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From primordial quantum fluctuations to the anisotropies of the cosmic microwave background radiation*

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These lecture notes cover mainly three connected topics. In the first part we give a detailed treatment of cosmological perturbation theory. The second part is devoted to cosmological inflation and the generation of primordial fluctuations. In part three it will be shown how these initial perturbation evolve and produce the temperature anisotropies of the cosmic microwave background radiation. Comparing the theoretical prediction for the angular power spectrum with the increasingly accurate observations provides important cosmological information (cosmological parameters, initial conditions).

Contents

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Introduction

Cosmology is going through a fruitful and exciting period. Some of the developments are definitely also of interest to physicists outside the fields of astrophysics and cosmology.

These lectures cover some particularly fascinating and topical subjects. A central theme will be the current evidence that the recent ($z < 1$) Universe is dominated by an exotic nearly homogeneous dark energy density with *negative* pressure. The simplest candidate for this unknown so-called *Dark Energy* is a cosmological term in Einstein's field equations, a possibility that has been considered during all the history of relativistic cosmology. Independently of what this exotic energy density is, one thing is certain since a long time: The energy density belonging to the cosmological constant is not larger than the cosmological critical density, and thus *incredibly small by particle physics standards*. This is a profound mystery, since we expect that all sorts of *vacuum energies* contribute to the effective cosmological constant.

Since this is such an important issue it should be of interest to see how convincing the evidence for this finding really is, or whether one should remain sceptical. Much of this is based on the observed temperature fluctuations of the cosmic microwave background radiation (CMB). A detailed analysis of the data requires a considerable amount of theoretical machinery, the development of which fills most space of these notes.

Since this audience consists mostly of diploma and graduate students, whose main interests are outside astrophysics and cosmology, I do not presuppose that you had already some serious training in cosmology. However, I do assume that you have some working knowledge of general relativity (GR). As a source, and for references, I usually quote my recent textbook [1].

In an opening chapter those parts of the Standard Model of cosmology will be treated that are needed for the main parts of the lectures. More on this can be found at many places, for instance in the recent textbooks on cosmology [2–6].

In Part I we will develop the somewhat involved cosmological perturbation theory. The formalism will later be applied to two main topics: (1) The generation of primordial fluctuations during an inflationary era. (2) The evolution of these perturbations during the linear regime. A main goal will be to determine the CMB power spectrum.

0 Essentials of Friedmann-Lemaître models

For reasons explained in the Introduction I treat in this opening chapter some standard material that will be needed in the main parts of these notes. In addition, an important topical subject will be discussed in some detail, namely the Hubble diagram for Type Ia supernovas that gave the first evidence for an accelerated expansion of the 'recent' and future universe. Most readers can directly go to Sect. 0.2, where this is treated.

0.1 Friedmann-Lemaître spacetimes

There is now good evidence that the (recent as well as the early) Universe[1]is – on large scales – surprisingly homogeneous and isotropic. The most impressive support for this comes from extended redshift surveys of galaxies and from the truly remarkable isotropy of the cosmic microwave background (CMB). In the Two Degree Field (2dF) Galaxy Redshift Survey,[2] completed in 2003, the redshifts of about 250'000 galaxies have been measured. The distribution of galaxies out to 4 billion light years shows that there are huge clusters, long filaments, and empty voids measuring over 100 million light years across. But the map also shows that there are *no larger structures*. The more extended Sloan Digital Sky Survey (SDSS) has already produced very similar results, and will in the end have spectra of about a million galaxies[3].

One arrives at the Friedmann (Lemaître-Robertson-Walker) spacetimes by postulating that for each observer, moving along an integral curve of a distinguished four-velocity field u, the Universe looks spatially isotropic. Mathematically, this means the following: Let $Iso_x(M)$ be the group of local isometries of a Lorentz manifold (M, g), with fixed point $x \in M$, and let $SO_3(u_x)$ be the group of all linear transformations of the tangent space $T_x(M)$ which leave the 4-velocity u_x invariant and induce special orthogonal transformations in the subspace orthogonal to u_x, then

$$\{T_x\phi : \ \phi \in Iso_x(M), \ \phi_\star u = u\} \supseteq SO_3(u_x)$$

($\phi_\star$ denotes the push-forward belonging to ϕ; see [1, p. 550]). In [7] it is shown that this requirement implies that (M, g) is a Friedmann spacetime, whose structure we now recall. Note that (M, g) is then automatically homogeneous.

A *Friedmann spacetime* (M, g) is a warped product of the form $M = I \times \Sigma$, where I is an interval of $\mathbb{R}$, and the metric g is of the form

$$g = -dt^2 + a^2(t)\gamma, \tag{1}$$

such that (Σ, γ) is a Riemannian space of constant curvature $k = 0, \pm 1$. The distinguished time t is the *cosmic time*, and $a(t)$ is the *scale factor* (it plays the role of the warp factor (see Appendix B of [1])). Instead of t we often use the *conformal time* η, defined by $d\eta = dt/a(t)$. The velocity field is perpendicular to the slices of constant cosmic time, $u = \partial/\partial t$.

1 By *Universe* I always mean that part of the world around us which is in principle accessible to observations. In my opinion the 'Universe as a whole' is not a scientific concept. When talking about *model universes*, we develop on paper or with the help of computers, I tend to use lower case letters. In this domain we are, of course, free to make extrapolations and venture into speculations, but one should always be aware that there is the danger to be drifted into a kind of 'cosmo-mythology'.

2 Consult the Home Page: http://www.mso.anu.edu.au/2dFGRS .

3 For a description and pictures, see the Home Page: http://www.sdss.org/sdss.html .

0.1.1 Spaces of constant curvature

For the space (Σ, γ) of constant curvature[4] the curvature is given by

$$R^{(3)}(X,Y)Z = k\,[\gamma(Z,Y)X - \gamma(Z,X)Y]\,; \tag{2}$$

in components:

$$R^{(3)}_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{3}$$

Hence, the Ricci tensor and the scalar curvature are

$$R^{(3)}_{jl} = 2k\gamma_{jl}\ ,\ R^{(3)} = 6k. \tag{4}$$

For the curvature two-forms we obtain from (3) relative to an orthonormal triad $\{\theta^i\}$

$$\Omega^{(3)}_{ij} = \frac{1}{2} R^{(3)}_{ijkl}\,\theta^k \wedge \theta^l = k\,\theta_i \wedge \theta_j \tag{5}$$

($\theta_i = \gamma_{ik}\theta^k$). The simply connected constant curvature spaces are in n dimensions the (n+1)-sphere S^{n+1} ($k = 1$), the Euclidean space ($k = 0$), and the pseudo-sphere ($k = -1$). Non-simply connected constant curvature spaces are obtained from these by forming quotients with respect to discrete isometry groups. (For detailed derivations, see [8].)

0.1.2 Curvature of Friedmann spacetimes

Let $\{\bar{\theta}^i\}$ be any orthonormal triad on (Σ, γ). On this Riemannian space the first structure equations read (we use the notation in [1]; quantities referring to this 3-dim. space are indicated by bars)

$$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j = 0. \tag{6}$$

On (M, g) we introduce the following orthonormal tetrad:

$$\theta^0 = dt,\ \theta^i = a(t)\bar{\theta}^i. \tag{7}$$

From this and (6) we get

$$d\theta^0 = 0,\ d\theta^i = \frac{\dot{a}}{a}\theta^0 \wedge \theta^i - a\,\bar{\omega}^i{}_j \wedge \bar{\theta}^j. \tag{8}$$

Comparing this with the first structure equation for the Friedmann manifold implies

$$\omega^0{}_i \wedge \theta^i = 0,\ \omega^i{}_0 \wedge \theta^0 + \omega^i{}_j \wedge \theta^j = \frac{\dot{a}}{a}\theta^i \wedge \theta^0 + a\,\bar{\omega}^i{}_j \wedge \bar{\theta}^j, \tag{9}$$

whence

$$\boxed{\omega^0{}_i = \frac{\dot{a}}{a}\,\theta^i,\ \omega^i{}_j = \bar{\omega}^i{}_j.} \tag{10}$$

The worldlines of *comoving observers* are integral curves of the four-velocity field $u = \partial_t$. We claim that these are geodesics, i.e., that

$$\nabla_u u = 0. \tag{11}$$

To show this (and for other purposes) we introduce the basis $\{e_\mu\}$ of vector fields dual to (7). Since $u = e_0$ we have, using the connection forms (10),

$$\nabla_u u = \nabla_{e_0} e_0 = \omega^\lambda{}_0(e_0)e_\lambda = \omega^i{}_0(e_0)e_i = 0.$$

[4] For a detailed discussion of these spaces I refer – for readers knowing German – to [8] or [9].

0.1.3 Einstein equations for Friedmann spacetimes

Inserting the connection forms (10) into the second structure equations we readily find for the curvature 2-forms $\Omega^{\mu}{}_{\nu}$:

$$\Omega^0{}_i = \frac{\ddot{a}}{a}\theta^0 \wedge \theta^i,\ \Omega^i{}_j = \frac{k+\dot{a}^2}{a^2}\theta^i \wedge \theta^j. \tag{12}$$

A routine calculation leads to the following components of the Einstein tensor relative to the basis (7)

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \tag{13}$$

$$G_{11} = G_{22} = G_{33} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}, \tag{14}$$

$$G_{\mu\nu} = 0\ (\mu \neq \nu). \tag{15}$$

In order to satisfy the field equations, the symmetries of $G_{\mu\nu}$ imply that the energy-momentum tensor *must* have the perfect fluid form (see [1, Sect. 1.4.2]):

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}, \tag{16}$$

where u is the comoving velocity field introduced above.

Now, we can write down the field equations (including the cosmological term):

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho + \Lambda, \tag{17}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi Gp - \Lambda. \tag{18}$$

Although the 'energy-momentum conservation' does not provide an independent equation, it is useful to work this out. As expected, the momentum 'conservation' is automatically satisfied. For the 'energy conservation' we use the general form (see (1.37) in [1])

$$\nabla_u \rho = -(\rho + p)\nabla \cdot u. \tag{19}$$

In our case we have for the *expansion rate*

$$\nabla \cdot u = \omega^{\lambda}{}_0(e_{\lambda})u^0 = \omega^i{}_0(e_i),$$

thus with (10)

$$\nabla \cdot u = 3\frac{\dot{a}}{a}. \tag{20}$$

Therefore, Eq. (19) becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{21}$$

For a given equation of state, $p = p(\rho)$, we can use (21) in the form

$$\frac{d}{da}(\rho a^3) = -3pa^2 \tag{22}$$

to determine ρ as a function of the scale factor a. Examples: 1. For free massless particles (radiation) we have $p = \rho/3$, thus $\rho \propto a^{-4}$. 2. For dust ($p = 0$) we get $\rho \propto a^{-3}$.

With this knowledge the *Friedmann equation* (17) determines the time evolution of $a(t)$.

Exercise. Show that (18) follows from (17) and (21).

As an important consequence of (17) and (18) we obtain for the acceleration of the expansion

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a + \frac{1}{3}\Lambda a. \tag{23}$$

This shows that as long as $\rho+3p$ is positive, the first term in (23) is decelerating, while a positive cosmological constant is repulsive. This becomes understandable if one writes the field equation as

$$G_{\mu\nu} = \kappa(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}) \qquad (\kappa = 8\pi G), \tag{24}$$

with

$$T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}. \tag{25}$$

This vacuum contribution has the form of the energy-momentum tensor of an ideal fluid, with energy density $\rho_\Lambda = \Lambda/8\pi G$ and pressure $p_\Lambda = -\rho_\Lambda$. Hence the combination $\rho_\Lambda + 3p_\Lambda$ is equal to $-2\rho_\Lambda$, and is thus negative. In what follows we shall often include in ρ and p the vacuum pieces.

0.1.4 Redshift

As a result of the expansion of the Universe the light of distant sources appears redshifted. The amount of redshift can be simply expressed in terms of the scale factor $a(t)$.

Consider two integral curves of the average velocity field u. We imagine that one describes the worldline of a distant comoving source and the other that of an observer at a telescope (see Fig. 1). Since light is propagating along null geodesics, we conclude from (1) that along the worldline of a light ray $dt = a(t)d\sigma$, where $d\sigma$ is the line element on the 3-dimensional space (Σ, γ) of constant curvature $k = 0, \pm 1$. Hence the integral on the left of

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{source}^{obs.} d\sigma, \tag{26}$$

between the time of emission (t_e) and the arrival time at the observer (t_o), is independent of t_e and t_o. Therefore, if we consider a second light ray that is emitted at the time $t_e + \Delta t_e$ and is received at the time $t_o + \Delta t_o$, we obtain from the last equation

$$\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a(t)} = \int_{t_e}^{t_o} \frac{dt}{a(t)}. \tag{27}$$

For a small Δt_e this gives

$$\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)}.$$

The observed and the emitted frequences ν_o and ν_e, respectively, are thus related according to

$$\frac{\nu_o}{\nu_e} = \frac{\Delta t_e}{\Delta t_o} = \frac{a(t_e)}{a(t_o)}. \tag{28}$$

The redshift parameter z is defined by

$$z := \frac{\nu_e - \nu_o}{\nu_o}, \tag{29}$$

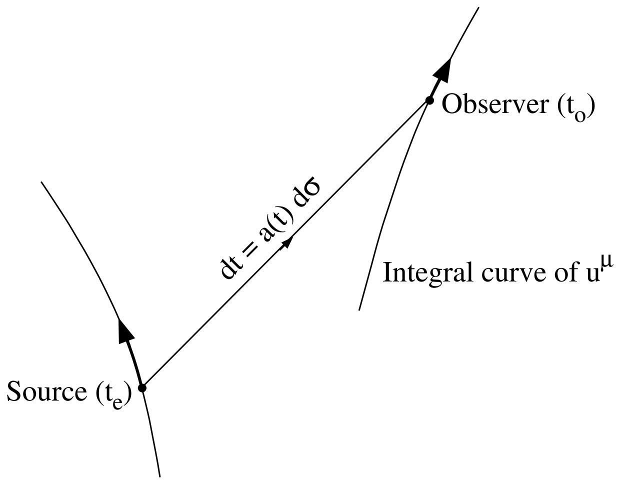

Fig. 1 Redshift for Friedmann models.

and is given by the key equation

$$\boxed{1+z=\frac{a(t_o)}{a(t_e)}.} \tag{30}$$

One can also express this by the equation $\nu \cdot a = const$ along a null geodesic.

0.1.5 Cosmic distance measures

We now introduce a further important tool, namely operational definitions of three different distance measures, and show that they are related by simple redshift factors.

If D is the physical (proper) extension of a distant object, and δ is its angle subtended, then the *angular diameter distance* D_A is defined by

$$D_A := D/\delta. \tag{31}$$

If the object is moving with the proper transversal velocity $V_\perp$ and with an apparent angular motion $d\delta/dt_0$, then the *proper-motion distance* is by definition

$$D_M := \frac{V_\perp}{d\delta/dt_0}. \tag{32}$$

Finally, if the object has the intrinsic luminosity $\mathcal{L}$ and $\mathcal{F}$ is the received energy flux then the *luminosity distance* is naturally defined as

$$D_L := (\mathcal{L}/4\pi\mathcal{F})^{1/2}. \tag{33}$$

Below we show that these three distances are related as follows

$$\boxed{D_L = (1+z)D_M = (1+z)^2 D_A.} \tag{34}$$

It will be useful to introduce on (Σ, γ) 'polar' coordinates (r, ϑ, φ), such that

$$\gamma = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2,\ d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2. \tag{35}$$

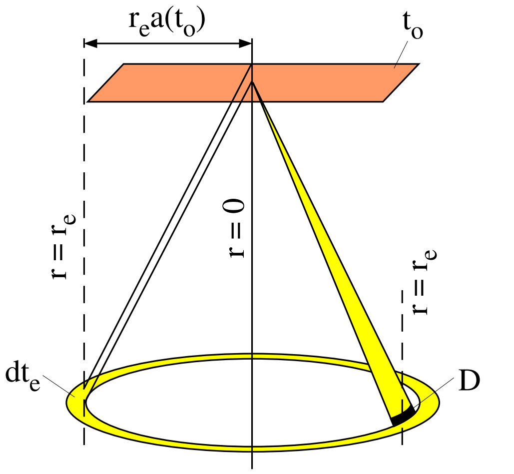

Fig. 2 Spacetime diagram for cosmic distance measures.

One easily verifies that the curvature forms of this metric satisfy (5). (This follows without doing any work by using in [1] the curvature forms (3.9) in the ansatz (3.3) for the Schwarzschild metric.)

To prove (34) we show that the three distances can be expressed as follows, if r_e denotes the comoving radial coordinate (in (35)) of the distant object and the observer is (without loss of generality) at $r = 0$.

$$D_A = r_e a(t_e),\ D_M = r_e a(t_0),\ D_L = r_e a(t_0)\frac{a(t_0)}{a(t_e)}. \tag{36}$$

Once this is established, (34) follows from (30).

From Fig. 2 and (35) we see that

$$D = a(t_e) r_e \delta, \tag{37}$$

hence the first equation in (36) holds.

To prove the second one we note that the source moves in a time dt_0 a proper transversal distance

$$dD = V_\perp dt_e = V_\perp dt_0 \frac{a(t_e)}{a(t_0)}.$$

Using again the metric (35) we see that the apparent angular motion is

$$d\delta = \frac{dD}{a(t_e) r_e} = \frac{V_\perp dt_0}{a(t_0) r_e}.$$

Inserting this into the definition (32) shows that the second equation in (36) holds. For the third equation we have to consider the observed energy flux. In a time dt_e the source emits an energy $\mathcal{L} dt_e$. This energy is redshifted to the present by a factor $a(t_e)/a(t_0)$, and is now distributed by (35) over a sphere with proper area $4\pi(r_e a(t_0))^2$ (see Fig. 2). Hence the received flux (*apparent luminosity*) is

$$\mathcal{F} = \mathcal{L} dt_e \frac{a(t_e)}{a(t_0)} \frac{1}{4\pi(r_e a(t_0))^2} \frac{1}{dt_0},$$

thus

$$\mathcal{F} = \frac{\mathcal{L} a^2(t_e)}{4\pi a^4(t_0) r_e^2}.$$

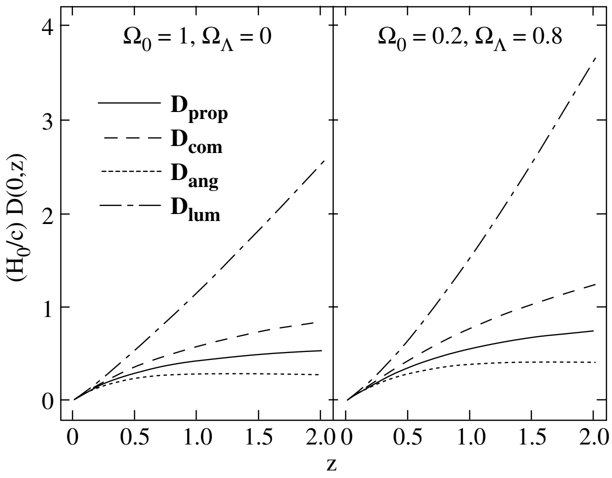

Fig. 3 Cosmological distance measures as a function of source redshift for two cosmological models. The angular diameter distance $D_{ang} \equiv D_A$ and the luminosity distance $D_{lum} \equiv D_L$ have been introduced in this section. The other two will be introduced later.

Inserting this into the definition (33) establishes the third equation in (36). For later applications we write the last equation in the more transparent form

$$\boxed{\mathcal{F} = \frac{\mathcal{L}}{4\pi(r_e a(t_0))^2}\frac{1}{(1+z)^2}.} \tag{38}$$

The last factor is due to redshift effects.

Two of the discussed distances as a function of z are shown in Fig. 3 for two Friedmann models with different cosmological parameters. The other two distance measures will be introduced later (Sect. 3.2).

0.2 Luminosity-redshift relation for Type Ia supernovas

A few years ago the Hubble diagram for Type Ia supernovas gave, as a big surprise, the first serious evidence for a currently accelerating Universe. Before presenting and discussing critically these exciting results, we develop on the basis of the previous section some theoretical background. (For the benefit of readers who start with this section we repeat a few things.)

0.2.1 Theoretical redshift-luminosity relation

We have seen that in cosmology several different distance measures are in use, which are all related by simple redshift factors. The one which is relevant in this section is the *luminosity distance* D_L. We recall that this is defined by

$$D_L = (\mathcal{L}/4\pi\mathcal{F})^{1/2}, \tag{39}$$

where $\mathcal{L}$ is the intrinsic luminosity of the source and $\mathcal{F}$ the observed energy flux.

We want to express this in terms of the redshift z of the source and some of the cosmological parameters. If the comoving radial coordinate r is chosen such that the Friedmann- Lemaître metric takes the form

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \quad k = 0, \pm 1, \tag{40}$$

then we have

$$\mathcal{F}dt_0 = \mathcal{L}dt_e \cdot \frac{1}{1+z} \cdot \frac{1}{4\pi(r_e a(t_0))^2}.$$

The second factor on the right is due to the redshift of the photon energy; the indices $0, e$ refer to the present and emission times, respectively. Using also $1 + z = a(t_0)/a(t_e)$, we find in a first step:

$$D_L(z) = a_0(1+z)r(z) \quad (a_0 \equiv a(t_0)). \tag{41}$$

We need the function $r(z)$. From

$$dz = -\frac{a_0}{a}\frac{\dot{a}}{a}dt, \quad dt = -a(t)\frac{dr}{\sqrt{1-kr^2}}$$

for light rays, we see that

$$\frac{dr}{\sqrt{1-kr^2}} = \frac{1}{a_0}\frac{dz}{H(z)} \quad (H(z) = \frac{\dot{a}}{a}). \tag{42}$$

Now, we make use of the Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho. \tag{43}$$

Let us decompose the total energy-mass density ρ into nonrelativistic (NR), relativistic (R), Λ, quintessence (Q), and possibly other contributions

$$\rho = \rho_{NR} + \rho_R + \rho_\Lambda + \rho_Q + \cdots . \tag{44}$$

For the relevant cosmic period we can assume that the "energy equation"

$$\frac{d}{da}(\rho a^3) = -3pa^2 \tag{45}$$

also holds for the individual components $X = NR, R, \Lambda, Q, \cdots$. If $w_X \equiv p_X/\rho_X$ is constant, this implies that

$$\rho_X a^{3(1+w_X)} = const. \tag{46}$$

Therefore,

$$\rho = \sum_X \left(\rho_X a^{3(1+w_X)}\right)_0 \frac{1}{a^{3(1+w_X)}} = \sum_X (\rho_X)_0 (1+z)^{3(1+w_X)}. \tag{47}$$

Hence the Friedmann equation (43) can be written as

$$\frac{H^2(z)}{H_0^2} + \frac{k}{H_0^2 a_0^2}(1+z)^2 = \sum_X \Omega_X (1+z)^{3(1+w_X)}, \tag{48}$$

where Ω_X is the dimensionless density parameter for the species X,

$$\Omega_X = \frac{(\rho_X)_0}{\rho_{\text{crit}}}, \tag{49}$$

where ρ_{crit} is the critical density:

$$\begin{aligned}\rho_{\text{crit}} &= \frac{3H_0^2}{8\pi G} \\ &= 1.88 \times 10^{-29}\, h_0^2\, g\, cm^{-3}\end{aligned} \tag{50}$$

$$= 8 \times 10^{-47} h_0^2 \, GeV^4.$$

Here h_0 is the *reduced Hubble parameter*

$$h_0 = H_0/(100 \, km \, s^{-1} \, Mpc^{-1}) \tag{51}$$

and is close to 0.7. Using also the curvature parameter $\Omega_K \equiv -k/H_0^2 a_0^2$, we obtain the useful form

$$\boxed{H^2(z) = H_0^2 E^2(z; \Omega_K, \Omega_X),} \tag{52}$$

with

$$E^2(z; \Omega_K, \Omega_X) = \Omega_K (1+z)^2 + \sum_X \Omega_X (1+z)^{3(1+w_X)}. \tag{53}$$

Especially for $z = 0$ this gives

$$\Omega_K + \Omega_0 = 1, \quad \Omega_0 \equiv \sum_X \Omega_X. \tag{54}$$

If we use (52) in (42), we get

$$\int_0^{r(z)} \frac{dr}{\sqrt{1 - kr^2}} = \frac{1}{H_0 a_0} \int_0^z \frac{dz'}{E(z')} \tag{55}$$

and thus

$$r(z) = \mathcal{S}(\chi(z)), \tag{56}$$

where

$$\chi(z) = \frac{1}{H_0 a_0} \int_0^z \frac{dz'}{E(z')} \tag{57}$$

and

$$\mathcal{S}(\chi) = \begin{cases} \sin\chi & : \ k = 1 \\ \chi & : \ k = 0 \\ \sinh\chi & : \ k = 1. \end{cases} \tag{58}$$

Inserting this in (41) gives finally the relation we were looking for

$$D_L(z) = \frac{1}{H_0} \mathcal{D}_L(z; \Omega_K, \Omega_X), \tag{59}$$

with

$$\mathcal{D}_L(z; \Omega_K, \Omega_X) = (1+z) \frac{1}{|\Omega_K|^{1/2}} \mathcal{S}\left(|\Omega_K|^{1/2} \int_0^z \frac{dz'}{E(z')} \right) \tag{60}$$

for $k = \pm 1$. For a flat universe, $\Omega_K = 0$ or equivalently $\Omega_0 = 1$, the “Hubble-constant-free” luminosity distance is

$$\mathcal{D}_L(z) = (1+z) \int_0^z \frac{dz'}{E(z')}. \tag{61}$$

Astronomers use as logarithmic measures of $\mathcal{L}$ and $\mathcal{F}$ the *absolute and apparent magnitudes* [5], denoted by M and m, respectively. The conventions are chosen such that the *distance modulus* $m - M$ is related to D_L as follows

$$m - M = 5 \log \left(\frac{D_L}{1\ Mpc} \right) + 25. \tag{62}$$

Inserting the representation (59), we obtain the following relation between the apparent magnitude m and the redshift z:

$$m = \mathcal{M} + 5 \log \mathcal{D}_L(z; \Omega_K, \Omega_X), \tag{63}$$

where, for our purpose, $\mathcal{M} = M - 5 \log H_0 + 25$ is an uninteresting fit parameter. The comparison of this theoretical *magnitude redshift relation* with data will lead to interesting restrictions for the cosmological Ω-parameters. In practice often only Ω_M and Ω_Λ are kept as independent parameters, where from now on the subscript M denotes (as in most papers) nonrelativistic matter.

The following remark about *degeneracy curves* in the Ω-plane is important in this context. For a fixed z in the presently explored interval, the contours defined by the equations $\mathcal{D}_L(z; \Omega_M, \Omega_\Lambda) = const$ have little curvature, and thus we can associate an approximate slope to them. For $z = 0.4$ the slope is about 1 and increases to 1.5-2 by $z = 0.8$ over the interesting range of Ω_M and Ω_Λ. Hence even quite accurate data can at best select a strip in the Ω-plane, with a slope in the range just discussed. This is the reason behind the shape of the likelihood regions shown later (Fig. 5).

In this context it is also interesting to determine the dependence of the *deceleration parameter*

$$q_0 = -\left(\frac{a\ddot{a}}{\dot{a}^2} \right)_0 \tag{64}$$

on Ω_M and Ω_Λ. At an any cosmic time we obtain from (23) and (47)

$$-\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2} \frac{1}{E^2(z)} \sum_X \Omega_X (1+z)^{3(1+w_X)} (1 + 3w_X). \tag{65}$$

For $z = 0$ this gives

$$q_0 = \frac{1}{2} \sum_X \Omega_X (1 + 3w_X) = \frac{1}{2} (\Omega_M - 2\Omega_\Lambda + \cdots). \tag{66}$$

The line $q_0 = 0$ ($\Omega_\Lambda = \Omega_M/2$) separates decelerating from accelerating universes at the present time. For given values of Ω_M, Ω_Λ, etc, (65) vanishes for z determined by

$$\Omega_M (1+z)^3 - 2\Omega_\Lambda + \cdots = 0. \tag{67}$$

This equation gives the redshift at which the deceleration period ends (coasting redshift).

Redshift dependent w for quintessence. In quintessence models the ratio $w_Q = p_Q/\rho_Q$ is often allowed to be redshift dependent. Then the function $E(z)$ in (53) gets modified. To see how, start from the energy equation (45) and write this as

$$\frac{d \ln(\rho_Q a^3)}{d \ln(1+z)} = 3w_Q.$$

[5] Beside the (bolometric) magnitudes m, M, astronomers also use magnitudes $m_B,\ m_V,\ \ldots$ referring to certain wavelength bands B (blue), V (visual), and so on.

This gives

$$\rho_Q(z) = \rho_{Q0}(1+z)^3 \exp\left(\int_0^{\ln(1+z)} 3w_Q(z')d\ln(1+z')\right)$$

or

$$\rho_Q(z) = \rho_{Q0} \exp\left(3\int_0^{\ln(1+z)} (1+w_Q(z'))d\ln(1+z')\right). \tag{68}$$

Hence, we have to perform on the right of (53) the following substitution:

$$\Omega_Q(1+z)^{3(1+w_Q)} \to \Omega_Q \exp\left(3\int_0^{\ln(1+z)} (1+w_Q(z'))d\ln(1+z')\right). \tag{69}$$

0.2.2 Type Ia supernovas as standard candles

It has long been recognized that supernovas of type Ia are excellent standard candles and are visible to cosmic distances [10] (the record is at present at a redshift of about 1.7). At relatively closed distances they can be used to measure the Hubble constant, by calibrating the absolute magnitude of nearby supernovas with various distance determinations (e.g., Cepheids). There is still some dispute over these calibration resulting in differences of about 10% for H_0. (For a review see, e.g., [11]; a recent paper in an ongoing research project is [12].)

In 1979 Tammann [13] and Colgate [14] independently suggested that at higher redshifts this subclass of supernovas can be used to determine also the deceleration parameter. In recent years this program became feasible thanks to the development of new technologies which made it possible to obtain digital images of faint objects over sizable angular scales, and by making use of big telescopes such as Hubble and Keck.

There are two major teams investigating high-redshift SNe Ia, namely the 'Supernova Cosmology Project' (SCP) and the 'High-Z Supernova search Team' (HZT). Each team has found a large number of SNe, and both groups have published almost identical results. (For up-to-date information, see the home pages [15] and [16].)

Before discussing these, a few remarks about the nature and properties of type Ia SNe should be made. Observationally, they are characterized by the absence of hydrogen in their spectra, and the presence of some strong silicon lines near maximum. The immediate progenitors are most probably carbon-oxygen white dwarfs in close binary systems, but it must be said that these have not yet been clearly identified.[6]

In the standard scenario a white dwarf accretes matter from a nondegenerate companion until it approaches the critical Chandrasekhar mass and ignites carbon burning deep in its interior of highly degenerate matter. This is followed by an outward-propagating nuclear flame leading to a total disruption of the white dwarf. Within a few seconds the star is converted largely into nickel and iron. The dispersed nickel radioactively decays to cobalt and then to iron in a few hundred days. A lot of effort has been invested to simulate these complicated processes. Clearly, the physics of thermonuclear runaway burning in degenerate matter is complex. In particular, since the thermonuclear combustion is highly turbulent, multidimensional simulations are required. This is an important subject of current research. (One gets a good impression of the present status from several articles in [17]. See also the recent review [18].) The theoretical uncertainties are such that, for instance, predictions for possible evolutionary changes are not reliable.

It is conceivable that in some cases a type Ia supernova is the result of a merging of two carbon-oxygen-rich white dwarfs with a combined mass surpassing the Chandrasekhar limit. Theoretical modelling indicates, however, that such a merging would lead to a collapse, rather than a SN Ia explosion. But this issue is still debated.

[6] This is perhaps not so astonishing, because the progenitors are presumably faint compact dwarf stars.

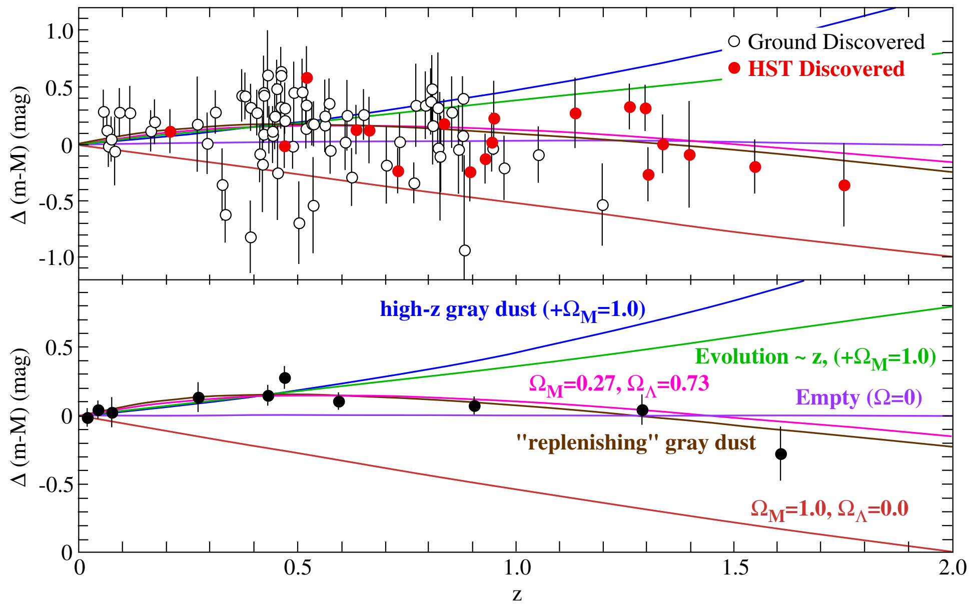

Fig. 4 Distance moduli relative to an empty uniformly expanding universe (residual Hubble diagram) for SNe Ia; see text for further explanations. (Adapted from [25], Fig. 7.).

In view of the complex physics involved, it is not astonishing that type Ia supernovas are not perfect standard candles. Their peak absolute magnitudes have a dispersion of 0.3–0.5 mag, depending on the sample. Astronomers have, however, learned in recent years to reduce this dispersion by making use of empirical correlations between the absolute peak luminosity and light curve shapes. Examination of nearby SNe showed that the peak brightness is correlated with the time scale of their brightening and fading: slow decliners tend to be brighter than rapid ones. There are also some correlations with spectral properties. Using these correlations it became possible to reduce the remaining intrinsic dispersion, at least in the average, to $\simeq 0.15\,\mathrm{mag}$. (For the various methods in use, and how they compare, see [19, 25], and references therein.) Other corrections, such as Galactic extinction, have been applied, resulting for each supernova in a corrected (rest-frame) magnitude. The redshift dependence of this quantity is compared with the theoretical expectation given by Eqs. (62) and (60).

0.2.3 Results

After the classic papers [20–22] on the Hubble diagram for high-redshift type Ia supernovas, published by the SCP and HZT teams, significant progress has been made (for reviews, see [23] and [24]). I discuss here the main results presented in [25]. These are based on additional new data for $z > 1$, obtained in conjunction with the GOODS (Great Observatories Origins Deep Survey) Treasury program, conducted with the Advanced Camera for Surveys (ACS) aboard the Hubble Space Telescope (HST).

The quality of the data and some of the main results of the analysis are shown in Fig. 4. The data points in the top panel are the distance moduli relative to an empty uniformly expanding universe, $\Delta(m-M)$, and the redshifts of a "gold" set of 157 SNe Ia. In this 'reduced' Hubble diagram the filled symbols are the HST-discovered SNe Ia. The bottom panel shows weighted averages in fixed redshift bins.

These data are consistent with the "cosmic concordance" model ($\Omega_M = 0.3,\ \Omega_\Lambda = 0.7$), with $\chi^2_{\rm dof} = 1.06$). For a flat universe with a cosmological constant, the fit gives $\Omega_M = 0.29\pm^{0.13}_{0.19}$ (equivalently, $\Omega_\Lambda = 0.71$). The other model curves will be discussed below. Likelihood regions in the $(\Omega_M, \Omega_\Lambda)$-plane, keeping only these parameters in (62) and averaging H_0, are shown in Fig. 5. To demonstrate the progress,

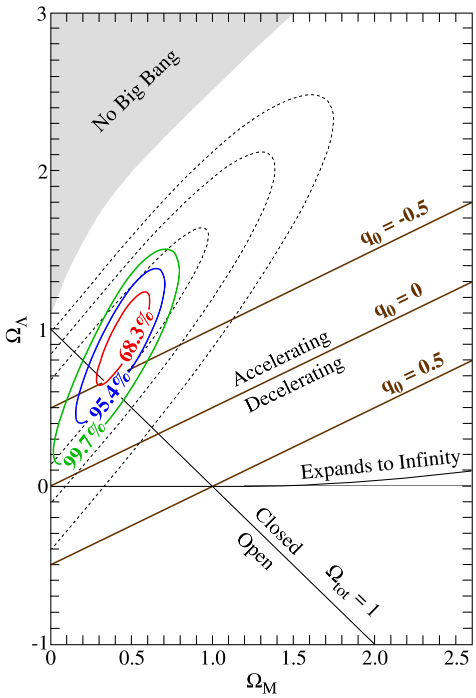

Fig. 5 Likelihood regions in the $(\Omega_M, \Omega_\Lambda)$-plane. The dotted contours are old results from 1998. (Adapted from [25, Fig. 8]).

old results from 1998 are also included. It will turn out that this information is largely complementary to the restrictions we shall obtain from the CMB anisotropies.

0.2.4 Systematic uncertainties

Possible systematic uncertainties due to astrophysical effects have been discussed extensively in the literature. The most serious ones are (i) *dimming* by intergalactic dust, and (ii) *evolution* of SNe Ia over cosmic time, due to changes in progenitor mass, metallicity, and C/O ratio. I discuss these concerns only briefly (see also [23,25]).

Concerning extinction, detailed studies show that high-redshift SN Ia suffer little reddening; their B-V colors at maximum brightness are normal. However, it can a priori not be excluded that we see distant SNe through a grey dust with grain sizes large enough as to not imprint the reddening signature of typical interstellar extinction. One argument against this hypothesis is that this would also imply a larger dispersion than is observed. In Fig. 4 the expectation of a simple grey dust model is also shown. The new high redshift data reject this monotonic model of astrophysical dimming. Eq. (67) shows that at redshifts $z \geq (2\Omega_\Lambda/\Omega_M)^{1/3} - 1 \simeq 1.2$ the Universe is *decelerating*, and this provides an almost unambiguous signature for Λ, or some effective equivalent. There is now strong evidence for a transition from a deceleration to acceleration at a redshift $z = 0.46 \pm 0.13$.

The same data provide also some evidence against a simple luminosity evolution that could mimic an accelerating Universe. Other empirical constraints are obtained by comparing subsamples of low-redshift SN Ia believed to arise from old and young progenitors. It turns out that there is no difference within the measuring errors, *after* the correction based on the light-curve shape has been applied. Moreover, spectra of high-redshift SNe appear remarkably similar to those at low redshift. This is very reassuring. On the other hand, there seems to be a trend that more distant supernovas are bluer. It would, of course, be helpful if evolution could be predicted theoretically, but in view of what has been said earlier, this is not (yet) possible.

In conclusion, none of the investigated systematic errors appear to reconcile the data with $\Omega_\Lambda = 0$ and $q_0 \geq 0$. But further work is necessary before we can declare this as a really established fact.

To improve the observational situation a satellite mission called SNAP ("Supernovas Acceleration Probe") has been proposed [29]. According to the plans this satellite would observe about 2000 SNe within a year and much more detailed studies could then be performed. For the time being some scepticism with regard to the results that have been obtained is still not out of place, but the situation is steadily improving.

Finally, I mention a more theoretical complication. In the analysis of the data the luminosity distance for an ideal Friedmann universe was always used. But the data were taken in the real inhomogeneous Universe. This may not be good enough, especially for high-redshift standard candles. The simplest way to take this into account is to introduce a filling parameter which, roughly speaking, represents matter that exists in galaxies but not in the intergalactic medium. For a constant filling parameter one can determine the luminosity distance by solving the Dyer-Roeder equation. But now one has an additional parameter in fitting the data. For a flat universe this was recently investigated in [30].

0.3 Thermal history below 100 MeV

A. Overview

Below the transition at about 200 MeV from a quark-gluon plasma to the confinement phase, the Universe was initially dominated by a complicated dense hadron soup. The abundance of pions, for example, was so high that they nearly overlapped. The pions, kaons and other hadrons soon began to decay and most of the nucleons and antinucleons annihilated, leaving only a tiny baryon asymmetry. The energy density is then almost completely dominated by radiation and the stable leptons ($e^{\pm}$, the three neutrino flavors and their antiparticles). For some time all these particles are in thermodynamic equilibrium. For this reason, only a few initial conditions have to be imposed. The Universe was never as simple as in this lepton era. (At this stage it is almost inconceivable that the complex world around us would eventually emerge.)

The first particles which freeze out of this equilibrium are the weakly interacting neutrinos. Let us estimate when this happened. The coupling of the neutrinos in the lepton era is dominated by the reactions:

$$e^- + e^+ \leftrightarrow \nu + \bar{\nu},\ e^{\pm} + \nu \to e^{\pm} + \nu,\ e^{\pm} + \bar{\nu} \to e^{\pm} + \bar{\nu}.$$

For dimensional reasons, the cross sections are all of magnitude

$$\sigma \simeq G_F^2 T^2, \tag{70}$$

where G_F is the Fermi coupling constant ($\hbar = c = k_B = 1$). Numerically, $G_F m_p^2 \simeq 10^{-5}$. On the other hand, the electron and neutrino densities n_e, n_ν are about T^3. For this reason, the reaction rates Γ for ν-scattering and ν-production per electron are of magnitude $c \cdot v \cdot n_e \simeq G_F^2 T^5$. This has to be compared with the expansion rate of the Universe

$$H = \frac{\dot{a}}{a} \simeq (G\rho)^{1/2}.$$

Since $\rho \simeq T^4$ we get

$$H \simeq G^{1/2} T^2 \tag{71}$$

and thus

$$\frac{\Gamma}{H} \simeq G^{-1/2} G_F^2 T^3 \simeq (T/10^{10}\ K)^3. \tag{72}$$

This ration is larger than 1 for $T > 10^{10}\ K \simeq 1\ MeV$, and the neutrinos thus remain in thermodynamic equilibrium until the temperature has decreased to about 1 MeV. But even below this temperature the neutrinos remain Fermi distributed,

$$n_\nu(p)dp = \frac{1}{2\pi^2} \frac{1}{e^{p/T_\nu} + 1} p^2 dp\ , \tag{73}$$

as long as they can be treated as massless. The reason is that the number density decreases as a^{-3} and the momenta with a^{-1}. Because of this we also see that the neutrino temperature T_ν decreases after decoupling as a^{-1}. The same is, of course true for photons. The reader will easily find out how the distribution evolves when neutrino masses are taken into account. (Since neutrino masses are so small this is only relevant at very late times.)

B. Chemical potentials of the leptons

The equilibrium reactions below 100 MeV, say, conserve several additive quantum numbers[7], namely the electric charge Q, the baryon number B, and the three lepton numbers L_e, L_μ, L_τ. Correspondingly, there are five independent chemical potentials. Since particles and antiparticles can annihilate to photons, their chemical potentials are oppositely equal: $\mu_{e^-} = -\mu_{e^+}$, etc. From the following reactions

$$e^- + \mu^+ \to \nu_e + \bar{\nu}_\mu,\ e^- + p \to \nu_e + n,\ \mu^- + p \to \nu_\mu + n$$

we infer the equilibrium conditions

$$\mu_{e^-} - \mu_{\nu_e} = \mu_{\mu^-} - \mu_{\nu_\mu} = \mu_n - \mu_p. \tag{74}$$

As independent chemical potentials we can thus choose

$$\boxed{\mu_p,\ \mu_{e^-},\ \mu_{\nu_e},\ \mu_{\nu_\mu},\ \mu_{\nu_\tau}.} \tag{75}$$

Because of local electric charge neutrality, the charge number density n_Q vanishes. From observations (see subsection E) we also know that the baryon number density n_b is much smaller than the photon number density ($\sim$ entropy density s_γ). The ratio n_B/s_γ remains constant for adiabatic expansion (both decrease with a^{-3}; see the next section). Moreover, the lepton number densities are

$$n_{L_e} = n_{e^-} + n_{\nu_e} - n_{e^+} - n_{\bar{\nu}_e},\ n_{L_\mu} = n_{\mu^-} + n_{\nu_\mu} - n_{\mu^+} - n_{\bar{\nu}_\mu},\ etc. \tag{76}$$

Since in the present Universe the number density of electrons is equal to that of the protons (bound or free), we know that after the disappearance of the muons $n_{e^-} \simeq n_{e^+}$ (recall $n_B \ll n_\gamma$), thus $\mu_{e^-} (= -\mu_{e^+}) \simeq 0$. It is conceivable that the chemical potentials of the neutrinos and antineutrinos can not be neglected, i.e., that n_{L_e} is not much smaller than the photon number density. In analogy to what we know about the baryon density we make the reasonable *asumption* that the lepton number densities are also much smaller than s_γ. Then we can take the chemical potentials of the neutrinos equal to zero ($|\mu_\nu|/kT \ll 1$). With what we said before, we can then put the five chemical potentials (75) equal to zero, because the charge number densities are all odd in them. Of course, n_B does not really vanish (otherwise we would not be here), but for the thermal history in the era we are considering they can be ignored.

Exercise. Suppose we are living in a degenerate $\bar{\nu}_e$-see. Use the current mass limit for the electron neutrino mass coming from tritium decay to deduce a limit for the magnitude of the chemical potential μ_{ν_e}.

C. Constancy of entropy

Let $\rho_{\rm eq}, p_{\rm eq}$ denote (in this subsection only) the total energy density and pressure of all particles in thermodynamic equilibrium. Since the chemical potentials of the leptons vanish, these quantities are only functions of the temperature T. According to the second law, the differential of the entropy $S(V,T)$ is given by

$$dS(V,T) = \frac{1}{T}[d(\rho_{\rm eq}(T)V) + p_{\rm eq}(T)dV]. \tag{77}$$

[7] Even if B, L_e, L_μ, L_τ should not be strictly conserved, this is not relevant within a Hubble time H_0^{-1}.

This implies

$$d(dS) = 0 = d\left(\frac{1}{T}\right) \wedge d(\rho_{\rm eq}(T)V) + d\left(\frac{p_{\rm eq}(I)}{T}\right) \wedge dV$$
$$= -\frac{\rho_{\rm eq}}{T^2} dT \wedge dV + \frac{d}{dT}\left(\frac{p_{\rm eq}(T)}{T}\right) dT \wedge dV,$$

i.e., the Maxwell relation

$$\boxed{\frac{dp_{\rm eq}(T)}{dT} = \frac{1}{T}[\rho_{\rm eq}(T) + p_{\rm eq}(T)].} \tag{78}$$

If we use this in (77), we get

$$dS = d\left[\frac{V}{T}(\rho_{\rm eq} + p_{\rm eq})\right],$$

so the entropy density of the particles in equilibrium is

$$\boxed{s = \frac{1}{T}[\rho_{\rm eq}(T) + p_{\rm eq}(T)].} \tag{79}$$

For an adiabatic expansion the entropy in a comoving volume remains constant:

$$S = a^3 s = \text{const.} \tag{80}$$

This constancy is equivalent to the energy equation (21) for the equilibrium part. Indeed , the latter can be written as

$$a^3 \frac{dp_{\rm eq}}{dt} = \frac{d}{dt}[a^3(\rho_{\rm eq} + p_{\rm eq})],$$

and by (79) this is equivalent to $dS/dt = 0$.

In particular, we obtain for massless particles ($p = \rho/3$) from (78) again $\rho \propto T^4$ and from (79) that $S =$ constant implies $T \propto a^{-1}$.

Exercise. Assume that all components are in equilibrium and use the results of this subsection to show that the temperature evolution is for $k = 0$ given by

$$\frac{dT}{dt} = -\sqrt{24\pi G}\,\frac{\sqrt{\rho(T)}}{\frac{d}{dT}\left[\ln \frac{dp}{dT}\right]}.$$

Once the electrons and positrons have annihilated below $T \sim m_e$, the equilibrium components consist of photons, electrons, protons and – after the big bang nucleosynthesis – of some light nuclei (mostly He^4). Since the charged particle number densities are much smaller than the photon number density, the photon temperature T_γ still decreases as a^{-1}. Let us show this formally. For this we consider beside the photons an ideal gas in thermodynamic equilibrium with the black body radiation. The total pressure and energy density are then (we use units with $\hbar = c = k_B = 1$; n is the number density of the non-relativistic gas particles with mass m):

$$p = nT + \frac{\pi^2}{45}T^4, \;\rho = nm + \frac{nT}{\gamma - 1} + \frac{\pi^2}{15}T^4 \tag{81}$$

($\gamma = 5/3$ for a monoatomic gas). The conservation of the gas particles, $na^3 = \text{const.}$, together with the energy equation (22) implies, if $\sigma := s_\gamma/n$,

$$\frac{d\ln T}{d\ln a} = -\left[\frac{\sigma+1}{\sigma+1/3(\gamma-1)}\right].$$

For $\sigma \ll 1$ this gives the well-known relation $T \propto a^{3(\gamma-1)}$ for an adiabatic expansion of an ideal gas.

We are however dealing with the opposite situation $\sigma \gg 1$, and then we obtain, as expected, $a \cdot T = const.$

Let us look more closely at the famous ratio n_B/s_γ. We need

$$s_\gamma = \frac{4}{3T}\rho_\gamma = \frac{4\pi^2}{45}T^3 = 3.60 n_\gamma,\ n_B = \rho_B/m_p = \Omega_B\rho_{\text{crit}}/m_p. \tag{82}$$

From the present value of $T_\gamma \simeq 2.7\ K$ and (50), $\rho_{\text{crit}} = 1.12 \times 10^{-5}\ h_0^2(m_p/\text{cm}^3)$, we obtain as a measure for the baryon asymmetry of the Universe

$$\boxed{\frac{n_B}{s_\gamma} = 0.75 \times 10^{-8}(\Omega_B h_0^2).} \tag{83}$$

It is one of the great challenges to explain this tiny number. So far, this has been achieved at best qualitatively in the framework of grand unified theories (GUTs).

D. Neutrino temperature

During the electron-positron annihilation below $T = m_e$ the a-dependence is complicated, since the electrons can no more be treated as massless. We want to know at this point what the ratio T_γ/T_ν is after the annihilation. This can easily be obtained by using the constancy of comoving entropy for the photon-electron-positron system, which is sufficiently strongly coupled to maintain thermodynamic equilibrium.

We need the entropy for the electrons and positrons at $T \gg m_e$, long before annihilation begins. To compute this note the identity

$$\int_0^\infty \frac{x^n}{e^x-1}dx - \int_0^\infty \frac{x^n}{e^x+1}dx = 2\int_0^\infty \frac{x^n}{e^{2x}-1}dx = \frac{1}{2^n}\int_0^\infty \frac{x^n}{e^x-1}dx,$$

whence

$$\int_0^\infty \frac{x^n}{e^x+1}dx = (1-2^{-n})\int_0^\infty \frac{x^n}{e^x-1}dx. \tag{84}$$

In particular, we obtain for the entropies s_e, s_γ the following relation

$$s_e = \frac{7}{8}s_\gamma\ (T \gg m_e). \tag{85}$$

Equating the entropies for $T_\gamma \gg m_e$ and $T_\gamma \ll m_e$ gives

$$(T_\gamma a)^3\big|_{before}\left[1 + 2\times\frac{7}{8}\right] = (T_\gamma a)^3\big|_{after} \times 1,$$

because the neutrino entropy is conserved. Therefore, we obtain

$$(aT_\gamma)\big|_{after} = \left(\frac{11}{4}\right)^{1/3}(aT_\gamma)\big|_{before}. \tag{86}$$

But $(aT_\nu)|_{after} = (aT_\nu)|_{before} = (aT_\gamma)\big|_{before}$, hence we obtain the important relation

$$\boxed{\left(\frac{T_\gamma}{T_\nu}\right)\bigg|_{after} = \left(\frac{11}{4}\right)^{1/3} = 1.401.} \tag{87}$$

E. Epoch of matter-radiation equality

In the main parts of these lectures the epoch when radiation (photons and neutrinos) have about the same energy density as non-relativistic matter (Dark Matter and baryons) plays a very important role. Let us determine the redshift, $z_{\rm eq}$, when there is equality.

For the three neutrino and antineutrino flavors the energy density is according to (84)

$$\rho_\nu = 3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma. \tag{88}$$

Using

$$\frac{\rho_\gamma}{\rho_{\rm crit}} = 2.47 \times 10^{-5} h_0^{-2} (1+z)^4, \tag{89}$$

we obtain for the total radiation energy density, ρ_r,

$$\frac{\rho_r}{\rho_{\rm crit}} = 4.15 \times 10^{-5} h_0^{-2} (1+z)^4, \tag{90}$$

Equating this to

$$\frac{\rho_M}{\rho_{\rm crit}} = \Omega_M (1+z)^3 \tag{91}$$

we obtain

$$\boxed{1 + z_{\rm eq} = 2.4 \times 10^4 \Omega_M h_0^2.} \tag{92}$$

Only a small fraction of Ω_M is baryonic. There are several methods to determine the fraction Ω_B in baryons. A traditional one comes from the abundances of the light elements. This is treated in most texts on cosmology. (German speaking readers find a detailed discussion in my lecture notes [9], which are available in the internet.) The comparison of the straightforward theory with observation gives a value in the range $\Omega_B h_0^2 = 0.021 \pm 0.002$. Other determinations are all compatible with this value. In Part III we shall obtain Ω_B from the CMB anisotropies. The striking agreement of different methods, sensitive to different physics, strongly supports our standard big bang picture of the Universe.

Part I

Cosmological perturbation theory

Introduction

The astonishing isotropy of the cosmic microwave background radiation provides direct evidence that the early universe can be described in a good first approximation by a Friedmann model[8]. At the time of recombination deviations from homogeneity and isotropy have been very small indeed ($\sim 10^{-5}$). Thus there was a long period during which deviations from Friedmann models can be studied perturbatively, i.e., by linearizing the Einstein and matter equations about solutions of the idealized Friedmann-Lemaître models.

Cosmological perturbation theory is a very important tool that is by now well developed. Among the various reviews I will often refer to [31], abbreviated as KS84. Other works will be cited later, but the

[8] For detailed treatments, see for instance the recent textbooks on cosmology [2–6]. For GR I usually refer to [1].

present notes should be self-contained. Almost always I will provide detailed derivations. Some of the more lengthy calculations are deferred to appendices.

The formalism, developed in this part, will later be applied to two main problems: (1) The generation of primordial fluctuations during an inflationary era. (2) The evolution of these perturbations during the linear regime. A main goal will be to determine the CMB power spectrum as a function of certain cosmological parameters. Among these the fractions of *Dark Matter* and *Dark Energy* are particularly interesting.

1 Basic equations

In this chapter we develop the model independent parts of cosmological perturbation theory. This forms the basis of all that follows.

1.1 Generalities

For the unperturbed Friedmann models the metric is denoted by $g^{(0)}$, and has the form

$$g^{(0)} = -dt^2 + a^2(t)\gamma = a^2(\eta)\left[-d\eta^2 + \gamma\right]; \tag{1.1}$$

γ is the metric of a space with constant curvature K. In addition, we have matter variables for the various components (radiation, neutrinos, baryons, cold dark matter (CDM), etc). We shall linearize all basic equations about the unperturbed solutions.

1.1.1 Decomposition into scalar, vector, and tensor contributions

We may regard the various perturbation amplitudes as time dependent functions on a three-dimensional Riemannian space (Σ, γ) of constant curvature K. Since such a space is highly symmetric, we can perform two types of decompositions.

Consider first the set $\mathcal{X}(\Sigma)$ of smooth vector fields on Σ. This module can be decomposed into an orthogonal sum of 'scalar' and 'vector' contributions

$$\mathcal{X}(\Sigma) = \mathcal{X}^S \bigoplus \mathcal{X}^V, \tag{1.2}$$

where $\mathcal{X}^S$ consists of all gradients and $\mathcal{X}^V$ of all vector fields with vanishing divergence.

More generally, we have for the p-forms $\bigwedge^p(\Sigma)$ on Σ the orthogonal decomposition[9]

$$\bigwedge^p(\Sigma) = d\bigwedge^{p-1}(\Sigma) \bigoplus ker\delta, \tag{1.3}$$

where the last summand denotes the kernel of the co-differential δ (restricted to $\bigwedge^p(\Sigma)$).

Similarly, we can decompose a symmetric tensor $t \in \mathcal{S}(\Sigma)$ (= set of all symmetric tensor fields on Σ) into 'scalar', 'vector', and 'tensor' contributions:

$$t_{ij} = t_{ij}^S + t_{ij}^V + t_{ij}^T, \tag{1.4}$$

where

$$t_{ij}^S = Tr(t)\gamma_{ij} + \left(\nabla_i\nabla_j - \tfrac{1}{3}\gamma_{ij}\triangle\right) f, \tag{1.5}$$

[9] This is a consequence of the Hodge decomposition theorem. The scalar product in $\bigwedge^p(\Sigma)$ is defined as

$$(\alpha, \beta) = \int_\Sigma \alpha \wedge \star\beta;$$

see also Sect. 13.9 of [1].

$$t_{ij}^V = \nabla_i \xi_j + \nabla_j \xi_i, \tag{1.6}$$

$$t_{ij}^T \; : \; Tr(t^T) = 0; \; \nabla \cdot t^T = 0. \tag{1.7}$$

In these equations f is a function on Σ and ξ^i a vector field with vanishing divergence. One can show that these decompositions are respected by the covariant derivatives. For example, if $\xi \in \mathcal{X}(\Sigma), \; \xi = \xi_* + \nabla f, \; \nabla \cdot \xi_* = 0$, then

$$\triangle \xi = \triangle \xi_* + \nabla \left[\triangle f + 2Kf \right] \tag{1.8}$$

(prove this as an exercise). Here, the first term on the right has a vanishing divergence (show this), and the second (the gradient) involves only f. For other cases, see Appendix B of [31]. Is there a conceptual proof based on the isometry group of (Σ, γ)?

1.1.2 Decomposition into spherical harmonics

In a second step we perform a harmonic decomposition. For $K = 0$ this is just Fourier analysis. The spherical harmonics $\{Y\}$ of (Σ, γ) are in this case the functions $Y(\mathbf{x};\mathbf{k}) = \exp(i\mathbf{k} \cdot \mathbf{x})$ (for $\gamma = \delta_{ij} dx^i dx^j$). The *scalar* parts of vector and symmetric tensor fields can be expanded in terms of

$$Y_i : = -k^{-1} \nabla_i Y, \tag{1.9}$$

$$Y_{ij} : = k^{-2} \nabla_i \nabla_j Y + \tfrac{1}{3} \gamma_{ij} Y, \tag{1.10}$$

and $\gamma_{ij} Y$.

There are corresponding complete sets of spherical harmonics for $K \neq 0$. They are eigenfunctions of the Laplace-Beltrami operator on (Σ, γ):

$$(\triangle + k^2) Y = 0. \tag{1.11}$$

Indices referring to the various modes are usually suppressed. By making use of the Riemann tensor of (Σ, γ) one can easily derive the following identities:

$$\begin{aligned}
\nabla_i Y^i &= kY, \\
\triangle Y_i &= -(k^2 - 2K) Y_i, \\
\nabla_j Y_i &= -k(Y_{ij} - \tfrac{1}{3} \gamma_{ij} Y), \\
\nabla^j Y_{ij} &= \tfrac{2}{3} k^{-1} (k^2 - 3K) Y_i, \\
\nabla_j \nabla^m Y_{im} &= \tfrac{2}{3} (3K - k^2) \left(Y_{ij} - \tfrac{1}{3} \gamma_{ij} Y \right), \\
\triangle Y_{ij} &= -(k^2 - 6K) Y_{ij}, \\
\nabla_m Y_{ij} - \nabla_j Y_{im} &= \frac{k}{3} \left(1 - \frac{3K}{k^2} \right) (\gamma_{im} Y_j - \gamma_{ij} Y_m).
\end{aligned} \tag{1.12}$$

Exercise. Verify some of the relations in (1.12).

The main point of the harmonic decomposition is, of course, that different modes in the linearized approximation do not couple. Hence, it suffices to consider a generic mode.

For the time being, we consider only scalar perturbations. Tensor perturbations (gravity modes) will be studied later. For the harmonic analysis of vector and tensor perturbations I refer again to [31].

1.1.3 Gauge transformations, gauge invariant amplitudes

In GR the diffeomorphism group of spacetime is an invariance group. This means that we can replace the metric g and the matter fields by their pull-backs $\phi^\star(g)$, etc., for any diffeomorphism ϕ, without changing the physics. For small-amplitude departures in

$$g = g^{(0)} + \delta g, \; etc, \tag{1.13}$$

we have, therefore, the *gauge freedom*

$$\delta g \to \delta g + L_\xi g^{(0)}, \; etc., \tag{1.14}$$

where ξ is any vector field and L_ξ denotes its Lie derivative. (For further explanations, see [1, Sect. 4.1]). These transformations will induce changes in the various perturbation amplitudes. It is clearly desirable to write all independent perturbation equations in a manifestly *gauge invariant* manner. In this way one can, for instance, avoid misinterpretations of the growth of density fluctuations, especially on superhorizon scales. Moreover, one gets rid of uninteresting gauge modes.

I find it astonishing that it took so long until the gauge invariant formalism was widely used.

1.1.4 Parametrization of the metric perturbations

The most general *scalar* perturbation of the metric can be parametrized as follows

$$\delta g = a^2(\eta)\left[-2A d\eta^2 - 2B_{,i}\, dx^i d\eta + (2D\gamma_{ij} + 2E_{|ij})dx^i dx^j\right]. \tag{1.15}$$

The functions $A(\eta, x^i),\ B,\ D,\ E$ are the scalar perturbation amplitudes; $E_{|ij}$ denotes $\nabla_i\nabla_j E$ on (Σ, γ). Thus the true metric is

$$\boxed{g = a^2(\eta)\left\{-(1+2A)d\eta^2 - 2B_{,i}\, dx^i d\eta + [(1+2D)\gamma_{ij} + 2E_{|ij}]dx^i dx^j\right\}.} \tag{1.16}$$

Let us work out how A, B, D, E change under a gauge transformation (1.14), provided the vector field is of the 'scalar' type[10]:

$$\xi = \xi^0\partial_0 + \xi^i\partial_i,\; \xi^i = \gamma^{ij}\xi_{|j}. \tag{1.17}$$

(The index 0 refers to the conformal time η.) For this we need ('$\equiv d/d\eta$)

$$L_\xi a^2(\eta) = 2aa'\xi^0 = 2a^2\mathcal{H}\xi^0,\; \mathcal{H} := a'/a,$$

$$L_\xi d\eta = dL_\xi\eta = (\xi^0)'d\eta + \xi^0{}_{|i}dx^i,$$

$$L_\xi dx^i = dL_\xi x^i = d\xi^i = \xi^i{}_{,j}\, dx^j + (\xi^i)'d\eta = \xi^i{}_{,j}\, dx^j + \xi'^{|i}d\eta,$$

implying

$$L_\xi\left(a^2(\eta)d\eta^2\right) = 2a^2\left\{(\mathcal{H}\xi^0 + (\xi^0)')d\eta^2 + \xi^0{}_{|i}dx^i d\eta\right\},$$

$$L_\xi\left(\gamma_{ij}dx^i dx^j\right) = 2\xi_{|ij}dx^i dx^j + 2\xi'_{|i}dx^i d\eta.$$

This gives the transformation laws:

$$A \to A + \mathcal{H}\xi^0 + (\xi^0)',\; B \to B + \xi^0 - \xi',\; D \to D + \mathcal{H}\xi^0,\; E \to E + \xi. \tag{1.18}$$

[10] It suffices to consider this type of vector fields, since vector fields from $\mathcal{X}^V$ do not affect the scalar amplitudes; check this.

From this one concludes that the following *Bardeen potentials*

$$\Psi = A - \frac{1}{a}\left[a(B+E')\right]', \tag{1.19}$$

$$\Phi = D - \mathcal{H}(B+E'), \tag{1.20}$$

are gauge invariant.

Note that the transformations of A and D involve *only* ξ^0. This is also the case for the combinations

$$\chi := a(B+E') \to \chi + a\xi^0 \tag{1.21}$$

and

$$\kappa := \frac{3}{a}(\mathcal{H}A - D') - \frac{1}{a^2}\triangle\chi \to \tag{1.22}$$

$$\kappa + \frac{3}{a}\left[\mathcal{H}(\mathcal{H}\xi^0 + (\xi^0)') - (\mathcal{H}\xi^0)'\right] - \frac{1}{a^2}\triangle\xi^0. \tag{1.23}$$

Therefore, it is good to work with A, D, χ, κ. This was emphasized in [32]. Below we will show that χ and κ have a simple geometrical meaning. Moreover, it will turn out that the perturbation of the Einstein tensor can be expressed completely in terms of the amplitudes A, D, χ, κ.

Exercise. The most general vector perturbation of the metric is obviously of the form

$$(\delta g_{\mu\nu}) = a^2(\eta)\begin{pmatrix} 0 & \beta_i \\ \beta_i & H_{i|j} + H_{j|i} \end{pmatrix},$$

with $B_i{}^{|i} = H_i{}^{|i} = 0$. Derive the gauge transformations for β_i and H_i. Show that H_i can be gauged away. Compute $R^0{}_j$ in this gauge. Result:

$$R^0{}_j = \frac{1}{2}\left(\triangle\beta_j + 2K\beta_j\right).$$

1.1.5 Geometrical interpretation

Let us first compute the scalar curvature $R^{(3)}$ of the slices with constant time η with the induced metric

$$g^{(3)} = a^2(\eta)\left[(1+2D)\gamma_{ij} + 2E_{|ij}\right]dx^i dx^j. \tag{1.24}$$

If we drop the factor a^2, then the Ricci tensor does not change, but $R^{(3)}$ has to be multiplied afterwards with a^{-2}.

For the metric $\gamma_{ij} + h_{ij}$ the *Palatini identity* (Eq. (4.20) in [1])

$$\delta R_{ij} = \frac{1}{2}\left[h^k{}_{i|jk} - h^k{}_{k|ij} + h^k{}_{j|ik} - \triangle h_{ij}\right] \tag{1.25}$$

gives

$$\delta R^i{}_i = h^{ij}{}_{|ij} - \triangle h \; (h := h^i_i), \; h_{ij} = 2D\gamma_{ij} + 2E_{|ij}.$$

We also use

$$h = 6D + 2\triangle E, \; E^{|ij}{}_{|ij} = \nabla_j(\triangle\nabla^j E) = \nabla_j(\nabla^j\triangle E - 2K\nabla^j E)$$
$$= \triangle^2 E - 2K\triangle E$$

(we used $(\nabla_i \triangle - \triangle \nabla_i) f = -R^{(0)}_{ij} \nabla^j f$, for a function f). This implies

$$h^{ij}{}_{|ij} = 2\triangle D + 2(\triangle^2 E - 2K\triangle E),$$
$$\delta R^i{}_i = -4D - 4K\triangle E),$$

whence

$$\delta R = \delta R^i{}_i + h^{ij} R^{(0)}_{ij} = -4\triangle D + 12KD.$$

This shows that D determines the scalar curvature perturbation

$$\boxed{\delta R^{(3)} = \frac{1}{a^2}(-4\triangle D + 12KD).} \tag{1.26}$$

Next, we compute the second fundamental form[11] K_{ij} for the time slices. We shall show that

$$\boxed{\kappa = \delta K^i{}_i,} \tag{1.27}$$

and

$$K_{ij} - \tfrac{1}{3} g_{ij} K^l{}_l = -\left(\chi_{|ij} - \tfrac{1}{3}\gamma_{ij}\triangle\chi\right). \tag{1.28}$$

Derivation. In the following derivation we make use of Sect. 2.9 of [1] on the $3+1$ formalism. According to Eq. (2.287) of this reference, the second fundamental form is determined in terms of the lapse α, the shift $\beta = \beta^i \partial_i$, and the induced metric $\bar{g}$ as follows (dropping indices)

$$K = -\frac{1}{2\alpha}(\partial_t - L_\beta)\bar{g}. \tag{1.29}$$

To first order this gives in our case

$$K_{ij} = -\frac{1}{2a(1+A)}\left[a^2(1+2D)\gamma_{ij} + 2a^2 E_{|ij}\right]' - aB_{|ij}. \tag{1.30}$$

(Note that $\beta_i = -a^2 B_{,i}$, $\beta^i = -\gamma^{ij} B_{,j}$.)
In zeroth order this gives

$$K^{(0)}_{ij} = -\frac{1}{a}\mathcal{H} g^{(0)}_{ij}. \tag{1.31}$$

Collecting the first order terms gives the claimed equations (1.27) and (1.28). (Note that the trace-free part must be of first order, because the zeroth order vanishes according to (1.31).)

Conformal gauge. According to (1.18) and (1.21) we can always chose the gauge such that $B = E = 0$. This so-called *conformal Newtonian (or longitudinal) gauge* is often particularly convenient to work with. Note that in this gauge

$$\chi = 0,\ A = \Psi,\ D = \Phi,\ \kappa = \frac{3}{a}(\mathcal{H}\Psi - \Phi').$$

[11] This geometrical concept is introduced in Appendix A of [1].

1.1.6 Scalar perturbations of the energy-momentum tensor

At this point we do not want to specify the matter model. For a convenient parametrization of the scalar perturbations of the energy-momentum tensor $T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}$, we define the four-velocity u^μ as a normalized timelike eigenvector of $T^{\mu\nu}$:

$$T^\mu{}_\nu u^\nu = -\rho u^\mu, \tag{1.32}$$

$$g_{\mu\nu} u^\mu u^\nu = -1. \tag{1.33}$$

The eigenvalue ρ is the *proper energy-mass density*.

For the unperturbed situation we have

$$u^{(0)0} = \frac{1}{a},\ u^{(0)}_0 = -a,\ u^{(0)i} = 0,\ T^{(0)0}{}_0 = -\rho^{(0)},\ T^{(0)i}{}_j = p^{(0)}\delta^i{}_j,\ T^{(0)0}{}_i = 0. \tag{1.34}$$

Setting $\rho = \rho^{(0)} + \delta\rho,\ u^\mu = u^{(0)\mu} + \delta u^\mu$, etc, we obtain from (1.33)

$$\delta u^0 = -\frac{1}{a}A,\ \delta u_0 = -aA. \tag{1.35}$$

The first order terms of (1.32) give, using (1.34),

$$\delta T^\mu{}_0 u^{(0)0} + \delta^\mu{}_0 u^{(0)0}\delta\rho + \left(T^{(0)\mu}{}_\nu + \rho^{(0)}\delta^\mu{}_\nu\right)\delta u^\nu = 0.$$

For $\mu = 0$ and $\mu = i$ this leads to

$$\delta T^0{}_0 = -\delta\rho, \tag{1.36}$$

$$\delta T^i{}_0 = -a(\rho^{(0)} + p^{(0)})\delta u^i. \tag{1.37}$$

From this we can determine the components of $\delta T^0{}_j$:

$$\begin{aligned}
\delta T^0{}_j &= \delta\left[g^{0\mu} g_{j\nu} T^\nu{}_\mu\right] \\
&= \delta g^{0k} g^{(0)}_{ij} T^{(0)i}{}_k + g^{(0)00}\delta g_{0j} T^{(0)0}{}_0 + g^{(0)00} g^{(0)}_{ij}\delta T^i{}_0 \\
&= \left(-\frac{1}{a^2}\gamma^{ki}B_{|i}\right)(a^2\gamma_{ij})p^{(0)}\delta^i{}_k + \left(-\frac{1}{a^2}\right)(-a^2 B_{|j})(-\rho^{(0)}) - \gamma_{ij}\delta T^i{}_0.
\end{aligned}$$

Collecting terms gives

$$\delta T^0{}_j = a(\rho^{(0)} + p^{(0)})\underbrace{\left[\gamma_{ij}\delta u^i - \frac{1}{a}B_{|j}\right]}_{a^{-2}\delta u_j}. \tag{1.38}$$

Scalar perturbations of δu^i can be represented as

$$\delta u^i = \frac{1}{a}\gamma^{ij} v_{|j}. \tag{1.39}$$

Inserting this above gives

$$\delta T^0{}_j = (\rho^{(0)} + p^{(0)})(v - B)_{|j}. \tag{1.40}$$

The scalar perturbations of the spatial components $\delta T^i{}_j$ can be represented as follows

$$\delta T^i{}_j = \delta^i{}_j\,\delta p + p^{(0)}\left(\Pi^{|i}{}_{|j} - \frac{1}{3}\delta^i{}_j\,\triangle\Pi\right). \tag{1.41}$$

Let us collect these formulae (dropping (0) for the unperturbed quantities $\rho^{(0)}$, etc):

$$\begin{aligned}
&\delta u^0 = -\frac{1}{a}A,\ \delta u_0 = -aA,\ \delta u^i = \frac{1}{a}\gamma^{ij}v_{|j}\ \Rightarrow \delta u_i = a(v-B)_{|i};\\
&\delta T^0{}_0 = -\delta\rho,\\
&\delta T^0{}_i = (\rho+p)(v-B)_{|i},\ \delta T^i{}_0 = -(\rho+p)\gamma^{ij}v_{|j},\\
&\delta T^i{}_j = \delta p\,\delta^i{}_j + p\left(\Pi^{|i}{}_{|j} - \frac{1}{3}\delta^i{}_j\,\triangle\Pi\right).
\end{aligned} \tag{1.42}$$

Sometimes we shall also use the quantity

$$\mathcal{Q} := a(\rho+p)(v-B),$$

in terms of which the energy flux density can be written as

$$\delta T^0{}_i = \frac{1}{a}\mathcal{Q}_{,i}\ ,\ (\Rightarrow T^t{}_i = \mathcal{Q}_{,i}). \tag{1.43}$$

For fluids one often decomposes δp as

$$p\pi_L := \delta p = c_s^2\delta\rho + p\Gamma, \tag{1.44}$$

where c_s is the sound velocity

$$c_s^2 = \dot{p}/\dot{\rho}. \tag{1.45}$$

Γ measures the deviation between $\delta p/\delta\rho$ and $\dot{p}/\dot{\rho}$.

As for the metric we have four perturbation amplitudes:

$$\boxed{\delta := \delta\rho/\rho\ ,\ v\ ,\ \Gamma\ ,\ \Pi\ .} \tag{1.46}$$

Let us see how they change under gauge transformations:

$$\delta T^\mu{}_\nu \to \delta T^\mu{}_\nu + (L_\xi T^{(0)})^\mu{}_\nu,\ (L_\xi T^{(0)})^\mu{}_\nu = \xi^\lambda T^{(0)\mu}{}_{\nu,\lambda} - T^{(0)\lambda}{}_\nu\xi^\mu{}_{,\lambda} + T^{(0)\mu}{}_\lambda\xi^\lambda{}_{,\nu}. \tag{1.47}$$

Now,

$$(L_\xi T^{(0)})^0{}_0 = \xi^0 T^{(0)0}{}_{0,0} = \xi^0(-\rho)',$$

hence

$$\delta\rho \to \delta\rho + \rho'\xi^0\ ;\ \delta \to \delta + \frac{\rho'}{\rho}\xi^0 = \delta - 3(1+w)\mathcal{H}\xi^0 \tag{1.48}$$

($w := p/\rho$). Similarly ($\xi^i = \gamma^{ij}\xi_{|j}$):

$$(L_\xi T^{(0)})^0{}_i = 0 - T^{(0)j}{}_i\xi^0{}_{|j} + T^{(0)0}{}_0\xi^0{}_{,i} = -\rho\xi^0{}_{|i} - p\xi^0{}_{|i};$$

so

$$v - B \to (v - B) - \xi^0. \tag{1.49}$$

Finally,

$$(L_\xi T^{(0)})^i{}_j = p'\delta^i{}_j\xi^0,$$

hence

$$\delta p \to \delta p + p'\xi^0, \tag{1.50}$$

$$\Pi \to \Pi. \tag{1.51}$$

From (1.44), (1.48) and (1.50) we also obtain

$$\Gamma \to \Gamma. \tag{1.52}$$

We see that $\Gamma,\ \Pi$ are gauge invariant. Note that the transformation of δ and $v - B$ involve only ξ^0, while v transforms as

$$v \to v - \xi'.$$

For $\mathcal{Q}$ we get

$$\mathcal{Q} \to \mathcal{Q} - a(\rho + p)\xi^0. \tag{1.53}$$

We can introduce various gauge invariant quantities. It is useful to adopt the following notation: For example, we use the symbol $\delta_\mathcal{Q}$ for that gauge invariant quantity which is equal to δ in the gauge where $\mathcal{Q} = 0$, thus

$$\delta_\mathcal{Q} = \delta - \frac{3}{a\rho}\mathcal{H}\mathcal{Q} = \delta - 3(1 + w)\mathcal{H}(v - B). \tag{1.54}$$

Similarly,

$$\delta_\chi = \delta + 3\frac{(1 + w)\mathcal{H}}{a}\chi = \delta + 3\mathcal{H}(1 + w)(B + E'); \tag{1.55}$$

$$V := (v - B)_\chi = v - B + a^{-1}\chi = v + E' = \frac{1}{a}\left(\chi + \frac{1}{\rho + p}\mathcal{Q}\right); \tag{1.56}$$

$$\mathcal{Q}_\chi = \mathcal{Q} + (\rho + p)\chi = a(\rho + p)V. \tag{1.57}$$

Another important gauge invariant amplitude, often called the *curvature perturbation* (see (1.26)), is

$$\mathcal{R} := D_\mathcal{Q} = D + \mathcal{H}(v - B) = D_\chi + \mathcal{H}(v - B)_\chi = D_\chi + \mathcal{H}V. \tag{1.58}$$

1.2 Explicit form of the energy-momentum conservation

After these preparations we work out the consequences $\nabla \cdot T = 0$ of Einstein's field equations for the metric (1.16) and $T^\mu{}_\nu$ as given by (1.34) and (1.42). The details of the calculations are presented in Appendix A of this chapter.

The energy equation reads (see (1.238)):

$$(\rho\delta)' + 3\mathcal{H}\rho\delta + 3\mathcal{H}p\pi_L + (\rho + p)\left[\triangle(v + E') + 3D'\right] = 0 \tag{1.59}$$

or, with $(\rho\delta)'/\rho = \delta' - 3\mathcal{H}(1+w)\delta$ and (1.56),

$$\delta' + 3\mathcal{H}(c_s^2 - w)\delta + 3\mathcal{H}w\Gamma = -(1+w)(\triangle V + 3D'). \tag{1.60}$$

This gives, putting an index χ, the gauge invariant equation

$$\delta'_\chi + 3\mathcal{H}(c_s^2 - w)\delta_\chi + 3\mathcal{H}w\Gamma = -(1+w)(\triangle V + 3D'_\chi). \tag{1.61}$$

Conversely, Eq. (1.60) follows from (1.61): the χ-terms cancel, as is easily verified by using the zeroth order equation

$$w' = -3(c_s^2 - w)(1+w)\mathcal{H}, \tag{1.62}$$

that is easily derived from the Friedman equations in Sect. 0.1.3. From the definitions it follows readily that the last factor in (1.60) is equal to $-(a\kappa - 3\mathcal{H}A - \triangle(v-B))$.

The momentum equation becomes (see (1.244)):

$$[(\rho+p)(v-B)]' + 4\mathcal{H}(\rho+p)(v-B) + (\rho+p)A + p\pi_L + \tfrac{2}{3}(\triangle + 3K)p\Pi = 0. \tag{1.63}$$

Using (1.44) in the form

$$p\pi_L = \rho(c_s^2\delta + w\Gamma), \tag{1.64}$$

and putting the index χ at the perturbation amplitudes gives the gauge invariant equation

$$[(\rho+p)V]' + 4\mathcal{H}(\rho+p)V + (\rho+p)A_\chi + \rho c_s^2\delta_\chi + \rho w\Gamma + \tfrac{2}{3}(\triangle + 3K)p\Pi = 0 \tag{1.65}$$

or[12]

$$V' + (1-3c_s^2)\mathcal{H}V + A_\chi + \frac{c_s^2}{1+w}\delta_\chi + \frac{w}{1+w}\Gamma + \frac{2}{3}(\triangle + 3K)\frac{w}{1+w}\Pi = 0. \tag{1.66}$$

For later use we write (1.63) also as

$$(v-B)' + (1-3c_s^2)\mathcal{H}(v-B) + A + \frac{c_s^2}{1+w}\delta + \frac{w}{1+w}\Gamma + \frac{2}{3}(\triangle + 3K)\frac{w}{1+w}\Pi = 0 \tag{1.67}$$

(from which (1.66) follows immediately).

1.3 Einstein equations

A direct computation of the first order changes $\delta G^\mu{}_\nu$ of the Einstein tensor for (1.15) is complicated. It is much simpler to proceed as follows: Compute first $\delta G^\mu{}_\nu$ in the *longitudinal gauge* $B = E = 0$. (That these gauge conditions can be imposed follows from (1.18).) Then we write the perturbed Einstein equations in a gauge invariant form. It is then easy to rewrite these equations without imposing any gauge conditions, thus obtaining the equations one would get for the general form (1.15).

$\delta G^\mu{}_\nu$ is computed for the longitudinal gauge in Appendix B to this chapter. Let us first consider the component $\mu = \nu = 0$ (see Eq. (1.256)):

$$\begin{aligned}\delta G^0{}_0 &= \frac{2}{a^2}\left[3\mathcal{H}(\mathcal{H}A - D') + (\triangle + 3K)D\right] \\ &= 2\left[3H(HA - \dot{D}) + \frac{1}{a^2}(\triangle + 3K)D\right].\end{aligned} \tag{1.68}$$

[12] Note that $h := \rho + p$ satisfies $h' = -3\mathcal{H}(1+c_s^2)h$.

Since $\delta T^0{}_0 = -\delta\rho$ (see (1.42)), we obtain the perturbed Einstein equation in the longitudinal gauge

$$3H(HA - \dot{D}) + \frac{1}{a^2}(\triangle + 3K)D = -4\pi G\rho\delta. \tag{1.69}$$

Since in the longitudinal gauge $\chi = 0$ and

$$\kappa = 3(HA - \dot{D}), \tag{1.70}$$

we can write (1.69) as follows

$$\frac{1}{a^2}(\triangle + 3K)D + H\kappa = -4\pi G\rho\delta. \tag{1.71}$$

Obviously, the gauge invariant form of this equation is

$$\frac{1}{a^2}(\triangle + 3K)D_\chi + H\kappa_\chi = -4\pi G\rho\delta_\chi, \tag{1.72}$$

because it reduces to (1.71) for $\chi = 0$. Recall in this connection the remark in Sect.1.1.4 that the gauge transformations of the amplitudes A, D, χ, κ involve only ξ^0. Therefore, $A_\chi,\ D_\chi,\ \kappa_\chi$ are *uniquely* defined; the same is true for δ_χ (see (1.55)).

From (1.72) we can now obtain the generalization of (1.71) in *any gauge*. First note that as a consequence of

$$A_\chi = A - \dot{\chi},\ D_\chi = D - H\chi \tag{1.73}$$

(verify this), we have, using also (1.22),

$$\begin{aligned}\kappa_\chi &= 3(HA_\chi - \dot{D}_\chi) = 3(HA - \dot{D}) + 3\dot{H}\chi \\ &= \kappa + \left(3\dot{H} + \frac{1}{a^2}\triangle\right)\chi.\end{aligned} \tag{1.74}$$

From this, (1.73) and (1.55) one readily sees that (1.72) is equivalent to

$$\boxed{\frac{1}{a^2}(\triangle + 3K)D + H\kappa = -4\pi G\rho\delta \text{ (any gauge)},} \tag{1.75}$$

in *any gauge*.

For the other components we proceed similarly. In the longitudinal gauge we have (see Eqs. (1.257) and (1.70)):

$$\delta G^0{}_j = -\frac{2}{a^2}(\mathcal{H}A - D')_{,j} = -\frac{2}{a}(H - \dot{D})_{,j} = -\frac{2}{3a}\kappa_{,j}, \tag{1.76}$$

$$\delta T^0{}_j = (\rho + p)(v - B)_{,j} = \frac{1}{a}\mathcal{Q}_{,j}. \tag{1.77}$$

This gives, up to an (irrelevant) spatially homogeneous term,

$$\kappa = -12\pi G\mathcal{Q} \text{ (long. gauge)}. \tag{1.78}$$

The gauge invariant form of this is

$$\kappa_\chi = -12\pi G\mathcal{Q}_\chi. \tag{1.79}$$

Inserting here (1.74), (1.57), and using the unperturbed equation

$$\dot{H} = \frac{K}{a^2} - 4\pi G(\rho + p) \tag{1.80}$$

(derive this), one obtains in any gauge

$$\boxed{\kappa + \frac{1}{a^2}(\triangle + 3K)\chi = -12\pi G\mathcal{Q} \text{ (any gauge)}.} \tag{1.81}$$

Next, we use (1.258):

$$\frac{a^2}{2}\delta G^i{}_j = \delta^i{}_j\Big[(2\mathcal{H}' + \mathcal{H}^2)A + \mathcal{H}A' - D'' \\ -2\mathcal{H}D' + KD + \frac{1}{2}\triangle(A+D)\Big] - \frac{1}{2}(A+D)^{|i}{}_{|j}. \tag{1.82}$$

This implies

$$\frac{a^2}{2}\left(\delta G^i{}_j - \frac{1}{3}\delta^i{}_j\,\delta G^k{}_k\right) = -\frac{1}{2}\left[(A+D)^{|i}{}_{|j} - \frac{1}{3}\delta^i{}_j(A+D)^{|k}{}_{|k}\right]. \tag{1.83}$$

Since

$$\delta T^i{}_j - \tfrac{1}{3}\delta^i{}_j\delta T^k{}_k = p\left[\Pi^{|i}{}_{|j} - \tfrac{1}{3}\delta^i{}_j\triangle\Pi\right]$$

we get following field equation for $S := A + D$

$$S^{|i}{}_{|j} - \tfrac{1}{3}\delta^i{}_j\triangle S = -8\pi G a^2 p\left(\Pi^{|i}{}_{|j} - \tfrac{1}{3}\delta^i{}_j\triangle\Pi\right).$$

Modulo an irrelevant homogeneous term (use the harmonic decomposition) this gives in the longitudinal gauge

$$A + D = -8\pi G a^2 p\Pi \tag{1.84}$$

The gauge invariant form is

$$A_\chi + D_\chi = -8\pi G a^2 p\Pi, \tag{1.85}$$

from which we obtain with (1.73) in any gauge

$$\boxed{\dot{\chi} + H\chi - A - D = 8\pi G a^2 p\Pi \text{ (any gauge)}.} \tag{1.86}$$

Finally, we consider the combination

$$\frac{1}{2}(\delta G^i{}_i - \delta G^0{}_0) = 3\left\{2(\dot{H} + H^2)A + H\dot{A} - \ddot{D} - 2H\dot{D}\right\} + \frac{1}{a^2}\triangle A.$$

Since

$$\frac{1}{2}(\delta T^i{}_i - \delta T^0{}_0) = \frac{1}{2}\rho\Big[\underbrace{(1 + 3c_s^2)\delta + 3w\Gamma}_{\delta + 3w\pi_L}\Big]$$

we obtain in the longitudinal gauge the field equation

$$6\dot{H}A + 6H^2A + 3H\dot{A} - 3\ddot{D} - 6H\dot{D} = -\frac{1}{a^2}\triangle A + 4\pi G(1 + 3s_s^2)\rho\delta + 12\pi G p\Gamma. \tag{1.87}$$

The gauge invariant form is obviously

$$6\dot{H}A_\chi + 6H^2A_\chi + 3H\dot{A}_\chi - 3\ddot{D}_\chi - 6H\dot{D}_\chi = -\frac{1}{a^2}\triangle A_\chi + 4\pi G(1+3s_s^2)\rho\delta_\chi + 12\pi Gp\Gamma. \quad (1.88)$$

or

$$3(HA_\chi - \dot{D}_\chi)^{\cdot} + 6H(HA_\chi - \dot{D}_\chi) = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A_\chi + 4\pi G(1+3c_s^2)\rho\delta_\chi + 12\pi Gp\Gamma.$$

With (1.74) we can write this as

$$\dot{\kappa}_\chi + 2H\kappa_\chi = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A_\chi + 4\pi G(1+3c_s^2)\rho\delta_\chi + 12\pi Gp\Gamma. \quad (1.89)$$

In an arbitrary gauge we obtain (the χ-terms cancel)

$$\boxed{\dot{\kappa} + 2H\kappa = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A \underbrace{+4\pi G(1+3c_s^2)\rho\delta + 12\pi Gp\Gamma}_{4\pi G\rho(\delta + 3w\pi_L)}.} \quad (1.90)$$

Intermediate summary

This exhausts the field equations. For reference we summarize the results obtained so far. First, we collect the equations that are valid in any gauge (indicating also their origin). As perturbation amplitudes we use A, D, χ, κ (metric functions) and $\delta, \mathcal{Q}, \Pi, \Gamma$ (matter functions), because these are either gauge invariant or their gauge transformations involve only the component ξ^0 of the vector field ξ^μ.

- definition of κ:

$$\kappa = 3(HA - \dot{D}) - \frac{1}{a^2}\triangle\chi; \quad (1.91)$$

- $\delta G^0{}_0$:

$$\frac{1}{a^2}(\triangle + 3K)D + H\kappa = -4\pi G\rho\delta; \quad (1.92)$$

- $\delta G^0{}_j$:

$$\kappa + \frac{1}{a^2}(\triangle + 3K)\chi = -12\pi G\mathcal{Q}; \quad (1.93)$$

- $\delta G^i{}_j - \frac{1}{3}\delta^i{}_j\,\delta G^k{}_k$:

$$\dot{\chi} + H\chi - A - D = 8\pi Ga^2 p\Pi; \quad (1.94)$$

- $\delta G^i{}_i - \delta G^0{}_0$:

$$\dot{\kappa} + 2H\kappa = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A \underbrace{+4\pi G(1+3c_s^2)\rho\delta + 12\pi Gp\Gamma}_{4\pi G\rho(\delta + 3w\pi_L)}; \quad (1.95)$$

- $T^{0\nu}{}_{;\nu}$ (Eq. (1.60)):

$$\dot{\delta} + 3H(c_s^2 - w)\delta + 3Hw\Gamma = (1+w)(\kappa - 3HA) - \frac{1}{\rho a^2}\triangle\mathcal{Q} \tag{1.96}$$

or

$$(\rho\delta)^{\cdot} + 3H\rho(\delta + \underbrace{w\pi_L}_{c_s^2\delta + w\Gamma}) = (\rho+p)(\kappa - 3HA) - \frac{1}{a^2}\triangle\mathcal{Q}; \tag{1.97}$$

- $T^{i\nu}{}_{;\nu} = 0$ (Eq. (1.63)):

$$\dot{\mathcal{Q}} + 3H\mathcal{Q} = -(\rho+p)A - p\pi_L - \frac{2}{3}(\triangle + 3K)p\Pi. \tag{1.98}$$

These equations are, of course, not all independent. Putting an index χ or $\mathcal{Q}$, etc at the perturbation amplitudes in any of them gives a gauge invariant equation. We write these down for $A_\chi, D_\chi, \cdots$ (instead of $\mathcal{Q}_\chi$ we use V; see also (1.61) and (1.66)):

$$\kappa_\chi = 3(HA_\chi - \dot{D}_\chi); \tag{1.99}$$

$$\frac{1}{a^2}(\triangle + 3K)D_\chi + H\kappa_\chi = -4\pi G\rho\delta_\chi; \tag{1.100}$$

$$\kappa_\chi = -12\pi G\mathcal{Q}_\chi; \tag{1.101}$$

$$A_\chi + D_\chi = -8\pi G a^2 p\Pi; \tag{1.102}$$

$$\dot{\kappa}_\chi + 2H\kappa_\chi = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A_\chi + \underbrace{4\pi G(1+3c_s^2)\rho\delta_\chi + 12\pi G p\Gamma}_{4\pi G\rho(\delta_\chi + 3w\pi_L)}; \tag{1.103}$$

$$\dot{\delta}_\chi + 3H(c_s^2 - w)\delta_\chi + 3Hw\Gamma = -3(1+w)\dot{D}_\chi - \frac{1+w}{a}\triangle V; \tag{1.104}$$

$$\dot{V} + (1-3c_s^2)HV = -\frac{1}{a}A_\chi - \frac{1}{a}\left[\frac{c_s^2}{1+w}\delta_\chi + \frac{w}{1+w}\Gamma + \frac{2}{3}(\triangle + 3K)\frac{w}{1+w}\Pi\right]. \tag{1.105}$$

Harmonic decomposition

We write these equations once more for the amplitudes of harmonic decompositions, adopting the following conventions. For those amplitudes which enter in $g_{\mu\nu}$ and $T_{\mu\nu}$ without spatial derivatives (i.e., A, D, δ, Γ) we set

$$A = A_{(k)}Y_{(k)} \text{ ,etc ;} \tag{1.106}$$

those which appear only through their gradients (B, v) are decomposed as

$$B = -\frac{1}{k}B_{(k)}Y_{(k)} \text{ , etc ,} \tag{1.107}$$

and, finally,, we set for E and Π, entering only through second derivatives,

$$E = \frac{1}{k^2}E_{(k)}Y_{(k)} \; (\Rightarrow \triangle E = -E_{(k)}Y_{(k)}). \tag{1.108}$$

The reason for this is that we then have, using the definitions (1.9) and (1.10),

$$B_{|i} = B_{(k)}Y_{(k)i}, \; \Pi_{|ij} - \tfrac{1}{3}\gamma_{ij}\triangle\Pi = \Pi_{(k)}Y_{(k)ij}. \tag{1.109}$$

The spatial part of the metric in (1.16) then becomes

$$g_{ij}dx^i dx^j = a^2(\eta)\left[\gamma_{ij} + 2\left(D - \tfrac{1}{3}E\right)\gamma_{ij}Y + 2EY_{ij}\right]dx^i dx^j. \tag{1.110}$$

The basic equations (1.91)-(1.98) imply for $A_{(k)}, B_{(k)}$, etc[13], dropping the index (k),

$$\kappa = 3(HA - \dot{D}) + \frac{k^2}{a^2}\chi, \tag{1.111}$$

$$-\frac{k^2 - 3K}{a^2}D + H\kappa = -4\pi G\rho\delta, \tag{1.112}$$

$$\kappa - \frac{k^2 - 3K}{a^2}\chi = -12\pi G\mathcal{Q}, \tag{1.113}$$

$$\dot{\chi} + H\chi - A - D = 8\pi Ga^2 p\Pi/k^2, \tag{1.114}$$

$$\dot{\kappa} + 2H\kappa = \left(\frac{k^2}{a^2} - 3\dot{H}\right)A \underbrace{+4\pi G(1+3c_s^2)\rho\delta + 12\pi Gp\Gamma}_{4\pi G\rho(\delta + 3w\pi_L)}, \tag{1.115}$$

$$(\rho\delta)^{\cdot} + 3H\rho(\delta + \underbrace{w\pi_L}_{c_s^2\delta + w\Gamma}) = (\rho + p)(\kappa - 3HA) + \frac{k^2}{a^2}\mathcal{Q}, \tag{1.116}$$

$$\dot{\mathcal{Q}} + 3H\mathcal{Q} = -(\rho + p)A - p\pi_L + \frac{2}{3}\frac{k^2 - 3K}{k^2}p\Pi. \tag{1.117}$$

For later use we also collect the gauge invariant Eqs. (1.99)–(1.105) for the Fourier amplitudes:

$$\kappa_\chi = 3(HA_\chi - \dot{D}_\chi), \tag{1.118}$$

$$-\frac{k^2 - 3K}{a^2}D_\chi + H\kappa_\chi = -4\pi G\rho\delta_\chi, \tag{1.119}$$

$$\kappa_\chi = -12\pi G\mathcal{Q}_\chi \, \left(\mathcal{Q}_\chi = -\frac{a}{k}(\rho + p)V\right), \tag{1.120}$$

$$k^2(A_\chi + D_\chi) = -8\pi Ga^2 p\Pi, \tag{1.121}$$

$$\dot{\kappa}_\chi + 2H\kappa_\chi = \left(\frac{k^2}{a^2} - 3\dot{H}\right)A_\chi \underbrace{+4\pi G(1+3c_s^2)\rho\delta_\chi + 12\pi Gp\Gamma}_{4\pi G\rho(\delta_\chi + 3w\pi_L)}, \tag{1.122}$$

$$\dot{\delta}_\chi + 3H(c_s^2 - w)\delta_\chi + 3Hw\Gamma = -3(1+w)\dot{D}_\chi - (1+w)\frac{k}{a}V, \tag{1.123}$$

$$\dot{V} + (1 - 3c_s^2)HV = \frac{k}{a}A_\chi + \frac{c_s^2}{1+w}\frac{k}{a}\delta_\chi + \frac{w}{1+w}\frac{k}{a}\Gamma - \frac{2}{3}\frac{w}{1+w}\frac{k^2 - 3K}{k^2}\frac{k}{a}\Pi. \tag{1.124}$$

[13] We replace χ by $\chi_{(k)}Y_{(k)}$, where according to (1.21) $\chi_{(k)} = -(a/k)(B - k^{-1}E')$; Eq. (1.111) is then just the translation of (1.22) to the Fourier amplitudes, with $\kappa \to \kappa_{(k)}Y_{(k)}$. Similarly, $\mathcal{Q} \to \mathcal{Q}_{(k)}Y_{(k)}$, $\mathcal{Q}_{(k)} = -(1/k)a(\rho + p)(v - B)_{(k)}$.

Alternative basic systems of equations

From the basic equations (1.91)–(1.105) we now derive another set which is sometimes useful, as we shall see. We want to work with $\delta_{\mathcal{Q}}, V$ and D_χ.

The energy equation (1.96) with index $\mathcal{Q}$ gives

$$\dot{\delta}_{\mathcal{Q}} + 3H(c_s^2 - w)\delta_{\mathcal{Q}} + 3Hw\Gamma = (1+w)(\kappa_{\mathcal{Q}} - 3HA_{\mathcal{Q}}). \tag{1.125}$$

Similarly, the momentum equation (1.98) implies

$$A_{\mathcal{Q}} = -\frac{1}{1+w}\left[c_s^2\delta_{\mathcal{Q}} + w\Gamma + \frac{2}{3}(\triangle + 3K)w\Pi\right]. \tag{1.126}$$

From (1.93) we obtain

$$\kappa_{\mathcal{Q}} + \frac{1}{a^2}(\triangle + 3K)\chi_{\mathcal{Q}} = 0. \tag{1.127}$$

But from (1.56) we see that

$$\chi_{\mathcal{Q}} = aV, \tag{1.128}$$

hence

$$\kappa_{\mathcal{Q}} = -\frac{1}{a}(\triangle + 3K)V. \tag{1.129}$$

Now we insert (1.126) and (1.129) in (1.125) and obtain

$$\boxed{\dot{\delta}_{\mathcal{Q}} - 3Hw\delta_{\mathcal{Q}} = -(1+w)\frac{1}{a}(\triangle + 3K)V + 2H(\triangle + 3K)w\Pi.} \tag{1.130}$$

Next, we use (1.105) and the relation

$$\delta_\chi = \delta_{\mathcal{Q}} + 3(1+w)HV, \tag{1.131}$$

which follows from (1.54), to obtain

$$\dot{V} + HV = -\frac{1}{a}A_\chi - \frac{1}{a(1+w)}\left[c_s^2\delta_{\mathcal{Q}} + w\Gamma + \frac{2}{3}(\triangle + 3K)w\Pi\right]. \tag{1.132}$$

Here we make use of (1.102), with the result

$$\boxed{\dot{V} + HV = \tfrac{1}{a}D_\chi - \tfrac{1}{a(1+w)}\left[c_s^2\delta_{\mathcal{Q}} + w\Gamma - 8\pi Ga^2(1+w)p\Pi + \tfrac{2}{3}(\triangle + 3K)w\Pi\right]} \tag{1.133}$$

From (1.99), (1.101), (1.102) and (1.57) we find

$$\boxed{\dot{D}_\chi + HD_\chi = 4\pi Ga(\rho + p)V - 8\pi Ga^2Hp\Pi.} \tag{1.134}$$

Finally, we replace in (1.100) δ_χ by $\delta_{\mathcal{Q}}$ (making use of (1.131)) and κ_χ by V according to (1.101), giving the Poisson-like equation

$$\boxed{\frac{1}{a^2}(\triangle + 3K)D_\chi = -4\pi G\rho\delta_{\mathcal{Q}}.} \tag{1.135}$$

The system we were looking for consists of (1.130), (1.133), (1.134) and (1.135).

From these equations we now derive an interesting expression for $\dot{\mathcal{R}}$. Recall (1.58):

$$\mathcal{R} = D_{\mathcal{Q}} = D_\chi + aHV = D_\chi + \dot{a}V. \tag{1.136}$$

Thus

$$\dot{\mathcal{R}} = \dot{D}_\chi + \ddot{a}V + \dot{a}\dot{V}.$$

On the right of this equation we use for the first term (1.134), for the second the following consequence of the Friedmann equations (17) and (23)

$$\ddot{a} = -\frac{1}{2}(1+3w)a\left(H^2 + \frac{K}{a^2}\right), \tag{1.137}$$

and for the last term we use (1.133). The result becomes relatively simple for $K = 0$ (the V-terms cancel):

$$\dot{\mathcal{R}} = -\frac{H}{1+w}\left[c_s^2\delta_{\mathcal{Q}} + w\Gamma + \frac{2}{3}w\triangle\Pi\right].$$

Using also (1.135) and the Friedmann equation (17) (for $K = 0$) leads to

$$\dot{\mathcal{R}} = \frac{H}{1+w}\left[\frac{2}{3}c_s^2\frac{1}{(Ha)^2}\triangle D_\chi - w\Gamma - \frac{2}{3}w\triangle\Pi\right]. \tag{1.138}$$

This is an important equation that will show, for instance, that $\mathcal{R}$ remains constant on superhorizon scales, provided Γ and Π can be neglected.

As another important application, we can derive through elimination a second order equation for $\delta_{\mathcal{Q}}$. For this we perform again a harmonic decomposition and rewrite the basic equations (1.130), (1.133), (1.134) and (1.135) for the Fourier amplitudes:

$$\dot{\delta}_{\mathcal{Q}} - 3Hw\delta_{\mathcal{Q}} = -(1+w)\frac{k}{a}\frac{k^2-3K}{k^2}V - 2H\frac{k^2-3K}{k^2}w\Pi, \tag{1.139}$$

$$\dot{V} + HV = -\frac{k}{a}D_\chi + \frac{1}{1+w}\frac{k}{a}\left[c_s^2\delta_{\mathcal{Q}} + w\Gamma - 8\pi G(1+w)\frac{a^2}{k^2}p\Pi - \frac{2}{3}\frac{k^2-3K}{k^2}w\Pi\right] \tag{1.140}$$

$$\frac{k^2-3K}{a^2}D_\chi = 4\pi G\rho\delta_{\mathcal{Q}}, \tag{1.141}$$

$$\dot{D}_\chi + HD_\chi = -4\pi G(\rho+p)\frac{a}{k}V - 8\pi GH\frac{a^2}{k^2}p\Pi. \tag{1.142}$$

Through elimination one can derive the following important second order equation for $\delta_{\mathcal{Q}}$ (including the Λ term)

$$\ddot{\delta}_{\mathcal{Q}} + (2 + 3c_s^2 - 6w)H\dot{\delta}_{\mathcal{Q}} + \left[c_s^2\frac{k^2}{a^2} - 4\pi G\rho(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} + (3c_s^2 - 5w)\Lambda\right]\delta_{\mathcal{Q}} = \mathcal{S}, \tag{1.143}$$

where

$$\mathcal{S} = -\frac{k^2-3K}{a^2}w\Gamma - 2\left(1 - \frac{3K}{k^2}\right)Hw\dot{\Pi} - \left(1 - \frac{3K}{k^2}\right)\left[-\frac{1}{3}\frac{k^2}{a^2} + 2\dot{H} + (5 - 3c_s^2/w)H^2\right]2w\Pi. \tag{1.144}$$

This is obtained by differentiating (1.139), and eliminating V and $\dot{V}$ with the help of (1.139) and (1.140). In addition one has to use several zeroth order equations. We leave the details to the reader. Note that $\mathcal{S} = 0$ for $\Gamma = \Pi = 0$.

1.4 Extension to multi-component systems

The *phenomenological* description of multi-component systems in this section follows closely the treatment in [31].

Let $T^{\mu}_{(\alpha)\nu}$ denote the energy-momentum tensor of species (α). The total $T^{\mu}{}_{\nu}$ is assumed to be just the sum

$$T^{\mu}{}_{\nu} = \sum_{(\alpha)} T^{\mu}_{(\alpha)\nu}, \tag{1.145}$$

and is, of course, 'conserved'. For the unperturbed background we have, as in (1.34),

$$T^{(0)}_{(\alpha)\mu}{}^{\nu} = (\rho^{(0)}_{\alpha} + p^{(0)}_{\alpha}) u^{(0)}_{\mu} u^{(0)\nu} + p^{(0)}_{\alpha} \delta_{\mu}{}^{\nu}, \tag{1.146}$$

with

$$\left(u^{(0)\mu}\right) = \left(\frac{1}{a}, \mathbf{0}\right). \tag{1.147}$$

The divergence of $T^{\mu}_{(\alpha)\nu}$ does, in general, not vanish. We set

$$T^{\nu}_{(\alpha)\mu;\nu} = Q_{(\alpha)\mu}; \ \sum_{\alpha} Q_{(\alpha)\mu} = 0. \tag{1.148}$$

The unperturbed $Q_{(\alpha)\mu}$ must be of the form

$$Q^{(0)}_{(\alpha)\mu} = \left(-aQ^{(0)}_{\alpha}, \mathbf{0}\right), \tag{1.149}$$

and we obtain from (1.148) for the background

$$\dot{\rho}^{(0)}_{\alpha} = -3H(\rho^{(0)}_{\alpha} + p^{(0)}_{\alpha}) + Q^{(0)}_{\alpha} = -3H(1 - q^{(0)}_{\alpha}) h_{\alpha}, \tag{1.150}$$

where

$$h_{\alpha} = \rho^{(0)}_{\alpha} + p^{(0)}_{\alpha}, \ q^{(0)}_{\alpha} := Q^{(0)}_{\alpha}/(3Hh_{\alpha}). \tag{1.151}$$

Clearly,

$$\rho^{(0)} = \sum_{\alpha} \rho^{(0)}_{\alpha}, \ p^{(0)} = \sum_{\alpha} p^{(0)}_{\alpha}, \ h := \rho^{(0)} + p^{(0)} = \sum_{\alpha} h_{\alpha}, \tag{1.152}$$

and (1.148) implies

$$\sum_{\alpha} Q^{(0)}_{\alpha} = 0 \ \Leftrightarrow \ \sum_{\alpha} h_{\alpha} q^{(0)}_{\alpha} = 0. \tag{1.153}$$

We again consider only *scalar perturbations*, and proceed with each component as in Sect.1.1.6. In particular, Eqs. (1.32), (1.33), (1.42) and (1.44) become

$$T^{\mu}_{(\alpha)\nu} u^{\nu}_{(\alpha)} = -\rho_{(\alpha)} u^{\mu}_{(\alpha)}, \tag{1.154}$$

$$g_{\mu\nu} u^{\mu}_{(\alpha)} u^{\nu}_{(\alpha)} = -1, \tag{1.155}$$

$$\delta u^{0}_{(\alpha)} = -\frac{1}{a} A, \ \delta u^{i}_{(\alpha)} = \frac{1}{a} \gamma^{ij} v_{\alpha|j} \ \Rightarrow \delta u_{(\alpha)i} = a(v_{\alpha} - B)_{|i},$$

$$\delta T^0_{(\alpha)0} = -\delta\rho_\alpha,$$
$$\delta T^0_{(\alpha)j} = h_\alpha(v_\alpha - B)_{|j},\ T^i_{(\alpha)0} = -h_\alpha\gamma^{ij}v_{\alpha|j},$$
$$\delta T^i_{(\alpha)j} = \delta p_\alpha \delta^i{}_j + p_\alpha\left(\Pi^{|i}_{\alpha|j} - \frac{1}{3}\delta^i{}_j\triangle\Pi_\alpha\right),$$
$$\delta p_\alpha = c^2_\alpha\delta\rho_\alpha + p_\alpha\Gamma_\alpha \equiv p_\alpha\pi_{L\alpha},\ c^2_\alpha := \dot{p}_\alpha/\dot{\rho}_\alpha. \tag{1.156}$$

In (1.156) and in what follows the index (0) is dropped.

Summation of these equations give ($\delta_\alpha := \delta\rho_\alpha/\rho_\alpha$):

$$\rho\delta = \sum_\alpha \rho_\alpha\delta_\alpha, \tag{1.157}$$

$$hv = \sum_\alpha v_\alpha, \tag{1.158}$$

$$p\pi_L = \sum_\alpha \pi_{L\alpha}, \tag{1.159}$$

$$p\Pi = \sum_\alpha p\Pi_\alpha. \tag{1.160}$$

The only new aspect is the appearance of the perturbations $\delta Q_{(\alpha)\mu}$. We decompose $Q_{(\alpha)\mu}$ into energy and momentum transfer rates:

$$Q_{(\alpha)\mu} = Q_\alpha u_\mu + f_{(\alpha)\mu},\ u^\mu f_{(\alpha)\mu} = 0. \tag{1.161}$$

Since u_i and $f_{(\alpha)i}$ are of first order, the orthogonality condition in (1.161) implies

$$f_{(\alpha)0} = 0. \tag{1.162}$$

We set (for scalar perturbations)

$$\delta Q_{(\alpha)} = Q^{(0)}_\alpha\varepsilon_\alpha, \tag{1.163}$$
$$f_{(\alpha)j} = \mathcal{H}h_\alpha f_{\alpha|j}, \tag{1.164}$$

with two perturbation functions $\varepsilon_\alpha,\ f_\alpha$ for each component. Now, recall from (1.42) that

$$\delta u_0 = -aA,\ \delta u_i = a(v - B)_{|i}.$$

Using all this in (1.161) we obtain

$$\delta Q_{(\alpha)0} = -aQ^{(0)}_\alpha(\varepsilon_\alpha + A), \tag{1.165}$$
$$\delta Q_{(\alpha)j} = a\left[Q^{(0)}_\alpha(v - B) + Hh_\alpha f_\alpha\right]_{|i}. \tag{1.166}$$

The constraint in (1.148) can now be expressed as

$$\sum_\alpha Q^{(0)}_\alpha\varepsilon_\alpha = 0,\ \sum_\alpha h_\alpha f_\alpha = 0 \tag{1.167}$$

(we have, of course, made use of (1.153)).

From now on we drop the index (0).

We turn to the gauge transformation properties. As long as we do not use the zeroth-order energy equation (1.150), the transformation laws for $\delta_\alpha, v_\alpha, \pi_{L\alpha}, \Pi_\alpha$ remain the same as those in Sect. 1.1.6 for δ, v, π_L, and Π. Thus, using (1.150) and the notation $w_\alpha = p_\alpha/\rho_\alpha$, we have

$$\begin{aligned} \delta_\alpha &\to \delta_\alpha + \frac{\rho'}{\rho}\xi^0 = \delta_\alpha - 3(1+w_\alpha)\mathcal{H}(1-q_\alpha)\xi^0, \\ v_\alpha - B &\to (v_\alpha - B) - \xi^0, \\ \delta p_\alpha &\to \delta p_\alpha + p'_\alpha \xi^0, \\ \Pi_\alpha &\to \Pi_\alpha, \\ \Gamma_\alpha &\to \Gamma_\alpha. \end{aligned} \tag{1.168}$$

The quantity $\mathcal{Q}$, introduced below (1.42), will also be used for each component:

$$\delta T^0_{(\alpha)i} =: \frac{1}{a}\mathcal{Q}_{\alpha|i}, \Rightarrow \mathcal{Q} = \sum_\alpha \mathcal{Q}_{\alpha|i}. \tag{1.169}$$

The transformation law of $\mathcal{Q}_\alpha$ is

$$\mathcal{Q}_\alpha \to \mathcal{Q}_\alpha - a h_\alpha \xi^0. \tag{1.170}$$

For each α we define gauge invariant density perturbations $(\delta_\alpha)_{\mathcal{Q}_\alpha}$, $(\delta_\alpha)_\chi$ and velocities $V_\alpha = (v_\alpha - B)_\chi$. Because of the modification in the first of Eq. (1.168), we have instead of (1.54)

$$\Delta_\alpha := (\delta_\alpha)_{\mathcal{Q}_\alpha} = \delta_\alpha - 3\mathcal{H}(1+w_\alpha)(1-q_\alpha)(v_\alpha - B). \tag{1.171}$$

Similarly, adopting the notation of [31, Eq. (1.55)] generalizes to

$$\Delta_{s\alpha} := (\delta_\alpha)_\chi = \delta_\alpha + 3(1+w_\alpha)(1-q_\alpha)H\chi. \tag{1.172}$$

If we replace in (1.171) $v_\alpha - B$ by $v - B$ we obtain another gauge invariant density perturbation

$$\Delta_{c\alpha} := (\delta_\alpha)_{\mathcal{Q}} = \delta_\alpha - 3\mathcal{H}(1+w_\alpha)(1-q_\alpha)(v - B), \tag{1.173}$$

which reduces to δ_α for the *comoving gauge*: $v = B$.

The following relations between the three gauge invariant density perturbations are useful. Putting an index χ on the right of (1.171) gives

$$\Delta_\alpha = \Delta_{s\alpha} - 3\mathcal{H}(1+w_\alpha)(1-q_\alpha)V_\alpha. \tag{1.174}$$

Similarly, putting χ as an index on the right of (1.173) implies

$$\Delta_{cs} = \Delta_{s\alpha} - 3\mathcal{H}(1+w_\alpha)(1-q_\alpha)V. \tag{1.175}$$

For V_α we have, as in (1.56),

$$V_\alpha = v_\alpha + E'. \tag{1.176}$$

From now on we use similar notations for the total density perturbations:

$$\Delta := \delta_{\mathcal{Q}},\ \Delta_s := \delta_\chi\ (\Delta \equiv \Delta_c). \tag{1.177}$$

Let us translate the identities (1.157)-(1.160). For instance,

$$\sum_\alpha \rho_\alpha \Delta_{c\alpha} = \sum \alpha\rho_\alpha\delta_\alpha + 3\mathcal{H}(v-B)\sum_\alpha h_\alpha(1-q_\alpha) = \rho\delta + 3\mathcal{H}(v-B)h = \rho\Delta.$$

We collect this and related identities:

$$\rho\Delta = \sum_\alpha \rho_\alpha \Delta_{c\alpha} \tag{1.178}$$

$$= \sum_\alpha \rho_\alpha \Delta_\alpha - a \sum_\alpha Q_\alpha V_\alpha, \tag{1.179}$$

$$\rho\Delta_s = \sum_\alpha \rho_\alpha \Delta_{s\alpha}, \tag{1.180}$$

$$hV = \sum_\alpha h_\alpha V_\alpha, \tag{1.181}$$

$$p\Pi = \sum_\alpha p_\alpha \Pi_\alpha. \tag{1.182}$$

We would like to write also $p\Gamma$ in a manifestly gauge invariant form. From (using (1.157), (1.159) and (1.156))

$$p\Gamma = p\pi_L - c_s^2 \rho\delta = \sum_\alpha \underbrace{p_\alpha \pi_{L\alpha}}_{c_\alpha^2 \rho_\alpha \delta_\alpha + p_\alpha \Gamma_\alpha} - c_s^2 \sum_\alpha \rho_\alpha \delta_\alpha = \sum_\alpha p_\alpha \Gamma_\alpha + \sum_\alpha (c_\alpha^2 - c_s^2)\rho_\alpha \delta_\alpha$$

we get

$$p\Gamma = p\Gamma_{int} + p\Gamma_{\rm rel}, \tag{1.183}$$

with

$$p\Gamma_{int} = \sum_\alpha p_\alpha \Gamma_\alpha \tag{1.184}$$

and

$$p\Gamma_{\rm rel} = \sum_\alpha (c_\alpha^2 - c_s^2)\rho_\alpha \delta_\alpha. \tag{1.185}$$

Since $p\Gamma_{int}$ is obviously gauge invariant, this must also be the case for $p\Gamma_{\rm rel}$. We want to exhibit this explicitly. First note, using (1.152) and (1.150), that

$$c_s^2 = \frac{p'}{\rho'} = \sum_\alpha \frac{p'_\alpha}{\rho'} = \sum_\alpha c_\alpha^2 \frac{\rho'_\alpha}{\rho'} = \sum_\alpha c_\alpha^2 \frac{h_\alpha}{h}(1 - q_\alpha), \tag{1.186}$$

i.e.,

$$c_s^2 = \bar{c}_s^2 - \sum_\alpha \frac{h_\alpha}{h} q_\alpha c_\alpha^2, \tag{1.187}$$

where

$$\bar{c}_s^2 = \sum_\alpha \frac{h_\alpha}{h} c_\alpha^2. \tag{1.188}$$

Now we replace δ_α in (1.185) with the help of (1.173) and use (1.186), with the result

$$p\Gamma_{\rm rel} = \sum_\alpha (c_\alpha^2 - c_s^2)\rho_\alpha \Delta_{c\alpha}. \tag{1.189}$$

One can write this in a physically more transparent fashion by using once more (1.186), as well as (1.152) and (1.153),

$$p\Gamma_{\rm rel} = \sum_{\alpha,\beta}(c_\alpha^2 - c_\beta^2)\frac{h_\beta}{h}(1-q_\beta)\rho_\alpha\Delta_{c\alpha},$$

or

$$p\Gamma_{\rm rel} = \frac{1}{2}\sum_{\alpha,\beta}(c_\alpha^2 - c_\beta^2)\frac{h_\alpha h_\beta}{h}(1-q_\alpha)(1-q_\beta) \cdot \left[\frac{\Delta_{c\alpha}}{(1+w_\alpha)(1-q_\alpha)} - \frac{\Delta_{c\beta}}{(1+w_\beta)(1-q_\beta)}\right]. \tag{1.190}$$

For the special case $q_\alpha = 0$, for all α, we obtain

$$p\Gamma_{\rm rel} = \frac{1}{2}\sum_{\alpha,\beta}(c_\alpha^2 - c_\beta^2)\frac{h_\alpha h_\beta}{h}S_{\alpha\beta}\,; \tag{1.191}$$

$$S_{\alpha\beta} := \frac{\Delta_{c\alpha}}{1+w_\alpha} - \frac{\Delta_{c\beta}}{1+w_\beta}. \tag{1.192}$$

The gauge transformation properties of ε_α, f_α are obtained from

$$\delta Q_{(\alpha)\mu} \to \delta Q_{(\alpha\mu)} + \xi^\lambda Q_{(\alpha)\mu,\lambda} + Q_{(\alpha)\lambda}\xi^\lambda{}_{,\mu}. \tag{1.193}$$

For $\mu = 0$ this gives, making use of (1.149) and (1.165),

$$\varepsilon_\alpha + A \to \varepsilon_\alpha + A + \xi^0\frac{(aQ_\alpha)'}{aQ_\alpha} + (\xi^0)'\,.$$

Recalling (1.18), we obtain

$$\varepsilon_\alpha \to \varepsilon_\alpha + \frac{(Q_\alpha)'}{Q_\alpha}\xi^0. \tag{1.194}$$

For $\mu = i$ we get

$$\delta Q_{(\alpha)i} \to \delta Q_{(\alpha)i} + Q_{(\alpha)0}\xi^0{}_i,$$

thus

$$v - B + Hh_\alpha f_\alpha \to v - B + Hh_\alpha f_\alpha - \xi^0.$$

But according to (1.49) $v - B$ transforms the same way, whence

$$\boxed{f_\alpha \to f_\alpha.} \tag{1.195}$$

We see that the following quantity is a gauge invariant version of ε_α

$$E_{c\alpha} := (\varepsilon_\alpha)_{\mathcal{Q}} = \varepsilon_\alpha + \frac{(Q_\alpha)'}{Q_\alpha}(v - B). \tag{1.196}$$

We shall also use

$$E_\alpha := (\varepsilon_\alpha)_{\mathcal{Q}_\alpha} = \varepsilon_\alpha + \frac{(Q_\alpha)'}{Q_\alpha}(v_\alpha - B) = E_{c\alpha} + \frac{(Q_\alpha)'}{Q_\alpha}(V_\alpha - V) \tag{1.197}$$

and

$$E_{s\alpha} := (\varepsilon_\alpha)_\chi = \varepsilon_\alpha - \frac{\dot{Q}_\alpha}{Q_\alpha}\chi. \tag{1.198}$$

Beside

$$F_{c\alpha} := f_\alpha \tag{1.199}$$

we also make use of

$$F_\alpha := F_{c\alpha} - 3q_\alpha(V_\alpha - V). \tag{1.200}$$

In terms of these gauge invariant amplitudes the constraints (1.167) can be written as (using (1.153))

$$\sum_\alpha Q_\alpha E_{c\alpha} = 0, \tag{1.201}$$

$$\sum_\alpha Q_\alpha E_\alpha = \sum_\alpha (Q_\alpha)' V_\alpha, \tag{1.202}$$

$$\sum_\alpha h_\alpha F_{c\alpha} = 0. \tag{1.203}$$

Dynamical equations

We now turn to the dynamical equations that follow from

$$\delta T^\nu_{(\alpha)\mu;\nu} = \delta Q_{(\alpha)\mu}, \tag{1.204}$$

and the expressions for $\delta T^\nu_{(\alpha)\mu;\nu}$ and $\delta Q_{(\alpha)\mu}$ given in (1.156), (1.165) and (1.166). Below we write these in a harmonic decomposition, making use of the formulae in Appendix A for $\delta T^\nu_{(\alpha)\mu;\nu}$ (see (1.235) and (1.243)). In the harmonic decomposition Eqs. (1.165) and (1.166) become

$$\delta Q_{(\alpha)0} = -aQ_\alpha(\varepsilon_\alpha + A)Y, \tag{1.205}$$

$$\delta Q_{(\alpha)j} = a\left[Q_\alpha(v - B) + Hh_\alpha f_\alpha\right] Y_j. \tag{1.206}$$

From (1.235) we obtain, following the conventions adopted in the harmonic decompositions and using the last line in (1.156),

$$(\rho_\alpha\delta_\alpha)' + 3\frac{a'}{a}\rho_\alpha\delta_\alpha + 3\frac{a'}{a}p_\alpha\pi_{L\alpha} + h_\alpha(kv_\alpha + 3D' - E') = aQ_\alpha(A + \varepsilon_\alpha). \tag{1.207}$$

In the longitudinal gauge we have $\Delta_{s\alpha} = \delta_\alpha, V_\alpha = v_\alpha, E_{s\alpha} = \varepsilon_\alpha, E = 0$, and (see (1.73) $A = A_\chi, D = D_\chi$. We also note that, according to the definitions (1.19), (1.20), the Bardeen potentials can be expressed as

$$\boxed{A_\chi = \Psi,\ D_\chi = \Phi.} \tag{1.208}$$

Eq. (1.207) can thus be written in the following gauge invariant form

$$(\rho_\alpha\Delta_{s\alpha})' + 3\frac{a'}{a}\rho_\alpha\Delta_{s\alpha} + 3\frac{a'}{a}p_\alpha\left(\frac{c_\alpha^2}{w_\alpha}\Delta_{s\alpha} + \Gamma_\alpha\right) + h_\alpha(kV_\alpha + 3\Phi') = aQ_\alpha(\Psi + E_{s\alpha}). \tag{1.209}$$

Similarly, we obtain from (1.243) the momentum equation

$$[h_\alpha(v_\alpha - B)]' + 4\frac{a'}{a}h_\alpha(v_\alpha - B) - kh_\alpha A - kp_\alpha\pi_{L\alpha} + \frac{2}{3}\frac{k^2 - 3K}{k}p_\alpha\Pi_\alpha$$

$$= a[Q_\alpha(v - B) + \frac{\dot{a}}{a} h_\alpha f_\alpha]. \tag{1.210}$$

The gauge invariant form of this is (remember that f_α is gauge invariant)

$$(h_\alpha V_\alpha)' + 4\frac{a'}{a} h_\alpha V_\alpha - k p_\alpha \left(\frac{c_\alpha^2}{w_\alpha} \Delta_{s\alpha} + \Gamma_\alpha \right)$$
$$-k h_\alpha \Psi + \frac{2}{3} \frac{k^2 - 3K}{k} p_\alpha \Pi_\alpha = a[Q_\alpha V + \frac{\dot{a}}{a} h_\alpha f_\alpha]. \tag{1.211}$$

Eqs. (1.209) and (1.211) constitute our basic system describing the dynamics of matter. It will be useful to rewrite the momentum equation by using

$$(h_\alpha V_\alpha)' = h_\alpha V_\alpha' + V_\alpha h_\alpha',\ h_\alpha' = \rho_\alpha'(1 + c_s^2) = -3\frac{a'}{a}(1 - q_\alpha)(1 + c_\alpha^2) h_\alpha.$$

Together with (1.151) and (1.200) we obtain

$$V_\alpha' - 3\frac{a'}{a}(1 - q_\alpha)(1 + c_\alpha^2) V_\alpha + 4\frac{a'}{a} V_\alpha - k \frac{p_\alpha}{h_\alpha} \left(\frac{c_\alpha^2}{w_\alpha} \Delta_{s\alpha} + \Gamma_\alpha \right)$$
$$-k\Psi + \frac{2}{3} \frac{k^2 - 3K}{k} \frac{p_\alpha}{h_\alpha} \Pi_\alpha = a[\frac{Q_\alpha}{h_\alpha} V + \frac{\dot{a}}{a} f_\alpha] = \frac{a'}{a}(F_\alpha + 3 q_\alpha V_\alpha)$$

or

$$V_\alpha' + \frac{a'}{a} V_\alpha = k\Psi + \frac{a'}{a} F_\alpha + 3\frac{a'}{a}(1 - q_\alpha) c_\alpha^2 V_\alpha$$
$$+k \left[\frac{c_\alpha^2}{1 + w_\alpha} \Delta_{s\alpha} + \frac{w_\alpha}{1 + w_\alpha} \Gamma_\alpha \right] - \frac{2}{3} \frac{k^2 - 3K}{k} \frac{w_\alpha}{1 + w_\alpha} \Pi_\alpha. \tag{1.212}$$

Here we use (1.174) in the harmonic decomposition, i.e.,

$$\Delta_\alpha = \Delta_{s\alpha} + 3(1 + w_\alpha)(1 - q_\alpha) \frac{a'}{a} \frac{1}{k} V_\alpha, \tag{1.213}$$

and finally get

$$V_\alpha' + \frac{a'}{a} V_\alpha = k\Psi + \frac{a'}{a} F_\alpha + k \left[\frac{c_\alpha^2}{1 + w_\alpha} \Delta_\alpha + \frac{w_\alpha}{1 + w_\alpha} \Gamma_\alpha \right] - \frac{2}{3} \frac{k^2 - 3K}{k} \frac{w_\alpha}{1 + w_\alpha} \Pi_\alpha. \tag{1.214}$$

In applications it is useful to have an equation for $V_{\alpha\beta} := V_\alpha - V_\beta$. We derive this for $q_\alpha = \Gamma_\alpha = 0$ ($\Rightarrow \Gamma_{int} = 0,\ F_\alpha = F_{c\alpha} = f_\alpha$). From (1.214) we get

$$V_{\alpha\beta}' + \frac{a'}{a} V_{\alpha\beta} = \frac{a'}{a} F_{\alpha\beta} + k \left[\frac{c_\alpha^2}{1 + w_\alpha} \Delta_\alpha - \frac{c_\beta^2}{1 + w_\beta} \Delta_\beta \right] - \frac{2}{3} \frac{k^2 - 3K}{k} \Pi_{\alpha\beta}, \tag{1.215}$$

where

$$\Pi_{\alpha\beta} = \frac{w_\alpha}{1 + w_\alpha} \Pi_\alpha - \frac{w_\beta}{1 + w_\beta} \Pi_\beta. \tag{1.216}$$

Beside (1.213) we also use (1.175) in the harmonic decomposition,

$$\Delta_{c\alpha} = \Delta_{s\alpha} + 3(1 + w_\alpha)(1 - q_\alpha) \frac{a'}{a} \frac{1}{k} V, \tag{1.217}$$

to get

$$\Delta_\alpha = \Delta_{c\alpha} + 3(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}(V_\alpha - V). \tag{1.218}$$

From now on we consider only a two-component system α, β. (The generalization is easy; see [31].) Then $V_\alpha - V = (h_\beta/h)V_{\alpha\beta}$, and therefore the second term on the right of (1.215) is (remember that we assume $q_\alpha = 0$)

$$\begin{aligned} &k\left[\frac{c_\alpha^2}{1+w_\alpha}\Delta_\alpha - \frac{c_\beta^2}{1+w_\beta}\Delta_\beta\right] \\ &\quad = k\left[\frac{c_\alpha^2}{1+w_\alpha}\Delta_{c\alpha} - \frac{c_\beta^2}{1+w_\beta}\Delta_{c\beta}\right] + 3\frac{a'}{a}\left(c_\alpha^2 V_{\alpha\beta}\frac{h_\beta}{h} + c_\beta^2 V_{\alpha\beta}\frac{h_\alpha}{h}\right) \end{aligned} \tag{1.219}$$

At this point we use the identity[14]

$$\frac{\Delta_{c\alpha}}{1+w_\alpha} = \frac{\Delta}{1+w} + \frac{h_\beta}{h}S_{\alpha\beta}. \tag{1.220}$$

Introducing also the abbreviation

$$c_z^2 := c_\alpha^2\frac{h_\beta}{h} + c_\beta^2\frac{h_\alpha}{h} \tag{1.221}$$

the right hand side of (1.219) becomes $k(c_\alpha^2 - c_\beta^2)\frac{\Delta}{1+w} + kc_z^2 S_{\alpha\beta} + 3\frac{a'}{a}c_z^2 V_{\alpha\beta}$. So finally we arrive at

$$\begin{aligned} &V'_{\alpha\beta} + \frac{a'}{a}(1-3c_z^2)V_{\alpha\beta} \\ &\quad = k(c_\alpha^2 - c_\beta^2)\frac{\Delta}{1+w} + kc_z^2 S_{\alpha\beta} + \frac{a'}{a}F_{\alpha\beta} - \frac{2}{3}\frac{k^2-3K}{k}\Pi_{\alpha\beta}. \end{aligned} \tag{1.222}$$

For the generalization of this equation, without the simplifying assumptions, see (II.5.27) in [31].

Under the same assumptions we can simplify the energy equation (1.209). Using

$$\left(\frac{\rho_\alpha\Delta_{s\alpha}}{h_\alpha}\right)' = \frac{1}{h_\alpha}(\rho_\alpha\Delta_{s\alpha})' - \frac{h'_\alpha}{h_\alpha}\frac{\rho_\alpha}{h_\alpha}\Delta_{s\alpha}, \quad \frac{h'_\alpha}{h_\alpha}\frac{\rho_\alpha}{h_\alpha} = -3\frac{a'}{a}(1+c_\alpha^2)\frac{1}{1+w_\alpha}$$

in (1.209) yields

$$\boxed{\left(\frac{\Delta_{s\alpha}}{1+w_\alpha}\right)' = -kV_\alpha - 3\Phi'.} \tag{1.223}$$

From this, (1.217) and the defining equation (1.192) of $S_{\alpha\beta}$ we obtain the useful equation

$$\boxed{S'_{\alpha\beta} = -kV_{\alpha\beta}.} \tag{1.224}$$

[14] From (1.192) we obtain for an arbitrary number of components (making use of (1.178))

$$\sum_\beta \frac{h_\beta}{h}S_{\alpha\beta} = \frac{\Delta_{c\alpha}}{1+w_\alpha} - \sum_\beta \underbrace{\frac{h_\beta}{h}\frac{1}{1+w_\beta}}_{\rho_\beta/h}\Delta_{c\beta} = \frac{\Delta_{c\alpha}}{1+w_\alpha} - \frac{\rho}{h}\Delta = \frac{\Delta_{c\alpha}}{1+w_\alpha} - \frac{\Delta}{1+w}.$$

It is sometimes useful to have an equation for $(\Delta_{c\alpha}/(1+w_\alpha))'$. From (1.217) and (1.223) (for $q_\alpha = 0$) we get

$$\left(\frac{\Delta_{c\alpha}}{1+w_\alpha}\right)' = -kV_\alpha - 3\Phi' + 3\left(\frac{a'}{a}\frac{1}{k}V\right)'.$$

For the last term make use of (1.137), (1.140) and (1.121). If one uses also the following consequence of (1.118) and (1.120)

$$\frac{a'}{a}\Psi - \Phi' = 4\pi G\rho a^2(1+w)k^{-1}V = \frac{3}{2}\left[\left(\frac{a'}{a}\right)^2 + K\right](1+w)k^{-1}V \tag{1.225}$$

one obtains after some manipulations

$$\left(\frac{\Delta_{c\alpha}}{1+w_\alpha}\right)' = -kV_\alpha + 3\frac{K}{k}V + 3\frac{a'}{a}c_s^2\frac{\Delta}{1+w} + 3\frac{a'}{a}\frac{w}{1+w}\Gamma - 3\frac{a'}{a}\frac{w}{1+w}\frac{2}{3}\left(1-\frac{3K}{k^2}\right)\Pi. \tag{1.226}$$

1.5 Appendix to Chapter 1

In this Appendix we give derivations of some results that were used in previous sections.

A. Energy-momentum equations

In what follows we derive the explicit form of the perturbation equations $\delta T^\mu{}_{\nu;\mu} = 0$ for scalar perturbations, i.e., for the metric (1.16) and the energy-momentum tensor given by (1.34) and (1.42).

Energy equation

From

$$T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu{}_{\mu\lambda}T^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu}T^\mu{}_\lambda \tag{1.227}$$

we get for $\nu = 0$:

$$\delta(T^\mu{}_{0;\mu}) = \delta T^\mu{}_{0,\mu} + \delta\Gamma^\mu{}_{\mu\lambda}T^\lambda{}_0 + \Gamma^\mu{}_{\mu\lambda}\delta T^\lambda{}_0 - \delta\Gamma^\lambda{}_{\mu 0}T^\mu{}_\lambda - \Gamma^\lambda{}_{\mu 0}\delta T^\mu{}_\lambda \tag{1.228}$$

(quantities without a δ in front are from now on the zeroth order contributions). On the right we have more explicitly for the first three terms

$$\begin{aligned}
\delta T^\mu{}_{0,\mu} &= \delta T^i{}_{0,i} + \delta T^0{}_{0,0},\\
\delta\Gamma^\mu{}_{\mu\lambda}T^\lambda{}_0 &= \delta\Gamma^\mu{}_{\mu 0}T^0{}_0 = \delta\Gamma^i{}_{i0}T^0{}_0 + \delta\Gamma^0{}_{00}T^0{}_0,\\
\Gamma^\mu{}_{\mu\lambda}\delta T^\lambda{}_0 &= \Gamma^\mu{}_{\mu 0}\delta T^0{}_0 + \Gamma^\mu{}_{\mu i}\delta T^i{}_0 = 4\mathcal{H}\delta T^0{}_0 + \Gamma^j{}_{ji}\delta T^i{}_0;
\end{aligned}$$

we used some of the unperturbed Christoffel symbols:

$$\Gamma^0{}_{00} = \mathcal{H},\ \Gamma^0{}_{0i} = \Gamma^i{}_{00} = 0,\ \Gamma^0{}_{ij} = \mathcal{H}\gamma_{ij},\ \Gamma^i{}_{0j} = \mathcal{H}\delta^i{}_j,\ \Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk}, \tag{1.229}$$

where $\bar{\Gamma}^i{}_{jk}$ are the Christoffel symbols for the metric γ_{ij}. With these the other terms become

$$\begin{aligned}
-\delta\Gamma^\lambda{}_{\mu 0}T^\mu{}_\lambda &= -\delta\Gamma^0{}_{\mu 0}T^\mu{}_0 - \delta\Gamma^i{}_{\mu 0}T^\mu{}_i = -\delta\Gamma^0{}_{00}T^0{}_0 - \delta\Gamma^i{}_{j0}T^j{}_i,\\
-\Gamma^\lambda{}_{\mu 0}\delta T^\mu{}_\lambda &= -\Gamma^0{}_{\mu 0}\delta T^\mu{}_0 - \Gamma^i{}_{\mu 0}\delta T^\mu{}_i = -\mathcal{H}\delta T^0{}_0 - \mathcal{H}\delta T^i{}_i.
\end{aligned}$$

Collecting terms gives

$$\delta(T^{\mu}{}_{0;\mu}) = (\delta T^{i}{}_{0})_{|i} + \delta T^{0}{}_{0,0} - \mathcal{H}\delta T^{i}{}_{i} + 3\mathcal{H}\delta T^{0}{}_{0} - (\rho + p)\delta\Gamma^{i}{}_{i0}. \tag{1.230}$$

We recall part of (1.42)

$$\delta T^{0}{}_{0} = -\delta\rho,\ \delta T^{i}{}_{0} = -(\rho + p)v^{|i},\ \delta T^{i}{}_{j} = \delta p\delta^{i}{}_{j} + p\Pi^{i}{}_{j}, \tag{1.231}$$

where

$$\Pi^{i}{}_{j} := \Pi^{|i}{}_{|j} - \tfrac{1}{3}\delta^{i}{}_{j}\,\triangle\Pi. \tag{1.232}$$

Inserting this gives

$$\delta(T^{\mu}{}_{0;\mu}) = -\delta\rho_{,0} - (\rho + p)\triangle v - 3\mathcal{H}(\delta\rho + \delta p) - (\rho + p)\delta\Gamma^{i}{}_{i0}. \tag{1.233}$$

We need $\delta\Gamma^{i}{}_{i0}$. In a first step we have

$$\delta\Gamma^{i}{}_{i0} = \tfrac{1}{2}g^{ij}(\delta g_{ij,0} + \delta g_{j0,i} - \delta g_{i0,j}) + \tfrac{1}{2}\delta g^{i\nu}(g_{\nu i,0} + g_{\nu 0,i} - \delta g_{i0,\nu}),$$

so

$$\delta\Gamma^{i}{}_{i0} = \frac{1}{2}\left(\frac{1}{a^2}\gamma^{ij}\delta g_{ij,0} + \delta g^{ij}(a^2)_{,0}\gamma_{ij}\right).$$

Inserting here (1.16), i.e.,

$$\delta g_{ij} = 2a^2(D\gamma_{ij} + E_{|ij}),\ \delta g^{ij} = -2a^2(D\gamma^{ij} + E^{|ij}),$$

gives

$$\delta\Gamma^{i}{}_{i0} = (3D + \triangle E)'. \tag{1.234}$$

Hence (1.233) becomes

$$-\delta(T^{\mu}{}_{0;\mu}) = (\delta\rho)' + 3\mathcal{H}(\delta\rho + \delta p) + (\rho + p)[\triangle(v + E') + 3D'], \tag{1.235}$$

giving the *energy equation*:

$$\boxed{(\delta\rho)' + 3\mathcal{H}(\delta\rho + \delta p) + (\rho + p)[\triangle(v + E') + 3D'] = 0} \tag{1.236}$$

or

$$(\delta\rho)^{\cdot} + 3H(\delta\rho + \delta p) + (\rho + p)[\triangle(v + \dot{E}) + 3\dot{D}] = 0. \tag{1.237}$$

We rewrite (1.236) in terms of $\delta := \delta\rho/\rho$, using also (1.44) and (1.56),

$$(\rho\delta)' + 3\mathcal{H}\rho\delta + 3\mathcal{H}p\pi_L + (\rho + p)[\triangle V + 3D'] = 0. \tag{1.238}$$

Momentum equation

For $\nu = i$ Eq. (1.227) gives

$$\delta(T^{\mu}{}_{i;\mu}) = \delta T^{\mu}{}_{i,\mu} + \delta\Gamma^{\mu}{}_{\mu\lambda}T^{\lambda}{}_{i} + \Gamma^{\mu}{}_{\mu\lambda}\delta T^{\lambda}{}_{i} - \delta\Gamma^{\lambda}{}_{\mu i}T^{\mu}{}_{\lambda} - \Gamma^{\lambda}{}_{\mu i}\delta T^{\mu}{}_{\lambda}. \tag{1.239}$$

On the right we have more explicitly, again using (1.229),

$$\begin{aligned}
\delta T^{\mu}{}_{i,\mu} &= \delta T^{j}{}_{i,j} + \delta T^{0}{}_{i,0},\\
\delta\Gamma^{\mu}{}_{\mu j}T^{\lambda}{}_{i} &= \delta\Gamma^{\mu}{}_{\mu j}T^{j}{}_{i} = \delta\Gamma^{0}{}_{0j}T^{j}{}_{i} + \delta\Gamma^{k}{}_{kj}T^{j}{}_{i},\\
\Gamma^{\mu}{}_{\mu\lambda}\delta T^{\lambda}{}_{i} &= \Gamma^{\mu}{}_{\mu 0}\delta T^{0}{}_{i} + \Gamma^{\mu}{}_{\mu j}\delta T^{j}{}_{i} = 4\mathcal{H}\delta T^{0}{}_{i} + \Gamma^{k}{}_{kj}\delta T^{j}{}_{i},\\
-\delta\Gamma^{\lambda}{}_{\mu i}T^{\mu}{}_{\lambda} &= -\delta\Gamma^{0}{}_{\mu i}T^{\mu}{}_{0} - \delta\Gamma^{j}{}_{\mu i}T^{\mu}{}_{j} = -\delta\Gamma^{0}{}_{0i}T^{0}{}_{0} - \delta\Gamma^{j}{}_{ki}T^{k}{}_{j},\\
-\Gamma^{\lambda}{}_{\mu i}\delta T^{\mu}{}_{\lambda} &= -\Gamma^{0}{}_{\mu i}\delta T^{\mu}{}_{0} - \Gamma^{j}{}_{\mu i}\delta T^{\mu}{}_{j} = -\mathcal{H}\gamma_{ij}\delta T^{j}{}_{0} - \mathcal{H}\delta T^{0}{}_{i} - \Gamma^{j}{}_{ki}\delta T^{k}{}_{j}.
\end{aligned}$$

Collecting terms gives

$$\delta(T^{\mu}{}_{i;\mu}) = (\delta T^{j}{}_{i})_{|j} + \delta T^{0}{}_{i,0} + 3\mathcal{H}\delta T^{0}{}_{i} - \mathcal{H}\gamma_{ij}\delta T^{j}{}_{0} + (\rho+p)\delta\Gamma^{0}{}_{0i}. \tag{1.240}$$

One readily finds

$$\delta\Gamma^{0}{}_{0i} = (A - \mathcal{H}B)_{|i} \tag{1.241}$$

We insert this and (1.231) into the last equation and obtain

$$\delta(T^{\mu}{}_{i;\mu}) = \left\{\delta p + (\rho+p)'(v-B) + (\rho+p)[(v-B)' + 4\mathcal{H}(v-B) + A]\right\}_{|i} + p\Pi^{j}{}_{i|j}.$$

From (1.232) we obtain ($R(\gamma)_{ij}$ denotes the Ricci tensor for the metric γ_{ij})

$$\Pi^{j}{}_{i|j} = \Pi^{|j}{}_{|ij} - \tfrac{1}{3}\Pi_{|i} = \Pi^{|j}{}_{|ji} + R(\gamma)_{ij}\Pi^{|j} - \tfrac{1}{3}\Pi_{|i} = \left[\tfrac{2}{3}(\triangle + 3K)\Pi\right]_{|i}. \tag{1.242}$$

As a result we see that $\delta(T^{\mu}{}_{i;\mu})$ is equal to ∂_i of the function

$$[(\rho+p)(v-B)]' + 4\mathcal{H}(\rho+p)(v-B) + (\rho+p)A + p\pi_L + \tfrac{2}{3}(\triangle + 3K)p\Pi, \tag{1.243}$$

and the momentum equation becomes explicitly ($h = \rho + p$)

$$\boxed{[h(v-B)]' + 4\mathcal{H}h(v-B) + hA + p\pi_L + \tfrac{2}{3}(\triangle + 3K)p\Pi = 0.} \tag{1.244}$$

B. Calculation of the Einstein tensor for the longitudinal gauge

In the longitudinal gauge the metric is equal to $g_{\mu\nu} + \delta g_{\mu\nu}$, with

$$g_{00} = -a^2,\ g_{0i} = 0,\ \delta\Gamma^{0}{}_{ij} = [2\mathcal{H}(D-A)+D']\gamma_{ij},\ g^{00} = -a^{-2},\ g^{0i} = 0,\ g^{ij} = a^{-2}\gamma^{ij}; \tag{1.245}$$

$$\begin{aligned}
&\delta g_{00} = -2a^2 A,\ \delta g_{0i} = 0,\ \delta g_{ij} = 2a^2 D\gamma_{ij},\\
&\delta g^{00} = 2a^{-2}A,\ \delta g^{0i} = 0,\ \delta g^{ij} = -2a^{-2}D\gamma^{ij}.
\end{aligned} \tag{1.246}$$

The unperturbed Christoffel symbols have been given before in (1.229).

Next we need

$$\delta\Gamma^{\mu}{}_{\alpha\beta} = \tfrac{1}{2}\delta g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) + \tfrac{1}{2}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}). \tag{1.247}$$

For example, we have

$$\delta\Gamma^0{}_{00} = \tfrac{1}{2}2a^{-2}A(-a^2)' + \tfrac{1}{2}(-a^2)(-2a^2A)' = A'.$$

Some of the other components have already been determined in Sect.A. We list, for further use, all $\delta\Gamma^\mu{}_{\alpha\beta}$:

$$\begin{aligned} &\delta\Gamma^0{}_{00} = A',\ \delta\Gamma^0{}_{0i} = A_{,i},\ \delta\Gamma^0{}_{ij} = [2\mathcal{H}(D-A)+D']\gamma_{ij}, \\ &\delta\Gamma^i{}_{00} = A^{,i},\ \delta\Gamma^i{}_{0j} = D'\delta^i{}_j,\ \delta\Gamma^i{}_{jk} = D_{,k}\delta^i{}_j + D_{,j}\delta^i{}_k - D^{,i}\delta_{jk} \end{aligned} \tag{1.248}$$

(indices are raised with γ^{ij}).

For $\delta R_{\mu\nu}$ we have the general formula

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^\lambda{}_{\nu\mu} - \partial_\nu\delta\Gamma^\lambda{}_{\lambda\mu} + \delta\Gamma^\sigma{}_{\nu\mu}\Gamma^\lambda{}_{\lambda\sigma} + \Gamma^\sigma{}_{\nu\mu}\delta\Gamma^\lambda{}_{\lambda\sigma} - \delta\Gamma^\sigma{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\sigma{}_{\lambda\mu}\delta\Gamma^\lambda{}_{\nu\sigma}. \tag{1.249}$$

We give the details for δR_{00},

$$\delta R_{00} = \partial_\lambda\delta\Gamma^\lambda{}_{00} - \partial_0\delta\Gamma^\lambda{}_{\lambda 0} + \delta\Gamma^\sigma{}_{00}\Gamma^\lambda{}_{\lambda\sigma} + \Gamma^\sigma{}_{00}\delta\Gamma^\lambda{}_{\lambda\sigma} - \delta\Gamma^\sigma{}_{\lambda 0}\Gamma^\lambda{}_{0\sigma} - \Gamma^\sigma{}_{\lambda 0}\delta\Gamma^\lambda{}_{0\sigma}. \tag{1.250}$$

The individual terms on the right are:

$$\begin{aligned} \partial_\lambda\delta\Gamma^\lambda{}_{00} &= (\delta\Gamma^0{}_{00})' + (\delta\Gamma^i{}_{00})_{,i} = A'' + A^{,i}{}_{,i}, \\ -\partial_0\delta\Gamma^\lambda{}_{\lambda 0} &= -A'' - 3D'', \\ \delta\Gamma^\sigma{}_{00}\Gamma^\lambda{}_{\lambda\sigma} &= \delta\Gamma^0{}_{00}\Gamma^\lambda{}_{\lambda 0} + \delta\Gamma^i{}_{00}\Gamma^\lambda{}_{\lambda i} = 4\mathcal{H}A' + \bar\Gamma^l{}_{li}A^{,i}, \\ \Gamma^\sigma{}_{00}\delta\Gamma^\lambda{}_{\lambda\sigma} &= \Gamma^0{}_{00}\delta\Gamma^\lambda{}_{\lambda 0} + \Gamma^i{}_{00}\delta\Gamma^\lambda{}_{\lambda i} = \mathcal{H}(A' + 3D'), \\ -\delta\Gamma^\sigma{}_{\lambda 0}\Gamma^\lambda{}_{0\sigma} &= -\delta\Gamma^0{}_{\lambda 0}\Gamma^\lambda{}_{00} - \delta\Gamma^i{}_{\lambda 0}\Gamma^\lambda{}_{0i} = -\mathcal{H}(A' + 3D'), \\ -\Gamma^\sigma{}_{\lambda 0}\delta\Gamma^\lambda{}_{0\sigma} &= -\Gamma^0{}_{\lambda 0}\delta\Gamma^\lambda{}_{00} - \Gamma^i{}_{\lambda 0}\delta\Gamma^\lambda{}_{0i} = -\mathcal{H}(A' + 3D'). \end{aligned}$$

Summing up gives the desired result

$$\boxed{\delta R_{00} = \triangle A + 3\mathcal{H}A' - 3D'' - 3\mathcal{H}D'.} \tag{1.251}$$

Similarly one finds (unpleasant exercise)

$$\delta R_{0j} = 2(\mathcal{H}A - D')_{,j}\ , \tag{1.252}$$

$$\delta R_{ij} = -(A+D)_{|ij} + \big[-\triangle D - (4\mathcal{H}^2 + 2\mathcal{H}')A - \mathcal{H}A' \tag{1.253}$$

$$+ (4\mathcal{H}^2 + 2\mathcal{H}')D - 5\mathcal{H}D' + D''\big]\,\gamma_{ij}. \tag{1.254}$$

Using also the zeroth order expressions for the Ricci tensor

$$R_{00} = -3\mathcal{H}',\ R_{ij} = [\mathcal{H}' + 2\mathcal{H}^2 + 2K]\gamma_{ij},\ R_{0i} = 0, \tag{1.255}$$

one finds for the Einstein tensor[15]

$$\delta G^0{}_0 = \frac{2}{a^2}[3\mathcal{H}(\mathcal{H}A - D') + \triangle D + 3KD], \tag{1.256}$$

[15] Note that $\delta R^\mu{}_\nu = \delta g^{\mu\lambda}R_{\lambda\nu} + g^{\mu\lambda}\delta R_{\lambda\nu}$.

$$\delta G^0{}_j = -\frac{2}{a^2}[\mathcal{H}A - D']_{,j} \ , \tag{1.257}$$

$$\delta G^i{}_j = \frac{2}{a^2}\Big\{(2\mathcal{H}' + \mathcal{H}^2)A + \mathcal{H}A' - D'' \\ -2\mathcal{H}D' + KD + \frac{1}{2}\triangle(A+D)\Big\}\delta^i{}_j - \frac{1}{a^2}(A+D)^{|i}{}_{|j}. \tag{1.258}$$

These results can be derived less tediously with the help of the '3+1 formalism', developed, for instance, in Sect. 2.9 of [1]. This was sketched in [33].

C. Summary of notation and basic equations

Notation in cosmological perturbation theory is a nightmare. Unfortunately, we had to introduce lots of symbols and many equations. For convenience, we summarize the adopted notation and indicate the location of the most important formulae. Some of them are repeated for further reference.

Recapitulation of the basic perturbation equations

For *scalar* perturbations we use the following gauge invariant amplitudes:
metric: $\Psi,\ \Phi$ (Bardeen potentials)

$$\Psi \equiv A_\chi,\ \Phi \equiv D_\chi; \tag{1.259}$$

total energy-momentum tensor $T^{\mu\nu}$: $\Delta,\ V$; instead of Δ we also use

$$\Delta_s = \Delta - 3(1+w)H\frac{a}{k}V. \tag{1.260}$$

The basic equations, derived from Einstein's field equations, and some of the consequences, can be summarized in the harmonic decomposition as follows:
• constraint equations:

$$(k^2 - 3K)\Phi = 4\pi G\rho a^2\Delta, \tag{1.261}$$

$$\dot\Phi - H\Psi = -4\pi G(\rho + p)\frac{a}{k}V; \tag{1.262}$$

• relevant dynamical equation:

$$\Phi + \Psi = -8\pi G\frac{a^2}{k^2}p\Pi; \tag{1.263}$$

• energy equation:

$$\dot\Delta - 3Hw\Delta = -\left(1 - \frac{3K}{k^2}\right)\left[(1+w)\frac{k}{a}V + 2Hw\Pi\right]; \tag{1.264}$$

• momentum equation:

$$\dot V + HV = \frac{k}{a}\Psi + \frac{1}{1+w}\frac{k}{a}\left[c_s^2\Delta + w\Gamma - \frac{2}{3}\frac{k^2 - 3K}{k^2}w\Pi\right]. \tag{1.265}$$

If Δ is replaced in (1.264) and (1.265) by Δ_s these equations become

$$\dot\Delta_s + 3H(c_s^2 - w)\Delta_s = -3(1+w)\dot\Phi - (1+w)\frac{k}{a}V - 3Hw\Gamma, \tag{1.266}$$

$$\dot{V} + (1 - 3c_s^2)HV = \frac{k}{a}\Psi + \frac{c_s^2}{1+w}\frac{k}{a}\Delta_s + \frac{w}{1+w}\frac{k}{a}\Gamma - \frac{2}{3}\frac{w}{1+w}\frac{k^2 - 3K}{k^2}\frac{k}{a}\Pi. \tag{1.267}$$

multi-component systems:

$$T^\mu{}_\nu = \sum_{(\alpha)} T^\mu_{(\alpha)\nu},\ T^\nu_{(\alpha)\mu;\nu} = Q_{(\alpha)\mu},\ \sum_\alpha Q_{(\alpha)\mu} = 0. \tag{1.268}$$

• additional *unperturbed* quantities, beside $\rho_\alpha, , p_\alpha, h_\alpha, c_\alpha, : Q_\alpha, q_\alpha$, satisfy:

$$\rho = \sum_\alpha \rho_\alpha,\ p = \sum_\alpha p_\alpha,\ h := \rho + p = \sum_\alpha h_\alpha, \tag{1.269}$$

$$Q_\alpha = 3Hh_\alpha q_\alpha,\ \sum_\alpha Q_\alpha = 0,\ \sum_\alpha h_\alpha q_\alpha = 0, \tag{1.270}$$

$$\dot{\rho}_\alpha = -3H(1 - q_\alpha)h_\alpha. \tag{1.271}$$

• *perturbations*: gauge invariant amplitudes: $\Delta_\alpha, \Delta_{s\alpha}, \Delta_{c\alpha}, \Pi_\alpha, \Gamma_\alpha$,

$$\rho\Delta = \sum_\alpha \rho_\alpha \Delta_{c\alpha} \tag{1.272}$$

$$= \sum_\alpha \rho_\alpha \Delta_\alpha - a \sum_\alpha Q_\alpha V_\alpha, \tag{1.273}$$

$$\rho\Delta_s = \sum_\alpha \rho_\alpha \Delta_{s\alpha}, \tag{1.274}$$

$$hV = \sum_\alpha h_\alpha V_\alpha, \tag{1.275}$$

$$p\Pi = \sum_\alpha p_\alpha \Pi_\alpha, \tag{1.276}$$

$$p\Gamma = p\Gamma_{\rm int} + p\Gamma_{\rm rel}, \tag{1.277}$$

$$p\Gamma_{\rm int} = \sum_\alpha p_\alpha \Gamma_\alpha, \tag{1.278}$$

$$p\Gamma_{\rm rel} = \sum_\alpha (c_\alpha^2 - c_s^2)\rho_\alpha \Delta_{c\alpha} \tag{1.279}$$

or

$$p\Gamma_{\rm rel} = \frac{1}{2}\sum_{\alpha,\beta}(c_\alpha^2 - c_\beta^2)\frac{h_\alpha h_\beta}{h}(1 - q_\alpha)(1 - q_\beta) \cdot \left[\frac{\Delta_{c\alpha}}{(1+w_\alpha)(1-q_\alpha)} - \frac{\Delta_{c\beta}}{(1+w_\beta)(1-q_\beta)}\right]; \tag{1.280}$$

for the special case $q_\alpha = 0$, for all α:

$$p\Gamma_{\rm rel} = \frac{1}{2}\sum_{\alpha,\beta}(c_\alpha^2 - c_\beta^2)\frac{h_\alpha h_\beta}{h} S_{\alpha\beta}\ ; \tag{1.281}$$

$$S_{\alpha\beta} := \frac{\Delta_{c\alpha}}{1+w_\alpha} - \frac{\Delta_{c\beta}}{1+w_\beta}. \tag{1.282}$$

• additional gauge invariant perturbations from $\delta Q_{(\alpha)\mu}$:
energy: $E_\alpha, E_{c\alpha}, E_{s\alpha}$; momentum: $F_\alpha, F_{c\alpha}$; constraints:

$$\sum_\alpha Q_\alpha E_{c\alpha} = 0, \tag{1.283}$$

$$\sum_\alpha Q_\alpha E_\alpha = \sum_\alpha (Q_\alpha)' V_\alpha, \tag{1.284}$$

$$\sum_\alpha h_\alpha F_{c\alpha} = 0. \tag{1.285}$$

• *dynamical equations* for $q_\alpha = \Gamma_\alpha = 0$ ($\Rightarrow \Gamma_{\text{int}} = 0$); some of the equations below hold only for two-component systems;

$$\left(\frac{\Delta_{s\alpha}}{1+w_\alpha}\right)' = -kV_\alpha - 3\Phi'; \tag{1.286}$$

Eq. (1.226) for $K = 0$:

$$\left(\frac{\Delta_{c\alpha}}{1+w_\alpha}\right)' = -kV_\alpha + 3\frac{a'}{a}c_s^2\frac{\Delta}{1+w} + 3\frac{a'}{a}\frac{w}{1+w}\Gamma - 3\frac{a'}{a}\frac{w}{1+w}\frac{2}{3}\Pi; \tag{1.287}$$

$$V_\alpha' + \frac{a'}{a}V_\alpha = k\Psi + \frac{a'}{a}F_\alpha + k\frac{c_\alpha^2}{1+w_\alpha}\Delta_\alpha - \frac{2}{3}\frac{k^2-3K}{k}\Pi_\alpha; \tag{1.288}$$

for $V_{\alpha\beta} := V_\alpha - V_\beta$:

$$V_{\alpha\beta}' + \frac{a'}{a}(1-3c_z^2)V_{\alpha\beta} = k(c_\alpha^2 - c_\beta^2)\frac{\Delta}{1+w} + kc_z^2 S_{\alpha\beta} + \frac{a'}{a}F_{\alpha\beta} - \frac{2}{3}\frac{k^2-3K}{k}\Pi_{\alpha\beta}, \tag{1.289}$$

relation between $S_{\alpha\beta}$ and $V_{\alpha\beta}$:

$$S_{\alpha\beta}' = -kV_{\alpha\beta}. \tag{1.290}$$

When working with $\Delta_{s\alpha}$ it is natural to substitute in (1.288) Δ_α with the help of (1.174) in terms of $\Delta_{s\alpha}$:

$$V_\alpha' + \frac{a'}{a}(1-3c_\alpha^2)V_\alpha = k\Psi + \frac{a'}{a}F_\alpha + k\frac{c_\alpha^2}{1+w_\alpha}\Delta_{s\alpha} - \frac{2}{3}\frac{k^2-3K}{k}\frac{w_\alpha}{1+w_\alpha}\Pi_\alpha, \tag{1.291}$$

2 Some applications of cosmological perturbation theory

In this chapter we discuss some applications of the general formalism. More relevant applications will follow in later parts.

Before studying realistic multi-component fluids, we consider first the simplest case when one component, for instance CDM, dominates. First, we study, however, a general problem

Let us write down the basic equations (1.139)–(1.142) in the notation adopted later ($A_\chi = \Psi, D_\chi = \Phi, \delta_Q = \Delta$):

$$\dot\Delta - 3Hw\Delta = -(1+w)\frac{k}{a}\frac{k^2-3K}{k^2}V - 2H\frac{k^2-3K}{k^2}w\Pi, \tag{2.1}$$

$$\dot V + HV = -\frac{k}{a}\Phi + \frac{1}{1+w}\frac{k}{a}\left[c_s^2\Delta + w\Gamma - 8\pi G(1+w)\frac{a^2}{k^2}p\Pi - \frac{2}{3}\frac{k^2-3K}{k^2}w\Pi\right], \tag{2.2}$$

$$\frac{k^2 - 3K}{a^2}\Phi = 4\pi G\rho\Delta, \tag{2.3}$$

$$\dot{\Phi} + H\Phi = -4\pi G(\rho + p)\frac{a}{k}V - 8\pi GH\frac{a^2}{k^2}p\Pi. \tag{2.4}$$

Recall also (1.121):

$$\Phi + \Psi = -8\pi G\frac{a^2}{k^2}p\Pi. \tag{2.5}$$

Note that $\Phi = -\Psi$ for $\Pi = 0$.

From these perturbation equations we derived through elimination the second order equation (1.143) for Δ, which we repeat for $\Pi = 0$ (vanishing anisotropic stresses) and $\Gamma = 0$ (vanishing entropy production):

$$\ddot{\Delta} + (2 + 3c_s^2 - 6w)H\dot{\Delta} + \left[c_s^2\frac{k^2}{a^2} - 4\pi G\rho(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} + (3c_s^2 - 5w)\Lambda\right]\Delta = 0. \tag{2.6}$$

Sometimes it is convenient to write this in terms of the conformal time for the quantity $\rho a^3\Delta$. Making use of $(\rho a^3)^{\cdot} = -3Hw(\rho a^3)$ (see (0.22)) one finds

$$(\rho a^3\Delta)'' + (1 + 3c_s^2)\mathcal{H}(\rho a^3\Delta)' + \left[(k^2 - 3K)c_s^2 - 4\pi G(\rho + p)a^2\right](\rho a^3\Delta) = 0. \tag{2.7}$$

Similarly, one can derive a second order equation for Φ:

$$\ddot{\Phi} + (4 + 3c_s^2)H\dot{\Phi} + \left[c_s^2\frac{k^2}{a^2} + 8\pi G\rho(c_s^2 - w) - 2(1 + 3c_s^2)\frac{K}{a^2} + (1 + c_s^2)\Lambda\right]\Phi = 0. \tag{2.8}$$

Remarkably, for $p = p(\rho)$ this can be written as [34]

$$\frac{\rho + p}{H}\left[\frac{H^2}{a(\rho + p)}\left(\frac{a}{H}\Phi\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\Phi = 0 \tag{2.9}$$

(Exercise).

2.1 Non-relativistic limit

It is instructive to first consider a one-component non-relativistic fluid. The non-relativistic limit of the second order equation (2.6) is

$$\ddot{\Delta} + 2H\dot{\Delta} = 4\pi G\rho\Delta - c_s^2\left(\frac{k}{a}\right)^2\Delta. \tag{2.10}$$

From this basic equation one can draw various conclusions.

The Jeans criterion

One sees from (2.10) that gravity wins over the pressure term $\propto c_s^2$ for $k < k_J$, where

$$k_J^2\left(\frac{c_s}{a}\right)^2 = 4\pi G\rho \tag{2.11}$$

defines the *comoving Jeans wave number*. The corresponding *Jeans length* (wave length) is

$$\lambda_J = \frac{2\pi}{k_J} a = \left(\frac{\pi c_s^2}{G\rho}\right)^{1/2}, \ \frac{\lambda_J}{2\pi} \simeq \frac{c_s}{H}. \tag{2.12}$$

For $\lambda < \lambda_J$ we expect that the fluid oscillates, while for $\lambda \gg \lambda_J$ an over-density will increase.

Let us illustrate this for a polytropic equation of state $p = const\ \rho^\gamma$. We consider, as a simple example, a matter dominated Einstein-de Sitter model ($K = 0$), for which $a(t) \propto t^{2/3},\ H = 2/(3t)$. Eq. (2.10) then becomes (taking ρ from the Friedmann equation, $\rho = 1/(6\pi G t^2)$)

$$\ddot{\Delta} + \frac{4}{3t}\dot{\Delta} + \left(\frac{L^2}{t^{2\gamma-2/3}} - \frac{2}{3t^2}\right)\Delta = 0, \tag{2.13}$$

where L^2 is the constant

$$L^2 = \frac{t^{2\gamma-2/3} c_s^2 k^2}{a^2}. \tag{2.14}$$

The solutions of (2.27)are

$$\Delta_\pm(t) \propto t^{-1/6} J_{\mp 5/6\nu}\left(\frac{L t^{-\nu}}{\nu}\right), \ \nu := \gamma - \frac{4}{3} > 0. \tag{2.15}$$

The Bessel functions J oscillate for $t \ll L^{1/\nu}$, whereas for $t \gg L^{1/\nu}$ the solutions behave like

$$\Delta_\pm(t) \propto t^{-\frac{1}{6} \pm \frac{5}{6}}. \tag{2.16}$$

Now, $t > L^{1/\nu}$ signifies $c_s^2 k^2/a^2 < 6\pi G\rho$. This is essentially again the Jeans criterion $k < k_J$. At the same time we see that

$$\Delta_+ \propto t^{2/3} \propto a, \tag{2.17}$$

$$\Delta_- \propto t^{-1}. \tag{2.18}$$

Thus the *growing mode increases like the scale factor*.

2.2 Large scale solutions

Consider, as an important application, wavelengths *larger than the Jeans length*, i.e., $c_s(k/aH) \ll 1$. Then we can drop the last term in equation (2.9) and solve for Φ in terms of quadratures:

$$\Phi(t, \mathbf{k}) = C(\mathbf{k})\frac{H}{a}\int_0^t \frac{a(\rho+p)}{H^2}dt + \frac{H}{a}d(\mathbf{k}). \tag{2.19}$$

We write this differently by using in the integrand the following background equation (implied by (1.80))

$$\frac{a(\rho+p)}{H^2} = \left(\frac{a}{H}\right)^{\cdot} - a\left(1 - \frac{K}{\dot{a}^2}\right).$$

With this we obtain

$$\Phi(t, \mathbf{k}) = C(\mathbf{k})\left[1 - \frac{H}{a}\int_0^t a\left(1 - \frac{K}{\dot{a}^2}\right)dt\right] + \frac{H}{a}d(\mathbf{k}). \tag{2.20}$$

Let us work this out for a mixture of dust ($p = 0$) and radiation ($p = \frac{1}{3}\rho$). We use the 'normalized' scale factor $\zeta := a/a_{\rm eq}$, where $a_{\rm eq}$ is the value of a when the energy densities of dust (CDM) and radiation are equal. Then (see Sect. 0.1.3)

$$\rho = \frac{1}{2}\zeta^{-4} + \frac{1}{2}\zeta^{-3},\ p = \frac{1}{6}\zeta^{-4}. \tag{2.21}$$

Note that

$$\zeta' = kx\zeta,\ x := \frac{Ha}{k}. \tag{2.22}$$

From now on we assume $K = 0,\ \Lambda = 0$. Then the Friedmann equation gives

$$H^2 = H_{\rm eq}^2 \frac{\zeta + 1}{2}\zeta^{-4}, \tag{2.23}$$

thus

$$x^2 = \frac{\zeta + 1}{2\zeta^2}\frac{1}{\omega^2},\ \omega := \frac{1}{x_{\rm eq}} = \left(\frac{Ha}{k}\right)_{\rm eq}. \tag{2.24}$$

In (2.11) we need the integral

$$\frac{H}{a}\int_0^t a dt = Ha_{\rm eq}\frac{1}{\zeta}\int_0^\eta \zeta^2 d\eta = \frac{\sqrt{\zeta+1}}{\zeta^3}\int_0^\zeta \frac{\zeta^2}{\sqrt{\zeta+1}}d\zeta.$$

As a result we get for the growing mode

$$\Phi(\zeta, \mathbf{k}) = C(\mathbf{k})\left[1 - \frac{\sqrt{\zeta+1}}{\zeta^3}\int_0^\zeta \frac{\zeta^2}{\sqrt{\zeta+1}}d\zeta\right]. \tag{2.25}$$

From (2.3) and the definition of x we obtain

$$\Phi = \frac{3}{2}x^2\Delta, \tag{2.26}$$

hence with (2.15)

$$\Delta = \frac{4}{3}\omega^2 C(\mathbf{k})\frac{\zeta^2}{\zeta+1}\left[1 - \frac{\sqrt{\zeta+1}}{\zeta^3}\int_0^\zeta \frac{\zeta^2}{\sqrt{\zeta+1}}d\zeta\right]. \tag{2.27}$$

The integral is elementary. One finds that Δ is proportional to

$$U_g = \frac{1}{\zeta(\zeta+1)}\left[\zeta^3 + \frac{2}{9}\zeta^2 - \frac{8}{9}\zeta - \frac{16}{9} + \frac{16}{9}\sqrt{\zeta+1}\right]. \tag{2.28}$$

This is a well-known result.

The decaying mode corresponds to the second term in (2.11), and is thus proportional to

$$U_d = \frac{1}{\zeta\sqrt{\zeta+1}}. \tag{2.29}$$

Limiting approximations of (2.19) are

$$U_g = \begin{cases} \frac{10}{9}\zeta^2 & : \ \zeta \ll 1 \\ \zeta & : \ \zeta \gg 1. \end{cases} \tag{2.30}$$

For the potential $\Phi \propto x^2\Delta$ the growing mode is given by

$$\Phi(\zeta) = \Phi(0)\frac{9}{10}\frac{\zeta+1}{\zeta^2}U_g. \tag{2.31}$$

Thus

$$\Phi(\zeta) = \Phi(0)\begin{cases} 1 & : \ \zeta \ll 1 \\ \frac{9}{10} & : \ \zeta \gg 1. \end{cases} \tag{2.32}$$

In particular, Φ stays *constant both in the radiation and in the matter dominated eras*. Recall that this holds only for $c_s(k/aH) \ll 1$. We shall later study Eq. (2.9) for arbitrary scales.

2.3 Solution of (2.6) for dust

Using the Poisson equation (2.3) we can write (2.9) in terms of Δ

$$\frac{1+w}{a^2H}\left[\frac{H^2}{a(\rho+p)}\left(\frac{a^3\rho}{H}\Delta\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\Delta = 0. \tag{2.33}$$

For dust this reduces to (using $\rho a^3 = const$)

$$\left[a^2H^2\left(\frac{\Delta}{H}\right)^{\cdot}\right]^{\cdot} = 0. \tag{2.34}$$

The general solution of this equation is

$$\Delta(t,\mathbf{k}) = C(\mathbf{k})H(t)\int_0^t \frac{dt'}{a^2(t')H^2(t')} + d(k)H(t). \tag{2.35}$$

This result can also be obtained in Newtonian perturbation theory. The first term gives the growing mode and the second the decaying one.

Let us rewrite (2.35) in terms of the redshift z. From $1+z = a_0/a$ we get $dz = -(1+z)Hdt$, so by (0.52)

$$\frac{dt}{dz} = -\frac{1}{H_0(1+z)E(z)},\ H(z) = H_0E(z). \tag{2.36}$$

Therefore, the growing mode $D_g(z)$ can be written in the form

$$D_g(z) = \frac{5}{2}\Omega_M E(z)\int_z^\infty \frac{1+z'}{E^3(z')}dz'. \tag{2.37}$$

Here the normalization is chosen such that $D_g(z) = (1+z)^{-1} = a/a_0$ for $\Omega_M = 1,\ \Omega_\Lambda = 0$. This growth function is plotted in Fig. 7.12 of [5].

2.4 A simple relativistic example

As an additional illustration we now solve (2.7) for a single perfect fluid with $w = const,\ K = \Lambda = 0$. For a flat universe the background equations are then

$$\rho' + 3\frac{a'}{a}(1+w)\rho = 0,\ \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho.$$

Inserting the ansatz

$$\rho a^2 = A\eta^{-\nu},\ a = a_0(\eta/\eta_0)^\beta$$

we get

$$\frac{\beta}{\eta^2} = \frac{8\pi G}{3}A\eta^{-\nu}\ \Rightarrow \nu = 2,\ A = \frac{3}{8\pi G}\beta^2.$$

The energy equation then gives $\beta = 2/(1+3w)$ $(= 1$ if radiation dominates). Let $x := k\eta$ and

$$f := x^{\beta-2}\Delta \propto \rho a^3\Delta.$$

Also note that $k/(aH) = x/\beta$. With all this we obtain from (2.7) for f

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + c_s^2 - \frac{\beta(\beta+1)}{x^2}\right] f = 0. \tag{2.38}$$

The solutions are given in terms of Bessel functions:

$$f(x) = C_0 j_\beta(c_s x) + D_0 n_\beta(c_s x). \tag{2.39}$$

This implies acoustic oscillations for $c_s x \gg 1$, i.e., for $\beta(k/aH) \gg 1$ (subhorizon scales). In particular, if the radiation dominates ($\beta = 1$)

$$\Delta \propto x[C_0 j_1(c_s x) + D_0 n_1(c_s x)], \tag{2.40}$$

and the growing mode is soon proportional to $x\cos(c_s x)$, while the term going with $\sin(c_s x)$ dies out.

On the other hand, on superhorizon scales ($c_s x \ll 1$) one obtains

$$f \simeq Cx^\beta + Dx^{-(\beta+1)},$$

and thus

$$\begin{aligned}
\Delta &\simeq Cx^2 + Dx^{-(2\beta-1)},\\
\Phi &\simeq \frac{3}{2}\beta^2(C + Dx^{-(2\beta+1)},\\
V &\simeq \frac{3}{2}\beta\left(-\frac{1}{\beta+1}Cx + Dx^{-2\beta}\right).
\end{aligned} \tag{2.41}$$

We see that the growing mode behaves as $\Delta \propto a^2$ in the radiation dominated phase and $\Delta \propto a$ in the matter dominated era.

The characteristic Jeans wave number is obtained when the square bracket in (2.7) vanishes. This gives

$$\lambda_J = \left(\frac{\pi c_s^2}{Gh}\right)^{1/2},\ h = \rho + p. \tag{2.42}$$

Exercise. Derive the exact expression for V.

In Part III we shall study more complicated coupled fluid models that are important for the evolution of perturbations before recombination. In the next part the general theory will be applied in attempts to understand the generation of primordial perturbations from original quantum fluctuations.

Part II
Inflation and generation of fluctuations

3 Inflationary scenario

3.1 Introduction

The horizon and flatness problems of standard big bang cosmology are so serious that the proposal of a very early accelerated expansion, preceding the hot era dominated by relativistic fluids, appears quite plausible. This general qualitative aspect of 'inflation' is now widely accepted. However, when it comes to concrete model building the situation is not satisfactory. Since we do not know the fundamental physics at superhigh energies not too far from the Planck scale, models of inflation are usually of a phenomenological nature. Most models consist of a number of scalar fields, including a suitable form for their potential. Usually there is no direct link to fundamental theories, like supergravity, however, there have been many attempts in this direction. For the time being, inflationary cosmology should be regarded as an attractive scenario, and not yet as a theory.

The most important aspect of inflationary cosmology is that *the generation of perturbations on large scales from initial quantum fluctuations is unavoidable and predictable*. For a given model these fluctuations can be calculated accurately, because they are tiny and cosmological perturbation theory can be applied. And, most importantly, these predictions can be *confronted with the cosmic microwave anisotropy measurements*. We are in the fortunate position to witness rapid progress in this field. The results from various experiments, most recently from WMAP, give already strong support of the basic predictions of inflation. Further experimental progress can be expected in the coming years.

In what follows I shall mainly concentrate on this aspect. It is, I think, important to understand in sufficient detail how the involved calculations are done, and which aspects are the most generic ones for inflationary models. We shall learn a lot in the coming years, thanks to the confrontation of the theory with precise observations.

3.2 The horizon problem and the general idea of inflation

I begin by describing the famous horizon puzzle (topic belonging to Chap. 0), which is a very serious causality problem of standard big bang cosmology.

Past and future light cone distances

Consider our past light cone for a Friedmann spacetime model (Fig. 3.1). For a radial light ray the differential relation $dt = a(t)dr/(1-kr^2)^{1/2}$ holds for the coordinates (t,r) of the metric (0.40). The proper radius of the past light sphere at time t (cross section of the light cone with the hypersurface $\{t = \text{const}\}$) is

$$l_p(t) = a(t)\int_0^{r(t)} \frac{dr}{\sqrt{1-kr^2}}, \tag{3.1}$$

where the coordinate radius is determined by

$$\int_0^{r(t)} \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{dt'}{a(t')}. \tag{3.2}$$

Hence,

$$l_p(t) = a(t)\int_t^{t_0} \frac{dt'}{a(t')}. \tag{3.3}$$

We rewrite this in terms of the redhift variable, using (2.36),

$$l_p(z) = \frac{1}{H_0(1+z)}\int_0^z \frac{dz'}{E(z')}. \tag{3.4}$$

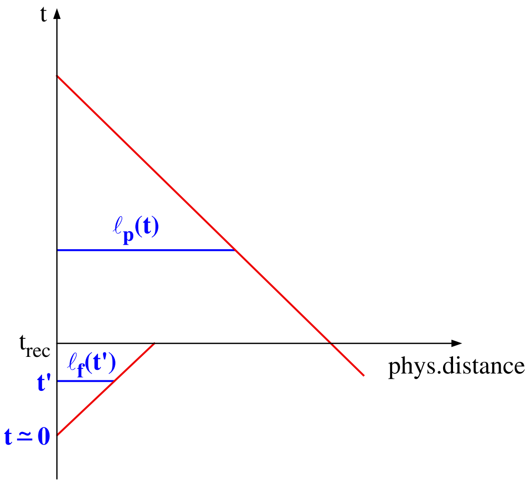

Fig. 3.1 Spacetime diagram illustrating the horizon problem.

Similarly, the extension $l_f(t)$ of the forward light cone at time t of a very early event ($t \simeq 0,\ z \simeq \infty$) is

$$l_f(t) = a(t) \int_0^t \frac{dt'}{a(t')} = \frac{1}{H_0(1+z)} \int_z^\infty \frac{dz'}{E(z')}. \tag{3.5}$$

For the present Universe (t_0) this becomes what is called the *particle horizon distance*

$$D_{\rm hor} = H_0^{-1} \int_0^\infty \frac{dz'}{E(z')}, \tag{3.6}$$

and gives the size of the *observable Universe*.

Analytical expressions for these distances are only available in special cases. For orientation we consider first the Einstein-de Sitter model ($K = 0,\ \Omega_\Lambda = 0,\ \Omega_M = 1$), for which $a(t) = a_0(t/t_0)^{2/3}$ and thus

$$D_{\rm hor} = 3t_0 = 2H_0^{-1},\ l_f(t) = 3t,\ \frac{l_p}{l_f} = \left(\frac{t_0}{t}\right)^{1/3} - 1 = \sqrt{1+z} - 1. \tag{3.7}$$

For a flat Universe a good fitting formula for cases of interest is (Hu and White)

$$D_{\rm hor} \simeq 2H_0^{-1} \frac{1 + 0.084 \ln \Omega_M}{\sqrt{\Omega_M}}. \tag{3.8}$$

It is often convenient to work with 'comoving distances', by rescaling distances referring to time t (like $l_p(t), l_f(t)$) with the factor $a(t_0)/a(t) = 1 + z$ to the present. We indicate this by the superscript c. For instance,

$$l_p^c(z) = \frac{1}{H_0} \int_0^z \frac{dz'}{E(z')}. \tag{3.9}$$

This distance is plotted in Fig. 3 of Chap. 0 as $D_{\rm com}(z)$. Note that for $a_0 = 1:\ l_f^c(\eta) = \eta,\ l_p^c(\eta) = \eta_0 - \eta$. Hence (3.5) gives the following relation between η and z:

$$\eta = \frac{1}{H_0} \int_z^\infty \frac{dz'}{E(z')}.$$

The number of causality distances on the cosmic photosphere

The number of causality distances at redshift z between two antipodal emission points is equal to $l_p(z)/l_f(z)$, and thus the ratio of the two integrals on the right of (3.4) and (3.5). We are particularly interested in this ratio at the time of last scattering with $z_{\rm rec} \simeq 1100$. Then we can use for the numerator a flat Universe with non-relativistic matter, while for the denominator we can neglect in the standard hot big bang model Ω_K and Ω_Λ. A reasonable estimate is already obtained by using the simple expression in (3.7), i.e., $z_{\rm rec}^{1/2} \approx 30$. A more accurate evaluation would increase this number to about 40. The length $l_f(z_{\rm rec})$ subtends an angle of about 1 degree (exercise). How can it be that there is such a large number of causally disconnected regions we see on the microwave sky all having the same temperature? This is what is meant by the *horizon problem* and was a troublesome mystery before the invention of inflation.

Vacuum-like energy and exponential expansion

This causality problem is potentially avoided, if $l_f(t)$ would be increased in the very early Universe as a result of different physics. If a vacuum-like energy density would dominate, the Universe would undergo an *exponential expansion*. Indeed, in this case the Friedmann equation is

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{\rm vac}, \; \rho_{\rm vac} \simeq const, \tag{3.10}$$

and has the solutions

$$a(t) \propto \begin{cases} \cosh\, H_{\rm vac}t & : \; k=1 \\ e^{H_{\rm vac}t} & : \; k=0 \\ \sinh\, H_{\rm vac}t & : \; k=1, \end{cases} \tag{3.11}$$

with

$$H_{\rm vac} = \sqrt{\frac{8\pi G}{3}\rho_{\rm vac}}\;. \tag{3.12}$$

Assume that such an exponential expansion starts for some reason at time t_i and ends at the *reheating time* t_e, after which standard expansion takes over. From

$$a(t) = a(t_i)e^{H_{\rm vac}(t-t_i)} \; (t_i < t < t_e), \tag{3.13}$$

for $k=0$ we get

$$l_f^c(t_e) \simeq a_0 \int_{t_i}^{t_e} \frac{dt}{a(t)} = \frac{a_0}{H_{\rm vac}a(t_i)}\left(1 - e^{-H_{\rm vac}\Delta t}\right) \simeq \frac{a_0}{H_{\rm vac}a(t_i)},$$

where $\Delta t := t_e - t_i$. We want to satisfy the condition $l_f^c(t_e) \gg l_p^c(t_e) \simeq H_0^{-1}$ (see (3.8), i.e.,

$$a_i H_{\rm vac} \ll a_0 H_0 \;\Leftrightarrow\; \frac{a_i}{a_e} \ll \frac{a_0 H_0}{a_e H_{\rm vac}} \tag{3.14}$$

or

$$e^{H_{\rm vac}\Delta t} \gg \frac{a_e H_{\rm vac}}{a_0 H_0} = \frac{H_{\rm eq}a_{\rm eq}}{H_0 a_0}\,\frac{H_{\rm vac}a_e}{H_{\rm eq}a_{\rm eq}}.$$

Here, eq indicates the values at the time $t_{\rm eq}$ when the energy densities of non-relativistic and relativistic matter were equal. We now use the Friedmann equation for $k=0$ and $w := p/\rho = {\rm const}$. From (0.46) it follows that in this case

$$Ha \propto a^{-(1+3w)/2},$$

and hence we arrive at

$$e^{H_{\rm vac}\Delta t} \gg \left(\frac{a_0}{a_{\rm eq}}\right)^{1/2}\left(\frac{a_{\rm eq}}{a_e}\right) = (1+z_{\rm eq})^{1/2}\left(\frac{T_e}{T_{\rm eq}}\right) = (1+z_{\rm eq})^{-1/2}\frac{T_{\rm Pl}}{T_0}\frac{T_e}{T_{\rm Pl}}, \tag{3.15}$$

where we used $aT = \text{const}$. So the number of e-folding periods during the inflationary period, $\mathcal{N} = H_{\rm vac}\Delta t$, should satisfy

$$\mathcal{N} \gg \ln\left(\frac{T_{\rm Pl}}{T_0}\right) - \frac{1}{2}\ln z_{\rm eq} + \ln\left(\frac{T_e}{T_{\rm Pl}}\right) \simeq 70 + \ln\left(\frac{T_e}{T_{\rm Pl}}\right). \tag{3.16}$$

For a typical GUT scale, $T_e \sim 10^{14}\ GeV$, we arrive at the condition $\mathcal{N} \gg 60$.

Such an exponential expansion would also solve the *flatness problem*, another worry of standard big bang cosmology. Let me recall how this problem arises.

The Friedmann equation (0.17) can be written as

$$(\Omega^{-1} - 1)\rho a^2 = -\frac{3k}{8\pi G} = \text{const.},$$

where

$$\Omega(t) := \frac{\rho(t)}{3H^2/8\pi G} \tag{3.17}$$

(ρ includes vacuum energy contributions). Thus

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\frac{\rho_0 a_0^2}{\rho a^2}. \tag{3.18}$$

Without inflation we have

$$\rho = \rho_{\rm eq}\left(\frac{a_{\rm eq}}{a}\right)^4 \ (z > z_{\rm eq}), \tag{3.19}$$

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^3 \ (z < z_{\rm eq}). \tag{3.20}$$

According to (0.47) $z_{\rm eq}$ is given by

$$1 + z_{\rm eq} = \frac{\Omega_M}{\Omega_R} \simeq 10^4\ \Omega_0 h_0^2. \tag{3.21}$$

Exercise: Derive the estimate on the right of (3.21).

For $z > z_{\rm eq}$ we obtain from (3.18) and (3.19)

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\frac{\rho_0 a_0^2}{\rho_{\rm eq} a_{\rm eq}^2}\frac{\rho_{\rm eq} a_{\rm eq}^2}{\rho a^2} = (\Omega_0^{-1} - 1)(1+z_{\rm eq})^{-1}\left(\frac{a}{a_{\rm eq}}\right)^2 \tag{3.22}$$

or

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)(1+z_{\rm eq})^{-1}\left(\frac{T_{\rm eq}}{T}\right)^2 \simeq 10^{-60}(\Omega_0^{-1} - 1)\left(\frac{T_{\rm Pl}}{T}\right)^2. \tag{3.23}$$

Let us apply this equation for $T = 1\,\text{MeV},\ \Omega_0 \simeq 0.2 - 0.3$. Then $\mid \Omega - 1 \mid \leq 10^{-15}$, thus the Universe was already incredibly flat at modest temperatures, not much higher than at the time of nucleosynthesis.

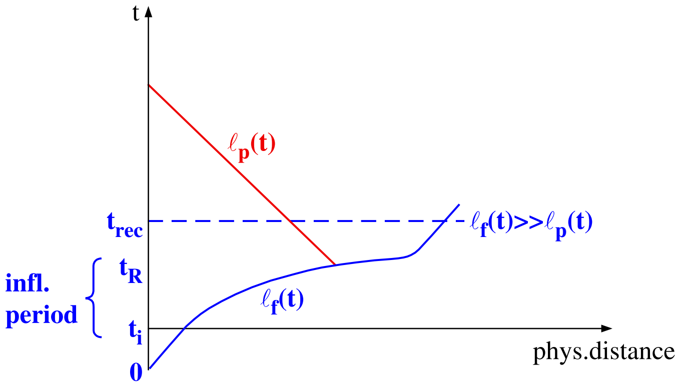

Fig. 3.2 Past and future light cones in models with an inflationary period.

Such a fine tuning must have a physical reason. This is naturally provided by inflation, because our observable Universe could originate from a small patch at t_e. (A tiny part of the Earth surface is also practically flat.)

Beside the horizon scale $l_f(t)$, the *Hubble length* $H^{-1}(t) = a(t)/\dot{a}(t)$ plays also an important role. One might call this the "microphysics horizon", because this is the maximal distance microphysics can operate coherently in one expansion time. It is this length scale which enters in basic evolution equations, such as the equation of motion for a scalar field (see Eq. (3.30) below).

We sketch in Figs. 3.2–3.4 the various length scales in inflationary models, that is for models with a period of accelerated (e.g., exponential) expansion. From these it is obvious that there can be – at least in principle – a *causal generation mechanism for perturbations*. This topic will be discussed in great detail in later parts of these lectures.

Exponential inflation is just an example. What we really need is an early phase during which the *comoving Hubble length decreases* (Fig. 3.4). This means that (for Friedmann spacetimes)

$$\boxed{\left(H^{-1}(t)/a\right)^{\cdot} < 0.} \tag{3.24}$$

This is the *general definition of inflation*; equivalently, $\ddot{a} > 0$ (accelerated expansion). For a Friedmann model Eq. (0.23) tells us that

$$\ddot{a} > 0 \Leftrightarrow p < -\rho/3. \tag{3.25}$$

This is, of course, not satisfied for 'ordinary' fluids.

Assume, as another example, *power-law inflation*: $a \propto t^p$. Then $\ddot{a} > 0 \Leftrightarrow p > 1$.

3.3 Scalar field models

Models with $p < -\rho/3$ are naturally obtained in scalar field theories. Most of the time we shall consider the simplest case of *one* neutral scalar field φ minimally coupled to gravity. Thus the Lagrangian density is assumed to be

$$\mathcal{L} = \frac{M_{pl}^2}{16\pi} R[g] - \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi), \tag{3.26}$$

where $R[g]$ is the Ricci scalar for the metric g. The scalar field equation is

$$\Box\varphi = V_{,\varphi}, \tag{3.27}$$

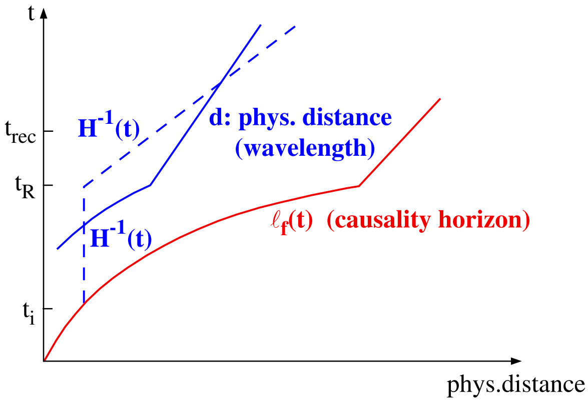

Fig. 3.3 Physical distance (e.g. between clusters of galaxies) and Hubble distance, and causality horizon in inflationary models.

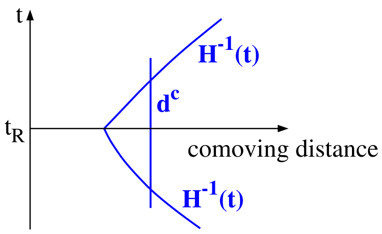

Fig. 3.4 Part of Fig. 3.3 expressed in terms of comoving distances.

and the energy-momentum tensor in the Einstein equation

$$G_{\mu\nu} = \frac{8\pi}{M_{\rm Pl}^2} T_{\mu\nu} \tag{3.28}$$

is

$$T_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi + g_{\mu\nu}\mathcal{L}_\varphi \tag{3.29}$$

($\mathcal{L}_\varphi$ is the scalar field part of (3.26)).

We consider first Friedmann spacetimes. Using previous notation, we obtain from (0.1)

$$\sqrt{-g} = a^3\sqrt{\gamma},\ \Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = -\frac{1}{a^3}(a^3\dot\varphi)^\cdot + \frac{1}{a^2}\triangle_\gamma\varphi.$$

The field equation (3.27) becomes

$$\boxed{\ddot\varphi + 3H\dot\varphi - \frac{1}{a^2}\triangle_\gamma\varphi = -V_{,\varphi}(\varphi).} \tag{3.30}$$

Note that the expansion of the Universe induces a ‘friction’ term. In this basic equation one also sees the appearance of the Hubble length. From (3.29) we obtain for the energy density and the pressure of the scalar field

$$\rho_\varphi = T_{00} = \frac{1}{2}\dot\varphi^2 + V + \frac{1}{2a^2}(\nabla\varphi)^2, \tag{3.31}$$

$$p_\varphi = \frac{1}{3}T^i{}_i = \frac{1}{2}\dot\varphi^2 - V - \frac{1}{6a^2}(\nabla\varphi)^2. \tag{3.32}$$

(Here, $(\nabla\varphi)^2$ denotes the squared gradient on the 3-space (Σ,γ).)

Suppose the gradient terms can be neglected, and that φ is during a certain phase slowly varying in time, then we get

$$\rho_\varphi \approx V,\ p_\varphi \approx -V. \tag{3.33}$$

Thus $p_\varphi \approx -\rho_\varphi$, as for a cosmological term.

Let us ignore for the time being the spatial inhomogeneities in the previous equations. Then these reduce to

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi}(\varphi) = 0; \tag{3.34}$$

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V,\ p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V. \tag{3.35}$$

Beside (3.34) the other dynamical equation is the Friedmann equation

$$\boxed{H^2 + \frac{K}{a^2} = \frac{8\pi}{3M_{\rm Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right].} \tag{3.36}$$

Eqs. (3.34) and (3.36) define a nonlinear dynamical system for the dynamical variables $a(t), \varphi(t)$, which can be studied in detail (see, e.g., [35]).

Let us ignore the curvature term K/a^2 in (3.36). Differentiating this equation and using (3.34) shows that

$$\dot{H} = -\frac{4\pi}{M_{\rm Pl}^2}\dot{\varphi}^2. \tag{3.37}$$

Regard H as a function of φ, then

$$\frac{dH}{d\varphi} = -\frac{4\pi}{M_{\rm Pl}^2}\dot{\varphi}. \tag{3.38}$$

This allows us to write the Friedmann equation as

$$\left(\frac{dH}{d\varphi}\right)^2 - \frac{12\pi}{M_{\rm Pl}^2}H^2(\varphi) = -\frac{32\pi^2}{M_{\rm Pl}^4}V(\varphi). \tag{3.39}$$

For a given potential $V(\varphi)$ this is a differential equation for $H(\varphi)$. Once this function is known, we obtain $\varphi(t)$ from (3.38) and $a(t)$ from (3.37).

3.3.1 Power-law inflation

We now proceed in the reverse order, assuming that $a(t)$ follows a power law

$$a(t) = \text{const. } t^p. \tag{3.40}$$

Then $H = p/t$, so by (3.37)

$$\dot{\varphi} = \sqrt{\frac{p}{4\pi}}M_{\rm Pl}\frac{1}{t},\ \varphi(t) = \sqrt{\frac{p}{4\pi}}M_{\rm Pl}\ln(t) + \text{const.},$$

hence

$$H \propto \exp\left(-\sqrt{\frac{4\pi}{p}}\frac{\varphi}{M_{\rm Pl}}\right). \tag{3.41}$$

Using this in (3.39) leads to an exponential potential

$$V(\varphi) = V_0\exp\left(-4\sqrt{\frac{\pi}{p}}\frac{\varphi}{M_{\rm Pl}}\right). \tag{3.42}$$

3.3.2 Slow-roll approximation

An important class of solutions is obtained in the slow-roll approximation (SLA), in which the basic Eqs. (3.34) and (3.36) can be replaced by

$$H^2 = \frac{8\pi}{3M_{\rm Pl}^2} V(\varphi), \tag{3.43}$$

$$3H\dot{\varphi} = -V_{,\varphi}. \tag{3.44}$$

A necessary condition for their validity is that the *slow-roll parameters*

$$\varepsilon_V(\varphi) := \frac{M_{\rm Pl}^2}{16\pi}\left(\frac{V_{,\varphi}}{V}\right)^2, \tag{3.45}$$

$$\eta_V(\varphi) := \frac{M_{\rm Pl}^2}{8\pi}\frac{V_{,\varphi\varphi}}{V} \tag{3.46}$$

are small:

$$\varepsilon_V \ll 1, \ | \ \eta_V \ | \ll 1. \tag{3.47}$$

These conditions, which guarantee that the potential is flat, are, however, not sufficient (for details, see Sect. 5.1.2).

The simplified system (3.43) and (3.44) implies

$$\boxed{\dot{\varphi}^2 = \frac{M_{\rm Pl}^2}{24\pi}\frac{1}{V}\left(V_{,\varphi}\right)^2.} \tag{3.48}$$

This is a differential equation for $\varphi(t)$.

Let us consider potentials of the form

$$V(\varphi) = \frac{\lambda}{n}\varphi^n. \tag{3.49}$$

Then Eq. (3.48) becomes

$$\boxed{\dot{\varphi}^2 = \frac{n^2 M_{\rm Pl}^2}{24\pi}\frac{1}{\varphi^2}V.} \tag{3.50}$$

Hence, (3.43) implies

$$\frac{\dot{a}}{a} = -\frac{4\pi}{nM_{\rm Pl}^2}(\varphi^2)^{\cdot},$$

and so

$$\boxed{a(t) = a_0 \exp\left[\frac{4\pi}{nM_{\rm Pl}^2}(\varphi_0^2 - \varphi^2(t))\right].} \tag{3.51}$$

We see from (3.50) that $\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi)$ for

$$\varphi \gg \frac{n}{4\sqrt{3\pi}} M_{\rm Pl}. \tag{3.52}$$

Consider first the example $n = 4$. Then (3.50) implies

$$\frac{\dot{\varphi}}{\varphi} = \sqrt{\frac{\lambda}{6\pi}} M_{\rm Pl} \;\Rightarrow \varphi(t) = \varphi_0 \exp\left(-\sqrt{\frac{\lambda}{6\pi}} M_{\rm Pl}\, t\right). \tag{3.53}$$

For $n \neq 4$:

$$\varphi(t)^{2-n/2} = \varphi_0^{2-n/2} + t\left(2 - \frac{n}{2}\right)\sqrt{\frac{n\lambda}{24\pi}} M_{\rm Pl}^{3-n/2}. \tag{3.54}$$

For the special case $n = 2$ this gives, using the notation $V = \frac{1}{2}m^2\varphi^2$, the simple result

$$\varphi(t) = \varphi_0 - \frac{mM_{\rm Pl}}{2\sqrt{3\pi}} t. \tag{3.55}$$

Inserting this into (3.51) provides the time dependence of $a(t)$.

3.4 Why did inflation start?

Attempts to answer this and related questions are *very speculative* indeed. A reasonable direction is to imagine random initial conditions and try to understand how inflation can emerge, perhaps generically, from such a state of matter. A.Linde first discussed a scenario along these lines which he called *chaotic inflation*. In the context of a single scalar field model he argued that typical initial conditions correspond to $\frac{1}{2}\dot{\varphi}^2 \sim \frac{1}{2}(\partial_i\varphi)^2 \sim V(\varphi) \sim 1$ (in Planckian units). The chance that the potential energy dominates in some domain of size $> \mathcal{O}(1)$ is presumably not very small. In this situation inflation could begin and $V(\varphi)$ would rapidly become even more dominant, which ensures continuation of inflation. Linde concluded from such considerations that chaotic inflation occurs under rather natural initial conditions. For this to happen, the form of the potential $V(\varphi)$ can even be a simple power law of the form (3.49). Many questions remain, however, open.

The chaotic inflationary Universe will look on very large scales – much larger than the present Hubble radius – extremely inhomogeneous. For a review of this scenario I refer to [36]. A much more extended discussion of inflationary models, including references, can be found in [4].

4 Cosmological perturbation theory for scalar field models

To keep this Chapter independent of the previous one, let us begin by repeating the set up of Sect. 3.3.

We consider a minimally coupled scalar field φ, with Lagrangian density

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi) \tag{4.1}$$

and corresponding field equation

$$\Box\varphi = U_{,\varphi}. \tag{4.2}$$

As a result of this the energy-momentum tensor

$$T^\mu{}_\nu = \partial^\mu\varphi\partial_\nu\varphi - \delta^\mu{}_\nu\left(\frac{1}{2}\partial^\lambda\varphi\partial_\lambda\varphi + U(\varphi)\right) \tag{4.3}$$

is covariantly conserved. In the general multi-component formalism (Sect. 1.4) we have, therefore, $Q_\varphi = 0$.

The unperturbed quantities ρ_φ, etc, are

$$\rho_\varphi = -T^0{}_0 = \frac{1}{2a^2}(\varphi')^2 + U(\varphi), \tag{4.4}$$

$$p_\varphi = \frac{1}{3}T^i{}_i = \frac{1}{2a^2}(\varphi')^2 - U(\varphi), \tag{4.5}$$

$$h_\varphi = \rho_\varphi + p_\varphi = \frac{1}{a^2}(\varphi')^2. \tag{4.6}$$

Furthermore,

$$\rho'_\varphi = -3\frac{a'}{a}h_\varphi. \tag{4.7}$$

It is not very sensible to introduce a "velocity of sound" c_φ.

4.1 Basic perturbation equations

Now we consider small deviations from the ideal Friedmann behavior:

$$\varphi \to \varphi_0 + \delta\varphi,\ \rho_\varphi \to \rho_\varphi + \delta\rho,\ etc. \tag{4.8}$$

(The index 0 is only used for the unperturbed field φ.) Since $L_\xi\varphi_0 = \xi^0\varphi'_0$ the gauge transformation of $\delta\varphi$ is

$$\delta\varphi \to \delta\varphi + \xi^0\varphi'_0. \tag{4.9}$$

Therefore,

$$\delta\varphi_\chi = \delta\varphi - \frac{1}{a}\varphi'_0\chi = \delta\varphi - \varphi'_0(B + E') \tag{4.10}$$

is gauge invariant (see (1.21)). Further perturbations are

$$\delta T^0{}_0 = -\frac{1}{a^2}\left[-\varphi_0'^2 A + \varphi'_0\delta\varphi' + U_{,\varphi}a^2\delta\varphi\right], \tag{4.11}$$

$$\delta T^0{}_i = -\frac{1}{a^2}\varphi'_0\delta\varphi_{,i}, \tag{4.12}$$

$$\delta T^i{}_j = -\frac{1}{a^2}[\varphi_0'^2 A - \varphi'_0\delta\varphi' + U_{,\varphi}a^2\delta\varphi]\delta^i{}_j. \tag{4.13}$$

This gives (recall (1.43))

$$\delta\rho = \frac{1}{a^2}[-\varphi_0'^2 A + \varphi'_0\delta\varphi' + a^2U_{,\varphi}\delta\varphi], \tag{4.14}$$

$$\delta p = p\pi_L = \frac{1}{a^2}[\varphi'_0\delta\varphi' - \varphi_0'^2 A - a^2U_{,\varphi}\delta\varphi], \tag{4.15}$$

$$\Pi = 0,\ \mathcal{Q} = -\dot\varphi_0\delta\varphi. \tag{4.16}$$

Einstein equations

We insert these expressions into the general perturbation equations (1.91)–(1.98) and obtain

$$\kappa = 3(HA - \dot D) - \frac{1}{a^2}\triangle\chi, \tag{4.17}$$

$$\frac{1}{a^2}(\triangle + 3K)D + H\kappa = -4\pi G[\dot\varphi_0\delta\dot\varphi - \dot\varphi_0^2 A + U_{,\varphi}\delta\varphi], \tag{4.18}$$

$$\kappa + \frac{1}{a^2}(\triangle + 3K)\chi = 12\pi G\dot\varphi_0\delta\varphi, \tag{4.19}$$

$$A + D = \dot{\chi} + H\chi \tag{4.20}$$

Eq. (1.95) is in the present notation

$$\dot{\kappa} + 2H\kappa = -\left(\frac{1}{a^2}\triangle + 3\dot{H}\right)A + 4\pi G[\delta\rho + 3\delta p],$$

with

$$\delta\rho + 3\delta p = 2(-2\dot{\varphi}_0^2 A + 2\dot{\varphi}_0\delta\dot{\varphi} - U_{,\varphi}\delta\varphi).$$

If we also use (recall (1.80))

$$\dot{H} = -4\pi G\dot{\varphi}_0^2 + \frac{K}{a^2}$$

we obtain

$$\dot{\kappa} + 2H\kappa = -\left(\frac{\triangle + 3K}{a^2} + 4\pi G\dot{\varphi}_0^2\right)A + 8\pi G(2\dot{\varphi}_0\delta\dot{\varphi} - U_{,\varphi}\delta\varphi). \tag{4.21}$$

The two remaining equations (1.97) and (1.98) are:

$$(\delta\rho)^{\cdot} + 3H(\delta\rho + \delta p) = (\rho + p)(\kappa - 3HA) - \frac{1}{a^2}\triangle\mathcal{Q}, \tag{4.22}$$

and

$$\dot{\mathcal{Q}} + 3H\mathcal{Q} = -(\rho + p)A - \delta p, \tag{4.23}$$

with the expressions (4.14) – (4.16). Since these last two equations express energy-momentum 'conservation', they are not independent of the others if we add the field equation for φ; we shall not make use of them below.

Eqs. (4.17)–(4.21) can immediately be written in a gauge invariant form:

$$\kappa_\chi = 3(HA_\chi - \dot{D}_\chi), \tag{4.24}$$

$$\frac{1}{a^2}(\triangle + 3K)D_\chi + H\kappa_\chi = -4\pi G[\dot{\varphi}_0\delta\dot{\varphi}_\chi - \dot{\varphi}_0^2 A_\chi + U_{,\varphi}\delta\varphi_\chi], \tag{4.25}$$

$$\kappa_\chi = 12\pi G\dot{\varphi}_0\delta\varphi_\chi, \tag{4.26}$$

$$A_\chi + D_\chi = 0 \tag{4.27}$$

$$\dot{\kappa}_\chi + 2H\kappa_\chi = -\left(\frac{\triangle + 3K}{a^2} + 4\pi G\dot{\varphi}_0^2\right)A_\chi + 8\pi G(2\dot{\varphi}_0\delta\dot{\varphi}_\chi - U_{,\varphi}\delta\varphi_\chi). \tag{4.28}$$

From now on we set $\mathbf{K} = \mathbf{0}$. Use of (4.27) then gives us the following four basic equations:

$$\kappa_\chi = 3(\dot{A}_\chi + HA_\chi), \tag{4.29}$$

$$\frac{1}{a^2}\triangle A_\chi - H\kappa_\chi = 4\pi G[\dot{\varphi}_0\delta\dot{\varphi}_\chi - \dot{\varphi}_0^2 A_\chi + U_{,\varphi}\delta\varphi_\chi], \tag{4.30}$$

$$\kappa_\chi = 12\pi G\dot{\varphi}_0\delta\varphi_\chi, \tag{4.31}$$

$$\dot{\kappa}_\chi + 2H\kappa_\chi = -\frac{1}{a^2}\triangle A_\chi - 4\pi G\dot{\varphi}_0^2 A_\chi + 8\pi G(2\dot{\varphi}_0\delta\dot{\varphi}_\chi - U_{,\varphi}\delta\varphi_\chi). \tag{4.32}$$

Recall also

$$4\pi G\dot{\varphi}_0^2 = -\dot{H}. \tag{4.33}$$

From (4.29) and (4.31) we get

$$\boxed{\dot{A}_\chi + HA_\chi = 4\pi G\dot{\varphi}_0\delta\varphi_\chi.} \tag{4.34}$$

The difference of (4.32) and (4.30) gives (using also (4.29))

$$(\dot{A}_\chi + HA_\chi)^{\cdot} + 3H(\dot{A}_\chi + HA_\chi) = 4\pi G(\dot{\varphi}_0\delta\dot{\varphi}_\chi - U_{,\varphi}\delta\varphi_\chi)$$

i.e.,

$$\boxed{\ddot{A}_\chi + 4H\dot{A}_\chi + (\dot{H} + 3H^2)A_\chi = 4\pi G(\dot{\varphi}_0\delta\dot{\varphi}_\chi - U_{,\varphi}\delta\varphi_\chi).} \tag{4.35}$$

Beside (4.34) and (4.35) we keep (4.30) in the form (making use of (4.33))

$$\boxed{\frac{1}{a^2}\triangle A_\chi - 3H\dot{A}_\chi - (\dot{H} + 3H^2)A_\chi = 4\pi G(\dot{\varphi}_0\delta\dot{\varphi}_\chi + U_{,\varphi}\delta\varphi_\chi).} \tag{4.36}$$

Scalar field equation

We now turn to the φ equation (4.2). Recall (the index 0 denotes in this subsection the t-coordinate)

$$g_{00} = -(1+2A),\ g_{0j} = -aB_{,j},\ g_{ij} = a^2[\gamma_{ij} + 2D\gamma_{ij} + 2E_{|ij}];$$
$$g^{00} = -(1-2A),\ g^{0j} = -\frac{1}{a}B^{,j},\ g^{ij} = \frac{1}{a^2}[\gamma^{ij} - 2D\gamma^{ij} - 2E^{|ij}];$$
$$\sqrt{-g} = a^3\sqrt{\gamma}(1 + A + 3D + \triangle E.$$

Up to first order we have (note that $\partial_j\varphi$ and g^{0j} are of first order)

$$\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{00}\dot{\varphi})^{\cdot} + \frac{1}{a^2}\triangle\delta\varphi - \frac{1}{a}\dot{\varphi}_0\triangle B.$$

Using the zeroth order field equation (3.34), we readily find

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\triangle + U_{,\varphi\varphi}\right)\delta\varphi =$$
$$(\dot{A} - 3\dot{D} - \triangle\dot{E} + 3HA - \frac{1}{a}\triangle B)\dot{\varphi}_0 - (3H\dot{\varphi}_0 + 2U_{,\varphi})A.$$

Recalling the definition of κ,

$$\kappa = 3(HA - \dot{D}) - \frac{1}{a}\triangle(B + a\dot{E}),$$

we finally obtain the perturbed field equation in the form

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\triangle + U_{,\varphi\varphi}\right)\delta\varphi = (\kappa + \dot{A})\dot{\varphi}_0 - (3H\dot{\varphi}_0 + 2U_{,\varphi})A. \tag{4.37}$$

By putting the index χ at all perturbation amplitudes one obtains a gauge invariant equation. Using also (4.29) one arrives at

$$\boxed{\delta\ddot{\varphi}_\chi + 3H\delta\dot{\varphi}_\chi + \left(-\frac{1}{a^2}\triangle + U_{,\varphi\varphi}\right)\delta\varphi_\chi = 4\dot{\varphi}_0\dot{A}_\chi - 2U_{,\varphi}A_\chi.} \tag{4.38}$$

Our basic – but not independent – equations are (4.34), (4.35), (4.36) and (4.38).

4.2 Consequences and reformulations

In (1.58) we have introduced the curvature perturbation (recall also (4.16))

$$\mathcal{R} := D_{\mathcal{Q}} = D_{\chi} - \frac{H}{\dot{\varphi}_0}\delta\varphi_{\chi} = D - \frac{H}{\dot{\varphi}_0}\delta\varphi. \tag{4.39}$$

It will turn out to be convenient to work also with

$$u = -z\mathcal{R},\ z := \frac{a\dot{\varphi}_0}{H}, \tag{4.40}$$

thus

$$u = a\left[\delta\varphi_{\chi} - \frac{\dot{\varphi}_0}{H}D_{\chi}\right] = a\left[\delta\varphi - \frac{\dot{\varphi}_0}{H}D\right]. \tag{4.41}$$

This amplitude will play an important role, because we shall obtain from the previous formulae the simple equation

$$\boxed{u'' - \triangle u - \frac{z''}{z}u = 0.} \tag{4.42}$$

This is a Klein-Gordon equation with a time-dependent mass.

We next rewrite the basic equations in terms of the conformal time:

$$\triangle A_{\chi} - 3\mathcal{H}A'_{\chi} - (\mathcal{H}' + 3\mathcal{H}^2)A_{\chi} = 4\pi G(\varphi'_0\delta\varphi'_{\chi} + U_{,\varphi}a^2\delta\varphi_{\chi}), \tag{4.43}$$

$$A'_{\chi} + \mathcal{H}A_{\chi} = 4\pi G\varphi'_0\delta\varphi_{\chi}, \tag{4.44}$$

$$A''_{\chi} + 3\mathcal{H}A'_{\chi} + (\mathcal{H}' + 2\mathcal{H}^2)A_{\chi} = 4\pi G(\varphi'_0\delta\varphi'_{\chi} - U_{,\varphi}a^2\delta\varphi_{\chi}), \tag{4.45}$$

$$\delta\varphi''_{\chi} + 2\mathcal{H}\delta\varphi'_{\chi} - \triangle\delta\varphi_{\chi} + U_{,\varphi\varphi}a^2\delta\varphi_{\chi} = 4\varphi'_0A'_{\chi} - 2U_{,\varphi}a^2A_{\chi}. \tag{4.46}$$

Let us first express u (or $\mathcal{R}$) in terms of A_{χ}. From (4.40), (4.39) we obtain in a first step

$$4\pi Gzu = 4\pi Gz^2A_{\chi} + 4\pi G\frac{z^2\mathcal{H}}{\varphi'_0}\delta\varphi_{\chi}.$$

For the first term on the right we use the unperturbed equation (see (4.33))

$$4\pi G\varphi_0^{\prime 2} = \mathcal{H}^2 - \mathcal{H}', \tag{4.47}$$

and in the second term we make use of (4.44). Collecting terms gives

$$\boxed{4\pi Gzu = \left(\frac{a^2A_{\chi}}{\mathcal{H}}\right)'.} \tag{4.48}$$

Next, we derive an equation for A_{χ} alone. For this we subtract (4.43) from (4.45) and use (4.44) to express $\delta\varphi_{\chi}$ in terms of A_{χ} and A'_{χ}. Moreover we make use of (4.47) and the unperturbed equation (3.34),

$$\varphi''_0 + 2\mathcal{H}\varphi'_0 + U_{,\varphi}(\varphi_0)a^2 = 0. \tag{4.49}$$

Detailed derivation: The quoted equations give

$$A''_\chi + 6\mathcal{H}A'_\chi - \triangle A_\chi + 2(\mathcal{H}' + 2\mathcal{H}^2)A_\chi =$$

$$-8\pi G U_{,\varphi} a^2 \delta\varphi_\chi = \frac{2}{\varphi'_0}(\varphi''_0 + 2\mathcal{H}\varphi'_0)(A'_\chi + \mathcal{H}A_\chi),$$

thus

$$A''_\chi + 2(\mathcal{H} - \varphi''_0/\varphi'_0)A'_\chi - \triangle A_\chi + 2(\mathcal{H}' - \mathcal{H}\varphi''_0/\varphi'_0)A_\chi = 0.$$

Rewriting the coefficients of A_χ, A'_χ slightly, we obtain the important equation:

$$\boxed{A''_\chi + 2\frac{(a/\varphi'_0)'}{a/\varphi'_0}A'_\chi - \triangle A_\chi + 2\varphi'_0(\mathcal{H}/\varphi'_0)'A_\chi = 0.} \tag{4.50}$$

Now we return to (4.48) and write this, using (4.47), as follows:

$$\frac{u}{z} = A_\chi + \frac{A'_\chi + \mathcal{H}A_\chi}{L}, \tag{4.51}$$

where

$$L = 4\pi G\frac{z^2\mathcal{H}}{a^2} = 4\pi G(\varphi'_0)^2/\mathcal{H} = \mathcal{H} - \mathcal{H}'/\mathcal{H}. \tag{4.52}$$

Differentiating (4.51) implies

$$\left(\frac{u}{z}\right)' = A'_\chi + \frac{A''_\chi + (\mathcal{H}A_\chi)'}{L} - \frac{A'_\chi + \mathcal{H}A_\chi}{L^2}L'$$

or, making use of (4.52) and (4.50),

$$L\left(\frac{u}{z}\right)' = (\mathcal{H} - \mathcal{H}'/\mathcal{H})A'_\chi - 2\frac{(a/\varphi'_0)'}{a/\varphi'_0}A'_\chi + \triangle A_\chi$$

$$-2\varphi'_0(\mathcal{H}/\varphi'_0)'A_\chi + (\mathcal{H}A_\chi)' - (A'_\chi + \mathcal{H}A_\chi)\frac{(\varphi_0'^2/\mathcal{H})'}{\varphi_0'^2/\mathcal{H}}.$$

From this one easily finds the simple equation

$$\boxed{4\pi G\frac{\mathcal{H}z^2}{a^2}\left(\frac{u}{z}\right)' = \triangle A_\chi.} \tag{4.53}$$

Finally, we derive the announced Eq. (4.42). To this end we rewrite the last equation as

$$\triangle A_\chi = 4\pi G\frac{\mathcal{H}}{a^2}(zu' - z'u),$$

from which we get

$$\triangle A'_\chi = 4\pi G\left(\frac{\mathcal{H}}{a^2}\right)'(zu' - z'u) + 4\pi G\frac{\mathcal{H}}{a^2}(zu'' - z''u).$$

Taking the Laplacian of (4.51) gives

$$4\pi G\frac{\mathcal{H}}{a^2}z\triangle u = L\triangle A_\chi + \triangle A'_\chi + \mathcal{H}\triangle A_\chi.$$

Combining the last two equations and making use of (4.52) shows that indeed (4.42) holds.

Summarizing, we have the basic equations

$$u'' - \triangle u - \frac{z''}{z} u = 0, \tag{4.54}$$

$$\triangle A_\chi = 4\pi G \frac{\mathcal{H}}{a^2} (zu' - z'u), \tag{4.55}$$

$$\left(\frac{a^2 A_\chi}{\mathcal{H}} \right)' = 4\pi G z u. \tag{4.56}$$

We now discuss some important consequences of these equations. The first concerns the curvature perturbation $\mathcal{R} = -u/z$ (original definition in (4.39)). In terms of this quantity Eq. (4.55) can be written as

$$\frac{\dot{\mathcal{R}}}{H} = \frac{1}{1 - \mathcal{H}'/\mathcal{H}^2} \frac{1}{(aH)^2} (-\triangle A_\chi). \tag{4.57}$$

The right-hand side is of order $(k/aH)^2$, hence very small on scales much larger than the Hubble radius. It is common practice to use the terms "Hubble length" and "horizon" interchangeably, and to call length scales satisfying $k/aH \ll 1$ to be *super-horizon*. (This can cause confusion; 'super-Hubble' might be a better term, but the jargon can probably not be changed anymore.)

We have studied $\dot{\mathcal{R}}$ already at the end of Sect.1.3. I recall (1.138):

$$\dot{\mathcal{R}} = \frac{H}{1+w} \left[\frac{2}{3} c_s^2 \frac{1}{(Ha)^2} \triangle D_\chi - w\Gamma - \frac{2}{3} w \triangle \Pi \right]. \tag{4.58}$$

This general equation also holds for our scalar field model, for which $\Pi = 0,\ D_\chi = -A_\chi$. The first term on the right in (4.58) is again small on super-horizon scales. So the non-adiabatic piece $p\Gamma = \delta p - c_s^2 \delta\rho$ must also be small on large scales. This means that the perturbations are **adiabatic**. We shall show this more directly further below, by deriving the following expression for Γ:

$$\boxed{p\Gamma = -\frac{U_{,\varphi}}{6\pi G H \dot{\varphi}} \frac{1}{a^2} \triangle A_\chi.} \tag{4.59}$$

After inflation, when relativistic fluids dominate the matter content, Eq. (4.58) still holds. The first term on the right is small on scales larger than the *sound horizon*. Since Γ and Π are then not important, we see that for super-horizon scales $\mathcal{R}$ *remains constant also after inflation*. This will become important in the study of CMB anisotropies.

Later, it will be useful to have a handy expression of A_χ in terms of $\mathcal{R}$. According to (1.58) and (1.57) we have

$$\mathcal{R} = D_\chi + \frac{\mathcal{H}}{a(\rho + p)} \mathcal{Q}. \tag{4.60}$$

We rewrite this by combining (1.99) and (1.101)

$$\mathcal{R} = D_\chi - \frac{\mathcal{H}}{4\pi G a^2 (\rho + p)} (\mathcal{H} A_\chi - D_\chi'). \tag{4.61}$$

At this point we specialize again to $K = 0$, and use (1.80) in the form

$$4\pi G a^2 (\rho + p) = \mathcal{H}^2 (1 - \mathcal{H}'/\mathcal{H}^2)$$

and obtain

$$\boxed{\mathcal{R} = D_\chi - \frac{1}{\varepsilon\mathcal{H}}(\mathcal{H}A_\chi - D'_\chi),} \tag{4.62}$$

where

$$\varepsilon := 1 - \mathcal{H}'/\mathcal{H}^2. \tag{4.63}$$

If $\Pi = 0$ then $D_\chi = -A_\chi$, so

$$\boxed{-\mathcal{R} = A_\chi + \frac{1}{\varepsilon\mathcal{H}}(\mathcal{H}A_\chi + A'_\chi),} \tag{4.64}$$

I claim that for a constant $\mathcal{R}$

$$A_\chi = -\left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}. \tag{4.65}$$

We prove this by showing that (4.65) satisfies (4.64). Differentiating the last equation gives by the same equation and (4.63) our claim.

As a special case we consider (always for $K = 0$) $w = \mathrm{const}$. Then, as shown in Sect.2.4,

$$a = a_0(\eta/\eta_0)^\beta,\ \beta = \frac{2}{3w+1}. \tag{4.66}$$

Thus

$$\frac{\mathcal{H}}{a^2}\int a^2 d\eta = \frac{\beta}{2\beta+1},$$

hence

$$\boxed{A_\chi = -\frac{3(w+1)}{3w+5}\mathcal{R}.} \tag{4.67}$$

This will be important later.

Derivation of (4.59): By definition

$$p\Gamma = \delta p - c_s^2\delta\rho,\ c_s^2 = \dot{p}/\dot{\rho} \Rightarrow p\Gamma = \frac{\dot{\rho}\delta p - \dot{p}\delta\rho}{\dot{\rho}}. \tag{4.68}$$

Now, by (4.7) and (4.5)

$$\dot{\rho} = -3H\dot{\varphi}^2,\ \dot{p} = \dot{\varphi}(\ddot{\varphi} - U_{,\varphi}) = -\dot{\varphi}(3H\dot{\varphi} + 2U_{,\varphi}),$$

and by (4.14) and (4.15)

$$\delta\rho = -\dot{\varphi}^2 A + \dot{\varphi}\delta\dot{\varphi} + U_{,\varphi}\delta\varphi\ ,\ \delta p = \dot{\varphi}\delta\dot{\varphi} - \dot{\varphi}^2 A - U_{,\varphi}\delta\varphi.$$

With these expressions one readily finds

$$p\Gamma = -\frac{2}{3}\frac{U_{,\varphi}}{H\dot{\varphi}}[-\ddot{\varphi}\delta\varphi + \dot{\varphi}(\delta\dot{\varphi} - \dot{\varphi}A)]. \tag{4.69}$$

Up to now we have not used the perturbed field equations. The square bracket on the right of the last equation appears in the combination (4.18)-$H\cdot$ (4.19) for the right hand sides. Since the right hand side of (4.69) must be gauge invariant, we can work in the gauge $\chi = 0$, and obtain (for $K = 0$) from (4.18),(4.19)

$$\frac{1}{a^2}\triangle A = 4\pi G[-\ddot{\varphi}\delta\varphi + \dot{\varphi}(\delta\dot{\varphi} - \dot{\varphi}A)],$$

thus (4.59) since in the longitudinal gauge $A = A_\chi$.

Application. We return to Eq. (4.57) and use there (4.59) to obtain

$$\boxed{\dot{\mathcal{R}} = 4\pi G\frac{\rho p}{\dot{U}}\Gamma.} \tag{4.70}$$

As a result of (4.59) Γ is small on super-horizon scales, and hence (4.70) tells us that $\mathcal{R}$ is almost constant (as we knew before).

The crucial conclusion is that the perturbations are *adiabatic*, which is not obvious (I think). For multi-field inflation this is, in general, not the case (see, e.g., [38]).

5 Quantization, primordial power spectra

The main goal of this Chapter is to derive the primordial power spectra that are generated as a result of quantum fluctuations during an inflationary period.

5.1 Power spectrum of the inflaton field

For the quantization of the scalar field that drives the inflation we note that the equation of motion (4.42) for the scalar perturbation (4.41),

$$u = a\left[\delta\varphi_\chi - \frac{\dot{\varphi}_0}{H}D_\chi\right] = a\left[\delta\varphi_\chi + \frac{\varphi_0'}{\mathcal{H}}A_\chi\right], \tag{5.1}$$

is the Euler-Lagrange equation for the effective action

$$S_{eff} = \frac{1}{2}\int d^3x d\eta\left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{5.2}$$

The normalization is chosen such that S_{eff} reduces to the correct action when gravity is switched off. (In [37] this action is obtained by considering the quadratic piece of the full action with Lagrange density (3.26), but this calculation is extremely tedious.)

The effective Lagrangian of (5.1) is

$$\mathcal{L} = \frac{1}{2}\left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{5.3}$$

This is just a free theory with a time-dependent mass $m^2 = -z''/z$. Therefore the quantization is straightforward. Once u is quantized the quantization of $\Psi = A_\chi$ is then also fixed (see Eq. (4.55)).

The canonical momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial u'} = u', \tag{5.4}$$

and the canonical commutation relations are the usual ones:

$$\left[\hat{u}(\eta,\mathbf{x}),\hat{u}(\eta,\mathbf{x}')\right] = \left[\hat{\pi}(\eta,\mathbf{x}),\hat{\pi}(\eta,\mathbf{x}')\right] = 0,\ \left[\hat{u}(\eta,\mathbf{x}),\hat{\pi}(\eta,\mathbf{x}')\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{5.5}$$

Let us expand the field operator $\hat{u}(\eta, \mathbf{x})$ in terms of eigenmodes $u_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}$ of Eq. (4.42), for which

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0. \tag{5.6}$$

The time-independent normalization is chosen to be

$$u_k^* u_k' - u_k u_k'^* = -i. \tag{5.7}$$

In the decomposition

$$\hat{u}(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int d^3k \left[u_k(\eta) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta) \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \tag{5.8}$$

the coefficients $\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger$ are annihilation and creation operators with the usual commutation relations:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{5.9}$$

With the normalization (5.7) these imply indeed the commutation relations (5.5). (Translate (5.8) with the help of (4.55) into a similar expansion of Ψ, whose mode functions are determined by $u_k(\eta)$.)

The modes $u_k(\eta)$ are chosen such that at very short distances ($k/aH \to \infty$) they approach the plane waves of the gravity free case with positive frequences

$$u_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{-ik\eta} \ (k/aH \gg 1). \tag{5.10}$$

In the opposite long-wave regime, where k can be neglected in (5.6), we see that the *growing mode* solution is

$$u_k \propto z \ (k/aH \ll 1), \tag{5.11}$$

i.e., u_k/z and thus $\mathcal{R}$ is constant on super-horizon scales. This has to be so on the basis of what we saw in Sect. 4.2. The power spectrum is conveniently defined in terms of $\mathcal{R}$. We have (we do not put a hat on $\mathcal{R}$)

$$\mathcal{R}(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int \mathcal{R}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k, \tag{5.12}$$

with

$$\mathcal{R}_{\mathbf{k}}(\eta) = \left[\frac{u_k(\eta)}{z} \hat{a}_{\mathbf{k}} + \frac{u_k^*(\eta)}{z} \hat{a}_{-\mathbf{k}}^\dagger \right]. \tag{5.13}$$

The *power spectrum* is defined by (see also Appendix A)

$$\langle 0 | \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'}^\dagger | 0 \rangle =: \frac{2\pi^2}{k^3} P_{\mathcal{R}}(k) \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{5.14}$$

From (5.13) we obtain

$$\boxed{P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \frac{|u_k(\eta)|^2}{z^2}.} \tag{5.15}$$

Below we shall work this out for the inflationary models considered in Chap. 4. Before, we should address the question why we considered the two-point correlation for the Fock vacuum relative to our choice of modes $u_k(\eta)$. A priori, the initial state could contain all kinds of excitations. These would, however, be redshifted away by the enormous inflationary expansion, and the final power spectrum on interesting scales, much larger than the Hubble length, should be largely independent of possible initial excitations. (This point should, perhaps, be studied in more detail.)

5.1.1 Power spectrum for power law inflation

For power law inflation one can derive an exact expression for (5.15). For the mode equation (5.6) we need z''/z. To compute this we insert in the definition (4.40) of z the results of Sect. 3.3.1, giving immediately $z \propto a(t) \propto t^p$. In addition (3.40) implies $t \propto \eta^{1/1-p}$, so $a(\eta) \propto \eta^{p/1-p}$. Hence,

$$\boxed{\frac{z''}{z} = \left(\nu^2 - \frac{1}{4}\right)\frac{1}{\eta^2}}, \tag{5.16}$$

where

$$\nu^2 - \frac{1}{4} = \frac{p(2p-1)}{(p-1)^2}. \tag{5.17}$$

Using this in (5.6) gives the mode equation

$$\boxed{u_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right)u_k = 0.} \tag{5.18}$$

This can be solved in terms of Bessel functions. Before proceeding with this we note two further relations that will be needed later. First, from $H = p/t$ and $a(t) = a_0 t^p$ we get

$$\eta = -\frac{1}{aH}\frac{1}{1 - 1/p}. \tag{5.19}$$

In addition,

$$\frac{z}{a} = \frac{\dot{\varphi}}{H} = \sqrt{\frac{p}{4\pi}}\frac{M_{\mathrm{Pl}}/t}{(p/t)} = \frac{1}{\sqrt{4\pi p}}M_{\mathrm{Pl}},$$

so

$$\boxed{\varepsilon := -\frac{\dot{H}}{H^2} = \frac{1}{p} = \frac{4\pi}{M_{\mathrm{Pl}}^2}\frac{z^2}{a^2}.} \tag{5.20}$$

Let us now turn to the mode equation (5.18). According to [39, 9.1.49], the functions $w(z) = z^{1/2}\mathcal{C}_\nu(\lambda z)$, $\mathcal{C}_\nu \propto H_\nu^{(1)}, H_\nu^{(2)}, ...$ satisfy the differential equation

$$w'' + \left(\lambda^2 - \frac{\nu^2 - 1/4}{z^2}\right)w = 0. \tag{5.21}$$

From the asymptotic formula for large z ([39, 9.2.3])

$$H_\nu^{(1)} \sim \sqrt{\frac{2}{\pi z}}e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \; (-\pi < \arg z < \pi), \tag{5.22}$$

we see that the correct solutions are

$$u_k(\eta) = \frac{\sqrt{\pi}}{2}e^{i(\nu + \frac{1}{2})\frac{\pi}{2}}(-\eta)^{1/2}H_\nu^{(1)}(-k\eta). \tag{5.23}$$

Indeed, since $-k\eta = (k/aH)(1 - 1/p)^{-1}$, $k/aH \gg 1$ means large $-k\eta$, hence (5.23) satisfies (5.10). Moreover, the Wronskian is normalized according to (5.7) (use 9.1.9 in [39]).

In what follows we are interested in modes which are well outside the horizon: $(k/aH) \ll 1$. In this limit we can use (9.1.9 in [39])

$$iH_\nu^{(1)}(z) \sim \frac{1}{\pi}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} \quad (z \to 0) \tag{5.24}$$

to find

$$u_k(\eta) \simeq 2^{\nu-3/2} e^{i(\nu-1/2)\pi/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\eta)^{-\nu+1/2}. \tag{5.25}$$

Therefore, by (5.19) and (5.20)

$$|u_k| = 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1-\varepsilon)^{\nu-1/2} \frac{1}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{-\nu+1/2}. \tag{5.26}$$

The form (5.26) will turn out to hold also in more general situations studied below, however, with a different ε. We write (5.26) as

$$|u_k| = C(\nu) \frac{1}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{-\nu+1/2}, \tag{5.27}$$

with

$$C(\nu) = 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1-\varepsilon)^{\nu-1/2} \tag{5.28}$$

(recall $\nu = \frac{3}{2} + \frac{1}{p-1}$).

The power spectrum is thus

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} \left|\frac{u_k(\eta)}{z^2}\right|^2 = \frac{k^3}{2\pi^2} \frac{1}{z^2} C^2(\nu) \frac{1}{2k} \left(\frac{k}{aH}\right)^{1-2\nu}. \tag{5.29}$$

For z we could use (5.20). There is, however, a formula which holds more generally: From the definition (4.40) of z and (3.38) we get

$$\boxed{z = -\frac{M_{\rm Pl}^2}{4\pi} \frac{a}{H} \frac{dH}{d\varphi}.} \tag{5.30}$$

Inserting this in the previous equation we obtain for the power spectrum on super-horizon scales

$$P_{\mathcal{R}}(k) = C^2(\nu) \frac{4}{M_{\rm Pl}^4} \frac{H^4}{(dH/d\varphi)^2} \left(\frac{k}{aH}\right)^{3-2\nu}. \tag{5.31}$$

For power-law inflation a comparison of (5.20) and (5.30) shows that

$$\frac{M_{\rm Pl}^2}{4\pi} \frac{(dH/d\varphi)^2}{H^2} = \frac{1}{p} = \varepsilon. \tag{5.32}$$

The asymptotic expression (5.31), valid for $k/aH \ll 1$, remains, as we know, constant in time[16]. Therefore, we can evaluate it at *horizon crossing* $k = aH$:

$$\boxed{P_{\mathcal{R}}(k) = C^2(\nu) \frac{4}{M_{\rm Pl}^4} \frac{H^4}{(dH/d\varphi)^2}\bigg|_{k=aH}.} \tag{5.33}$$

[16] Let us check this explicitly. Using (5.32) we can write (5.31) as

$$P_{\mathcal{R}}(k) = C^2(\nu) \frac{1}{\pi M_{\rm Pl}^2} \frac{H^2}{\varepsilon} \left(\frac{k}{aH}\right)^{3-2\nu},$$

and we thus have to show that $H^2(aH)^{2\nu-3}$ is time independent. This is indeed the case since $aH \propto 1/\eta,\ H = p/t,\ t \propto \eta^{1/(1-p)} \Rightarrow\ H \propto \eta^{-1/(1-p)}$.

We emphasize that this is *not* the value of the spectrum at the moment when the scale crosses outside the Hubble radius. We have just rewritten the asymptotic value for $k/aH \ll 1$ in terms of quantities at horizon crossing.

Note also that $C(\nu) \simeq 1$. The result (5.33) holds, as we shall see below, also in the slow-roll approximation.

5.1.2 Power spectrum in the slow-roll approximation

We now define two slow-roll parameters and rewrite them with the help of (3.37) and (3.38):

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi}{M_{\rm Pl}^2}\frac{\dot{\varphi}^2}{H^2} = \frac{M_{\rm Pl}^2}{4\pi}\left(\frac{dH/d\varphi}{H(\varphi)}\right)^2, \tag{5.34}$$

$$\delta = -\frac{\ddot{\varphi}}{H\dot{\varphi}} = \frac{M_{\rm Pl}^2}{4\pi}\frac{d^2H/d\varphi^2}{H} \tag{5.35}$$

($|\ \varepsilon\ |, |\ \delta\ | \ll 1$ in the slow-roll approximation). These parameters are approximately related to ε_U, η_U introduced in (3.45) and (3.46), as we now show. From (3.36) for $K = 0$ and (3.37) we obtain

$$H^2(1 - \frac{\varepsilon}{3}) = \frac{8\pi}{3M_{\rm Pl}^2}U(\varphi). \tag{5.36}$$

For small $|\ \varepsilon\ |$ we obtain from this the following approximate expressions for the slow-roll parameters:

$$\varepsilon \simeq \frac{M_{\rm Pl}^2}{16\pi}\left(\frac{U_{,\varphi}}{U}\right)^2, \tag{5.37}$$

$$\delta \simeq \frac{M_{\rm Pl}^2}{8\pi}\frac{U_{,\varphi\varphi}}{U} - \frac{M_{\rm Pl}^2}{16\pi}\left(\frac{U_{,\varphi}}{U}\right)^2. \tag{5.38}$$

(In the literature the letter η is often used instead of δ, but η is already occupied for the conformal time.)

We use these small parameters to approximate various quantities, such as the effective mass z''/z.

First, we note that (5.34) and (5.30) imply the relations[17]

$$\varepsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{4\pi}{M_{\rm Pl}^2}\frac{z^2}{a^2}. \tag{5.39}$$

According to (5.35) we have $\delta = 1 - \varphi''/\varphi'\mathcal{H}$. For the last term we obtain from the definition $z = a\varphi'/\mathcal{H}$

$$\frac{\varphi''}{\varphi'\mathcal{H}} = \frac{z'}{z\mathcal{H}} - (1 - \mathcal{H}'/\mathcal{H}^2).$$

Hence

$$\delta = 1 + \varepsilon - \frac{z'}{z\mathcal{H}}. \tag{5.40}$$

Next, we look for a convenient expression for the conformal time. From (5.39) we get

$$\frac{\varepsilon}{a\mathcal{H}}da = \varepsilon d\eta = d\eta - (\mathcal{H}'/\mathcal{H}^2)d\eta = d\eta + d\left(\frac{1}{\mathcal{H}}\right),$$

[17] Note also that

$$\frac{\ddot{a}}{a} \equiv \dot{H} + H^2 = (1-\varepsilon)H^2,$$

so $\ddot{a} > 0$ for $\varepsilon < 1$.

so

$$\eta = -\frac{1}{\mathcal{H}} + \int \frac{\varepsilon}{a\mathcal{H}} da. \tag{5.41}$$

Now we determine z''/z to first order in ε and δ. From (5.40), i.e., $z'/z = \mathcal{H}(1+\varepsilon-\delta)$, we get

$$\frac{z''}{z} - \left(\frac{z'}{z}\right)^2 = (\varepsilon' - \delta')\mathcal{H} + (1+\varepsilon-\delta)\mathcal{H}',$$

hence

$$z''/z = \mathcal{H}^2 \left[\frac{\varepsilon' - \delta'}{\mathcal{H}} + (1+\varepsilon-\delta)(2-\delta)\right]. \tag{5.42}$$

We can consider ε', δ' as of second order: For instance, by (5.39)

$$\varepsilon' = \frac{4\pi}{M_{\rm Pl}^2} \frac{2zz'}{a^2} - 2\varepsilon\mathcal{H}$$

or

$$\varepsilon' = 2\mathcal{H}\varepsilon(\varepsilon - \delta). \tag{5.43}$$

Treating ε, δ as constant, Eq. (5.41) gives $\eta = -(1/\mathcal{H}) + \varepsilon\eta$, thus

$$\eta = -\frac{1}{\mathcal{H}} \frac{1}{1-\varepsilon}. \tag{5.44}$$

This generalizes (5.19), in which $\varepsilon = 1/p$ (see (5.20)). Using this in (5.42) we obtain to first order

$$\frac{z''}{z} = \frac{1}{\eta^2}(2 + 2\varepsilon - 3\delta).$$

We write this as (5.16), but with a different ν:

$$\frac{z''}{z} = \left(\nu^2 - \frac{1}{4}\right)\frac{1}{\eta^2}\,,\ \nu := \frac{1+\varepsilon-\delta}{1-\varepsilon} + \frac{1}{2}. \tag{5.45}$$

As a result of all this we can immediately write down the power spectrum in the slow-roll approximation. From the derivation it is clear that the formula (5.33) still holds, and the same is true for (5.28). Since ν is close to 3/2 we have $C(\nu) \simeq 1$. In sufficient approximation we thus finally obtain the important result:

$$\boxed{P_{\mathcal{R}}(k) = \frac{4}{M_{\rm Pl}^4} \frac{H^4}{(dH/d\varphi)^2}\bigg|_{k=aH} = \frac{1}{\pi M_{\rm Pl}^2} \frac{H^2}{\varepsilon} \left(\frac{k}{aH}\right)^{3-2\nu}.} \tag{5.46}$$

This spectrum is *nearly scale-free*. This is evident if we use the formula (5.31), from which we get

$$\boxed{n - 1 := \frac{d\ln P_{\mathcal{R}(k)}}{d\ln k} = 3 - 2\nu = 2\delta - 4\varepsilon,} \tag{5.47}$$

so n is *close to unity*.

Exercise. Show that (5.47) follows also from (5.46).

Solution: In a first step we get

$$n-1=\frac{d}{d\varphi}\ln\left[\left.\frac{H^4}{(dH/d\varphi)^2}\right|_{k=aH}\right]\frac{d\varphi}{d\ln k}.$$

For the last factor we note that $k=aH$ implies

$$d\ln k=\frac{da}{a}+\frac{dH}{H}\Rightarrow\frac{d\ln k}{d\varphi}=\frac{H}{\dot\varphi}+\frac{dH/d\varphi}{H}$$

or, with (3.37),

$$\frac{d\ln k}{d\varphi}=\frac{4\pi}{M_{\rm Pl}^2}\frac{H}{dH/d\varphi}\left[\frac{M_{\rm Pl}^2}{4\pi}\left(\frac{dH/d\varphi}{H}\right)^2-1\right].$$

Hence, using (5.34),

$$\frac{d\varphi}{d\ln k}=\frac{M_{\rm Pl}^2}{4\pi}\frac{dH/d\varphi}{H}\frac{1}{\varepsilon-1}.$$

Therefore,

$$n-1=\frac{M_{\rm Pl}^2}{4\pi}\frac{dH/d\varphi}{H}\frac{1}{\varepsilon-1}\left[4\frac{dH/d\varphi}{H}-2\frac{d^2H/d\varphi^2}{dH/d\varphi}\right]=\frac{1}{\varepsilon-1}(4\varepsilon-2\delta)$$

by (5.34) and (5.35).

5.1.3 Power spectrum for density fluctuations

Let $P_\Phi(k)$ be the power spectrum for the Bardeen potential $\Phi=D_\chi$. The latter is related to the density fluctuation Δ by the Poisson equation (2.3),

$$k^2\Phi=4\pi G\rho a^2\Delta. \tag{5.48}$$

Recall also that for $\Pi=0$ we have $\Phi=-\Psi$ $(=-A_\chi)$, and according to (4.67) the following relation for a period with $w=\text{const}$.

$$\boxed{\Phi=\frac{3(w+1)}{3w+5}\mathcal{R},} \tag{5.49}$$

and thus

$$P_\Phi^{1/2}(k)=\frac{3(w+1)}{3w+5}P_{\mathcal{R}}^{1/2}(k). \tag{5.50}$$

Inserting (5.46) gives for the *primordial* spectrum on super-horizon scales

$$P_\Phi(k)=\left[\frac{3(w+1)}{3w+5}\right]^2\frac{4}{M_{\rm Pl}^4}\left.\frac{H^4}{(dH/d\varphi)^2}\right|_{k=aH}. \tag{5.51}$$

From (5.48) we obtain

$$\Delta(k)=\frac{2(w+1)}{3w+5}\left(\frac{k}{aH}\right)^2\mathcal{R}(k), \tag{5.52}$$

and thus for the power spectrum of Δ:

$$P_\Delta(k) = \frac{4}{9}\left(\frac{k}{aH}\right)^4 P_\Phi(k) = \frac{4}{9}\left[\frac{3(w+1)}{3w+5}\right]^2\left(\frac{k}{aH}\right)^4 P_{\mathcal{R}}(k). \tag{5.53}$$

During the plasma era until recombination the primordial spectra (5.46) and (5.51) are modified in a way that will be studied in Part III of these lectures. The modification is described by the so-called *transfer function*[18] $T(k,z)$, normalized such that $T(k) \simeq 1$ for $(k/aH) \ll 1$. Including this, we have in the (dark) matter dominated era (in particular at the time of recombination)

$$P_\Delta(k) = \frac{4}{25}\left(\frac{k}{aH}\right)^4 P_{\mathcal{R}}^{\rm prim}(k) T^2(k), \tag{5.54}$$

where $P_{\mathcal{R}}^{\rm prim}(k)$ denotes the primordial spectrum ((5.46) for our simple model of inflation).

Remark. Using the fact that $\mathcal{R}$ is constant on super-horizon scales allows us to establish the relation between $\Delta_H(k) := \Delta(k,\eta)\mid_{k=aH}$ and $\Delta(k,\eta)$ on these scales. From (5.52) we see that

$$\Delta(k,\eta) = \left(\frac{k}{aH}\right)^2 \Delta_H(k). \tag{5.55}$$

In particular, if $\mid \mathcal{R}(k) \mid \propto k^{n-1}$, thus $\mid \Delta(k,\eta)\mid^2 = Ak^{n+3}$, then

$$\boxed{\mid \Delta_H(k) \mid^2 = Ak^{n-1},} \tag{5.56}$$

and this is *independent* of k for $n = 1$. In this case the density fluctuation for each mode at horizon crossing has the same magnitude. This explains why the case $n = 1$ – also called the *Harrison-Zel'dovich spectrum* – is called *scale free*.

5.2 Generation of gravitational waves

In this section we determine the power spectrum of gravitational waves by quantizing tensor perturbations of the metric.

These are parametrized as follows

$$g_{\mu\nu} = a^2(\eta)[\gamma_{\mu\nu} + 2H_{\mu\nu}], \tag{5.57}$$

where $a^2(\eta)\gamma_{\mu\nu}$ is the Friedmann metric ($\gamma_{\mu 0} = 0,\ \gamma_{ij}$: metric of (Σ,γ)), and $H_{\mu\nu}$ satisfies the *transverse traceless* (TT) gauge conditions

$$H_{00} = H_{0i} = H^i{}_i = H_i{}^j{}_{|j} = 0. \tag{5.58}$$

The tensor perturbation amplitudes H_{ij} remain invariant under gauge transformations (1.14). Indeed, as in Sect. 1.14, one readily finds

$$L_\xi g^{(0)} = 2a^2(\eta)\left\{-(\mathcal{H}\xi^0 + (\xi^0)')d\eta^2 + (\xi_i' - \xi^0{}_{|i})dx^i d\eta + (\mathcal{H}\gamma_{ij}\xi^0 + \xi_{i|j})dx^i dx^j\right\}.$$

Decomposing ξ^μ into scalar and vector parts gives the scalar and vector contributions of $L_\xi g^{(0)}$, but there are obviously *no* tensor contributions.

[18] For more on this, see Sect. 6.2.4, where the z-dependence of $T(k,z)$ is explicitly split off.

The perturbations of the Einstein tensor belonging to $H_{\mu\nu}$ are derived in the Appendix to this Chapter. The result is:

$$\delta G^0{}_0 = \delta G^0{}_j = \delta G^i{}_0 = 0,$$
$$\delta G^i{}_j = \frac{1}{a^2}\left[(H^i{}_j)'' + 2\frac{a'}{a}(H^i{}_j)' + (-\triangle + 2K)H^i{}_j\right]. \tag{5.59}$$

We claim that the quadratic part of the Einstein-Hilbert action is

$$S^{(2)} = \frac{M_{\rm Pl}^2}{16\pi}\int\left[(H^i{}_k)'(H^k{}_i)' - H^i{}_{k|l}H^k{}_i{}^{|l} - 2KH^i{}_kH^k{}_i\right]a^2(\eta)d\eta\sqrt{\gamma}d^3x. \tag{5.60}$$

(Remember that the indices are raised and lowered with γ_{ij}.) Note first that $\sqrt{-g}d^4x = \sqrt{\gamma}a^4(\eta)d\eta d^3x+$ quadratic terms in H_{ij}, because H_{ij} is traceless. A direct derivation of (5.60) from the Einstein-Hilbert action would be extremely tedious (see [37]). It suffices, however, to show that the variation of (5.60) is just the linearization of the general variation formula (see Sect. 2.3 of [1])

$$\delta S = -\frac{M_{\rm Pl}^2}{16\pi}\int G^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}d^4x \tag{5.61}$$

for the Einstein-Hilbert action

$$S = \frac{M_{\rm Pl}^2}{16\pi}\int R\sqrt{-g}d^4x. \tag{5.62}$$

Now, we have after the usual partial integrations,

$$\delta S^{(2)} = -\frac{M_{\rm Pl}^2}{8\pi}\int\left[\frac{(a^2H^i{}_k)')'}{a^2} + (-\triangle + 2K)H^i{}_k\right]\delta H^k{}_i a^2(\eta)d\eta\sqrt{\gamma}d^3x.$$

Since $\delta H^k{}_i = \frac{1}{2}\delta g^k{}_i$ this is, with the expression (5.59), indeed the linearization of (5.61).

We absorb in (5.60) the factor $a^2(\eta)$ by introducing the rescaled perturbation

$$P^i{}_j(x) := \left(\frac{M_{\rm Pl}^2}{8\pi}\right)^{1/2}a(\eta)H^i{}_j(x). \tag{5.63}$$

Then $S^{(2)}$ becomes, after another partial integration,

$$\boxed{S^{(2)} = \frac{1}{2}\int\left[(P^i{}_k)'(P^k{}_i)' - P^i{}_{k|l}P^k{}_i{}^{|l} + \left(\frac{a''}{a} - 2K\right)P^i{}_kP^k{}_i\right]d\eta\sqrt{\gamma}d^3x.} \tag{5.64}$$

In what follows we take again $K = 0$. Then we have the following Fourier decomposition: Let $\epsilon_{ij}(\mathbf{k},\lambda)$ be the two polarization tensors, satisfying

$$\epsilon_{ij} = \epsilon_{ji},\ \epsilon^i{}_i = 0,\ k^i\epsilon_{ij}(\mathbf{k},\lambda) = 0,\ \epsilon_i{}^j(\mathbf{k},\lambda)\epsilon_j{}^i(\mathbf{k},\lambda)^* = \delta_{\lambda\lambda'},$$
$$\epsilon_{ij}(-\mathbf{k},\lambda) = \epsilon^*_{ij}(\mathbf{k},\lambda), \tag{5.65}$$

then

$$P^i{}_j(\eta,\mathbf{x}) = (2\pi)^{-3/2}\int d^3k\sum_\lambda v_{\mathbf{k},\lambda}(\eta)\epsilon^i{}_j(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.66}$$

The field is now quantized by interpreting $v_{\mathbf{k},\lambda}(\eta)$ as the operator

$$\hat{v}_{\mathbf{k},\lambda}(\eta) = v_k(\eta)\hat{a}_{\mathbf{k},\lambda} + v_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k},\lambda}, \tag{5.67}$$

where $v_k(\eta)\epsilon_{ij}(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}$ satisfies the field equation[19] corresponding to the action (5.64), that is (for $K = 0$)

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0. \tag{5.68}$$

(Instead of z''/z in (5.6) we now have the "mass" a''/a.)

In the long-wavelength regime the growing mode now behaves as $v_k \propto a$, hence v_k/a remains constant.

Again we have to impose the normalization (5.7):

$$v_k^* v_k' - v_k v_k'^* = -i, \tag{5.69}$$

and the asymptotic behavior

$$v_k(\eta) \sim \frac{1}{\sqrt{2k}} e^{-ik\eta} \ (k/aH \gg 1). \tag{5.70}$$

The decomposition (5.66) translates to

$$H^i{}_j(\eta,\mathbf{x}) = (2\pi)^{-3/2} \int d^3k \sum_\lambda \hat{h}_{\mathbf{k},\lambda}(\eta)\epsilon^i{}_j(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.71}$$

where

$$\hat{h}_{\mathbf{k},\lambda}(\eta) = \left(\frac{8\pi}{M_{\rm Pl}^2}\right)^{1/2} \frac{1}{a}\hat{v}_{\mathbf{k},\lambda}(\eta). \tag{5.72}$$

We define the *power spectrum of gravitational waves* by

$$\frac{2\pi^2}{k^3} P_g(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}') = \sum_\lambda \langle 0|\hat{h}_{\mathbf{k},\lambda}\hat{h}^\dagger_{\mathbf{k}',\lambda}|0\rangle \tag{5.73}$$

thus

$$\sum_\lambda \langle 0|\hat{v}_{\mathbf{k},\lambda}\hat{v}^\dagger_{\mathbf{k}',\lambda}|0\rangle = \frac{M_{\rm Pl}^2 a^2}{8\pi}\frac{2\pi^2}{k^3} P_g(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{5.74}$$

Using (5.67) for the left-hand side we obtain instead of (5.15)[20]

$$\boxed{P_g(k) = 2\frac{8\pi}{M_{\rm Pl}^2 a^2}\frac{k^3}{2\pi^2}|v_k(\eta)|^2.} \tag{5.75}$$

The factor 2 on the right is due to the two polarizations.

[19] We ignore possible tensor contributions to the energy-momentum tensor

[20] In the literature one often finds an expression for $P_g(k)$ which is 4 times larger, because the power spectrum is defined in terms of $h_{ij} = 2H_{ij}$.

5.2.1 Power spectrum for power-law inflation

For the modes $v_k(\eta)$ we need a''/a. From

$$\frac{a''}{a} = (a\mathcal{H})'/a = \mathcal{H}^2 + \mathcal{H}' = 2\mathcal{H}^2 \left[1 - \frac{1}{2}(1 - \mathcal{H}'/\mathcal{H}^2)\right]$$

and (5.39) we obtain the generally valid formula

$$\frac{a''}{a} = 2\mathcal{H}^2(1 - \varepsilon/2). \tag{5.76}$$

For power-law inflation we had $\varepsilon = 1/p,\ a(\eta) \propto \eta^{p/(1-p)}$, thus

$$\mathcal{H} = \frac{p}{p-1}\frac{1}{\eta}$$

and hence

$$\frac{a''}{a} = \left(\mu^2 - \frac{1}{4}\right)\frac{1}{\eta^2},\ \mu := \frac{3}{2} + \frac{1}{p-1}. \tag{5.77}$$

This shows that for power-law inflation $v_k(\eta)$ is identical to $u_k(\eta)$. Therefore, we have by Eq. (5.27)

$$|v_k| = C(\mu)\frac{1}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{-\mu+1/2}, \tag{5.78}$$

with

$$C(\mu) = 2^{\mu-3/2}\frac{\Gamma(\mu)}{\Gamma(3/2)}(1-\varepsilon)^{\mu-1/2}. \tag{5.79}$$

Inserting this in (5.75) gives

$$P_g(k) = \frac{16\pi}{M_{\rm Pl}^2}\frac{k^3}{2\pi^2}\frac{1}{a^2}C^2(\mu)\frac{1}{2k}\left(\frac{k}{aH}\right)^{1-2\mu}. \tag{5.80}$$

or

$$P_g(k) = C^2(\mu)\frac{4}{\pi}\left(\frac{H}{M_{\rm Pl}}\right)^2\left(\frac{k}{aH}\right)^{1-2\mu}. \tag{5.81}$$

Alternatively, we have

$$\boxed{P_g(k) = C^2(\mu)\frac{4}{\pi}\left.\frac{H^2}{M_{\rm Pl}^2}\right|_{k=aH}.} \tag{5.82}$$

5.2.2 Slow-roll approximation

From (5.76) and (5.44) we obtain again the first equation in (5.77), but with a different μ:

$$\mu = \frac{1}{1-\varepsilon} + \frac{1}{2}. \tag{5.83}$$

Hence $v_k(\eta)$ is equal to $u_k(\eta)$ if ν is replaced by μ. The formula (5.82), with $C(\mu)$ given by (5.79), remains therefore valid, but now μ is given by (5.83), where ε is the slow-roll parameter in (5.34) or (5.39). Again $C(\mu) \simeq 1$.

The power index for tensor perturbations,

$$n_T(k) := \frac{d \ln P_g(k)}{d \ln k}, \tag{5.84}$$

can be read off from (5.81):

$$\boxed{n_T \simeq -2\varepsilon,} \tag{5.85}$$

showing that the *power spectrum is almost flat*[21].

Consistency equation

Let us collect some of the important formulas:

$$A_S(k) := \frac{2}{5} P_{\mathcal{R}}^{1/2}(k) = \frac{4}{5} \left. \frac{H^2}{M_{\rm Pl}^2 |dH/d\varphi|} \right|_{k=aH}, \tag{5.86}$$

$$A_T(k) := \frac{1}{5} P_g^{1/2}(k) = \frac{2}{5\sqrt{\pi}} \left. \frac{H}{M_{\rm Pl}} \right|_{k=aH}, \tag{5.87}$$

$$n - 1 = 2\delta - 4\varepsilon, \tag{5.88}$$

$$n_T = -2\varepsilon. \tag{5.89}$$

The relative amplitude of the two spectra (scalar and tensor) is thus given by

$$\boxed{\frac{A_T^2}{A_S^2} = \varepsilon \left(\frac{P_g}{P_{\mathcal{R}}} = 4\varepsilon \right).} \tag{5.90}$$

More importantly, we obtain the *consistency condition*

$$\boxed{n_T = -2 \frac{A_T^2}{A_S^2},} \tag{5.91}$$

which is characteristic for inflationary models. In principle this can be tested with CMB measurements, but there is a long way before this can be done in practice.

5.2.3 Stochastic gravitational background radiation

The spectrum of gravitational waves, generated during the inflationary era and stretched to astronomical scales by the expansion of the Universe, contributes to the background energy density. Using the results of the previous section we can compute this.

I first recall a general formula for the effective energy-momentum tensor of gravitational waves. (For detailed derivations see Sect. 4.4 of [1].)

By 'gravitational waves' we mean propagating ripples in curvature on scales much smaller than the characteristic scales of the background spacetime (the Hubble radius for the situation under study). For

[21] The result (5.86) can also be obtained from (5.82). Making use of an intermediate result in the solution of the Exercise on p.88 and (5.34), we get

$$n_T = \frac{d \ln H^2}{d\varphi} \frac{d\varphi}{d \ln k} = \frac{2\varepsilon}{\varepsilon - 1} \simeq -2\varepsilon.$$

sufficiently high frequency waves it is meaningful to associate them – in an *averaged* sense – an energy-momentum tensor. Decomposing the full metric $g_{\mu\nu}$ into a background $\bar{g}_{\mu\nu}$ plus fluctuation $h_{\mu\nu}$, the effective energy-momentum tensor is given by the following expression

$$T^{(GW)}_{\alpha\beta} = \frac{1}{32\pi G} \left\langle h_{\mu\nu|\alpha} h^{\mu\nu}{}_{|\beta} \right\rangle, \tag{5.92}$$

if the gauge is chosen such that $h^{\mu\nu}{}_{|\nu} = 0,\ h^{\mu}{}_{\mu} = 0$. Here, a vertical stroke indicates covariant derivatives with respect to the background metric, and $\langle \cdots \rangle$ denotes a four-dimensional average over regions of several wave lengths.

For a Friedmann background we have in the TT gauge for $h_{\mu\nu} = 2H_{\mu\nu} : h_{\mu 0} = 0,\ h_{ij|0} = h_{ij,0}$, thus

$$T^{GW)}_{00} = \frac{1}{8\pi G} \left\langle \dot{H}_{ij} \dot{H}^{ij} \right\rangle. \tag{5.93}$$

As in (5.71) we perform (for $K = 0$) a Fourier decomposition

$$H_{ij}(\eta, \mathbf{x}) = (2\pi)^{-3/2} \int d^3k \sum_{\lambda} h_{\lambda}(\eta, \mathbf{k}) \epsilon_{ij}(\mathbf{k}, \lambda) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.94}$$

The gravitational background energy density, ρ_g, is obtained by taking the space-time average in (5.93). At this point we regard $h_{\lambda}(\eta, \mathbf{k})$ as a random field, indicated by a hat (since it is on macroscopic scales equivalent to the original quantum field $\hat{h}_{\lambda}(\eta, \mathbf{k})$), and replace the spatial average by the *stochastic average* (for which we use the same notation). Clearly, this is only justified if some *ergodicity* property holds. This issue will appear again in Part III, and we shall devote Appendix C for some clarifications.

If we adopt this procedure we obtain, anticipating the δ-function in (5.96),

$$\rho_g = \frac{1}{8\pi G a^2 (2\pi)^3} \int d^3k d^3k' \sum_{\lambda} \left\langle \left\langle \hat{h}'_{\lambda}(\eta, \mathbf{k}) \hat{h}'^{\star}_{\lambda}(\eta, \mathbf{k}') \right\rangle \right\rangle. \tag{5.95}$$

Here, the average on the right includes also an average over several periods. (As always, a dot denotes the derivative with respect to the cosmic time t, thus $\dot{h} = h'/a$.) Using (5.72) and (5.67) we obtain for the statistical average

$$\sum_{\lambda} \left\langle \hat{h}'_{\lambda}(\eta, \mathbf{k}) \hat{h}'^{*}_{\lambda}(\eta, \mathbf{k}') \right\rangle = 2 |h'_k(\eta)|^2 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{5.96}$$

where (see (5.72))

$$h_k(\eta) = \left(\frac{8\pi}{M^2_{\mathrm{Pl}}} \right)^{1/2} \frac{1}{a} v_k(\eta). \tag{5.97}$$

Thus

$$\rho_g = \frac{2}{8\pi G a^2 (2\pi)^3} \int d^3k \left\langle |h'_k(\eta)|^2 \right\rangle, \tag{5.98}$$

where from now on $\langle \cdots \rangle$ denotes the *average over several periods*. For the spectral density this gives

$$k \frac{d\rho_g(k)}{dk} = \frac{k^3}{G a^2 (2\pi)^3} \left\langle |h'_k(\eta)|^2 \right\rangle. \tag{5.99}$$

If η_i is some early time, we can write

$$|h'_k(\eta)|^2 = \left| \frac{h'_k(\eta)}{h_k(\eta_i)} \right|^2 \frac{\pi^2}{k^3} P_g(k, \eta_i), \tag{5.100}$$

where $P_g(k,\eta_i)$ is the power spectrum at η_i (for which we may take (5.75)). Hence we obtain

$$\boxed{k\frac{d\rho_g(k)}{dk} = \frac{M_{\rm Pl}^2}{8\pi a^2}\left\langle\left|\frac{h_k'(\eta)}{h_k(\eta_i)}\right|^2\right\rangle P_g(k,\eta_i).} \tag{5.101}$$

When the radiation is well inside the horizon, we can replace h_k' by kh_k.

The differential equation (5.68) reads in terms of $h_k(\eta)$

$$h'' + 2\frac{a'}{a}h' + k^2 h = 0. \tag{5.102}$$

For the matter dominated era ($a(\eta)\propto\eta^2$) this becomes

$$h'' + \frac{4}{\eta}h' + k^2 h = 0.$$

Using 9.1.53 of [39] one sees that this is satisfied by $j_1(k\eta)/k\eta$. Furthermore, by 10.1.4 of the same reference, we have $3j_1(x)/x \to 1$ for $x\to 0$ and

$$\left(\frac{j_1(x)}{x}\right)' = -\frac{1}{x}j_2(x) \;\to 0\ (x\to 0).$$

So the correct solution is

$$\frac{h_k(\eta)}{h_k(0)} = 3\frac{j_1(k\eta)}{k\eta} \tag{5.103}$$

if the modes cross inside the horizon during the matter dominated era. Note also that

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}. \tag{5.104}$$

For modes which enter the horizon earlier, we introduce a transfer function $T_g(k)$ by

$$\frac{h_k(\eta)}{h_k(0)} =: 3\frac{j_1(k\eta)}{k\eta}T_g(k), \tag{5.105}$$

that has to be determined numerically from the differential equation (5.102). We can then write the result (5.101) as

$$k\frac{d\rho_g(k)}{dk} = \frac{M_{\rm Pl}^2}{8\pi}\frac{k^2}{a^2}P_g^{\rm prim}(k)|T_g(k)|^2\left\langle\left[\frac{3j_1(k\eta)}{k\eta)}\right]^2\right\rangle, \tag{5.106}$$

where $P_g^{\rm prim}(k)$ denotes the primordial power spectrum. This holds in particular at the present time η_0 ($a_0 = 1$). Since the time average $\langle\cos^2 k\eta\rangle = \frac{1}{2}$, we finally obtain for $\Omega_g(k) := \rho_g(k)/\rho_{\rm crit}$ (using $\eta_0 = 2H_0^{-1}$)

$$\boxed{\frac{d\Omega_g(k)}{d\ln k} = \frac{3}{8}P_g^{\rm prim}(k)|T_g(k)|^2\frac{1}{(k\eta_0)^2}.} \tag{5.107}$$

Here one may insert the inflationary result (5.82), giving

$$\frac{d\Omega_g(k)}{d\ln k} = \frac{3}{2\pi}\left.\frac{H^2}{M_{\rm Pl}^2}\right|_{k=aH}|T_g(k)|^2\frac{1}{(k\eta_0)^2}. \tag{5.108}$$

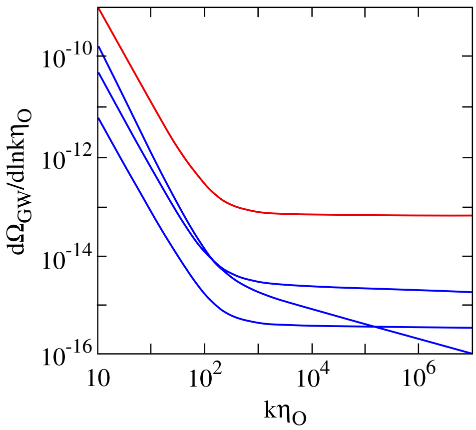

Fig. 5.1 Differential energy density (5.108) of the stochastic background of inflation-produced gravitational waves. The normalization of the upper curve, representing the scale-invariant limit, is arbitrary. The blue curves are normalized to the COBE quadrupole, and show the result for $n_T = -0.003,\ -0.03$, and -0.3. (Adapted from [40].)

5.2.4 Numerical results

Since the normalization in (5.82) can not be predicted, it is reasonable to choose it, for illustration, to be equal to the observed CMB normalization at large scales. (In reality the tensor contribution is presumably only a small fraction of this; see (5.90).) Then one obtains the result shown in Fig. 5.1, taken from [40]. This shows that the spectrum of the stochastic gravitational background radiation is predicted to be flat in the interesting region, with $d\Omega_g/d\ln(k\eta_0) \sim 10^{-14}$. Unfortunately, this is too small to be detectable by the future LISA interferometer in space.

Exercise. Consider a massive free scalar field ϕ (mass m) and discuss the quantum fluctuations for a de Sitter background (neglecting gravitational back reaction). Compute the power spectrum as a function of conformal time for $m/H < 3/2$.

Hint: Work with the field $a\phi$ as a function of conformal time.

Remark: This exercise was solved at an astonishingly early time (∼ 1940) by E. Schrödinger.

5.3 Appendix to Chapter 5: Einstein tensor for tensor perturbations

In this Appendix we derive the expressions (5.59) for the tensor perturbations of the Einstein tensor.

The metric (5.57) is conformal to $\tilde{g}_{\mu\nu} = \gamma_{\mu\nu} + 2H_{\mu\nu}$. We first compute the Ricci tensor $\tilde{R}_{\mu\nu}$ of this metric, and then use the general transformation law of Ricci tensors for conformally related metrics (see Eq. (2.264) of [1]).

Let us first consider the simple case $K = 0$, that we considered in Sect. 5.2. Then $\gamma_{\mu\nu}$ is the Minkowski metric. In the following computation of $\tilde{R}_{\mu\nu}$ we drop temporarily the tildes.

The Christoffel symbols are immediately found (to first order in $H_{\mu\nu}$)

$$\Gamma^{\mu}{}_{00} = \Gamma^{0}{}_{0i} = 0,\ \Gamma^{0}{}_{ij} = H'_{ij},\ \Gamma^{i}{}_{0j} = (H^{i}{}_{j})',$$
$$\Gamma^{i}{}_{jk} = H^{i}{}_{j,k} + H^{i}{}_{k,j} - H_{jk}{}^{,i}. \tag{5.109}$$

So these vanish or are of first order small. Hence, up to higher orders,

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu}. \tag{5.110}$$

Inserting (5.109) and using the TT conditions (5.58) readily gives

$$R_{00} = 0,\ R_{0i} = 0, \tag{5.111}$$

$$R_{ij} = \partial_\lambda \Gamma^\lambda{}_{ij} - \partial_j \Gamma^\lambda{}_{\lambda i} = \partial_0 \Gamma^0{}_{ij} + \partial_k \Gamma^k{}_{ij} - \partial_j \Gamma^0{}_{0i} - \partial_j \Gamma^k{}_{ki}$$
$$= H''_{ij} + (H^k{}_{i,j} + H^k{}_{j,i} - H_{ij}{}^{,k})_{,k}.$$

Thus

$$R_{ij} = H''_{ij} - \triangle H_{ij}. \tag{5.112}$$

Now we use the quoted general relation between the Ricci tensors for two metrics related as $g_{\mu\nu} = e^f \tilde{g}_{\mu\nu}$. In our case $e^f = a^2(\eta)$, hence

$$\tilde{\nabla}_\mu f = 2\mathcal{H}\delta_{\mu 0},\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \partial_\mu(2\mathcal{H}\delta_{\nu 0}) - \Gamma^\lambda{}_{\mu\nu} 2\mathcal{H}\delta_{\lambda 0}$$
$$= 2\mathcal{H}'\delta_{\mu 0}\delta_{\nu 0} - 2\mathcal{H}H'_{\mu\nu},\ \tilde{\triangle} f = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \tilde{\nabla}_\nu f = 2\mathcal{H}'.$$

As a result we find

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + (-2\mathcal{H}' + 2\mathcal{H}^2)\delta_{\mu 0}\delta_{\nu 0} + (\mathcal{H}' + 2\mathcal{H}^2)\tilde{g}_{\mu\nu} + 2\mathcal{H}H'_{\mu\nu}, \tag{5.113}$$

thus

$$\delta R_{00} = \delta R_{0i} = 0,$$
$$\delta R_{ij} = H''_{ij} - \triangle H_{ij} + 2(\mathcal{H}' + 2\mathcal{H}^2)H_{ij} + 2\mathcal{H}H'_{ij}. \tag{5.114}$$

From this it follows that

$$\delta R = g^{(0)\mu\nu}\delta R_{\mu\nu} + \delta g^{\mu\nu} R^{(0)}_{\mu\nu} = 0. \tag{5.115}$$

The result (5.59) for the Einstein tensor is now easily obtained.

Generalization to $K \neq 0$

The relation (5.113) still holds. For the computation of $\tilde{R}_{\mu\nu}$ we start with the following general formula for the Christoffel symbols (again dropping tildes).

$$\delta\Gamma^\mu{}_{\alpha\beta} = \gamma^{\mu\nu}(H_{\nu\alpha|\beta} + H_{\nu\beta|\alpha} - H_{\alpha\beta|\nu}) \tag{5.116}$$

(see [1, Eq. (2.93)]). For the computation of the covariant derivatives $H_{\alpha\beta|\mu}$ with respect to the unperturbed metric $\gamma_{\mu\nu}$, we recall the unperturbed Christoffel symbols (1.229) with $a \to 1$,

$$\Gamma^0{}_{00} = \Gamma^0{}_{i0} = \Gamma^i{}_{00} = \Gamma^0{}_{ij} = \Gamma^i{}_{0j} = 0,\ \Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk}. \tag{5.117}$$

One readily finds

$$H_{\mu 0|\nu} = 0,\ H_{ij|0} = H'_{ij},\ H_{ij|k} = H_{ij\|k}, \tag{5.118}$$

where the double stroke denotes covariant differentiation on (Σ, γ). Therefore,

$$\delta\Gamma^0{}_{00} = \delta\Gamma^0{}_{i0} = \delta\Gamma^i{}_{00} = 0,\ \delta\Gamma^0{}_{ij} = H'_{ij},\ \delta\Gamma^i{}_{0j} = (H^i{}_j)'$$
$$\delta\Gamma^i{}_{jk} = H^i{}_{j\|k} + H^i{}_{k\|j} - H_{jk}{}^{\|i}. \tag{5.119}$$

With these expressions we can compute $\delta R_{\mu\nu}$,using the formula (1.249). The first of the following two equations

$$\delta R_{00} = 0,\ \delta R_{0i} = 0 \tag{5.120}$$

is immediate, while one finds in a first step $\delta R_{0i} = H^k{}_{j\|k}$, and this vanishes because of the TT condition. A bit more involved is the computation of the remaining components. From (1.249) we have

$$\begin{aligned}\delta R_{ij} &= \partial_\lambda \delta\Gamma^\lambda{}_{ij} - \partial_j \delta\Gamma^\lambda{}_{\lambda i} + \delta\Gamma^\sigma{}_{ji}\Gamma^\lambda{}_{\lambda\sigma} + \Gamma^\sigma{}_{ji}\delta\Gamma^\lambda{}_{\lambda\sigma} - \delta\Gamma^\sigma{}_{\lambda i}\Gamma^\lambda{}_{j\sigma} - \Gamma^\sigma{}_{\lambda i}\delta\Gamma^\lambda{}_{j\sigma} \\ &= H''_{ij} + \partial_l \delta\Gamma^l{}_{ij} - \partial_j \delta\Gamma^l{}_{li} + \delta\Gamma^s{}_{ji}\Gamma^l{}_{ls} + \Gamma^s{}_{ji}\delta\Gamma^l{}_{ls} - \delta\Gamma^s{}_{li}\Gamma^l{}_{js} - \Gamma^s{}_{li}\delta\Gamma^l{}_{js}.\end{aligned}$$

But

$$\delta\Gamma^l{}_{ls} = H^l{}_{l\|s} + H^l{}_{s\|l} - H_{ls}{}^{\|l} = 0,$$

so

$$\delta R_{ij} = H''_{ij} + \partial_l \delta\Gamma^l{}_{ij} + \delta\Gamma^s{}_{ji}\Gamma^l{}_{ls} - \delta\Gamma^s{}_{li}\Gamma^l{}_{js} - \Gamma^s{}_{li}\delta\Gamma^l{}_{js} = H''_{ij} + (\delta\Gamma^l{}_{ij})_{\|l}$$

or

$$\delta R_{ij} = H''_{ij} + H^l{}_{i\|jl} + H^l{}_{j\|il} - H_{ij\|l}{}^{\|l}. \tag{5.121}$$

In order to impose the TT conditions , we make use of the Ricci identity[22]

$$H^l{}_{i\|jl} = H^l{}_{i\|jl} + 3KH_{ij},$$

giving

$$\delta R_{ij} = H''_{ij} + 6KH_{ij} - \triangle H_{ij}. \tag{5.122}$$

Part III

Microwave background anisotropies

Introduction

Investigations of the cosmic microwave background have presumably contributed most to the remarkable progress in cosmology during recent years. Beside its spectrum, which is Planckian to an incredible degree, we also can study the temperature fluctuations over the "cosmic photosphere" at a redshift $z \approx 1100$. Through these we get access to crucial cosmological information (primordial density spectrum, cosmological parameters, etc). A major reason for why this is possible relies on the fortunate circumstance that the fluctuations are tiny ($\sim 10^{-5}$) at the time of recombination. This allows us to treat the deviations from homogeneity and isotropy for an extended period of time perturbatively, i.e., by linearizing the Einstein and matter equations about solutions of the idealized Friedmann-Lemaître models. Since the physics is effectively *linear*, we can accurately work out the *evolution* of the perturbations during the early phases of the Universe, given a set of cosmological parameters. Confronting this with observations, tells us a lot about the cosmological parameters as well as the initial conditions, and thus about the physics of the very early Universe. Through this window to the earliest phases of cosmic evolution we can, for instance, test general ideas and specific models of inflation.

Let me add in this introduction some qualitative remarks, before we start with a detailed treatment. Long before recombination (at temperatures $T > 6000K$, say) photons, electrons and baryons were so strongly

[22] On (Σ, γ) we have:

$$H^l{}_{i\|jl} - H^l{}_{i\|jl} = R^l{}_{slj}H^s{}_i + R_i{}^s{}_{lj}H^l{}_s = 3KH_{ij}.$$

coupled that these components may be treated together as a single fluid. In addition to this there is also a dark matter component. For all practical purposes the two interact only gravitationally. The investigation of such a two-component fluid for small deviations from an idealized Friedmann behavior is a well-studied application of cosmological perturbation theory, and will be treated in Chapter 6.

At a later stage, when decoupling is approached, this approximate treatment breaks down because the mean free path of the photons becomes longer (and finally 'infinite' after recombination). While the electrons and baryons can still be treated as a single fluid, the photons and their coupling to the electrons have to be described by the general relativistic Boltzmann equation. The latter is, of course, again linearized about the idealized Friedmann solution. Together with the linearized fluid equations (for baryons and cold dark matter, say), and the linearized Einstein equations one arrives at a complete system of equations for the various perturbation amplitudes of the metric and matter variables. Detailed derivations will be given in Chapter 7. There exist widely used codes e.g. CMBFAST [43], that provide the CMB anisotropies – for given initial conditions – to a precision of about 1%. A lot of qualitative and semi-quantitative insight into the relevant physics can, however, be gained by looking at various approximations of the basic dynamical system.

Let us first discuss the temperature fluctuations. What is observed is the temperature autocorrelation:

$$C(\vartheta) := \left\langle \frac{\Delta T(\mathbf{n})}{T} \cdot \frac{\Delta T(\mathbf{n}')}{T} \right\rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l P_l(\cos\vartheta),$$

where ϑ is the angle between the two directions of observation $\mathbf{n}, \mathbf{n}'$, and the average is taken ideally over all sky. The *angular power spectrum* is by definition $\frac{l(l+1)}{2\pi} C_l \ \ versus \ \ l \ \ (\vartheta \simeq \pi/l)$.

A characteristic scale, which is reflected in the observed CMB anisotropies, is the sound horizon at last scattering, i.e., the distance over which a pressure wave can propagate until decoupling. This can be computed within the unperturbed model and subtends about half a degree on the sky for typical cosmological parameters. For scales larger than this sound horizon the fluctuations have been laid down in the very early Universe. These have first been detected by the COBE satellite. The (gauge invariant brightness) temperature perturbation $\Theta = \Delta T/T$ is dominated by the combination of the intrinsic temperature fluctuations and gravitational redshift or blueshift effects. For example, photons that have to climb out of potential wells for high-density regions are redshifted. We shall show in Sect. 8.5 that these effects combine for adiabatic initial conditions to $\frac{1}{3}\Psi$, where Ψ is one of the two gravitational Bardeen potentials. The latter, in turn, is directly related to the density perturbations. For scale-free initial perturbations and almost vanishing spatial curvature the corresponding angular power spectrum of the temperature fluctuations turns out to be nearly flat (Sachs-Wolfe plateau; see Fig. 8.1).

On the other hand, inside the sound horizon before decoupling, acoustic, Doppler, gravitational redshift, and photon diffusion effects combine to the spectrum of small angle anisotropies. These result from gravitationally driven synchronized acoustic oscillations of the photon-baryon fluid, which are damped by photon diffusion (Sect. 8.2).

A particular realization of $\Theta(\mathbf{n})$, such as the one accessible to us (all sky map from our location), cannot be predicted. Theoretically, Θ is a random field $\Theta(\mathbf{x}, \eta, \mathbf{n})$, depending on the conformal time η, the spatial coordinates, and the observing direction $\mathbf{n}$. Its correlation functions should be rotationally invariant in $\mathbf{n}$, and respect the symmetries of the background time slices. If we expand Θ in terms of spherical harmonics,

$$\Theta(\mathbf{n}) = \sum_{lm} a_{lm} Y_{lm}(\mathbf{n}),$$

the random variables a_{lm} have to satisfy[23]

$$\langle a_{lm} \rangle = 0, \quad \langle a^{\star}_{lm} a_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l(\eta),$$

[23] A formal proof of this can easily be reduced to an application of Schur's Lemma for the group $SU(2)$ (Exercise).

where the $C_l(\eta)$ depend only on η. Hence the correlation function at the present time η_0 is given by the previous expression with $C_l = C_l(\eta_0)$, and the bracket now denotes the statistical average. Thus,

$$C_l = \frac{1}{2l+1}\left\langle \sum_{m=-l}^{l} a^\star_{lm} a_{lm} \right\rangle .$$

The standard deviations $\sigma(C_l)$ measure a fundamental uncertainty in the knowledge we can get about the C_l's. These are called *cosmic variances*, and are most pronounced for low l. In simple inflationary models the a_{lm} are Gaussian distributed, hence

$$\frac{\sigma(C_l)}{C_l} = \sqrt{\frac{2}{2l+1}}.$$

Therefore, the limitation imposed on us (only one sky in one universe) is small for large l.

Exercise. Derive the last equation.

Solution: The claim is a special case of the following general fact: Let $\xi_1, \xi_2, ..., \xi_n$ be independent Gaussian random variables with mean 0 and variance 1, and let

$$\zeta = \frac{1}{n}\sum_{i=1}^{n} \xi_i^2.$$

Then the variance and standard deviation of ζ are

$$var(\zeta) = \frac{2}{n},\ \sigma(\zeta) = \sqrt{\frac{2}{n}}.$$

To show this, we use the equation of Bienaymé

$$var(\zeta) = \frac{1}{n^2}\sum_{i=1}^{n} var(\xi_i^2),$$

and the following formula for the variance for each ξ_i^2:

$$var(\xi^2) = \langle\xi^4\rangle - \langle\xi^2\rangle^2 = 1\cdot 3 - 1 = 2$$

(the even moments of ξ are $m_{2k} = 1\cdot 3\cdot \cdots (2k-1)$).

Alternatively, we can use the fact that $\sum_{i=1}^{n}\xi_i^2$ is χ_n^2-distributed, with distribution function ($p = n/2,\ \lambda = 1/2$):

$$f(x) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x}$$

for $x > 0$, and 0 otherwise. This gives the same result.

6 Tight coupling phase

Long before recombination, photons, electrons and baryons are so strongly coupled that these components may be treated as a single fluid, indexed by r in what follows. Beside this we have to include a CDM component for which we we use the index d (for 'dust' or dark). For practical purposes these two fluids interact only gravitationally.

6.1 Basic equations

We begin by specializing the basic equations, derived in Part I and collected in Sect.1.5.C to the situation just described. Beside neglecting the spatial curvature ($K = 0$), we may assume $q_\alpha = \Gamma_\alpha = 0,\ E_\alpha = F_\alpha = 0$ (no energy and momentum exchange between r and d). In addition, it is certainly a good approximation to neglect in this tight coupling era the anisotropic stresses Π_α. Then $\Psi = -\Phi$ and since $\Gamma_{\rm int} = 0$ the amplitude Γ for entropy production is proportional to

$$S := S_{dr} = \frac{\Delta_{cd}}{1+w_d} - \frac{\Delta_{cr}}{1+w_r}, \ \frac{w}{1+w}\Gamma = \frac{h_d h_r}{h^2}(c_d^2 - c_r^2)S. \tag{6.1}$$

We also recall the definition (1.221)

$$c_z^2 = \frac{h_r}{h}c_d^2 + \frac{h_d}{h}c_r^2. \tag{6.2}$$

The energy and momentum equations are

$$\Delta' - 3\frac{a'}{a}w\Delta = -k(1+w)V, \tag{6.3}$$

$$V' + \frac{a'}{a}V = k\Psi + k\frac{c_s^2}{1+w}\Delta + k\frac{w}{1+w}\Gamma. \tag{6.4}$$

By (1.290) the derivative of S is given by

$$S' = -kV_{dr}, \tag{6.5}$$

and that of V_{dr} follows from (1.289):

$$V_{dr}' + \frac{a'}{a}(1-3c_z^2)V_{dr} = k(c_d^2 - c_r)\frac{\Delta}{1+w} + kc_z^2 S. \tag{6.6}$$

In the constraint equation (1.261) we use the Friedmann equation for $K = 0$,

$$\frac{8\pi G\rho}{3H^2} = 1, \tag{6.7}$$

and obtain

$$\Phi = -\Psi = \frac{3}{2}\left(\frac{Ha}{k}\right)^2\Delta. \tag{6.8}$$

It will be convenient to introduce the comoving wave number in units of the Hubble length $x := Ha/k$ and the renormalized scale factor $\zeta := a/a_{\rm eq}$, where $a_{\rm eq}$ is the scale factor at the ‘equality time’ (see Sect. 0.3.E). Then the last equation becomes

$$\boxed{\Phi = -\Psi = \frac{3}{2}x^2\Delta.} \tag{6.9}$$

Using $\zeta' = kx\zeta$ and introducing the operator $D := \zeta d/d\zeta$ we can write (6.3) as

$$\boxed{(D - 3w)\Delta = -\frac{1}{x}(1+w)V.} \tag{6.10}$$

Similarly, (6.4) (together with (6.1)) gives

$$\boxed{(D+1)V = \frac{\Psi}{x} + \frac{c_s^2}{x}\frac{\Delta}{1+w} + \frac{1}{x}\frac{h_d h_r}{h^2}(c_d^2 - c_r^2)S.} \tag{6.11}$$

We also rewrite (6.5) and (6.6)

$$\boxed{DS = -\frac{1}{x}V_{dr},} \tag{6.12}$$

$$\boxed{(D + 1 - 3c_z^2)V_{dr} = \frac{1}{x}(c_d^2 - c_r)\frac{\Delta}{1+w} + \frac{1}{x}c_z^2 S.} \tag{6.13}$$

It will turn out to be useful to work alternatively with the equations of motion for V_α and

$$X_\alpha := \frac{\Delta_{c\alpha}}{1+w_\alpha}\,(\alpha = r, d). \tag{6.14}$$

From (1.288) we obtain

$$V'_\alpha + \frac{a'}{a}V_\alpha = k\Psi + k\frac{c_\alpha^2}{1+w_\alpha}\Delta_\alpha, \tag{6.15}$$

Here, we replace Δ_α by $\Delta_{c\alpha}$ with the help of (1.174) and (1.175), implying (in the harmonic decomposition)

$$\Delta_\alpha = \Delta_{c\alpha} + 3(1+w_\alpha)\frac{a'}{a}\frac{1}{k}(V_\alpha - V). \tag{6.16}$$

We then get

$$V'_\alpha + \frac{a'}{a}(1 - 3c_\alpha^2)V_\alpha = k\Psi + kc_\alpha^2 X_\alpha - 3\frac{a'}{a}c_\alpha^2 V. \tag{6.17}$$

From (1.287) we find, using (6.1),

$$X'_\alpha = -kV_\alpha + 3\frac{a'}{a}c_s^2\frac{\Delta}{1+w} + 3\frac{a'}{a}\frac{h_d h_r}{h^2}(c_d^2 - c_r^2)S. \tag{6.18}$$

Rewriting the last two equations as above, we arrive at the system

$$(D + 1 - 3c_\alpha^2)V_\alpha = \frac{\Psi}{x} + \frac{c_\alpha^2}{x}X_\alpha - 3c_\alpha^2 V, \tag{6.19}$$

$$DX_\alpha = -\frac{V_\alpha}{x} + 3c_s^2\frac{\Delta}{1+w} + 3\frac{h_d h_r}{h^2}(c_d^2 - c_r^2)S. \tag{6.20}$$

This system is closed, since by (6.1), (1.272) and (1.275)

$$S = X_d - X_r,\ \frac{\Delta}{1+w} = \sum_\alpha \frac{h_\alpha}{h}X_\alpha,\ V = \sum_\alpha \frac{h_\alpha}{h}V_\alpha. \tag{6.21}$$

Note also that according to (1.220)

$$\frac{\Delta}{1+w} = X_r + \frac{h_d}{h}S = X_d - \frac{h_r}{h}S. \tag{6.22}$$

From these basic equations we now deduce second order equations for the pair (Δ, S), respectively, for X_α $(\alpha = r, d)$. For doing this we note that for any function f, $f' = (a'/a)Df$, in particular (using (1.80) and (1.62))

$$Dx = -\frac{1}{2}(3w+1)x,\ Dw = -3(1+w)(c_s^2 - w). \tag{6.23}$$

The result of the somewhat tedious but straightforward calculation is [41]:

$$D^2\Delta + \left[\frac{1-3w}{2} + 3c_s^2 - 6w\right] D\Delta + \left[\frac{c_s^2}{x^2} - 3w + 9(c_s^2 - w) + \frac{3}{2}(3w^2 - 1)\right]\Delta = \frac{1}{x^2}\frac{h_r h_d}{\rho h}(c_r^2 - c_d^2)S, \tag{6.24}$$

$$D^2 S + \left[\frac{1-3w}{2} - 3c_z^2\right] DS + \frac{c_z^2}{x^2} S = \frac{c_r^2 - c_d^2}{x^2(1+w)}\Delta \tag{6.25}$$

for the pair Δ, S, and

$$D^2 X_\alpha + \left[\frac{1-3w}{2} - 3c_\alpha^2\right] DX_\alpha + \left\{\frac{c_\alpha^2}{x^2} - \frac{h_\alpha}{h}\left[\frac{3}{2}(1+w) + \frac{3}{2}(1-3w)c_\alpha^2 + 9c_\alpha^2(c_s^2 - c_\alpha^2) + 3Dc_s^2\right]\right\} X_\alpha = 3\frac{h_\beta}{h}\left[(c_\beta^2 - c_\alpha^2)D + \frac{1+w}{2} + \frac{1-3w}{2}c_\beta^2 + 3c_\beta^2(c_s^2 - c_\beta^2) + Dc_\beta^2\right] X_\beta \tag{6.26}$$

for the pair X_α.

Alternative system for tight coupling limit

Instead of the first order system (6.17), (6.18) one may work with similar equations for the amplitudes $\Delta_{s\alpha}$ and V_α. From (1.291) we obtain instead of (6.17) for $\Pi_\alpha = F_\alpha = 0$

$$V_\alpha' + \frac{a'}{a}(1 - 3c_\alpha^2)V_\alpha = k\Psi + k\frac{c_\alpha^2}{1+w_\alpha}\Delta_{s\alpha}. \tag{6.27}$$

Beside this we have Eq. (1.286)

$$\left(\frac{\Delta_{s\alpha}}{1+w_\alpha}\right)' = -kV_\alpha - 3\Phi'. \tag{6.28}$$

To this we add the following consequence of the constraint equations (1.261), (1.262) and the relations (1.260), (1.274), (1.275):

$$k^2\Psi = -4\pi G a^2 \sum_\alpha \left[\rho_\alpha \Delta_{s\alpha} + 3\frac{aH}{k}\rho_\alpha(1+w_\alpha)V_\alpha\right]. \tag{6.29}$$

Instead one can also use, for instance for generating numerical solutions, the following first order differential equation that is obtained similarly

$$k^2\Psi + 3\frac{a'}{a}\left(\Psi' + \frac{a'}{a}\Psi\right) = -4\pi G a^2 \sum_\alpha \rho_\alpha \Delta_{s\alpha}. \tag{6.30}$$

Adiabatic and isocurvature perturbations

These differential equations have to be supplemented with initial conditions. Two linearly independent types are considered for some very early stage, for instance at the end of the inflationary era:

- **adiabatic** perturbations: all $S_{\alpha\beta} = 0$, but $\mathcal{R} \neq 0$;

- **isocurvature** perturbations: some $S_{\alpha\beta} \neq 0$, but $\mathcal{R} = 0$.

Recall that $\mathcal{R}$ measures the spatial curvature for the slicing $\mathcal{Q} = 0$. According to the initial definition (1.58) of $\mathcal{R}$ and the Eqs. (6.9), (6.10) we have

$$\mathcal{R} = \Phi - xV = \frac{x^2}{1+w}\left[D + \frac{3}{2}(1-w)\right]\Delta. \tag{6.31}$$

Explicit forms of the two-component differential equations

At this point we make use of the equation of state for the two-component model under consideration. It is convenient to introduce a parameter c by

$$R := \frac{3\rho_b}{4\rho_\gamma} = \frac{\zeta}{c} \Rightarrow \frac{\Omega_d}{\Omega_b} = \frac{3c}{4} - 1. \tag{6.32}$$

We then have for various background quantities

$$\begin{aligned}
\frac{\rho_d}{\rho_{\mathrm{eq}}} &= \frac{1}{2}\left(1 - \frac{4}{3c}\right)\frac{1}{\zeta^3},\ p_d = 0,\\
\frac{\rho_r}{\rho_{\mathrm{eq}}} &= \frac{2}{3}\frac{\zeta + 3c/4}{c}\frac{1}{\zeta^4},\ \frac{p_r}{\rho_{\mathrm{eq}}} = \frac{1}{6}\frac{1}{\zeta^4},\\
\frac{\rho}{\rho_{\mathrm{eq}}} &= \frac{1}{2}(\zeta+1)\frac{1}{\zeta^4},\ \frac{p}{\rho_{\mathrm{eq}}} = \frac{1}{6}\frac{1}{\zeta^4},\\
\frac{h_r}{h} &= \frac{4}{3}\frac{\zeta + c}{c(\zeta + 4/3)},\ \frac{h_d}{h} = \left(1 - \frac{4}{3c}\right)\frac{\zeta}{\zeta + 4/3},\\
w &= \frac{1}{3(\zeta+1)},\ w_r = \frac{c}{4\zeta + 3c},\ w_d = 0,\\
c_d^2 &= 0,\ c_r^2 = \frac{1}{3}\frac{c}{\zeta + c},\ c_s^2 = \frac{4}{9}\frac{1}{\zeta + 4/3},\ c_z^2 = \frac{1}{3}\frac{(c - 4/3)\zeta}{(\zeta + c)(\zeta + 4/3)},\\
H^2 &= H_{\mathrm{eq}}^2\frac{\zeta+1}{2}\frac{1}{\zeta^4},\ x^2 = \frac{\zeta+1}{2\zeta^2}\frac{1}{\omega^2},\ \omega := \frac{1}{x_{\mathrm{eq}}} = \left(\frac{k}{aH}\right)_{\mathrm{eq}}.
\end{aligned} \tag{6.33}$$

Since we now know that the dark matter fraction is much larger than the baryon fraction, we write the basic equations only in the limit $c \to \infty$. (For finite c these are given in [41].) Eq.(6.26) leads to the pair

$$\begin{aligned}
&D^2 X_r + \left(\frac{1}{2}\frac{\zeta}{1+\zeta} - 1\right) D X_r\\
&+ \left\{\frac{2}{3}\frac{\omega^2\zeta^2}{1+\zeta} + \frac{4}{3}\frac{1}{\zeta + 4/3}\left[\frac{\zeta}{\zeta + 4/3} - 2\right]\right\} X_r = \left[\frac{3}{2}\frac{\zeta}{\zeta+1} - \frac{\zeta}{\zeta + 4/3} D\right] X_d,
\end{aligned} \tag{6.34}$$

$$\left\{D^2+\frac{1}{2}\frac{\zeta}{1+\zeta}D-\frac{3}{2}\frac{\zeta}{1+\zeta}\right\}X_d=\frac{4}{3}\frac{1}{\zeta+4/3}\left[D+2-\frac{\zeta}{\zeta+4/3}\right]X_r. \tag{6.35}$$

From (6.24) and (6.25) we obtain on the other hand

$$\begin{aligned} &D^2\Delta+\left(-1+\frac{5}{2}\frac{\zeta}{\zeta+1}-\frac{\zeta}{\zeta+4/3}\right)D\Delta \\ &+\left\{-2+\frac{3}{4}\zeta+\frac{1}{2}\left(\frac{\zeta}{\zeta+1}\right)^2-\frac{3\zeta^2}{\zeta+1}+\frac{9\zeta^2}{4(\zeta+4/3)}\right\}\Delta \\ &=\frac{8}{9}\omega^2\frac{\zeta^2}{(\zeta+1)^2(\zeta+4/3)}\left[\zeta S-(\zeta+1)\Delta\right], \end{aligned} \tag{6.36}$$

$$D^2S+\left(\frac{1}{2}\frac{1}{\zeta+1}-\frac{1}{\zeta+4/3}\right)\zeta DS+\frac{2}{3}\omega^2\frac{\zeta^3}{(\zeta+1)(\zeta+4/3)}S=\frac{2}{3}\omega^2\frac{\zeta^2}{\zeta+4/3}\Delta. \tag{6.37}$$

We also note that (6.31) becomes

$$\mathcal{R}=\frac{1}{2\omega^2}\frac{\zeta+1}{\zeta^2(\zeta+4/3)}\left[(\zeta+1)D+\frac{3}{2}\zeta+1\right]\Delta. \tag{6.38}$$

We can now define more precisely what we mean by the two types of primordial initial perturbations by considering solutions of our perturbation equations for $\zeta \ll 1$.

• *adiabatic* (or *curvature*) perturbations: growing mode behaves as

$$\begin{aligned} \Delta&=\zeta^2\left[1-\frac{17}{16}\zeta+\cdots\right]-\frac{\omega^2}{15}\zeta^4[1-\cdots], \\ S&=\frac{\omega^2}{32}\zeta^4\left[1-\frac{28}{25}\zeta+\cdots\right];\Rightarrow\mathcal{R}=\frac{9}{8\omega^2}(1+\mathcal{O}(\zeta)). \end{aligned} \tag{6.39}$$

• *isocurvature* perturbations: growing mode behaves as

$$\begin{aligned} \Delta&=\frac{\omega^2}{6}\zeta^3\left[1-\frac{17}{10}\zeta+\cdots\right], \\ S&=1-\frac{\omega^2}{18}\zeta^3\left[1-\cdots\right];\Rightarrow\mathcal{R}=\frac{1}{4}\zeta(1+\mathcal{O}(\zeta)). \end{aligned} \tag{6.40}$$

From (6.21) and (6.22) we obtain the relation between the two sets of perturbation amplitudes:

$$X_r=\frac{\zeta+1}{\zeta+4/3}\Delta-\frac{\zeta}{\zeta+4/3}S,\ X_d=\frac{\zeta+1}{\zeta+4/3}\Delta+\frac{4}{3}\frac{1}{\zeta+4/3}S, \tag{6.41}$$

$$\Delta=\frac{1}{\zeta+1}\left(\frac{4}{3}X_r+\zeta X_d\right),\ S=X_d-X_r. \tag{6.42}$$

6.2 Analytical and numerical analysis

The system of linear differential equations (6.34)–(6.37) has been discussed analytically in great detail in [41]. One learns, however, more about the physics of the gravitationally coupled fluids in a mixed analytical-numerical approach.

6.2.1 Solutions for super-horizon scales

For super-horizon scales ($x \gg 1$) Eq. (6.12) implies that S is constant. If the mode enters the horizon in the matter dominated era, then the parameter ω in (6.33) is small. For $\omega \ll 1$ Eq. (6.36) reduces to

$$D^2\Delta + \left(-1 + \frac{5}{2}\frac{\zeta}{\zeta+1} - \frac{\zeta}{\zeta+4/3}\right)D\Delta$$
$$+\left\{-2 + \frac{3}{4}\zeta + \frac{1}{2}\left(\frac{\zeta}{\zeta+1}\right)^2 - \frac{3\zeta^2}{\zeta+1} + \frac{9\zeta^2}{4(\zeta+4/3)}\right\}\Delta$$
$$= \frac{8}{9}\omega^2\frac{\zeta^3}{(\zeta+1)^2(\zeta+4/3)}S. \tag{6.43}$$

For *adiabatic* modes we are led to the homogeneous equation already studied in Sect. 2.1, with the two independent solutions U_g and U_d given in (2.28) and (2.29). Recall that the Bardeen potentials remain constant both in the radiation and in the matter dominated eras. According to (2.32) Φ decreases to 9/10 of the primordial value Φ^{prim}.

For *isocurvature* modes we can solve (6.41) with the Wronskian method, and obtain for the growing mode [41]

$$\Delta_{iso} = \frac{4}{15}\omega^2 S\zeta^3\frac{3\zeta^2+22\zeta+24+4(3\zeta+4)\sqrt{1+\zeta}}{(\zeta+1)(3\zeta+4)[1+(1+\zeta)^{1/2}]^4}. \tag{6.44}$$

thus

$$\Delta_{iso} \simeq \begin{cases} \frac{1}{6}\omega^2 S\zeta^3 & : \ \zeta \ll 1 \\ \frac{4}{15}\omega^2 S\zeta & : \ \zeta \gg 1. \end{cases} \tag{6.45}$$

6.2.2 Horizon crossing

We now study the behavior of adiabatic modes more closely, in particular what happens in horizon crossing.

Crossing in radiation dominated era

When the mode enters the horizon in the radiation dominated phase we can neglect in (6.36) the term proportional to S for $\zeta < 1$. As long as the radiation dominates ζ is small, whence (6.36) gives in leading order

$$(D^2 - D - 2)\Delta = -\frac{2}{3}\omega^2\zeta^2\Delta. \tag{6.46}$$

(This could also be directly obtained from (6.24), setting $c_s^2 \simeq 1/3,\ w \simeq 1/3$.) Since $D^2 - D = \zeta^2 d^2/d\zeta^2$ this perturbation equation can be written as

$$\left[\zeta^2\frac{d^2}{d\zeta^2} + \left(\frac{2}{3}\omega^2\zeta^2 - 2\right)\right]\Delta = 0. \tag{6.47}$$

Instead of ζ we choose as independent variable the comoving sound horizon r_s times k. We have

$$r_s = \int c_s d\eta = \int c_s \frac{d\eta}{d\zeta}d\zeta,$$

with $c_s \simeq 1/\sqrt{3},\ d\zeta/d\eta = kx\zeta = aH\zeta = (aH)/(aH)_{\mathrm{eq}}(k/\omega)\zeta \simeq (k/\omega\sqrt{2})$, thus $\zeta \simeq (k/\sqrt{2}\omega)\eta$ and

$$u := kr_s \simeq \sqrt{\frac{2}{3}}\omega\zeta \simeq k\eta/\sqrt{3}. \tag{6.48}$$

Therefore, (6.45) is equivalent to

$$\boxed{\left[\frac{d^2}{du^2} + \left(1 - \frac{2}{u^2}\right)\right]\Delta = 0.} \tag{6.49}$$

This differential equation is well-known. According to 9.1.49 of [39] the functions $w(x) \propto x^{1/2}\mathcal{C}_\nu(\lambda x)$, $\mathcal{C}_\nu \propto H_\nu^{(1)}, H_\nu^{(2)}$, satisfy

$$w'' + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) w = 0. \tag{6.50}$$

Since $j_\nu(x) = \sqrt{\pi/2x}J_{\nu+1/2}(x),\ n_\nu(x) = \sqrt{\pi/2x}Y_{\nu+1/2}(x)$, we see that Δ is a linear combination of $uj_1(u)$ and $un_1(u)$:

$$\Delta(\zeta) = Cuj_1(u) + Dun_1(u);\ u = \sqrt{\frac{2}{3}}\omega\zeta\ \left(u = kr_s = \frac{k\eta}{\sqrt{3}}\right). \tag{6.51}$$

Now,

$$xj_1(x) = \frac{1}{x}\sin x - \cos x,\ xn_1(x) = -\frac{1}{x}\cos x - \sin x. \tag{6.52}$$

On super-horizon scales $u = kr_s \ll 1$, and $uj_1(u) \approx u \propto a$, while $un_1(u) \approx -1/u \propto 1/a$. Thus the first term in (6.49) corresponds to the growing mode. If we only keep this, we have

$$\Delta(\zeta) \approx C\left(\frac{1}{u}\sin u - \cos u\right). \tag{6.53}$$

Once the mode is deep within the Hubble horizon only the cos-term survives. This is an important result, because if this happens long before recombination we can use for adiabatic modes the *initial condition*

$$\boxed{\Delta(\eta) \propto cos[kr_s(\eta)].} \tag{6.54}$$

We conclude that all adiabatic modes are temporally correlated (*synchronized*), while they are spatially uncorrelated (random phases). This is one of the basic reasons for the appearance of acoustic peaks in the CMB anisotropies. Note also that, as a result of (6.9) and (6.33), $\Phi \propto \Delta/\zeta^2 \propto \Delta/u^2$, i.e.,

$$\Psi = 3\Psi^{(prim)}\left[\frac{\sin u - u\cos u}{u^3}\right]. \tag{6.55}$$

Thus: If the mode enters the horizon during the radiation dominated era, its *potential begins to decay*.

As an exercise show that for isocurvature perturbations the cos in (6.52) has to be replaced by the sin (out of phase).

We could have used in the discussion above the system (6.34) and (6.35). In the same limit it reduces to

$$\left(D^2 - D - 2 + \frac{2}{3}\omega^2\zeta^2\right)X_r \simeq 0,\ D^2X_d \simeq (D+2)X_r. \tag{6.56}$$

As expected, the equation for X_r is the same as for Δ. One also sees that X_d is driven by X_r, and is growing logarithmically for $\omega \gg 1$.

The previous analysis can be improved by not assuming radiation domination and also including baryons (see [41]). It turns out that for $\omega \gg 1$ the result (6.54) is not much modified: The cos-dependence remains, but with the exact sound horizon; only the amplitude is slowly varying in time $\propto (1+R)^{-1/4}$.

Since the matter perturbation is driven by the radiation, we may use the potential (6.55) and work out its influence on the matter evolution. It is more convenient to do this for the amplitude Δ_{sd} (instead of Δ_{cd}), making use of the equations (6.27) and (6.28) for $\alpha = d$:

$$\Delta'_{sd} = -kV_d - 3\Phi', \; V'_d = -\frac{a'}{a}V_d - k\Phi. \tag{6.57}$$

Let us eliminate V_d:

$$\Delta''_{sd} = -V'_d - 3\Phi'' = \frac{a'}{a}kV_d + k^2\Phi - 3\Phi'' = \frac{a'}{a}(-\Delta'_{sd} - 3\Phi') + k^2\Phi - 3\Phi''.$$

The resulting equation

$$\Delta''_{sd} + \frac{a'}{a}\Delta'_{sd} = k^2\Phi - 3\Phi'' - 3\frac{a'}{a}\Phi' \tag{6.58}$$

can be solved with the Wronskian method. Two independent solutions of the homogeneous equation are $\Delta_{sd} = \mathrm{const.}$ and $\Delta_{sd} = \ln(a)$. These determine the Green's function in the standard manner. One then finds in the radiation dominated regime (for details, see [5, p. 198])

$$\Delta_{sd}(\eta) = A\Phi^{\mathrm{prim}}\ln(Bk\eta), \tag{6.59}$$

with $A \simeq 9.0, \; B \simeq 0.62$.

Matter dominated approximation

As a further illustration we now discuss the matter dominated approximation. For this ($\zeta \gg 1$) the system (6.34),(6.35) becomes

$$\left(D^2 - \frac{1}{2}D + \frac{2}{3}\omega^2\zeta\right)X_r = \left(-D + \frac{3}{2}\right)X_d, \tag{6.60}$$

$$\left(D^2 + \frac{1}{2}D - \frac{3}{2}\right)X_d = 0. \tag{6.61}$$

As expected, the equation for X_d is independent of X_r, while the radiation perturbation is driven by the dark matter. The solution for X_d is

$$X_d = A\zeta + B\zeta^{-3/2}. \tag{6.62}$$

Keeping only the growing mode, (6.60) becomes

$$\frac{d}{d\zeta}\left(\zeta\frac{dX_r}{d\zeta}\right) - \frac{1}{2}\frac{dX_r}{d\zeta} + \frac{2}{3}\omega^2\left(X_r - \frac{3A}{4\omega^2}\right) = 0. \tag{6.63}$$

Substituting

$$X_r =: \frac{3A}{4\omega^2} + \zeta^{-3/4}f(\zeta),$$

we get for $f(\zeta)$ the following differential equation

$$f'' = -\left(\frac{3}{16}\frac{1}{\zeta^2} + \frac{2}{3}\frac{\omega^2}{\zeta}\right)f. \tag{6.64}$$

For $\omega \gg 1$ we can use the WKB approximation

$$f = \frac{\zeta^{1/4}}{\sqrt{\omega}} \exp\left(\pm i\sqrt{\frac{8}{3}}\omega\zeta^{1/2}\right),$$

implying the following oscillatory behavior of the radiation

$$X_r = \frac{3A}{4\omega^2} + B\frac{1}{\sqrt{\omega\zeta}} \exp\left(\pm i\sqrt{\frac{8}{3}}\omega\zeta^{1/2}\right). \tag{6.65}$$

A look at (6.42) shows that this result for X_d, X_r implies the constancy of the Bardeen potentials in the matter dominated era.

6.2.3 Sub-horizon evolution

For $\omega \gg 1$ one may expect on physical grounds that the dark matter perturbation X_d eventually evolves independently of the radiation. Unfortunately, I can not see this from the basic equations (6.34), (6.35). Therefore, we choose a different approach, starting from the alternative system (6.27)–(6.29). This implies

$$\Delta'_{sd} = -kV_d - 3\Phi', \tag{6.66}$$

$$V'_d = -\frac{a'}{a}V_d - k\Phi, \tag{6.67}$$

$$k^2\Phi = 4\pi G a^2[\rho_d\Delta_{sd} + \cdots]. \tag{6.68}$$

As an approximation, we drop in the last equation the radiative[24] and velocity contributions that have not been written out. Then we get a closed system which we again write in terms of the variable ζ:

$$D\Delta_{sd} = -\frac{1}{x}V_d - 3D\Phi, \tag{6.69}$$

$$DV_d = -V_d - \frac{1}{x}\Phi, \tag{6.70}$$

$$\Phi \simeq \frac{3}{4}\frac{1}{\omega^2}\frac{1}{\zeta}\Delta_{sd}. \tag{6.71}$$

In the last equation we used $\rho_d = (\zeta/\zeta+1)\rho$, (6.7) and the expression (6.33) for x^2.

For large ω we can easily deduce a second order equation for Δ_{sd}: Applying D to (6.69) and using (6.70) gives

$$\begin{aligned} D^2\Delta_{sd} &= -\frac{1}{x}DV_d + \frac{1}{x^2}(Dx)V_d - 3D^2\Phi \\ &= \frac{1}{x^2}\Phi + \frac{1}{2}(1-3w)\frac{1}{x}V_d - 3D^2\Phi \\ &= \frac{1}{x^2}\Phi - \frac{1}{2}(1-3w)D\Delta_{sd} - \frac{3}{2}(1-3w)D\Phi - 3D^2\Phi. \end{aligned}$$

Because of (6.71) the last two terms are small, and we end up (using again (6.33)) with

$$\left\{D^2 + \frac{1}{2}\frac{\zeta}{1+\zeta}D - \frac{3}{2}\frac{\zeta}{1+\zeta}\right\}\Delta_{sd} = 0, \tag{6.72}$$

[24] The growth in the matter perturbations implies that eventually $\rho_d\Delta_{sd} > \rho_r\Delta_{sr}$ even if $\Delta_{sd} < \Delta_{sr}$.

known in the literature as the *Meszaros equation*. Note that this agrees, as was to be expected, with the homogeneous equation belonging to (6.35).

The Meszaros equation can be solved analytically. On the basis of (6.62) one may guess that one solution is linear in ζ. Indeed, one finds that

$$X_d(\zeta) = D_1(\zeta) = \zeta + 2/3 \tag{6.73}$$

is a solution. A linearly independent solution can then be found by quadratures. It is a general fact that $f(\zeta) := \Delta_{sd}/D_1(\zeta)$ must satisfy a differential equation which is first order for f'. One readily finds that this equation is

$$\left(1 + \frac{3\zeta}{2}\right) f'' + \frac{1}{4\zeta(\zeta+1)}[21\zeta^2 + 24\zeta + 4]f' = 0.$$

The solution for f' is

$$f' \propto (\zeta + 2/3)^{-2}\zeta^{-1}(\zeta+1)^{-1/2}.$$

Integrating once more provides the second solution of (6.72)

$$D_2(\zeta) = D_1(\zeta)\ln\left[\frac{\sqrt{1+\zeta}+1}{\sqrt{1+\zeta}-1}\right] - 2\sqrt{1+\zeta}. \tag{6.74}$$

For late times the two solutions approach to those found in (6.62).

The growing and the decaying solutions D_1, D_2 have to be superposed such that a match to (6.59) is obtained.

6.2.4 Transfer function, numerical results

According to (2.31), (2.32) the early evolution of Φ on super-horizon scales is given by[25]

$$\Phi(\zeta) = \Phi^{(\mathrm{prim})}\frac{9}{10}\frac{\zeta+1}{\zeta^2}U_g \simeq \frac{9}{10}\Phi^{(\mathrm{prim})}\ ,\ for\ \zeta \gg 1. \tag{6.75}$$

At sufficiently late times in the matter dominated regime all modes evolve identically with the *growth function* $D_g(\zeta)$ given in (2.37). I recall that this function is normalized such that it is equal to a/a_0 when we can still ignore the dark energy (at $z > 10$, say). The growth function describes the evolution of Δ, thus by the Poisson equation (2.3) Φ grows with $D_g(a)/a$. We therefore define the *transfer function* $T(k)$ by (we choose the normalization $a_0 = 1$)

$$\Phi(k,a) = \Phi^{(\mathrm{prim})}\frac{9}{10}\frac{D_g(a)}{a}T(k) \tag{6.76}$$

for late times. This definition is chosen such that $T(k) \to 1$ for $k \to 0$, and does not depend on time.

At these late times $\rho_M = \Omega_M a^{-3}\rho_{\mathrm{crit}}$, hence the Poisson equation gives the following relation between Φ and Δ

$$\Phi = \left(\frac{a}{k}\right)^2 4\pi G\rho_M\Delta = \frac{3}{2}\frac{1}{ak^2}H_0^2\Omega_M\Delta.$$

Therefore, (6.76) translates to

$$\boxed{\Delta(a) = \frac{3}{5}\frac{k^2}{\Omega_M H_0^2}\Phi^{(\mathrm{prim})}D_g(a)T(k).} \tag{6.77}$$

[25] The origin of the factor 9/10 is best seen from the constancy of $\mathcal{R}$ for super-horizon perturbations, and Eq. (4.67).

The transfer function can be determined by solving numerically the pair (6.24), (6.25) of basic perturbation equations. One can derive even a reasonably good analytic approximation by putting our previous results together (for details see again [5, Sect. 7.4]). For a CDM model the following accurate fitting formula to the numerical solution in terms of the variable $\tilde{q} = k/k_{\text{eq}}$, where k_{eq} is defined such that the corresponding value of the parameter ω in (6.33) is equal to 1 (i.e., $k_{\text{eq}} = a_{\text{eq}}H_{\text{eq}} = \sqrt{2\Omega_M}H_0/\sqrt{a_{\text{eq}}}$, using (0.52)) was given in [42]:

$$T_{BBKS}(\tilde{q}) = \frac{\ln(1 + 0.171\tilde{q})}{0.171\tilde{q}}[1 + 0.284\tilde{q} + (1.18\tilde{q})^2 + (0.399\tilde{q})^3 + (0.490\tilde{q})^4]^{-1/4}. \qquad (6.78)$$

Note that $\tilde{q}$ depends on the cosmological parameters through the combination[26] $\Omega_M h_0$, usually called the *shape parameter* Γ. In terms of the variable $q = k/(\Gamma h_0 Mpc^{-1})$ (6.78) can be written as

$$T_{BBKS}(q) = \frac{\ln(1 + 2.34q)}{2.34q}[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4}. \qquad (6.79)$$

This result for the transfer function is based on a simplified analysis. The tight coupling approximation is no more valid when the decoupling temperature is approached. Moreover, anisotropic stresses and baryons have been ignored. We shall reconsider the transfer function after having further developed the basic theory in the next chapter. It will, of course, be very interesting to compare the theory with available observational data. For this one has to keep in mind that the linear theory only applies to sufficiently large scales. For late times and small scales it has to be corrected by numerical simulations for nonlinear effects.

For a given primordial power spectrum, the transfer function determines the power spectrum after the 'transfer regime' (when all modes evolve with the growth function D_g). From (6.77) we obtain for the power spectrum of Δ

$$P_\Delta(z) = \frac{9}{25}\frac{k^4}{\Omega_M^2 H_0^4}P_\Phi^{(\text{prim})}D_g^2(z)T^2(k). \qquad (6.80)$$

We choose $P_\Phi^{(\text{prim})} \propto k^{n-1}$ and the amplitude such that

$$\boxed{P_\Delta(z) = \delta_H^2\left(\frac{k}{H_0}\right)^{3+n}T^2(k)\left(\frac{D_g(z)}{D_g(0)}\right)^2.} \qquad (6.81)$$

Note that $P_\Delta(0) = \delta_H^2$ for $k = H_0$. The normalization factor δ_H has to be determined from observations (e.g. from CMB anisotropies at large scales). Comparison of (6.80) and (6.81) and use of (5.50) implies

$$P_{\mathcal{R}}^{(\text{prim})}(k) = \frac{9}{4}P_\Phi^{(\text{prim})}(k) = \frac{25}{4}\delta_H^2\left(\frac{\Omega_M}{D_g(0)}\right)^2\left(\frac{k}{H_0}\right)^{n-1}. \qquad (6.82)$$

Exercise. Write the equations (6.27)-(6.30) in explicit form, using (6.33) in the limit when baryons are neglected ($c \to \infty$). (For a truncated subsystem this was done in (6.69) – (6.71)). Solve the five first order differential equations (6.27), (6.28) for $\alpha = d, r$ and (6.30) numerically. Determine, in particular, the transfer function defined in (6.76). (A standard code gives this in less than a second.)

7 Boltzmann equation in GR

For the description of photons and neutrinos before recombination we need the general relativistic version of the Boltzmann equation.

[26] since k is measured in units of $h_0\ Mpc^{-1}$ and $a_{\text{eq}} = 4.15 \times 10^{-5}/(\Omega_M h_0^2)$.

7.1 One-particle phase space, Liouville operator for geodesic spray

For what follows we first have to develop some kinematic and differential geometric tools. Our goal is to generalize the standard description of Boltzmann in terms of one-particle distribution functions.

Let g be the metric of the spacetime manifold M. On the cotangent bundle $T^*M = \bigcup_{p\in M} T^*_p M$ we have the natural symplectic 2-form ω, which is given in natural bundle coordinates[27] (x^μ, p_ν) by

$$\omega = dx^\mu \wedge dp_\mu. \tag{7.1}$$

(For an intrinsic description, see Chap. 6 of [44].) So far no metric is needed. The pair (T^*M, ω) is always a symplectic manifold.

The metric g defines a natural diffeomorphism between the tangent bundle TM and T^*M which can be used to pull ω back to a symplectic form ω_g on TM. In natural bundle coordinates the diffeomorphism is given by $(x^\mu, p^\alpha) \mapsto (x^\mu, p_\alpha = g_{\alpha\beta}p^\beta)$, hence

$$\omega_g = dx^\mu \wedge d(g_{\mu\nu}p^\nu). \tag{7.2}$$

On TM we can consider the "Hamiltonian function"

$$L = \frac{1}{2} g_{\mu\nu}p^\mu p^\nu \tag{7.3}$$

and its associated Hamiltonian vector field X_g, determined by the equation

$$i_{X_g}\omega_g = dL. \tag{7.4}$$

It is not difficult to show that in bundle coordinates

$$X_g = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \tag{7.5}$$

(Exercise). The Hamiltonian vector field X_g on the symplectic manifold (TM, ω_g) is the *geodesic spray*. Its integral curves satisfy the canonical equations:

$$\frac{dx^\mu}{d\lambda} = p^\mu, \tag{7.6}$$

$$\frac{dp^\mu}{d\lambda} = -\Gamma^\mu{}_{\alpha\beta}p^\alpha p^\beta. \tag{7.7}$$

The *geodesic flow* is the flow of the vector field X_g.

Let Ω_{ω_g} be the volume form belonging to ω_g, i.e., the Liouville volume

$$\Omega_{\omega_g} = const\ \omega_g \wedge \cdots \wedge \omega_g,$$

or $(g = \det(g_{\alpha\beta}))$

$$\begin{aligned}\Omega_{\omega_g} &= (-g)(dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) \wedge (dp^0 \wedge dp^1 \wedge dp^2 \wedge dp^3)\\ &\equiv (-g)dx^{0123} \wedge dp^{0123}.\end{aligned} \tag{7.8}$$

The *one-particle phase space* for particles of mass m is the following submanifold of TM:

$$\Phi_m = \{v \in TM \mid v \text{ future directed, } g(v,v) = -m^2\}. \tag{7.9}$$

[27] If x^μ are coordinates of M then the dx^μ form in each point $p \in M$ a basis of the cotangent space $T^*_p M$. The *bundle coordinates* of $\beta \in T^*_p M$ are then (x^μ, β_ν) if $\beta = \beta_\nu dx^\nu$ and x^μ are the coordinates of p. With such bundle coordinates one can define an atlas, by which T^*M becomes a differentiable manifold.

This is invariant under the geodesic flow. The restriction of X_g to Φ_m will also be denoted by X_g. Ω_{ω_g} induces a volume form Ω_m (see below) on Φ_m, which is also invariant under X_g:

$$L_{X_g}\Omega_m = 0. \tag{7.10}$$

Ω_m is determined as follows (known from Hamiltonian mechanics): Write Ω_{ω_g} in the form

$$\Omega_{\omega_g} = -dL \wedge \sigma,$$

(this is always possible, but σ is not unique), then Ω_m is the pull-back of Ω_{ω_g} by the injection $i:\ \Phi_m \to TM$,

$$\Omega_m = i^*\sigma. \tag{7.11}$$

While σ is not unique (one can, for instance, add a multiple of dL), the form Ω_m is independent of the choice of σ (show this). In natural bundle coordinates a possible choice is

$$\sigma = (-g)dx^{0123} \wedge \frac{dp^{123}}{(-p_0)},$$

because

$$-dL \wedge \sigma = [-g_{\mu\nu}p^\mu dp^\nu + \cdots] \wedge \sigma = (-g)dx^{0123} \wedge g_{\mu 0}p^\mu dp^0 \wedge \frac{dp^{123}}{p_0} = \Omega_{\omega_g}.$$

Hence,

$$\Omega_m = \eta \wedge \Pi_m, \tag{7.12}$$

where η is the volume form of (M, g),

$$\eta = \sqrt{-g}dx^{0123}, \tag{7.13}$$

and

$$\Pi_m = \sqrt{-g}\frac{dp^{123}}{|p_0|}, \tag{7.14}$$

with $p^0 > 0$, and $g_{\mu\nu}p^\mu p^\nu = -m^2$.

We shall need some additional tools. Let Σ be a hypersurface of Φ_m transversal to X_g. On Σ we can use the volume form

$$vol_\Sigma = i_{X_g}\Omega_m \mid \Sigma. \tag{7.15}$$

Now we note that the 6-form

$$\omega_m := i_{X_g}\Omega_m \tag{7.16}$$

on Φ_m is closed,

$$d\omega_m = 0, \tag{7.17}$$

because

$$d\omega_m = di_{X_g}\Omega_m = L_{X_g}\Omega_m = 0$$

(we used $d\Omega_m = 0$ and (7.10)). From (7.12) we obtain

$$\omega_m = (i_{X_g}\eta) \wedge \Pi_m + \eta \wedge i_{X_g}\Pi_m. \tag{7.18}$$

In the special case when Σ is a "time section", i.e., in the inverse image of a spacelike submanifold of M under the natural projection $\Phi_m \to M$, then the second term in (7.18) vanishes on Σ, while the first term is on Σ according to (7.5) equal to $i_p\eta \wedge \Pi_m,\ p = p^\mu \partial/\partial x^\mu$. Thus, we have on a time section[28] Σ

$$\boxed{\mathrm{vol}_\Sigma = \omega_m \mid \Sigma = i_p\eta \wedge \Pi_m.} \tag{7.19}$$

Let f be a one-particle distribution function on Φ_m, defined such that the number of particles in a time section Σ is

$$N(\Sigma) = \int_\Sigma f\omega_m. \tag{7.20}$$

The particle number current density is

$$n^\mu(x) = \int_{P_m(x)} f p^\mu \Pi_m, \tag{7.21}$$

where $P_m(x)$ is the fiber over x in Φ_m (all momenta with $\langle p,p\rangle = -m^2$). Similarly,, one defines the energy-momentum tensor, etc.

Let us show that

$$n^\mu{}_{;\mu} = \int_{P_m} \left(L_{X_g} f\right) \Pi_m. \tag{7.22}$$

We first note that (always in Φ_m)

$$d(f\omega_m) = \left(L_{X_g} f\right) \Omega_m. \tag{7.23}$$

Indeed, because of (7.17) the left-hand side of this equation is

$$df \wedge \omega_m = df \wedge i_{X_g}\Omega_m = \left(i_{X_g} df\right) \wedge \Omega_m = \left(L_{X_g} f\right) \Omega_m.$$

Now, let D be a domain in Φ_m which is the inverse of a domain $\bar{D} \subset M$ under the projection $\Phi_m \to M$. Then we have on the one hand by (7.18), setting $i_X\eta \equiv X^\mu \sigma_\mu$,

$$\int_{\partial D} f\omega_m = \int_{\partial \bar{D}} \sigma_\mu \int_{P_m(x)} p^\mu f \Pi_m = \int_{\partial \bar{D}} \sigma_\mu n^\mu = \int_{\partial \bar{D}} i_n \eta = \int_{\bar{D}} (\nabla \cdot n)\eta.$$

On the other hand, by (7.23) and (7.12)

$$\int_{\partial D} f\omega_m = \int_D d(f\omega_m) = \int_D \left(L_{X_g} f\right) \Omega_m = \int_{\bar{D}} \eta \int_{P_m(x)} \left(L_{X_g} f\right) \Pi_m.$$

Since $\bar{D}$ is arbitrary, we indeed obtain (7.22).

The proof of the following equation for the energy-momentum tensor

$$T^{\mu\nu}{}_{;\nu} = \int_{P_m} p^\mu \left(L_{X_g} f\right) \Pi_m \tag{7.24}$$

can be reduced to the previous proof by considering instead of n^ν the vector field $N^\nu := v_\mu T^{\mu\nu}$, where v_μ is geodesic in x.

[28] Note that in Minkowski spacetime we get for a constant time section $vol_\Sigma = dx^{123} \wedge dp^{123}$.

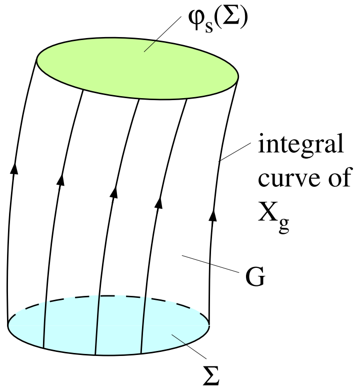

Fig. 7.1 Picture for the proof of (7.25).

7.2 The general relativistic Boltzmann equation

Let us first consider particles for which collisions can be neglected (e.g. neutrinos at temperatures much below 1 MeV). Then the conservation of the particle number in a domain that is comoving with the flow ϕ_s of X_g means that the integrals

$$\int_{\phi_s(\Sigma)} f\omega_m,$$

Σ as before a hypersurface of Φ_m transversal to X_g, are independent of s. We now show that this implies the *collisionless Boltzmann equation*

$$\boxed{L_{X_g} f = 0.} \tag{7.25}$$

The proof of this expected result proceeds as follows. Consider a 'cylinder' $\mathcal{G}$, sweping by Σ under the flow ϕ_s in the interval $[0, s]$ (see Fig. 7.1), and the integral

$$\int_{\mathcal{G}} L_{X_g} f\Omega_m = \int_{\partial\mathcal{G}} f\omega_m$$

(we used Eq. (7.23)). Since $i_{X_g}\omega_m = i_{X_g}(i_{X_g}\Omega_m) = 0$, the integral over the mantle of the cylinder vanishes, while those over Σ and $\phi_s(\Sigma)$ cancel (conservation of particles). Because Σ and s are arbitrary, we conclude that (7.25) must hold.

From (7.22) and (7.23) we obtain, as expected, the conservation of the particle number current density: ${n^\mu}_{;\mu} = 0$.

With collisions, the Boltzmann equation has the symbolic form

$$\boxed{L_{X_g} f = C[f]}, \tag{7.26}$$

where $C[f]$ is the "collision term". For the general form of this in terms of the invariant transition matrix element for a two-body collision, see (B.9). In Appendix B we also work this out explicitly for photon-electron scattering.

By (7.24) and (7.26) we have

$$T^{\mu\nu}{}_{;\nu} = Q^\mu, \tag{7.27}$$

with

$$Q^\mu = \int_{P_m} p^\mu C[f]\Pi_m. \tag{7.28}$$

7.3 Perturbation theory (generalities)

We consider again small deviations from Friedmann models, and set correspondingly

$$f = f^{(0)} + \delta f. \tag{7.29}$$

How does δf change under a gauge transformation? At first sight one may think that we simply have $\delta f \to \delta f + L_{T\xi} f^{(0)}$, where $T\xi$ is the lift of the vector field ξ, defining the gauge transformation, to the tangent bundle. (We recall that $T\xi$ is obtained as follows: Let ϕ_s be the flow of ξ and consider the flow $T\phi_s$ on TM, $T\phi_s =$ tangent map. Then $T\xi$ is the vector field belonging to $T\phi_s$.) Unfortunately, things are not quite as simple, because f is only defined on the one-particle subspace of TM, and this is also perturbed when the metric is changed. One way of getting the right transformation law is given in [33]. Here, I present a more pedestrian, but simpler derivation.

First, we introduce convenient independent variables for the distribution function. For this we choose an adapted orthonormal frame $\{e_{\hat{\mu}},\ \hat{\mu} = 0,1,2,3\}$ for the perturbed metric (1.16), which we recall

$$g = a^2(\eta)\left\{-(1+2A)d\eta^2 - 2B_{,i}\,dx^i d\eta + [(1+2D)\gamma_{ij} + 2E_{|ij}]dx^i dx^j\right\}. \tag{7.30}$$

$e_{\hat{0}}$ is chosen to be orthogonal to the time slices $\eta = const$, whence

$$e_{\hat{0}} = \frac{1}{\alpha}\left(\partial_\eta + \beta^i \partial_i\right),\ \alpha = 1 + A,\ \beta_i = B_{,i}. \tag{7.31}$$

This is indeed normalized and perpendicular to ∂_i. At the moment we do not need explicit expressions for the spatial basis $e_{\hat{i}}$ tangential to $\eta = const$.

From

$$p = p^{\hat{\mu}} e_{\hat{\mu}} = p^\mu \partial_\mu$$

we see that $p^{\hat{0}}/\alpha = p^0$. From now on we consider massless particles and set[29] $q = p^{\hat{0}}$, whence

$$q = a(1+A)p^0. \tag{7.32}$$

Furthermore, we use the unit vector $\gamma^i = p^{\hat{i}}/q$. Then the distribution function can be regarded as a function of η, x^i, q, γ^i, and this we shall adopt in what follows. For the case $K = 0$, which we now consider for simplicity, the unperturbed tetrad is $\{\frac{1}{a}\partial_\eta,\ \frac{1}{a}\partial_i\}$, and for the unperturbed situation we have $q = ap^0,\ p^i = p^0\gamma^i$.

As a further preparation we interpret the Lie derivative as an infinitesimal coordinate change. Consider the infinitesimal coordinate transformation

$$\bar{x}^\mu = x^\mu - \xi^\mu(x), \tag{7.33}$$

then to first order in ξ

$$\left(L_\xi g\right)_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) - g_{\mu\nu}(x), \tag{7.34}$$

and correspondingly for other tensor fields. One can verify this by a direct comparison of the two sides. For the simplest case of a function F,

$$\bar{F}(x) - F(x) = F(x+\xi) - F(x) = \xi^\mu \partial_\mu F = L_\xi F.$$

Under the transformation (7.33) and its extension to TM the p^μ transform as

$$\bar{p}^\mu = p^\mu - \xi^\mu{}_{,\nu} p^\nu.$$

[29] This definition of q is only used in the present subsection. Later, after eqn. (7.62), q will denote the comoving momentum aq.

We need the transformation law for q. From

$$\bar{q} = a(\bar{\eta})[1 + \bar{A}(\bar{x})]\bar{p}^0$$

and the transformation law (1.18) of A,

$$A \to A + \frac{a'}{a}\xi^0 + \xi^{0\prime},$$

we get

$$\bar{q} = a(\eta)[1 - \mathcal{H}\xi^0][1 + A(x)\mathcal{H}\xi^0 + \xi^{0\prime}][p^0 - \xi^0{}_{,\nu}p^\nu].$$

The last square bracket is equal to $p^0(1 - \xi^{0\prime} - \xi^0{}_{,i}\gamma^i)$. Using also (7.32) we find

$$\bar{q} = q - q\xi^0{}_{,i}\gamma^i. \tag{7.35}$$

Since the unperturbed distribution function $f^{(0)}$ depends only on q and η, we conclude from this that

$$\delta f \to \delta f + q\frac{\partial f^{(0)}}{\partial q}\xi^0{}_{,i}\gamma^i + \xi^0 f^{(0)\prime}. \tag{7.36}$$

Here, we use the equation of motion for $f^{(0)}$. For massless particles this is an equilibrium distribution that is stationary when considered as a function of the *comoving momentum* aq. This means that

$$\frac{\partial f^{(0)}}{\partial \eta} + \frac{\partial f^{(0)}}{\partial q}q' = 0$$

for $(aq)' = 0$, i.e., $q' = -\mathcal{H}q$. Thus,

$$f^{(0)\prime} - \mathcal{H}q\frac{\partial f^{(0)}}{\partial q} = 0. \tag{7.37}$$

If this is used in (7.36) we get

$$\boxed{\delta f \to \delta f + q\frac{\partial f^{(0)}}{\partial q}[\mathcal{H}\xi^0 + \xi^0{}_{,i}\gamma^i]}. \tag{7.38}$$

Since this transformation law involves only ξ^0, we can consider various gauge invariant distribution functions, such as $(\delta f)_\chi,\ (\delta f)_\mathcal{Q}$. From (1.21), $\chi \to \chi + a\xi^0$, we find

$$\mathcal{F}_s := (\delta f)_\chi = \delta f - q\frac{\partial f^{(0)}}{\partial q}[\mathcal{H}(B + E') + \gamma^i(B + E')_{,i}]. \tag{7.39}$$

$\mathcal{F}_s$ reduces to δf in the longitudinal gauge, and we shall mainly work with this gauge invariant perturbation. In the literature sometimes $\mathcal{F}_c := (\delta f)_\mathcal{Q}$ is used. Because of (1.49), $v - B \to (v - B) - \xi^0$, we obtain $\mathcal{F}_c$ from (7.39) in replacing $B + E'$ by $-(v - B)$:

$$\mathcal{F}_c := (\delta f)_\mathcal{Q} = \delta f + q\frac{\partial f^{(0)}}{\partial q}[\mathcal{H}(v - B) + \gamma^i(v - B)_{,i}]. \tag{7.40}$$

Since by (1.56) $(v - B) + (B + E') = V$, we find the relation

$$\mathcal{F}_c = \mathcal{F}_s + q\frac{\partial f^{(0)}}{\partial q}[\mathcal{H}V + \gamma^i V_{,i}]. \tag{7.41}$$

Instead of v, V we could also use the baryon velocities v_b, V_b.

7.4 Liouville operator in the longitudinal gauge

We want to determine the action of the Liouville operator $\mathcal{L} := L_{X_g}$ on $\mathcal{F}_s$. The simplest way to do this is to work in the longitudinal gauge $B = E = 0$.

In this section we do not assume a vanishing K. It is convenient to introduce an adapted orthonormal tetrad

$$e_0 = \frac{1}{a(1+A)}\partial_\eta,\ e_i = \frac{1}{a(1+D)}\hat{e}_i, \tag{7.42}$$

where $\hat{e}_i$ is an orthonormal basis for the unperturbed space (Σ, γ). Its dual basis will be denoted by $\hat{\vartheta}^i$, and that of e_μ by θ^μ. We have

$$\theta^0 = (1+A)\bar{\theta}^0,\ \theta^i = (1+D)\bar{\theta}^i, \tag{7.43}$$

where

$$\bar{\theta}^0 = a(\eta)d\eta,\ \bar{\theta}^i = a(\eta)\hat{\vartheta}^i. \tag{7.44}$$

Connection forms. The unperturbed connection forms have been obtained in Sect. 0.1.2. In the present notation they are

$$\bar{\omega}^i{}_0 = \bar{\omega}^0{}_i = \frac{a'}{a^2}\bar{\theta}^i,\ \bar{\omega}^i{}_j = \hat{\omega}^i{}_j, \tag{7.45}$$

where $\hat{\omega}^i{}_j$ are the connection forms of (Σ, γ) relative to $\hat{\vartheta}^i$.

For the determination of the perturbations $\delta\omega^\mu{}_\nu$ of the connection forms we need $d\theta^\mu$. In the following calculation we make use of the first structure equations, both for the unperturbed and the actual metric. The former, together with (7.45), implies that the first term in

$$d\theta^0 = (1+A)d\bar{\theta}^0 + dA \wedge \bar{\theta}^0$$

vanishes. Using the notation $dA = A'd\eta + A_{|i}\bar{\theta}^i = A_{|\mu}\bar{\theta}^\mu$ we obtain

$$d\theta^0 = A_{|i}\bar{\theta}^i \wedge \bar{\theta}^0. \tag{7.46}$$

Similarly,

$$d\theta^i = (1+D)d\bar{\theta}^i + dD \wedge \bar{\theta}^i = (1+D)[-\bar{\omega}^i{}_j \wedge \bar{\theta}^j - \bar{\omega}^i{}_0 \wedge \bar{\theta}^0] + D_{|j}\bar{\theta}^j \wedge \bar{\theta}^i + D_{|0}\bar{\theta}^0 \wedge \bar{\theta}^i. \tag{7.47}$$

On the other hand, inserting $\omega^\mu{}_\nu = \bar{\omega}^\mu{}_\nu + \delta\omega^\mu{}_\nu$ into $d\theta^\mu = -\omega^\mu{}_\nu \wedge \theta^\nu$, and comparing first orders, we obtain the equations

$$-\delta\omega^0{}_i \wedge \bar{\theta}^i - \underbrace{\bar{\omega}^0{}_i \wedge (D\bar{\theta}^i)}_{0} = -A_{|i}\bar{\theta}^0 \wedge \bar{\theta}^i, \tag{7.48}$$

$$\begin{aligned} &-\delta\omega^i{}_0 \wedge \bar{\theta}^0 - \delta\omega^i{}_j \wedge \bar{\theta}^j - \bar{\omega}^i{}_0 \wedge A\bar{\theta}^0 - \bar{\omega}^i{}_j \wedge D\bar{\theta}^j = \\ &\quad -D\bar{\omega}^i{}_j \wedge \bar{\theta}^j - D\bar{\omega}^i{}_0 \wedge \bar{\theta}^0 + D_{|j}\bar{\theta}^j \wedge \bar{\theta}^i + D_{|0}\bar{\theta}^0 \wedge \bar{\theta}^i. \end{aligned} \tag{7.49}$$

Eq. (7.48) requires

$$\delta\omega^0{}_i = A_{|i}\bar{\theta}^0 + (\propto \bar{\theta}^i). \tag{7.50}$$

Let us try the guess

$$\delta\omega^i{}_j = -D_{|i}\bar{\theta}^j + D_{|j}\bar{\theta}^i \tag{7.51}$$

and insert this into (7.49). This gives

$$-\delta\omega^i{}_0 \wedge \bar{\theta}^0 - A\bar{\omega}^i{}_0 \wedge \bar{\theta}^0 = -D\bar{\omega}^i{}_0 \wedge \bar{\theta}^0 + D_{|0}\bar{\theta}^0 \wedge \bar{\theta}^i, \tag{7.52}$$

and this is satisfied if the last term in (7.50) is chosen according to

$$\delta\omega^0{}_i = A_{|i}\bar{\theta}^0 - (A-D)\bar{\omega}^0{}_i + \frac{1}{a}D'\bar{\theta}^i. \tag{7.53}$$

Since the first structure equations are now all satisfied (to first order) our guess (7.51) is correct, and we have determined all $\delta\omega^\mu{}_\nu$.

From (7.45) and (7.53) we get to first order

$$\omega^i{}_0 = \left[\frac{a'}{a^2}(1-A) + \frac{1}{a}D'\right]\theta^i + A_{|i}\theta^0. \tag{7.54}$$

We shall not need $\omega^i{}_j$ explicitly, except for the property $\omega^i{}_j(e_0) = 0$, which follows from (7.45) and (7.51).

We take the spatial components p^i of the momenta p relative to the orthonormal tetrad $\{e_\mu\}$ as independent variables of f (beside x). Then

$$\boxed{\mathcal{L}f = p^\mu e_\mu(f) - \omega^i{}_\alpha(p)p^\alpha \frac{\partial f}{\partial p^i}}\ (p = p^\mu e_\mu). \tag{7.55}$$

Derivation. Eq. (7.55) follows from (7.5) and the result of the following consideration.

Let $X = \sum_{i=1}^{n+1} \xi^i \partial_i$ be a vector field on a domain of $\mathbf{R}^{n+1}$ and let Σ be a hypersurface in $\mathbf{R}^{n+1}$, parametrized by

$$\varphi : U \subset \mathbf{R}^n \to \mathbf{R}^{n+1},\ (x^1, \cdots x^n) \mapsto (x^1, \cdots x^n, g(x^1, \cdots x^n)),$$

to which X is tangential. Furthermore, let f be a function on Σ, which we regard as a function of $x^1, \cdots, x^n$. I claim that

$$X(f) = \sum_{i=1}^n \xi^i \frac{\partial(f\circ\varphi)}{\partial x^i}. \tag{7.56}$$

This can be seen as follows: Extend f in some manner to a neighborhood of Σ (at least locally). Then

$$X(f) \mid \Sigma = \sum_{i=1}^n \left(\xi^i \frac{\partial f}{\partial x^i} + \xi^{n+1}\frac{\partial f}{\partial x^{n+1}}\right)\Bigg|_{x^{n+1}=g(x^1,\cdots x^n)}. \tag{7.57}$$

Now, we have on Σ : $dg - dx^{n+1} = 0$ and thus $\langle dg - dx^{n+1}, X\rangle = 0$ since X is tangential. Using (7.57) this implies

$$\xi^{n+1} = \sum_{i=1}^n \xi^i \frac{\partial g}{\partial x^i},$$

whence (7.57) gives by the chain rule

$$X(f) \mid \Sigma = \sum_{i=1}^{n} \xi^i \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial x^{n+1}} \frac{\partial g}{\partial x^i} \right) = \sum_{i=1}^{n} \xi^i \frac{\partial (f \circ \varphi)}{\partial x^i}.$$

This fact was used in (7.55) for the vector field

$$X_g = p^\mu e_\mu - \omega^\mu{}_\alpha(p) p^\alpha \frac{\partial}{\partial p^\mu}. \tag{7.58}$$

$\mathcal{L}f$ to first order. For $\mathcal{L}f$ we need

$$p^\mu e_\mu(f) = p^0 \frac{1}{a}(1-A)f' + p^i e_i(f) = p^0 \frac{1}{a}(1-A)f' + p^i \frac{1}{a}\hat{e}_i(\delta f)$$

and

$$\begin{aligned} \omega^i{}_\alpha(p)p^\alpha \frac{\partial}{\partial p^i} &= \omega^i{}_0(p)p^0 \frac{\partial}{\partial p^i} + \omega^i{}_j(p)p^j \frac{\partial}{\partial p^i} \\ &= [\omega^i{}_0(e_0)p^0 + \omega^i{}_0(\mathbf{p})]p^0 \frac{\partial}{\partial p^i} + [\omega^i{}_j(e_0)p^0 + \omega^i{}_j(\mathbf{p})]p^j \frac{\partial}{\partial p^i}. \end{aligned}$$

From (7.54) we get $\omega^i{}_0(e_0) = A^{|i}$, and

$$\omega^i{}_0(\mathbf{p}) = \left[\frac{a'}{a^2}(1-A) + \frac{1}{a}D' \right] p^i.$$

Furthermore, the Gauss equation implies $\omega^i{}_j(\mathbf{p}) = \tilde{\omega}^i{}_j(\mathbf{p})$, where $\tilde{\omega}^i{}_j$ are the connection forms of the spatial metric (see Appendix A of [1]).

As an intermediate result we obtain

$$\begin{aligned} \mathcal{L}f &= (1-A)\frac{p^0}{a}f' + \frac{p^i}{a}\hat{e}_i(\delta f) \\ &\quad - \left[\tilde{\omega}^i{}_j(\mathbf{p})p^j + (p^0)^2 A^{|i} + \frac{p^0}{a}D'p^i + p^0\frac{a'}{a^2}(1-A)p^i \right] \frac{\partial f}{\partial p^i}. \end{aligned} \tag{7.59}$$

From now on we use as independent variables $\eta, x^i, p, \gamma^i = p^i/p$ ($p = [\sum_i (p^i)^2]^{1/2}$). We have

$$\frac{\partial f}{\partial p^i} = \frac{p_i}{p}\frac{\partial f}{\partial p} + \frac{1}{p}\left(\delta^l{}_i - p_i p^l/p^2 \right) \frac{\partial f}{\partial \gamma^l}. \tag{7.60}$$

Contracting this with $\tilde{\omega}^i{}_j(\mathbf{p})p^j$, appearing in (7.59), the first term on the right in (7.60) gives no contribution (antisymmetry of $\tilde{\omega}^i{}_j$), and since $\partial f/\partial \gamma^l$ is of first order we can replace $\tilde{\omega}^i{}_j$ by the connection forms of the unperturbed metric $a^2\gamma_{ij}$; these are the same as the connection forms $\hat{\omega}^i{}_j$ of γ_{ij} relative to $\hat{\vartheta}^i$. What remains is thus

$$\hat{\omega}^i{}_j(\mathbf{p})\frac{p^j}{p}\left(\delta^l{}_i - p_i p^l/p^2 \right)\frac{\partial \delta f}{\partial \gamma^l} = \hat{\omega}^i{}_j(\mathbf{p})\frac{p^j}{p}\frac{\partial \delta f}{\partial \gamma^i} = \frac{p}{a}\gamma^j\gamma^k\hat{\Gamma}^i{}_{jk}\frac{\partial \delta f}{\partial \gamma^i}.$$

Inserting this and (7.60) into (7.59) gives in zeroth order for the Liouville operator

$$(\mathcal{L}f)^{(0)} = \frac{p^0}{a}\left(f^{(0)'} - \mathcal{H}p\frac{\partial f^{(0)}}{\partial p} \right),$$

and the first order contribution is

$$-A(\mathcal{L}f)^{(0)} + \frac{p^0}{a}(\delta f)' + \frac{p^i}{a}\hat{e}_i(\delta f) - \frac{p}{a}\gamma^j\gamma^k\hat{\Gamma}^i{}_{jk}\frac{\partial \delta f}{\partial \gamma^i} - \frac{(p^0)^2}{ap}\hat{e}_i(A)p^i\frac{\partial f^{(0)}}{\partial p} - \frac{p^0}{a}D'p\frac{\partial f^{(0)}}{\partial p} - \frac{p^0}{a}\mathcal{H}p\frac{\partial \delta f}{\partial p}.$$

Therefore, we obtain for the Liouville operator, up to first order,

$$\frac{a}{p^0}\mathcal{L}f = (1-A)\left(f^{(0)\prime} - \mathcal{H}p\frac{\partial f^{(0)}}{\partial p}\right) + (\delta f)' - \mathcal{H}p\frac{\partial \delta f}{\partial p} + \frac{p^i}{p^0}\hat{e}_i(\delta f) - \frac{p}{p^0}\gamma^j\gamma^k\hat{\Gamma}^i{}_{jk}\frac{\partial \delta f}{\partial \gamma^i} - p\left[D' + \frac{p^0}{p}\gamma^i\hat{e}_i(A)\right]\frac{\partial f^{(0)}}{\partial p}. \tag{7.61}$$

As a first application we consider the collisionless Boltzmann equation for $m = 0$. In zeroth order we get the equation (7.37) (q in that equation is our present p). The perturbation equation becomes

$$(\delta f)' - \mathcal{H}p\frac{\partial \delta f}{\partial p} + \gamma^i\hat{e}_i(\delta f) - \gamma^j\gamma^k\hat{\Gamma}^i{}_{jk}\frac{\partial \delta f}{\partial \gamma^i} - \left[D' + \gamma^i\hat{e}_i(A)\right]p\frac{\partial f^{(0)}}{\partial p} = 0. \tag{7.62}$$

It will be more convenient to write this in terms of the *comoving momentum*, which we denote by $q,\ q = ap$. (This slight change of notation is unfortunate, but should not give rise to confusions, because the equations at the beginning of Sect. 7.3, with the earlier meaning $q \equiv p$, will no more be used. But note that (7.38)-(7.41) remain valid with the present meaning of q.) Eq. (7.62) then becomes

$$\boxed{(\partial_\eta + \gamma^i\hat{e}_i)\delta f - \hat{\Gamma}^i{}_{jk}\gamma^j\gamma^k\frac{\partial \delta f}{\partial \gamma^i} - \left[D' + \gamma^i\hat{e}_i(A)\right]q\frac{\partial f^{(0)}}{\partial q} = 0.} \tag{7.63}$$

It is obvious how to write this in gauge invariant form

$$\boxed{(\partial_\eta + \gamma^i\hat{e}_i)\mathcal{F}_s - \hat{\Gamma}^i{}_{jk}\gamma^j\gamma^k\frac{\partial \mathcal{F}_s}{\partial \gamma^i} = \left[\Phi' + \gamma^i\hat{e}_i(\Psi)\right]q\frac{\partial f^{(0)}}{\partial q}.} \tag{7.64}$$

(From this the collisionless Boltzmann equation follows in any gauge; write this out.)

In the special case $K = 0$ we obtain for the Fourier amplitudes, with $\mu := \hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}$,

$$\boxed{\mathcal{F}_s' + i\mu k\mathcal{F}_s = \left[\Phi' + ik\mu\Psi\right]q\frac{\partial f^{(0)}}{\partial q}.} \tag{7.65}$$

This equation can be used for neutrinos as long as their masses are negligible (the generalization to the massive case is easy).

7.5 Boltzmann equation for photons

The collision term for photons due to Thomson scattering on electrons will be derived in Appendix B. We shall find that in the longitudinal gauge, ignoring polarization effects (to be discussed later),

$$C[f] = x_e n_e \sigma_T p\left[\langle\delta f\rangle - \delta f - q\frac{\partial f^{(0)}}{\partial q}\gamma^i\hat{e}_i(v_b) + \frac{3}{4}Q_{ij}\gamma^i\gamma^j\right]. \tag{7.66}$$

On the right, $x_e n_e$ is the unperturbed free electron density (x_e = ionization fraction), σ_T the Thomson cross section, and v_b the scalar velocity perturbation of the baryons. Furthermore, we have introduced the spherical averages

$$\langle \delta f \rangle = \frac{1}{4\pi} \int_{S^2} \delta f \, d\Omega_\gamma, \tag{7.67}$$

$$Q_{ij} = \frac{1}{4\pi} \int_{S^2} [\gamma_i \gamma_j - \frac{1}{3}\delta_{ij}] \delta f \, d\Omega_\gamma. \tag{7.68}$$

(Because of the tight coupling of electrons and ions we can take $v_e = v_b$.)

Since the left-hand side of (7.63) is equal to $(a/p_0)\mathcal{L}f$, the linearized Boltzmann equation becomes

$$(\partial_\eta + \gamma^i \hat{e}_i)\delta f - \hat{\Gamma}^i{}_{jk}\gamma^j\gamma^k \frac{\partial \delta f}{\partial \gamma^i} - \left[D' + \gamma^i \hat{e}_i(A)\right] q \frac{\partial f^{(0)}}{\partial q}$$
$$= a x_e n_e \sigma_T \left[\langle \delta f \rangle - \delta f - q \frac{\partial f^{(0)}}{\partial q} \gamma^i \hat{e}_i(v_b) + \frac{3}{4} Q_{ij}\gamma^i\gamma^j \right]. \tag{7.69}$$

This can immediately be written in a gauge invariant form, by replacing

$$\delta f \to \mathcal{F}_s, \; v_b \to V_b, \; A \to \Psi, \; D \to \Phi. \tag{7.70}$$

In our applications to the CMB we work with the gauge invariant *brightness temperature* perturbation

$$\Theta_s(\eta, x^i, \gamma^j) = \int \mathcal{F}_s q^3 dq \Big/ \; 4 \int f^{(0)} q^3 dq. \tag{7.71}$$

(The factor 4 is chosen because of the Stephan-Boltzmann law, according to which $\delta\rho/\rho = 4\delta T/T$.) It is simple to translate the Boltzmann equation for $\mathcal{F}_s$ to a kinetic equation for Θ_s. Using

$$\int q \frac{\partial f^{(0)}}{\partial q} q^3 dq = -4 \int f^{(0)} q^3 dq$$

we obtain for the convective part (from the left-hand side of the Boltzmann equation for $\mathcal{F}_s$)

$$\Theta_s' + \gamma^i \hat{e}_i(\Theta_s) - \hat{\Gamma}^i{}_{jk}\gamma^j\gamma^k \frac{\partial \Theta_s}{\partial \gamma^i} + \Phi' + \gamma^i \hat{e}_i(\Psi).$$

The collision term gives

$$\dot{\tau}(\theta_0 - \Theta_s + \gamma^i \hat{e}_i V_b + \frac{1}{16}\gamma^i\gamma^j \Pi_{ij}),$$

with $\dot{\tau} = x_e n_e \sigma_T a/a_0, \; \theta_0 = \langle \Theta_s \rangle$ (spherical average), and

$$\frac{1}{12}\Pi_{ij} := \frac{1}{4\pi} \int [\gamma_i\gamma_j - \frac{1}{3}\delta_{ij}] \Theta_s \, d\Omega_\gamma. \tag{7.72}$$

The basic equation for Θ_s is thus

$$(\Theta_s + \Psi)' + \gamma^i \hat{e}_i(\Theta_s + \Psi) - \hat{\Gamma}^i{}_{jk}\gamma^j\gamma^k \frac{\partial}{\partial \gamma^i}(\Theta_s + \Psi) =$$
$$(\Psi' - \Phi') + \dot{\tau}(\theta_0 - \Theta_s + \gamma^i \hat{e}_i V_b + \frac{1}{16}\gamma^i\gamma^j \Pi_{ij}). \tag{7.73}$$

In a mode decomposition we get for $K = 0$ (I drop from now on the index s on Θ):

$$\boxed{\Theta' + ik\mu(\Theta + \Psi) = -\Phi' + \dot{\tau}[\theta_0 - \Theta - i\mu V_b - \frac{1}{10}\theta_2 P_2(\mu)]} \tag{7.74}$$

(recall $V_b \to -(1/k)V_b$). The last term on the right comes about as follows. We expand the Fourier modes $\Theta(\eta, k^i, \gamma^j)$ in terms of Legendre polynomials

$$\Theta(\eta, k^i, \gamma^j) = \sum_{l=0}^{\infty}(-i)^l \theta_l(\eta, k) P_l(\mu), \quad \mu = \hat{\boldsymbol{k}} \cdot \boldsymbol{\gamma}, \tag{7.75}$$

and note that

$$\frac{1}{16}\gamma^i\gamma^j\Pi_{ij} = -\frac{1}{10}\theta_2 P_2(\mu) \tag{7.76}$$

(Exercise). The expansion coefficients $\theta_l(\eta, k)$ in (7.75) are the *brightness moments*[30]. The lowest three have simple interpretations. We show that in the notation of Chap. 1:

$$\theta_0 = \frac{1}{4}\Delta_{s\gamma},\ \theta_1 = V_\gamma,\ \theta_2 = \frac{5}{12}\Pi_\gamma. \tag{7.77}$$

Derivation of (7.77). We start from the general formula (see Sect. 7.1)

$$T^\mu_{(\gamma)\nu} = \int p^\mu p_\nu f(p)\frac{d^3p}{p^0} = \int p^\mu p_\nu f(p) p dp\, d\Omega_\gamma. \tag{7.78}$$

According to the general parametrization (1.156) we have

$$\delta T^0_{(\gamma)0} = -\delta\rho_\gamma = -\int p^2 \delta f(p) p dp\, d\Omega_\gamma. \tag{7.79}$$

Similarly, in zeroth order

$$T^{(0)0}_{(\gamma)\ \ 0} = -\rho^{(0)}_\gamma = -\int p^2 f^{(0)}(p) p dp\, d\Omega_\gamma. \tag{7.80}$$

Hence,

$$\frac{\delta\rho_\gamma}{\rho^{(0)}_\gamma} = \frac{\int q^3 \delta f\ dq\ d\Omega_\gamma}{\int q^3 f^{(0)} dq\ d\Omega_\gamma}. \tag{7.81}$$

In the longitudinal gauge we have $\Delta_{s\gamma} = \delta\rho_\gamma/\rho^{(0)}_\gamma$, $\mathcal{F}_s = \delta f$ and thus by (7.71) and (7.75)

$$\Delta_{s\gamma} = 4\frac{1}{4\pi}\int \Theta\ d\Omega_\gamma = 4\theta_0.$$

Similarly,

$$T^i_{(\gamma)0} = -h_\gamma v^{|i}_\gamma = \int p^i p_0 \delta f p dp\ d\Omega_\gamma$$

[30] In the literature the normalization of the θ_l is sometimes chosen differently: $\theta_l \to (2l+1)\theta_l$.

or

$$v_\gamma^{|i} = \frac{3}{4\rho_\gamma^{(0)}} \int \gamma^i \delta f p^3 dp\, d\Omega_\gamma. \tag{7.82}$$

With (7.80) and (7.71) we get

$$V_\gamma^{|i} = \frac{3}{4\pi} \int \gamma^i \Theta\, d\Omega_\gamma. \tag{7.83}$$

For the Fourier amplitudes this gauge invariant equation gives ($V_\gamma \to -(1/k)V_\gamma$)

$$-iV_\gamma \hat{k}^i = \frac{3}{4\pi} \int \gamma^i \Theta\, d\Omega_\gamma$$

or

$$-iV_\gamma = \frac{3}{4\pi} \int \mu \Theta\, d\Omega_\gamma.$$

Inserting here the decomposition (7.75) leads to the second relation in (7.77).

For the third relation we start from (1.156) and (7.79)

$$\delta T^i_{(\gamma)j} = \delta p_\gamma \delta^i{}_j + p_\gamma^{(0)} \left(\Pi^{|i}_{\gamma|j} - \frac{1}{3}\delta^i{}_j \triangle \Pi_\gamma \right) = \int p^i p_j \delta f p\, dp\, d\Omega_\gamma.$$

From this and (7.79) we see that $\delta p_\gamma = \frac{1}{3}\delta\rho_\gamma$, thus $\Gamma_\gamma = 0$ (no entropy production with respect to the photon fluid). Furthermore, since $p_\gamma^{(0)} = \frac{1}{3}\rho_\gamma^{(0)}$ we obtain with (7.72)

$$\Pi^{|i}_{\gamma|j} - \frac{1}{3}\delta^i{}_j \triangle \Pi_\gamma = 4 \cdot 3 \frac{1}{4\pi} \int [\gamma^i \gamma_j - \frac{1}{3}\delta^i{}_j] \Theta\, d\Omega_\gamma = \Pi^i{}_j.$$

In momentum space ($\Pi_\gamma \to (1/k^2)\Pi_\gamma$) this becomes

$$-(\hat{k}^i \hat{k}_j - \frac{1}{3})\Pi_\gamma = \Pi^i{}_j$$

or, contracting with $\gamma_i \gamma^j$ and using (7.76), the desired result.

Hierarchy for moment equations

Now we insert the expansion (7.75) into the Boltzmann equation (7.74). Using the recursion relations for the Legendre polynomials,

$$\mu P_l(\mu) = \frac{l}{2l+1} P_{l-1}(\mu) + \frac{l+1}{2l+1} P_{l+1}(\mu), \tag{7.84}$$

we obtain

$$\sum_{l=0}^{\infty} (-i)^l \theta_l' P_l + ik \sum_{l=0}^{\infty} (-i)^l \theta_l \left[\frac{l}{2l+1} P_{l-1} + \frac{l+1}{2l+1} P_{l+1} \right] + ik\Psi P_1$$
$$= -\Phi' P_0 - \dot{\tau} \left[\sum_{l=1}^{\infty} (-i)^l \theta_l P_l - iV_b P_1 - \frac{1}{10}\theta_2 P_2 \right].$$

Comparing the coefficients of P_l leads to the following hierarchy of ordinary differential equations for the brightness moments $\theta_l(\eta)$:

$$\theta_0' = -\frac{1}{3}k\theta_1 - \Phi', \tag{7.85}$$

$$\theta_1' = k\Big(\theta_0 + \Psi - \frac{2}{5}\theta_2\Big) - \dot\tau(\theta_1 - V_b), \tag{7.86}$$

$$\theta_2' = k\Big(\frac{2}{3}\theta_1 - \frac{3}{7}\theta_3\Big) - \dot\tau\frac{9}{10}\theta_2, \tag{7.87}$$

$$\theta_l' = k\Big(\frac{l}{2l-1}\theta_{l-1} - \frac{l+1}{2l+3}\theta_{l+1}\Big), \quad l > 2. \tag{7.88}$$

At this point it is interesting to compare the first moment equation (7.86) with the phenomenological equation (1.212) for γ:

$$V_\gamma' = k\Psi + \frac{1}{4}\Delta_{s\gamma} - \frac{1}{6}k\Pi_\gamma + \mathcal{H}F_\gamma. \tag{7.89}$$

On the other hand, (7.86) can be written with (7.77) as

$$V_\gamma' = k\Psi + \frac{1}{4}\Delta_{s\gamma} - \frac{1}{6}k\Pi_\gamma - \dot\tau(V_\gamma - V_b). \tag{7.90}$$

The two equations agree if the phenomenological force F_γ is given by

$$\boxed{\mathcal{H}F_\gamma = -\dot\tau(V_\gamma - V_b).} \tag{7.91}$$

From the general relation (1.203) we then obtain

$$F_b = -\frac{h_\gamma}{h_b}F_\gamma = -\frac{4\rho_\gamma}{3\rho_b}F_\gamma. \tag{7.92}$$

7.6 Tensor contributions to the Boltzmann equation

Considering again only the case $K = 0$, the metric (5.57) for tensor perturbations becomes

$$g_{\mu\nu} = a^2(\eta)[\eta_{\mu\nu} + 2H_{\mu\nu}], \tag{7.93}$$

where the $H_{\mu\nu}$ satisfy the TT gauge conditions (5.58). An adapted orthonormal tetrad is

$$\theta^0 = a(\eta)d\eta,\ \theta^i = a(\delta^i{}_j + H^i{}_j)dx^j. \tag{7.94}$$

Relative to this the connection forms are (Exercise):

$$\omega^0{}_i = \frac{a'}{a^2}\theta^i + \frac{1}{a}H_{ij}'\theta^j,\ \omega^i{}_j = \frac{1}{2a}(H^i{}_{k,j} - H_{jk}{}^{,i})\theta^k. \tag{7.95}$$

For $\mathcal{L}f$ we get from (7.55) to first order

$$\begin{aligned}\mathcal{L}f &= \frac{p^0}{a}f' + p^i\frac{1}{a}\hat e_i(f) + \omega^i{}_0(\mathbf{p})p^0\frac{\partial f}{\partial p^i} + \omega^i{}_j(\mathbf{p})p^j\frac{\partial f}{\partial p^i}\\ &= \frac{p^0}{a}\left[f' + \frac{p^i}{p^0}\partial_i f + H_{ij}'p^j\frac{\partial f}{\partial p^i}\right].\end{aligned}$$

Passing again to the variables η, x^i, p, γ^i we obtain instead of (7.61)

$$\frac{a}{p^0}\mathcal{L}f = f^{(0)\prime} - \mathcal{H}p\frac{\partial f^{(0)}}{\partial p} + (\delta f)' - \mathcal{H}p\frac{\partial \delta f}{\partial p} + \frac{p^i}{p^0}\partial_i(\delta f) + H'_{ij}\gamma^i\gamma^j p\frac{\partial f^{(0)}}{\partial p}. \tag{7.96}$$

Instead of (7.63) we now obtain the following collisionless Boltzmann equation

$$\boxed{(\partial_\eta + \gamma^i\partial_i)\delta f + H'_{ij}\gamma^i\gamma^j q\frac{\partial f^{(0)}}{\partial q} = 0.} \tag{7.97}$$

For the temperature (brightness) perturbation this gives

$$\boxed{(\partial_\eta + \gamma^i\partial_i)\Theta = H'_{ij}\gamma^i\gamma^j.} \tag{7.98}$$

This describes the influence of tensor modes on Θ. The evolution of these tensor modes is described according to (5.59) by

$$H''_{ij} + 2\mathcal{H}H'_{ij} - \triangle H_{ij} = 0, \tag{7.99}$$

if we neglect tensor perturbations of the energy-momentum tensor. We shall study the implications of the last two equations for the CMB fluctuations in Sect. 8.6.

8 The physics of CMB anisotropies

We have by now developed all ingredients for a full understanding of the CMB anisotropies. In the present chapter we discuss these for the CDM scenario and primordial initial conditions suggested by inflation (derived in Part II). Other scenarios, involving for instance topological defects, are now strongly disfavored.

We shall begin by collecting all independent perturbation equations, derived in previous chapters. There are fast codes that allow us to solve these equations very accurately, given a set of cosmological parameters. It is, however, instructive to discuss first various qualitative and semi-quantitative aspects. Finally, we shall compare numerical results with observations, and discuss what has already come out of this, which is a lot. In this connection we have to include some theoretical material on polarization effects, because WMAP has already provided quite accurate data for the so-called E-polarization.

The B-polarization is much more difficult to get, and is left to future missions (Planck satellite, etc). This is a very important goal, because accurate data will allow us to determine the power spectrum of the gravity waves.

For further reading I recommend Chap. 8 of [5] and the the two research articles [45, 46]. For a well written review and extensive references, see [48].

8.1 The complete system of perturbation equations

For references in later sections, we collect below the complete system of (independent) perturbation equations for scalar modes and $K = 0$ (see Sects. 1.5.C and 7.5). Let me first recall and add some notation.

Unperturbed *background* quantities: ρ_α, p_α denote the densities and pressures for the species $\alpha = b$ (baryons and electrons), γ (photons), c (cold dark matter); the total density is the sum $\rho = \sum_\alpha \rho_\alpha$, and the same holds for the total pressure p. We also use $w_\alpha = p_\alpha/\rho_\alpha, w = p/\rho$. The sound speed of the baryon-electron fluid is denoted by c_b, and R is the ratio $3\rho_b/4\rho_\gamma$.

Here is the list of gauge invariant *scalar perturbation* amplitudes:

- $\delta_\alpha := \Delta_{s\alpha}, \delta := \Delta_s$: density perturbations ($\delta\rho_\alpha/\rho_\alpha, \delta\rho/\rho$ in the longitudinal gauge); clearly: $\rho\,\delta = \sum \rho_\alpha \delta_\alpha$.
- V_α, V : velocity perturbations; $\rho(1+w)V = \sum_\alpha \rho_\alpha(1+w_\alpha)V_\alpha$.
- θ_l, N_l : brightness moments for photons and neutrinos.
- Π_α, Π : anisotropic pressures; $\Pi = \Pi_\gamma + \Pi_\nu$. For the lowest moments the following relations hold:

$$\delta_\gamma = 4\theta_0, \;\; V_\gamma = \theta_1, \;\; \Pi_\gamma = \frac{12}{5}\theta_2, \tag{8.1}$$

  and similarly for the neutrinos.
- Ψ, Φ: Bardeen potentials for the metric perturbation.

As *independent* amplitudes we can choose: $\delta_b, \delta_c, V_b, V_c, \Phi, \Psi, \theta_l, N_l$. The basic evolution equations consist of three groups.

- *Fluid equations*:

$$\delta_c' = -kV_c - 3\Phi', \tag{8.2}$$

$$V_c' = -aHV_c + k\Psi; \tag{8.3}$$

$$\delta_b' = -kV_b - 3\Phi', \tag{8.4}$$

$$V_b' = -aHV_b + kc_b^2\delta_b + k\Psi + \dot{\tau}(\theta_1 - V_b)/R. \tag{8.5}$$

- *Boltzmann hierarchies* for photons (Eqs. (7.85)–(7.88)) (and the collisionless neutrinos):

$$\theta_0' = -\frac{1}{3}k\theta_1 - \Phi', \tag{8.6}$$

$$\theta_1' = k\Big(\theta_0 + \Psi - \frac{2}{5}\theta_2\Big) - \dot{\tau}(\theta_1 - V_b), \tag{8.7}$$

$$\theta_2' = k\Big(\frac{2}{3}\theta_1 - \frac{3}{7}\theta_3\Big) - \dot{\tau}\frac{9}{10}\theta_2, \tag{8.8}$$

$$\theta_l' = k\Big(\frac{l}{2l-1}\theta_{l-1} - \frac{l+1}{2l+3}\theta_{l+1}\Big), \quad l > 2. \tag{8.9}$$

- *Einstein equations* : We only need the following algebraic ones for each mode:

$$k^2\Phi = 4\pi Ga^2\rho\Big[\delta + 3\frac{aH}{k}(1+w)V\Big], \tag{8.10}$$

$$k^2(\Phi + \Psi) = -8\pi Ga^2 p\,\Pi. \tag{8.11}$$

In arriving at these equations some approximations have been made which are harmless [31], except for one: We have ignored polarization effects in Thomson scattering. For quantitative calculations these have to be included. Moreover, polarization effects are highly interesting, as I shall explain later. We shall take up this topic in Sect. 8.7.

8.2 Acoustic oscillations

In this section we study the photon-baryon fluid. Our starting point is the following approximate system of equations. For the baryons we use (8.4) and (8.5), neglecting the term proportional to c_b^2. We truncate the photon hierarchy, setting $\theta_l = 0$ for $l \geq 3$. So we consider the system of first order equations:

$$\theta_0' = -\frac{1}{3}k\theta_1 - \Phi', \tag{8.12}$$

[31] In the notation of Sect. 1.4 we have set $q_\alpha = \Gamma_\alpha = 0$, and are thus ignoring certain intrinsic entropy perturbations within individual components.

$$\theta_1' = k\Big(\theta_0 + \Psi - \frac{2}{5}\theta_2\Big) - \dot{\tau}(\theta_1 - V_b), \tag{8.13}$$

$$\delta_b' = -kV_b - 3\Phi', \tag{8.14}$$

$$V_b' = -aHV_b + kc_b^2\delta_b + k\Psi + \dot{\tau}(\theta_1 - V_b)/R, \tag{8.15}$$

and (8.8). This is, of course, not closed (Φ and Ψ are "external" potentials).

As long as the mean free path of photons is much shorter than the wavelength of the fluctuation, the optical depth through a wavelength $\sim \dot{\tau}/k$ is large[32]. Thus the evolution equations may be expanded in the small parameter $k/\dot{\tau}$.

In lowest order we obtain $\theta_1 = V_b,\ \theta_l = 0$ for $l \geq 2$, thus $\delta_b' = 3\theta_0'$ $(= 3\delta_\gamma'/4)$.

Going to first order, we can replace in the following form of (8.15)

$$\theta_1 - V_b = \dot{\tau}^{-1} R \left[V_b' + \frac{a'}{a}\theta_1 - k\Psi \right] \tag{8.16}$$

on the right V_b by θ_1:

$$\theta_1 - V_b = \dot{\tau}^{-1} R \left[\theta_1' + \frac{a'}{a}\theta_1 - k\Psi \right]. \tag{8.17}$$

We insert this in (8.13), and set in *first order* also $\theta_2 = 0$:

$$\theta_1' = k(\theta_0 + \Psi) - R\left[\theta_1' + \frac{a'}{a} V_b - k\Psi \right]. \tag{8.18}$$

Using $a'/a = R'/R$, we obtain from this

$$\theta_1' = \frac{1}{1+R}k\theta_0 + k\Psi - \frac{R'}{1+R}\theta_1. \tag{8.19}$$

Combining this with (8.12), we obtain by eliminating θ_1 the *driven oscillator equation*:

$$\boxed{\theta_0'' + \frac{R}{1+R}\frac{a'}{a}\theta_0' + c_s^2 k^2 \theta_0 = F(\eta),} \tag{8.20}$$

with

$$c_s^2 = \frac{1}{3(1+R)},\ F(\eta) = -\frac{k^2}{3}\Psi - \frac{R}{1+R}\frac{a'}{a}\Phi' - \Phi''. \tag{8.21}$$

According to (1.186) and (1.187) c_s is the velocity of sound in the approximation $c_b \approx 0$. It is suggestive to write (8.20) as ($m_{\rm eff} \equiv 1 + R$)

$$(m_{\rm eff}\theta_0')' + \frac{k^2}{3}(\theta_0 + m_{\rm eff}\Psi) = -(m_{\rm eff}\Phi')'. \tag{8.22}$$

This equation provides a lot of insight, as we shall see. It may be interpreted as follows: The change in momentum of the photon-baryon fluid is determined by a competition between pressure restoring and gravitational driving forces.

Let us, in a first step, ignore the time dependence of $m_{\rm eff}$ (i.e., of the baryon-photon ratio R), then we get the forced harmonic oscillator equation

$$m_{\rm eff}\theta_0'' + \frac{k^2}{3}\theta_0 = -\frac{k^2}{3}m_{\rm eff}\Psi - (m_{\rm eff}\Phi')'. \tag{8.23}$$

[32] Estimate $\dot{\tau}/k$ as a function of redshift $z > z_{\rm rec}$ and (aH/k).

The effective mass $m_{\rm eff} = 1 + R$ accounts for the inertia of baryons. Baryons also contribute gravitational mass to the system, as is evident from the right hand side of the last equation. Their contribution to the pressure restoring force is, however, negligible.

We now ignore in (8.23) also the time dependence of the gravitational potentials Φ, Ψ. With (8.21) this then reduces to

$$\theta_0'' + k^2 c_s^2 \theta_0 = -\tfrac{1}{3} k^2 \Psi. \tag{8.24}$$

This simple harmonic oscillator under constant acceleration provided by gravitational infall can immediately be solved:

$$\theta_0(\eta) = [\theta_0(0) + (1+R)\Psi]\cos(kr_s) + \frac{1}{kc_s}\dot{\theta}_0(0)\sin(kr_s) - (1+R)\Psi, \tag{8.25}$$

where $r_s(\eta)$ is the comoving sound horizon $\int c_s d\eta$.

We know (see (6.54)) that for *adiabatic* initial conditions there is only a cosine term. Since we shall see that the "effective" temperature fluctuation is $\Delta T = \theta_0 + \Psi$, we write the result as

$$\Delta T(\eta, k) = [\Delta T(0,k) + R\Psi]\cos(kr_s(\eta)) - R\Psi. \tag{8.26}$$

Discussion

In the radiation dominated phase ($R = 0$) this reduces to $\Delta T(\eta) \propto \cos kr_s(\eta)$, which shows that the oscillation of θ_0 is displaced by gravity. The zero point corresponds to the state at which gravity and pressure are balanced. The displacement $-\Psi > 0$ yields hotter photons in the potential well since gravitational infall not only increases the number density of the photons, but also their energy through gravitational blue shift. However, well after last scattering the photons also suffer a redshift when climbing out of the potential well, which precisely cancels the blue shift. Thus the effective temperature perturbation we see in the CMB anisotropies is indeed $\Delta T = \theta_0 + \Psi$, as we shall explicitely see later.

It is clear from (8.25) that a characteristic wave-number is $k = \pi / r_s(\eta_{\rm dec})$ $\approx \pi / c_s \eta_{\rm dec}$. A spectrum of k-modes will produce a sequence of peaks with wave numbers

$$k_m = m\pi / r_s(\eta_{\rm dec}), \quad m = 1, 2, \dots \, . \tag{8.27}$$

Odd peaks correspond to the compression phase (temperature crests), whereas even peaks correspond to the rarefaction phase (temperature troughs) inside the potential wells. Note also that the characteristic length scale $r_s(\eta_{\rm dec})$, which is reflected in the peak structure, is determined by the underlying unperturbed Friedmann model. This comoving sound horizon at decoupling depends on cosmological parameters, but not on Ω_Λ. Its role will further be discussed below.

Inclusion of baryons not only changes the sound speed, but gravitational infall leads to greater compression of the fluid in a potential well, and thus to a further displacement of the oscillation zero point (last term in (8.25)). This is not compensated by the redshift after last scattering, since the latter is not affected by the baryon content. As a result all peaks from compression are enhanced over those from rarefaction. Hence, the relative heights of the first and second peak is a *sensitive measure of the baryon content*. We shall see that the inferred baryon abundance from the present observations is in complete agreement with the results from big bang nucleosynthesis.

What is the influence of the slow evolution of the effective mass $m_{\rm eff} = 1 + R$? Well, from the adiabatic theorem we know that for a slowly varying $m_{\rm eff}$ the ratio energy/frequency is an adiabatic invariant. If A denotes the amplitude of the oscillation, the energy is $\frac{1}{2} m_{\rm eff} \omega^2 A^2$. According to (8.21) the frequency $\omega = kc_s$ is proportional to $m_{\rm eff}^{-1/2}$. Hence $A \propto \omega^{-1/2} \propto m_{\rm eff}^{1/4} \propto (1+R)^{-1/4}$.

Photon diffusion. In *second order* we do no more neglect θ_2 and use in addition (8.8),

$$\theta_2' = k\Big(\frac{2}{3}\theta_1 - \frac{3}{7}\theta_3\Big) - \dot{\tau}\frac{9}{10}\theta_2, \tag{8.28}$$

with $\theta_3 \simeq 0$. This gives in leading order

$$\theta_2 \simeq \frac{20}{27}\dot{\tau}^{-1}k\theta_1. \tag{8.29}$$

If we neglect in the Euler equation for the baryons the term proportional to a'/a, then the first order equation (8.17) reduces to

$$V_b = \theta_1 - \dot{\tau}^{-1}R[\theta_1' - k\Psi]. \tag{8.30}$$

We use this in (8.16) without the term with a'/a, to get

$$\theta_1 - V_b = \dot{\tau}^{-1}R[\theta_1' - k\Psi] - \frac{R^2}{\dot{\tau}^2}(\theta_1'' - k\Psi'). \tag{8.31}$$

This is now used in (8.13) with the approximation (8.29) for θ_2. One finds

$$(1+R)\theta_1' = k[\theta_0 + (1+R)\Psi] - \frac{8}{27}\frac{k^2}{\dot{\tau}} + \frac{R^2}{\dot{\tau}}(\theta_1'' - k\Psi'). \tag{8.32}$$

In the last term we use the first order approximation of this equation, i.e.,

$$(1+R)(\theta_1' - k\Psi) = k\theta_0,$$

and obtain

$$(1+R)\theta_1' = k[\theta_0 + (1+R)\Psi] - \frac{8}{27}\frac{k^2}{\dot{\tau}} + \frac{k}{\dot{\tau}}\frac{R^2}{1+R}\theta_0'. \tag{8.33}$$

Finally, we eliminate in this equation θ_1' with the help of (8.12). After some rearrangements we obtain

$$\theta_0'' + \frac{k^2}{3\dot{\tau}}\left[\frac{R^2}{(1+R)^2} + \frac{8}{9}\frac{1}{1+R}\right]\theta_0' + \frac{k^2}{3(1+R)}\theta_0 = -\frac{k^2}{3}\Psi - \Phi'' - \frac{8}{27}\frac{k^2}{3\dot{\tau}}\frac{1}{1+R}\Phi'. \tag{8.34}$$

The term proportional to θ_0' in this equation describes the *damping due to photon diffusion*. Let us determine the characteristic damping scale.

If we neglect in the homogeneous equation the time dependence of all coefficients, we can make the ansatz $\theta_0 \propto \exp(i\int \omega d\eta)$. (We thus ignore variations on the time scale $a/\dot{a}$ with those corresponding to the oscillator frequency ω.) The dispersion law is determined by

$$-\omega^2 + i\frac{\omega}{3}\frac{k^2}{\dot{\tau}}\left[\frac{R^2}{(1+R)^2} + \frac{8}{9}\frac{1}{1+R}\right] + \frac{k^2}{3}\frac{1}{1+R} = 0,$$

giving

$$\omega = \pm kc_s + i\frac{k^2}{6}\frac{1}{\dot{\tau}}\frac{R^2 + \frac{8}{9}(1+R)}{(1+R)^2}. \tag{8.35}$$

So acoustic oscillations are damped as $\exp[-k^2/k_D^2]$, where

$$k_D^2 = \frac{1}{6}\int \frac{1}{\dot{\tau}}\frac{R^2 + \frac{8}{9}(1+R)}{(1+R)^2}d\eta. \tag{8.36}$$

This is sometimes written in the form

$$k_D^2 = \frac{1}{6}\int \frac{1}{\dot\tau}\,\frac{R^2 + \frac{4}{5}f_2^{-1}(1+R)}{(1+R)^2}\,d\eta. \tag{8.37}$$

Our result corresponds to $f_2 = 9/10$. In some books and papers one finds $f_2 = 1$. If we would include polarization effects, we would find $f_2 = 3/4$. The damping of acoustic oscillations is now clearly observed.

Sound horizon. The sound horizon determines according to (8.27) the position of the first peak. We compute now this important characteristic scale.

The comoving sound horizon at time η is

$$r_s(\eta) = \int_0^\eta c_s(\eta')d\eta'. \tag{8.38}$$

Let us write this as a redshift integral, using $1 + z = a_0/a(\eta)$, whence by (0.52) for $K \neq 0$

$$d\eta = -\frac{1}{a_0}\frac{dz}{H(z)} = -|\Omega_K|^{1/2}\frac{dz}{E(z)}. \tag{8.39}$$

Thus

$$r_s(z) = |\Omega_K|^{1/2}\int_z^\infty c_s(z')\frac{dz'}{E(z')}. \tag{8.40}$$

This is seen at present under the (small) angle

$$\theta_s(z) = \frac{r_s(z)}{r(z)}, \tag{8.41}$$

where $r(z)$ is given by (0.56) and (0.57):

$$r(z) = \mathcal{S}\left(|\Omega_K|^{1/2}\int_0^z \frac{dz'}{E(z')}\right). \tag{8.42}$$

Before decoupling the sound velocity is given by (8.21), with

$$R = \frac{3}{4}\frac{\Omega_b}{\Omega_\gamma}\frac{1}{1+z}. \tag{8.43}$$

We are left with two explicit integrals. For $z_{\rm dec}$ we can neglect in (8.40) the curvature and Λ terms. The integral can then be done analytically, and is in good approximation proportional to $(\Omega_M)^{-1/2}$ (exercise). Note that (8.42) is closely related to the angular diameter distance to the last scattering surface (see (0.34) and (0.60)). A numerical calculation shows that $\theta_s(z_{\rm dec})$ depends mainly on the curvature parameter Ω_K. For a typical model with $\Omega_\Lambda = 2/3,\ \Omega_b h_0^2 = 0.02,\ \Omega_M h_0^2 = 0.16,\ n = 1$ the parameter sensitivity is approximately [48]

$$\frac{\Delta\theta_s}{\theta_s} \approx 0.24\frac{\Delta(\Omega_M h_0^2)}{\Omega_M h_0^2)} - 0.07\frac{\Delta(\Omega_b h_0^2)}{\Omega_b h_0^2} + 0.17\frac{\Delta\Omega_\Lambda}{\Omega_\Lambda} + 1.1\frac{\Delta\Omega_{tot}}{\Omega_{tot}}.$$

8.3 Formal solution for the moments θ_l

We derive in this section a useful integral representation for the brightness moments at the present time. The starting point is the Boltzmann equation (7.74) for the brightness temperature fluctuations $\Theta(\eta, k, \mu)$,

$$(\Theta + \Psi)' + ik\mu(\Theta + \Psi) = \Psi' - \Phi' + \dot{\tau}[\theta_0 - \Theta - i\mu V_b - \frac{1}{10}\theta_2 P_2(\mu)]. \tag{8.44}$$

This is of the form of an inhomogeneous linear differential equation

$$y' + g(x)y = h(x),$$

whose solution can be written as (variation of constants)

$$y(x) = e^{-G(x)}\left\{y_0 + \int_{x_0}^{x} h(x')e^{G(x')}dx'\right\},$$

with

$$G(x) = \int_{x_0}^{x} g(u)du.$$

In our case $g = ik\mu + \dot{\tau},\ h = \dot{\tau}[\theta_0 + \Psi - i\mu V_b - \frac{1}{10}\theta_2 P_2(\mu)] + \Psi' - \Phi'$. Therefore, the present value of $\Theta + \Psi$ can formally be expressed as

$$(\Theta + \Psi)(\eta_0, \mu; k) = \int_0^{\eta_0} d\eta\Big[\dot{\tau}(\theta_0 + \Psi - i\mu V_b - \frac{1}{10}\theta_2 P_2) + \Psi' - \Phi'\Big]e^{-\tau(\eta,\eta_0)}e^{ik\mu(\eta-\eta_0)}, \tag{8.45}$$

where

$$\tau(\eta, \eta_0) = \int_{\eta}^{\eta_0} \dot{\tau}d\eta \tag{8.46}$$

is the *optical depth*. The combination $\dot{\tau}e^{-\tau}$ is the (conformal) *time visibility function*. It has a simple interpretation: Let $p(\eta, \eta_0)$ be the probability that a photon did not scatter between η and today (η_0). Clearly, $p(\eta - d\eta, \eta_0) = p(\eta, \eta_0)(1 - \dot{\tau}d\eta)$. Thus $p(\eta, \eta_0) = e^{-\tau(\eta,\eta_0)}$, and the visibility function times $d\eta$ is the probability that a photon last scattered between η and $\eta + d\eta$. The visibility function is therefore *strongly peaked* near decoupling. This is very useful, both for analytical and numerical purposes.

In order to obtain an integral representation for the multipole moments θ_l, we insert in (8.45) for the μ-dependent factors the following expansions in terms of Legendre polynomials:

$$e^{-ik\mu(\eta_0-\eta)} = \sum_l (-i)^l(2l+1)j_l(k(\eta_0 - \eta))P_l(\mu), \tag{8.47}$$

$$-i\mu e^{-ik\mu(\eta_0-\eta)} = \sum_l (-i)^l(2l+1)j_l'(k(\eta_0 - \eta))P_l(\mu), \tag{8.48}$$

$$(-i)^2 P_2(\mu)e^{-ik\mu(\eta_0-\eta)} = \sum_l (-i)^l(2l+1)\frac{1}{2}[3j_l'' + j_l]P_l(\mu). \tag{8.49}$$

Here, the first is well-known. The others can be derived from (8.47) by using the recursion relations (7.84) for the Legendre polynomials and the following ones for the spherical Bessel functions

$$lj_{l-1} - (l+1)j_{l+1} = (2l+1)j_l', \tag{8.50}$$

or by differentiation of (8.47) with respect to $k(\eta_0 - \eta)$. Using the definition (7.75) of the moments θ_l, we obtain for $l \geq 2$ the following useful formula:

$$\frac{\theta_l(\eta_0)}{2l+1} = \int_0^{\eta_0} d\eta e^{-\tau(\eta)} \Big[(\dot{\tau}\theta_0 + \dot{\tau}\Psi + \Psi' - \Phi')j_l(k(\eta_0 - \eta)) + \dot{\tau}V_b j_l' + \dot{\tau}\frac{1}{20}\theta_2(3j_l'' + j_l)\Big]. \quad (8.51)$$

Sudden decoupling approximation. In a reasonably good approximation we can replace the visibility function by the δ-function, and obtain with $\Delta\eta \equiv \eta_0 - \eta_{\rm dec},\ V_b(\eta_{\rm dec}) \simeq \theta_1(\eta_{\rm dec})$ the instructive result

$$\frac{\theta_l(\eta_0, k)}{2l+1} \simeq [\theta_0 + \Psi](\eta_{\rm dec}, k)j_l(k\Delta\eta) + \theta_1(\eta_{\rm dec}, k)j_l'(k\Delta\eta) + ISW + Quad. \quad (8.52)$$

Here, the quadrupole contribution (last term) is not important. ISW denotes the *integrated Sachs-Wolfe effect*:

$$ISW = \int_0^{\eta_0} d\eta(\Psi' - \Phi')j_l(k(\eta_0 - \eta)), \quad (8.53)$$

which only depends on the time variations of the Bardeen potentials between recombination and the present time.

The interpretation of the first two terms in (8.52) is quite obvious: The first describes the fluctuations of the *effective* temperature $\theta_0+\Psi$ on the cosmic photosphere, as we would see them for free streaming between there and us, if the gravitational potentials would not change in time. (Ψ includes blue- and redshift effects.) The dipole term has to be interpreted, of course, as a Doppler effect due to the velocity of the baryon-photon fluid. It turns out that the integrated Sachs-Wolfe effect enhances the anisotropy on scales comparable to the Hubble length at recombination.

In this approximate treatment we have to know – beside the ISW – only the effective temperature $\theta_0 + \Psi$ and the velocity moment θ_1 at decoupling. The main point is that Eq. (8.52) provides a good understanding of the physics of the CMB anisotropies. Note that the individual terms are all gauge invariant. In gauge dependent methods interpretations would be ambiguous.

8.4 Angular correlations of temperature fluctuations

The system of evolution equations has to be supplemented by initial conditions. We can not hope to be able to predict these, but at best their statistical properties (as, for instance, in inflationary models). Theoretically, we should thus regard the brightness temperature perturbation $\Theta(\eta, x^i, \gamma^j)$ as a random field. Of special interest is its angular correlation function at the present time η_0. Observers measure only one realization of this, which brings unavoidable *cosmic variances* (see the Introduction to Part III).

For further elaboration we insert (7.75) into the Fourier expansion of Θ, obtaining

$$\Theta(\eta, \boldsymbol{x}, \boldsymbol{\gamma}) = (2\pi)^{-3/2}\int d^3k \sum_l \theta_l(\eta, k)G_l(\boldsymbol{x}, \boldsymbol{\gamma}; \boldsymbol{k}), \quad (8.54)$$

where

$$G_l(\boldsymbol{x}, \boldsymbol{\gamma}; \boldsymbol{k}) = (-i)^l P_l(\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma})\exp(i\boldsymbol{k}\cdot\boldsymbol{x}). \quad (8.55)$$

With the addition theorem for the spherical harmonics the Fourier transform is thus

$$\Theta(\eta, \boldsymbol{k}, \boldsymbol{\gamma}) = \sum_{lm} Y_{lm}(\boldsymbol{\gamma})\frac{4\pi}{2l+1}\theta_l(\eta, k)\ (-i)^l Y_{lm}^*(\hat{\boldsymbol{k}}). \quad (8.56)$$

This has to be regarded as a stochastic field of **k** (parametrized by γ). The randomness is determined by the statistical properties at an early time, for instance after inflation. If we write Θ as (dropping η) $\mathcal{R}(\boldsymbol{k}) \times (\Theta(\boldsymbol{k},\boldsymbol{\gamma})/\mathcal{R}(\boldsymbol{k}))$, the second factor evolves deterministically and is independent of the initial amplitudes, while the stochastic properties are completely determined by those of $\mathcal{R}(\boldsymbol{k})$. In terms of the power spectrum of $\mathcal{R}(\boldsymbol{k})$,

$$\langle \mathcal{R}(\boldsymbol{k})\mathcal{R}^*(\boldsymbol{k'})\rangle = \frac{2\pi^2}{k^3} P_{\mathcal{R}}(k)\delta^3(\boldsymbol{k}-\boldsymbol{k'}) \tag{8.57}$$

(see (5.14)), we thus have for the correlation function in momentum space

$$\langle \Theta(\boldsymbol{k},\boldsymbol{\gamma})\Theta^*(\boldsymbol{k'},\boldsymbol{\gamma'})\rangle = \frac{2\pi^2}{k^3} P_{\mathcal{R}}(k)\delta^3(\boldsymbol{k}-\boldsymbol{k'}) \frac{\Theta(k,\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma})}{\mathcal{R}(k)} \frac{\Theta^*(k,\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma'})}{\mathcal{R}^*(k)}. \tag{8.58}$$

Because of the δ-function the correlation function in **x**-space is

$$\langle \Theta(\boldsymbol{x},\boldsymbol{\gamma})\Theta(\boldsymbol{x},\boldsymbol{\gamma'})\rangle = \int \frac{d^3k}{(2\pi)^3} \int d^3k' \langle \Theta(\boldsymbol{k},\boldsymbol{\gamma})\Theta(\boldsymbol{k'},\boldsymbol{\gamma'})\rangle. \tag{8.59}$$

Inserting here (8.56) and (8.58) finally gives

$$\langle \Theta(\boldsymbol{x},\boldsymbol{\gamma})\Theta(\boldsymbol{x},\boldsymbol{\gamma'})\rangle = \frac{1}{4\pi}\sum_l (2l+1)C_l P_l(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma'}), \tag{8.60}$$

with

$$\boxed{\frac{(2l+1)^2}{4\pi}C_l = \int_0^\infty \frac{dk}{k}\left|\frac{\theta_l(k)}{\mathcal{R}(k)}\right|^2 P_{\mathcal{R}}(k).} \tag{8.61}$$

Instead of $\mathcal{R}(k)$ we could, of course, use another perturbation amplitude. Note also that we can take $\mathcal{R}(k)$ and $P_{\mathcal{R}}(k)$ at any time. If we choose an early time when $P_{\mathcal{R}}(k)$ is given by its primordial value, $P_{\mathcal{R}}^{(\mathrm{prim})}(k)$, then the ratios inside the absolute value, $\theta_l(k)/\mathcal{R}(k)$, are two-dimensional *CMB transfer functions*.

8.5 Angular power spectrum for large scales

The *angular power spectrum* is defined as $l(l+1)C_l$ versus l. For large scales, i.e., small l, observed first with COBE, the first term in Eq. (8.52) dominates. Let us have a closer look at this so-called Sachs-Wolfe contribution.

For large scales (small k) we can neglect in the first equation (8.6) of the Boltzmann hierarchy the term proportional to k: $\theta_0' \approx -\Phi' \approx \Psi'$, neglecting also Π (i.e., θ_2) on large scales. Thus

$$\theta_0(\eta) \approx \theta_0(0) + \Psi(\eta) - \Psi(0). \tag{8.62}$$

To proceed, we need a relation between $\theta_0(0)$ and $\Psi(0)$. This can be obtained by looking at superhorizon scales in the tight coupling limit, using the results of Sect. 6.1. (Alternatively, one can investigate the Boltzmann hierarchy in the radiation dominated era.)

From (7.77) and (1.175) or (1.217) we get (recall $x = Ha/k$)

$$\theta_0 = \frac{1}{4}\Delta_{s\gamma} = \frac{1}{4}\Delta_{c\gamma} - xV.$$

The last term can be expressed in terms of Δ, making use of (6.10) for $w = 1/3$,

$$xV = -\frac{3}{4}x^2(D-1)\Delta.$$

Moreover, we have from (6.41)

$$\frac{3}{4}\Delta_{c\gamma} = \frac{\zeta+1}{\zeta+4/3}\Delta - \frac{\zeta}{\zeta+4/3}S.$$

Putting things together, we obtain for $\zeta \ll 1$

$$\theta_0 = \frac{3}{4}\left[x^2(D-1) + \frac{1}{4}\right]\Delta - \frac{1}{4}\zeta S, \tag{8.63}$$

thus

$$\theta_0 \simeq \frac{3}{4}x^2(D-1)\Delta - \frac{1}{4}\zeta S, \tag{8.64}$$

on superhorizon scales ($x \gg 1$).

For adiabatic perturbations we can use here the expansion (6.39) for $\omega \ll 1$ and get with (6.9)

$$\theta_0(0) \simeq \frac{3}{4}x^2\Delta = -\frac{1}{2}\Psi(0). \tag{8.65}$$

For isocurvature perturbations, the expansion (6.40) gives

$$\theta_0(0) = \Psi(0) = 0. \tag{8.66}$$

Hence, the initial condition for the effective temperature is

$$(\theta_0 + \Psi)(0) = \begin{cases} \frac{1}{2}\Psi(0) & : \ (\text{adiabatic}) \\ 0 & : \ (\text{isocurvature}). \end{cases} \tag{8.67}$$

If this is used in (8.62) we obtain

$$\theta_0(\eta) = \Psi(\eta) - \frac{3}{2}\Psi(0) \text{ for adiabatic perturbations.}$$

On large scales (2.32) gives for $\zeta \gg 1$, in particular for $\eta_{\rm rec}$,

$$\Psi(\eta) = \frac{9}{10}\Psi(0). \tag{8.68}$$

Thus we obtain the result (Sachs-Wolfe)

$$\boxed{(\theta_0 + \Psi)(\eta_{\rm dec}) = \frac{1}{3}\Psi(\eta_{\rm dec})} \text{ for } \textit{adiabatic} \text{ perturbations.} \tag{8.69}$$

On the other hand, we obtain for isocurvature perturbations with (8.66) $\theta_0(\eta) = \Psi(\eta)$, thus

$$\boxed{(\theta_0 + \Psi)(\eta_{\rm dec}) = 2\Psi(\eta_{\rm dec})} \text{ for } \textit{isocurvature} \text{ perturbations.} \tag{8.70}$$

Note the factor 6 difference between the two cases. The Sachs-Wolfe contribution to the θ_l is therefore

$$\frac{\theta_l^{\rm SW}(k)}{2l+1} = \begin{cases} \frac{1}{3}\Psi(\eta_{\rm dec})j_l(k\Delta\eta) & : \ (\textit{adiabatic}) \\ 2\Psi(\eta_{\rm dec})j_l(k\Delta\eta) & : \ (\textit{isocurvature}). \end{cases} \tag{8.71}$$

We express at this point $\Psi(\eta_{\rm dec})$ in terms of the primordial values of $\mathcal{R}$ and S. For adiabatic perturbations $\mathcal{R}$ is constant on superhorizon scales (see (1.138)), and according to (4.67) we have in the matter dominated era $\Psi = -\frac{3}{5}\mathcal{R}$. On the other hand, for isocurvature perturbations the entropy perturbation S is constant on superhorizon scales (see Sect. 6.2.1), and for $\zeta \gg 1$ we have according to (6.45) and (6.9) $\Psi = -\frac{1}{5}S$. Hence we find

$$\boxed{(\theta_0 + \Psi)(\eta_{\rm dec}) = -\frac{1}{5}(\mathcal{R}^{(\rm prim)} + 2S^{(\rm prim)}).} \tag{8.72}$$

The result (8.71) inserted into (8.61) gives the the dominant Sachs-Wolfe contribution to the coefficients C_l for large scales (small l). For adiabatic initial fluctuations we obtain with (8.72)

$$\boxed{C_l^{\rm SW} = \frac{4\pi}{25}\int_0^\infty \frac{dk}{k}\,|j_l(k\Delta\eta)|^2\,P_{\mathcal{R}}^{(\rm prim)}(k).} \tag{8.73}$$

Here we insert (6.82) and obtain

$$C_l^{\rm SW} \simeq \pi H_0^{1-n}\delta_H^2\left(\frac{\Omega_M}{D_g(0)}\right)^2\int_0^\infty \frac{dk}{k^{2-n}}\,|j_l(k\Delta\eta)|^2\,. \tag{8.74}$$

The integral can be done analytically. Eq. 11.4.34 in [39] implies as a special case

$$\int_0^\infty t^{-\lambda}[J_\mu(at)]^2dt = \frac{\Gamma(\frac{2\mu-\lambda+1}{2})}{2^\lambda a^{1-\lambda}\Gamma(\mu+1)\Gamma(\frac{\lambda+1}{2})} \times {}_2F_1\left(\frac{2\mu-\lambda+1}{2}, \frac{-\lambda+1}{2}; \mu+1; 1\right). \tag{8.75}$$

Since

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x)$$

the integral in (8.74) is of the form (8.75). If we also use Eq. 15.1.20 of the same reference,

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

we obtain

$$\int_0^\infty t^{n-2}[j_l(ta)]^2dt = \frac{\pi}{2^{4-n}a^{n-1}}\,\frac{\Gamma(3-n)}{[\Gamma(\frac{4-n}{2})]^2}\,\frac{\Gamma(\frac{2l+n-1}{2})}{\Gamma(\frac{2l+5-n}{2})} \tag{8.76}$$

and thus

$$\boxed{C_l^{\rm SW} \simeq 2^{n-4}\pi^2(H_0\eta_0)^{1-n}\delta_H^2\left(\frac{\Omega_M}{D_g(0)}\right)^2\frac{\Gamma(3-n)}{[\Gamma(\frac{4-n}{2})]^2}\,\frac{\Gamma(\frac{2l+n-1}{2})}{\Gamma(\frac{2l+5-n}{2})}.} \tag{8.77}$$

For a *Harrison-Zel'dovich spectrum* ($n = 1$) we get

$$l(l+1)C_l^{\rm SW} = \frac{\pi}{2}\delta_H^2\left(\frac{\Omega_M}{D_g(0)}\right)^2. \tag{8.78}$$

Because the right-hand side is a constant one usually plots the quantity $l(l+1)C_l$ (often divided by 2π).

8.6 Influence of gravity waves on CMB anisotropies

In this section we study the effect of a stochastic gravitational wave background on the CMB anisotropies. According to Sect. 5.2 such a background is unavoidably produced in inflationary models.

A. Basic equations. We consider only the case $K = 0$. Let us first recall some basic formulae from Sects. 5.2 and 7.6. The metric for tensor modes is of the form

$$g = a^2(\eta)[-d\eta^2 + (\delta_{ij} + 2H_{ij})dx^i dx^j]. \tag{8.79}$$

For a mode $H_{ij} \propto \exp(i\boldsymbol{k}\cdot\boldsymbol{x})$, the tensor amplitudes satisfy

$$H^i{}_i = 0,\ H^i{}_j k^j = 0. \tag{8.80}$$

The tensor perturbations of the energy-momentum tensor can be parametrized as follows

$$\delta T^0{}_0 = 0,\ \delta T^0{}_i = 0,\ \delta T^i{}_j = \Pi^i_{(T)j}, \tag{8.81}$$

where $\Pi^i_{(T)j}$ satisfies in **k**-space

$$\Pi^i_{(T)i} = 0,\ \Pi^i_{(T)j}k^j = 0. \tag{8.82}$$

According to (5.59) the Einstein equations reduce to

$$H''_{ij} + 2\frac{a'}{a}H'_{ij} + k^2 H_{ij} = 8\pi G a^2 \Pi_{(T)ij}. \tag{8.83}$$

The Boltzmann equation (7.98) becomes in the metric (8.79)

$$\Theta' + ik\mu\Theta = -H'_{ij}\gamma^i\gamma^j. \tag{8.84}$$

The solution of this equation in terms of H_{ij} is

$$\Theta(\eta_0, \boldsymbol{k}, \boldsymbol{\gamma}) = -\int_0^{\eta_0} H'_{ij}(\eta_0, \boldsymbol{k})\gamma^i\gamma^j e^{-ik\mu(\eta_0-\eta)}d\eta. \tag{8.85}$$

For the photon contribution to $\Pi^i_{(T)j}$ we obtain as in Sect. 7.5

$$\Pi^i_{(T)\gamma j} = 12\int[\gamma^i\gamma_j - \frac{1}{3}\delta^i{}_j]\Theta\ \frac{d\Omega_\gamma}{4\pi}. \tag{8.86}$$

To this one should add the neutrino contribution, but in what follows we can safely neglect the source $\Pi^i_{(T)\gamma j}$ in the Einstein equation (8.83).

B. Harmonic decompositions. We decompose H_{ij} as in Sect. 5.2:

$$H_{ij}(\eta, \boldsymbol{k}) = \sum_{\lambda=\pm 2} h_\lambda(\eta, \boldsymbol{k})\epsilon_{ij}(\boldsymbol{k}, \lambda), \tag{8.87}$$

where the polarization tensor satisfies (5.65). If $\mathbf{k} = (0, 0, k)$ then the x,y components of $\epsilon_{ij}(\boldsymbol{k}, \lambda)$ are

$$(\epsilon_{ij}(\boldsymbol{k}, \lambda)) = \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix},\ \lambda = \pm 2. \tag{8.88}$$

One easily verifies that for this choice of **k**

$$\epsilon_{ij}(\lambda)(\gamma^i\gamma^j - \frac{1}{3}\delta^{ij}) = \frac{4}{\sqrt{2}}\sqrt{\frac{\pi}{15}}Y_{2\lambda}(\boldsymbol{\gamma}),\ \lambda = \pm 2. \tag{8.89}$$

If we insert this and the expansion

$$e^{-ik\mu(\eta_0-\eta)} = 4\pi\sum_{L,M}(-i)^L j_l(k(\eta_0-\eta))Y^*_{LM}(\hat{\boldsymbol{k}})Y_{LM}(\boldsymbol{\gamma}) \tag{8.90}$$

in (8.85) we obtain for each polarization λ the expansion (dropping the variable η)

$$\Theta_\lambda(\boldsymbol{k},\boldsymbol{\gamma}) = \sum_{l,m} a^{(\lambda)}_{lm}(k)Y_{lm}(\boldsymbol{\gamma}), \tag{8.91}$$

with

$$\begin{aligned} a^{(\lambda)}_{lm}(k) &= \int Y^*_{lm}(\boldsymbol{\gamma})\Theta_\lambda(\boldsymbol{k},\boldsymbol{\gamma})d\Omega_\gamma \\ &= -\int_0^{\eta_0} d\eta h'_\lambda(\eta,k)4\pi\sum_{L,M}(-i)^L j_l(k(\eta_0-\eta))Y^*_{LM}(\hat{\boldsymbol{k}}) \\ &\quad\times\frac{4}{\sqrt{2}}\sqrt{\frac{\pi}{15}}\int Y^*_{lm}(\boldsymbol{\gamma})Y_{2\lambda}(\boldsymbol{\gamma})Y_{LM}(\boldsymbol{\gamma})d\Omega_\gamma. \end{aligned} \tag{8.92}$$

Since **k** points in the 3-direction we have $Y^*_{LM}(\hat{\boldsymbol{k}}) = \delta_{M0}\sqrt{\frac{2L+1}{4\pi}}$. If we also use the spherical integral

$$\int Y^*_{lm}Y_{2\lambda}Y_{L0}d\Omega = \left[\frac{(2l+1)5(2L+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & 2 & L \\ 0 & 0 & 0\end{pmatrix}(-1)^m\begin{pmatrix} l & 2 & L \\ -m & \lambda & 0\end{pmatrix}$$

we obtain

$$a^{(\lambda)}_{lm} = -\sqrt{\frac{8\pi}{3}}\int_0^{\eta_0} d\eta h'_\lambda(\eta,k)(2l+1)^{1/2}\sum_{L=l,l\pm 2} j_L(k(\eta_0-\eta))(-i)^l X_{L,\lambda}\delta_{m\lambda},$$

where

$$(-i)^l X_{L,\lambda} := (-i)^L(2L+1)\begin{pmatrix} l & 2 & L \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & 2 & L \\ -m & \lambda & 0\end{pmatrix}.$$

Note that this is invariant under $\lambda\to-\lambda$. With a table of Clebsch-Gordan coefficients one readily finds

$$\begin{aligned} X_{l,\lambda} &= -\sqrt{\frac{3}{2}}\,[(l+2)(l+1)l(l-1)]^{1/2}\,\frac{1}{(2l+3)(2l-1)}, \\ X_{l+2,\lambda} &= -\sqrt{\frac{3}{8}}[\cdots]^{1/2}\frac{1}{(2l+3)(2l+1)}, \\ X_{l-2,\lambda} &= -\sqrt{\frac{3}{8}}[\cdots]^{1/2}\frac{1}{(2l+1)(2l-1)}, \end{aligned}$$

and thus

$$\sum_{L=l,l\pm 2} j_L X_{L,\lambda} = -\sqrt{\frac{3}{8}}\left[\frac{(l+2)!}{(l-2)!}\right]^{1/2}$$

$$\times \left[\frac{j_{l+2}}{(2l+3)(2l+1)} + 2\frac{j_l}{(2l+3)(2l-1)} + \frac{j_{l-2}}{(2l+1)(2l-1)} \right].$$

Using twice the recursion relation

$$\frac{j_l(x)}{x} = \frac{1}{2l+1}(j_{l-1} + j_{l+1}),$$

shows that the last square bracket is equal to $j_l(k(\eta_0 - \eta))/[k(\eta_0 - \eta)]^2$. Thus we find

$$a_{lm}^{(\lambda)}(k) = \sqrt{\pi}(-i)^l \left[\frac{(l+2)!}{(l-2)!}\right]^{1/2} \times \int_0^{\eta_0} d\eta h'_\lambda(\eta, k)(2l+1)^{1/2} \frac{j_l(k(\eta_0 - \eta))}{[k(\eta_0 - \eta)]^2} \delta_{m\lambda}. \tag{8.93}$$

Recall that so far the wave vector is assumed to point in the 3-direction. For an arbitrary direction $a_{lm}^{(\lambda)}(\boldsymbol{k})$ is determined by (see (8.91) and use the fact that $a_{lm}^{(\lambda)}(k)$ is proportional to $\delta_{m\lambda}$)

$$\sum_m a_{lm}^{(\lambda)}(\boldsymbol{k}) Y_{lm}(\boldsymbol{\gamma}) = a_{l\lambda}^{(\lambda)}(k) Y_{l\lambda}(R^{-1}(\hat{\boldsymbol{k}})\boldsymbol{\gamma}),$$

where $R(\hat{\boldsymbol{k}})$ rotates (0,0,1) to $\hat{\boldsymbol{k}}$. Let $D^l_{m\lambda}(\hat{\boldsymbol{k}})$ be the corresponding representation matrices[33]. Since

$$Y_{l\lambda}(R^{-1}(\hat{\boldsymbol{k}})\boldsymbol{\gamma}) = \sum_m D^l_{m\lambda}(\hat{\boldsymbol{k}}) Y_{lm}(\boldsymbol{\gamma}),$$

we obtain

$$a_{lm}^{(\lambda)}(\boldsymbol{k}) = a_{l\lambda}^{(\lambda)}(k) D^l_{m\lambda}(\hat{\boldsymbol{k}}), \tag{8.94}$$

where $a_{l\lambda}^{(\lambda)}(k)$ is given by (8.93) for $m = \lambda$.

C. The coefficients C_l for tensor modes. For the computation of the C_l's due to gravitational waves we proceed as in Sect. 8.4 for scalar modes. On the basis (8.91) and (8.94) we can write

$$\Theta_\lambda(\eta, \boldsymbol{k}, \boldsymbol{\gamma}) = h_\lambda(\eta_i, \boldsymbol{k}) \sum_{l,m} \frac{a_{lm}^{(\lambda)}(k)}{h_\lambda(\eta_i, k)} D^l_{m\lambda}(\hat{\boldsymbol{k}}) Y_{lm}(\boldsymbol{\gamma}), \tag{8.95}$$

where η_i is some very early time, e.g., at the end of inflation. A look at (8.93) shows that the factor $a_{lm}^{(\lambda)}(k)/h_\lambda(\eta_i, k)$ involves only $h'_\lambda(\eta, k)/h_\lambda(\eta_i, k)$, and is thus independent of the initial amplitude of h_λ and also independent of λ (see paragraph D below). The stochastic properties are entirely located in the first factor of (8.95). Its correlation function is given in terms of the primordial power spectrum $P_g^{(\mathrm{prim})}(k)$ of the gravitational waves:

$$\sum_\lambda \langle h_\lambda(\eta_i, \boldsymbol{k}) h^*_\lambda(\eta_i, \boldsymbol{k}') \rangle = \frac{2\pi^2}{k^3} P_g^{(\mathrm{prim})}(k) \delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{8.96}$$

[33] The Euler angles are $(\varphi, \vartheta, 0)$,where (ϑ, φ) are the polar angles of $\hat{\boldsymbol{k}}$.

(see also (5.73)). With this and the orthogonality properties of the representation matrices

$$\int d\Omega_{\mathbf{k}} D^{l}_{m\lambda}(\hat{\boldsymbol{k}}) D^{l'*}_{m'\lambda'}(\hat{\boldsymbol{k}}) = \frac{2l+1}{4\pi}\delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}, \tag{8.97}$$

we obtain at the present time

$$\langle\Theta(\boldsymbol{x},\boldsymbol{\gamma})\Theta(\boldsymbol{x},\boldsymbol{\gamma'})\rangle = \frac{1}{4\pi}\sum_l (2l+1) C_l^{GW} P_l(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma'}), \tag{8.98}$$

with

$$C_l^{GW} = \frac{4\pi}{2l+1}\int_0^\infty \frac{dk k^2}{(2\pi)^3}\frac{2\pi^2}{k^3} P_g^{(\mathrm{prim})}(k)\left|\frac{a_{lm}^{(\lambda)}(k)}{h_\lambda(\eta_i,k)}\right|^2 .$$

Finally, inserting here (8.93) gives our main result

$$\boxed{C_l^{GW} = \pi\frac{(l+2)!}{(l-2)!}\int_0^\infty \frac{dk}{k} P_g^{(\mathrm{prim})}(k)\left|\int_{\eta_i\approx 0}^{\eta_0} d\eta \frac{h'(\eta,k)}{h(\eta_i,k)}\frac{j_l(k(\eta_0-\eta))}{[k(\eta_0-\eta)]^2}\right|^2 .} \tag{8.99}$$

Note that the tensor modes (8.95) are in $\hat{\boldsymbol{k}}$-space orthogonal to the scalar modes, which are proportional to $D^l_{m0}(\hat{\boldsymbol{k}})$.

D. The modes $h_\lambda(\eta,k)$. In the Einstein equations (8.83) we can safely neglect the anisotropic stresses $\Pi_{(T)ij}$. Then $h_\lambda(\eta,k)$ satisfies the homogeneous linear differential equation

$$h'' + 2\frac{a'}{a}h' + k^2 h = 0. \tag{8.100}$$

At very early times, when the modes are still far outside the Hubble horizon, we can neglect the last term in (8.100), whence h is *frozen*. For this reason we solve (8.100) with the initial condition $h'(\eta_i,k) = 0$. Moreover, we are only interested in growing modes.

This problem was already discussed in Sect. 5.2.3. For modes which enter the horizon during the matter dominated era we have the analytic solution (5.103),

$$\frac{h_k(\eta)}{h_k(0)} = 3\frac{j_1(k\eta)}{k\eta}. \tag{8.101}$$

For modes which enter the horizon earlier, we use again a transfer function $T_g(k)$:

$$\frac{h_k(\eta)}{h_k(0)} =: 3\frac{j_1(k\eta)}{k\eta}T_g(k), \tag{8.102}$$

that has to be determined by solving the differential equation numerically.

On large scales (small l), larger than the Hubble horizon at decoupling, we can use (8.101). Since

$$\left(\frac{j_1(x)}{x}\right)' = -\frac{1}{x}j_2(x), \tag{8.103}$$

we then have

$$\frac{h'(\eta,k)}{3h(0,k)} = -k\frac{j_2(x)}{x},\; x := k\eta. \tag{8.104}$$

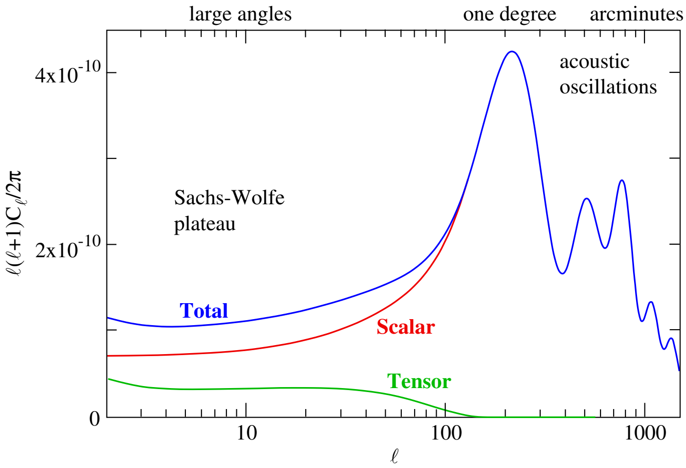

Fig. 8.1 Theoretical angular power spectrum for adiabatic initial perturbations and typical cosmological parameters. The scalar and tensor contributions to the anisotropies are also shown.

Using this in (8.99) gives

$$C_l^{GW} = 9\pi \frac{(l+2)!}{(l-2)!} \int_0^\infty \frac{dk}{k} P_g^{(\mathrm{prim})}(k) I_l^2(k), \tag{8.105}$$

with

$$I_l(k) = \int_0^{x_0} dx \frac{j_l(x_0 - x) j_2(x)}{(x_0 - x)^2 x}, \; x_0 := k\eta_0. \tag{8.106}$$

Remark. Since the power spectrum is often defined in terms of $2H_{ij}$, the pre-factor in (8.105) is the 4 times smaller.

For inflationary models we obtained for the power spectrum Eq. (5.82),

$$P_g(k) \simeq \frac{4}{\pi} \frac{H^2}{M_{\mathrm{Pl}}^2}\bigg|_{k=aH}, \tag{8.107}$$

and the power index

$$n_T \simeq -2\varepsilon. \tag{8.108}$$

For a flat power spectrum the integrations in (8.105) and (8.106) can perhaps be done analytically, but I was not able to do achieve this.

E. Numerical results A typical theoretical CMB spectrum is shown in Fig. 8.1. Beside the scalar contribution in the sense of cosmological perturbation theory, considered so far, the tensor contribution due to gravity waves is also plotted.

Parameter dependences are discussed in detail in [47] (see especially Fig. 1 of this reference).

8.7 Polarization

A polarization map of the CMB radiation provides important additional information to that obtainable from the temperature anisotropies. For example, we can get constraints about the epoch of reionization. Most importantly, future polarization observations may reveal a stochastic background of gravity waves, generated in the very early Universe. In this section we give a brief introduction to the study of CMB polarization.

The mechanism which partially polarizes the CMB radiation is similar to that for the scattered light from the sky. Consider first scattering at a single electron of unpolarized radiation coming in from all directions. Due to the familiar polarization dependence of the differential Thomson cross section, the scattered radiation is, in general, polarized. It is easy to compute the corresponding Stokes parameters. Not surprisingly, they are not all equal to zero if and only if the intensity distribution of the incoming radiation has a non-vanishing quadrupole moment. The Stokes parameters Q and U are proportional to the overlap integral with the combinations $Y_{2,2} \pm Y_{2,-2}$ of the spherical harmonics, while V vanishes.) This is basically the reason why a CMB polarization map traces (in the tight coupling limit) the quadrupole temperature distribution on the last scattering surface.

The polarization tensor of an all sky map of the CMB radiation can be parametrized in temperature fluctuation units, relative to the orthonormal basis $\{d\vartheta, \sin\vartheta \; d\varphi\}$ of the two sphere, in terms of the Pauli matrices as $\Theta \cdot 1 + Q\sigma_3 + U\sigma_1 + V\sigma_2$. The Stokes parameter V vanishes (no circular polarization). Therefore, the polarization properties can be described by the following symmetric trace-free tensor on S^2:

$$(\mathcal{P}_{ab}) = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}. \tag{8.109}$$

As for gravity waves, the components Q and U transform under a rotation of the 2-bein by an angle α as

$$Q \pm iU \to e^{\pm 2i\alpha}(Q \pm iU), \tag{8.110}$$

and are thus of spin-weight 2. $\mathcal{P}_{ab}$ can be decomposed uniquely into *'electric'* and *'magnetic'* parts:

$$\mathcal{P}_{ab} = E_{;ab} - \tfrac{1}{2} g_{ab}\Delta E + \tfrac{1}{2}(\varepsilon_a{}^c B_{;bc} + \varepsilon_b{}^c B_{;ac}). \tag{8.111}$$

Expanding here the scalar functions E and B in terms of spherical harmonics, we obtain an expansion of the form

$$\mathcal{P}_{ab} = \sum_{l=2}^{\infty} \sum_m \left[a^E_{(lm)} Y^E_{(lm)ab} + a^B_{(lm)} Y^B_{(lm)ab} \right] \tag{8.112}$$

in terms of the tensor harmonics:

$$Y^E_{(lm)ab} := N_l \left(Y_{(lm);ab} - \tfrac{1}{2} g_{ab} Y_{(lm);c}{}^c \right), \quad Y^B_{(lm)ab} := \tfrac{1}{2} N_l (Y_{(lm);ac}\varepsilon^c{}_b + a \leftrightarrow b), \tag{8.113}$$

where $l \geq 2$ and

$$N_l \equiv \left(\frac{2(l-2)!}{(l+2)!} \right)^{1/2}.$$

Equivalently, one can write this as

$$Q + iU = \sqrt{2} \sum_{l=2}^{\infty} \sum_m \left[a^E_{(lm)} + i a^B_{(lm)} \right] {}_2Y_l^m, \tag{8.114}$$

where ${}_sY_l^m$ are the spin-s harmonics:

$${}_sY_l^m = \sqrt{\frac{2l+1}{4\pi}} D^l_{-s,m}(\vartheta, \varphi, 0).$$

The multipole moments $a^E_{(lm)}$ and $a^B_{(lm)}$ are random variables, and we have equations analogous to those of the temperature fluctuations, with

$$C_l^{TE} = \frac{1}{2l+1} \sum_m \langle a_{lm}^{\Theta\star} a_{lm}^E \rangle, \quad etc. \tag{8.115}$$

(We have now put the superscript Θ on the a_{lm} of the temperature fluctuations.) The C_l's determine the various angular correlation functions. For example, one easily finds

$$\langle \Theta(\boldsymbol{n}) Q(\boldsymbol{n'}) \rangle = \sum_l \; C_l^{TE} \frac{2l+1}{4\pi} N_l P_l^2(\cos\vartheta) \tag{8.116}$$

(the last factor is the associated Legendre function P_l^m for $m = 2$).

For the space-time dependent Stokes parameters Q and U of the radiation field we can perform a normal mode decomposition analogous to

$$\Theta(\eta, \boldsymbol{x}, \boldsymbol{\gamma}) = (2\pi)^{-3/2} \int d^3k \sum_l \theta_l(\eta, k) G_l(\boldsymbol{x}, \boldsymbol{\gamma}; \boldsymbol{k}), \tag{8.117}$$

where

$$G_l(\boldsymbol{x}, \boldsymbol{\gamma}; \boldsymbol{k}) = (-i)^l P_l(\hat{\boldsymbol{k}} \cdot \boldsymbol{\gamma}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}). \tag{8.118}$$

If, for simplicity, we again consider only scalar perturbations this reads

$$Q \pm iU = (2\pi)^{-3/2} \int d^3k \sum_l (E_l \pm iB_l) \,{}_{\pm 2}G_l^0, \tag{8.119}$$

where

$${}_sG_l^m(\boldsymbol{x}, \boldsymbol{\gamma}; \boldsymbol{k}) = (-i)^l \left(\frac{2l+1}{4\pi} \right)^{1/2} {}_sY_l^m(\boldsymbol{\gamma}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{8.120}$$

if the mode vector **k** is chosen as the polar axis. (Note that G_l in (8.118) is equal to ${}_0G_l^0$.)

The Boltzmann equation implies a coupled hierarchy for the moments θ_l, E_l, and B_l [49,50]. It turns out that the B_l vanish for scalar perturbations. Non-vanishing magnetic multipoles would be a unique signature for a spectrum of gravity waves. We give here, without derivation, the equations for the E_l:

$$E_l' = k \left\{ \frac{(l^2-4)^{1/2}}{2l-1} E_{l-1} - \frac{[(l+1)^2-4]^{1/2}}{2l+1} E_{l+1} \right\} - \dot{\tau}(E_l + \sqrt{6} P \delta_{l,2}), \tag{8.121}$$

where

$$P = \frac{1}{10} [\theta_2 - \sqrt{6} E_2]. \tag{8.122}$$

The analog of the integral representation (8.51)is

$$\frac{E_l(\eta_0)}{2l+1} = -\frac{3}{2} \sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{\eta_0} d\eta e^{-\tau(\eta)} \dot{\tau} P(\eta) \frac{j_l(k(\eta_0 - \eta))}{(k(\eta_0 - \eta))^2}. \tag{8.123}$$

For large scales the first term in (8.122) dominates, and the E_l are thus determined by θ_2.

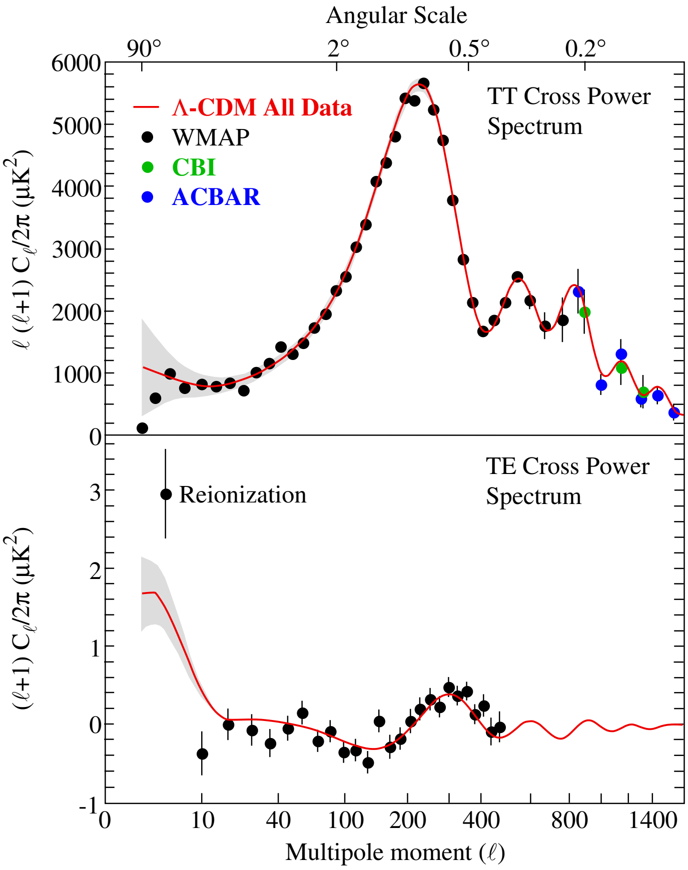

Fig. 8.2 Temperature-temperature (TT) and temperature-polarization TE power spectra. The best fit ΛCDM model is also shown. (Adapted from [52].)

For large l we may use the tight coupling approximation in which $E_2 = -\sqrt{6} \Rightarrow P = \theta_2/4$. In the sudden decoupling approximation, the present electric multipole moments can thus be expressed in terms of the brightness quadrupole moment on the last scattering surface and spherical Bessel functions as

$$\frac{E_l(\eta_0, k)}{2l+1} \simeq \frac{3}{8}\theta_2(\eta_{\rm dec}, k)\frac{l^2 j_l(k\eta_0)}{(k\eta_0)^2}. \tag{8.124}$$

Here one sees how the observable E_l's trace the quadrupole temperature anisotropy on the last scattering surface. In the tight coupling approximation the latter is proportional to the dipole moment θ_1.

8.8 Observational results

In recent years several experiments gave clear evidence for multiple peaks in the angular temperature power spectrum at positions expected on the basis of the simplest inflationary models and big bang nucleosynthesis [51]. These results have been confirmed and substantially improved by WMAP [52] (see Fig. 8.2).

In spite of the high accuracy of the data, it is not possible to extract unambiguously cosmological parameters, because there are intrinsic degeneracies, especially when tensor modes are included. These can only be lifted if other cosmological information is used. Beside the supernova results, use has been made for instance of the available information for the galaxy power spectrum (in particular from the 2-degree-Field Galaxy Redshift Survey (2dFGRS)), and limits for the Hubble parameter. For example, if one adds to the CMB data the well-founded constraint $H_0 \geq 50\ km/s/Mpc$, then the total density parameter $\Omega_{\rm tot}$ has to be in the range $0.98 < \Omega_{\rm tot} < 1.08$ (95 %) (see Fig. 8.3). The Universe is thus *spatially almost flat*. (For further evidence, see Fig. 8.4.) In what follows we therefore always assume $K = 0$.

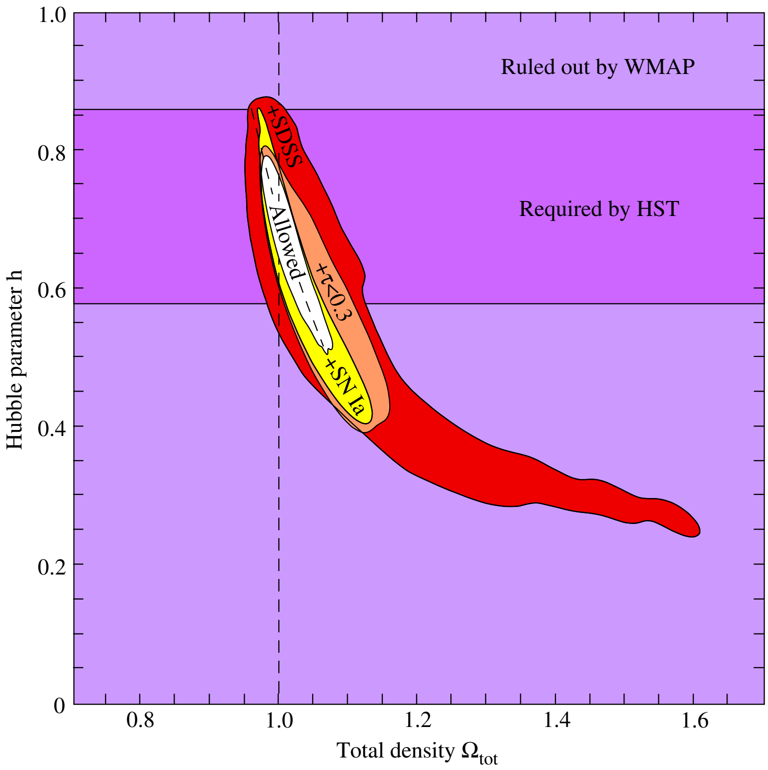

Fig. 8.3 95% constraints in the $(\Omega_{\rm tot}, h_0)$ plane. The WMAP data alone, when analyzed with with a 7-parameter curved model, allows only the banana-shaped region. This becomes considerably smaller if $h_0 > 0.5$ is imposed. Additional information reduces the allowed region even further. (Adapted from [47, Fig. 7].)

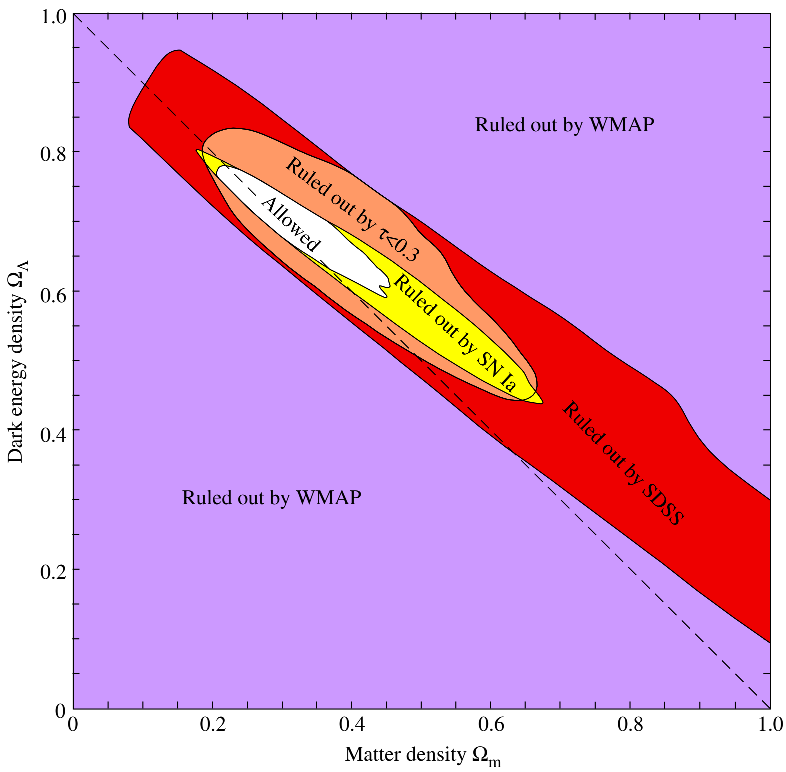

Fig. 8.4 95% constraints in the $(\Omega_M, \Omega_\Lambda)$ plane. These are based on the same models and data as in Fig. 8.3. (Adapted from [47, Fig. 9].)

Table 1

Parameter	CMB alone	CMB and 2dFGRS
$\Omega_b h_0^2$	0.024 ± 0.001	0.023 ± 0.001
$\Omega_M h_0^2$	0.14 ± 0.02	0.134 ± 0.006
h_0	0.72 ± 0.05	0.71 ± 0.04
Ω_b	0.047 ± 0.006	$\simeq$ same
Ω_M	0.29 ± 0.07	$\simeq$ same

Table 1 is extracted from the extended analysis [53] of the WMAP data and other cosmological information. It shows the 68% confidence ranges for some of the cosmological parameters for two types of fits, assuming a ΛCDM model. In the first only the CMB data are used (but tensor modes are included), while in the second these data are combined with the 2dFGRS power spectrum (assuming adiabatic, Gaussian initial conditions described by power laws).

Note that there is little difference between the two columns. The age of the Universe for these parameters is close to 14 Gyr. Another interesting result coming from the rise of the temperature-polarization correlation function at large scales (small l) in Fig. 2 is that reionization of the Universe has set in surprisingly early –, at a redshift of $z_r = 17 \pm 5$, with a corresponding optical depth $\tau = 0.17 \pm 0.06$.

Before the new results possible admixtures of isocurvature modes were not strongly constraint. But now the measured temperature-polarization correlations imply that the primordial fluctuations were primarily *adiabatic*. Admixtures of isocurvature modes do not improve the fit.

One worry is that the quadrupole amplitude (C_2) measured by WMAP is lower than expected according to the best fit ΛCDM model [28]. This issue has led to lots of discussions. A reanalysis [54] of the effects of Galactic cuts indicates that this discrepancy is not particularly significant, being in the region of a few percent. This issue may look differently, once the second year WMAP data have been analyzed. (We are still eagerly waiting for seeing this.)

WMAP has determined the amplitude of the primordial power spectrum:

$$P_{\mathcal{R}}^{(\mathrm{prim})}(k) \simeq 2.95 \times 10^{-9}\ \mathrm{A},\ A = 0.6\text{–}1 \tag{8.125}$$

(depending on the model). Using (5.46) this implies

$$\frac{1}{\pi M_{\mathrm{Pl}}^2} \frac{H^2}{\varepsilon} \simeq (2-3) \times 10^{-9}, \tag{8.126}$$

hence the Hubble parameter during inflation is

$$H \simeq (0.9 - 1.2) \times 10^{15} \varepsilon^{1/2}\ \mathrm{GeV}. \tag{8.127}$$

With (5.36) this gives

$$U^{1/4} \simeq (6.3 - 7.1) \times 10^{16} \varepsilon^{1/4}\ \mathrm{GeV}. \tag{8.128}$$

The WMAP data constrain the ratio $P_g/P_{\mathcal{R}}$, and hence by (5.90) also ε : $\varepsilon < 0.08$. Therefore, we can conclude that the energy scale of inflation has to satisfy the bound

$$U^{1/4} < 3.8 \times 10^{16}\ \mathrm{GeV}. \tag{8.129}$$

A positive detection of the B - mode in the CMB polarization would provide a lower bound for $U^{1/4}$.

8.9 Concluding remarks

A wide range of astronomical data support the following 'concordance' ΛCDM model: The Universe is spatially flat and dominated by vacuum energy density and weakly interacting cold dark matter. Furthermore, the primordial fluctuations are adiabatic and nearly scale invariant, as predicted in simple inflationary models. Deviations from Gaussian statistics are small[34].

A vacuum energy with density parameter $\Omega_\Lambda \simeq 0.7$ is so surprising that it should be examined whether this conclusion is really unavoidable. Since we do not have a tested theory predicting the spectrum of primordial fluctuations, it appears reasonable to consider a wider range of possibilities than simple power laws. An instructive illustration in this direction has been given in [56], by constructing an Einstein-de Sitter model with $\Omega_\Lambda = 0$, fitting the CMB data as well as the power spectrum of 2dFGRS. In this the Hubble constant is, however, required to be rather low: $H_0 \simeq 46\ km/s/Mpc$. The authors argue that this cannot definitely be excluded, because 'physical' methods lead mostly to relatively low values of H_0. In order to be consistent with matter fluctuations on cluster scales they add relic neutrinos with degenerate masses of order eV or a small contribution of quintessence with zero pressure ($w = 0$). In addition, they have to ignore the direct evidence for an accelerating Universe from the Hubble-diagram for distant Type Ia supernovas, on the basis of remaining systematic uncertainties. There is also the question whether the model is compatible with the observed large scale structure.

We do not discuss here other recent proposals. It is very likely that the present concordance model will survive. Additional evidence is steadily accumulating. But the mysteries of Dark Matter and Dark Energy will remain with us for a long time.

Note Added in Proof. Shortly before this paper was published, the improved WMAP data after three years of integration became available [astro-ph/0603449; astro-ph/0603450]. There have also been significant improvements in other astronomical data (high redshift supernovae, galaxy clustering, etc.). It is most remarkable that a six parameter cosmological ΛCDM model is able to fit a rich body of astronomical observations. An exciting result is that the WMAP data match the basic inflationary model predictions, and is even well fit by the simplest model $V \propto \varphi^2$.

Appendices

A Random fields, power spectra, filtering

Let $\xi(\mathbf{x})$ a random field on $\mathbb{R}^3$, and $\hat{\xi}(\mathbf{k})$ its Fourier transform, normalized according to

$$\xi(\mathbf{x}) = (2\pi)^{-3/2} \int \hat{\xi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k. \tag{A.1}$$

In our applications $\xi(\mathbf{x})$ will be, for instance, the field of density fluctuations $\delta(\mathbf{x})$ at a fixed time.

In practice $\hat{\xi}(\mathbf{k})$ will be distributional (generalized random field).

Correlation function and power spectrum

In our cosmological applications we shall often assume that the different **k**- modes are uncorrelated:

$$\langle \hat{\xi}(\mathbf{k}) \hat{\xi}(\mathbf{k})^* \rangle = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \mathcal{P}(\mathbf{k}). \tag{A.2}$$

Note that $\left|\hat{\xi}(\mathbf{k})\right|^2$ is not defined. (One might, therefore, prefer to work in a finite volume with periodic boundary conditions.)

[34] The search for non-Gaussian behavior is a topical subject. Such deviations are also expected in inflationary models, when non-linear corrections are taken into account. For an extended recent review see [55].

The function $\mathcal{P}(\mathbf{k})$ is the *power spectrum* belonging to $\xi(\mathbf{x})$. This is also the Fourier transform of the correlation function:

$$C_\xi(\mathbf{x}-\mathbf{x}') = \langle \xi(\mathbf{x})\xi(\mathbf{x}')\rangle = \frac{1}{(2\pi)^3}\int \mathcal{P}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} d^3k. \tag{A.3}$$

Filtering

Let W be a window function (filter) and define the filtered ξ by

$$\xi_W = \xi \star W. \tag{A.4}$$

With our convention we have for the Fourier transforms

$$\hat{\xi}_W = (2\pi)^{3/2}\hat{\xi}\,\hat{W}. \tag{A.5}$$

Therefore,

$$\mathcal{P}_{\xi_W}(\mathbf{k}) = (2\pi)^3 \left|\hat{W}(\mathbf{k})\right|^2 \mathcal{P}_\xi(\mathbf{k}). \tag{A.6}$$

With (A.3) this gives, in particular,

$$\langle \xi_W^2(\mathbf{x})\rangle = \int \left|\hat{W}(\mathbf{k})\right|^2 \mathcal{P}_\xi(\mathbf{k}) d^3k. \tag{A.7}$$

Example

For W we choose a top-hat:

$$W(\mathbf{x}) = \frac{1}{V}\theta(R-|\mathbf{x}|),\; V = \frac{4\pi}{3}R^3, \tag{A.8}$$

where θ is the Heaviside function. The Fourier transform is readily found to be

$$\hat{W}(\mathbf{k}) = (2\pi)^{-3/2}\tilde{W}(kR),\; \tilde{W}(kR) := \frac{3(\sin kR - kR\cos kR)}{(kR)^3}. \tag{A.9}$$

Thus,

$$\mathcal{P}_{\xi_W}(\mathbf{k}) = \left|\tilde{W}(kR)\right|^2 \mathcal{P}_\xi(\mathbf{k}). \tag{A.10}$$

For a spherically symmetric situation we get from (A.7)

$$\langle \xi_W^2(\mathbf{x})\rangle = \frac{1}{2\pi^2}\int \left|\tilde{W}(kR)\right|^2 \mathcal{P}_\xi(k) k^2 dk \tag{A.11}$$

(independent of $\mathbf{x}$).

For this reason one often works with the following definition of the power spectrum

$$P_\xi(k) := \frac{k^3}{2\pi^2}\mathcal{P}_\xi(k). \tag{A.12}$$

Then the last equation becomes

$$\langle \xi_W^2(\mathbf{x})\rangle = \int \left|\tilde{W}(kR)\right|^2 P_\xi(k)\frac{dk}{k}. \tag{A.13}$$

If ξ is the density fluctuation field $\delta(\mathbf{x})$, the filtered fluctuation σ_R^2 on the scale R is

$$\sigma_R^2 = \int \left|\tilde{W}(kR)\right|^2 P_\delta(k)\frac{dk}{k}. \tag{A.14}$$

B Collision integral for Thomson scattering

The main goal of this Appendix is the derivation of equation (7.66) for the collision integral in the Thomson limit.

When we work relative to an orthonormal tetrad the collision integral has the same form as in special relativity. So let first consider this case.

Collision integral for two-body scattering

In SR the Boltzmann equation (7.26) reduces to

$$p^\mu \partial_\mu f = C[f] \tag{B.1}$$

or

$$\partial_t f + v^i \partial_i f = \frac{1}{p^0} C[f]. \tag{B.2}$$

In order to find the explicit expression for $C[f]$ things become easier if the following non-relativistic normalization of the one-particle states $|p, \lambda\rangle$ is adopted:

$$\langle p', \lambda' | p, \lambda \rangle = (2\pi)^3 \delta_{\lambda,\lambda'} \delta^{(3)}(\mathbf{p}' - \mathbf{p}). \tag{B.3}$$

(Some readers may even prefer to discretize the momenta by using a finite volume with periodic boundary conditions.) Correspondingly, the one-particle distribution functions f are normalized according to

$$\int f(p) \frac{g d^3 p}{(2\pi)^3} = n, \tag{B.4}$$

where g is the statistical weight (= 2 for electrons and photons), and n is the particle number density.

The S-matrix element for a 2-body collision $p, q \to p', q'$ has the form (suppressing polarization indices)

$$\langle p', q' | S - 1 | p, q \rangle = -i (2\pi)^4 \delta^{(4)}(p' + q' - p - q) \langle p', q' | T | p, q \rangle. \tag{B.5}$$

Because of our non-invariant normalization we introduce the Lorentz invariant matrix element M by

$$\langle p', q' | T | p, q \rangle = \frac{M}{(2p^0 2q^0 2p'^0 2q'^0)^{1/2}}. \tag{B.6}$$

The transition probability per unit time and unit volume is then (see, e.g., Sect. 64 of [57])

$$dW = (2\pi)^4 \frac{1}{2p^0 2q^0} |M|^2 \delta^{(4)}(p' + q' - p - q) \frac{d^3 p'}{(2\pi)^3 2p'^0} \frac{d^3 q'}{(2\pi)^3 2q'^0}. \tag{B.7}$$

Since we ignore in the following polarization effects, we average $|M|^2$ over all polarizations (helicities) of the initial and final particles. This average is denoted by $\overline{|M|^2}$. Per polarization we still have the formula (B.7), but with $|M|^2$ replaced by $\overline{|M|^2}$. From time reversal invariance we conclude that $\overline{|M|^2}$ remains invariant under $p, q \leftrightarrow p', q'$.

With the standard arguments we can now write down the collision integral. For definiteness we consider Compton scattering $\gamma(p) + e^-(q) \to \gamma(p') + e^-(q')$ and denote the distribution functions of the photons and electrons by $f(p)$ and $f_{(e)}(q)$, respectively. In the following expression we neglect the Pauli suppression factors $1 - f_{(e)}$, since in our applications the electrons are highly non-degenerate. Explicitly, we have

$$\frac{1}{p^0} C[f] = \frac{1}{2p^0} \int \frac{2d^3 q}{(2\pi)^3 2q'^0} \frac{2d^3 q'}{(2\pi)^3 2q'^0} \frac{2d^3 p'}{(2\pi)^3 2p'^0} (2\pi)^4 \overline{|M|^2} \delta^{(4)}(p' + q' - p - q)$$

$$\times \left\{ (1+f(p))\, f(p')f_{(e)}(q') - \big(1+f(p')\big)\, f(p)f_{(e)}(q) \right\} . \tag{B.8}$$

At this point we return to the normalization of the one-particle distributions adopted in Sect. 7.1. This amounts to the substitution $f \to 4\pi^3 f$. Performing this in (B.1) and (B.8) we get for the collision integral

$$\begin{aligned} C[f] = \frac{1}{16\pi^2} \int \frac{d^3q}{q^0}\,\frac{d^3q'}{q'^0}\,\frac{d^3p'}{p'^0}\,\overline{|M|^2}\delta^{(4)}(p'+q'-p-q) \\ \times \left\{ \big(1+4\pi^3 f(p)\big)\, f(p')f_{(e)}(q') - \big(1+4\pi^3 f(p')\big)\, f(p)f_{(e)}(q) \right\} . \end{aligned} \tag{B.9}$$

The invariant function $\overline{|M|^2}$ is explicitly known, and can for instance be expressed in terms of the Mandelstam variables s, t, u (see Sect. 86 of [57]).

The integral with respect to d^3q' can trivially be done

$$C[f] = \frac{1}{16\pi^2} \int \frac{d^3q}{q^0}\,\frac{1}{q'^0}\,\frac{d^3p'}{p'^0}\,\delta(p'^0+q'^0-p^0-q^0)\overline{|M|^2} \times \{\cdots\}. \tag{B.10}$$

The integral with respect to $\mathbf{p}'$ can most easily be evaluated by going to the rest frame of q^μ. Then

$$\int d^3p' \frac{1}{p'^0 q'^0}\delta(p'^0+q'^0-p^0-q^0)\cdots = \int d\Omega_{\hat{\mathbf{p}}'} \int d|\mathbf{p}'|\frac{|\mathbf{p}'|}{q'^0}\delta(m+q'^0-p^0-q^0)\cdots .$$

We introduce the following notation: With respect to the rest system of q^μ let $\omega := p^0 = |\mathbf{p}|$, $\omega' := p'^0 = |\mathbf{p}'|$, $E' = \sqrt{\mathbf{q}'^2 + m^2}$. Then the last integral is equal to

$$\frac{\omega'}{E'}\,\frac{1}{|1+\partial E'/\partial\omega'|} = \frac{\omega'^2}{m\omega}.$$

In getting the last expression we have used energy and momentum conservation.

So far we are left with

$$C[f] = \frac{1}{16\pi^2 m} \int \frac{d^3q}{q^0} \int d\Omega_{\hat{\mathbf{p}}'}\,\frac{\omega'^2}{\omega}\,\overline{|M|^2} \times \{\cdots\}. \tag{B.11}$$

In the rest system of q^μ the following expression for $\overline{|M|^2}$ can be found in many books (for a derivation, see [58])

$$\overline{|M|^2} = 3\pi m^2 \sigma_T \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin\vartheta \right], \tag{B.12}$$

where ϑ is the scattering angle in that frame. For an arbitrary frame, the combination $d\Omega_{\hat{\mathbf{p}}'}\,\frac{\omega'^2}{\omega}\,\overline{|M|^2}$ has to be treated as a Lorentz invariant object.

At this point we take the non-relativistic limit $\omega/m \to 0$, in which $\omega' \simeq \omega$ and $C[f]$ reduces to the simple expression

$$C[f] = \frac{3}{16\pi}\sigma_T \omega n_e \int d\Omega_{\hat{\mathbf{p}}'}(1+\cos^2\vartheta)[f(p') - f(p)]. \tag{B.13}$$

Derivation of (7.66)

In Sect. 7.4 the components p^μ of the four-momentum p refer to the tetrad e_μ defined in (7.42). Relative to this[35] we introduced the notation $p^\mu = (p, p\gamma^i)$. The electron four-velocity is according to (1.156) given to first order by

$$u_{(e)} = \frac{1}{a}(1-A)\partial_\eta + \frac{1}{a}\gamma^{ij} v_{(e)|j}\partial_j = e_0 + v^i_{(e)} e_i;\ v^i_{(e)} = v_{(e)i} = \hat{e}_i(v_{(e)}). \tag{B.14}$$

Now ω in (B.13) is the energy of the four-momentum p in the rest frame of the electrons, thus

$$\omega = -\langle p, u_{(e)}\rangle = p[1 - \hat{e}_i(v_{(e)})\gamma^i]. \tag{B.15}$$

Similarly,

$$\omega' = -\langle p', u_{(e)}\rangle = p'[1 - \hat{e}_i(v_{(e)})\gamma'^i]. \tag{B.16}$$

Since in the non-relativistic limit $\omega' = \omega$, we obtain the relation

$$p'[1 - \hat{e}_i(v_{(e)})\gamma'^i] = p[1 - \hat{e}_i(v_{(e)})\gamma^i]. \tag{B.17}$$

Therefore, to first order

$$\begin{aligned} f(p', \gamma'^i) &= f^{(0)}(p') + \delta f(p', \gamma'^i) \\ &= f^{(0)}(p) + \frac{\partial f^{(0)}}{\partial p}(p' - p) + \delta f(p, \gamma'^i) \\ &= f^{(0)}(p) + p\frac{\partial f^{(0)}}{\partial p}\hat{e}_i(v_{(e)})(\gamma'^i - \gamma^i) + \delta f(p, \gamma'^i). \end{aligned} \tag{B.18}$$

Remember that the surface element $d\Omega_{\hat{\mathbf{p}}'}$ in (B.13) also refers to the rest system. This is related to the surface element $d\Omega_{\gamma'}$ by[36]

$$d\Omega_{\hat{\mathbf{p}}'} = \left(\frac{p'}{\omega'}\right)^2 d\Omega_{\gamma'} = [1 + 2\hat{e}_i(v_{(e)})\gamma'^i] d\Omega_{\gamma'}. \tag{B.19}$$

Inserting (B.18) and (B.19) into (B.13) gives to first order, with the notation of Sect. 7.5,

$$C[f] = n_e\sigma_T p\left[\langle\delta f\rangle - \delta f - p\frac{\partial f^{(0)}}{\partial p}\hat{e}_i(v_{(e)})\gamma^i + \frac{3}{4}Q_{ij}\gamma^i\gamma^j\right], \tag{B.20}$$

that is the announced equation (7.66).

This approximation suffices completely for our applications. The first order corrections to the Thomson limit have also been worked out [59].

C Ergodicity for (generalized) random fields

In Sect. 5.2.3 we have replaced a spatial average by a stochastic average. Since this is often done in cosmology, we add some remarks about what is behind this procedure.

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35 Without specifying the gauge one can easily generalize the following relative to the tetrad defined by (7.31).

36 Under a Lorentz transformation, the surface element for photons transforms as

$$d\Omega = (\omega'/\omega)^2 d\Omega'$$

(exercise).

Mathematical remarks on generalized random fields

Let ϕ be a generalized random field. Each 'smeared' $\phi(f)$ is a random variable on some probability space $(\Omega, \mathcal{F}, \mu)$. Often one can choose $\Omega = \mathcal{S}'(\mathbb{R}^D)$, $\mathcal{F}$: σ-algebra generated by cylindrical sets, and $\phi(f)$ the 'coordinate function'

$$\phi(f)(\omega) = \langle \omega, f \rangle, \ \omega \in \mathcal{S}'(\mathbb{R}^D), \ f \in \mathcal{S}(\mathbb{R}^D).$$

Notation: We use the letter ϕ for elements of Ω and interpret $\phi(f)$ as the coordinate function: $\phi \mapsto \langle \phi, f \rangle$.

Let τ_a denote the translation of $\mathbb{R}^D$ by a. This induces translations of Ω, as well as random variables such as $A = \phi(f_1) \cdots \phi(f_n)$, which we all denote by the same symbol τ_a. Assume that μ is an invariant measure on $(\Omega, \mathcal{F})$ which is also *ergodic*: For any measurable subset $M \in \Omega$ which is invariant under translations $\mu(M)$ equals 0 or 1. Then the following **Birkhoff ergodic theorem** holds: *"spatial average (of individual realization)= stochastic average"*, i.e., μ-almost always

$$\lim_{\Lambda \uparrow \mathbf{R}^D} \frac{1}{|\Lambda|} \int_\Lambda \tau_a A \, da = \langle A \rangle_\mu, \tag{C.1}$$

where Λ is a finite hypercube, and the right-hand side denotes the stochastic average of the random variable A.

Generalized random fields on a torus. Often it is convenient to work on a "big" torus T^D with volume $V = L^D$. Then $\Omega = \mathcal{D}'(T^D)$ (periodic distributions), etc. The Fourier transform and cotransform are topological isomorphisms between $\mathcal{D}(T^D)$ and $\mathcal{S}(\Delta^D)$, $\Delta^D := (2\pi/L)^D \mathbb{Z}^D$, the rapidly decreasing (tempered) sequences[37]. These provide, in turn, isomorphisms between $\mathcal{D}'(T^D)$ and $\mathcal{S}'(\Delta^D)$. Each (periodic) distribution $S \in \mathcal{D}'(T^D)$ can be expanded in a convergent Fourier series

$$S = \frac{1}{\sqrt{V}} \sum_{k \in \Delta^D} c_k(S) \chi_k, \ \chi_k(x) := \frac{1}{\sqrt{V}} e^{ik \cdot x} \tag{C.2}$$

(χ_k regarded as a distribution), where

$$c_k(S) = \langle S, e^{-ik \cdot x} \rangle. \tag{C.3}$$

Written symbolically,

$$S(x) = \frac{1}{V} \sum_{k \in \Delta^D} S_k e^{ik \cdot x}, \ S_k = \int S(x) e^{-ik \cdot x} \, dx. \tag{C.4}$$

Let us consider the correlation functions $\langle \phi(f) \phi(g) \rangle_\mu$. In terms of the Fourier expansion for $\phi(x)$, we have

$$\langle \phi(x) \phi(y) \rangle_\mu = \frac{1}{V^2} \sum_{k,k'} \langle \phi_k \phi_k'^* \rangle e^{i(k \cdot x - k' \cdot y)}.$$

This is only translationally invariant if the tempered sequences ϕ_k are uncorrelated,

$$\langle \phi_k \phi_k'^* \rangle = \delta_{kk'} \langle |\phi_k|^2 \rangle.$$

[37] For proofs of this and some other statements below, see W. Schempp and B. Dressler, *Einführung in die harmonische Analyse* (Teubner, 1980), Sect. I.8.

Then

$$\langle\phi(x)\phi(y)\rangle_\mu = \frac{1}{V}\sum_k \frac{\langle|\phi_k|^2\rangle}{V} e^{ik\cdot(x-y)}. \tag{C.5}$$

By definition, the power spectrum $P_\phi(k)$ of the generalized random field ϕ is the Fourier transform of the correlation function (distribution)

$$\langle\phi(x)\phi(y)\rangle_\mu = \frac{1}{V}\sum_{k\in\Delta^D} P_\phi(k) e^{ik\cdot(x-y)}. \tag{C.6}$$

Therefore,

$$P_\phi(k) = \frac{1}{V}\langle|\phi_k|^2\rangle_\mu. \tag{C.7}$$

If the measure is ergodic with respect to translations τ_a, we obtain μ-almost always the same result if we take for a particular realization of $\phi(x)$ its *spatial average*. This follows from Birkhoff's ergodic theorem, stated above, together with the following well-known theorem of H. Weyl:

Theorem (H. Weyl). Let f be a continuous function on the torus T^D, then

$$\lim_{\Lambda\uparrow\mathbf{R}^D} \frac{1}{|\Lambda|}\int_\Lambda f\circ\tau_a \, da = \int_{T^D} f \, d\lambda, \tag{C.8}$$

where λ is the invariant normalized measure on T^D.

For a proof I refer to Arnold's "Mathematical methods of classical mechanics", Sect. 51.

A discrete example for ergodic random fields

Proving ergodicity is usually very difficult. Below we give an example of a discrete random Gaussian field, for which this can be established without much effort.

Let $\Omega = \mathbb{R}^{\mathbb{Z}^D}$, and consider the discrete random field $\phi_x(\omega) = \omega_x$, where $\omega : \mathbb{Z}^D \to \mathbb{R}$, and ω_x denotes the value of ω at site $x \in \mathbb{Z}^D$. We assume that the random field ϕ_x is Gaussian, and that the underlying probability measure μ is invariant under translations. Then the correlation function $C(x-y) = \langle\phi_x\phi_y\rangle$ depends only on the difference $x - y$. Being of positive type, we have by the Bochner-Herglotz theorem a representation of the form

$$C(x) = \int_{T^D} e^{ik\cdot x} \, d\sigma(k), \tag{C.9}$$

where σ is a positive measure.

Now we can formulate an interesting fact:

Theorem (Fomin, Maruyama). (1) The random field ϕ_x is ergodic (i.e., the probability measure μ is ergodic relative to discrete translations τ_a), if and only if the measure σ is nonatomic. (2) The translations are mixing if σ is absolutely continuous with respect to λ.

For a proof, see Cornfeld, Fomin, and Sinai, *Ergodic Theory*, Springer (Grundlehren, 245), Sect. 14.2. (I was able to simplify this proof somewhat.)

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