Strocchi, 1970). ∎

Lemma 4.1.3 *Let D be a (complex) linear space with positive semi-definite sesquilinear form $(.\mid.)$ and let $\hat{B}_1, \hat{B}_2 \in L(D,D)$ fulfill the condition*
$$\left(\Phi' \mid \hat{B}_1\Phi\right) = \left(\hat{B}_2\Phi' \mid \Phi\right) \quad \forall\,\Phi,\Phi' \in D\,.$$
Then
$$\hat{B}_1 D_{00} \subset D_{00} \stackrel{\text{def}}{=} \{\Phi \in D : (\Phi \mid \Phi) = 0\}\,.$$

Proof: By means of **Schwartz**' inequality (see (Strocchi and Wightman, 1974); Lemma 2.2 and application). ∎

Corollary 4.1.4 *Unless all $\hat{F}^{\mu\nu}_{\hat{A}}(x)$ vanish, conditions (i)–(x) imply that $(.\mid.)$ can be neither*[8] *positive definite on D_F nor positive semi-definite on D_A if*
$$\left(\Phi \mid \partial_\mu \hat{F}^{\mu\nu}_{\hat{A}}(x)\,\Phi\right) = 0 \quad \forall\,\Phi \in D_F\,.$$

[8]By (4.9)–(4.13) positive definiteness on D_F would imply $\partial_\mu \hat{F}^{\mu\nu}_{\hat{A}}(x) = 0$, in contradiction to Theorem 4.1.2. The case of positive semi-definiteness on D_A may be reduced to the case of positive definiteness via factorization, by Lemma 4.1.3.

Gupta-Bleuler Generalization

Gupta (Gupta, 1950) and Bleuler (Bleuler, 1950) realized that, fortunately, it is sufficient to postulate positive semi-definiteness of $(. \mid .)$ on D_F :

$$(\Phi \mid \Phi) \geq 0 \quad \forall \Phi \in D_F \,. \tag{4.16}$$

Then assumptions (i)–(ix) guarantee that factorization[9] of $D_F, \hat{V}(a,\Lambda)\bigwedge D_F$ and $\hat{F}^{\mu\nu}_{\hat{A}}(x)\bigwedge D_F$ by

$$D_{00} \stackrel{\text{def}}{=} \{\Phi \in D_F : \, (\Phi \mid \Phi) = 0\} \,, \tag{4.17}$$

gives a Wightman theory of the free electromagnetic field (compare last part of 4.1.3).

4.1.3 Gupta-Bleuler Construction

Quantized Electromagnetic Potentials

As domain D_A for the quantized potentials we choose the set of all truncated sequences

$$A_{\text{GB}} = \{A_0, A_1, \ldots, A_n, 0, 0, \ldots\} \,,$$

where

$$A_0 \in \mathbb{C} \,, \quad A_n \mathrel{\widehat{=}} \{a^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1, \ldots, \mathbf{p}_n)\}_{\mu_j=0,1,2,3} \,,$$

and

$$a^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1, \ldots, \mathbf{p}_n) = a^{\mu_{\pi 1},\ldots,\mu_{\pi n}}(\mathbf{p}_{\pi 1}, \ldots, \mathbf{p}_{\pi n}) \in \mathcal{S}\left(\mathbb{R}^{3n}\right) \quad \forall \pi \in \mathrm{S}_n \,. \tag{4.18}$$

As inner product we choose

$$\left\langle A_{\text{GB}} \mid \check{A}_{\text{GB}} \right\rangle \stackrel{\text{def}}{=} \overline{A_0}\check{A}_0 + \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=0}^{3} \int \overline{a^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1, \ldots, \mathbf{p}_n)}\check{a}^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1, \ldots, \mathbf{p}_n) \frac{d\mathbf{p}_1}{2\,|\mathbf{p}_1|} \cdots \frac{d\mathbf{p}_n}{2\,|\mathbf{p}_n|} \,. \tag{4.19}$$

Then $\mathcal{H}$ is chosen to be the completion of D_A with respect to this inner product. Similarly to (2.23) resp. (2.32) we define annihilation operators $\hat{a}^\mu(\mathbf{p})$ by[10]

$$\begin{aligned} \left(\hat{a}^\mu(\mathbf{p})A_{\text{GB}}\right)_0 &\stackrel{\text{def}}{=} a^\mu(\mathbf{p}) \\ \left(\hat{a}^\mu(\mathbf{p})A_{\text{GB}}\right)^{\mu_1,\ldots,\mu_{n-1}}\Big(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}\Big) &\stackrel{\text{def}}{=} \sqrt{n}\, a^{\mu,\mu_1,\ldots,\mu_{n-1}}\Big(\mathbf{p}, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}\Big) \quad \text{for } n > 1 \end{aligned} \tag{4.20}$$

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[9] Factorization is possible thanks to Lemma 4.1.3.

[10] We use the identification $(A_{\text{GB}})^{\mu_1,\ldots\mu_\nu}(\mathbf{p}_1, \ldots, \mathbf{p}_\nu) = a^{\mu_1,\ldots\mu_\nu}(\mathbf{p}_1, \ldots, \mathbf{p}_\nu)\,.$

resp. creation operators $\hat{a}^\mu(\mathbf{p})^\dagger$ by[11]

$$\begin{aligned}
\left(\hat{a}^\mu(\mathbf{p})^\dagger A_{\mathrm{GB}}\right)_0 &\stackrel{\mathrm{def}}{=} 0\,,\\
\left(\hat{a}^\mu(\mathbf{p})^\dagger A_{\mathrm{GB}}\right)^{\mu_1}(\mathbf{p}_1) &\stackrel{\mathrm{def}}{=} -2\,|\mathbf{p}_1|\,g^{\mu\mu_1}\delta(\mathbf{p}-\mathbf{p}_1)A_0\,,\\
\left(\hat{a}^\mu(\mathbf{p})^\dagger A_{\mathrm{GB}}\right)^{\mu_1,\dots,\mu_{n+1}}(\mathbf{p}_1,\dots,\mathbf{p}_{n+1}) &\stackrel{\mathrm{def}}{=} -\sqrt{\tfrac{1}{n+1}}\sum_{j=1}^{n+1} 2\,|\mathbf{p}_j|\,g^{\mu\mu_j}\delta(\mathbf{p}-\mathbf{p}_j)\cdot\\
&\quad\cdot a^{\mu_1,\dots,\not{\mu_j},\dots,\mu_{n+1}}(\mathbf{p}_1,\dots,\not{\mathbf{p}_j},\dots,\mathbf{p}_{n+1})\,.
\end{aligned} \tag{4.21}$$

Then, thanks to the commutation relations[12]

$$\begin{aligned}
\left[\hat{a}^\mu(\mathbf{p}),\hat{a}^{\mu'}(\mathbf{p}')^\dagger\right]_- &= -2\,|\mathbf{p}|\,g^{\mu\mu'}\delta(\mathbf{p}-\mathbf{p}')\,,\\
\left[\hat{a}^\mu(\mathbf{p}),\hat{a}^{\mu'}(\mathbf{p}')\right]_- &= \left[\hat{a}^\mu(\mathbf{p})^\dagger,\hat{a}^{\mu'}(\mathbf{p}')^\dagger\right]_- = 0
\end{aligned} \tag{4.22}$$

the quantized potentials[13]

$$\hat{A}^\mu_{\mathrm{GB}}(x) \stackrel{\mathrm{def}}{=} \hat{A}^{(+)\mu}_{\mathrm{GB}}(x) + \hat{A}^{(-)\mu}_{\mathrm{GB}}(x)\,, \tag{4.23}$$

$$\hat{A}^{(+)\mu}_{\mathrm{GB}}(x) \stackrel{\mathrm{def}}{=} \sqrt{\zeta}\,(2\pi)^{-3/2}\int_{p_0=+|\mathbf{p}|} \hat{a}^\mu(\mathbf{p})e^{-\frac{i}{\hbar}p^\mu x_\mu}\,\frac{d\mathbf{p}}{2\,|\mathbf{p}\,|}\,, \tag{4.24}$$

$$\hat{A}^{(-)\mu}_{\mathrm{GB}}(x) \stackrel{\mathrm{def}}{=} \sqrt{\zeta}\,(2\pi)^{-3/2}\int_{p_0=-|\mathbf{p}|} \hat{a}^\mu(-\mathbf{p})^\dagger e^{-\frac{i}{\hbar}p^\mu x_\mu}\,\frac{d\mathbf{p}}{2\,|\mathbf{p}\,|}\,, \tag{4.25}$$

obeying condition (i) of 4.1.2, fulfill the commutation relations[14]

$$\begin{aligned}
\left[\hat{A}^{(+)\mu}_{\mathrm{GB}}(x),\hat{A}^{(-)\mu'}_{\mathrm{GB}}(x')\right]_- &= -\zeta\,g^{\mu\mu'}i\Delta_0^{(+)}(x-x')\,,\\
\left[\hat{A}^{(+)\mu}_{\mathrm{GB}}(x),\hat{A}^{(+)\mu'}_{\mathrm{GB}}(x')\right]_- &= \left[\hat{A}^{(-)\mu}_{\mathrm{GB}}(x),\hat{A}^{(-)\mu'}_{\mathrm{GB}}(x')\right]_- = 0\,,
\end{aligned} \tag{4.26}$$

$$\left[\hat{A}^\mu_{\mathrm{GB}}(x),\hat{A}^{\mu'}_{\mathrm{GB}}(x')\right]_- = -\zeta\,g^{\mu\mu'}i\Delta_0(x-x')\,. \tag{4.27}$$

(4.27) directly implies condition (v) of 4.1.2. By (4.23)–(4.25) also condition (vi) of 4.1.2 is fulfilled. According to (4.20)–(4.25) we **have to** define

$$(A_{\mathrm{GB}}\mid A'_{\mathrm{GB}}) \stackrel{\mathrm{def}}{=} \langle A_{\mathrm{GB}}\mid \hat{\eta}A'_{\mathrm{GB}}\rangle\,, \tag{4.28}$$

where

$$\begin{aligned}
(\hat{\eta}A_{\mathrm{GB}})^{\mu_1,\dots,\mu_n}(\mathbf{p}_1,\dots,\mathbf{p}_n) &\stackrel{\mathrm{def}}{=} (-g_{\mu_1\nu_1})\cdots(-g_{\mu_n\nu_n})a^{\nu_1,\dots,\nu_n}(\mathbf{p}_1,\dots,\mathbf{p}_n)\,,\\
(\hat{\eta}A_{\mathrm{GB}})_0 &\stackrel{\mathrm{def}}{=} A_0\,,
\end{aligned} \tag{4.29}$$

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[11]The use of $-g^{\mu\mu_1}$ instead of $\delta_{\mu\mu_j}$ is necessary to yield the vector transformation property (4.6) with (4.31) for the Gupta-Bleuler potentials $\hat{A}^\mu(x) = \hat{A}^\mu_{\mathrm{GB}}(x)$.

[12]Compare (2.34).

[13]Compare (2.39), (2.40), and (2.43). For the physically correct choice of ζ see Exercise 61.

[14]Compare (2.46), (2.47), and (2.62).

in order to fulfill (4.3) and $(\Omega \mid \Omega) = 1$ for the vector

$$\Omega \stackrel{\text{def}}{=} \{1, 0, 0, \ldots\} \,. \tag{4.30}$$

Since $\hat{\eta}$ is unitary[15] w.r.t. $\langle . \mid . \rangle$ the sesquilinear form $(. \mid .)$, defined this way, fulfills condition (ii) of 4.1.2, indeed. By

$$\begin{array}{l} \left(\hat{V}(a,\Lambda)A_{\text{GB}}\right)_0 \stackrel{\text{def}}{=} A_0 \,, \\ \left(\hat{V}(a,\Lambda)A_{\text{GB}}\right)^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n) \\ \stackrel{\text{def}}{=} e^{i(p_1+\ldots+p_n)a}\Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}a^{\nu_1,\ldots,\nu_n}\left(\overrightarrow{\Lambda^{-1}p_1},\ldots,\overrightarrow{\Lambda^{-1}p_n}\right)_{|p_j^0=|\mathbf{p}_j|} \end{array} \tag{4.31}$$

(compare (2.21)), since (4.19) and (4.28) imply

$$(A_{\text{GB}} \mid A'_{\text{GB}}) = \overline{A_0}A'_0 + \sum_{n=1}^{\infty}\int \overline{a^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n)}a'_{\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n)\,\frac{d\mathbf{p}_1}{2\,|\mathbf{p}_1|}\cdots\frac{d\mathbf{p}_n}{2\,|\mathbf{p}_n|}\,, \tag{4.32}$$

we get a representation of $\mathcal{P}_+^{\uparrow}$ fulfilling conditions (iii), (iv), and (viii) of 4.1.2.

Warning: For $\mathbf{v} \neq 0$, the operators $\hat{V}(\Lambda_{\mathbf{v}}, 0)$ are **unbounded** w.r.t. $\langle . \mid . \rangle$.

Proof For every $\mathbf{v} \neq 0$ there is a *1-photon state* vector $\hat{A}_{\text{GB}}$ fulfilling

$$\begin{array}{l} \left\langle \hat{A}_{\text{GB}} \mid \hat{A}_{\text{GB}} \right\rangle = 1 \,, \\ \left\| \hat{V}\left(\Lambda_{\mathbf{v}},0\right) A_{\text{GB}} \right\|^2 \stackrel{\text{def}}{=} \left\langle \hat{V}\left(\Lambda_{\mathbf{v}},0\right)\hat{A}_{\text{GB}} \mid \hat{V}\left(\Lambda_{\mathbf{v}},0\right)\hat{A}_{\text{GB}} \right\rangle = 1+\epsilon \,, \quad \epsilon > 0 \,. \end{array}$$

Then, for

$$\begin{array}{l} (A^N_{\text{GB}})^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n) \\ \stackrel{\text{def}}{=} \begin{cases} \sqrt{\dfrac{\epsilon}{N}\dfrac{1}{(1+\epsilon)^{n+1}}}\,\hat{a}^{\mu_1}(\mathbf{p}_1)\cdots\hat{a}^{\mu_n}(\mathbf{p}_n) & \text{if } n < 1+N^2\dfrac{1+\epsilon}{\epsilon}\,, \\ 0 & \text{else}\,, \end{cases} \end{array}$$

we have[16]

$$\left\|A^N_{\text{GB}}\right\|^2 < \frac{1}{N}\,, \quad \left\|\hat{V}\left(\Lambda_{\mathbf{v}},0\right)A^N_{\text{GB}}\right\|^2 > N\,. \quad \blacksquare$$

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[15] W.r.t. $(. \mid .)$, of course, $\hat{\eta}$ is unbounded.

[16] Note that $\sum_{n=0}^{\infty}\frac{\epsilon}{(1+\epsilon)^{n+1}} = 1$.

Field Operators

In view of (4.15) the simplest possibility to fulfill also condition (vii) of 4.1.2 is[17]

$$D_F \stackrel{\text{def}}{=} \left\{ \Phi \in D_A : \partial_\mu \hat{A}_{\text{GB}}^{(+)\mu}(x)\Phi = 0 \right\} . \tag{4.33}$$

Then[18]

$$\begin{aligned} D_F &= D_{\text{Coul}} + D_{00} , \quad \text{where} \\ D_{\text{Coul}} &\stackrel{\text{def}}{=} \left\{ \Phi \in D_F : \hat{A}_{\text{GB}}^{(+)0}(x)\Phi = 0 \right\} , \end{aligned} \tag{4.34}$$

and

$$(\Phi \mid \Phi') = 0 \quad \forall \, \Phi \in D_F , \, \Phi' \in D_{00} . \tag{4.35}$$

Since $(. \mid .)$ and $\langle . \mid . \rangle$ coincide on D_{Coul} this shows that $(. \mid .)$ is positive semi-definite on D_F .

Sketch of proof for (4.34):

$$A_{\text{GB}} - \hat{P}_{\text{Coul}} A_{\text{GB}} \in D_{00} \quad \text{with} \quad \hat{P}_{\text{Coul}} = \prod_{j=1}^{\infty} (1 - \hat{T}_j) ,$$

where

$$\begin{aligned} &\left(\hat{T}_j A_{\text{GB}} \right)^{\mu_1, \ldots, \mu_n} (\mathbf{p}_1, \ldots, \mathbf{p}_n) \\ &\stackrel{\text{def}}{=} \begin{cases} \dfrac{p_j^{\mu_j}}{p_j^0} \, a^{\mu_1, \ldots, \overbrace{0}^{j.\text{ place}}, \ldots, \mu_n} (\mathbf{p}_1, \ldots, \mathbf{p}_n)_{|_{p_j^0 = |\mathbf{p}_j|}} & \text{for } j = 1, \ldots, n , \\ 0 & \text{for } j > n . \end{cases} \end{aligned}$$

■

Sketch of proof for (4.35):

$$\begin{aligned} \langle A_{\text{GB}} \mid A_{\text{GB}} \rangle &= \left(\hat{P}_{\text{Coul}} A_{\text{GB}} \mid \hat{P}_{\text{Coul}} A_{\text{GB}} \right) \; \forall \, A_{\text{GB}} \in D_F , \\ \rightsquigarrow \quad A_{\text{GB}} \in D_{00} &\iff A_{\text{GB}} = (1 - \hat{P}_{\text{Coul}}) A_{\text{GB}} . \end{aligned}$$

■

Exercise 60 Show that $\partial_\mu \hat{A}_{\text{GB}}^{(+)\mu}(x)\Phi$ commutes with all the smeared field operators $\hat{F}_{\text{BG}}^{\alpha\beta}(x)$ on D_A and hence the latter leave D_F invariant.

One easily checks[19] that for every $\Phi \in D_{\text{Coul}}$ and for every $\epsilon > 0$ there is a $\Phi' \in \mathcal{Z}_F^{\text{Coul}}$ with $(\Phi - \Phi' \mid \Phi - \Phi') < \epsilon$, where $\mathcal{Z}_F^{\text{Coul}}$ denotes the smallest linear subspace of

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[17] Then $\partial_\mu \hat{A}_{\text{GB}}^{\mu}(\varphi) D_F \subset D_{00} \quad \forall \, \varphi \in \mathcal{S}(\mathbb{R}^4)$.

[18] D_{00} was defined in (4.17).

[19] Note that

$$\mathbf{p} \cdot \mathbf{a} \implies \mathbf{a} = \mathbf{p} \times \frac{\mathbf{a} \times \mathbf{p}}{|\mathbf{p}|^2} .$$

D_{Coul} containing Ω and being invariant under all $\hat{F}^{jk}_{\hat{A}_{\mathrm{GB}}}(x)$ with $j,k \in \{1,2,3\}$. This is because in D_{Coul} the topology induced by $(. \mid .)$ is equivalent to that induced by

$$\||A_{\mathrm{GB}}\||^2 \stackrel{\mathrm{def}}{=} |A_0|^2 + \sum_{n=1}^{\infty} \sum_{\mu_1,\ldots,\mu_n=1}^{3} \int |a^{\mu_1,\ldots,\mu_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n)|^2 \, \frac{\mathrm{d}\mathbf{p}_1}{2\,|\mathbf{p}_1|} \cdots \frac{\mathrm{d}\mathbf{p}_n}{2\,|\mathbf{p}_n|} \, .$$

Therefore (4.34)/(4.35) imply that also condition (x) of 4.1.2 is fulfilled for $\hat{F}^{\mu\nu}_{\hat{A}_{\mathrm{GB}}}(x)$.

Transition to the Wightman Theory

Let D' denote the set of all equivalence classes

$$D' \stackrel{\mathrm{def}}{=} \{[A_{\mathrm{GB}}] : A_{\mathrm{GB}} \in D_F\} \tag{4.36}$$

of D_F corresponding to the equivalence relation

$$A_{\mathrm{GB}} \sim A'_{\mathrm{GB}} \quad \stackrel{\mathrm{def}}{\Longleftrightarrow} \quad A_{\mathrm{GB}} - A'_{\mathrm{GB}} \in D_{00} \, . \tag{4.37}$$

Then, according to (4.35),

$$\langle [A_{\mathrm{GB}}] \mid [A'_{\mathrm{GB}}] \rangle \stackrel{\mathrm{def}}{=} (A_{\mathrm{GB}} \mid A'_{\mathrm{GB}}) \tag{4.38}$$

does not depend on the choice of representatives $A_{\mathrm{GB}}, A'_{\mathrm{GB}} \in D_F$ and, by condition (ii) of 4.1.2 and (4.34)/(4.35), and defines a positive definite inner product on D' . Hence, the completion $\mathcal{H}'$ of D' w.r.t. $\langle . \mid . \rangle$ is a **Hilbert** space. The appropriate representation of $\mathcal{P}_+^{\uparrow}$ results from continuous extension of the operators[20]

$$\hat{V}'(a,\Lambda)\,[A_{\mathrm{GB}}] \stackrel{\mathrm{def}}{=} \left[\hat{V}(a,\Lambda) A_{\mathrm{GB}}\right] \tag{4.39}$$

onto all of $\mathcal{H}'$. Then, up to a constant factor,

$$\Omega' \stackrel{\mathrm{def}}{=} [\Omega] \tag{4.40}$$

is the only element of $\mathcal{H}'$ that is invariant under all $\hat{V}'(a,\Lambda)$. The spectrum condition follows from condition (viii) of 4.1.2. To summarize:

> For $\mathcal{H}'$, $\hat{V}'(a,\Lambda)$, and Ω' all requirements of the zeroth Wightman axiom are fulfilled.

Next, we define the field operators:[21]

$$\hat{F}'^{\mu\nu}(\varphi)\Phi \stackrel{\mathrm{def}}{=} \left[\hat{F}^{\mu\nu}_{\hat{A}_{\mathrm{GB}}}(\varphi) A_{\mathrm{GB}}\right] \quad \text{for } \Phi = [A_{\mathrm{GB}}] \in D' \, . \tag{4.41}$$

Now all the Wightman axioms for the free electromagnetic field are fulfilled, as may be easily derived from conditions (i)–(x) of 4.1.2.

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[20] Definition (4.39) is allowed by (4.11), (4.5) and Lemma 4.1.3.

[21] By (4.11), (4.3), and Lemma 4.1.3 this definition is allowed. The $\hat{F}^{\mu\nu}_{\hat{A}_{\mathrm{GB}}}(x)$ were defined in (4.13).

Corollary 4.1.5 *There is a Wightman theory of the free electromagnetic field, given by* $\mathcal{H}'$, $\hat{V}'(a,\Lambda)$, Ω', *and* $\hat{F}'^{\mu\nu}(x)$, *with*

$$\left\langle \Omega' \mid \hat{F}'^{\mu_1\nu_1}(x_1)\cdots\hat{F}'^{\mu_n\nu_n}(x_n)\Omega'\right\rangle = \left(\Omega \mid \hat{F}^{\mu_1\nu_1}_{\hat{A}_{\rm GB}}(x_1)\cdots\hat{F}^{\mu_n\nu_n}_{\hat{A}_{\rm GB}}(x_n)\Omega\right)$$

for all $\mu_1,\nu_1,\ldots,\mu_n,\nu_n \in \{0,\ldots,3\}$.

Final remark: $\partial_\mu \hat{A}^{(-)\mu}_{\rm GB}(x) = 0$ does **not** hold on all of D_F . Hence

$$\partial \hat{F}^{\mu\nu}_{\hat{A}_{\rm GB}}(x) \neq 0$$

(compare Theorem 4.1.2). However, the following statements are true:

$$\begin{aligned}
\partial^\nu \epsilon_{\mu\nu\alpha\beta}\hat{F}^{\alpha\beta}_{\hat{A}_{\rm GB}}(x) &\underset{(4.13)}{=} 0 \quad \text{on } D_A\,,\\
\partial_\mu \hat{F}^{\mu\nu}_{\hat{A}_{\rm GB}}(x) &\underset{(4.9)/(4.13)}{=} \partial^\nu\partial_\mu \hat{A}^\mu_{\rm GB}(x)\,,\\
\partial_\mu \left\langle A_{\rm GB} \mid \hat{F}^{\mu\nu}_{\hat{A}_{\rm GB}}(x) \mid A'_{\rm GB}\right\rangle &\underset{(4.10)}{=} 0 \quad \forall\, A_{\rm GB}, A'_{\rm GB} \in D_F\,.
\end{aligned}$$

4.1.4 Gupta-Bleuler Observables

Agreement: By ***Gupta-Bleuler observable*** we will always mean a $(.\mid.)$-hermitian operator $\hat{A} \in L(D_A, D_A)$ leaving D_F invariant for which $\hat{A}'$, defined by

$$\hat{A}'\left[A_{\rm GB}\right] \stackrel{\rm def}{=} \left[\hat{A}\, A_{\rm GB}\right] \quad \forall\, A_{\rm GB} \in D_F$$

(recall Lemma 4.1.3), is an essentially selfadjoint operator of the Wightman theory constructed as described in 4.1.3. We will call Gupta-Bleuler observables $\hat{A}$ and $\hat{B}$ ***equivalent*** if they induce the same transformation of the equivalence classes:

$$\hat{A} \sim \hat{B} \stackrel{\rm def}{\Longleftrightarrow} \left(\left[\hat{A}\, A_{\rm GB}\right] = \left[\hat{B}\, A_{\rm GB}\right] \quad \forall\, A_{\rm GB} \in D_F\right)\,.$$

Gauge Transformations

By ***gauge transformation in***[22] $\{\mathcal{H}, \langle.\mid.\rangle, \hat{\eta}, D_A, \Omega\}$ we will mean transition from one quantized potential $\hat{\mathbf{A}}(\mathbf{x},t)$ to another quantized potential $\hat{A}'^{\mu}(x)$ without changing the n-point functions of the corresponding field strength operators:

$$\left(\Omega \mid \hat{F}^{\mu_1\nu_1}_{\hat{A}}(x_1)\cdots\hat{F}^{\mu_n\nu_n}_{\hat{A}}(x_n)\right) = \left(\Omega \mid \hat{F}^{\mu_1\nu_1}_{\hat{A}'}(x_1)\cdots\hat{F}^{\mu_n\nu_n}_{\hat{A}'}(x_n)\right)\,.$$

[22] For a more general classification of gauge transformations see (Strocchi and Wightman, 1974).

Certainly, the latter is guaranteed if the connection between $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{A}'^\mu(x)$ is given by $\hat{A}'^\mu(x) = \hat{\mathbf{A}}(\mathbf{x},t) + \partial^\mu \hat{\chi}(x)$, where $\hat{\chi}(x)$ is a $(.\mid.)$-hermitian tempered field in $\{\mathcal{H}, \langle.\mid.\rangle\}$ with invariant domain D_A. Not quite that simple is the gauge transformation

$$\hat{A}^\mu_{\mathrm{GB}}(x) \longrightarrow \hat{A}^\mu_{\mathrm{Coul}}(x) \stackrel{\mathrm{def}}{=} \hat{A}^\mu_{\mathrm{GB}}(x) + \hat{\chi}^\mu(x) \tag{4.42}$$

with

$$\begin{aligned} \hat{\chi}^\mu(x) &\stackrel{\mathrm{def}}{=} \sqrt{\zeta}\,(2\pi)^{-3/2} \int_{p^0=|\mathbf{p}|} \left(\hat{c}^\mu(\mathbf{p})e^{-ip^\mu x_\mu} + \hat{c}^\mu(\mathbf{p})^\dagger e^{+ip^\mu x_\mu}\right) \frac{d\mathbf{p}}{2p^0}\,, \\ \hat{c}^\mu(\mathbf{p}) &\stackrel{\mathrm{def}}{=} \begin{cases} -\hat{a}^0(\mathbf{p}) & \text{for } \mu = 0\,, \\ -\dfrac{p^\mu}{|\mathbf{p}|^2} \displaystyle\sum_{j=1}^3 p^j \hat{a}^j(\mathbf{p}) & \text{for } \mu = 1,2,3\,. \end{cases} \end{aligned} \tag{4.43}$$

A simple calculation shows that

$$\hat{F}^{jk}_{\hat{A}_{\mathrm{GB}}}(x) = \hat{F}^{jk}_{\hat{A}_{\mathrm{Coul}}}(x) \quad \text{for } j,k \in \{1,2,3\}$$

and[23]

$$\left(\hat{F}^{0j}_{\hat{A}_{\mathrm{GB}}}(\varphi) - \hat{F}^{0j}_{\hat{A}_{\mathrm{Coul}}}(\varphi)\right) D_F \subset D_{00} \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^4)\,. \tag{4.44}$$

Hence, indeed, (4.42)/(4.43) is a gauge transformation. The index "Coul" is to indicate validity of the equations

$$\hat{A}^0_{\mathrm{Coul}}(x) = 0\,, \tag{4.45}$$

$$\sum_{j=1}^3 \partial_j \hat{A}^j_{\mathrm{Coul}}(x) = 0\,. \tag{4.46}$$

Restricting the $\hat{A}^j_{\mathrm{Coul}}(x)$ to the domain D_{Coul} (compare (4.34)), which they leave invariant and on which $(.\mid.) = \langle.\mid.\rangle$, we get quantized potentials of the free electromagnetic field in a pre-Hilbert space with positive definite metric (***radiation gauge***, the special case (4.45) of the ***Coulomb gauge***, characterized by (4.46)).

The price to be paid for positive definiteness of the metric is invalidity of the covariance condition (4.6) – for **every** representation $\hat{V}(a,\Lambda)$ – as well as of the locality condition (4.8). This follows from (4.45) and the structure of the 2-point function[24]

$$\left(\Omega \mid \hat{A}^j_{\mathrm{Coul}}(x)\hat{A}^k_{\mathrm{Coul}}(y)\right) = i\zeta \left(\delta_{jk} - \frac{\partial^j \partial^k}{\Delta}\right) \Delta_0^{(+)}(x-y)\,. \tag{4.47}$$

(4.44) shows that, for $j \in \{1,2,3\}$ and real-valued $\varphi \in \mathcal{S}(\mathbb{R}^4)$, $\hat{F}^{0j}_{\mathrm{GB}}(\varphi)$ and $\hat{F}^{0j}_{\mathrm{Coul}}(\varphi)$ are equivalent Gupta-Bleuler observables, although not identical.

[23]Note that $p^0\,\hat{c}^k(\mathbf{p}) - p^k\,\hat{c}^0(\mathbf{p}) = -\frac{p^k}{p^0}\,p_\mu \hat{a}^\mu(\mathbf{p})$ for $k \in \{1,2,3\}$ and that, by (4.33) and (4.26), $\partial_\mu \hat{A}^\mu_{\mathrm{GB}}(\varphi) D_F \subset D_{00}$.

[24]The $\frac{\partial^j \partial^k}{\Delta}$-term spoils commutativity.

Special Observables

Using the commutation relations (4.22), one easily checks that

$$\hat{P}^\alpha_{\rm GB} = \int_{p^0=|\mathbf{p}|} p^\alpha \left(-\hat{a}^\mu(\mathbf{p})^\dagger\, \hat{a}_\mu(\mathbf{p})\right) \frac{\mathrm{d}\mathbf{p}}{2p^0}\,. \tag{4.48}$$

is the generator of space-time translations:

$$i\left[\hat{A}_{\rm GB}{}^\alpha, \hat{A}_{\rm GB}^{(\pm)\mu}(x)\right]_- = \partial^\alpha \hat{A}_{\rm GB}^{(\pm)\mu}(x)\,. \tag{4.49}$$

Therefore, $\hat{P}^\alpha_{\rm GB}$ is the Gupta-Bleuler observable of linear 4-momentum specified by

$$\hat{P}^\alpha_{\rm GB}\,\Omega = 0\,. \tag{4.50}$$

This is consistent with the interpretation of

$$\frac{-1}{2\,|\mathbf{p}|}\hat{a}^\mu(\mathbf{p})^\dagger\, \hat{a}_\mu(\mathbf{p})$$

as a Gupta-Bleuler observable for

$$\lim_{\tilde{V}_\mathbf{p}\to\{\mathbf{p}\}} \frac{\text{number of } \textbf{physical} \text{ photons with momentum } \mathbf{p}' \in \tilde{V}_\mathbf{p}}{\text{momentum space volume } \tilde{V}_\mathbf{p}}\,.$$

Remark: Given $\varphi \in S(\mathbb{R}^3)$, we always have

$$\int \hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p})\,\varphi(\mathbf{p})\,\mathrm{d}\mathbf{p}\; D \subset D \quad \text{for } D = D_{\rm F} \text{ as well as for } D = D_{00}\,,$$

but **not** $\hat{a}^\mu(\mathbf{p})^\dagger\, \hat{a}_\mu(\mathbf{p}) = 0$ on D_{00}.

Exercise 61 Show that the Gupta-Bleuler observables $c\,\hat{P}^0_{\rm GB}$ and

$$\hat{E}_{\rm GB}(x^0) \stackrel{\rm def}{=} \frac{\epsilon_0'\,c'^2}{2}\int :\Big(g_{\alpha\beta}g_{\gamma\delta}\hat{F}^{\alpha\gamma}_{\hat{A}_{\rm GB}}(x)\hat{F}^{\beta\delta}_{\hat{A}_{\rm GB}}(x) + 4\,g_{\mu\nu}\hat{F}^{0\mu}_{\hat{A}_{\rm GB}}(x)\hat{F}^{0\nu}_{\hat{A}_{\rm GB}}(x)\Big):\mathrm{d}\mathbf{x}$$

are equivalent (**not** identical) if

$$\zeta = \frac{1}{\epsilon_0'}\left(\frac{\hbar}{c'}\right)^2\,. \tag{4.51}$$

In the following ζ will always be assumed given by (4.51), with ϵ_0' and c' specified by the used system of units[25] (see Appendix A.3 of (Lücke, edyn)).

[25] In Heaviside units: $\epsilon_0' = c' = 1$.

Similarly to (4.49) one may show that

$$\begin{aligned} \hat{\mathbf{J}}_{\mathrm{GB}} &= \hat{\mathbf{L}}_{\mathrm{GB}} + \hat{\mathbf{S}}_{\mathrm{GB}}\,, \quad \text{where} \\ \hat{\mathbf{L}}_{\mathrm{GB}} &\stackrel{\text{def}}{=} \int_{p^0=|\mathbf{p}|} -\hat{a}^\mu(\mathbf{p})^\dagger\, \mathbf{p} \times \frac{1}{i} \boldsymbol{\nabla}_{\mathbf{p}}\, \hat{a}_\mu(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{2p^0}\,, \\ \hat{\mathbf{S}}_{\mathrm{GB}} &\stackrel{\text{def}}{=} i \int_{p^0=|\mathbf{p}|} \hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{2p^0}\,, \end{aligned} \tag{4.52}$$

is a generator of spatial rotations:

$$\begin{aligned} \left[\hat{\mathbf{L}}_{\mathrm{GB}}, \hat{A}^{(\pm)\mu}_{\mathrm{GB}}(x)\right] &= i\,\mathbf{x} \times \boldsymbol{\nabla}_{\mathbf{x}} \hat{A}^{(\pm)\mu}_{\mathrm{GB}}(x)\,, \\ \left[\mathbf{e}\cdot\hat{\mathbf{S}}_{\mathrm{GB}}, \overrightarrow{\hat{A}^{(\pm)}_{\mathrm{GB}}}(x)\right] &= i\,\mathbf{e} \times \overrightarrow{\hat{A}^{(\pm)}_{\mathrm{GB}}}(x)\,. \end{aligned} \tag{4.53}$$

Hence $\hat{\mathbf{J}}_{\mathrm{GB}}$ is the Gupta-Bleuler observable of total angular momentum specified by

$$\hat{\mathbf{J}}_{\mathrm{GB}}\Omega = 0\,.$$

Warning: $\hat{\mathbf{L}}_{\mathrm{GB}}$ and $\hat{\mathbf{S}}_{\mathrm{GB}}$ themselves are **not** Gupta-Bleuler observables since they do not leave D_F invariant!

This is consistent with the interpretation of

$$\frac{1}{2\,|\mathbf{p}|}\left(-\hat{a}^\mu(\mathbf{p})^\dagger\, \mathbf{p} \times \frac{1}{i}\boldsymbol{\nabla}_{\mathbf{p}}\, \hat{a}_\mu(\mathbf{p}) + i\hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p})\right)$$

as Gupta-Bleuler observable of

$$\lim_{\tilde{V}_{\mathbf{p}}\to\{\mathbf{p}\}} \frac{\text{total angular momentum of all photons with momentum } \mathbf{p}' \in \tilde{V}_{\mathbf{p}}}{\text{momentum space volume } \tilde{V}_{\mathbf{p}}\,.}$$

Hence, the so-called ***helicity operator***

$$\hat{\lambda}_{\mathrm{GB}} \stackrel{\text{def}}{=} i \int_{p^0=|\mathbf{p}|} \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\left(\hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p})\right) \frac{\mathrm{d}\mathbf{p}}{2p^0} \tag{4.54}$$

is a Gupta-Bleuler observable for the component of angular momentum along three-momentum.

n-photon states are called those represented by the elements of D_A which are of the form

$$A_{\mathrm{GB}} = \{0, \ldots, 0, A_n, 0, \ldots\}$$

(compare 4.1.3). The Gupta-Bleuler observable for the number of **physical** photons is

$$\hat{N}_{\mathrm{GB}} \stackrel{\text{def}}{=} -\int_{p^0=|\mathbf{p}|} \hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{2p^0} \tag{4.55}$$

(recall the comment to (4.48)).

Remark: The expectation values for the field strengths in physical n-photon states all vanish!

Every 1-photon state $A_{\mathrm{GB}} = \{0, A_1, 0, 0, \ldots\}$ has a unique decomposition

$$A_{\mathrm{GB}} = A^{\mathrm{tr}}_{\mathrm{GB}} + A^{\mathrm{lo}}_{\mathrm{GB}} + A^{\mathrm{ti}}_{\mathrm{GB}}$$

with:

$$\begin{aligned}
&A^{\mathrm{tr}}_{\mathrm{GB}} \in D_{\mathrm{Coul}}\,, && \text{i.e. } p^j \left(A^{\mathrm{tr}}_{\mathrm{GB}}\right)^j(\mathbf{p}) = \left(A^{\mathrm{tr}}_{\mathrm{GB}}\right)^0(\mathbf{p}) = 0\,,\\
&A^{\mathrm{lo}}_{\mathrm{GB}} \in D_{00}\,, && \text{i.e. } \left(A^{\mathrm{lo}}_{\mathrm{GB}}\right)^j(\mathbf{p}) = \frac{p^j}{|\mathbf{p}|}\left(A^{\mathrm{lo}}_{\mathrm{GB}}\right)^0(\mathbf{p})\,,\\
&\left(A^{\mathrm{ti}}_{\mathrm{GB}}\right)^j(\mathbf{p}) = 0 \quad \text{for } j = 1,2,3\,.
\end{aligned}$$

One says:

$A^{\mathrm{tr}}_{\mathrm{GB}}$ describes a ***transverse photon***, $A^{\mathrm{lo}}_{\mathrm{GB}}$ a ***longitudinal photon***, and $A^{\mathrm{ti}}_{\mathrm{GB}}$ a ***time-like photon***.

Of course, the 1-photon state vector A_{GB} corresponds to a ***physical state*** only if $A^{\mathrm{ti}}_{\mathrm{GB}} = 0$. In this case A_{GB} and $A^{\mathrm{tr}}_{\mathrm{GB}}$ correspond to the same physical state.

The commutation relations

$$i\left[\hat{a}^0(\mathbf{p})^\dagger, \hat{\lambda}_{\mathrm{GB}}\right]_- = 0\,, \quad i\left[\mathbf{e}\cdot\hat{\mathbf{a}}(\mathbf{p})^\dagger, \hat{\lambda}_{\mathrm{GB}}\right]_- = \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\left(\mathbf{e}\times\hat{\mathbf{a}}(\mathbf{p})^\dagger\right) \tag{4.56}$$

show[26] that the longitudinal (as well as the time-like) 1-photon state vectors are eigenvectors of $\hat{\lambda}_{\mathrm{GB}}$ with eigenvalue 0. Moreover, we see that the transversal 1-photon state vectors of the form

$$\begin{aligned}
&\int_{p^0=|\mathbf{p}|} \varphi(\mathbf{p})\left(\mathbf{e}_1(\mathbf{p})\cdot\hat{\mathbf{a}}(\mathbf{p})^\dagger - i\sigma\mathbf{e}_2(\mathbf{p})\cdot\hat{\mathbf{a}}(\mathbf{p})^\dagger\right)\frac{d\mathbf{p}}{2p^0}\,\Omega\,,\\
&\text{where: } \left\{\mathbf{e}_1(\mathbf{p}), \mathbf{e}_2(\mathbf{p}), \frac{\mathbf{p}}{|\mathbf{p}|}\right\} \text{ right-handed orthonormal basis of } \mathbb{R}^3\ \forall\,\mathbf{p}\neq 0\,,
\end{aligned} \tag{4.57}$$

with $\sigma \in \{+1, -1\}$ are eigenvectors of $\hat{\lambda}_{\mathrm{GB}}$ with Eigenvalue σ, hence correspond to physical 1-photon states with ***helicity*** σ. Obviously, every transversal 1-photon state vector may be written as a linear combination of vectors of the form (4.57).

4.2 The Quantized Free Dirac Field

4.2.1 Lorentz Transformations Characterized via Complex 2 × 2-Matrices

Every selfadjoint complex 2×2-matrix $\tilde{X}$ may be written in the form

$$\tilde{X} = x^\mu \tau_\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \tag{4.58}$$

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[26] Note that $\hat{\lambda}_{\mathrm{GB}}\,\Omega = 0$.

with suitable $x \in \mathbb{R}^4$, where

$$\boxed{\tau_0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \tau_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; \tau_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \; \tau_3 \stackrel{\text{def}}{=} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.} \tag{4.59}$$

Exercise 62 Show that[27]

$$A^2 = \mathrm{Tr}(A)\, A - \det(A)\, \mathbb{1}_2$$

holds for all complex 2×2-matrices A.

Since[28]

$$\mathrm{Tr}(\tau_\mu \tau_\nu) = 2\, \delta_{\mu\nu} \tag{4.60}$$

the coefficients in (4.58) are

$$x^\mu = \frac{1}{2}\, \mathrm{Tr}\Big(\tau_\mu \tilde{X}\Big) . \tag{4.61}$$

Moreover,[29]

$$\det\Big(A \tilde{X} A^*\Big) = \det(\tilde{X}) = x^\mu x_\mu \quad \forall\, A \in \mathrm{SL}(2, \mathbb{C}) \tag{4.62}$$

implies

$$x'^\mu x'_\mu = x^\mu x_\mu \quad \text{if } \tilde{X}' = A \tilde{X} A^* .$$

Hence, by (4.61), for every $A \in \mathrm{SL}(2, \mathbb{C})$

$$x^\mu \longmapsto x'^\mu = (\Lambda_A)^\mu{}_\nu\, x^\nu , \tag{4.63}$$

where

$$\boxed{(\Lambda_A)^\mu{}_\nu \stackrel{\text{def}}{=} \frac{1}{2}\, \mathrm{Tr}(\tau_\mu A \tau_\nu A^*) ,} \tag{4.64}$$

is a restricted Lorentz transformation,[30] $\Lambda_A \in L_+^\uparrow$, depending continuously on A. Note that $A \mapsto \Lambda_A$ is a representation of $\mathrm{SL}(2, \mathbb{C})$:

$$\Lambda_A \Lambda_B = \Lambda_{AB} \quad \forall A, B \in \mathrm{SL}(2, \mathbb{C}) . \tag{4.65}$$

[27]This is a special case of a well-known theorem by Caley stating for arbitrary $n \in \mathbb{N}$ that

$$\Big(c_M(z) \stackrel{\text{def}}{=} \det(M - \lambda\, \mathbb{1}_n) \quad \text{for } z \in \mathbb{C}\Big) \implies c_M(M) = 0$$

holds for all $n \times n$-matrices M.

[28]This is a simple consequence of:

$$\frac{1}{2}\, [\tau^\mu, \tau^\nu]_+ = \begin{cases} \delta_{\mu\nu} & \text{if } \mu, \nu \in \{1, 2, 3\}, \\ \tau^\mu & \text{if } \nu = 0 \end{cases}$$

(compare (4.68)).

[29]As usual, $\mathrm{SL}(2, \mathbb{C})$ denotes the group of all complex 2×2-matrices X with $\det X = 1$.

[30]Here we identify the active Lorentz transformations with their matrix realizations w.r.t. some fixed orthonormal inertial system. The $A \in \mathrm{SL}(2, \mathbb{C})$ with $\Lambda_A = \Lambda$ for given $\Lambda \in L_+^\uparrow$ are determined in (Macfarlane, 1962). That (4.64) defines **restricted** Lorentz transformations follows from the fact that for every $A \in \mathrm{SL}(2, \mathbb{C})$ there is a continuous path connecting Λ_A with $\Lambda_{\tau_0} = \mathbb{1}_4$.

The subgroup of all unitary elements in $\mathrm{SL}(2,\mathbb{C})$ is[31]

$$\mathrm{SU}(2) = \left\{ U_{\boldsymbol{\varphi}} : \boldsymbol{\varphi} \in \mathbb{R}^3 \,,\, |\varphi| \le 2\pi \right\} \;, \quad U_{\boldsymbol{\varphi}} \stackrel{\text{def}}{=} \exp\left(-i\,\boldsymbol{\tau}\cdot\frac{\boldsymbol{\varphi}}{2}\right), \tag{4.66}$$

where the components of $\boldsymbol{\tau}$ are the ***Pauli matrices***

$$\tau^j \stackrel{\text{def}}{=} -\tau_j \quad \text{for } j \in \{1,2,3\} \;, \tag{4.67}$$

fulfilling[32]

$$\tau^j \tau^k = \delta_{jk} + i \sum_{l=1}^{3} \epsilon_{jkl} \tau^l \tag{4.68}$$

and hence

$$(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})^2 = |\boldsymbol{\varphi}|^2 \quad \forall\, \boldsymbol{\varphi} \in \mathbb{R}^3 \,.$$

Exercise 63 Prove

$$U_{\boldsymbol{\varphi}} = \mathbb{1}_2 \cos\frac{|\boldsymbol{\varphi}|}{2} - i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\sin\frac{|\boldsymbol{\varphi}|}{2} \,,$$

(4.66), and:[33]

$$U_{\boldsymbol{\varphi}'} = U_{\boldsymbol{\varphi}} \iff \left(\boldsymbol{\varphi}' = \boldsymbol{\varphi} \,\vee\, |\boldsymbol{\varphi}'| = |\boldsymbol{\varphi}| = 2\pi\right) \,.$$

Now we may easily show (recall Exercise 63), that for the unitary elements

$$U_{\boldsymbol{\varphi}} \in \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$$

[31]Note that $\det\left(e^{iJ}\right) = e^{i\,\mathrm{Tr}(J)}$ for $J = J^*$, and: $\mathrm{Tr}(J) = 0 \underset{(4.67)}{\Longrightarrow} J = -\boldsymbol{\tau}\cdot\frac{\boldsymbol{\varphi}}{2}$.

[32]As usual, we define

$$\epsilon_{jkl} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } (j,k,l) \text{ is an even permutation of } (1,2,3)\,, \\ -1 & \text{if } (j,k,l) \text{ is an odd permutation of } (1,2,3)\,, \\ \;\;0 & \text{else}\,. \end{cases}$$

Hence

$$\left\{ x^0 + x^1(i\tau^1) + x^2(i\tau^2) + x^3(i\tau^3) : x \in \mathbb{R}^4 \right\} \,,$$

considered as algebra over $\mathbb{R}^1$, is isomorphic to the algebra of *quaternions*, generated by $\hat{\imath} \stackrel{\text{def}}{=} i\tau^1$ and $\hat{\jmath} \stackrel{\text{def}}{=} i\tau^2$ (as *Clifford algebra*; see (Choquet-Bruhat et al., 1978, S. 63/64)).

[33]This shows that SU(2) (w.r.t. its natural topology) – contrary to the rotation group – is *simply connected*.

the associated transformations $\Lambda_{U_{\boldsymbol{\varphi}}}$ are spatial rotations:[34]

$$\Lambda_{U_{\boldsymbol{\varphi}}} = \exp\left(\frac{1}{2}\sum_{j,k,l=1}^{3} \epsilon_{jkl}\mathbb{T}^{jk}\varphi^l\right) , \quad \text{where:}$$

$$\mathbb{T}^{23} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}_{|\varphi=0} , \tag{4.69}$$

$$\mathbb{T}^{13} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \mathbb{T}^{12} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\mathbb{T}^{jk} \stackrel{\text{def}}{=} -\mathbb{T}^{jk} \quad \text{for } j > k \,.$$

Obviously, $\Lambda_{U_{\boldsymbol{\varphi}}}$ is the matrix (w.r.t. the right-handed basis in which $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$) of a right-handed rotation, by the angle $|\boldsymbol{\varphi}|$, around an axis oriented along $\boldsymbol{\varphi}$. Therefore:

$$\Lambda_{U_{\boldsymbol{\varphi}}} = \Lambda_{U_{\boldsymbol{\varphi}'}} \iff U_{\boldsymbol{\varphi}'} \in \left\{+U_{\boldsymbol{\varphi}}, -U_{\boldsymbol{\varphi}}\right\} .$$

Exercise 64 Prove that every **positive** hermitian element of SL$(2,\mathbb{C})$ is of the form

$$\begin{aligned} H_{\mathbf{v}} \stackrel{\text{def}}{=} \exp\left(-\frac{\chi_{\mathbf{v}}}{2}\boldsymbol{\tau}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right) &= \cosh\left(\frac{\chi_{\mathbf{v}}}{2}\right)\hat{1} - \sinh\left(\frac{\chi_{\mathbf{v}}}{2}\right)\boldsymbol{\tau}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \sqrt{\frac{\mathfrak{v}^0+1}{2}}\left(\tau^0 - \boldsymbol{\tau}\cdot\frac{\mathbf{v}}{1+1/\mathfrak{v}^0}\right) \end{aligned} \tag{4.70}$$

with suitable $\mathbf{v}\in\mathbb{R}^3$, $|\mathbf{v}|<1$, where:[35]

$$\chi_{\mathbf{v}} \stackrel{\text{def}}{=} \tanh^{-1}|\mathbf{v}| \geq 0 \,, \quad \mathfrak{v}^0 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1-|\mathbf{v}|^2}} \,.$$

Moreover, show for arbitrary $\mathbf{v},\mathbf{v}'\in\mathbb{R}^3$ with $|\mathbf{v}|,|\mathbf{v}'|<1$ that

$$H_{\mathbf{v}'} = H_{\mathbf{v}} \iff \mathbf{v}' = \mathbf{v} \,.$$

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[34] It is sufficient to check the generators. The mapping $U_{\boldsymbol{\varphi}} \mapsto \Lambda_{U_{\boldsymbol{\varphi}}}$ has all the properties of a ***covering mapping*** (Pontrjagin, 1958, Def. 45). Therefore (recall Footnote 33), SU(2) is the ***universal covering group*** of the rotation group.

[35] Note that $\tanh\frac{\chi_{\mathbf{v}}}{2} = \frac{\tanh\chi_{\mathbf{v}}}{1+\sqrt{1-\tanh^2\chi_{\mathbf{v}}}} = \frac{|\mathbf{v}|}{1+1/\mathfrak{v}^0}$, $\cosh = \frac{1}{\sqrt{1-\tanh^2}}$.

Similarly (recall Exercise 64), for positive hermitian $A \in \mathrm{SL}(2,\mathbb{C})$ the Λ_A correspond to Lorentz boosts:

$$\Lambda_{H_{\mathbf{v}}} = \exp\left(\chi_{\mathbf{v}} \sum_{j=1}^{3} \mathbb{T}^{0j} \frac{v^j}{|\mathbf{v}|}\right), \qquad \text{where:}$$

$$\mathbb{T}^{01} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}\chi} \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{|\chi=0}, \tag{4.71}$$

$$\mathbb{T}^{02} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{T}^{03} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 65 Prove (4.69) and (4.71).

The so-called ***polar decomposition*** (for invertible[36] A)

$$A = \underbrace{A\sqrt{A^{-1}{A^*}^{-1}}}_{\text{unitary}} \underbrace{\sqrt{A^*A}}_{\text{pos. herm.}} \tag{4.72}$$

together with (4.69) and (4.71) shows:[37]

$$\{\Lambda_A \,:\, A \in \mathrm{SL}(2,\mathbb{C})\} = \{\Lambda_{A_{\boldsymbol{\varphi}}}\Lambda_{H_{\mathbf{v}}} \,:\, \boldsymbol{\varphi}, \mathbf{v} \in \mathbb{R}^3,\ |\mathbf{v}| < 1\} = L_+^\uparrow \tag{4.73}$$

(for the last equality see, e.g., equations (2.39) and (2.37) in (Lücke, rel)).

4.2.2 Relativistic Covariance in General

Consider any relativistic quantum theory with state space $\mathcal{H}$. Even if the relativistic symmetries are realized as Wigner symmetries, there is no reason, why these should correspond to a true representation of $\mathcal{P}_+^\uparrow$. In this case, by Wigner's theorem (Theorem 1.2.1), one may choose for every (a,Λ) a unitary operator $\hat{U}(a,\Lambda)$ such that[38]

$$\left\{\lambda\,\hat{U}(a_1,\Lambda_1)\hat{U}(a_2,\Lambda_2)\Psi \,:\, \lambda \in \mathbb{C}\right\} = \left\{\lambda\,\hat{U}\Big((a_1,\Lambda_1)\circ(a_2,\Lambda_2)\Big)\Psi \,:\, \lambda \in \mathbb{C}\right\}$$

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[36]For singular A the unitary operator $A\sqrt{A^{-1}{A^*}^{-1}}$ has to be replaced by a suitable isometric operator (see, e.g. (Bratteli and Robinson, 1979, p. 39).)

[37]Note that $\det(A) = 1 \implies \det\left(\sqrt{A^*A}\right) = 1$.

[38]Here $(a_1,\Lambda_1)\circ(a_2,\Lambda_2) \stackrel{\text{def}}{=} (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)$ is the group operation of $\mathcal{P}_+^\uparrow$.

holds for every pair of Poincaré transformations $(a_1,\Lambda_1),(a_2,\Lambda_2)\in\mathcal{P}_+^\uparrow$ and all $\Psi\in\mathcal{H}$, since the action of $\hat{U}(a,\Lambda)$ is to be interpreted in the sense of (2.6) (with $\hat{f}$ replaced by an arbitrary element of $\mathcal{H}$). But this only implies existence of a phase function $\varphi((a_1,\Lambda_1),(a_2,\Lambda_2))\in\mathbb{R}$ with

$$\hat{U}\left((a_1,\Lambda_1)\circ(a_2,\Lambda_2)\right)=e^{i\varphi((a_1,\Lambda_1),(a_2,\Lambda_2))}\hat{U}(a_1,\Lambda_1)\hat{U}(a_2,\Lambda_2)\,. \qquad (4.74)$$

Exercise 66 Show that the phase function φ in (4.74) is a ***2-cocycle*** w.r.t. the trivial representation $\pi(a,\Lambda)=1$ of $\mathcal{P}_+^\uparrow$ in $\mathbb{R}$, i.e. it fulfills the condition $\delta^{(2)}\varphi=0$, where $\delta^{(n)}$ denotes the ***coboundary operator*** defined by

$$\begin{aligned}(\delta^{(n)}f)(g_1,\ldots,g_{n+1}) &\stackrel{\text{def}}{=} \pi(g_1)f(g_2,\ldots,g_{n+1})+\\ &+\sum_{\nu=1}^{n}(-1)^\nu f(g_1,\ldots,g_\nu\circ g_{\nu+1},\ldots,g_{n+1})+(-1)^{n+1}f(g_1,\ldots,g_n)\end{aligned}$$

((Van Est, 1953, Eq. 25)). Moreover, show that φ may be eliminated by suitable change of phase $(\hat{U}(g)\longrightarrow e^{ih(g)}\hat{U}(g))$ iff φ is a ***1-coboundary***, i.e. of the form $\varphi=\delta^{(1)}h$.

In this case, we still have

$$\left\{\lambda\,\hat{U}(a_1,\Lambda_1)\hat{U}(a_2,\Lambda_2)\,:\,\lambda\in\mathbb{C}\right\}=\left\{\lambda\,\hat{U}\Big((a_1,\Lambda_1)\circ(a_2,\Lambda_2)\Big)\,:\,\lambda\in\mathbb{C}\right\}$$

for every pair of Poincaré transformations $(a_1,\Lambda_1),(a_2,\Lambda_2)\in\mathcal{P}_+^\uparrow$, i.e.

$$(a,\Lambda)\longmapsto\mathcal{U}(a,\Lambda)\stackrel{\text{def}}{=}\left\{\lambda\,\hat{U}(a,\Lambda)\,:\,\lambda\in\mathbb{C}\right\}$$

is a unitary ***ray representation*** of $\mathcal{P}_+^\uparrow$.

Fortunately, according to Bargmann (Bargmann, 1954) the following holds:[39]

Theorem 4.2.1 *Let $\mathcal{U}(a,\Lambda)$ be a continuous*[40] *ray representation of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$. then there is a continuous unitary representation $\hat{U}(a,A)$ of*[41] *$iSL(2,\mathbb{C})$ in $\mathcal{H}$ with:*

$$\mathcal{U}(a,\Lambda)=\left\{\lambda\,\hat{U}(a,A):\lambda\in\mathbb{C}\right\}\quad\forall\,(a,A)\in iSL(2,\mathbb{C})\,.$$

[39]See also (Varadarajan, 2007, Sect. VIII.5) and, for 1 + 2 dimensions, (Grigore, 1993). Ray representations of the Galilei group may always be considered as true representations of some central extension of the Galilei group (Levy-Leblond, 1963).

[40]Here continuity is to be understood in the sense of Condition (iii) of Definition 1.1.4.

[41]As usual, iSL(2, ℂ) denotes the group

$$\mathrm{iSL}(2,\mathbb{C})=\left\{(a,A)\,:\,A\in\mathrm{SL}(2,\mathbb{C})\,,\ a\in\mathbb{R}^4\right\}$$

In this sense, $\mathrm{iSL}(2,\mathbb{C})$ is more fundamental than $\mathcal{P}_+^\uparrow$:

> If the symmetries which express – in the sense of special relativity – the equivalence of all inertial systems are Wigner symmetries then they are given by some[42] continuous unitary representation of $\mathrm{iSL}(2,\mathbb{C})$.

4.2.3 Dirac Particles

Massive Spin-$\frac{1}{2}$ Representations of iSL$(2,\mathbb{C})$

The simplest non-scalar unitary representations[43] of $\mathrm{iSL}(2,\mathbb{C})$ are of the form[44]

$$\left(\hat{U}(a,A)\chi\right)(\mathbf{p}) \stackrel{\text{def}}{=} e^{(ip^\mu a_\mu)} A\,\chi\left(\overrightarrow{\Lambda_{A^{-1}}p}\right)_{|p^0=\omega_\mathbf{p}} \quad ; \quad \omega_\mathbf{p} \stackrel{\text{def}}{=} \sqrt{m^2+\mathbf{p}^{\,2}}\,, \tag{4.75}$$

where m is some fixed **positive** mass and the representation space is the set of all $\mathbb{C}^2$-valued wave functions $\chi(\mathbf{p})$ with finite Hilbert space norm

$$\|\chi\| \stackrel{\text{def}}{=} \sqrt{\int_{p^0=\omega_\mathbf{p}} \chi(\mathbf{p})^* \frac{p^\mu\sigma_\mu}{m}\chi(\mathbf{p})\frac{\mathrm{d}\mathbf{p}}{2p^0}}\,, \quad \sigma_\mu \stackrel{\text{def}}{=} \tau^\mu\,. \tag{4.76}$$

Since

$$A^*A \stackrel{\text{i.a.}}{\neq} \mathbb{1}_2\,,$$

the term $p^\mu\sigma_\mu$ fulfilling

$$A^*p^\mu\sigma_\mu A = \left(\Lambda_{A^{-1}}p\right)^\mu\sigma_\mu \tag{4.77}$$

is needed to make the representation (4.75) unitary.

Sketch of proof for (4.77): With $\sigma^\mu \stackrel{\text{def}}{=} g^{\mu\nu}\sigma_\nu = \tau_\mu$ we have

$$\begin{aligned}
\mathrm{Tr}\left(\tau_\lambda\left(\Lambda_{A^{-1}}p\right)^\mu\sigma_\mu\right) &= \mathrm{Tr}\left(\tau_\lambda p_\mu \left(\Lambda_A\right)^\mu{}_\nu\sigma^\nu\right)\\
&\underset{(4.60)}{=} 2p_\mu\left(\Lambda_A\right)^\mu{}_\lambda\\
&\underset{(4.64)}{=} \mathrm{Tr}\left(p_\mu\sigma^\mu A\tau_\lambda A^*\right)\\
&= \mathrm{Tr}\left(\tau_\lambda A^* p^\mu\sigma_\mu A\right) \quad \text{for } \lambda=0,1,2,3\,.
\end{aligned}$$

Together with (4.60) this implies (4.77). ∎

Exercise 67 ______ *Draft, November 9, 2007* ______

with multiplication

$$(a_1,A_1)\circ(a_2,A_2) = (a_1+\Lambda_{A_1}a_2, A_1A_2)\,.$$

Since $\mathrm{SU}(2)$ is simply connected (recall Footnote 33), (4.71) shows that the same is true for $\mathrm{iSL}(2,\mathbb{C})$.

[42] Of course, not all continuous unitary representations of $\mathrm{iSL}(2,\mathbb{C})$ are physically relevant.

[43] See, e.g. (Streater and Wightman, 1989, p. 15) for a characterization of all irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

[44] Note that $\Lambda_{A^{-1}} \underset{(4.65)}{=} (\Lambda_A)^{-1}$.

Show that[45] $p^\mu p_\mu > 0 \implies p^\mu \sigma_\mu \geq 0$ and that the representation (4.75) of $\mathrm{iSL}(2,\mathbb{C})$ is irreducible.

According to the action of $\Lambda_{U_{\varphi \mathbf{e}}}$, the components of the the operator $\hat{\mathbf{J}}$ defined by (4.75) and

$$\mathbf{e}\cdot\hat{\mathbf{J}} \stackrel{\text{def}}{=} i\frac{\mathrm{d}}{\mathrm{d}\varphi}\hat{U}\left(0, U_{\varphi \mathbf{e}}\right)_{|\varphi=0} = \mathbf{e}\cdot\Big(\underbrace{i\mathbf{p}\times\boldsymbol{\nabla}_{\mathbf{p}}}_{\stackrel{\text{def}}{=}\hat{\mathbf{L}}} + \underbrace{\frac{1}{2}\boldsymbol{\tau}}_{\stackrel{\text{def}}{=}\hat{\mathbf{S}}}\Big) \tag{4.78}$$

are interpreted as observables of total angular momentum.

Exercise 68 For $\hat{\mathbf{S}}$ given by (4.78) prove

$$A_{\boldsymbol{\varphi}}\left(\mathbf{e}\cdot\hat{\mathbf{S}}\right)A_{\boldsymbol{\varphi}}^{-1} = \left(\hat{D}_{\boldsymbol{\varphi}}\,\mathbf{e}\right)\cdot\hat{\mathbf{S}} \quad \forall\, \mathbf{e}\in\mathbb{R}^3\,, \tag{4.79}$$

where $\hat{D}_{\boldsymbol{\varphi}}$ denotes right-handed rotation by the angle $|\boldsymbol{\varphi}|$ around an axis oriented along $\boldsymbol{\varphi}$, and use this to determine the eigenstates of $\mathbf{e}\cdot\hat{\mathbf{S}}$ for arbitrary $\mathbf{e}\in\mathbb{R}^3$.

(4.78) becomes especially simple for $\mathbf{e} = \frac{\mathbf{p}}{|\mathbf{p}|}$:

$$\hat{h} \stackrel{\text{def}}{=} \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\hat{\mathbf{J}} = \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\hat{\mathbf{S}} \qquad \textit{\textbf{helicity operator}}\,. \tag{4.80}$$

Obviously,

$$\hat{h}\,\chi_\pm(\mathbf{p}) = \pm\frac{1}{2}\chi_\pm(\mathbf{p}) \tag{4.81}$$

holds for

$$\begin{aligned}
\chi_+\Big(|\mathbf{p}|(\cos\vartheta\cos\varphi\,\mathbf{e}_1 + \sin\vartheta\sin\varphi\,\mathbf{e}_2 + \cos\vartheta\,\mathbf{e}_3)\Big) &\stackrel{\text{def}}{=} A_{\varphi\mathbf{e}_3}A_{\vartheta\mathbf{e}_2}\begin{pmatrix}1\\0\end{pmatrix} \\
&= \begin{pmatrix} +e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ +e^{+i\frac{\varphi}{2}}\sin\frac{\vartheta}{2}\end{pmatrix}, \\
\chi_-\Big(|\mathbf{p}|(\cos\vartheta\cos\varphi\,\mathbf{e}_1 + \sin\vartheta\sin\varphi\,\mathbf{e}_2 + \cos\vartheta\,\mathbf{e}_3)\Big) &\stackrel{\text{def}}{=} A_{\varphi\mathbf{e}_3}A_{\vartheta\mathbf{e}_2}\begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix} -e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \\ +e^{+i\frac{\varphi}{2}}\cos\frac{\vartheta}{2}\end{pmatrix}
\end{aligned} \tag{4.82}$$

(compare Exercise 68).

The helicity survives in the limit $\frac{m}{|\mathbf{p}|} \to 0$ (to be discussed at the end of this section) in which, however, the representation of $\mathrm{iSL}(2,\mathbb{C})$ becomes reducible (neutrinos!).

[45] Check $\det(p^\mu\sigma_\mu)$ and $\mathrm{Tr}(p^\mu\sigma_\mu)$.

The wave functions may be written in the form

$$\chi(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p})\,\chi_\sigma(\mathbf{p})\,, \tag{4.83}$$

where

$$\chi_+(\mathbf{p}) \stackrel{\text{def}}{=} H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad \chi_-(\mathbf{p}) \stackrel{\text{def}}{=} H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{4.84}$$

Then, by (4.77), the norm (4.76) becomes

$$\|\chi\| = \sqrt{\sum_{\sigma=\pm} \int |b_\sigma(\mathbf{p})|^2 \, \frac{d\mathbf{p}}{2\omega_\mathbf{p}}}\,. \tag{4.85}$$

Moreover,

$$(4.75), (4.83) \implies \left(\hat{U}(a,A)\chi\right)(\mathbf{p}) = \sum\nolimits_{\sigma=\pm} b'_\sigma(\mathbf{p})\,\chi_\sigma(\mathbf{p})\,, \text{ where:}$$

$$\begin{pmatrix} b'_+(\mathbf{p}) \\ b'_-(\mathbf{p}) \end{pmatrix} = e^{ip^\mu a_\mu} \left(H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}}\, A\, H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right) \begin{pmatrix} b_+(\mathbf{p}') \\ b_-(\mathbf{p}') \end{pmatrix}\,, \quad p' \stackrel{\text{def}}{=} \left(\Lambda_A^{-1} p\right)_{|_{p^0=\omega_\mathbf{p}}}\,. \tag{4.86}$$

Proof of (4.86):

$$\begin{aligned}
A\,\chi(\mathbf{p}') &\underset{(4.83)}{=} \sum_\sigma b_\sigma(\mathbf{p}')\, A\, \chi_\sigma(\mathbf{p}') \\
&\underset{(4.84)}{=} \sum_\sigma b_\sigma(\mathbf{p}')\, H_{\frac{+\mathbf{p}}{\omega_\mathbf{p}}} H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}}\, A\, H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}}\, \chi_\sigma(0) \\
&= \sum_\sigma b_\sigma(\mathbf{p}')\, H_{\frac{+\mathbf{p}}{\omega_\mathbf{p}}} \sum_{\sigma'} \left(H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}}\, A\, H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right)_{\sigma'\sigma} \chi_{\sigma'}(0) \\
&\underset{(4.84)}{=} \sum_{\sigma'} \left(\sum_\sigma \left(H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}}\, A\, H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right)_{\sigma'\sigma} b_\sigma(\mathbf{p}') \right) \chi_{\sigma'}(\mathbf{p})\,. \quad \blacksquare
\end{aligned}$$

Exercise 69 Show for arbitrary $A \in \mathrm{SL}(2,\mathbb{C})$ that[46]

$$p' \stackrel{\text{def}}{=} \left(\Lambda_A^{-1} p\right)_{|_{p^0=\omega_\mathbf{p}}} \implies H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}}\, A\, H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \text{ unitary}\,,$$

as to be expected (compare, e.g., Sect. 2.4.3 of (Lücke, rel)).

[46]**Hint:** First, show that (4.70) implies

$$\left(H_{\mathbf{p}/p^0}\right)^2_{|_{p^0=\omega_\mathbf{p}}} = p^\mu \tau_\mu / m \quad \forall\, \mathbf{p} \in \mathbb{R}^3$$

and therefore

$$H_{\frac{-\mathbf{p}'}{\omega_{\mathbf{p}'}}} = \sqrt{\left(H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}} A\right)^{-1} \left(H_{\frac{-\mathbf{p}}{\omega_\mathbf{p}}} A\right)^{*-1}}\,,$$

(see proof of (4.94)). Then recall the polar decomposition (4.72).

Positive Frequency Wave Functions

The transition

$$\chi(\mathbf{p}) \longrightarrow (2\pi)^{-\frac{3}{2}} \int_{p^0=\omega_{\mathbf{p}}} \chi(\mathbf{p})\, e^{-ip^\mu x_\mu} \frac{d\mathbf{p}}{2p^0}$$

would give a usable configuration space version of the above representation (local transformation behavior). However, **technically** more convenient (for later inclusion of anti-particles) is another, unitarily equivalent, representation in the Hilbert space of 4-component momentum space wave functions

$$\hat{\Psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p})\, \omega_\sigma^{(+)}(\mathbf{p}) \tag{4.87}$$

with norm

$$\left\|\hat{\Psi}^{(+)}\right\| \stackrel{\text{def}}{=} \sqrt{\sum_{\sigma=\pm} \int |b_\sigma(\mathbf{p})|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}} \tag{4.88}$$

where

$$\omega_+^{(+)}(\mathbf{p}) \stackrel{\text{def}}{=} \frac{S\left(H_{\frac{\mathbf{p}}{\omega_p v}}\right)}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega_-^{(+)}(\mathbf{p}) \stackrel{\text{def}}{=} \frac{S\left(H_{\frac{\mathbf{p}}{\omega_p v}}\right)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \tag{4.89}$$

namely:

$$\left(\hat{U}(a,A)\hat{\Psi}^{(+)}\right)(\mathbf{p}) \stackrel{\text{def}}{=} e^{ip^\mu a_\mu}\, S(A)\, \hat{\Psi}^{(+)}\left(\overrightarrow{\Lambda_{A^{-1}} p}\right)_{|p^0=\omega_{\mathbf{p}}}, \tag{4.90}$$

where

$$S(A) \stackrel{\text{def}}{=} \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}. \tag{4.91}$$

Exercise 70 Show that

$$(4.90), (4.87) \implies \left(\hat{U}(a,A)\hat{\Psi}^{(+)}\right)(\mathbf{p}) = \sum_{\sigma=\pm} b'_\sigma(\mathbf{p})\, \omega_\sigma^{(+)}(\mathbf{p})$$

holds with $b'_\sigma(\mathbf{p})$ given by (4.86).

(4.88) may be written in the form

$$\left\|\hat{\Psi}^{(+)}\right\| = \sqrt{\int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \frac{p_\mu \alpha^\mu}{m} \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}}, \tag{4.92}$$

where

$$\alpha^\mu \stackrel{\text{def}}{=} \begin{pmatrix} \tau_\mu & 0 \\ 0 & \tau^\mu \end{pmatrix}, \tag{4.93}$$

since now the generalization

$$S(A)^*\, p_\mu \alpha^\mu S(A) = \left(\Lambda_{A^{-1}} p\right)_\mu \alpha^\mu \tag{4.94}$$

of (4.77) holds.

Sketch of proof for (4.94):

$$\begin{aligned}\mathrm{Tr}\left(\tau_\lambda \left(\Lambda_{A^{-1}} p\right)^\mu \tau_\mu\right) &\underset{(4.60))}{=} 2\left(\Lambda_{A^{-1}}\right)^\lambda{}_\nu\, p^\nu \\ &\underset{(4.64))}{=} \mathrm{Tr}\left(p^\nu \tau_\lambda A^{-1} \tau_\nu A^{*-1}\right) .\end{aligned}$$

$$\underset{(4.61)}{\leadsto} A^{-1} p^\nu \tau_\nu A^{*-1} = \left(\Lambda_{A^{-1}} p\right)^\mu \tau_\mu .$$

Together with (4.77) and (4.93) this implies (4.94). ∎

With

$$\gamma^0 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma^j \stackrel{\text{def}}{=} \gamma^0 \alpha^j = \begin{pmatrix} 0 & \tau^j \\ \tau_j & 0 \end{pmatrix} \tag{4.95}$$

(4.94) becomes equivalent to

$$S\left(A^{-1}\right) \gamma^\mu S(A) = \left(\Lambda_A\right)^\mu{}_\nu\, \gamma^\nu , \tag{4.96}$$

thanks to

$$S(A)^* \gamma^0 = \gamma^0 S(A)^{-1} . \tag{4.97}$$

By (4.90), (4.96) is equivalent to

$$\left[\gamma^\mu p_\mu, \hat{U}(a, A)\right]_- = 0 . \tag{4.98}$$

Since $\hat{U}(a, A)$ is irreducible and[47]

$$\det\left(\gamma^\mu p_\mu\right) \underset{(4.95)}{=} \det\left(\alpha^\mu p_\mu\right) = m^4 ,$$

(4.98) implies:[48]

$$\left(\gamma^\mu p_\mu - m\right) \hat{\Psi}^{(+)}(\mathbf{p})_{|p^0 = \omega_\mathbf{p}} = 0 . \tag{4.99}$$

Conversely, (4.99) implies that $\hat{\Psi}^{(+)}(\mathbf{p})$ is of the form (4.87).

Sketch of proof:

$$\begin{aligned}(4.99) \quad &\iff \left(S\left(H_\mathbf{v}\right)^{-1} \left(\gamma^\mu p_\mu - m\right) \hat{\psi}^{(+)}(\mathbf{p}) = 0\right) \\ &\underset{(4.96)}{\iff} \left((\gamma^0 - 1) S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right)^{-1} \hat{\psi}^{(+)}(\mathbf{p}) = 0\right) \\ &\underset{(4.95)}{\iff} S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right)^{-1} \hat{\psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(0) \\ &\underset{(4.89)}{\iff} \hat{\psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p}) . \quad ∎\end{aligned}$$

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[47] **Warning:** The matrix $\gamma^\mu p_\mu$ is **not** selfadjoint (compare (4.109)).

[48] The correct sign may be easily determined by checking the special case $\mathbf{p} = 0$.

Therefore, the set of admitted configuration space wave functions

$$\Psi^{(+)}(x) \stackrel{\text{def}}{=} \sqrt{2m}\,(2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})\, e^{-ipx}\, \frac{\mathrm{d}\mathbf{p}}{2p^0} \tag{4.100}$$

coincides with the set of all normalizable **positive frequency** solutions of the ***Dirac equation***

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0\,. \tag{4.101}$$

Thanks to the factor $\sqrt{2m}$ in (4.100) we have

$$\left\|\hat{\Psi}^{(+)}\right\|^2 = \int \Psi^{(+)}(x)^*\, \Psi^{(+)}(x)\, \mathrm{d}\mathbf{x} \quad \forall\, x^0 \in \mathbb{R}\,. \tag{4.102}$$

Proof:

$$\begin{aligned}
&\int \Psi^{(+)}(x)^* \Psi^{(+)}(x)\, \mathrm{d}\mathbf{x} \\
&\underset{(4.100)}{=} 2m \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^*\, \hat{\Psi}^{(+)}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{(2p^0)^2} \\
&\underset{(4.99)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \gamma^\mu p_\mu \hat{\Psi}^{(+)}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{(2p^0)^2} + \int_{p^0=\omega_{\mathbf{p}}} \left(\gamma^\mu p_\mu \hat{\Psi}^{(+)}(\mathbf{p})\right)^* \hat{\Psi}^{(+)}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{(2p^0)^2} \\
&\underset{(4.109)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^*\, \gamma^0\, \hat{\Psi}^{(+)}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{2p^0} \\
&\underset{(4.99)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^*\, \frac{\gamma^0\gamma^\mu p_\mu}{m}\, \hat{\Psi}^{(+)}(\mathbf{p})\, \frac{\mathrm{d}\mathbf{p}}{2p^0}\,.
\end{aligned}$$

By (4.95) and (4.92), this implies the statement. ∎

Exercise 71 Show that

$$S(U_{\boldsymbol{\varphi}}) = \exp\left(-\frac{i}{4}\sum_{j,k,l=1}^{3} \epsilon_{jkl}\gamma^j\gamma^k\varphi^l\right), \quad S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) = \sqrt{\frac{\omega_{\mathbf{p}}+m}{2m}}\,\gamma^0\left(\gamma^0 + \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{\omega_{\mathbf{p}}+m}\right),$$

and

$$\gamma^1\gamma^3\,\overline{S(A)} = S\left(A^{*-1}\right)\gamma^1\gamma^3 \quad \forall\, A \in \mathrm{SL}(2,\mathbb{C})\,,$$

where $\overline{S}$ denotes the matrix resulting from substituting the entries of the matrix S by their complex conjugates (i.e. $\overline{S} = S^{*\mathrm{T}}$).

Discrete Symmetries

The so-called ***parity operator*** $\mathfrak{P}$, describing total spatial reflection of the state, is fixed – up to some irrelevant constant phase factor – by[49]

$$\mathfrak{P}^{-1}\,\hat{U}\,(a,A)\,\mathfrak{P} = \hat{U}\,\left(\mathbb{P}a, A^{*-1}\right)\,, \quad \mathbb{P}x \stackrel{\text{def}}{=} (x^0, -\mathbf{x})\,, \tag{4.103}$$

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[49]For the necessity of both conditions see, e.g., (Martin and Spearman, 1970, Chapter 5 §1).

and the requirement of unitarity (positive energy[50]). Note that

$$\mathbb{P}\Lambda_A\mathbb{P} = \mathbb{T}\Lambda_A\mathbb{T} = \Lambda_{A^{*-1}}\ \forall\, A \in \mathrm{SL}(2,\mathbb{C})\,,$$

where

$$\mathbb{T}x \stackrel{\text{def}}{=} (-x^0, \mathbf{x})\,.$$

The natural choice is $\mathfrak{P}^2 = \hat{1}$, i.e.:

$$\left(\mathfrak{P}\Psi^{(+)}\right)(x) = \gamma^0\Psi^{(+)}(\mathbb{P}x)\,, \tag{4.104}$$

by (4.97).

Exercise 72 Show that

$$\gamma^0\omega_\sigma^{(+)}(-\mathbf{p}) = \omega_\sigma^{(+)}(\mathbf{p})$$

and, therefore,

$$\begin{aligned} \Psi^{(+)}(x) &= \sqrt{2m}\,(2\pi)^{-3/2}\int_{p^0=\omega_\mathbf{p}}\sum_{\sigma=\pm} b_\sigma(\mathbf{p})\,\omega_\sigma^{(+)}(\mathbf{p})\,e^{-ipx}\,\frac{d\mathbf{p}}{2p^0}\\ \Longrightarrow\quad \left(\mathfrak{P}\Psi^{(+)}\right)(x) &= \sqrt{2m}\,(2\pi)^{-3/2}\int_{p^0=\omega_\mathbf{p}}\sum_{\sigma=\pm} b_\sigma(-\mathbf{p})\,\omega_\sigma^{(+)}(\mathbf{p})\,e^{-ipx}\,\frac{d\mathbf{p}}{2p^0}\,. \end{aligned}$$

Similarly the time reversal operator $\mathfrak{T}$ is fixed – up to some irrelevant constant phase factor – by

$$\mathfrak{T}^{-1}\hat{U}\,(a, A)\,\mathfrak{T} = \hat{U}\left(\mathbb{T}a, A^{*-1}\right)$$

and the condition of **anti**-unitarity.[51] The usual choice is:[52]

$$\left(\mathfrak{T}\,\Psi^{(+)}\right)(x) = i\gamma^1\gamma^3\,\overline{\Psi^{(+)}(\mathbb{T}x)} \tag{4.105}$$

(recall Exercise 71).

Exercise 73 Show that

$$\gamma^1\gamma^3\,\overline{\omega_\sigma^{(\sigma')}(+\mathbf{p})} = -\sigma\sigma'\,\omega_{-\sigma}^{(+)}(-\mathbf{p})$$

and, therefore,

$$\begin{aligned} \Psi^{(+)}(x) &= \sqrt{2m}\,(2\pi)^{-3/2}\int_{p^0=\omega_\mathbf{p}}\sum_{\sigma=\pm} b_\sigma(\mathbf{p})\,\omega_\sigma^{(+)}(\mathbf{p})\,e^{-ipx}\,\frac{d\mathbf{p}}{2p^0}\\ \Longrightarrow\quad \left(\mathfrak{T}\Psi^{(+)}\right)(x) &= \sqrt{2m}\,(2\pi)^{-3/2}\int_{p^0=\omega_\mathbf{p}}\sum_{\sigma=\pm} i\sigma\,\overline{b_{-\sigma}(-\mathbf{p})}\,\omega_\sigma^{(+)}(\mathbf{p})\,e^{-ipx}\,\frac{d\mathbf{p}}{2p^0}\,. \end{aligned}$$

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[50]By (4.103), anti-unitarity of $\mathfrak{P}$ would imply $\mathfrak{P}^{-1}\hat{P}^0\mathfrak{P} = -\hat{P}^0$ and thus $\left\langle \mathfrak{P}\Psi \mid \hat{P}^0\mathfrak{P}\Psi\right\rangle = \overline{\left\langle \Psi \mid \mathfrak{P}^{-1}\hat{P}^0\mathfrak{P}\Psi\right\rangle} = -\left\langle \Psi \mid \hat{P}^0\Psi\right\rangle$ for the Hamiltonian $\hat{P}^0 = -i\frac{d}{dx^0}\hat{U}\left((x^0,0,0,0), \mathbb{1}_4\right)$.

[51]However, because of anti-linearity of $\mathfrak{T}$, Schur's lemma is **not directly** applicable. Note that, for the same reason, $\mathfrak{T}^2 = \hat{1}$ does not depend on the choice of phase factor.

[52]In the representation (4.95): $i\gamma^1\gamma^3 = -\begin{pmatrix}\tau^2 & 0\\ 0 & \tau^2\end{pmatrix}$.

Defining

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\alpha^1\alpha^2\alpha^3\,, \tag{4.106}$$

and noting that (4.93) and (4.68) imply

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & +\mathbb{1}_2 \end{pmatrix} \quad \text{in the representation (4.95)}\,,$$

we get the following relations:

$$S\left(A^{-1}\right)\gamma^5 S(A) = \gamma^5\,, \tag{4.107}$$

$$[\gamma^\mu, \gamma^\nu]_+ = \begin{cases} +2 & \text{if } \mu = \nu \in \{0,5\}\,, \\ -2 & \text{if } \mu = \nu \in \{1,2,3\}\,, \\ 0 & \text{else}\,, \end{cases} \tag{4.108}$$

$$(\gamma^\mu)^* = \begin{cases} +\gamma^\mu & \text{for } \mu \in \{0,5\}\,, \\ -\gamma^\mu & \text{for } \mu \in \{1,2,3\}\,, \end{cases} \tag{4.109}$$

$$\hat{\mathbf{S}}\cdot\mathbf{e} \stackrel{\text{def}}{=} i\frac{\mathrm{d}}{\mathrm{d}\varphi}S(U_{\varphi\mathbf{e}})_{|\varphi=0} = \frac{1}{2}\gamma^5\gamma^0\,\boldsymbol{\gamma}\cdot\mathbf{e}\,, \tag{4.110}$$

$$\left[\hat{\mathbf{S}}, \gamma^5\right]_- = 0\,. \tag{4.111}$$

Exercise 74 Show that

$$V_+^\mu(x) \stackrel{\text{def}}{=} \Psi^{(+)}(x)^*\gamma^0\gamma^\mu\Psi^{(+)}(x) \tag{4.112}$$

is an ordinary current density and

$$V_-^\mu(x) \stackrel{\text{def}}{=} \Psi^{(+)}(x)^*\gamma^0\gamma^5\gamma^\mu\Psi^{(+)}(x) \tag{4.113}$$

an ordinary ***axial*** current density, i.e:[53]

$$\begin{array}{lcl} V_\pm^\mu(x^0,\mathbf{x}) & \stackrel{\hat{U}(a,A)}{\longrightarrow} & (\Lambda_A)^\mu{}_\nu\, V_\pm^\nu\left(\Lambda_A^{-1}(x-a)\right)\,, \\ V_\pm^\mu(x^0,\mathbf{x}) & \stackrel{\hat{P}}{\longrightarrow} & \pm g_{\mu\nu}V_\pm^\nu(x^0,-\mathbf{x})\,, \\ V_\pm^\mu(x^0,\mathbf{x}) & \stackrel{\hat{T}}{\longrightarrow} & +g_{\mu\nu}V_\pm^\nu(-x^0,\mathbf{x}) \end{array} \tag{4.114}$$

Final remark: Physically relevant are only the **general** relations between the γ-matrices. Transformations of the type

$$\gamma^\mu \longrightarrow \gamma'^\mu = M\gamma^\mu M^{-1}\,,$$

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[53] A more complete listing is given in (Itzykson and Zuber, 1980b, Sect. 3-4-4).

where M is any unitary 4×4-matrix, are always allowed.[54] E.g., for

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} +\mathbb{1}_2 & +\mathbb{1}_2 \\ -\mathbb{1}_2 & +\mathbb{1}_2 \end{pmatrix} , \quad M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} +\mathbb{1}_2 & -\mathbb{1}_2 \\ +\mathbb{1}_2 & +\mathbb{1}_2 \end{pmatrix}$$

we get the standard representation

$$\gamma'^0 = -\gamma^5 \,, \; \gamma'^5 = \gamma^0 \,, \; \gamma'^j = \gamma^j \quad \text{for } j = 1, 2, 3 \tag{4.115}$$

of (Bjorken and Drell, 1964).

The Limit $\frac{m}{|\mathbf{p}|} \to 0$

According to (4.110), (4.99) is equivalent to

$$\gamma^5 \hat{\mathbf{S}} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \, \hat{\Psi}^{(+)}(\mathbf{p}) = \frac{1}{2|\mathbf{p}|} \left(\omega_{\mathbf{p}} - m\gamma^0 \right) \hat{\Psi}^{(+)}(\mathbf{p}) \,. \tag{4.116}$$

By (4.108) and (4.111) the latter is equivalent to validity of the following two equations:[55]

$$\begin{aligned} \hat{h} \left(1 + \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) &= +\frac{\omega_{\mathbf{p}}}{2|\mathbf{p}|} \left(1 + \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) - \frac{m}{2|\mathbf{p}|}\gamma^0 \left(1 - \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) \,, \\ \hat{h} \left(1 - \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) &= -\frac{\omega_{\mathbf{p}}}{2\,|\mathbf{p}|} \left(1 - \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) + \frac{m}{2|\mathbf{p}|}\gamma^0 \left(1 + \gamma^5\right) \hat{\Psi}^{(+)}(\mathbf{p}) \,. \end{aligned} \tag{4.117}$$

This implies

$$\left. \begin{aligned} \hat{h}\hat{\Psi}^{(+)}_{\mathrm{R}}(\mathbf{p}) &= +\frac{1}{2}\hat{\Psi}^{(+)}_{\mathrm{R}}(\mathbf{p}) \\ \hat{h}\hat{\Psi}^{(+)}_{\mathrm{L}}(\mathbf{p}) &= -\frac{1}{2}\hat{\Psi}^{(+)}_{\mathrm{L}}(\mathbf{p}) \end{aligned} \right\} \text{ for } \frac{m}{|\mathbf{p}|} \to 0 \,, \tag{4.118}$$

where

$$\begin{aligned} \hat{\Psi}^{(+)}_{\mathrm{R}}(\mathbf{p}) &\stackrel{\text{def}}{=} \tfrac{1}{2}(1 + \gamma^5)\, \hat{\Psi}^{(+)}(\mathbf{p}) \,, \\ \hat{\Psi}^{(+)}_{\mathrm{L}}(\mathbf{p}) &\stackrel{\text{def}}{=} \tfrac{1}{2}(1 - \gamma^5)\, \hat{\Psi}^{(+)}(\mathbf{p}) \,. \end{aligned}$$

Exercise 75 Show that

$$\gamma^0\gamma^\mu = \left(\frac{1 + \gamma^5}{2} \right)^* \gamma^0\gamma^\mu \frac{1 + \gamma^5}{2} + \left(\frac{1 - \gamma^5}{2} \right)^* \gamma^0\gamma^\mu \frac{1 - \gamma^5}{2}$$

and hence

$$V^\mu_+(x) = \Psi^{(+)}_{\mathrm{R}}(x)^*\gamma^0\gamma^\mu\Psi^{(+)}_{\mathrm{R}}(x) + \Psi^{(+)}_{\mathrm{L}}(x)^*\gamma^0\gamma^\mu\Psi^{(+)}_{\mathrm{L}}(x) \,.$$

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[54] Of course, $S(A)$ (recall Exercise 71) has to be defined accordingly, as well as the $\omega^{(+)}_{\pm}(0)$:

$$\gamma^0\omega^{(+)}_{\pm}(0) = \omega^{(+)}_{\pm}(0) \,, \quad \hat{S}^3\omega^{(+)}_{\pm}(0) = \pm\frac{1}{2}\omega^{(+)}_{\pm}(0) \,.$$

[55] The first of these relations results by adding (4.116) and (4.116) multiplied by γ^5 (from the left). Recall that the helicity operator is $\hat{h} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \hat{\mathbf{S}}$.

Note that, in the representation (4.95), the equations

$$\hat{\Psi}^{(+)}_{\mathrm{R}}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_{\mathrm{R}}(\mathbf{p}) \end{pmatrix} , \quad \hat{\Psi}^{(+)}_{\mathrm{L}}(\mathbf{p}) = \begin{pmatrix} \chi_{\mathrm{L}}(\mathbf{p}) \\ 0 \end{pmatrix} ,$$

with corresponding **2-component** spinors $\chi_{\mathrm{R}}(\mathbf{p}),\ \chi_{\mathrm{L}}(\mathbf{p})$, and

$$\hat{h} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\tau} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} & 0 \\ 0 & \boldsymbol{\tau} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \end{pmatrix}$$

hold. Hence, for $m = 0$, the equations (4.118) are equivalent to the two so-called ***Weyl equations***

$$\begin{gathered} \partial_0 \Phi_{\mathrm{R}}(x) = -\boldsymbol{\tau} \cdot \boldsymbol{\nabla}_{\mathbf{x}} \Phi_{\mathrm{L}}(x) \,, \quad \partial_0 \Phi_{\mathrm{L}}(x) = +\boldsymbol{\tau} \cdot \boldsymbol{\nabla}_{\mathbf{x}} \Phi_{\mathrm{L}}(x) \,, \\ \text{where: } \Phi_{\mathrm{R(L)}}(x) \stackrel{\mathrm{def}}{=} (2\pi)^{-\frac{3}{2}} \int_{p^0 = \omega_{\mathbf{p}}} \chi_{\mathrm{R(L)}}(\mathbf{p}) e^{-i p^\mu x_\mu} \frac{d\mathbf{p}}{2p^0} \,. \end{gathered} \tag{4.119}$$

4.2.4 Quantized Dirac Field

Positive Frequency Part of the Dirac Field

Similarly to the electromagnetic field the Dirac field is quantized by replacement of the amplitudes $b_\sigma(\mathbf{p})$ in (4.87) by corresponding annihilation operators $\hat{b}_\sigma(\mathbf{p})$ (of particles with linear momentum $\mathbf{p}$ and $\mathbf{e}_3$-component $\sigma\frac{1}{2}$ of the internal angular momentum in the **center of mass system** of the particle):

$$\hat{\Psi}^{(+)}(x) = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \int_{p^0 = \omega_{\mathbf{p}}} \underbrace{\sum_{\sigma = \pm} \hat{b}_\sigma(\mathbf{p})\, \omega^{(+)}_\sigma(\mathbf{p})}_{\stackrel{\mathrm{def}}{=} \hat{\hat{\Psi}}^{(+)}(\mathbf{p})}\; e^{-i p^\mu x_\mu} \frac{d\mathbf{p}}{2p^0} \tag{4.120}$$

(compare (4.100)). Now, however, the $\hat{b}_\sigma(\mathbf{p})$ (respecting the ***Pauli-principle***) act in a space

$$\mathcal{H}_0 \stackrel{\mathrm{def}}{=} \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$$

the n-particle components $\mathcal{H}^{(n)}$ of which are spanned by totally **anti**-symmetric wave functions b_n :

$$b_n(\mathbf{p}_{\pi 1}, \sigma_{\pi 1}; \ldots; \mathbf{p}_{\pi 1}, \sigma_{\pi 1}) = \operatorname{sign}(\pi)\, b_n(\mathbf{p}_1, \sigma_1; \ldots; \mathbf{p}_n, \sigma_n) \ \text{ for all } \pi \in S_n \,. \tag{4.121}$$

The inner product is given by natural generalization of the norm (4.88):

$$\|b_n\| \stackrel{\mathrm{def}}{=} \sqrt{\sum_{\sigma_1, \ldots \sigma_n = \pm} \int \|b_n(\mathbf{p}_1, \sigma_1; \ldots; \mathbf{p}_n, \sigma_n)\|^2 \prod_j \frac{d\mathbf{p}_j}{2\omega_{\mathbf{p}_j}}} \,. \tag{4.122}$$

Since, the $\hat{b}_\sigma(\mathbf{p})$ are given by

$$\begin{aligned} \left(\hat{b}_\sigma(\mathbf{p})\,b\right)_0 &\stackrel{\text{def}}{=} b_1(\mathbf{p},\sigma)\,, \\ \left(\hat{b}_\sigma(\mathbf{p})\,b\right)_{n-1}(\mathbf{p}_1,\sigma_1;\ldots;\mathbf{p}_{n-1},\sigma_{n-1}) &\stackrel{\text{def}}{=} \sqrt{n}\,b_n(\mathbf{p},\sigma;\mathbf{p}_1,\sigma_1;\ldots;\mathbf{p}_n,\sigma_{n-1}) \end{aligned} \tag{4.123}$$

on their natural domain of definition[56] we now have the **anti**-commutation relations

$$\begin{aligned} \left[\hat{b}_\sigma(\mathbf{p}),\hat{b}_{\sigma'}(\mathbf{p}')^*\right]_+ &= 2\omega_{\mathbf{p}}\,\delta_{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{p}')\,, \\ \left[\hat{b}_\sigma(\mathbf{p}),\hat{b}_{\sigma'}(\mathbf{p}')\right]_+ &= 0\,. \end{aligned} \tag{4.124}$$

All linear relations of the 1-particle theory also hold for the quantized field. The field operator for the current density produced by all **particles** (not anti-particles) of charge[57] q is

$$\hat{j}_q^{(+)\mu}(x) \stackrel{\text{def}}{=} q\,\hat{\Psi}^{(+)}(x)^*\,\gamma^0\gamma^\mu\,\hat{\Psi}^{(+)}(x) \tag{4.125}$$

and transforms like a 4-vector field (compare Exercise 74) under the natural extension $\hat{U}_0(a,A)$ of $\hat{U}(a,A)$ to all of $\mathcal{H}_0$, fixed – together with the representation of space-time reflections – by the requirements

$$\hat{U}_0(a,A)^{-1}\hat{\Psi}^{(+)}(x)\hat{U}_0(a,A) = S(A)\hat{\Psi}^{(+)}\left(\Lambda_{A^{-1}}(x-a)\right)\,, \tag{4.126}$$

$$\mathfrak{P}^{-1}\hat{\Psi}^{(+)}(x)\mathfrak{P} = \gamma^0\hat{\Psi}^{(+)}(x^0,-\mathbf{x})\,, \tag{4.127}$$

$$\mathfrak{T}^{-1}\hat{\Psi}^{(+)}(x)\mathfrak{T} = i\gamma^1\gamma^3\hat{\Psi}^{(+)}(-x^0,\mathbf{x})^{*\,\mathrm{T}}\,, \tag{4.128}$$

and

$$\hat{U}_0(a,A)\Omega_0 = \mathfrak{P}\Omega_0 = \mathfrak{T}\Omega_0 = \Omega_0 \quad \left(\stackrel{\text{def}}{=} 1 \in \mathcal{H}^{(0)} = \mathbb{C}\right)\,. \tag{4.129}$$

consistent with the described 1-particle theory. Due to[58]

$$\omega_\sigma^{(+)}(\mathbf{p})^*\gamma^0\omega_{\sigma'}^{(+)}(\mathbf{p}) = \delta_{\sigma\sigma'} \tag{4.130}$$

and

$$\int \hat{\Psi}^{(+)}(x)^*\,\gamma^0\gamma^\mu\,\hat{\Psi}^{(+)}(x)\,d\mathbf{x} = \int_{p^0=\omega_{\mathbf{p}}} \hat{\hat{\Psi}}^{(+)}(\mathbf{p})^*\,\gamma^0\,\hat{\hat{\Psi}}^{(+)}(\mathbf{p})\,\frac{d\mathbf{p}}{2p^0}$$

(compare proof of (4.102)) the corresponding total charge is

$$\hat{Q}^{(+)} \stackrel{\text{def}}{=} \int \hat{j}_q^{(+)0}(x)\,d\mathbf{x} = q\int_{p^0=\omega_{\mathbf{p}}}\sum_{\sigma=\pm}\hat{b}_\sigma^*(\mathbf{p})\hat{b}_\sigma(\mathbf{p})\,\frac{d\mathbf{p}}{2p^0}\,, \tag{4.131}$$

[56]The natural invariant domain of definition for $\hat{b}_\sigma(\mathbf{p})$ is characterized by the conditions

$$b_n^{\sigma_1,\ldots,\sigma_n}(\mathbf{p}_1,\ldots,\mathbf{p}_n) \stackrel{\text{def}}{=} b_n(\mathbf{p}_1,\sigma_1;\ldots;\mathbf{p}_n,\sigma_n) \in \mathcal{S}(\mathbb{R}^{3n}) \quad \text{for fixed } (\sigma_1,\ldots,\sigma_n)\in\{+,-\}^n$$

and

$$b_n = 0 \quad \text{for sufficiently large } n\,.$$

[57]q need not be the electric charge!

[58](4.130) is trivial for $\mathbf{p}=0$ and therefore, by (4.89) and (4.97), also valid for $\mathbf{p}\neq 0$.

independent of $x^0 \in \mathbb{R}$.

The theory with only positive frequencies has (among others) the following disadvantages:

(P1): The current density (4.125) violates Einstein's causality principle, i.e:

$$x \times y \;\not\Longrightarrow\; \left[\hat{j}_q^{(+)\alpha}(x), \hat{j}_q^{(+)\beta}(y)\right]_- = 0\,.$$

(P2): In general, minimal coupling with an exterior field does not allow a solution for which both the incoming and the outgoing free Dirac field have vanishing negative frequency parts.[59]

Local Operator Field

Obviously problem (P2) requires the additional introduction of negative frequency solutions $\hat{\Psi}^{(-)}(x)$ of the Dirac field operator equation. These can be constructed using the 4-spinors

$$\omega_\sigma^{(-)}(\mathbf{p}) \stackrel{\text{def}}{=} \gamma^5 \omega_{-\sigma}^{(+)}(-\mathbf{p})\,. \tag{4.132}$$

Exercise 76 Show the following:[60]

1.

$$\omega_+^{(-)}(\mathbf{p}) = S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right)^{-1}\begin{pmatrix}0\\-1\\0\\+1\end{pmatrix} = S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right)^{-1}\begin{pmatrix}-1\\0\\+1\\0\end{pmatrix}, \quad \omega_-^{(-)}(\mathbf{p}) \tag{4.133}$$

in the representation (4.95).

2.

$$S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right) S\left(H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}\right) = \frac{p_\mu\gamma^\mu}{m}\gamma^0\,.$$

3.

$$\sum_{\sigma,\sigma'=\pm}\left(\omega_\sigma^{\sigma'}(\sigma'\mathbf{p})\right)_r \overline{\left(\omega_\sigma^{\sigma'}(\sigma'\mathbf{p})\right)_{r'}} = \left(\frac{p_\mu\gamma^\mu}{m}\gamma^0\right)_{r,r'}, \quad \text{with } p^0 = \omega_\mathbf{p}\,.$$

4.

$$\sum_{\sigma,\sigma'=\pm}\sigma'\left(\omega_\sigma^{\sigma'}(\sigma'\mathbf{p})\right)_r \overline{\left(\omega_\sigma^{\sigma'}(\sigma'\mathbf{p})\right)_{r'}} = \left(\gamma^0\right)_{r,r'}\,.$$

[59] This is related to ***Klein's paradox*** (see (Telegdi, 1995) and references given there).

[60] **Hint:** To prove the second statement, exploit (4.94) for the special case $\mathbf{p} = 0\,,\; A = H_{\frac{\mathbf{p}}{\omega_\mathbf{p}}}$.

Then

$$\hat{\Psi}^{(-)}(x) = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \int_{p^0=-\omega_{\mathbf{p}}} \overbrace{\sum_{\sigma=\pm} \hat{d}^*_\sigma(-\mathbf{p})\omega^{(-)}_\sigma(\mathbf{p})}^{\hat{\hat{\Psi}}^{(-)}(\mathbf{p}) \stackrel{\text{def}}{=}}\, e^{-ip^\mu x_\mu}\, \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \tag{4.134}$$

is indeed a solution of the Dirac equation:

$$(\gamma^\mu p_\mu - m)\,\hat{\hat{\Psi}}^{(-)}(\mathbf{p})_{|p^0=-\omega_{\mathbf{p}}} = 0\,.$$

Sketch of proof:

$$\begin{aligned}
&S\Big(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\Big)\,(\gamma^\mu p_\mu - m)_{|p^0=-\omega_{\mathbf{p}}}\,\omega^{(-)}_\sigma(\mathbf{p})\\
&\underset{(4.133)}{=} S\Big(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\Big)\,(\gamma^\mu p_\mu - m)_{|p^0=-\omega_{\mathbf{p}}}\,S\Big(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}}\Big)\omega^{(-)}_\sigma(0)\\
&\underset{(4.96)}{=} \left(\Big(\Lambda_{H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}}}\Big)^\mu{}_\nu\,\gamma^\nu p_\mu - m\right)_{|p^0=-\omega_{\mathbf{p}}}\omega^{(-)}_\sigma(0)\\
&= -\left(\gamma^\mu\Big(\Lambda_{H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}}(\omega_{\mathbf{p}},-\mathbf{p})\Big)_\mu + m\right)\omega^{(-)}_\sigma(0)\\
&= -m\,(\gamma^0+1)\omega^{(-)}_\sigma(0)\\
&= 0\,. \quad \blacksquare
\end{aligned}$$

With (4.132) and (4.132) one can show that the equations (4.126)–(4.128) hold also for

$$\hat{\Psi}(x) \stackrel{\text{def}}{=} \hat{\Psi}^{(+)}(x) + \hat{\Psi}^{(-)}(x) \tag{4.135}$$

instead of $\hat{\Psi}^{(+)}(x)$ **if** the

$$\breve{\hat{b}}_\sigma(\mathbf{p}) \stackrel{\text{def}}{=} \sigma \hat{d}_\sigma(\mathbf{p})$$

transform the same way as the $\hat{b}_\sigma(\mathbf{p})$ do.

Hence, in principle, one could identify the operators $\breve{\hat{b}}_\sigma(\mathbf{p})$ with the operators $\hat{b}_\sigma(\mathbf{p})$. However, a solution of the problems mentioned above will be achieved[61] only if the $\hat{d}_\sigma(\mathbf{p})$ the are interpreted as annihilation operators for **anti**-particles, with invariant domain $\breve{D}_0 \subset \breve{\mathcal{H}}_0$ (compare Section 2.4.1). Then the operators

$$\hat{b}_\sigma(\mathbf{p}) = \hat{b}_\sigma(\mathbf{p}) \otimes \hat{1}\,, \quad \hat{d}_\sigma(\mathbf{p}) = (-1)^{\hat{Q}^{(+)}/q}\,\hat{1} \otimes \hat{d}_\sigma(\mathbf{p})\,,$$

well-defined on

$$D_{\mathrm{D}} \stackrel{\text{def}}{=} D_0 \otimes \breve{D}_0 \subset \mathcal{H}_{\mathrm{D}} \stackrel{\text{def}}{=} \mathcal{H}_0 \otimes \breve{\mathcal{H}}_0\,,$$

fulfill the anti-commutation relations

$$\Big[\hat{d}_\sigma(\mathbf{p}), \hat{b}_{\sigma'}(\mathbf{p}')\Big]_+ = \Big[\hat{d}_\sigma(\mathbf{p})^*, \hat{b}_{\sigma'}(\mathbf{p}')\Big]_+ = 0 \tag{4.136}$$

[61] See, e.g., (Seiler, 1978).

in addition to (4.124) and the corresponding relations for the $\hat{d}_\sigma(\mathbf{p})$. From this, using the results of Exercise 76, one easily derives the anti-commutation relations

$$\begin{aligned} \left[\left(\hat{\Psi}^{(\sigma)}(x)\right)_r, \left(\hat{\Psi}^{(\sigma')}(x')\right)_{r'}\right]_+ &= 0\,, \\ \left[\left(\hat{\Psi}^{(\sigma)}(x)\right)_r, \left(\hat{\Psi}^{(\sigma')}(x')\right)^*_{r'}\right]_+ &= \delta_{\sigma\sigma'}\left((i\gamma^\mu\partial_\mu + m)\gamma^0\right)_{rr'} i\Delta_m^{(\sigma)}(x-x')\,. \end{aligned} \tag{4.137}$$

Thus the solution (4.135) of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\,\hat{\Psi}(x) = 0\,, \tag{4.138}$$

is a ***local Fermi field***, i.e.:

$$x \times x' \implies \left[\left(\hat{\Psi}(x)\right)_r, \left(\hat{\Psi}(x')\right)^{(*)}_{r'}\right]_+ = 0\,. \tag{4.139}$$

Moreover, the anti- commutation relations (4.137) imply that

$$\hat{\jmath}^\mu_q(x) \stackrel{\text{def}}{=} q:\hat{\Psi}(x)^*\gamma^0\gamma^\mu\hat{\Psi}(x): \tag{4.140}$$

is a ***local Bose field***, i.e.:

$$x \times x' \implies \left[\hat{\jmath}^\alpha_q(x), \hat{\jmath}^\beta_q(x')\right]_- = 0\,. \tag{4.141}$$

Normal ordering $:\,.\,:$ in (4.140) means that Fermi creation and annihilation operators have to be **anti**-commute, if necessary, irrespective of the actual anti-commutation relations until no creation operator is on the right of any annihilation operator.[62]

$\hat{\jmath}^\mu_q(x)$ is interpreted as observable of the total current density since, e.g., the following relations hold:

$$\int \hat{\jmath}^0_q(x)\,d\mathbf{x} = \hat{Q}_q \stackrel{\text{def}}{=} q\int_{p^0=\omega_\mathbf{p}} \sum_{\sigma=\pm}\left(\hat{b}^*_\sigma(\mathbf{p})\hat{b}_\sigma(\mathbf{p}) - \hat{d}^*_\sigma(\mathbf{p})\hat{d}_\sigma(\mathbf{p})\right)\frac{d\mathbf{p}}{2p^0}\,, \tag{4.142}$$

$$\partial_\mu\hat{\jmath}^\mu_q(x) = 0\,, \tag{4.143}$$

$$\hat{\jmath}^\mu_q(x) = \hat{\jmath}^\mu_q(x)^*\,. \tag{4.144}$$

Since

$$\Delta_m(0,\mathbf{x}) = 0\,, \quad \left(\partial_0\Delta(x)\right)_{|_{x^0=0}} = -\delta(\mathbf{x})\,, \tag{4.145}$$

the anti-commutation relations (4.137) imply the canonical anti-commutation relations

$$\begin{aligned} \left[\left(\hat{\Psi}(x^0,\mathbf{x})\right)_r, \left(\hat{\Psi}(x^0,\mathbf{x}')\right)_{r'}\right]_+ &= 0\,, \\ \left[\left(\hat{\Psi}(x^0,\mathbf{x})\right)_r, \left(\hat{\Psi}(x^0,\mathbf{x}')\right)^*_{r'}\right]_+ &= \delta_{rr'}\delta(\mathbf{x}-\mathbf{x}')\,. \end{aligned} \tag{4.146}$$

[62] Another effect of normal ordering, besides making $\hat{\jmath}^\mu_q(x)$ well-defined (as operator-valued distribution, not just a quadratic form), is that the quantized current density – contrary to the classical one – is no longer positive (compare remark on Corollary 2.2.14).

Therefore,[63]

$$\hat{P}^\mu_{\mathrm{D}} \stackrel{\text{def}}{=} \int :\hat{\Psi}(x)^* i\partial^\mu \hat{\Psi}(x): \mathrm{d}\mathbf{x} \tag{4.147}$$

is the generator of space-time translations, i.e.

$$i\left[\hat{P}^\mu_{\mathrm{D}}, \hat{\Psi}(x)\right]_- = \partial^\mu \hat{\Psi}(x)\,. \tag{4.148}$$

Final remark: Note that the **local** Dirac formalism is physically consistent only since anti-particles actually exist and are different from the corresponding particles (e.g. having charge of different sign). Contrary to this, local **Bose** fields may well describe particles (e.g. photons) that are identical with their anti-particles.

The Limit $\frac{m}{|\mathbf{p}|} \to 0$

Particle or anti-particles in states which can be created by applying (smeared versions of)

$$\hat{\Psi}_{\mathrm{R}}(x) = \frac{1}{2}(1+\gamma^5)\hat{\Psi}(x) \tag{4.149}$$

or $\hat{\Psi}_{\mathrm{R}}(x)^*$ to the vacuum vector are called ***right handed***. Similarly, particle or anti-particles in states which can be created by applying (smeared versions of)

$$\hat{\Psi}_{\mathrm{L}}(x) = \frac{1}{2}(1-\gamma^5)\hat{\Psi}(x) \tag{4.150}$$

or $\hat{\Psi}_{\mathrm{R}}(x)^*$ to the vacuum vector are called ***left handed***.

The transition $\frac{m}{|\mathbf{p}|} \to 0$ may be performed as discussed at the end of 4.2.3:

$$\begin{aligned} \text{for } m=0: \quad & \left[\hat{\Psi}_{\mathrm{R}}(x), \hat{h}\right]_+ = +\tfrac{1}{2}\hat{\Psi}_{\mathrm{R}}(x) \\ & \left[\hat{\Psi}_{\mathrm{L}}(x), \hat{h}\right]_+ = -\tfrac{1}{2}\hat{\Psi}_{\mathrm{L}}(x) \\ \text{where now: } \quad & \hat{h} \stackrel{\text{def}}{=} \int \mathrm{d}x :\hat{\Psi}(x)^* \left(\frac{1}{2}\gamma^5\gamma^0\gamma^j\right) \frac{i\partial^j}{|\boldsymbol{\nabla}_{\mathbf{x}}|}\hat{\Psi}(x):. \end{aligned} \tag{4.151}$$

(compare (4.110) and (4.117)). This implies:

$$\begin{array}{ll} \text{helicity of } \textit{right handed} \text{ massless particles:} & +1/2 \\ \text{helicity of } \textit{left handed} \text{ massless particles:} & -1/2 \\ \text{helicity of } \textit{left handed} \text{ massless } \textbf{anti-} \text{ particles:} & +1/2 \\ \text{helicity of } \textit{right handed} \text{ massless } \textbf{anti-} \text{ particles:} & -1/2 \end{array} \tag{4.152}$$

[63] Note that $[\hat{A}\hat{B},\hat{C}]_- = \hat{A}[\hat{B},\hat{C}]_+ - [\hat{A},\hat{C}]_+\hat{B}$.

4.3 The S-Matrix of Quantum Electrodynamics (QED)

Let us choose Heaviside units (in addition to $\hbar = c = 1$). Then, by (4.51) and Footnote 25,

$$\zeta = 1\,. \tag{4.153}$$

4.3.1 Naive Interaction Picture of QED

Asymptotic Description

For simplicity we consider the interaction of the electromagnetic field with the electron-positron field, only. Then, in the Gupta-Bleuler formalism, the interacting system of quantum electrodynamics is asymptotically identified (in the sense of 2.3.1) with the following 'free' system:

The basic Hilbert space (containing unphysical degrees of freedom) is

$$\mathcal{H} = \mathcal{H}_{\mathrm{D}} \otimes \mathcal{H}_{\mathrm{GB}}\,, \tag{4.154}$$

where $\mathcal{H}_{\mathrm{D}}$ is the Hilbert space of the free electron-positron system as described in 4.2.4 and $\mathcal{H}_{\mathrm{GB}}$ is the Hilbert space of the Gupta-Bleuler description of the quantized free electromagnetic field given in 4.1.3. The Dirac field $\hat{\Psi}(x)$ will now be identified with $\hat{\Psi}(x) \otimes \hat{1}$ and the Gupta-Bleuler field $\hat{A}^\mu_{\mathrm{GB}}(x)$ with $\hat{1} \otimes \hat{A}^\mu_{\mathrm{GB}}(x)$. Then the vacuum state vector

$$\Omega = \Omega_{\mathrm{D}} \otimes \Omega_{\mathrm{GB}}$$

of the total system is cyclic w.r.t. the fields $\hat{A}^\mu_{\mathrm{GB}}(x)$, $\hat{\Psi}(x)_r$ and $\hat{\Psi}(x)^\dagger_l$, well-defined as tempered field operators with invariant domain[64]

$$D = D_{\mathrm{D}} \otimes D_A\,.$$

Here, of course, $\Omega_{\mathrm{D}} \stackrel{\mathrm{def}}{=} \Omega_0 \otimes \breve{\Omega}_0$ denotes the vacuum state vector of the Dirac theory and Ω_{GB} the vacuum state vector of the Gupta-Bleuler formalism (also denoted by Ω in 4.1.3). The new vacuum state vector is invariant under the representation

$$\hat{U}(a,A) = \hat{U}_{\mathrm{D}}(a,A) \otimes \hat{V}\,(a,\Lambda_A)$$

of $\mathrm{iSL}(2,\mathbb{C})$ (compare 4.2.2), where $\hat{U}_{\mathrm{D}}(a,A) \stackrel{\mathrm{def}}{=} \hat{U}_0(a,A) \times \breve{\hat{U}}_0(a,A)$ denotes the representation of $\mathrm{iSL}(2,\mathbb{C})$ for the Dirac theory and $\hat{V}(a,\Lambda)$ the representation of $\mathcal{P}^\uparrow_+$ given in 4.1.3 . It is unitary w.r.t. the indefinite inner product

$$(\Phi_1 \mid \Phi_2) \stackrel{\mathrm{def}}{=} \left\langle \Phi_1 \mid \hat{1} \otimes \hat{\eta}\, \Phi_2 \right\rangle\,. \tag{4.155}$$

[64] Here $\otimes$ denotes the algebraic tensor product whereas in (4.154), of course, the topological tensor product of Hilbert spaces has to be taken.

with $\hat{\eta}$ defined by (4.29).

The subset of D describing physical states is

$$D_F = \left\{ \Phi \in D : \overline{\partial_\mu \hat{A}^{(+)\mu}_{\mathrm{GB}}(x)}\Phi = 0 \right\}$$

(compare (4.33)). Elements of D_F describe the same physical state if their difference is in

$$D_{00} = \{\Phi \in D_F : (\Phi \mid \Phi) = 0\}$$

(compare (4.17)). For **physical** states the expectation values[65] of the quantized **free** electromagnetic field

$$\hat{F}^{\mu\nu}_{\mathrm{GB}}(x) = \partial^\mu \hat{A}^\nu_{\mathrm{GB}}(x) - \partial^\nu \hat{A}^\mu_{\mathrm{GB}}(x)$$

(compare (4.13)) fulfill the **free** Maxwell equations

$$\partial_\mu \left(\Phi \mid \hat{F}^{\mu\nu}_{\mathrm{GB}}(x)\, \Phi \right) = 0 \quad \forall\, \Phi \in D_F\,.$$

(compare final remark of 4.1.3) although the current density (4.27) of the free Dirac field does not vanish on D_F. This means that the **free electromagnetic field operators describe only the radiative part** – not the field dragged along by the asymptotic charged particles (and contributing to their physical mass).

The Hamiltonian of the FS is

$$\hat{H}_0 = \hat{P}^0_{\mathrm{D}} + \hat{P}^0_{\mathrm{GB}} \tag{4.156}$$

with $\hat{P}^0_{\mathrm{D}}$ resp. $\hat{P}^0_{\mathrm{GB}}$ given by (4.147) resp. (4.48).

Formal Minimal Coupling

By (4.153) and (4.145), the commutation relations (4.27) imply the canonical commutation relations

$$\begin{aligned} \left[\hat{A}^\mu_{\mathrm{GB}}(x), \left(\frac{\partial}{\partial x'^0} \hat{A}^{\mu'}_{\mathrm{GB}}(x') \right)_{|_{x'^0 = x^0}} \right]_- &= -i\, g^{\mu\mu'}\, \delta(\mathbf{x} - \mathbf{x}') \\ \left[\hat{A}^\mu_{\mathrm{GB}}(x), \hat{A}^\mu_{\mathrm{GB}}(x^0, \mathbf{x}') \right]_- = \left[\frac{\partial}{\partial x'^0} \hat{A}^{\mu'}_{\mathrm{GB}}(x), \left(\frac{\partial}{\partial x'^0} \hat{A}^{\mu'}_{\mathrm{GB}}(x') \right)_{|_{x'^0 = x^0}} \right]_- &= 0\,. \end{aligned} \tag{4.157}$$

These and the canonical anti-commutation relations (4.146) for the Dirac Field (together with (4.27) and (4.137)) imply[66] that a **formal** solution of the fundamental differential equations

$$\begin{aligned} \Box\, \hat{A}^\mu_{\mathrm{int}}(x) = \hat{\jmath}^\mu_{\mathrm{int}}(x) &\overset{\text{‘def’}}{=} e^{i\hat{H}x^0}\, \hat{\jmath}^\mu_{-e}(0, \mathbf{x})\, e^{-i\hat{H}x^0}\,, \\ \left(i\gamma^\mu \left(\partial_\mu - ie\, g_{\mu\nu} \hat{A}^\nu_{\mathrm{int}}(x) \right) - m \right) \hat{\Psi}_{\mathrm{int}}(x) &= 0\,, \end{aligned} \tag{4.158}$$

[65] Recall that expectation values are given via $(. \mid .)$, in the Gupta-Bleuler formalism

[66] Recall Section 3.1.2

of QED is given by

$$\begin{aligned} \hat{A}^{\mu}_{\mathrm{int}}(x) &\stackrel{\mathrm{def}}{=} e^{i\hat{H}x^0}\,\hat{A}^{\mu}_{\mathrm{GB}}(0,\mathbf{x})\,e^{-i\hat{H}x^0}\,, \\ \hat{\Psi}_{\mathrm{int}}(x) &\stackrel{\mathrm{def}}{=} e^{i\hat{H}x^0}\,\hat{\Psi}(0,\mathbf{x})\,e^{-i\hat{H}x^0}\,, \end{aligned} \tag{4.159}$$

with

$$\hat{H} = \hat{P}^0_{\mathrm{D}} + \hat{P}^0_{\mathrm{GB}} + \hat{H}_{\mathrm{int}}\,, \tag{4.160}$$

where

$$\hat{H}_{\mathrm{int}} = \int g_{\mu\nu} : \hat{\jmath}^{\mu}_{-e}(0,\mathbf{x})\hat{A}^{\nu}_{\mathrm{GB}}(0,\mathbf{x}) : d\mathbf{x}\,. \tag{4.161}$$

Remark: Here, e is the modulus of the electron charge in **Heaviside units** (see Appendix A.3.4 von (Lücke, edyn)). Note that the definition of $\hat{\jmath}_{\mathrm{int}}(x)$ is only a formal one. $\hat{\jmath}^{\mu}_{q}(x)$ was defined in (4.140), but now * has to be replaced by †, of course. The transition from (4.138)/(4.9) to (4.158) is called ***minimal coupling*** .

Exercise 77 Show that

$$\left[\hat{\jmath}^{\mu}_{-\mathrm{e}}(0,\mathbf{x}'),\hat{\Psi}(0,\mathbf{x})\right]_{-} = +e\,\delta(\mathbf{x}-\mathbf{x}')\,\gamma^0\gamma^\mu\,\hat{\Psi}(0,\mathbf{x})\,.$$

By (4.156), the operator (3.9) is

$$\hat{H}_{\mathrm{I}}(x^0) = \int g_{\mu\nu} : \hat{\jmath}^{\mu}_{-e}(x)\hat{A}^{\nu}_{\mathrm{GB}}(x) : d\mathbf{x}\,,$$

i.e. we have to set[67]

$$\boxed{\hat{S}_1(x) = ie\,g_{\mu\nu} : \hat{\Psi}(x)^\dagger\gamma^0\gamma^\mu\hat{\Psi}(x)\hat{A}^{\nu}_{\mathrm{GB}}(x) :} \tag{4.162}$$

in (3.26) (and let $g \longrightarrow 1$) **if** the time ordering is suitably defined.

Transition Probabilities

If the actual state of the IS looks for $t \longrightarrow -\infty$ like the state of the FS described by Φ then the probability for a positive outcome of an ideal test whether the IS is in a state looking for $t \longrightarrow +\infty$ like the state of the FS described by Φ' is the ***transition probability***

$$p(\Phi \longrightarrow \Phi') \stackrel{\mathrm{def}}{=} \left|\left(\Phi' \mid \hat{S}_0\,\Phi\right)\right|^2 \tag{4.163}$$

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[67]Note that, by Lemma 4.1.3,

$$\Phi \in D_{00} \implies \int \hat{S}_1(x)\,g(x)\,dx\,\Phi \in \overline{D_{00}}^{\langle\cdot|\cdot\rangle}$$

holds for all $g \in \mathcal{S}(\mathbb{R}^4)$.

(recall Section 2.3.1). In practice one is only interested in states Φ, Φ' of the form

$$\int \hat{c}_{k_1}(\mathbf{p}_1)^\dagger \cdots \hat{c}_{k_N}(\mathbf{p}_N)^\dagger \, \varphi(\mathbf{p}_1, \ldots, \mathbf{p}_N) \, \mathrm{d}\mathbf{p}_1 \cdots \mathrm{d}\mathbf{p}_N \, \Omega \,, \quad \varphi \in \mathcal{S}\left(\mathbb{R}^{3N}\right) ,$$

where the $\hat{c}_k(\mathbf{p})^\dagger$ are suitable creation operators. Then the essential task is to calculate the ***scattering amplitudes***

$$\left(\hat{c}_{j_1}(\mathbf{p}'_1)^\dagger \cdots \hat{c}_{j_{N'}}(\mathbf{p}'_{N'})^\dagger \, \Omega \mid \left(\hat{S}_0 - \hat{1}\right) \hat{c}_{k_1}(\mathbf{p}_1)^\dagger \cdots \hat{c}_{k_N}(\mathbf{p}_N)^\dagger \, \Omega\right) . \tag{4.164}$$

4.3.2 General Perturbation Theory

Generalization of WICK's Theorem

To every type of field appearing in $\hat{S}_1(x)$ and its adjoint (if not identical to the field itself) one assign a characteristic type of line; e.g. wavy lines for photons, simple lines with upward orientation for electrons etc.

The ***transition matrix***

$$\hat{S}_0 - \hat{1} = \sum_{n=1}^{\infty} \int T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{4.165}$$

(compare (3.26)) is evaluated by first writing the expressions

$$T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) - \left\langle \Omega \right| T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) \Omega \Big\rangle$$

as linear combinations of normally ordered products. This involves so-called ***internal contractions*** of pairs of field operators appearing in $\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)$ and depending on different variables x_ν. These contractions will be characterized by joining typical lines attached the operators to be contracted. For instance, the (dashed) line in

$$:\hat{\Phi}_{11}(x_1) \cdots \underbrace{\hat{\Phi}_{j1}(x_1) \cdots :: \hat{\Phi}_{12}(x_2) \cdots :: \hat{\Phi}_{1n}(x_n) \cdots \hat{\Phi}_{kn}}(x_n) \cdots :$$

means internal contraction of the pair of field operators $\hat{\Phi}_{j_1}(x_1), \hat{\Phi}_{k_n}(x_n)$ (corresponding to the line type), i.e. this pair has to be replaced by its ***propagator***[68]

$$\left(\Omega \,\middle|\, T\left(\hat{\Phi}_{j_1}(x_1)\hat{\Phi}_{k_n}(x_n)\right)\Omega\right) .$$

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[68]The definition of the propagators implies that only contractions of pairs with fitting line types can be different from zero.

If $\hat{\Phi}_{j_1}(x_1)$ and $\hat{\Phi}_{k_n}(x_n)$ are both Fermi fields with an odd number of Fermi fields in between, a factor -1 has to be applied in addition. Here the time ordering T is defined as the least singular ***covariant*** operation fulfilling the condition

$$\boxed{\begin{aligned} T\left(\hat{\Phi}(x)\hat{\Phi}'(x')\right) &\stackrel{\text{def}}{=} \begin{cases} \hat{\Phi}(x)\hat{\Phi}'(x') & \text{for } x^0 > x'^0\,,\\ \sigma\hat{\Phi}'(x')\hat{\Phi}(x) & \text{for } x^0 < x'^0\,,\end{cases}\\ \text{where:}\quad \sigma &\stackrel{\text{def}}{=} \begin{cases} -1 & \text{if both } \hat{\Phi} \text{ and } \hat{\Phi}' \text{ are Fermi fields}\,,\\ +1 & \text{else}\,.\end{cases}\end{aligned}} \tag{4.166}$$

For simplicity we assume that $\hat{S}_1(x)$, which should be a Bose field,[69] is a monomial[70] of field operators. Then one may prove[71]

$$\begin{aligned} & T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right) - \left\langle\Omega\right| T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right)\Omega\Big\rangle\\ = \;& :\hat{S}_1(x_1)\cdots\hat{S}_1(x_n):\\ & +:\hat{S}_1(x_1)\cdots\hat{S}_1(\underbrace{x_j)\cdots\cdots\hat{S}_1(x_k})\cdots\hat{S}_1(x_n):\\ & \qquad + \text{ all other 1-fold contractions}\\ & +:\cdots\hat{S}_1(x_{j_1})\cdots\hat{S}_1(x_{j_2})\cdots\cdots\hat{S}_1(x_{j_3})\cdots\hat{S}_1(x_{j_4})\cdots\cdots:\\ & \qquad + \text{ all other 2-fold contractions}\\ & +:\cdots\hat{S}_1(x_{j_1})\cdots\hat{S}_1(x_{j_2})\cdots\cdots\hat{S}_1(x_{j_3})\cdots\cdots:\\ & \qquad + \text{ all other } \textbf{incomplete} \text{ multiple contractions}\,. \end{aligned}$$

for pairwise different x^0_ν by straightforward generalization of the techniques used in Section 3.2.1.

Let $\hat{K}(x_1,\ldots,x_n)$ be any of the incompletely contracted $:\hat{S}_1(x_1)\cdots\hat{S}_1(x_n):$-terms (maybe $:\hat{S}_1(x_1)\cdots\hat{S}_1(x_n):$ itself) and assume that products of propagators are suitably defined, if necessary.[72] Then its contribution to (4.164) is evaluated by rewriting

$$\hat{c}_{j_{N'}}(\mathbf{p}'_{N'})\cdots\hat{c}_{j_1}(\mathbf{p}'_1)\,\hat{K}(x_1,\ldots,x_n)\,\hat{c}^\dagger_{k_1}(\mathbf{p}_1)\cdots\hat{c}^\dagger_{k_N}(\mathbf{p}_N) \tag{4.167}$$

as the $K(x_1,\ldots,x_n;\mathbf{p}_1,\ldots,\mathbf{p}_N;\mathbf{p}'_1,\ldots,\mathbf{p}'_{N'})$-fold of $\hat{1}$ plus some linear combination of normally ordered field products. Then, by (4.165), the contribution of $\hat{K}(x_1,\ldots,x_n)$ to (4.164) is given by

$$\frac{1}{n!}\int K(x_1,\ldots,x_n;\mathbf{p}_1,\ldots,\mathbf{p}_N;\mathbf{p}'_1,\ldots,\mathbf{p}'_{N'})\,dx_1\cdots dx_n\,,$$

where, by straightforward generalization of Wick's theorem,

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[69]Note that the time ordering of the $\hat{S}_1(x_\nu)$-factors used in (3.26) resp. (4.165) was that for Bose fields.

[70]The necessary modifications for polynomials are obvious.

[71]As in 3.2.1, all field operators have to be formally considered as different in the contraction schemes. Recall $\left\langle\Omega\mid\hat{S}_0\,\Omega\right\rangle = \langle\Omega\mid\Omega\rangle$.

[72]Actually, as discussed in Section 3.2.2 for the $\lambda\hat{\Phi}^4$-theory, physically correct definition of these products is the task of renormalization theory.

$$
\begin{aligned}
&K(x_1,\ldots,x_n;\mathbf{p}_1,\ldots,\mathbf{p}_N;\mathbf{p}'_1,\ldots,\mathbf{p}'_{N'}) \\
&= \hat{c}_{j_{N'}}(\mathbf{p}'_{N'})\cdots\hat{c}_{j_1}(\mathbf{p}'_1)\,:\hat{K}(x_1,\ldots\ldots\ldots,x_n):\hat{c}^\dagger_{k_1}(\mathbf{p}_1)\cdots\hat{c}^\dagger_{k_N}(\mathbf{p}_N)
\end{aligned}
\tag{4.168}
$$

+ all other **complete** external contractions .

External contractions are those involving at least one of the operators $\hat{c}_k(\mathbf{p})^\dagger$ or $\hat{c}_j(\mathbf{p}')$ and for which the corresponding 2-point function (without time-ordering) is used instead of the propagator.

Feynman Diagrams with External Momenta

The nontrivial terms contribution to (4.168) can be represented by Feynman diagrams G of the following type:

1. G consists vertices (corresponding to the variables x_ν) numbered $1,\ldots,V_G > 0$ and lines representing internal or external contractions, the free ends of the latter being provided with a unique characterization of the operator ($\hat{c}_k(\mathbf{p})^\dagger$ or $\hat{c}_j(\mathbf{p}')$) to be contracted.

2. Every vertex is connected with a family of contraction lines corresponding to the family of field factors building $\hat{S}_1(x)$.

3. Every line representing an internal contraction forms a direct link between two vertices.

4. Every line representing an external contraction has at least one free end.

5. A free end of a line representing an external contraction has to be the lowest point of this line if it corresponds to a creations operator.

6. A free end of a line representing an external contraction has to be the highest point of this line if it corresponds to an annihilation operator.

As demonstrated in 3.2.3 for the $\lambda\hat{\Phi}^4$-theory one can show that the terms in (4.168) corresponding to diagrams with vacuum subdiagrams may be skipped.

The admitted diagrams are heuristically interpreted as follows:

- Every lower end of an exterior line represents an incoming particle corresponding to the attached information.

- Every higher end of an exterior line represents an outgoing particle corresponding to the attached information.

- Every internal line represent a ***virtual*** particle corresponding to the line type.

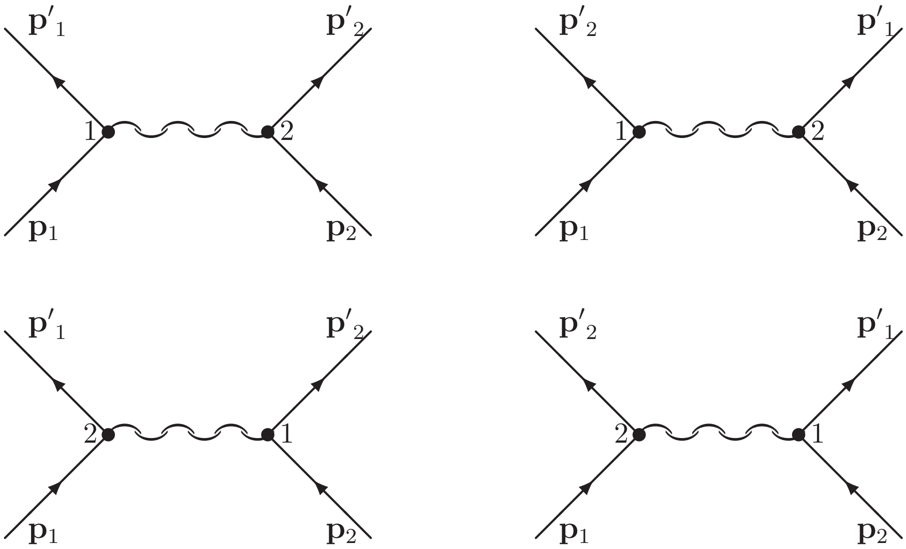

Figure 4.1: Møller scattering[73]

- Every vertex represents an event where the particles corresponding to the connected lines interact with each other being annihilated or created respectively.

The four diagrams of QED sketched in Figure 4.1, e.g., describe scattering of two electrons with incoming momenta $\mathbf{p}_1, \mathbf{p}_2$ and outgoing momenta $\mathbf{p}_1', \mathbf{p}_2'$.

Similarly, the four diagrams of Figure 4.2 describe scattering of an electron having initial momentum $\mathbf{p}_1$ and final momentum $\mathbf{p}_1'$ with a positron having initial momentum $\mathbf{p}_2$ and final momentum $\mathbf{p}_2'$.

4.3.3 The Feynman Rules of QED

In QED, i.e. if $\hat{S}_1(x)$ is given by (4.162), every vertex is connected with exactly three lines:[74] a wavy photon line corresponding to $\hat{A}_{\mathrm{GB}}$, an incoming solid fermion line corresponding to $\hat{\Psi}$, and an outgoing solid fermion line corresponding to $\overline{\hat{\Psi}} = \hat{\Psi}^\dagger \gamma^0$. This is sketched in Figure 4.3.

The propagators corresponding to internal lines connecting the indices j and $k > j$ are those given by Figure 4.4. If one is interested only in **linearly polarized**

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[73] Actually, the spin states should also be indicated at the ends of the external lines.

[74] Actually, since $\hat{S}_1(x)$ is a sum of monomials (recall Footnote 70) should be considered as a family of lines with indices to be summed over according to the Feynman rules formulated below.

[75] The orientation is relevant only for external lines.

[76] Die notation r_j, l_k for the indices is to indicate the original position (left/right) of the contracted field operators relative to each other. For the definition of the bosonic propagator recall (3.28)/(3.30) and (4.26). For the fermionic propagator recall (4.137).

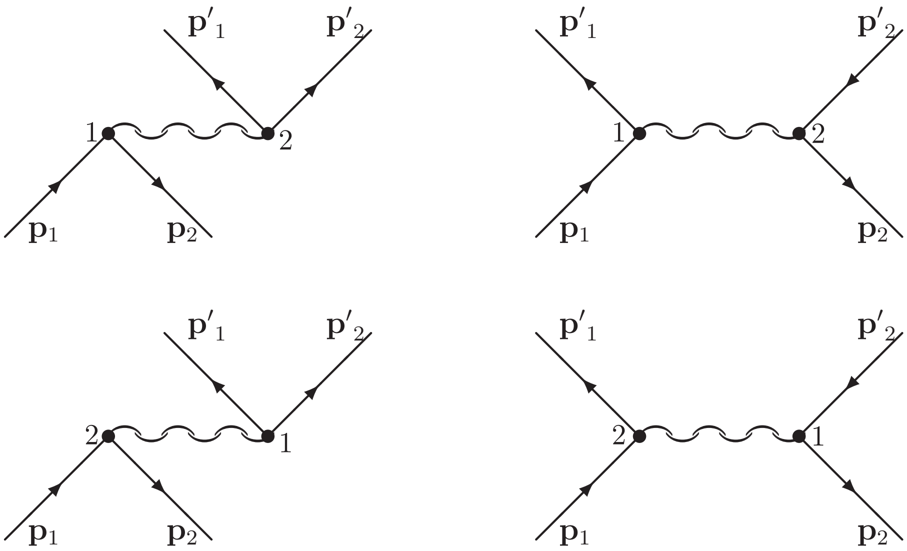


Figure 4.2: Bhabba scattering

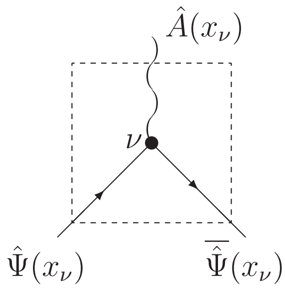


Figure 4.3: A typical[75] vertex of QED

$$\widehat{=} \left\langle \Omega \mid T\left(\left(\hat{\Psi}(x_j)\right)_{r_j} \left(\hat{\Psi}(x_k)^\dagger \gamma^0\right)_{l_k} \right) \Omega \right\rangle \stackrel{\text{def}}{=} +i(2\pi)^{-4} \lim_{\epsilon \to +0} \int \mathrm{d}q \frac{(\gamma^\mu q_\mu + m)_{r_j l_k}}{q^2 - m^2 + i\epsilon} e^{-iq(x_j - x_k)}$$

$$\widehat{=} \left\langle \Omega \mid T\left(\left(\hat{\Psi}(x_j)^\dagger \gamma^0\right)_{l_j} \left(\hat{\Psi}(x_k)\right)_{r_k} \right) \Omega \right\rangle \stackrel{\text{def}}{=} -i(2\pi)^{-4} \lim_{\epsilon \to +0} \int \mathrm{d}q \frac{(\gamma^\mu q_\mu + m)_{r_k l_j}}{q^2 - m^2 + i\epsilon} e^{-iq(x_k - x_j)}$$

$$\widehat{=} \left\langle \Omega \mid T\left(\hat{A}^{\mu_j}(x_j) \hat{A}^{\mu_k}(x_k) \right) \Omega \right\rangle \stackrel{\text{def}}{=} -g^{\mu_j \mu_k}\, i(2\pi)^{-4} \lim_{\epsilon \to +0} \int \frac{\mathrm{d}k}{k^\nu k_\nu + i\epsilon} e^{\pm ik(x_j - x_k)}$$

Figure 4.4: Internal Lines[76] of QED ($j < k$)

asymptotic 'particles' then the $\hat{c}_j(\mathbf{p})$ used in (4.164) are Dirac operators $\hat{b}_\sigma(\mathbf{p}), \hat{d}_\sigma(\mathbf{p})$ (compare (4.2.4)) or **transversal** photon operators[77]

$$\hat{a}_\epsilon(\mathbf{p}) \stackrel{\text{def}}{=} \boldsymbol{\epsilon}(\mathbf{p}) \cdot \sum_{j=1}^{3} \mathbf{e}_j\, \hat{a}^j(\mathbf{p})$$

with

$$\epsilon^0(\mathbf{p}) = 0\,,\ \boldsymbol{\epsilon}(\mathbf{p}) = \boldsymbol{\epsilon}(\mathbf{p})^*\,,\ |\boldsymbol{\epsilon}(\mathbf{p})| = 1\,,\ \mathbf{p} \cdot \boldsymbol{\epsilon}(\mathbf{p}) = 0\,. \tag{4.169}$$

Then the 2-point functions corresponding to external lines are those of Figure 4.5.

As in 3.2.3, any two diagrams G_1, G_2 are considered as ***equal*** if they differ only by their diagramatical realization resp. are called ***equivalent*** ($G_1 \cong G_2$) if they differ only by the distribution of their vertex indices.

If all integrals over internal momenta exist[78] (absence of ultraviolet divergences) then $\hat{A}_G$ can be evaluated by the following naive ***Feynman rules*** of QED:

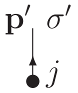
1. Assign suitable momenta to every line of G and then replace it by a corre-

Draft, November 9, 2007

[77]Here $\mathbf{e}_j$ and $\hat{a}^j(\mathbf{p})$ are to be understood in the sense of (4.57).
[78]This is the case if G is a so-called ***tree diagram***, i.e. if it does not contain closed loops.

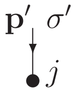
$$\widehat{=} \quad \left\langle \Omega \right| \hat{a}_{\epsilon'}(\mathbf{k}') \hat{A}^{\mu_j}(x_j) \Omega \Big\rangle = (2\pi)^{-\frac{3}{2}} \epsilon'^{\mu_j}(\mathbf{p}')\, e^{+ik'x_j}|_{k'^0=|\mathbf{k}'|}$$

$$\widehat{=} \quad \left\langle \Omega \right| \hat{A}^{\mu_j}(x_j) \hat{a}^\dagger_{\epsilon}(\mathbf{k}) \Omega \Big\rangle = (2\pi)^{-\frac{3}{2}} \epsilon^{\mu_j}(\mathbf{p})\, e^{-ikx_j}|_{k^0=|\mathbf{k}|}$$



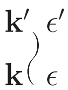
$$\widehat{=} \quad \left\langle \Omega \right| \hat{b}_{\sigma'}(\mathbf{p}') \left(\hat{\Psi}(x_j)^\dagger \gamma^0 \right)_l \Omega \Big\rangle = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \left(\omega^{(+)}_{\sigma'}(\mathbf{p}')^* \gamma^0 \right)_l e^{+ip'x_j}|_{p'^0=\omega_{\mathbf{p}''}}$$

$$\widehat{=} \quad \left\langle \Omega \right| \left(\hat{\Psi}(x_j) \right)_r \hat{b}_{\sigma}(\mathbf{p})^\dagger \Omega \Big\rangle = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \left(\omega^{(+)}_{\sigma}(\mathbf{p}) \right)_r e^{-ipx_j}|_{p^0=\omega_{\mathbf{p}}}$$

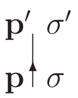


$$\widehat{=} \quad \left\langle \Omega \right| \hat{d}_{\sigma'}(\mathbf{p}') \left(\hat{\Psi}(x_j) \right)_r \Omega \Big\rangle = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \left(\omega^{(-)}_{\sigma'}(-\mathbf{p}') \right)_r e^{+ip'x_j}|_{p'^0=\omega_{\mathbf{p}'}}$$

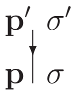
$$\widehat{=} \quad \left\langle \Omega \right| \left(\hat{\Psi}(x_j)^\dagger \gamma^0 \right)_l \hat{d}^\dagger_{\sigma}(\mathbf{p}) \Omega \Big\rangle = \sqrt{2m}\,(2\pi)^{-\frac{3}{2}} \left(\omega^{(-)}_{\sigma}(-\mathbf{p})^* \gamma^0 \right)_l e^{-ipx_j}|_{p^0=\omega_{\mathbf{p}}}$$



$$\widehat{=} \quad \left\langle \Omega \right| \hat{a}_{\epsilon'}(\mathbf{k}') \hat{a}_{\epsilon}(\mathbf{k})^\dagger \Omega \Big\rangle = 2|\mathbf{k}|\, \boldsymbol{\epsilon}'(\mathbf{k}') \cdot \boldsymbol{\epsilon}(\mathbf{k}')\, \delta(\mathbf{k} - \mathbf{k}')$$



$$\widehat{=} \quad \left\langle \Omega \right| \hat{b}_{\sigma'}(\mathbf{p}') \hat{b}_{\sigma}(\mathbf{p})^\dagger \Omega \Big\rangle = 2\omega_{\mathbf{p}}\, \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}')$$

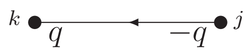


$$\widehat{=} \quad \left\langle \Omega \right| \hat{d}_{\sigma'}(\mathbf{p}') \hat{d}_{\sigma}(\mathbf{p})^\dagger \Omega \Big\rangle = 2\omega_{\mathbf{p}}\, \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}')$$

Figure 4.5: External lines of QED

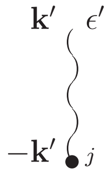
sponding factor:

$$-i(2\pi)^{-4}\epsilon(k-j)\lim_{\epsilon\to+0}\frac{(\gamma^\nu q_\nu+m)_{|r_k l_j}}{q^2-m^2+i\epsilon} \quad \widehat{=}$$

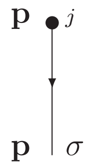


$$-g^{\mu_j\mu_k}\, i(2\pi)^{-4}\lim_{\epsilon\to+0}\frac{1}{k^2+i\epsilon} \quad \widehat{=}$$

$$(2\pi)^{-\frac{3}{2}}\epsilon'^{\mu_j}(\mathbf{k}') \quad \widehat{=}$$



$$\sqrt{2m}(2\pi)^{-\frac{3}{2}}\left(\omega_\sigma^{(-)}(-\mathbf{p})^*\gamma^0\right)_{l_j} \quad \widehat{=}$$



etc.

2. For $j=1,\ldots,V_G$ replace vertex j by the factor[79]

$$-ie\left(\gamma_{\mu_j}\right)_{l_j r_j}(2\pi)^4\delta(P)$$

where

$$P \stackrel{\text{def}}{=} \begin{cases}\text{sum of all 4-momenta}^{80}\text{ assigned to line ends}\\ \text{connected with vertex } j\,.\end{cases}$$

3. Take the product of all factors and sum over all indices r_j, l_j, μ_j and integrate over all momenta assigned to internal lines (only one integration per internal line).

4. Finally multiply by $\sigma_G \in \{+1,-1\}$ to be determined as follows:

   Write down the corresponding contraction scheme, e.g.

$$\hat{c}_{j_{N'}}\left(\mathbf{p}'_{N'}\right)\cdots\hat{c}_{j_1}\left(\mathbf{p}'_1\right):\hat{S}_1(x_1)\cdots\cdots\hat{S}\left(x_{V_G}\right):\hat{c}^\dagger_{k_1}(\mathbf{p}_1)\cdots\hat{c}^\dagger_{k_N}(\mathbf{p}_N)\,,$$

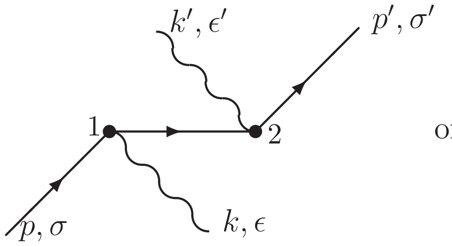
   and rearrange the operators such that all contracted pairs become direct neighbors without changing the relative order of the operators forming any such pair. Then σ_G is the signum of the overall permutation resulting this way.

[79] The δ-function results from integration over $x_1,\ldots,x_{V_G}$.

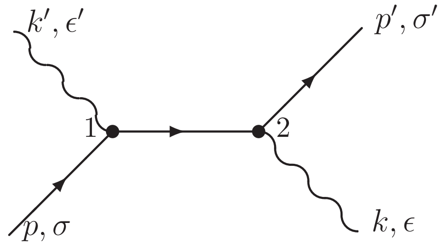
[80] Here, the 3-momenta assigned to external lines have to lifted to the mass shell of the asymptotic 'particle' type.

4.3.4 Example: Compton Scattering

For ***Compton scattering***, i.e. for electron-photon scattering in second order[81] of perturbation theory, only graphs equivalent to

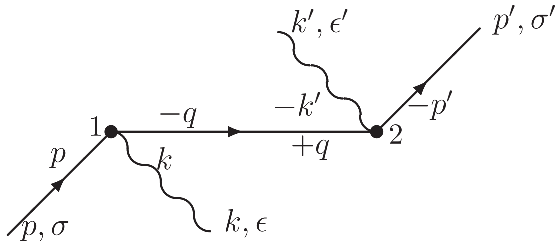


or



are relevant. The factors corresponding to the graph

$G_1 \;=$



are the following:

$$\sqrt{2m}\,(2\pi)^{-3/2}\left(\omega_\sigma^{(+)}(\mathbf{p})\right)_{r_1}$$

$$(2\pi)^{-3/2}\epsilon^{\mu_1}(\mathbf{k})\,,$$

$$-ie\,(\gamma_{\mu_1})_{l_1 r_1}\,(2\pi)^4\,\delta(p+k-q)\,,$$

$$-i\,(2\pi)^{-4}\lim_{\epsilon\to+0}\frac{(\gamma^\nu q_\nu + m)_{r_2 l_1}}{q^2 - m^2 + i\epsilon}$$

$$-ie\,(\gamma_{\mu_2})_{l_2 r_2}\,(2\pi)^4\,\delta(q-p'-k')\,,$$

$$(2\pi)^{-3/2}\epsilon'^{\mu_2}(\mathbf{k}')\,,$$

$$\sqrt{2m}\,(2\pi)^{-3/2}\left(\omega_{\sigma'}^{(+)}(\mathbf{p}')^*\gamma^0\right)_{l_2}\;.$$

The signum of the permutation mapping

$$\hat b_{\sigma'}(\mathbf{p}')\hat\Psi(x_1)^\dagger\hat\Psi(x_1)\hat\Psi(x_2)^\dagger\hat\Psi(x_2)\hat b_\sigma^\dagger(\mathbf{p})$$

onto

$$\left(\hat\Psi(x_1)\hat b_\sigma^\dagger(\mathbf{p})\right)\left(\hat\Psi(x_1)^\dagger\hat\Psi(x_2)\right)\left(\hat b_{\sigma'}(\mathbf{p}')\hat\Psi(x_2)^\dagger\right)$$

[81] This means that only contributions with $V_G \le 2$ are considered.

is $\sigma_{G_1} = -1$. Therefore

$$\begin{aligned}\hat{A}_{G_1} &= \frac{me^2}{2i\pi^2} \lim_{\epsilon\to+0} \int \delta(p+k-q)\delta(q-p'-k')\omega^{(+)}_{\sigma'}(\mathbf{p}')^* \gamma^0 \not{\epsilon}'(\mathbf{k}') \cdot \\ &\qquad \cdot \frac{\not{q}+m}{q^2-m^2+i\epsilon} \not{\epsilon}(\mathbf{k})\omega^{(+)}_{\sigma}(\mathbf{p})\,\mathrm{d}q \\ &= \frac{me^2}{2i\pi^2} \delta(p+k-p'-k')\omega^{(+)}_{\sigma'}(\mathbf{p}')^*\gamma^0 \not{\epsilon}'(\mathbf{k}') \frac{\not{p}+\not{k}+m}{2p\cdot k} \not{\epsilon}(\mathbf{k})\omega^{(+)}_{\sigma}(\mathbf{p})\,,\end{aligned}$$

where we use the usual conventions

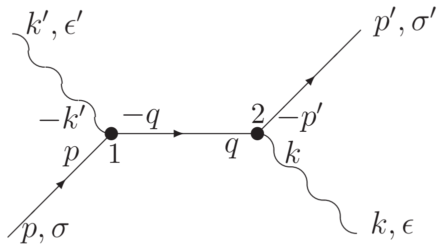
$$\not{x} \stackrel{\text{def}}{=} \gamma^\mu x_\mu\,, \qquad \not{p} \stackrel{\text{def}}{=} \gamma^\mu p_\mu \quad \text{etc.}$$

Similarly we get

$$\hat{A}_{G_2} = \frac{me^2}{2i\pi^2} \delta(p+k-p'-k')\omega^{(+)}_{\sigma'}(\mathbf{p}')^*\gamma^0 \not{\epsilon}(\mathbf{k}) \frac{\not{p}-\not{k}'+m}{-2p\cdot k'} \not{\epsilon}'(\mathbf{k}')\omega^{(+)}_{\sigma}(\mathbf{p})$$

for

$G_2 =$



and hence

$$\begin{aligned}&\left\langle \hat{b}_{\sigma'}(\mathbf{p}')\hat{a}_{\epsilon'}(\mathbf{k}')\Omega \mid \left(\hat{S}_0 - \hat{1}\right)_{\text{2. order}} \hat{b}_{\sigma}(\mathbf{p})^\dagger \hat{a}_{\epsilon}(\mathbf{k})^\dagger\Omega \right\rangle \\ &= \frac{me^2}{2i\pi^2}\delta(p+k-p'-k')\,\omega^{(+)}_{\sigma'}(\mathbf{p}')^*\Gamma(p,k,k')\omega^{(+)}_{\sigma}(\mathbf{p})\,; \quad \text{where:} \\ &\Gamma(p,k,k') \stackrel{\text{def}}{=} \gamma^0\left(\not{\epsilon}'(\mathbf{k}')\frac{\not{p}+\not{k}+m}{2p\cdot k}\not{\epsilon}(\mathbf{k}) + \not{\epsilon}(\mathbf{k})\frac{\not{p}-\not{k}'+m}{-2p\cdot k'}\not{\epsilon}'(\mathbf{k}')\right)\,.\end{aligned} \tag{4.170}$$

The cross section for unpolarized incoming and outcoming electrons[82] is

$$\begin{aligned}&\sigma(p,k,\epsilon,\epsilon') \\ &= \frac{(2\pi)^2}{8p\cdot k}\sum_{\sigma,\sigma'=\pm}\int\left|\frac{\left\langle \hat{b}_{\sigma'}(\mathbf{p}')\hat{a}_{\epsilon'}(\mathbf{k}')\Omega \mid \left(\hat{S}_0-\hat{1}\right)_{\text{2. order}} \hat{b}_{\sigma}(\mathbf{p})^\dagger\hat{a}_{\epsilon}(\mathbf{k})^\dagger\Omega\right\rangle}{\delta(p'+k'-p-k)}\right|^2 \times \\ &\qquad\qquad \times\,\delta(p'+k'-p-k)\,\frac{\mathrm{d}\mathbf{k}'}{2k'^0}\frac{\mathrm{d}\mathbf{p}'}{2p'^0}\,,\end{aligned} \tag{4.171}$$

[82]This means summation of the polarizations of the outgoing electron and averaging over the the polarizations of the incoming electron. Therefore, we need an additional factor $\frac{1}{2}$ compared to the formula of Exercise 53.

where p resp. k is the momentum of the incoming electron resp. photon and ϵ resp. ϵ' is the polarization of the incoming resp. outgoing photon. Note that all 4-momenta have to be on the corresponding mass shell:

$$k^0 = |\mathbf{k}| \ , \ k'^0 = |\mathbf{k}'| \ , \quad p^0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^{\,2} + m^2} \, , \ p'^0 = \omega_{\mathbf{p}'} \, .$$

In order to evaluate (4.171) we have to calculate

$$I \stackrel{\text{def}}{=} \int \sum_{\sigma,\sigma'=\pm} \left| \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \, \Gamma(p,k,k') \, \omega_{\sigma}^{(+)}(\mathbf{p}) \right|^2 \delta(p' + k' - p - k) \, \frac{d\mathbf{p}'}{2\omega_{\mathbf{p}'}} \frac{d\mathbf{k}'}{2|\mathbf{k}'|} \, . \tag{4.172}$$

By $\mathbf{p} = 0$ and $\epsilon^0(\mathbf{k}) = \epsilon'^0(\mathbf{k}') = 0$, (4.108) gives

$$(\not p + m) \ \not\epsilon^{(\prime)}(\mathbf{p}) \omega_{\sigma}^{(+)}(\mathbf{p}) = - \ \not\epsilon^{(\prime)}(\mathbf{p}) \, (\not p - m) \, \omega_{\sigma}^{(+)}(\mathbf{p}) = 0 \, .$$

With

$$\left(\omega'^* \hat\Gamma \omega \right)^{*\mathrm{T}} = (\omega'^*)^* \left(\hat\Gamma^* \right)^{\mathrm{T}} \omega^{*\mathrm{T}} = \left(\omega^* \hat\Gamma^* \omega' \right)^{\mathrm{T}}$$

this implies

$$\left| \omega'^* \hat\Gamma \omega^* \right|^2 = \left(\omega'^* \hat\Gamma \omega \right) \left(\omega^* \hat\Gamma \omega' \right) \, .$$

In the standard representation (4.115) the latter together with

$$\boxed{\sum_{\sigma=\pm} \left(\omega_{\sigma}^{(+)}(\mathbf{p}) \right)_r \left(\omega_{\sigma}^{(+)}(\mathbf{p})^* \gamma^0 \right)_l = \left(\frac{\not p + m}{2m} \right)_{rl}} \tag{4.173}$$

implies

$$I = \int \frac{d\mathbf{p}'}{2\omega_{\mathbf{p}'}} \frac{d\mathbf{k}'}{2\,|\mathbf{k}'|} \, \delta(p' + k' - p - k) \, \mathrm{Tr} \left(\gamma^0 \, \hat\Gamma(p,k,k')^* \frac{\not p' + m}{2m} \right) \, ,$$

where

$$\hat\Gamma(p,k,k') \stackrel{\text{def}}{=} \gamma^0 \left(\frac{\not\epsilon'(\mathbf{k}') \ \not k \ \not\epsilon(\mathbf{k})}{2\,|\mathbf{k}|\, m} + \frac{\not\epsilon(\mathbf{k}) \ \not k' \ \not\epsilon'(\mathbf{k}')}{2\,|\mathbf{k}'|\, m} \right) \tag{4.174}$$

and hence

$$\hat\Gamma(p,k,k')^* = \gamma^0 \left(\frac{\not\epsilon(\mathbf{k}) \ \not k \ \not\epsilon'(\mathbf{k}')}{2\,|\mathbf{k}|\, m} + \frac{\not\epsilon'(\mathbf{k}') \ \not k' \ \not\epsilon(\mathbf{k})}{2\,|\mathbf{k}'|\, m} \right) \tag{4.175}$$

(because of $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^*$ and $\gamma^0 = \gamma^{0*}$).

In the laboratory system, i.e. for

$$\mathbf{p} = 0 \, , \quad p^0 = m \, ,$$

the conditions

$$p' + k' = p + k \ , \quad p'^2 = m^2 \ , \quad k'^2 = 0$$

are well known to imply the so-called ***Compton condition***

$$\boxed{k'^0 = |\mathbf{k}'| = \frac{|\mathbf{k}|}{1 + \frac{|\mathbf{k}|}{m}(1-\cos\vartheta)}\,, \quad \cos\vartheta \stackrel{\text{def}}{=} \frac{\mathbf{k}}{|\mathbf{k}|}\cdot\frac{\mathbf{k}'}{|\mathbf{k}'|}\,.} \tag{4.176}$$

Therefore p' and k' are uniquely fixed, in the laboratory system, by $\mathbf{k}$ and the direction of $\mathbf{k}'$. Hence, there is a function $f(|\mathbf{k}|, (\vartheta,\varphi))$ with

$$f(|\mathbf{k}|, (\vartheta,\varphi)) = \mathrm{Tr}\left(\gamma^0\hat{\gamma}(p,k,k')^* \frac{\not{p}' + m}{2m}\right)\,. \tag{4.177}$$

(ϑ,φ) are polar angles of $\mathbf{k}'$ (with $\vartheta = 0$ for $\mathbf{k} \parallel \mathbf{k}'$). For $\mathbf{p} = 0$, therefore,

$$\begin{aligned} I &= \int \frac{|\mathbf{k}'|^2\,\mathrm{d}\,|\mathbf{k}'|\,\mathrm{d}\omega}{2\,|\mathbf{k}'|} \int \mathrm{d}p'\,\theta(p'^0)\delta(p'^2 - m^2)\delta(p'+k'-p-k) f(k,\vartheta,\varphi) \\ &= \mathrm{d}\omega\,\mathrm{d}|\mathbf{k}'|\,\frac{|\mathbf{k}'|}{2}\theta(p^0+k^0-k'^0)\delta\left((p+k-k')^2 - m^2\right) f(k,\vartheta,\varphi) \\ &= \int \mathrm{d}\omega \int_0^{p^0+k^0} \mathrm{d}|\mathbf{k}'|\,\frac{|\mathbf{k}'|}{2}\delta(2m\,|\mathbf{k}| - |\mathbf{k}'|\,(2m + 2\,|\mathbf{k}|\,(1-\cos\vartheta)))\, f(k,\vartheta,\varphi) \\ &\qquad\qquad\qquad\qquad \text{(upper boundary redundant due to (4.176))} \\ &= \int \mathrm{d}\omega\, \frac{m\,|\mathbf{k}|}{4\,(m + |\mathbf{k}|\,(1-\cos\vartheta))^2} f(k,\vartheta,\varphi)\,. \end{aligned}$$

By (4.176), this implies

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{|\mathbf{k}'|^2}{4m\,|\mathbf{k}|} f(k,\vartheta,\varphi)\,. \tag{4.178}$$

(4.177) and (4.172)/(4.175), on the other hand, imply

$$\begin{aligned} f(k,\vartheta,\varphi) &= \mathrm{Tr}\left(\frac{\not{p}'+m}{2m}\left(\frac{\not{\epsilon}'(k')\ \not{k}\ \not{\epsilon}(k)}{2\,|\mathbf{k}|\,m} + \frac{\not{\epsilon}(k)\ \not{k}'\ \not{\epsilon}'(k')}{2\,|\mathbf{k}'|\,m}\right)\times\right. \\ &\qquad \left.\times\,\frac{\not{p}+m}{2m}\left(\frac{\not{\epsilon}(k)\ \not{k}\ \not{\epsilon}'(k')}{2\,|\mathbf{k}|\,m} + \frac{\not{\epsilon}'(k')\ \not{k}'\ \not{\epsilon}(k)}{2\,|\mathbf{k}'|\,m}\right)\right)\,. \end{aligned} \tag{4.179}$$

Calculation of this trace is easily done by computer algebra. In Section 17.7 of the REDUCE[83] manual there is already a listing of the corresponding program:

```
ON DIV;
MASS K= 0, KP= 0, P= MC, PP= MC; VECTOR EP,E;
MSHELL K,KP,P,PP;
LET P.EP= 0, P.E= 0, P.PP= MC**2+K.KP, P.K= MC*NK,P.KP=
    MC*NKP, PP.EP= -KP.EP, PP.E= K.E, PP.K= MC*NKP, PP.KP=
    MC*NK, K.EP= 0, K.KP= MC*(NK-NKP), KP.E= 0, EP.EP= -1, E.E=-1;
(G(L,PP) + MC)/(2*MC)*(G(L,E,EP,K)/(2*K.P) + G(L,EP,E,KP)/(2*KP.P))
  * (G(L,P) + MC)/(2*MC)*(G(L,K,EP,E)/(2*K.P) + G(L,KP,E,EP)/(2*KP.P))$
WRITE "1/4 Trace = ",WS;
```


[83]For MATHEMATICA the package TRACER by M. Jamin and M.E. Lautenbacher (Jamin and Lautenbacher, 1993) is useful in this context.

This gives the following result for (4.179):

`1/4 Trace =` $\mathrm{MC}^{(-2)} * (1/2*\mathrm{EP.E}^2 + 1/8*\mathrm{NKP}*\mathrm{NK}^{(-1)} + 1/8*\mathrm{NKP}^{(-1)}*\mathrm{NK} - 1/4)$

The explicit meaning of this REDUCE message is:

$$\frac{1}{4}f(k,\vartheta,\varphi) = (mc)^{-2}\left(\frac{1}{2}(\epsilon'\cdot\epsilon)^2 + \frac{1}{8}\frac{|\mathbf{k}'|}{|\mathbf{k}|} + \frac{1}{8}\frac{|\mathbf{k}|}{|\mathbf{k}'|} - \frac{1}{4}\right).$$

Since we use natural units this, together with (4.178), proves the so-called ***Klein-Nishina formula***:

$$\frac{\mathrm{d}}{\mathrm{d}\Omega}\sigma(p,k,\epsilon,\epsilon')_{|\mathbf{p}=0} = \frac{1}{4}(r_0)^2\left(\frac{|\mathbf{k}'|}{|\mathbf{k}|}\right)^2\left(\frac{|\mathbf{k}'|}{|\mathbf{k}|} + \frac{|\mathbf{k}|}{|\mathbf{k}'|} - 2 + 4\left(\epsilon'(\mathbf{k}')\cdot\epsilon(\mathbf{k})\right)^2\right),$$

$$\begin{aligned} \text{where:}\quad r_0 &\stackrel{\text{def}}{=} \text{classical radius of the electron} \\ &= \frac{\mathrm{e}^2}{4\pi m} \quad \text{in natural units in the Heaviside system} \\ &\,\widehat{=}\, 2,82\ldots\cdot 10^{-13}\mathrm{cm}\,. \end{aligned}$$

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