HETEROCLINIC SOLUTIONS BETWEEN STATIONARY POINTS AT DIFFERENT ENERGY LEVELS

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1. Introduction

In the past ten years, there has been a considerable development of tools and techniques in the calculus of variations to study homoclinic and heteroclinic solutions of Hamiltonian systems. See e.g. [3, 10, 9, 11, 6]. A particular problem that has received much attention is

$$-\ddot{x} = W_x(t, x) \tag{1.1}$$

where $x \in \mathbb{R}^n$, W is 1-periodic in t, and has at least two time independent global maxima in x. An important special case arises in model problems of multiple pendulum type where W is periodic in the components of x. A typical result for (1.1) is the existence of a solution heteroclinic from ξ to η where ξ and η are a pair of time independent global maxima of W.

Suppose that $W(t, x) = a(t)V(x)$. The main goal of this paper is to present a simple minimization method to find heteroclinic connections between isolated critical points of V, say 0 and ξ, which are local maxima but do not necessarily have the same value of V. In particular for a class of positive slowly oscillating periodic functions a, it will be shown that if $\delta = |V(0) - V(\xi)|$ is sufficiently small and another technical condition is satisfied, then there exist a pair of solutions of (1.1), Q^+ heteroclinic from 0 to ξ and Q^- heteroclinic from ξ to 0. Note that when $V(0) \neq V(\xi)$, a cannot be constant. Indeed if a is constant, conservation of energy then implies $V(Q^+(-\infty)) = V(0) = V(Q^+(\infty)) = V(\xi)$.

Two major cases where the technical condition is satisfied are (i) when $n = 1$ and 0 and ξ are adjacent local maxima of V and (ii) when 0 is a global maximum and ξ a local maximum of V.

Once the basic pair of heteroclinics has been found, the same minimization ideas can be used to obtain further heteroclinics as well as homoclinic solutions of (1.1). These are solutions which start at 0 or ξ at $t = -\infty$, oscillate back and forth between neighborhoods of 0 and ξ a finite number of times before terminating at 0 or ξ at $t = \infty$. Indeed there are infinitely many such solutions characterized by the amount of time they spend near 0 and ξ between transition states. Moreover by a limit process, there are solutions of (1.1) which perform infinitely many such transitions.

More generally if V has several local maxima, ξ_i, $1 \le i \le N$, and the appropriate technical condition is satisfied, then the above results yield heteroclinics Q_i^+ from ξ_i to ξ_{i+1}, and Q_i^- from ξ_{i+1} to ξ_i, $1 \le i \le N-1$. Let (P_k) be any finite formal chain constructed from $\{Q_i^+, Q_j^- \mid 1 \le i, j \le N-1\}$, i.e. $P_{k+1}(-\infty) = P_k(\infty)$, $1 \le k \le K$. Such a chain will be called an augmented chain. E.g. in the previous paragraph, the augmented chain consists of $Q^\pm$ followed by $Q^\mp$, etc. As an extension of the above results, there are infinitely many actual heteroclinics Q of (1.1) with $Q(-\infty) = P_1(-\infty)$, $Q(\infty) = P_K(\infty)$ and Q spends long time intervals near $P_k(\infty)$, $1 \le k \le K-1$.

When there are enough points ξ_i, e.g. of order δ^{-1}, the difference $|V(P_1(-\infty)) - V(P_k(\infty))|$ can be of order 1. Indeed an example will be given for $n = 1$ where there is a sequence $(\xi_i)_{i\in\mathbb{Z}}$ with $\xi_i \to \pm\infty$ as $i \to \pm\infty$, and $V(\xi_i) \to \pm\infty$ as $i \to \pm\infty$. In that sense what is being done here is reminiscent of Arnold diffusion and the variational approach to it by Bessi [1], the recent work of Mather on orbits of infinite energy which shadow a family of periodics of increasing energy [7], and other recent work of Bolotin and Treschev [2] and of Delshams, de la Llave, and Seara [5] that was inspired by [7]. See also [4] and [8] which have some ideas in common with the current work.

The basic heteroclinic $Q^\pm$ will be obtained in §2. Then §3 treats the case when V has several local maxima. The results on homoclinics and heteroclinics associated with the augmented chains will be given as a special case of this setting. Lastly §4 gives some examples.

2. Basic heteroclinics

In this section, it will be shown how to construct a heteroclinic solution of (HS) which joins a pair of equilibrium points for the system, the equilibria corresponding to slightly different values of the potential.

Consider

$$\ddot{q} + a(t)V_\delta'(q) = 0, \qquad V_\delta'(x) = \frac{\partial V_\delta}{\partial x}(x) \tag{HS}$$

where V_δ is a function having (at least) two isolated local maxima, one at 0 and one at ξ, with $0 = V(0) > V(\xi) = -\delta$. More precisely, assume:

- **($\mathbf{V}_1$)** $V_\delta \in C^1(\mathbb{R}^n, \mathbb{R})$, $\delta \in [0, \delta_0]$ and V_δ continuous in δ;
- **($\mathbf{V}_2$)** There is an $r_0 > 0$ such that $0 = V_\delta(0) > V_\delta(x)$ for all $x \in \overline{B_{r_0}(0)} \setminus \{0\}$, $\delta \in [0, \delta_0]$.

Let $\mathcal{R}_0$ be the connected component of $\{\, x \in \mathbb{R}^n \mid V_\delta(x) \le 0 \,\}$ which contains 0, and, for $h < 0$,

$$\mathcal{R}_h = \{\, x \in \mathbb{R}^n \mid V_\delta(x) \le h \,\} \bigcap \mathcal{R}_0.$$

Further assume

- **($\mathbf{V}_3$)** There is a $\xi \in \mathcal{R}_0 \setminus \{0\}$ such that $V_\delta(\xi) = -\delta > V_\delta(x)$ for all $x \in \overline{B_{r_0}(\xi)} \setminus \{\xi\}$ and $\delta \in [0, \delta_0]$.

Fixing $\underline{a}, \overline{a} > 0$, the function a in (HS) is required to belong to the set

$$\begin{aligned} \mathcal{A} = \{\, a \in C(\mathbb{R}, \mathbb{R}) \mid 0 < \underline{a} \le \min a < \max a \le \overline{a} \\ \text{and there is a minimal } T = T(a) > 0 \text{ such that } a(t+T) = a(t) \,\}. \end{aligned}$$

More restrictions will be imposed on a later. The variational formulation of the problem can now be introduced. Let $E = W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$, with

$$\|q\|^2 = |q(0)|^2 + \int_{\mathbb{R}} |\dot{q}(t)|^2 \, dt.$$

Given $m_1 + 1 < m_2$, $\eta_1 \in B_{r_0}(0)$ and $\eta_2 \in B_{r_0}(\xi)$, set

$$\begin{aligned} \gamma(m_1, m_2, \eta_1, \eta_2) = \{\, q \in E \mid q(m_1) = \eta_1, \\ q(t) \in \mathcal{R}_0 \text{ for all } t \in [m_1, m_2],\ q(m_2) = \eta_2 \,\}. \end{aligned}$$

Let $L(t, q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - a(t)V_\delta(q)$. For $q \in \gamma(m_1, m_2, \eta_1, \eta_2)$, define

$$I_0(q) = \int_{m_1}^{m_2} L(t, q(t), \dot{q}(t)) \, dt.$$

Next an additional hypothesis (V_4) will be made. It is a technical condition needed to obtain the main existence result of this section, Theorem 2.5. In §4, examples will be given of when (V_4) is satisfied. E.g. an important special case to keep in mind is when 0 is a global maximum for V_δ.

(V_4) There is a $h < 0$ such that
  (a) 0 and ξ are path-connected in $\mathcal{D}_h \equiv B_{r_0}(0) \cup \mathcal{R}_h \cup B_{r_0}(\xi)$;
  (b) for all $m_2 - m_1 > 1$, $\delta \in [0, \delta_0]$, $a \in \mathcal{A}$, $\eta_1 \in B_{r_0}(0)$ and $\eta_2 \in B_{r_0}(\xi)$, whenever Q_0 is a minimizer of I_0 in $\gamma(m_1, m_2, \eta_1, \eta_2)$, then $Q_0(t) \in \mathcal{D}_h$ for all $t \in [m_1, m_2]$.

Remark 2.1. Note that $B_{r_0}(0) \cup B_{r_0}(\xi) \subset \mathcal{R}_0$. It is straightforward to show that the minimum in $\gamma(m_1, m_2, \eta_1, \eta_2)$ always exists.

We now define

$$\begin{aligned}\Gamma(m_1, m_2) = \big\{\, q \in E \;\big|\, q(-\infty) = 0,\ &|q(t)| \le r_0 \text{ for all } t \le m_1,\\ &q(t) \in \mathcal{R}_0 \text{ for all } t \in [m_1, m_2],\\ &|q(t) - \xi| \le r_0 \text{ for all } t \ge m_2,\ q(+\infty) = \xi \,\big\}.\end{aligned}$$

Observe that $\Gamma(m_1, m_2)$ is not empty by assumption (V_4). The heteroclinics we seek will lie in $\Gamma(m_1, m_2)$. For $q \in \Gamma(m_1, m_2)$, let

$$\mathcal{L}_\delta(q) = \begin{cases} L(t, q(t), \dot q(t)) & t < m_2 \\ L(t, q(t), \dot q(t)) - \delta a(t) & t \ge m_2. \end{cases}$$

Define

$$I(q) = \int_{\mathbb{R}} \mathcal{L}_\delta(q)\, dt$$

and

$$c(m_1, m_2) = \inf_{\Gamma(m_1, m_2)} I(q). \tag{2.2}$$

The next lemma makes the first step towards the main existence theorem of this section. In what follows, it will always be assumed that (V_1)–(V_4) are satisfied.

Lemma 2.3. *There is a $\overline{c} \in \mathbb{R}$ such that $0 < c(m_1, m_2) \le \overline{c} - 1 \le \overline{c}$ for all $\delta \in [0, \delta_0]$, $a \in \mathcal{A}$ and $m_2 - m_1 \ge 1$. Moreover there is a function $Q \in \Gamma(m_1, m_2)$ such that $I(Q) = c(m_1, m_2)$ and $Q(t) \in \mathcal{D}_h$ for all t.*

Proof. Since $q \in \Gamma(m_1, m_2)$ implies that $q(t) \in \mathcal{R}_0$ for all t, it follows that $V_\delta(q(t)) \le 0$ for all t. On the other hand for all $t \ge m_2$, $q(t) \in \overline{B_{r_0}(\xi)}$ and thus the assumptions (V_2) and (V_3), imply that

$$V_\delta(q(t)) \le -\delta \qquad \text{for all } t \ge m_2.$$

Hence $\mathcal{L}_\delta(q) \ge 0$ for all t and therefore $c(m_1, m_2) > 0$.

The existence of $\overline{c}$ follows by taking a function $q \in E$ such that $q(t) = 0$ for all $t \le -\frac{1}{2}$, $q(t) = \xi$ for all $t \ge \frac{1}{2}$, and $q(t) \in \mathcal{R}_0$ for all t. Then $\tilde q(t) = q(t - m_2 + 1/2) \in \Gamma(m_1, m_2)$. Note that, for such a $\tilde q$,

$$I(\tilde q) = \int_{m_2 - 1}^{m_2} [\frac{1}{2} |\dot{\tilde q}|^2 - a(t) V_\delta(\tilde q)]\, dt \le \int_{-1/2}^{1/2} [\frac{1}{2} |\dot q|^2 - \overline{a} V_\delta(q)]\, dt.$$

Setting

$$\overline{c} = 1 + \sup_{\delta \in [0, \delta_0]} \int_{-1/2}^{1/2} [\frac{1}{2} |\dot q|^2 - \overline{a} V_\delta(q)]\, dt \ge 1 + I(\tilde q),$$

the bound on $c(m_1, m_2)$ follows.

To show that $c(m_1, m_2)$ is achieved, take a minimizing sequence (q_k) for I. Then, for all $t \in [m_1, m_2]$, setting $q_k = q$ we have, for k large,

$$\begin{aligned} |q(t)| &\le |q(m_1)| + |q(t) - q(m_1)| \\ &\le r_0 + \Big(\int_{m_1}^{t} |\dot q|^2\, ds\Big)^{1/2} (t - m_1)^{1/2} \\ &\le r_0 + \sqrt{2\overline{c}(m_2 - m_1)}. \end{aligned}$$

Hence, since $q(t) \in \overline{B}_{r_0}(0)$ for all $t \le m_1$ and $q(t) \in \overline{B}_{r_0}(\xi)$ for all $t \ge m_2$,

$$|q(t)| \le 3r_0 + \sqrt{2\overline{c}(m_2 - m_1)} \qquad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

Now by (2.4) and the form of I, we deduce that (q_k) is bounded in H^1_{loc}. Consequently there exists a subsequence, still denoted (q_k), which converges weakly in $W^{1,2}_{\mathrm{loc}}$ and strongly in L^∞_{loc} to $Q \in \Gamma(m_1, m_2)$. Standard arguments show that such a Q is a minimizer of I in $\Gamma(m_1, m_2)$.

To show that $Q(t) \in \mathcal{D}_h$, it is enough to observe that:

(1) $Q(t) \in B_{r_0}(0) \subset \mathcal{D}_h$ for all $t \le m_1$;
(2) $Q\big|_{[m_1,m_2]}$ minimizes I_0 in $\gamma(m_1, m_2, Q(m_1), Q(m_2))$, and hence $Q(t) \in \mathcal{D}_h$ for all $t \in [m_1, m_2]$ by assumption (V_4);
(3) $Q(t) \in B_{r_0}(\xi) \subset \mathcal{D}_h$ for all $t \ge m_2$.

□

Now the main result of this section can be stated. The proof of the theorem will be carried out in a series of Lemmas.

Theorem 2.5. *Let V satisfy (V_1)–(V_4). Then there is an $\mathcal{A}^* \subset \mathcal{A}$ such that for each $a \in \mathcal{A}^*$, there exists a $\delta_2 = \delta_2(a) \le \delta_0$ and a corresponding solution of (HS) heteroclinic from 0 to ξ and a solution heteroclinic from ξ to 0.*

Proof. A solution will be obtained in $\Gamma(m_1, m_2)$ for appropriate choices of $m_2 - m_1$. Recall that $m_2 - m_1 \ge 1$. More assumptions will be made later on $m_2 - m_1$. Let Q be a minimizer for I over $\Gamma(m_1, m_2)$. By Lemma 2.3, $Q(t) \in \mathcal{D}_h$ for all $t \in [m_1, m_2]$. Then $Q(t) \notin \partial\mathcal{R}_0$ for $t \in [m_1, m_2]$. Consequently $Q(t)$ is a solution of (HS) for $t \in [m_1, m_2]$.

The function Q is a solution of (HS) for all $t < m_1$ whenever $Q(t) \notin \partial B_{r_0}(0)$, and also for $t > m_2$ whenever $Q(t) \notin \partial B_{r_0}(\xi)$. Hence in order to prove the theorem, it only need be shown that $Q(t) \notin \partial B_{r_0}(0)$ for $t < m_1$ and that that $Q(t) \notin \partial B_{r_0}(\xi)$ for $t > m_2$. For $0 < \rho < r_0$, let

$$\beta_1(\rho) = \min_{\substack{x \in \overline{B}_{r_0}(0) \setminus B_\rho(0) \\ 0 \le \delta \le \delta_0}} -V_\delta(x), \qquad \beta_2(\rho) = \min_{\substack{x \in \overline{B}_{r_0}(\xi) \setminus B_\rho(\xi) \\ 0 \le \delta \le \delta_0}} (-V_\delta(x) - \delta),$$

and take $\beta(\rho) = \min\{\beta_1(\rho), \beta_2(\rho)\} > 0$. With $h < 0$ given by (V_4), take ρ_1 so small that

$$-h > \beta(\rho) \tag{2.6}$$

for all $\rho < \rho_1$. Then, for all $x \in \mathcal{D}_h \setminus (B_\rho(0) \cup B_\rho(\xi))$, it follows that $-V_\delta(x) \ge \beta(\rho)$.

Lemma 2.7. *Let $t^* = 2\overline{c}/\underline{a}\beta(\rho)$. Then there is a $\overline{t} \in [m_2, m_2 + t^*]$ such that $Q(\overline{t}) \in B_\rho(\xi)$ and a $\underline{t} \in [m_2 - t^*, m_2]$ such that $Q(\underline{t}) \in B_\rho(\xi) \cup B_\rho(0)$. Similarly there is an $\overline{s} \in [m_1, m_1 + t^*]$ such that $Q(\overline{s}) \in B_\rho(0) \cup B_\rho(\xi)$, and an $\underline{s} \in [m_1 - t^*, m_1]$ such that $Q(\underline{s}) \in B_\rho(0)$.*

Proof. To show the existence of $\bar{t}$, note that if $\bar{t}$ does not exist, $Q(t) \in \overline{B}_{r_0}(\xi) \backslash B_\rho(\xi)$, $t \in [m_2, m_2 + t^*]$ and therefore

$$\bar{c} \geq I(Q) \geq \int_{m_2}^{m_2+t^*} -a(t)(V_\delta(Q(t)) + \delta)\, dt \geq \underline{a} t^* \beta(\rho) = 2\bar{c}.$$

Similarly if $\underline{t}$ does not exist, $Q(t) \in \mathcal{D}_h \setminus (B_\rho(0) \cup B_\rho(\xi))$ for all $t \in [m_2 - t^*, m_2]$. Then $-V_\delta(Q(t)) \geq \beta(\rho)$ for all $t \in [m_2 - t^*, m_2]$, and a contradiction is obtained by arguing as before. The existence of $\bar{s}$, $\underline{s}$ follow in a similar way. □

Let

$$\tilde{V}_\delta(x) = \begin{cases} V_\delta(x) & x \in B_{r_0}(0) \\ V_\delta(x) + \delta & x \in B_{r_0}(\xi) \end{cases}$$

and define $\varphi(\rho)$ in the following way:

$$\varphi(\rho) = \sup \Big\{ \frac{1}{2} \int_0^1 |\dot{q}(t)|^2 \, dt - \bar{a} \int_0^1 \tilde{V}_\delta(q(t))\, dt \;\Big|\; \Big| \; \delta \in [0, \delta_0],\ q(t) = \eta_1 + t(\eta_2 - \eta_1),\ \eta_1, \eta_2 \in B_\rho(0),\ \text{or } \eta_1, \eta_2 \in B_\rho(\xi) \Big\}. \tag{2.8}$$

Henceforth assume that r_0 is so small that $\varphi(r_0) < 1/2$. One immediately sees that $\varphi(\rho) \to 0$, as $\rho \to 0$, and arguing as in Lemma 2.3, one can show that

$$\Big| \int_{-\infty}^{\underline{s}} \mathcal{L}_\delta(Q)\, dt \Big|, \ \Big| \int_{\bar{t}}^{\infty} \mathcal{L}_\delta(Q)\, dt \Big| \leq \varphi(\rho). \tag{2.9}$$

For what follows $s << r_0$ means s is small compared to r_0.

Lemma 2.10. *For $\rho \leq \rho_2 \ll r_0$, $Q(t) \in B_{r_0}(\xi)$ for $t \geq \bar{t}$ and $Q(t) \in B_{r_0}(0)$ for $t \leq \underline{s}$.*

Proof. The first assertion is a consequence of Lemma 2.7, (2.9) and the fact that the cost as measured by I of going from $\partial B_\rho(\xi)$ to $\partial B_{r_0}(\xi)$ exceeds $\gamma \gg \varphi(\rho)$ for some constant γ depending on r_0. The second statement follows by the same reasoning. □

Lemma 2.11. *There is a $\delta_1 \leq \delta_0$ such that if $\delta \leq \delta_1$ and $Q(\underline{t}) \in B_\rho(\xi)$, then $Q(t) \in B_{r_0}(\xi)$ for $t \in [\underline{t}, \bar{t}]$ and if $Q(\bar{s}) \in B_\rho(0)$, $Q(t) \in B_{r_0}(0)$ for $t \in [\underline{s}, \bar{s}]$.*

Proof. It is already known that $Q(\bar{t}) \in B_\rho(\xi)$. Assume $Q(\underline{t}) \in B_\rho(\xi)$, and $Q(\tau) \notin B_{r_0}(\xi)$ for some $\tau \in (\underline{t}, \bar{t})$. Then, as in Lemma 2.10, $\int_{\underline{t}}^{\bar{t}} \mathcal{L}_\delta(Q)\, dt \geq \gamma = \gamma(r_0)$. Let

$$\overline{Q}(t) = \begin{cases} Q(t) & t \leq \underline{t} \\ \text{linear} & \underline{t} \leq t \leq \underline{t} + 1 \\ \xi & t \geq \underline{t} + 1 \end{cases}$$

Then by the minimality of Q in $\Gamma_{r_0}(m_1, m_2)$,

$$\begin{aligned}\gamma &\le \int_{\underline{t}}^{+\infty} \mathcal{L}_\delta(Q)\,dt \le \int_{\underline{t}}^{+\infty} \mathcal{L}_\delta(\overline{Q})\,dt \\ &= \int_{\underline{t}}^{m_2} \mathcal{L}_\delta(\overline{Q})\,dt + \int_{m_2}^{+\infty} \mathcal{L}_\delta(\overline{Q})\,dt \\ &\le \int_{\underline{t}}^{\underline{t}+1} [\frac{1}{2}\left|\dot{\overline{Q}}(t)\right|^2 - a(t)V_\delta(\overline{Q})]\,dt - \int_{\underline{t}+1}^{m_2} a(t)V_\delta(\xi)\,dt \\ &\quad + \int_{m_2}^{+\infty} [-a(t)V_\delta(\xi) - \delta a(t)]\,dt \\ &\le \varphi(\rho) + \overline{a}\delta(m_2 - \underline{t}) \le \varphi(\rho) + \overline{a}\delta t^* \le \varphi(\rho) + \frac{2\overline{ac}}{\underline{a}\beta(\rho)}\delta.\end{aligned}$$

Taking $\delta = \delta(\rho)$ sufficiently small and recalling that $\varphi(\rho) << \gamma(r_0)$ yields a contradiction. Therefore $Q(t) \in B_{r_0}(\xi)$ for $t \in [\underline{t}, \overline{t}]$ and similarly for the s case. □

Lemma 2.12. *Suppose $\delta \le \delta_1$. Assume $Q(\tilde{t}) \in \partial B_{r_0}(\xi)$ for some $\tilde{t} \in [m_2, \overline{t}]$. Then $Q(t) \in B_{r_0}(0)$ for all $t \le \underline{t}$. Similarly, if $Q(\tilde{s}) \in \partial B_{r_0}(0)$ for some $\tilde{s} \in [\underline{s}, m_1]$, then $Q(t) \in B_{r_0}(\xi)$ for all $t \ge \overline{s}$.*

Proof. The first part of the lemma follows by observing that $Q(\underline{t}) \in B_\rho(\xi)$ is not possible via Lemma 2.11. Then, arguing as in Lemma 2.10, shows that $Q(t) \in B_{r_0}(0)$ for all $t \le \underline{t}$. Again the s case is proved in the same way. □

So far $a \in \mathcal{A}$ and $\rho \le \min\{\rho_1, \rho_2\}$ are free. Further choose ρ so that

$$\frac{4\overline{a}}{\underline{a}}\varphi(\rho) \le \frac{1}{4}(\overline{a} - \underline{a})\frac{d^2}{2\overline{c}}\,|h| \tag{2.13}$$

where

$$d = \operatorname{dist}(B_{r_0}(0), B_{r_0}(\xi)),$$

and h is given by (V_4). With ρ now fixed, choose $a \in \mathcal{A}^*$ where

$$\mathcal{A}^* = \left\{\, a \in \mathcal{A} \;\middle|\; \min_{[-t^*,t^*]} a - \max_{[-t^*-\theta, t^*-\theta]} a \ge \frac{1}{2}(\overline{a} - \underline{a}) \text{ for some } \theta \in (0,T) \,\right\}. \tag{2.14}$$

This condition will be satisfied for T sufficiently large and a which oscillates slowly between its maximum and minimum. The simplest examples of $a \in \mathcal{A}^*$ occur when $a(t) = b(\varepsilon t)$ for $b \in \mathcal{A}$ and $0 < \varepsilon$ sufficiently small.

The significance of $\mathcal{A}^*$ is that if e.g. $Q(t) \in \partial B_{r_0}(\xi)$ for some $t \in [m_2, \overline{t}]$, by the previous lemma, the transition of Q from $B_\rho(0)$ to $B_\rho(\xi)$ occurs in $[m_2 - t^*, m_2 + t^*]$, an interval in which a is relatively large. But heuristically, the minimizer of I in $\Gamma(m_1, m_2)$ should not undergo a transition when a is relatively large; rather it should occur when a is relatively small. In the next lemma, a comparison function argument exploits this idea.

Lemma 2.15. *Let $a \in \mathcal{A}^*$, $m_1, m_2 \in T\mathbb{N}$, $m_2 - m_1 \ge T(a) + t^*$. Then, for $\delta > 0$ small, $Q(t) \in \partial B_{r_0}(\xi)$ for some $t \in [m_2, \overline{t}]$ is not possible, and also $Q(t) \in \partial B_{r_0}(0)$ for some $t \in [\underline{s}, m_1]$ is not possible.*

Proof. Suppose $Q(t) \in \partial B_{r_0}(\xi)$ for some $t \in [m_2, \overline{t}]$. Then by Lemma 2.12 $Q(t) \in B_{r_0}(0)$ for all $t \le \underline{t}$. Let $\theta \in (0,T)$ be such that

$$\min_{[-t^*,t^*]} a - \max_{[-t^*-\theta, t^*-\theta]} a \ge \frac{1}{2}(\overline{a} - \underline{a}). \tag{2.16}$$

We claim that $\tau_{-\theta}Q(\cdot) \equiv Q(\cdot + \theta) \in \Gamma(m_1, m_2)$. Indeed, for all $t \ge m_2$, $t + \theta \ge t \ge m_2$ implies $Q(t + \theta) \in \overline{B_{r_0}(\xi)}$ while $t \le m_1$ implies that $t + \theta \le m_1 + \theta \le$

$m_2 - T - t^* + \theta \le m_2 - t^* \le \underline{t}$, so that $Q(t+\theta) \in \overline{B_{r_0}(0)}$ for all $t \le m_1$ follows from Lemma 2.12. Hence by the minimality of Q,

$$
\begin{aligned}
0 \ge I(Q) - I(\tau_{-\theta}Q) = &\int_{-\infty}^{m_2} (a(t) - a(t-\theta))(-V_\delta(Q))\,dt \\
&+ \int_{m_2}^{\infty} (a(t) - a(t-\theta))(-V_\delta(Q) - \delta)\,dt - \delta \int_{m_2}^{m_2+\theta} a(t-\theta)\,dt.
\end{aligned} \tag{2.17}
$$

By Lemma 2.12, $Q(\underline{t}) \in B_\rho(0)$. Therefore as in (2.9),

$$
\begin{aligned}
\Big| \int_{-\infty}^{\underline{t}} (a(t) - a(t-\theta))(-V_\delta(Q))\,dt \Big| &\le 2\overline{a} \int_{-\infty}^{\underline{t}} -V_\delta(Q)\,dt \\
&\le \frac{2\overline{a}}{\underline{a}} \int_{-\infty}^{\underline{t}} L(Q)\,dt \le \frac{2\overline{a}}{\underline{a}} \varphi(\rho).
\end{aligned} \tag{2.18}
$$

Similarly,

$$
\Big| \int_{\overline{t}}^{\infty} (a(t) - a(t-\theta))(-V_\delta(Q) - \delta) dt \Big| \le \frac{2\overline{a}}{\underline{a}} \varphi(\rho). \tag{2.19}
$$

The last term on the right in (2.17) can simply be estimated by

$$
\delta \int_{m_2}^{m_2+\theta} a(t-\theta)\,dt \le \delta \overline{a} T. \tag{2.20}
$$

Since $a \in \mathcal{A}^*$,

$$
\begin{aligned}
&\int_{\underline{t}}^{m_2} (a(t) - a(t-\theta))(-V_\delta(Q))\,dt + \int_{m_2}^{\overline{t}} (a(t) - a(t-\theta))(-V_\delta(Q) - \delta)\,dt \\
&= \int_{\underline{t}}^{\overline{t}} (a(t) - a(t-\theta))(-V_\delta(Q))\,dt - \delta \int_{m_2}^{\overline{t}} (a(t) - a(t-\theta))\,dt \\
&\ge \frac{1}{2}(\overline{a} - \underline{a}) \int_{\underline{t}}^{\overline{t}} (-V_\delta(Q))\,dt - 2\delta t^* \overline{a}.
\end{aligned} \tag{2.21}
$$

Now, if $\underline{t}_1 = \sup\{\, t \mid Q(t) \in B_{r_0}(0) \,\}$ and $\overline{t}_1 = \inf\{\, t \mid Q(t) \in B_{r_0}(\xi) \,\}$, by ($V_4$), it follows that $Q(t) \in \mathcal{R}_h$ for all $t \in [\underline{t}_1, \overline{t}_1]$ and

$$
d \le \Big| \int_{\underline{t}_1}^{\overline{t}_1} \dot{Q}\,dt \Big| \le (\overline{t}_1 - \underline{t}_1)^{1/2} \Big(\int_{\underline{t}_1}^{\overline{t}_1} |\dot{Q}|^2\,dt \Big)^{1/2} \le (\overline{t}_1 - \underline{t}_1)^{1/2} (2\overline{c})^{1/2}. \tag{2.22}
$$

This last inequality implies

$$
\overline{t} - \underline{t} \ge \overline{t}_1 - \underline{t}_1 \ge \frac{d^2}{2\overline{c}}. \tag{2.23}
$$

Therefore

$$
-\int_{\underline{t}}^{\overline{t}} V_\delta(Q)\,dt \ge -\int_{\underline{t}_1}^{\overline{t}_1} V_\delta(Q)\,dt \ge \frac{d^2}{2\overline{c}} \left| h \right|. \tag{2.24}
$$

Hence by (2.17)–(2.24), (2.6), (2.13) and the definition of t^*,

$$
\delta \overline{a} \Big(T + \frac{4\overline{c}}{\underline{a}\beta(\rho)} \Big) \ge 4 \frac{\overline{a}}{\underline{a}} \varphi(\rho). \tag{2.25}
$$

Consequently for $\delta = \delta(\rho)$ suitably small, Q is not a minimizer of I, a contradiction.

Finally to prove that $Q(t) \notin \partial B_{r_0}(0)$ for $t \in [\underline{s}, m_1]$, note that if to the contrary, $Q(t) \in \partial B_{r_0}(0)$ for some such t, then $Q(t) \in B_{r_0}(\xi)$ for $t \ge \overline{s}$. Consider $\tau_T Q$. For $t \le m_1$, $\tau_T Q(t) \in \overline{B}_{r_0}(0)$. For $t \ge m_2$, $t - T \ge m_2 - T \ge m_1 + t^* \ge \overline{s}$ so

$\tau_T Q(t) \in \overline{B}_{r_0}(\xi)$. Also $Q(t) \in \mathcal{R}_0$ for all t implies the same for $\tau_T Q(t)$. Hence $\tau_T Q \in \Gamma(m_1, m_2)$. Therefore as in (2.17),

$$
\begin{aligned}
0 &\geq I(Q) - I(\tau_T Q) \\
&= \int_{-\infty}^{m_2} (a(t) - a(t+T)(-V_\delta(Q))dt \\
&+ \int_{m_2}^{\infty} (a(t) - a(t+T)(-V_\delta(Q) - \delta)dt \\
&+ \delta \int_{m_2 - T}^{m_2} a(t+T)dt > 0
\end{aligned} \tag{2.26}
$$

since the first two terms vanish due to the periodicity of a. Thus (2.26) shows this case is impossible. □

Lemma 2.27. *Q is a solution of (HS) heteroclinic from 0 to ξ.*

Proof. It has already been noted that Q is a solution of (HS) provided $Q(t) \notin \partial B_{r_0}(0)$, for $t \leq m_1$ and $Q(t) \notin \partial B_{r_0}(\xi)$ for $t \geq m_2$. This is now an immediate consequence of Lemma 2.10 and Lemma 2.15. Standard arguments then show that Q is actually an heteroclinic solution of (HS). □

Remark 2.28. Similarly there is a solution of (HS) heteroclinic from ξ to 0.

The above observations end the proof of Theorem 2.5. □

3. Multi-bump solutions

Suppose that V_δ has several local maxima, e.g. at $\xi_0 = 0, \xi_1, \ldots, \xi_N$ and that $|V_\delta(\xi_{i-1}) - V_\delta(\xi_i)|$ is small, $1 \leq i \leq N$. Then the arguments of §2 can be extended to show that (HS) has solutions heteroclinic from 0 to ξ_N and which spend at least prescribed amounts of time near the points ξ_i, $1 \leq i \leq N-1$. In order to simplify the presentation, assume (V_1), (V_2), (V_4) and

(V_3') There is a $\xi \in \mathcal{R}_0 \setminus \{0\}$ such that $V_\delta(x+\xi) = V_\delta(x) - \delta$ for all $x \in \mathbb{R}^n$ and $\delta \in [0, \delta_0]$.

Note that (V_3') implies (V_3), so that all the results of Section 2 hold in this setting. Moreover we have that $y \in \mathcal{R}_0 + j\xi$ implies that $y = x + j\xi$ with $x \in \mathcal{R}_0$ so using (V_3'),

$$V_\delta(y) = V_\delta(x + j\xi) = V_\delta(x) - j\delta \leq -j\delta \text{ for all } y \in \mathcal{R}_0 + j\xi.$$

Given $N \in \mathbb{N}$, and $\vec{m} \in \mathbb{R}^{2N}$ such that $m_{j+1} - m_j \geq 2$, let $m_0 = -\infty$, $m_{2N+1} = +\infty$ and

$$
\begin{aligned}
\Gamma(\vec{m}) = \big\{ q \in E \bigm| & q(-\infty) = 0, \\
& q(t) \in \mathcal{R}_0 + \ell\xi \text{ for all } t \geq m_{2\ell+1},\ \ell = 0, \ldots, N-1, \\
& q(t) \in \overline{B_{r_0}(\ell\xi)} \text{ for all } t \in [m_{2\ell}, m_{2\ell+1}],\ \ell = 0, \ldots, N \\
& \qquad \text{and} \quad q(+\infty) = N\xi \big\}.
\end{aligned}
$$

If $q \in \Gamma(\vec{m})$, define

$$
\mathcal{L}_\delta(q) = \begin{cases} L(t, q(t), \dot{q}(t)) - \ell\delta a(t) & m_{2\ell} \leq t < m_{2\ell+2},\ \ell = 0, 1, \ldots, N-1 \\ L(t, q(t), \dot{q}(t)) - N\delta a(t) & t \geq m_{2N} \end{cases}
$$

It is immediate to check that $\mathcal{L}_\delta(q) \geq 0$ for all $t \in \mathbb{R}$ if $q \in \Gamma(\vec{m})$. Indeed, for $m_{2\ell} \leq t \leq m_{2\ell+2}$, we have that $q(t) \in B_{r_0}(\ell\xi) \cup (\mathcal{R}_0 + \ell\xi)$. Since our assumptions imply that $B_{r_0}(\ell\xi) \subset (\mathcal{R}_0 + \ell\xi)$, we deduce that

$$V_\delta(q(t)) \leq -\ell\delta \quad \text{for all } m_{2\ell} \leq t \leq m_{2\ell+2}$$

so that

$$\mathcal{L}_\delta(q) = \frac{1}{2}\left|\dot{q}(t)\right|^2 - a(t)V_\delta(q(t)) - \ell\delta a(t) \geq \ell\delta a(t) - \ell\delta a(t) \geq 0$$

for this range of values of t. Define

$$I(q) = \int_{-\infty}^{\infty} \mathcal{L}_\delta(q)\, dt,$$

and

$$c_{\vec{m}} = \inf_{\Gamma(\vec{m})} I(q).$$

Lemma 3.1. *Let $\bar{c}$ be given by Lemma 2.3. Then for all $\delta \in [0, \delta_0]$ and $a \in \mathcal{A}$, it follows that $c_{\vec{m}} \leq N\bar{c}$ and there is $Q \in \Gamma(\vec{m})$ such that $I(Q) = c_{\vec{m}}$. Moreover for $\ell = 0, 1, \ldots, N-1$*

$$\int_{m_{2\ell}}^{m_{2\ell+3}} \mathcal{L}_\delta(Q)\, dt \leq \bar{c}. \tag{3.2}$$

Proof. The existence of a minimizer Q of I follows as in §2. To get the estimates, let q be the function defined in the proof of Lemma 2.3. Set

$$p(t) = \begin{cases} q(t - m_{2\ell+2} + 1/2) + \ell\xi & m_{2\ell} \leq t \leq m_{2\ell+2},\ 0 \leq \ell \leq N-1 \\ q(t) = N\xi, & t \geq m_{2N} \end{cases}$$

Then $p \in \Gamma(\vec{m})$ and $c_{\vec{m}} \leq I(p) \leq N\bar{c}$.

To prove (3.2), consider the function $\overline{Q} \in \Gamma(\vec{m})$ defined as

$$\overline{Q}(t) = \begin{cases} Q(t) & t \leq m_{2\ell} \\ \text{linear} & m_{2\ell} \leq t \leq m_{2\ell} + 1 \\ p(t) & m_{2\ell} + 1 \leq t \leq m_{2\ell+3} - 1 \\ \text{linear} & m_{2\ell+3} - 1 \leq t \leq m_{2\ell+3} \\ Q(t) & t \geq m_{2\ell+3} \end{cases}$$

Then

$$0 \leq I(\overline{Q}) - I(Q) = \int_{m_{2\ell}}^{m_{2\ell+3}} [\mathcal{L}_\delta(\overline{Q}) - \mathcal{L}_\delta(Q)]\, dt.$$

Hence by Lemma 2.3, and recalling that $\varphi(\rho_0) < 1/2$,

$$\begin{aligned} \int_{m_{2\ell}}^{m_{2\ell+3}} \mathcal{L}_\delta(Q)\, dt &\leq \int_{m_{2\ell}}^{m_{2\ell+3}} \mathcal{L}_\delta(\overline{Q}) \\ &= \int_{m_{2\ell}}^{m_{2\ell+1}} \mathcal{L}_\delta(\overline{Q})\, dt + \int_{m_{2\ell+2}-1}^{m_{2\ell+2}} \mathcal{L}_\delta(\overline{Q})\, dt + \int_{m_{2\ell+3}-1}^{m_{2\ell+3}} \mathcal{L}_\delta(\overline{Q})\, dt \\ &\leq \bar{c} - 1 + 2\varphi(r_0) \leq \bar{c} \end{aligned}$$

and the result follows. □

Lemma 3.3. *Let Q be a minimizer of I in $\Gamma(\vec{m})$ given by Lemma 3.1.*

Then, for $\ell = 0, 1, \ldots, N-1$,

$$Q(t) \in B_{r_0}(\ell\xi) \cup (\mathcal{R}_h + \ell\xi) \cup B_{r_0}((\ell+1)\xi) = \mathcal{D}_h + \ell\xi \quad \textit{for all } t \in [m_{2\ell+1}, m_{2\ell+2}].$$

Proof. Set

$$q(t) = Q\big|_{[m_{2\ell+1}, m_{2\ell+2}]}(t) - \ell\xi$$

and observe that

$$q(t) \in \gamma(m_{2\ell+1}, m_{2\ell+2}, q(m_{2\ell+1}), q(m_{2\ell+2}))$$

and

$$\mathcal{L}_\delta(Q) = L(t, Q(t), \dot{Q}(t)) - \ell\delta a(t) = L(t, q(t), \dot{q}(t)), \text{ for all } t \in [m_{2\ell+1}, m_{2\ell+2}].$$

Hence q minimizes I_0 over $\gamma(m_{2\ell+1}, m_{2\ell+2}, q(m_{2\ell+1}), q(m_{2\ell+2}))$. The lemma then follows from assumption (V_4). □

Lemma 3.4. *Let t^* be as in Lemma 2.7. Suppose $m_{2\ell+1} - m_{2\ell} \geq 2t^*$. Then, for $\ell = 1, 2, \dots, N$, there is a $\overline{t}_\ell \in [m_{2\ell}, m_{2\ell} + t^*]$, $\underline{t}_\ell \in [m_{2\ell} - t^*, m_{2\ell}]$, $\overline{s}_\ell \in [m_{2\ell-1}, m_{2\ell-1} + t^*]$, and $\underline{s}_\ell \in [m_{2\ell-1} - t^*, m_{2\ell-1}]$ such that*

$$Q(\overline{t}_\ell) \in B_\rho(\ell\xi) \qquad Q(\underline{t}_\ell) \in B_\rho((\ell-1)\xi) \cup B_\rho(\ell\xi)$$
$$Q(\overline{s}_\ell) \in B_\rho((\ell-1)\xi) \cup B_\rho(\ell\xi) \qquad Q(\underline{s}_\ell) \in B_\rho((\ell-1)\xi).$$

Moreover, setting $\overline{t}_0 = -\infty$, $\underline{s}_{N+1} = +\infty$, then for $\ell = 0, \dots, N$

$$0 \leq \int_{\overline{t}_\ell}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(Q)\, dt \leq \varphi(\rho) \to 0 \qquad \text{as } \rho \to 0 \tag{3.5}$$

and

$$Q(t) \in B_{r_0}(\ell\xi) \qquad t \in [\overline{t}_\ell, \underline{s}_{\ell+1}]. \tag{3.6}$$

Proof. The proof of the first part is very similar to that of Lemma 2.7. Indeed, suppose $\overline{t}_\ell$ does not exist. Then $Q(t) \in B_{r_0}(\ell\xi) \setminus B_\rho(\ell\xi)$ for all $t \in [m_{2\ell}, m_{2\ell} + t^*]$ and, using Lemma 3.1

$$\overline{c} \geq \int_{m_{2\ell}}^{m_{2\ell}+t^*} -a(t)(V_\delta(Q(t)) + \ell\delta)\, dt \geq t^* \underline{a}\beta(\rho) = 2\overline{c}.$$

The estimates (3.5) follow as in Lemma 2.7 using the arguments of Lemma 2.3, and $q(t) \in B_{r_0}(\ell\xi)$ for $t \in [\overline{t}_\ell, \underline{s}_\ell]$ since, as in Lemma 2.10, the cost of going from $\partial B_\rho(\ell\xi)$ to $\partial B_{r_0}(\ell\xi) \geq \gamma \gg \varphi(\rho)$. □

Now the main theorem of this section can be stated.

Theorem 3.7. *Let ρ satisfy*

$$8\varphi(\rho) \leq \frac{1}{4}(\overline{a} - \underline{a})\frac{d^2}{2\overline{c}}|h|. \tag{3.8}$$

and define $t^* = 2\overline{c}/\underline{a}\beta(\rho)$ and $\mathcal{A}^*$ as in (2.14). Then for all $a \in \mathcal{A}^*$, there is a $\delta_3 \leq \delta_0$ such that for all $0 < \delta \leq \delta_3$, and for all $\vec{m} \in \mathbb{R}^{2N}$ which satisfy

$$m_j \in T\mathbb{Z} \qquad j = 1, \dots, 2N \tag{3.9}$$
$$m_{2\ell+1} - m_{2\ell} \geq 2t^* + 2 \qquad \ell = 1, \dots, N-1 \tag{3.10}$$
$$m_{2\ell} - m_{2\ell-1} \geq 2t^* + T(a) + 1 \qquad \ell = 1, \dots, N \tag{3.11}$$

(HS) *has a heteroclinic solution $Q \in \Gamma(\vec{m})$.*

Proof. Set $m_0 = -\infty$, $m_{2N+1} = \infty$. Let Q be the minimizer of I over $\Gamma(\vec{m})$. It is immediate that such a function is a solution of (HS)

- for all $t \in [m_{2\ell+1}, m_{2\ell+2}]$, $\ell = 0, \dots, N-1$, by Lemma 3.3;
- for all $t \in [m_{2\ell}, m_{2\ell+1}]$ such that $Q(t) \notin \partial B_{r_0}(\ell\xi)$, $\ell = 0, \dots, N$.

So, to prove the theorem, it only need be shown that $Q(t) \notin \partial B_{r_0}(\ell\xi)$ for all $t \in [m_{2\ell}, m_{2\ell+1}]$ and $\ell = 0, \dots, N$. This will be done for $\ell = 1, \dots, N-1$. The cases of $\ell = 0$ and $\ell = N$ are treated in a similar but simpler fashion and will be omitted.

By (3.6) of Lemma 3.4, it is known that $Q(t) \in B_{r_0}(\ell\xi)$ for all $t \in [\overline{t}_\ell, \underline{s}_{\ell+1}] \subset [m_{2\ell}, m_{2\ell+1}]$. Thus it remains to verify that $Q(\tau) \in \partial B_{r_0}(\ell\xi)$ for some $\tau \in [m_{2\ell}, \overline{t}_\ell]$ or $\tau \in [\underline{s}_{\ell+1}, m_{2\ell+1}]$ is not possible. Assume to the contrary that $Q(\tau) \in \partial B_{r_0}(\ell\xi)$ for some $\tau \in [m_{2\ell}, \overline{t}_\ell]$. Then first of all, since $Q(\overline{t}_\ell) \in B_\rho(\ell\xi)$, $Q(\tau) \in \partial B_{r_0}(\ell\xi)$

and $Q(\underline{t}_\ell) \in B_\rho((\ell-1)\xi) \cup B_\rho(\ell\xi)$, the arguments of Lemma 2.11 and Lemma 2.12 imply that $Q(\underline{t}_\ell) \in B_\rho((\ell-1)\xi)$ and hence

$$Q(t) \in B_{r_0}((\ell-1)\xi) \qquad \text{for all } t \in [\bar{t}_{\ell-1}, \underline{t}_\ell]. \tag{3.12}$$

Let $\theta \in (0, T(a))$ be as in §2 and define $\tilde{Q}(t)$ as follow:

$$\tilde{Q}(t) = \begin{cases} Q(t) & t \le \bar{t}_{\ell-1} \\ (\ell-1)\xi & \bar{t}_{\ell-1}+1 \le t \le \underline{t}_\ell - \theta - 1 \\ Q(t+\theta) & \underline{t}_\ell - \theta \le t \le \bar{t}_\ell - \theta \\ \ell\xi & \bar{t}_\ell - \theta + 1 \le t \le \underline{s}_{\ell+1} - 1 \\ Q(t) & t \ge \underline{s}_{\ell+1} \\ \text{linear} & \text{otherwise} \end{cases}$$

To verify that $\widetilde{Q}$ is well defined, note that

- $\bar{t}_{\ell-1}+1 \le \underline{t}_\ell - \theta - 1$, since by (3.10)–(3.11), $\underline{t}_\ell - \bar{t}_{\ell-1} \ge m_{2\ell} - m_{2\ell-2} - 2t^* = (m_{2\ell} - m_{2\ell-1}) + (m_{2\ell-1} - m_{2\ell-2}) - 2t^* \ge 2t^* + T(a) + 3 \ge \theta + 2$;
- $\bar{t}_\ell - \theta + 1 \le \underline{s}_{\ell+1} - 1$, since $\underline{s}_{\ell+1} - \bar{t}_\ell \ge m_{2\ell+1} - m_{2\ell} - 2t^* \ge 2$.

We claim that $\tilde{Q} \in \Gamma(\vec{m})$. Since $Q \in \Gamma(\vec{m})$, by the definition of $\widetilde{Q}$, it must be verified that

(a) $\widetilde{Q} \in \overline{B}_{r_0}((\ell-1)\xi), \quad t \in [\bar{t}_{\ell-1}, m_{2\ell-1}]$,
(b) $\widetilde{Q} \in \mathcal{R}_0 + (\ell-1)\xi, \quad t \in [m_{2\ell-1}, m_{2\ell}]$,
(c) $\widetilde{Q} \in \overline{B}_{r_0}(\ell\xi), \quad t \in [m_{2\ell}, \underline{s}_{\ell+1}]$.

Using the definition of $\widetilde{Q}$, (a) follows from (3.11), (b) from (V_2)–(V_3') and the fact that if $t \ge m_{2\ell-1}$, then $t+\theta \ge m_{2\ell-1}$, and (c) from (3.10) and the fact that if $t \ge m_{2\ell}$, then $t + \theta \ge m_{2\ell}$.

Since $\widetilde{Q} \in \Gamma(\vec{m})$, arguing as in §2,

$$0 \le I(\widetilde{Q}) - I(Q) = \int_{\underline{t}_\ell - \theta}^{\bar{t}_\ell - \theta} \mathcal{L}_\delta(\widetilde{Q})dt - \int_{\underline{t}_\ell}^{\bar{t}_\ell} \mathcal{L}_\delta(Q)dt + \mathcal{R} \tag{3.13}$$

where

$$\mathcal{R} = \int_{\bar{t}_{\ell-1}}^{\underline{t}_\ell - \theta} \mathcal{L}_\delta(\widetilde{Q})dt + \int_{\bar{t}_\ell - \theta}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(\widetilde{Q})dt - \int_{\bar{t}_{\ell-1}}^{\underline{t}_\ell} \mathcal{L}_\delta(Q)dt - \int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(Q)dt. \tag{3.14}$$

By earlier arguments,

$$\Big| \int_{\bar{t}_{\ell-1}}^{\underline{t}_\ell - \theta} \mathcal{L}_\delta(\widetilde{Q})dt \Big| \le 2\varphi(\rho). \tag{3.15}$$

Since $Q(\bar{t}_{\ell-1}) \in B_\rho((\ell-1)\xi)$ and $Q(\underline{t}_\ell) \in B_\rho((\ell-1)\xi)$, the minimality of Q and simple comparison arguments as e.g. in Lemma 3.1 imply

$$\Big| \int_{\bar{t}_{\ell-1}}^{\underline{t}_\ell} \mathcal{L}_\delta(Q)dt \Big| \le 2\varphi(\rho) \tag{3.16}$$

and similarly

$$\Big| \int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(Q)dt \Big| \le 2\varphi(\rho). \tag{3.17}$$

The function $\mathcal{L}_\delta(\cdot)$ has a jump discontinuity (by $-\delta a$) at $t = m_{2\ell}$ so some care must be taken with this value of t. The jump in $\mathcal{L}_\delta(Q)$ occurs in the integral over $[\underline{t}_\ell, \bar{t}_\ell]$. If $\bar{t}_\ell - \theta \ge m_{2\ell}$, the jump in $\mathcal{L}_\delta(\widetilde{Q})$ occurs in the integral over $[\underline{t}_\ell - \theta, \bar{t}_\ell - \theta]$. Hence

$$\int_{\bar{t}_\ell - \theta}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(\widetilde{Q})dt \le 2\varphi(\rho) \tag{3.18}$$

as for (3.15) and by (3.13)–(3.18),

$$(3.19)\quad 0 \le \int_{\underline{t}_\ell-\theta}^{\bar{t}_\ell-\theta} \mathcal{L}_\delta(\widetilde{Q})dt - \int_{\underline{t}_\ell}^{\bar{t}_\ell} \mathcal{L}_\delta(Q)dt + 8\varphi(\rho)$$

$$\le \int_{\underline{t}_\ell}^{\bar{t}_\ell} (a(t-\theta)-a(t))(-V_\delta(Q(t))-(\ell-1)\delta)dt + \delta\overline{a}\theta + 8\varphi(\rho).$$

On the other hand, if $\bar{t}_\ell - \theta \le m_{2\ell}$.

$$(3.20)\quad \int_{\bar{t}_\ell-\theta}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(\tilde{Q})\,dt = \int_{\bar{t}_\ell-\theta}^{\underline{s}_{\ell+1}} [\frac{1}{2}\left|\dot{\tilde{Q}}\right|^2 - a(t)(V_\delta(\tilde{Q})+\ell\delta)]\,dt + \int_{\bar{t}_\ell-\theta}^{m_{2\ell}} \delta a(t)\,dt$$

$$\le 2\varphi(\rho) + \overline{a}(m_{2\ell}-\bar{t}_\ell+\theta) \le 2\varphi(\rho) + \overline{a}\delta\theta$$

and

$$0 \le I(\tilde{Q}) - I(Q) \le \int_{\underline{t}_\ell-\theta}^{\bar{t}_\ell-\theta} \mathcal{L}_\delta(\tilde{Q})\,dt - \int_{\underline{t}_\ell}^{\bar{t}_\ell} \mathcal{L}_\delta(Q)\,dt + 8\varphi(\rho) + \overline{a}\delta\theta$$

$$(3.21)\quad \le \int_{\underline{t}_\ell}^{\bar{t}_\ell} (a(t-\theta)-a(t))(-V_\delta(Q(t))-(\ell-1)\delta)\,dt$$

$$- \int_{m_{2\ell}}^{\bar{t}_\ell} \delta a(t)\,dt + \delta\overline{a}\theta + 8\varphi(\rho),$$

and equation (3.19) holds also in this case.

Then, by the same arguments used in equations (2.21)–(2.24) we find that

$$0 \le I(\tilde{Q}) - I(Q) \le -\frac{1}{2}(\overline{a}-\underline{a})\frac{d^2}{2\overline{c}}|h| + \delta\overline{a}T(a) + 8\varphi(\rho),$$

a contradiction for δ small via (3.8).

To complete the proof of Theorem 3.7, it remains to show that $Q(\tau) \in \partial B_{r_0}(\ell\xi)$ for some $\tau \in [\underline{s}_{\ell+1}, m_{2\ell+1}]$ is impossible.

This involves a comparison function argument based on a combination of the case just carried out and the last part of the proof of Lemma 2.15. Arguing as earlier,

$$(3.22)\quad Q(t) \in B_{r_0}((\ell+1)\xi),\ t \in [\overline{s}_{\ell+1}, \underline{s}_{\ell+2}].$$

Suppose $k \in \mathbb{N}$ satisfies

$$(3.23)\quad (k+1)T + 2t^* + 2 > m_{2\ell+2} - m_{2\ell+1} \ge kT + 2t^* + 2.$$

Let $\overline{\theta} = kT - \theta$ with $\theta \in (0,T)$ as earlier. Define

$$\widetilde{Q}(t) = \begin{cases} Q(t), & t \le \bar{t}_\ell \\ \ell\xi, & \bar{t}_{\ell+1} \le t \le \underline{s}_{\ell+1} + \overline{\theta} - 1 \\ Q(t-\overline{\theta}), & \underline{s}_{\ell+1} + \overline{\theta} \le t \le \overline{s}_{\ell+1} + \overline{\theta} \\ (\ell+1)\xi, & \overline{s}_{\ell+1} + \overline{\theta} + 1 \le t \le \underline{t}_{\ell+1} - 1 \\ Q(t), & t \ge \underline{t}_{\ell+1} \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then (3.23) and earlier arguments show $\widetilde{Q}$ is well defined and $\widetilde{Q} \in \Gamma(\vec{m})$. As in (3.13)

$$(3.24)\quad 0 = I(\widetilde{Q}) - I(Q) = \int_{\underline{s}_{\ell+1}+\overline{\theta}}^{\overline{s}_{\ell+1}+\overline{\theta}} \mathcal{L}_\delta(\widetilde{Q})\,dt - \int_{\underline{s}_{\ell+1}}^{\overline{s}_{\ell+1}} \mathcal{L}_\delta(Q)\,dt + \mathcal{R}$$

where now

$$\mathcal{R} = \int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}+\bar{\theta}} \mathcal{L}_\delta(\widetilde{Q})\,dt + \int_{\bar{s}_{\ell+1}+\bar{\theta}}^{\underline{t}_{\ell+1}} \mathcal{L}_\delta(\widetilde{Q})\,dt - \int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(Q)\,dt - \int_{\bar{s}_{\ell+1}}^{\underline{t}_{\ell+1}} \mathcal{L}_\delta(Q)\,dt \tag{3.25}$$

No jumps of $\mathcal{L}_\delta(\cdot)$ are involved here so a similar but simpler argument than in (3.15)–(3.21), leads to

$$\int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}+\bar{\theta}} \mathcal{L}_\delta(\widetilde{Q})\,dt \le 2\varphi(\rho)$$

$$\int_{\bar{s}_{\ell+1}+\bar{\theta}}^{\underline{t}_{\ell+1}} \mathcal{L}_\delta(\widetilde{Q})\,dt \le 2\varphi(\rho) + \delta(\underline{t}_{\ell+1} - \bar{s}_{\ell+1} - \bar{\theta})$$

$$\int_{\bar{t}_\ell}^{\underline{s}_{\ell+1}} \mathcal{L}_\delta(Q)\,dt \ge 0$$

$$\int_{\bar{s}_{\ell+1}}^{\underline{t}_{\ell+1}} \mathcal{L}_\delta(Q)\,dt \ge \delta(m_{2\ell+2} - \bar{s}_{\ell+1}),$$

so that

$$0 \le \int_{\underline{s}_{\ell+1}}^{\bar{s}_{\ell+1}} (a(t+\bar{\theta}) - a(t))(-V_\delta(Q(t)) - \ell\delta)\,dt + 4\varphi(\rho). \tag{3.26}$$

But $a(t+\bar{\theta}) = a(t+kT-\theta) = a(t-\theta)$ so as earlier

$$0 \le I(\widetilde{Q}) - I(Q) \le -\frac{1}{2}(\bar{a} - \underline{a})\frac{d^2}{2\bar{c}}|h| + 8\varphi(\rho) \tag{3.27}$$

contrary to the choice of ρ. □

Remark 3.28. As was noted earlier, it is not necessary that (V_3') holds, i.e. $\xi_i = i\xi$ and $V_\delta(\xi_{i-1}) - V_\delta(\xi_i) = \delta$. The argument of Theorem 3.7 applies whenever there are points $\xi_0, \dots, \xi_N$ such that $|V_\delta(\xi_{i-1}) - V_\delta(\xi_i)|$ is sufficiently small, $1 \le i \le N$, and each ξ_i is a (strict) local maximum.

Remark 3.29. As a special case of Theorem 3.7, suppose the setting of Theorem 2.5 obtains. Set $\xi_0 = 0$, $\xi_1 = \xi$, $\xi_{2i} = \xi_0$, and $\xi_{2i+1} = \xi_1$, $i > 0$. Then by Theorem 3.7, there exist solutions of (HS) which are homoclinic to 0 if N is odd and heteroclinic from 0 to ξ if N is even and which spend the time interval $[m_{2i}, m_{2i+1}]$ near ξ_i. These are the simplest examples of the augmented chains mentioned in the Introduction.

Remark 3.30. By a limiting procedure, one can allow $(\xi_i)_{i\in\mathbb{N}}$ or $(\xi_i)_{i\in\mathbb{Z}}$ provided that δ is independent of the number of points. Indeed for the case of $(\xi_i)_{i\in\mathbb{Z}}$ and corresponding $m \in (\mathbb{Z}\backslash\{0\})^\infty$, set $\ell_k = (m_{-(2k+1)}, m_{-1}, m_1, \dots, m_{2k+1}) \in \mathbb{Z}^{4k+2}$. Then by Theorem 3.7, there exists a solution Q_{ℓ_k} of (HS) heteroclinic from ξ_{-k} to ξ_k. It is not difficult to get L^∞ bounds for Q_{ℓ_k} in each interval $[m_i, m_{i+1}]$ as in Lemma 2.3 or Lemma 3.1. Then (HS) gives bounds for Q_{ℓ_k} in C^2_{loc} independently of k. These bounds imply the existence of the limit solution.

4. On the assumption (V_4)

In this section some examples will be given for which (V_4) is valid. The first example is one-dimensional.

Assume $V_\delta(x) = V_0(x) + \delta W(x)$, where

($\mathbf{W}_1$) $V_0 \in C^2(\mathbb{R}, \mathbb{R})$ is 1-periodic;
($\mathbf{W}_2$) $V_0(0) = V_0'(0) = 0$, $V_0''(0) < 0$, $V(x) < 0$ for all $x \notin \mathbb{Z}$;
($\mathbf{W}_3$) $W \in C^2(\mathbb{R}, \mathbb{R})$, $W(x+1) = W(x) - 1$, for all $x \in \mathbb{R}$;

($\mathbf{W}_4$) $W(0) = W'(0) = 0$;

Proposition 4.1. *Suppose V_0 satisfies (W_1)–(W_2) and W satisfies (W_3)–(W_4). Then there is a $\delta_0 > 0$ such that*

$$V_\delta(x) = V_0(x) + \delta W(x)$$

satisfies (V_1), (V_2), (V_3') and (V_4) for all $\delta \in (0, \delta_0)$.

Proof. It is clear that (V_1) and (V_3') hold for all $\delta > 0$ if we take $\xi = 1$. Take $r_1 > 0$ and C_1, $C_2 > 0$ such that, for all $|x| \le r_1$,

$$V_0(x) \le -C_1 |x|^2, \qquad |W(x)| \le C_2 |x|^2.$$

Then

$$V_\delta(x) \le -(C_1 - \delta C_2)|x|^2 \qquad \text{for all } |x| \le r_1$$

so (V_2) holds for any $r_0 < r_1$ if $\delta_0 < C_1/C_2$.

Suppose further that

$$\delta_0 \sup_{[-1,1]} |W(x)| \le \frac{1}{2} \inf_{[r_1, 1-r_1]} |V_0(x)|.$$

Then one can check that $V_\delta(x) < 0$ for all $\delta < \delta_0$, and for all $x \in [-1 + r_1, -r_1] \cup [r_1, 1 - r_1]$. Using assumption ($\mathrm{W}_3$), one deduces that

$$[-1 + r_1, +\infty] \subset \mathcal{R}_0,$$

Now choose $r_0 < r_1$ such that the cost of going from $-r_0$ to $-r_1$ is greater then the cost of going from $-r_0$ to 0. (This can be done as in the proof of Lemma 2.10.) Let $h < 0$ be such that

$$0 > h > \sup\{ V_\delta(x) \mid x \in [-1 + r_1, -r_0] \cup [r_0, 1 - r_0],\ \delta \in [0, \delta_0] \}. \tag{4.2}$$

Then

$$[-1 + r_1, 2 - r_1] \subset \overline{B}_{r_0}(0) \cup \mathcal{R}_h \cup \overline{B}_{r_0}(1) \equiv \mathcal{D}_h, \tag{4.3}$$

so that 0 and 1 are path connected in $\mathcal{D}_h$ for all $h_0 < h < 0$ and (V_4)(a) follows.

Assume that such an h does not satisfy (V_4)(b). Then there is $-r_0 < \eta_1 < r_0$, $\xi - r_0 < \eta_2 < \xi + r_0$, $Q_0 \in \gamma(m_1, m_2)$, a minimizer for I_0 and $\tau \in [m_1, m_2]$ such that $Q_0(\tau) \in \partial \mathcal{D}_h$. By (4.3)

$$\partial \mathcal{D}_h \subset (-\infty, -1 + r_1] \cup [1 + r_1, +\infty).$$

Hence there is a number $m_1 \le \tau \le m_2$ such that $Q_0(\tau) = -1 + r_1$ (or $Q(\tau) = 1 + r_1$). Since $Q(m_1) = \eta_1 > -r_0$ and $Q(m_2) = \eta_2 > -r_0$, one has a contradiction with our choice of r_0. Thus (V_4) has been established for all h satisfying (4.2). $\square$

Next using Proposition 4.1, a somewhat artificial example of a potential in higher dimensions which satisfies our assumptions can be given. Fix $V_0\colon \mathbb{R} \to \mathbb{R}$ satisfying (W_1)-(W_2) and $W\colon \mathbb{R} \to \mathbb{R}$ satisfying (W_3)-(W_4). We know, from Proposition 4.1 that there is δ_0 such that $V_0 + \delta W$ satisfies (V_1), (V_2), (V_3') and (V_4) for all $\delta \in [0, \delta_0]$. Then take r_1 and r_0 as in the proof of Proposition 4.1. We then know that (V_1), (V_2), (V_3') and (V_4) hold.

Set

$$\mu_1 = \inf\Big\{ \int_0^\tau \big(\frac{1}{2}|\dot q|^2 - \underline{a} V_\delta(q)\big)\, dt \ \Big| \ \Big| \ \tau \ge 0,\ \delta \in [0, \delta_0],\ q(0) \ge -r_0,\ q(\tau) \le -1 + r_1,\ q \in W^{1,2}(0, \tau) \Big\}. \tag{4.4}$$

Let φ be as in (2.8). Take r_0 eventually smaller so that

$$\varphi(r_0) + r_0^2 \le r_1.$$

Then take $R\colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ such that

(W_5) $R \in C^2(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{R})$ and $R(x+1,y) = R(x,y)$ for all $(x,y)\in\mathbb{R}\times\mathbb{R}^{n-1}$;
(W_6) $R(0,0) = \nabla R(0,0) = 0$, $R''_{yy}(0,0) < 0$, $R(x,y) < 0$ for all $y \neq 0$;
(W_7) $R(x,y) \geq -\mu \geq -\frac{\mu_1}{2a}$ for all x, $|y| \leq r_0$;
(W_8) $R(x,0) \geq R(x,y)$ for all $x\in\mathbb{R}$, $|y|\geq r_0$ and $\sup_{|y|\geq r_0} R(x,y) < 0$.

We will show that

$$V_\delta(x,y) = V_0(x) + \delta W(x) + R(x,y)$$

satisfies, (V_1)-(V_2)-(V_3')-(V_4) for all $0 < \delta < \delta_0$. Indeed (V_1) and (V_2) follows as in Proposition 4.1, while (V_3') (with $\xi = (1,0)$) is a direct consequence of (W_1), (W_3) and (W_5).

To prove (V_4), observe that for all h satisfying (4.2) and

$$0 > h > \sup\big\{\, R(x,y) \;\big|\; -r_0 \leq |x| \leq r_0,\ |y| \geq r_0 \,\big\}$$

it follows from (W_6) that

$$[-1+r_1, 2-r_1]\times\mathbb{R}^{n-1} \subset \mathcal{D}_h,$$

so that (V_4)(a) holds.

In order to prove (V_4)(b), assume it does not hold. Then there is a $\eta_1 \in B_{r_0}(0)$, $\eta_2 \in B_{r_0}(\xi)$, $Q_0 = (x(t),y(t)) \in \gamma(m_1,m_2,\eta_1,\eta_2)$, a minimizer for I_0 and $\tau \in [m_1,m_2]$ such that $Q_0(\tau)\in\partial\mathcal{D}_h$. By (4.3)

$$\partial\mathcal{D}_h \subset \big((-\infty,-1+r_1]\cup[2-r_1,+\infty)\big)\times\mathbb{R}^{n-1}.$$

Hence there is a number $m_1 \leq t_0 \leq m_2$ such that $x(t_0) \leq -1+r_1$. (The case $x(t_0)\geq 2-r_1$ can be dealt with similarly). Then there is $t_1 > t_0$ such that $x(t)\geq 0$ for all $t \geq t_1$ and $t_2 \geq t_1$ such that $|y(t)|\geq\rho$ for all $t_1 < t \leq t_2$.

Define a new function $\overline{Q}\in\gamma(m_1,m_2,\eta_1,\eta_2)$ as follow:

$$\overline{Q}(t) = (\overline{x}(t),\overline{y}(t)) = \begin{cases} \eta_1 & t = m_1 \\ \text{linear} & m_1 \leq t \leq m_1+1 \\ (0,0) & m_1+1 \leq t \leq t_1 \\ (x(t),0) & t_1 \leq t \leq t_2 - 1 \\ (x(t),\text{linear}) & t_2-1 \leq t \leq t_2 \\ (x(t),y(t)) & t_2 \leq t \leq m_2 \end{cases}$$

Note that we can assume $t_1 \geq m_1 + 1$, and that minor modifications are required if $t_1 \geq t_2 - 1$. Estimating $I_0(Q) - I_0(\overline{Q})$:

$$\begin{aligned}
0 &\geq I_0(Q) - I_0(\overline{Q}) \\
&= \int_{m_1}^{m_2} \big(\tfrac{1}{2}(|\dot{x}|^2 + |\dot{y}|^2) - a(t)(V_\delta(x) + R(x,y))\big)\, dt \\
&\quad - \int_{m_1}^{m_2} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 + |\dot{\overline{y}}|^2) - a(t)(V_\delta(\overline{x}) + R(\overline{x},\overline{y}))\big)\, dt \\
&= \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{x}|^2 + |\dot{y}|^2) - a(t)(V_\delta(x) + R(x,y))\big)\, dt \\
&\quad - \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 + |\dot{\overline{y}}|^2) - a(t)(V_\delta(\overline{x}) + R(\overline{x},\overline{y}))\big)\, dt \\
&= \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{x}|^2 - a(t)V_\delta(x)\big)\, dt - \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 - a(t)V_\delta(\overline{x})\big)\, dt \\
&\quad + \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{y}|^2 - a(t)R(x,y)\big)\, dt - \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 - a(t)R(\overline{x},\overline{y})\big)\, dt \\
&= \int_{m_1}^{t_1} \big(\tfrac{1}{2}(|\dot{x}|^2 - a(t)V_\delta(x)\big)\, dt - \int_{m_1}^{t_1} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 - a(t)V_\delta(\overline{x})\big)\, dt \\
&\quad + \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{y}|^2 - a(t)R(x,y)\big)\, dt - \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 - a(t)R(\overline{x},\overline{y})\big)\, dt
\end{aligned}$$

Let us now observe that

$$\int_{m_1}^{t_1} \big(\tfrac{1}{2}(|\dot{x}|^2 - a(t)V_\delta(x)\big)\, dt \geq 2\mu_1$$

while, by the choice of φ,

$$\int_{m_1}^{t_1} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 - a(t)V_\delta(\overline{x})\big)\, dt = \int_{m_1}^{m_1+1} \big(\tfrac{1}{2}(|\dot{\overline{x}}|^2 - a(t)V_\delta(\overline{x})\big)\, dt \leq \varphi(r_0).$$

We also have that

$$\begin{aligned}
\int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{y}|^2 - a(t)R(x,y)\big)\, dt &\geq \int_{m_1+1}^{t_2-1} \big(\tfrac{1}{2}(|\dot{y}|^2 - a(t)R(x,y)\big)\, dt \\
&\geq -\int_{m_1+1}^{t_2-1} a(t)R(x,y)\, dt
\end{aligned}$$

and

$$\int_{m_1}^{m_1+1} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 - a(t)R(\overline{x},\overline{y})\big)\, dt \leq \int_{m_1}^{m_1+1} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 + \overline{a}\mu\big)\, dt \leq \frac{1}{2}r_0^2 + \overline{a}\mu$$

$$\int_{t_2-1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 + a(t)R(\overline{x},\overline{y})\big)\, dt \leq \frac{1}{2}r_0^2 + \overline{a}\mu.$$

We deduce that

$$\begin{aligned}
&\int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{y}|^2 - a(t)R(x,y)\big)\, dt - \int_{m_1}^{t_2} \big(\tfrac{1}{2}(|\dot{\overline{y}}|^2 - a(t)R(\overline{x},\overline{y})\big)\, dt \\
&\quad \geq \int_{m_1+1}^{t_2-1} \big(-a(t)R(x,y) + a(t)R(x,0)\big)\, dt - r_0^2 - 2\overline{a}\mu \\
&\quad \geq -r_0^2 - 2\overline{a}\mu
\end{aligned}$$

Combining these inequalities yields:

$$0 \geq I_0(Q) - I_0(\overline{Q}) \geq 2\mu_1 - \varphi(r_0) - r_0^2 - 2\overline{a}\mu,$$

a contradiction which shows that (V_4)(b) holds.

For our next example, suppose V_δ satisfies (V_1)–(V_3) and in addition:

(V_5) $V_\delta(x) < 0$ for all $x \in \mathbb{R}^n \backslash \{0, \xi\}$, $\delta \in [0, \delta_0]$.

By (V_5), 0 is a global maximum for V_δ. Now $\mathcal{R}_0 = \mathbb{R}^n$. The next proposition shows that (V_4) is valid for this setting.

Proposition 4.5. *If V_δ satisfies (V_1)–(V_3) and (V_5), then (V_4) also holds.*

By (V_5), (V_4)(a) is satisfied. To verify (V_4)(b), observe that as in the proof of Lemma 2.3, along a minimizing sequence for I_0 in $\gamma(m_1, m_2, \eta_1, \eta_2)$,

$$|q(t)| \leq r_0 + \sqrt{2\overline{c}(m_2 - m_1)} \equiv R$$

as in Lemma 2.3. Choose $h_0 < 0$ such that

$$V_\delta(x) \leq h_0, \quad x \in \overline{B}_R(0) \backslash (B_{\frac{r_0}{2}}(0) \cup B_{\frac{r_0}{2}}(\xi))$$

and $\delta \in [0, \delta_0]$. Then if $h = h_0/2$, (V_4)(b) holds.

Remark 4.6. Note that there may be several values of ξ for which (V_3) is satisfied possibly with different (small) values of δ.

We conclude with a couple of examples to which Theorem 3.7 and Remark 3.30 apply. Suppose $n = 1$, e.g. (W_1)–(W_4) hold. Then Proposition 4.1 and Theorem 2.5 show there is a solution, Q_1 of (HS) heteroclinic from 0 to 1 for each small δ. Similarly there are solutions Q_j, of (HS) heteroclinic from $j-1$ to j. By the argument of Proposition 4.1 again together with Theorem 3.7, there are heteroclinic solutions of (HS) from j to k for any $j, k \in \mathbb{Z}$ as well as solutions going from $-\infty$ to ∞ via Remark 3.30. Moreover there are augmented chain type solutions in the spirit of the Introduction and Remark 3.29.

A variant of these arguments shows $V_\delta(x) = (1+\delta)(\cos(x) - 1) + \delta x$ has heteroclinics as in the previous paragraph.

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