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Yolanda S. Stander

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Special thanks to my partner, Rudi van der Linde, who was the first person to read this book. It would not be possible without his help and support. I would also like to mention two extraordinary people who taught me a lot about risk management: Jurie Bester and Satyajit Das.

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CHAPTER 1

Introduction

The success of any financial institution in the trading environment rests on one important issue: The successful implementation of the building blocks that make up such an institution. In this book we consider one of the most important building blocks, namely yield curves.

Yield curves are commonly used in the pricing and valuation environments to assist in the subjective and objective decision-making processes respectively. In a pricing environment yield curves assist dealmakers to determine the current level of market interest rates, and can offer information on what price to quote for interest rate instruments. Yield curves that are used for pricing are usually subjective, in that dealmakers add spreads to the yield curve to cover factors like transaction costs, credit risk, liquidity issues, and a little extra to make the deal profitable.

In a valuation environment, yield curves are used to value the deals in a *portfolio*. A portfolio is considered to be a set of instruments bought or sold in the market. The valuation yield curves have to be objective and accurate reflections of market rates to ensure that the estimates of future profits are correct. The numbers are usually compiled by accountants, who are typically interested in the factors to which profits or losses can be attributed.

The valuation yield curves are also used in the risk management process to determine the current market risk of any portfolio. The risk management function is separate from the dealmakers, because risk managers have to be objective when determining the risks to which the financial institution is exposed.

In short, an incorrect yield curve may lead to incorrect valuations and risk numbers, which in turn may lead to either inadequate or too large economic capital reserves. These respectively increase the solvency risk of the institution, or inhibit the dealmakers from entering into more lucrative deals.

In Section 1.1 we discuss some background theory regarding yield curves. We consider the various shapes a yield curve can take, and discuss the various theories that attempt to explain yield curve shapes. In Section 1.2 we consider the main approaches to modeling yield curves. Finally in Section 1.3 we give a brief layout of the rest of the book.

1.1 YIELD CURVE BACKGROUND

Interest rates usually differ according to the tenor of the instrument. For example, the return on a one-year instrument will differ from the return on a two-year instrument. This is the concept behind the term structure of interest rates. A yield curve describes both the shape and level of the term structure of interest rates at any time.

The four main shapes that yield curves usually take are shown in Figure 1.1. They are positively sloped, negatively sloped, humped, and flat curves (Douglas, 1988; McEnally, 1987). It is interesting to note the different shapes that can occur as a result of market dynamics, since a naïve view of

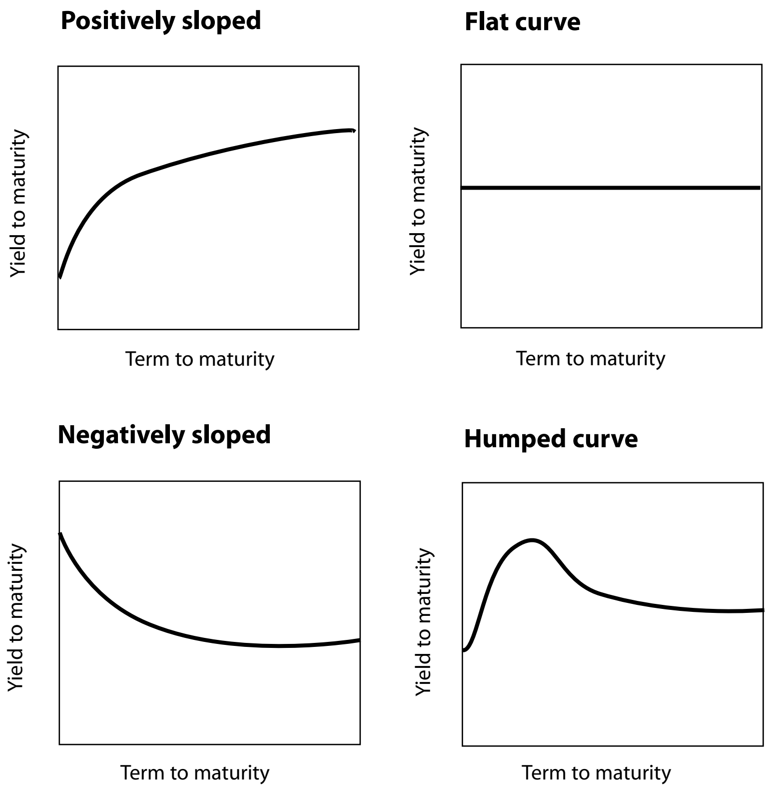


Figure 1.1 Different shapes of the yield curve

yield curves could be that only upward-sloping yield curves are possible, as investors want to be compensated more for money that is invested for longer periods.

In the attempt to explain the shape of the term structure, three prominent theories have been proposed: the market segmentation theory, the pure expectations theory and the liquidity premium theory. These theories are generally not exclusively accepted by market participants; they are rather regarded as influences on the shape of the yield curve.

According to the *market segmentation theory*, institutions have preferences for securities that match their needs, and the shape of the term structure is primarily determined by the supply and demand of the instruments and the interaction of the institutions in the market. For instance, consider two groups of investors, where the one group buys short-term securities and the other group buys long-term securities. The two groups may influence the prices of the securities in which they are investing, since they simultaneously increase the demand for the instruments and also reduce their effective liquidity by holding on to the securities. The interaction of the two groups thus creates an interest rate differential between securities with different maturities (Bodie, Kane, and Marcus, 1996).

The *pure expectations theory* acknowledges that maturity preferences exist, but theorizes that investors may take advantage of the expectations about future interest rates to such an extent that they neutralize the maturity preferences and create yield differences for securities with differing maturities. Investors require a premium to hold instruments with maturities different from their investment horizons. The expectations theory depends heavily on the assumption that investors are indifferent to maturities as long as they obtain the highest total return over the investment period (Bodie *et al.*, 1996; Douglas, 1988).

The *liquidity premium theory* differs from the expectations theory in that the notion that investors are indifferent to the maturity of securities is rejected. The assumption is that investors will prefer short-term securities, because they have smaller interest-rate risk. The greater demand for short-term securities increases the liquidity of the instruments, which means their yields are lower than those of long-term securities. The yield curve will be upward sloping when short-term securities have lower yields than long-term securities (Douglas, 1988; Fisch, 1997).

1.2 MAIN APPROACHES TO MODELING YIELD CURVES

The traditional way of deriving yield curves is to apply *regression-type yield curve models*. Consider a scatter plot of the yields to maturity against the terms to maturity of a series of bonds. The yield curve is the function that best fits through all the points on the scatter plot. The resulting yield curve is called a *par rate curve*, which has to be bootstrapped to get a *zero rate curve*. Please refer to Chapter 2 for a detailed discussion.

One of the main problems with the regression-type models is that the "coupon effect" is not taken into account. The coupon effect refers to the fact that different bonds with the same term to maturity may have very different yields to maturity because of differences in their coupon rates. To get around the "coupon effect" issue some practitioners use duration on the x-axis instead of term to maturity (McLeod, 1990). However, it is difficult to interpret the duration yield curve because it is unclear to which periods the different rates apply.

Another approach to yield curve modeling is to assume a functional form for the discount function. The discount function is used to calculate the fitted prices of a series of instruments. The parameters of the discount function are estimated by setting up an optimization routine that minimizes the differences between the actual and fitted instrument prices. These models are referred to as *empirical yield curve models*, and were developed to overcome some of the shortcomings of the regression-type yield curve models.

Some of the empirical models are highly parameterized, which may be problematic since it is usually difficult to find adequate estimates for the parameters. The empirical models are the most commonly used models in practice because they tend to show a good fit of the underlying instruments.

The final approach to modeling yield curves is known as the *dynamic asset pricing approach*. With this approach we take a dynamic view of both the shape of the term structure and its evolution over time, by making explicit assumptions regarding the stochastic processes of the factors driving the interest rates (Yao, 1998).

In Chapter 3 we consider one of the major approaches to dynamic asset pricing, namely *equilibrium models*. Equilibrium models make assumptions about economic variables by taking phenomena such as mean reversion and volatility roll-down into account. *Mean reversion* refers to the idea that interest rates have a drift rate that forces them to converge to some long-term average level, and *volatility roll-down* refers to the fact that volatility is a decreasing function of maturity.

Equilibrium models are only briefly considered in Chapter 3, because zero curves derived from equilibrium models do not fit the observed data very well. These models are not flexible enough to allow for the various shapes of yield curves.

1.3 THE LAYOUT OF THE BOOK

The book is divided into a number of chapters that discuss the various aspects of yield curve modeling.

It is important to choose the correct instruments when any yield curve is derived. The instruments dictate the way the yield curve should be derived, as well as the type of instruments that can be valued from the derived yield curve. There are different types of rates that have to be allowed for; for instance:

- the yields-to-maturity of coupon-paying bonds are *par rates*
- the rates derived from forward rate agreements (FRAs) are *forward rates*
- the Treasury bill rates are usually *discount rates*
- zero-coupon bonds prices are discount factors so that the rates derived from them are *zero rates*
- the interest rate swap rates quoted in the market are *par rates*.

In Chapter 2 we define the various types of interest rates. We show the logical derivation of the various formulae and discuss practical examples on the use of the formulae. We also highlight possible pitfalls. The formulae and concepts discussed in Chapter 2 form the basic building blocks in the process of deriving any yield curve, and are used extensively in later chapters.

There are three main types of yield curve models, namely regression-type models, empirical models, and equilibrium models (please refer to the discussion in Section 1.2). In Chapter 3 we consider some of the most popular models that are proposed in the literature. In this book we focus on the empirical models.

Other important issues that have to be taken into account when deriving yield curves are:

- the compounding of the instrument rates, for example whether it is a simple rate or a continuously compounded rate
- daycount conventions
- allowing for public holidays and business day rules
- the credit quality of the instrument
- the liquidity of the instrument
- interpolation and extrapolation techniques.

These issues are considered in Chapter 4.

We begin Chapter 5 with a discussion on how to fit the empirical yield curve models discussed in Chapter 3 to coupon-paying bonds. There are various important points that are addressed, like:

- setting up the optimization routine
- choosing initial parameter estimates for the yield curve function
- choosing the appropriate yield curve function
- which bonds to include in the process
- how to test whether the fitted yield curve model is adequate.

Any attempt to model the term structure introduces the practical problem of how to fill the gaps in the maturity spectrum that arise because of incomplete and imperfect financial markets. To fill these gaps there is a trade-off between smoothness and reliability. The "noise" should be removed from the data, while still allowing for genuine "bends" in the term structure. This process is discussed in detail in Chapter 5.

In Chapter 5 we also consider how to derive zero rate curves from FRAs and interest rate futures. Finally we discuss a way in which to derive a term structure for interest rates that have no quoted term structure, for instance the bank prime rate.

The idea of Chapter 5 is to give the reader practical ideas on how to derive yield curves from the various types of instruments, and to explain the issues that should be taken into account. After working through this chapter it should be clear that the most important part in deriving a yield curve is to understand the underlying instrument.

In Chapter 6 we consider the relationship between nominal interest rates, real interest rates, and inflation rates. We consider some inflation-linked securities, and show how to derive a real curve from these instruments. We also discuss a few interesting examples on how to derive an inflation term structure, and how to use the inflation term structure to derive real interest rates where there are no inflation-linked securities available in the market.

In Chapter 7 we explore the factors that drive credit, liquidity, and country risk premiums, and we consider various ways in which to measure these premiums using yield curves. It is important for financial institutions to be able to understand the different sources of risk and to quantify it.

In Chapter 8 we consider the types of risks that have to be minimized when a yield curve model is put together. We briefly consider the types of yield curve risks that have to be allowed for, as well as how to measure interest rate risk. We also discuss the effect incorrect yield curves can have on the risk measures. We further explore some examples where operational and model risk may occur with respect to yield curves, and we discuss ways to minimize the risks. We end the chapter with a brief discussion on liquidity risk.

This book will give the reader insight into the techniques that can be used to model yield curves in our incomplete and imperfect financial markets. It is assumed that the reader has a basic understanding of the financial instruments that are available in the market. Various practical solutions are provided and possible pitfalls highlighted. The objective of this book is to be a practical guide on yield curves that is easy to follow, and that presents techniques that are simple to implement.

CHAPTER 2

Concepts and Terminology

In this chapter a few basic types of interest rates and the relationships between them are defined. The formulae denoting these relationships are derived by using very straightforward arguments; this involves the breaking up of instruments into their underlying cash flows. The formulae and concepts discussed in this chapter form the basic building blocks for the derivation of yield curves, and are used extensively in later chapters.

In Section 2.1 zero interest rates are discussed. It is important to understand the different ways in which rates can be compounded, because it has a major effect on the instruments valued with the rate. In Section 2.2 discount factors are discussed. Discount factors play an important role in deriving empirical yield curves from bonds, but that is discussed in more detail in Chapters 3 and 5. Another important concept, discussed in Section 2.3, is that of forward interest rates. A forward interest rate represents the return that is expected to be achieved between two dates in the future. An intuitive way in which to derive forward rates is shown. In Sections 2.4 and 2.5 the concepts of yield to maturity and par interest rates are discussed. These two types of rates are in fact the same in that they are implicitly internal rates of return: the yield to maturity is the internal rate of return of a bond, and a par rate the internal rate of return of an interest rate swap. In later chapters we do not refer to a yield to maturity curve, but rather just a par curve derived from bond yields. In Section 2.6 we obtain the formulae to derive zero rates from par rates. Finally in Section 2.7 the concept of implied interest rates is discussed by considering the relationship between domestic and foreign interest rates and the relevant exchange rates.

The information in this chapter is adopted from various sources, which include Deacon and Derry (1994a), McCutcheon and Scott (1994), Svensson (1994), Anderson *et al.* (1997), Waggoner (1997), McCulloch and Kochin (1998), and Cairns (2004).

2.1 ZERO INTEREST RATES

The concept of *time value of money* refers to the fact that one unit of money, invested today, will be worth more (or less) at the end of the investment period. The difference between the value of the investment today and the value of the investment at some future date is due to the fact that a zero nominal interest rate is earned on the investment. For example, say an investor deposits an amount P into a bank account today for a period n years where a nominal zero interest rate of i per annum, compounded annually, is earned. After the first year, the investment will be worth:

$$FV_1 = P(1 + i)$$

.

This amount FV_1 will now earn interest for the next year, so at the end of two years the investment will be worth:

$$FV_2 = FV_1(1 + i) = P(1 + i)(1 + i) = P(1 + i)^2$$

and so on until at the end of n years the investment will have a value of

$$FV_n = P(1 + i)^n \qquad (2.1)$$

where FV_n is the future value of the investment at the end of year n.

The zero interest rate is a *nominal annual rate* that is compounded annually, or in short, a *naca* rate. The formulae derived above assume that the rate at which interest is earned and reinvested remains constant over the investment period. This assumption is necessary to find the relationship between rates that are compounded differently.

In the rest of this section we explore the different ways in which zero interest rates can be compounded. By using similar arguments to those used to derive (2.1), we derive the formulae necessary to convert between two rates that are compounded differently.

2.1.1 Rates compounded at regular intervals

The relationship between a nominal rate that is compounded m times a year and a nominal rate that is compounded n times a year can be expressed as:

$$\left(1 + \frac{i_{(m)}}{m}\right)^m = \left(1 + \frac{i_{(n)}}{n}\right)^n \qquad (2.2)$$

where

$i_{(m)}$ = nominal annual rate compounded m times per annum

$i_{(n)}$= nominal annual rate compounded n times per annum
m = number of compounding periods per annum
n = number of compounding periods per annum.

By solving for $i_{(m)}$ in (2.2) we find that to convert a rate compounded n times a year, to a rate compounded m times a year, we do the following:

$$i_{(m)} = m\left[\left(1+\frac{i_{(n)}}{n}\right)^{\frac{n}{m}} - 1\right]$$

In practice the most commonly used compounding periods are annually, semi-annually, quarterly, and monthly. The corresponding interest rates are referred to as *naca*, *nacs*, *nacq* and *nacm* rates respectively.

2.1.2 Continuous compounding

Continuous compounding assumes that an investment amount grows continuously at a fixed specified rate throughout the investment period. The relationship between a continuously compounded rate i and a nominal rate compounded n times a year, $i_{(n)}$, is:

$$e^i = \left(1+\frac{i_{(n)}}{n}\right)^n \tag{2.3}$$

where $e \approx 2.718$ is the base of the natural logarithm. To convert the nominal rate to a continuously compounded rate, we would rewrite (2.3) as follows:

$$i = n \times \ln\left(1+\frac{i_{(n)}}{n}\right)$$

2.1.3 Simple interest

Simple interest is defined as an investment where interest is earned on the principal amount only, in other words no "interest on interest" is earned. To illustrate this, say an amount P is invested and simple interest is earned at a rate of r per annum. After n years the investment will be worth:

$$FV_n = P(1+nr) \tag{2.4}$$

where FV_n denotes the future value of the investment. By setting (2.4) equal to (2.1) we see that the relationship between a simple rate and an annually compounded rate can be written as:

$$(1 + nr) = (1 + i)^n$$

A more generalized formula to illustrate the relationship between a simple interest rate r and a nominal rate compounded m times a year is as follows:

$$1 + r\frac{d}{DC} = \left(1 + \frac{i_{(m)}}{m}\right)^{\frac{d \times m}{DC}} \qquad (2.5)$$

where d is the number of days in the investment period and DC is the number of days that there is assumed to be in a year (which depends on the daycount convention used).

The daycount convention differs between countries, for example when working with US dollar rates we usually use 360 days, and for sterling (GBP) rates we usually use 365 days. However, the daycount convention may differ depending on the type of instrument. Please refer to Chapter 4 for a detailed discussion.

By rearranging (2.5), the simple interest rate is obtained from the compounded rate as follows:

$$r = \frac{DC}{d}\left[\left(1 + \frac{i_{(m)}}{m}\right)^{\frac{d \times m}{DC}} - 1\right]$$

Alternatively, the compounded rate can be calculated from the simple rate as follows:

$$i_{(m)} = m\left[\left(1 + r\frac{d}{DC}\right)^{\frac{DC}{d \times m}} - 1\right]$$

Another important concept is the calculation of the *effective rate of interest*. The effective rate reflects the total interest amount an investor receives over a period, assuming that interest is received periodically and then immediately reinvested at the original rate. The effective rate is always greater than the nominal rate by an amount that increases as the interest earned is reinvested more frequently. To calculate an effective annual rate i, from a nominal annual rate compounded m times a year $i_{(m)}$, the following relationship is used:

$$1 + i = \left(1 + \frac{i_{(m)}}{m}\right)^m \qquad (2.6)$$

which is analogous to the previously shown relationship between simple and compounding interest rates given by (2.5).

2.1.4 Discount rates

Some money market instruments like treasury bills and bankers acceptances are generally issued and traded at a discount rate. The discount rate of interest is used to calculate the amount an investor must invest today in order to receive a prespecified amount at maturity. The difference between the maturity value and the initial investment amount is the discount amount. The discount amount, when compared with the maturity value, results in the *discount rate*.

The relationship between simple and discount rates is as follows:

$$\left(1 + i\frac{d}{DC}\right) = \left(1 - r\frac{d}{DC}\right)^{-1} \tag{2.7}$$

where

i = simple interest rate
r = discount rate
d = number of days in the investment period
DC = number of days in the year—this depends on the daycount convention applicable.

To derive a simple rate from a discount rate, we rearrange (2.7) as follows:

$$i = \frac{r}{1 - r \times \frac{d}{DC}}$$

and similarly, to derive the discount rate from a simple rate, we have that:

$$r = \frac{i}{1 + i \times \frac{d}{DC}}$$

2.2 DISCOUNT FACTORS

The discount factor df_m denotes the price at time t of a zero-coupon bond that pays 1 at the maturity T where the term to maturity is calculated as $m = T - t$. Discount factors can take on values in the interval [0; 1].

The plot of a set of discount factors against the relevant terms to maturity is called the *discount factor curve*, or the *discount function*. The discount function is considered to be the continuous analog to a set of discount factors and is denoted by $\delta(m)$. A discount factor df_m can be regarded as a discrete point on the continuous discount function $\delta(m)$, thus $df_m = \delta(m)$ (Deacon and Derry, 1994a). The discount function typically has a negative exponential shape with $df_0 = 1$. An example of a discount function is given in Figure 2.1.

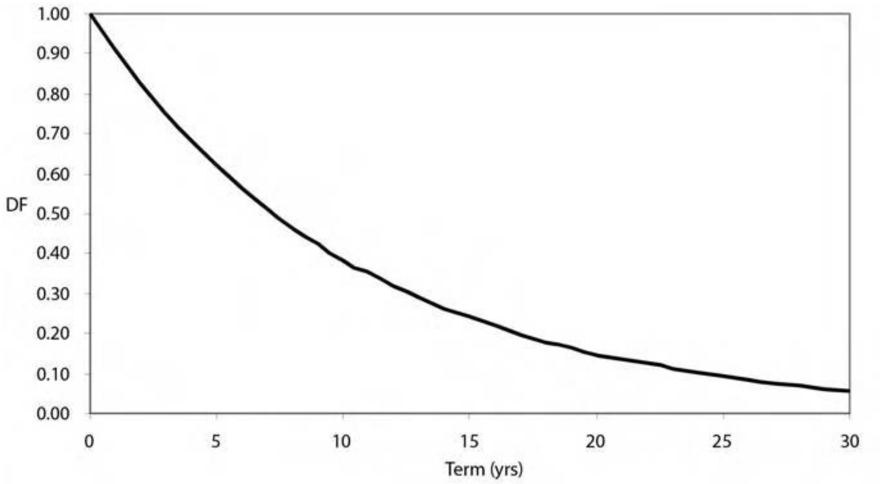


Figure 2.1 Example of a discount function

Discount factors are used to calculate the present value of an investment. Say the investor expects to receive a series of cash flows in the future. To calculate what the total value of these cash flows is today, the investor has to discount each of the cash flows back to today with the appropriate discount factor.

The compounding of the rate has to be taken into account when a discount factor is calculated from a zero interest rate. It is done as follows:

$$df_m = \begin{cases} \dfrac{1}{(1+\frac{i}{n})^{mn}} & , i \text{ compounded } n \text{ times per annum} \\ \dfrac{1}{(1+im)} & , i \text{ a simple rate} \\ e^{-im} & , i \text{ a continuously compounded rate} \end{cases} \tag{2.8}$$

where i is the annual zero rate compounded as specified in each case, and m the period in years for which a discount factor is calculated.

2.3 FORWARD RATES

A *forward interest rate* represents the return that is expected to be achieved between two dates in the future. Forward rates are calculated from zero rates using the arbitrage-free condition, which holds that an investment that earns interest of i_1 from t_0 up to time t_1 and is then reinvested at an interest rate of $f_{1,2}$ for the remainder of the period up to time t_2, should have the same value at the end of the investment period (t_2) as an investment of the same funds that earns interest of i_2 from t_0 up to time t_2. This concept is depicted graphically in Figure 2.2 and mathematically in (2.9).

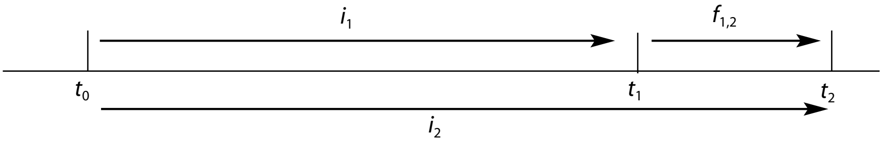


Figure 2.2 An investment that earns interest of i_1 up to time t_1 and is then reinvested to earn interest of $f_{1,2}$ up to time t_2, should have the same value as an investment of the same funds that earns interest of i_2 from t_0 to time t_2

$$\left[1 + i_1 \frac{t_1 - t_0}{DC}\right]\left[1 + f_{1,2} \frac{t_2 - t_1}{DC}\right] = \left[1 + i_2 \frac{t_2 - t_0}{DC}\right] \tag{2.9}$$

Equation (2.9) is valid when i_1 and i_2 are simple zero rates and DC refers to the number of days in the year as dictated by the daycount convention. By solving for $f_{1,2}$ in (2.9), we have that:

$$f_{1,2} = \left(\frac{\left[1 + i_2 \frac{t_2 - t_0}{DC}\right]}{\left[1 + i_1 \frac{t_1 - t_0}{DC}\right]} - 1 \right) \times \frac{DC}{t_2 - t_1}$$

where $f_{1,2}$ is the simple forward rate between times t_1 and t_2.

Using the same arguments, a forward rate can be calculated between any two periods, with the forward rate compounded in any appropriate way. For example, to calculate a simple forward rate from zero rates that are compounded m times a year, the formula needs to be adjusted slightly, to get the following relationship:

$$\left[1 + \frac{i_1}{m}\right]^{\frac{m(t_1 - t_0)}{DC}}\left[1 + f_{1,2} \frac{t_2 - t_1}{DC}\right] = \left[1 + \frac{i_2}{m}\right]^{\frac{m(t_2 - t_0)}{DC}} \tag{2.10}$$

where i_1 and i_2 are now compounded m times per year. The forward rate can easily be determined by solving for $f_{1,2}$ as follows:

$$f_{1,2} = \left(\frac{\left[1 + \frac{i_2}{m}\right]^{\frac{m(t_2 - t_0)}{DC}}}{\left[1 + \frac{i_1}{m}\right]^{\frac{m(t_1 - t_0)}{DC}}} - 1 \right) \times \frac{DC}{t_2 - t_1}$$

By using the relationship between discount factors and zero interest rates given by (2.8), it is clear that simple forward interest rates can easily be calculated from discount factors by rewriting the relationship as given by (2.10) as follows:

$$\frac{1}{df_{t1}} \left[1 + f_{1,2} \frac{t_2 - t_1}{DC} \right] = \frac{1}{df_{t2}}$$

where df_t denotes the discount factor for a period of t years. By solving for the forward rate we get the following relationship:

$$f_{1,2} = \left[\frac{df_{t1}}{df_{t2}} - 1 \right] \times \frac{DC}{t_2 - t_1} \tag{2.11}$$

The difference between zero rates and forward rates can be briefly stated as follows: Zero rates describe interest rates over periods from the current date to a given future date, while forward rates describe the interest rates applicable between two dates in the future.

Two other concepts which are important are the instantaneous forward interest rate and the mean forward interest rate. The instantaneous forward interest rate $\rho(m)$ is given by the following equation:

$$\rho(m) = -\frac{d}{dm} \ln\delta(m) = -\frac{\delta'(m)}{\delta(m)} \tag{2.12}$$

where $\delta(m)$ is the continuous discount function defined in Section 2.2. The instantaneous forward rate can be used to derive the mean forward interest rate $f_{1,2}$ over the interval $[t_1,t_2]$ as follows:

$$f_{1,2} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \rho(m) dm = \frac{1}{t_2 - t_1} \ln\frac{\delta(t_1)}{\delta(t_2)}$$

which corresponds with (2.11), the formulae derived for discrete time.

Under certain assumptions forward rates can be interpreted as indicating market expectations of future short-term interest rates (Svensson, 1994).

2.4 YIELD TO MATURITY

The yield to maturity is the internal rate of return of a bond if it is held until maturity. This means the yield to maturity is the constant interest rate that makes the present value of all the future coupon payments and the redemption payment equal to the price of the bond (Fabozzi, 1993). It can be explained as a complex average of spot rates.

In general the price of a regular fixed coupon-paying bond can be written as:

$$P = \sum_{j=1}^{n} \frac{C_j}{(1 + y/2)^{2 \times m_j}} \tag{2.13}$$

where

P = the all-in price of the bond

n = number of outstanding cash flows of the bond

y = yield-to-maturity of the bond

m_j = the term in years from the settlement date until cash flow C_j

C_j = jth cash flow of the bond (the last cash flow will be the redemption payment plus the coupon payment).

In some markets, bonds are quoted in terms of their prices. So to determine the yield to maturity we need to solve iteratively for the yield to maturity y in (2.13). Conversely there are other markets where bonds are quoted and traded in terms of their yields to maturity, and (2.13) can then be used to calculate the price of the bond.

2.5 PAR RATES

A *par rate* is defined as the average rate at which a bond should be priced if the bond is to be issued at par. In other words, given the coupon rate and the par rate, the bond has a value of 100 per 100 nominal (Cairns, 2004). The same concept is used to value the fixed leg of an interest rate swap, therefore the terms "par" and "swap" rates refer to the same type of rate. The average rate is also known as the *internal rate of return*.

Let us first consider the concept of an internal rate of return. Say we have an instrument that pays three cash flows CF_1, CF_2, and CF_3, one cash flow every year. The present value of this instrument is:

$$PV = CF_1 \times df_1 + CF_2 \times df_2 + CF_3 \times df_3 = \frac{CF_1}{(1+i_1)} + \frac{CF_2}{(1+i_2)^2} + \frac{CF_3}{(1+i_3)^3} \quad (2.14)$$

where df_1, df_2 and df_3 denote the one-year, two-year, and three-year discount factors respectively, and i_1, i_2 and i_3 denote the zero rates applicable to a one-year, two-year, and three-year period respectively. To determine an internal rate of return, we need to determine an average rate r_3 over the three-year period that can be used to discount all three cashflows so that we will get to the same present value PV as in (2.14). This means that we have to solve for the average rate r_3 in the following:

$$PV = \frac{CF_1}{(1+r_3)} + \frac{CF_2}{(1+r_3)^2} + \frac{CF_3}{(1+r_3)^3} \quad (2.15)$$

so that r_3 represents the internal rate of return for the three-year period. Alternatively we can write (2.15) as follows:

$$0 = -CF_0 + \frac{CF_1}{(1+r_3)} + \frac{CF_2}{(1+r_3)^2} + \frac{CF_3}{(1+r_3)^3} \quad (2.16)$$

In other words in this transaction the investor invests an initial amount of CF_0 (equal to the PV in equation (2.15)) and receives three cash flows CF_1, CF_2, and CF_3.

We have seen that the internal rate of return is the average rate the investment earns over the whole investment period. In a similar way we can define an average rate that is paid by an investment. Consider the case where the three yearly cash flows are determined by the three interest rates f_1, f_2, and f_3, where the cash flow at year 1 is determined from f_1, the cash flow at year 2 is determined from f_2, and the cash flow at year 3 is determined from f_3. The present value of the transaction can be written as:

$$PV = (f_1 \times df_1) + (f_2 \times df_2) + (f_3 \times df_3) \quad (2.17)$$

We are now interested in calculating the fixed average rate that can be paid each year such that we get the same present value as in (2.17). Mathematically this is written as:

$$PV = (r_3 \times df_1) + (r_3 \times df_2) + (r_3 \times df_3) \quad (2.18)$$

where r_3 denotes the average three-year rate for which we need to solve iteratively. Similarly to (2.16), we can now rewrite (2.18) as follows:

$$0 = -CF_0 + (r_3 \times df_1) + (r_3 \times df_2) + (r_3 \times df_3)$$

where $CF_0 = PV$. This can be interpreted as the investor that invests an initial amount of CF_0 and receives three cash flows calculated from the average rate r_3. The same concept is used when we value the fixed leg of an interest rate swap. A one-year swap rate r_1 is the average rate calculated from the following:

$$0 = -1 + (1 + r_1) \times df_1 \tag{2.19}$$

where df_1 denotes the one-year discount factor. This means by investing a nominal amount of 1, the investor is paid back the nominal plus interest of r_1 after one year. Similarly a two-year swap rate r_2 is the average rate calculated from:

$$0 = -1 + r_2 \times df_1 + (1 + r_2) \times df_2 \tag{2.20}$$

where df_2 denotes the two-year discount factor. By investing the nominal amount of 1, the investor receives interest of r_2 every year, and on the expiry date the investor is paid back the nominal as well. In a similar way we can set up the equations for each of the yearly swap rates up to r_n which denotes the n-year swap rate. We will now set up these equations and show how they can be used to iteratively convert zero rates to par rates.

Consider the situation where we have a zero yield curve available that is indicative of current market interest rates. Say a dealer would like to use the zero curve to determine appropriate quotes for interest rate swaps. The yearly zero rates with maturities up to n years are interpolated off the zero yield curve and are denoted by z_i, $i = 1,...,n$. Please refer to Chapter 4 for a discussion on interpolation techniques. The zero rates are converted to yearly discount factors by using (2.8) which are then denoted by df_i; the yearly swap rates which we need to calculate are denoted by r_i.

By using the same argument that is used to derive (2.19) and (2.20), we find the following set of equations:

$$0 = -1 + (1 + r_1) \times df_1$$
$$\Rightarrow \quad 1 = (1 + r_1)\, df_1$$

so that we can solve for the one-year swap rate as follows:

$$r_1 = \frac{1 - df_1}{df_1}$$

The rest of the swap rates are extracted iteratively with similar arguments, as follows:

$$1 = (r_2)df_1 + (1 + r_2)df_2 \quad \therefore \; r_2 = \frac{1 - df_2}{df_1 + df_2}$$

$$1 = (r_3)df_1 + (r_3)df_2 + (1 + r_3)df_3 \; \therefore \; r_3 = \frac{1 - df_3}{df_1 + df_2 + df_3}$$

and so on, so that in general the formula to extract the swap rates from the zero rates, is:

$$\therefore \; r_k = \frac{1 - df_k}{\sum_{j=1}^{k} df_j} \tag{2.21}$$

where $k = 1,...,n$. However, in practice we know that the swap rates will not apply to a fixed number of years due to business-day rules and daycount conventions (please refer to Chapter 4 for a detailed discussion). In other words, the one-year swap rate may apply to a period of 1.01 years, the two-year swap rate may apply to a period of 2.03 years, and so on. We have to allow for this when we derive swap rates to ensure that more accurate results are obtained.

Consider the same dealer that has the zero rate curve available. This dealer knows that when a swap is quoted in the market, the one-year swap rate has a term-to-maturity of t_1; the two-year swap rate has a term-to-maturity date of t_2, and so on. In order to take the swap maturity dates into account, we first have to interpolate the zero curve to get a zero rate at each of these swap maturity dates. Each of these interpolated zero rates is then converted to a discount factor that is used in the following recursive formulae (using similar arguments as before) to derive the simple swap rates:

$$1 = \; (1 + r_1[t_1 - t_0])df_1 \quad \therefore \; r_1 = \frac{1 - df_1}{[t_1 - t_0]df_1}$$

$$1 = \; (r_2[t_1 - t_0])df_1 + (1 + r_2[t_2 - t_1])df_2 \quad \therefore \; r_2 = \frac{1 - df_2}{[t_1 - t_0]df_1 + [t_2 - t_1]df_2}$$

and so on, so that in general the formula to extract the simple swap rates from zero rates, is:

$$\therefore \; r_k = \frac{1 - df_k}{\sum_{j=1}^{k} [t_j - t_{j-1}]df_j} \tag{2.22}$$

where t_k is the term in years from the curve date to the swap maturity date of contract k. We take t_0 to be the value date.

The calculations to derive a par curve from a zero curve are illustrated with a simple example in Section 2.5.1.

2.5.1 Example: Deriving naca par rates from yearly zero rates

In this example we assume that we have a zero naca curve, with rates at yearly tenors. From this curve we would like to derive par rates at the fixed yearly tenors using (2.21). Table 2.1 shows the assumed zero rates. The discount factors are derived from the zero rates by using (2.8). From these discount factors the par rates are derived as follows:

$$r_1 = \frac{1 - df_1}{df_1} = \frac{1 - 0.976}{0.976} = 2.45\%$$

$$r_2 = \frac{1 - df_2}{df_1 + df_2} = \frac{1 - 0.951}{0.976 + 0.951} = 2.56\%$$

and so on to get swap rates out to 15 years.

Figure 2.3 shows a comparison of the derived par rates with the zero rates. It is interesting to see that the differences between the two curves are minimal at the short end and then increase at the longer end. This par curve can now be used by dealers to set prices in the interest rate swap market.

Table 2.1 Deriving naca par rates from yearly naca zero rates by assuming fixed periods

Tenor	k	Zero rate $i_{(1)}$	df_k	$\sum_{j=1}^{k} df_j$	Par rate r_k
0 year	0	2.40	1.000		2.40
1 year	1	2.45	0.976	0.976	2.45
2 year	2	2.56	0.951	1.927	2.56
3 year	3	2.70	0.923	2.850	2.70
4 year	4	2.85	0.894	3.744	2.84
5 year	5	2.95	0.865	4.608	2.94
6 year	6	3.16	0.830	5.438	3.13
7 year	7	3.60	0.781	6.219	3.53
8 year	8	4.10	0.725	6.944	3.96
9 year	9	4.57	0.669	7.613	4.35
10 year	10	5.00	0.614	8.227	4.69
11 year	11	5.23	0.571	8.797	4.88
12 year	12	5.12	0.549	9.347	4.82
13 year	13	4.87	0.539	9.886	4.66
14 year	14	4.60	0.533	10.418	4.48
15 year	15	4.50	0.517	10.935	4.42

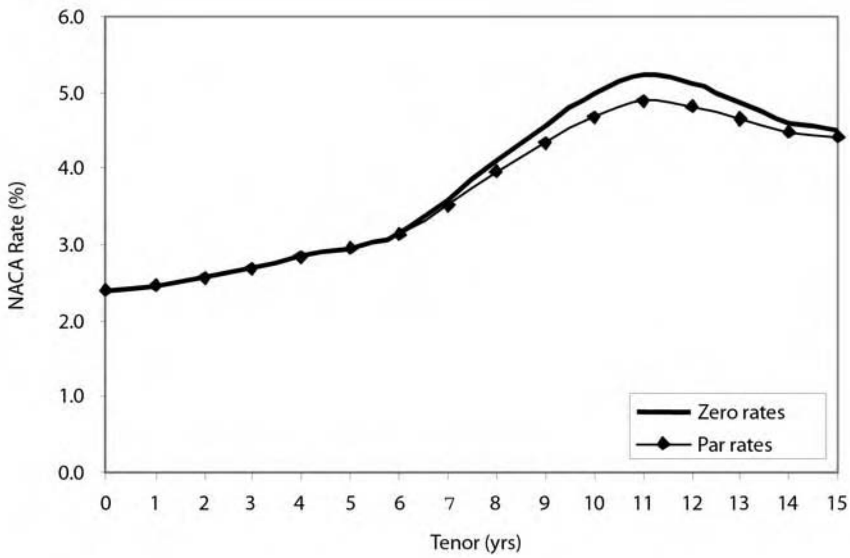


Figure 2.3 Comparison of zero and par rate curves

2.6 THE "BOOTSTRAPPING" TECHNIQUE

The technique to derive zero rates from swap rates is called *bootstrapping*. Using the same arguments as in Section 2.5, we set up the equations with which to value the fixed leg of an interest rate swap. The only difference is that we now solve for the discount factors and not the swap rates.

2.6.1 Zero rates from naca swap rates

Say naca swap rates with maturities up to n years are available and denoted by r_i, $i = 1,\dots,n$. We assume the swap rates apply to exact yearly periods, so the discount factors are derived as follows:

$$1 = (1 + r_1)df_1 \quad \therefore\ df_1 = \frac{1}{(1 + r_1)}$$

$$1 = (r_2)df_1 + (1 + r_2)df_2 \quad \therefore\ df_2 = \frac{1 - r_2 df_1}{(1 + r_2)}$$

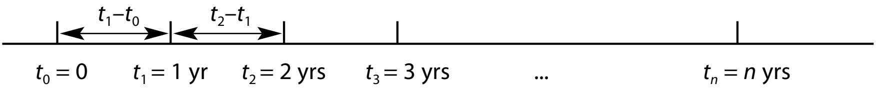
$$1 = (r_3)df_1 + (r_3)df_2 + (1 + r_3)df_3 \quad \therefore\ df_3 = \frac{1 - r_3(df_1 + df_2)}{(1 + r_3)}$$

and so on, so that in general the formula to extract the discount factors is:

$$df_k = \frac{1 - r_k \sum_{j=1}^{k-1} df_j}{1 + r_k} \tag{2.23}$$

where $k = 1,...,n$ and df_k is the k-year discount factor. The zero rates are then calculated from these discount factors using the relationship given by (2.8).

As was mentioned in Section 2.5, we have the situation in practice that the periods to which different swap rates apply are usually different due to market conventions relating to business-day rules and daycount conventions. It is necessary to allow for the actual periods to which the swap rates apply in order to derive the zero rates more accurately. The simplest way to take this into account when deriving the bootstrap formulae, is to assume that yearly naca swap rates with maturities up to n years are available in the market.



Let t_k, $k = 1,...,n$, denote the actual term (in years) from the value date to the maturity date, corresponding with the swap rate r_k. We take t_0 to be the value date. The discount factors are then calculated using these formulae:

$$1 = (1 + r_1)^{t_1 - t_0} df_1 \therefore df_1 = \frac{1}{(1 + r_1)^{t_1 - t_0}}$$

$$1 = [(1 + r_2)^{t_1 - t_0} - 1]df_1 + (1 + r_2)^{t_2 - t_1} df_2 \therefore df_2 = \frac{1 - [(1 + r_2)^{t_1 - t_0} - 1]df_1}{(1 + r_2)^{t_2 - t_1}}$$

so that in general:

$$df_k = \frac{1 - \sum_{j=1}^{k-1}[(1 + r_k)^{t_j - t_{j-1}} - 1]df_j}{(1 + r_k)^{t_k - t_{k-1}}} \tag{2.24}$$

where $k = 1,...,n$ and df_k is the k-year discount factor.

2.6.2 Zero rates from nacq swap rates

Say yearly swap contracts with quarterly resets are quoted in the market. These swaps rates are nacq. To derive zero rates from these swap rates, there are two possible approaches. The first approach is to convert all the swap rates to naca rates and then just use (2.24) to derive the discount factors. The second approach involves interpolating the nacq swap rates to get an interpolated swap rate value at each quarter. These swap rates are then bootstrapped using (2.24) as before, except that the bootstrap formula now has to take into account the fact that the rates are compounded quarterly.

To derive the formulae for the second approach, let t_k denote the actual term (in years) from the value date to the maturity date corresponding with each of the quarterly swap rates r_k. The quarterly discount factors are then calculated using these formulae:

$$1 = \left(1 + \frac{r_1}{4}\right)^{4(t_1-t_0)} df_1 \therefore df_1 = \frac{1}{\left(1 + \frac{r_1}{4}\right)^{4(t_1-t_0)}}$$

$$1 = \left[\left(1 + \frac{r_2}{4}\right)^{4(t_1-t_0)} - 1\right] df_1 + \left(1 + \frac{r_2}{4}\right)^{4(t_2-t_1)} df_2 \therefore df_2 = \frac{1 - \left[\left(1 + \frac{r_2}{4}\right)^{4(t_1-t_0)} - 1\right] df_1}{\left(1 + \frac{r_2}{4}\right)^{4(t_2-t_1)}}$$

and so on, so that in general:

$$df_k = \frac{1 - \sum_{j=1}^{k-1}\left[\left(1 + \frac{r_k}{4}\right)^{4(t_j-t_{j-1})} - 1\right] df_j}{\left(1 + \frac{r_k}{4}\right)^{4(t_k-t_{k-1})}} \tag{2.25}$$

where k denotes each of the quarters and df_k is the discount factor for the contract with maturity t_k.

2.6.3 Zero rates from simple swap rates

It is straightforward to extend the bootstrap formulae to allow for simple swap rates. The discount factors are calculated by:

$$df_k = \frac{1 - r_k \sum_{j=1}^{k-1}(t_j - t_{j-1})\, df_j}{(1 + r_k(t_j - t_{j-1}))} \tag{2.26}$$

where r_k are the simple swap rates, all other symbols as defined before.

2.6.4 Generalized bootstrap formulae

It is clear from 2.6.1 and 2.6.2 that in each of the instances the same type of approach is followed. When naca swap rates are quoted, discount factors are calculated for yearly periods. When nacs swap rates are quoted, discount factors for semi-annual periods are calculated. In general, when swap rates are quoted that are compounded m times a year, we can calculate m-period discount factors.

When swap rates are quoted in the market, it is usually for yearly contracts, in other words we have contracts for one year, two years, three years and so on. However, these swaps will have reset frequencies that usually correspond with the compounding frequency of the quoted swap rate. Say we have a two-year swap contract with quarterly resets, then usually the swap rate will also be quarterly compounded. To use the methodology described in this section, we have to interpolate to get swap rates for each of the interim contract maturities, in other words for the 1.25 year, 1.5 year, 1.75 year, and so on, contracts. In general, the bootstrap formula for swap rates compounded m times a year, is:

$$df_k = \frac{1 - \sum_{j=1}^{k-1}\left[\left(1 + \frac{r_k}{m}\right)^{m(t_j - t_{j-1})} - 1\right] df_j}{\left(1 + \frac{r_k}{m}\right)^{m(t_k - t_{k-1})}} \tag{2.27}$$

where k denotes each of the m-periods; df_k is the k-period discount factor; t_k is the time in years until the k^{th} reset and r_k is the swap rate for each of the m periods.

An application of the bootstrap formulae is shown in the example in Section 2.6.5.

2.6.5 Example: Bootstrapping nacs swap rates quoted yearly

Table 2.2 shows the mid swap rates quoted for yearly contracts. These contracts have semi-annual resets and the swap rates are nacs. We are interested in using these swap rates to derive a zero curve that will be representative of the current interest rates in the market. We assume the curve date is 24 June 2004 and that an actual/360 daycount convention is applicable. The specifics of this daycount convention can be found in Chapter 4.

Following the discussion in this section, there are three ways in which we can bootstrap these swap rates to get to zero rates.

Table 2.2 Semi-annually compounded swap rates as quoted for yearly contracts

Tenor	Maturity date	Mid rate
6 month	28 Dec 2004	1.600
1 year	24 Jun 2005	2.100
2 year	26 Jun 2006	3.000
3 year	25 Jun 2007	3.600
4 year	24 Jun 2008	4.000
5 year	24 Jun 2009	4.300
6 year	24 Jun 2010	4.600
7 year	24 Jun 2011	4.800
8 year	25 Jun 2012	5.000
9 year	24 Jun 2013	5.100
10 year	24 Jun 2014	5.200

Approach 1

The quotes are for yearly contracts, so we convert the swap rates to naca rates and then apply the bootstrap formulae that assume exactly yearly periods. We derive the discount factors with (2.23). In this case r_k denotes the k-year swap rate compounded annually. We have that:

$$df_1 = \frac{1}{(1+r_1)} = \frac{1}{(1+0.02111)} = 0.979$$

$$df_2 = \frac{1-r_2 df_1}{(1+r_2)} = \frac{1\text{–}0.03022(0.979)}{(1+0.03022)} = 0.942$$

$$df_3 = \frac{1-r_3(df_1+df_2)}{(1+r_3)} = \frac{1\text{–}0.03632(0.979+0.942)}{(1+0.03632)} = 0.898$$

and so on to get discount factors out to 10 years. The calculations are summarized in Table 2.3.

The problem with this approach is that we have an actual/360 daycount convention. This means that we should take the actual number of days from the curve date to the swap maturity date into account. For instance, the one-year swap will have an actual term to maturity of approximately 365/360 ≈ 1.014 years whereas with this approach we assume it applies to 360/360 = 1 year.

Table 2.3 Bootstrapping by assuming the swap rates apply to exactly yearly periods

	k	r_k	$\sum_{j=1}^{k-1} df_j$	df_k
1 year	1	2.111		0.979
2 year	2	3.022	0.979	0.942
3year	3	3.632	1.921	0.898
4 year	4	4.040	2.819	0.852
5 year	5	4.346	3.671	0.805
6 year	6	4.653	4.476	0.757
7 year	7	4.858	5.233	0.711
8 year	8	5.062	5.944	0.665
9 year	9	5.165	6.609	0.626
10 year	10	5.268	7.236	0.588

Approach 2

We convert the swap rates to be compounded annually and take the actual contract maturity dates into account by using (2.27) to derive the discount factors. In this application r_k denotes the k-year swap rate compounded annually (m = 1). We have that:

$$df_1 = \frac{1}{(1+r_1)^{(t_1-t_0)}} = \frac{1}{(1+0.02111)^{1.0}} = 0.979$$

$$df_2 = \frac{1-[(1+r_2)^{(t_1-t_0)}-1]df_1}{(1+r_2)^{(t_2-t_1)}} = \frac{1-[(1+0.03022)^{1.0}-1](0.979)}{(1+0.03022)^{1.02}} = 0.941$$

$$df_3 = \frac{1-[(1+r_3)^{(t_1-t_0)}-1]df_1-[(1+r_3)^{(t_2-t_1)}-1]df_2}{(1+r_3)^{(t_3-t_2)}}$$

$$= \frac{1-[(1+0.03632)^{1.0}-1](0.979)-[(1+0.03632)^{1.02}-1](0.941)}{(1+0.03632)^{1.01}} = 0.896$$

and so on, the results summarized in Table 2.4.

Table 2.4 Bootstrapping by taking the swap maturity dates into account

Tenor	k	$t_k - t_{k-1}$	r_k	$\sum_{j=1}^{k-1}[(1+r_k)^{(t_j-t_{j-1})}-1]df_j$	df_k
1 year	1	1.0	2.111		0.979
2 year	2	1.02	3.022	0.030	0.941
3 year	3	1.01	3.632	0.071	0.896
4 year	4	1.01	4.040	0.115	0.850
5 year	5	1.01	4.346	0.162	0.803
6 year	6	1.01	4.653	0.211	0.753
7 year	7	1.01	4.858	0.257	0.708
8 year	8	1.02	5.062	0.305	0.661
9 year	9	1.01	5.165	0.346	0.622
10 year	10	1.01	5.268	0.386	0.583

Approach 3

We use linear interpolation to get a swap rate for each semi-annual period out to ten years. We then bootstrap the semi-annual rates by taking the actual contract maturity into account by using (2.27). In this application r_k denotes the semi-annual swap rate (m = 2). Deriving the discount factors recursively, we have that:

$$df_1 = \frac{1}{\left(1+\frac{r_1}{2}\right)^{2(t_1-t_0)}} = \frac{1}{\left(1+\frac{0.016}{2}\right)^{2(0.52)}} = 0.992$$

$$df_2 = \frac{1-\left[\left(1+\frac{r_2}{2}\right)^{2(t_1-t_0)}-1\right]df_1}{\left(1+\frac{r_2}{2}\right)^{2(t_2-t_1)}} = \frac{1-\left[\left(1+\frac{0.021}{2}\right)^{2(0.52)}-1\right]0.992}{\left(1+\frac{0.021}{2}\right)^{2(0.49)}} = 0.979$$

and so on, with the results summarized in Table 2.5.

Table 2.5 Bootstrapping nacs swap rates and taking the actual swap maturity dates into account

Tenor	Maturity date	k	$t_k - t_{k-1}$	r_k	$\sum_{j=1}^{k-1}\left[\left(1+\frac{r_k}{2}\right)^{(t_j-t_{j-1})}-1\right]df_j$	df_k
0.5 year	28 Dec 2004	1	0.52	1.600		0.992
1 year	24 Jun 2005	2	0.49	2.100	0.011	0.979
1.5 year	24 Dec 2005	3	0.51	2.549	0.025	0.962
2 year	26 Jun 2006	4	0.51	3.000	0.045	0.941
2.5 year	26 Dec 2006	5	0.51	3.302	0.065	0.920
3 year	25 Jun 2007	6	0.50	3.600	0.088	0.896
3.5 year	25 Dec 2007	7	0.51	3.801	0.110	0.873
4 year	24 Jun 2008	8	0.51	4.000	0.133	0.850
4.5 year	24 Dec 2008	9	0.51	4.150	0.156	0.826
5 year	24 Jun 2009	10	0.51	4.300	0.180	0.803
5.5 year	24 Dec 2009	11	0.51	4.450	0.204	0.778
6 year	24 Jun 2010	12	0.51	4.600	0.229	0.753
6.5 year	24 Dec 2010	13	0.51	4.700	0.252	0.730
7 year	24 Jun 2011	14	0.51	4.800	0.275	0.708
7.5 year	24 Dec 2011	15	0.51	4.900	0.299	0.684
8 year	25 Jun 2012	16	0.51	5.000	0.322	0.661
8.5 year	25 Dec 2012	17	0.51	5.050	0.342	0.641
9 year	24 Jun 2013	18	0.50	5.100	0.362	0.622
9.5 year	24 Dec 2013	19	0.51	5.150	0.382	0.602
10 year	24 Jun 2014	20	0.51	5.200	0.402	0.583

With each of the three approaches, we get discount factors out to 10 years that are converted to zero rates. Table 2.6 shows the naca zero rates. It is interesting to see what big differences there are between the zero rates from Approaches 1 and 2. It is clearly necessary to take the actual swap maturity dates into account, because the differences between the two sets of zero rates are as big as 8 basis points (bps) at the longer maturities. There are no big differences between the zero rates derived using Approaches 2 and 3. The biggest difference is around 0.4 bps and may be due to the fact that linear interpolation was used that does not allow for curvature.

Table 2.6 Zero rates calculated from the discount factors obtained from each of the bootstrapping approaches. These rates are compounded annually.

Tenor	Approach 1	Approach 2	Approach 3
1 year	2.08	2.11	2.11
2 year	2.99	3.04	3.04
3 year	3.61	3.67	3.67
4 year	4.03	4.10	4.10
5 year	4.36	4.42	4.43
6 year	4.69	4.76	4.77
7 year	4.92	4.99	4.99
8 year	5.15	5.23	5.23
9 year	5.26	5.34	5.34
10 year	5.38	5.46	5.46

This example illustrates the differences in zero curves when actual swap maturity dates are taken into account, compared with the case where fixed periods are assumed. In practice these differences may lead to big valuation errors. Say the zero curve is derived from the swaps by incorrectly assuming fixed periods as in Approach 1. Now consider a scenario where a dealer enters into an interest rate swap. By definition the value of this deal should be approximately zero on the trade date; however when this deal is valued with the zero curve, the deal will show a value. When this value is not favorable to the trader, there will be complaints.

The effect of assuming fixed periods are usually not very significant when actual/365 daycount conventions are followed. However, it is significant when the market follows an actual/360 daycount convention.

2.7 IMPLIED INTEREST RATES

Some countries may have undesirable political and economic issues that oblige them to provide incentives for investors to invest money. This may be done by paying investors a spread above the interest rates. This spread is known as the *country risk premium.* The country risk premium is discussed in detail in Chapter 7. In this section we investigate a way in which investors can calculate an implied rate from domestic interest rates and exchange rates to determine what interest rate they are actually earning on an investment.

A nominal amount N is invested in the domestic currency for d days at the domestic interest rate of i_d. The future value of this investment is:

$$N\left[1 + i_d \frac{d}{DC_d}\right]$$

where DC_d is the number of days assumed in a year according to the domestic daycount convention.

Say the same nominal amount N is converted at the spot exchange rate X_s to a foreign currency amount N/X_s. When this amount is invested in a foreign market for the same investment period of d days, at an interest rate of i_f, then the future value of the investment (expressed in the units of the foreign currency) will be:

$$\frac{N}{X_s}\left[1 + i_f \frac{d}{DC_f}\right]$$

where DC_f is the number of days assumed in a year according to the foreign daycount convention. After d days the investment is then converted back to the domestic currency at the exchange rate at that time, which is expected to be X_f, the forward exchange rate.

In order to prevent arbitrage between the two markets, we must have that:

$$N\left[1 + i_d \frac{d}{DC_d}\right] = \frac{N}{X_s}\left[1 + i_f \frac{d}{DC_f}\right] \times X_f \tag{2.28}$$

where

N = nominal amount (in units of the domestic currency)
X_s = spot exchange rate
X_f = forward exchange rate
i_f = foreign interest rate (this is a simple rate)
d = number of days in the investment period
i_d = domestic interest rate (this is a simple rate)
DC_d = number of days in the year as specified by the daycount convention of the domestic market
DC_f = number of days in the year as specified by the daycount convention of the foreign market.

Using this relationship, it is trivial to solve for the required implied rate. Typically the domestic interest rate i_d and exchange rates X_s and X_f are known because they are traded in the market, then we just solve for i_f as follows:

$$i_f = \left(\frac{X_s}{X_f}\left[1 + i_d \frac{d}{DC_d}\right] - 1\right) \times \frac{DC_f}{d}$$

assuming that the interest rates are simple rates. When the investment period is longer than one year, we typically would not be working with simple rates. Using the same argument as before, we get to the following relationship:

$$N\left[1+\frac{i_d}{m}\right]^{m\frac{d}{DC_d}} = \frac{N}{X_s}\left[1+\frac{i_f}{n}\right]^{n\frac{d}{DC_f}} \times X_f \tag{2.29}$$

assuming that the domestic rate is compounded m times per year and the foreign rate n times per year. Solving for i_f we get:

$$i_f = \left[\left(\frac{X_s}{X_f}\left[1+\frac{i_d}{m}\right]^{m\frac{d}{DC_d}}\right)^{\frac{DC_f}{nd}} - 1\right] \times n$$

all the symbols as defined before. With similar arguments it is possible to derive an implied domestic rate or implied forward exchange rates.

These formulae provide a way with which to recognize arbitrage opportunities between the foreign exchange market and the interest rate market. However, the practitioner may face a problem where there may no forward exchange rates quoted in the market that are liquid enough for the longer investment periods. These formulae cannot be used to imply rates in such circumstances. Another option is to use basis swaps.

The example in Section 2.7.1 illustrates the calculations of implied interest rates.

2.7.1 Example: Calculating an implied domestic zero curve

When the foreign interest rates as well as the exchange rates are known, we can infer a domestic zero rate from the relationship given by (2.29) which is as follows:

$$i_d = m \times \left(\left[\frac{X_f}{X_s}\left[1+\frac{i_f}{n}\right]^{\frac{nd}{DC_f}}\right]^{\frac{DC_d}{md}} - 1\right) \tag{2.30}$$

Table 2.7 shows the foreign zero rates (assumed to be naca) at various dates as well as the spot and forward foreign exchange rates. We assume the foreign rates follows an actual/360 daycount convention, whereas the domestic rates follow an actual/365 daycount convention. From these rates we then infer annually compounded domestic rates from (2.30) as follows:

$$i_{30} = 1 \times\left(\left[\frac{6.33}{6.3}\left[1+\frac{2.37}{100}\right]^{\frac{30}{360}}\right]^{\frac{365}{30}} - 1\right) = 8.5\%$$

$$i_{61} = 1 \times\left(\left[\frac{6.37}{6.3}\left[1+\frac{2.01}{100}\right]^{\frac{61}{360}}\right]^{\frac{365}{61}} - 1\right) = 9.01\%$$

and so on until we have a rate at each tenor.

Table 2.7 Deriving an implied domestic curve

	End date	d	Foreign rate i_f	X_f	Implied domestic i_d
Spot date:	05 Dec 2004	30	2.37	6.33	8.50
05 Nov 2004	05 Jan 2005	61	2.01	6.37	9.01
FX spot: (X_s)	05 Feb 2005	92	2.00	6.40	8.61
6.30	05 Mar 2005	120	2.30	6.41	7.86
	05 Apr 2005	151	2.60	6.42	7.43
	05 May 2005	181	2.88	6.43	7.25
	05 Jun 2005	212	3.00	6.44	7.02
	05 July 2005	242	3.10	6.45	6.87
	05 Aug 2005	273	3.20	6.46	6.77
	05 Sep 2005	304	3.21	6.47	6.61
	05 Oct 2005	334	3.23	6.48	6.50
	05 Nov 2005	365	3.25	6.50	6.58

Figure 2.4 shows a comparison of the foreign and implied domestic curves (out to one year). It should be noted that the implied curve is very sensitive to the foreign exchange quotes and may not always appear this smooth.

Another important fact to note is that it is implicitly assumed that we are working with spot curves. To get a curve for the value date, both these curves need to be discounted from the spot date to the curve date. This, however, is a concept that will be discussed in Chapter 5.

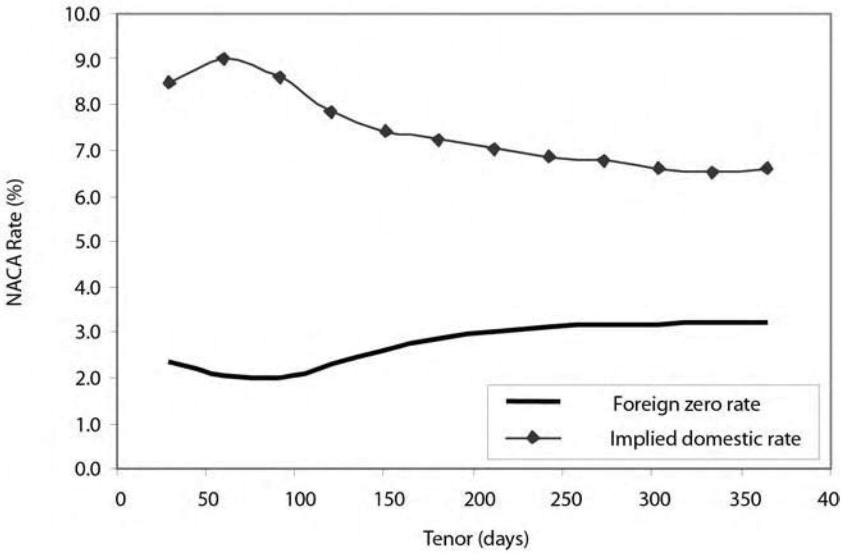


Figure 2.4 Comparison of foreign and implied domestic zero curves

2.8 CONCLUDING REMARKS

In this chapter we considered the various types of interest rates and showed how they are related to each other, using simple arbitrage arguments. It was in various cases necessary to break the instruments up into their underlying cash flows in order to, for instance, derive the relationship between a zero rate and a par rate. It is important that the way rates are compounded as well as the periods to which the rates apply are taken into account, because it has a big effect on the resulting yield curve and can have a big effect on the portfolio value when this yield curve is used to value the instruments in that portfolio.

Incorrect yield curves will lead to incorrect valuations which in turn lead to unexplained profits and losses. Our goal is to always value the portfolios as accurately as possible to prevent this. Unexplained profits and losses indicate that the instruments are not properly understood, which in turn means that they are not properly hedged. This uncertainty increases the risk of the instruments, which means that more capital has to be reserved to be able to handle big losses that can occur in times of stress. A large capital reserve inhibits the dealer from taking on more positions, which reduces the dealer’s ability to make big profits.

The aforementioned argument illustrates the importance of getting the basics right, like deriving the yield curve correctly and valuing the instruments accurately, because this has an effect on everything in the financial institution.

CHAPTER 3

Yield Curve Models

This chapter provides a basic overview of the three types of yield curve models known as regression-type models, empirical models, and equilibrium models. In Section 3.1 various regression-type models are discussed. Regression modeling is the most traditional method of fitting a yield curve, where a function is simply fitted to the yields to maturity of regular coupon-paying bonds. The fitted curve is considered to be a *par yield curve*, which can be converted to a *zero curve* by using the methods described in Chapter 2. One of the main problems with the regression-type models is that the ”coupon effect” is not taken into account. The coupon effect refers to the fact that different bonds with the same term to maturity may have very different yields to maturity because of differences in their coupon rates.

Empirical yield curve models, which were developed to overcome some of the shortcomings of the regression-type yield curve models, are discussed in Section 3.2. Empirical yield curve models usually specify a functional form for the discount function. Various ways in which to fit empirical yield curve models bonds are discussed in Chapter 5.

In Section 3.3 equilibrium models are discussed. The models that are discussed have closed-form solutions and can be fitted to bond prices in a similar way to empirical yield curve models. However, these models are usually not flexible enough in practice to provide an adequate fit of the underlying instruments used to derive the yield curve.

3.1 REGRESSION-TYPE MODELS

The traditional way of deriving a yield curve is to consider a scatter plot of the yields to maturity against the terms to maturity of a series of bonds. The yield curve is then the function that best fits all the points on the graph.

Regression techniques are used to estimate the parameters of the models. Typically we would estimate the function parameters by minimizing the squared differences between the actual and fitted yields to maturity, in other words:

$$\min \sum_{j=1}^{n} (r_i - \hat{r}_i)^2 \tag{3.1}$$

where

r_i = actual yield to maturity of bond i (derived from the market price of the bond)
$\hat{r}_i$ = fitted yield to maturity of bond i as determined by the specified function
n = number of bonds used to derive the par yield curve.

We shall take a brief look at some of the regression models that are discussed in the literature.

3.1.1 Bradley–Crane model

The Bradley–Crane model is formulated as follows (McEnally, 1987):

$$\ln(1 + r_i) = \beta_0 + \beta_1 t_i + \beta_2 \ln(t_i) \tag{3.2}$$

where

r_i = yield to maturity of bond i
t_i = term to maturity of bond i (in years)
β = regression parameters to be estimated.

This curve is not defined when t_i is 0.

The Bradley–Crane model is a very simplistic model and is not able to allow for the different types of shapes a yield curve can take on. An example of a poor fit between bond yields and the Bradley–Crane model is shown in Figure 3.1, where the Bradley–Crane model is fitted to the bond yields given in Table 3.1. It is clear that the actual yield curve is shaped like a spoon, in other words it is curved, and the fitted curve does not allow for the curvature.

3.1.2 Elliot–Echols model

The Elliot–Echols model is a three-dimensional yield curve and has the following form (McEnally, 1987):

$$\ln(1 + r_i) = \beta_0 + \beta_1 \frac{1}{t_i} + \beta_2 t_i + \beta_3 C_i \tag{3.3}$$

Table 3.1 Bond yields that will be used to derive a regression-type yield curve

Bond code	Term to maturity	Yield to maturity
BOND1	1.8	2.250
BOND2	2.9	2.500
BOND3	4.2	3.098
BOND4	4.5	3.300
BOND5	4.7	3.320
BOND6	6.0	3.900
BOND7	7.0	4.150
BOND8	7.5	4.210
BOND9	8.0	4.260
BOND10	8.5	4.420
BOND11	9.0	4.470
BOND12	9.2	4.397
BOND13	16.5	4.250
BOND14	21.7	4.100

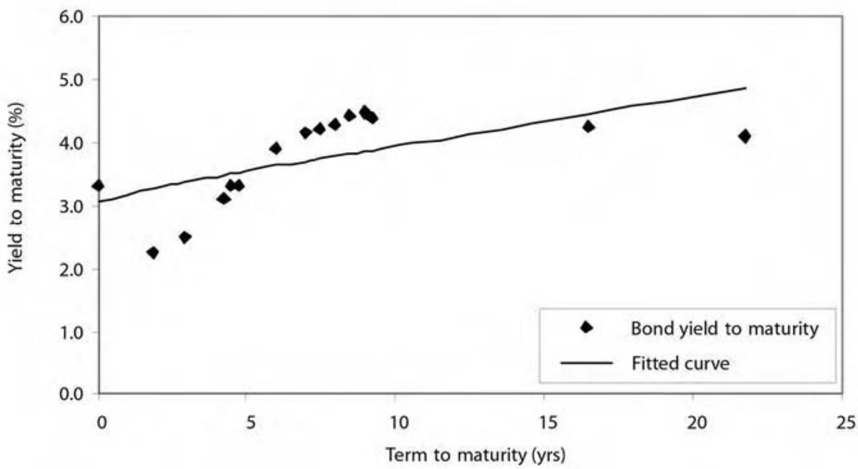


Figure 3.1 An example of the Bradley–Crane curve fitted to the bond yields in Table 3.1

where

r_i = yield to maturity of bond i

t_i = term to maturity of bond i (in years)

C_i = the coupon rate of bond i

β = regression parameters to be estimated.

This model attempts to take the effect coupons have on the yields to maturity into account. However, market participants will not be in favour of this model because a three-dimensional yield curve is unnecessarily complicated. Furthermore, this model is not flexible enough to allow for the different shapes the yield curve can take on.

3.1.3 Dobbie–Wilkie model

The Dobbie–Wilkie model has the following functional form (Dobbie and Wilkie, 1978 and 1979):

$$r_i = \beta_0 + \beta_1 e^{-\alpha_1 t_i} + \beta_2 e^{-\alpha_2 t_i} \tag{3.4}$$

where
r_i = yield to maturity of bond i
t_i = term to maturity of bond i (in years)
α_i = the non-linear regression parameters with $i = 1$ and 2
β_j = the linear regression parameters to be estimated with $j = 0$, 1, and 2.

This model was used by some practitioners, but it was found that it is susceptible to "catastrophic jumps." This term is used to describe the event when the least squares fit jumps from one set of parameters to another quite different set of values (Cairns, 1998; Feldman *et al.*, 1998).

3.1.4 Ayres–Barry model

The Ayres–Barry model is given by:

$$r_i = r_\infty + e^{-\beta(t_i - t_0)}(r_0 - r_\infty) \tag{3.5}$$

where
r_i = yield to maturity of bond i
r_∞ = yield on a perpetuity (or long-dated instrument)
t_i = term to maturity of bond i (in years)
t_0 = term of shortest stock in the fit (in years)
β = fitted parameter
r_0 = yield on shortest stock in the fit.

Paterson (1996) adjusts this model by allowing both r_∞ and r_0 to be fitted parameters together with ß, and finds that this resulted in an improvement in the goodness of fit. This model is not as flexible as the Dobbie–Wilkie model discussed in Section 3.1.3, and has the same shortcomings as the Bradley–Crane and Elliot–Echols models.

3.1.5 The Super-Bell model

The Super-Bell model was developed by Bell Canada Limited in the 1960s. It is a regression-type model which specifies a functional form for a par curve as (Bolder and Stréliski, 1999):

$$r_i = \beta_0 + \beta_1 t_i + \beta_2 t_i^2 + \beta_3 t_i^3 + \beta_4 \sqrt{t_i} + \beta_5 \ln(t_i) + \beta_6 C_i + \beta_7 C_i t_i \qquad (3.6)$$

where
r_i = yield to maturity of bond i
t_i = term to maturity of bond i (in years)
C_i = the coupon rate of bond i
β_j = regression parameters to be estimated with $j = 0, 1, ..., 7$.

To fit the model, the regression parameters of (3.6) are estimated. The fitted par yields are then used in a second regression analysis, where the parameters of (3.6) are estimated again, but ignoring the parts depended on the coupon rate, in other words estimating only the following terms:

$$\hat{r}_i = \beta_0 + \beta_1 t_i + \beta_2 t_i^2 + \beta_3 t_i^3 + \beta_4 \sqrt{t_i} + \beta_5 \ln(t_i)$$

One of the biggest shortcomings of this model is that the zero curve can only be derived for discrete points in time, which means that additional assumptions are needed to interpolate between the discrete points. In some situations this model leads to forward rate curves with very strange shapes, and it does not always fit the underlying bonds adequately (Bolder and Stréliski, 1999).

3.1.6 McLeod model

This model was developed in the 1990s and was used as the official South African yield curve model until 2003. Cluster analysis is used to divide the bonds into different groups according to their terms to maturity. A cubic spline function is then fitted through these node points (McLeod, 1990).

The process to do the cluster analysis is as follows:

- Five knot points $\kappa_1,...,\kappa_5$ are provided to enable the cluster analysis to begin. These knot points denote the terms to maturity (in years), and are independent of the data and the date of analysis.
- The two bonds whose terms to maturity are closest to each of the knot points are chosen. It is possible that a situation can occur where a bond contributes to more than one cluster. The bond is then treated as an additional bond and added to the process. For each bond the following value is calculated:

$$g_{i,j} = \frac{W_i}{(t_i - \kappa_j)^2}\text{, i} = 1,\ldots,m; j = 1,\ldots,5$$

where m is the number of bonds used to derive the curve; t_i is the term to maturity of bond i, and W_i the weight attached to bond i. The two bonds with the highest $g_{i,j}$ value are chosen for each cluster.

- Each of the five clusters now contains two bonds. For each cluster the weighted average term to maturity (AD_j) and the weighted average yield to maturity (AY_j) are calculated. The bonds get weighted with the nominal amount issued, which is done as follows:

$$AD_j = \frac{t_{1,j}W_{1,j} + t_{2,j}W_{2,j}}{W_{1,j} + W_{2,j}} \text{ and } AY_j = \frac{r_{1,j}W_{1,j} + r_{2,j}W_{2,j}}{W_{1,j} + W_{2,j}}; \text{ j} = 1,\ldots,5 \quad (3.7)$$

where $t_{1,j}$ and $r_{1,j}$ denote the term to maturity and yield to maturity respectively of the bond closest to κ_j, and $t_{2,j}$ and $r_{2,j}$ denote the term to maturity and yield to maturity respectively of the bond second closest to κ_j.

- The remaining bonds are now allocated to clusters by comparing their yields and terms to maturity to the weighted averages that were calculated in the previous step. The remaining bonds are not allowed to contribute to more than one cluster. To enforce this, the following quantity is calculated:

$$h_{z,j} = (t_z - AD_j)^2 + \frac{1}{2}(r_z - AY_j)^2; j = 1,\ldots 5$$

where z denotes the index by which we loop through the remaining bonds. The idea is to allocate each remaining bond to that cluster where it shows the lowest $h_{z,j}$ value.

- The center point of each cluster is now determined by calculating the weighted average term to maturity and the weighted average yield to maturity similarly to the way it was done in (3.7).
- The weighted average terms to maturity AD_j form the x coordinates, and the weighted average yields to maturity AY_j form the y coordinates when fitting the cubic spline.

Please refer to Chapter 4 for a detailed discussion on the cubic spline interpolation function.

The problem with this model is that the resulting par yield curve is dependent on the chosen knot points to such an extent that the individual par yields may not be fitted closely enough. A further disadvantage is that because a cubic spline is used, the curve does not converge to a constant level at the long end.

3.1.7 Concluding remarks

Using the yields to maturity of coupon-paying bonds to represent the term structure of interest rates is not entirely correct. First, the yield to maturity can be seen as an average of the zero rates up to maturity, and second, for a given term structure of zero rates, the yield to maturity of a bond will depend on its coupon rate (known as the “coupon effect”), which is why two coupon-bearing bonds that mature at the same date generally have different yields to maturity if they have different coupon rates. The reason is that, all else being equal, a higher coupon rate implies that the share of early payment increases, which gives more weight to the short spot rate in the determination of the yield to maturity (Svensson, 1994).

The McLeod model attempts to address the issue where different bonds with the same term to maturity do not necessarily have the same yield to maturity. However, this model also has some shortcomings, as discussed in Section 3.1.6.

A drawback of the regression-type models is that when fitting a curve through the yields to maturity, we do not explicitly restrict payments due on the same date to be discounted at the same rate (Anderson *et al.*, 1997). To see the effect of this, consider the example of two bonds, A and B, where bond A is maturing in one period’s time and bond B in two periods. The prices of these two bonds are calculated as follows:

$$\text{Price of bond A} = \frac{C_A + R_A}{(1 + y_A)}$$

$$\text{Price of bond B} = \frac{C_B}{(1 + y_B)} + \frac{C_B + R_B}{(1 + y_B)^2}$$

where C refers to the coupon payment, R to the redemption payment and y to the yield to maturity of each specific bond. The first coupon payment of bond B is thus not restricted to being discounted at the same rate as bond A, even though they are due at exactly the same time. Thus, when estimating a yield curve, the assumption needs to be made that the yield on bond A is used to discount the first coupon on bond B, and that the yield on bond B reflects the difference in rates between periods 1 and 2.

Another drawback of regression-type models is that they do not show an adequate fit of the data, because of their over-simplified presentations of the term structure of market rates. When deriving a yield curve, it is necessary that the curve fits the yields as smoothly as possible; that the model is easy to implement, and that the model is easily adaptable to changing conditions in the market.

A problem when estimating the regression parameters with ordinary least squares arguments is that we may find that the parameter estimates are very volatile from day to day, since the regression assumptions are not necessarily met.

The regression-type models will not be considered or discussed in any amount of detail because of their shortcomings as discussed above. Empirical yield curve models are an improvement on regression-type models, and are discussed in detail in the next section.

3.2 EMPIRICAL YIELD CURVE MODELS

The models discussed in this section were developed to overcome some of the shortcomings of the regression-type yield curve models. In general, the idea is to use an appropriate mathematical discount function and estimate the parameters thereof, with the constraint that when each of the bond's cash flows are discounted with it, the fitted bond price will equal the current market price. Note that the price on bond i ($i = 1,...,N$) can be written as:

$$P_i = \sum_{j=1}^{n_i} C_{i,j}\, df_{m_{i,j}} \tag{3.8}$$

where

P_i = the all-in price of bond i

n_i = number of outstanding cash flows of bond i

N = total number of bonds used to derive the curve

df_m = discount function for a term m

$m_{i,j}$ = term in years from the value date until cash flow $C_{i,j}$

$C_{i,j}$ = jth cash flow of bond i (the last cash flow will be the redemption payment plus the coupon payment).

In this section we implicitly assume that the value date corresponds with the settlement date. The validity of this assumption is, however, discussed in Chapter 5.

Empirical yield curve models are expressed in two ways:

- An explicit mathematical function for the discount function is given, for instance the exponential polynomial model discussed in Section 3.2.2.
- In terms of basis functions, for instance the McCulloch–Kochin model discussed in Section 3.2.5.

Basis functions are defined by decomposing the discount function into a set of k linearly independent functions that can be written as (Anderson *et al*., 1997; McCulloch, 1971):

$$df_m = \alpha_0 + \sum_{j=1}^{k} \alpha_j f_j(m) \tag{3.9}$$

where the αs are the parameters to be estimated and $f_j(m)$ denotes the basis functions. They are complex functions of the term of the discount factor m.

The form of the basis function $f_j(m)$ and the value of k are very important for the quality of the fit. There are two main rules when choosing the basis functions (McCulloch, 1971):

- $f_j(m)$ must be continuously differentiable
- $f_j(0)$ must be equal to 0.

We now consider the various empirical yield curve models that are proposed in the literature. It is important to note that all these models are functions of the term denoted by m.

3.2.1 Polynomial model

The polynomial model is the most basic empirical yield curve model. This model is defined in terms of the following basis functions:

$$f_j(m) = m^j; j = 1,2,...k \tag{3.10}$$

Substituting (3.10) into (3.9) yields the following equation for the discount function:

$$df_m = \alpha_0 + \sum_{j=1}^{k} \alpha_j m^j$$

by taking α_0 as one. The discount function df_m is a kth-degree polynomial.

Some drawbacks to using a polynomial are:

- This formulation does not have the ability to give more weight to values of m which are more likely to occur.
- It would take an extremely high-order polynomial to fit both the long and short end of the curve, and this would result in the polynomial taking on extreme values between observations at the long end.
- Some practitioners also found that the polynomial conforms too strongly to shapes at the far end of the curve while smoothing over shapes implied by data at the near end.
- The function is not monotonically decreasing, which is a requirement for a discount function (McCulloch, 1971; Coleman, Fisher, and Ibbotson, 1992).

3.2.2 Exponential polynomial model

The exponential polynomial model is defined as:

$$df_m = \exp\left[-\sum_{j=1}^{k} \alpha_j m^j\right] \tag{3.11}$$

where the αs are the parameters to be estimated and k denotes the degree of the polynomial.

In practice a third-degree polynomial is usually sufficient (Chambers *et al.*, 1984). The Nelson–Siegel and Cairns models that are discussed in Section 3.2.10 are variations on this model.

3.2.3 Bernstein polynomial model

The Bernstein polynomial model is defined in terms of basis functions as follows:

$$f_j(m) = \begin{cases} 0 & j = 0 \\ \sum_{r=0}^{k-j} (-1)^{r+1} \binom{k-j}{r} \dfrac{m^{j+r}}{j+r} & j = 1,\ldots,k \end{cases} \tag{3.12}$$

where k denotes the degree of the polynomial and the α_j parameters of (3.9) are restricted to be greater than or equal to 0. By letting $\alpha_0 = 1$ it is ensured that $df_0 = 1$.

The advantage of this function over conventional polynomial functions is that it estimates the derivatives better, which is important since the forward rate is a function of the first derivative of the discount function (Schaefer, 1981; Anderson *et al.*, 1997).

3.2.4 McCulloch cubic spline model

Say we have N bonds available to derive a yield curve and we let m_N denote the term to maturity of the longest-term bond (in years). We divide the maturity range $[0, m_N]$ into subintervals by specifying k knot points $\kappa_1, \kappa_2,\ldots,\kappa_k$ such that $\kappa_1 = 0$ and $\kappa_k = m_N$. A separate function for df_m is then fitted to each subinterval.

The McCulloch cubic spline model is defined in terms of basis functions and is given by (McCulloch, 1975; Anderson *et al.*, 1997):

$$f_j(m) = \begin{cases} 0 & \text{for } m < \kappa_{j-1} \\ \dfrac{(\mathrm{m} - \kappa_{j-1})^3}{6(\kappa_j - \kappa_{j-1})} & \text{for } \kappa_{j-1} \leq m < \kappa_j \\ \dfrac{c^2}{6} + \dfrac{ce + e^2}{2} + \dfrac{e^3}{6(\kappa_{j+1} - \kappa_j)} & \text{for } \kappa_j \leq m < \kappa_{j+1} \\ (\kappa_{j+1} - \kappa_{j-1})\left[\dfrac{2\kappa_{j+1} - \kappa_j - \kappa_{j-1}}{6} + \dfrac{m - \kappa_{j+1}}{2}\right] & \text{for } \kappa_{j+1} \leq m \end{cases} \tag{3.13}$$

when $j < k$ and with $c = \kappa_j - \kappa_{j-1}$ and $e = m - \kappa_j$. When $j = k$ we have that $f_j(m) = m$ for all m. Please note that k denotes the chosen number of knot points.

It is not an easy task to determine the number of knot points. The greater the number of knot points, the better the fit of the discount function will be. However, if too many knot points are chosen, the discount function may conform too closely to outliers and the resulting curve will not be smooth enough. The spline may weave around the exponential function, which will result in highly unstable forward rates. A suggestion is to choose the number of knots as the nearest integer of $\sqrt{N}$, and choose the knot points so that approximately an equal number of bonds fall between any pair of knot points (McCulloch, 1971).

A drawback to using this model is that at times the estimated discount function may slope upward at the long end of the curve, which leads to negative forward rates. A solution is to impose the additional constraint of a negative slope everywhere in the spline approximation. Even though this will serve to prevent negative forward rates, it will not, however, lead to more stable forward-rate curves. It is impossible to adequately fit the exponential form of the discount function with a splines model (Shea, 1984; Vasicek and Fong, 1982; Deacon and Derry, 1994a; Bliss, 1997).

The cubic splines approach can be shown to be mathematically the smoothest function that fits the data points, is continuous, and is twice differentiable at the knot points (Van Deventer and Imai, 1997). Adams and van Deventer (1994) define smoothness as follows:

$$Z = \int_0^T [f''(s)]^2 ds$$

where f is the function used to smooth the observed data. Bigger values of Z indicate a more erratic or a less smooth function f. Should the objective be to produce the smoothest possible yield curve, then the cubic spline model should be used to fit the yields to maturity, since it produces the smoothest yield curve. If the objective is the smoothest possible discount bond price function, a cubic spline fitted to the zero-coupon bond prices produces the smoothest curve by the above-mentioned smoothness criterion.

A problem when fitting cubic splines directly to the yields of bonds is that the resulting forward-rate curve is not twice differentiable at the knot points. This means that it is not smooth (according to the definition of smoothness above) and that the forward rate curve tends to be volatile particularly at the longer maturities (Van Deventer and Imai, 1997).

3.2.5 The McCulloch–Kochin quadratic-natural spline model

McCulloch and Kochin (1998) suggest the quadratic-natural spline model for which the basis functions are defined as follows:

$$f_j(m) = \theta_j(m) - \frac{\theta''_j(m_N)}{\theta''_{n+1}(m_N)}\,\theta_{n+1}(m) \qquad ,j = 1,...,k \tag{3.14}$$

where $\theta_1(m) = m$, $\theta_2(m) = m^2$ and $\theta_j(m) = \max(0, m - \kappa_{j-2})^3$ for $j = 3,..., k+1$. Let κ denote the knot points, m_N the longest term to maturity (in years) and k the number of knot points. We have that $\theta''_j(m)$ denotes the second derivative of θ with respect to m.

McCulloch and Kochin define $\varphi(m)$ as the log of the discount function, which may be constructed as:

$$\varphi(m) = \sum_{j=1}^{k} \alpha_j f_j(m) \tag{3.15}$$

so that the discount function is defined as:

$$df_m = \exp\left[-\sum_{j=1}^{k} \alpha_j f_j(m)\right] \tag{3.16}$$

From (3.15) and (3.16) it follows that $\varphi(m) = -\log[df_m]$. By imposing the restriction that $\varphi'(m_N) = 0$ at the longest observed maturity, the function can be extrapolated by a straight line to infinity. This restriction implies that the discount function is a pure exponential decay beyond this maturity. Another restriction is that $\varphi''(m) = 0$, $m \in [0, \kappa_1]$ where κ_1 is the first knot. This restriction allows the spot and forward curves to have any slope at $m = 0$ and only constrains them to be linear out to the first knot.

The number of knots can be chosen to be $\sqrt{N}$ where N is the number of bonds used to derive the yield curve. The positions of the knots are chosen so that an equal number of bonds fall between two adjacent knots.

3.2.6 The Vasicek–Fong exponential spline model

Vasicek and Fong (1982) use exponential splines that produce a smooth and continuous forward rate curve. They suggested that a transform should be applied to the argument m of the discount function df_m, where the transform has the following form:

$$\mathrm{m} = -\left(\frac{1}{\alpha}\right)\ln(1 - x), 0 \le x \le 1$$

The goal is to transform the discount function from an exponential function of m to a linear function of x. It can be shown that using a cubic spline to estimate both the transformed discount function and the parameter α is equivalent to estimating the discount function by a third-order exponential spline. Between each pair of knots the discount function takes the following form:

$$df_m = b_0 + b_1 e^{-\alpha m} + b_2 e^{-2\alpha m} + b_3 e^{-3\alpha m} \quad (3.17)$$

Vasicek and Fong (1982) claim that the advantage of using this approach is that no complicated non-linear estimation techniques need to be used. Shea (1985) shows that this is not the case. He furthermore shows that the Vasicek–Fong model is no more likely to yield stable forward interest rate structures than ordinary polynomial spline models. He recommends using polynomial spline models in preference to exponential splines since the resulting term-structure estimates are very close to those obtained by the polynomial spline method but are computationally not as cumbersome.

3.2.7 Carriere model

Carriere (1998) points out that the discount function df_m can be viewed as a survival function, which means that the parametric survival functions that actuaries have used in modeling losses can be used as price models. His model is formulated as follows.

Let $\kappa_0,\kappa_1,\ldots,\kappa_n$ denote the knot points of the spline function restricted to the interval [0, 1]. The formula for a polynomial q-spline is then given by:

$$df_m = \sum_{j=0}^{q}\phi_j[1 - v(m)]^j + \sum_{i=1}^{n-1}\xi_i \max(0,1 - \upsilon(m) - \kappa_j)^q \quad (3.18)$$

where $\{\phi_0;\ldots;\phi_q;\xi_1;\ldots;\xi_{n-1}\}$ denote the set of unknown parameters that have to be estimated and $\upsilon(m)$ denotes any parametric discount function. This spline is a piecewise polynomial of degree q with q–1 continuous derivatives at the knots $\kappa_0,\kappa_1,\ldots,\kappa_n$ where $\kappa_0 = 0$ and $\kappa_n = 1$.

With this model $df_m \geq 0$ is restricted to be a non-increasing function. A further restriction is that $df_0 = 1$, which implies that $\phi_0 = 1$, and to get $df_m = 0$ when $1 - \upsilon(m) = 1$ we must have that:

$$\sum_{j=0}^{q}\phi_j + \sum_{i=1}^{n-1}\xi_i(1 - \kappa_i)^q = 0$$

Note that that the spline is linear in the parameters.

There are two sets of parameters to be estimated: the parameters of the $\upsilon(m)$ function and that of the spline df_m. Carriere found that when these two sets of parameters are estimated simultaneously, the implied forward rates took on strange shapes. For this reason he suggests that the parameters of

$\upsilon(m)$ must be estimated first. These should then be used as constants when the remaining parameters are estimated.

Table 3.2 Two-parameter functions used by Carriere

Function	$\upsilon(m)$
Weibull	$\exp\left[-\left(\frac{m}{\mu}\right)^{\frac{\mu}{\sigma}}\right]$, $\mu > 0, \sigma > 0$
Gompertz	$\exp\left[e^{-\frac{\mu}{\sigma}}\left(1 - e^{\frac{m}{\sigma}}\right)\right]$, $\mu \in \Re, \sigma > 0$

Some of the analyzed $\upsilon(m)$ functions are the transformed gamma, Weibull, Gompertz, and lognormal functions. Carriere shows that the best-fitting two-parameter model is the Weibull model, but cautions that it shows odd behaviour at the shorter maturities. Other functional forms he considers are the Nelson–Siegel, Vasicek, and the Cox, Ingersoll, and Ross models. The Weibull and Gompertz functions are shown in Table 3.2.

3.2.8 The Malan model

Malan (1999) proposes the following model:

$$df_m = 1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 y^3 + b_5 z^3 + b_6 w^3 \tag{3.19}$$

where

$x = 1 - e^{-\alpha m}$

$$y = \begin{cases} 0 & , m < \kappa_1 \\ 1 - e^{-\alpha(m-\kappa_1)} & , \text{otherwise} \end{cases}$$

$$z = \begin{cases} 0 & , m < \kappa_2 \\ 1 - e^{-\alpha(m-\kappa_2)} & , \text{otherwise} \end{cases}$$

$$w = \begin{cases} 0 & , m < \kappa_3 \\ 1 - e^{-\alpha(m-\kappa_3)} & , \text{otherwise} \end{cases}$$

The function she suggests consists of three knot points κ_1, κ_2, and κ_3 given in years. The issues around specifying knot points are discussed in Chapter 5.

She estimates the parameters by fixing the value of α and then using generalized least squares to obtain the b_i parameter values. A one-dimensional function minimization procedure is then implemented to obtain a value for α where the b_i values are now fixed.

This model is a variation of the model suggested by Vasicek and Fong (1982), which is discussed in Section 3.2.6.

3.2.9 B-spline model

Steeley (1991) defines the B-spline model in terms of basis functions as:

$$B_p^g(m) = \sum_{l=p}^{p+g+1} \left[\prod_{h=p,\, h\neq l}^{p+g+1} \frac{1}{(m_h - m_l)} \right] \max(0, m - m_l)^g \quad , \quad -\infty < \mathrm{m} < \infty \tag{3.20}$$

where the subscript p indicates that $B_p^g(m)$ is non-zero only if m is in the interval $[m_p, m_{p+g+1}]$ and g indicates the degree of the spline. For instance, a linear B-spline function ($g = 1$) would be given by:

$$B_p^1(m) = \sum_{l=p}^{p+2} \left[\prod_{h=p,\, h\neq l}^{p+2} \frac{1}{(m_h - m_l)} \right] \max(0, m - m_l), \quad -\infty < \mathrm{m} < \infty$$

which is non-zero over the interval $[m_p, m_{p+2}]$. This function takes on the values:

$$B_p^1(m) = \begin{cases} 0 & m \le m_p \\ \dfrac{(m - m_p)}{(m_{p+1} - m_p)(m_{p+2} - m_p)} & m_p < m \le m_{p+1} \\ \dfrac{(m_{p+2} - m)}{(m_{p+2} - m_p)(m_{p+2} - m_{p+1})} & m_{p+1} < m \le m_{p+2} \\ 0 & m_{p+2} < m \end{cases}$$

To obtain a smooth forward rate, a function of at least order three should be used.

Steeley (1991) also gives the following recurrence relation:

$$B_p^g(m) = \frac{(m - m_p)\, B_p^{g-1}(m) + (m_{p+g+1} - m)\, B_{p+1}^{g-1}(m)}{(m_{p+g+1} - m_p)}$$

which holds for all real values of m. The discount function between any two knots m_p and m_{p+1} is then defined as:

$$df_m = \sum_{j=p-g}^{p} \alpha_j B_j^g(m)$$

where $m_p \le m < m_{p+1}$ (Steeley, 1991; Anderson *et al.*, 1997).

3.2.10 Nelson and Siegel type models

An important aspect of the Nelson–Siegel model is the fact that it attempts to explicitly model the implied forward-rate curve. The chosen functional form allows the forward-rate curve to take on a number of sensible shapes (Anderson *et al.*, 1997; Cairns, 1998). The Nelson–Siegel model for the forward rate is:

$$\rho(m) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1}\exp\left(-\frac{m}{\tau_1}\right) \tag{3.21}$$

from which the discount function can be derived as:

$$df_m = \exp\left[-m\left(\beta_0 + (\beta_1 + \beta_2)\frac{\tau_1}{m}\left[1 - \exp\left(-\frac{m}{\tau_1}\right)\right] - \beta_2 \exp\left[-\frac{m}{\tau_1}\right]\right)\right]$$

where both β_0 and τ_1 must be positive.

The forward rate function (3.21) consists of three components. The first component is a constant β_0, which specifies the long rate to which the forward rates converge, and that is why it is necessary to restrict this parameter to be positive. The second component is an exponential term $\beta_1 \exp\left(-\frac{m}{\tau_1}\right)$ which is monotonically decreasing (or increasing if β_1 is negative), and the third component is a term that generates a hump-shape (or U-shape if β_2 is negative).

When term to maturity approaches zero, the forward rate approaches the value $\beta_0+\beta_1$ (Anderson *et al.*, 1997).

Svensson (1994) increases the flexibility of the Nelson and Siegel model by adding a fourth term and so allowing for a second hump-shape or U-shape. The Svensson model of the forward rate is written as:

$$\rho(m) = \beta_0 + \beta_1\exp\left(-\frac{m}{\tau_1}\right) + \beta_2\left[\frac{m}{\tau_1}\exp\left(-\frac{m}{\tau_1}\right)\right] + \beta_3\left[\frac{m}{\tau_2}\exp\left(-\frac{m}{\tau_2}\right)\right] \tag{3.22}$$

where β_0, τ_1 and τ_2 must be positive.

Another model that has the same form as the Nelson and Siegel model is the exponential model (Anderson *et al.*, 1997). The forward rate is described by the following function:

$$\rho(m) = \sum_{j=0}^{n}\beta_j e^{-c_j m}$$

where $c_j > 0$ for all $j \geq 1$. Conventionally $c_j > c_{j+1}$ for all j. A similar model is suggested by Cairns (1998) to fit the forward-rate curve of the UK securities market. He specifies the model as follows:

$$\rho(m) = \beta_0 + \sum_{j=1}^{4}\beta_j e^{-c_j m}$$

which is equivalent to fitting the discount function:

$$df_m = \exp\left[-\beta_0 m - \sum_{j=1}^{4} \beta_j \frac{1 - e^{-c_j m}}{c_j}\right] \tag{3.23}$$

By fixing the c_j as constants in the model, only five parameters need to be estimated, and Cairns argues that this reduces the risk of multiple solutions. He considers this model to be an improvement to the Nelson–Siegel and Svensson models as this model allows up to three turning points and does not exhibit the catastrophic jumps to which the other two models are subject. Cairns (1998) discusses how the c_j values are to be chosen. He uses $c_j = (0{,}2; 0{,}4; 0{,}8; 1{,}6)$.

3.2.11 Piecewise constant forward rate model

Coleman *et al.* (1992) approximate the forward curve by fixing it to be constant over certain maturity ranges $\rho(m) = f_j$ for $\kappa_{j-1} < m \leq \kappa_j$. This method is equivalent to fitting exponential splines to estimate the discount function, for example for $\kappa_2 < m \leq \kappa_3$:

$$df_m = \exp[-f_1\kappa_1 + f_2(\kappa_2 - \kappa_1) + f_3(m - \kappa_2)] \tag{3.24}$$

where κ_i denote the knot points. Coleman *et al.* estimate $\{f_j\}$ using non-linear least squares and maximum likelihood estimation. They find that the function adequately approximates the different shapes the yield curve can take on. This model yields continuous discount functions, but the first derivative is discontinuous.

3.2.12 Concluding remarks

Some of the empirical models are very highly parameterized, and they can also be very difficult to implement, since it may be difficult to find adequate parameter estimates. The empirical models, however, do show a good fit when fitted to bonds. In Chapter 5 we consider some examples that illustrate how to implement the empirical models.

3.3 EQUILIBRIUM MODELS

Equilibrium models propose theories about the nature of the stochastic process that drives interest rates, and deduce a characterization of the term structure in an efficiently operating market (Vasicek and Fong, 1982). Zero curves derived from these models depend only on a few parameters. The problem is that these zero curves do not fit the observed data on bond yields and prices very well. Actual yield curves typically exhibit more varied shapes than those justified by equilibrium models.

The most basic equilibrium models are one-factor models where it is assumed that a single economic variable explains the level of the yield curve. In this section three equilibrium models, for which a functional form for the discount function can be derived, are examined. These equilibrium models are then used as descriptive models in the sense that they can be fitted to a single day's yield curve data. These three models are chosen because they are the most basic models and thus not too difficult to implement.

3.3.1 Ito's lemma

It is important to know Ito's lemma in order to understand how the differential equation for the discount factor, which is a function of the interest rate, is determined. It is assumed that the interest rate r follows an Ito process.

Suppose we have a variable x that follows an Ito process:

$$dx = a(x,t)dt + b(x,t)dZ \tag{3.25}$$

where dZ is a Wiener process and a and b are functions of x and t.

A Wiener process has two basic properties. First, dZ is related to dt by the equation $dZ = \varepsilon\sqrt{dt}$ where ε is a random variable from a standard normal distribution. Second, the values for dZ for any two different short intervals of time dt are independent.

From (3.25) we see that x has a drift rate of a and variance of b^2. According to Ito's lemma we know that a function G of x and t then follows the process:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2\right)dt + \frac{\partial G}{\partial x} b dZ \tag{3.26}$$

where dZ is the same Wiener process as in equation (3.25) (Hull, 1997).

3.3.2 Vasicek model

Vasicek (1977) explains the levels of yields by making use of a model with a single exogenous factor. His model produces a zero-coupon (spot) yield curve that is arbitrage-free and driven by the current level of short-term rates. He defines the short-term rate process as:

$$dr = \alpha(\gamma - r)dt + \sigma dZ \tag{3.27}$$

where

r = the risk-free money-market rate
γ = the expected long-term level of the money-market rate
α = the mean reversion rate
σ = the volatility of the money-market rate
dZ = a normally distributed stochastic term.

The model incorporates what is known as *mean reversion*. In practice it was found that interest rates appeared to be pulled back to some long-run average level over time, and this is referred to as mean reversion. When r is high it will have a negative drift; when r is low it has a positive drift.

Using Ito's lemma and the process proposed for r (3.27), we find:

$$dP = \mu P dt + \rho P dZ \tag{3.28}$$

where $\mu = \frac{1}{P}\left(\frac{\partial P}{\partial r}\alpha(\gamma - r) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2\right)$ and $\rho = \sigma\left(\frac{\partial P}{\partial r}\frac{1}{P}\right)$

We interpret μ as the instantaneous rate of return on the bond; and ρ as the sensitivity of the bond price to changes in the spot rate (Vasicek, 1977). Under the assumption that the expected return on a bond is equal to the risk-free rate plus an allowance for risk, it holds that $\mu = r + a\rho$ where a denotes the market price of risk and all the other symbols are defined as before. For the sake of simplicity it is assumed that a is constant.

It can be shown that the differential equation of (3.28) has the following solution (and thus the functional form of the discount function):

$$df_m = \exp\left\{\frac{1}{\alpha}(1 - e^{-\alpha m})(r_\infty - r) - mr_\infty - \frac{\rho^2}{4\alpha^3}(1 - e^{-\alpha m})^2\right\} \tag{3.29}$$

where $r_\infty = \gamma + \frac{a\rho}{\alpha} - \frac{\rho^2}{2\alpha^2}$

and all the other symbols defined as before. The zero-rate curve starts at the current level of the short-term rate and approaches an asymptote of r_∞ at longer maturities.

One problem with Vasicek's model is that negative short-term rates are allowed. Cox, Ingersoll, and Ross (1985) proposed an alternative one-factor model for which the short-term rates are always non-negative. The Cox–Ingersoll–Ross model is discussed in the next section.

3.3.3 Cox, Ingersoll, and Ross model

Cox, Ingersoll, and Ross (1985) propose the following risk-neutral process for r:

$$dr = \alpha(\gamma - r)dt + \sigma r^{\beta} dZ \tag{3.30}$$

It has the same mean-reverting drift as Vasicek's formula, but the stochastic term has volatility proportional to r^{β}. They suggest $\beta = 0.5$, as this would prevent the interest rate from being negative. When $\beta = 0$ the proposed model is exactly the same as the Vasicek model, which may lead to negative interest rates.

It can be shown that the discount function has the following form when $\beta = 0.5$:

$$df_m = \left[\frac{2ae^{m(\alpha+\theta+a)/2}}{(\alpha+\theta+a)(e^{am}-1)+2a}\right]^{2a\gamma/\sigma^2} \exp\left[-r\left\{\frac{2\,(e^{am}-1)}{(\alpha+\theta+a)(e^{am}-1)+2a}\right\}\right] \tag{3.31}$$

where $a = \sqrt{(\alpha+\theta)^2 + 2\sigma^2}$ and θ is the market price of risk.

3.3.4 Duffie–Kan model

The Duffie–Kan model assumes that the short-term rate follows this process:

$$dr = (a - br)dt + \sqrt{\mu^2 + rv^2}\, dZ$$

where a, b, μ and v are the four parameters to be estimated. The formula for the discount function is then:

$$df_m = \exp\left[-\frac{2(e^{\gamma m}-1)}{((\gamma+b)(e^{\gamma m}-1)+2\gamma)}\left[r + \frac{\mu^2}{v^2}\right] + 2\,\frac{b\mu^2 + av^2}{v^4} \times \ln\left(\frac{2\gamma e^{(b+\gamma)m/2}}{(\gamma+b)(e^{\gamma m}-1)+2\gamma}\right) + \frac{\mu^2 m}{v^2}\right] \tag{3.32}$$

where $\gamma = \sqrt{b^2 + 2v^2}$ (Brousseau, 2002).

3.3.5 Concluding remarks

Yield curves exhibit more varied shapes in practice than those justified by equilibrium models. We have only considered one-factor models since the complexity of the model increases when additional factors are taken into account, which makes them more difficult to implement. The three models that were discussed in Section 3.3 were chosen because they have closed-form solutions available for the discount factor. This allows them to be fitted to bond yields similarly to the empirical yield curve models.

CHAPTER 4

Practical Issues

4.1 OVERVIEW

This chapter should be used as a checklist when deriving any yield curve. Various issues are discussed that practitioners have to take into account when deriving interest rate curves.

In Section 4.2 the various daycount conventions are discussed, along with practical examples of how to allow for them when deriving a yield curve.

In Section 4.3 the importance of knowing the quoting convention of the instruments used to derive the yield curve is discussed. This refers to how the instrument rate is compounded as well as whether it is a zero, par, or forward rate.

Sections 4.4 and 4.5 show how to allow for public holidays and business day rules. It is important to note that these rules cannot be standardized because they differ from market to market.

Sections 4.6 and 4.7 discuss credit and liquidity issues which may lead to distortions in the resulting yield curve. In Section 4.8 we mention various points that have to be considered before loading any curve into a subsystem.

In Sections 4.9 and 4.10 the various ways in which curves can be interpolated and extrapolated are discussed. Interpolation methods include linear, cubic splines, log-linear, and exponential. Extrapolation methods include both keeping the rates constant and regression methods. Some examples show why it is important to be consistent in the way interpolation methods are applied to different curves.

4.2 DAYCOUNT CONVENTIONS

It is important to know which daycount convention is associated with each instrument, because it can have a major impact on the valuation of the instrument as well as on the curve that is derived from that instrument. The daycount convention indicates how the number of days between any two dates should be determined.

The most common daycount conventions are 30/360, 30/360 European, actual/365 and actual/actual. Typically the first number denotes the assumption about the number of days in the month; the second number denotes the assumption regarding the number of days in a year. For example with a 30/360 daycount convention the number of days between two dates are determined based on the assumption that each month contains 30 days and that the number of days in the year is 360. Similarly with the actual/365 convention, the number of days between the two dates is taken as the actual number of days, assuming 365 days in a year.

4.2.1 Actual/actual daycount convention

There are various definitions for the actual/actual daycount convention (ISDA, 1998). With the ISDA definition, the actual number of days is used as numerator, and 366 is used as a denominator to determine the portion of the calculation period that falls within a leap year; the actual number of days is used as numerator and 365 is used as denominator to determine the portion of the calculation period that does not fall in the leap year.

Consider the example shown in Figure 4.1. The investment period is from 5 November 2003 to 5 March 2004, so with the ISDA definition the investment will earn interest for the period t (in years), determined as $t = (57/365) + (64/366) = 0.331$ years.

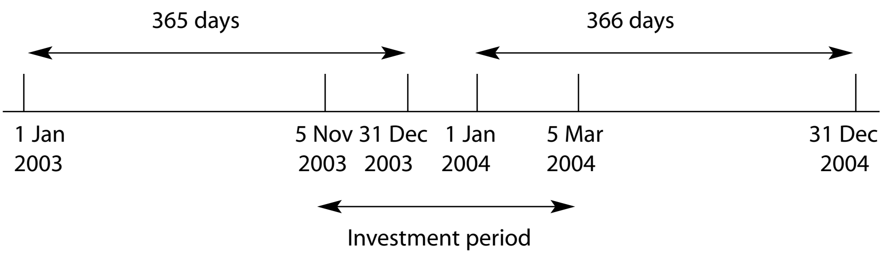


Figure 4.1 Illustrating the determination of the interest accrual period with the actual/actual daycount convention

Another definition is the AFB method, where the denominator is 365 if the investment period does not contain 29 February, or 366 if it does contain 29 February. Considering Figure 4.1, we would determine the investment period with this approach as $t = 121/366 = 0.3306$ years.

4.2.2 30/360 daycount convention

The 30/360 daycount convention is also known as the U.S. (NASD) method. The following rules are followed to determine the number of days between two dates:

- If the starting date is the 31st of a month, it becomes equal to the 30th of the same month.
- If the ending date is the 31st of a month and the starting date is earlier than the 30th of a month, the ending date becomes equal to the 1st of the next month; otherwise the ending date becomes equal to the 30th of the same month.

4.2.3 30/360 European daycount convention

The only rule that is followed with the 30/360 European daycount convention is if either of the two dates falls on the 31st of a month it is changed to the 30th of the same month.

4.2.4 Actual/360 and actual/365 daycount conventions

The only difference between the actual/360 and actual/365 daycount conventions is that 360 and 365 are used as the denominators respectively. In both cases the actual number of days between the two dates are used as the numerator.

4.2.5 Example

Consider the scenario where we invest 100 units of currency at a simple interest rate of 3 percent for the period 29 January 2004 to 31 December 2004. Table 4.1 shows the number of days in this investment period for each of the daycount conventions.

Table 4.1 An illustration of the differences between daycount conventions

Daycount convention	Begin date	End date	Number of days	Future value of the investment
30/360	29 Jan 2004	31 Dec 2004	332	376.67
30/360 European	29 Jan 2004	31 Dec 2004	331	375.83
Actual/365	29 Jan 2004	31 Dec 2004	337	376.99
Actual/360	29 Jan 2004	31 Dec 2004	337	380.83
AFB actual/actual	29 Jan 2004	31 Dec 2004	337	376.23

To calculate the future value of the investment (FV), we use:

$$FV = 100\left(1 + 3\% \times \frac{d}{DC}\right)$$

where d denotes the number of days in the investment period and DC denotes the number of days (according to the daycount convention) assumed to be in the year. We can see that the differences between the investment values can be quite significant when we are investing millions.

4.3 INSTRUMENT QUOTING CONVENTION

Yield curves are very important when one wants to be competitive in the financial market, because without them most instruments cannot be valued. It is important not only to have a yield curve that reflects the market as accurately as possible, but also to use the correct curves to value different types of deals.

Quoting convention refers to how the instrument rate is compounded as well as whether it is a zero, par, or forward rate. In Chapter 2 the relationship between the different types of rates is discussed. Making an incorrect assumption about the type of rate would mean that the wrong approach was used to derive zero rates, and that would result in a yield curve that was completely wrong or inappropriate.

4.4 HOLIDAYS AND WEEKENDS

The importance of taking holidays and weekends into account when determining the period to which any rate applies is illustrated with a simple example.

Consider when we derive a yield curve from instruments with various tenors that all have the same settlement date. This settlement date is called the *spot date,* and is usually two business days after the value date (depending on the specific market convention). When we derive a yield curve from these instruments, we have a *spot curve*. In other words, the one-month rate typically applies to the period one month beginning from the spot date. Similarly the three-month rate applies to the period three months beginning from the spot date. Figure 4.2 illustrates the periods to which the different rates apply.

Usually we are interested in finding a curve from the value date onwards. In this situation we then have to discount the whole curve back with the number of days between the spot date and the value date, so we can have what is called a “today curve.”

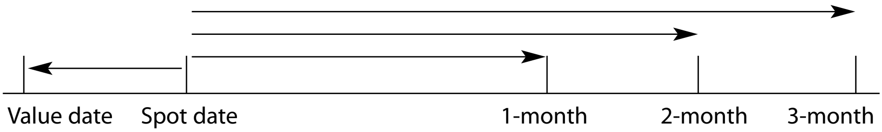


Figure 4.2 The rates used to derive the curve all apply from the spot date onwards. To get a today curve, we have to discount the whole curve from the spot date back to the value date.

Now consider the case when the value date falls on a Thursday. If just two business days are added to the value date, the spot date is Monday. However, when the Monday is a public holiday, the spot date would actually be Tuesday. By taking Monday as the spot date, we would then discount the curve by four days to get a today curve, whereas in actual fact it should be discounted with five days. With this example our spot date will be perfect, but our today curve will be wrong, and we will show unexpected profit and loss numbers if the trades in the subsystem are valued with this (incorrect) today curve.

It is important to ensure that the underlying assumptions are correct, and that the market conventions for each instrument used to derive a yield curve are known. Even when instruments with the same underlying currency are used, there may still be different settlement conventions and assumptions regarding public holidays. There are cases when public holidays are not necessarily market holidays, so that settlements can still take place on these dates. The information on the exact settlement date should be specified in the contract that defines any instrument.

4.5 BUSINESS DAY RULES

A business day rule is typically applied when the settlement date of an instrument falls on a public holiday or weekend. Depending on the rule applicable, the settlement date of the instrument is moved either forward or backward. The examples discussed in Section 4.4 typically assumed that the settlement date is moved forward.

There are various rules that have to be allowed for, such as:

- the following business day rule
- the modified-following business day rule
- the preceding business day rule
- the modified-preceding business day rule.

These rules are usually associated with a specific instrument, which means that we cannot just define general rules, but also have to allow that the different instruments follow different rules.

4.5.1 Following business day rule

When the settlement date falls on a weekend or public holiday, the date is adjusted to be the next business day that follows this date.

4.5.2 Modified following business day rule

When the settlement date falls on a weekend or public holiday, the date is adjusted to be the next business day that follows the settlement date. However, if the next business day is not in the same month, the settlement date is adjusted to be the business day just prior to the settlement date.

4.5.3 Preceding business day rule

When the settlement date falls on a weekend or public holiday, the date is adjusted to be the business day just prior to this date.

4.5.4 Modified preceding business day rule

When the settlement date falls on a weekend or public holiday, the date is adjusted to be the business day just prior to this date. If the date is not within the same month, the date is adjusted to be the first business day following the settlement date.

4.5.5 Example

Say we are working in a market where the settlement date is determined on a t+3 basis. This means that three business days are added to the value date to determine the settlement date. Table 4.2 shows an example where the different types of business day rules are applied to the value date 25 November 2004.

Table 4.2 Example showing the application of the various business day rules

Business day rule	Value date	Settlement date	Holidays
Following	25 Nov 2004	01 Dec 2004	30 Nov 2004
Modified following	25 Nov 2004	29 Nov 2004	
Preceding	25 Nov 2004	29 Nov 2004	
Modified preceding	25 Nov 2004	29 Nov 2004	

4.6 CREDIT QUALITY OF THE INSTRUMENTS

Deriving a yield curve with instruments of different credit quality will result in a curve that may not representative of the market. For example a risk-free curve derived from government bonds will always be below a more risky corporate bond curve, because investors require compensation for the additional risk that they take on. Not all instruments can be valued with the same yield curve because of the differences in the instruments. For instance we cannot value a money market instrument off a risk-free curve, because we will clearly under-value the trade.

Let us consider an example:

Table 4.3 Bonds available to derive a yield curve

Bond	Term to maturity	Yield to maturity
A001	0	5.5
A002	1	5.1
A003	2	5.0
A004	3	5.4
A005	3.5	5.9
A006	5	6.1
A007	6	6.3
A008	7	6.4
A009	8	6.7
A010	10	6.3
A011	15	5.6
A012	20	5.2

Say we have the bonds in Table 4.3 available to derive a yield curve. There are various approaches to follow, but first we consider the case where we want to derive a curve under the assumption that all bonds should be priced perfectly off this curve.

Figure 4.3 (overleaf) shows a *perfect-fit par curve*, which means that all the bond yields lie on the fitted par curve. The problem is that this curve is not very smooth, so typically we can take the same combination of bond yields and fit another curve with the constraints that the curve has to be smooth and fits all the bonds as closely as possible. The curve is then known as the *best-fit curve*, and is shown in Figure 4.4 (also overleaf). We can see how the par curve either over-estimates or under-estimates the bond yields in the five to ten-year area.

Say we find some more information regarding the bonds that are used to derive the yield curve. We find that all the bonds are government bonds,

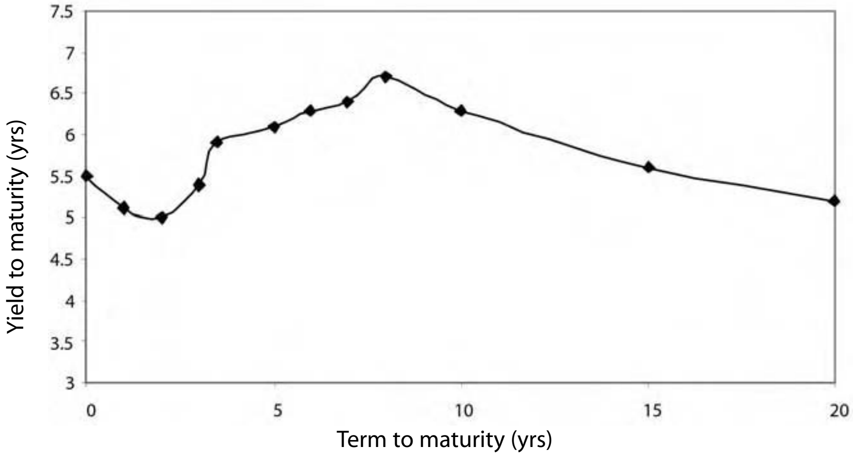


Figure 4.3 A perfect-fit curve

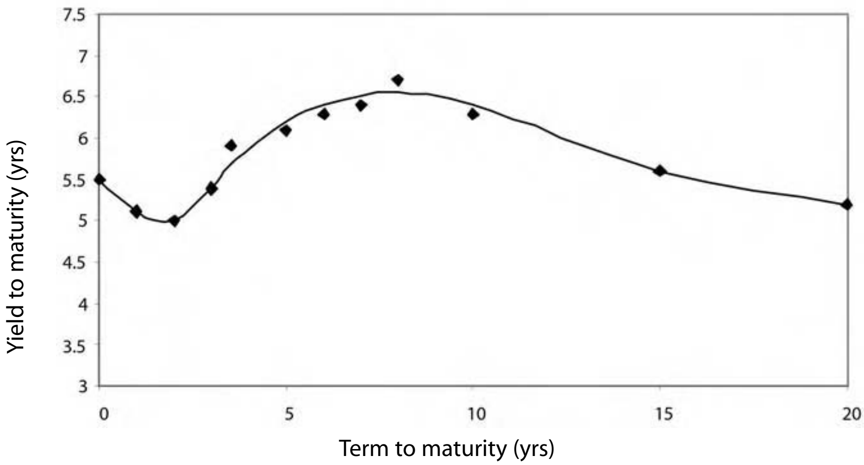


Figure 4.4 Best-fit curve

except two that are actually corporate bonds (bonds A005 and A009 in Table 4.3). Figure 4.5 shows the perfect-fit curve derived from the government bonds only. Clearly our yield curve is now much smoother. Additionally we also now know that it is a risk-free curve. Should we wish to include the two corporate bonds when deriving the yield curve, we will need to adjust the two bonds with credit spreads. Ways to estimate credit spreads are discussed in Chapter 7.

The point of this example is to show the effect on the final yield curve, depending on the underlying assumptions that are made when the curve is derived, using the same set of bonds. The example also shows how

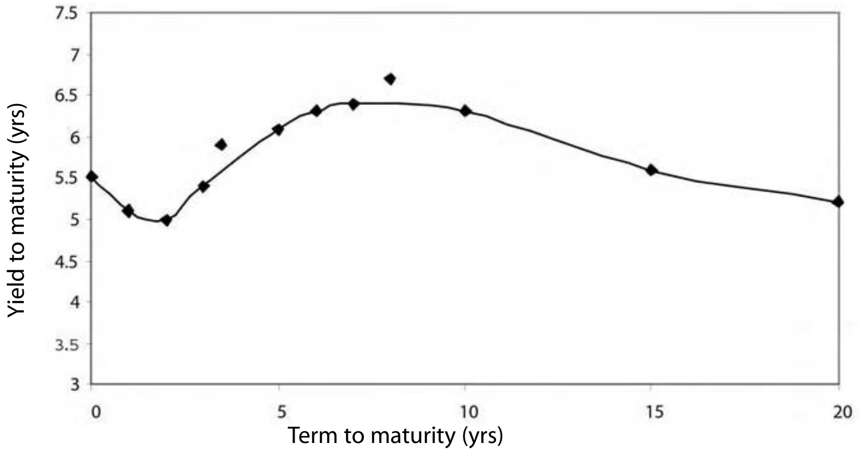


Figure 4.5 Perfect-fit par yield curve derived from only the government bonds

distorted a yield curve can be, should instruments with different credit qualities be used to derive the same curve.

An additional problem may be that not enough short-term instruments of a certain credit quality are available to derive a curve. For example, using risky instruments at the short end and risk-free instruments at the long end of a curve, means there would be a definite change in the level of the curve where the two parts of the curves are put together. We can solve this problem by estimating credit spreads and adjusting the yields of the short-term instruments downward. Please refer to Chapter 7 for a detailed discussion on the various techniques available to estimate credit spreads.

4.7 LIQUIDITY OF THE INSTRUMENTS

A curve has to be derived from liquid instruments to ensure that it is representative of current market rates. Distortions in the curve may be caused when combinations of liquid and illiquid instruments are used in its derivation.

Consider very liquid market instruments. Typically the rates of the liquid instruments will be lower than those of the less liquid instruments, because investors require additional compensation to invest in less liquid instruments. This means less liquid instruments should be priced lower, or put another way, have higher returns.

Among the features that influence the liquidity of bonds are benchmarks, the amount in issue, futures contracts, ex-dividend effects, and market segmentation.

Benchmarks are bonds issued with convenient terms to maturity. Because of this they attract investors, and this greater demand drives their prices up. This has the effect that their yields lie below the yields of similar bonds that are less liquid.

The *nominal amount* issued of a bond also plays an important role in the bond's liquidity. The characteristics of those who purchased the bonds are also important, because some investors may not wish to actively trade the bond. This effectively causes the bond to be less liquid. It is important to use turnover in conjunction with the nominal amount when considering the liquidity of the bond.

A *futures contract* on a bond signifies that a bond will be bought or sold on a future date, at a price that is determined on the day the contract is written. This contract on a bond means that the seller has to buy the specific bond in order to be able to deliver the bond to the purchaser of the futures contract. Effectively this increases the liquidity of the bond, as there is greater demand for it. As explained above, this results in the yield of this bond lying below that of the other bonds.

When a bond is traded in an *ex-interest period*, the yield at which it is traded will lie above the *cum-interest* bonds in the market. Effectively this again distorts the estimated yield curve.

The existence of "preferred habitats" which segment the market may drive up the prices of the "preferred" securities. For instance, banks mostly require short-dated (and high-coupon) securities. They simultaneously increase the demand for these bonds and reduce their effective liquidity.

When tax is payable on income (the coupons received) but not on capital gains, investors will tend to prefer lower-coupon bonds, resulting in these bonds being more liquid.

Another quantity that can be considered an indication of the liquidity of a bond is the *bid/ask spread* of the quoted prices (Persaud, 2003).

The effect of liquidity can be shown with examples similar to that given in Section 4.6, where the credit quality of an instrument is discussed. Please refer to Chapter 7 for a detailed discussion on how the liquidity premium of instruments can be estimated.

4.8 SUBSYSTEM REQUIREMENTS

In a trading environment, all deals are booked into a subsystem. It is important to understand which yield curve to load into the subsystem, when this subsystem is also used to value the deals and to calculate the risk numbers. Some issues to address are:

- whether the bid, offer, or mid rates of the instruments be used to derive the curve
- whether it should be a today or a spot curve

- how the rates should be compounded
- whether the curves should be saved as discount factors or as zero rates.

4.9 INTERPOLATION TECHNIQUES

Once a yield curve has been derived, only the rates at certain tenors are saved to a database. It is not very sensible to save a rate for each of the possible tenors out to, say, 30 years since it will take up too much space electronically and will not add any real value. In practice, a certain number of tenors are decided on and the rates at these tenors are stored. When a rate is needed for a specific tenor that is not stored in the database, the rate has to be interpolated. Various interpolation techniques are discussed in the next sections.

4.9.1 Constant interpolation

Constant interpolation refers to the situation where the interpolated value is assumed to be equal to the value at the previous node. This means that if a curve is constructed with constant interpolation, the curve is a step function.

4.9.2 Linear interpolation

Linear interpolation assumes a linear function between any two known node points (x_1, y_1) and (x_2, y_2). Say we are interested in interpolating a value for y at node point x, where $y_1 \leq y \leq y_2$ and $x_1 \leq x \leq x_2$. Graphically this is shown in Figure 4.6 (overleaf).

The linear interpolation formula is derived from the following relationship:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

which means that we want the slope of the linear function to be equal in the intervals $[x_1; x_2]$ and $[x; x_1]$. Solving for y we get:

$$y = y_1 + (x - x_1)\frac{(y_2 - y_1)}{(x_2 - x_1)} \tag{4.1}$$

which is then used as the linear interpolation formula.

4.9.3 Cubic spline interpolation

To construct a cubic spline function we define three knot points κ_1, κ_2, and κ_3. At these knot points the cubic spline takes on values f_1, f_2, and f_3. Graphically this is depicted in Figure 4.7 (overleaf).

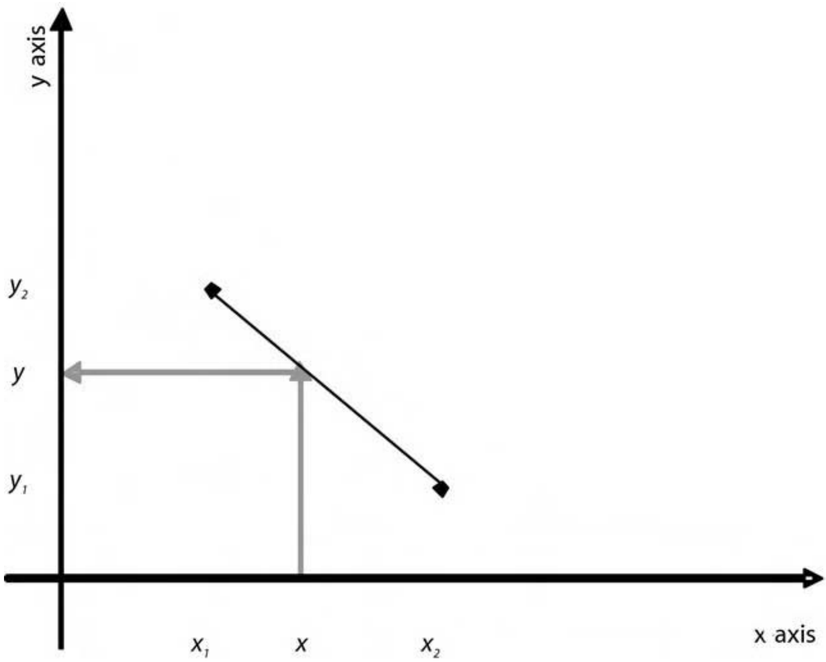


Figure 4.6 Linear interpolating a value for y between two nodes x_1 and x_2

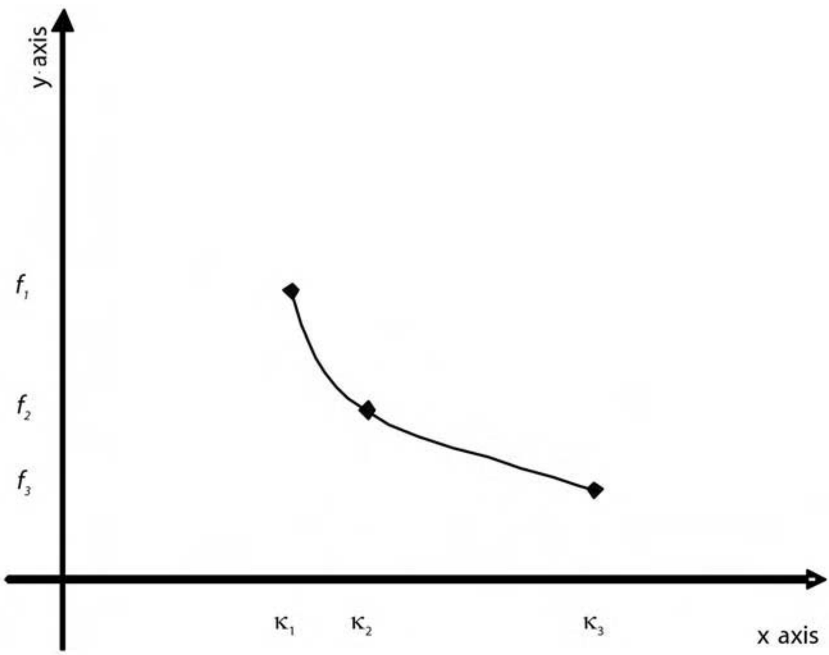


Figure 4.7 Cubic spline interpolation

The cubic polynomial function between knot points is denoted by

$$F(x) = \begin{cases} a_1 + a_2(x - \kappa_1) + a_3(x - \kappa_1)^2 + a_4(x - \kappa_1)^3 & , x \in [\kappa_1;\kappa_2] \\ b_1 + b_2(x - \kappa_2) + b_3(x - \kappa_2)^2 + b_4(x - \kappa_2)^3 & , x \in [\kappa_2;\kappa_3] \end{cases} \quad (4.2)$$

where $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ are the parameters to be solved for. The first derivative of this function is given by:

$$F'(x) = \begin{cases} a_2 + 2a_3(x - \kappa_1) + 3a_4(x - \kappa_1)^2 & , x \in [\kappa_1;\kappa_2] \\ b_2 + 2b_3(x - \kappa_2) + 3b_4(x - \kappa_2)^2 & , x \in [\kappa_2;\kappa_3] \end{cases}$$

and the second derivative by:

$$F''(x) = \begin{cases} 2a_3 + 6a_4(x - \kappa_1) & , x \in [\kappa_1;\kappa_2] \\ 2b_3 + 6b_4(x - \kappa_2) & , x \in [\kappa_2;\kappa_3] \end{cases}$$

The parameters are solved by setting the following conditions:

- We know that $F(\kappa_1) = f_1$; $F(\kappa_2) = f_2$;and $F(\kappa_3) = f_3$.
- To get a function that is twice continuously differentiable at the interior knot point κ_2, we set the first and second derivatives to be equal which means that $a_2 + 2a_3(\kappa_2 - \kappa_1) + 3a_4(\kappa_2 - \kappa_1)^2 - b_2 = 0$ and $2a_3 + 6a_4(\kappa_2 - \kappa_1) - 2b_3 = 0$.
- The final two conditions are that $F''(\kappa_1) = 0$ and $F''(\kappa_3) = 0$.

The simplest way to solve for these parameters is to put these conditions into matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & i_1 & i_1^2 & i_1^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i_2 & i_2^2 & i_2^3 \\ 0 & 1 & 2i_1 & 3i_1^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6i_1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6i_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.3)$$

where $i_1 = \kappa_2 - \kappa_1$ and $i_2 = \kappa_3 - \kappa_2$. We then just solve for the parameters with simple matrix manipulation. To illustrate these calculations, say we know a function has the following function values at the given knot points:

i	knot point κ_i	function value f_i
1	7	3
2	4	5
3	3	8

and we are interested in interpolating a value at $x = 3.5$. Substituting the known values into the parameters of the matrix given in (4.3) yields:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -3 & 9 & -27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -6 & 27 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -18 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so it follows that our interpolated value is $F(3.5) = 6.39$. Please refer to Bolder and Gusba (2002) for a detailed discussion.

4.9.4 Log-linear interpolation

Log-linear interpolation is used when we need to interpolate between two nodes that lie on an exponential curve, like a discount function. Say we are interested in interpolating a value for y at node point x, where $y_1 \leq y \leq y_2$ and $x_1 \leq x \leq x_2$. This is illustrated by Figure 4.8.

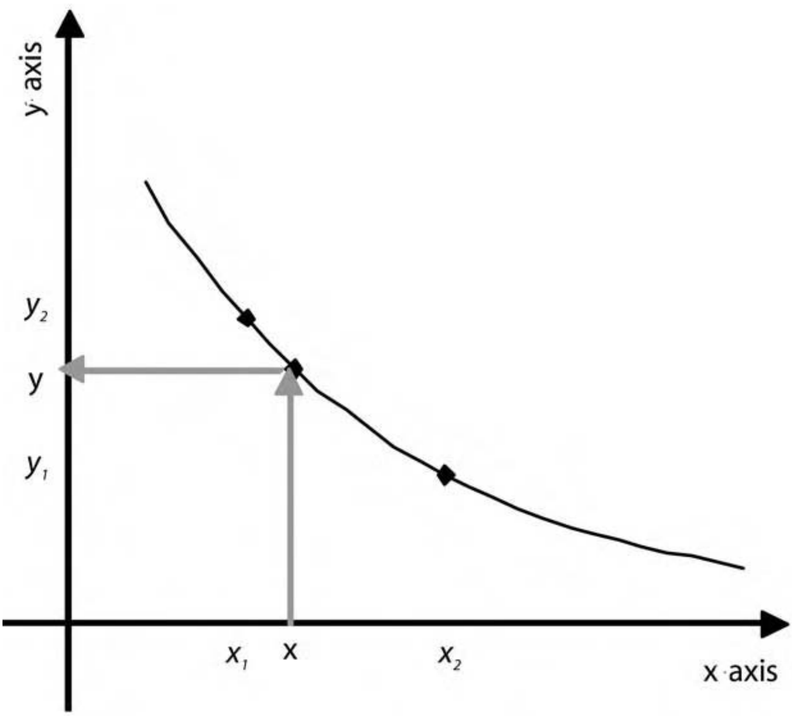


Figure 4.8 Log-linear interpolation

By taking the log of the y values, we convert the exponential curve to a linear curve as illustrated by Figure 4.9, which means that we can use (4.1) as follows (Haug, 1997):

$$\ln y = \ln y_1 + (x - x_1)\frac{(\ln y_2 - \ln y_1)}{(x_2 - x_1)}$$

and then solving for y gives:

$$y = y_1 \exp\left(\frac{x_2 - x}{x_2 - x_1}\right) \times y_2 \exp\left(\frac{x - x_1}{x_2 - x_1}\right) \tag{4.4}$$

This means we can use (4.4) to interpolate between y values that fall on an exponential curve.

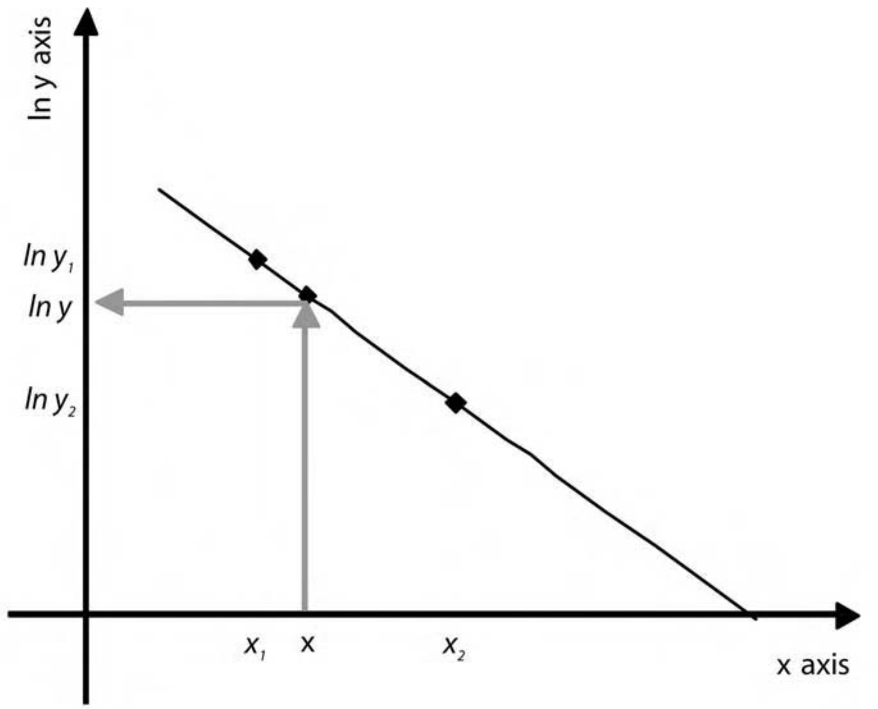


Figure 4.9 Log-linear interpolation: the exponential curve is converted to a linear curve

4.9.5 Exponential interpolation

The *exponential interpolation* function is derived with similar arguments to those for the log-linear interpolation function.

We are interested in interpolating a value for y at node point x, where $y_1 \le y \le y_2$ and $x_1 \le x \le x_2$, where the y values can be modeled with a log function. By taking the exponent of the y values, we convert the logarithmic curve to a linear curve, which means that we can use (4.1) as follows:

$$e^y = e^{y_1} + (x - x_1)\frac{(e^{y_2} - e^{y_1})}{(x_2 - x_1)}$$

and then solving for y gives:

$$y = \ln\left[e^{y_1} + (x - x_1)\frac{(e^{y_2} - e^{y_1})}{(x_2 - x_1)}\right] \tag{4.5}$$

4.9.6 Example

Table 4.4 shows an example of yield curve values at certain tenors. We are interested in comparing the different approaches to interpolate values at those tenors that are not available.

Table 4.4 Yield curve values

Term (yrs)	Interest rate (%)
0	4.0
1	3.6
3	3.0
5	3.3
10	5.0
15	5.0
20	4.0
25	2.0
30	3.0

Figure 4.10 shows a comparison of the curves when these rates are interpolated with constant, linear, and cubic spline interpolation. Some interesting facts are illustrated:

- Depending on the technique used, we get very different interpolated interest rate curves. This means that two people who are determining the value of the same portfolio may get two totally different portfolio values, should they use two different interpolation techniques.
- Linear interpolation does not allow for curvature. The adjacent points of the curve are joined by a straight line. At the short end of the curve,

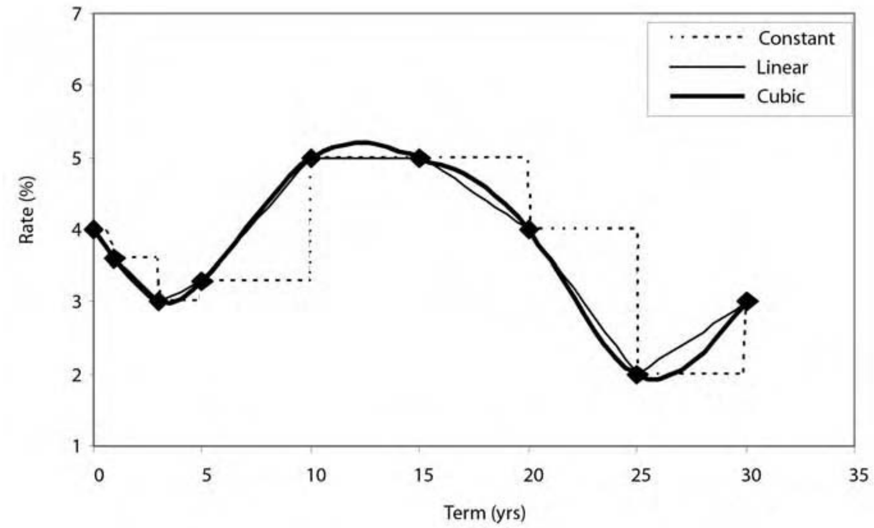


Figure 4.10 A comparison of different interpolation techniques

where more rates are available, the cubic spline and linear interpolated rates are very close.

- Constant interpolation should only be used if a lot of tenor points are available and we do not want to make any assumptions regarding the shape the curve can take on between two adjacent points.
- Cubic spline interpolation may allow for too much curvature, which means that at times this technique may over-fit the data points so that the resulting curve is not smooth enough. A good example of a curve that allows for too much curvature is shown in Figure 4.3.

4.10 EXTRAPOLATION TECHNIQUES

Extrapolation refers to the scenario where a curve has to be extended, but there are no market rates available at those tenors. Typically we may have a yield curve out to 30 years, but we need a 35-year rate to determine the value of a certain deal. The various extrapolation techniques are discussed in this section. All examples will use the data in Table 4.5 (overleaf).

4.10.1 Constant zero rates

The yield curve is extrapolated by keeping the value at the last node constant. This is similar to the constant interpolation technique that was discussed in Section 4.9.1.

Table 4.5 A zero and par yield curve. All rates are naca.

Term (yrs)	Zero rates (%)	Par rates (%)
0	1.01	1.01
1	2.05	2.05
2	2.925	2.912335
3	3.5325	3.501142
4	3.9825	3.929518
5	4.318	4.243466
6	4.5775	4.482263
7	4.783	4.668366
8	4.945	4.813153
9	5.078	4.93032
10	5.19	5.027499
11	5.279877	5.104989
12	5.37	5.179812
13	5.433333	5.233583
14	5.496667	5.285238
15	5.56	5.334915

Figure 4.11 shows the zero rates and one-year forward rates (calculated from the zero rates using the techniques discussed in Chapter 2). With this extrapolation technique the zero curve from the 15-year point is constant (data from Table 4.5). It is immediately evident that the main problem with this approach is that the discontinuity at the node on the curve where the

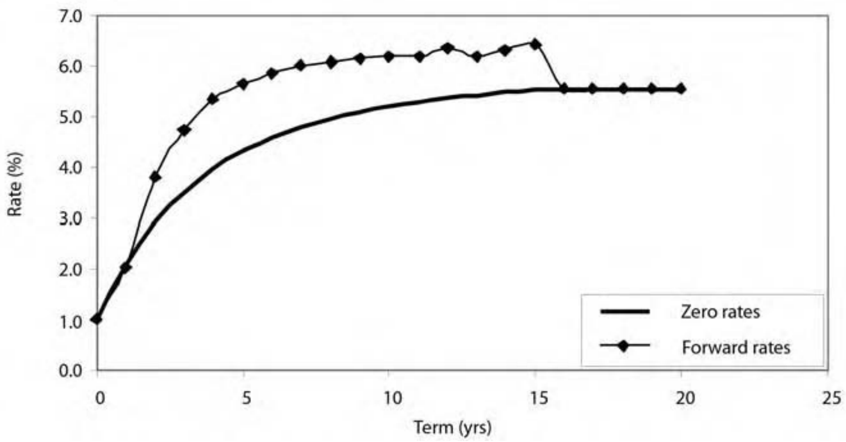


Figure 4.11 Comparison of the zero and forward rate curves using constant zero rate extrapolation

extrapolated curve starts (the 15-year point in this case) leads to discontinuities in the forward rate curve. However, most practitioners prefer this method, because no assumptions regarding the shape of the curve at the long end are made, and we still have a zero curve that converges to a level.

4.10.2 Constant forward rates

Another approach to extrapolate a yield curve is to keep the last forward rate constant. Figure 4.12 illustrates this technique using the data in Table 4.5 and keeping the one-year forward rate, calculated from the zero rates, constant. The zero rates from the 16-year point onwards are then derived from the extrapolated forward rates.

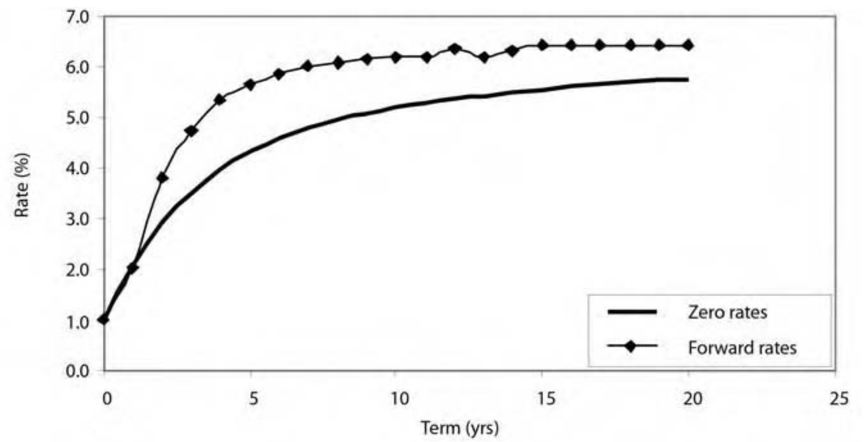


Figure 4.12 Comparison of the zero and forward rate curves using constant forward rate extrapolation

The problem with this approach is that the forward rate curve is smooth, but the zero rate curve does not converge to a fixed level. We will either get very high or very low zero rates at the very long maturities. In this specific example we see that our curve is upward sloping.

Another issue to keep in mind with this approach is that we have to decide which forward rates to keep constant. There will be a difference in the extrapolated curve when the three-month forward rates are kept constant and when the one-year forward rates are kept constant.

4.10.3 Constant par rates

To extrapolate the yield curve we can also keep the swap (or par) rates constant. Figure 4.13 shows the zero and forward rate curves calculated from the extrapolated par rates (data in Table 4.5). In this case the zero rate curve

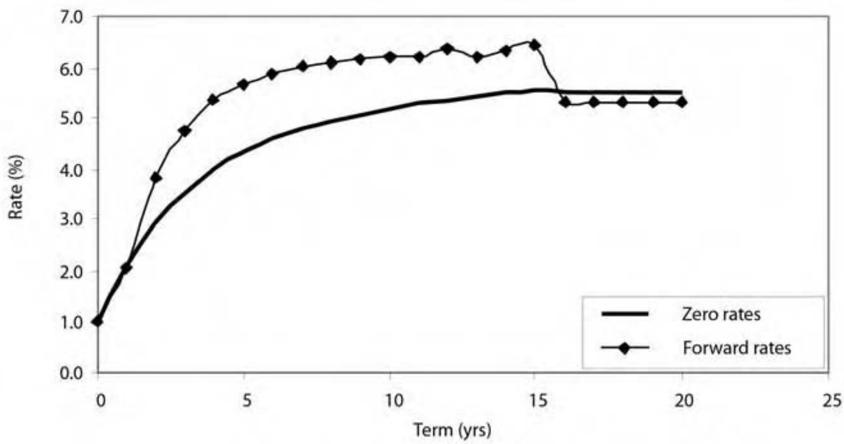


Figure 4.13 Comparison of the zero and forward rate curves using constant par rate extrapolation

is now downward-sloping at the long end and there is a discontinuity in the forward curve.

4.10.4 Regression techniques

To extrapolate a yield curve with regression techniques, we make use of the linear interpolation function described in Section 4.9.2. We basically estimate the slope between the last two node points of the given zero curve with:

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \tag{4.6}$$

where y_i denotes the zero rate and x_i denotes the term in years. The idea is then to just keep this slope constant when extrapolating the rest of the curve with:

$$y = y_1 + (x - x_1)m \tag{4.7}$$

where all the symbols are defined as in section 4.9.2. Figure 4.14 shows the zero and forward rate curves when applying this technique to the zero rates in Table 4.5 and then calculating the forward rates from the zero rates. It is interesting to see that in this case both the zero and forward rate curves are upward-sloping at the long end.

4.11 CONCLUDING REMARKS

We have seen the effects holidays and weekends, daycount conventions, instrument quoting conventions, and business-day rules have on the

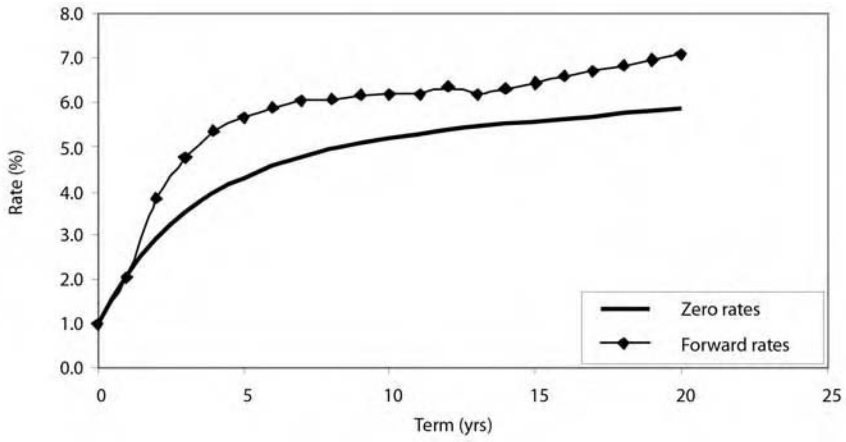


Figure 4.14 Comparison of the zero and forward rate curves when extrapolating the zero rates with regression techniques

valuation of an instrument, and thus also on the yield curve derived from this instrument. Another two very important issues are the credit quality and liquidity of the instruments. We have seen some examples that show how distorted a yield curve can be when the credit quality of the instruments is not taken into account. Similar remarks apply to illiquid instruments. Ways in which to estimate the credit and liquidity premium are discussed in Chapter 7.

Another interesting point to note is how one set of data can be used to get very different estimates of long-term rates, depending on the interpolation and extrapolation techniques used. It is important that the practitioner carefully considers the various techniques and the underlying assumptions made. It is obviously important to apply techniques consistently. For instance, consider the situation when two curves are derived from different instruments at the short end, but their long-term instruments are the same. Depending on the interpolation or extrapolation technique used at the long end, we may find that the long ends of the curves do not move together when there is a market move in interest rates. This means that the risk manager will pick up *basis risk*. Remember that basis risk refers to the situation where an instrument is valued off one curve and its hedge is valued off another curve. When the two curves do not move together, it means the hedge may not be effective any more, and that means the portfolio is more risky. However, in this example the basis risk that the risk manager will pick up is actually model risk, because the two curves are the same at the long end and theoretically should move together. Basis risk and model risk are discussed in detail in Chapter 8.

It is possible that a situation may occur where the portfolio will show a bigger profit when a specific interpolation or extrapolation technique is applied to the yield curve used to value the portfolio. It will obviously be very tempting to choose the method that will result in the biggest profit in the portfolio. However an inappropriate technique, even when applied consistently, may still have some undesired after-effects.

CHAPTER 5

Yield Curves in Practice

In this chapter we explore how to derive yield curves from the various types of instruments available in the market.

In Section 5.1 we see how to fit the yield curve models discussed in Chapter 3 to coupon-paying bonds. There are various important points in this process that have to be considered. These include how the optimization routine should be set up; choosing initial parameter estimates for the yield curve function; choosing the appropriate yield curve function; which bonds to include in the process; and how to test whether the fitted yield curve model is adequate.

In Section 5.2 we derive a curve from forward rate agreements (FRAs). FRAs are usually used to derive the midsection of a swap-type yield curve. The difference between FRAs and futures is discussed in Section 5.3. We also consider the various issues that complicate the model when deriving a curve from futures contracts.

In Section 5.4 we consider a way in which to derive a term structure for interest rates that have no quoted term structure, for instance the bank prime rate. We use the concept of an averaging swap, and apply a technique similar to the bootstrap technique discussed in Chapter 2.

The idea with this chapter is to give the reader practical ideas on how to derive yield curves from the various types of instruments, and outline the issues that should be taken into account. After reading this chapter it should be clear that the most important part in deriving a yield curve is to understand how an instrument is valued. By decomposing the instrument into its relevant cash flows it should be reasonably simple to put together a yield curve model.

5.1 BOND CURVES

In this section we discuss the various issues relating to deriving yield curves from bonds using the functions discussed in Chapter 3. We begin by considering ways in which to set up the optimization routine which is used to estimate the parameters of the yield curve function.

5.1.1 Setting up an optimization routine

Say we have a series of N bonds, and we know the price of bond i can be written as:

$$P_i = \sum_{j=1}^{n_i} C_{i,j} df_{m_{i,j}} \tag{5.1}$$

where $i = 1,\ldots,N$ and

P_i = the all-in price of bond i

n_i = number of outstanding cash flows of bond i

N = total number of bonds used to derive the curve

df_m = discount function for a term m

$m_{i,j}$ = term in years from the settlement date until cash flow $C_{i,j}$

$C_{i,j}$ = jth cash flow of bond i (the last cash flow will be the redemption payment plus the coupon payment).

The idea is to choose an appropriate mathematical function for the discount factors and then estimate the parameters, with the constraint that when each of the bond's cash flows are discounted with this function, the fitted bond price will equal the current market price.

The information on the bonds that are used to derive the curves in this

Table 5.1 Bond information that is used to derive the curves in Section 5.1.1

Bond	Yield	Coupon	Maturity date
A001	2.250	1.125	30 Jun 2005
A002	2.500	2	31 July 2006
A003	3.098	3	15 Nov 2007
A004	3.300	3	15 Feb 2008
A005	3.320	2.625	15 May 2008
A006	3.900	3.25	15 Aug 2009
A007	4.150	5.75	15 Aug 2010
A008	4.210	5	15 Feb 2011
A009	4.260	5	15 Aug 2011
A010	4.420	4.875	15 Feb 2012
A011	4.470	4.375	15 Aug 2012
A012	4.397	4	15 Nov 2012
A013	4.250	3.5	15 Feb 2020
A014	4.100	3.45	15 May 2025

section is shown in Table 5.1. We know that the settlement date of these bonds is calculated on a $t + 3$ basis and that we are working with a 30/360 daycount convention. The modified following business day rule is applied. Coupons are paid yearly on all bonds, and we work with a nominal of 1 million.

The value date is 25 August 2003, so we know the settlement date is 28 August 2003. The short-term rate used to discount the bonds from the settlement date to the value date is 3.32% SMP. All instruments are assumed to be of the same credit quality.

The first step is to determine the cash flow dates of each of the bonds. Table 5.2 shows the cash flows for bond A003. We can see how some of the dates had to be adjusted when they fall on a weekend (the first column in Table 5.2). The next step is to determine what cash flow will take place on the cash flow date. When the cash flow date is not the maturity date, the cash flow is:

$$C_{i,j} = (\text{Coupon rate of bond } i \text{ x Nominal})/100$$

and on the maturity date the cash flow is:

$$C_{i,j} = \text{Nominal} + (\text{Coupon rate of bond } i \text{ x Nominal})/100$$

The present value of each of these cash flows is then calculated by multiplying each cash flow with the appropriate discount function calculated from the yield to maturity of the bond. As is shown in Table 5.2, the actual all-in price is then the sum of these present-valued cash flows. To get the all-in price of the bond on the value date, we discount the bond back from the settlement date to the value date using the short-term rate.

Table 5.2 The cash flows of bond A003 and determining the actual all-in price

Cash flow date	Cash flow $C_{i,j}$	Term $m_{i,j}$	Present value $C_{i,j}$ x df_m
15 Nov 2002	30,000		
17 Nov 2003	30,000	0.22	29,800
15 Nov 2004	30,000	1.21	28,909
15 Nov 2005	30,000	2.21	28,041
15 Nov 2006	30,000	3.21	27,198
15 Nov 2007	1,030,000	4.21	905,748
Actual all-in price (settlement)			1,019,696
Actual all-in price (value date)			1,019,414

In this example the Cairns model will be used as the functional form for the discount function. It is important to remember that the model is a function of the term m. The Cairns function is given by:

$$df_m = \exp\left[-\beta_0 m - \sum_{i=1}^{4}\beta_i \frac{1-e^{-c_i m}}{c_i}\right] \tag{5.2}$$

where $\{\beta_0,\ldots,\beta_4,c_1,\ldots,c_4\}$ are the parameters to be estimated.

To determine the fitted price of the bond with the Cairns function we create the cash flows similarly to the method shown in Table 5.2, except that we now have a column where the discount factor is given as a function of the term (in years). This is shown in Table 5.3 in column 4. The present value of each cash flow is then calculated as the cash flow multiplied with the relevant discount factor calculated from the Cairns function given by (5.2). We obviously need to assume starting values for the Cairns model parameters, otherwise we will not get any values for the discount factors. The fitted all-in price on the settlement date is then the sum of all these present-valued cash flows. The all-in price on the value date is again calculated by discounting the all-in price on the settlement day with three days using the short-term rate.

The parameters of the discount function are now estimated by an iterative technique where we minimize the weighted squared differences (WSE) between the actual and fitted all-in prices. The WSE is defined by:

Table 5.3 The cash flows of bond A003 and determining the fitted all-in price

Cash flow date	**Cash flow** $C_{i,j}$	**Term** $m_{i,j}$	**Discount factor** df_m	**Present value**
15 Nov 2002	30,000			
17 Nov 2003	30,000	0.22	$\exp[-0.22^*\beta_0$ $-\beta_1(1-\exp(-c_1{}^*0.22))/c_1$ $-\beta_2(1-\exp(-c_2{}^*0.22))/c_2$ $-\beta_3(1-\exp(-c_3{}^*0.22))/c_3$ $-\beta_4(1-\exp(-c_4{}^*0.22))/c_4]$	$C_{i,1}$ x df_m
15 Nov 2004	30,000	1.21		
15 Nov 2005	30,000	2.21	⋮	⋮
15-Nov 2006	30,000	3.21		
15 Nov 2007	1,030,000	4.21	repeated in each column	$C_{i,5}$ x df_m
			Fitted all-in price (settlement)	$C_{i,1}$ x df_m + … + $C_{i,5}$ x df_m

$$WSE = \sum_{i=1}^{N} \frac{1}{W_i} \left(P_i - \hat{P}_i\right)^2 \tag{5.3}$$

where W_i is the weight of bond i, $\hat{P}_i$ denotes the fitted all-in price and N is the number of bonds used to derive the yield curve.

By setting these calculations up in a spreadsheet, we can see how the discount function values change as the optimization routine runs through different combinations of parameter values in an attempt to minimize the WSE. Table 5.4 shows the estimated parameters when we take $w_i = 1/N$, in other words, when all bonds carry the same weight.

Table 5.4 Estimated parameters of the Cairns function

β_0	0.03
β_1	1.06
β_2	−2.52
β_3	1.53
β_4	−0.06
c_1	0.36
c_2	0.50
c_3	0.67
c_4	1.87

Figure 5.1 shows the bond yields given in Table 5.1 with the fitted par rate curve derived from the Cairns discount function. The short-term rate is used to anchor the curve at the very short end of the curve.

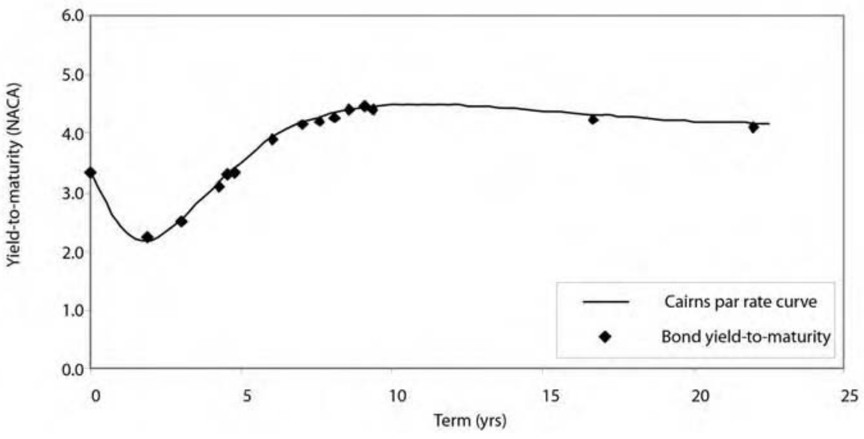


Figure 5.1 A par yield curve derived using the Cairns model and the bonds specified in Table 5.1

We can see that the par curve fits the bond yields very closely, but not perfectly. This type of curve would usually be referred to as a best-fit curve. Should we wish to have a perfect-fit curve, we will have to add more terms to the Cairns function or use another type of yield curve function. A perfect-fit curve will be volatile in that every little hump or trough will be shown. With a best-fit curve we smooth the curve to get a more average curve.

5.1.2 Initial parameter estimates

The parameters of the empirical yield curve models are estimated with iterative techniques. All iterative procedures require starting values for the parameters. If good starting values are chosen, the iterative technique will usually converge to results much faster, and may also show a global optimum value instead of wrongly converging to a local optimum.

The simplest way to choose starting values is to consider various different combinations of parameters, then choose that combination where the resulting curve has the correct shape. This can be a tedious task the first time a specific yield curve function is fitted. However, once the parameters are adequately estimated, these parameter estimates are used as the starting values for the next day, and so on.

With some of the yield curve models discussed in Chapter 3, we specify the range in which the parameters can fall. Some of the model specifications also involve specifying whether the parameters can assume negative values, and so on. For example, the Nelson and Siegel model for the discount function is specified as:

$$df_m = \exp\left[-m\left(\beta_0 + (\beta_1 + \beta_2)\frac{\tau_1}{m}\left[1 - \exp\left(-\frac{m}{\tau_1}\right)\right] - \beta_2 \exp\left[-\frac{m}{\tau_1}\right]\right)\right]$$

where β_0 and τ_1 must be positive. We know that β_0 specifies the long rate to which the forward rates converge, so we restrict this parameter to be positive and we can choose an initial parameter estimate as the level of the long-term bond yields.

5.1.3 Other estimation functions

In Section 5.1.1 the WSE function is minimized to estimate the model parameters. In this section we look an alternative function that can be minimized, which incorporates a smoothness quantity.

Waggoner (1997) considered minimizing the function:

$$\sum_{i=1}^{N} [P_i - \hat{P}_i]^2 + \lambda \int_0^{K_k} [\psi''(t)]^2 dt \tag{5.4}$$

where

P_i = the actual all-in price of bond i

$\hat{P}_i$ = the fitted all-in price of bond i

N = total number of bonds used to derive the curve

k = total number of knot points specified

λ = roughness penalty

κ = the knot points

ψ = the function to be fitted to the bonds.

In our case where we fit the discount function, the function which minimizes (5.4) over the space of all twice continuously differentiable functions will be a cubic spline with the specified knot points.

Minimizing (5.4) is a trade-off between minimizing the first term which measures the goodness of fit, and the second term that measures smoothness. The positive constant λ determines the trade-off between goodness of fit and smoothness, and is called the *roughness penalty*. For large values of λ, the flexibility of the function is the same across all regions. Waggoner (1997) argues that the function should be more flexible at the short end than at the long end, which results in his proposal of a modified smoothed spline, and means the function ψ is estimated by minimizing:

$$\sum_{i=1}^{N} [P_i - \hat{P}_i]^2 + \int_0^{K_k} \lambda(t)[\psi''(t)]^2 dt \qquad (5.5)$$

where all symbols are as defined before. He finds that by allowing the roughness penalty λ to be a function of the term to maturity of the bond, the volatility at the long end of the curve is reduced while the flexibility at the short end of the curve remains. He specified the roughness penalty as follows:

$$\lambda(t) = \begin{cases} 0.1 & ,0 \le t \le 1 \\ 100 & ,1 \le t \le 10 \\ 100{,}000 & ,t > 10 \end{cases}$$

which is a step function that corresponds with the maturities of the three sets of instruments he uses in his analysis. This function ensures that the short end of the curve will be more volatile, whereas the long end has a much higher λ and thus will be much smoother. By testing the out-of-sample performance of the fitted curves, it is possible to ascertain whether the roughness penalty specified is correct. If the out-of-sample fit is not adequate, it is an indication that another roughness penalty should be specified.

5.1.4 Perfect fit and smoothness

It has to be decided whether the bonds are to be fitted perfectly, as this will have an effect on the smoothness of the curve. When there are a lot of bonds in a certain maturity range, all with different coupons and yields to maturity, it is referred to as "noisy" data, which means that a perfect-fit curve will not be smooth in that maturity range. Should we wish to get an idea of the average level of the market interest rates, we would typically prefer a best-fit curve and smooth over the noise in the data. However, when the curve is used for valuation purposes, the practitioner would prefer that the bonds be priced perfectly off the curve. In this situation the analyst typically has two choices: either choose fewer bonds to derive the curve and thereby enforce smoothness, or have a curve that is not very smooth but fits the bonds perfectly.

In the optimization routine we would usually also add the constraint that the forward rate curve should be smooth and the forward rates should not be allowed to be negative. The smoothness constraint of the forward curve usually depends on the model; typically we would prefer a model that has no discontinuities, as the forward rate curve would then be smooth. Ensuring non-negative forward rates typically involves constraining the values the parameters can take on. This will differ from model to model. It is usually never a good idea to constrain a model too much, because then the optimization routine will struggle to find adequate parameter estimates. In this case the starting values (discussed in Section 5.1.2) are even more important.

5.1.5 Knot points

Various yield curve models that are discussed in Chapter 3 require knot points to be chosen before the model parameters can be estimated.

There are various issues to consider when determining the number of knot points, the most important being the trade-off between goodness of fit and smoothness. The greater the number of knot points, the better the fit of the discount function will be. However, if too many knot points are chosen, the discount function may conform too closely to outliers and the resulting curve will not be smooth enough. This can be seen clearly in Figure 5.2, where the cubic spline function was fitted to the bonds in Table 5.5. In this example the number of knots is chosen to be the same as the number of bonds used to derive the curve. The actual knots were chosen to correspond with the terms to maturity of each of the bonds.

Another idea is to choose the number of knots to be $\sqrt{N}$ where N is the number of bonds used to derive the yield curve. The knots are chosen such that an equal number of bonds are between each subsequent knot (McCulloch, 1971; McCulloch and Kochin, 1998). Remember that most knot points are specified in terms of terms to maturity. We shall refer to this approach to choosing knot points as the *McCulloch approach.*

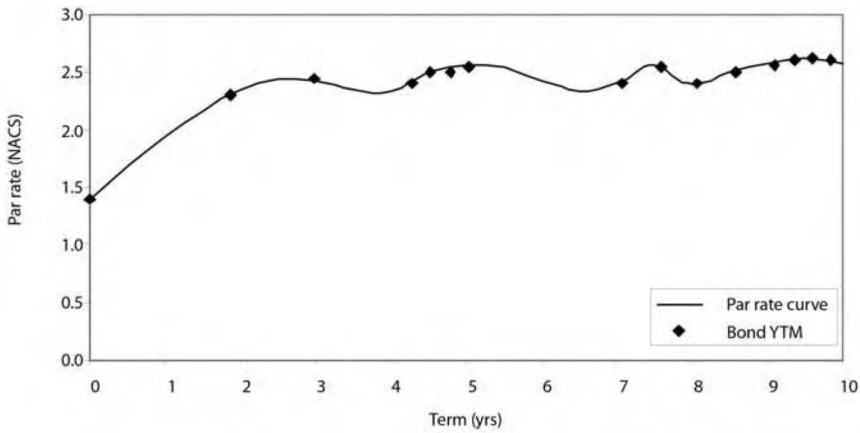


Figure 5.2 Cubic spline curve derived from the bonds in Table 5.5

Figure 5.3 shows a comparison of the perfect fit curve shown in Figure 5.2 and the curve derived by choosing knot points with the McCulloch approach. The knot points are {0; 4.2; 5; 8; 9.7}. It is clear that the curve does not fit the bonds adequately. This is mainly because the bonds are not spread evenly across maturities, but tend to form clusters.

Figure 5.4 (overleaf) shows the case where the knot points are chosen to correspond with the bond clusters. In this case the knot points are chosen as {0; 1.8; 4.2; 5; 6.9} and we can see that the resulting curve is now much smoother than the perfect fit curve, yet still fits the bonds adequately. If the knot points are chosen to correspond with the bond clusters, the curve seems to fit the bond yields more closely than when the knots are chosen

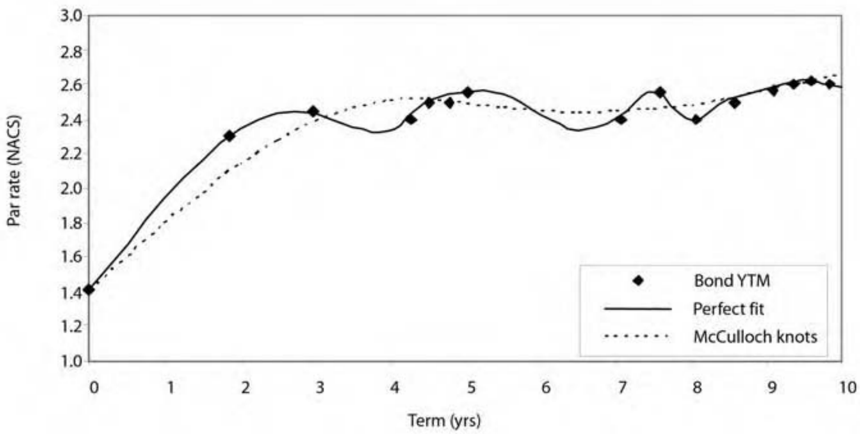


Figure 5.3 Comparison of cubic spline curves derived from the bonds in Table 5.5, but with different sets of knot points

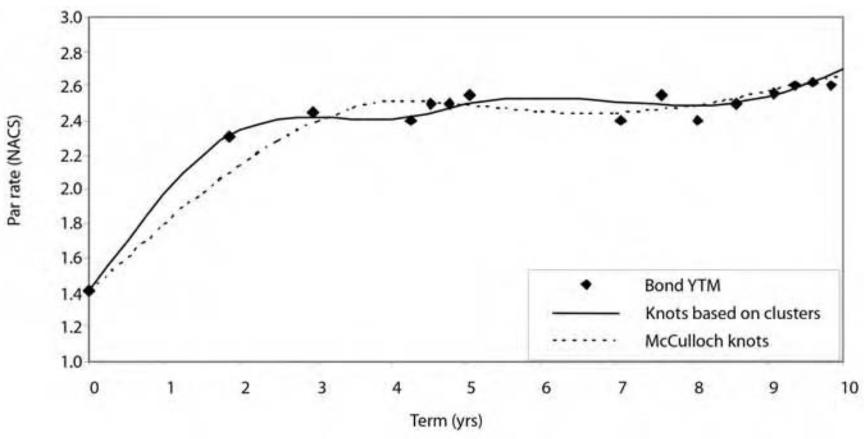


Figure 5.4 Comparison of cubic spline curves derived from the bonds in Table 5.5, but with different sets of knot points

with the McCulloch approach. It is quite interesting to see that the biggest difference between these two curves is around 20 bps. This is a very high number, especially when we consider that in each case the same set of bonds is used to derive the curve.

Consider the way in which the risk of an interest rate portfolio is determined. In practice we would add 1 bp to the yield curve and see what the effect is on the portfolio value. This gives us an idea of the sensitivity of the portfolio value to yield curve moves, and is usually referred to as the *present value of a basis point move* or PV01. One way to determine the stress loss of the portfolio is to stress the PV01s with stress factors that are calculated from yield curve moves. (Please refer to Chapter 8 for a more complete discussion.) In the example above we see that by just changing the knot points, we get a 20 bp difference in the curve. This move is not a result of market moves in the underlying bonds, but is purely caused by the way the knot points are chosen. The problem is that our risk measure will pick up these moves in the yield curve, and this results in incorrect estimates of the risk of the portfolio.

The examples discussed in this section show how sensitive spline models are to the number and location of the knot points.

Another interesting issue is when a market only has a limited number of bonds available that can be used to derive a yield curve. This may be because there are not enough liquid bonds or not enough bonds of the same credit quality. Over time the number of bonds used to derive the curve is thus kept constant, but the term to maturity of each bond decreases. A bond will then move from one interval to the next, where the interval refers to the area between any two knot points. In situations like this it is necessary to constantly update the knot points, as this may affect the parameter estimates of the yield curve model and also how well the curve fits the bonds. The effects of changing knot points on the parameter estimates are discussed in Section 5.1.8.

The bonds in Table 5.5 have settlement dates that are determined on a $t + 3$ basis. This specific market follows a 30/360 daycount convention and a modified following business day rule. Coupons are paid semi-annually on all bonds. The value date is 25 August 2003, so we know the settlement date is 28 August 2003. The short-term rate that is used to discount the bonds from the settlement date to the value date is 1.4% SMP. All these instruments are assumed to be of the same credit quality.

5.1.6 Parsimonious function

In fitting any yield curve there is a trade-off between goodness of fit, smoothness, and a parsimonious function. When deciding on the number of parameters, we have to consider the same issues as those discussed in Section 5.1.5, where we considered them in the context of knot points. In other words, it may be possible that we are over-fitting the underlying bonds when using a too highly parameterized function, so that the resulting yield curve may show troughs and humps that cannot be explained by the actual bond yields.

When the function is too highly parameterized, we may not be able to estimate consistent parameters and then we may have trouble with

Table 5.5 Bond information that is used to derive the curves in Section 5.1.5

Bond	Yield	Coupon	Maturity date
A001	2.30	1.125	30 Jun 2005
A002	2.45	1.5	31 Jul 2005
A003	2.40	3	15 Nov 2007
A004	2.50	3	15 Feb 2008
A005	2.50	2.625	15 May 2008
A006	2.55	3.25	15 Aug 2008
A007	2.40	5.75	15 Aug 2010
A008	2.55	5	15 Feb 2011
A009	2.40	5	15 Aug 2011
A010	2.50	4.875	15 Feb 2012
A011	2.56	4.375	15 Aug 2012
A012	2.60	4	15 Nov 2012
A013	2.62	3.875	15 Feb 2013
A014	2.60	3.625	15 May 2013

"catastrophic" jumps (a topic discussed in Section 5.1.8). A general rule is not to use a function with more parameters than we have instruments to which we want to fit the function.

When the function is too parsimonious, the resulting function will not be flexible enough to allow for the different shapes the yield curve can take on. It will typically produce an overly smooth curve which will not fit the bonds adequately.

5.1.7 Weights for the WSE

The weighted squared error (WSE) defined by equation (5.3) is the function we wish to minimize when estimating the parameters of the yield curve model. Obviously the bigger the weight we choose for any particular bond, the closer the yield curve function will fit that specific bond.

There are various reasons that we might want to use weights. Consider for instance the situation where we have noisy data to which we would like to fit a smooth curve. This is the situation in the example in Section 5.1.5, where the bonds do not cover the whole maturity range, but are clustered around certain maturity dates. Say we suspect some of the bonds are more liquid than others. (This would explain the volatility in the yields to maturity, even though their maturity dates are close to each other.) Then we can weight them according to some liquidity measure. An example of a liquidity measure is the amount issued of a particular bond. We would want bonds that are more liquid to have a greater weight, because the liquid bond yields more closely reflect the actual level of the market rates.

In a market where the short-term instruments are more liquid than the long-term instruments, we can use the inverse of the instrument's modified duration as a liquidity weight. This will ensure that the short-term instruments carry the higher weight. This is a good approach, because the liquidity weights are chosen objectively, which is preferable when the yield curve is used for valuation purposes.

One problem when choosing different weights for the various bonds is that it is possible to manipulate the yield curve model to get a curve that is most advantageous in the sense that when a portfolio is valued off it, the portfolio will show big potential profits. Thus, by applying weights we get a very objective curve and not one that is necessarily a true reflection of market rates.

When a yield curve is used for pricing purposes by a trader, the weights applied do not need to be objective. Traders could adjust the weights as they see fit and depending on where they think the market is most liquid or under-valued. By trying different weights they could get a feel of what rate would be most appropriate when they want to trade in any instrument. The flip side of this is that when they price a deal off one curve, but the risk manager values the deal off another more objective curve, the trader may show a loss. To prevent situations like that the trader may prefer to price a deal off the same curve that is used to value the deal. Obviously a trader will not be too worried when he/she shows a profit.

In order to be as objective as possible, most practitioners use equal weights across all bonds.

When trading with a counterparty, there usually is a legal agreement in place which specifies which curve will be used to settle the deal at maturity (for example the ISDA agreements when trading in interest rate swaps). These agreements almost force the issue of transparency, and usually refer to a yield curve that is published by a trusted outside source like an exchange. When we value a portfolio with an objective yield curve, and the agreements refer to another curve that is completely different, there may be big differences between the expected profits our portfolio shows when the deals are valued off our curve, and the actual earnings made at maturity when settling the deal off the curve specified in the agreement.

Another important reason to be objective when deriving yield curves is that the external audit process will typically pick up subjectivity; this can be perceived as manipulating the results. This may cause the auditors to write unfavourable reports to upper management, and there is also a reputation risk that has to be considered.

5.1.8 Catastrophic jumps

Estimating the parameters of empirical yield curve models usually involves the minimization or maximization of a function like the weighted least squares function discussed in Section 5.1.1.

A yield curve model with linear parameters, like the regression-type models discussed in Chapter 3, will have a single global minimum. However, most yield curve models are non-linear so that the function may have more than one local minimum (Draper and Smith, 1981: 465–6). The location of the global minimum might jump when the function is evaluated from day to day, resulting in a completely new set of parameter estimates. When such a jump occurs, the resulting effect is that the yield curve will also jump in an equally obvious way, and the curve may start to lose credibility, because the moves will not be explainable by moves in the underlying bonds (Cairns and Pritchard, 1999).

When a yield curve model requires knot points to be specified, it is necessary to ensure that the knots are chosen in such a way that a specific bond will not jump from one interval to the next, where an interval refers to the gap between two successive knot points. Consider the situation where a bond has a yield substantially different from the rest of the portfolio, and that bond has a large weight in the optimization routine. (Please refer to Section 5.1.7 for more details.) When the bond moves to a different interval there may be a significant change in the parameter estimates, in other words a "catastrophic" jump. User intervention is thus necessary at the start of each month or quarter, when the bonds and the weights that are used to derive the yield curve are reviewed, to prevent a situation like this from occurring.

Using a different number of knots from one day to the next may cause huge differences in the parameter estimates which are not warranted by actual bond moves, as was shown with an example in Section 5.1.5.

When deciding on a yield curve model, it is important to back-test the model through time to see whether the model parameters are subject to these catastrophic jumps. One way to test whether a model is subject to catastrophic jumps is to choose 100 different starting points at random for optimization, and see whether the optimization routine converges to the same minimum in each case (Cairns, 1998).

5.1.9 Selecting the bonds

When choosing the bonds that will be used to derive a yield curve, there are several issues that have to be taken into account. These include:

- The credit rating of the instruments that will be priced off the yield curve. For example when we need a risk-free curve we would typically use government bonds to derive the curve.
- We would typically only use the most liquid bonds, because then we know the yield curve is representative of market interest rates. The amount issued, turnover, or the bid/ask spread of the bond can be used as an indication of the liquidity of the bond.
- Only bonds for which prices are actively quoted in the market by a minimum number of market makers should be considered. If a bond is included and with no current price available, it may cause distortions in the yield curve.
- Depending on the market perception of their value, the prices (and thus the yields) of bonds with special features like callable, puttable, and convertible bonds will differ from those of the rest of the bonds in the market.
- The duration of a bond is defined as the weighted average time to cash flows. A very general formula for duration is (Fabozzi, 1993):

$$D_t = \frac{1}{P}\left[\frac{1C}{(1+y)^1} + \frac{2C}{(1+y)^2} + \dots + \frac{n(C+R)}{(1+y)^n}\right]$$

where

P = all-in price of the bond per 100 nominal

C = coupon of the bond
R = redemption value of the bond per 100 nominal
n = number of coupon payments of the bond
y = yield to maturity of the bond.

Depending on the shape of the yield curve, the duration of a bond may be an indication of which bond has the higher value. For example, if the yield curve is upward sloping, then (ignoring other factors like income tax) high-coupon bonds having a shorter duration than low coupon bonds will usually be preferred. This may lead to high-coupon bonds being more in demand, which indicates that they will also be priced higher than lower-coupon bonds with the same term. The implication is that these higher-coupon bonds will have lower yields than the rest of the bonds in the market.

5.1.10 Comparison of yield curve models

In this section we compare three of the yield curve models that are discussed in Chapter 3. The models are the McCulloch–Kochin quadratic natural spline; the Carriere–Gompertz model, and a restricted version of the Cairns model. The idea is that in the process of comparing the different models, we also illustrate the way the different types of functions are specified. Finally we show some examples of the parameter estimates.

The yield curve functions are fitted to the bonds listed in Table 5.6.

Table 5.6 Bond information used to derive the curves in Section 5.1.1

Bond	Yield	Coupon	Maturity date
A001	2.500	1.125	30 Jun 2005
A002	2.750	2	31 July 2006
A003	3.200	3	15 Nov 2007
A004	3.500	3	15 Feb 2008
A005	3.200	2.625	15 May 2008
A006	3.900	3.25	15 Aug 2009
A007	4.150	5.75	15 Aug 2010
A008	4.210	5	15 Feb 2011
A009	4.600	5	15 Aug 2011
A010	4.300	4.875	15 Feb 2012
A011	4.500	4.375	15 Aug 2012
A012	4.200	4	15 Nov 2012
A013	4.200	3.5	15 Feb 2020
A014	4.000	3.45	15 May 2025

The settlement date of these bonds are calculated on a $t + 3$ basis and a 30/360 daycount convention is applicable. The modified following business day rule is applied. Coupons are paid yearly on all bonds.

The value date is 25 August 2003 and the short-term rate that is used to discount the bond prices from the settlement date to the value date is 1.80% SMP. All instruments are assumed to be of the same credit quality.

McCulloch–Kochin quadratic natural spline

The first model we consider is the McCulloch–Kochin quadratic natural spline, where the basis functions are defined as follows:

$$f_j(m) = \theta_j(m) - \frac{\theta''_j(m_N)}{\theta''_{n+1}(m_N)}\,\theta_{n+1}(m) \qquad , j = 1,...,k \tag{5.6}$$

where
$\theta_1(m) = m$
$\theta_2(m) = m^2$
$\theta_j(m) = \max(0, m - \kappa_{j-2})^3$ for $j = 3,..., k + 1$
and
κ = the knot points
m_N = the term to maturity (in years) of the longest-term bond
k = number of knot points
$\theta''_j(m)$ = the second derivative of θ with respect to m.

The discount function is then defined as:

$$df_m = \exp\left[-\sum_{j=1}^{k}\alpha_j f_j(m)\right] \tag{5.7}$$

We choose five knot points with $\kappa_5 = m_N$. These knot points are all specified in terms of years. Table 5.7 shows how (5.6) is used to determine the basis functions. Please note that $\{\alpha_1; \alpha_2; \alpha_3; \alpha_4; \alpha_5\}$ in (5.7) are the parameters of the discount function that are estimated. The parameters are estimated with the approach discussed in Section 5.1.1, in other words by minimizing the weighted squared differences between the actual and fitted bond prices; we use equal weights for all bonds.

The chosen knot points and estimated parameters of the McCulloch–Kochin quadratic natural spline model are given in Table 5.8.

Carriere–Gompertz model

The second model we consider is the Carriere–Gompertz model given by:

$$df_m = \sum_{j=0}^{q} \phi_j[1 - v(m)]^j + \sum_{i=1}^{n-1} \xi_j \max(0,1 - v(m) - \kappa_i)^q \tag{5.8}$$

Table 5.7 Illustration of how to create the McCulloch–Kochin basis functions when five knot points are chosen

j	$\theta_j(m)$	$\theta_j'(m)$	$\theta_j''(m)$	$f_j(m)$
1	m	1	0	m
2	m^2	$2m$	2	$m^2 - 2\max(0,m-\kappa_1)^3 / 6\max(0,\kappa_5-\kappa_1)$
3	$\max(0,m-\kappa_1)^3$	$3\times\max(0,m-\kappa_1)^2$	$6\times\max(0,m-\kappa_1)$	$\max(0,m-\kappa_1)^3 - \max(0,\kappa_5-\kappa_1) \times \max(0,m-\kappa_2)^3 / \max(0,\kappa_5-\kappa_2)$
4	$\max(0,m-\kappa_2)^3$	$3\times\max(0,m-\kappa_2)^2$	$6\times\max(0,m-\kappa_2)$	$\max(0,m-\kappa_2)^3 - \max(0,\kappa_5-\kappa_2) \times \max(0,m-\kappa_3)^3 / \max(0,\kappa_5-\kappa_3)$
5	$\max(0,m-\kappa_3)^3$	$3\times\max(0,m-\kappa_3)^2$	$6\times\max(0,m-\kappa_3)$	$\max(0,m-\kappa_3)^3 - \max(0,\kappa_5-\kappa_3) \times \max(0,m-\kappa_4)^3 / \max(0,\kappa_5-\kappa_4)$
6	$\max(0,m-\kappa_4)^3$	$3\times\max(0,m-\kappa_4)^2$	$6\times\max(0,m-\kappa_4)$	

Table 5.8 Knot points and estimated parameters of the McCulloch–Kochin model fitted to the bonds in Table 5.6

Parameters		Knot points	
α_1	0.01799	κ_1	0.25
α_2	0.00207	κ_2	4
α_3	0.00041	κ_3	7
α_4	−0.00114	κ_4	9
α_5	0.00063	κ_5	22.0

where $\{\kappa_0;\kappa_1;...;\kappa_n\}$ denote the knot points of the spline function restricted to the interval [0;1] and $\{\phi_0;...;\phi_q;\xi_1;...;\xi_{n-1}\}$denote the parameters to be estimated for the spline function. We have that $v(m)$ is the Gompertz function given by:

$$v(m) = \exp\left[e^{-\frac{\mu}{\sigma}}\left(1-e^{\frac{m}{\sigma}}\right)\right] \tag{5.9}$$

where μ and σ are two further parameters that have to be estimated. By choosing q as 3 and the number of knot points n as 5, we fit the model to the bonds in Table 5.6. In this case an iterative process is necessary to estimate the parameters:

- Initial values are chosen for $\{\phi_1;...;\phi_3;\xi_1;...;\xi_4\}$and $\{\mu;\sigma\}$.
- Estimate only $\{\mu;\sigma\}$ by minimizing the equally weighted squared differences between the fitted and actual all-in prices of the bonds.
- Keep the estimates of $\{\mu;\sigma\}$ now fixed and estimate only $\{\phi_1;...;\phi_3;\xi_1;...;\xi_4\}$.
- Repeat this process until both sets of parameters converge.

Table 5.9 shows the specified knot points and the estimated parameters of the Carriere–Gompertz yield curve model. Please note that with this model the knot points are specified in terms of the discount function $v(m)$ and are thus restricted to [0; 1].

Table 5.9 Parameter estimates and knot points of the Carriere–Gompertz model

Parameters		Knot points	
ϕ_1	−0.43	κ_0	0
ϕ_2	−1.21	κ_1	0.1
ϕ_3	−5.78	κ_2	0.2
ξ_1	10.98	κ_3	0.3
ξ_2	7.82	κ_4	0.6
ξ_3	−17.34	κ_5	1.0
ξ_4	−31.36		
μ	−78.19		
σ	71.58		

Restricted Cairns model

The third model we consider is the restricted Cairns model. The model is considered in Section 5.1.1 and is given by (5.2). However, in this section we restrict the model in the sense that the only parameters that are estimated are $\{\beta_0;...;\beta_4\}$. We choose fixed constant values for $\{c_1;...;c_4\}$. Table 5.10 shows the estimated parameter values as well as the fixed constants for $\{c_1;...;c_4\}$.

Comparing the models

Figure 5.5 shows a comparison of the par yield curves derived from the McCulloch–Kochin, Carriere–Gompertz and restricted Cairns models. At

Table 5.10 Parameters of the restricted Cairns model

β_0	0.026
β_1	0.211
β_2	–0.392
β_3	0.203
β_4	–0.031
c_1	0.200
c_2	0.400
c_3	0.800
c_4	1.600

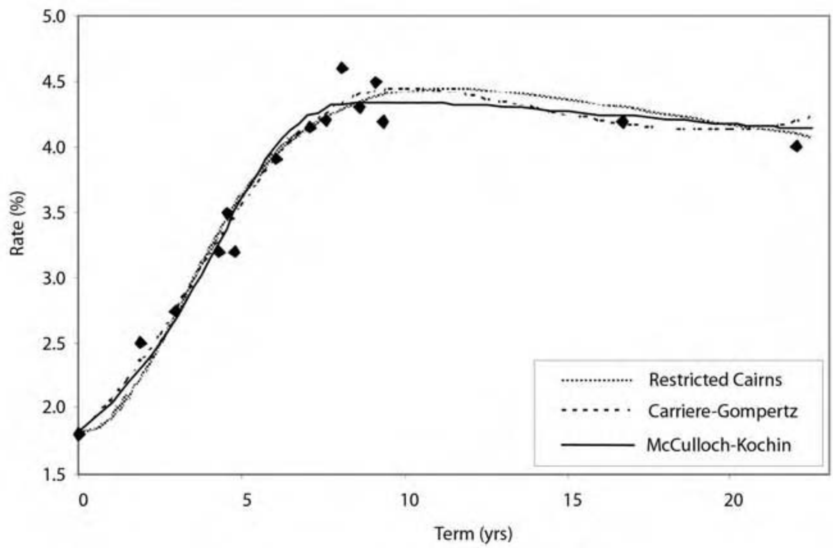


Figure 5.5 A plot of the bond yields and the par curves from each of the yield curve models fitted to the data in Table 5.6

the long end of the curve we see that the Carriere–Gompertz model does not converge to a level. This model may be too highly parameterized in this example, because it seem to allow for a lot of curvature but does not seem to fit the bond yields very closely. The restricted Cairns and McCulloch–Kochin models both show a very good fit of the bonds, even though the par curves from these two models are very different. This is an interesting situation, because it is difficult to decide which model is superior; in other words which curve more accurately reflects the current level of market interest rates.

Perhaps we can argue that the McCulloch–Kochin model in this example does not allow for enough curvature. However, a lot depends on the choice of knot points. Perhaps by adding more knot points to the McCulloch–Kochin model we would find it will showed more curvature. When we consider model complexity as one of the factors when choosing the superior model, the restricted Cairns model is definitely the superior model. With the Cairns model no knot points have to be determined and only five parameters are estimated. It is also a very simple function that clearly allows for adequate curvature.

5.1.11 Testing the final yield curve

The final step in the yield curve derivation process is to test whether the curve fits the bonds adequately and is an adequate reflection of the market interest rates. This is done by deriving a yield curve from a set of bonds; it is referred to as the *in-sample fit*. We then take a bond that is not part of the sample (but has similar characteristics) and calculate its price using the curve. This is referred to as the *out-of-sample fit*. If the out-of-sample fit is adequate, we know that our yield curve is correct.

5.2 FORWARD RATE AGREEMENT CURVE

To derive a yield curve from forward rate agreement (FRA) rates, it is important to remember that these rates are forward rates. Table 5.11 shows a typical example of the rates available in the market that can be used to

Table 5.11 Typical deposit and FRA rates quoted in the market

Short-term rate	Bid	Offer
1 month	1.04	1.14
2 month	1.12	1.22
3 month	1.23	1.33
FRA contract	**Bid**	**Offer**
1 x 4	1.5	1.53
2 x 5	1.72	1.74
3 x 6	1.892	1.922
4 x 7	2.073	2.103
5 x 8	2.26	2.28
6 x 9	2.427	2.457
7 x 10	2.579	2.609
8 x 11	2.725	2.755
9 x 12	2.88	2.9

derive a yield curve. These rates are all assumed to be simple annual rates, and we assume an actual/360 daycount convention is applicable.

The FRA rates are interpreted as follows: the 1 x 4 FRA is the three-month rate, applicable one month from the spot date; the 2 x 5 FRA is the three-month rate applicable two months from the spot date, and so on. Figure 5.6 shows a graphical depiction of this.

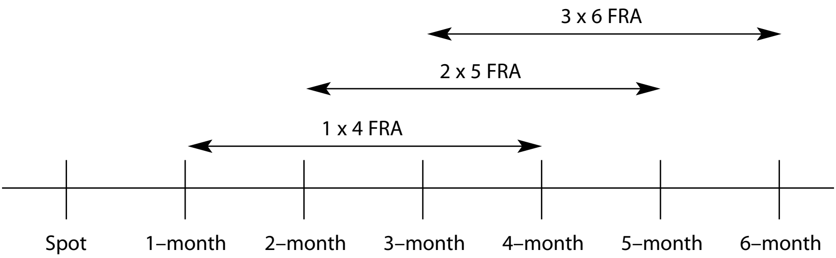


Figure 5.6 An example of the periods to which the FRA rates apply

To derive a zero curve from the FRAs we follow these steps:

- Calculate the mid rates. A *mid rate* is the average of the bid and offer rates.
- Determine the period to which all the short-term rates apply. For example when we are looking at the one-month rate, we need to determine the number of days in the one-month period.
- Convert all the short-term mid rates to discount factors. Typically we would use deposit rates, which are zero rates.
- Determine the number of days in each of the forward periods. We are working with an actual/360 daycount convention, so even though we are using three-month FRA rates, the number of days in each forward period will usually differ.
- Calculate the forward discount factors. This is the discount factor for each of the forward periods. For example, to calculate the forward discount factor for the 2 x 5 FRA, we have:

$$df_{2,5} = \frac{1}{1 + it} = \frac{1}{1 + 1.73 \times \frac{91}{360}} = 0.996$$

  where i denotes the mid FRA rate and t the period in years in the forward period.
- Now determine the number of days between the spot date and the end of the FRA period. This will form the period to which the zero rate applies that is derived from each FRA. For example, we will derive a six-month zero rate from the 3 x 6 FRA.

- From Figure 5.6 we can see that to get a discount factor for the period from spot to four months, we need to multiply the forward discount factor $df_{1,4}$ with the one-month discount factor calculated from the short-term rates. Similarly, to get a discount factor for the period from spot to five months, we need to multiply the forward discount factor $df_{2,5}$ with the two-month discount factor derived from the short-term rate. These calculations are shown in Table 5.12.

Table 5.12 Demonstrating the calculations to derive a zero curve from FRAs

Short-term rate	Mid rate (SMP)	Number of days from spot		Discount factor (DF)	
1 month	1.09	30		0.999	
2 month	1.17	62		0.998	
3 month	1.28	91		0.997	
FRA contract	**Mid rate (SMP)**	**Forward period**	**Forward DF**	**Number of days from spot**	**DF**
1 x 4	1.515	90	0.996	120	0.995
2 x 5	1.73	91	0.996	153	0.994
3 x 6	1.907	89	0.995	180	0.992
4 x 7	2.088	92	0.995	212	0.990
5 x 8	2.27	91	0.994	244	0.988
6 x 9	2.442	92	0.994	272	0.986
7 x 10	2.594	92	0.993	304	0.984
8 x 11	2.74	92	0.993	336	0.981
9 x 12	2.89	92	0.993	364	0.979

Once all the discount factors are calculated, it is easy to convert them to zero rates using the techniques described in Chapter 2. Figure 5.7 shows the final zero curve.

Usually we would extend this curve by bootstrapping swap rates and stitching it to the long end of the curve. The bootstrapping technique is discussed in Chapter 2.

5.3 FUTURES CURVE

Interest rate futures can be used as an alternative to FRAs to derive the middle area of the swap curve. The differences between FRAs and futures are:

- FRAs have a fixed time horizon to settlement and the contracts settle at maturity.

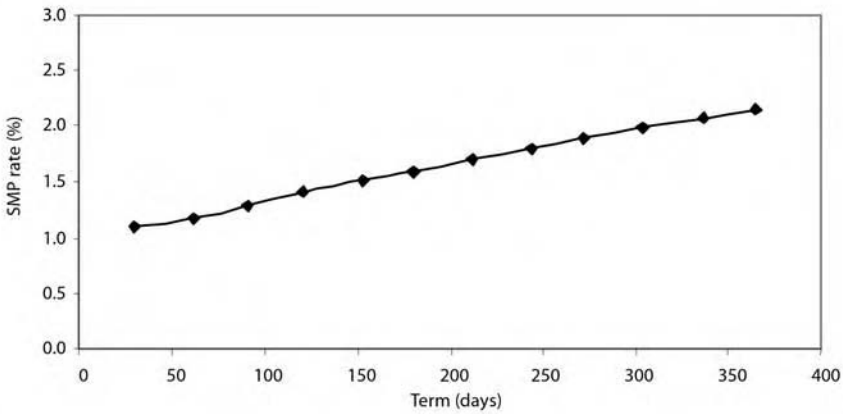


Figure 5.7 The zero curve derived from the instruments in Table 5.11

- Futures contracts have a fixed settlement date but are marked to market daily.

In practice the FRAs for most currencies suffer from lack of liquidity, whereas futures contracts are exchange traded, which make them more liquid and transparent. An interesting discussion on the relationship between futures and forwards can be found in Das (2004: 307–15).

5.3.1 Convexity adjustment

The *convexity adjustment* is the adjustment of the theoretical forward interest rate that is necessary to allow for the impact of funding or reinvesting the cash flows resulting from the futures contract being marked to market periodically. The convexity adjustment is positively correlated to the futures contract maturity, and gradually increases with term to maturity.

Kirikos and Novak (1997) discuss a way in which to adjust Eurodollar futures prices with the convexity bias. At maturity the value of the futures contract is:

$$F_t = 100(1 - f_{t,T}) \tag{5.10}$$

where

F_t = price of the futures contract at maturity t
$f_{t,T}$ = Libor observed at time t for a deposit maturing at time T
t = maturity date of the futures contract
T = maturity date of the underlying Libor deposit.

where $f_{t,T}$ is calculated as:

$$f_{t,T} = \frac{DC}{T-t}\left(\frac{1}{df_{t,T}} - 1\right) \tag{5.11}$$

and DC denotes the number of days assumed in the year as specified by the daycount convention and $df_{t,T}$ the discount factor applicable to the period from t to T. It is important to know the relationship between forward prices and futures prices to be able to derive a yield curve. The convexity bias is estimated as

$$c_{t,T} = (1 - e^{-Z})\left(100 - F_{quoted} + 100\frac{DC}{T-t}\right) \tag{5.12}$$

where

$$Z = \Lambda + \Phi$$

$$\Lambda = \sigma^2\left(\frac{1 - e^{-2a(t-t_0)/DC}}{2a}\right)\left(\frac{1 - e^{-a(T-t)/DC}}{a}\right)^2$$

$$\Phi = \frac{\sigma^2}{2a^3}(1 - e^{-a(T-t)/DC})(1 - e^{-a(t-t_0)/DC})^2$$

with $c_{t,T}$ denoting the convexity bias for the futures contract which has maturity t and whose underlying Libor deposit has maturity T, t_0 the value date, F_{quoted} the current quoted market price of the futures contract, σ the annual volatility and a the mean reversion factor. The quoted futures price is then adjusted as follows:

$$F_{adjusted} = F_{quoted} + c_{t,T} \tag{5.13}$$

so that the implied forward rate can be derived from the adjusted futures price $F_{adjusted}$, as follows:

$$r_{t,T} = 100 - F_{adjusted} \tag{5.14}$$

where $r_{t,T}$ denotes the simple implied forward rate between t and T. Please refer to Kirikos and Novak (1997) for a detailed discussion.

Another way to estimate the convexity adjustment is given by Hull (1997), and is specified as follows:

$$c_{t,T} = B(t,T)\frac{DC}{T-t}[B(t,T)(1 - e^{-2a(t-t_0)/DC}) + 2aB(t_0,t)^2]\frac{\sigma^2}{4a} \tag{5.15}$$

where

$c_{t,T}$ = convexity adjustment for the futures rate applicable to period t to T
t_0 = value date

t = maturity date of the futures contract
T = maturity date of the underlying Libor deposit
a = mean reversion
σ = annual volatility
DC = the assumed number of days in the year as prescribed by the market daycount convention
$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)/DC})$
$B(t_0,t) = \frac{1}{a}(1 - e^{-a(t-t_0)/DC})$

When we assume that the mean reversion factor is zero, then the convexity adjustment can be estimated by:

$$c_{t,T} = \tfrac{1}{2}\,\sigma^2 \times t \times T \qquad (5.16)$$

with all symbols as defined before. In this case the implied forward rate is calculated from the futures rate as:

$$r_{t,T} = r_{futures} - c_{t,T} \qquad (5.17)$$

where $r_{t,T}$ is the implied forward rate and $r_{futures}$ the rate derived from the futures price. Both these rates are assumed to be continuously compounded.

In both the approaches discussed above to estimate the convexity adjustment, we need estimates of the volatility and the mean reversion rate. Various ways in which to estimate these quantities are discussed in Sections 5.3.2 and 5.3.3 respectively.

Please refer to Piterbarg and Renedo (2004) for an interesting approach where a volatility smile is used to determine the convexity adjustment.

5.3.2 Interest rate volatility

There are several methods for estimating the volatility of the short-term interest rate. The most basic methods that are usually employed in practice are discussed in this section.

Some issues that have to be addressed when estimating historical volatility are:

- which data series to use: do we use closing prices, bid/ask prices, opening prices, high or low prices?
- the frequency of the data points: whether to use daily, weekly or monthly data
- the number of observations to use
- whether the assumption that we can use past observations to estimate future volatility is correct.

The most common approach is to use the closing prices, because then we capture the effect of a full-day move. We assume that all relevant information that became available in the market through the course of a day is captured in the closing price. Should close to open prices be used we measure overnight volatility, and that is an indication of the reaction to information released overnight. Please refer to Das (2004: 448–84) for a complete discussion.

Hull (1997) describes estimating volatility from historical data, which basically entails calculating the standard deviation of the daily moves in the interest rate and then scaling it to get an annualized volatility with the $\sqrt{t}$ rule where t denotes the number of business days. According to this rule we can scale an one-day volatility by $\sqrt{252}$ to get an annual volatility. This obviously assumes 252 business days in the year. However, we know that theoretically this rule only applies when we have a series of data points that is independent and from the same distribution (Diebold *et al.*, 1997). Typically that is not the case with financial time series, but in practice this fact is usually ignored and the rule is applied anyway.

A class of models that is very well documented is the GARCH type. An ARMA(p,q) time series model is fitted to a return series r_t so we have a model of the following form:

$$r_t - \sum_{i=1}^{p} \phi_i r_{t-i} = \varepsilon_t - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

where $\{\varepsilon_t\}$ is a white noise process and ϕs and θs the parameters to be estimated (Wei, 1990). Typically it is assumed that $\{\varepsilon_t\}$ has a constant variance of σ^2. The GARCH(P,Q) model is an extension of this model in that it is assumed that the variance is time-dependent and that the variance can be modeled by a process of the following form:

$$\sigma_t^2 - \sum_{i=1}^{P} \lambda_i \sigma_{t-i}^2 = \sum_{j=1}^{Q} \rho_j \varepsilon_{t-j}^2 \qquad (5.18)$$

where λs and ρs are the parameters to be estimated (Engle, 1982; Bollerslev, 1986, 1987). ARMA models with GARCH innovations offer a parsimonious and flexible description of the conditional mean and variance dynamics in a time series (Diebold and Lopez, 1995). GARCH models capture the *volatility clustering* that is usually evident in a financial return series. Volatility clustering refers to the situation where large returns in the market tend to be followed by large returns, of either sign.

Figure 5.8 shows the return series of the yields of the South African R150 bond. This is a government bond and the data is available daily from the Bond Exchange of South Africa (or from the website www.bondex.co.za). We can clearly see the clusters of periods where the market shows more volatility.

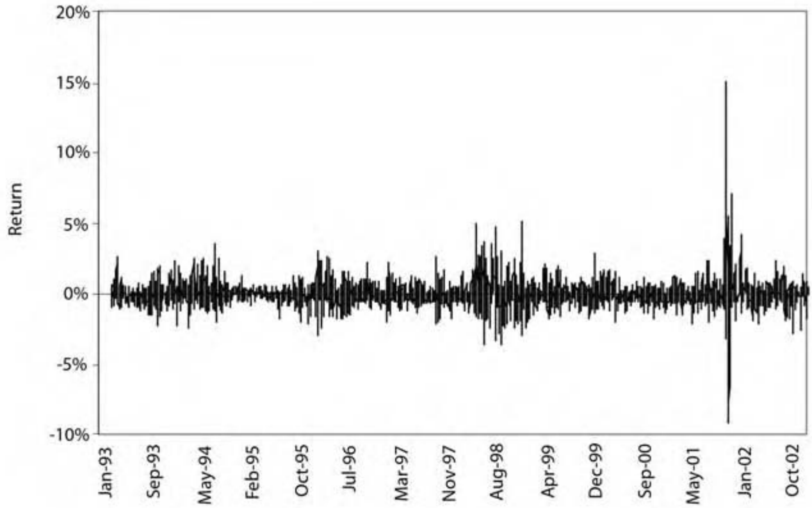


Figure 5.8 The daily return series of the South African R150 bond yield

The problem with GARCH-type models is that it is quite a process to estimate the parameters and to determine which model fits the data adequately. However, there are various software packages available that can be used to this end, for instance MATLAB (please look at www.mathworks.com for more information).

Diebold and Lopez (1995) discuss examples where GARCH models perform well in an in-sample test, but do not show adequate out-of-sample performance. Contrary to this study Anderson and Bollerslev (1997) show that GARCH models can provide good out-of-sample performance should the model be specified correctly.

A simplified version of (5.18) is known as the exponentially weighted-moving average model, where the volatility is estimated from the return series as follows (Mina and Xiao, 2001):

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1 + \lambda)r_t^2 \tag{5.19}$$

where λ is known as the decay factor. Typically when calculating a one-day volatility we use a λ of 0.94. The volatility estimate changes every day as new market information is taken into account, and the model allows for the stochastic nature of volatility.

The problem with the GARCH-type models is that when there are big moves in the short-term rate, the volatility will show a big jump. When a convexity adjustment is calculated from this volatility, the convexity adjustment may also show a big increase across all tenors. This increase may not necessarily be explainable by actual market moves.

Finally it is also possible to calculate an implied volatility from option prices. The problem is that the relevant options may not be readily available in the market.

5.3.3 Mean reversion

Ron (2000) discusses a way in which historical data can be used to estimate a mean reversion rate. He considers the Vasicek (1977) process of the short-term rate (please refer to Chapter 3 for a more detailed discussion on this process) which is given by :

$$dr = \alpha(\gamma - r)dt + \sigma dZ$$

where

r = the risk-free money-market rate
γ = the expected long-term level of the money-market rate
α = the mean reversion rate
σ = the volatility of the money-market rate
dZ = a normally distributed stochastic term.

He considers a discretized version of this process of the following form:

$$r_t - r_{t-1} = \beta_0 + \beta_1 r_{t-1} + e_t \tag{5.20}$$

where the parameters β_0 and β_1 are estimated in a regression analysis. The mean reversion factor is then assumed to be equal to $-\beta_1$.

Another way to set up the regression model is to note that the convexity formulae are based on the Hull–White model of the term structure of interest rates. A discrete form of this continuous model is:

$$r_t - r_{t-1} = a(\overline{r} - r_{t-1}) + e_t \tag{5.21}$$

where r_t is the interest rate at time t; $\overline{r}$ the long-run average of the short-term rate, and a the mean reversion rate.

According to Ron (2000), a typical range of values for the mean reversion rate is 0.001 when there is very little mean reversion, to 0.1 which indicates a stronger mean reversion rate. He also states that to simplify the model we could assume a constant value for the mean reversion rate. Alternatively it is possible to use the values published by vendors like Bloomberg.

5.3.4 Example of a futures curve

In this section we consider an example of how to derive a zero curve from futures prices. Table 5.13 shows the futures prices that we have available

Table 5.13 Example of futures prices

Futures maturity date *t*	**Maturity date of the underlying Libor deposit *T***	**Eurodollar futures closing price F_{quoted}**
17 Mar 2004	16 Jun 2004	98.84
16 Jun 2004	15 Sep 2004	98.72
15 Sep 2004	15 Dec 2004	98.465
15 Dec 2004	16 Mar 2005	98.1
16 Mar 2005	15 Jun 2005	97.71
15 Jun 2005	21 Sep 2005	97.315
21 Sep 2005	21 Dec 2005	96.97
21 Dec 2005	15 Mar 2006	96.675
15 Mar 2006	21 Jun 2006	96.46
21 Jun 2006	20 Sep 2006	96.245
20 Sep 2006	20 Dec 2006	96.05
20 Dec 2006	21 Mar 2007	95.85
21 Mar 2007	20 Jun 2007	95.71

for value date 15 January 2004. We also show the futures maturity dates as well as the maturity date of the underlying Libor deposits. In practice we have a problem in that there are gaps between contract maturity dates. For example the maturity of the June futures contract does not necessarily coincide with the maturity date of the underlying deposit of the March contract. To be able to derive a yield curve, we usually make the assumption that there are no gaps and we adjust the futures contract maturity dates to match. For instance, in Table 5.13 the March contract has a futures maturity date of 17 March and the June contract has a futures maturity date of 16 June. We now assume that the underlying Libor deposit of the March contract also expires on 16 June so that it coincides with the June futures maturity date.

We shall estimate the convexity adjustment with the method proposed by Kirikos and Novak (1997) which is discussed in Section 5.3.1. We assume a mean reversion rate of 3%, a volatility of 2% and an actual/360 daycount convention. The calculations are summarized in Table 5.14 (overleaf).

It is important to note that the implied forward rates that we obtain by using (5.14) are simple rates. The process from this point onwards to derive a zero curve is similar to the approach described in Section 5.2 where a zero curve is derived from FRAs. Table 5.15 (also overleaf) shows the calculations to convert the forward rates to zero rates. A short-term rate is necessary. We assume a simple rate of 1.2%, and this rate is applicable to the period from the value date 15 January 2004 to 17 March 2004. This assumption makes the calculations easier.

Table 5.14 Illustrating the estimation of the convexity adjustment with the approach suggested by Kirikos and Novak (1997)

Futures maturity date t	Λ	Φ	Z	Convexity adjustment $c_{t,T}$	Adjusted futures price $F_{adjusted}$	Implied forward rate $r_{t,T}$
17 Mar 2004	0.000004	0.000001	0.000006	0.0023	98.8423	1.1577
16 Jun 2004	0.000011	0.000009	0.000020	0.0078	98.7278	1.2722
15 Sep 2004	0.000017	0.000023	0.000040	0.0157	98.4807	1.5193
15 Dec 2004	0.000023	0.000042	0.000065	0.0260	98.1260	1.8740
16 Mar 2005	0.000029	0.000068	0.000097	0.0386	97.7486	2.2514
15 Jun 2005	0.000040	0.000107	0.000148	0.0546	97.3696	2.6304
21 Sep 2005	0.000041	0.000140	0.000181	0.0721	97.0421	2.9579
21 Dec 2005	0.000040	0.000169	0.000209	0.0901	96.7651	3.2349
15 Mar 2006	0.000060	0.000245	0.000305	0.1131	96.5731	3.4269
21 Jun 2006	0.000058	0.000285	0.000343	0.1369	96.3819	3.6181
20 Sep 2006	0.000064	0.000343	0.000407	0.1626	96.2126	3.7874
20 Dec 2006	0.000069	0.000407	0.000476	0.1904	96.0404	3.9596
21 Mar 2007	0.000074	0.000476	0.000550	0.2200	95.9300	4.0700

Table 5.15 Converting the forward rates implied from the futures prices to zero rates

t	T	Forward rate $r_{t,T}$ (SMP)	Period between t and T (yrs)	$df_{t,T}$	$df_{t0,T}$	Zero rate (SMP)
15 Jan 2004	17 Mar 2004	1.2	0.172	0.998	0.998	1.20
17 Mar 2004	16 Jun 2004	1.1577	0.253	0.997	0.995	1.18
16 Jun 2004	15 Sep 2004	1.2722	0.253	0.997	0.992	1.21
15 Sep 2004	15 Dec 2004	1.5193	0.253	0.996	0.988	1.30
15 Dec 2004	16 Mar 2005	1.8740	0.253	0.995	0.983	1.43
16 Mar 2005	15 Jun 2005	2.2514	0.253	0.994	0.978	1.58
15 Jun 2005	21 Sep 2005	2.6304	0.272	0.993	0.971	1.76
21 Sep 2005	21 Dec 2005	2.9579	0.253	0.993	0.964	1.92
21 Dec 2005	15 Mar 2006	3.2349	0.233	0.993	0.956	2.08
15 Mar 2006	21 Jun 2006	3.4269	0.272	0.991	0.948	2.24
21 Jun 2006	20 Sep 2006	3.6181	0.253	0.991	0.939	2.39
20 Sep 2006	20 Dec 2006	3.7874	0.253	0.991	0.930	2.53
20 Dec 2006	21 Mar 2007	3.9596	0.253	0.990	0.921	2.66
21 Mar 2007	20 Jun 2007	4.0700	0.253	0.990	0.912	2.79

In Table 5.15 we define $df_{t,T}$ as the discount factor applicable to the period from t to T. This means that to get the discount factor for the period from 15 January 2004 to 17 March 2004, we have:

$$df_{15\text{ Jan }04,\ 17\text{ Mar }04} = \frac{1}{1 + 1.2 \times \frac{(17\text{ Mar }04 - 15\text{ Jan }04)}{360}} = 0.998$$

or to calculate the discount factor for the period from 17 March 2004 to 16 June 2004, we have:

$$df_{17\text{ Mar }04,16\text{ June }04} = \frac{1}{1 + 1.1577 \times \frac{(16\text{ June }04 - 17\text{ Mar }04\ )}{360}} = 0.997$$

and so on. This is how we get to column 5 in Table 5.15. These discount factors are forward discount factors. The period to which these discount factors apply are graphically depicted in Figure 5.9.

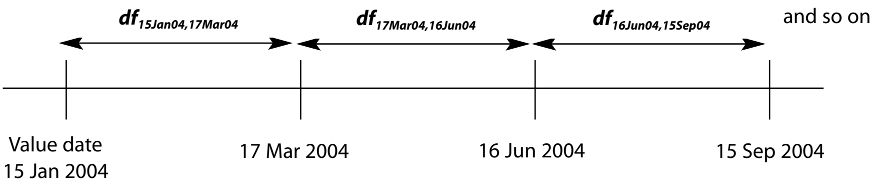


Figure 5.9 The periods to which the forward discount factors in Table 5.15 apply

The next step is get a discount factor in each case applicable from the value date t_0 out to T. It is clear from Figure 5.9 that we need to do a recursive calculation. The discount factor for the period from the value date up to 16 June 2004 is calculated as:

$$df_{15\text{ Jan }04,16\text{ June }04} = df_{15\text{ Jan }04,17\text{ Mar }04} \times df_{17\text{ Mar }04,16\text{ June }04} \tag{5.22}$$

The next discount factor for the period from the value date up to 15 September 2004 is calculated using (5.22) as follows:

$$df_{15\text{ Jan }04,15\text{ Sep }04} = df_{15\text{ Jan }04,16\text{ June }04} \times df_{16\text{ June }04,15\text{ Sep }04}$$

and so on. This is column 6 in Table 5.15. From these discount factors the zero rates are derived as always using the relationships discussed in Chapter 2. The zero rates in Table 5.15 are simple rates.

Figure 5.10 shows a plot of the forward and zero rate curves derived

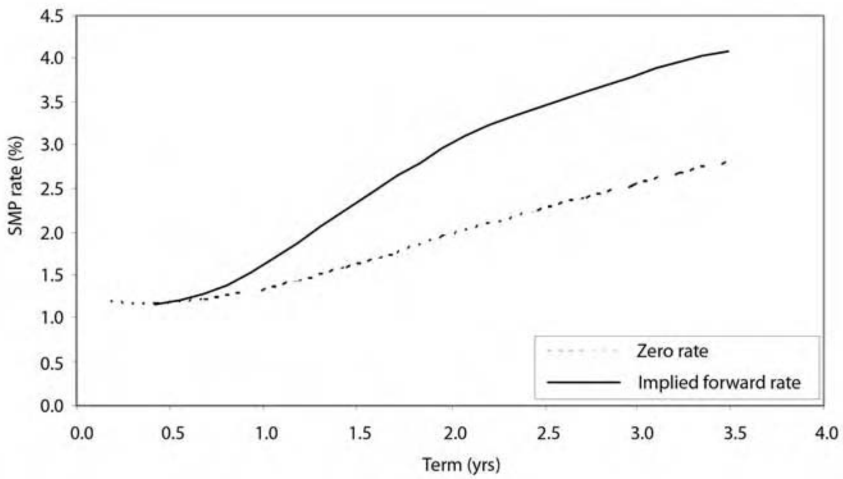


Figure 5.10 The implied forward and zero rates curves derived from the futures prices in Table 5.13

from the futures prices. It is interesting to see how much these two curves differ. The resulting zero curve is however very smooth.

Figure 5.11 shows a comparison of zero curves derived from the futures in Table 5.15 by just assuming different volatility estimates. It is interesting to see that the volatility estimate has the biggest effect at the long end of the curves. We know the convexity adjustment is an increasing function of the tenor of the contract; if we assume higher volatility the convexity adjustment increases at a higher rate. The biggest difference between the zero curves is 27 bps, which is quite significant.

This example highlights one of the problems when deriving a zero curve

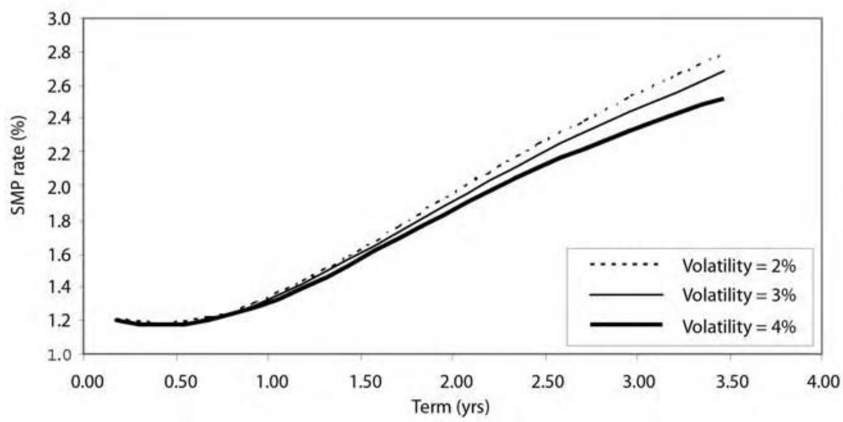


Figure 5.11 The zero rate curves derived from the futures prices in Table 5.13 assuming different volatility when deriving the convexity bias

from futures. We see how dependent the resulting zero curve is on the volatility estimate. The volatility estimates may show big jumps from day to day, for instance when the exponential weighted model given by (5.19) is used. These jumps lead to jumps in the convexity adjustment, which in turn lead to big moves in the zero curves. The changes in the zero curves may not necessarily be explainable by moves in the market, but may purely be an artefact of an incorrectly specified model. This is an example of *model risk*, and will be discussed in more detail in Chapter 8.

5.3.5 General comments on futures curves

Futures are highly liquid exchange-traded contracts, and this makes them attractive to use when deriving yield curves. The disadvantage is that we need to estimate a convexity adjustment, which in turn means we need to estimate volatility and a mean reversion factor. This increases the complexity of the model, and also increases the possibility for making mistakes: in other words increasing model risk.

What is interesting is that in practice the convexity adjustment is observable, in that we can compare zero rates derived from FRAs and interest rate swaps with rates derived from futures contracts. The differences between these two sets of rates are the observed convexity adjustment.

Consider the example of a dealer that trades in interest rate swaps. The official yield curve that is used to value his/her portfolio is derived from interest rate futures. Say the estimated convexity adjustment is used to adjust the futures and then derive futures curve. If the estimated convexity adjustment does not correspond with the observed convexity adjustment it may give the dealer an opportunity to arbitrage the official yield curve.

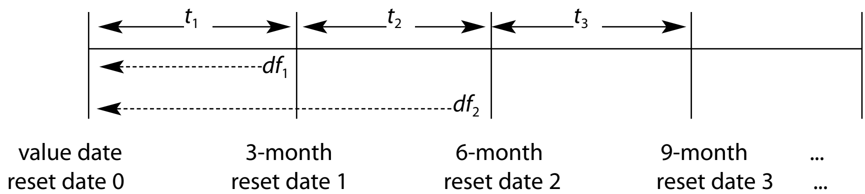
This leads to the interesting question of whether a futures curve should be used to derive a yield curve. In order to prevent the situation sketched above, we may wish to take the market observed convexity as our "estimated" convexity adjustment. However, that will lead to the same zero curve as when FRAs are used directly to derive the zero curve. Perhaps the answer lies in which instruments are valued with the final yield curve; or perhaps the liquidity of the FRAs. FRAs are assumed to be less liquid than futures, which means that there is a further liquidity premium which can explain the difference between observed and estimated convexity adjustments. Liquidity premiums and ways to estimate them are discussed in Chapter 8.

5.4 AVERAGING SWAP CURVE

Basis swaps between different money market or short-term indexes entail floating-to-floating interest rate swaps. Please refer to Das (2004, 2730: 77) for a comprehensive discussion on the various types of basis swaps. An example is where the investor enters a deal to swap the bank prime rate adjusted with a margin, with the three-month Libor rate. With an averaging swap we have predefined averaging periods where the index rate, in this example the bank prime rate, is sourced every day. At the end of the averaging period, the average prime rate is calculated and then the average rate, adjusted with the margin, is used to determine one leg of the swap. The problem is how to value this leg of the swap, because there is no term structure of prime rates available. In this section we consider a way in which to derive a term structure of average prime rates which can be used to value this type of deals.

5.4.1 Deriving the formulae

Consider a basis swap that pays at three-monthly intervals:



We define the following variables:

P_j = the average prime rate during the period t_j (these are simple annual forward rates)

t_j = the period in years between two reset dates, in other words t_j =[reset date(j) – reset date(j–1)]/365 assuming an actual/365 daycount convention

df_j = discount factor calculated off the zero curve derived from swap rates for the period from the value date to reset date (j)

M_j = margin below the specific prime rate payable (margin below simple annual rates).

To derive the average prime rates we use arguments similar to those used in Chapter 2 where the bootstrap formulae are derived. The discount factors are calculated from the zero rates for which (average prime – margin) is swapped.

We know that average prime minus a margin is paid. To get to the average prime forward rates, we get a set of equations and solve for the average forward prime rate as follows:

Three-month rate: $1 = [1 + (P_1 - M_1)t_1]df_1 \quad \therefore \; P_1 = M_1 + \left[\dfrac{1 - df_1}{t_1 df_1}\right]$

Six-month rate: $1 = (P_1 - M_2)t_1 df_1 + [1 + (P_2 - M_2)t_2]df_2$

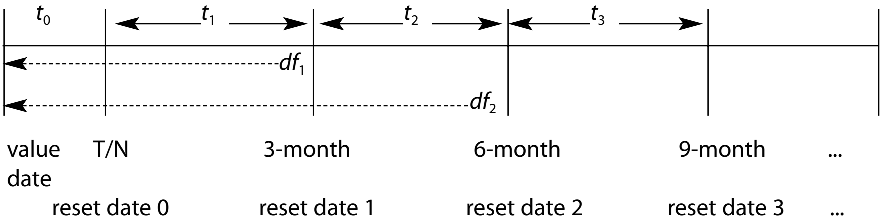
$$\therefore \; P_2 = \left[\frac{1 - df_2 \; - P_1 t_1 df_1 + \sum_{j=1}^{2} M_2 t_j df_j}{t_2 df_2}\right]$$

and so on, so that in general we have that

$$P_n = \left[\frac{1 - df_n \; - \sum_{j=1}^{n-1} P_j t_j df_j + \sum_{j=1}^{n} M_n t_j df_j}{t_n df_n}\right] \tag{5.23}$$

It is important that P_i, $i = 1,\ldots,n$ denotes average forward prime rates. This means we still need to convert these rates to average zero prime rates.

However, using this approach the current prime rate is not taken into account. We will assume that the current prime rate P_0 is the average rate from today to the next business day. To take the current prime rate into account, consider the following figure:



The variables are exactly as defined before, except for t_1 which is now the period from the first business day (after the value date) out to the first swap date. To get to the average forward prime rates and taking the current prime rate P_0 into account, we get a set of equations as follows:

Three-month rate: $1 = [1 + (P_0 - M_1)t_0 + (P_1 - M_1)t_1]df_1$

$$\therefore \; P_1 = M_1 + \left[\frac{1 - df_1 - \; (P_0 - M_1)t_0 df_1}{t_1 df_1}\right]$$

Six-month rate:

$$1 = [(P_0 - M_2)t_0 + (P_1 - M_2)t_1]df_1 + [1 + (P_2 - M_2)t_2]df_2$$

$$\therefore \; P_2 = \left[\frac{1 - df_2 \; - (P_0 - M_2)t_0 df_1 \; - P_1 t_1 df_1 + \sum_{j=1}^{2} M_2 t_j df_j}{t_2 df_2}\right]$$

and in general we have that

$$P_n = \left[\frac{1 - df_n - (P_0 - M_n)t_0 df_1 - \sum_{j=1}^{n-1} P_j t_j df_j + \sum_{j=1}^{n} M_n t_j df_j}{t_n df_n}\right] \tag{5.24}$$

Once we have a set of average forward prime rates we need to derive the average zero rate curve. This is done by defining

$$T_j = \frac{reset\ date(j) - \text{value date}}{365}$$

and PZ_j as the zero rate compounded m times per annum with a tenor of T_j years. The zero rates can then be extracted from the forward rates by solving iteratively for PZ_j in the following way:

$$\left(1 + \frac{PZ_0}{m}\right)^{mT_0} = (1 + P_0 t_0)$$

$$\left(1 + \frac{PZ_1}{m}\right)^{mT_1} = (1 + P_1 t_1)\left(1 + \frac{PZ_0}{m}\right)^{mT_0}$$

$$\left(1 + \frac{PZ_2}{m}\right)^{mT_2} = (1 + P_2 t_2)\left(1 + \frac{PZ_1}{m}\right)^{mT_1}$$

and so on, so that in general the zero rates are calculated with:

$$PZ_n = m \times \left\{\left[\left(1 + \frac{PZ_{n-1}}{m}\right)^{mT_{n-1}} (1 + P_n t_n)\right]^{\frac{1}{mT_n}} - 1\right\} \tag{5.25}$$

with all symbols as defined before. Please remember that PZ_j denotes the average zero prime rate, which means that this curve can now be used to value a prime/Libor averaging basis swap.

The formulae derived in this section are illustrated in the example discussed in Section 5.4.2.

5.4.2 Example of an average prime zero curve

Consider a country where the current bank prime rate is 14.25% SMP. Table 5.16 shows the margins below prime at which the basis swaps are quoted. These basis swaps have quarterly resets.

We take the term structure of the basis swap spreads into account by interpolating a spread at each tenor we need. Also, we assume an actual/365 daycount convention and a regular following business day rule. The term structure for the prime rate is derived following these steps:

Table 5.16 Margins quoted for prime/money market basis swaps

Tenor	Margin below prime
1 year	3.78
2 years	3.85
3 years	3.90
4 years	3.95
5 years	3.92

- Determine the tenor points for which rates have to be derived. All dates are determined by adjusting them with the following business day rule.
- Calculate the period in years from the value date up to the date at each tenor and denote this by T_j. For instance, in Table 5.17 T_2 = (20 June 2005 – 20 December 2004)/365 = 0.5 years.
- Calculate the term in years between two adjacent tenor points, in other words, $t_j = T_j - T_{j-1}$.
- Calculate discount factors df_j off the zero curve that is assumed appropriate to discount the swap leg. In this case the discount factors are calculated off the swap curve derived from market quotes corresponding with the other leg of this basis swap. In this example the discount factors are assumed to be known. They are given in Table 5.17 (overleaf).
- Interpolate the margins in Table 5.16 to get a margin at every tenor. In this example linear interpolation is applied. To extrapolate for tenors less than one year, we just keep the one-year value constant.
- We can now calculate the average simple forward prime rates with (5.24). For example to get P_1 we have:

$$P_1 = M_1 \times \left[\frac{1 - df_1 - (P_0 - M_1)t_0 df_1}{t_1 df_1}\right]$$

$$= 3.78 + 100 \times \left[\frac{1 - 0.977 - (14.25 - 3.78)/100 \times 0.003 \times 0.977}{0.25 \times 0.977}\right]$$

$$= 13.40$$

Remember that the values in Table 5.17 are rounded.

- We can now calculate the average prime zero rates using (5.25). In this example quarterly compounded zero rates are calculated. For example, PZ_1 in Table 5.17 is calculated as follows:

$$PZ_1 = 400 \times \left\{\left[\left(1 + \frac{14.5}{400}\right)^{4 \times 0.003}\left(1 + \frac{13.4}{100} \times 0.25\right)\right]^{\frac{1}{4 \times 0.25}} - 1\right\}$$

$$= 13.41$$

Table 5.17 Deriving a term structure for the bank prime rate

Value date	20 Dec 2004						
Dates	***j***	**Term from value date** T_j	**Term between tenors** t_j	df_j	**Margin below prime** M_j	**Average forward prime rate** P_j	**Average zero prime rate** PZ_j
20 Dec 2004		0.0		1.000	3.78	14.25	14.50
21 Dec 2004	0	0.0	0.003	1.000	3.78	14.25	14.50
21 Mar 2005	1	0.2	0.25	0.977	3.78	13.40	13.41
20 Jun 2005	2	0.5	0.25	0.954	3.78	13.20	13.31
20 Sep 2005	3	0.8	0.25	0.936	3.78	11.55	12.72
20 Dec 2005	4	1.0	0.25	0.915	3.78	12.72	12.72
20 Mar 2006	5	1.2	0.25	0.895	3.80	12.84	12.74
20 Jun 2006	6	1.5	0.25	0.878	3.81	11.64	12.56
20 Sep 2006	7	1.8	0.25	0.860	3.83	12.25	12.51
20 Dec 2006	8	2.0	0.25	0.842	3.85	12.69	12.53
20 Mar 2007	9	2.2	0.25	0.823	3.86	13.31	12.62
20 Jun 2007	10	2.5	0.25	0.804	3.87	13.56	12.71
20 Sep 2007	11	2.8	0.25	0.784	3.89	13.79	12.81
20 Dec 2007	12	3.0	0.25	0.766	3.90	13.90	12.90
20 Mar 2008	13	3.2	0.25	0.747	3.91	13.98	12.99
20 Jun 2008	14	3.5	0.25	0.729	3.93	14.13	13.07
22 Sep 2008	15	3.8	0.26	0.710	3.94	14.34	13.15
22 Dec 2008	16	4.0	0.25	0.692	3.95	14.66	13.25
20 Mar 2009	17	4.2	0.24	0.674	3.94	14.56	13.32
22 Jun 2009	18	4.5	0.26	0.656	3.93	14.66	13.40
21 Sep 2009	19	4.8	0.25	0.639	3.93	14.76	13.47
21 Dec 2009	20	5.0	0.25	0.622	3.92	14.71	13.53

Figure 5.12 shows the average forward and zero prime rate curves. This is an interesting example which shows how we can derive a curve using the definition of the instrument that will be priced off this curve.

5.5 CONCLUDING REMARKS

The first step in deriving any curve is to decide what that curve will be used for, because that will determine what type of instrument should be used to derive it. When we need a curve to value interest rate swaps, we would typically use interest rate swaps to derive it. When we need a risk-free curve, we would use government bonds to derive the curve. Bonds issued by the governing body are usually assumed to have no risk of default.

It is important to establish where the curve will be used: a curve used in a valuation environment will differ from a curve in a pricing environment. In

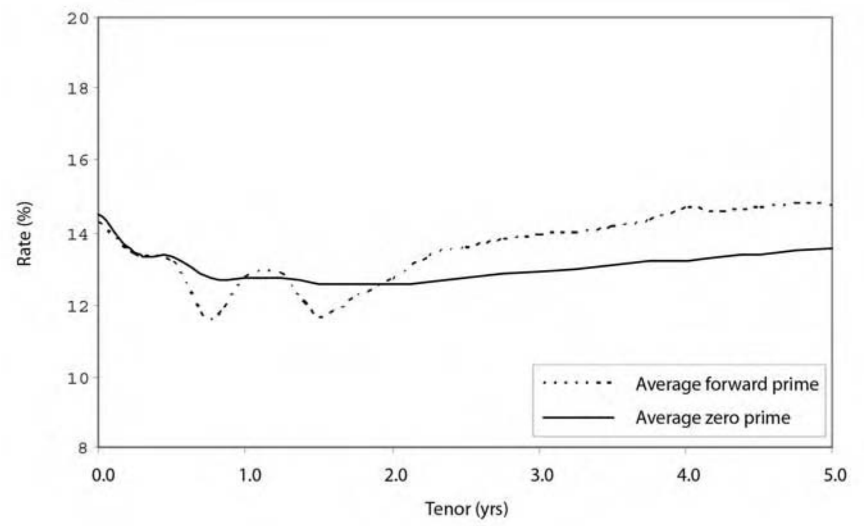


Figure 5.12 A comparison of the average forward and zero prime rate curves that are derived in Table 5.17

a valuation environment we need an objective curve that accurately represents the current market interest rates. The curve will usually be a perfect-fit curve, which means that market instruments are priced accurately from this curve.

The next important decision is which quoted rate to use, in other words bid or offer rates. When we need a curve for valuation purposes, we would usually derive the curve with mid rates. The mid rates are the average of the market bid and offer rates. A dealer that needs a curve to determine where to quote prices in the market will rather use the bid prices when deriving the curve. Alternatively the mid prices can be used, but then the final curve will have to be adjusted with a spread to get the correct bid prices.

The type of instrument dictates the way we decompose the instrument into its underlying cash flows, and thus which type of approach to follow when deriving a yield curve from it. There are various intricacies that have to be allowed for, which can complicate a model significantly. A good example is when we derive a futures curve and we need to estimate a convexity adjustment. There is a lot of uncertainty in the model, because not only is there uncertainty in the way the convexity adjustment formulae are derived, but the formulae usually also require further assumptions regarding interest rate volatility and mean reversion.

The choice of the yield curve model is very important. We have to choose a yield curve model that converges to a fixed level at the longer maturities, because that ensures that long-term rate estimates behave adequately. Say we have a set of bonds and the longest-term bond has a term to maturity of 20 years. When we fit a yield curve to the bonds and the yield curve is upward-sloping after the 20-year point, we will find that long-term rates calculated from the function will be extremely high. Similarly when the yield curve function is downward-sloping at the long end after the 20-year point, the long-term rates calculated from the function will converge to zero and may even go negative, which is also undesirable. In truth we do not know what the actual rates are after the 20-year point, so a better idea is to just choose a function that converges to the level of the rate at the 20-year point. This will ensure that we keep the longest-term rate (that is known in the market) constant and do not make any additional assumptions regarding the very long-term rates. Please refer to Chapter 4 for a discussion on the extrapolation techniques.

Finally a curve also has to be derived from instruments with similar liquidity and credit quality, otherwise there will be distortions in the curve and it will not adequately reflect market rates.

In deriving any type of yield curve, there are a lot of decisions that have to be made. Even the simplest decision, for instance which interpolation technique to use, can have an adverse effect on the final yield curve. Incorrect decisions or over-complicating yield curve models lead to model risk. Model risk is discussed in detail in Chapter 8.

CHAPTER 6

Real Yield Curves

In this chapter the relationships between nominal interest rates, real interest rates, and inflation rates are shown. Various ways in which to derive real yield curves are discussed.

We begin this chapter by considering the various types of inflation-linked securities, in Section 6.1. It is important to understand the security to be able to derive a real yield curve from it. In Section 6.2 we discuss various ways in which to derive an inflation term structure, and how to use the inflation term structure to derive real rates where there are no inflation-linked securities available in the market. Finally in Section 6.3 we show an example where a real yield curve is derived from an inflation-linked security by allowing for indexation lags.

6.1 INFLATION-LINKED SECURITIES

Inflation-indexed securities are designed to protect investors against the effect of inflation. There are various inflation-linked securities available in the market. In this section we briefly describe how the cash flows of the most basic inflation-indexed securities are determined. When we know how to determine the cash flows of a security and there are market quotes available for the security in the market, then we can derive a yield curve from it.

The information in this section was obtained from Deacon, Derry, and Mirfendereski (2004).

6.1.1 Capital-indexed bonds

A capital-indexed bond (CIB) has a fixed real coupon rate, but the redemption value varies with inflation. The cash flows of this bond are determined as follows:

$$C_j = \begin{cases} r \times \dfrac{I_j}{I_0} & , j < n \\ 100 \times \dfrac{I_j}{I_0} + r \times \dfrac{I_j}{I_0} & , j = n \end{cases} \tag{6.1}$$

where

C_j = cash flow at time j, $j = 1,...,n$
r = fixed real coupon rate
I_0 = index value on the issue date of the bond
I_j = index value at time j
n = time of the final cash flow of the bond.

A typical problem with these securities is that we need estimates of the future index values. This will be addressed in a later section. To illustrate how the cash flows for the CIB are determined, consider a ten-year CIB that pays an annual real coupon of 3 percent with assumed index values as shown in Table 6.1.

Table 6.1 Cash flows of a ten-year CIB that pays an annual real coupon of 3 percent

Cash flow no.	Fixed real coupon	Index value I_t	I_t / I_0	Coupon portion	Redemption portion	Final cash flow
0		103				
1	3	110	1.07	3.20		3.20
2	3	115	1.12	3.35		3.35
3	3	116	1.13	3.38		3.38
4	3	118	1.15	3.44		3.44
5	3	127	1.23	3.70		3.70
6	3	135	1.31	3.93		3.93
7	3	136	1.32	3.96		3.96
8	3	140	1.36	4.08		4.08
9	3	142	1.38	4.14		4.14
10	3	144	1.40	4.19	139.81	144.00

6.1.2 Interest-indexed bonds

An interest-indexed bond (IIB) pays a fixed real coupon plus an indexation of the fixed redemption value on each cash flow date. The redemption value at maturity, however, is not adjusted. The cash flows for these bonds are calculated as:

$$C_j = \begin{cases} 100 \times \left(\frac{I_j}{I_{j-1}} - 1 \right) + r & , j < n \\ 100 \times \frac{I_j}{I_{j-1}} + r & , j = n \end{cases} \tag{6.2}$$

where all symbols are as defined before. Table 6.2 illustrates the cash flows of a ten-year IIB that pays an annual coupon of 3 percent. It is interesting to see how the cash flow of this bond differ from the CIB that is shown in Table 6.1.

Table 6.2 Cash flows of a ten-year IIB that pays an annual real coupon of 3 percent

Cash flow no.	Fixed real coupon	Index value I_t	$I_t\ I_{t-1}$	Coupon portion	Redemption portion	Final cash flow
0		103				
1	3	110	1.07	3.00	6.80	9.80
2	3	115	1.05	3.00	4.55	7.55
3	3	116	1.01	3.00	0.87	3.87
4	3	118	1.02	3.00	1.72	4.72
5	3	127	1.08	3.00	7.63	10.63
6	3	135	1.06	3.00	6.30	9.30
7	3	136	1.01	3.00	0.74	3.74
8	3	140	1.03	3.00	2.94	5.94
9	3	142	1.01	3.00	1.43	4.43
10	3	144	1.01	3.00	101.41	104.41

6.1.3 Current pay bonds

A current pay bond (CPB) pays an inflation-adjusted coupon as well as an indexation of the fixed redemption value. The redemption value at maturity is not adjusted for inflation. The cash flows of this bond are determined as follows:

$$C_j = \begin{cases} 100 \times \left(\frac{I_j}{I_{j-1}} - 1 \right) + r \times \left(\frac{I_j}{I_{j-1}} \right) & , j < n \\ 100 \times \frac{I_j}{I_{j-1}} + r \times \frac{I_j}{I_{j-1}} & , j = n \end{cases} \tag{6.3}$$

where all the symbols are as defined before. Table 6.3 illustrates the cash flows of a ten-year CPB with an annual real coupon of 3 percent.

Table 6.3 Cash flows of a ten-year CPB that pays an annual real coupon of 3 percent

Cash flow no.	Fixed real coupon	Index value I_t	$I_t\, I_{t-1}$	Coupon portion	Redemption portion	Final cash flow
0		103				
1	3	110	1.07	3.20	6.80	10.00
2	3	115	1.05	3.14	4.55	7.68
3	3	116	1.01	3.03	0.87	3.90
4	3	118	1.02	3.05	1.72	4.78
5	3	127	1.08	3.23	7.63	10.86
6	3	135	1.06	3.19	6.30	9.49
7	3	136	1.01	3.02	0.74	3.76
8	3	140	1.03	3.09	2.94	6.03
9	3	142	1.01	3.04	1.43	4.47
10	3	144	1.01	3.04	101.41	104.45

6.1.4 Indexed annuity bond

Index annuity bonds (IAB) has a fixed base annuity payment and a variable payment to compensate the investor for inflation. The cash flows of this instrument are determined as follows:

$$C_j = B \times \frac{I_j}{I_0} \qquad \text{, for all } j = 1,\ldots,n \tag{6.4}$$

where B denotes the base payment. The base payment is determined as follows:

$$B = R/\left(\frac{1-(1+r)^{-n}}{r}\right) \tag{6.5}$$

where r is the annual real interest rate for a period of n years and R is the redemption value of the bond. Table 6.4 shows an example of a ten-year AIB with the base payment calculated based on a real rate of 3 percent.

6.1.5 Indexed zero-coupon bonds

Indexed zero-coupon bonds (IZCB) pays a single inflation-adjusted amount on the redemption date. This cash flow is determined as follows:

$$C_n = 100 \times \frac{I_n}{I_0} \tag{6.6}$$

with all symbols as defined before.

Table 6.4 Cash flows of a ten-year AIB based on an annual real rate of 3 percent

Cash flow no.	Base payment	Index value	I_t/I_0	Final cash flow
0		103		
1	11.7	110	1.07	12.52
2	11.7	115	1.12	13.09
3	11.7	116	1.13	13.20
4	11.7	118	1.15	13.43
5	11.7	127	1.23	14.45
6	11.7	135	1.31	15.37
7	11.7	136	1.32	15.48
8	11.7	140	1.36	15.93
9	11.7	142	1.38	16.16
10	11.7	144	1.40	16.39

6.2 FORECASTING INDEX VALUES

There are various possible indices to which a bond can be linked. In this chapter the focus is on the consumer price index (CPI) which is the most widely used index.

From the discussion in Section 6.1 it is clear that when deriving a yield curve, and pricing inflation-linked securities for that matter, part of the process is to estimate future values of the inflation index. In this process some issues about index values that have to be taken into account are:

- The index definition may change over time. This means that should we wish to fit a time series model to the historical index series to predict future index values, our time series model has to incorporate these changes.
- When an index is determined, not all the components are necessarily sampled and included in the calculations each time. For instance, to calculate the CPI index the housing sub-index may not be included every time it is published because the calculation would be too onerous.
- The index value will show a seasonality effect that has to be taken into account.
- The index value may not be published regularly and so most markets use a lagged index value in the calculation of the inflation-indexed securities.

There are various ways in which to estimate future index values. These approaches are discussed in Sections 6.2.1 to 6.2.6.

6.2.1 A time series approach

One idea is to fit a time series model to the historical index series. The problem is that various factors have to be taken into account (as discussed above), and that could complicate the model significantly. The analyst may also wish to take other variables into account when estimating future inflation rates, for instance the exchange rate. Please refer to Schwert (1986) and Barr and Campbell (1996) for some examples of inflation time series models.

6.2.2 Breakeven inflation rates

Another interesting approach is to use the concept of breakeven inflation rates. Deacon *et al.* (2004) show that we can take a set of nominal and inflation-linked bonds with the same maturities, and then define the expected inflation yield as the difference between the nominal bond yield to maturity and the real yield on the index-linked bond. In other terms:

$$\pi = i - r \tag{6.7}$$

where π is the expected inflation rate, i the nominal yield to maturity, and r the real yield.

With (6.7) we obtain a rough estimate of the average expected inflation value over the specific period. In other words, when we have a nominal and real yield with maturity of three years, then the expected inflation rate denotes the average inflation rate over this whole three-year period. The inflation number should not be interpreted as the expected inflation in three years' time. To get a term structure of annual inflation rates we use an interpolation function to find an average inflation rate at each tenor, then use a bootstrap procedure to calculate yearly inflation rates (as discussed in Chapter 2).

6.2.3 The interest rate parity approach

Jarrow and Yildirim (2003) suggest using the interest rate parity equation (discussed in Chapter 2), which links the forward exchange rate to the spot exchange rate:

$$X_f = X_s \times \left(\frac{1 + i_d}{1 + i_f}\right)^t \tag{6.8}$$

where
X_f = forward exchange rate
X_s = spot exchange rate

i_d = annually compounded domestic interest rate
i_f = annually compounded foreign interest rate
t = time to maturity in years (assuming similar daycount conventions for the domestic and foreign interest rates).

Jarrow and Yildirim suggest that real yields correspond to foreign rates; the nominal yields correspond to the domestic rates; and the inflation rate corresponds to the spot exchange rate. This means we can rewrite (6.8) as:

$$CPI_t = CPI_0 \times \left(\frac{1+i}{1+r}\right)^t \tag{6.9}$$

where
CPI_t = forecasted CPI value
CPI_0 = current CPI value
i = annually compounded nominal interest rate for a period of t years
r = annually compounded real interest rate for a period of t years
t = time to maturity in years (assuming similar daycount conventions for the nominal and real interest rates).

Depending on the market we are working with, we do not necessarily have real yields available at all maturities. Some markets have only a few inflation-indexed bonds available that can be used in an analysis like this.

Consider for example the South African market, which has only four government-issued inflation-linked bonds available. These bonds are quoted in terms of their yields, and the closing prices are published daily by the Bond Exchange of South Africa (refer to www.bondex.co.za). For this market we can use the yields of the four inflation-linked bonds, together with the yields of similar maturity nominal bonds in (6.9), to infer future CPI values. We will have five points on a curve: the CPI value today plus the four inferred CPI values. A suitable function can then be fitted to these CPI values to form a whole term structure of future index values.

6.2.4 Fisher identity

Consider a market where there are no short-term inflation-linked instruments available that can be used to derive a curve. An interesting idea is to use the Fisher identity, which provides a relationship between nominal and real interest rates as follows (Deacon *et al.*, 2004):

$$(1+i) = (1+r)(1+\pi)(1+\kappa) \tag{6.10}$$

where
i = nominal annual interest rate

r = real annual interest rate
κ = risk premium
π = expected inflation rate.

The risk premium reflects the uncertainty of future inflation. Section 6.3.1 shows an example where the Fisher identity is applied. The main drawback of using (6.10) is that further assumptions regarding the risk premium have to be made. The practitioner has to decide whether a term structure of risk premiums is appropriate, or whether a constant value can be used.

Similar to the breakeven inflation rate, the inflation rate derived from this approach is interpreted as the average inflation rate over the period.

Deacon and Derry (1994b) discuss an interesting application of (6.10). Consider an index-linked bond that is priced as follows:

$$P = \sum_{i=1}^{n} \frac{C(1+\pi_0)}{(1+r)^i(1+\pi^a)} + \frac{R(1+\pi_0)}{(1+r)^n(1+\pi^a)} \tag{6.11}$$

where
P = price of the index-linked bond
C = annual real coupon
R = redemption payment
π_0 = known inflation rate
π^a = assumed average future inflation rate over the life of the bond
r = real yield to maturity.

Deacon and Derry derive a term structure of inflation rates with the following steps:

- Assume a risk premium of zero in (6.10) and fix the average inflation rate π^a in (6.11) at 3 percent (this can be any fixed percentage).
- Derive a nominal par curve with the regression techniques discussed in Chapter 3 using a set of coupon-paying bonds. Convert the par rates to zero and forward interest rates.
- Determine the real yields from the market prices of the index-linked bonds.
- Fit a real par yield curve to these real yields by applying one of the regression models discussed in Chapter 3. From the real par yield curve a real zero and forward rate curve is derived.
- By using the derived nominal and real forward interest rates for similar periods in (6.10), it is possible to derive an initial forward inflation rate curve. The forward inflation rate between time j–1 and j is denoted by π_j.
- The forward inflation curve is then converted into an average inflation curve by solving for π^a_t in the following relationship:

$$(1 + \pi_t^a)^t = \prod_{j=1}^{t}(1 + \pi_j)$$

where π_t^a is the average inflation rate for the period t years.

- Inflation rates are interpolated for each of the index-linked gilts from the average inflation rate curve, and then substituted into (6.11).
- The real yields on the index-linked bonds are now re-estimated and again a par real curve is fitted to these yields.
- Following the same steps, an inflation term structure is produced.
- These steps are repeated until there is convergence between the inflation rates produced by (6.10) and the real yields derived from the prices of the bonds given by (6.11).

6.2.5 Survey methods

It is possible to use published forecasts. Typically a data vendor may ask economists from various financial institutions to predict index values. The median of the various forecasts can then be used when valuing the instrument. The problem with this approach is that survey respondents have no incentive to answer accurately. However if the results are published stating the organization that made the specific forecast, the answers should be more appropriate.

A drawback of this approach is that it will usually reflect short-term expectations of inflation. Because of the time it may take to compile the surveys, these may not reflect current inflationary expectations. Please refer to Deacon and Derry (1994b) for a more detailed discussion.

6.2.6 Constant extrapolation

The simplest approach is to just keep the current inflation rate constant. That way no assumptions regarding future inflation need to be made.

6.2.7 General comments on the various approaches

In Sections 6.2.2, 6.2.3, and 6.2.4 we mentioned matching the maturities of nominal and inflation-linked bonds to infer an inflation rate. The problem with this approach is that we do not take into account the different coupon structures of the bonds. Two bonds with the same term to maturity may pay different coupon rates, so that their yields to maturity will be different. To take this coupon effect into account, we can rather match the bonds by duration. However, the problem with duration is that it is difficult to interpret the period to which the implied inflation rate applies (Deacon and Derry, 1994b).

Another important issue to take note of is the type of nominal bond used in the calculations. Bonds with special features, for instance callable bonds, will have different characteristics from regular nominal bonds, which result in differences in their yields. The yield differences may distort the implied inflation rate.

6.3 TERM STRUCTURE OF REAL RATES

In a perfect market the derivation of a real interest zero rate curve is not significantly different from the construction of a nominal interest zero rate curve. In this perfect market we have reasonably liquid inflation-indexed securities with a sufficient range of differing maturities. In practice, however, this is not yet the case, and this restricts the approaches that can be followed to derive a real interest rate curve.

In Section 6.3.1 we look at an example of how to derive the short end of the real curve when there are no liquid short-term inflation-linked instruments available in the market. To derive the long end of the curve we assume there are at least some index-linked bonds available, and in Section 6.3.2 we see how to derive a curve from the index-linked bonds by taking indexation lags into account.

6.3.1 The short end of the curve

Say there are no liquid short-term inflation-linked instruments in the market. We only have a nominal zero curve available, as well as a term structure of inflation rates which was derived using one of the approaches discussed in Section 6.2. We are going to use the Fisher identity discussed in Section 6.2.4 to derive the real rates.

Depending on the approach used to estimate the inflation term structure (for instance using (6.7) or (6.9)), it is possible that the resulting inflation term structure already incorporates the risk premium as defined in (6.10). With this argument it is then appropriate to use a risk premium of zero. However, Deacon *et al.* (2004) allude to the fact that the inflation rate risk premium and bond convexity will bias the inflation term structure in opposite directions, to the extent that they may partially offset each other. This means that we need to incorporate a risk premium. One method of doing this is to use historical risk premium estimates and just keep the risk premium fixed in the model. These premiums can then be updated as new research becomes available. The problem is that this adds another level of uncertainty to the model.

Table 6.5 shows a term structure of nominal rates that are compounded semi-annually, and inflation rates compounded annually. When assuming a

Table 6.5 Deriving a real curve from nominal rates and an inflation term structure

Value date: 05 April 2004

Date	Term (yrs)	Nominal rate (NACS)	Inflation rate (NACA)	Real rate (NACS)
05 Apr 2004	0.00	7.84	6.47	1.42
06 Apr 2004	0.00	7.84	6.47	1.42
01 May 2004	0.07	7.92	7.00	1.01
01 Jun 2004	0.16	8.03	7.80	0.36
01 July 2004	0.24	8.13	9.80	−1.37
01 Aug 2004	0.32	8.16	9.40	−0.98
01 Sep 2004	0.41	8.18	7.23	1.04
01 Oct 2004	0.49	8.20	6.47	1.77
01 Nov 2004	0.58	8.24	6.46	1.82
01 Dec 2004	0.66	8.28	6.92	1.42
01 Jan 2005	0.74	8.32	6.67	1.70
01 Feb 2005	0.83	8.37	6.01	2.38
01 Mar 2005	0.90	8.41	5.44	2.96
01 Apr 2005	0.99	8.46	5.28	3.16
01 July 2005	1.24	8.62	6.62	2.04
01 Oct 2005	1.49	8.79	5.99	2.80
01 Jan 2006	1.74	8.96	6.11	2.85
01 Apr 2006	1.99	9.13	5.39	3.71

flat risk premium term structure of zero, then (6.10) has to be adjusted as follows to derive the semi-annually compounded real rates:

$$\left(1+\frac{r}{2}\right)^2 = \frac{\left(1+\frac{i}{2}\right)^2}{(1+\pi)} \tag{6.12}$$

where all symbols are as defined before. Solving for the real rate then gives:

$$r = 2 \times \left[\left(\frac{\left(1+\frac{i}{2}\right)^2}{(1+\pi)}\right)^{\frac{1}{2}} - 1\right]$$

which is the formula that is used to derive the real rate given in Table 6.5.

In this example an actual/365 daycount convention is used and no business day rule is applied.

Implicit in this example is the fact that the nominal rates are zero rates with the required credit quality. In other words, say we need a risk-free real

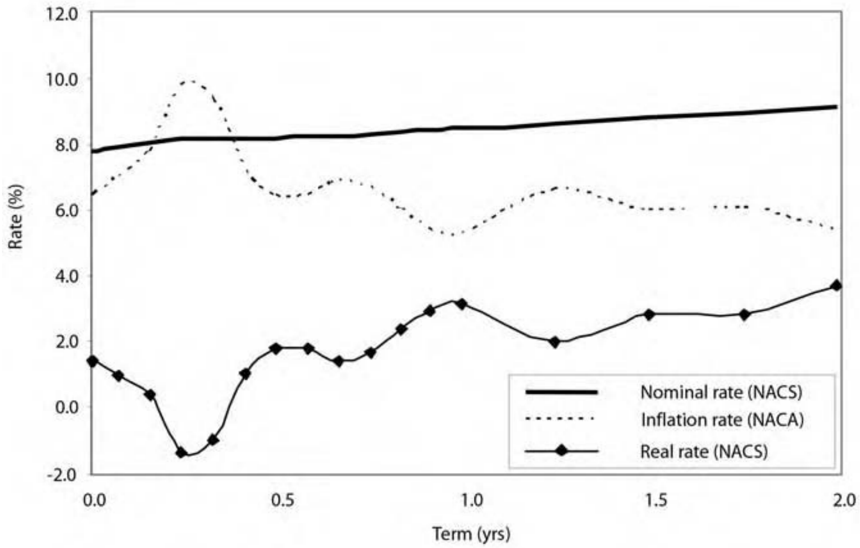


Figure 6.1 A comparison of the real rate curve with the nominal and inflation rate curves from which it is derived

rate curve: then we use risk-free nominal rates. Credit and liquidity premiums are discussed in detail in Chapter 7.

It is interesting to see that some of the implied real rates are negative because of the high inflation rates relative to the given nominal rates. Figure 6.1 shows a comparison of the nominal, inflation, and real rate term structures. The nominal rate curve is quite flat, so the main volatility in the real rate curve can be explained by the volatile nature of the inflation rate curve.

An analyst may prefer to have a smoother real zero curve, because then the forward rates derived from the zero curve will also be smoother. To get a smoother real curve, we can smooth either the inflation rate curve or the real rate curve directly. In this example we smooth the real curve. A plot of a smoothed real curve is shown in Figure 6.2.

To derive the smoothed curve in Figure 6.2 we took the actual overnight, one-month, and then the six-monthly real rates out to two years. The values between these tenor points are interpolated using the cubic splines approach. The resulting curve is much smoother, but the problem is that depending on the initial tenors chosen, we can get a completely different real curve.

Another way to smooth the real curve is to fit a function through all the real rates simultaneously to get a best-fit curve. However, in the end it may be too subjective to actually smooth the real curve. Remember that if the curve is used for valuation purposes, we need to have an objective curve (please refer to the discussion in Chapter 1).

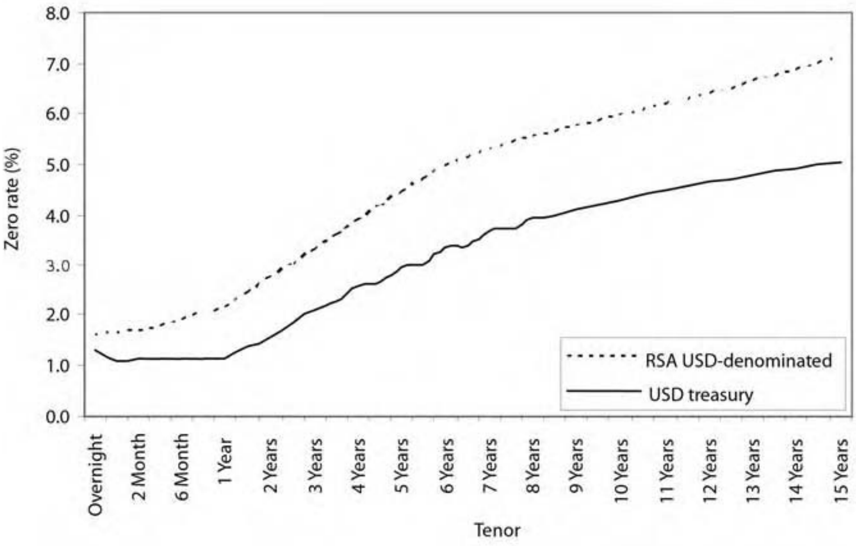


Figure 6.2 The actual real curve compared with a smoothed version

There is a lot of uncertainty in this model:

- We make assumptions when deriving a nominal curve (please refer to the discussion in Chapter 5).
- We have to make assumptions in order to derive an inflation term structure.
- We assume a flat risk premium term structure.
- We assume that the Fisher identity given by (6.10) actually holds, and that it is appropriate to derive real interest rates from this identity.

When working in a market where there are not enough instruments available, the assumptions are necessary to get at least some idea of the level and shape of the term structure of real rates.

6.3.2 The long end of the curve

To derive a real yield curve for longer maturities we use inflation-indexed securities. Consider the price of a nominal bond paying a regular annual nominal coupon:

$$P_N = \sum_{j=1}^{n} \frac{C_N}{(1+y_{tj})^{t_j}} + \frac{R_N}{(1+y_{tn})^{t_n}} \tag{6.13}$$

where

P_N= the all-in price of the nominal bond

n = number of outstanding cash flows of the bond
y_{tj} = nominal annual zero rate for a period t_j
t_j = the term in years from the settlement date until cash flow j
C_N = nominal annual coupon of the bond
R_N = the nominal redemption payment at the maturity of the bond.

The price of a CIB paying real annual coupons can be written similarly:

$$P_R = \sum_{j=1}^{n} \frac{C_R \times \frac{I_{tj}}{I_0}}{(1+r_{tj})^{t_j} \times \frac{I_{tj}}{I_0}} + \frac{R_R \times \frac{I_{tn}}{I_0}}{(1+r_{tn})^{t_n} \times \frac{I_{tn}}{I_0}}$$

$$= \sum_{j=1}^{n} \frac{C_R \prod_{i=1}^{j}(1+\pi_{ti})}{(1+r_{tj})^{t_j} \prod_{i=1}^{j}(1+\pi_{ti})} + \frac{R_R \prod_{i=1}^{n}(1+\pi_{ti})}{(1+r_{tn})^{t_n} \prod_{i=1}^{n}(1+\pi_{ti})}$$

$$= \sum_{j=1}^{n} \frac{C_R}{(1+r_{tj})^{t_j}} + \frac{R_R}{(1+r_{tn})^{t_n}} \tag{6.14}$$

where
P_R = the all-in price of the real bond
n = number of outstanding cash flows of the bond
r_{tj} = real annual zero rate that apply to a period t_j
t_j = the term in years from the settlement date until cash flow j
I_{tj} = CPI value at time t_j
π_{ti} = inflation rate between time t_{i-1} and t_i
C_N = real annual coupon of the bond
R_N = the real redemption payment at the maturity of the bond.

In (6.14) it is assumed that there is no indexation lag. In practice there usually is a lag in the publication of the index value. This lag is due to the time it takes to sample the data and calculate indices like the CPI. This lag will differ between markets. Deacon *et al.* (2004) discuss how various countries deal with this issue. The interested reader can also refer to Evans (1998) and Anderson and Sleath (2001) for a discussion on UK index-linked securities.

In this section we consider an example of a market where there are three inflation-linked securities available and that in this market (6.14) is adjusted as follows:

$$P_R = \frac{I_t}{I_0}\left[\sum_{j=1}^{n} \frac{C_R}{(1+r_{tj})^{tj}} + \frac{R_R}{(1+r_{tn})^{tn}}\right] \tag{6.15}$$

where I_t is the most recently published CPI value and I_0 is the index value fixed on the issue date of the bond; all other symbols as defined before. Table 6.6 shows the information of these bonds. The bonds pay semi-annual coupons. We assume that the bonds are traded in terms of yields (and not prices); the yields are semi-annually compounded. The value date is assumed to be 20 December 2004.

Table 6.6 Information on CIBs

Name	Maturity	Reference index	Yield	Coupon rate
CP01	15 Mar 2010	118	3.2	3.0
CP02	15 Feb 2012	115	4.4	4.0
CP03	15 Oct 2014	116	4.7	4.2

To derive a yield curve from these bonds, we note that we only have long-term bonds. This means the first issue to be addressed is how to find a short-term security which can anchor the yield curve at the short end. The discussion in the previous section showed that there can be a lot of volatility when real rates are inferred using the Fisher identity, so to prevent this, we only use the information available in the market. We only know the current inflation rate, so using this rate and the corresponding short-term nominal rate, we infer a short-term real rate with (6.10) by assuming a risk premium of zero. This rate is 2.2% and is compounded semi-annually.

Table 6.7 (overleaf) shows the cash flows of the CIB CP01; the most recent published CPI value is 121.6. An actual/365 daycount convention is used and no business day rule is applied. Also, the settlement date of the bond is assumed to correspond with the value date. The cash flows of the bond are determined exactly as described by (6.15). We determine the current market prices of the bonds by using the quoted yields given in Table 6.6.

The next step is to choose a functional form for the discount function (similarly to the method described in Chapter 5 when deriving a yield curve from nominal bonds). The fitted function in this example is assumed to be a function of n+1 parameters given by $\beta_0,...,\beta_n$ and a function of the term of the specific cash flow, namely t_i, $i = 1,..., m$ where m denotes the number of cash flows and the function is denoted by $f(\beta_0,..., \beta_n|t_i)$.

By multiplying each of the cash flows now with the fitted discount factors and adding them all, we have the fitted price of the inflation-linked bonds. This process is repeated for each of the bonds in Table 6.6. To estimate the parameters of the chosen function $f(\beta_0,..., \beta_n|t_i)$, we use an optimization routine to minimize the weighted-squared differences

Table 6.7 Cash flows of CIB CP01 given in Table 6.6

			Bond:	CP01			
			Current index value (I_t):	121.6			
			Reference index (I_0):	118			
Cash flow date	**Fixed real coupon**	**I_t/I_0**	**Cash flow**	**DF from YTM**	**Cash flow × DF**	**Fitted DF**	**Cash flow × fitted DF**
15 Mar 2005	1.5	1.03	1.55	0.99	1.53	$f(\beta_0,\ldots,\beta_n \mid t_1)$	$f(.\mid t_1)$ x 1.55
15 Sep 2005	1.5	1.03	1.55	0.98	1.51		
15 Mar 2006	1.5	1.03	1.55	0.96	1.49	:	:
15 Sep 2006	1.5	1.03	1.55	0.95	1.46		
15 Mar 2007	1.5	1.03	1.55	0.93	1.44		
15 Sep 2007	1.5	1.03	1.55	0.92	1.42		
15 Mar 2008	1.5	1.03	1.55	0.90	1.40		
15 Sep 2008	1.5	1.03	1.55	0.89	1.38		
15 Mar 2009	1.5	1.03	1.55	0.88	1.35		
15 Sep 2009	1.5	1.03	1.55	0.86	1.33		
15 Mar 2010	1.5	1.03	104.60	0.85	88.81	$f(\beta_0,\ldots,\beta_n \mid t_{11})$	$f(.\mid t_{11})$ x 104.60
		Actual value:			103.12		↑ Add all the values to get fitted price

between the actual and fitted prices. Please refer to Chapter 5 for a detailed discussion on how this process is set up.

Figure 6.3 shows the real curve derived from the inflation-indexed bonds in Table 6.6 using a cubic spline function. The curve is very smooth, even at the short end where we have no instruments available and had to make assumptions to derive a short-term rate that can anchor the curve. The problem is that we have no instruments up to the six-year point, which means that the area on the curve between the implied overnight rate and the six-year point is purely derived from the fitted function. Depending on the function, the curve may show very different shapes. At least we know that we can price inflation-linked securities accurately from this derived real curve.

Because of the indexation lag, this real rate is only an approximation of the actual real rate in the market (Evans, 1998).

This is a very simplified example. Consider the case where indexation takes place as is given by (6.11); then we will have to estimate an average inflation interest rate, which leads to some extra complexity. It is important to model the inflation-linked security as specified by the governing body of the specific country.

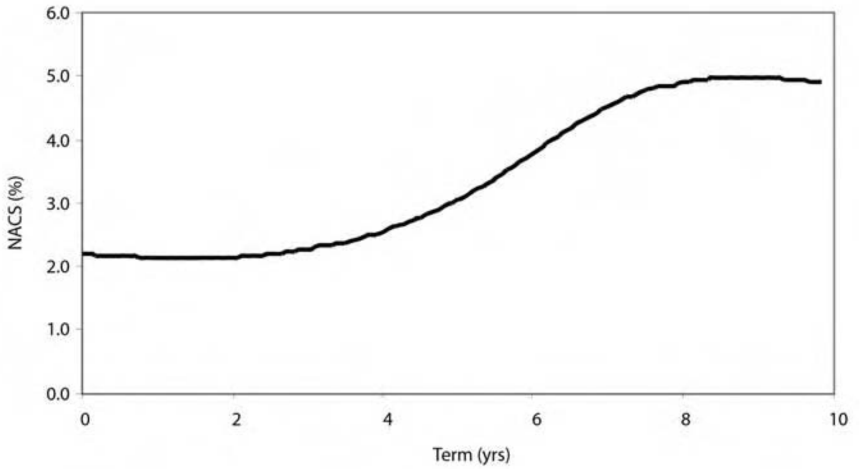


Figure 6.3 Real rate curve derived from the inflation-linked bonds in Table 6.6

6.4 CONCLUDING REMARKS

The instruments discussed in Section 6.1 give a broad overview of how indexation is applied to value the different types of index-linked instruments. However, the indexation usually also differs across countries, so when putting together a yield curve model, the analyst has to ensure that the instruments are modeled as prescribed by the specific country's governing body (Das, 2004; Deacon *et al.*, 2004).

In Section 6.2 various techniques to forecast index values, specifically the CPI and thus the inflation rate, are discussed. It is important to forecast these values so we are able to price the index-linked instruments. The forecast inflation rates can also be used to derive a term structure of real rates when no actual short-term inflation-linked instruments are available in the market.

To illustrate a possible problem with implying the real rates from inflation forecasts: consider the case where we imply a real curve out to two years with the Fisher identity. Assume the shortest-dated inflation-linked instrument that we have available has a term to maturity of three years. This means that in the process of deriving the real yield curve, we price the three-year inflation-linked instrument with the implied real rates out to two years, and real rates from the interpolating function between the two and three-year period. The problem is that as time progresses, the optimization routine used to derive the yield curve will have difficulty pricing this instrument accurately, because a bigger part of the value of the bond will be determined by the fixed implied real rates and only a very small portion will depend on the interpolating function. In a case like this the analyst should infer a shorter part of the real curve, and rather let the interpolating function fill in the gaps.

In Section 6.3.2 we show that to derive a real curve from inflation-linked securities is similar to the approach followed to derive a yield curve from nominal bonds. The problem, however, is that there are usually only a limited number of available instruments, and gaps in the term structure require additional assumptions which are difficult to validate. The liquidity in the underlying market may also be limited, which means that it may be necessary to allow for liquidity spreads in the model. Liquidity spreads are discussed in detail in Chapter 7.

CHAPTER 7

Estimating Credit, Liquidity, and Country Risk Premiums

It is important to financial institutions to understand the different sources of risk inherent in any transaction and to be able to quantify it. Traders need to understand the dynamics of credit, liquidity, and country premiums, otherwise they will not be able to put successful hedges in place.

In this chapter we explore the factors that drive credit, liquidity, and country risk premiums, and consider various ways in which to measure these premiums.

7.1 CREDIT SPREADS

7.1.1 Overview

Government bonds are usually considered to be risk-free. Any non-government bond is expected to have some risk attached to it, and typically pays higher yields. The higher yields compensate investors for taking on additional default and liquidity risks. Consider two bonds that are identical, except one is a risky bond and the other a risk-free government bond. The difference between the yields of these two bonds is a spread that consists of two components, one attributable to credit risk and the other to liquidity risk. The difficulty comes in estimating these two components separately.

Amato and Remolona (2003) show that spreads on corporate bonds are much wider than would be needed to meet the expected default losses. They argue that this is because in practice investors cannot diversify all unexpected losses, which means that the credit spread also has to allow for undiversifiable risk.

There are various factors that affect the credit spread. The *credit rating* of the bond plays a role, because this is an indication of the credit quality of the bond. The better the credit rating, the lower we would expect the credit spread to be (Amato and Remolona, 2003). The *credit spread* of the instrument may also be affected by the instrument tenor. We can argue that longer-term instruments have higher credit spreads, because there is more uncertainty about the future. However Annaert and Ceuster (1999) argue that the credit spread term structure is not necessarily upward sloping; various curve shapes are possible. Another important factor to remember is that the credit spreads differ between different types of instruments. The spread above a coupon bond will not necessarily be the same as above a zero coupon bond, because the credit spread term structure is not necessarily flat.

There are various reasons that practitioners are interested in estimating these spreads. For instance, financial institutions may set the credit spread on a loan high enough to ensure the target return is achieved on the economic capital that is set aside for the transactions. Another example is that a trader of corporate bonds may hedge with derivatives based on government bonds. It is obviously important the trader understands the dynamics of the credit spread with respect to the interest rate term structure for the hedge to be successful. Please refer to Annaert and Ceuster (1999) for a complete discussion.

Say we wish to derive a risk-free curve, but there are not enough liquid risk-free instruments in the market to derive it. Then it is possible to take a risky instrument, adjust it with a credit spread, and use the adjusted instrument when deriving the risk-free curve.

7.1.2 Estimating a credit spread for a single instrument

In this section we will focus on estimating credit spreads, so in this discussion we assume that all instruments have the same liquidity.

Consider the bonds shown in Table 7.1. The bond code RF indicates a risk-free bond, and CB indicates a corporate bond. The two corporate bonds have the same issuer and the same credit rating. All bonds have coupons that are paid semi-annually, and they are settled on a $t + 3$ basis. The bonds are assumed to go ex-coupon one month before each coupon.

Table 7.1 Bond information

Bond code	Maturity	Coupon rate	Yield to maturity
RF001	28 Feb 2008	10.00	8.710
CB002	1 Jun 2008	11.00	9.180
RF002	31 Aug 2010	13.00	9.100
RF003	15 Sep 2015	13.50	9.645
CB001	1 Aug 2020	13.50	9.855
RF004	21 Dec 2026	10.50	9.000

The value date is 21 December 2004. The overnight rate used is 7.278% compounded semi-annually.

We would like to derive a risk-free curve from the bonds in Table 7.1. The problem is that we only have four bonds available, and the gap in the terms to maturity between RF003 and RF004 is quite big, so the idea is to add the corporate bond CB001 to the process. We will now consider the various approaches to estimate a credit spread for this corporate bond.

Estimating a credit spread: approach 1

We are interested in estimating a credit spread for the corporate bond CB001. Say we have an identical risk-free bond available, then we can estimate the credit spread as the difference between the bond yields to maturity. In other words:

$$cs = y_{cb} - y_{rf} \tag{7.1}$$

where

cs = credit spread above the risk-free rate
y_{rf} = yield to maturity of the risk-free bond
y_{cb} = yield to maturity of the corporate bond.

It is important to note that the credit spread derived from (7.1) is a spread above a yield to maturity (or par rate). The credit spread can be interpreted as the average spread over the specific tenor (which is equal to the bond's term to maturity). When we are interested in adjusting a risky zero rate with a credit spread, we will obviously have to adjust the credit spread to be a spread above zero rates.

In practice (and in this example), corporate bonds will rarely have a risk-free bond available in the market that exactly matches their cash flow structures. We have to make additional assumptions should we wish to follow this approach. In this example two assumptions are necessary:

- We assume a flat credit spread term structure, in other words the credit spread for all the corporate bonds (with the same issuer) is the same across all tenors.
- The second corporate bond that is not part of the process to derive this curve, CB002, has a maturity that is very close to the risk-free bond RF001. Ignoring the differences in coupon rates, we can calculate the differences between these two bonds' yields to maturity using (7.1). In this case the credit spread will then be 9.18 – 8.71 = 47 bps.

Because we are assuming a flat credit spread term structure, we can now argue that the credit spread estimated from CB002 can be used to adjust the yield of CB001. Once the yield to maturity of CB001 has been adjusted, we can continue to derive the yield curve from the (now) five risk-free bonds.

However, the assumptions made are quite bold. We know that a yield to maturity is the internal rate of return for a bond if it is held till maturity (please refer to Chapter 2). This means that term to maturity and coupon rates definitely play a role in the level of the yield to maturity. The assumptions made with this approach will never be adequate, and it is therefore not recommended to use a credit spread derived from the yield of one bond, to adjust the yield of another bond if the two bonds do not have the same cash flow structure.

Estimating a credit spread: approach 2

One way to overcome the inadequacies of approach 1 is to fit a yield curve to the risk-free bond yields using the regression techniques discussed in Chapter 3. However, instead of using term to maturity, we use the duration of each bond on the x axis to counteract the effects the different coupon rates of the risk-free bonds have on the yield (Annaert and Ceuster, 1999). We follow these steps:

- Plot the risk-free yields against the duration of each bond.
- Decide on one of the regression-type yield curve models and estimate the parameters so the fitted function will go through all the points on the curve.
- Interpolate a yield (from the fitted curve) with a duration similar to the duration of the corporate bond CB001 for which we need the credit spread. This is an estimate of the risk-free version of the corporate bond.
- Calculate the credit spread with (7.1); it is the difference between the yield on the corporate bond CB001 and the interpolated risk-free yield.

Figure 7.1 shows the yield curve fitted with a cubic spline. The corporate bond CB001 has a duration of approximately 7.5 years, so interpolating a value from the curve at this duration, we get a synthetic risk-free yield of 9.599%. The estimated credit spread is then 9.855% – 9.599% = 26 bps.

The problem with approach 2 is that it is very dependent on the interpolating function used.

Estimating a credit spread: approach 3

Another approach to estimate a credit spread is to create a synthetic government bond that exactly matches the cash flow structure of the

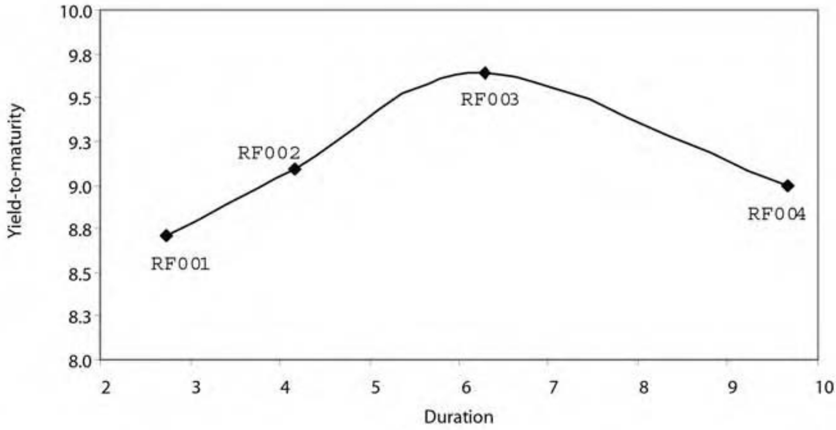


Figure 7.1 A plot of the yield to maturity against duration of the risk-free bonds in Table 7.1

corporate bond. This is done by deriving a zero curve from the risk-free bonds using one of the empirical models discussed in Chapter 3. From the empirical yield curve we can now price a risk-free bond with the same coupon rate and cash flow dates as the corporate bond. Once we have a price for the synthetic bond, we solve iteratively to find the yield to maturity of the synthetic bond. The yield to maturity represents the risk-free equivalent of the corporate bond, so the credit spread can be calculated using (7.1) (Leake, 2003; Longstaff, Mithal, and Neis, 2004).

Figure 7.2 shows the curve derived from the risk-free bonds. Following the procedure described above, the synthetic yield to maturity is found to

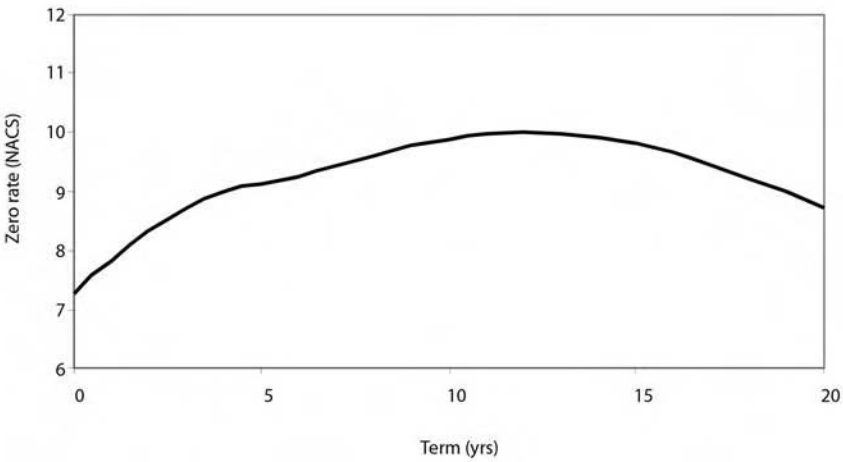


Figure 7.2 Zero curve derived from the risk-free bonds in Table 7.1

be 9.607% which means the credit spread for CB002 is 9.855% – 9.607% = 25 bps.

These calculations again depend on the interpolating function used, but it should be more accurate than approaches 1 and 2. The reason we want to add the corporate bond to this process has to be kept in mind: there is a big gap at the long end of the curve between the terms to maturity of RF003 and RF004. The gap is filled by the interpolating function, so we have to rely on the function being stable in this process.

There is an interesting problem with the way this approach has been applied in this example. The risk-free zero curve is used to derive a synthetic yield for CB002, and this yield is then used to derive the spread for the corporate bond. This means the adjusted corporate bond will now always lie on the zero curve, because of the way in which the credit spread is estimated, and will thus not really help with the problem of filling the maturity gap at the long end.

Another idea is to use the shorter-term corporate bond CB002 in the procedure, and again assume a flat credit spread term structure. The spread of CB002 is then used to adjust CB001. However, we again have the problem that the cash flow structures of the two corporate bonds are not the same.

It may be more appropriate to estimate a spread above zero rates, because then we get rid of the "cash flow structure" issue.

Estimating a credit spread: approach 4

In approach 4 we estimate a credit spread above zero rates. This is done by considering the way a government bond can be priced off a risk-free zero curve:

$$P = \sum_{j=1}^{n} C_j \left(1 + \frac{i_j}{m}\right)^{-m \times t_j} \tag{7.2}$$

where

P = the all-in price of bond

n = number of outstanding cash flows of bond

i_j = risk-free zero interest rate applicable to tenor j

t_j = term between the settlement date and the cash flow date

m = number of times a year the zero rate is compounded

C_j = jth cash flow of the bond; the last cash flow will be the redemption payment plus the coupon payment.

Say we would like to price a corporate bond using the same approach as (7.2). The corporate bond has to be adjusted with a credit spread, so we can adjust (7.2) as follows:

$$P = \sum_{j=1}^{n} C_j \left(1 + \frac{i_j + cs}{m}\right)^{-m \times t_j} \tag{7.3}$$

where *cs* denotes the credit spread. We assume here that the credit spread is fixed across all tenors. Remember that the zero curve is a risk-free curve, so to price the corporate bond we need to add a credit spread to the risk-free rates.

To estimate the credit spread from (7.3), we follow these steps:

- Derive a risk-free zero curve from all the risk-free bonds in Table 7.1.
- Model the shorter corporate bond CB002 as shown in Table 7.2 by interpolating interest rates i_j, $j = 1,...,7$ from the risk-free zero curve.
- We know the current market price of CB002, so using the market price and the fitted price obtained in Table 7.2, we iteratively solve for a credit spread with the constraint that the fitted and actual corporate bond prices must be exactly equal. In this example the credit spread was found to be approximately 40 bps. This is now an average credit spread above zero rates.
- We assume a flat credit spread term structure, in other words, the credit spread derived from CB002 is adequate to adjust the longer-term corporate bond CB001.

Table 7.2 Modeling the CB002 corporate bond in Table 7.1 to estimate the credit spread

Cash flow date	Cash flow no. j	Term to settlement date t_j	Cash flow per 100 nominal C_j	Interpolated zero rate $i_j + cs$	PV
1 Jun 2005	1	0.447	55,000	$r_1 = i_1 + cs$	$C_1 / (1+r_1/2)^{2 \times t_1}$
1 Dec 2005	2	0.948	55,000		
1 Jun 2006	3	1.447	55,000		
1 Dec 2006	4	1.948	55,000	:	:
1 Jun 2007	5	2.447	55,000		
1 Dec 2007	6	2.948	55,000		
1 Jun 2008	7	3.449	1,055,000	$r_7 = i_7 + cs$	$C_7 / (1+r_7/2)^{2 \times t_7}$
			Price on settlement date:		add values in column
			Price on value date:		PV settlement price 3 business days to value date

Please note that the credit spread derived from this approach is the average spread above the risk-free zero rates. This approach should give answers very close to approach 3. The difference is that we do not have the coupon-effect problem in approach 4, because it is a spread above zero rates.

In Section 7.1.4 we discuss the term structure of credit spreads, then we take a closer look at how to interpret this spread.

Comparison of yield curves

The final step in the example is to derive the risk-free yield curve from the now five risk-free bonds. Remember we have four risk-free bonds and then the long-term corporate bond CB001 which is adjusted with the estimated spread, so this bond can now also be considered to be "risk-free." It is important to know that we have to model CB001 similarly to the approach shown in Table 7.2 when deriving the risk-free curve. The only difference is that we "know" the credit spread and can just continue to derive the yield curve as if we have five risk-free bonds following the techniques discussed in Chapter 5.

It is extremely interesting to see how the credit spreads differ, depending on the method applied. Table 7.3 shows a summary of the credit spreads obtained from each of the approaches. In all cases we are working with semi-annually compounded rates.

Table 7.3 Summary of the estimated credit spreads from the four different approaches

Approach	Spread (bps)
1	47
2	26
3	25
4	40

Figure 7.3 shows a comparison between two zero curves derived from the risk-free bonds and the corporate bond CB001 adjusted with the credit spreads derived from approaches 3 and 4. The differences between the two curves in Figure 7.3 may have a big impact on instruments that are valued off this curve, and this highlights the importance of the approach chosen to estimate a credit spread.

This example focuses mainly on deriving a credit spread for coupon-paying bonds. In a market in which actively traded government and corporate zero-coupon bonds are available, the credit spread can be derived

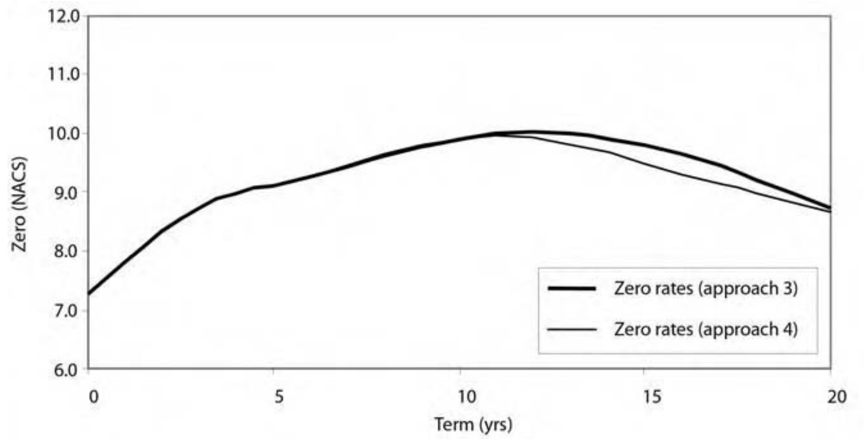


Figure 7.3 Comparison of zero curves when applying approaches 3 and 4 to estimate a credit spread for CB002

directly from the differential of the zero rates implied from these zero-coupon bonds. In this case we then have a credit spread above a zero rate.

7.1.3 Interpretation of the credit spread

The type of instrument used to derive the credit spread is one of the factors that has an effect on the level of the credit spread. Annaert and Ceuster (1999) discuss the fact that credit spreads differ between different types of instruments, even though the different credit spreads may move together.

We have to be consistent in the application of the credit spread; a spread derived from par rates should not be used to adjust zero rates. Also, the compounding of the rate of the instrument is also very important. Consider the following example.

We have a risk-free zero rate of 7.5% which is compounded semi-annually. The credit spread above this zero rate is 40 bps, so that the risky rate is 7.5% + 40/100 = 7.9%.

Say we need risky rates that are compounded quarterly to be able to value a specific instrument. By converting the risky rate, we get a quarterly compounded rate of 7.823%. However, when we convert the risk-free rate and the credit spread individually to quarterly rates, we get a risk-free rate of 7.431% and a spread of 38 bps. When we now calculate the quarterly compounded risky rate, we get 7.431% + 38/100 = 7.813%. This means there is a 1 bp difference between the two quarterly compounded risky rates purely because of the approach followed to do the conversion.

The example illustrates that we cannot just convert a credit spread by using the approaches discussed in Chapter 2 for interest rates and add that to the converted risk-free rate. The reason for this can be seen from the following relationship:

$$\left(1 + \frac{i_m + cs_m}{m}\right)^m = \left(1 + \frac{i_n + cs_n}{n}\right)^n \tag{7.4}$$

where

i_m = risk-free rate compounded m times per annum

cs_m = credit spread above a risk-free rate that is compounded m times per annum.

This means that when we have a credit spread cs_m available above a rate that compounds m times a year, we can find the credit spread above a rate compounded n times a year by solving for cs_n in (7.4). In simple terms, when the credit spread is derived for semi-annual rates, then this spread should only be used to adjust semi-annual rates, because the spread above annual rates will usually be different.

7.1.4 Deriving a credit spread curve

In Section 7.1.2 various techniques are discussed that can be used to calculate a single credit spread for a certain instrument. To derive a credit spread term structure, we can use the same techniques, except that we now need corporate bonds with a range of different maturities.

The idea is to estimate a credit spread at each tenor and then just use an interpolating technique to fit a function to these spreads to get a credit spread curve. However, there are various factors that have to taken into account:

- The corporate bonds used in the process must have the same credit rating. We discussed in Section 7.1.1 how credit spreads differ depending on the rating.
- The corporate bonds must have the same features. For example, we know that credit spreads of coupon bonds will differ from credit spreads on zero coupon bonds.
- We have to make sure that the credit spreads are all based on rates compounded similarly. If not, we need to use (7.4) to convert the spreads to the same compounding factor.
- The instruments must have the same liquidity premium.

Approach 4 discussed in Section 7.1.2 will be used to estimate a credit spread for each of the corporate bonds in Table 7.1. We follow these steps:

- Derive a zero curve from all the risk-free bonds.

Table 7.4 Credit spreads estimated using approach 3 in Section 7.1.2

Bond	Tenor	Credit spread
Overnight	0	0
CB002	3	40
CB001	16	25

- Model each corporate bond with the technique shown in Table 7.2.
- Iteratively solve to find the credit spread for each bond separately.

Table 7.4 shows the estimated spread for each of the corporate bonds. The overnight rate is assumed to have a credit spread of zero. This at least helps us to anchor the curve. Figure 7.4 shows the credit spread term structure. These are spreads above semi-annual zero rates. It is interesting to see that in this case our credit spread actually decreases with tenor.

The spreads calculated with approach 4 indicate the average spreads that are appropriate for the tenors. For example, the three-year spread is a fixed average credit spread of 40 bps across all tenors. This means this credit spread term structure is similar to the par rate concept explained in Chapter 2. To get "zero" credit spreads, we thus have to bootstrap the average spreads.

A practical problem with the example is that we have only two corporate bonds available, so a big area has to be interpolated. This means the interpolating function plays an important role in the process. Please refer to Chapter 4 for a discussion of the effect different interpolating functions have on a curve.

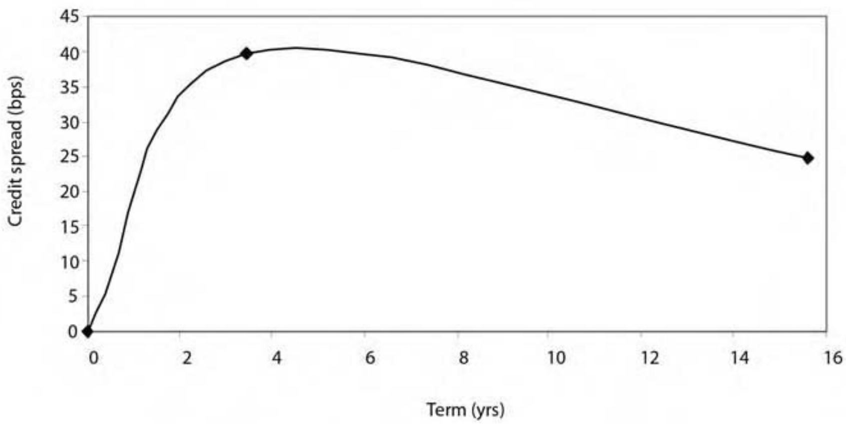


Figure 7.4 Credit spread term structure estimated from the corporate bonds in Table 7.1

A better way to obtain a "zero" credit spread term structure is to assume a functional form for the credit spread. The function will typically be one of the empirical yield curve models discussed in Chapter 3.

Say we have a risky zero rate r_t, a risk-free zero rate rf_t, and a credit spread cs_t, all with tenor t years and compounded continuously. The relationship between the discount factors derived from r_t and rf_t respectively is as follows:

$$\begin{aligned} df^r_t &= \exp[-r_t \times t] \\ &= \exp[-(rf_t + cs_t) \times t] \\ &= \exp[-rf_t \times t] + \exp[-cs_t \times t] \\ &= df^{rf}_t \times df^{cs}_t \end{aligned} \tag{7.5}$$

where df^r_t, df^{rf}_t, and df^{cs}_t denote the discount factors derived from r_t, rf_t, and cs_t respectively.

In Chapter 5 we show how to derive a yield curve by assuming a functional form for the discount function, estimating the parameters of the function by minimizing the weighted squared differences between bond prices. From (7.5) it is clear that we can follow a similar approach for corporate bonds, except that we have to assume an additional function for the credit spread discount function, because we are actually interested in estimating a credit spreads term structure above zero rates. Using (7.5), we see that we can derive the all-in price of a corporate bond as follows:

$$P = \sum_{j=1}^{n} C_j \times df^r_{m_j} = \sum_{j=1}^{n} C_j \times df^{rf}_{m_j} \times df^{cs}_{m_j} \tag{7.6}$$

where all symbols are as defined before and

P = the all-in price of the corporate bond
n = number of outstanding cash flows of the corporate bond
m_j = term in years from the value date until cash flow C_j
C_j = jth cash flow of the corporate bond (the last cash flow will be the redemption payment plus the coupon payment).

To derive the credit spread term structure, we first derive a risk-free curve and then model the corporate bond with (7.6). Table 7.5 shows the cash flows per 100 nominal of a corporate bond paying a semi-annual coupon. The annual coupon rate is 9%. The value date is assumed to be 21 December 2004.

The risk-free discount factor in Table 7.5 is calculated from the risk-free curve and is known. In this example we assume a functional form for the credit spread discount function is $f(\beta_0,\beta_1,\beta_2|t)$ which denote the facts that the chosen function has three parameters that have to be estimated, namely

Table 7.5 Cash flows for a corporate bond paying a semi-annual coupon of 9 percent

Cash flow date	Cash flow C_j	Term m_j	Risk-free discount factor df^{rf}	Credit spread discount factor df^{cs}	Risky discount factor df_m	Present value
1 Jun 2005	4.5	m_1	$df^{rf}(1)$	$f(\beta_0,\beta_1,\beta_3 \mid m_1)$	$df_{m1} = df^{rf}(1) \times f(\beta_0,\beta_1,\beta_3 \mid m_1)$	C_1 x df_{m1}
1 Dec 2005	4.5	m_2	$df^{rf}(2)$	$f(\beta_0,\beta_1,\beta_3 \mid m_2)$		
1 Jun 2006	4.5	m_3				
1 Dec 2006	4.5					
…	…	…	…	…	…	…
1 Jun 2010	4.5					
1 Dec 2010	104.5	m_{12}			repeated in each column	C_{12} x df_{m12}
					Fitted all-in price:	$C_1\ df_{m1}$ + ... + $C_{12}\ df_{m12}$

$\{\beta_0,\beta_1,\beta_2\}$ and that the function depends on t, the term from the value date to the cash flow date (in years). The exact function is not important, because we just want to illustrate how the modeling should be done. The parameters of the credit spread discount function are estimated by minimizing the weighed difference between the actual and fitted all-in prices. Please refer to Chapter 5 for a complete discussion on this procedure.

Another approach to derive a zero credit spread curve is to derive two zero interest rate curves, one from risk-free instruments and one from risky instruments (like corporate bonds). The credit spread curve is then derived as the difference between the corporate zero curve and the risk-free zero curve, calculated at each tenor. The tenors at which the credit spreads are calculated have to be fixed. The obvious problem with this approach is that a lot depend on the interpolation functions used and the number of instruments available to derive the two zero curves.

7.1.5 Credit default swaps

The value of credit derivatives is driven by the credit risk of institutions (other than the counterparties to the credit derivative transaction itself). The principal feature of credit derivatives is that they provide the means to isolate and trade credit risk with the purpose of replicating, transferring, and/or hedging credit risk. Credit default swaps are the most common type of credit derivative. The transaction operates as follows (Das, 2004):

- The protection buyer pays a periodic fee to the protection seller on an identified reference entity.
- If there is a credit event in respect of the reference entity, the protection seller makes an agreed payment to the protection buyer to cover any loss suffered because of credit exposure to the reference entity.
- If there is a no credit event, there are no payments by the protection seller.

An interesting idea is to use the credit default swap premium as an estimate of the credit spread.

Up to this point we have assumed that all bonds have the same liquidity. In practice this is not the case. Government bond markets are larger and more liquid than corporate bond markets, which implies that in addition to the credit spread, investors also require an additional liquidity premium as compensation (Annaert and Ceuster,1999).

Longstaff *et al.* (2004) calculate credit spreads from corporate and risk-free bonds using approach 3 discussed in Section 7.1.2. They measure the default component of these spreads by following these steps:

- They define r_t as the risk-free rate, λ_t the intensity process governing default, and γ_t a liquidity process. Each of these processes is assumed to be stochastic, and they evolve independently of each other.
- They assume an intensity process λ_t as follows:

$$d\lambda = (\alpha - \beta\lambda)dt + \sigma\sqrt{\lambda}dZ_\lambda \tag{7.7}$$

  where α, β and σ are positive constants to be estimated, and Z_λ a standard Brownian motion.
- The liquidity process γ_t is given by

$$d\gamma = \eta dZ_\gamma \tag{7.8}$$

  where η is a positive constant to be estimated and Z_λ a standard Brownian motion.
- They argue that the liquidity process plays a role when valuing corporate bonds, but not when valuing credit default swaps. With this argument they show that when valuing a credit default swap with maturity T years and assuming the premium s is paid continuously, the protection buyer will pay the following:

$$PB_T = E\left[s\int_0^T \exp\left(-\int_0^t (r_k + \lambda_k)dk\right)dt\right] \tag{7.9}$$

and the protection seller will pay the following:

$$PS_T = E\left[W \int_0^T \exp\left(-\int_0^t (r_k + \lambda_k)dk\right)dt\right] \tag{7.10}$$

where it is assumed that 1–W of the par value of the bond is recovered when a default occurs. Please note that E[.] denotes the expected future value. By setting (7.9) equal to (7.10), it is possible to solve for the credit default swap premium that is paid. Using market quotes for credit default swaps, the next step is to estimate the parameters of the default component given by (7.7), with the constraint that this default component must fit the credit default swap spread accurately.

- Assuming a continuous coupon c is paid, the corporate bond is valued as:

$$\begin{aligned} CB_T = {} & E\left[c \int_0^T \exp\left(-\int_0^t (r_k + \lambda_k + \gamma_k)dk\right)dt\right] \\ & + E\left[\exp\left(-\int_0^T (r_t + \lambda_t + \gamma_t)dt\right)\right] \\ & + E\left[(1-W)\int_0^T \lambda_t \exp\left(-\int_0^t (r_k + \lambda_k + \gamma_k)dk\right)dt\right] \end{aligned} \tag{7.11}$$

Thus by using the parameters of the intensity process and still assuming a zero liquidity process, (7.11) will give us the price at which the corporate bond will trade should there be no liquidity issues. Longstaff *et al.* call this the *liquidity-adjusted corporate bond.*

- The default component is now obtained by solving iteratively for the yield on this liquidity-adjusted corporate bond and subtracting from this the yield of a risk-free bond with similar cash flows (approach 3 in Section 7.1.2).

Longstaff *et al.* (2004) show that there is quite a big difference between the estimated default component and the credit default swap premium, which means that the credit default swap premium cannot directly be used as an estimate of the default component of a corporate bond yield spread above the risk-free yield.

7.1.6 Summary

In Section 7.1.2 various techniques are discussed to estimate credit spreads by way of an example. To summarize these techniques:

- In approach 1 we calculate the credit spread as the difference between the yields to maturity of a corporate and risk-free bond that have similar maturities. We ignore the fact that their cash flow structures may differ.

- With approach 2 we plot the yield to maturity of the risk-free bonds against duration and interpolate a risk-free yield off the curve with a similar duration to the corporate bond. The credit spread is estimated as the difference between the corporate bond yield and the synthetic risk-free yield.
- With the third approach a synthetic government bond is created that exactly matches the cash flow structure of the non-government bond. The synthetic bond is priced off the risk-free zero curve and then we solve iteratively to find the corresponding risk-free yield to maturity of the synthetic bond. The credit spread is the difference between the corporate bond yield and the synthetic risk-free yield.
- Approach 4 shows a way in which an average credit spread above zero rates can be estimated. This spread is fixed across all tenors. In Section 7.1.4 we see that a credit spread curve derived with this technique is similar to the par-rate concept discussed in Chapter 2. This means we need to bootstrap the average credit spread curve to get to a "zero" spread curve. The credit spreads derived from approach 4 should be similar to those derived from approach 3.
- When we have zero coupon bonds available, it is possible to directly derive credit spreads above zero rates by just calculating the spread between the zero rate from the corporate zero-coupon bond and the zero rate from the risk-free zero-coupon bond.
- It is also possible to derive a zero credit spread curve by deriving two zero interest rate curves: one from risk-free instruments and one from risky instruments (like corporate bonds) and then calculating the spread curve between the two zero curves.
- In Section 7.1.5 we show how to derive the credit spread from credit default swaps.

There are various assumptions underlying each of the approaches to estimate credit spreads. It is important to note that we get very different estimates of credit spreads using the same dataset, depending on the assumptions we are willing to make.

7.2 LIQUIDITY PREMIUM

7.2.1 Overview

By defining corporate spreads as the difference between the yield of a corporate bond and that of a risk-free bond with similar cash flows, we know the spreads consist of two main components. The two components compensate investors for the credit quality and liquidity of the corporate bond. Longstaff *et al.* (2004) find that the credit risk component accounts for the majority of the corporate spread across all ratings.

There are various market variables that can be used as an indication of the liquidity of an instrument, for example Fleming (2001), Longstaff *et al.* (2004), and De Jong and Driessen (2004):

- The *bid/ask spread* of the instrument: the bigger the spread, the less liquid is the instrument. However, a drawback of the bid/ask spread is that the bid and offer quotes are usually only appropriate for a limited period and a specific quantity of the instrument.
- The *rise in price* that occurs with a buyer-initiated trade. This is measured as the slope of the regression of price changes against net trade volume. It is referred to as the *Kyle lamda*.
- *Trading frequency*, which reflects the number of trades executed within a specified interval, not taking into account the trade size. High trading frequency reflects a more liquid market.
- The *notional amount outstanding* measures the general availability of the bond in the market. The larger issues are usually more liquid.
- The *age of the bond*, which uses the notion of *on-the-run* and *off-the-run* bonds. On-the-run bonds refer to the most recently issued securities. The measure assumes that on-the-run bonds are much more liquid than off-the-run bonds.
- The *term to maturity* of the bond: longer-term bonds are usually less liquid than short-term bonds.
- The rating of the institution that issued the bond may also play a role. The higher the rating, the higher the liquidity of the bond should be.

In the next section we consider various ways in which to estimate liquidity spreads from market instruments. In all cases we ignore the effects of tax.

7.2.2 Estimating liquidity spreads

In this section we consider various ways in which to estimate liquidity spreads depending on the instruments available in the market.

Estimating a liquidity spread: approach 1

When we work in a market where credit derivatives are actively traded, it is possible to use the approach suggested by Longstaff *et al.* (2004), where credit default swaps are used to estimate a default and non-default portion of the corporate bond spread. This approach is discussed in Section 7.1.5. The non-default portion of the corporate spread is simply the difference between the total corporate spread and the default component. It is assumed here that taxes do not play a significant role, so the non-default portion is assumed to be attributable to the liquidity of the instrument.

Estimating a liquidity spread: approach 2

Another way to estimate the liquidity premium is to compare the spreads between the yields of on-the-run and off-the-run bonds with similar cash flows and credit quality. We expect on-the-run bonds to be more liquid and thus have a smaller liquidity spread. An interesting problem may arise when we compare an off-the-run benchmark bond with an on-the-run bond, because this spread is not easy to interpret (Fleming, 2001). Remember that benchmark bonds are issued at convenient points along the yield curve so they attract investors, and thus are liquid. By comparing an off-the-run benchmark with an on-the-run bond, we may be comparing two liquid instruments and not get an appropriate estimate of the liquidity spread.

With approach 2 we need a lot of bonds with similar credit quality in the market. Usually it will also be difficult to find instruments with similar cash flow characteristics, which means that interpolation techniques have to be applied, and this adds to the uncertainty in the model.

Estimating a liquidity spread: approach 3

Consider a market where there are only five benchmark government bonds available. We know these bonds are liquid, and they are considered to be risk-free. We can compare these benchmark bonds with other government bonds (with similar features) and estimate liquidity premiums with similar techniques to those discussed in Section 7.1, where credit spreads are derived. The differences between these two sets of bonds should reflect the liquidity premium, because both sets of bonds are risk-free.

We assume that the liquidity premium estimated from government bonds applies directly to corporate bonds. However, when we have corporate bonds that are liquid, we can follow a similar approach to estimate the liquidity premium. The point is that we need to compare instruments with the same credit quality, but with different liquidity.

7.2.3 Summary

Some research indicates that tax effects should be taken into account when the non-default component of the corporate spread is estimated (Elton *et al.*, 2001). However, Longstaff *et al.* (2004) analyzed the non-default component of corporate spreads to test for tax effects. They found only weak evidence that tax affects the spread, and concluded that liquidity measures play a more significant role. We ignored the tax effect when estimating the liquidity premium.

7.3 COUNTRY RISK PREMIUM

7.3.1 Overview

Investors in emerging markets are, among other things, exposed to economic and political phenomena that are not generally present in more developed markets. This is usually referred to *country* or *political risk* (Clark, 2002). The investors require a premium to compensate them for the additional risk, and this premium is called the *country risk premium*.

Borio and Packer (2004) explain the three main views on country risk. The first view is known as "debt intolerance," which refers to the reduced debt-bearing capacity of emerging market economies because of their history of economic mismanagement. High inflation and past defaults are indications of deeper economic problems which may discourage investors to invest in these countries. The second view is known as "original sin," and argues that countries less able to borrow in their own currency should be riskier, because exchange rate depreciations should make it increasingly difficult to pay external debts. Similar to this is the third view which is known as "currency mismatches." The net debt positions in a foreign currency that is sensitive to exchange rate fluctuations can make countries more vulnerable, and increase the costs in a crisis.

Borio and Packer (2004) calculate the average of the foreign currency sovereign ratings published by Moody and Standard & Poor, as a measure of country risk. They do a regression analysis by recoding the ratings numerically, with AAA (Aaa) equal to 17, down to CCC+ (Caa1) which is set equal to 1. They consider a variety of explanatory variables including:

- macroeconomic variables like the per capita GDP, inflation, and real GDP growth
- debt burden as measured by external debt/exports
- government finance measured by public debt/GDP
- political, socioeconomic variables like corruption and political risk
- dummy variables indicating the history of defaults and the percentage of time high inflation has occurred
- financial development of the country measured by the foreign exchange spot and derivatives turnover/GDP.

They find that economic and structural determinants, per capita GDP, measures of corruption and political risk, and proxies for a history of economic mismanagement account for most of the variation in country risk.

Techniques on how to estimate the country risk premium with yield curves are discussed in the next section.

7.3.2 Estimating country risk premiums

To estimate country risk premiums, we estimate the spreads between the yields of risk-free bonds issued in a foreign currency and the yields of risk-free instruments issued in that country (Damodaran, 1998; Bernoth, von Hagen and Schuknecht, 2004; De Estudios *et al*., 2003; Clark, 2002). By doing this we ensure that credit risk does not play a role and that we get a pure estimate of the country risk premium. The problem however, is that this spread may be affected by a liquidity premium.

It is important to note that the same approaches as are used to estimate credit spreads (please refer to Section 7.1) can be used to estimate country risk premiums. The only differences are in the instruments that are used, as well as the assumptions made regarding credit quality and liquidity.

Consider a simple example where the country risk premium for South Africa is calculated. In this example we analyze the spread between the South African US$-denominated bonds and the US$ Treasury strips curve. This is done by following these steps:

- Derive a zero curve from the RSA (Republic of South Africa) US$-denominated bonds.
- Derive a zero curve from the US$ Treasury strips.
- Interpolate these two zero curves to get values at specified tenors. The spread between the two zero rates at each tenor is an indication of the country risk premium.

Table 7.6 shows the information on the four RSA US$-denominated bonds. These bonds pay semi-annual coupons. To derive the zero curve from these

Table 7.6 Republic of South Africa US$-denominated bond information

Maturity date	Bond code	Coupon
17 Oct 2006	AC89	8.375
19 May 2009	AE46	9.125
25 Apr 2012	AG93	7.375
23 Jun 2017	AD62	8.5

bonds, we used a cubic spline function. The BBA Libor rates are used to anchor the curve at the short end. The zero rates derived from the US$ Treasury strips are linearly interpolated.

Figure 7.5 shows the RSA US$-denominated and US$ Treasury strips zero curves for 10 April 2003, with data obtained from Bloomberg. It is interesting to see how the spread between these two curves widens as the maturity increases.

The problem with this example is that we know these RSA US$-denominated bonds are not very liquid, so a part of the estimated country risk premium also captures a liquidity premium. We could estimate the liquidity premium with one of the approaches discussed in Section 7.2; however, it would mean that we need more liquid RSA US$-denominated instruments which we do not have available. Another approach would be to just use different US$ Treasury instruments and calculate a liquidity spread from them; adjust the US$ Treasury strips curve upward with this liquidity spread; and then calculate the country risk premium as before from the adjusted strips curve. This means that we are assuming that the liquidity premiums estimated from US$ Treasury instruments are the same between countries, and this may be too large an assumption.

Usually when we estimate the country risk premium, we repeat this analysis with a historical data series. We then get a series of country risk premium estimates at each tenor. A descriptive statistic like the average or

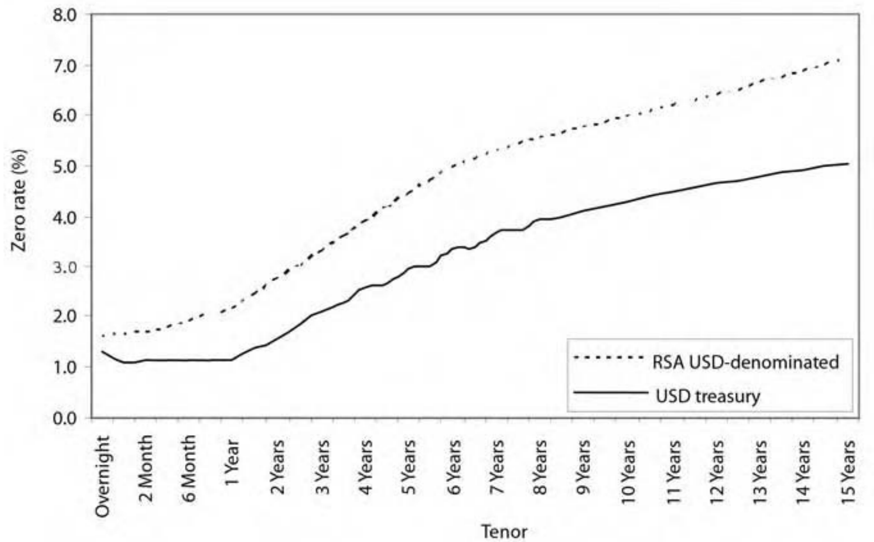


Figure 7.5 Comparison of the zero curve derived from Republic of South Africa US$-denominated bonds, and the zero curve derived from US$ Treasury strips

median spread calculated at each tenor is then taken to be the country risk premium for that tenor.

An alternative way to estimate the country risk premium is to use the relationship between domestic interest rates and foreign interest rates discussed in Chapter 2. The interest rates from two countries are linked by the spot and forward exchange rates, and the relationship is as follows:

$$\left[1 + i_d \frac{d}{DC_d}\right] = \frac{X_f}{X_s}\left[1 + i_f \frac{d}{DC_f}\right] \tag{7.12}$$

where

X_s = spot exchange rate
X_f = forward exchange rate
i_f = foreign interest rate (this is a simple rate)
d = number of days in the investment period
i_d = domestic interest rate (this is a simple rate)
DC_d = number of days in the year as specified by the daycount convention of the domestic market
DC_f = number of days in the year as specified by the daycount convention of the foreign market.

When the exchange rates X_s and X_f are actively traded in the market, and the foreign interest rate i_f is known, we can solve for the domestic interest rate i_d in (7.12). The spread between this implied domestic interest rate i_d and an interest rate quoted in the domestic market with a similar tenor is an estimate of the country risk premium. The problem with this approach is that the forward exchange rates quoted in the market may not be liquid enough for tenors greater than one year, which means the resulting spread will then also have a liquidity portion.

7.4 CONCLUDING REMARKS

The ways in which credit, liquidity, and country risk premiums are estimated are very similar. The main difference is the credit quality and liquidity of the instruments used in the analysis.

To estimate credit spreads we mainly considered the differences between yields on corporate and risk-free bonds with similar cash flow structures. However we showed that credit default swaps can also be used in the process.

When deriving liquidity premiums, we consider the spreads between two sets of instruments with the same credit quality. We then assume that the differences between the yields of these two sets of instruments can be explained by a liquidity premium.

To estimate country risk premiums, we determine the spreads between the yields of risk-free bonds issued in a foreign currency and the yields of risk-free instruments issued in the foreign country. The main idea is to get rid of the credit risk issue when estimating the country risk premium. In Section 7.3 we consider an example where a zero curve derived from South African US$-denominated bonds is compared to a US$ Treasury strips zero curve. The main problem, however, is that a portion of the estimated country risk premium can be attributed to the liquidity premium, and it is difficult to separate it from the country risk premium.

CHAPTER 8

Risk Issues

When any yield curve model is created, there are some risks that have to be measured, managed and possibly be prevented.

In Section 8.1 interest rate risk management is discussed. It is important to understand what types of interest rate risks there are and how to measure them, because this helps to identify potential problems in a yield curve model.

Sections 8.2 and 8.3 covers operational risk and model risk respectively. Some examples are discussed showing where these risks may occur with respect to yield curves. In practice model risk is considered to be a subsection of operational risk.

Liquidity risk is discussed in Section 8.4. Some examples show how to get a feel for the liquidity of the various instruments that are considered when deriving a yield curve.

8.1 INTEREST RATE RISK

8.1.1 Definition

When we have a portfolio with interest rate instruments, there are usually three types of risks for which allowance has to be made: parallel risk, pivot risk, and basis risk.

Parallel risk refers to the loss that a portfolio can incur when the whole curve moves up or down by a certain number of basis points. With *pivot risk* we make allowance for the fact that different parts of the yield curve may move in different directions, for instance the short-term rates may move down and the long-term rates may move up from one day to the next. Finally we have to consider the situation when a traded security and its hedge are valued from two different curves. With *basis risk* we allow for the fact that these two curves may not necessarily move together. This means the hedge may not always be effective, which then increases the risk of the portfolio.

8.1.2 Measuring parallel and pivot risk

There are various ways in which to measure parallel risk. A primitive way is to stress the whole yield curve up and down, using a stress factor, to determine the effect on the portfolio. Typically the risk manager looks at various scenarios for the stress factor, for instance stressing the whole yield curve from –200 bps to 200 bps in steps of 50 bps. Depending on how a portfolio is structured, a move of 50 bps may result in a bigger loss than a move of 200 bps.

However, as is discussed in Section 8.1.4, the stress factors across tenors usually differ quite significantly. Another way to measure the risk is to consider each of the maturity buckets individually. Each tenor on the yield curve that corresponds to a maturity bucket is stressed individually with one basis point, which yields a PV01 value per maturity bucket. The PV01 is the effect the one basis point move has on the portfolio value. The idea is then to multiply the PV01 in each bucket by the stress factor for that bucket. By setting risk limits per maturity bucket as well as a cumulative bucket risk limit, this approach captures parallel and pivot risk. Remember that risk limits denote the maximum amount of risk that may be taken on through normal trading operations. The risk limits are determined by the board and senior management, and are monitored by the risk management function (Das, 2004).

Not all instruments in a portfolio are valued off the same curve, so typically we calculate a PV01 value for each curve in the maturity bucket. Figure 8.1 shows an example where the value of the portfolio

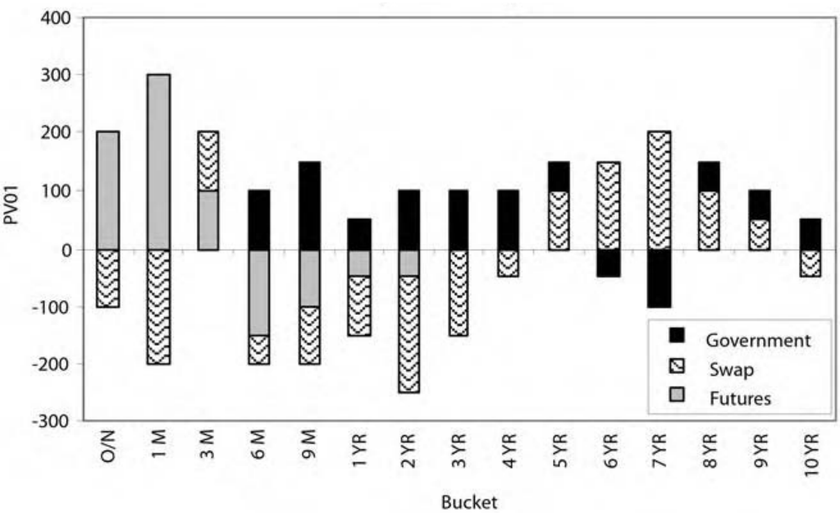


Figure 8.1 PV01 per yield curve per maturity bucket

depends on three curves: a curve derived from interest rate swaps, a curve derived from futures, and finally a curve derived from government bonds. The PV01 for each maturity bucket for each curve is calculated as discussed above. To determine the parallel and pivot risk of the portfolio, we stress the PV01 of each curve in each bucket with the appropriate stress factors. The way in which the stress factors are determined is discussed in 8.1.4. With a risk limit per bucket, traders are forced to put trades in the portfolio that will ensure that the bucket risk numbers always fall within these limits.

Consider a portfolio as a complex function of a yield curve, denoted by $P(r_1,...,r_n)$ where r_i denotes the interest rate at tenor $i = 1,...,n$. It is important to note that in general:

$$[\mathrm{P}(r_1 + 1bp,\ldots, r_n) - \mathrm{P}(r_1,\ldots, r_n)] \times SF_1 \neq \mathrm{P}(r_1 + SF_1,\ldots, r_n) - \mathrm{P}(r_1,\ldots, r_n) \tag{8.1}$$

where SF_i is the stress factor for bucket i.

From (8.1) it is clear that stressing a maturity bucket by one basis point, determining the possible loss (by subtracting the actual portfolio value) and then multiplying by the stress factor, does not give the same value as just stressing the interest rate in the bucket by the stress factor and then calculating the possible loss. This means that the approach discussed above is just an approximation of the risk per bucket, and will not be appropriate when there are a lot of options in the portfolio. In such a case it may be more appropriate to stress each bucket with the stress factor.

There are various other approaches that can be used to determine pivot risk, sometimes also referred to as non-parallel risk. For instance Phoa (2000) uses principal component analysis to identify the types of yield curve shifts that occur most often.

8.1.3 Determining basis risk

When a trade and its hedge are valued off different yield curves, the two curves may move in different directions from one day to the next. This means that the hedge may not be effective, so that the risk of the portfolio is higher. This type of scenario should be captured by the *basis risk* measure.

In order to measure basis risk, we would typically look at the PV01 per maturity bucket per yield curve as is shown by Figure 8.1. The risk manager has to estimate the proportion of positions and their relative hedges that may exhibit basis risk. There are various ways in which to estimate this quantity, which will be referred to as the matched portion.

Basis risk: approach 1

The most basic way to calculate the matched portion is to add all the positive PV01s and negative PV01s separately across all buckets. The minimum of the total positive and absolute of the total negative PV01s is an estimate of the matched portion. The calculation is illustrated for the data of Figure 8.1 that is captured in Table 8.1. The PV01s were calculated per bucket per curve as is discussed in Section 8.1.2. To determine the matched portion we have that:

- the total positive PV01 is 600 + 700 + 800 = 2100
- the total negative PV01 is – 350 – 1 000 – 150 = –1500
- the absolute of the total negative PV01 is thus 1500

so that it follows that the matched portion = min(2100; 1500) = 1500. The matched portion has to be stressed with a stress factor assumed to be appropriate.

The problem with this approach is immediately apparent. We do not allow for different stress factors for different buckets, even though we know in practice they are very different.

Table 8.1 Matched portion across buckets by calculating total positive and total negative PV01 per curve (illustrated by Figure 8.1)

Maturity bucket	Futures	Swap	Government
O/N	200	–100	0
1 month	300	–200	0
3 month	100	100	0
6 month	–150	–50	100
9 month	–100	–100	150
1 year	–50	–100	50
2 year	–50	–200	100
3 year	0	–150	100
4 year	0	–50	100
5 year	0	100	50
6 year	0	150	–50
7 year	0	200	–100
8 year	0	100	50
9 year	0	50	50
10 year	0	–50	50
Positive PV01	600	700	800
Negative PV01	–350	–1000	–150
Matched portion	1500		

Basis risk: approach 2

A more accurate way to determine basis risk is to calculate the matched portion per maturity bucket as is shown in Table 8.2. The matched portion is calculated as follows:

- Calculate the total positive PV01 for each bucket. This is column 5, "Bucket positive PV01" in Table 8.2.
- Calculate the total negative PV01 for each bucket. This is column 6, "Bucket negative PV01" in Table 8.2.
- The matched portion per bucket is then the min(bucket positive PV01; absolute of the bucket negative PV01). This is column 7, "Bucket matched portion" in Table 8.2.

The total matched portion is then 1050, which is obtained by adding all the bucket matched portions in column 7. We now have two approaches available to us to determine the basis risk: we can either multiply this total matched portion with a stress factor, or multiply each of the bucket matched portions with a different stress factor. The latter should be more accurate, because then we allow for a term structure of stress factors. In

Table 8.2 Matched portion per bucket (illustrated by Figure 8.1)

Maturity bucket	Futures	Swap	Government	Bucket positive PV01	Bucket negative PV01	Bucket matched portion
O/N	200	–100	0	200	–100	100
1 month	300	–200	0	300	–200	200
3 month	100	100	0	200	0	0
6 month	–150	–50	100	100	–200	100
9 month	–100	–100	150	150	–200	150
1 year	–50	–100	50	50	–150	50
2 year	–50	–200	100	100	–250	100
3 year	0	–150	100	100	–150	100
4 year	0	–50	100	100	–50	50
5 year	0	100	50	150	0	0
6 year	0	150	–50	150	–50	50
7 year	0	200	–100	200	–100	100
8 year	0	100	50	150	0	0
9 year	0	50	50	100	0	0
10 year	0	–50	50	50	–50	50
					Total:	1050

other words we allow for the fact that stress factors will usually differ across buckets.

The next issue that has to be addressed is that by following the second approach, we have PV01s calculated from three yield curves. and the basis risk stress factors may differ significantly for different combinations of curves. Please refer to Section 8.1.4 for a discussion on how the stress factors are calculated.

Basis risk: approach 3

To allow for the differences in basis risk stress factors calculated from different combinations of yield curves, we can follow these steps to determine the basis risk:

- Determine the biggest positive PV01 and biggest negative PV01 in each bucket. For example, in Table 8.2 for the six-month bucket we have the biggest positive PV01 of 100 which is contributed by the Government curve. The biggest negative PV01 is –150 which is contributed by the Futures curve.
- Calculate the matched portion as the minimum of the positive PV01 and the absolute of the negative PV01. For the six-month bucket in Table 8.2 we will then have that the matched portion = min(100, 150) = 100.
- Multiply the matched portion in each bucket with the basis risk stress factor calculated from the two curves that contributed to the matched portion. We see that for the six-month bucket in Table 8.2 the two relevant curves are the Government and Futures curves.

This is the first iteration. Following the exact same procedure, we now calculate the matched portion of the two sets of PV01s that contribute second most to each bucket. This is done as follows:

- Determine the second biggest positive PV01 and second biggest negative PV01 in each bucket. For example, in Table 8.2 for the six-month bucket we only have one positive PV01, which was used in the previous step. This means we do not have a second biggest positive PV01, so this value is taken to be 0. The second biggest negative PV01 is –50 which is contributed by the Swap curve.
- Calculate the matched portion as the minimum of the second biggest positive PV01 and the absolute of the second biggest negative PV01. For the six-month bucket in Table 8.2 we will then have that the matched portion = min(0, 50) = 0.
- Multiply the matched portion in each bucket with the basis risk stress factor calculated from the relative two curves' price series.

We continue this process until all the PV01s across all buckets have been matched.

A possible problem with this approach is that we do not determine the matched portions across different buckets. It may be possible that a trader has a three-year instrument that is hedged with a two-year instrument. This can be allowed for by following the iterative procedure described as in approach 3 above.

We considered the example where there are only three curves, which means that in each bucket we have a third PV01 portion that is not matched. In this case we can consider the correlation between buckets and then start matching the positive and negative PV01s in the two buckets that shows the highest correlation, then the second highest correlation, and so on, until all the PV01s have been matched across buckets and stressed with the appropriate stress factor.

8.1.4 Calculating stress factors

Stress factors are the quantities by which we stress the various parts of the yield curve to determine the effect different yield curve moves may have on the value of the portfolio.

The stress factor for parallel and pivot risk is determined by calculating the differences in the yield curve values over time. Obviously we need a database of historical yield curves. We know that the yield curve does not always move by the same amount at each tenor. There are various reasons for this, for instance the liquidity of the instruments in a certain maturity range.

To allow for the fact that rates at different tenors on the yield move by different amounts, we divide the yield curve into maturity buckets and determine the move in each maturity bucket. The steps to follow are:

- Decide on the appropriate maturity buckets, for instance {overnight; one month; three month; six month; nine month; one to ten years}. The buckets must be expressed in terms of days, for instance the one-month rate corresponds to 30 days, the one-year rate corresponds to 360 days, and so on, depending on the daycount convention.
- Interpolate to get values from the yield curve at each of the specified tenors (maturity buckets). Typically the values on the yield curve may not exactly correspond with the specified maturity buckets because of the specific daycount conventions and business day rules that were taken into account when the curve was derived. For instance, the one-month rate might refer to a 30-day period today whereas tomorrow it might refer to a 33-day period.
- Calculate the daily differences in the yield curve values at each tenor:

$$d_{i,t} = r_{i,t} - r_{i,t-1} \tag{8.2}$$

where
$d_{i,t}$ = daily difference at maturity bucket i for day t
$r_{i,t}$ = rate at maturity bucket i for day t
i = denotes the maturity bucket.

This assumes we are working with a one-day holding period. Typically practitioners who work with a ten-day holding period would calculate ten-day differences. A *holding period* refers to the period that it is assumed it would take to get out of any deal in the specific market.

- Calculate the high percentile of the differences $d_{i,t}$ in each bucket. This is usually specified at the 99 percent confidence level for risk management purposes. These percentiles are the stress factors for each of the maturity buckets.

These stress factors capture the parallel moves at each tenor of the yield curve as well as the fact that the different sections of the curve may not necessarily move together: in other words this is the *pivot risk*. The stress factors have to be calculated for each yield curve that is used to value the portfolio.

The next step in the process is to determine how the different curves move with respect to each other. This is done as follows:

- Calculate the moves at each tenor with (8.2) for each curve.
- Calculate the differences in the moves between any two curves $cd_{i,t}$ at tenor i on day t, in other words:

$$cd_{i,t} = d^A_{i,t} - d^B_{i,t} \tag{8.3}$$

where $d^A_{i,t}$ and $d^B_{i,t}$ denotes the daily differences for curves A and B respectively, calculated from (8.2).

- Calculate the high percentile of $cd_{i,t}$ for each maturity bucket. These percentiles are the stress factors to be used when we calculate basis risk.

In a situation like approach 1 discussed in Section 8.1.3, where we need one stress factor for all the buckets, we will typically use the average of the $cd_{i,t}$ values as the stress factor.

With this calculation it is obvious that if the two curves always move together, the basis risk stress factor will be very small.

8.1.5 Risk implications when deriving yield curves

In this section we consider various examples where we may find inappropriate risk numbers because of problems in the way yield curves are derived or used.

Example 1

Say we derive a yield curve from bonds that are subject to "catastrophic" jumps, in other words, changes in the yield curve that cannot be explained by changes in the underlying bonds that were used to derive the curve. This means that we may not necessarily capture actual market events when the stress numbers are calculated. Our stress factors will be much higher than is appropriate. By using the stress factors to determine interest rate risk we will overestimate the actual market risk.

Example 2

Consider a situation where the yield curve is derived from coupon-paying bonds, but does not fit the underlying bonds perfectly. We refer to this curve as a *best-fit curve*. The actual changes in the bond prices will not be accurately captured by the best-fit yield curve. By comparing the best-fit yield curve with a perfect-fit curve, we may find very big basis risk stress numbers. These numbers may be counterintuitive in that the practitioner knows the two markets move together more than is shown by this basis risk stress number.

Example 3

Consider the example where two curves are derived with the same instruments out to one year, but with different instruments from thereon. A situation can arise where an instrument is valued off one curve, and its hedge is valued off the second curve. We know that there is theoretically zero basis risk when these two instruments have terms to maturity of less that one year. However, the risk manager may pick up basis risk because the one curve may be interpolated with a cubic spline and the other with linear interpolation. This example illustrates the importance of applying consistent interpolation methods.

Example 4

An interesting situation occurs when we are saving the same yield curve to two different subsystems. Say the first subsystem requires yearly rates out to ten years and the second subsystem requires half-yearly rates out to ten years. The subsystems employ the same interpolation techniques and business day rules. Risk managers could find basis risk if they were not aware that the curves in these two subsystems are the same. This is because the curve with the yearly tenors has to be interpolated to obtain the half-yearly tenors, which means we need to make an assumption about the shape of the curve for a longer period than when we look at the curve with the half-yearly rates. It is important that we ensure that the tenor points correspond when deriving curves from similar instruments and saving them to different subsystems.

8.2 OPERATIONAL RISK

8.2.1 Definition

Operational risk covers a variety of risks that are not covered by market and credit risk. The focus is on measuring the risk of financial losses due to a failure of internal processes or systems that may arise from human error, market conditions, or natural disasters. Please refer to Das (2004) for a discussion on some of the definitions of operational risk as published by various organizations.

The focus on operational risk is driven by the fact that institutions that were required to hold capital against credit and market risk in the past are now forced to hold additional capital against operational risk. The main reason for this additional charge is that it is widely accepted that operational risk is something that has to be managed in financial institutions. The need for this type of management is stressed by the occurrences of a number of well-documented operational risk losses. There are also potentially large increases in operational risks because of the increasing growth in the financial industry (Das, 2004; Jorion, 2003).

Operational risk covers systems, personnel, legal, and regulatory risks. In the next section operational risk with the focus on yield curve modeling is discussed.

8.2.2 Yield curve operational risk examples

Systems risk includes processing risk. Typically this can occur as a result of:

- incorrect rates loaded into a subsystem
- delays or failures in the process of generating the yield curves
- outdated technological platforms
- system downtime.

Personnel risk is related to staff turnover and experience. Staff are trained at an expense. However should they not be trained adequately, they may not have the necessary knowledge and skills to reduce errors in the yield curve models. These risks are discussed in more detail in Section 8.3 under model risk. It is always important to have a back-up person to perform key tasks, which is why most institutions have a policy that forces each staff member to take mandatory leave during certain periods. This ensures that key-person dependency is reduced, and in a way it can also be seen as a type of audit process, because the back-up person should pick up inconsistencies or possible errors.

Legal risk with respect to yield curves may occur when a contract with a counterparty refers to a prespecified yield curve that will be used at the contract expiry date to settle the transaction. However, an unexpected loss may be incurred when a different yield curve is used to value the transaction.

Regulatory risk may occur when models are back-tested over time and errors are found. These errors may lead to the regulators increasing the capital requirements because they do not have trust in the internal systems of the financial institution. The result of this may also be *reputation risk* for that institution.

Other more practical examples of where operational risk can occur through human error are:

- Consider a yield curve that is derived from futures prices in a spreadsheet that contains electronic links to the published data vendor pages. It is important for the rates administrator to ensure that the links are rebuilt when the futures contract expires, since incorrect links may point to invalid or outdated data, which in turn will lead to problematic yield curve outputs.
- It may be impossible to build electronic links to a system, so the rates of the underlying instruments have to be input manually into the yield curve model. This increases the possibility of human error.

The main operational risk with respect to yield curve modeling is *model risk*, which is discussed in detail in Section 8.3.

8.3 MODEL RISK

8.3.1 Definition

Model risk is an aspect of operational risk, and reflects possible financial losses owing to inaccurate or incorrect models. Das (2004) gives some examples of model risk which include the following:

- Inaccurate models may lead to transactions being valued incorrectly. The financial loss is realized when transactions are closed out in the market.
- Inaccurate models may lead to inaccurate hedges being established. The loss from hedging errors is shown as the difference between the hedge and the position being hedged.
- Incorrect models may lead to incorrect estimates of the risk underlying a transaction. Examples of assumptions that lead to incorrect risk estimates are assumptions regarding the distribution of price changes, inadequate specifications of the number of risk factors, errors in risk decomposition of positions, and inaccurate volatility/correlations. Incorrect estimates of the risk of a portfolio may have an effect on the capital being held against the positions, which may create *solvency risk*. At a regulatory level this means that the required back-testing of internal models will perform poorly. This can lead the regulator to impose higher capital charges.

It is important to note that an incorrect model may not necessarily result in a financial loss for the institution. There may be instances when the same model is used by all market participants, even when there are known shortcomings in the model.

In the next section we discuss model risk specifically with respect to yield curves.

8.3.2 Yield curve model risk examples

Model risk may arise in a yield curve context due to the failure of the yield curve model to capture actual market conditions correctly or the inappropriate application of a model. There are several different types of model risk (Das, 2004): failure in model design, implementation, input, and application.

Examples of failure in model design include:

- The failure to incorporate all relevant variables in the model through a lack of understanding of the underlying instrument. For example when deriving a curve from futures, the future rate has to be adjusted with a convexity adjustment. If a convexity adjustment is not allowed for, the level of the forward rates is not estimated adequately, and this may lead to the incorrect valuation of the portfolio.
- Incorrect assumptions regarding variables, for instance assuming a fixed volatility when estimating the convexity adjustment instead of allowing for a volatility smile.
- The model may be overly complex. By adding factors to a model that have to be estimated, we increase the complexity and uncertainty in the

model. For example, the convexity adjustment adds more uncertainty to the yield curve model because it has to be estimated from the underlying risk factors that are assumed to follow specific stochastic processes. It is usually necessary to estimate volatility and mean-reversion parameters, which add to the complexity of the model.

- Failure to specify the correct relationship between the variables.
- A lack of understanding of trading liquidity may lead to the mis-specification of the model. It is important to always use the most liquid instruments available to derive a curve. These instruments must have market quotes available daily, or distortions will be introduced into the yield curve. It is also important to obtain rate quotes from a reputable source, or the resulting yield curve may not be an accurate reflection of market rates. This can lead to valuation problems and an inaccurate representation of the risk in the transactions.
- Incorrect setup of the mathematical model when deriving a yield curve. The yield curve model fitted to the bonds may be subject to "catastrophic" jumps, which means that the moves in the yield curve cannot be explained by actual moves in the underlying instruments. This is an indication that the yield curve model is not specified correctly to capture actual market events.
- Failure of the model to produce an accurate yield curve. For instance, when a descriptive yield curve model is fitted to bonds, an iterative technique is necessary to estimate the model parameters. The optimization routine may only find a local optimal value and not necessary a globally optimal value. The starting values chosen in the optimization routine play an important role here. This means that the process of estimating parameters increases model risk, because the optimization routine may fail.
- Incorrect assumptions regarding instruments may lead to incorrect models. We have seen how the procedures differ when fitting yield curves to different types of market instruments (please refer to Chapter 5). It is important to allow for the different factors like daycount conventions, business day rules, credit and liquidity, and the quoting convention of the instruments, as is discussed in Chapter 4.
- In Section 8.1.5 the effect that incorrect yield curve models have on the interest rate risk of a portfolio is discussed. The errors will have an effect not only on the stress factors (please refer to Section 8.1.4), but also on the quantification of the risk (please refer to Sections 8.1.2 and 8.1.3). We have seen how it is possible to pick up basis risk as a result of interpolation techniques that are not applied consistently across subsystems.

Model risk caused by incorrect model implementation refers to failures in software implementation, for instance programming errors. It also refers to failures in the process of back-testing the model under various stress scenarios to ensure the model gives consistent results. For instance, it is important that the yield curve model allows for public holidays. On public holidays there may be no quotes available for certain instruments, and the model has to allow for this. This also leads to the concept of model risk through the failure to have correct inputs.

Finally it is important to ensure that any yield curve is used the correct way. When the end-user does not understand the credit and liquidity quality, or the instruments from which the yield curve is derived, the curve may be used to value instruments for which it was not intended. For example, it is usually not appropriate to value interest rate swaps with a curve derived from government bonds.

8.3.3 Minimizing model risk

Model risk is minimized by ensuring that all models are validated by an independent body. It is important to note that this includes all models developed within the organization as well as those procured from external sources.

The model is validated by performing the following checks:

The correctness of the model has to be ensured on an ongoing basis, based on the latest available research in the market. When there are significant changes in the market, for instance the way a certain type of instrument is valued, the valuation model has to be updated to reflect the changes.

Where a historic relationship is assumed in a model, it is important to review the relationship regularly to ensure that there are no structural, market, or economic changes over the observation period that may impact the model's integrity. For instance, when correlation is an input to the model, the correlation need not necessarily be estimated daily, but it must be updated when there are big changes in the market. Another example is when knot points are set for a yield curve model. These knot points are usually fixed based on the maturities of the underlying bonds. However, these knot points have to be adjusted over time as the terms to maturity of the underlying bonds decrease over time.

The model input has to be checked for the following:

- The data are obtained from a reliable source and available daily. A secondary data source must always be specified. This source must then be used in the event that the data is not available from the primary data source, the data vendor.
- The data have to be used consistently. For instance, when the same instrument quote is used to derive two different yield curves, it is important to ensure that the rate is fixed for both curves at the same

time, otherwise the risk manager may pick up incorrect basis risk.

- A central department should take responsibility for the capturing and validation of the data (for instance the timing of closing prices). When each department in an organization is allowed to fix its own rates, it may be very difficult to determine a correct aggregate risk report for the organization.
- The necessary formatting and conversion of the data has to done before it is entered into the model. For instance, when simple rates are quoted in the market, but the yield curve model needs continuously compounded rates as input, the rates must be converted before input into the model.
- The inputs to the model must be relevant and used correctly. For instance, it is important to know whether the rates must be input as a percentage, in other words 3 to indicate 3 percent, or whether the model requires an input of 0.03.

As part of the validation process, it is important to authorize the purpose for which the yield curve may be used, in other words whether it may be used for pricing or valuation purposes; which instruments may be valued with the model; and for which divisions it would be appropriate to use the specific model.

To ensure model accuracy, a yield curve model can be compared with yield curves published by an external party. For instance, we can compare the derived swap curve with one published by a data vendor. It is also necessary to test the model under various stress scenarios to see whether it allows for all conceivable situations.

The documentation of all models is very important. It is important to describe the purpose of the model (valuation, pricing, risk management); the primary assumptions and limitations; descriptions of any revisions; data requirements; and details of any back-testing/stress testing that may have been conducted on the model.

The final step in minimizing model risk is to set a change control procedure in place. All changes to existing models should be subject to rigorous testing, and all relevant departments should approve the modifications to the model.

8.4 LIQUIDITY RISK

Liquidity risk refers to the loss incurred as a result of the inability of an instrument to trade. When there is more than one instrument available to derive a curve, the more liquid one is usually chosen, since the resulting yield curve is more representative of current market interest rates. Typical indicators of the liquidity of an instrument are:

- the amount issued (for a bond)
- turnover
- the bid/ask spread.

It is very important to include only those instruments for which prices are actively quoted in the market by a minimum number of market makers. Distortions may be caused in the yield curve if the current market price is not available.

Another indication that an instrument is fairly liquid is when it is both exchange-traded and available over-the-counter (OTC). These instruments are usually the more standard ones, with basic documentation that is widely accepted and understood.

8.5 CONCLUDING REMARKS

In Section 8.1 we considered the impact that incorrect yield curves may have on the measurement of interest rate risk. The problem is that the stress factors calculated from yield curve moves may be overstated, which means that we will overestimate the risk of a portfolio. Consider a situation where the performance of a trader is measured on a risk–return basis. The trader requires a very high return for the least amount of risk, because then the performance measure will prove the trader to be effective, and the trader will receive greater bonuses. If the risk is overestimated on certain positions, the risk–return measure will lead the trader to not take on more of these "risky" positions. In short, the incorrect risk measure may have an effect on the trader's decisions about which instruments to include in a portfolio. Using a similar argument, a risk measure that underestimates the risk will cause the trader's portfolio to be more risky than expected, which in turn may lead to great financial losses in times of stress.

We also considered various areas where operational risk and model risk occur in yield curve models. The major portion of the risk is however due to model risk, and in Section 8.3.3 we considered ways in which to minimize model risk.

The issues around liquidity risk are briefly discussed in Section 8.4. There is a more detailed discussion on how to estimate the liquidity premium to allow for this risk in Chapter 7.

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