$$|\Delta F| = |F[\varphi(x) + \psi(x)] - F[\varphi(x)]| < \epsilon$$

whenever ψ is of class $C^{(n)}$, or continuous and of class $C^{(n)}$ except possibly at a finite number of x-values, and

$$|\psi(x)| < \delta, \ |\psi'(x)| < \delta, \ \ldots\ldots, \ |\psi^{(n)}(x)| < \delta \qquad (a \leqq x \leqq b). \qquad (10)$$

Further F has a derivative at any value $x = \xi$ which is approached with order $2n$; that is, if ψ does not change sign and vanishes except on an interval of length less than h including $x = \xi$, and if furthermore

$$|\psi(x)| < \delta, \ |\psi'(x)| < \delta, \ \ldots\ldots, \ |\psi^{(2n)}(x)| < \delta, \qquad (11)$$

then the limit

$$F'[\varphi(x), \xi] = \lim_{\substack{\delta=0 \\ h=0}} \frac{\Delta F}{\sigma}$$

exists. Further the absolute value of the quotient $\Delta F/\delta h$ will be bounded for all choices of $\delta > 0$, $h > 0$, $\psi(x)$ satisfying the relations (11) and such that the values $(x, y, y', \ldots, y^{(n)})$ on the arc $y = \varphi + \psi$, $a \leqq x \leqq b$, are all in the neighborhood R.

[1] Volterra, *Leçons sur les equations intégrales*, ch. 1, art. 5: or his *Leçons sur les fonctions des lignes*, ch. 1, art. 2.

[2] Volterra, arts. VII and 2, 3, respectively, of the chapters referred to above.

[3] See Jordan, *Cours d'Analyse*, vol. 1, p. 247.

[4] See Fischer, A generalization of Volterra's derivative of a function of a curve, *Amer. J. Math.*, 35, 385 (1913).

A CLASSIFICATION OF QUADRATIC VECTOR FUNCTIONS

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There is probably no chapter of mathematics more worthy of attention, or more neglected at present, than the theory of vector functions. In the case of the linear vector function, it is true, a good deal has been found out in one way or another, and this by some of the very greatest of mathematicians. First investigated in detail by Hamilton[1] and again appearing as Cayley's matrix of the third order,[2] the linear vector function is essentially the same as the Grassmann open product[3] and the Gibbs dyadic.[4] In Germany the nonion or three-square matrix bears the name Tensor,[5] a word used by others in a different sense. On the other hand, we may make a clean sweep of all these operational concepts and, if it pleases us, define a vector function as a set of three algebraic forms X,Y,Z, homogeneous polynomials in three variables x,y,z, with nothing left of the original idea of a vector as a directed quantity except a definite order in writing the three forms X,Y,Z. The occurrence of the same mathematical entity under such a variety of names and algorisms is perhaps the natural consequence of its fundamental character.

From whatever point of view we prefer to start, it is well known in the linear case that a convenient classification of types may be made with reference to the axes of the function; an axis of a linear vector function ϕ of a vector ρ being a direction of ρ such that $\phi\rho$ and ρ are parallel, or $\phi\rho = g\rho$, where g is a mere number. In the language of algebraic forms this is the same as saying that an axis is a point, in homogeneous coördinates, satisfying the equations

$$yZ - zY = 0, \; zX - xZ = 0, \; xY - yX = 0. \tag{1}$$

In the longer work of which the present paper is an outline, a similar basis is taken for a classification of types of *quadratic* vector functions $F\rho$ of the vector ρ. Related mathematical problems which, by reason of their close kinship, suggest the study of vectors of higher degree are numerous. For example, if x,y,z, and X,Y,Z, denote points respectively in a first plane and in a transformed plane, the vector $F\rho$ obviously enough defines a geometric point-transformation. The worker who limits himself, however, to such an interpretation in homogeneous coördinates will lose sight of the conveniences of vector addition. We may with equal ease let $F\rho$ define a transformation in space of three dimensions with the origin invariant.

As another application, the properties of $F\rho$, by reason of their invariant character with reference to change of coördinate axes, are intimately connected with the whole theory of a set of three algebraic forms. That the study of the linear vector function led to the discovery of various invariants belonging to one function or to a system of several such functions, is well known.[6]

Again, the student of certain types of differential equation will find that the notion of a vector function comes readily into his work. The very appearance of equations like

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}, \tag{2}$$

where X, Y, and Z are algebraic forms as already explained; or like

$$(yZ - zY)dx + (zX - xZ)dy + (xY - yX)dz = 0; \tag{3}$$

suggests translation into some sort of vectorial language. With special reference to the quadratic case, there exist several long and interesting, although not very recent, French monographs, notably one by Darboux,[7] in which he takes advantage of the close relation of equations (2) and (3) with each other and with the equations, in non-homogeneous forms,

$$\frac{dy}{dx} = \frac{Y}{X}, \; z = 1, \; Z = 0. \tag{4}$$

It follows that a vector function offers a ready tool for inquiring into the nature of the functions defined by any equation of the type $dy/dx = R$, where R is a rational function of x and y. Darboux does not get much beyond an examination of a great variety of cases where (3) possesses one or more algebraic integrals, or else can be brought to depend on a Riccati equation. He does show very clearly the wide range of even this problem, indicating the very general character of the function which would satisfy (4) when X and Y are quadratics set down at random. For instance, the most general hypergeometric function satisfies a second order equation which is a resolvent for a very special case of (4) in the Riccati form. Darboux uses no vector algebra as such, but he brings out and uses a fact which, translated into vector language, is as follows: The addition to a vector $F\rho$ of another vector of the form ρt, where t is a scalar variable, does not alter the axes of $F\rho$. This is geometrically evident. Analytically expressed it means that if $X,Y,Z,$ satisfy (1) when a certain set of values of x,y,z is given the equations will still be satisfied by

$$X + tx, \; Y + ty, \; Z + tz,$$

written instead of X,Y,Z. This can be verified directly. In fact the variable t disappears automatically from (3). Roughly speaking, the connection of ideas consists in this, that if (3) has been completely solved (which requires a certain number of particular solutions), then both (2) and (4) can be solved by quadratures. In the quadratic case, the scalar t takes the form $S\delta\rho$, that is, it depends upon a single constant vector δ. Now if, for a value of δ, we can find a solution of (4), or (what is much the same here), of the partial differential equation

$$X\frac{\partial u}{\partial x} + Y\frac{\partial u}{\partial y} + Z\frac{\partial u}{\partial z} = 0, \tag{5}$$

this solution will be a particular solution of (3). Stated another way, all the different functions defined by (5) when all possible values are given to the vector δ can be found by quadratures when (3) has been solved completely; these functions constitute a family or set possessing some group properties,—just how far they form a group has not been investigated, so far as I am aware.

The axes of the vector function correspond to the singular points of (3), if we interpret in homogeneous coördinates. It is well known that these are $n^2 + n + 1$ in number,[8] when n is the degree of the forms X,Y,Z. If $n = 2$ we thus have seven axes, in general.

The necessity for careful examination and classification of types of quadratic vector functions appears from the fact that many differential equations like (4) do not yield vectors of the most general kind, having all seven axes distinct, but possess multiple or coincident axes of all orders up to seven. To take a simple example, if $X = xy$, $Y = yz$ and $Z = zx$, the vectors **i**, **j**, and **k** are all double axes, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ a single axis, that is, in homogeneous coördinates, the points (1,0,0,), (0,1,0,), and (0,0,1), are higher singularities of (3), and (1,1,1,) is an ordinary singularity. By a proper choice of t, that is of δ, we can add a term $\rho S\delta\rho$ which shall make $Z = 0$, and have the equation (4) as

$$\frac{dy}{dx} = \frac{y\,(1 - x)}{x\,(y - x)};$$

the value of t being $-x$, and that of δ being **i**. The variables x,y,z themselves correspond to particular solutions of (3). Four particular solutions are needed, however, to complete the solution by quadratures; hence the rest of the functions of the family cannot be found by mere quadratures.

Again, a quadratic vector may have more than seven axes, but if so it has an infinite number, and equation (3) may be divided through by a scalar variable. Take for example one of the simplest types furnished by the technique of vector algebra, viz., $V\rho V\alpha\rho$, or in Gibbs' notation $\rho \times (\alpha \times \rho)$. This vector may be expanded as

$$\rho S\alpha\rho - \alpha\rho^2,$$

which differs from the vector $\alpha(x^2 + y^2 + z^2)$ only by the term in ρ, having no effect on the axes. Hence any element of the minimal cone $\rho^2 = 0$ is an axis, and α is the only other axis.

The most general quadratic vector function may be very elegantly defined by a sum of triads, that is, a triadic. A single triad $\alpha\beta\gamma$ multiplied (dot product) by and into ρ is the same as the Hamiltonian vector $\beta S\alpha\rho S\gamma\rho$. Evidently β is an axis. Also, any vector at right angles either to α or to γ is an axis. As a less special example, the vector $Vq\rho s\rho t$, where q,s,t are constant quaternions, has important geometrical applications; Hamilton showed that its properties include those of the most general cubic cone.[9] This vector cannot be so simply expressed in any other algorism. It has, in general, all its axes distinct; two of them are on the minimal cone $\rho^2 = 0$, and are easily found. The others are determined by an equation of the fifth degree.

In developing a classification of various types, I have made comparatively slight use of the technical processes of vector algebra, and have based my subclasses on configurations of the axes rather than on the possibility of simple algebraic expression. It appears that, *in the most general type, a normal form of vector is easily obtained in terms of the axes themselves.* If there are double axes, but no higher axis, there is still no particular difficulty, although the normal forms are less simple. It is shown that there is only one kind of triple axis; obviously, a quadratic vector can have at most two of these. A normal form of vector with two triple axes is developed, and has a number of properties in the way of symmetry. An axis of the fourth order, on the other hand, may be of two kinds. Quadratic vectors with an axis of higher order than the third fall naturally into two families, according as the axis is of the first, or of the second kind. An axis of the first kind is shown to correspond to a double point common to all three of the cubic curves defined by equations (1), if we interpret in homogeneous coördinates. The second kind is shown to depend on a partial differential equation satisfied by the vector $F\rho$. This differential condition depends in part on the results of my former papers, where the properties of a differential vector have been developed.[10]

Tests for the existence of axes of any order up to, and including the fourth have been given for vectors of any degree whatever. In the quadratic case, normal forms are given including all possible types.

The existence of over one hundred special types makes it very desirable to have, on the formal side, the means of covering in one comparatively simple algebraic expression as many of these types as possible,—and in such a way that their properties are easily correlated. The largest number of advantages for this purpose appears to be possessed by the form $V\phi\rho\theta\rho$, where ϕ and θ denote linear vector functions. In Gibbs' notation, this is the same as determining our vector function by the cross product of two dyadics. Besides compactness of expression, this vector product offers the following advantages:

1. It is easily interpreted as the most general birational quadratic point transformation in a plane.

2. It differs from a quadratic vector of the most general type only by a term in ρ, which, as already shown, does not alter the axes.

3. The properties of a quadratic vector are made to depend on those of linear vectors.

4. For the study of differential equations it is especially suited since it is a vector product. Equation (3) takes the factored form $S\phi\rho\theta\rho V\rho d\rho = 0$, or in Gibbs' notation $(\phi\rho \times \theta\rho).(\rho \times d\rho) = 0$.

5. Three of its axes are zeros, that is, for three values of ρ the vector vanishes in all its components.

It appears, therefore, as a problem of importance to determine how far the various possible configurations of axes, in special types of quadratic vectors, are included among possible configurations of the axes of $V\phi\rho\theta\rho$. This is the same as the problem of determining with what exceptions a quadratic vector of whatever type or sub-type can be written in the form

$$V\phi\rho\theta\rho + \rho S\delta\rho.$$

At this point of the investigation a certain difficulty presented itself. It is easy so to chose δ that the resulting quadratic vector shall have three zeros, distinct or multiple; if, then, the vector does not fall into a uniplanar, i.e., a binomial form, it is possible to factor vectorially into $V\phi\rho\theta\rho$. But a binomial quadratic vector cannot be so factored, hence the necessity of examining a very large number of choices of δ to find those which do not yield a binomial. For most types where such a vector δ can be found, I have contented myself with giving the value of the resulting δ, since its accuracy, when found, is easy to verify. In the cases where no value of δ can be found, I have, of course, demonstrated the impossibility.

The final result is, on the whole, highly satisfactory. It appears that *the form* $V\phi\rho\theta\rho + \rho S\delta\rho$ *includes all types of quadratic vectors except two simple sub-types both belonging to the family having a higher axis of the second kind.* Normal forms for these two very exceptional types have been given.

The existence of these exceptions is due to the fact that, as the form of the vector grows more and more restricted, the possible choices of δ, which avoid the binomial, decrease in number. Thus in general there are thirty-five possible values of δ; but if the determinant of the components of a set of three axes vanishes, the number falls to thirty-one; and the occurrence of multiple axes also reduces the number. The wonder appears to be, not that there are exceptions, but that there are so few, and these so simple.

Various properties of quadratic vector functions appear by virtue of the normal forms which characterize their types. Some of these bring out properties of the differential equations (3) and (4). For example, it is very easy to determine, when the type of the vector is known, whether these equations can be solved by quadratures. Again, it appears that the most general equation (4) never corresponds to the vector function of the most general type, but to a slightly restricted type. A consideration of the details of these normal forms would carry the discussion beyond the limits of the present paper.

[1] W. R. Hamilton, *Lectures on Quaternions*, p. 480.

[2] Arthur Cayley, *London, Phil. Trans. R. Soc.*, **48**, 17 (1858).

[3] Hermann Grassmann, *Gesammelte Werke* (Die Lineale Ausdehnungslehre, 1844, § 172).

[4] Gibbs-Wilson, *Vector Analysis* (1902). A dyadic is a sum of dyad terms, where each dyad is a product of two vectors. When multiplied into the vector ρ by dot, i.e., scalar, multiplication, a single dyad is equivalent to a Hamiltonian term of the form $\beta S\alpha\rho$.

[5] W. Voigt, *Die fundamentalen Eigenschaften der Krystalle* (1898).

[6] Hamilton, *loc. cit.*, Lecture VII. A Hamiltonian invariant is a function of three vectors which is unaltered when these vectors are varied.

[7] Darboux, Mémoire sur les Equations Différentielles Algébriques, *Bul. Sc. Math., Paris*, **13**, 83 (1878).

[8] Clebsch, *Leçons sur la Geometrie* (Tr. Benoist), t. 2, p. 113; *Vorlesungen über Geometrie*, Bd. 1, 390, 1001.

[9] Hamilton, *Elements of Quaternions*, Art. 415.

[10] F. L. Hitchcock, *Phil. Mag.*, Ser. 6, **3**, 576 (1902); Ibid., **4**, 187 (1903).

ON THE RADIAL VELOCITIES OF FIVE NEBULAE IN THE MAGELLANIC CLOUDS

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In the course of observations on the velocities of approach and recession (radial velocities) of southern nebulae whose spectra contain bright lines, made with spectrographs attached to the 37-inch reflecting telescope of the D. O. Mills Expedition from the Lick Observatory, at Cerro San Cristobal, Santiago, Chile, we have found that the following five nebulae have high radial velocities. The results depend upon the observed positions of the Hβ hydrogen line and the nebulium lines at 5007A and 4959A, in each case. The velocities are not corrected for the solar motion.

Nebula	*Right ascension*	*Declination*	*Radial velocity km/sec.*	*Number of plates*
N. G. C. 1644	1h 6.2m	−73° 44′	+158	3
N. G. C. 1714	4 52.0	−67 06	+301	2
N. G. C. 2111	5 52.6	−69 33	+268	2
N. G. C. 1743	4 54.6	−69 21	+254	1
N. G. C. 2070	5 39.4	−69 09	+276	1