Beyond flat-space quantum field theory

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Abstract

We examine the quantum field theory of scalar field in non-Minkowski spacetimes. We first develop a model of a uniformly accelerating particle detector and demonstrate that it will detect a thermal spectrum of particles when the field is in an "empty" state (according to inertial observers). We then develop a formalism for relating field theories in different coordinate systems (Bogolubov transformations), and apply it to compare comoving observers in Minkowski and Rindler spacetimes. Rindler observers are found to see a hot bath of particles in the Minkowski vacuum, which confirms the particle detector result. The temperature is found to be proportional to the proper acceleration of comoving Rindler observers. This is generalized to 2D black hole spacetimes, where the Minkowski frame is related to Kruskal coordinates and the Rindler frame is related to conventional (t, r) coordinates. We determine that when the field is in the Kruskal (Hartle-Hawking) vacuum, conventional observers will conclude that the black hole acts as a blackbody of temperature $\kappa/2\pi k_{\mathrm{B}}$ (k_{B} is Boltzmann's constant). We examine this result in the context of static particle detectors and thermal Green's functions derived from the 4D Euclidean continuation of the Schwarzschild manifold. Finally, we give a semi-qualitative 2D account of the emission of scalar particles from a ball of matter collapsing into a black hole (the Hawking effect).

1 Introduction

The special theory of relativity postulates that all inertial reference frames are equivalent. That is, the laws of physics are symmetric under the Lorentz group, which consists of all the proper Lorentz rotations. In quantum field theory, one usually makes the additional demand that physical systems be invariant under four dimensional translations, which has the net result of making the Poincaré group a symmetry of fundamental interactions. In other words, quantum fields look the same to all inertial observers. However, these symmetries are far too restrictive if quantum field theory is to be understood in the sense of general relativity. The principle of general covariance asserts that the laws of physics be invariant under arbitrary coordinate transformations. All observers are to be treated on an equal footing, regardless of how they are moving.

This paper attempts to examine what changes in the standard formulation of quantum field theory when one allows for arbitrary, non-inertial observers. We begin by constructing a simple model of a particle detector moving along an arbitrary world line $x(\tau)$ and discover that the particle content of the "vacuum" is entirely dependent on the state of motion of the detector, which is the so-called "Unruh effect". This motivates us to study the notion of the vacuum in more detail, which leads to the conclusion that the ground state as defined in Minkowski coordinates $|0_{\mathrm{M}}\rangle$ need not be the same as in arbitrary coordinates. To quantify the difference between field theories in difference coordinate systems, we introduce Bogolubov transformations between "plane-wave" expansions of quantum fields. We re-examine the case of accelerating observers by studying the Rindler spacetime where spatially comoving observers are in fact uniformly accelerating. The black hole case is then considered in analogy to the Rindler spacetime, and the temperature of external black holes is derived. We re-derive this result using a static particle detector model and thermal Green's functions derived from the Euclidean analogue of Schwarzschild space. We end off by giving a semi-qualitative account of the Hawking radiation emitted from a collapsing ball of matter in 2D and black hole evapouration. We will be using the standard metric signature of quantum field theory $(+ - --)$, and will often work with two dimensional models.

2 Accelerating particle detectors

In this example, we will consider a scalar field propagating in a $1+3$ spacetime[1]. The generalization to higher spins, while straightforward, would just clutter the notion and cloud the ideas. We begin by constructing a model for a detector that can be used to determine if there are any particles in a given quantum state of the field $\phi(x)$. Our particle detector will be a quantum system with energy levels $\{E_n\}_{n=0}^{\infty}$ and a non-interacting Hamiltonian H_d. The standard field Hamiltonian

[1] We follow section 3.3 in Birrell & Davies [1] and Unruh's paper [2].

will be denoted by H_f. The detector will move along a worldline $x(\tau)$, where τ is the proper time, and it will be coupled to the scalar field via a (small) monopole moment operator m_{S} (the subscript indicates that we are dealing with a Schrödinger operator). In the Schrödinger representation, the interacting Hamiltonian is

$$H(\tau) = H_f + H_d + m_{\mathrm{S}}\phi(x(\tau)). \tag{1}$$

$H(\tau)$ reduces to $H_0 \equiv H_f + H_d$ if the coupling m_{s} goes to zero. The eigenstates of H_0 are given by

$$|\mathbf{k}, n\rangle \equiv |\mathbf{k}\rangle \otimes |n\rangle, \tag{2}$$

$$H_f|\mathbf{k}\rangle = \omega_k|\mathbf{k}\rangle, \tag{3}$$

$$H_d|n\rangle = E_n|n\rangle, \tag{4}$$

where $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$. The ground state of the H_0 operator may be written as $|0_{\mathrm{M}}, 0\rangle$, where $|0_{\mathrm{M}}\rangle$ is the standard Minkowski vacuum field configuration. We suppose that in the distant past $\tau \to -\infty$, the system is in the ground state. We wish to calculate the probability amplitude that the system will be found in another eigenstate $|\mathbf{k}, n\rangle$ of H_0 at some later time τ. If an observer traveling with the detector initially prepares the device in the ground state in the distant past and makes a non-zero measurement of the energy in the future, she will conclude that the detector absorbed energy from the field. That is, she will have detected a particle excitation of the field. Because the Hamiltonian is an explicit function of time, we do not expect energy to be conserved in this system.

We will calculate the required probability amplitude to first order in the monopole moment m_{s}. It is easiest to first work in the Schrödinger picture and then partially convert the result into the Heisenberg form . The state vector at some arbitrary time $|\psi\rangle_\tau$ can be expanded in terms of eigenstates of H_0:

$$|\psi\rangle_\tau = \sum_{\mathbf{k},n} c_{\mathbf{k},n}(\tau)|\mathbf{k}, n\rangle_\tau, \tag{5}$$

where we have chosen a box normalization. The probability amplitude of measuring the state of the system to be $|\mathbf{k}, n\rangle$ at some time τ is $c_{\mathbf{k},n}(\tau)$. $H(\tau)$ governs the time evolution of $|\psi\rangle_\tau$ while H_0 governs the time evolution of $|\mathbf{k}, n\rangle_\tau$ via their respective Schödinger equations. Now, we can take an explicit time derivative of $|\psi\rangle_\tau$ to get

$$-i\sum_{\mathbf{k},n}\left(H_0 + m_{\mathrm{S}}\phi_{\mathrm{s}}\right)c_{\mathbf{k},n}|\mathbf{k}, n\rangle_\tau = \sum_{\mathbf{k},n}\left(\dot{c}_{\mathbf{k},n} - iH_0 c_{\mathbf{k},n}\right)|\mathbf{k}, n\rangle_\tau, \tag{6}$$

where we have indicated that $\phi_{\mathrm{s}} = \phi_{\mathrm{s}}(x(\tau))$ is to be understood as a Schrödinger operator. Also, $\dot{c}_{\mathbf{k},n} = dc_{\mathbf{k},n}/d\tau$. Taking the inner product with ${}_\tau\langle\mathbf{p}, r|$ and making use of orthonormality gives

$$\dot{c}_{\mathbf{p},r} = -im_{\mathrm{S}}\sum_{\mathbf{k},n} c_{\mathbf{k},n}\left[{}_\tau\langle\mathbf{p}, r|\phi_{\mathrm{s}}|\mathbf{k}, n\rangle_\tau\right]. \tag{7}$$

Now, we replace the Schrödinger vectors by their Heisenberg counterparts using

$$|\mathbf{k}, n\rangle_\tau = e^{-i(\omega_k + E_n)\tau}|\mathbf{k}, n\rangle = e^{-iE_n\tau}e^{-iH_f\tau}|\mathbf{k}, n\rangle, \tag{8}$$

and

$${}_\tau\langle\mathbf{p}, r| = \langle\mathbf{p}, r|e^{+i(\omega_p + E_r)\tau} = \langle\mathbf{p}, r|e^{+iE_r\tau}e^{+iH_f\tau}. \tag{9}$$

However, to zeroth order in m_S

$$\phi(x(\tau)) = e^{iH_f\tau}\phi_\mathrm{s}(x(\tau))e^{-iH_f\tau}, \tag{10}$$

since $\phi_\mathrm{s}(x(\tau))$ must commute with H_d. Now, the zeroth order solution to (7) corresponding to the initial condition that $|\psi\rangle_\tau$ is the ground state is

$$c^{(0)}_{\mathbf{p},m} = \delta_{\mathbf{p},\mathbf{0}}\delta_{m,0}. \tag{11}$$

Putting this zeroth order solution into (7) and integrating with respect to time yields our final result

$$c^{(1)}_{\mathbf{k},n}(\tau) = -i\langle n|m_\mathrm{S}|0\rangle \int_{-\infty}^{\tau} d\tau' e^{i(E_n - E_0)\tau'}\langle\mathbf{k}|\phi(x(\tau'))|0_\mathrm{M}\rangle. \tag{12}$$

for $\mathbf{k} \neq \mathbf{0}$ and $n \neq 0$. The matrix element $\xi_n \equiv -i\langle n|m|0\rangle$ depends on the details of the detector structure, and will hence remain unspecified.

Equation (12) represents the probability amplitude that the system will make a transition from the ground state to the excited state at some arbitrary time τ. What is the probability that there will be one particle of 4-momentum k^α in the final state? It's straightforward to calculate

$$\langle 1_\mathbf{k}|\phi(x(\tau))|0_\mathrm{M}\rangle = e^{+ik\cdot x(\tau)}, \tag{13}$$

using the standard expansion for $\phi(x)$, $|1_\mathbf{k}\rangle = a^\dagger_\mathbf{k}|0_\mathrm{M}\rangle$ and $[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$ in the box normalization. Let's first consider the case when the detector is moving along an inertial (non-accelerating) trajectory,

$$x^\alpha(\tau) = \frac{\tau\,(1, \mathbf{v})}{\sqrt{1 - v^2}}, \tag{14}$$

where $\mathbf{v}$ is a constant vector such that $v^2 = \mathbf{v}\cdot\mathbf{v} < 1$. Hence.

$$c^{(1)}_{1_\mathbf{k},n}(\tau) = \xi_n \int_{-\infty}^{\tau} d\tau' \exp\left\{i\left[(E_n - E_0) + \frac{\omega_k - \mathbf{v}\cdot\mathbf{k}}{\sqrt{1 - v^2}}\right]\tau'\right\} \tag{15}$$

Taking the limit as $\tau \to \infty$ we get a delta-function,

$$c^{(1)}_{1_\mathbf{k},n}(\infty) = 2\pi\xi_n\delta\left[(E_n - E_0) + \frac{\omega_k - \mathbf{v}\cdot\mathbf{k}}{\sqrt{1 - v^2}}\right]. \tag{16}$$

However, $E_n > E_0$ and $\omega_k > |\mathbf{k}||\mathbf{v}| > \mathbf{k}\cdot\mathbf{v}$. Hence, the argument of the delta function is strictly positive and the transition is forbidden on energy grounds. Therefore, inertial observers will measure the particle content of the field to be zero in the distant future, just as it was in the distant past.

How about non-inertial observers? It's easier in this case to calculate the probability P_n that the detector will be found in the n^{th} eigenstate as $\tau \to \infty$:

$$
\begin{aligned}
P_n &= \sum_{\mathbf{k}} |c^{(1)}_{\mathbf{k},n}(\infty)|^2 \\
&= |\xi_n|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E(\tau-\tau')} \langle 0_{\mathrm{M}}|\phi(x(\tau))\phi(x(\tau'))|0_{\mathrm{M}}\rangle,
\end{aligned}
\tag{17}
$$

where $\Delta E \equiv E_n - E_0$ and we have made use of $1 = \sum_{\mathbf{k}} |\mathbf{k}\rangle\langle\mathbf{k}|$. Now,

$$
\begin{aligned}
\langle 0_{\mathrm{M}}|\phi(x(\tau))\phi(x(\tau'))|0_{\mathrm{M}}\rangle &= \sum_{\mathbf{k},\mathbf{p}} \frac{\langle 0_{\mathrm{M}}|a_{\mathbf{p}} e^{-ip\cdot x(\tau)} a^\dagger_{\mathbf{k}} e^{+ik\cdot x(\tau')}|0_{\mathrm{M}}\rangle}{2V\sqrt{\omega_k\omega_p}} \\
&= \sum_{\mathbf{k},\mathbf{p}} \frac{e^{-ip\cdot x(\tau)} e^{+ik\cdot x(\tau')}\delta_{\mathbf{k},\mathbf{p}}}{2V\sqrt{\omega_k\omega_p}} \\
&\to \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\omega_k} e^{-ik\cdot[x(\tau)-x(\tau')]} \\
&= i\Delta^+[x(\tau)-x(\tau')],
\end{aligned}
\tag{18}
$$

where the limit in the third line is taken for $V \to \infty$ and $\Delta^+(x)$ is the standard plus Green's Function for the Klein-Gordon field, also known as the Wightman Green's function. Let's specialize to the massless case where,

$$
\Delta^+(x) = D^+(x) = \frac{i}{(2\pi)^2}\frac{1}{(x-i\eta)^2}.
\tag{19}
$$

This expression is to be understood in the limit of $\eta \to 0$, where $\eta^\alpha = (\eta, 0, 0, 0)$ is a small future pointing vector.

Let's evaluate the Green's function for the inertial path (14). We have

$$
\begin{aligned}
[x(\tau) - x(\tau') - i\eta]^2 &= \frac{(\tau - \tau' - i\eta\sqrt{1-v^2})^2 - (\tau-\tau')^2 v^2}{1-v^2} \\
&= (\tau - \tau' - i\epsilon)^2,
\end{aligned}
$$

where $\epsilon = \eta/(1-v^2)^{1/2}$. Then, (17) becomes

$$
P_n = -\frac{|\xi_n|^2}{(2\pi)^2}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\zeta \frac{e^{-i\Delta E\zeta}}{(\zeta - i\epsilon)^2},
\tag{20}
$$

where we have made the change of variable $\zeta = \tau - \tau'$. This expression is formally infinite because of the integration over τ, but may be regulated by defining the transition probability per unit time as $p_n = P_n / \int_{-\infty}^{\infty} d\tau$. Since $\Delta E > 0$, we perform the ζ-integral by completing the contour in the lower-half plane. But the only pole on the integrand is at $\zeta = i\epsilon$ in the upper-half plane which means the integral is zero. Hence, $p_n = 0$ for inertial observers, confirming our previous result that such observers do not detect any particles.

Now, consider a detector following a hyperbolic trajectory

$$x^\mu(\tau) = \alpha \left[\sinh(\tau/\alpha), 0, 0, \cosh(\tau/\alpha)\right]. \tag{21}$$

It's easy to verify that the magnitude of the detector's proper acceleration is $\sqrt{a^\mu a_\mu} = \alpha^{-1}$, where $a^\mu = d^2 x^\mu / d\tau^2$. That is, the detector's acceleration is measured to be a constant in an instantaneously comoving frame. Now,

$$\begin{aligned}
[x(\tau) - x(\tau') - i\eta]^2 &= \alpha^2 \left[\sinh\left(\frac{\tau}{\alpha}\right) - \sinh\left(\frac{\tau'}{\alpha}\right) - i\eta\right]^2 \\
&\quad -\alpha^2 \left[\cosh\left(\frac{\tau}{\alpha}\right) - \cosh\left(\frac{\tau'}{\alpha}\right)\right]^2 \\
&= 4\alpha^2 \sinh^2\left(\frac{\tau - \tau'}{2\alpha}\right)\left(1 - \frac{i\epsilon}{\alpha}\right)^2 \\
&= 4\alpha^2 \sinh^2\left(\frac{\tau - \tau'}{2\alpha} - \frac{i\epsilon}{\alpha}\right),
\end{aligned} \tag{22}$$

where

$$\epsilon = \alpha\eta \cosh^2\left(\frac{\tau + \tau'}{2\alpha}\right). \tag{23}$$

Then, p_n becomes

$$p_n = -\frac{|\xi_n|^2}{(4\pi\alpha)^2} \int_{-\infty}^{\infty} e^{-i\Delta E \zeta} \operatorname{csch}^2\left(\frac{\zeta}{2\alpha} - \frac{i\epsilon}{\alpha}\right) d\zeta. \tag{24}$$

Now,

$$\operatorname{csch}^2\left(\frac{\zeta - 2i\epsilon}{2\alpha}\right) = \sum_{k=-\infty}^{\infty} \frac{4\alpha^2}{(\zeta - 2i\epsilon + 2\pi i k\alpha)^2} \tag{25}$$

Putting (25) into (24) yields

$$p_n = -\frac{|\xi_n|^2}{4\pi^2} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\Delta E \zeta} d\zeta}{(\zeta - 2i\epsilon + 2\pi i k\alpha)^2}. \tag{26}$$

Now, as mentioned above, to do the integral we need to complete the contour in the lower-half plane. However, each term in the integrand has a second-order pole at

$\zeta = i(2\epsilon - 2\pi k\alpha)$. Hence, there will only be contributions for $k = 1 \dots \infty$. For those values of k, the $2i\epsilon$ term is irrelevant and we can use residue theory to get

$$\int_{-\infty}^{\infty} \frac{e^{-i\beta\zeta} d\zeta}{(\zeta + i\gamma)^2} = -2\pi\beta e^{-i\beta\gamma}, \quad \beta, \gamma > 0, \tag{27}$$

and

$$p_n = \frac{|\xi_n|^2}{2\pi} \frac{E_n - E_0}{e^{2\pi\alpha(E_n - E_0)} - 1}. \tag{28}$$

This is the probability per unit time that the detector will absorb an amount of energy $E_n - E_0$ from the "vacuum"-state $|0_{\mathrm{M}}\rangle$. We can interpret the $|\xi_n|^2(E_n - E_0)$ term as the sensitivity of the detector at this particular energy. But the probability p_n should go like the sensitivity of the detector to particles of energy $E_n - E_0$ times the relative number of such particles in the field, which in this case would be given by

$$\frac{1}{e^{2\pi\alpha(E_n - E_0)} - 1}. \tag{29}$$

Compare this to the expression from statistical mechanics for the occupation number of the energy levels between E and $E + dE$ when a gas of bosons is in equilibrium with a heat bath of temperature T:

$$\frac{1}{e^{(E-\mu)/k_{\mathrm{B}}T} - 1}, \tag{30}$$

where k_{B} is Boltzmann's constant. The similarity of the two expression leads us to conclude that an observer traveling along with the particle detector will conclude that she is moving through a hot gas of bosons with a temperature of

$$T = 1/2\pi\alpha k_{\mathrm{B}}. \tag{31}$$

When she makes the identification $E = E_n - E_0$, she will conclude that the chemical potential μ of the gas is zero. That is, the bosons are massless. The conclusion is that uniformly accelerating observers will not see the quantum field in it's ground state, but will rather see a thermal excitation of the field with a temperature proportional to their acceleration. It should be clear that energy is not conserved in this situation because the initial state has $E = E_0$ while the final state has $E > E_0$. Where did the energy come from? The standard answer is that the agent responsible for accelerating the detector must do work on the field ϕ. Then, when the detector interacts with the field, the energy is used to excite the detector out of the ground state.

3 Bogolubov Transformations

In reading the previous section, the reader might wonder whether or not the fact that the uniformly accelerating observer detects a bath of thermal bosons depends on the details of the detector model adopted. It turns out that the conclusions are independent of the detector, and are rather based in the difference between the natural coordinate systems used by inertial and non-inertial observers[2].

To make the last statement more concrete, let us consider a flat $1+1$ spacetime with the line element

$$ds^2 = dt^2 - dx^2. \tag{32}$$

We have suppressed the two spurious spatial dimensions found in the last section. The coordinate system represented by (32) is well suited to inertial observers because we can always use 2D Lorentz transformations such that any inertial observer moves on a $x =$ constant trajectory. Consider the transformation

$$\begin{aligned} t &= a^{-1} e^{a\xi} \sinh(a\eta) \\ x &= a^{-1} e^{a\xi} \cosh(a\eta), \end{aligned} \tag{33}$$

which casts the line element in the form

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2), \tag{34}$$

which is the defining relation for the 2D Rindler spacetime. The coordinate system represented by (34) is well suited to observers moving on $\xi =$ constant trajectories, given by

$$x^\mu(\eta) = (t(\eta), x(\eta)) = a^{-1} e^{a\xi} \left(\sinh(a\eta), \cosh(a\eta)\right). \tag{35}$$

But the proper time for $\xi =$ constant observers is $\tau = e^{a\xi}\eta$. Hence, by comparing (21) with (35), we conclude the spatially comoving observers in the Rindler spacetime are uniformly accelerating with a proper acceleration $\alpha^{-1} = ae^{-a\xi}$.

We can reasonably assume that observers will tend to construct quantum field theories in coordinate systems where they are comoving; that is, in their own rest frames. So, inertial observers will attempt to formulate a quantum description of the field ϕ in the (t, x) coordinate system, while uniformly accelerating observers will attempt to do the same in the (η, ξ) system. The question is, how are the two representations related?

The answer is given in terms of Bogolubov transformation between the "plane-wave" decompositions of ϕ in different coordinate systems. Although we will return to the Minkowski and Rindler spacetimes in the next section, we now work in a general curved manifold of dimension n. We will assume that the spacetime admits the existence of a timelike Killing vector field, which will allow us to make a sensible definition of positive frequency modes. We also demand that the manifold be globally hyperbolic, which makes the initial value problem for the field ϕ tractable. The relativistic generalization of the Klein-Gordon equation is

$$\left(\nabla^\alpha \nabla_\alpha + m^2 + \zeta R\right)\phi = 0, \tag{36}$$

[2]We draw on section 3.2 of Birrell & Davies [1] and section 14.2 of Wald's relativity text [3] for the discussion in this section.

where m is the mass, R is the Ricci scalar and ζ is a constant that defines the coupling of of the field to the curvature of the manifold. The $\zeta = 0$ case is referred to as minimally coupled, while the $\zeta = [(n-2)/(n-1)]/4$ case is referred to conformally coupled because the massless wave equation is invariant under conformal transformations ($g_{\alpha\beta} \to \Omega g_{\alpha\beta}, \phi \to \Omega^{1-n/2}\phi$). In 2D, the minimal and conformal coupling cases coincide, which allows for considerable simplification in the solution of the wave equation for conformally flat spaces.

The wave equation will in general involve a number of mode solutions $\{u_i, u_j^*\}$ which are eigenfunctions of the Lie derivative operator

$$\pounds_\xi u_i = -i\omega_i u_i \tag{37}$$

$$\pounds_\xi u_i^* = +i\omega_i u_i^*, \tag{38}$$

where ξ^α is a timelike Killing vector and $\omega_i > 0$. The label i is used to schematically tell the difference between modes and may be continuous or discrete. The modes $\{u_i\}$ are said to be of positive frequency, while the modes $\{u_i^*\}$ are of negative frequency. We define the scalar product between two functions as

$$(\phi_1, \phi_2) = -i \int_\Sigma \phi_1(x)(\overrightarrow{\partial_\alpha} - \overleftarrow{\partial_\alpha})\phi_2^*(x) d\Sigma^\alpha \tag{39}$$

Where Σ is a spacelike (Cauchy) surface. On can show that the value of the scalar product (ϕ_1, ϕ_2) is independent of the surface Σ. Also, note that

$$(\phi_2, \phi_1) = -(\phi_1^*, \phi_2^*). \tag{40}$$

The mode solutions $\{u_i, u_j^*\}$ are orthonormal in the sense

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i^*, u_j) = 0. \tag{41}$$

In n-D Minkowski space, it is easy to verify that the mode solutions to the minimally coupled wave equation

$$u_{\mathbf{k}}(x) = [2\omega_k(2\pi)^{n-1}]^{-1/2} e^{-i\omega_k t + \mathbf{k}\cdot\mathbf{x}}, \quad \omega_k = \sqrt{|\mathbf{k}|^2 + m^2}, \tag{42}$$

satisfy the relations (37) and (41) in the limit where $\mathbf{k}$ is a continuous label, with $\xi = \partial/\partial t$ and Σ equal to a surface of constant time. To achieve quantization of the field, we expand the field operator $\phi(x)$ in terms of the mode functions $\{u_i, u_j^*\}$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right], \tag{43}$$

and impose the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{44}$$

with all other commutators vanishing.

To derive the modes $\{u_i\}$, we have solved the Klein-Gordon equation (36) in a particular coordinate system x^μ. What happens if we solve (36) in a different coordinate system $\bar{x}^\nu = \bar{x}^\nu(x^\mu)$? We will, in general, have a new set of modes $\{\bar{u}_i, \bar{u}_j^*\}$ in the $\bar{x}^\nu$ system, which satisfy the same relations that $\{u_i, u_j^*\}$ do. We can expand $\phi(x)$ in this new set as

$$\phi(x) = \sum_i \left[\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)\right], \tag{45}$$

and quantize the field by demanding

$$[\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}. \tag{46}$$

Using orthonormality, we get

$$(\phi, u_i) = a_i = \sum_j \left[\bar{a}_j(\bar{u}_j, u_i) + \bar{a}_j^\dagger(\bar{u}_j^*, u_i)\right], \tag{47}$$

and

$$(\bar{u}_j, \phi) = \bar{a}_j = \sum_i \left[a_i(\bar{u}_j, u_i) + a_i^\dagger(\bar{u}_j, u_i^*)\right]. \tag{48}$$

We define the Bogolubov coefficients as

$$\alpha_{ij} = (\bar{u}_i, u_j) \tag{49}$$
$$-\beta_{ij} = (\bar{u}_i, u_j^*), \tag{50}$$

to get

$$a_i = \sum_j \left(\alpha_{ji}\bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger\right) \tag{51}$$
$$\bar{a}_j = \sum_i \left(\alpha_{ji} a_i - \beta_{ji} a_i^\dagger\right). \tag{52}$$

Consider the vacuum states defined by the two mode decompositions:

$$a_i|0\rangle = 0, \quad \bar{a}_i|\bar{0}\rangle = 0. \tag{53}$$

Define the number operators as usual:

$$N_i = \sum_i a_i^\dagger a_i, \quad \bar{N}_i = \sum_i \bar{a}_i^\dagger \bar{a}_i. \tag{54}$$

We now ask the question: if the field is in the quantum state $|0\rangle$, what is the expectation value of $\bar{N}$? That is, if observers in the x^μ system use their preferred mode decomposition $\{u_i, u_j^*\}$ to determine that the field is devoid of quanta, what will observers in the $x^{\nu\prime}$ system using the $\{\bar{u}_i, \bar{u}_j^*\}$ modes measure the particle content of the field to be? We have

$$\begin{aligned}\langle 0|\bar{N}|0\rangle &= \sum_{ijk}\langle 0|\left(\alpha_{ij}^* a_j^\dagger - \beta_{ij}^* a_j\right)\left(\alpha_{ik}a_k - \beta_{ik}a_j^\dagger\right)|0\rangle \\ &= \sum_{ij}|\beta_{ij}|^2. \end{aligned} \tag{55}$$

Therefore, $\langle 0|\bar{N}|0\rangle \neq 0$ in general and observers in different coordinate systems will not agree on a vacuum state of the field. That is, what one observer claims to be the vacuum will in general not be the vacuum in different frames of reference, which is exactly what we saw in the previous section for the case of inertial versus non-inertial observers. However, the two vacuum states will be equivalent if $\beta_{ij} \equiv 0$. This can be understood by noting that the definitions (49) imply that

$$\bar{u}_j = \sum_i \left(\alpha_{ji}u_i + \beta_{ji}u_i^*\right). \tag{56}$$

Hence, $\beta_{ij} \equiv 0$ implies that it is possible to expand the positive frequency modes in one frame $\bar{u}_j$ in terms of only the positive frequency modes in the other frame u_i.

4 Rindler versus Minkowski frames – The Unruh effect

Having developed the machinery that lets us distinguish between quantum field theories in arbitrary coordinate systems, let us return to the 2D Minkowski and Rindler spacetimes[3]. We write the Minkowski metric as

$$ds^2 = dt^2 - dx^2 = d\bar{u}\, d\bar{v}. \tag{57}$$

where $\bar{u} = t - r$ and $\bar{v} = t + r$. The transformation to Rindler coordinates (33) can be written as

$$\begin{aligned}\bar{u} &= -a^{-1}e^{-au} \\ \bar{v} &= +a^{-1}e^{+av}, \end{aligned} \tag{58}$$

where $u = \eta - \xi$ and $v = \eta + \xi$. The range of η and ξ (u and v) is assumed to be between $-\infty$ and ∞. However, the transformation (58) makes it clear that the Rindler coordinates only cover the $\bar{u} < 0$ and $\bar{v} > 0$ portion of the tx-plane. We denote the region covered by the transformation (58) as R, its positions is depicted in figure 1. We can cover the $u > 0, v < 0$ region, which we call L, by a different Rindler coordinate patch, defined by the transformation

[3] We follow section 4.5 of Birrell & Davies [1] and Unruh's paper [2].

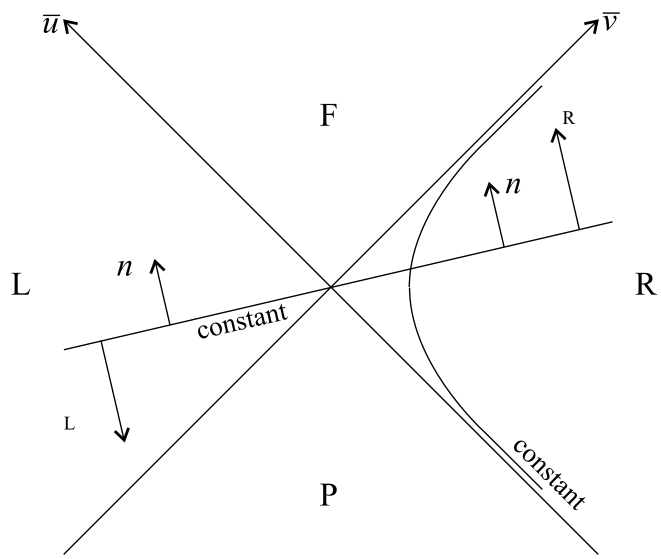

Figure 1: The structure of the Rindler coordinate patches needed to cover the Minkowski manifold

$$\begin{aligned} \bar{u} &= +a^{-1}e^{-au} \\ \bar{v} &= -a^{-1}e^{+av}. \end{aligned} \tag{59}$$

In both L and R, the Rindler metric has the line element

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) = e^{2a\xi}du\,dv. \tag{60}$$

The regions L and R are causally disconnected, while the $\bar{u} = 0$ and $\bar{v} = 0$ lines act as horizons in that comoving Rindler observers never cross into the F and P region. So, we could formulate a field theory restricted to just L or R and be done with it. But we want to relate the accelerated observers to their inertial counterparts, and the Minkowski modes extend over the whole manifold. So, we should try to formulate the Rindler field theory in as much of the manifold as *necessary*. Looking at the Bogolubov transformations of the previous section, we see that we only need to evaluate the different modes on a single spacelike Cauchy surface to calculate $\alpha_{kk'}$ and $\beta_{kk'}$. We can see from figure 1 that lines of constant η are spacelike 1D surfaces that extend across the entire manifold. We will use these surfaces to evaluate the scalar products between mode solutions (39), so we don't really need access to the F and P regions and will not worry about them anymore[4].

To make contact with the results of section 2, we will limit our analysis to massless scalar fields. The Rindler spacetime is conformally related to ordinary

[4]We could define two new patches that would cover F and P by merely fiddling with the signs in the transformation (58).

Minkowski space, which makes the wave equation in both L and R trivially easy to calculate:

$$0 = \nabla^\alpha \nabla_\alpha \phi = e^{-2a\xi} \left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2} \right) \phi, \tag{61}$$

with mode solutions

$$e^{i(k\xi \pm \omega\eta)}, \quad \omega \equiv |k|. \tag{62}$$

Now, it's easy to see that

$${}^{\mathrm{L}}\zeta^\alpha = \partial^\alpha \eta, \quad {}^{\mathrm{R}}\zeta^\alpha = \partial^\alpha \eta, \tag{63}$$

are timelike and normal to surfaces of constant η in L and R respectively. Since the metric coefficients are independent of η, ${}^{\mathrm{L}}\zeta^\alpha$ and ${}^{\mathrm{R}}\zeta^\alpha$ must be Killing vectors as well. However, ${}^{\mathrm{R}}\zeta^\alpha$ is future pointing while ${}^{\mathrm{L}}\zeta^\alpha$ is past pointing. We therefore wish to write down solutions of (61) that are positive frequency with respect to $-{}^{\mathrm{L}}\zeta^\alpha$ and ${}^{\mathrm{R}}\zeta^\alpha$ in L and R respectively. Such a solution is given in terms of distributions:

$$u_k(\eta, \xi) = {}^{\mathrm{R}}u_k(\eta, \xi) + {}^{\mathrm{L}}u_k(\eta, \xi), \tag{64}$$

where

$${}^{\mathrm{R}}u_k(\eta, \xi) = \Theta(+x - |t|)(4\pi\omega)^{-1/2} e^{i(k\xi - \omega\eta)} \tag{65}$$

$${}^{\mathrm{L}}u_k(\eta, \xi) = \Theta(-x - |t|)(4\pi\omega)^{-1/2} e^{i(k\xi + \omega\eta)}. \tag{66}$$

We justify the normalization of the basis modes by calculating various scalar products between (65) and (66). We choose one of the lines of constant η as the Cauchy surface Σ in the formula (39). The future pointing unit normal to the surface is $n^\alpha = \pm e^{-a\xi} \partial^\alpha \eta$, where the plus sign is taken in R while the minus sign is taken in L. The surface element is $d\Sigma = e^{a\xi} d\xi$. Some care must be exercised in setting up the integrals because the coordinate ξ runs from $-\infty$ to ∞ in both L and R. The quantities we need are:

$$({}^{\mathrm{R}}u_k, {}^{\mathrm{R}}u_{k'}) = -\frac{i}{4\pi\sqrt{\omega\omega'}} \int_{-\infty}^{\infty} e^{i(k\xi - \omega\eta)} (\overrightarrow{\partial_\eta} - \overleftarrow{\partial_\eta}) e^{-i(k\xi - \omega\eta)} d\xi \tag{67}$$

$$= \delta(k - k'), \tag{68}$$

$$({}^{\mathrm{L}}u_k, {}^{\mathrm{L}}u_{k'}) = -\frac{i}{4\pi\sqrt{\omega\omega'}} \int_{-\infty}^{\infty} e^{i(k\xi + \omega\eta)} (\overleftarrow{\partial_\eta} - \overrightarrow{\partial_\eta}) e^{-i(k\xi + \omega\eta)} d\xi \tag{69}$$

$$= \delta(k - k'), \tag{70}$$

and

$$({}^{\mathrm{L}}u_k, {}^{\mathrm{R}}u_{k'}) = 0, \tag{71}$$

which follows from the fact that the product ${}^{\mathrm{L}}u_k {}^{\mathrm{R}}u_{k'}$ is zero over all of Σ.

It is clear that $\{{}^{\rm R}u_k, {}^{\rm R}u_k^*\}$ is complete in R, while $\{{}^{\rm L}u_k, {}^{\rm L}u_k^*\}$ is complete in L. Hence, we can expand an arbitrary solution to the massless scalar equation as

$$\phi = \int_{-\infty}^{\infty} dk \left({}^{\rm L}b_k {}^{\rm L}u_k + {}^{\rm R}b_k {}^{\rm R}u_k + {}^{\rm L}b_k^\dagger {}^{\rm L}u_k^* + {}^{\rm R}b_k^\dagger {}^{\rm R}u_k^* \right). \tag{72}$$

We we've essentially done is write down two disjoint field theories in the L and R regions. This makes sense when one considers that fact that the two regions are causally separated and the preferred set of Rindler observers are confined to either L or R. The natural split between spacetime regions is reflected in the different sets of annihilation/creation operators that act in each domain. In fact, when the standard commutators

$$[{}^{\rm L}b_k, {}^{\rm L}b_{k'}^\dagger] = \delta(k - k'), \tag{73}$$

and

$$[{}^{\rm R}b_k, {}^{\rm R}b_{k'}^\dagger] = \delta(k - k'), \tag{74}$$

with $[{}^{\rm L}b_k, {}^{\rm L}b_{k'}] = 0$, $[{}^{\rm R}b_k, {}^{\rm R}b_{k'}] = 0$, etc. . . ., we find that the commutators between operators in the left and regions vanish[5]. The vacuum of this mode decomposition is defined by

$${}^{\rm L}b_k|0_{\rm R}\rangle = {}^{\rm R}b_k|0_{\rm R}\rangle = 0, \tag{75}$$

where $|0_{\rm R}\rangle$ denotes the Rindler vacuum. The action of ${}^{\rm L}b_k^\dagger$ on the vacuum is to create a particle in region L, while the action of ${}^{\rm R}b_k^\dagger$ is to create a particle in region R.

Now, let us introduce another set of mode solutions

$$v_k = [2\sinh(\pi\omega/a)]^{-1/2} \left(e^{\pi\omega/2a}\, {}^{\rm R}u_k + e^{-\pi\omega/2a}\, {}^{\rm L}u_{-k}^* \right) \tag{76}$$

$$\bar{v}_k = [2\sinh(\pi\omega/a)]^{-1/2} \left(e^{-\pi\omega/2a}\, {}^{\rm R}u_{-k}^* + e^{\pi\omega/2a}\, {}^{\rm L}u_k \right). \tag{77}$$

Using the scalar products (67) – (71), we see that v_k and $\bar{v}_k$ are appropriately normalized. Because they involve combinations of positive and negative frequency modes, v_k and $\bar{v}_k$ are of indeterminate frequency in the (η, ξ) coordinate system. Also, since $v_k^* \neq \bar{v}_k$, we need to expand a real field ϕ in terms of the entire set $\{v_k, \bar{v}_k, v_k^*, \bar{v}_k^*\}$:

$$\phi = \int_{-\infty}^{\infty} dk \left(d_k^{(1)} v_k + d_k^{(2)} \bar{v}_k + d_k^{(1)\dagger} v_k^* + d_k^{(2)\dagger} \bar{v}_k^* \right). \tag{78}$$

The vacuum state with respect to these modes is defined by

$$d_k^{(1)}|\bar{0}\rangle = d_k^{(2)}|\bar{0}\rangle = 0. \tag{79}$$

[5] This can be demonstrated by using the scalar products (67) – (71) to express ${}^{\rm L,R}b_k$ as a superposition of the field operator ϕ evaluated at different positions in L and R respectively. But because L and R are causally separate, the commutator $[\phi(x), \phi(x')] = 0$ for $x \in L$ and $x' \in R$. Hence, the commutator between *any* L and R creation/annihilation operators must be zero.

It can be shown (see footnote following equation (90)) that the equal time commutation relations for ϕ and $\dot{\phi}$ imply that $[d_k^{(1)}, d_{k'}^{(2)\dagger}] = 0$. This means that the action of $d_k^{(1)\dagger}$ on the vacuum is to create a particle of type 1, while the action of $d_k^{(2)\dagger}$ is to create a particle of type 2.

Why have we introduced these new modes? The reason is that they share the same vacuum state as the usual Minkowski plane waves; that is

$$|\bar{0}\rangle = |0_{\mathrm{M}}\rangle. \tag{80}$$

To see why this is the case, lets express v_k in Minkowski coordinates:

$$v_k = \frac{1}{N_\omega}\begin{cases} (a\bar{u})^{+i\omega/a}\left\{\begin{array}{ll} e^{+\pi\omega/2a}(-1)^{+i\omega/a}, & x \in \mathrm{R} \\ e^{-\pi\omega/2a}, & x \in \mathrm{L} \end{array}\right\} & k > 0 \\ (a\bar{v})^{-i\omega/a}\left\{\begin{array}{ll} e^{+\pi\omega/2a}, & x \in \mathrm{R} \\ e^{-\pi\omega/2a}(-1)^{-i\omega/a}, & x \in \mathrm{L} \end{array}\right\} & k < 0 \end{cases}, \tag{81}$$

where $N_\omega^2 = 8\pi\omega\sinh(\pi\omega/a)$. Despite its appearance, v_k is continuous across the transition between L and R if we choose a branch cut in the upper-half of the complex $\bar{u}$ and $\bar{v}$ planes. That is, we choose $-1 = e^{-i\pi}$ and $+1 = e^0$. Then

$$\begin{aligned} e^{\pi\omega/2a}(-1)^{i\omega/a} &= e^{-\pi\omega/2a}e^{\pi\omega/a}(-1)^{i\omega/a} \\ &= e^{-\pi\omega/2a}(-e^{-i\pi})^{i\omega/a} \\ &= e^{-\pi\omega/2a}, \end{aligned}$$

and

$$\begin{aligned} e^{-\pi\omega/2a}(-1)^{-i\omega/a} &= e^{\pi\omega/2a}e^{-\pi\omega/a}(-1)^{-i\omega/a} \\ &= e^{\pi\omega/2a}(-e^{-i\pi})^{-i\omega/a} \\ &= e^{\pi\omega/2a}. \end{aligned}$$

We can expand v_k in terms of Minkowski plane waves using a Bogolubov transformation:

$$v_k = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{\sqrt{2\omega'}}\left(\alpha_{kk'}e^{-i\omega' t+ik'x} + \beta_{kk'}e^{i\omega' t-ik'x}\right). \tag{82}$$

This expression is valid in the L and R regions only. Let us evaluate v_k along the $\bar{v} = 0$ Rindler horizon. Putting $t = -x$ and $t = \bar{u}/2$ in (82), we get

$$v_k\Big|_{\bar{v}=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{\sqrt{2\omega'}}\left(\alpha_{kk'}e^{-i(\omega'+k')\bar{u}/2} + \beta_{kk'}e^{i(\omega'+k')\bar{u}/2}\right). \tag{83}$$

Now, we multiply by $e^{-i\omega\bar{u}}$ (with $\omega > 0$) and integrate with respect to $\bar{u}$. We get

$$\int_{-\infty}^{\infty} d\bar{u}\, e^{-i\omega\bar{u}}v_k\Big|_{\bar{v}=0} = \int_{-\infty}^{\infty}\frac{dk'}{\sqrt{2\omega'}}\left\{\alpha_{kk'}\delta\left[\omega + (\omega'+k')/2\right]\right. \tag{84}$$

$$\left. +\beta_{kk'}\delta\left[\omega - (\omega'+k')/2\right]\right\}. \tag{85}$$

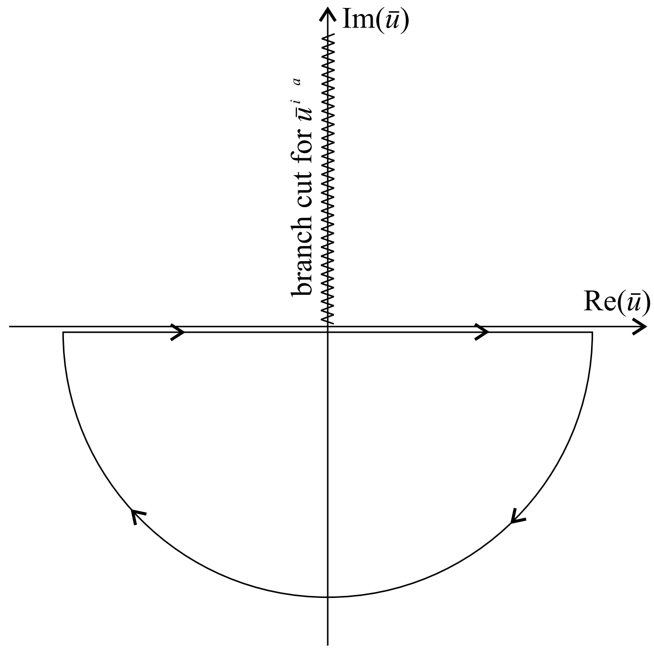

Figure 2: The contour in the complex $\bar{u}$ plane used to show that $\beta_{kk'} = 0$

Since $\omega' = |k'|$, only the second delta function will contribute. Hence,

$$\beta_{k\omega} = \sqrt{2\omega} \int_{-\infty}^{\infty} d\bar{u}\, e^{-i\omega\bar{u}} v_k \Big|_{\bar{v}=0}. \tag{86}$$

By integrating over the $\bar{u} = 0$ surface, we get in a similar fashion

$$\beta_{k-\omega} = \sqrt{2\omega} \int_{-\infty}^{\infty} d\bar{v}\, e^{-i\omega\bar{v}} v_k \Big|_{\bar{u}=0}. \tag{87}$$

Now, for $k < 0$, v_k is identically equal to zero on the $\bar{v} = 0$ surface. So, $\beta_{k\omega} = 0$ for $k < 0$. Similarly, $v_k \equiv 0$ on the $\bar{u} = 0$ surface for $k > 0$, so $\beta_{k-\omega} = 0$ for $k > 0$. For $k > 0$,

$$\beta_{k\omega} \propto \int_{-\infty}^{\infty} d\bar{u}\, e^{-i\omega\bar{u}} (a\bar{u})^{i\omega/a}. \tag{88}$$

To do the integral, we need to complete the contour in the lower-half plane as shown in figure 2. But, the principle branch of $(a\bar{u})^{i\omega/a}$ has already been chosen to be in the upper-half of the complex $\bar{u}$ plane. So, the integral is zero and $\beta_{k\omega} = 0$ for $k > 0$. In an entirely analogous fashion, we can show from (87) that $\beta_{k\omega} = 0$ for $k > 0$. Therefore,

$$\beta_{kk'} = 0 \tag{89}$$

for all k and k'. We can repeat this analysis for the $\bar{v}_k$ modes and come up with the same conclusion, both v_k and $\bar{v}_k$ can be expressed as a superposition of positive frequency Minkowski modes. By the comments of the previous section, this means that the vacuum defined by the modes $\{v_k, \bar{v}_k\}$ is the same as the Minkowski vacuum. Hence, (80) is proved.

Now, we return to the expansion of ϕ in terms of the original Rindler modes $\{{}^{\mathrm{L}}u_k, {}^{\mathrm{L}}u_k\}$, given by (72). By taking a scalar product of both sides with ${}^{\mathrm{L}}u_k$ and ${}^{\mathrm{L}}u_k$, we get that ${}^{\mathrm{L}}b_k = ({}^{\mathrm{L}}u_k, \phi)$ and ${}^{\mathrm{R}}b_k = ({}^{\mathrm{R}}u_k, \phi)$. Now, if we evaluate the scalar products with respect to the expansion of ϕ in terms of $\{v_k, \bar{v}_k\}$, given by (78), we get

$$
\begin{aligned}
{}^{\mathrm{L}}b_k &= [2\sinh(\pi\omega/a)]^{-1/2}[e^{\pi\omega/2a}d_k^{(2)} + e^{-\pi\omega/2a}d_{-k}^{(1)\dagger}] && (90)\\
{}^{\mathrm{R}}b_k &= [2\sinh(\pi\omega/a)]^{-1/2}[e^{\pi\omega/2a}d_k^{(1)} + e^{-\pi\omega/2a}d_{-k}^{(2)\dagger}]. && (91)
\end{aligned}
$$

This is the Bogolubov transformation between the creation and annihilation operators associated with the Rindler and Minkowski modes[6].

So, having at last derived the Bogolubov transformation between the accelerating and non-accelerating frames, we can ask the question: if the field in the Minkowski vacuum state $|0_{\mathrm{M}}\rangle$, how many particles of momentum k will an accelerating observer detect? Assuming that the observer is traveling in the R region, she will use $\{{}^{\mathrm{R}}b_k, {}^{\mathrm{R}}b_k^\dagger\}$ as her creation and annihilation operators. The expected number of particles is

$$
N_k = \langle 0_{\mathrm{M}}|{}^{\mathrm{R}}b_k^\dagger {}^{\mathrm{R}}b_k|0_{\mathrm{M}}\rangle = \frac{e^{-\pi\omega/a}}{2\sinh(\pi\omega/a)} = \frac{1}{e^{2\pi\omega/a}-1}. \qquad (92)
$$

But this is just a thermal Planck spectrum for massless bosons! Therefore, the accelerating observer will detect a thermal spectrum of particles in the Minkowski vacuum. According to the Tolman relation, the local temperature measured by an observer whose velocity is parallel to the timelike Killing vector ${}^{\mathrm{R}}\zeta$ is

$$
T = \frac{a g_{00}^{-1/2}}{2\pi k_{\mathrm{B}}} = \frac{a e^{-a\xi}}{2\pi k_{\mathrm{B}}} = \frac{1}{2\pi\alpha k_{\mathrm{B}}}, \qquad (93)
$$

where $\alpha^{-1} = e^{a\xi}/a$ is the observer's proper acceleration. This is exactly the same result as we obtained in the section 2, which confirms the conclusion that accelerating observers detect particles in states that inertial observers would find empty. This is the so-called Unruh effect.

[6] Having derived this relation, it's easy to use the commutators between ${}^{\mathrm{L}}b_k$ and ${}^{\mathrm{R}}b_k$ to derive the commutators between $d_k^{(1)}$ and $d_k^{(2)}$. Such a calculation would confirm that $d_k^{(1,2)\dagger}$ creates particles of type $(1, 2)$.

5 The thermal character of black holes

Without too much effort, we can extend the results of the last section to the situation around a stationary black hole of mass M[7]. Let's consider a two dimensional model of a black hole with the line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) du\, dv, \tag{94}$$

where $u = t - r_*$ and $v = t + r_*$ with

$$r_* = r + 2M \ln(r/2M - 1). \tag{95}$$

Consider the transformation to Kruskal coordinates

$$U = -4Me^{-u/4M}, \quad V = 4Me^{v/4M}, \tag{96}$$

which casts the line element in the form

$$ds^2 = \frac{2M}{r} e^{-r/2M} dU\, dV. \tag{97}$$

We define a Kruskal time and spatial coordinate by

$$T = (U + V)/2, \quad X = (U - V)/2. \tag{98}$$

Exploiting the conformal triviality of the massless Klein-Gordon equation in two dimension, we obtain the scalar wave equation for the field ϕ in both coordinate systems

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2}\right) \phi = 0, \tag{99}$$

$$\left(\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2}\right) \phi = 0. \tag{100}$$

Comparing (96) with (59), (99) with (61), and (100) with the wave equation in Minkowski space, we see that the problem of relating a quantum field theory in standard Schwarzschild coordinates to one in Kruskal coordinates is the mathematically equivalent to the problem of relating Rindler and Minkowski formulations of scalar field theory. That is, provided we make the identifications

$$\begin{aligned} a &\to 1/4M, \\ \eta &\to t, \\ \xi &\to r_*, \\ t &\to T, \\ x &\to X. \end{aligned} \tag{101}$$

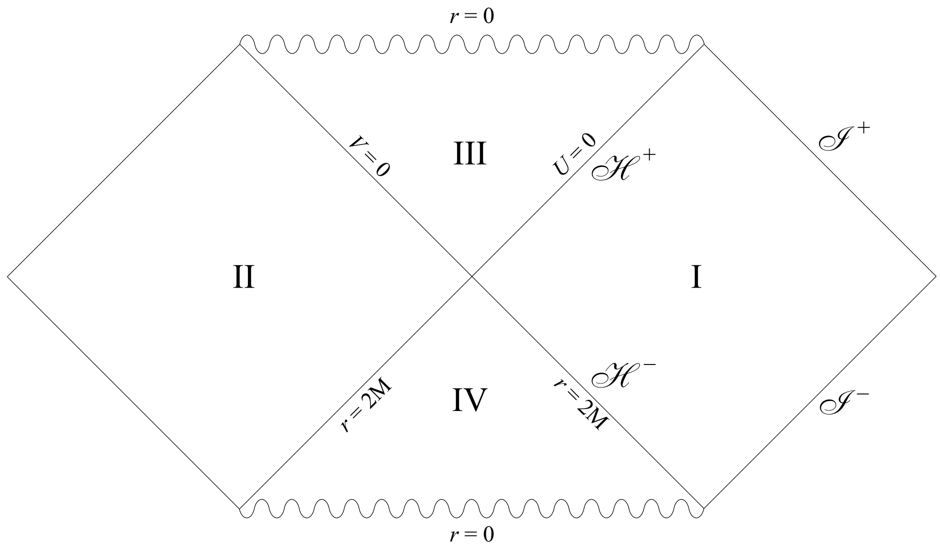

Figure 3: A Penrose-Carter diagram of the maximally extended Schwarzschild manifold

The reason that the two problems are so similar to one another becomes apparent when one compares figure 1 with the Penrose-Carter diagram shown in figure 3. We have already discussed how the $\bar{u} = 0$ and $\bar{v} = 0$ lines act as horizons for Rindler observers that separate the L, R, F and P regions. In the black hole manifold, the $U = 0$ and $V = 0$ lines form barriers between the I, II, III and IV regions and mark the position of the event horizon. The coordinates (η, ξ), which cover L and R in separate coordinate patches, are entirely analogous to the ordinary Schwarzschild coordinates (t, r_*), which cover I and II in separate patches. An observer living in region I (that is, *our* portion of the extended Schwarzschild manifold) will tend to construct a field theory in the (t, r_*) coordinates, which cover the only part of the manifold that he can truly regard as "physical". His mode solutions are then well defined in the range $r_* \in (-\infty, \infty)$ or $r \in (2M, \infty)$. But it may well be that the true vacuum state of the scalar field is the one associated with the entire manifold, defined by $UV < 1$. We denote the vacuum associated with (t, r_*) observers as $|0_\mathrm{S}\rangle$, the so-called Boulware vacuum, while the vacuum associated with (U, V) observers is called $|0_\mathrm{K}\rangle$, the so-called Hartle-Hawking or Israel vacuum.

The Bogolubov transformation between the creation and annihilation operators in the two coordinate systems is given by (90). For obvious reasons, we re-label ${}^\mathrm{L}b_k \to b_k^\mathrm{II}$ and ${}^\mathrm{R}b_k \to b_k^\mathrm{I}$. Since the Rindler spacetime is associated with the ordinary Schwarzschild coordinates, the action of $b_k^{\mathrm{I}\dagger}$ on the Boulware vacuum creates

[7] For the first part of this section, we draw on section 8.3 of Birrell & Davies [1].

a particle in region I, while the action of $b_k^{\mathrm{II}\dagger}$ is to create a particle in region II. In a similar fashion, we see that $\{d_k^{(1)}, d_k^{(2)}\}$ operators annihilate the Hartle-Hawking vacuum and $\{d_k^{(1)\dagger}, d_k^{(2)\dagger}\}$ create particles on the extended manifold.

Just as we had before, an observer using Schwarzschild modes to analyze the field in the Hartle-Hawking vacuum will find a thermal spectrum of field quanta, with a characteristic temperature

$$T_0 = \kappa/2\pi k_{\mathrm{B}}, \tag{102}$$

where $\kappa = 1/4M$ is the surface gravity of the black hole. The number of particles in each mode is given by

$$N_k = \frac{1}{e^{\omega/k_{\mathrm{B}}T_0} - 1}, \tag{103}$$

which is the characteristic spectrum of blackbody radiation. We can verify that the radiation is thermal by examining the Bogolubov transformation in more detail[8]. We can invert (90) to obtain:

$$d_k^{(1)} = [2\sinh(4M\pi\omega)]^{-1/2}\,(e^{2M\pi\omega} b_k^{\mathrm{I}} - e^{-2M\pi\omega} b_{-k}^{\mathrm{II}\dagger}) \tag{104}$$

$$d_k^{(2)} = [2\sinh(4M\pi\omega)]^{-1/2}\,(e^{2M\pi\omega} b_k^{\mathrm{II}} - e^{-2M\pi\omega} b_{-k}^{\mathrm{I}\dagger}). \tag{105}$$

Now, working with a box normalization, we have that

$$\begin{aligned}
0 &= [2\sinh(4M\pi\omega)]^{1/2}\,\langle 0_{\mathrm{S}}|d_k^{(1)} d_{-k}^{(2)}|0_{\mathrm{K}}\rangle \\
&= \langle 0_{\mathrm{S}}|(e^{2M\pi\omega} b_k^{\mathrm{I}} - e^{-2M\pi\omega} b_{-k}^{\mathrm{II}\dagger})(e^{2M\pi\omega} b_{-k}^{\mathrm{II}} - e^{-2M\pi\omega} b_k^{\mathrm{I}\dagger})|0_{\mathrm{K}}\rangle \\
&= e^{4M\pi\omega}\langle 0_{\mathrm{S}}|b_k^{\mathrm{I}} b_{-k}^{\mathrm{II}}|0_{\mathrm{K}}\rangle - \langle 0_{\mathrm{S}}|b_k^{\mathrm{I}} b_k^{\mathrm{I}\dagger}|0_{\mathrm{K}}\rangle \\
&= e^{4M\pi\omega}\langle 1_k^{\mathrm{I}}, 1_{-k}^{\mathrm{II}}|0_{\mathrm{K}}\rangle - \langle 0_{\mathrm{S}}|0_{\mathrm{K}}\rangle.
\end{aligned}$$

The notation is that $|1_k^{\mathrm{I}}, 1_{-k}^{\mathrm{II}}\rangle$ is a state where there is one particle with momentum k in region I and one particle with momentum $-k$ in region II. We define $Z^{-1} \equiv \langle 0_{\mathrm{S}}|0_{\mathrm{K}}\rangle$. We then get

$$\langle 1_k^{\mathrm{I}}, 1_{-k}^{\mathrm{II}}|0_{\mathrm{K}}\rangle = Z^{-1}\,e^{-4M\pi\omega}. \tag{106}$$

Working inductively (and tediously), we obtain

$$\langle n_k^{\mathrm{I}}, m_{-k'}^{\mathrm{II}}|0_{\mathrm{K}}\rangle = Z^{-1}\,e^{-n_k(4M\pi\omega)}\,\delta_{kk'}\delta_{n_k m_{-k'}}. \tag{107}$$

Where $|n_k^{\mathrm{I}}\rangle = (b_k^{\mathrm{I}\dagger})^n|0_{\mathrm{S}}\rangle/(n!)^{1/2}$ and $|n_k^{\mathrm{II}}\rangle = (b_k^{\mathrm{II}\dagger})^n|0_{\mathrm{S}}\rangle/(n!)^{1/2}$. This means that the Hartle-Hawking vacuum is made up of Boulware states such that there are the same number of particles in region I and II with equal and opposite momenta, i.e. the

[8]We loosely follow Unruh [2].

particles occur in pairs. Keeping the condition on k and k' in mind, we can expand the Hartle-Hawking vacuum in terms of Boulware particle states:

$$\begin{aligned}
|0_{\rm K}\rangle &= \prod_k \sum_{n_k} \sum_{m_{-k}} |n^{\rm I}_k, m^{\rm II}_{-k}\rangle\langle n^{\rm I}_k, m^{\rm II}_{-k}|0_{\rm K}\rangle \\
&= Z^{-1}\prod_k \sum_{n_k} e^{-n_k(4M\pi\omega)}|n^{\rm I}_k, n^{\rm II}_{-k}\rangle \\
&= Z^{-1}\prod_k \sum_{n_k} \frac{\left[e^{-4M\pi\omega} b^{\rm I\dagger}_k b^{\rm II\dagger}_{-k}\right]^{n_k}}{n_k!}|0_{\rm S}\rangle \\
&= Z^{-1}\prod_k \exp\left[e^{-4M\pi\omega} b^{\rm I\dagger}_k b^{\rm II\dagger}_{-k}\right]|0_{\rm S}\rangle. \qquad (108)
\end{aligned}$$

Now, if the quantum state is in the Hartle-Hawking vacuum, we can write the probability that an observer using Boulware operators in region I will find n_{k_1} particles of momentum k_1 *and* n_{k_2} particles of momentum k_2 *and* n_{k_3} particles of momentum k_3, etc..., as

$$\left|\left(\prod_{m=1}^{\infty}\langle n^{\rm I}_{k_m}, n^{\rm II}_{-k_m}|\right)|0_{\rm K}\rangle\right|^2 = Z^{-2}\prod_{m=1}^{\infty} e^{-n_{k_m}(8M\pi\omega)} = \prod_{m=1}^{\infty} P(n_{k_m}), \qquad (109)$$

where $P(n_{k_m})$ is the probability of finding *only* n_{k_m} particles of momentum k_m in region I. Since the total probability observing the state to be characterized by $\{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\}$ is the product of observing each of the n_{k_m} separately, we conclude that the occupation number of any two momentum eigenstates are independent statistical variables. Since the probabilities are uncorrelated, we are indeed dealing with thermal blackbody radiation.

Since blackbody radiation is in thermal equilibrium with its surroundings, observers restricted to region I will conclude that the scalar field is in thermal equilibrium with the black hole region $r < 2M$. Hence, the temperature of the black hole is indeed given by (102). An immediate consequence of this fact comes from the first law of black hole mechanics, which states that for any physical process involving a black hole

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J, \qquad (110)$$

where δM is a change in the black hole mass, δA is a change in the black hole surface area, Ω_H is the angular velocity and δJ is the change in angular momentum. Substituting the temperature into this expression yields:

$$\delta M = T_0\, \delta\left(\frac{k_{\rm B} A}{4}\right) + \Omega_H \delta J. \qquad (111)$$

This bears a remarkable resemblance to the first law of thermodynamics if one identifies the mass of the black hole with the total energy of the system. Then, one is forced to the conclusion that the *entropy* of the black hole is given by

$$S = k_{\mathrm{B}} A/4, \tag{112}$$

i.e. one-fourth of the surface area of the horizon. The reason why black holes have an intrinsic temperature and entropy is open to much debate, but the presence of the horizon seem to play a pivotal rôle. Observers who use Kruskal coordinates have access to the entire manifold while the event horizon limits the observations of those in the Schwarzschild region I. When the field is in the pure Hartle-Hawking vacuum state, Schwarzschild observers can only see the part of $|0_{\mathrm{K}}\rangle$ that overlaps the region I part of the Fock space. That is, observers in region I lack enough information to decompose the state of the field into a complete set of particle state because they are ignorant of the particle content in region II. This loss of information is characteristic of entropy, and entropy is characteristic of physical systems at finite temperature. This is hardly a rigorous derivation of black hole entropy, and a full statistical accounting of the internal modes of a black hole is as of yet forthcoming [3, 2, 4].

What is the local temperature as measured by a particle detector at a constant Schwarzschild radius $r = R$, i.e. observers with 4-velocities parallel to the static Killing field[9]? For such an detector, the increment in proper time τ is related to the increment in coordinate time t by

$$d\tau = (1 - 2M/R)^{1/2}\, dt, \tag{113}$$

while the trajectory in Kruskal coordinates is

$$U(t) = -4M\, e^{R_*/2M} e^{-t/4M}, \quad V(t) = 4M\, e^{R_*/2M} e^{t/4M}. \tag{114}$$

The response of the detector is given by equation (17) with $|0_{\mathrm{M}}\rangle$ replaced by $|0_{\mathrm{K}}\rangle$ and $x(\tau)$ and $x(\tau')$ evaluated in Kruskal coordinates. To evaluate the detector response, we need the massless Wightman Green's function $D^+(U, V; U', V')$ for the Kruskal modes. This is given by

$$\begin{aligned}
D^+(U,V;U',V') &= \langle 0_{\mathrm{K}}|\phi(U,V)\phi(U',V')|0_{\mathrm{K}}\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\omega}e^{-i\omega T+ikX}e^{i\omega T'-ikX'} \\
&= \frac{1}{4\pi}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{dk}{k}\left[e^{-ik(U-U'-i\epsilon)}+e^{-ik(V-V'-i\epsilon)}\right] \\
&= \frac{1}{4\pi}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{dk}{k}\left[e^{-ik(U-U'-i\epsilon)}-e^{-k}\right]+ \\
&\quad \frac{1}{4\pi}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{dk}{k}\left[e^{-ik(V-V'-i\epsilon)}-e^{-k}\right]+\frac{1}{2\pi}\int_0^{\infty}\frac{e^{-k}dk}{k} \\
&= -\frac{1}{4\pi}\lim_{\epsilon\to 0}\ln\left(U-U'-i\epsilon\right)\left(V-V'-i\epsilon\right)+\infty.
\end{aligned} \tag{115}$$

[9]We follow section 8.3 of Birrell & Davies [1]

The infinite constant in (115) comes from the sum of $\left(\int_0^\infty dk\, e^{-k}/k\right)/4\pi$ and the complex term $\ln(-1)/4\pi$. It represents the divergence of the Fourier space Green's function for low values of k: the so-called "infrared divergence". Infinite constants are not uncommon in quantum field theories, and the standard procedure is to ignore them. Doing so and dropping the limit notation, we can plug (114) into the expression (115) to obtain

$$D^+(U(t),V(t);U(t'),V(t')) = -\frac{1}{2\pi}\ln\left[8Me^{R_*/2M}\sinh\left(\frac{t-t'}{8M}-i\epsilon\right)\right], \tag{116}$$

where we have redefined ϵ as appropriate. Hence, in terms of the proper time interval $\Delta\tau \equiv \tau - \tau'$:

$$D^+(\Delta\tau) = \text{constant} - \frac{1}{2\pi}\ln\sinh\left(\frac{1}{2}\tilde{\kappa}\Delta\tau - i\epsilon\right), \tag{117}$$

where

$$\tilde{\kappa} = \kappa(1-2M/R)^{-1/2}. \tag{118}$$

Putting (117) into the formula for the detector response and dividing through by the infinite time integral, we obtain

$$p_n = -\frac{|\xi_n|^2}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\Delta E\tau}\ln\sinh\left(\frac{1}{2}\tilde{\kappa}\Delta\tau - i\epsilon\right), \tag{119}$$

where we have noted that the fact that $\Delta E = E_n - E_0 > 0$ implies that the constant term in $D^+(\tau)$ integrates to zero. Now, using the identity

$$\sinh x = x\prod_{m=1}^{\infty}(m\pi - ix)(m\pi + ix)/(m\pi)^2, \tag{120}$$

we can write

$$\begin{aligned}
-\ln\sinh\left(\frac{\tilde{\kappa}\tau}{2} - i\epsilon\right) &= -\ln\left(\frac{\tilde{\kappa}\tau}{2\epsilon} - i\right) + \sum_{m=1}^{\infty}\ln(m\pi)^2 \\
&\quad - \ln\prod_{m=1}^{\infty}\sigma e^{\gamma}\left(m\pi - \frac{i\tilde{\kappa}\tau}{2}\right) - \ln\prod_{m=1}^{\infty}\sigma e^{\gamma}\left(m\pi + \frac{i\tilde{\kappa}\tau}{2}\right).
\end{aligned}$$

Here, we have defined $\epsilon = \sigma^2 e^{2\gamma}$, where γ is Euler's constant. When this expression is inserted into (119), the first term integrates to zero. This is because the $e^{-i\Delta E\tau}$ factor forces us to complete the contour in the lower half plane. But branch of the logarithm must radiate from $\tau = 2i\epsilon/\tilde{\kappa}$ in the upper half plane, which makes the integrand analytic in the lower-half plane. The second term is a series of constants, which integrate to zero for the reasons mentioned above. We can the last two terms can be dealt with by using the identity:

$$\int_0^\infty \frac{e^{-i\omega x}}{\omega(e^{\beta\omega}-1)}d\omega = -\lim_{\sigma\to 0}\ln\prod_{m=1}^{\infty}\sigma e^{\gamma}(\beta m + ix), \tag{121}$$

which yields

$$\begin{aligned} p_n &= \frac{|\xi_n|^2}{2\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} d\tau \frac{e^{-i\Delta E\tau}(e^{i\omega\tilde{\kappa}\tau/2}+e^{-i\omega\tilde{\kappa}\tau/2})}{\omega(e^{\pi\omega}-1)} \\ &= \frac{|\xi_n|^2\tilde{\kappa}}{4\pi\Delta E}\frac{1}{e^{2\pi\Delta E/\tilde{\kappa}}-1}. \end{aligned} \tag{122}$$

Here again we see a characteristic Planck spectrum with a local temperature given by

$$T = \tilde{\kappa}/2\pi k_{\mathrm{B}} = T_0(1-2M/R)^{-1/2}. \tag{123}$$

This is in agreement with the Tolman relation for the local temperature observed by comoving observes in a static spacetime (93). This temperature diverges at the horizon, which can be understood by noting that the proper acceleration $(a^\mu a_\mu)^{1/2}$ required to maintain a constant height above a black hole is

$$M/[R^2(1-2M/R)^{1/2}].$$

This acceleration blows up as $R \to 2M$ and hence the temperature measured by constant-R observers is infinite. However, observers at spatial infinity $R\to\infty$ have zero acceleration, yet they still observe the field to have a finite temperature. We see a combination of two effects here: both the acceleration of the Killing observers and the intrinsic temperature of the field contribute to the detector response (122).

All of our results center on the analysis of 2-dimensional black holes. However, they all generalize to 4-dimensions in an approximate fashion. The full scalar wave equation $\nabla^\alpha\nabla_\alpha\Phi = 0$ in the 4-dimensional Schwarzschild solution is given by:

$$-\frac{\partial^2 f_{kl}}{\partial r_*^2} + V_l(r_*)f_{kl} = \omega^2 f_{kl} \tag{124}$$

where we have

$$V_l(r_*) = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\left[l(l+1)+\frac{2M}{r}\right]. \tag{125}$$

and

$$\Phi_{klm}(r,\theta,\phi) = \frac{e^{\pm i\omega t}f_{kl}(r_*)Y_{lm}(\theta,\phi)}{r}. \tag{126}$$

Equation (124) is merely a one-dimensional Schrödinger equation for the wavefunction $f_{kl}(r_*)$ propagating in the presence of a potential $V(r_*)$ with an energy of ω^2. Since it is impossible to invert the relationship between r and r_*, it is impossible to express the potential in terms of the natural radial coordinate. Hence, no exact solution of (124) is known. However, the asymptotics are fairly easy to unravel. As $r_* \sim r \to \infty$, $V_l(r_*) \to l(l+1)/r_*^2$ which implies an approximate solution $f_{kl}(r_*) \propto r_* j_l(\omega r_*) \sim \sin(\omega r_* - l\pi/2)$ or $f_{kl}(r_*) \propto r_* n_l(\omega r_*) \sim -\cos(\omega r_* - l\pi/2)$ where j_l and n_l are spherical Bessel functions of the first and second kind respectively. At the other end, we have that as $r_* \to -\infty$, $e^{r_*/2M} \to r/2M - 1$ and the

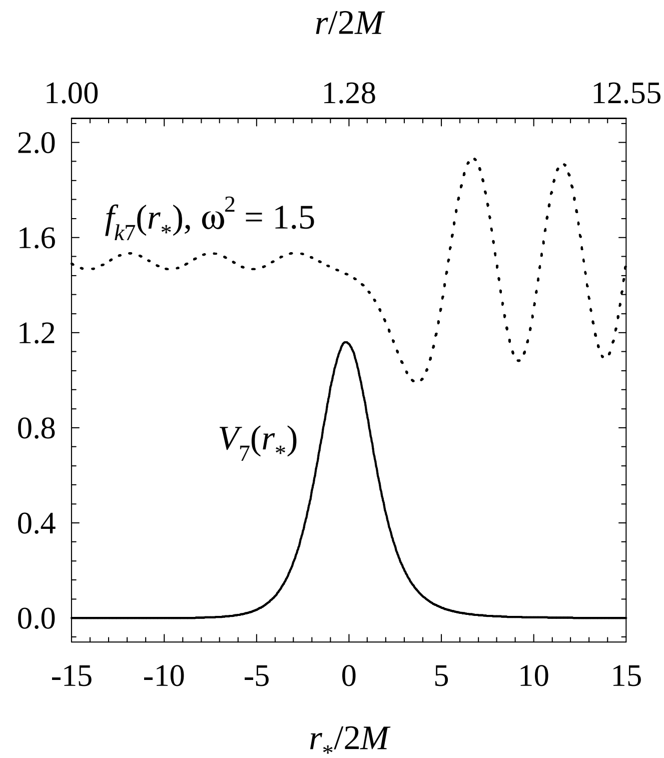

Figure 4: The potential $V_7(r_*)$ in the scalar wave equation for the $M = 1$ case. We also plot a solution of the wave equation $f_{k7}(r_*)$ with $\omega^2 = 1.5$, obtained from numerical methods. We employ geometric units in this figure.

potential falls off exponentially in r_*. This gives solutions like $f_{kl} \propto e^{\pm ikr_*}$ for the region immediately outside the black hole. As viewed in the (t, r, θ, ϕ) coordinates, the wavefunction will undergo infinite frequency oscillations near the event horizon which correspond to an infinite blueshift. These approximate results are easily verified by numerical work. In figure 4, we plot $V_7(r_*)$ versus r_* for the $M = 1$ case. The asymptotically flat behaviour of the potential is readily apparent from this plot. In the same plot, we also show a solution for $f_{k7}(r_*)$ for $M = 1$ and $\omega^2 = 1.5$ obtained from fourth order Runge-Kutta numerical methods[10]. We confirm that that f_{k0} behaves like a free wave for $r_* \to \pm\infty$. The upshot of all if this is that we can write that the mode solutions of (124) that are positive frequency with respect to $\partial/\partial t$ are given by

$$u_{klm}(x) \sim \frac{e^{-i\omega t + ikr_*} Y_{lm}(\theta, \phi)}{\sqrt{2\omega}(2\pi)^2 r_*}, \tag{127}$$

[10]We employ standard geometrical units. In conventional units, $M = 1$ corresponds to a mass of 1.3×10^{27} kg, or 68% of the mass of Jupiter. On the other hand, $\omega^2 = 1.5$ corresponds to a frequency of 58 MHz, or 2.4×10^{-7} eV.

for large values of r_*. In a completely analogous fashion, one obtains the asymptotic form of the mode solutions in Kruskal coordinates $\bar{u}_{klm}$ and can then construct approximate Bogolubov transformations between the field operators in the two reference frames. The result is the same as in the two dimensional case, and we find that the field has the same temperature as before.

It is a little dismaying that we cannot do the four-dimensional calculation in an exact manner because we do not have solutions for the mode functions in a closed form. However, there is another very elegant way of determining the temperature of field around black holes that is based on thermal Green's functions [1, 3, 5]. Using this method, we can determine the temperature of 4D black holes exactly and demonstrate that the thermodynamics of such systems is a consequence of the structure of Riemannian space. To so this, we need to introduce the concept of thermal Green's functions. Consider for example the Wightman Green's function

$$G^+(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle. \tag{128}$$

This is really a zero temperature expectation value because it assumes the field to be in the vacuum state exclusively. But, if the field is in the presence of a heat bath of temperature $k_{\mathrm{B}}T = 1/\beta$, we expect the the probability that the n^{th} eigenstate is occupied to be given by

$$p_n = e^{-\beta E_n}/Z, \quad Z = \sum_n e^{-\beta E_n}, \tag{129}$$

where the label n is used to schematically label all of the energy eigenstates. If we replace the vacuum state $|0\rangle$ in (128) with the appropriate thermal state, we get the Wightman function at a finite temperature T:

$$\begin{aligned} G_T^+(x, x') &= Z^{-1} \sum_n e^{-\beta E_n} \langle \psi_n|\phi(x)\phi(x')|\psi_n\rangle \\ &= Z^{-1} \sum_n \langle \psi_n|\phi(x)\phi(x')e^{-\beta H}|\psi_n\rangle. \end{aligned} \tag{130}$$

Here, $|\psi_n\rangle$ is the energy eigenstate of the Hamiltonian H with eigenvalue E_n. Now, using the Heisenberg equation of motion, we can consider an "imaginary time translation" of one of the arguments of the Green's function:

$$t' \to t' - i\beta. \tag{131}$$

Under such a translation, the field operator transforms as

$$\phi(t' - i\beta, \mathbf{x}) = e^{\beta H}\phi(t', \mathbf{x})e^{-\beta H}. \tag{132}$$

Hence we may write:

$$G_T^+(t, \mathbf{x}; t', \mathbf{x}') = Z^{-1} \sum_n \langle \psi_n|\phi(t, \mathbf{x})e^{-\beta H} e^{\beta H}\phi(t', \mathbf{x}')e^{-\beta H}|\psi_n\rangle$$

$$
\begin{aligned}
&= Z^{-1}\sum_n \langle\psi_n|\phi(t,\mathbf{x})e^{-\beta H}\phi(t'-i\beta,\mathbf{x}')|\psi_n\rangle \\
&= Z^{-1}\sum_{nm} \langle\psi_n|\phi(t,\mathbf{x})e^{-\beta H}|\psi_m\rangle\langle\psi_m|\phi(t'-i\beta,\mathbf{x}')|\psi_n\rangle \\
&= Z^{-1}\sum_m \langle\psi_m|\phi(t'-i\beta,\mathbf{x}')\phi(t,\mathbf{x})e^{-\beta H}|\psi_m\rangle \\
&= G_T^-(t,\mathbf{x};t'-i\beta,\mathbf{x}'). \qquad (133)
\end{aligned}
$$

We can analytically continue this result by rotating the time arguments of the Green's functions by a complex angle of $\pi/2$. Defining $\tau = it$, we get

$$G_T^+(\tau,\mathbf{x};\tau',\mathbf{x}') = G_T^-(\tau,\mathbf{x};\tau'+\beta,\mathbf{x}'). \qquad (134)$$

In a similar fashion, we get

$$G_T^+(\tau+\beta,\mathbf{x};\tau',\mathbf{x}') = G_T^-(\tau,\mathbf{x};\tau',\mathbf{x}'). \qquad (135)$$

The thermal Feynman propagator is given by

$$
\begin{aligned}
G_T^F(t,\mathbf{x};t',\mathbf{x}') &= Z^{-1}\sum_n e^{-\beta E_n}\langle\psi_n|T\{\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\}|\psi_n\rangle \\
&= \Theta(t-t')G_T^+(t,\mathbf{x};t',\mathbf{x}') + \Theta(t'-t)G_T^-(t,\mathbf{x};t',\mathbf{x}'), \qquad (136)
\end{aligned}
$$

where $T\{\cdots\}$ is the usual time-ordered product. The analytic continuation of the thermal Feynman propagator is clearly

$$G_T^F(\tau,\mathbf{x};\tau',\mathbf{x}') = \Theta(\tau-\tau')G_T^+(\tau,\mathbf{x};\tau',\mathbf{x}') + \Theta(\tau'-\tau)G_T^-(\tau,\mathbf{x};\tau',\mathbf{x}') \qquad (137)$$

Let's consider the case $0 < \tau - \tau' < \beta$. Then

$$
\begin{aligned}
G_T^F(\tau,\mathbf{x};\tau'+\beta,\mathbf{x}') &= G_T^-(\tau,\mathbf{x};\tau'+\beta,\mathbf{x}') \\
&= G_T^+(\tau,\mathbf{x};\tau',\mathbf{x}') \\
&= G_T^F(\tau,\mathbf{x};\tau',\mathbf{x}').
\end{aligned}
$$

Also, for $0 < \tau' - \tau < \beta$ we get

$$G_T^F(\tau+\beta,\mathbf{x};\tau',\mathbf{x}') = G_T^F(\tau,\mathbf{x};\tau',\mathbf{x}'). \qquad (138)$$

Hence, we see that the thermal Feynman propagator is periodic in both of the imaginary time arguments, $\tau = it$ and $\tau' = it'$, with a period of $\beta = 1/k_{\mathrm{B}}T$ if $|\tau - \tau'| < \beta$. This characteristic is shared by some of the other two-point thermal Green's functions, as can easily be seen by their definitions in terms of G_T^+ and G_T^-.

Now, let us consider a "Euclidean" Schwarzschild solution obtained by performing a Wick rotation $t \to i\tau$ on the conventional solution:

$$ds_E^2 = -\left(1-\frac{2M}{r}\right)d\tau^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\Omega^2, \qquad (139)$$

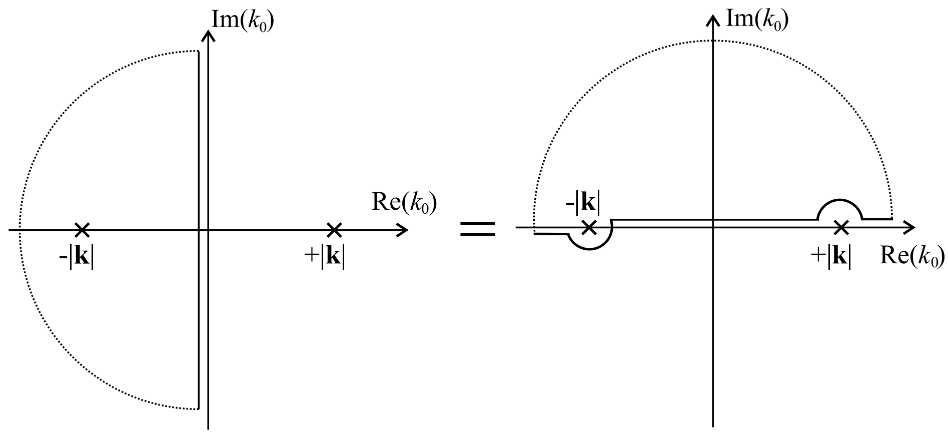

Figure 5: Equivalent contours of integration for the calculation of the massless Euclidean Green's function in the complex k_0 plane

where the E subscript stands for "Euclidean". Define a new radial coordinate

$$R = 4M(1 - 2M/r)^{1/2}, \tag{140}$$

to cast the metric in the form

$$ds_E^2 = -R^2 d\left(\frac{\tau}{4M}\right)^2 - \left(\frac{r}{2M}\right)^4 dR^2 - r^2 d\Omega. \tag{141}$$

This metric has a coordinate singularity at $R = 0$ that can be properly understood by identifying $\tau/4M$ with an angular coordinate of period 2π. Hence, the topology of the Euclidean manifold is $\mathbb{R}^2 \times S^2$.

Now, it turns out that there is only one type of Green's function that needs to be considered for the metric (139). This can be seen for the $M = 0$ case by considering the massless Green's function $D_E(x, x')$, which satisfies

$$\nabla^\alpha \nabla_\alpha D_E(x, x') = -\delta(x - x'). \tag{142}$$

This equation can be solved in Fourier space and inverted to give

$$D_E(\tau, \mathbf{x}; \tau', \mathbf{x}') = \frac{1}{(2\pi)^4} \int d\kappa\, d\mathbf{k}\, \frac{e^{i[\kappa(\tau-\tau') + \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}}{\kappa^2 + \mathbf{k}^2} \tag{143}$$

The integrand has poles at $\kappa = \pm i|\mathbf{k}|$. Because there are no poles on the real κ axis, we do not need to modify our contour of integration to avoid singularities like in the Lorentzian case. This means the Green's function is unique. Now, we continue D_E back to the usual Lorentzian space, we write

$$D_E(-it,\mathbf{x};-it',\mathbf{x}') = \frac{1}{(2\pi)^4}\int d\kappa\, d\mathbf{k}\, \frac{e^{\kappa(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\kappa^2+\mathbf{k}^2}$$
$$= \frac{i}{(2\pi)^4}\int d\mathbf{k}\int_{-i\infty}^{i\infty} dk_0\, \frac{e^{-ik_0(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k_0^2-\mathbf{k}^2},$$

where we have switched integration variables according to $k_0 = i\kappa$. The contour used to calculate the k_0 integral is shown in the lefthand part of figure 5. Notice that the poles now occur for real values of the integration variable k_0. We can deform the contour to the standard one for the Feynman Green's function, which is shown on the right hand side of figure 5, by rotating it 90° clockwise. Hence, when the Euclidean Green's function is analytically continued into the Lorentzian manifold, we recover the Feynman propagator. This result holds equally well for $M \neq 0$. Because of the topology of the Euclidean manifold, we see that $G_E(x,x')$ must be periodic in τ and τ' with period $2\pi/\kappa$, where $\kappa = 1/4M$ is the surface gravity. When $G_E(x,x')$ is analytically continued to the Lorentzian Schwarzschild manifold, the resulting Feynman propagator will be periodic under imaginary time translations $t \to t \pm i\beta$. Hence, the quantum field represented by this Feynman propagator is in in thermal equilibrium with a heat bath of temperature $T_0 = \kappa/2\pi$.

There is one ambiguous point that deserves mention, namely, what state does the thermal Feynman propagator actually represent? Is it $|0_\mathrm{S}\rangle$, $|0_\mathrm{K}\rangle$ or some other state? There is no general prescription for answering this question except for explicit calculation of the propagator in various coordinate systems and comparison with the expression derived from $G_E(x,x')$. Such a calculation would reveal that the Feynman propagator so derived is indeed the one that is obtained by calculating $\langle 0_\mathrm{K}|T\{\phi(x)\phi(x')\}|0_\mathrm{K}\rangle$ in Schwarzschild coordinates [5]. This confirms our previous result that $|0_\mathrm{K}\rangle$ is a state where the quantum field is in equilibrium with the black hole, that is, the black hole emits as much radiation as it absorbs. But what of black hole radiation, where the central body emits particles that travel to infinity? In order to understand this effect we must consider a more realistic model of a black hole, namely, one that forms from gravitational collapse as opposed to being there for all values of t.

6 The Hawking effect

The black holes considered in the last sections are no physically realistic because they exist for all times. True astrophysical black holes are likely formed from the gravitational collapse of some matter distribution. We would like to determine how a massless scalar field ϕ behaves in such a spacetime and what are the consequences of this behaviour when the field theory is quantized. This discussion will be semi-qualitative[11]; the interested reader is directed to more rigourous accounts in section 8.1 of Birrell & Davies [1] and Unruh's paper [2].

[11] Largely because the details are beyond the originally intended scope of this paper

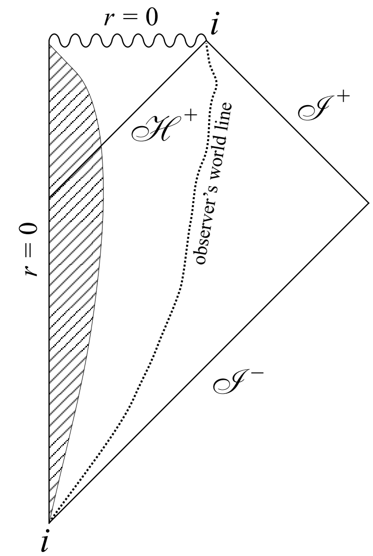

Figure 6: A Penrose diagram of the gravitational collapse of a matter distribution into a black hole

As usual, we will model our spacetime in two-dimensions in order to exploit the conformal simplicity of the system. The spacetime we imagine is depicted in figure 6. The matter distribution is assumed to be confined to a ball $r < R(\tau)$ where $R(\tau)$ is the world line of the boundary Σ. Outside the ball we assume a line element of the form

$$ds_+^2 = C(r)\,dt^2 - \frac{1}{C(r)}\,dr^2 = C(r)\,du\,dv, \tag{144}$$

where the precise form of $C(r)$ is not really important, but is in actuality given by the Schwarzschild factor $C(r) = 1 - 2M/r$. The important point is that C goes to zero for some non-zero value of r, i.e. the exterior metric has an event horizon that may occur inside or outside the ball. The u and v coordinates are given by

$$u = t - (r^* - R_0^*), \tag{145}$$
$$v = t + (r^* - R_0^*). \tag{146}$$

In general, radial coordinates adorned with a star are related to their unstarred counterparts by the solution of the equation

$$\frac{dr^*}{dr} = \frac{1}{C(r)}. \tag{147}$$

In the Schwarzschild case,

$$R_0^* = R_0 + 2M\ln(R_0/2M - 1), \tag{148}$$

where R_0 is a constant. We assume that prior to $\tau = t = 0$ the ball of matter is static with a radius of R_0. The surface of the ball is hence described by $u = v$ for $\tau < 0$. Inside the ball, we take a metric of the form

$$ds_-^2 = A(U,V)\left(d\tau^2 - dr^2\right) = A(U,V)\,dU\,dV. \tag{149}$$

Again, we do not specify the function $A(U,V)$, which will in general depend on the details of the collapse. The U and V coordinates are given by

$$U = \tau - (r - R_0), \tag{150}$$
$$V = \tau + (r - R_0). \tag{151}$$

We find the relation between t and τ by equating the induced metric on the surface of the ball as viewed from the inside with the induced metric as viewed from the outside. From the outside, the induced metric is

$$ds_\Sigma^2 = C\,dt^2 - \frac{\dot{R}^2}{C}\,d\tau^2, \tag{152}$$

where $\dot{R} = dR/d\tau$ and we evaluate C at $R(\tau)$. From the inside

$$ds_\Sigma^2 = A(1 - \dot{R}^2)\,d\tau^2, \tag{153}$$

where A is evaluated at τ and $R(\tau)$. Setting these expressions equal to each other, we get

$$C\frac{dt}{d\tau} = \sqrt{AC\left(1 - \dot{R}^2\right) + \dot{R}^2}. \tag{154}$$

Now, define the coordinate transformation between the (u,v) and (U,V) coordinates by

$$U = \alpha(u), \quad v = \beta(V). \tag{155}$$

Then

$$\alpha'(u) = \frac{dU}{du} = \frac{(1-\dot{R})\,d\tau}{dt - \dot{R}/C\,d\tau} = \frac{C(1-\dot{R})}{C(dt/d\tau) - \dot{R}}, \tag{156}$$

and

$$\beta'(V) = \frac{dv}{dV} = \frac{dt + \dot{R}/C\,d\tau}{(1+\dot{R})\,d\tau} = \frac{C(dt/d\tau) + \dot{R}}{C(1+\dot{R})}, \tag{157}$$

where A and C are evaluated at $r = R(\tau)$. Now, for times before $\tau = 0$ when $\dot{R} = 0$, we have that

$$\alpha'(u) = \sqrt{\frac{C}{A}}, \quad \beta'(V) = \sqrt{\frac{A}{C}}, \tag{158}$$

which implies, by the inverse function theorem, that α and β are functional inverses of one another. Also, the requirement that the ball of matter be static before the collapse implies that A must only be a function of r and is hence constant on the

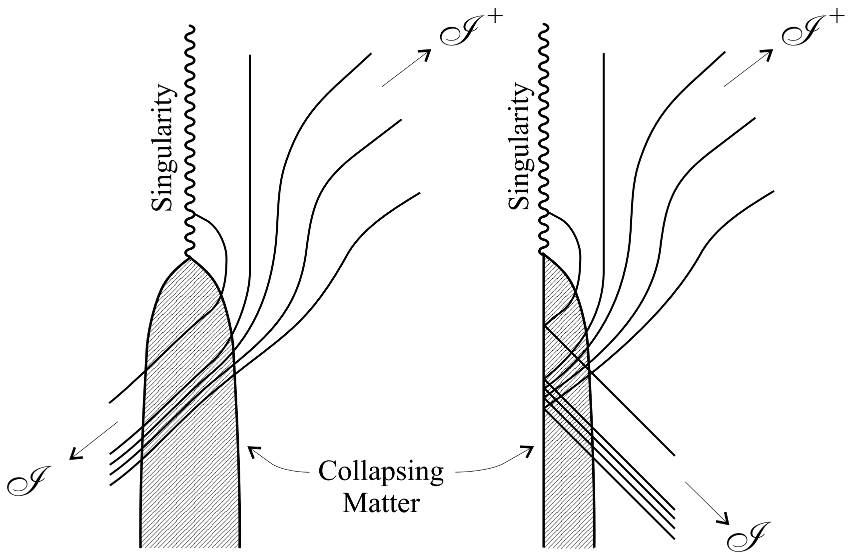

Figure 7: The propagation of null geodesics through the center of a ball of collapsing matter (*left*). The ray γ generates the event horizon. In order to simulate this situation in 2D, we reflect null rays at $r = 0$ (*right*).

boundary. $C(r)$ must also be constant, so we get that $\alpha(u) = ku$ and $\beta(V) = V/k$ before the collapse, where k is a constant.

Now, before writing down the mode solutions for our model, we need to address how we are planning to simulate the spherical symmetry of the 4D spacetime in our 2D manifold. The problem is depicted in the lefthand side of figure 7. Here, we see how null geodesics propagating in from past null infinity travel through the ball and proceed to future null infinity. This diagram, in essence, stretches from $r = -\infty$ to $r = \infty$ because it depicts both sides of the star. However, our metric only covers one-half of the total space because r must run from 0 to ∞ ($C(r)$ would certainly have strange properties if r were allow to be negative). To remedy the situation, we will reflect null rays in the line $r = 0$, as in the righthand side of figure 7. This will appropriately mimic light rays that travel through the star. But, what effect does this have on the scalar field ϕ? Since we assume the field is massless, null geodesics are perpendicular to the surfaces of constant phase. Hence, ϕ is the field describing the reflected rays. In analogy with electromagnetism, the condition of total reflection at $r = 0$ means that ϕ should vanish at the origin of the radial coordinate[12]. That is, ϕ ought to vanish along the line:

$$U - V = 2R_0. \tag{159}$$

Now, the conformally trivial wave equations for ϕ are

$$0 = \frac{\partial^2\phi}{\partial u \partial v} = \frac{\partial^2\phi}{\partial U \partial V}. \tag{160}$$

A solution that satisfies the boundary condition (159) is

$$\phi_k = \frac{i}{(4\pi\omega)^{1/2}}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}). \tag{161}$$

Observe that if $U - V = 2R_0$, then

$$\beta[\alpha(u) - 2R_0] = \beta(U - 2R_0) = \beta(V) = v, \tag{162}$$

and ϕ_k vanishes. Also observe that before the collapse

$$\beta[\alpha(u) - 2R_0] = \beta(ku - 2R_0) = u - 2R_0/k, \tag{163}$$

where $k = \sqrt{C/A}$ evaluated on the static boundary. The mode solutions hence reduce to

$$\phi_k = \frac{i}{(4\pi\omega)^{1/2}}(e^{-i\omega v} - e^{i\varphi}e^{-i\omega u}), \tag{164}$$

where φ is an unimportant constant phase. This is explicitly positive frequency with respect to the Killing field $\partial/\partial t$ and hence represents the same vacuum state used by observers using standard mode solutions $\sim e^{-i(\omega t + kr)}$ before the collapse. But, after the collapse, the complicated form of $\beta[\alpha(u) - 2R_0]$ virtually guarantees that ϕ_k will not be strictly positive frequency with respect to $\partial/\partial t$. So, observers in the future will measure the particle content of the field described by ϕ_k to be non-zero with respect to their standard mode solutions. But these particle we not there in the past, so observers will conclude that as the matter collapses, it emits a flux of particles. This is the Hawking effect

In practical calculations, it is easier to work with a form of ϕ_k that reduces to regular mode solutions in the asymptotic future and has a complicated form in the asymptotic past. Computation of the Bogolubov transformation between such modes and standard modes reveals that the temperature of the thermal radiation at late times is the same as in the case of the external black holes in the previous

[12] Another way to think about this boundary condition is to note that in 4D, the presence of the centrifugal barrier near $r = 0$ drives wavefunctions to either go to zero or diverge. In order to keep solutions regular, one usually demands the solution vanish at the origin, which is akin to choosing to expand the field in spherical Bessel functions as opposed to spherical Neumann functions. Because we have no centrifugal barrier in 2D, we need to simulate its presence by imposing the $\phi(r = 0) = 0$ boundary condition, which is directly responsible for the thermal radiation from collapsing matter.

section [1, 2]. More interesting is an analysis of the stress-energy tensor associated with the field ϕ_k reveals that there is a flux of *positive* energy escaping to infinity while there is a flux of *negative* energy entering the black hole[13]. Clearly, this quantum field doesn't obey the classical energy conditions. Nominally, this result doesn't mean a thing to the black hole because we have assumed that the presence of the field ϕ does not affect the underlying spacetime. But if we take the back-reaction of the metric caused by the field into account, the fact that the black hole is being lit up be a stream of negative mass radiation implies that it's own mass must decrease. This is the phenomena of black hole evapouration, where black holes radiate away their mass in a finite amount of time after their creation. No one really knows what happens to the curvature singularity at $r = 0$ when $M \to 0$, though speculation ranges from the formation of a naked singularity or some sort of explosive realignment of spacetime. This situation is shown in the Penrose diagram presented in figure 8. We can estimate the time it takes for the black hole to evapourate completely by using the Stefan-Boltzmann law for the total radiative flux dE/dt from a perfect blackbody [3]: $dE/dt \propto AT^4$, where A is the black hole area and T is it's temperature. Since $A \propto M^2$ and $T \propto 1/M$ and the energy of the black hole is simply it's mass, we get

$$\frac{dM}{dt} \propto -\frac{1}{M^2}. \tag{165}$$

Putting in all of the units and constants, we see that the time scale for the evapouration of a solar mass object is $\sim 10^{71}$ seconds, which is much larger than the age of the universe. So we don't really need to worry about many large black hole explosions happening in the immediate vicinity anytime soon. However, small black holes ($M \sim 10^{14}$ grams) created in the primordial universe might be undergoing the final stages of evapouration right now. Since a black hole becomes quite hot as it's mass decreases, one would expect these black holes to be adding a significant amount of radiation to the the high energy γ-ray background. No such contribution has been observed, suggesting there cannot be many mini-black holes in the universe.

7 Conclusions

We have presented a model of a particle detector in flat space and have argued that inertial and accelerating observers will disagree as to the particle content of the Minkowski vacuum $|0_{\mathrm{M}}\rangle$.

A general formalism for quantum field theory in curved space was discussed, and we demonstrated how creation and annihilation operators behave under coordinate changes using Bogolubov transformation.

Using Bogolubov transformations, we considered the quantum theory of a massless scalar field in a Rindler spacetimes, and determined that comoving Rindler

[13]The derivation of this result is complicated and has a lot to do with the thorny issue of the renormalization of $T_{\alpha\beta}$ in curved space. See chapter 6 of Birrell & Davies [1]

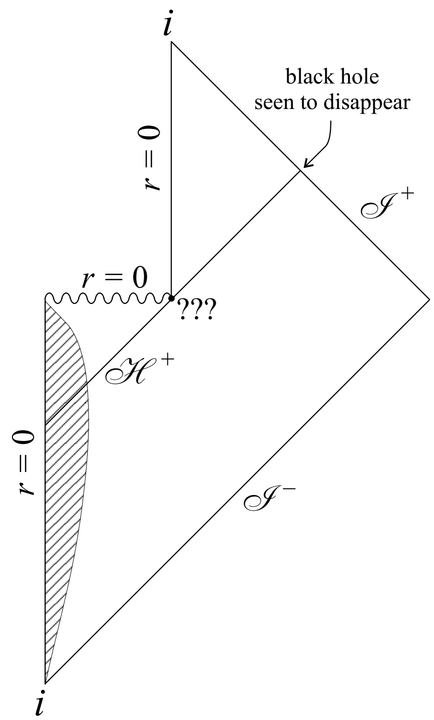

Figure 8: A Penrose diagram of an evapourating black hole. The black hole horizon intersects future null infinity at a finite time, which means that observers could possibly see the curvature singularity at $r = 0$ (marked with ???). The region above the singularity is presumably flat Minkowski space.

observers will detect a thermal spectrum of particles in the Minkowski vacuum. The temperature is proportional to the observer's proper acceleration. This is the Unruh effect.

We generalized this result to a 2D model of the Schwarzschild spacetime, and showed that Rindler observers are analogous to observers following constant r trajectories in that they observe a thermal spectrum of particles with a temperature proportional to the surface gravity. From this and the first law of black hole mechanics, we derived the "entropy" of a black hole. We considered particle detectors following static Killing orbits and showed that they will measure a field temperature consistent with the Tolman relation. We have demonstrated how the same conclusions can be drawn from considering a Euclidean continuation of the Schwarzschild metric into imaginary times $\tau = it$, where we discovered that the Feynman propagator must be periodic in τ. The phenomena of black hole temperature is sometimes called the curved space Unruh effect.

Finally, we presented a semi-qualitative derivation of black hole radiance by considering the 2D gravitational collapse of a ball of matter. We have shown that observers that initially measure the field to be in the vacuum state will later measure the field to contain a thermal spectrum of particles. It was argued that the flux of particles from the black hole must result in the mass decreasing all the way down to zero, which is the phenomena known as black hole evapouration. This is the Hawking effect.

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