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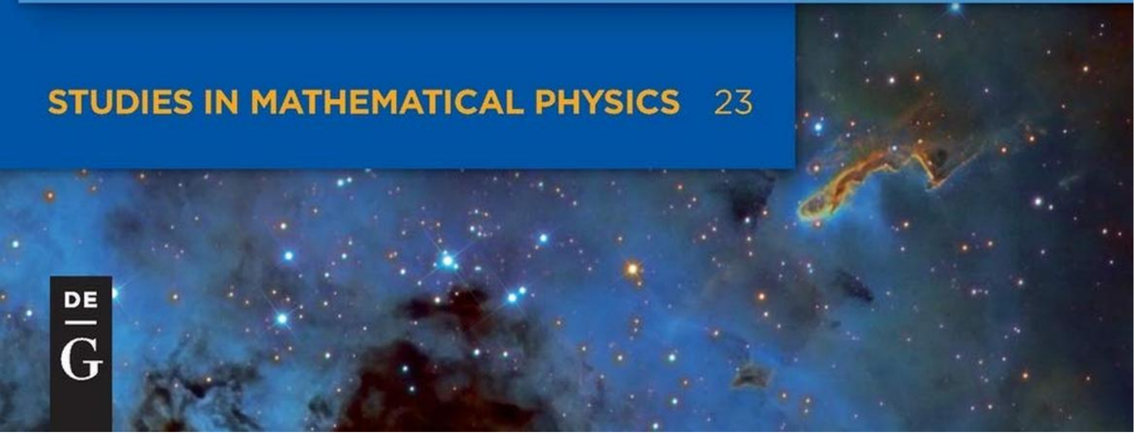

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Preface

The purpose of the book is the study of non-perturbative effects in gauge theories constituting the Standard Model (SM). Nowadays we have excellent basis for understanding elementary particles physics. Two main constituents of the SM – the quantum chromodynamics (QCD) and the electro-weak theory (EWT) are renormalizable theories with firmly established rules of the perturbation theory calculations. The excellent agreement of these calculations in the regions of their applicability with the totality of the experimental data remains no place for any doubt in adequacy of the QCD and the EWT as genuine theories. However in both theories there are aspects, which can not be achieved by the perturbation theory calculations. In QCD it is a description of low momenta hadron physics. In EWT the problem of the initial symmetry breaking also causes troubles, even in case of final confirmation of the status of the recently discovered state with mass $\simeq 125.7$ GeV as a genuine Higgs scalar particle. In any case, the ability of consistent dealing with the non-perturbative effects in both theories is extremely desirable. There are also widely discussed problems of hierarchy and of naturalness, which demand to be born in mind.

As for non-perturbative effects, there are the widely known and elaborated lattice calculations with supercomputers, which are successfully applied to QCD and to some aspects of the EWT. While paying tribute to achievements of this powerful tool, we would state, that analytic methods, proposed and elaborated in the book allow to move up further and obtain possible values of the main physical parameters, which define the corresponding theories.

As for general problems, for example, that of hierarchy and of naturalness, they are attentively considered in a number of directions of theoretical studies. First of all, there is a conception, that all the problems will be solved in the framework of the so called New Physics. The quest for the New Physics means looking for possibilities to extend the SM by adding new interactions and new particles. The option includes numerous possibilities: the supersymmetry, the superstrings, the extra dimensions of the space-time and many others. While these directions of studies deserve overall attention, it is not less necessary to long for a solution of the main contemporary problems of the particles physics in the framework of the SM by taking into account the non-perturbative effects. As a matter of fact these effects can by no means be neglected at all.

There are persuasive grounds of unavoidable necessity of introduction of the so called effective interactions in theories of the SM for phenomenological description of the non-perturbative effects. The first, while the most instructive, example of such interaction is provided by the famous Nambu–Jona-Lasinio model. The model was elaborated in the time of very active development of results having been obtained in the course of solving the problem of superconductivity in works by Bardeen, Cooper, Schriffer and by Bogoliubov, who have used his compensation approach in microscopic theory of the superconductivity. That was also the time of the beginning of my scientific activity. After participation in a work on the spontaneous generation of a mass in a chiral invariant model, which was done under the guidance of Nikolai Nikolaevich Bogoliubov, I had undertook attempts to apply the same method to the problem of spontaneous generation of Nambu-like interactions. I have discussed these attempts with N.N. Bogoliubov, but his reaction was at the first sight discouraging. He said: "The compensation equation in your case is a functional one, and to understand, if there is a non-trivial solution is a hopeless problem." But after few moments he added: "A dog knows, though, what if something will turn out?" In any case, in connection with results, which constitute the basis of the book, which follows this Preface, I would express sincere gratitude to the outstanding scientist Nikolai Nikolaevich Bogoliubov.

I am also to recall with gratitude other outstanding physicist, which influenced a lot on the essence of the approach considered in the book, Lev Davidovich Landau. L. D. Landau has taught me all the course of theoretical physics at the Moscow University, starting of the classical electrodynamics up to the quantum field theory. The later course was delivered by prof. Landau the only one time, just for our year of the study (1959–1960). In addition to a knowledge of the theory, I have got from L. D. Landau his scientific philosophy. He was a staunch supporter of the Ockham's razor principle. That is he was very cautious in admitting new subjects and notions. Of course, it was not pleasant to hear from him in the lecture roughly negative reference to Dirac monopole. Such new proposals he prefer to call "pathology". In any case it was his conviction.

With establishing of the Standard Model as the genuine theory of elementary particles, I came to appreciate the Landau position with more vigor. The influence of his conviction push me to formate my attitude to the problem of the so called effects beyond the Standard Model. There are evidently such effects. However, the Ockham principle tells us: first try to find an explanation in the framework of the existing Standard Model and only in case of a total defeat start to invent new notions and principles. In doing this attempt it is necessary to apply new methods and approaches. One of these approaches is proposed and developed in the book. This approach is based on the N.N. Bogoliubov compensation method.

In a sense, the proposal advocated in the book is based on the conviction, that before introduction of a completely new notion, one have to examine all the possibilities of explanation of the totality of facts in the framework of firmly established notions. In our case this means to look for achievements in the framework of the Standard Model. The application of the method is inevitably connected with an approximate scheme. The specific approximation will be formulated in the proper place. We shall mostly ground our conclusions on the qualitative and sometimes even in quantitative agreement of results with the real physics. In this connection I recollect again an episode from my early years in the scientific activity. Then I had the favorable opportunity of having contacts with the prominent physicist I. Ya. Pomeranchuk. Once I had given a talk at his seminar, which was related to a discussion of the so called bootstrap method (then having been just new). One of participants of the seminar had remarked: “But this is a model!” Pomeranchuk replied in an instructive tone: “A model, which explains the totality of data is just the theory.” Following this wisdom I would consider the comparison of results of the approach, proposed and developed in the book, with the real physics as a possible check of an applicability of the approach. I do invite a reader to follow the logics and the results of the compensation approach in application to effective interactions of the Standard Model and to acquire the own opinion on its validity.

I would express the deep gratitude to my colleagues, coauthors of works, which contribute to the substance of the book, R. N. Faustov, A. T. Filippov, A. N. Tavkhelidze, M. K. Volkov, I. V. Zaitsev. The collaboration with the colleagues and the experience, acquired from numerous discussions with them, assists greatly in creation of the book.

I would also recall with the sincere gratitude prominent physicist Abraham Pais, who had strongly supported my studies on a self damping in non-renormalizable equations, which in the present book supply the main technical tool for an investigation of the phenomenon of a spontaneous generation of effective interactions.

The intercourse with my colleagues E. E. Boos, O. A. Khrustalev, V. A. Matveev, V. A. Rubakov, V. I. Savrin, B. V. Struminsky has influenced a lot on my vision of physical problems and so also promotes the work on the book.

I do express the most deep gratitude to my wife Larisa Lomonosova for encouraging assistance in all my undertakings, especially in respect to this book.

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Contents

1 Elementary particles and fields

1.1 Conventions and notations

The main source of our knowledge on elementary particles is provided by experiments at high energies, which are performed at accelerators and colliders. Energy in particle physics is measured in eV (electron-volt) and in its multiples. The commonly used nowadays are the following units

$$\mathrm{MeV} = 10^6\ \mathrm{eV}, \quad \mathrm{GeV} = 10^9\ \mathrm{eV}, \quad \mathrm{TeV} = 10^{12}\ \mathrm{eV}. \tag{1.1}$$

We shall use these units throughout the book. In calculations and estimates we shall use natural system of units, in which two the most important constants, the Planck constant $\hbar$ and the velocity of the light c both are unity

$$\hbar = 1, \quad c = 1. \tag{1.2}$$

Then mass M, momentum $\mathbf{p}$, temperature T are measured also in eV. Both time t and distance x are measured in inverse eV. There are the following coefficients of transition from eV to the usual units

$$\frac{1}{\mathrm{GeV}} = 1.973269602(77) \cdot 10^{-14}\ \mathrm{cm}, \qquad \frac{1}{\mathrm{GeV}} = 6.58211889(26) \cdot 10^{-25}\ \mathrm{s}. \tag{1.3}$$

Effective cross-sections are measured in barns ($\mathrm{b} = 10^{-24}\ \mathrm{cm}^2$) and in its parts: millibarn, microbarn, nanobarn, picobarn, femtobarn

$$\begin{aligned} \mathrm{mb} &= 10^{-27}\ \mathrm{cm}^2, \quad \mu\mathrm{b} = 10^{-30}\ \mathrm{cm}^2, \quad \mathrm{nb} = 10^{-33}\ \mathrm{cm}^2, \\ \mathrm{pb} &= 10^{-36}\ \mathrm{cm}^2, \quad \mathrm{fb} = 10^{-39}\ \mathrm{cm}^2. \end{aligned} \tag{1.4}$$

Throughout the book we use kinematics of the four-dimensional Minkovsky space, that means metric tensor g_{ij} being diagonal and

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1. \tag{1.5}$$

A scalar product of two vectors is defined by the following notation

$$(p\,q) = p_i\, q_j\, g_{ij} = p_0 q_0 - \mathbf{pq}, \tag{1.6}$$

where by bold letters we designate spatial parts of four-vectors. We always shall mean summation on recurring indices. Bearing in mind a simplification of the notations we shall not explicitly write contravariant indices. For example, we shall write the product of two tensors of the second rank in the following form

$$F \cdot F = F_{\mu\nu}\, F_{\mu\nu}, \tag{1.7}$$

that, as a matter of fact, means

$$F \cdot F = F_{\mu\nu} F_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} . \tag{1.8}$$

In all cases, when it does not cause misunderstanding we shall use notations like (1.7). The exception will be necessarily made in Chapter 8, where some aspects of the gravitational interaction will be discussed.

When considering experimental data we shall use well-known parameter s for an invariant energy squared

$$s = (p_1 + p_2)^2 , \tag{1.9}$$

where p_1 and p_2 are initial four-momenta of colliding particles. Other kinematic variable will be explicated in appropriate places below.

The physics of elementary particles is described in the framework of the quantum field theory. The presentation in the book supposes sufficient knowledge of this theory. We shall refer mostly to books [1] and [2], in which a reader will find necessary details of problems being considered below.

In the course of evaluations we shall encounter loop integrals. As a rule we shall evaluate these integrals using the well-known Wick rotation, which will be explicitly explained in the proper place. This procedure of transition to the four-dimensional Euclid space may be used in some cases by default.

1.2 Particles and interactions

In the Nature there exist numerous particles and different interactions act between them. Physicists usually distinguish interactions to be strong, electromagnetic, weak and gravitational. According to involvement of particles in these interactions and also to their peculiar properties particles divide into hadrons, leptons and gauge bosons. A special role plays the Higgs boson.

The gravitational interaction is the most universal. All the particles are involved in the gravitation.

By definition leptons are particles with half-integer spins, which additionally participate only in weak and electromagnetic interactions. The most familiar representative of the class is the electron.

Gauge bosons are presented by the photon, the graviton, the electroweak bosons W, Z and the mediators of the strong interaction – the gluons.

The most numerous class of particles are hadrons, which participate strong interaction as well as weak and electromagnetic ones. The most familiar representatives are proton and neutron, which constitute atomic nuclei. However these hadrons are not elementary to the same level as, for example, the electron. It became clear, that hadrons are composite particles, constituting of elementary quarks.

There are six quarks and six leptons to describe the totality of our knowledge on variety of particles. They are presented in Table 1.1, where Q is an electric charge of a

Table 1.1. Fundamental quarks and leptons

$\frac{Q}{e}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	−1	1	0
g1	u	d	$\bar{d}$	$\bar{u}$	ν_e	e	$\bar{e}$	$\bar{\nu}_e$
g2	c	s	$\bar{s}$	$\bar{c}$	ν_μ	μ	$\bar{\mu}$	$\bar{\nu}_\mu$
g3	t	b	$\bar{b}$	$\bar{t}$	ν_τ	τ	$\bar{\tau}$	$\bar{\nu}_\tau$

particle and $\mathrm{e} = (1.60217649 \pm 0.00000004)10^{-19}\,C$ is the elementary electric charge (the electron charge magnitude). Here elementary objects with fractional charges are the well-known quarks and antiquarks while the right-hand part of the Table presents leptons and antileptons. We see from Table 1.1, that quarks and leptons are gathering into three quite similar rows, which usually are called generations. We number generations in the first column. So, e.g. u- and d-quarks and the electron represent the first generation, whereas the t-quark and the τ-lepton correspond to the third one.

The mediator fields will be introduced in the proper places below.

Now the main difference between quarks and leptons consists in their relation to the strong interaction. Namely quarks participate in strong interaction while leptons do not. The both take part in electromagnetic interaction, which we consider in the next section.

Quarks and leptons in Table 1.1 have all spin 1/2. Quarks form bound states, which are the observable strongly interacting particles, usually being called hadrons. The variety of hadrons is divided in two large classes: baryons and mesons. Baryons have half-integer spin, thus they consist of odd number of quarks, first of all of three quarks. Mesons always have integer spin, so they consist of even number of quarks, first of all of quark and antiquark. Thus baryons and antibaryons look like

$$B_{abc} \to q_a\, q_b\, q_c, \quad \bar{B}^{abc} \to \bar{q}^{\,a}\, \bar{q}^{\,b}\, \bar{q}^{\,c}. \tag{1.10}$$

In the same way mesons are represented in the following form

$$M_a^b \to q_a\, \bar{q}^{\,b}. \tag{1.11}$$

After introduction of conception of quarks [3] there was magnificent success in understanding of systematics of hadrons. For example the proton and the neutron are represented in terms of quarks of Table 1 as follows

$$P \to u\,u\,d, \quad N \to u\,d\,d. \tag{1.12}$$

The most light hadrons – π-mesons:

$$\pi^+ \to u\,\bar{d}, \quad \pi^- \to d\,\bar{u}, \quad \pi^+ \to \frac{u\,\bar{u} - d\bar{d}}{\sqrt{2}}. \tag{1.13}$$

In (1.12, 1.13) we see correspondence with electric charges in Table 1.1. We need also prescribe quarks some other quantum numbers. We know that baryons including the proton and the neutron possess conserving baryon quantum number B. Defining $B(P) = 1$ for the proton, we have $B(\bar{P}) = -1$ and thus according to (1.10) for each quark $B(q) = \frac{1}{3}$ and for each antiquark $B(\bar{q}) = -\frac{1}{3}$.

Quarks have also individual quantum numbers, which are conserved in the strong interaction For quarks of the second and of the third generations there are special designations:

$$\begin{aligned}
&c \to Charm(Ch), \quad s \to Strangeness(S), \\
&t \to Topness(Tp), \quad b \to Beauty(Be), \\
&Ch(c) = 1, \quad Ch(\bar{c}) = -1, \quad S(s) = -1, \quad S(\bar{s}) = 1, \\
&Tp(t) = 1, \quad Tp(\bar{t}) = -1, \quad Bt(b) = -1, \quad Bt(\bar{b}) = 1.
\end{aligned} \tag{1.14}$$

What concerns the first pair of quarks u, d it comes out that they are quite close in mass so the are considered as a doublet of isotopic spin I.

Let us present quantum numbers of quarks in Table 1.2 and those of leptons in Table 1.3.

Here we use unit GeV for a mass of a quark. In view of convenience we use system of units with speed of light $c = 1$ and Planck constant $\hbar = 1$. This system is widely used in application of quantum field theory to the physics of elementary particles.

Let us draw attention to closeness of masses of the u and d quarks. So we have two states, which differ only by their electric charges. Thus in respect to strong interaction we may consider this two quarks as components of a doublet corresponding to isotopic symmetry. Thus this pair has isotopic spin $I = \frac{1}{2}$ and proton p corresponds to the third

Table 1.2. Quantum numbers of quarks

quark	B	I, I_3	S	Ch	Bt	Tp	J^P	Mass GeV
u	$\frac{1}{3}$	$\frac{1}{2}, \frac{1}{2}$	0	0	0	0	$\frac{1}{2}^+$	0.0025 ± 0.0007
d	$\frac{1}{3}$	$\frac{1}{2}, -\frac{1}{2}$	0	0	0	0	$\frac{1}{2}^+$	0.005 ± 0.001
s	$\frac{1}{3}$	$0, 0$	-1	0	0	0	$\frac{1}{2}^+$	$0.101^{+0.029}_{-0.021}$
c	$\frac{1}{3}$	$0, 0$	0	1	0	0	$\frac{1}{2}^+$	$1.27^{+0.07}_{-0.11}$
t	$\frac{1}{3}$	$0, 0$	0	0	1	0	$\frac{1}{2}^+$	173.18 ± 0.94
b	$\frac{1}{3}$	$0, 0$	0	0	0	-1	$\frac{1}{2}^+$	$4.20^{+0.17}_{-0.07}$
$\bar{u}$	$-\frac{1}{3}$	$\frac{1}{2}, -\frac{1}{2}$	0	0	0	0	$\frac{1}{2}^-$	0.0025 ± 0.0007
$\bar{d}$	$-\frac{1}{3}$	$\frac{1}{2}, \frac{1}{2}$	0	0	0	0	$\frac{1}{2}^-$	0.005 ± 0.001
$\bar{s}$	$-\frac{1}{3}$	$0, 0$	1	0	0	0	$\frac{1}{2}^-$	$0.101^{+0.029}_{-0.021}$
$\bar{c}$	$-\frac{1}{3}$	$0, 0$	0	-1	0	0	$\frac{1}{2}^-$	$1.27^{+0.07}_{-0.11}$
$\bar{t}$	$-\frac{1}{3}$	$0, 0$	0	0	-1	0	$\frac{1}{2}^-$	173.18 ± 0.94
$\bar{b}$	$-\frac{1}{3}$	$0, 0$	0	0	0	1	$\frac{1}{2}^-$	$4.20^{+0.17}_{-0.07}$

Table 1.3. Quantum numbers of leptons

lepton	B	$\frac{Q}{e}$	J	Mass MeV
ν_e	0	0	$\frac{1}{2}$	$< 2 \cdot 10^{-6}$
ν_μ	0	0	$\frac{1}{2}$	< 0.19
ν_τ	0	0	$\frac{1}{2}$	< 18.2
e	0	-1	$\frac{1}{2}$	0.510998928(11)
μ	0	-1	$\frac{1}{2}$	105.6583715(35)
τ	0	-1	$\frac{1}{2}$	1776.82 ± 0.16

projection $I_3 = \frac{1}{2}$ and neutron n corresponds to $I_3 = -\frac{1}{2}$. Hadrons which are consisting of quarks according to (1.10), (1.11) also has isotopic spins.

In view of illustration for isotopic spin we present in Table 1.4 the most important examples of hadrons [4] with their quantum numbers

Table 1.4. Light hadrons

Hadr.	$\frac{Q}{e}$	I, I_3	Mass GeV	Mean life s Width MeV	J^{PC}	Composit.
p	1	$\frac{1}{2}, \frac{1}{2}$	0.938272	∞	$\frac{1}{2}^+$	uud
n	0	$\frac{1}{2}, -\frac{1}{2}$	0.939565	885.7(8)	$\frac{1}{2}^+$	udd
Δ^{++}	2	$\frac{3}{2}, \frac{3}{2}$	1.2319	$6.04(3)10^{-24}$	$\frac{3}{2}^+$	uuu
Δ^{+}	1	$\frac{3}{2}, \frac{1}{2}$	1.2316	$5.92(8)10^{-24}$	$\frac{3}{2}^+$	uud
Δ^{0}	0	$\frac{3}{2}, -\frac{1}{2}$	1.2331	$5.60(6)10^{-24}$	$\frac{3}{2}^+$	udd
Δ^{-}	-1	$\frac{3}{2}, -\frac{3}{2}$	1.232	$5.72(6)10^{-24}$	$\frac{3}{2}^+$	ddd
π^-	1	1,-1	0.139570	$2.6033(5)\ 10^{-8}$	0^-	$\bar{u}d$
π^0	0	1, 0	0.134976	$8.4(5)10^{-17}$	0^{-+}	$\frac{\bar{u}u-\bar{d}d}{\sqrt{2}}$
π^+	-1	1, 1	0.139570	$2.6033(5)\ 10^{-8}$	0^-	$\bar{d}u$
σ	0	0, 0	0.40-0.55	$(4\text{-}7)10^{-24}$ 400–700	0^{++}	$\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}$
η	0	0, 0	0.547853	$0.507(0.027)10^{-18}$	0^{-+}	$\cos\theta_\eta \frac{\bar{u}u+\bar{d}d-2\bar{s}s}{\sqrt{6}} + \sin\theta_\eta \frac{\bar{u}u+\bar{d}d+\bar{s}s}{\sqrt{3}}$
$\rho^\pm$	1	1,-1	0.7751(3)	$4.38(3)10^{-24}$ 147.8(9)	1^-	$\bar{u}d$
ρ^0	0	1, 0	0.7753(3)	$4.37(3)10^{-24}$ 149.1(8)	1^{--}	$\frac{\bar{u}u-\bar{d}d}{\sqrt{2}}$
ω	0	0, 0	0.7827(1)	$7.75(7)10^{-23}$ 8.49(8)	1^{--}	$\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}$
ϕ	0	0, 0	1.0195(2)	$1.545(3)10^{-22}$	1^{--}	$\bar{s}s$

The first six rows represent baryons and the rest represent mesons. There are evident isotopic multiplets with approximately equal masses. Inside the multiplets electric charges of particles depend on the third component I_3 of an isotopic spin. This dependence is the following

$$\frac{Q}{\mathrm{e}} = \frac{B}{2} + I_3, \tag{1.15}$$

where B is baryon number, which is 1 for baryons and 0 for mesons in Table 1.4. On the other hand we may obtain electric charge of a state with the aid of Table 1.1. For example, we have

$$\begin{aligned} u\,u\,u &\to Q = 2\,\mathrm{e}, & u\,u\,s &\to Q = e, \\ u\,c\,s &\to Q = \mathrm{e}, & u\,d\,s &\to Q = 0, \\ u\,\bar{s} &\to Q = \mathrm{e}, & d\,\bar{s} &\to Q = 0, \\ s\,\bar{u} &\to Q = -\mathrm{e}, & s\,\bar{d} &\to Q = 0. \end{aligned} \tag{1.16}$$

Here the last two rows represent K-mesons and anti-K-mesons.

We would draw attention to the mean life column. Here there is only one stable particle p, while all other ones decay. States with very short decay time (Δ, ρ, ...) are usually called resonances and instead of mean life width Γ of a resonance is presented. The connection of these values is very simple

$$\frac{0.65822 \times 10^{-24}}{\Gamma(\mathrm{GeV})} = \tau(s). \tag{1.17}$$

Table 1.4 also illustrates the very important quality of quarks – the existence of a new quantum number, which acquires the name “color”. We see here states Δ^{++} $(u\,u\,u)$ and Δ^- $(d\,d\,d)$ which consist of identical quarks in symmetric spin state which corresponds to total spin $3/2$. However, the well-known Pauli principle prescribes states consisting of identical fermions to be antisymmetric. This contradiction at the time of formulation of the quark theory causes a lot of discussions. As a result of these discussions the conception of the new quantum number was formulated [5–9], which equips each quark of the three in these states with distinct quantum number, which differs it from the two other ones. Thus we have three states of each quark and anti-quark:

$$q_a^\alpha, \quad \bar{q}_\beta^b, \quad \alpha, \beta = (1, 2, 3), \tag{1.18}$$

where a and b denote different sorts of quarks (see Table 1.2). Then a baryon is now composed of quarks with different colors, that is,

$$B_{abc} = \frac{1}{3!}\,\epsilon_{\alpha\beta\delta}\,q_a^\alpha\,q_b^\beta\,q_c^\delta. \tag{1.19}$$

Introduction of color became inevitable, when after formulation of the constituent quark model of hadrons the necessity of a study of symmetry problems of hadron states was appreciated. In the course of the study of the problem the particular problem of magnetic moments of nucleons p, n have proved to be extremely important. At that time it was not at all understood, why magnetic moments of the proton and of the neutron do not obey the Dirac result for a spin $\frac{1}{2}$ particle

$$\mu = \frac{e}{2M}, \tag{1.20}$$

where e is the electric charge of the particle and M is its mass and the direction of the magnetic moment coincides with that of the spin. There are quite another values for nucleons [4]

$$\mu_P = \frac{e}{2\,M_P}(2.792847337 \pm 0.000000029), \tag{1.21}$$

$$\mu_N = \frac{e}{2\,M_P}(-\,1.91304272 \pm 0.0000004529). \tag{1.22}$$

According to (1.20) one would expect inside the brackets of the formula $\simeq$ 1 for (1.21) and $\simeq$ 0 for (1.22). This consideration, for example, allows to L. D. Landau to assert, that nucleons are not Dirac particles.

However, after introduction of the quark conception, law (1.20) has to be applied not to nucleons, but to their constituents, namely to quarks. Then one has to show, how this law really works. So let us follow the reasoning here arising.

According to Table 1.4 let us write the quark composition of the proton with spin direction up in the following form

$$P_\uparrow = \frac{1}{\sqrt{2}}(u_\uparrow d_\downarrow - d_\uparrow u_\downarrow)u_\uparrow. \tag{1.23}$$

The composition (1.23) guarantees spin $\frac{1}{2}$ of the system and isotopic spin also $\frac{1}{2}$. However term (1.23) is not all the story, because one may commute quark constituents. There are six independent commutations, which result in the following representations in case of symmetric and antisymmetric combination for the complete wave function of the proton $\Psi_{P_\uparrow}$

$$\begin{aligned}\Psi^{sym}_{P_\uparrow} &= \frac{1}{\sqrt{18}}(2(u_\uparrow u_\uparrow d_\downarrow + u_\uparrow d_\downarrow u_\uparrow + d_\downarrow u_\uparrow u_\uparrow) \\ &\quad - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow - u_\uparrow d_\uparrow u_\downarrow - d_\uparrow u_\uparrow u_\downarrow - u_\downarrow d_\uparrow u_\uparrow - d_\uparrow u_\downarrow u_\uparrow). \\ \Psi^{asym}_{P_\uparrow} &= \frac{1}{\sqrt{6}}(u_\downarrow u_\uparrow d_\uparrow - u_\uparrow u_\downarrow d_\uparrow + u_\uparrow d_\uparrow u_\downarrow - d_\uparrow u_\uparrow u_\downarrow - u_\downarrow d_\uparrow u_\uparrow + d_\uparrow u_\downarrow u_\uparrow).\end{aligned} \tag{1.24}$$

Then we apply Dirac relation (1.20) to quark states in relation (1.24), that is

$$u_\uparrow \to \frac{e}{2m}\frac{2}{3}, \quad u_\downarrow \to -\frac{e}{2m}\frac{2}{3}, \quad d_\uparrow \to -\frac{e}{2m}\frac{1}{3}, \quad d_\downarrow \to \frac{e}{2m}\frac{1}{3}, \tag{1.25}$$

where m is average mass of u and d quarks and a triple combination in (1.24) means sum of corresponding terms in (1.25). According to these rules we obtain for the magnetic moment of the proton

$$\langle\Psi^{sym}_{P_\uparrow}|\mu|\Psi^{sym}_{P_\uparrow}\rangle = \frac{e}{2\,m}\,\frac{1}{18}\left(12\left(\frac{4}{3}+\frac{1}{3}\right)+6\left(-\frac{1}{3}\right)\right) = \frac{e}{2\,m}, \tag{1.26}$$

$$\langle\Psi^{asym}_{P_\uparrow}|\mu|\Psi^{asym}_{P_\uparrow}\rangle = \frac{e}{2\,m}\,\frac{1}{6}\left(6\left(-\frac{1}{3}\right)\right) = -\frac{e}{2\,m}\,\frac{1}{3}. \tag{1.27}$$

For the neutron we have with change $u \leftrightarrow d$

$$\begin{aligned}\Psi^{sym}_{N_\uparrow} &= \frac{1}{\sqrt{18}}(2(d_\uparrow d_\uparrow u_\downarrow + d_\uparrow u_\downarrow d_\uparrow + u_\downarrow d_\uparrow d_\uparrow) \\ &\qquad - d_\uparrow d_\downarrow u_\uparrow - d_\downarrow d_\uparrow u_\uparrow - d_\uparrow u_\uparrow d_\downarrow - u_\uparrow d_\uparrow d_\downarrow - d_\downarrow u_\uparrow d_\uparrow - u_\uparrow d_\downarrow d_\uparrow), \\ \Psi^{asym}_{N_\uparrow} &= \frac{1}{\sqrt{6}}(d_\downarrow d_\uparrow u_\uparrow - d_\uparrow d_\downarrow u_\uparrow + d_\uparrow u_\uparrow d_\downarrow - u_\uparrow d_\uparrow d_\downarrow - d_\downarrow u_\uparrow d_\uparrow + u_\uparrow d_\downarrow d_\uparrow).\end{aligned} \tag{1.28}$$

Again using rules (1.25) we obtain

$$\langle \Psi^{sym}_{N_\uparrow} | \mu | \Psi^{sym}_{N_\uparrow} \rangle = \frac{e}{2m}\frac{1}{18}\left(12\left(-\frac{2}{3} - \frac{2}{3}\right) + 6\left(\frac{2}{3}\right)\right) = -\frac{e}{2m}\frac{2}{3}, \tag{1.29}$$

$$\langle \Psi^{asym}_{N_\uparrow} | \mu | \Psi^{asym}_{N_\uparrow} \rangle = \frac{e}{2\,m}\frac{1}{6}\left(6\left(\frac{2}{3}\right)\right) = \frac{e}{2\,m}\frac{2}{3}. \tag{1.30}$$

Thus we obtain following results for magnetic moments in the symmetric case

$$\mu^{sym}_P = \frac{e}{2\,m}, \quad \mu^{sym}_N = -\frac{e}{2\,m}\left(\frac{2}{3}\right). \tag{1.31}$$

For the antisymmetric case there are quite different results

$$\mu^{asym}_P = -\frac{e}{2\,m}\left(\frac{1}{3}\right), \quad \mu^{asym}_N = \frac{e}{2\,m}\left(\frac{2}{3}\right). \tag{1.32}$$

For ratios of the nucleon magnetic moments we have

$$\left(\frac{\mu_P}{\mu_N}\right)_{sym} = -\frac{3}{2}, \quad \left(\frac{\mu_P}{\mu_N}\right)_{asym} = -\frac{1}{2}. \tag{1.33}$$

Only the first option is close to the real ratio, following from experimental values (1.21, 1.22)

$$\left(\frac{\mu_P}{\mu_N}\right)_{exp} = -1.46. \tag{1.34}$$

However in the initial formulation of the quark constituent model it was necessary to choose antisymmetric combination due to the Pauli principle. The only way to reconcile quark model with results (1.21, 1.22) was just to introduce the new quantum number color and to construct baryons according to rule (1.19). Now the fulfillment of the Pauli principle is guaranteed by antisymmetric factor $\epsilon^{\alpha\beta\gamma}$. This conclusion was first made in work [5].

Relations (1.21, 1.22, 1.32) make it possible to estimate parameter m. We have from values for the proton and the neutron respectively

$$m = 336\,\text{MeV}, \quad m = 327\,\text{MeV},$$

that leads to the following average value

$$m \simeq 331.5\ \text{MeV}. \tag{1.35}$$

This value is quite different from that presented in Table 1.2. We return to this problem below and here only would mention, that the mass, which enters into expressions (1.32) is usually called "the constituent quark mass" and that in Table 1.2 is called "the current quark mass".

The physics of elementary particles is described in terms of quantum field theory (QFT). In the present book we suppose, that a reader is acquainted with the main points of QFT, e. g, with courses [1, 2]. The current situation in this science is described by the so-called Standard Model, consisting of Quantum Chromodynamics (QCD) and the Electro-Weak Theory (EWT). We would briefly describe these parts. The first theory, which was elaborated was the Quantum Electrodynamics (QED). It describes interaction of spinor charged particles bearing the electric charge e with photons.

1.3 Quantum electrodynamics

The interaction of charged spinor fields, e. g., those presented in Table 1.1 with photons is described by the following Lagrangian

$$
\begin{aligned}
L = \sum_{k=1}^{N} \Big(\frac{\imath}{2} \left(\bar\psi_k \gamma_\mu \partial_\mu \psi_k - \partial_\mu \bar\psi_k \gamma_\mu \psi_k \right) - m_k \bar\psi_k \psi_k + \mathrm{e}\, Q_k \bar\psi_k \gamma_\mu A_\mu \psi_k \Big) \\
- \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu,
\end{aligned}
\tag{1.36}
$$

where $Q_k = -1$ for leptons e, μ, τ, $Q_k = \frac{2}{3}$ for up quarks u, c, t and $Q_k = -\frac{1}{3}$ for down quarks d, s, b. The lagrangian is invariant in respect to gauge transformations

$$
\begin{aligned}
&\psi \to U\,\psi, \quad \bar\psi \to \bar\psi\, U^{-1}, \\
&A_\mu \to U\, A_\mu\, U^{-1} + \frac{\imath}{\mathrm{e}} U\, \partial_\mu U^{-1} = A_\mu + \frac{\imath}{\mathrm{e}}\, \partial_\mu \phi(x). \\
&U = \exp\left(\imath\, \mathrm{e}\phi(x)\right), \quad U^{-1} = \exp\left(-\imath\, \mathrm{e}\phi(x)\right).
\end{aligned}
\tag{1.37}
$$

The quantity

$$
\alpha = \frac{\mathrm{e}^2}{4\,\pi}
\tag{1.38}
$$

is usually called the Sommerfeld fine structure constant. At low energies value of α is defined by the electron charge and is equal to

$$
\alpha = 0.007297352533 \pm 0.000000000027 \simeq \frac{1}{137.036}.
\tag{1.39}
$$

The charged fermion (electron) propagator with momentum p

$$
\frac{\imath\,(\hat p + m)}{(2\pi)^4 (p^2 - m^2 + \imath\,\epsilon)}.
\tag{1.40}
$$

The photon propagator with p and μ, ν being respectively its momentum and initial and final indices

$$\frac{-\imath g_{\mu\nu}}{(2\pi)^4(p^2+\imath\epsilon)}. \tag{1.41}$$

The charged fermion–photon vertex (index μ), Q being its charge in units $\boldsymbol{e}$, we have already mentioned, that for the electron $Q=-1$

$$\imath\,(2\pi)^4\,Q\,\mathrm{e}\,\gamma_\mu. \tag{1.42}$$

For the structure of the theory the momentum dependence of the effective charge (1.38) $\alpha(q^2)$ is very important. In QED this dependence was obtained in the early years of development of the quantum field theory by L. D. Landau and his collaborators [10, 11]

$$\alpha(q^2)=\frac{3\,\pi\,\alpha(\mu)}{3\,\pi-N\,\alpha(\mu)\,\ln(\frac{-q^2}{\mu^2})}, \tag{1.43}$$

where N is the number of elementary unit charge fermions. As a matter of fact we now know also fractionally charged fermions – six color quarks, so that $N=3+3\times 3((2/3)^2+(1/3)^2)=8$. One also has to take into account elementary charged $W^{\pm}$-bosons, which gives for $q^2\gg M_W^2$ negative contribution to this number N_{eff} : $N_{eff}=8-11/2=5/2$. In any case we have expression (1.43) with $N=N_{eff}$. For space-like $q^2\to-\infty$ we encounter the pole in expression (1.43). This pole is usually referred to as Landau pole. For $\mu\simeq M_z$ $\alpha(M_Z)=\frac{1}{129}$.

The existence of the pole makes a theory internally contradictory. As for QED, L. D. Landau himself in the issue dedicated to Niels Bohr [11] had first stated, that for a realistic number of the charged elementary fields $N\leq 20$ the pole was situated far beyond the Planck mass

$$M_P=\sqrt{\frac{1}{\kappa^2}}=(1.220892\pm 0.000061)10^{19}\,\mathrm{GeV}, \tag{1.44}$$

where κ^2 is the gravitational constant. So the pole presumably could be removed by quantum gravitation effects. As we have just remarked, nowadays $N_{eff}=5/2$ and thus this argument is valid. However we shall see, that for QCD the solution of the problem needs application of special efforts.

Quantum electrodynamics proves to be very precise theory. Results of calculations in QED agrees experimental results up to high orders of the perturbation theory. For example, the experimental value for anomalous magnetic moment of the electron

$$a_e=0.001159652187\pm 0.000000000004, \tag{1.45}$$

with total magnetic moment

$$\mu_e=(1+a_e)\frac{\mathrm{e}}{2\,m_e}.$$

The results agrees QED calculations within error bars.

Let us remind, that just calculation of a_e by J. Schwinger [13] was very important argument on behalf of quantum electrodynamics having been genuine theory of the electromagnetic interaction. In view of the forthcoming calculation in the book of the similar quantity for the muon with account of nonperturbative contributions we would present briefly the procedure of this calculation.

The Lorentz structure of the magnetic moment term is the following

$$\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu}, \quad \sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2\imath}, \tag{1.46}$$

that means additional term in the electron-photon vertex

$$\gamma_\rho + \imath \frac{\mathrm{e}\, a_e}{2\, m_e} \sigma_{\rho\lambda}\, k_\lambda. \tag{1.47}$$

Let us perform the calculation. The first approximation to the additional term to the well-known Dirac magnetic moment (1.20) corresponds to the triangle diagram presented in Figure 1.1 Now Figure 1.1 defines the following contribution to the vertex (in the Feynman gauge)

$$\frac{\mathrm{e}^3}{(2\,\pi)^4}\int \frac{\gamma_\mu(\hat{q}+\hat{p}_1+m)\gamma_\rho(\hat{q}+\hat{p}_2+m)\gamma_\mu\, dq}{q^2((q+p_1)^2-m^2)((q+p_2)^2-m^2)} \tag{1.48}$$
$$= \frac{\mathrm{e}^3}{(2\,\pi)^4}\int \frac{-2(\hat{q}+\hat{p}_2)\gamma_\rho(\hat{q}+\hat{p}_1)+4m(2\,q_\rho+p_{1\rho}+p_{2\rho})-2\,m^2\,\gamma_\rho}{q^2(q^2+2qp_1)(q^2+2qp_2)}dq,$$
$$p_1^2 = p_2^2 = m^2, \quad \bar{\psi}\hat{p}_1 = \bar{\psi}m \quad \hat{p}_2\psi = m\psi, \quad p_2 - p_1 = k.$$

It is easy to see, that the last term in the nominator contributes only charge structure γ_ρ, the last but one term does not contain gamma matrices at all, so only the first

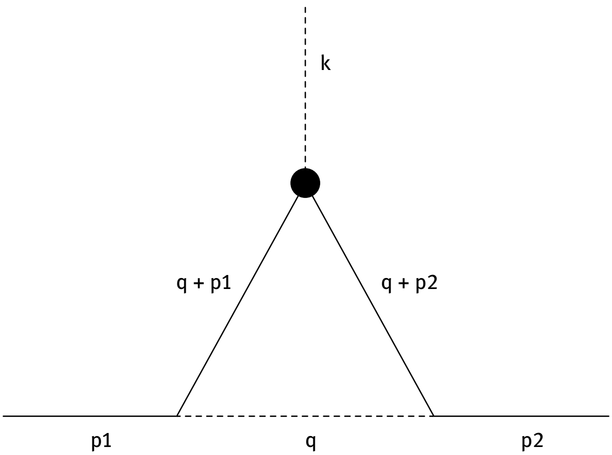


Fig. 1.1. Diagram corresponding to calculation of the anomalous magnetic moment of the electron. Dotted lines correspond to photons.

term may contribute additional structure (1.47). We perform calculations using the α-representation [1].

Namely, we represent denominators in (1.48) in the following form

$$\frac{1}{q^2} = \frac{1}{\imath}\int e^{\imath\alpha(q^2+\imath\epsilon)}\,d\alpha, \qquad \frac{1}{q^2+2qp_1} = \frac{1}{\imath}\int e^{\imath\beta(q^2+2qp_1+\imath\epsilon)}\,d\beta,$$
$$\frac{1}{q^2+2qp_2} = \frac{1}{\imath}\int e^{\imath\gamma(q^2+2qp_2+\imath\epsilon)}\,d\gamma. \tag{1.49}$$

Let us perform the following substitution of variables

$$\beta = \lambda\xi, \quad \gamma = \lambda\eta, \quad \alpha = \lambda(1-\xi-\eta). \tag{1.50}$$

Then the term under study takes the form

$$-\frac{2\,\imath\,\mathrm{e}^3}{(2\,\pi)^4}\int_0^\infty \lambda^2 d\lambda\int_0^1 d\xi\int_0^{1-\xi} d\eta\int dq(\hat{q}+\hat{p}_2)\gamma_\rho(\hat{q}+\hat{p}_1)e^{\imath\lambda(q^2+2\xi qp_1+2\eta qp_2+\imath\epsilon)}. \tag{1.51}$$

Now we use standard integrals [1]

$$\int dq\, e^{\imath(a\,q^2+2\,bq)} = \frac{\pi^2}{\imath a^2}\,e^{-\frac{\imath b^2}{a}},$$
$$\int dq\, q_\mu\, e^{\imath(a\,q^2+2\,bq)} = \frac{\imath\,b_\mu}{a}\,\frac{\pi^2}{a^2}\,e^{-\frac{\imath b^2}{a}}, \tag{1.52}$$
$$\int dq\, q_\mu\, q_\nu\, e^{\imath(a\,q^2+2\,bq)} = \frac{a\,g_{\mu\nu}-2\,\imath\,b_\mu\,b_\nu}{2a^2}\,\frac{\pi^2}{a^2}\,e^{-\frac{\imath b^2}{a}},$$

to obtain for expression (1.51) terms giving contribution to the magnetic moment structure in expression (1.47)

$$-\frac{\mathrm{e}^3}{8\pi^2}\,\frac{2\hat{p}_1\gamma_\rho\hat{p}_1+\hat{p}_1\gamma_\rho\hat{p}_2+\hat{p}_2\gamma_\rho\hat{p}_1+2\hat{p}_2\gamma_\rho\hat{p}_2}{12\,m^2} + O(k^2). \tag{1.53}$$

Note, that the remaining terms in the nominator of the last expression in (1.48) give contribution only to Lorentz structure γ_ρ and thus have to be included in renormalization of the vertex. They contain both ultraviolet and infrared divergences. Now, using relations presented in the last line of (1.48), we obtain from (1.53) the following expression

$$\frac{\mathrm{e}^3}{32\,\pi^2\,m}(\gamma_\rho\,\hat{k}-\hat{k}\,\gamma_\rho) \tag{1.54}$$

that with account of relations (1.46, 1.47) leads to the final result

$$a_e = \frac{\alpha}{2\,\pi} = 0.0011641\ldots, \tag{1.55}$$

where we have used value (1.39). Comparing the result with the experimental number (1.45) we see agreement up to terms of order of magnitude $(\alpha/\pi)^2$. The corresponding calculation of these terms and also of higher terms gives the final excellent agreement.

Result (1.55) in line with the Lamb shift calculations [14], as we have already mentioned, have proved QED to be the correct theory. Thus QED became the first and the most elaborated renormalizable quantum field theory. The subsequent development of the elementary particles physics leads to formulation of the theory of the strong interaction and the theory of the electro-weak interaction. Both was constructed in close analogy to QED. In the next sections we briefly describe these theories.

1.4 Quantum chromodynamics

The quantum chromodynamics (QCD) exploits color symmetry group $SU(3)_c$ for formulation of strong interaction of quarks. The corresponding Yang–Mills fields [15, 16] are named gluons. The main properties of the theory were first disclosed in [17–20].

QCD describes fundamental interaction of quarks and gluons. There are six sorts of quarks each bearing color

$$u,\ c,\ t, \tag{1.56}$$

with electric charge $Q = \frac{2e}{3}$, where e is the elementary electric charge

$$d,\ s,\ b, \tag{1.57}$$

with electric charge $Q = -\frac{2e}{3}$.

We start with QCD Lagrangian with N_f quarks with number of colors $N = 3$

$$\begin{aligned} L &= \sum_{k=1}^{N_f}\left[\frac{\imath}{2}\left(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\right) - m_k\bar\psi_k\psi_k + g\,\bar\psi_k\gamma_\mu t_a A^a_\mu\psi_k\right] - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu}\right), \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,f_{abc}A^b_\mu A^c_\nu. \end{aligned} \tag{1.58}$$

where we use the standard QCD notations including $F^a_{\mu\nu}$ for a Yang–Mills field [15]. Here f_{abc} are structure constants of the $SU(3)$ group and t_a, $(a = 1,\dots,8)$ are the well-known $SU(3)$ matrices 3×3 of the infinitesimal transformations with the following properties

$$\begin{gathered} t_a\,t_b - t_b\,t_a = \imath f_{abc}\,t_c, \quad Trace\,(t_a) = 0, \\ t_a\,t_b + t_b\,t_a = \frac{1}{N}\,\delta_{ab} + d_{abc}\,t_c, \\ Trace\,(t_a\,t_b) = \frac{1}{2}\,\delta_{ab}, \quad t_a\,t_a = \frac{N^2-1}{2\,N}\,I, \\ f_{bcm}\,f_{anm} + f_{acm}\,f_{nbm} + f_{abm}\,f_{cnm} = 0, \\ f_{lam}\,f_{mbn}\,f_{ncl} = -\frac{N}{2}\,f_{abc}, \quad f_{amn}\,f_{bmn} = N\,\delta_{ab}, \\ f_{lam}\,d_{mbn}\,d_{ncl} = \frac{N^2-4}{2\,N}\,f_{abc}, \quad f_{amn}\,f_{bmn} = N\,\delta_{ab}, \end{gathered} \tag{1.59}$$

where I is the unit matrix 3×3 and as usually we sum up on recurring indices. We also introduce number N, which marks dimension of a $SU(N)$ group in view of using the formulas both for $SU(2)$ and $SU(3)$. In QCD we have $N = 3$.

In what follows we in some cases shall consider m_k to be small enough and set them to zero.

Lagrangian (1.58) is invariant in respect to gauge transformations, which are described also by (1.37), where we have

$$U = \exp(\imath g\, t_a\, \theta^a(x)), \quad U^{-1} = \exp(-\imath g\, t_a\, \theta^a(x)), \quad \mathrm{e} \to g, \tag{1.60}$$

where $\theta^a(x)$ are eight parameters of the $SU(3)$ transformation, which depend on the point of the space.

It is also useful to rewrite expression (1.58) in terms of matrix quantities:

$$\mathbf{A}_\mu = A^a_\mu\, t_a, \quad \mathbf{F}_{\mu\nu} = F^a_{\mu\nu}\, t_a. \tag{1.61}$$

With this notations we have instead of (1.58)

$$L = \sum_{k=1}^{N_f} \left[\frac{\imath}{2} \left(\bar{\psi}_k \gamma_\mu \partial_\mu \psi_k - \partial_\mu \bar{\psi}_k \gamma_\mu \psi_k \right) - m_k \bar{\psi}_k \psi_k + g\, \bar{\psi}_k \gamma_\mu \mathbf{A}_\mu \psi_k \right] - \frac{1}{2}\, Trace(\mathbf{F}_{\mu\nu}\, F_{\mu\nu}),$$
$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - \imath g\, [\mathbf{A}_\mu, A_\nu]. \tag{1.62}$$

Let us present Feynman rules for QCD:

(1) The quark propagator is just usual fermion one (1.40)

(2) The gluon propagator differs from the photon one only by color Kronecker symbol

$$\frac{-\imath\, g_{\mu\nu}\, \delta^{a\,b}}{(2\pi)^4 (p^2 + \imath\,\epsilon)}. \tag{1.63}$$

(3) Quark-gluon vertex, μ and a being respectively Lorentz and color indices of the gluon (see also (1.42)

$$\imath\,(2\pi)^4\, g\, \gamma_\mu\, t^a. \tag{1.64}$$

(4) The three-gluon vertex with Lorentz indices, color indices and momenta of gluons being respectively (μ, a, k), (ν, b, p), (ρ, c, q)

$$(2\pi)^4\, g\, f_{abc}\, (g_{\mu\nu}(k - p)_\rho + g_{\nu\rho}(p - q)_\mu + g_{\rho\mu}(q - k)_\nu). \tag{1.65}$$

The rule of construction of the vertex is evident: the metrical tensor indices corresponds to momenta in brackets, which have the remaining Lorentz index.

(5) The four-gluon vertex with color and Lorentz indices (a, μ), (b, ν), (c, ρ), (d, σ)

$$-\imath\,(2\pi)^4\, g^2 (f_{abf}\, f_{cdf}(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}) + f_{acf}\, f_{bdf}(g_{\mu\nu}\, g_{\rho\sigma} - g_{\mu\sigma}\, g_{\nu\rho})$$
$$+ f_{adf}\, f_{bcf}(g_{\mu\nu}\, g_{\rho\sigma} - g_{\mu\rho}\, g_{\nu\sigma})). \tag{1.66}$$

There are also auxiliary color fields (ghosts), which are scalars, but give extra minus sign for closed loop as well as fermion fields. The propagator

$$\frac{\imath\,\delta^{a\,b}}{(2\pi)^4(p^2+\imath\,\epsilon)}, \tag{1.67}$$

and the ghost-gluon vertex with μ, b being the gluon Lorentz and color indices respectively and p, a being the momentum of the outgoing ghost, having color index a

$$-(2\pi)^4\, g\, f_{abc}\, p_\mu. \tag{1.68}$$

Running coupling in the three-loop approximation is the following

$$\begin{aligned}\alpha_s(\mu) = \frac{4\pi}{\beta_0\ln(\mu^2/\Lambda^2)}\Bigg[1 &- \frac{2\beta_1\,\ln(\ln(\mu^2/\Lambda^2))}{\beta_0^2\,\ln(\mu^2/\Lambda^2)}\\ &+\frac{4\beta_1^2}{\beta_0^4\,\ln^2(\mu^2/\Lambda^2)}\left(\left(\ln(\ln(\mu^2/\Lambda^2))-\frac{1}{2}\right)^2+\frac{\beta_2\beta_0}{8\beta_1^2}-\frac{5}{4}\right)\Bigg],\end{aligned} \tag{1.69}$$

where Λ is the famous QCD scale parameter and

$$\begin{aligned}&\beta_0 = 11-\frac{2\,N_f}{3},\quad \beta_1 = 51-\frac{19\,N_f}{3},\\ &\beta_2 = 2857-\frac{5033\,N_f}{9}+\frac{325\,N_f^2}{27},\end{aligned} \tag{1.70}$$

N_f is quark flavors number, which are involved for μ range in the problem under consideration. For example, we may take value of α_s at point $\mu = M_Z$ of the Z-boson mass. Here we have to take $N_f = 5$ (all quarks but the t-quark) and use experimental value [4]

$$\alpha_s(M_Z) = 0.1191 \pm 0.0028. \tag{1.71}$$

Using expressions (1.69, 1.70) we obtain for Λ in the region of five flavors

$$\Lambda_5 = 221^{+36}_{-32}\,\text{MeV}. \tag{1.72}$$

Coupling (1.69) decreases with $\mu^2 \to \infty$ that gives the well-known property of the asymptotic freedom. The extensive studies of QCD effects in the region of the asymptotic freedom, that is for high energies, show excellent agreement of perturbative calculations with experimental data. Thus perturbative QCD, as well as QED, proves to be an adequate theory in the region of its applicability. However, running coupling (1.69) evidently has a singularity at $\mu^2 = \Lambda^2$, which is quite analogous to the Landau pole in QED. A hope is usually expressed, that in the region of strong interaction there are some nonperturbative contributions, which somehow eliminate this singularity. But there was no explicit mechanism being formulated for such phenomenon. We shall return to this important problem below in the proper place.

In any case nonperturbative contributions in QCD are very important for low-momenta phenomena. In low-momenta region coupling (1.69) becomes large and thus perturbation theory calculations fails. What can we do with this problem? For momenta below some scale Λ_0 one can try to to write down a simplified effective Lagrangian, where contribution of heavier states are eliminated (one often says “integrated out”). The typical example of an effective interaction is provided by the Nambu–Jona-Lasinio model, which we shall intensively exploit below.

Nowadays the simulations of the theory on the lattice is considered as the most promising tool for nonperturbative problems in QCD. The technique was started by K. Wilson [21] and it shows increasing progress during the last decades. In this technique QCD is reformulated on a discrete space-time, an cubic lattice of sites with spacing a and 4-volume L^4. Gluon and quark fields are specified on the lattice sites and the path integral is computed numerically as a sum over field configurations. We are interested in limit $a \to 0$, so it is desirable to work with as small spacing a as possible. Such calculations become increasingly predictive. For example, they allow to construct behavior of running coupling $\alpha_s(Q)$ in low momenta region. In this direction there are several results, which sometimes do not agree one with each other. In any case, it is advisable to consider also other methods to consider nonperturbative effects.

The lattice method is also considered as a possible tool to ground the phenomenon of color confinement. Confinement is the basic property of the strong interaction and it consists in the assertion that only colorless states can be observable. No free quarks, no free gluons, no other color states. For example, consideration of effective potential between a quark and an antiquark with lattice techniques gives [22]

$$V_{q\bar{q}} \simeq \frac{4\,\alpha_s(r)}{3\,r} + \sigma\,r. \tag{1.73}$$

The term in effective potential (1.73), increasing at infinity, illustrates phenomenon of confinement. Indeed, a colored particle can not overcome the infinite barrier of potential (1.73). There is also temperature dependence of slope σ, such that after critical value T_c it becomes zero and the deconfinement phenomenon is expected. The estimation of value T_c gives $T_c \simeq 175$ MeV.

1.5 Bethe–Salpeter equation

Throughout the book we shall use mostly two types of equations. The first one will be a compensation equation, which provides essence of the method. This equation will be introduced and elaborated below. We also will often use Bethe–Salpeter equations for bound states in quantum field theory [23]. In view of absence of a description of the Bethe–Salpeter equation in the most popular standard courses on the quantum field theory we present here short explication of the necessary knowledge on this equation.

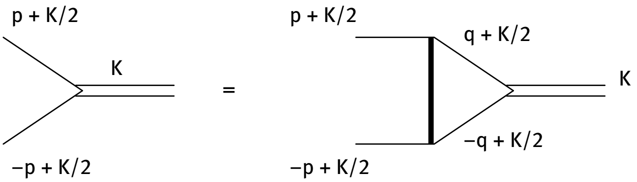


Fig. 1.2. Diagram representation of a Bethe–Salpeter equation a bound state. Momenta are written down by corresponding lines. Double line – the bound state X. Thick line corresponds to kernel $K(p,q,K)$.

The Bethe–Salpeter equation for two-particle bound state, which in diagram firm is shown in Figure 1.2, is an integral equation

$$\Phi(p,K) = \int dq\, K(p,q,K)\, \Delta\left(q + \frac{K}{2}\right)\, \Phi(q,K)\, \Delta\left(-q + \frac{K}{2}\right), \tag{1.74}$$

where for any momentum t, $\Delta(t)$ is a propagator of the corresponding particle, for example (1.40), and kernel $K(p,q,K)$ is to be constructed according to the following rules:

(1) The kernel is a sum of four-leg diagrams with the designated in Figure 1.2 distribution of momenta.
(2) All these diagrams are connected and two-particle irreducible, that is, they can not be reduced to unconnected diagram by cutting of two horizontal lines in the diagram. For example, among diagrams presented in Figure 1.4 there are three irreducible diagrams and two reducible ones.

Momentum K belongs to the bound state X and thus

$$K^2 = M_X^2. \tag{1.75}$$

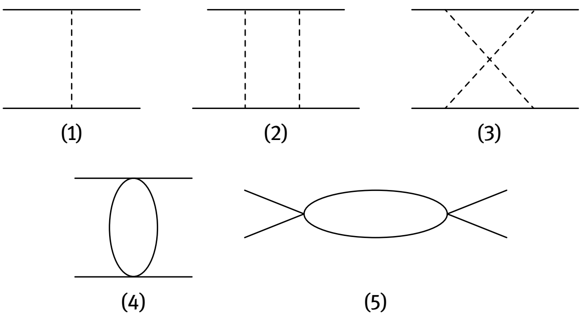


Fig. 1.3. Diagrams (1), (3), (4) are two-particle irreducible, but diagrams (2) and (5) are two-particle reducible.

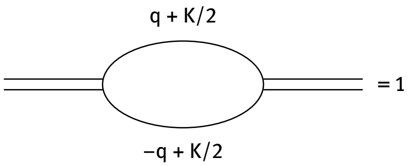


Fig. 1.4. Normalization condition for the Bethe–Salpeter wave function in diagram form. Triple vertices in the diagram correspond to wave function $\Phi(q, K)$.

In most case we shall make calculation after the Wick rotation in the momentum space $q \to q_E$

$$q_0 = \imath q_4, \quad q^2 \to -q_E^2. \tag{1.76}$$

Then in the Euclid space we have instead of (1.75)

$$K^2 = -M_X^2. \tag{1.77}$$

The normalization condition for Bethe–Salpeter wave function $\Phi(p, K)$ defines coupling constant in the effective triple vertex. For all cases to be considered below this condition reduces to calculation of simple loop diagram presented in Figure 1.4

The diagram in Figure 1.4 is to be calculated according to Feynman rules with $\Phi(q, K)$ in vertices and the result is to be developed in series by K^2. The coefficient at K^2 is to be equal to unity. Thus the condition have to be of the form

$$\frac{g_X^2}{(2\pi)^4} \int \Phi^2(q, K) N(q, K)\, dq, \tag{1.78}$$

where $N(q, K)$ is a function, which form depends on a problem under a study. In expression (1.78) wave function is normalized on the mass shell

$$\Phi(q, K) = 1, \quad q^2 = -m^2, \quad K^2 = -M_X^2,$$

that allows to consider g_X in expression (1.78) as an effective coupling constant.

1.6 Effective interactions

1.6.1 Preliminaries

We have briefly described in the previous sections gauge renormalizable theories, which comprise the contemporary theory of the main interactions of elementary particles. In the framework of the perturbation theory these theories have proved their efficiencies for description of phenomena in regions of their applicability. However these regions do not include all the variety of physical systems and conditions. For example, the behavior of running coupling (1.69) has a pole in the region of momenta around few hundreds of MeV. Just this region corresponds to formation of numerous particles and resonances consisting of quarks and gluons. Thus we encounter problem of the perturbation theory failing in this low-momenta region. Thus bearing in mind extensive evidence of the validity of QCD for large momenta region we are to admit, that in low-momenta region nonperturbative effects are inevitable. The existence of nonperturbative effects were the main and principal assumption of the method of sum rules in QCD [24], in which nonperturbative quantities e.g. vacuum averages of elementary fields were introduced. The gluon condensate, that is the following vacuum average of the normal product of gluon fields

$$V_2 = \left\langle : \frac{g^2}{4\,\pi^2}\, F^a_{\mu\nu}\, F^a_{\mu\nu} : \right\rangle . \tag{1.79}$$

It was shown (see [24] and numerous subsequent works) that this quantity decisively improves agreement of QCD sum rules with real physics provided this quantity is $V_2 = (0.012 \pm 0.002)\ \mathrm{GeV}^4$, whereas the perturbative value for this quantity is zero. The quark condensate

$$\langle \bar{q}q \rangle = \langle : \bar{\psi}\,\psi : \rangle \simeq -(0.23)^3\ \mathrm{GeV}^3 \tag{1.80}$$

was also shown to be present necessarily for adequate description of low-momenta region. These and similar quantities were used in the QCD sum rules method [24]. The purpose of the method is to find out how the high-momenta perturbative quantities and the low-momenta nonperturbative quantities could be considered simultaneously. In this method one consider a correlation quantity (correlator) comprising two or more of gauge invariant currents with quantum numbers of those states, which properties are under study:

$$\Pi(q^2) = \int dx\, e^{\imath qx} \langle 0|TJ(x), J(o)|0\rangle . \tag{1.81}$$

For $Q^2 = -q^2$ this quantity is described in two-fold way. On the one hand one uses Wilson operator decomposition[21], which application gives factorization of large and small virtual momenta. Contributions of large momenta are the subject for perturbative calculations, which in this region works quite efficiently due to the asymptotic freedom. Thus the coefficient functions are defined. On the other hand, contributions of the low-momenta region, which are nonperturbative are described by the so called condensates, which are represented as vacuum averages of local operators.

Thus such decomposition leads to the following representation of quantity (1.81)

$$\Pi(q^2) = C_1 + \sum_n C_n(q^2)\langle 0|O_n|0\rangle . \tag{1.82}$$

Here coefficients C_i are to be calculated in the framework of the perturbation theory (see [24–27]). Vacuum averages $\langle 0|O_n|0\rangle$ are purely nonperturbative. In the approach [24] they are extracted from sum rules and thus are defined from a phenomenology. For example phenomenological considerations lead to the following estimates of vacuum averages

$$V_2 = \left\langle \frac{\alpha_s}{\pi} F^a_{\mu\nu} F^a_{\mu\nu} \right\rangle \simeq 0.012\,\text{GeV}^4, \tag{1.83}$$
$$V_3 = \langle g^3 f_{abc} F^a_{\mu\nu} F^b_{\nu\rho} F^c_{\rho\mu} \rangle = (0.5 \pm 0.5)\,\text{GeV}^6,$$
$$\langle \bar{q} q \rangle \simeq -(0.23\,\text{GeV})^3.$$

In the perturbation theory all these parameters are zero. The possible connection of these nonperturbative parameters with the instanton [28] contribution was considered [24, 29, 30].

In what follows the method of calculation of these quantities in the framework of the compensation approach will be described in details.

On the other hand it becomes clear that there is another quite effective method, which serves for account of nonperturbative effects at low momenta. The most well-known and significant example of such method give the famous Nambu–Jona-Lasinio model [31, 32]. This model was the first example of a theory, which further acquires notation "an effective theory". This notation means that this theory somehow appears in the framework of the fundamental theory (QCD, EW, *etc.*) and it acts in the restricted region of the momentum space. The simplest possibility to describe this property is to introduce an auxiliary cut-off, that exactly was done in the initial works [31, 32]. Let us briefly present highlights of the model, which acquires real popularity. In more details of the NJL model the reader can see, e. g., in works [33–38]. For a recent review see [39].

1.6.2 The model NJL

For the formulation of the model we may start from the initial QCD Lagrangian (1.58). We know, that the first two masses of u and d quarks are small. What occurs if they are just zero? Then Lagrangian (1.58) is invariant under the so called chiral transformations

$$q \to e^{\imath\gamma_5 \tau_a \theta^a}\, q, \quad \bar{q} \to \bar{q}\, e^{\imath\gamma_5 \tau_a \theta^a}, \tag{1.84}$$

where θ^a are parameters of a $SU(2)$ isotopic transformation and

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}, \quad \bar{q}(x) = (\bar{u}(x), \bar{d}(x)), \tag{1.85}$$

are fields of u and d quarks.

This property is realized in the Nambu–Jona-Lasinio model [31, 32]. For description of the main properties of the NJL model we start from its simplest version. The initial four-quark $SU(2) \times SU(2)$-symmetric Lagrangian has the following form:

$$L = \bar{q}(x)(\imath\partial_\alpha \gamma_\alpha - m_0) q(x)$$
$$+ \frac{G}{2}\Big((\bar{q}(x)\, q(x))(\bar{q}(x)\, q(x)) - (\bar{q}(x)\, \tau^a\, \gamma_5 q(x))(\bar{q}(x)\, \tau^a\, \gamma_5 q(x)) \Big), \tag{1.86}$$

where m_0 is a small initial current mass of light quarks and G is the four-fermion coupling constant, τ^a – the Pauli matrices. Color is summed up in each product $\bar{q}O_i q$ with different inside operators O_i.

Let us find out if Lagrangian (1.86) is invariant in respect to the chiral transformations.

Let us illustrate this property by writing the first term of expansion of transformation of four-fermion terms in equation (1.86)

$$\bar{q}q\,\bar{q}q \to \bar{q}(1 + 2\imath\gamma_5\tau^a\theta^a)q\,\bar{q}(1 + 2\imath\gamma_5\tau^a\theta^a)q = \bar{q}q\,\bar{q}q + 4\imath\bar{q}\gamma_5\tau^a\theta^a q\,\bar{q}q,$$
$$\begin{aligned} \bar{q}\tau^b\gamma_5 q &\to \bar{q}(1 + \imath\gamma_5\tau^a\theta^a)\tau^b\gamma_5(1 + \imath\gamma_5\tau^a\theta^a)q \\ &= \bar{q}\tau^b\gamma_5 q + \imath\bar{q}(\tau^a\tau^b + \tau^b\tau^a)\theta^a\, q \;= \bar{q}\tau^b\gamma_5 q + 2\imath\bar{q}\,\theta^b q, \end{aligned} \tag{1.87}$$
$$\bar{q}\tau^b\gamma_5 q\,\bar{q}\tau^b\gamma_5 q \to (\bar{q}\tau^b\gamma_5 q)^2 + 4\imath\bar{q}\gamma_5\tau^b\theta^b q\,\bar{q}\,q.$$

Let us remind, that in our definition $\gamma_5^2 = 1$. Due to (1.87) terms proportional to θ^a cancel in the four-fermion part of Lagrangian(1.86). The kinetic term in (1.86) $\bar{q}\imath\partial_\alpha\gamma_\alpha\, q$ is also invariant due to anti commutativity of γ_α and γ_5. However the mass term $-\,m_0\bar{q}\,q$ evidently is not invariant under transformations (1.84). In view of this we for the beginning consider the case of chiral invariance, that means

$$m_0 = 0. \tag{1.88}$$

Then, following [31, 32], let us introduce evident Feynman rules and define ultraviolet cut-off Λ, which defines the region of applicability of the effective interaction (1.86,1.88).

First of all we consider the problem of a possibility of a spontaneous generation of quark mass. We start with Lagrangian (1.86) with $m_0 \to 0$. At this point we for the first time use the procedure, following from N. N. Bogoliubov compensation approach [40–42]. With small but nonzero m_0 we expect some value for mass m arising as a result of interaction in (1.86). So let us add to and subtract from expression (1.86) mass term $m\,\bar{q}\,q$ and rewrite (1.86) in the following form:

$$\begin{aligned} L &= L_0 + L_{int}, \\ L_0 &= \bar{q}(\imath\partial_\alpha\gamma_\alpha - m)q, \end{aligned} \tag{1.89}$$
$$L_{int} = \frac{G}{2}\Big((\bar{q}\,q)(\bar{q}\,q) - (\bar{q}\,\tau^a\,\gamma_5 q)(\bar{q}\,\tau^a\,\gamma_5 q)\Big) + (m - m_0)\,\bar{q}\,q. \tag{1.90}$$

Now in the interaction Lagrangian (1.90) there is quite improper mass term. Thus we have to guarantee real absence of this term due to a compensation condition to fulfill. This means the condition for all possible contributions of interaction (1.90) to the mass term giving zero as a result. For the first approximation, corresponding to diagrams of Figure 1.5 we have the following compensation equation:

$$-(m - m_0) + \frac{8\,N_c\,m\,G}{(2\,\pi)^4}\int \frac{d^4q\,\theta(\Lambda^2 - q^2)}{q^2 + m^2} = -(m - m_0) + \frac{3\,m\,G}{2\,\pi^2}\int_0^\Lambda \frac{y dy}{y + m^2} = 0. \tag{1.91}$$

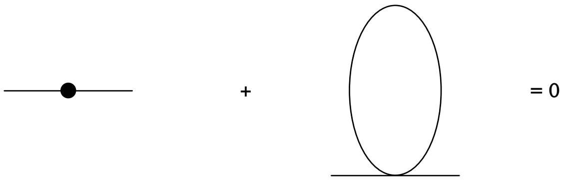


Fig. 1.5. Diagrams corresponding to compensation equation for spontaneous generation of mass in NJL model.

Here we have used Wick rotation in a transition to the four-dimensional Euclid momentum space. Let us remind, that this procedure means, that after all algebraic calculations in the usual Minkovsky space we perform in an integral under a consideration the following substitution: $(pq) \to -(pq)$ and $dq \to \imath\, dq$. In such simple variant all quantities are expressed in terms of two main integrals

$$I_1 = \frac{N_c}{16\pi^2}\int_0^\Lambda \frac{y\,dy}{y+m^2} = \frac{N_c}{16\pi^2}\left(\Lambda^2 - m^2 \ln\frac{\Lambda^2+m^2}{m^2}\right),$$
$$I_2 = \frac{N_c}{16\pi^2}\int_0^\Lambda \frac{y\,dy}{(y+m^2)^2}\left(1-\frac{m^4}{(y+m^2)^2}\right)$$
$$= \frac{N_c}{16\pi^2}\left(\ln\frac{\Lambda^2+m^2}{m^2} - \frac{\Lambda^2(\Lambda^2+2m^2)}{2(\Lambda^2+m^2)^2}\right). \tag{1.92}$$

Now we take limit $m_0 \to 0$. Thus we have from (1.91,1.92)

$$-m\,(1-8\,G I_1) = 0, \quad -m\left[1-\frac{3\,\mu}{2\pi^2}\left(\frac{\lambda}{\mu} - \ln\frac{\lambda+\mu}{\mu}\right)\right] = 0,$$
$$\mu = G m^2, \quad \lambda = G\Lambda^2. \tag{1.93}$$

Equation (1.93) evidently has the trivial solution

$$m = 0. \tag{1.94}$$

However, there is also a nontrivial solution, which occurs when the expression in the square brackets is zero. One readily gets convinced, that a nontrivial solution does exist for $\lambda \geq 6.58$. For example, with $\lambda = 8.87$ behavior of this expression is presented in Figure 1.6 and the solution is $\mu = 1$.

Let us consider the Bethe–Salpeter equation for isotopic vector pseudoscalar state with momentum $Q = 0$, presented in Figure 1.7. It reads

$$\gamma_5\,\tau_a\,g_{\pi qq} = \gamma_5\,\tau_a\,\frac{G\,8\,N_c}{(2\pi)^4}\int \frac{(q^2+m^2)\,d^4q}{(q^2+m^2)^2}\,g_{\pi qq}. \tag{1.95}$$

Here coefficient $8N_c$ is due to traces in isospin, color and spinor indices. We immediately get convinced, that this equation is equivalent to equation (1.93) for $m \neq 0$. This

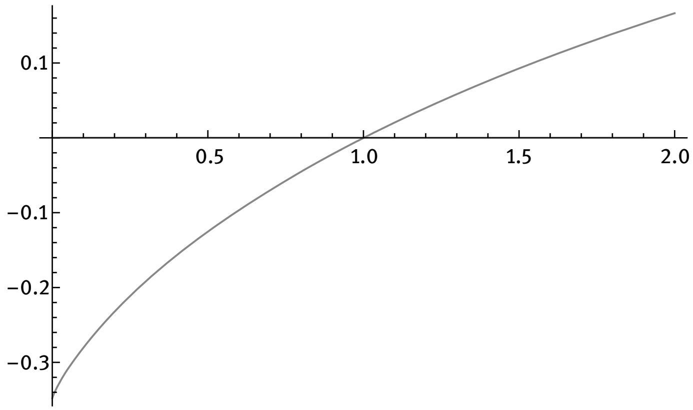


Fig. 1.6. Behavior of expression in square brackets of (1.93) for $\lambda = 8.87$ in dependence on μ.

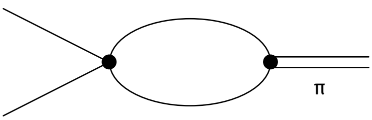


Fig. 1.7. Bethe–Salpeter equation for pseudoscalar state.

means, that for the nontrivial solution of (1.93) there exists a massless pseudoscalar state with unit isotopic spin. This is an inevitable consequence of the symmetry breaking according to Bogoliubov–Goldstone theorem [40, 43, 44]. Such state has semblance of pion, but the pion mass is not zero, although it is small in comparison to other hadron masses. Thus we have to use Lagrangian (1.86) with small, but not zero value for m_0. This means that we have not exact chiral invariance (1.84), but only initially approximated invariance. Then instead of (1.93) we have

$$-\frac{m_0}{m} + (1 - 8\,I_1) = -\frac{m_0}{m} + \left[1 - \frac{3\,\mu}{2\pi^2}\left(\frac{\lambda}{\mu} - \ln\frac{\lambda+\mu}{\mu}\right)\right] = 0,$$
$$\mu = G\,m^2, \quad \lambda = G\,\Lambda^2. \tag{1.96}$$

Qualitatively main features of a nontrivial solution are the same as for $m_0 = 0$. For example, for

$$\frac{m_0}{m} = \frac{1}{60}$$

a nontrivial solution exists provided $\lambda \geq 6.47$ and $\mu = 1$ for $\lambda = 8.747$. The Bethe–Salpeter equation for a small nonzero pion mass is now the following

$$g_{\pi qq} = \frac{8\,G\,N_c}{(2\pi)^4}\int\frac{d^4q}{(q^2+m^2)^2}\left(q^2 + m^2 + \frac{m_\pi^2}{2}\left(1 - \frac{m^4}{(q^2+m^2)^2}\right)\right) g_{\pi qq}. \tag{1.97}$$

From (1.97,1.95,1.96) we have

$$G m_\pi^2 = \mu_\pi = \frac{m_0}{4\,m\,I_2}. \tag{1.98}$$

Now vertices for the pion-quark-antiquark interaction and the $\sigma\,\bar{q}\,q$ interaction are correspondingly the following

$$g_{\pi qq}\,\bar{q}\gamma_5\,\tau_a\,q\,\pi^a, \quad g_{\sigma qq}\,\bar{q}\,q\,\sigma. \tag{1.99}$$

This coupling is defined by the normalization condition for Bethe–Salpeter wave function. We have for pion normalization correspondingly the following

$$4g^2_{\pi qq}\,I_2 = 1. \tag{1.100}$$

Let us in the same approximation calculate the π-decay constant f_π. The experimental value for this constant is the following [4]

$$f_\pi = 92.42 \pm 0.33 \text{ MeV}, \tag{1.101}$$

According to the evident one-loop diagram we have

$$f_\pi = 4\,g_{\pi qq}\,m\,I_2 = \frac{m}{g_{\pi qq}}, \tag{1.102}$$

where we have used normalization condition (3.28). Thus the well-known Goldberger–Treiman relation [45] naturally arises in NJL model. Now let us formulate the set of equations

$$\begin{aligned} I_2 &= \frac{1}{4g^2_{\pi qq}}, \quad 1 - 8\,I1 = \frac{m_0}{m}, \\ m^2_\pi &= \frac{\bar{g}^2_{\pi qq}\,m_0\,m}{\mu}, \quad \frac{g^2_\rho(M^2_\rho - 4\,m^2_\pi)^{3/2}}{48\,\pi\,M^2_\rho} = \Gamma_\rho, \\ g^2_\rho &= \sqrt{6}\,g_{\pi qq}, \quad \bar{g}_{\pi qq} = \sqrt{\frac{M^2_\rho + 6\,m^2}{M^2_\rho}}\,g_{\pi qq}. \end{aligned} \tag{1.103}$$

We also have evident relation for masses of π- and σ mesons, which one obtain from difference of one-loop contributions to corresponding meson mass square

$$m_\sigma = \sqrt{m^2_\pi + 4\,m^2}. \tag{1.104}$$

Let us fix the well-known values of f_π and m_π (average of charged ones and the neutral one) and the ρ-meson parameters

$$\begin{aligned} f_\pi &= 92.4\,\text{MeV}, \quad m_\pi = 138\,\text{MeV}, \\ M_\rho &= 775.4\,\text{MeV}, \quad \Gamma_\rho = 149.2\,\text{MeV}. \end{aligned} \tag{1.105}$$

and solve set of equations (1.103), that gives

$$\begin{aligned} \mu &= 0.5723, \quad \lambda = 8.0754, \\ m &= 320.08\,\text{MeV}, \quad m_0 = 2.838\,\text{MeV}. \end{aligned} \tag{1.106}$$

We use also the definition of the quark condensate in our notations (1.92,1.93)

$$\langle \bar{q}q \rangle = \lim_{\epsilon \to 0} \langle \bar{q}(\epsilon)\, q(0) \rangle = - \lim_{\epsilon \to 0} \; Trace\, G(\epsilon) = - \frac{4\, m^3\, I_1}{\mu},$$

where $G(x)$ is the quark propagator.

Taking into account expressions (1.103,1.104,1.106,1.107) we come to the following set of parameters

$$\begin{aligned} m_0 &= 2.84\,\mathrm{MeV}, & G &= \frac{1}{(425.1\,\mathrm{MeV})^2}, & \Lambda &= 1202.3\,\mathrm{MeV}, \\ m &= 320\,\mathrm{MeV}, & \langle \bar{q}q \rangle &= -(305.1\,\mathrm{MeV})^3, & m_\sigma &= 654.9\,\mathrm{MeV}, \\ g_{\pi qq} &= 2.44, & \bar{g}_{\pi qq} &= 3.46, & g_{\rho qq} &= 5.97. \end{aligned} \tag{1.107}$$

Results (1.106, 1.107) demonstrate satisfactory agreement with data. Value of the quark condensate is rather higher than its conventional value. But nevertheless value in (1.107) is within 20% accuracy off standard value $-\langle 250\,\mathrm{MeV} \rangle^3$. Note, that result (1.107) with high accuracy agrees the well-known Gell-Mann–Oaks–Renner relation [46]

$$m_\pi^2 f_\pi^2 = -2\, m_0\, \langle \bar{q}q \rangle. \tag{1.108}$$

Mass of the σ-meson rather exceeds its experimental value $400\,\mathrm{MeV} < m_\sigma < 550\,\mathrm{MeV}$ [4]. However there are arguments for real mass of the lightest scalar σ to have larger mass in interval 700–900 MeV [47], but due to effect of four-quark configurations in the scalar state its mass is shifted below. We shall return to this problem in what follows.

To conclude with NJL results we would state, that such a simple assumption gives quite adequate description of low mass hadron physics, which could not be achieved in the framework of a perturbation theory. The NJL theory is just very instructive example of effective nonperturbative theory. There are few significant features of an effective theory, which manifest themselves in NJL theory. The principal ones are the following:

(1) If one consider an effective theory being local it necessarily is nonrenormalizable.
(2) Thus the theory may make sense only in presence of a form-factor $F(q_i^2)$ being somehow defined. In the considered case there is simple cut-off, that means $F(q^2) = \theta(\Lambda^2 - q^2)$. Due to this an effective interaction acts in a restricted region of the momentum space.

Now we are sure that the genuine theory of strong interaction is QCD. But NJL from the first sight has nothing to do with this gauge theory. The natural question arises why the NJL approach can explain low mass hadron physics with such excellent precision? The only possible answer to this question is that NJL interaction appears in the framework of QCD due to a mechanism of spontaneous generation.

We are aware of a phenomenon of spontaneous breaking of an invariance. For example, this phenomenon leads to the important physical effects: superconductivity, superfluidity, ferromagnetism. In the framework of quantum statistical theory these effects were explained decades ago. One of the most powerful methods, which were applied for the explanations, was the compensation approach by N.N. Bogoliubov [40–42]. At the same time there were applications to QFT problems, one of the most important was just the Y. Nambu works, which was strongly influenced by works [40–42]. The essence of the physical effect is a spontaneous generation of a quark mass due to a nontrivial solution of equations (1.91, 1.88). As a result we obtain spontaneous violation of the chiral symmetry (1.84). We shall use the compensation approach also for more complicated problems, namely for studies of possibilities of spontaneous generation of effective interaction, similar to the NJL effective interaction, considered above.

The main principles of the approach will be presented in the third chapter.

2 The standard model

2.1 The electro-weak theory

Let us remind, that before the formulation of the EWT we had the QED, various models for strong interaction and the weak interaction with the following interaction Lagrangian

$$L_{int} = \frac{G_F}{\sqrt{2}} J_\alpha J_\alpha^\dagger, \tag{2.1}$$
$$J_\alpha = \bar{e}(1-\gamma_5)\gamma_\alpha \nu_e + \bar{\mu}(1-\gamma_5)\gamma_\alpha \nu_\mu + \bar{\tau}(1-\gamma_5)\gamma_\alpha \nu_\tau$$
$$+ \bar{d}\,'(1-\gamma_5)\gamma_\alpha u + \bar{s}\,'(1-\gamma_5)\gamma_\alpha c + \bar{b}\,'(1-\gamma_5)\gamma_\alpha t,$$

where

$$G_F = 1.16639(1)\cdot 10^{-5}\ \mathrm{GeV}^{-2}, \tag{2.2}$$

is the Fermi constant. We denote since now the spinor field of a particle by its symbol, e. g., $\mu = \psi_\mu$ denotes the spinor field of the muon. Here all the lepton and quarks are present, one have to bear in mind, that for quarks also summing over color indices is understood, so, in simple words, there are three terms for each quark pair. The prime up the down quarks means linear combinations according to the well-known Cabibbo–Kobayashi–Maskawa matrix [48, 49]: $D' = U\,D$, that is

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{\imath\delta} & c_1 c_2 s_3 + s_2 s_3 e^{\imath\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{\imath\delta} & c_1 s_2 s_3 - c_2 c_3 e^{\imath\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{2.3}$$

Here $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, θ_i are three mixing angles and δ is a phase, which corresponds to a CP violation.

The result of fit of totality of experimental data for absolute values of matrix elements of matrix (2.3) is the following [4]

$$\begin{pmatrix} 0.97427 \pm 0.00015 & 0.22534 \pm 0.00065 & 0.00352^{+0.00015}_{-0.00014} \\ 0.22520 \pm 0.00065 & 0.97344 \pm 0.00016 & 0.0412^{+0.001}_{0.0005} \\ 0.00867^{+0.00029}_{-0.00031} & 0.0404^{+0.0011}_{-0.0005} & 0.999146^{+0.000021}_{-0.000046} \end{pmatrix}. \tag{2.4}$$

Matrix element V_{12} of matrix V (2.4) is with high precision the well-known parameter $\sin\phi_c$, that is the sine of the Cabibbo angle

$$\sin\phi_c = 0.225, \tag{2.5}$$

where Cabibbo angle describes mixing of d and s quarks

$$d' = d\,\cos\phi_c + s\,\sin\phi_c, \quad s' = -d\,\sin\phi_c + s\,\cos\phi_c. \tag{2.6}$$

In dealing with four-fermion interactions similar to (2.1) the Fiertz transformation is very useful. In what follows it will be repeatedly applied. The transformation relates to change of order of spinor operators in four-fermions terms such as the following

$$\bar{\psi}_1\, O_k \psi_2\, \bar{\psi}_3\, O'_{k'} \psi_4 = \sum D^{kk'}_{ll'}\, \bar{\psi}_1\, O_l \psi_4\, \bar{\psi}_3\, O'_{l'} \psi_2, \tag{2.7}$$

where $O_k, \ldots, O'_{l'}$ are some matrices, e.g. the Dirac ones. In obtaining corresponding relations we proceed as follows. Let us take an example

$$\begin{aligned}\bar{\psi}_1 \gamma_m \psi_2\, \bar{\psi}_3 \gamma_m \psi_4 = a_s\, \bar{\psi}_1 \psi_4\, \bar{\psi}_3\, \psi_2 &+ a_p\, \bar{\psi}_1\, \gamma_5 \psi_4\, \bar{\psi}_3\, \gamma_5 \psi_2 + a_v \bar{\psi}_1 \gamma_m \psi_4 \bar{\psi}_3 \gamma_m \psi_2 \\ &+ a_a\, \bar{\psi}_1 \gamma_m \gamma_5 \psi_4 \bar{\psi}_3 \gamma_m \gamma_5 \psi_2 + a_t \bar{\psi}_1 \sigma_{\mu\nu} \psi_4 \bar{\psi}_3 \sigma_{\mu\nu} \psi_2,\end{aligned} \tag{2.8}$$

Let us write relation (2.9) using explicit indices

$$\begin{aligned}(\gamma_m)^\alpha_\beta\, (\gamma_m)^\lambda_\sigma = a_s\, I^\alpha_\sigma\, I^\lambda_\beta &+ a_p\, (\gamma_5)^\alpha_\sigma\, (\gamma_5)^\lambda_\beta + a_v\, (\gamma_n)^\alpha_\sigma\, (\gamma_n)^\lambda_\beta \\ &+ a_a (\gamma_m \gamma_5)^\alpha_\sigma\, (\gamma_m \gamma_5)^\lambda_\beta + a_t\, (\sigma_{\mu\nu})^\alpha_\sigma\, (\sigma_{\mu\nu})^\lambda_\beta.\end{aligned} \tag{2.9}$$

Now we multiply (2.9) by $\delta^\sigma_\alpha\, \delta^\beta_\lambda$ and obtain due to Trace of Dirac matrices but the unit one I being zero

$$16\, a_s = Trace(\gamma_m\, \gamma_m) = 16, \quad a_s = 1. \tag{2.10}$$

The next time we multiply (2.9) by $(\gamma_5)^\sigma_\alpha\, (\gamma_5)^\beta_\lambda$ and in the same way we obtain

$$16\, a_p = Trace(\gamma_5\, \gamma_m\, \gamma_5\, \gamma_m) = -16, \quad a_p = -1. \tag{2.11}$$

We proceed in the same way for combinations and obtain results being presented in Table 2.1, where we use the following notations

$$\begin{aligned}
&SS \to \bar{\psi}\, \psi\, \bar{\psi}\, \psi, && PP \to \bar{\psi}\gamma_5 \psi\, \bar{\psi}\gamma_5 \psi, && SP \to \bar{\psi}\, \psi\, \bar{\psi}\gamma_5 \psi, \\
&PS \to \bar{\psi}\gamma_5 \psi\, \bar{\psi}\, \psi, && VV \to \bar{\psi}\gamma_\mu \psi\, \bar{\psi}\gamma_\mu \psi && AA \to \bar{\psi}\gamma_\mu \gamma_5 \psi \bar{\psi}\gamma_\mu \gamma_5 \psi, \\
&VA \to \bar{\psi}\gamma_\mu \psi\, \bar{\psi}\gamma_\mu \gamma_5 \psi, && AV \to \bar{\psi}\gamma_\mu \gamma_5 \psi\, \bar{\psi}\gamma_\mu \psi, && \\
&TT \to \bar{\psi}\sigma_{\mu\nu} \psi\, \bar{\psi}\sigma_{\mu\nu} \psi, && \sigma_{\mu\nu} = \frac{\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu}{2\imath}. &&
\end{aligned} \tag{2.12}$$

We also have to take into account, that in the procedure of the Fiertz transformation there is an odd number of commutations of spinor operators. Because of these operators obey anticommutative relations, each such commutation leads to a change of sign. Thus we have additional minus sign in all the elements of Table 2.1.

It is also useful to have the analogous transformation with isotopic matrices, the coefficients of the transformation are presented in Table 2.2. We have also to take into account colors of quarks. Indeed in Lagrangian (2.1) in each quark term there is summation by colors which corresponds to colorless current J.

$$\bar{\psi}^\alpha \cdots \psi_\alpha, \tag{2.13}$$

Table 2.1. Coefficients of the Fiertz transformation

	SS	PP	SP	PS	VV	AA	VA	AV	TT
SS	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{8}$
PP	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{8}$
SP	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0
PS	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0
VV	-1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
AA	1	-1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
VA	0	0	-1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
AV	0	0	1	-1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
TT	3	3	0	0	0	0	0	0	$\frac{1}{2}$

Table 2.2. Coefficients of the isotopic Fiertz-like transformation

	II	$\tau\tau$
II	$\frac{1}{2}$	$\frac{1}{2}$
$\tau\tau$	$\frac{3}{2}$	$-\frac{1}{2}$

In what follows sometimes it will be necessary to take into account color indices in terms of (2.1) type. While performing the Fiertz transformation, color indices also change places. So we have relations

$$I^{\alpha}_{\beta}\, I^{\gamma}_{\delta} = a_{11}\, I^{\alpha}_{\delta}\, I^{\gamma}_{\beta} + a_{12}\, (t^a)^{\alpha}_{\delta}\, (t^a)^{\gamma}_{\beta},$$
$$(t^a)^{\alpha}_{\beta}\, (t^a)^{\gamma}_{\delta} = a_{21}\, I^{\alpha}_{\delta}\, I^{\gamma}_{\beta} + a_{22}\, (t^a)^{\alpha}_{\delta}\, (t^a)^{\gamma}_{\beta}. \tag{2.14}$$

Coefficients a_{ij} are calculated in the same way as in case of the usual Fiertz transformation. The result of the operation is described in Table 2.3

The very important element enters Lagrangian (2.1), namely, combination $(1-\gamma_5)$. It deserve more detailed attention. Note that we define here $\gamma_5 = -\imath\gamma_0\gamma_1\gamma_2\gamma_3$, so that $\gamma_5^2 = 1$. Let us construct two matrices

$$P_L = \frac{1+\gamma_5}{2}, \quad P_R = \frac{1-\gamma_5}{2}. \tag{2.15}$$

Table 2.3. Coefficients of the color Fiertz-like transformation (for colored quarks).

	II	$t^a\, t^a$
II	$\frac{1}{3}$	2
$t^a\, t^a$	$\frac{4}{9}$	$-\frac{1}{3}$

One easily sees, that these matrices possess all the properties, necessary for the complete set of projection operators, namely

$$P_i^2 = P_i, \quad P_L + P_R = 1, \quad P_L * P_R = P_R * P_L = 0. \tag{2.16}$$

Therefore these operators split the space of spinors into two subspaces

$$\Psi_L = P_L \Psi = \frac{1+\gamma_5}{2}\Psi, \quad \Psi_R = P_R \Psi = \frac{1-\gamma_5}{2}\Psi. \tag{2.17}$$

Thus operators P_L, P_R project the initial spinor on its left and right components. By Dirac conjugation equation (2.17) becomes

$$\bar{\Psi}_L = \bar{\Psi}\frac{1-\gamma_5}{2}, \quad \bar{\Psi}_R = \bar{\Psi}\frac{1+\gamma_5}{2}, \tag{2.18}$$

where we take into account that γ_5 anticommutes with γ_0 being present in the conjugated spinor. In other words, left particle corresponds to right antiparticle Let us consider free Lagrangian of the spinor field

$$\frac{\imath}{2}\left(\frac{\partial\bar{\psi}}{\partial x^\mu}\gamma_\mu\psi - \bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x^\mu}\right) - m\bar{\psi}\psi, \tag{2.19}$$

Using equations (2.17, 2.18) we rewrite it in terms of left and right components

$$\frac{\imath}{2}\left(\frac{\partial\bar{\psi}_L}{\partial x^\mu}\gamma_\mu\psi_L - \bar{\psi}_L\gamma_\mu\frac{\partial\psi_L}{\partial x^\mu} + \frac{\partial\bar{\psi}_R}{\partial x^\mu}\gamma_\mu\psi_R - \bar{\psi}_R\gamma_\mu\frac{\partial\psi_R}{\partial x^\mu}\right) - m\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right), \tag{2.20}$$

We see here that the first term divides in two independent parts, connected with left and right spinors respectively, but the mass term mixes these components. If an interaction is described in terms only of, say, left components and a mass is zero, right components are thoroughly decoupled and do not appear, if they are not present initially.

Now let us return to the would-be interaction (2.1). Such four-fermion interaction is nonrenormalizable. In particular it leads to cross-sections, which linearly rise with energy increasing. The situation may be improved if one introduces in addition to the photon three massive intermediate bosons: W^+, W^-, Z. Then instead of interaction (2.1) we have

$$L_{int} = -g_W(J_\alpha W_\alpha + J_\alpha^\dagger W_\alpha^\dagger), \tag{2.21}$$

and interaction of boson Z with a neutral current. We have the following relation

$$\frac{g_W^2}{M_W^2} = \frac{G_F}{\sqrt{2}}. \tag{2.22}$$

For the theory being renormalizable we have to introduce a gauge symmetry. It comes out that this goal is achieved provided the symmetry of the gauge theory of the electroweak interaction is $SU(2)\times U(1)$ [50, 51]. $SU(2)$ group is completely the same (in mathematical aspects) as the symmetry group, describing usual spin and isotopic spin. This is why, the symmetry can be named a weak isospin and its representations are labeled by values of this isospin I, which as usually , are either integer or half-integer. U(1) group corresponds to some symmetry, conserving a charge, similar to electric charge or hypercharge. The corresponding quantum number is named weak hypercharge Y. A representation of the electro-weak symmetry group is hence characterized by two numbers: (I, Y).

Now we have three Yang-Mills fields W^a_μ, corresponding to $SU(2)$ subgroup and one photon-like field B_μ connected with $U(1)$. All four of them are massless. The theory of interaction of such a vector field set with a conserved spinor current is renormalizable. But such theory has nothing to do with real physics. As we see from a brief review of W interactions W^+ W^- and Z as well have to be sufficiently massive and only the photon is to be massless. So we have a good theory, which is not physical because it is too good, too symmetric. A breaking of the symmetry is necessary and here lies the central point of the theory. Let us demonstrate how the famous Higgs mechanism works, which gives this symmetry breaking.

So we proceed to the Higgs phenomenon [52–54]. Let us consider the following lagrangian, describing interaction of a massless vector field A_μ and a complex scalar field ϕ

$$\frac{\partial\phi^\dagger}{\partial x^\mu}\frac{\partial\phi}{\partial x^\mu} - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \imath \mathrm{e}\left(\phi^\dagger\frac{\partial\phi}{\partial x^\mu} - \frac{\partial\phi^\dagger}{\partial x^\mu}\phi\right)A_\mu + \mathrm{e}^2 A_\mu A_\mu \phi^\dagger\phi, \quad (2.23)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is an electromagnetic field. The lagrangian is invariant under the following transformations

$$\phi \to e^{\imath\theta(x)}\phi, \quad \phi^\dagger \to e^{-\imath\theta(x)}\phi^\dagger, \quad A_\mu \to A_\mu + \frac{1}{\mathrm{e}}\frac{\partial\phi}{\partial x^\mu}. \quad (2.24)$$

If one introduce instead of fields ϕ, $\phi^\dagger$ two real fields ϕ_j, $j = 1, 2$

$$\phi = \frac{\phi_1 + \imath\phi_2}{\sqrt{2}}, \quad \phi^\dagger = \frac{\phi_1 - \imath\phi_2}{\sqrt{2}}, \quad (2.25)$$

then transformations (2.24) for scalar fields take the form of a two-dimensional rotation

$$\phi_1' = \cos\theta\phi_1 - \sin\theta\phi_2, \quad \phi_2' = \sin\theta\phi_1 + \cos\theta\phi_2. \quad (2.26)$$

Thus, the symmetry upon transformations $U(1)$ (2.24) means un fact invariance in respect to rotations of a plane. Transformations in the form (2.26) demonstrate independence of the theory on the direction in the plane with axes 1, 2. Let us ask the question, if a spontaneous breaking of the invariance could occur? For answering the question one has to add to Lagrangian (2.23) a small term, which breaks the invariance

$$\delta L = \epsilon\,\phi_1. \quad (2.27)$$

Now we lose equivalence of two axes, so the initial invariance is valid no longer. In particular if one considers vacuum expectation value of field ϕ_1, it is not zero, due to (2.27), whereas $\langle\phi_2\rangle = 0$. So we have

$$\langle\phi_1\rangle = \eta, \quad \langle\phi_2\rangle = 0. \tag{2.28}$$

In a usual theory for application of the perturbation theory being possible, such vacuum averages should be zero. So we redefine fields in the following way

$$\phi_1 = \eta + \xi, \quad \phi_2 = \psi. \tag{2.29}$$

Now vacuum averages of both fields ξ and ψ are zero and Lagrangian (2.23) takes the following form

$$\begin{aligned}
&\frac{1}{2}\,\partial_\mu\psi\,\partial_\mu\psi + \frac{1}{2}\,\partial_\mu\xi\,\partial_\mu\xi - \frac{m^2}{2}\,(\psi^2 + \xi^2) - m^2\xi\,\eta - \frac{m^2}{2}\,\eta^2 \\
&\quad - \frac{1}{4}\,F_{\mu\nu}F_{\mu\nu} + \mathrm{e}\,(\psi\partial_\mu\xi - \xi\partial_\mu\psi)A_\mu - \mathrm{e}\,\eta\,\partial_\mu\psi A_\mu + \epsilon\eta + \epsilon\xi \\
&\quad - \frac{\lambda}{4}\left(\psi^4 + 2\psi^2(\xi^2 + 2\xi\eta + \eta^2) + \xi^2 + 4\xi^3\eta + 6\xi^2\eta^2 + 4\xi\eta^3 + \eta^4\right) \\
&\quad + \frac{\mathrm{e}^2}{2}\,A_\mu A_\mu(\psi^2 + \xi^2 + 2\xi\eta + \eta^2).
\end{aligned} \tag{2.30}$$

We have now to demand vacuum average of redefined field ξ to be zero. But we know, that in case there is a term proportional to a field itself in a Lagrangian the corresponding average is not zero. So we look for the linear terms in ξ in (2.30) and obtain the following condition

$$\epsilon - m^2 - \lambda\,\eta^3 = 0. \tag{2.31}$$

At this point we switch off the initial breaking term, that is we put $\epsilon \to 0$ and obtain the following relation There are two solutions of this equation. The first one is the trivial one $\eta = 0$. The other one,

$$\eta^2 = -\frac{m^2}{\lambda}, \tag{2.32}$$

just breaks the initial invariance. If both nominator and denominator are positive, the solution does not exist. Condition $\lambda > 0$ is inevitable, because if it is not so, a theory becomes unstable. So we have to assume, that $m^2 = -m_0^2$ is negative, that is not so sweet, because means, that initial scalar particles are tachyons. However we shall see, that this is not leading to observable effects. Thus, let us consider symmetry breaking solution (2.32).

Let us look now at Lagrangian (2.30) and mark terms quadratic in scalar fields ξ, ψ:

$$\frac{1}{2}\,\partial_\mu\psi\,\partial_\mu\psi + \frac{1}{2}\,\partial_\mu\xi\,\partial_\mu\xi + \frac{m_0^2}{2}\,(\psi^2 + \xi^2) - \frac{\lambda\,\eta^2}{2}\,\psi^2 - \frac{3\,\lambda\,\eta^2}{2}\,\xi^2. \tag{2.33}$$

Using equation (2.32) we get convinced, that field ψ has now zero mass and field ξ acquires normal real mass

$$m_\psi = 0, \quad m_\xi = \sqrt{2}\,m_0. \tag{2.34}$$

The fact, that field ψ is massless occurs by no means by chance. It is consequence of the general theorem, which was proven by N. N. Bogoliubov in statistical physics [40] and was applied to particle physics by Goldstone [43, 44]. Bogoliubov–Goldstone theorem states, that in case of spontaneous symmetry breaking there is always a zero excitation. For QFT this means inevitable existence of zero-mass field, in our case ψ. The corresponding particles are usually called Goldstone particles.

Thus we have now vector field A_μ, which initially is massless, massless scalar field ψ and massive scalar field ξ. The appearance of the massless scalar field makes the model unrealistic, because this means an existence of a new Coulomb force, which is severely restricted by observations. However, as it is shown in works [52–54], there occurs a phenomenon, which leads to acquiring of a mass by the vector field and at the same time to a disappearance of the massless scalar from the physical spectrum. Indeed, let us consider terms in equation (2.30), which are quadratic in the vector field and in field ψ

$$-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\psi\,\partial_\mu\psi - e\eta\partial_\mu\psi A_\mu + \frac{e^2\,\eta^2}{2}A_\mu A_\mu. \tag{2.35}$$

One easily sees, that the last three terms are united into combination

$$\frac{e^2\,\eta^2}{2}\left(A_\mu - \frac{1}{e\,\eta}\partial_\mu\,\psi\right)\left(A_\mu - \frac{1}{e\,\eta}\partial_\mu\,\psi\right). \tag{2.36}$$

The expression in brackets is just a gauge transformation (2.24), so we can introduce new vector-potential

$$B_\mu = A_\mu - \frac{1}{e\,\eta}\partial_\mu\,\psi. \tag{2.37}$$

Of course, the electromagnetic field can be as well expressed in terms of B_μ: $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. As a result of this substitution field ψ disappears completely from equation (2.35), which is now expressed exclusively in terms of B

$$-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{e^2\,\eta^2}{2}B_\mu B_\mu, \tag{2.38}$$

that is, it corresponds to a vector field with the mass

$$M_B = e\,\eta. \tag{2.39}$$

It comes out, that at the beginning we have four field degrees of freedom: two degrees of the massless vector field, that corresponds to two possible states of its polarization, and two degrees of freedom – one per each scalar. As a result of the symmetry breaking we have massive vector field with three degrees of freedom, because a massive vector particle has three polarization states, and one massive scalar particle, which corresponds to the scalar field, which acquires nonzero vacuum average, i. e., former ϕ_1. It occurs so, that the degree of freedom of the massless scalar field is used for a creation of the additional (the third) degree of freedom of the vector field, so that it describes now massive spin-one particle. The physical sense of the Higgs effect is connected just with this fact.

The problem, what solution of the two ones is preferable and has to be realized is of the utmost importance. Here we can rely on a consideration of a classical interaction energy. Indeed, the fact, that vacuum average (2.28) is not zero, corresponds to an existence of classical constant field $\phi_1 = \eta$. Then in the expression for the energy all derivatives vanish and it now looks like

$$V(\phi_1) = \frac{m^2}{2}\phi_1^2 + \frac{\lambda}{4}\phi_1^4. \tag{2.40}$$

Here we take into account, that $L = T - V$, where kinetic energy T is just connected with terms with derivatives. The dependence of potential energy (2.40) on ϕ_1 drastically differs for two cases: $m^2 > 0$ and $m^2 < 0$. As we see from 6.5, a minimum of the potential energy for $m^2 > 0$ is evidently situated at $\phi_1 = 0$, whereas for $m^2 > 0$ it is situated at some finite point $\phi_1^2 = -m^2/\lambda$ and is negative. This means that for the second case there are two equilibrium points, but the trivial point $\phi_1 = 0$ is unstable. On the contrary, the nontrivial one corresponds to stable state.

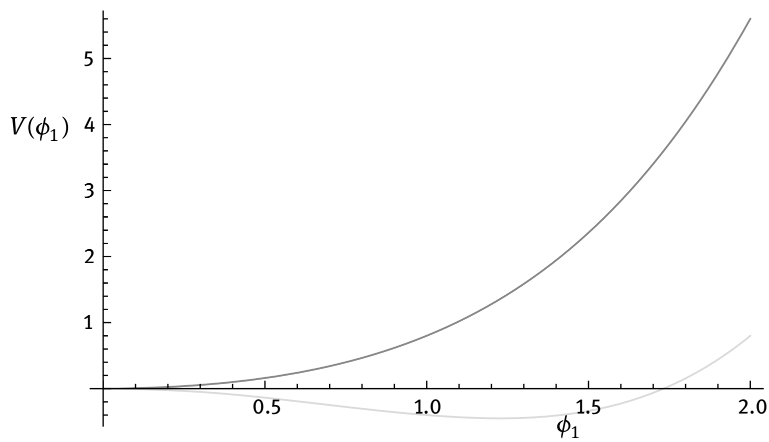


Fig. 2.1. Dependence of the effective potential energy on ϕ_1. The upper line corresponds to $m^2 > 0$, the lower one corresponds to $m^2 < 0$.

The stable state with minimal energy just corresponds to a nonzero vacuum average, that is it leads to the symmetry breaking with the properties being described. The position of the minimum of course exactly coincides with solution (2.32). Let us note also, that Figure 2.1 illustrates the statement, which was made above, concerning an instability of a theory with $\lambda < 0$. Indeed, in this case for $\phi_1 \to \infty$ a potential energy tends to the negative infinity, that evidently leads to instability.

It is important to note, that for the nontrivial solution the initial gauge invariance (2.24) is broken, because now we have the massive vector field B_μ. The Higgs phenomenon occurs also in nonabelian Yang–Mills theories [15]. The application to this problem is demonstrated in works [55, 56]. Let us proceed to the real situation in the EWT. As we have noted, for a description of the electro-weak interactions we have to use four gauge fields, corresponding to symmetry group $SU(2) \times U(1)$. So we have three W^a_μ, $a = 1, 2, 3$ and one B_μ, in other notations

$$W^+_\mu = \frac{W^1_\mu - \imath W^2_\mu}{\sqrt{2}}, \quad W^-_\mu = \frac{W^1_\mu + \imath W^2_\mu}{\sqrt{2}}, \quad W^0_\mu = W^3_\mu. \tag{2.41}$$

It is necessary to remind, that as a result of a breaking of the gauge invariance of the theory we have to obtain three massive vector fields: W^+_μ, W^-_μ, Z_μ and a massless photon A_μ. Now W^0 and B have the same quantum numbers, being as well the same as physical states Z and A are to have. So the last two can be expressed as linear combinations of the initial ones

$$W^0_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \quad B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu, \tag{2.42}$$

where θ_W is usually called Weinberg mixing angle. To create three masses we need at least three Goldstone bosons. We need also at least one scalar field which acquires nonzero vacuum average and finally becomes an observable spin-zero particle. So there is minimal number four for initial scalar fields. Thus we consider the standard set of scalar fields, which correspond to spinor representations of the group $SU(2)$, that is ϕ is a complex doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \phi^\dagger = (\phi_1^\dagger \;\; \phi_2^\dagger). \tag{2.43}$$

Then the Lagrangian describing the intermediate bosons and scalars (2.43) in the electro-weak theory is the following

$$\begin{gathered} L = -\frac{1}{4} W^a_{\mu\nu} W^a_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B_{\mu\nu} + (D_\mu\phi)^\dagger (D_\mu\phi) - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2, \\ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \\ W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon^{abc} W^b_\mu W^c_\nu. \end{gathered} \tag{2.44}$$

Charges of ϕ in respect to $SU(2) : g$ and $U(1) : g'$ correspond to the following long (covariant) derivative of ϕ

$$\begin{gathered} D_\mu \phi = \partial_\mu\phi + \imath g\, W^a_\mu \frac{\tau_a}{2} \phi + \frac{\imath}{2} g' B_\mu \phi, \\ (D_\mu \phi)^\dagger = \partial_\mu\phi^\dagger - \imath g\, W^a_\mu \frac{\tau^a}{2} \phi^\dagger - \frac{\imath}{2} g' B_\mu \phi^\dagger. \end{gathered} \tag{2.45}$$

From equation (2.45) we see, that ϕ corresponds to representation $(1/2, 1)$ of the $SU(2) \times U(1)$. Let us put again $m^2 < 0$, $m^2 = -m_0^2$, then Higgs phenomenon occurs. Let ϕ_2 has nonzero real vacuum expectation value

$$\langle \phi_2 \rangle = \frac{\eta}{\sqrt{2}}. \tag{2.46}$$

Let us introduce new notations, which generalizes (2.43)

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + \imath \psi_2 \\ \eta + \sigma + \imath \xi \end{pmatrix}, \quad \phi^\dagger = \begin{pmatrix} \frac{\psi_1 - \imath \psi_2}{\sqrt{2}} & \frac{\eta + \sigma - \imath \xi}{\sqrt{2}} \end{pmatrix}. \tag{2.47}$$

Similarly to how we deal with the fields substitution in Higgs model, we proceed here and obtain in place of equation (2.44)

$$\begin{aligned} &\frac{1}{2}\partial_\mu \psi_i \partial_\mu \psi_i + \frac{1}{2}\partial_\mu \sigma \partial_\mu \sigma + \frac{1}{2}\partial_\mu \xi \partial_\mu \xi + \frac{m_0^2}{2}(\psi_i^2 + \xi^2 + \sigma^2 + 2\eta\sigma + \eta^2) \\ &\quad - \frac{\lambda}{4}\Big((\psi_i^2 + \xi^2 + \sigma^2)^2 + 2\eta\sigma(\psi_i^2 + \xi^2 + \sigma^2) + 2\,\eta^2(\psi_i^2 + \xi^2 + 3\sigma^2) + 4\eta^3\sigma + \eta^4\Big) \\ &\quad - \frac{1}{4}\, W^i_{\mu\nu} W^i_{\mu\nu} - \frac{1}{4}\, W^0_{\mu\nu} W^0_{\mu\nu} - \frac{1}{4}\, B_{\mu\nu} B_{\mu\nu} \\ &\quad - \frac{g\eta}{2}\, \partial_\mu \psi_i\, W^i_\mu - \frac{\eta}{2}\, \partial_\mu \xi (g\, W^0_\mu - g' B_\mu) + \frac{g^2\eta^2}{8}\, W^i_\mu\, W^i_\mu \\ &\quad + \frac{\eta^2}{8}(g\, W^0_\mu - g' B_\mu)(g\, W^0_\mu - g' B_\mu) + \cdots \end{aligned} \tag{2.48}$$

Here $i = 1, 2$. By dots we designate numerous additional terms, which does not enter into definition of masses and mixing parameters of fields. From the beginning we assume the negative sign for the mass squared of the scalar field. The condition of the vacuum average of the redefined field σ be equal to zero is exactly the same, as equation (2.32) in the original Higgs model. Provided nontrivial solution is realized, fields ψ_i, ξ have zero masses and field σ acquires mass $\sqrt{2}\, m_0$. Namely this particle, which remains in the physical spectrum is called the Higgs boson H and we have

$$M_H = \sqrt{2}\, m_0 = \eta\, \sqrt{2\,\lambda}. \tag{2.49}$$

We see also from equation (2.48), that massless scalars again are united in combinations similar to that of equation (2.36) and the corresponding vector fields obtain masses. The charged W^i are redefined in the following way:

$$W^i_\mu \to W^i_\mu - \frac{1}{g\eta}\partial_\mu \psi^i, \tag{2.50}$$

and have now the following mass:

$$M_W = \frac{g\eta}{2}. \tag{2.51}$$

Neutral fields W^0_μ and B_μ form the following physical combination:

$$Z^i_\mu = \frac{g}{\sqrt{g^2 + g'^2}}\, W^0_\mu - \frac{g'}{\sqrt{g^2 + g'^2}}\, B_\mu - \frac{2}{\eta\, \sqrt{g^2 + g'^2}}\, \partial_\mu \xi, \tag{2.52}$$

and the neutral vector boson Z has the following mass:

$$M_Z = \frac{\eta\, \sqrt{g^2 + g'^2}}{2}. \tag{2.53}$$

Comparing equation (2.52) with equation (2.42), we see, that the Weinberg angle is defined in terms of the coupling constants in the following way:

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{2.54}$$

Again from equation (2.48) we also see, that the photon combination (2.42) remains massless. Thus we explicitly trace the action of the Higgs effect in the electro-weak theory and obtain masses of intermediate bosons and Higgs scalar, value of vacuum average η and expression of Weinberg mixing angle in terms of initial parameters of the theory: g, g', λ, m_0.

Let us obtain also relation, connecting the parameters of the theory with elementary electric charge e . We know, that charged $W^{\pm}$ are to have unit charge. Hence, the upper components of scalars (2.43) have also unit charge. So in long derivative (2.45) electromagnetic vector-potential has to enter with coefficient $\imath\mathbf{e}$. Substituting equation (2.42) into equation (2.45) we have

$$\mathrm{e} = \frac{gg'}{\sqrt{g^2+g'^2}} = g\sin\theta_W. \tag{2.55}$$

Let us proceed to a description of an interaction of vector and scalar fields with leptons and quarks. The interaction of the charged bosons $W^{\pm}$ with leptons and quarks are to be described by expression (2.21) with charged current (2.1). For the beginning let us take only the first term of the current, which contain the electron and the electron neutrino. We have to formulate a Lagrangian having exact $SU(2)\times U(1)$ symmetry. We know, that $W^{\pm}$ interact only with left-handed components of leptons, whereas a photon interacts with both components. Therefore, a minimal set of spin one-half fields consists of a doublet of left spinors and a right electron, which is singlet in respect to weak $SU(2)$.

$$\Psi_L = \frac{1+\gamma_5}{2}\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \Psi_R = \frac{1-\gamma_5}{2}e. \tag{2.56}$$

The spinor containing part of the Lagragian has the following form

$$\begin{aligned} Ł_{\psi} = {} & \frac{\imath}{2}(\bar{\Psi}_L\gamma_\alpha D_\alpha\Psi_L - D_\alpha\bar{\Psi}_L\gamma_\alpha\Psi_L) \\ & + \frac{\imath}{2}(\bar{\Psi}_R\gamma_\alpha D_\alpha\Psi_R - D_\alpha\bar{\Psi}_R\gamma_\alpha\Psi_R) + g_e(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L), \end{aligned} \tag{2.57}$$

where

$$D_\alpha\Psi_L = \partial_\alpha\Psi_L + \frac{\imath g\tau_b}{2}W^b_\alpha\Psi_L - \imath g_1 B_\alpha\Psi_L, \quad D_\alpha\Psi_R = \partial_\alpha\Psi_R - \imath g_2 B_\alpha\Psi_R, \tag{2.58}$$

and corresponding expressions for Dirac conjugated spinors. Here as usually the gauge interaction of a Yang-Mills triplet with a doublet is defined by the same gauge constant g, which enters equations (2.44, 2.45). In respect to $U(1)$ symmetry Ψ_L belongs to representation $(1/2, -2g_1/g')$ and Ψ_L to $(0, -2g_2/g')$ one. We will soon explicate values of hyper-charges. Note, that the right singlet does not interact with W field. It as easy to understand the fact, because Ψ_R has weak isospin zero, so adding two zero isospin state one never obtains isospin one of W. The last term in equation (2.57) is very important, because it provides an electron with a mass. Really, substituting equation (2.47) with account of (2.56) into this term we have

$$\Delta L_\psi = \frac{g_e \eta}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L) + \frac{g_e \sigma}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L) + \frac{\imath g_e \xi}{\sqrt{2}} (\bar{e}_L e_R - \bar{e}_R e_L), \tag{2.59}$$

The first term of this relation is just describing an electron mass, namely

$$m_e = \frac{g_e \eta}{\sqrt{2}} = \frac{\sqrt{2}\, g_e M_W}{g}, \tag{2.60}$$

where we have used equation (2.51) for the W mass. Hence, the symmetry breaking leads at the same time to appearing of a mass of the electron (and of other spin one-half particles). The second term in equation (2.59) is describing the interaction of the Higgs boson σ with an electron (and similar terms with other spinors). We come thus to a conclusion, that an interaction constant of Higgs boson with a spinor particle a is proportional to its mass

$$g_a = \frac{g\, m_a}{\sqrt{2}\, M_W}. \tag{2.61}$$

Let us return to the interaction with gauge bosons. Substituting equation (2.58) into equation (2.57) we have the following expression for the interaction Lagrangian

$$L_{int} = -\frac{g}{2} \bar{\Psi}_L \tau^b W^b_\mu \gamma_\mu \Psi_L + g_1 \bar{\Psi}_L B_\mu \gamma_\mu \Psi_L + g_2 \bar{\Psi}_R B_\mu \gamma_\mu \Psi_R. \tag{2.62}$$

First of all we demand the Lagrangian (2.62) to describe correctly the electromagnetic interaction. Using equation (2.42) we extract the photon interaction

$$\left(\frac{g}{2} \bar{e}_L \gamma_\alpha e_L \sin\theta_W + (g_1 \bar{e}_L \gamma_\alpha e_L + g_2 \bar{e}_R \gamma_\alpha e_R) \cos\theta_W\right) A_\alpha = \mathrm{e}\, (\bar{e}_L \gamma_\alpha e_L + \bar{e}_R \gamma_\alpha e_R) A_\alpha. \tag{2.63}$$

The right hand side of the expression describes the well-known electromagnetic interaction (1.36). Demanding independent terms in equation (2.63) be equal in both sides, we obtain the following conditions

$$\begin{gathered} \frac{g}{2} \sin\theta_W + g_1 \cos\theta_W = \mathrm{e}, \quad g_2 \cos\theta_W = \mathrm{e}, \\ -\frac{g}{2} \sin\theta_W + g_1 \cos\theta_W = 0. \end{gathered} \tag{2.64}$$

Solving the set of equations we obtain finally

$$g = \frac{\mathrm{e}}{\sin\theta_W}, \quad g_1 = \frac{\mathrm{e}}{2 \cos\theta_W}, \quad g_2 = \frac{\mathrm{e}}{\cos\theta_W}, \quad g' = g_2. \tag{2.65}$$

We encounter already the first relation and the last one gives values of weak hypercharges of the left and the right spinors respectively

$$\Psi_L \to \left(\frac{1}{2}, -1\right), \quad \Psi_R \to (0, -2). \tag{2.66}$$

Of course, we can also express g_i in terms of g

$$g_2 = 2\,g_1 = g\ \tan\theta_W. \tag{2.67}$$

Now from equation (2.62) we have interaction terms for W and Z

$$\Delta L_{int} = -\frac{g}{2\sqrt{2}}\left(\bar{e}\,\gamma_\mu(1+\gamma_5)\nu_e\,W_\mu + \bar{\nu}_e\,\gamma_\mu(1+\gamma_5)e\,W_\mu^\dagger\right) - \frac{g\,Z_\mu}{2\,\cos\theta_W}$$
$$\times\left(\bar{\nu}_e\,\gamma_\mu\frac{(1+\gamma_5)}{2}\nu_e + \bar{e}\,\gamma_\mu\left(-(1-2\sin^2\theta_W)\frac{(1+\gamma_5)}{2} + 2\,\sin^2\theta_W\frac{(1-\gamma_5)}{2}\right)e\right). \tag{2.68}$$

The first two terms coincide with expression (2.21) with current (2.1). Then we have the following relation

$$\frac{g}{2\sqrt{2}} = g_W. \tag{2.69}$$

This relation allows us to connect W mass with other parameters. Indeed, from (2.22) and (2.69) we have

$$M_W^2 = \frac{g^2\,\sqrt{2}}{8\,G_F} = \frac{e^2\,\sqrt{2}}{\sin^2\theta_W\,8\,G_F} = \frac{\pi\,\alpha}{\sqrt{2}\,\sin^2\theta_W\,G_F}, \tag{2.70}$$

where α is the fine structure constant (1.38). Here α, G_F and $\sin\theta_W$ were known from experiments before W discovery, so expression(2.70) served for prediction of W mass. Using equations (2.51, 2.53), we have

$$M_Z = \frac{M_W}{|\cos\theta_W|}, \tag{2.71}$$

The term in brackets by the Z_μ in equation (2.69) is the weak neutral current. It can be represented in the following form

$$J_\mu^0 = \frac{1}{2}\,\bar{\nu}_e\,\gamma_\mu(1+\gamma_5)\nu_e - \frac{1}{2}\,\bar{e}\,\gamma_\mu(1+\gamma_5)e + 2\ \sin^2\theta_W\,\bar{e}\gamma_\mu\,e = I_\mu^{3(V-A)} + 2\ \sin^2\theta_W\,J_\mu^{em}. \tag{2.72}$$

Now we can include in the theory also other elementary leptons and quarks of the Table 1.1. The expression for charge current J_ρ is already presented in equations (2.1, 2.3). As a matter of fact this means, that in addition to doublet (2.56), we introduce the following lepton and quark doublets

$$\Psi_{\mu L} = \frac{1+\gamma_5}{2}\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix},\quad \Psi_{\tau L} = \frac{1+\gamma_5}{2}\begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix},$$
$$\Psi_{UL} = \frac{1+\gamma_5}{2}\begin{pmatrix}u\\ d'\end{pmatrix},\quad \Psi_{CL} = \frac{1+\gamma_5}{2}\begin{pmatrix}c\\ s'\end{pmatrix}, \tag{2.73}$$
$$\Psi_{TL} = \frac{1+\gamma_5}{2}\begin{pmatrix}t\\ b'\end{pmatrix}.$$

All the lepton doublets are $(1/2,-1)$ and all the quark doublets are $(1/2,1/3)$ in respect to $SU(2)\times U(1)$. The prime at lower components of doublets is explicated by

Cabibbo–Kobayashi–Maskawa matrix (2.3). We are to have also right components for charged lepton and for all quarks. So we have the following set of right spinors in addition to (2.56)

$$
\begin{aligned}
&\Psi_{\mu R} = \frac{1-\gamma_5}{2}\,\mu, \quad \Psi_{\tau R} = \frac{1-\gamma_5}{2}\,\tau,\\
&\Psi_{UR} = \frac{1-\gamma_5}{2}\,u, \quad \Psi_{CR} = \frac{1-\gamma_5}{2}\,c, \quad \Psi_{TR} = \frac{1-\gamma_5}{2}\,t,\\
&\Psi_{DR} = \frac{1-\gamma_5}{2}\,d, \quad \Psi_{SR} = \frac{1-\gamma_5}{2}\,s, \quad \Psi_{BR} = \frac{1-\gamma_5}{2}\,b.
\end{aligned}
\tag{2.74}
$$

Here all leptons (the first line) are (0,-2), up quarks (the second line) are (0,4/3) and down quarks (the third line) are (0, -2/3).

These newly introduced objects interact with gauge bosons and scalars in the same way, as the electron and its neutrino, so that we have electro-weak interaction Lagrangian of W, Z, A with leptons and quarks in the following form:

$$
\begin{aligned}
L_{int} &= -\frac{g}{2\sqrt{2}}\,(J_\rho\, W_\rho + J_\rho^\dagger\, W_\rho^\dagger) - \frac{g}{2\cos\theta_W}\,J_\rho^0\, Z_\rho + e\,J_\rho^{em}\,A_\rho,\\
J_\rho^0 &= \frac{1}{2}\Big(\bar{\nu}_e\gamma_\rho(1+\gamma_5)\nu_e + \bar{\nu}_\mu\gamma_\rho(1+\gamma_5)\nu_\mu + \bar{\nu}_\tau\gamma_\rho(1+\gamma_5)\nu_\tau\\
&\quad + \bar{u}\gamma_\rho(1+\gamma_5)u + \bar{c}\gamma_\rho(1+\gamma_5)c + \bar{t}\gamma_\rho(1+\gamma_5)t\Big)\\
&\quad - \frac{1}{2}\Big(\bar{e}\gamma_\rho(1+\gamma_5)e + \bar{\mu}\gamma_\rho(1+\gamma_5)\mu + \bar{\tau}\gamma_\rho(1+\gamma_5)\tau\\
&\quad + \bar{d}\gamma_\rho(1+\gamma_5)d + \bar{s}\gamma_\rho(1+\gamma_5)s + \bar{b}\gamma_\rho(1+\gamma_5)b\Big) + 2\sin^2\theta_W\, J_\rho^{em},\\
J_\rho^{em} &= \bar{e}\gamma_\rho e + \bar{\mu}\gamma_\rho\mu + \bar{\tau}\gamma_\rho\tau - \frac{2}{3}\,(\bar{u}\gamma_\rho u + \bar{c}\gamma_\rho c + \bar{t}\gamma_\rho t) + \frac{1}{3}\,(\bar{d}\gamma_\rho d + \bar{s}\gamma_\rho s + \bar{b}\gamma_\rho b).
\end{aligned}
\tag{2.75}
$$

2.1.1 Feynman rules for the electro-weak interaction

Propagators for intermediate bosons, scalars ϕ and ghosts ω -n ξ-gauge:

$$
\begin{aligned}
W^\pm &: \frac{-\imath}{(2\pi)^4(k^2 - M_W^2 + \imath\epsilon)}\left(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2 - \xi M_W^2}\right),\\
Z &: \frac{-\imath}{(2\pi)^4(k^2 - M_Z^2 + \imath\epsilon)}\left(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2 - \xi M_Z^2}\right),\\
\phi^\pm &: \frac{\imath}{(2\pi)^4(k^2 - \xi M_W^2 + \imath\epsilon)}\\
\phi_2 &: \frac{\imath}{(2\pi)^4(k^2 - \xi M_Z^2 + \imath\epsilon)}\\
\phi_1 \equiv H &: \frac{\imath}{(2\pi)^4(k^2 - M_H^2 + \imath\epsilon)}\\
\omega^\pm &: \frac{-\imath}{(2\pi)^4(k^2 - \xi M_W^2 + \imath\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
\omega_Z &: \frac{-\imath}{(2\pi)^4(k^2 - \xi M_Z^2 + \imath\epsilon)} \\
\omega_\gamma &: \frac{-\imath}{(2\pi)^4(k^2 + \imath\epsilon)}
\end{aligned}
\tag{2.76}
$$

Here $\xi = 1$ corresponds to the Feynman gauge, $\xi = 0$ corresponds to the Landau gauge and the limit for $\xi \to \infty$ gives the unitary gauge without ghosts. Note that from here we denote the surviving scalar field ϕ_1 by H and name it the Standard Model Higgs scalar particle (Higgs boson).

Vertices for the boson interactions are the following, where incoming momenta are correspondingly presented in brackets

$$
\begin{aligned}
A_\mu\, W_\nu^+, W_\rho^-(k;\, p;\, q) &: (2\pi)^4\, e\,(g_{\mu\nu}(k_\rho - p_\rho) + g_{\nu\rho}(p_\mu - q_\mu) + g_{\rho\mu}(q_\nu - k_\nu)), \\
Z_\mu\, W_\nu^+ W_\rho^-(k;\, p,\, q) &: (2\pi)^4\, g\cos\theta_W(g_{\mu\nu}(k_\rho - p_\rho) + g_{\nu\rho}(p_\mu - q_\mu) + g_{\rho\mu}(q_\nu - k_\nu)), \\
W_\mu^a W_\nu^b W_\rho^c W_\sigma^d &: ig^2\,(2\pi)^4(\epsilon^{abn}\epsilon^{ncd}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + \epsilon^{acn}\epsilon^{nbd}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}) \\
&\quad + \epsilon^{adn}\epsilon^{nbc}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma})), \\
W_\mu^+ \bar{u} d &: (2\pi)^4 \frac{\imath g}{2\sqrt{2}}\gamma_\mu(1 + \gamma_5), \\
W_\mu^- \bar{d} u &: (2\pi)^4 \frac{\imath g}{2\sqrt{2}}\gamma_\mu(1 + \gamma_5), \\
Z_\mu \bar{u} u &: (2\pi)^4 \frac{\imath g}{4\cos\theta_W}\gamma_\mu\left((1 + \gamma_5) - \frac{8}{3}\sin^2\theta_W\right), \\
Z_\mu \bar{d} d &: (2\pi)^4 \frac{\imath g}{4\cos\theta_W}\gamma_\mu\left(-(1 + \gamma_5) + \frac{4}{3}\sin^2\theta_W\right), \\
Z_\mu \bar{\nu}_i \nu_i &: (2\pi)^4 \frac{\imath g}{4\cos\theta_W}\gamma_\mu(1 + \gamma_5), \\
Z_\mu \bar{l}_i l_i &: (2\pi)^4 \frac{\imath g}{4\cos\theta_W}\gamma_\mu(-(1 + \gamma_5) + 4\sin^2\theta_W), \\
HW_\mu^+ W_\nu^- &= \imath\,(2\pi)^4\, g\, M_W\, g_{\mu\nu}, \\
HZ_\mu Z_\nu &= \imath\,(2\pi)^4\, \frac{g\, M_W}{\cos\theta_W}\, g_{\mu\nu}, \\
H\Psi_j\Psi_j &= \imath\,(2\pi)^4\, \frac{g\, M_j}{\sqrt{2}\, M_W},
\end{aligned}
\tag{2.77}
$$

Now we can proceed to applications. At first we demonstrate simple tools to deal with W and Z decays in the lowest approximation, corresponding to the so called tree diagrams. In numerical estimates we use here values (the precise values are present in corresponding tables)

$$
\begin{aligned}
G_F &= 1.166\,\mathrm{GeV}^{-2}, \quad & \sin^2\theta_W &= 0.23, \\
M_W &= 80.4\,\mathrm{GeV}, & M_Z &= 91.2\,\mathrm{GeV}.
\end{aligned}
\tag{2.78}
$$

Let us start with W decays, which are described by vertices (2.77). Calculation of a decay into a lepton pair gives the following partial width provided we neglect lepton masses

$$\Gamma(W^+ \to l^+ \nu_l) = \frac{g^2 M_W}{48\pi} = \frac{G_F M_W^3}{6\sqrt{2}\pi} \simeq 227\,\text{MeV}. \tag{2.79}$$

The same expression multiplied by 3 (the number of colors) is describing the decay $W \to u\bar{d}$. We have three lepton channels, according to e, μ, τ and two quark channels $u\bar{d}$, $c\bar{s}$. So to obtain the total width one has to multiply (2.79) by $3 + 2 \cdot 3$, i. e.,

$$\Gamma_t(W) = \frac{9\, G_F M_W^3}{6\sqrt{2}\pi} = 9 \cdot 227\,\text{MeV} = 2.043\,\text{GeV}. \tag{2.80}$$

Branching ratios for different decay modes follow form the above and are 1/9 for each lepton mode and 1/3 for each quark mode. these simple results are valid with a considerable accuracy.

To deal with Z decays, we have to note, that (2.79) describes a partial width no matter which sign stands in brackets $(1 \pm \gamma_5)$. Contributions of left and right parts do not interfere in the approximation of zero spinor masses due to orthogonality property (2.16), so we have for a partial width

$$\Gamma(Z \to \Psi\bar{\Psi}) = \frac{G_F M_Z^3}{12\sqrt{2}\pi}(g_L^2 + g_R^2), \tag{2.81}$$

where $g_{L,R}$ are coefficients afore $(1 \pm \gamma_5)$ respectively in brackets of expressions (2.77) for a corresponding vertex. We calculate in this way the total width and obtain

$$\Gamma_t(Z) = \frac{G_F M_Z^3}{12\sqrt{2}\pi}\left(21 - 40s + \frac{160s^2}{3}\right) \simeq 2.43\,\text{GeV}.$$
$$s = \sin^2\theta_W. \tag{2.82}$$

Again simple estimate gives a satisfactory value ($\Gamma_{exp} = 2.4952 \pm 0.0023\,\text{GeV}$). It easy to obtain also branching ratios for different decay modes. For instance. invisible neutrino modes have each $G_L = 1$, $g_R = 0$, so for three neutrinos we have

$$\frac{\Gamma_{invis}}{\Gamma_{exp}} = \frac{3}{21 - 40s + \frac{160s^2}{3}} = 0.1994, \tag{2.83}$$

to be compared with the measured value 0.2000 ± 0.0008, that as is well-known, fix the number of light neutrinos to be just three.

Thus simple tree-level calculations give satisfactory agreement with data. However, the lowest order of perturbation theory may be the same in different physical theories. So, a real proof of of the validity of a theory can be achieved only after measurement of radiative corrections. Such was the history of a justification of QED. Now precision of experiments on electro-weak effects allows to measure also radiative corrections. The totality of data gives now precision tests of the Standard Model, the part of data are presented in Table 2.4. For proper comparison one have to take into account the the electro-weak corrections as well as the QCD corrections.

Let us discuss briefly the QCD corrections. First of all we have to take into account the fact, that all quantities, connected with color objects become running with Q^2. That is, we have running strong coupling constant $\alpha_s(Q^2)$ and running quark masses $m_q(Q^2)$. Secondly, all probabilities of quark – antiquark decay channels have to be multiplied by the famous factor

$$R_s = 1 + \frac{\alpha_s(M_Z^2)}{\pi} \tag{2.84}$$

This factor has to be introduced, e. g., in calculations of Z width. We have instead of (2.82) with account of (2.84) and $\alpha_S(M_Z) = 0.119$

$$\Gamma_t(Z) = \frac{G_F M_Z^3}{12\sqrt{2}\,\pi}\left(6 - 12s + 24s^2 + R_s\left(15 - 28s + \frac{88\,s^2}{3}\right)\right) = 2.494\,\text{GeV}, \tag{2.85}$$

that remarkably improves agreement with the experimental number. In the similar way other radiative corrections are calculated with necessary precision that allows to perform estimation of agreement of the theory with the experiment. The main points of the comparison are presented in Table 2.4. The notations there are the following:

hadronic peak cross-section at $\sqrt{s} = M_z$	σ_{had}
partial leptonic and hadronic widths	$\Gamma_\nu, \Gamma_e, \Gamma_\mu, \Gamma_\tau, \Gamma_u, \Gamma_d, \Gamma_s, \Gamma_c, \Gamma_b$
the total hadronic width	$\Gamma_h = \Gamma_u + \Gamma_d + \Gamma_s + \Gamma_c + \Gamma_b$
ratios	R_l, R_b, R_c
asymmetries	A_{FB}^b, A_{FB}^c

The notations here needs explications, which follows below:

$$\begin{gathered}\Gamma_{invis} = \Gamma_t(Z) - \Gamma_e - \Gamma_\mu - \Gamma_\tau - \Gamma_h, \\ R_l = \frac{\Gamma_h}{\Gamma_l}, \quad R_{b,c} = \frac{\Gamma_{b,c}}{\Gamma_h}, \quad \sigma_h = \frac{12\pi\Gamma_e\Gamma_h}{\Gamma_Z M_Z^2},\end{gathered} \tag{2.86}$$

and asymmetries are defined according to the following relation:

$$A_{FB}^b = \frac{\sigma(e^-e^+ \to b\bar{b})_F - \sigma(e^-e^+ \to b\bar{b})_B}{\sigma(e^-e^+ \to b\bar{b})_F + \sigma(e^-e^+ \to b\bar{b})_B}, \tag{2.87}$$

where σ_F corresponds to the b-quark flying inside the forward hemisphere in direction of the electron momentum.

2.1.2 Higgs scalar search

The existence of scalar Higgs particle is one of the main keystones of the electro-weak theory. Let us summarize properties of the particle. Remind, that its coupling are proportional to masses of particles. With M_H increasing its interaction becomes stronger.

For M_H in the range 114–130 GeV the dominant decay channel of the Higgs boson is $H \to \bar{b}\, b$. In the first approximation the partial width of a decay to a fermion pair is the following

$$\Gamma(H \to \bar{f} f) = \frac{N_C\, G_F}{4\,\pi\,\sqrt{2}} M_H\, m_f^2 \left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}}. \tag{2.88}$$

The channel of Higgs decay to vector boson pair corresponds to the following partial width

$$\Gamma(H \to VV) = \frac{G_F\, M_H^3}{16\,\pi\,\sqrt{2}} \delta_V \sqrt{1-4x}\,(1 - 4x + 12x^2), \quad x = \frac{M_V^2}{M_H^2}, \tag{2.89}$$

where $\delta_W = 2$, $\delta_Z = 1$.

The decay channels to $(\gamma\gamma)$, $(Z\gamma)$, $(gluon\, gluon)$ proceed via loop diagrams and the resulting widths are written down below

$$\Gamma(H \to \gamma\,\gamma) = \frac{G_F\, \alpha^2 M_H^3}{128\,\pi^3\,\sqrt{2}} |A_W(\tau_W) + \sum_f N_C\, Q_f^2\, A_f(\tau_f)|^2, \tag{2.90}$$

$$\Gamma(H \to g\,g) = \frac{G_F\, \alpha_s^2\, M_H^3}{64\,\pi^3\,\sqrt{2}} |\sum_{f=Q} A_f(\tau_f)|^2, \tag{2.91}$$

where $\tau_i = M_H^2/(4\, m_f^2)$ and

$$A_W(\tau) = -\frac{2\tau^2 + 3\tau + 3\,f(\tau)(2\tau - 1))}{\tau^2},$$
$$A_f(\tau) = \frac{2(\tau + f(\tau)(\tau - 1))}{\tau^2}, \tag{2.92}$$

where

$$f(\tau) = -\ \arcsin^2 \sqrt{\tau} \quad for \quad \tau \le 1,$$
$$f(\tau) = -\frac{1}{4}\,|\ln \frac{\tau + \sqrt{\tau - 1}}{\tau - \sqrt{\tau - 1}} - \imath\pi|^2 \quad for \quad \tau > 1. \tag{2.93}$$

For $H \to \gamma\,\gamma$ and for $H \to Z\,\gamma$ the W loop provides dominant contribution for not very large M_H.

The intensive search for the Higgs boson is performing at the LHC and recently it results in discovery [57, 58] of a resonance at mass $M = 125.7 \pm 0.6\, GeV$ with properties consistent with those predicted for the Higgs particle. We shall return to discussion of this discovery in the subsequent chapters.

2.2 Status of the standard model

We consider the Standard Model to be the reliable basis for description of elementary particles physics.

Let us present the main experimental data in comparison to the SM calculations according to [59] in Table 2.4. We have to compare the second column, representing experimental results, with the rest three, which present results of the overall fit in the framework of the electroweak theory. We see, that the agreement is satisfactory while there are only two points, in which there are discrepancies slightly more than 2 standard deviations. The are the forward-backward asymmetry $A_{FB}^{0,b}$ in decay $Z \to \bar{b}b$ and the relative probability of decay $Z \to \bar{b}b$. Nevertheless the overall fit is in good form, so we may conclude on behalf of the agreement of the theory and the experiment.

We have also to bear in mind, that the totality of experimental data on hadronic reactions at high energies including the recent data from the LHC experiments agrees the QCD predictions. In doing this comparison the behavior of the running coupling (1.69) with parameters (1.70) is used. The value of $\alpha_s(M_Z)$ quoted in (1.71) is taken just from data, presented in Table 2.4.

There is only one suspicious point in all the totality of data. This is the anomalous magnetic moment of the muon. We know, that a charged spin one-half particle has Dirac magnetic moment

$$D_m^0 = \frac{\mathrm{e}}{2\,m}$$

The radiative corrections give additional contributions $D_m = D_m^0(1 + a_m)$ to this quantity and according to measurements of this contribution to the electron [12] and the muon [60] we have

$$\begin{aligned} a_e^{exp} &= 11596521807.3(2.8) \cdot 10^{-13}, \\ a_\mu^{exp} &= 11659208.9(6.3) \cdot 10^{-10}. \end{aligned} \tag{2.94}$$

The first number for electron agrees calculations in the framework of QED with account of the electroweak and hadronic contributions. However the second number deviates from such calculations, which give the following contributions to a_μ

$$\begin{aligned} a_\mu^{QED} &= (11658418.853 \pm 0.037) \cdot 10^{-11}, \\ a_\mu^{EW} &= (154 \pm 2.0) \cdot 10^{-11}, \quad a_\mu^{hadron} = (6967 \pm 59) \cdot 10^{-11}. \\ a_\mu^{theor} &= a_\mu^{QED} + a_\mu^{EW} + a_\mu^{hadron}. \end{aligned} \tag{2.95}$$

Thus we have the following discrepancy with significance $2.9\,\sigma$

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{theor} = (249 \pm 87) \cdot 10^{-11}. \tag{2.96}$$

The other method of processing of data [61–63] gives rather larger significance of the effect

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{theor} = (349.3 \pm 82.3) \cdot 10^{-11}. \tag{2.97}$$

This would-be discrepancy may cause serious trouble for the perturbative electroweak theory. We shall consider possible nonperturbative contributions to a_μ below.

Table 2.4. Input values and fit results for the observable and parameters of the global electroweak fit. The first and the second columns list respectively the parameters used in the fit and their experimental values. In the third column the fit results are given

Parameter	Experiment	Fit result
M_H [GeV]	125.7 ± 0.4	94^{+25}_{-22}
M_W[GeV]	80.385 ± 0.015	80.377
M_Z[GeV]	91.1875 ± 0.0021	91.1874
σ^0_{had} [nb]	41.540 ± 0.037	41.478
R^0_l	20.767 ± 0.025	20.7142
$A^{0,l}_{FB}$	0.0171 ± 0.0010	0.01645
$\sin^2\theta^l_{eff}$	0.2324 ± 0.0012	0.2314
A_c	0.670 ± 0.027	0.668
A_b	0.923 ± 0.020	0.935
$A^{0,c}_{FB}$	0.0707 ± 0.0035	0.0742
$A^{0,b}_{FB}$	0.0992 ± 0.0016	0.1038
R^0_c	0.1721 ± 0.0030	0.1723
R^0_b	0.21629 ± 0.00066	0.21579
$\alpha_s(M_Z)$	—	0.1191 ± 0.0028

We present in Table 2.4 the comparison of experimental values of several parameters with the corresponding calculated values being obtained from the overall fit.

We see from Table 2.4 possible discrepancy in value of forward-backward asymmetry $A^{0,b}_{FB}$ in process $e^+\,e^- \to \bar{b}\,b$ at energy $\sqrt{s} = M_Z$

$$\Delta A^{0,b}_{FB} = \frac{A^{0,b}_{FB\,exp} - A^{0,b}_{FB\,th}}{A^{0,b}_{FB\,th}} = -0.044 \pm 0.016. \tag{2.98}$$

It is 2.8 s.d. effect, so the statistical significance in rather poor, however one may bear this problem in mind also.

There is also one very interesting problem, which speaking formally contradicts the Standard Model. We mean the problem of the neutrino oscillations.

In the electroweak Lagrangian (2.75) we assume neutrino masses to be zero. However, there is no convincing argument for this prescription, contrary to cases of gauge vector bosons (the photon and the gluons). In case masses of neutrinos are not strictly zero, they may be mixed one with the other analogously the case of lower quarks (d, s, b) mixing. This mixing is described by Cabibbo–Kobayashi–Maskawa matrix (2.3). Let us for simplicity consider case of two different neutrino states ν_e and ν_μ. Let us suppose, that states with definite masses are not these electron and muon neutrinos, but states, which we designate ν_1 and ν_2. Then in general we have the following relations

$$\nu_e = \cos\phi\,\nu_1 + \sin\phi\,\nu_2, \quad \nu_\mu = -\sin\phi\,\nu_1 + \cos\phi\,\nu_2, \tag{2.99}$$

where ϕ is a mixing angle. Let us consider time dependence of state ν_e, having momentum $\mathbf{p}$

$$|\nu_e(t)>= e^{-\imath E_1 t}\cos\phi\,|\nu_1> + e^{-\imath E_2 t}\sin\phi\,|\nu_2>, \tag{2.100}$$

where

$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2}, \quad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}. \tag{2.101}$$

Now by writing in expression $\nu_{1,2}$ in terms of ν_e, ν_μ according to (2.99, we have

$$|\nu_e(t)>= (e^{-\imath E_1 t}\cos^2\phi + e^{-\imath E_2 t}\sin^2\phi)\,|\nu_e> + \cos\phi\,\sin\phi\,(e^{-\imath E_2 t} - e^{-\imath E_1 t})|\nu_\mu>. \tag{2.102}$$

Now the probability to detect the muon neutrino ν_μ if for $t = 0$ we have pure neutrino electron state is the following

$$W(\nu_e \to \nu_\mu) = \frac{1}{2}\sin^2 2\phi\,\left(1 - \cos(E_1 - E_2)t\right). \tag{2.103}$$

Substituting relations (2.101) into expression (2.103) and using distance x instead of time t we have the following result for probability of $\nu_e \to \nu_\mu$ transition

$$\begin{aligned} W(\nu_e \to \nu_\mu) &= \frac{1}{2}\sin^2 2\phi\left(1 - \cos\frac{\Delta m^2 x}{2\mathbf{p}}\right), \\ &= \sin^2 2\phi\,\sin^2\frac{\Delta m^2 x}{4\mathbf{p}}, \quad \Delta m^2 = m_2^2 - m_1^2. \end{aligned} \tag{2.104}$$

The probability for ν_e to remain itself is evidently the following

$$W(\nu_e \to \nu_e) = 1 - \sin^2 2\phi\,\sin^2\frac{\Delta m^2 x}{4\mathbf{p}}. \tag{2.105}$$

The phenomenon of neutrino oscillations was predicted by B. Pontecorvo [64] and discovered first in atmospheric neutrino studies [65].

We have in interaction Lagrangian (2.75) three neutrinos. Thus for complete description of oscillations we have to use all three neutrino mixing (see, e. g. [66]), which is described by 3×3 matrix in the commonly adopted notations

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau' \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & e^{-\imath\delta}s_{13} \\ -s_{12}c_{23} - e^{\imath\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{\imath\delta}s_{12}s_{13}s_{23} & c_{13}s_{23} \\ -e^{\imath\delta}s_{13}c_{12}c_{23} + s_{12}s_{23} & -e^{\imath\delta}s_{12}s_{13}c_{23} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2.106}$$

Here, as well as in expression (2.3), $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

After the discovery [65] the phenomenon was extensively studied and nowadays we have considerable information on the parameters of neutrino oscillations, which may be found in [4]. Values of differences of the mass squared $\Delta\, m^2$ are of the most interest for the forthcoming discussion in the book. The available data are the following

$$
\begin{gathered}
\sin^2 2\theta_{12} = 0.857 \pm 0.024, \quad \sin^2 2\theta_{23} > 0.95, \\
\sin^2 2\theta_{13} = 0.095 \pm 0.010, \\
\Delta m^2_{12} = (7.50 \pm 0.20) \cdot 10^{-5}\,\mathrm{eV}^2, \\
\Delta m^2_{23} = 0.00232^{+0.00012}_{-0.00008}\,\mathrm{eV}^2.
\end{gathered} \tag{2.107}
$$

Strictly speaking nonzero masses of neutrinos and their mixing do not contradict to the ideology of the electroweak interaction.

In any case the Standard Model seems to be in good form now. The only possible deviations may be due to nonperturbative effects of the electroweak interaction. We shall return to these considerations in the forthcoming chapters.

However, there are problems of hierarchy and of naturalness. The hierarchy problem is connected with the danger of an instability of the Standard Model in relation to quantum corrections [67, 68]. This is connected with existence of the fundamental scalar field of the Higgs boson. Even the second order correction to the scalar field mass is quadratic divergent. We have already mentioned in Section 1.3, that natural scale of effective cut-off may be connected with Planck mass (1.44). But then with interaction of the Higgs scalar defined by rules (2.77) with other fields we see, that the largest contribution is given by the t-quark loop with $H\,\bar{t}t$ vertex

$$
\imath\,(2\pi)^4\,\frac{g\,M_t}{\sqrt{2}\,M_W}\,H\,\bar{\Psi}_t\,\Psi_t. \tag{2.108}
$$

One loop diagram gives the following contribution to $(\Delta M_H)^2$ after the Wick rotation

$$
(\Delta M_H)^2 = \frac{3\,g^2\,M_t^2}{8\,\pi^2\,M_W^2}\int_0^{\Lambda^2}\frac{y(y - M_t^2)}{(y + M_t^2)^2}dy \simeq \frac{3\,g^2\,M_t^2}{8\,\pi^2\,M_W^2}\,\Lambda^2 = 0.273\,\Lambda^2. \tag{2.109}
$$

where we take physical value $g = 0.65$ and masses from Tables 1.2 and 2.4. With typical scale of the electroweak interaction

$$
\eta = \frac{2\,M_W}{g} = 247.221 \pm 0.002\,\mathrm{GeV}, \tag{2.110}
$$

we obtain the corresponding value of Λ

$$
\Lambda = \eta\,\frac{2\pi M_W}{g M_t}\sqrt{\frac{2}{3}} \simeq 0.91\,\mathrm{TeV}. \tag{2.111}
$$

Thus we are to have the effective cut-off somewhat less than TeV. That means that on this level some New Physics have to manifest itself. However, all experiments devoted to searches for the New Physics give negative results at this scale. Thus the problem of hierarchy may cause troubles for good form of the Standard Model.

There is also important problem, which consists in the question, what is the principle, which defines values of numerous parameters entering the Standard Model. Indeed, we have two gauge couplings and the Weinberg mixing angle, which are defined, e. g., by the following relations

$$\alpha_s(M_z) = 0.116 \pm 0.001, \quad \frac{1}{\alpha(M_z)} = 127.916 \pm 0.015,$$
$$\sin^2\theta_W = 0.23108 \pm 0.00005. \tag{2.112}$$

We have masses of 6 quarks, 3 charged leptons and the W boson mass, which are presented in tables, and the Higgs boson mass, which we have, discussed above. Note, that the W boson mass with use of parameters (2.112) defines also the Z boson mass and vacuum average η (2.110). We have the Cabibbo–Kobayashi–Maskawa matrix (2.3) with four independent parameters. In addition it is necessary to have yet unknown masses of three neutrinos and four parameters of the neutrino mixing matrix (2.106). Thus we have at least 25 independent parameters, which defines the Standard Model in the framework of the perturbation theory, and no leading reasons yet on how to understand their experimental values. This problem represents special interest for interpretation of results to be considered in the book. It is also of considerable interest the problem of how to calculate nonperturbative parameters, such as condensates (1.83). We shall as well deal with these problems.

2.3 Properties of nonrenormalizable equations, instructive example

All the theories consisting the Standard Model but the gravitation belongs to the class of renormalizable interactions. However, the theories, which often used as effective theories are formally nonrenormalizable, while as a rule it is assumed, that such theories act in a restricted region of the momentum space. This could be achieved either by introduction of a cut-off, or by using nonlocal variants, which as a matter if fact lead to severe difficulties with maintaining of the unitarity and the locality. However for a long time a phenomenon of arising of self-consistent damping in nonrenormalizable theories is known. Let us consider an example of this phenomenon [69], which for our presentation is quite appropriate, because here we develop a method, which shall be used throughout all the book.

Let us take a theory of interaction of massive vector fields $A_\mu^{(k)}$ with massless scalar fields $\phi^{(i)}$, which is invariant in respect to transformation of some group, and the multiplet of vector fields corresponds to adjoint representation of the group. Thus we take the interaction Lagrangian in the following form

$$L_{int} = \lambda\, c_{ijk}\, \phi^{(i)}\, \frac{\partial \phi^{(j)}}{\partial x^\mu}\, A_\mu^{(k)}. \tag{2.113}$$

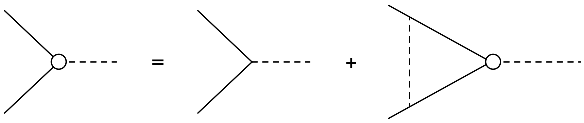


Fig. 2.2. Diagram representation of equation 2.115. Simple lines represent the scalar field and the dotted ones represent the vector field.

Her c_{ijk} are structure constant of the symmetry group. For example, for $SU(2)$ $c_{ijk} = \epsilon_{ijk}$, for $SU(3)$ $c_{ijk} = f_{ijk}$.

Let us consider the full vertex function of the interaction of the scalar field with the vector one. In doing this we use approximate equation for a vertex function, which graphically is shown in Figure 2.2. It is evident, that the approximation, which is used in this equation is equivalent to the well-known ladder approximation in Bethe–Salpeter equations. Let us introduce for the diagram in the left-hand side of the equation the following definition

$$(2\pi)^4 \lambda \Gamma_{(ijk)\,\mu} = (2\pi)^4 \lambda\, c_{(ijk)} \Gamma_\mu(p,k). \tag{2.114}$$

The equation for function $\Gamma_\mu(p,k)$ in the ladder approximation is the following

$$\Gamma_\mu(p,k) = 2\,p_\mu\,Z + \frac{a\,\lambda^2}{\imath\,(2\pi)^4} \int \frac{(p+q)^2 - k^2 - [(p^2-q^2)^2 - (k,p-q)^2]/m^2}{[(p-q)^2 - m^2](q-\frac{k}{2})^2(q+\frac{k}{2})^2}\,\Gamma_\mu(q,k)\,dq, \tag{2.115}$$

where m is the vector field mass and $a = 1$ for $SU(2)$ group, $a = 3/2$ for $SU(3)$ group.

The possible renormalization of the vertex needs introduction of factor Z before inhomogeneous term in the equation.

Let us consider equation (2.115) for $k_\mu = 0$. In this case we succeed in obtaining an exact solution of the equation. Vertex Γ_μ in this case has the following simple structure

$$\Gamma_\mu(p,0) = 2\,p_\mu\,F(p^2). \tag{2.116}$$

Let us perform Wick rotation in equation (2.115). This means the following substitute

$$dq \to \imath\,d^4q, \quad q^2,\,p^2,\,(pq) \to -q^2, -p^2, -(pq). \tag{2.117}$$

where now d^4q corresponds to integration in the four-dimensional Euclid space. In what follows we shall always use the Wick rotation (2.117) in studies of analogous equations.

Let us rewrite the equation in the following symbolic form

$$F = Z + I + K_0\,F + K'\,F, \tag{2.118}$$

where

$$I = \frac{a\lambda^2}{m^2 p^2} \int \frac{d^4q}{(2\pi)^4} \frac{2\,(pq)^2}{(q^2)^2} F(q^2), \tag{2.119}$$

$$K_0 F = \frac{a\lambda^2}{m^2 p^2} \int \frac{d^4q}{(2\pi)^4} \left[\frac{(p^2 - q^2)^2 (pq)}{(p-q)^2} - 2(pq)^2 \right] \frac{F(q^2)}{(q^2)^2},$$

$$K' F = \frac{4a\lambda^2}{p^2} \int \frac{d^4q}{(2\pi)^4} \frac{p^2 q^2 - (pq)^2}{(p-q)^2\,[(p-q)^2 + m^2]} \frac{F(q^2)}{(q^2)^2}.$$

Here the first line defines constant integral. Let us look for solution F in the form $F = F^0 + F'$, where the main part of the vertex function is defined from the following equation

$$F^0 = Z + I + K_0 F^0, \tag{2.120}$$

and F' is defined by

$$F' = K' F^0 + (K^0 + K') F'. \tag{2.121}$$

Here we study solution of equation (2.120). It is possible to show, that equation (2.121) may be solved by iterations and these iterations converge [70].

Let us consider equation (2.120). We use spherical coordinates of the four-dimensional Euclid space

$$\begin{gathered} q_0 = q\,\cos\theta, \quad q_1 = q\,\sin\theta\,\cos\phi, \\ q_2 = q\,\sin\theta\,\sin\phi\,\cos\psi, \quad q_2 = q\,\sin\theta\,\sin\phi\,\sin\psi, \\ d^4q = \frac{1}{2} q^2\,dq^2\,\sin^2\theta\,d\theta\,\sin\phi\,d\phi\,d\psi. \end{gathered} \tag{2.122}$$

We choose θ to be the angle between p and q, that is $(pq) = pq\cos\theta$. Then equation (2.120) can be rewritten in the following form (from here we omit zero in the superscript of F)

$$\begin{aligned} F(x) = {} & Z + I - \frac{2a\lambda^2}{m^2(2\pi)^3} \int_0^\infty dy F(y) \int_0^\pi \sin^2\theta \cos^2\theta\,d\theta \\ & + \frac{a\lambda^2}{m^2\,(2\pi)^3} \int_0^\infty y dy (x-y)^2 \frac{F(y)}{y^2} \int_0^\pi \frac{\sin^2\theta\,\sqrt{xy}\cos\theta\,d\theta}{x + y - 2\sqrt{xy}\cos\theta}, \end{aligned} \tag{2.123}$$

where $x = p^2$, $y = q^2$. Applying the simple, but very useful relation

$$\int_0^\pi \frac{\sin^2\theta\,d\theta}{x + y - 2\sqrt{xy}\cos\theta} = \frac{\pi}{2}\left[\frac{1}{x}\vartheta(x-y) + \frac{1}{y}\vartheta(y-x)\right], \tag{2.124}$$

where, as usually, $\vartheta(x) = 1$ for $x \geq 0$ and $\vartheta(x) = 0$ for $x < 0$, we obtain the following equation

$$F(x) = A + \frac{G}{12}\left[\frac{1}{x^2}\int_0^x y^2 dy\, F(y) - \frac{2}{x}\int_0^x y dy\, F(y) - 2\,x\int_x^\infty \frac{dy\, F(y)}{y} + x^2\int_x^\infty \frac{dy\, F(y)}{y^2}\right],$$

$$I = Z + \frac{G}{12}\int_0^\infty dy\, F(y), \quad G = \frac{3\,a\,\lambda^2}{8\,m^2\,\pi}. \tag{2.125}$$

Here we have introduced the following notation

$$A = Z + \frac{G}{12}\int_0^\infty dy\, F(y), \tag{2.126}$$

and demand renormalization constant Z to be chosen in such a way, that quantity A being finite. Finally constant (2.126) has to be defined by normalization condition

$$\Gamma_\mu(p,k) = 2\,p_\mu, \quad k^2 = -\,m^2, \quad \left(p \pm \frac{k}{2}\right)^2 = 0. \tag{2.127}$$

Let us proceed with equation (2.125). If one try to solve the equation by iterations, starting of zero approximation $F_0 = A$, already in the first approximation logarithmic divergent terms appear and in subsequent approximations appear power divergences with growing powers. This picture is appropriate to nonrenormalizable character of initial interaction (2.113). However, there exists the unique solution of the equation (2.125). The simplest way to show this is to reduce the equation (2.125) to a differential one by proper application of sequential differentiations. In such a way we are persuaded, that equation (2.125) leads to the following differential equation

$$\frac{d^2}{dx^2}\left[\frac{1}{x}\frac{d^2}{dx^2}(x^2\,F)\right] + \frac{G\,F}{x^2} = \frac{4\,A}{x^3}. \tag{2.128}$$

This equation reduces to the well-known Meijer equation [71], which in the standard form with substitution $z = G\,x$ is the following

$$\left[\left(z\frac{d}{dz} + 2\right)\left(z\frac{d}{dz} + 1\right)\left(z\frac{d}{dz} - 1\right)\left(z\frac{d}{dz} - 2\right) + z\right] F(z) = 4\,A. \tag{2.129}$$

Of course, not each solution of equation (2.129) satisfies integral equation, which with use of variable z is the following

$$F(z) = A + \frac{1}{12}\left[\frac{1}{z^2}\int_0^z t^2 dt\, F(t) - \frac{2}{z}\int_0^z t dt\, F(t) - 2\,z\int_z^\infty \frac{dt\, F(t)}{t} + z^2\int_z^\infty \frac{dt\, F(t)}{t^2}\right],$$

$$z = G\,x, \quad t = G\,y. \tag{2.130}$$

A solution of differential equation (2.129) has to fulfil boundary conditions at zero and at infinity. For us to be able to formulate these conditions and to show, that they can be fulfilled, we have to consider asymptotic behavior of solutions in this points.

A general solution of equation (2.129) has the following form

$$F(z) = F_0(z) + \sum_{i=1}^{4} C_i F_i(z), \tag{2.131}$$

where $F_0(z)$ is a particular solution of inhomogeneous equation (2.129), which evidently is the following

$$F_0(z) = \frac{4A}{z}. \tag{2.132}$$

Linearly independent solutions of homogenous equation F_i, $i = 1, 2, 3, 4$ are just the Meijer functions. In view of application to the present problem, as well as to numerous problems being discussed in what follows, we present here necessary properties of these functions:

$$F(x) = G_{pq}^{mn}\left(x\middle|_{b_1,\dots,b_q}^{a_1,\dots,a_p}\right), \tag{2.133}$$

is a solution of the following differential equation:

$$\left[(-1)^{p-m-n}\, x \prod_{j=1}^{p}(\delta - a_j + 1) - \prod_{j=1}^{q}(\delta - b_j)\right] F(x) = 0, \quad \delta = x\frac{d}{dx}. \tag{2.134}$$

Independent solutions are defined by proper choice of upper indices m, n of function (2.133).

An asymptotical behavior of a Meijer function for $x \to 0$ is defined by the following decomposition

$$\begin{aligned} G_{pq}^{mn}\left(x\middle|_{b_r}^{a_r}\right) &= \sum_{h=1}^{m} \frac{\prod_{j=1}^{\prime m} \Gamma(b_j - b_h) \prod_{j=1}^{n} \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + b_h - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_h)}\, x^{b_h} \\ &\quad \times F_{p,q-1}\left[{}^{1+b_h-a_1,\dots,1+b_h-a_p}_{1+b_h-b_1,\dots,*,\dots,1+b_h-b_q;\,(-1)^{p-m-n}x}\right], \\ &\quad p < q, \quad \text{or} \quad p = q \quad \text{and} \quad |x| < 1. \end{aligned} \tag{2.135}$$

Here

$$F_{p,q-1}\left[{}^{1+b_h-a_1,\dots,1+b_h-a_p}_{1+b_h-b_1,\dots,*,\dots,1+b_h-b_q;\,(-1)^{p-m-n}x}\right] = 1 + (-1)^{p-m-n} x \frac{(1 + b_h - a_1)\cdots(1 + b_h - a_p)}{(1 + b_h - b_1)\dots * \dots(1 + b_h - a_p)} + \cdots$$

is a generalized hypergeometric series with p upper indices and $q - 1$ lower indices, in which index with number h is omitted. The prime in the product of Γ-functions means, that the term with $j = h$ is also omitted. Note, that Meijer functions are defined for any relation between parameters. In case of difference of some parameters being integer, that leads to singularities in expression (2.135), one has to supply the parameter by a small increment ϵ, and perform in expression (2.135) limit $\epsilon \to 0$. This procedure leads to an appearance of logarithmic terms $\ln z, \ln^2 z$ *etc* in asymptotic (2.135).

For $x \to \infty$ we have

$$G_{pq}^{mn}\left(x\Big|_{b_r}^{a_r}\right) = \sum_{h=1}^{n} \frac{\prod_{j=1}^{'n} \Gamma(a_h - a_j) \prod_{j=1}^{m} \Gamma(1 + b_j - a_h)}{\prod_{j=n+1}^{p} \Gamma(1 + a_j - a_h) \prod_{j=n+1}^{p} \Gamma(a_h - b_j)} x^{a_h - 1}$$

$$\times F_{q,p-1}\left[{}^{1+b_1-a_h,\dots,1+b_q-a_h}_{1+a_1-a_h,\dots,*,\dots,1+a_p-a_h;\,(-1)^{q-m-n}x^{-1}}\right], \tag{2.136}$$

$$q < p, \quad \text{or} \quad p = q \quad \text{and} \quad |x| > 1.$$

From (2.135) we see, that we have at $x \to 0$ power behavior x^{b_h}, where b_h is the smallest parameter b_j in the decomposition (2.135), or $x^{b_h} \ln x$ in case of two coinciding parameters $b_i = b_j = b_h$. For $x \to \infty$ we have asymptotic $x^{a_h - 1}$ also with possible logarithms in case of coinciding parameters. By the way, Meijer functions are properly defined for all values of parameters including coinciding and differing by integer numbers. In this cases one has to introduce small difference ϵ, perform necessary evaluations and take the limit $\epsilon \to 0$.

For $x \to \infty$ G-function has power behavior provided $p < q$ either

$$n \geq 1, \quad m + n > \frac{p+q}{2}, \quad |arg\, x| < \left(m + n - \frac{p+q}{2}\right)\pi, \tag{2.137}$$

or

$$q = p + 1, \quad |arg\, x| - (m + n - p + 2k - 1) < \frac{\pi}{2}, \tag{2.138}$$

where k is an integer number. Provided $x \to \infty$ and $p < q$ and

$$m > \frac{p+q}{2}, \quad n = 0, \quad |arg\, x| < \left(m - \frac{p+q}{2}\right)\pi, \tag{2.139}$$

the G-function decreases exponentially. Under the same conditions for x, p, q the G-function increases exponentially at the following regions:
if $q \geq p = 2$ one has to take either

$$m + n > \frac{p+q}{2} \quad \text{and} \quad |arg\, x| - \left(m + n - \frac{p+q}{2}\right)\pi < \frac{\pi}{2}, \tag{2.140}$$

or

$$m + n \leq \frac{p+q}{2} \quad \text{for all} \quad arg\, x. \tag{2.141}$$

In case

$$m + n = \frac{p+q}{2}$$

G-function has oscillation asymptotic for real positive x

$$\frac{C}{x^c} \cos\left((q-p)\, x^{\frac{1}{q-p}} + \phi\right), \tag{2.142}$$

$$c = \frac{q - p - 1 + 2\sum_{i=1}^{p} a_i - 2\sum_{j=1}^{q} b_j}{2(q-p)}.$$

A power factor with the same c appears also before a decreasing exponent in case (2.139) and before an increasing one in case (2.140).

Let us apply this information to our problem of solution of integral equation (2.130) using solution (2.131, 2.132) of differential equation (2.129). We may choose the following set of linearly independent solutions F_i

$$
\begin{aligned}
F_1(z) &= G^{1\,0}_{0\,4}(z\,|\,1,-2,-1,\ 2),\\
F_2(z) &= G^{1\,0}_{0\,4}(z\,|\,2,-2,-1,\ 1),\\
F_3(z) &= G^{3\,0}_{0\,4}(z\,|-1,\ 1,\ 2-1),\\
F_4(z) &= G^{3\,0}_{0\,4}(z\,|-2,\ 1,\ 2,-2).
\end{aligned}
\tag{2.143}
$$

Let us note here, that in case of $p = 0$ (no upper indices) we write lower indices in one row.

First of all, let us formulate conditions for $z \to 0$ and for $z \to \infty$. It is easy to see, that in homogenous part of equation (2.129) $b_1 = -2$, $b_2 = -1$, $b_3 = 1$, $b_4 = 2$. Thus we have solutions with asymptotic z^{b_h} at $z \to 0$. Let us try asymptotic behavior

$$
F(z) \sim \frac{1}{z^2}. \tag{2.144}
$$

Substituting this behavior into integral equation (2.130) we reproduce only term with asymptotic $F \sim 1/z$. Thus behavior (2.144) is excluded. In the same way we get convinced, that a term with asymptotic behavior $1/z$ is to be excluded also.

On the other hand, solutions F_i, $i = \overline{1,4}$ for $z \to \infty$ with account of power term (2.142) have following behavior

$$
F_i \simeq z^{-\frac{3}{8}} \sum_k C_{ik}\ \exp[a_k\, 4\, z^{1/4}], \tag{2.145}
$$

where a_k are roots of equation $a^4 = -1$, that is

$$
a_1 = e^{\imath\pi/4}, \quad a_2 = e^{-\imath\pi/4}, \quad a_3 = -e^{\imath\pi/4}, \quad a_4 = -\,e^{-\imath\pi/4}. \tag{2.146}
$$

Integrals in equation (2.130) are to converge at infinity. However, due to properties at infinity (2.140) $F_1(z)$ and $F_2(z)$ exponentially increase at infinity. Therefore we may choose only $F_3(z)$ or $F_4(z)$. From these $F_3(z)$ behaves as $1/z$ for $z \to 0$, that is necessary for cancelation of singular behavior (2.132) of the particular solution of inhomogeneous equation (2.129). Thus we are rested with $F_3(z)$ and this fixes the following solution

$$
F(z) = A\left(\frac{4}{z} - 2\, G^{3\,0}_{0\,4}(z\,|-1,\ 1,\ 2,-2)\right). \tag{2.147}
$$

According to (2.135)

$$
G^{3\,0}_{0\,4}(z\,|-1,\ 1,\ 2,-2) = \frac{2}{z}\left(1 - \frac{z}{4}\right) + o(z), \tag{2.148}
$$

therefore undesirable terms containing $1/x$ cancel and $F(0) = A$. In the first approximation normalization condition (2.127) gives $A = 1$. Thus we obtain the unique solution (2.147) with $A = 1$ of equation (2.130).

It also instructive to rewrite solution (2.147) in another way. Let us multiply differential equation (2.129) by operator

$$z\frac{d}{dz},$$

then the equation becomes being a homogenous Meijer equation with set of parameters $b_i\,:\,-2,-1,\,0,\,1,\,2;\,a_i\,:\,0$. Taking into account all boundary and normalization conditions we again come to the unique answer

$$F(z) = 2\,G^{3\,1}_{1\,5}\big(z\,\big|^{0}_{0,\,1,\,2,-1,-2}\big). \tag{2.149}$$

Of course expression (2.149) is identic to (2.147) with $A = 1$. This is consequence of the useful general relation

$$G^{m\,1}_{1\,q+1}\big(z\,\big|^{0}_{0,\,b2,\dots,b_q,-1}\big) = \frac{1}{z}\frac{\prod_{j=2}^{m}\Gamma(1+b_j)}{\prod_{j=m+1}^{q}\Gamma(-b_j)} - G^{m\,0}_{0\,q}(z\,|-1,\,b_2,\dots,b_q). \tag{2.150}$$

Let us present also another useful relation, which is usually absent in textbooks. It will be applied below in the book.

$$G^{m\,1}_{1\,q+1}\big(z\,\big|^{1}_{1,\,b2,\dots,b_q,\,0}\big) = \frac{\prod_{j=2}^{m}\Gamma(b_j)}{\prod_{j=m+1}^{q}\Gamma(1-b_j)} - G^{m\,0}_{0\,q}(z\,|0,\,b_2,\dots,b_q). \tag{2.151}$$

We have demonstrated the method of solution of this equation in view, that in what follows we repeatedly will encounter similar equations and the method, described here, will be applied.

It seems also advisable to show the expression for an integral, containing a Meijer function and a variable power factor

$$\int x^{\alpha-1}\,G^{m\,n}_{p\,q}\left(x\,\Big|^{a_i}_{b_j}\right)dx = G^{m\,n+1}_{p+1\,q+1}\left(x\,\Big|^{1,\,a_i+\alpha}_{b_j+\alpha,\,0}\right). \tag{2.152}$$

We shall often encounter integrals of this type in what follows.

We would also present here few useful relations, which one encounters in calculations in the framework of approach being developed in the book

$$x^{\alpha}\,G^{m\,n}_{p\,q}\left(x\,\Big|^{a_i}_{b_j}\right) = G^{m\,n}_{p\,q}\left(x\,\Big|^{a_i+\alpha}_{b_j+\alpha}\right), \tag{2.153}$$

$$G^{m\,n}_{p\,q}\left(x\,\Big|^{a_1,\dots,a_{n-1},\,a,\,a_{n+1},\dots,\,a_p}_{b_1,\dots,b_m,\dots,\,b_{q-1},\,a}\right) = G^{m\,n-1}_{p-1\,q-1}\left(x\,\Big|^{a_1,\dots,\,a_{n-1},\,a_{n+1},\dots,a_p}_{b_1,\dots,b_m,\dots,b_{q-1}}\right), \tag{2.154}$$

$$G^{m\,n}_{p\,q}\left(x\,\Big|^{a_1,\dots,a_n,\,b,\,a_{n+2},\dots,\,a_p}_{b_1,\dots,b_{m-1},\,b,\,b_{m+1},\dots,\,b_q}\right) = G^{m-1\,n}_{p-1\,q-1}\left(x\,\Big|^{a_1,\dots,\,a_n,\,a_{n+2},\dots,\,a_p}_{b_1,\dots,\,b_{m-1},\,b_{m+1},\dots,\,b_q}\right), \tag{2.155}$$

$$\int_0^{\infty} G^{s\,t}_{u\,v}\left(x\,\Big|^{c_l}_{d_k}\right)G^{m\,n}_{p\,q}\left(\omega\,x\,\Big|^{a_i}_{b_j}\right) = G^{m+t,\,n+s}_{p+v,\,q+u}\left(\omega\,\Big|^{a_1,\dots,\,a_n,-d_1,\dots,-d_v,\,a_{n+1},\dots,\,a_p}_{b_1,\dots,\,b_m,-c_1,\dots,-c_u,\,b_{m+1},\dots,\,b_q}\right). \tag{2.156}$$

Relations (2.154, 2.155) means, that equal upper and lower indices cancel cross-wise. Other useful formulas may be found in textbook [72].

3 Bogoliubov compensation

3.1 Origin of the approach

The compensation approach was introduced and elaborated by N. N. Bogoliubov firstly in application to problems of statistical mechanics (see [40]). Let us illustrate the approach with the case of an ideal isotropic ferromagnetic. Let us start with the dynamical system with the following Hamiltonian (Heisenberg model)

$$H = -\frac{1}{2}\sum_{(f_1,f_2)} I(f_1 - f_2)(\mathbf{S}_{f_1}\cdot\mathbf{S}_{f_2}), \tag{3.1}$$

where (f) are space points corresponding to points of a crystal lattice, $\mathbf{S}_f$ are spin vectors with the usual commutation relations, $I(f_1 - f_2)$ are nonnegative numbers. We assume, that $I(f_1 - f_2)$ are positive for points f_1, f_2 being the nearest neighbors.

For this system each component of total spin $\mathbf{S} = \sum_{(f)} \mathbf{S}_f$ is the integral of the motion. We have also

$$S_i\,S_j - S_j\,S_i = \imath\,\epsilon_{ijk}\,S_k, \tag{3.2}$$

Now we have the following definition of an average $\langle\mathfrak{A}\rangle$ of a dynamical variable $\mathfrak{A}$

$$\langle\mathfrak{A}\rangle = \frac{Trace\,(\mathfrak{A}\,e^{-H/\theta})}{Trace\,(e^{-H/\theta})}. \tag{3.3}$$

Due to S_x commutes with H we have

$$Trace\,(S_y\,S_x e^{-H/\theta}) = Trace\,(S_y\,e^{-H/\theta}S_x) = Trace\,(S_x\,S_y e^{-H/\theta}), \tag{3.4}$$

and from (3.2) we immediately obtain

$$Trace\,(S_z\,e^{-H/\theta}) = 0. \tag{3.5}$$

The same relation is valid for S_x, S_y and so

$$Trace\left(\sum_{(f)}\mathbf{S}_\mathbf{f}\right) = Trace\,(\mathbf{S}) = 0. \tag{3.6}$$

Let us introduce magnetization vector of the unit volume $\mathfrak{M} = \mu/V\,\mathbf{S}$. Then we have from (3.3, 3.6)

$$\langle\mathfrak{M}\rangle = 0. \tag{3.7}$$

Thus the usual average of vector $\mathfrak{M}$ equals zero, that evidently corresponds to isotropy of the system in respect to the rotation group. Let us emphasize, that result (3.7) is valid for any temperature including those being below the Curie point. However we know that in the last case the magnetization vector differs from zero and its direction can be chosen arbitrary. Thus the state of statistical equilibrium is degenerated.

Let us switch on an external magnetic field $\nu\,\mathbf{e}$ ($\nu > 0$, $\mathbf{e}^2 = 1$) and change Hamiltonian (3.1) for the following expression

$$H_\nu = H + \nu\,(\mathbf{e}\cdot\mathfrak{M})\,V. \tag{3.8}$$

Then bearing in mind well-known properties of a ferromagnetic below the Curie point we see, that

$$\langle\mathfrak{M}\rangle = \mathbf{e}M_\nu, \tag{3.9}$$

where M_ν will tend to nonzero limit with intensity of the external field ν tends to zero. We have here instability of the usual averages with addition to Hamiltonian (3.1) of term $\nu\,(\mathbf{e}\cdot\mathfrak{M})\,V$ with infinitely small ν, which result in a finite increment of average $\langle\mathfrak{M}\rangle$.

Then Bogoliubov introduces the notion quasi-average. Let us take a dynamical quantity A, which consists of combinations of products

$$S^{\alpha_1}_{f_1}(t_1)\cdots S^{\alpha_r}_{f_r}(t_r),$$

and let us define quasi-average $\prec A\succ$ of this quantity by the following prescription

$$\prec A\succ = \lim_{\nu\to 0}\langle A\rangle_{\nu e}, \tag{3.10}$$

where $\langle A\rangle_{\nu e}$ is the usual average of A for hamiltonian (3.8).

In this way the existence of a degeneration immediately results in dependence of quasi-averages on direction of unit vector $\mathbf{e}$. It is evident also, that

$$\langle A\rangle = \int \prec A\succ\, d\mathbf{e}.$$

The conception of quasi-averages proves to be very effective in dealing with the most interesting phenomena such as the superfluidity and superconductivity. In particular, this approach quite adequately describes phenomenon of spontaneous breaking of an invariance. Such problems, as we have seen in Section 1.6.2, inevitably arise in problems of elementary particles interactions. In what follows we would discuss these problems in details.

3.2 Application to QFT

Now we would like to apply N. N. Bogoliubov quasi-averages method [40, 42], which is the most consistent and effective method of studying of a spontaneous symmetry breaking problems, to problems of the Quantum Field Theory.

An important point of the quasi-averages method is connected with a compensation equation [40, 42]. Bearing in mind numerous applications of these equations in the book, let us briefly formulate method of construction of the compensation equations. In the line of a study of a possible spontaneous symmetry breaking in quantum field theory problems in method [40] the following procedure is applied[1]. Let the initial Lagrangian

$$L = L_0 + L_{int}, \tag{3.11}$$

to possess some symmetry. Let us add to expression (3.11) some term ϵL_{br}, which breaks the initial symmetry. With this modification of the problem we perform evaluations of necessary quantities and we set $\epsilon \to 0$ only after these evaluations. Not always the results of such a procedure (quasi-averages) coincide with results, obtained in the framework of the initial symmetric problem (simply averages). In the line of these evaluations of quasi-averages one has to solve compensation equations. For instance, in a theory with the initial chiral symmetry fermions are to have zero masses. Let us use the following small increment which breaks the symmetry

$$\epsilon L_{br} = -\epsilon \bar{\psi} \psi. \tag{3.12}$$

Now let us add to the modified Lagrangian (3.12) a possible mass term and subtract the same. We have

$$L = L_0 - m \bar{\psi} \psi + L_{int} + m \bar{\psi} \psi - \epsilon \bar{\psi} \psi\,. \tag{3.13}$$

Let the fist two terms to be the new free Lagrangian while the three last terms now comprise the new interaction Lagrangian. Then we have to demand the new interaction does not contribute to the mass term, that is two-field Green function obtained from the modified interaction Lagrangian be zero on the mass shell. This condition is just the compensation equation of the problem. In the case under consideration this condition leads to equation

$$-m + \epsilon + \Sigma(m) = 0, \tag{3.14}$$

where $\Sigma(m)$ is mass operator on the mass shell of the modified free Lagrangian. In this equation one already can set $\epsilon \to 0$. As a rule (see e.g. [73]) mass operator $\Sigma(m)$ is proportional to m and trivial solution of the compensation equation $m = 0$ always exists. However a nontrivial solution $m \neq 0$ also may exist.

Thus the main principle of construction of a compensation equation consist in the procedure "add-subtract" of symmetry breaking terms, one of these terms being related to the free Lagrangian and the other one being related to the interaction Lagrangian. Then one has to compensate that term, which is to be zero in the corresponding problem. This principle will be applied below.

1 At first methods [40, 42] where applied to quantum field theory problems in work [73].

3.3 A spontaneous generation of the Nambu–Jona-Lasinio interaction

We have already discussed a spontaneous generation of a mass in Nambu–Jona-Lasinio model (Section 1.5.2). It just corresponds to the procedure described in the previous section. The natural question arises if one can achieve an effect of a spontaneous generation of an effective interaction itself. This possibility was first thought on by the author after the discovery of the phenomenon of a mass generation. After expiration of a long time I have returned to this problem and have developed the method of dealing with it in the framework of the compensation conception [40, 42]. The explication of the method is the main goal of the present book.

Now let us illustrate the first considerations on the problem with Nambu– Jona-Lasinio interaction taken as an example. We start with QCD Lagrangian with only light quark doublet (u, d) assuming $SU(2)$ isotopic invariance (see (1.85)). Thus we have from (1.58)

$$\begin{aligned} L &= \frac{\imath}{2}\left(\bar{q}\gamma_\mu\partial_\mu q - \partial_\mu\bar{q}\gamma_\mu q\right) - m_0\bar{q}q + g_s\bar{q}\gamma_\mu t_a A^a_\mu q - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu}\right), \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc}A^b_\mu A^c_\nu, \quad q = (u,\, d). \end{aligned} \tag{3.15}$$

Now we would like to find out, if the Nambu–Jona-Lasinio interaction (1.86) could be spontaneously generated in the problem (3.15). In doing this we again proceed with the add-subtract procedure:

$$L = L_0 + L_{int},$$

$$\begin{aligned} L_0 &= \bar{q}(x)(\imath\partial_\alpha\gamma_\alpha - m)q(x) \\ &\quad - \left(\frac{G_S}{2}\left(\bar{q}(x)q(x)\right)\left(\bar{q}(x)q(x)\right) - \frac{G_P}{2}\left(\bar{q}(x)\,\tau^a\,\gamma_5 q(x)\right)\left(\bar{q}(x)\,\tau^a\,\gamma_5 q(x)\right)\right) \\ &\quad - \frac{1}{4}(F^a_{0\mu\nu}F^a_{0\mu\nu}), \end{aligned} \tag{3.16}$$

$$\begin{aligned} L_{int} &= \bar{q}(x)\,(m - m_0)q(x) \\ &\quad + \left(\frac{G_S}{2}(\bar{q}(x)\,q(x))(\bar{q}(x)\,q(x)) - \frac{G_P}{2}(\bar{q}(x)\tau^a\gamma_5 q(x))(\bar{q}(x)\tau^a\gamma_5 q(x))\right) \\ &\quad - \frac{1}{4}(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\mu\nu}F^a_{0\mu\nu}) + g_s\bar{q}\gamma_\mu t_a A^a_\mu q. \end{aligned} \tag{3.17}$$

It is important, that we have introduced different couplings for scalar iso-scalar terms and pseudoscalar iso-vector terms due to introduction of mass m in both parts of the Lagrangian: L_0 and L_{int}. The presence of the mass evidently breaks the initial chiral invariance (1.84) (for $m_0 = 0$). We assume, that interactions with couplings $G_{S,P}$ act in restricted region of the momentum space, that is in four-dimensional Euclid space for $0 < q^2 < \Lambda^2$ with Λ being a cut-off to be defined in the course of the solution of the problem. This means that the chiral invariance is broken only for this specific region.

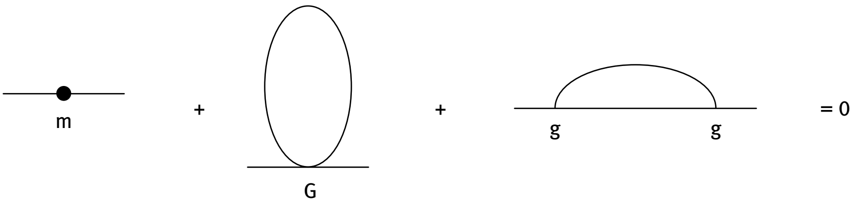


Fig. 3.1. Diagrams corresponding to compensation equation for spontaneous generation of mass in NJL model.

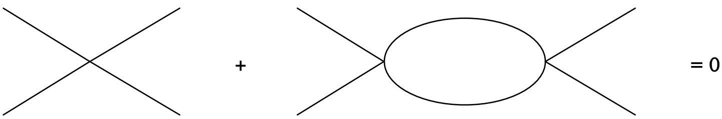


Fig. 3.2. Diagrams corresponding to compensation equation for spontaneous generation of interaction in NJL model.

Now let us formulate compensation equations. They are due to two reasons. The first one is due to compensation of the improper mass term in the newly defined interaction Lagrangian (3.17). The equation is similar to that being presented in graphic form in Figure 1.5 in Section 1.6.2. However in the present formulation of the problem it is necessary to add the contribution of gluon exchange into the mass, because this compensation equation exclude improper mass term from the interaction Lagrangian (see Figure 3.1).

The other two equations correspond to diagrams in Figure 3.2 presented here. Vertices in Figure 3.2 correspond to those in the newly defined free Lagrangian (3.16). These compensation equations exclude undesirable interaction terms from the free Lagrangian.

Here we restrict ourselves with one-loop terms in the equations. The first approximation, which gives constant solutions of equations, corresponds to an account of such one-loop diagrams. Provided we include in a consideration also vertical diagrams, it would correspond to necessity of a dependence on variable $(p-q)^2$, where p is initial momentum and q is the final one. We shall see further, that such terms have to be taken into account in the next approximation,described in Chapter 5

Calculations give the following set of compensation equations

$$\begin{aligned} G_P &= \frac{3\,G_P^2}{2\,\pi^2}\left(\Lambda^2 - m^2\,\ln\frac{\Lambda^2+m^2}{m^2}\right),\\ G_S &= \frac{3\,G_S^2}{2\,\pi^2}\left(\frac{\Lambda^2(\Lambda^2+3m^2)}{\Lambda^2+m^2} - 3\,m^2\,\ln\frac{\Lambda^2+m^2}{m^2}\right),\\ m - m_0 + \frac{m\,\alpha_s}{\pi}\ln\frac{\Lambda^2+m^2}{m^2} &= \frac{3\,m\,G_S}{2\,\pi^2}\left(\Lambda^2 - m^2\,\ln\frac{\Lambda^2+m^2}{m^2}\right). \end{aligned} \tag{3.18}$$

Note, that in calculation of the QCD term in the last equation we have used Landau gauge, because in this case there is no contribution to structure $\gamma + \mu p_\mu$, which would need renormalization of the wave functions of quarks.

We see that the first two equations have evident trivial solution

$$G_P = G_S = 0, \tag{3.19}$$

and in case $m_0 = 0$ the third one has trivial solution

$$m = 0. \tag{3.20}$$

However we look for a nontrivial solution, thus we divide the first and the second equation by G_P and G_S respectively and the third one by m. It is also easy to see, that expressions in brackets in the first and in the third equations coincide, so that after introduction of the following dimensionless variables

$$G_S\Lambda^2 = x, \quad G_P\Lambda^2 = y, \quad G_P m^2 = z, \tag{3.21}$$

we obtain the following set of equations:

$$\begin{gathered}\frac{3}{2\pi^2}\left(\frac{x(y+3z)}{y+z} - 3\frac{xz}{y}\ln\frac{y+z}{z}\right) = 1, \quad \epsilon = \frac{m_0}{m}, \\ \frac{3}{2\pi^2}\left(y - z\ln\frac{y+z}{z}\right) = 1, \quad \frac{x}{y} + \epsilon - \frac{\alpha_s}{\pi}\ln\frac{y+z}{z} = 1.\end{gathered} \tag{3.22}$$

Table 3.1. Physical parameters from solution of set (3.22) in dependence on strong coupling α_s for $m_0 = 5.5$ MeV, all variables in MeV but dimensionless x, y, z

α_s	x	y	z	m	$-\sqrt[3]{\langle\bar{q}q\rangle}$	m_π	$G_P^{-\frac{1}{2}}$	$G_S^{-\frac{1}{2}}$	Λ
0.6	12.24	8.24	0.625	255.4	237.2	131.1	323.2	265.1	927.5
0.7	13.00	8.40	0.717	263.3	233.4	128.0	310.9	249.9	901.2
0.8	13.74	8.56	0.807	270.3	230.5	125.6	301.0	237.6	880.6
0.9	14.46	8.70	0.893	276.7	228.1	123.7	292.9	227.2	864.0
1.0	15.16	8.83	0.976	282.6	226.2	122.1	286.1	218.4	850.3

Now for given α_s and ϵ (meaning fixation of current mass m_0) we look for solution of set (3.22). All variables in the set are dimensionless, so to get knowledge on values of initial variables m, G_S, G_P, Λ we have to introduce an additional relation. From equations (1.102, 1.101) with account of (3.21) we have

$$f_\pi = \frac{m\sqrt{3}(\ln\frac{y+z}{z} - \frac{y}{y+z})}{2\pi\sqrt{\ln\frac{y+z}{z} - \frac{3y}{4(y+z)} - \frac{yz}{4(y+z)^2}}}, \quad f_\pi = 92.42 \pm 0.33 \text{ MeV}. \tag{3.23}$$

Table 3.2. Physical parameters from solution of set (3.22) in dependence on strong coupling α_s for $m_0 = 6$ MeV, all variables in MeV but dimensionless x, y, z

α_s	x	y	z	m	$-\sqrt[3]{\langle\bar{q}q\rangle}$	m_π	$G_P^{-\frac{1}{2}}$	$G_S^{-\frac{1}{2}}$	Λ
0.6	12.23	8.23	0.623	255.3	237.3	137.0	323.5	265.5	928.1
0.7	12.98	8.40	0.715	263.1	233.5	133.8	311.1	250.2	901.7
0.8	13.72	8.55	0.805	270.2	230.6	131.2	301.2	237.8	881.0
0.9	14.44	8.70	0.891	276.6	228.2	129.2	293.1	227.5	864.3
1.0	15.14	8.83	0.974	282.5	226.2	127.5	286.3	218.6	850.5

Table 3.3. Physical parameters from solution of set (3.22) in dependence on strong coupling α_s for $m_0 = 6.5$ MeV, all variables in MeV but dimensionless x, y, z

α_s	x	y	z	m	$-\sqrt[3]{\langle\bar{q}q\rangle}$	m_π	$G_P^{-\frac{1}{2}}$	$G_S^{-\frac{1}{2}}$	Λ
0.6	12.21	8.23	0.621	255.1	237.3	142.7	323.8	265.8	928.8
0.7	12.97	8.40	0.713	263.0	233.6	139.3	311.3	250.5	902.2
0.8	13.70	8.55	0.803	270.0	230.6	136.6	301.4	238.1	881.4
0.9	14.42	8.69	0.888	276.5	228.2	134.5	293.3	227.7	864.6
1.0	15.13	8.83	0.972	282.4	226.2	132.8	286.4	218.8	850.8

Expression (3.23) corresponds to the well-known Goldberger–Treiman relation [37, 45]. For definition of the π-meson mass we shall use Gell-Mann–Oaks–Renner relation (1.108)

$$m_\pi^2 f_\pi^2 = -2\, m_0\, \langle\bar{q}q\rangle, \tag{3.24}$$

where quark condensate $\langle\bar{q}q\rangle$ is defined by (1.107). Due to this definition in our approximation it reads

$$\langle\bar{q}q\rangle = \frac{3\, m^3}{4\,\pi^2}\left(\frac{y}{z} - \ln\frac{y+z}{z}\right). \tag{3.25}$$

Using along with set (3.22) relations (3.23, 3.24, 3.25) we for each value of α_s has definite solution. From phenomenological results of Section 1.6.2 we expect value of cut-off Λ to be of order of magnitude $\simeq 1$ GeV. For such scale value of strong coupling α_s from (1.69) can be estimated to be $\alpha_s \simeq 0.8$. In the subsequent chapters we will get convinced, that such values are natural for an average strong coupling in the non-perturbative region. In any case here we present results for solution of the set for values of $\alpha_s = 0.8 \pm 0.2$. The results are shown in Tables 3.1, 3.2, 3.3 for values of $m_0 = 5.5,\ 6,\ 6.5$ MeV respectively.

Let us emphasize, that for $\alpha_s = 0$ a solution does not exist at all. Comparing results of the Tables with phenomenological results (1.106, 1.107), obtained in Section 1.6.2, we see qualitative agreement. As for experimental data, agreement is even better, e. g., for quark condensate $\langle\bar{q}q\rangle$. It seems, that the best agreement with both the pion mass and the quark condensate corresponds to $m_0 = 6.5$ MeV and

$$\alpha_s = 0.8 \pm 0.1. \tag{3.26}$$

As we shall see below, this value for strong coupling (3.26) in the nonperturbative region is consistent with its behavior with account of nonperturbative effects. However value for m_0 is rather higher than the estimated phenomenological value [4]

$$m_0 = 3.5^{+0.5}_{-0.2}\ \text{MeV}. \tag{3.27}$$

This deviation might be prescribed to approximation being used here. On the other hand we have to bear in mind, that result (3.27) corresponds to scale $\simeq 2$ GeV and for smaller scales we expect increasing of m_0.

Results being obtained allows to calculate other low-energy parameters. Let us consider the interaction of composite particles, such as π-meson or ρ-meson (see Table 1.4) with constituent quarks. This means existence of such effective interactions

$$\begin{gathered} g_{\sigma qq}\,\phi_\sigma\,\bar q\,q, \\ \imath\,g_{\pi qq}\,\phi^a_\pi\,\bar q\,\tau^a\,q, \\ g_{\rho qq}\,\phi^a_{\rho,\mu}\,\bar q\,\tau^a\,\gamma^\mu\,q. \end{gathered} \tag{3.28}$$

In the framework of the Nambu–Jona-Lasinio interaction coupling constants (3.28) are to be calculated with account of one-loop diagrams represented in Figure 3.3.

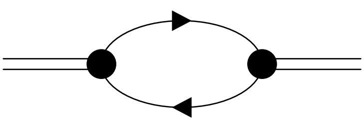


Fig. 3.3. Diagrams corresponding to calculation of meson-quark coupling constants in NJL model.

In the course of the diagrams evaluation we have to single out terms being proportional to p^2 and demand its coefficient being unity. Really it is normalization condition for bound state fields ϕ_π and ϕ_ρ. Straightforward calculations give for coupling constants the following results, which we express in terms of variables (3.21)

$$\begin{aligned} g_\sigma &= \frac{2\,\pi}{\sqrt{3\left(\ln\frac{y+z}{z} - \frac{y(13y^2+27yz+12z^2)}{12(y+z)^3}\right)}}, \\ g_\pi &= \frac{2\,\pi}{\sqrt{3\left(\ln\frac{y+z}{z} - \frac{z^2}{(y+z)^2}\right)}}, \\ g_\rho &= \frac{\sqrt{2}\,\pi}{\sqrt{\ln\frac{y+z}{z} - \frac{13}{32} + \frac{9z}{8(y+z)} - \frac{z^3}{24(y+z)^2}}}. \end{aligned} \tag{3.29}$$

For example, for $m_0 = 6.5$, $\alpha_s = 0.9$ we have

$$g_\sigma = 3.00, \quad g_\pi = 2.59, \quad g_\rho = 3.76. \tag{3.30}$$

The knowledge of coupling constants allows to estimate masses of resonances. For this purpose the procedure of "bosonization" was proposed. In simple words this procedure consists in the following identical transformation

$$\frac{G_S}{2}\,\bar{q}\,q\,\bar{q}q + g_\sigma\,\bar{q}\,q\,\phi_\sigma \;\equiv\; \frac{1}{2\,G_S}\,(G_S\,\bar{q}q + g_\sigma\,\phi_\sigma)^2 - \frac{g_\sigma^2\,\phi_\sigma^2}{2\,G_S}. \tag{3.31}$$

While considering the QFT S-matrix an effective Lagrangian is placed in an exponent. One may exclude quark fields by functional integration by $dq\,d\bar{q}$. The quadratic combination in (3.31) then enters the following functional integral

$$\int \exp\left[\frac{1}{2\,G_S}\,(G_S\,\bar{q}q + g_\sigma\,\phi_\sigma)^2\right] d\bar{q}dq. \tag{3.32}$$

Integral (3.32) is of Gauss type and it gives a normalization factor. Thus we are left with the last term of expression (3.31) which is essentially scalar field ϕ_σ mass term, corresponding to the mass of the σ-resonance. Indeed, the last term in expression (3.31) is exactly the mass term of scalar field ϕ_σ with mass

$$m_\sigma = \frac{g_\sigma}{\sqrt{G_S}}. \tag{3.33}$$

Substituting into (3.33) numbers from Table 3.3 and results (3.30) we have for $m_0 = 6.5$ MeV and $\alpha_s = 0.9$

$$m_\sigma = 701\ \text{MeV}. \tag{3.34}$$

The satisfactory agreement of numerous parameters with their measured values, starting only from two input quantities: m_0 and average α_s, testifies to the approach, which we exploit here.

However, the most interesting result of this section is the existence of the nontrivial solution itself, because we now do not make an assumption on existence of an effective interaction with arbitrary parameters, but obtain just the definite interaction with definite parameters. Fixing purely QCD parameters α_s and m_0 we calculate all other parameters without any arbitrariness.

We have to admit, that the approximation used is rather rude. We have assumed an existence of a cut-off Λ, which restricts a region of the effective interaction. The problem is how to justify such assumption. In view of this goal we consider a comparatively simple model, involving one scalar field. The formulation and study of the model will be performed in subsequent sections.

3.4 Justification of the model choice

We have discussed in Section 1.6.2 the problem of effective interactions, which proves to provide an adequate description of the most complicated items of elementary particles physics, e.g. low-energy hadron interactions. We have laid arguments on behalf of a possibility to describe nonperturbative effects in terms of effective interaction. Starting from these considerations we are to investigate this possibility and this will be the main body of the present book. For example, the famous Nambu–Jona-Lasinio effective interaction is shown to describe low-energy region of strong interactions. However we are sure, that the genuine theory of strong interaction is QCD, which seemingly has nothing to do with this effective interaction. Our aim is to find an approach, which gives possibility to consider spontaneous generation of effective interactions in the framework of firmly established gauge theories of the Standard Model. In doing this we rely on Bogoliubov compensation approach, which provides possibility of generation of quasi-averages. As a result nonperturbative effects was obtained in such important problems, as phenomena of superconductivity, superfluidity *etc.*

We are willing to study the phenomenon of a spontaneous generation of an effective theory. The first attempt to consider this problem was made in the previous section under assumption of existence of built-in cut-off Λ. In this way we have obtained satisfactory results.

We have already remarked that to be more rigorous one has to consider more complicated approximation, under which there is not necessary to introduce the cut-off from the beginning.

The problem is to find a model, in which the main features of the approach may be tested. Let us consider for the beginning elementary scalar fields. Their self-interaction leads to a nontrivial quantum field theory, in which we would study a possible generation of an effective interaction. The purpose of the present work is to consider a simple model, which would allow to have exact solutions of (approximate) compensation equations. Using these solutions one could study conditions under which the assumptions would be fulfilled. To some extent the model has to correspond to features of a renormalizable theory. Namely we achieve a simplicity by considering a scalar field. In view of coupling constants to have proper dimensions we choose dimensionality of the space-time to be six. Really, in this case the coupling constant of interaction $g\,\phi^3$ is dimensionless and interaction $G\,\phi^4$ has constant of inverse mass squared dimension, that corresponds to the dimension of a constant of a four-fermion interaction in four-dimensional space-time. Thus we can consider the theory being chosen as a model for NJL interaction (3.15). The model was considered in work [74].

So we introduce in the six-dimensional space-time a scalar field ϕ with initial scale-invariant Lagrangian

$$L = \frac{1}{2}\,g^{\mu\nu}\,\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + \frac{g_0}{3!}\,\phi^3. \tag{3.35}$$

Let us choose the natural signature with one time and five space axes. The transition from this space-time to Euclidean six-dimensional space is accompanied by the following substitutions

$$p^2 \to -p_E^2, \quad d^6 p \to \imath\, d_E^6 p. \tag{3.36}$$

It was important for us to find a model, which corresponds to the approach under consideration. So here we will not discuss physical meaning of a multi-dimensional theory and we consider the chosen variant as purely model one, as well as two-dimensional models are often considered.

Now we start with Lagrangian (3.35). Evident evaluations give one-loop renormalization group equation [1] for $g^2(\mu^2)$

$$\frac{d\,g^2(\mu^2)}{dL} = -\frac{3\,g^4}{4\,(4\,\pi)^3}, \qquad L = \log\frac{\mu^2}{\Lambda_3^2}, \tag{3.37}$$

Solution of equation (3.37) has a form

$$g^2(\mu^2) = g_0^2\left(1 + \frac{3\,g_0^2}{4\,(4\,\pi)^3}\,\log\frac{\mu^2}{\Lambda_3^2}\right)^{-1}. \tag{3.38}$$

Sometimes it is convenient to use parameter $\bar{h}(\mu^2)$ defined by the following relation

$$\bar{h}(\mu^2) = \frac{3\,g^2(\mu^2)}{4\,(4\,\pi)^3} = \left(\log\frac{\mu^2}{\Lambda_g^2}\right)^{-1}, \tag{3.39}$$

where for transition from Λ_3^2 to Λ_g^2 we have used the standard tool analogous to that in QCD:

$$\Lambda_g^2 = \Lambda_3^2\,\exp\left(-\frac{4\,(4\pi)^3}{3\,g_0^2}\right).$$

Thus we get convinced, that the theory (3.35) is an asymptotic free one and expression (3.39) makes sense for $\mu^2 \gg \Lambda_g^2$.

Note that in this theory there are quadratic divergences in the scalar field mass. It is the common feature of theories with elementary scalars. The problem of the mass of the scalar field will be considered in details later on.

3.5 Compensation equation in a six-dimensional scalar model

Let us have a massless scalar field of the six-dimensional space. The initial free Lagrangian possesses scale symmetry. We shall look for a solution, which breaks this symmetry, with the aid of the Bogoliubov compensation approach being formulated above in this chapter. Namely according to the rules of the approach we add to the Lagrangian the following small increment

$$-\,\epsilon\,\frac{\phi^4}{4!}.$$

Now the scale invariance is already broken and an appearance of nonlocal terms of the form

$$G \int \bar{F}(x, x_1, x_2, x_3, x_4)\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\, dx_1\, dx_2\, dx_3\, dx_4 \tag{3.40}$$

is possible. Here G is a dimensional coupling constant and $\bar{F}(x, x_1, x_2, x_3, x_4)$ is a function of four differences of coordinates $x - x_i$, which Fourier transform $F(p_1, p_2, p_3, p_4)$, where p_i are momenta of legs, represents a form-factor, defining range of interaction (3.40). We shall look for a solution, decreasing at momentum infinity and thus defining a region of action of the effective interaction.

Let us add to the initial Lagrangian such a term with an interaction of the forth power and subtract the same

$$L = \frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} - \frac{m^2}{2}\phi^2 - \frac{G}{4!} F \cdot \phi^4 - \frac{\epsilon}{4!}\phi^4 + \frac{g_0}{3!}\phi^3 + \frac{G}{4!} F \cdot \phi^4 + \frac{m^2}{2}\phi^2, \tag{3.41}$$

where we use abbreviated notation $-\, G F \cdot \phi^4$ instead of expression (3.40). Of course the presence of term (3.40) explicitly breaks the scale invariance, so we perform a procedure "add-subtract" for a mass term as well. Let us refer the forth power term with the plus sign to the interaction Lagrangian and the same term with the minus sign we refer to the free Lagrangian.

$$\begin{aligned} L_0 &= \frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} - \frac{m^2}{2}\phi^2 - \frac{G}{4!} F \cdot \phi^4 - \frac{\epsilon}{4!}\phi^4, \\ L_{int} &= \frac{g_0}{3!}\phi^3 + \frac{G}{4!} F \cdot \phi^4 + \frac{m^2}{2}\phi^2. \end{aligned} \tag{3.42}$$

According to the compensation approach the interaction term with the negative sign in (3.41, 3.42) has to be compensated. This means, that the new free Lagrangian leads to zero four-particle connected Green functions and as a final result contains only terms of the second power in fields. Thus performing evaluations with sign which is inherent to the term in the new free Lagrangian, we come to the compensation equation, which schematically looks in the following way: the first order term plus one-loop terms plus two-loop terms *etc*. Emphasize once more, that here one has to use term $+\, G\phi^4$ as an interaction Lagrangian. One has to equalize to zero the expansion obtained in such a way . This condition is an equation for function $F(p_1, p_2, p_3, p_4)$. We set $\epsilon \to 0$ after evaluations, in our case this means after compensation equations being obtained.

The equation explicitly differs from expansion in powers of interaction Lagrangian

$$L_{int} = \frac{G}{4!} F \cdot \phi^4, \tag{3.43}$$

in the sign of the interaction constant. In view of this note let us emphasize, that the procedure being described can be applied only to symmetry breaking terms of even powers in fields. For terms of odd powers, e.g. for three-linear ones, a fulfillment of a compensation equation leads to vanishing of connected Green function, which is defined by an interaction Lagrangian, because the two expansions in this case differ only in overall sign.

Note, that the presence of term $-G\,\phi^4$ in the new free Lagrangian may lead to appearance of connected Green functions of higher powers in ϕ, that is of the sixth power, of the eighth power etc. Generally speaking, one has to construct a chain set of compensation equations for all these Green functions. We start with an equation for the fourth power Green function and delay the problem of higher Green functions to be discussed and considered in the forthcoming studies.

Let us construct an approximated equation for the fourth power connected Green function. First of all we choose the following kinematics: both left legs have zero momenta and the right ones have momenta p and $-p$. We restrict ourselves by terms up to two-loop ones inclusively. Namely, we have the first order term – the point; three terms of the second order – simple loops, i.e. a horizontal one and two vertical ones with permuted left legs; in the third order we have a horizontal and two vertical two-loop chains and six terms "wine glass": horizontal wine glasses having bases to the left and to the right and vertical ones with bases up and down. The number of the last terms is to be counted twice due to permutations of the left-sided momenta p and $-p$. Generally speaking, in each vertex form-factor F is present. However we can solve only a linear version of the equation, which is obtained by keeping in the equation the first and the second order terms, the two-loop horizontal chain and the wineglass with the basing to the right. Contributions of the rest third order terms we shall consider later on. We proceed to the linear equation keeping form-factor $F(p,-p,0,0) \equiv F(p^2)$ in the first order term and in right-hand vertices of the horizontal loop of the second order, of the horizontal two-loop chain and of the wineglass in the third order. Other vertices in diagrams we consider to correspond to point-like interaction in which the form-factor is changed for its value at zero ($F(0) = 1$)

$$\frac{G}{4!}\,F(0)\,\phi^4 = \frac{G}{4!}\,\phi^4. \tag{3.44}$$

In vertical simple loops, which as well serve as a kernel of the integral equation, we substitute point-like vertices (3.44). Corresponding integrals diverge of course. In view of our search for decreasing solutions at momentum infinity for $F(p^2)$, we introduce some cut-off Λ, which existence is to be confirmed by results of a solution of the equation. In doing this we make the following substitution

$$\int_0^\infty dq^2 \;\to\; \Lambda^2.$$

For estimation of Λ order of magnitude we use the following definition

$$\Lambda^2 = \int_0^\infty F(y)\,dy, \tag{3.45}$$

where one of vertices is changed for the form-factor. For justification of the approach the problem of convergence of the integral in (3.45). We shall use the same cut-off Λ in logarithmic diverging integrals. A possible difference of an actual cut-off in these integrals from Λ leads to some change in constant term c, which enters into corresponding expressions. It will come clear, that the solution will not depend on a value of this constant. Thus the formulation of the equation in the framework of the accepted approximations does not contain arbitrary assumptions.

We consider the equation in six-dimensional Euclidean space with the aid of substitutions (3.36). In the course of evaluations one has to perform angle integrations in of functions $((p-q)^2)^{-1}$ and $\log(p-q)^2$ with powers of (pq). We have (for the logarithmic case see [75])

$$\int \frac{d\Omega_6}{p^2+q^2-2\,p\,q\,\cos\theta} = \frac{4\,\pi^3}{3}\left(\Theta(x-y)\left(\frac{3}{4x}-\frac{y}{4x^2}\right)+\Theta(y-x)\left(\frac{3}{4y}-\frac{x}{4y^2}\right)\right),$$
$$\int d\Omega_6 \ln(p^2+q^2-2pq\cos\theta)$$
$$= \frac{\pi^3}{12}\left(\Theta(x-y)\left(\frac{8y}{x}-\frac{y^2}{x^2}+12\ln x\right)\Theta(y-x)\left(\frac{8x}{y}-\frac{x^2}{y^2}+12\,\ln\,y\right)\right), \tag{3.46}$$
$$\int d\Omega_6 (pq)\ln(p^2+q^2-2pq\cos\theta)$$
$$= \frac{\pi^3}{18}\left(\Theta(x-y)\left(\frac{3y^2}{x}-6\,y-\frac{3y^3}{5x^2}\right)+\Theta(y-x)\left(\frac{3x^2}{y}-6\,x-\frac{3x^3}{5y^2}\right)\right), \quad x=p^2,\ y=q^2.$$

First of all let us calculate one-loop integral keeping terms of zero and the first orders in m^2. We have for one such vertical diagram ($x=p^2$, where p is the total momentum along the loop)

$$-\imath\,\frac{G^2\,\pi^3}{2\,(2\pi)^6}\left(\Lambda^2+\frac{1}{3}\,x\,\ln\left(\frac{x}{\Lambda^2}\right)+2\,m^2\,\ln\left(\frac{x}{\Lambda^2}\right)-c\,x\right), \tag{3.47}$$

where Λ is the square of the cut-off being mentioned and c is a constant, depending on a behavior of the form-factor.

Let us consider the linear compensation equation, obtained in agreement with the formulated rules (see Figure 3.4). The equation in this approximation has the following form

$$\begin{aligned} G\,F(p^2) = {}& \frac{G^2}{2(4\pi)^3}\left(3\,\Lambda^2+\frac{2}{3}\,p^2\,\ln\frac{p^2}{\Lambda^2}+4\,m^2\,\ln\frac{p^2}{\Lambda^2}-2\,c\,p^2\right)\\ &-\frac{G^3}{8(2\pi)^9}\int\left[\frac{1}{3}(p-q)^2\,\ln\frac{(p-q)^2}{\Lambda^2}+2\,m^2\,\ln\frac{(p-q)^2}{\Lambda^2}-c\,(p-q)^2\right]\\ &\times\frac{F(q^2)}{(q^2+m^2)^2}\,d^6q-\frac{3\,G^3\,\pi^3\,\Lambda^2}{2\,(2\pi)^{12}}\int\frac{F(q^2)}{(q^2+m^2)^2}\,d^6q. \end{aligned} \tag{3.48}$$

Firstly let us note, that trivial solution $G=0$ is evidently possible. In view of looking for a nontrivial solution we cancel the equation by G. Performing here angle integrations

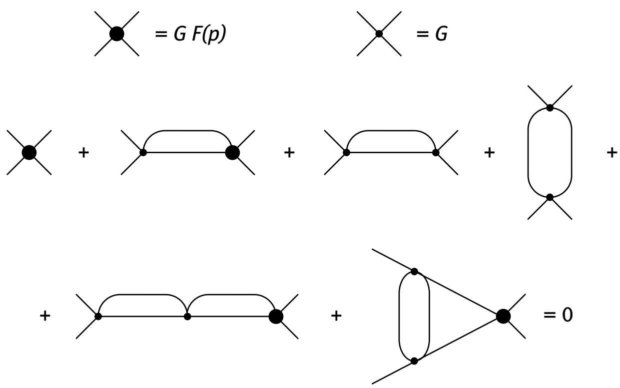


Fig. 3.4. The graphic representation of the linear compensation equation (3.48).

by using formulas (3.46) we obtain the following one-dimensional integral equation

$$
\begin{aligned}
F(x) = {} & \frac{G}{2(4\pi)^3}\left(3\Lambda^2 + \frac{2}{3}\,x\,\ln\frac{x}{\Lambda^2} + 4m^2\,\ln\frac{x}{\Lambda^2} - 2cx\right) \\
& - \frac{3\,G^2\Lambda^2}{4(4\pi)^6}\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy - \frac{G^2}{18(4\pi)^6}\left(-\frac{1}{20\,x^2}\int_0^x \frac{y^5\,F(y)}{(y+m^2)^2}\,dy\right. \\
& + \frac{3}{4x}\int_0^x \frac{y^4\,F(y)}{(y+m^2)^2}\,dy + 3\,\log x\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + 3\,x\,\log x\int_0^x \frac{y^2\,F(y)}{(y+m^2)^2}\,dy \\
& + 4\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy + \frac{3\,x^2}{4}\int_x^\infty \frac{y\,F(y)}{(y+m^2)^2}\,dy - \frac{x^3}{20}\int_x^\infty \frac{F(y)}{(y+m^2)^2}\,dy \\
& \left. + 3\int_x^\infty \frac{y^3\,\ln y\,F(y)}{(y+m^2)^2}\,dy + x\int_x^\infty \frac{(4+3\ln y)\,y^2\,F(y)}{(y+m^2)^2}\,dy\right) \\
& - \frac{G^2\,m^2}{12(4\pi)^6}\left(-\frac{1}{x^2}\int_0^x \frac{y^4\,F(y)}{(y+m^2)^2}\,dy + \frac{8}{x}\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy\right. \\
& + 12\,\ln x\int_0^x \frac{y^2\,F(y)}{(y+m^2)^2}\,dy + 12\int_x^\infty \frac{y^2\,\ln y\,F(y)}{(y+m^2)^2}\,dy \\
& \left. + 8\,x\int_x^\infty \frac{y\,F(y)}{(y+m^2)^2}\,dy - x^2\int_x^\infty \frac{F(y)}{(y+m^2)^2}\,dy\right) \\
& + \frac{G^2}{6\,(4\pi)^6}\left(\ln\Lambda^2 + 3\,c\right)\left(\int_0^\infty \frac{y^3\,F(y)}{(y+m^2)^2}\,dy\right. \\
& \left. + x\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy\right) + \frac{G^2\,m^2}{(4\pi)^6}\,\ln\Lambda^2\int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy.
\end{aligned}
\tag{3.49}
$$

A method of solution of equations of (3.49) type is developed in Section 2.3. Equation (3.49) is reduced to a differential one by sequential differentiations. Evident evaluation gives

$$\frac{d^4}{dx^4}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)$$
$$= -\beta\left(\frac{F(x)}{(x+m^2)^2} + 2m^2\left(x\frac{d^2}{dx^2}\frac{F(x)}{(x+m^2)^2} + 3\frac{d}{dx}\frac{F(x)}{(x+m^2)^2}\right)\right)m \quad \beta = \frac{2\,G^2}{(4\,\pi)^6}. \tag{3.50}$$

One easily see, that equation (3.50) can be rewritten in the form

$$\left(\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\right.$$
$$\left.\times\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}-3\right)+\beta x^2\right)F(x)$$
$$= 2\,\beta\, m^2 x\left(F(x) + x\frac{dF}{dx} - x^2\frac{d^2F}{dx^2}\right), \tag{3.51}$$

where two terms of expansion in m^2 are kept. From this form of the equation we immediately conclude, that for $x \to 0$ there are eight independent asymptotes, which coefficients we denote as follows

$$\frac{a_{-2}}{x^2},\quad \frac{a_{-1}}{x},\quad a_0,\quad a_{0l}\ln x,\quad a_1\,x,$$
$$a_{1l}x\ln x,\quad a_2\,x^2,\quad a_3\,x^3. \tag{3.52}$$

Eight independent asymptotes at infinity are the following

$$F_k(x) \simeq x^{-3/16}\exp\left(4\,(\beta x^2)^{1/8}\exp\left(\frac{\imath\,\pi\,(2k-1)}{8}\right)\right),\quad k = 1,\,2,\ldots,\,8. \tag{3.53}$$

Four of these asymptotes at infinity decrease exponentially ($k = 3,\,4,\,5,\,6$), and the rest four ones do increase.

Equation (3.51) is equivalent to the initial integral equation under definite boundary conditions being fulfilled. First of all we can use only solutions, decreasing at infinity. To obtain conditions at zero we have to substitute expression

$$F(x) = -\frac{x^2}{\beta}\frac{d^4}{dx^4}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)$$

in integrals of equation (3.49) and perform sequential integrations by parts. The results are presented in the appendix to this chapter.

Substituting expressions (3.87) into equation (3.49), we have

$$F(x) = F(x) - \frac{a_{-2}}{x^2} - \frac{a_{-1}}{x} - a_{0l}\ln x - a_{1l}x\ln x$$
$$+ \frac{G\pi^3}{2(2\pi)^6}\left(3\,\Lambda^2\left(1-\frac{G\,I}{2(4\,\pi)^3}\right)+\frac{2\,x}{3}\ln\left(\frac{x}{\Lambda^2}\right)-2\,c\,x\right) \tag{3.54}$$
$$+ x\left(\ln\Lambda^2+3\,c\right)a_{1l},\qquad I = \int_0^\infty \frac{y^2\,F(y)}{(y+m^2)^2}\,dy.$$

From here we obtain the following condition (independently on values of Λ^2 and c)

$$a_{-2} = 0, \quad a_{-1} = 0, \quad a_{0l} = \frac{2\,G\,m^2}{(4\pi)^3},$$
$$a_{1l} = \frac{G\pi^3}{3\,(2\pi)^6} = \frac{\sqrt{2}\,\beta}{6}, \tag{3.55}$$
$$I = \frac{2(4\pi)^3}{G} = \frac{2\,\sqrt{2}}{\sqrt{\beta}}. \tag{3.56}$$

The first four conditions (3.55) are boundary conditions for equation (3.51). A combination of four solutions decreasing at infinity with account of these boundary conditions gives the unique solution. It can be expressed in terms of well-known special functions for case $m^2 = 0$. Indeed, let us make the following substitution in equation (3.51)

$$z = \frac{\beta\,x^2}{2^8}, \tag{3.57}$$

which reduces the equation to the canonical form of Meijer equation [71] of the eighth order

$$\left(\left(z\frac{d}{dz}+1\right)\left(z\frac{d}{dz}+\frac{1}{2}\right)\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}-\frac{1}{2}\right)\right.$$
$$\left.\times\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-1\right)\left(z\frac{d}{dz}-\frac{3}{2}\right)+z\right)F(z) = 0. \tag{3.58}$$

Conditions (3.55) fix the solution. Firstly, four solutions, decreasing at infinity, always could be combined to set to zero three singular asymptotes at zero, i.e. to fulfill conditions $a_{-2} = a_{-1} = a_{0l} = 0$. Such property has the following Meijer function (see Section 2.3)

$$C \cdot G^{50}_{08}\left(z\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,\,-\frac{1}{2},-1\right).$$

The constant is defined by the coefficient before $\sqrt{z}\ln z$. For small z this Meijer function behaves as follows

$$G^{50}_{08}\left(z\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,\,-\frac{1}{2},-1\right) = \pi + \frac{16}{3}\sqrt{z}\,\ln z + \cdots. \tag{3.59}$$

Comparing the coefficient afore $\sqrt{z}\,\ln z$ with (3.55), we obtain

$$C = \frac{\sqrt{2}}{4}.$$

Performing integration (see [72]), we have in accordance with definition of I (3.54)

$$I = \int_0^\infty F(y)\,dy = \frac{\sqrt{2}}{4}\int_0^\infty G^{50}_{08}\left(\frac{\beta y^2}{2^8}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,\,-\frac{1}{2},-1\right)dy = \frac{2\sqrt{2}}{\sqrt{\beta}}, \tag{3.60}$$

that perfectly agrees with condition (3.55).

Thus, solution

$$F(x) = \frac{\sqrt{2}}{4} G^{50}_{08}\left(\frac{\beta x^2}{2^8} \mid \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, 0, 0, -\frac{1}{2}, -1\right), \tag{3.61}$$

fulfills all conditions (3.55), and consequently the initial equation (3.49), which is an approximate compensation equation. This solution is a nontrivial solution, which contains dimensional parameter G, and hence it leads to the initial scale symmetry breaking. Of course, as we have noted before, trivial solution $F(x) = 0$ is also possible. Note, that the boundary conditions are not dependent on value of the form-factor at zero. Equality $F(0) = 1$ will serve as an additional condition in what follows.

Let us take into account terms proportional to m^2. We shall look for a correction to the solution of equation (3.51) in the following form

$$F(x) = F_0(x) + \Delta F(x). \tag{3.62}$$

Substituting (3.62) into equation (3.51) we have the following equation in the first order in m^2

$$\begin{aligned}\Bigg(\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\\ \times\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}-3\right)+\beta x^2\Bigg)\,\Delta F(x)\\ = 2\,\beta m^2 x\left(F_0(x) + x\frac{dF_0}{dx} - x^2\frac{d^2F_0}{dx^2}\right).\end{aligned} \tag{3.63}$$

From equation (3.63) we can exactly define several terms of expansion of $\Delta F(x)$ for small x. Indeed let us consider the following expression

$$\begin{aligned}\bar{\Delta} F(x) &= \frac{2m^2}{x}\left(F_0(x) + x\frac{dF_0}{dx} - x^2\frac{d^2F_0}{dx^2}\right)\\ &= 2m^2\left(\frac{\pi}{2\sqrt{2}\,x} + \frac{2G}{3(4\pi)^3}\left(\ln(\sqrt{\beta}x) + 4\gamma - \frac{23}{6}\right) + \frac{\pi\sqrt{2}\,G^2}{96(4\pi)^6}\,x\ln x + O(x)\right),\end{aligned} \tag{3.64}$$

where $\gamma = 0.577215665\ldots$ is the Euler constant. Substituting expression (3.64) into equation (3.63), we get convinced, that it fulfills the equation up to terms of x^3 order, because the differential operator in the left-hand side nullifies the terms presented in (3.64) and subsequent terms up to the indicated order. We are interested just in the presented terms (3.64) because they refer to the boundary conditions. Indeed expression (3.64) contains terms $z^{-1/2}$, $\ln z$, $z^{1/2}\,\ln z$, which violate their boundary conditions. Hence we are to add to expression (3.64) a combination of solutions of the homogeneous equation to force the boundary conditions to be fulfilled. Finally we obtain

$$
\begin{aligned}
\Delta F(x) = \bar{\Delta} F(x) &- \frac{\pi^2 \mu}{8}\, G^{50}_{08}\left(\frac{\beta x^2}{2^8}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},-\frac{1}{2},\,0,\,0,-1\right)\\
&- \frac{2\,\mu}{3}\, G^{50}_{08}\left(\frac{\beta x^2}{2^8}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,0,\,0,\,\frac{1}{2},-\frac{1}{2},-1\right)\\
&- \pi\,\mu\left(\gamma + \ln\,2 - \frac{43}{48}\right) G^{50}_{08}\left(\frac{\beta x^2}{2^8}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right),
\end{aligned} \tag{3.65}
$$

$$
\mu = \frac{G\,m^2}{2\,(4\pi)^3}.
$$

From this expression we extract the exact value for $F(0)$. While doing this one has to bear in mind, that the presence of a term being proportional to $\ln x$ at $x \to 0$ is a consequence of an expansion in m^2 at $x \gg m^2$. Looking back at the corresponding evaluations we see, that for $x \to 0$ one has to change $\ln x$ for $\ln(4\,m^2)$. Now we have

$$
F(0) = \frac{\pi\sqrt{2}}{4} + \mu\left(4\ln Y + (16-\pi^2)\gamma + (14-\pi^2)\log 2 - \frac{122}{9} + \pi^2\frac{42}{48}\right). \tag{3.66}
$$

For $\mu = 0$ we obtain $F(0) = 1.11072$. Condition $F(0) = 1$ defines the value of μ, which is connected with mass (see (3.65))

$$
\mu = 0.005789. \tag{3.67}
$$

Thus the solution, which is found here, satisfies all the necessary conditions provided (3.67) is valid. Emphasize, that (3.67) defines the mass of the scalar field. Note, that the small value of (3.67) thoroughly justifies the account of only the first term of the expansion in m^2. We reject the second solution of condition $F(0) = 1$, which is of order of unity, due to to its inconsistence with the expansion of the solution in m^2.

We have mentioned already, that generally speaking one has to consider a total chain of compensation equations including connected Green functions with six, eight, etc. legs. Note, that corresponding equations will contain inhomogeneous parts, expressed in terms of Green functions of lower order, and homogeneous parts, being proportional to the corresponding form-factor, e.g. F_6 with six legs. Assuming our result the connected four-leg Green function be zero, we come to the conclusion, that inhomogeneous part of equation for F_6 is zero, so trivial solution $F_6 = 0$ inevitably exists. The analogous considerations lead to conclusions on possibility of existence of trivial solutions of all higher Green functions. One may, of course, study possibilities of existence of nontrivial solutions as well. However, the purpose of the present work is to show that even though one nontrivial solution does exist, so we rely on following variant: nontrivial solution for four-leg connected Green function and trivial solutions for all higher connected Green functions. The consideration of compensation equation for Green function with two legs, which defines mass of the scalar field will be performed particularly later on.

The next step of study should include nonlinear equation with account of all possible diagrams. However this problem evidently do not admit analytic solution. Approximate estimate of nonlinear corrections to the form-factor's value at zero will be obtained in what follows. Maybe future studies will be connected with numerical methods. We are convinced, that the experience achieved in finding of the nontrivial solution will help in formulation and realization of numerical methods. Presumably result (3.67), which means the existence of a solution only for definite relation between dimensional coupling constant and mass of scalar field, will be important.

3.6 Bethe–Salpeter equation and zero excitation

It is well-known, that a symmetry breaking is to be accompanied by an appearance of an excitation with zero mass [40, 42, 43]. Let us consider this problem in the same approximation. While constructing an equation for a bound state one has to keep in mind, that here genuine interaction (3.43) acts, that one, which is referred to the interaction Lagrangian and remains, of course, not compensated. Bethe–Salpeter equation for a massless bound state of two scalar fields in this case has the form

$$\begin{aligned}
\Psi(x) = {} & \frac{G\pi^3\Lambda'}{2(2\pi)^6} - \frac{G^2\pi^6\Lambda\Lambda'}{2\,(2\pi)^{12}} + \frac{G^2\pi^6}{18(2\pi)^{12}}\left(-\frac{1}{20\,x^2}\int_0^x y^3\,\Psi(y)\,dy\right. \\
& + \frac{3}{4x}\int_0^x y^2\,\Psi(y)\,dy + 3\,\ln x\int_0^x y\,\Psi(y)\,dy \\
& + 3\,x\ln x\int_0^x \Psi(y)\,dy + 4\int_0^x y\,\Psi(y)\,dy + 3\int_x^\infty y\,\ln y\,\Psi(y)\,dy \\
& \left.+\,x\int_x^\infty (4+3\ln y)\,\Psi(y)\,dy + \frac{3\,x^2}{4}\int_x^\infty \frac{\Psi(y)}{y}\,dy - \frac{x^3}{20}\int_x^\infty \frac{\Psi(y)}{y^2}\,dy\right), \\
\Lambda' = {} & \int_0^\infty \Psi(y)\,dy.
\end{aligned} \tag{3.68}$$

Comparing this equation (3.68) with compensation equation (3.49), we see the main difference in the sign before the kernel of the integral equation. Remind once more, that the compensation equation is the condition of vanishing of the total expansion in G in the modified free Lagrangian in expression (3.41) and therefore terms of the first and of the third orders are situated in the same part of equation, e.g. in the left-handed one, whereas in the Bethe–Salpeter equation the corresponding terms are situated in different parts of equation.

The sign before the kernel is very important. This means, that in a differential equation sign before β changes as well

$$\left(\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\right.$$
$$\left.\times\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}-3\right)-\beta x^2\right)\Psi(x)=0. \quad (3.69)$$

One easily see, that due to absence of term being proportional to $x \log x$ in the inhomogeneous part boundary conditions are the following

$$a_{-2} = a_{-1} = a_{0l} = a_{1l} = 0. \quad (3.70)$$

The change of sign before β leads to changing of asymptotes at infinity

$$\Psi_k(x) \simeq \exp\left(4\,(\beta x^2)^{1/8}\exp\left(\frac{\imath\pi k}{4}\right)\right), \qquad k = 1, 2, \ldots, 8. \quad (3.71)$$

Now we have three decreasing asymptotes ($k = 3, 4, 5$), two oscillating ones ($k = 2, 6$), and the remaining three are increasing. Using the first five solutions, which allow a definition of integrals at infinity, we fulfill four boundary conditions at zero (3.70). As a result we obtain the following solution of equation (3.68)

$$\Psi(x) = A\,G^{40}_{08}\left(\frac{\beta x^2}{2^8}\,|\,\frac{3}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, 0, -\frac{1}{2}, -1\right), \quad (3.72)$$

where constant A is defined by normalization condition of a Bethe–Salpeter wave function. Direct calculation leads to result $\Lambda' = 0$, so the inhomogeneous part of equation (3.68) vanishes. Thus we have shown, that the equation for a bound state with zero mass has a solution.

The solution being obtained proves the existence of zero mass excitation [40, 42, 43] in the model. Of course definition of a Bethe–Salpeter equation itself is possible only provided a nontrivial solution of a compensation equation to exist and thus interaction (3.43) to act. The obligatory correspondence between a nontrivial solution of a compensation equation and an existence of a zero excitation thoroughly corresponds to Bogoliubov quasi-averages approach [40].

It is interesting to note, that with taking into account of three-fold interaction $g\,\phi^3$ in the kernel of equation (3.68) the mass of the bound state becomes nonzero. One easily understands this, because interaction (3.68) itself leads to dimensional parameter Λ_3 being present and thus the scale invariance being already broken.

3.7 Compensation equation for scalar field mass

Let us look at interaction Lagrangian (3.42). The mass term there is quite improper. To solve the problem one has to formulate a compensation equation for Green function with two scalar legs. Let us consider this equation taking into account solution (3.62) and three-fold interaction. The compensation equation means nullification of total contribution of interaction (3.42) to the mass. In the first approximation the contribution of the four-fold interaction is described by the first order diagram "bubble" and that of the three-fold one is represented by simple one-loop diagram (see Figure 3.5).

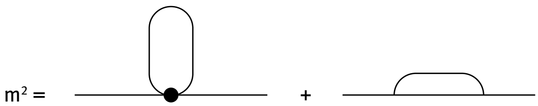


Fig. 3.5. Compensation equation for mass of the scalar field.

Putting momenta of the external legs to be zero, we have for "bubble" diagram just solution (3.62) in the vertex. As a result we obtain the following compensation equation for scalar mass

$$\begin{aligned} m^2 &= -\frac{G}{(2\pi)^6}\int \frac{F(q^2)\,d^6q}{q^2+m^2} - \frac{g^2}{(2\pi)^6}\int \frac{d^6q}{(q^2+m^2)^2} \\ &= -\frac{G}{2\,(4\pi)^3}\int_0^\infty y\,dy\,(F_0(y)+\Delta F(y)) \\ &\quad + \frac{Gm^2}{2\,(4\pi)^3}\int_0^\infty dy\,F_0(y) - \frac{g^2}{2\,(4\pi)^3}\int_0^\infty \frac{y^2\,dy}{(y+m^2)^2}, \end{aligned} \tag{3.73}$$

Here in "bubble" diagram we perform an expansion in m^2 and take into account the zeroth and the first orders of the expansion. By direct evaluation with the aid of expressions (3.61, 3.64, 3.65) we obtain that the zeroth order terms is zero and the first order term is equal to $3\,m^2$. The loop, which is described by the last term in (3.73), quadratically diverges. Note, that in the initial theory (3.35) we introduce some cut-off Λ_3, which corresponds to a physical limitation of a region of applicability of the theory. As a result we have the following compensation equation for the mass provided $m \ll \Lambda_3$

$$m^2 = 3\,m^2 - \frac{g^2}{2\,(4\pi)^3}\,\Lambda_3^2. \tag{3.74}$$

Emphasize, that for the trivial solution $G = 0$ the first term in the right-hand side of equation (3.74) is absent and we have a negative mass squared, i.e. a tachyon solution. For the nontrivial solution we have

$$m^2 = \frac{g^2}{4\,(4\pi)^3}\,\Lambda_3^2. \tag{3.75}$$

It is well-known, that a scalar tachyon leads to instability for small fields. Therefore the restoration of the normal sign of the mass squared, which is achieved provided the nontrivial solution is valid, corresponds to a transition to a more stable state.

So the value of the scalar mass is defined in terms of initial parameters of the theory g and Λ_3. The value of parameter Y (3.67) gives the relation of the mass and of the coupling constant G of the four-fold interaction. Thus all the parameters entering into the nontrivial solution are defined in terms of the initial ones.

Note that the initial cut-off Λ_3 corresponds to some boundary energy, which provides real physical cut-off of the corresponding integrals. In the physical four-dimensional space-time it may be for example the Planck energy $1.22 \cdot 10^{19}$ GeV. One should expect the expressions similar to (3.74) also would lead to relations, which connect the theory parameters with a boundary energy (e.g. the Planck one), which enters into logarithmic divergent terms.

The final result for effective Lagrangian of the theory after the symmetry breaking occurs is the following

$$L = \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\mu} - \frac{m^2}{2}\phi^2 + \frac{g_0}{3!}\phi^3 \qquad (3.76)$$
$$+ \frac{G}{4!}\int \bar{F}(x,x_1,x_2,x_3,x_4)\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,dx_1dx_2dx_3dx_4,$$

where form-factor F is the solution of the compensation equation.

3.8 Estimate of nonlinearity influence

Till now our results were obtained in the framework of the linear approximation. The decrease of the form-factor at infinity indicates an applicability region of the approximation. It evidently is incorrect for large momenta variables because the effective coupling constant becomes too small in comparison to constant G, which was used to define the kernel of the integral equation. We can roughly take into account an influence of a nonlinearity, using the following procedure.

Let equation (3.51) be valid for small x (we put $m^2 = 0$).

$$\frac{d^4}{dx^4}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) = -\beta\,\frac{F(x)}{x^2}, \quad \beta = \frac{2\,G^2}{(4\,\pi)^6}. \qquad (3.77)$$

We use this equation with the corresponding (3.55) for $x \le x_0$, whereas for $x \ge x_0$ one has to take into account a nonlinearity. Let us draw attention to the fact, that β is proportional to G^2 i.e. it contains the form-factor squared. Therefore for $x \ge x_0$ instead of (3.77) we use the following equation

$$\frac{d^4}{dx^4}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) = -\beta\,\frac{F^3(x)}{x^2}. \qquad (3.78)$$

In this approximation we have correct behavior of right-hand sides at small (3.77) and at very large (3.78) values of x. In the intermediate region there is a tear in the right hand side at $x = x_0$. This means that the eighth derivative tears at this point. As we shall see soon the form-factor and its derivatives up to the fifth order have to be continuous.

Let us introduce variable $y = \sqrt{\beta}\,x$. One easily sees that for $y \to \infty$ equation (3.78) defines the following decreasing asymptotic

$$F(y) \simeq \frac{b}{y^2} - \frac{6\,b^3}{5!\,7!\,y^4} + \frac{12\,b^5}{7!\,7!\,8!\,y^6} + \cdots, \tag{3.79}$$

where b is a constant. At the same time equation (3.77) with account of boundary conditions has the following solution in region $(0,\,y_0)$:

$$\begin{aligned} F(y) = {} & \frac{\sqrt{2}}{4}\,G^{50}_{08}\left(\frac{y^2}{256}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right) \\ & + C_1\,G^{30}_{08}\left(\frac{y^2}{256}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right) \\ & + C_2\,G^{30}_{08}\left(\frac{y^2}{256}\,|\,\frac{3}{2},\,1,\,0,\,\frac{1}{2},\,\frac{1}{2},\,0,-\frac{1}{2},-1\right) \\ & + C_3\,G^{10}_{08}\left(\frac{y^2}{2}56\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right) \\ & + C_4\,G^{10}_{08}\left(\frac{y^2}{256}\,|\,1,\,\frac{3}{2},\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right), \end{aligned} \tag{3.80}$$

where C_i are constants. The appearance of the additional terms with these coefficients multiplied by Meijer functions increasing at infinity is due to the fact, that now the decrease at infinity is provided by asymptotic (3.79) and thus in region $(0,\,y_0)$ we have to use all solutions of equation (3.77), which fulfill the boundary conditions at zero. The first line here is solution (3.61), which was obtained earlier. Let us begin a sequential account of the new terms starting from the zero approximation, in which in region $(0,\,y_0)$ we have this old solution, i.e. all $C_i = 0$. This solution is matched to solution (3.79) in point y_0. It will come clear, that in expression (3.79) an account of the first term is sufficient. Then from continuity of the function and of its first derivative we obtain the following set of equations

$$\begin{aligned} & \frac{\sqrt{2}}{4}\,G^{50}_{08}\left(\frac{y_0^2}{256}\,|\,\frac{3}{2},\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,0,-\frac{1}{2},-1\right) - \frac{b}{y_0^2} = 0, \\ & \frac{\sqrt{2}}{4}\,G^{50}_{08}\left(\frac{y_0^2}{256}\,|\,\frac{3}{2},\,1,\,1,\,\frac{1}{2},\,\frac{1}{2},\,0,-\frac{1}{2},-1\right) - \frac{b}{y_0^2} = 0, \end{aligned} \tag{3.81}$$

Solution of the set:

$$y_0 = 8.4980, \quad b = 7.5055. \tag{3.82}$$

The second term in asymptotic (3.79) at y_0 comprises $7.7 \cdot 10^{-6}$ times the first one, that justifies the account of the first term only. The value of the form-factor at zero does not change $F(0) = 1.1107$.

Now let us take into account two additional terms in (3.80) with coefficients C_1 and C_2, which for small y give larger contribution than the remaining two terms. In this case we have to match values of the function and of its derivatives up to the third order. One obtains the set of four equations with aid of rules of differentiation of Meijer functions [71]. Its solution reads

$$y_0 = 17.635, \quad b = 9.410, \quad C_1 = 0.0166, \quad C_2 = -0.0538. \tag{3.83}$$

The value of the form-factor at zero becomes

$$F(0) = \frac{\pi\sqrt{2}}{4} + \frac{C_2}{\pi} = 1.0936. \tag{3.84}$$

Now let us take into account terms with coefficients C_3, C_4. We consider them and deviations from solution (3.83) as well to be small. Then matching the function and its derivatives up to the fifth order, we obtain a set of six linear equations leading to the following solution

$$\begin{aligned} &\Delta y_0 = 1.457, \quad \Delta b = 1.032, \quad \Delta C_1 = -0.0094,\\ &\Delta C_2 = 0.0223, \quad C_3 = -0.0249, \quad C_4 = 0.0136. \end{aligned} \tag{3.85}$$

Substituting the last result into (3.84), we have

$$F(0) = 1.1007. \tag{3.86}$$

The sequence of numbers 1.1107, 1.0936, 1.1007 for value $F(0)$ demonstrates stability of the result in respect to contribution of nonlinear corrections

3.9 Conclusions of simple scalar model

Grounding on the results being obtained we conclude, that in the model under consideration a nontrivial solution does exist, which breaks the initial scale invariance and leads to a spontaneous appearance of effective interaction in Lagrangian (3.76), acting in a restricted region of the momenta space in accordance with the value of parameter G. Effective form-factor $F(p)$ decreases exponentially with oscillations for $p^2 \to \pm\infty$, i.e. both for space-like and time-like momenta. We confirm the existence of a zero mass excitation, which has to be present for an occurrence of spontaneous symmetry breaking.

We start with the renormalizable theory of a scalar field (in a six-dimensional space), and we obtain as a result the definite theory with interaction breaking scale symmetry. New dimensional parameters $G^{-1/2}$ and m are proportional to parameter Λ_3, which defines the initial asymptotically free interaction. Let us emphasize once more, that the interaction being obtained is an effective one, that first of all is reflected in a presence of form-factor $F(p)$, which is just the solution of the compensation equation. At momentum infinity the theory becomes asymptotically free again.

It is quite important, that the problem under consideration has a consistent solution only provided triple interaction $g\,\phi^3$ is acting. Really, albeit compensation equation (3.48) contains no contribution of this interaction, the nonzero scalar field mass appears only for $g \neq 0$. If it is not the case the value of form-factor at zero $F(0)$ is not unity. In general one can not exclude a possibility of condition $F(0) = 1$ being fulfilled for $m = 0$. However the experience obtained in considering the present problem shows that this condition could be fulfilled only provided the model has very peculiar properties. As a matter of fact the problem under consideration is defined not by compensation equation (3.48) only, but by set of equations (3.48, 3.73), which explicitly contains a contribution of triple interaction $g\,\phi^3$. The same conclusion we are to make after consideration of spontaneous generation of the Nambu–Jona-Lasinio effective interaction in Section 3.3

It should be noted, that a possibility of a nontrivial solution strongly depends on the choice of the theory. This may be demonstrated by comparison of different signatures of the six-dimensional space-time. Namely if one instead of signature $1 + 5$ will choose signature $3+3$, then in definition (3.36) of a transition to Euclidean coordinates the sign before $\imath\, d^6p$ changes. As a result all signs change for one-loop integrals. For four-fold interaction we restore all previous results by simple substitution $G \to -G$. However the one-loop integral with two three-fold vertices inevitably changes sign and relation (3.74) leads to tachyon mass. So we come to the conclusion, that for signature $3 + 3$ only the trivial solution $G = 0$ is stable.

Of course, we base our conclusions only on exact solutions of approximate equations. However it is possible, that qualitative properties of solutions, which manifest themselves in the model problem, will be quite useful in study of problems of spontaneous symmetry breaking in more realistic cases, when there is no hope for analytic solution of corresponding equations and what is possible to apply are just numerical methods. Attractive qualitative results are the existence of relations between parameters of the problem and the natural appearance of small parameter Y (3.67). The essential result is connected also with the conclusion on the stability of the nontrivial solution. The estimate of nonlinearity contribution, which does not lead to decisive change of properties of the solution, provides additional argument on behalf of the present approach.

The resulting theory is nonlocal and the question might arise, whether the general principles of unitarity and causality are here valid. The initial theory (3.35) quite corresponds to these principles. One should expect, that its solutions, nontrivial ones as well, have also to fulfill these conditions. Therefore one can consider the present example as a step in direction of formulating of a consistent nonlocal theory. Basing on results of the present work we may assume, that such theory can be consistent not for an arbitrary form-factor but for the one, which follows from a nontrivial solution of an initially local theory.

Without any doubt a possibility of spontaneous generation of an effective interaction, containing a dimensional parameter, is of great interest for studies of problems beyond the Standard Model. In particular, the phenomenon of a spontaneous generation of an effective interaction, provided it to occur in a genuine physical theory, e.g. in the electro-weak theory, might essentially promote our understanding of bases of the theory. A subsequent results in this direction and applications of the approach to spontaneous generation of effective interactions in gauge theories of the Standard Model will be presented below.

To conclude the present chapter we would state, that theory of massless scalar field ϕ with interaction $g\phi^3$ in six-dimensional space is considered. A possibility of initial scale invariance breaking, which results in a spontaneous arising of effective interaction $G\phi^4$, is studied by application of Bogoliubov quasi-averages approach. It is shown, that compensation equation for form-factor of this interaction in approximation up to the third order in G has a nontrivial solution. In the same approximation the Bethe–Salpeter equation for a zero-mass bound state of two scalar fields ϕ is shown to have a solution. The conditions imposed on form-factor value at zero and scalar field mass m fix the unique solution, which gives relations between parameters of interaction $g\phi^3$ and parameters G and m. Arguments are laid down in favor of a stability of the nontrivial solution.

3.10 Appendix

Here formulas of integration by parts of expressions entering in equation (3.49) are presented

$$\beta\int_0^x \frac{y^2\,F(y)}{(y+m^2)^2}\,dy = -x^2\,\frac{d^3}{dx^3}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) + 2x\,\frac{d^2}{dx^2}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) - 2\,\frac{d}{dx}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) + 12\,a_{1l}, \tag{3.87}$$

$$\beta\int_0^x \frac{y^3\,F(y)}{(y+m^2)^2}\,dy = -x^3\,\frac{d^3}{dx^3}\left(x^2\,\frac{d^4}{dx^4}\left(x^2F(x)\right)\right) + 3x^2\,\frac{d^2}{dx^2}\left(x^2\,\frac{d^4}{dx^4}\left(x^2F(x)\right)\right) - 6x\,\frac{d}{dx}\left(x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right)\right) + 6\,x^2\,\frac{d^4}{dx^4}\left(x^2\,F(x)\right) - 12\,a_{0l},$$

$$\beta \int_0^x \frac{y^4 F(y)}{(y+m^2)^2} dy = -x^4 \frac{d^3}{dx^3}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right)$$
$$+ 4x^3 \frac{d^2}{dx^2}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right) - 12x^2 \frac{d}{dx}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right)$$
$$+ 24x^3 \frac{d^4}{dx^4}\left(x^2 F(x)\right) - 24x^2 \frac{d^3}{dx^3}\left(x^2 F(x)\right) + 48\,x \frac{d^2}{dx^2}\left(x^2 F(x)\right)$$
$$- 48 \frac{d}{dx}\left(x^2 F(x)\right) + 48\, a_{-1},$$

$$\beta \int_0^x \frac{y^5 F(y)}{(y+m^2)^2} dy = -x^5 \frac{d^3}{dx^3}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right)$$
$$+ 5x^4 \frac{d^2}{dx^2}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right) - 20x^3 \frac{d}{dx}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right)$$
$$+ 60x^4 \frac{d^4}{dx^4}\left(x^2 F(x)\right) - 120x^3 \frac{d^3}{dx^3}\left(x^2 F(x)\right)$$
$$+ 360\,x^2 \frac{d^2}{dx^2}\left(x^2 F(x)\right) - 720x \frac{d}{dx}\left(x^2 F(x)\right) + 720x^2 F(x) - 720 a_{-2},$$

$$\beta \int_x^\infty \frac{F(y)}{(y+m^2)^2} dy = \frac{d^3}{dx^3}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right),$$

$$\beta \int_x^\infty \frac{y\,F(y)}{(y+m^2)^2} = x \frac{d^3}{dx^3}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right) - \frac{d^2}{dx^2}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right),$$

$$\beta \int_x^\infty \frac{y^2 \ln y\,F(y)}{(y+m^2)^2} dy = x^2 \ln x \frac{d^3}{dx^3}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right) - (2x \ln x + x)$$
$$\times \frac{d^2}{dx^2}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right) + (2 \ln x + 3) \frac{d}{dx}\left(x^2 \frac{d^4}{dx^4}\left(x^2 F(x)\right)\right)$$
$$- 2\,x \frac{d^4}{dx^4}\left(x^2 F(x)\right) + 2 \frac{d^3}{dx^3}\left(x^2 F(x)\right),$$

$$\beta\int_x^\infty \frac{y^2 F(y)}{(y+m^2)^2}dy = x^2\frac{d^3}{dx^3}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)$$
$$-2x\frac{d^2}{dx^2}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)+2\frac{d}{dx}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right),$$

$$\beta\int_x^\infty \frac{y^3\ln y\,F(y)}{(y+m^2)^2}dy = x^3\ln x\frac{d^3}{dx^3}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)-(3x^2\ln x+x^2)$$
$$\times\frac{d^2}{dx^2}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)+(6x\ln x+5x)\frac{d}{dx}\left(x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)\right)$$
$$-(6\ln x+11)x^2\frac{d^4}{dx^4}\left(x^2F(x)\right)+6x\frac{d^3}{dx^3}\left(x^2F(x)\right)-6\frac{d^2}{dx^2}\left(x^2F(x)\right).$$

4 Three-gluon effective interaction

In this chapter we begin to study nonperturbative effects in strong interactions. We proceed from initial QCD, the main points of which is presented in the first two chapters. We have already laid down, that there is unavoidable necessity to introduce nonperturbative quantities, e. ,g, condensate averages. In view to achieve a description of nonperturbative contributions we shall rely on the main idea of the present book and we shall look for effects of spontaneous generation of effective interactions. The results to be discussed in the chapter are mostly obtained in work [76].

4.1 Compensation equation

For the beginning we consider pure gluon QCD without quarks. We start with Lagrangian with gauge group $SU(3)$. That is we define the gauge sector to be color octet of gluons F^a_μ.

$$L = -\frac{1}{4}\,F^a_{\mu\nu}F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu F^a_\nu - \partial_\nu F^a_\mu + g\,f_{abc}F^b_\mu F^c_\nu. \tag{4.1}$$

where we use the standard notations. Let us consider a possibility of spontaneous generation of an effective interaction

$$-\frac{G}{3!}\cdot f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}, \tag{4.2}$$

which may be called an anomalous three-gluon interaction.

Here notation $\frac{G}{3!}\cdot f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}$ means corresponding nonlocal vertex in the momentum space

$$\begin{aligned}(2\pi)^4 G f_{abc}(&g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk)\\ &+ g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho)\\ &\times F(p,q,k)\,\delta(p+q+k) + \cdots\,,\end{aligned} \tag{4.3}$$

where $F(p,q,k)$ is a form-factor and $p,\mu,a;\;\; q,\nu,b;\;\; k,\rho,c$ are respectively incoming momenta, Lorentz indices and color indices of gluons.

In accordance to the Bogoliubov compensation approach [40–42] in application to QFT, which was discussed in Chapter 3, we look for a nontrivial solution of a compensation equation, which is formulated on the basis of the Bogoliubov procedure add-subtract. Namely let us write down the initial expression (4.1) in the following form

$$\begin{aligned} L &= L_0 + L_{int},\\ L_0 &= -\frac{1}{4}\,F^a_{\mu\nu}F^a_{\mu\nu} + \frac{G}{3!}\cdot f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}, \qquad (4.4)\\ L_{int} &= -\frac{G}{3!}\cdot f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}. \qquad (4.5)\end{aligned}$$

Here isotopic and color summation is performed inside of each quark bi-linear combination, and notation $-\frac{G}{3!}\cdot f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}$ means corresponding nonlocal vertex in the momentum space

$$(2\pi)^4\,G\,f_{abc}\,(g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk) \\ + g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho)\,F(p,q,k)\,\delta(p+q+k) + \cdots, \tag{4.6}$$

where $F(p,q,k)$ is a form-factor and $p,\mu,a;\;\; q,\nu,b;\;\; k,\rho,c$ are respectfully incoming momenta, Lorentz indices and color indices of gluons. We mean also that there are present four-gluon, five-gluon and six-gluon vertices according to expression for $F^a_{\mu\nu}$ (4.1). Note, that inclusion of total gluon term $F^a_{\mu\nu}\,F^a_{\mu\nu}$ in the new free Lagrangian (4.4) is performed in view of maintaining the gauge invariance of the approach.

Effective interaction (4.2) is called anomalous three-gluon interaction. Our interaction constant G is to be defined by the subsequent studies.

Let us consider expression (4.4) as the new free Lagrangian L_0, whereas expression (4.5) as the new interaction Lagrangian L_{int}. It is important to note, that we put into the new free Lagrangian the full quadratic in F term including boson self-interaction, because we prefer to maintain gauge invariance of the approximation being used. Indeed, we shall use both quadratic term from the last term in (4.4) and triple one from the last but one term of (4.4). Then compensation conditions (see for details Chapter 3) will consist in demand of full connected three-boson vertices of the structure (4.6), following from Lagrangian L_0, to be zero. This demand gives a nonlinear equation for form-factor F.

Such equations according to our terminology are compensation equations. In a study of these equations it is always evident the existence of a perturbative trivial solution (in our case $G = 0$), but, in general, a nonperturbative nontrivial solution may also exist. Just the quest of a nontrivial solution inspires the main interest in such problems. One can not succeed in finding an exact nontrivial solution in a realistic theory, therefore the goal of a study is a quest of an adequate approach, the first non-perturbative approximation of which describes the main features of the problem. Improvement of a precision of results is to be achieved by corrections to the initial first approximation.

Thus our task is to formulate the first approximation. Here the experience acquired in the course of performing works [74, 77, 78], as well as examples being considered above, could be helpful. Now in view of obtaining the first approximation we would make the following assumptions.

(1) In a compensation equation we restrict ourselves by terms with loop numbers 0, 1.
(2) We reduce thus obtained nonlinear compensation equation to a linear integral equation. It means that in loop terms only one vertex contains the form-factor, being defined above, while other vertices are considered to be point-like. In diagram form equation for form-factor F is presented in Figure 4.1. Here four-leg vertex correspond to an interaction of four gluons due to our effective three-field

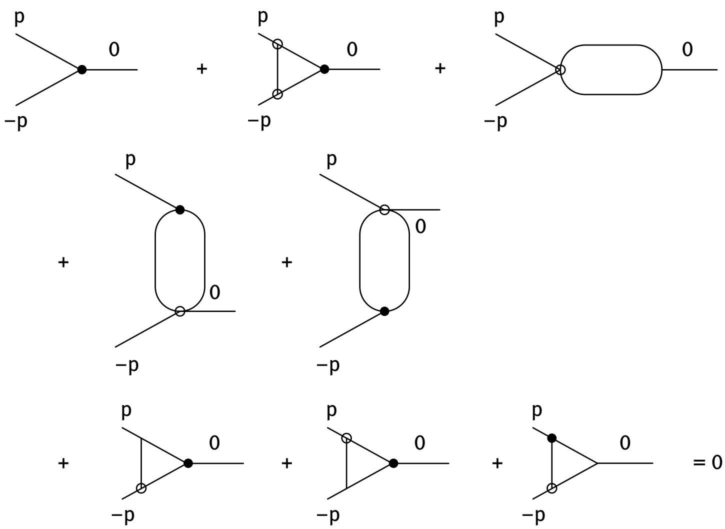


Fig. 4.1. Diagrams, describing the compensation equation for vertex (4.6), denoted by the black spot, open circles denote the same vertex without form-factor and also four-leg vertex (6.10) without the form-factor. Simple lines correspond to gluons.

interaction. In our approximation we take here a point-like vertex with an interaction constant being proportional to $g\,G$.

(3) We integrate by angular variables of the 4-dimensional Euclidean space. The necessary rules are presented in Section 5.8 and in paper [77].

Let us note that such approximation was previously used in works [77, 78] in the study of spontaneous generation of the effective Nambu–Jona-Lasinio interaction. It was shown in Chapter 3 that the accuracy of the approximation could be estimated to be $\simeq 10$–$15\,\%$. Thus we could hope for such accuracy in the present problem.

At first let us present the expression for four-gluon vertex

$$
\begin{aligned}
&\frac{V(p,m,\lambda;\ q,n,\sigma;\ k,r,\tau;\ l,s,\pi)}{\imath\,(2\,\pi)^4} = \qquad\qquad (4.7)\\
&gG\Big(f^{amn}f^{ars}\Big(U(k,l;\sigma,\tau,\pi,\lambda) - U(k,l;\lambda,\tau,\pi,\sigma) - U(l,k;\sigma,\pi,\tau,\lambda) + U(l,k;\lambda,\pi,\tau,\sigma)\\
&\quad + U(p,q;\pi,\lambda,\sigma,\tau) - U(p,q;\tau,\lambda,\sigma,\pi) - U(q,p;\pi,\sigma,\lambda,\tau) + U(q,p;\tau,\sigma,\lambda,\pi)\Big)\\
&\quad + f^{arn}\,f^{ams}\Big(U(p,l;\sigma,\lambda,\pi,\tau) - U(l,p;\sigma,\pi,\lambda,\tau) - U(p,l;\tau,\lambda,\pi,\sigma) + U(l,p;\tau,\pi,\lambda,\sigma)\\
&\quad + U(k,q;\pi,\tau,\sigma,\lambda) - U(q,k;\pi,\sigma,\tau,\lambda) - U(k,q;\lambda,\tau,\sigma,\pi) + U(q,k;\lambda,\sigma,\tau,\pi)\Big)\\
&\quad - f^{asn}\,f^{amr}\Big(U(k,p;\sigma,\tau,\lambda,\pi) - U(p,k;\sigma,\lambda,\tau,\pi) + U(p,k;\pi,\lambda,\tau,\sigma) - U(k,p;\pi,\tau,\lambda,\sigma)\\
&\quad - U(l,q;\tau,\pi,\sigma,\lambda) + U(l,q;\lambda,\pi,\sigma,\tau) - U(q,l;\lambda,\sigma,\pi,\tau) + U(q,l;\tau,\sigma,\pi,\lambda)\Big)\Big),\\
&U(k,l;\sigma,\tau,\pi,\lambda) = (k_\sigma\, l_\tau\, g_{\pi\lambda} - k_\sigma\, l_\lambda\, g_{\pi\tau} + k_\pi\, l_\lambda\, g_{\sigma\tau} - (kl) g_{\sigma\tau} g_{\pi\lambda}) F(k,l,-(k+l)).
\end{aligned}
$$

Here triad p, m, λ etc. means correspondingly incoming momentum, color index, Lorentz index of a gluon. Properties of structure constants f^{amn} of the $SU(3)$ group are explicated in Section 1.4.

Let us formulate compensation equations in this approximation. For free Lagrangian L_0 full connected three-boson vertices with Lorentz structure (4.6) are to vanish. One can succeed in obtaining analytic solutions for the following set of momentum variables (see Figure 4.1): left-hand legs have momenta p and $-p$, and a right-hand leg has zero momenta. However in our approximation we need form-factor F also for nonzero values of this momentum. We look for a solution with the following simple dependence on all three variables

$$F(p_1, p_2, p_3) = F\left(\frac{p_1^2 + p_2^2 + p_3^2}{2}\right), \tag{4.8}$$

Really, expression (4.8) is symmetric and it turns to $F(x)$ for $p_3 = 0$, $p_1^2 = p_2^2 = x$. We consider the representation (4.8) to be the first approximation and we postpone calculations of corresponding corrections for forthcoming studies. Now according to the rules being stated above we obtain the following equation for form-factor $F(x)$ due to diagram representation in Figure 4.1

$$\begin{aligned}
F(x) = &-\frac{G^2 N}{64\pi^2}\left(\int_0^Y F(y)\,y\,dy - \frac{1}{12x^2}\int_0^x F(y)\,y^3\,dy\right.\\
&\left.+\frac{1}{6x}\int_0^x F(y)\,y^2\,dy + \frac{x}{6}\int_x^Y F(y)\,dy - \frac{x^2}{12}\int_x^Y \frac{F(y)}{y}\,dy\right)\\
&+\frac{GgN}{16\pi^2}\int_0^Y F(y)\,dy + \frac{GgN}{24\pi^2}\left(\int_{3x/4}^x \frac{(3x-4y)^2(2y-3x)}{x^2(x-2y)}F(y)\,dy\right.\\
&\left.+\int_x^Y \frac{(5x-6y)}{(x-2y)}F(y)\,dy\right) + \frac{GgN}{32\pi^2}\left(\int_x^Y \frac{3(x^2-2y^2)}{8(2y-x)^2}F(y)\,dy\right.\\
&+\int_{3x/4}^x \frac{3(4y-3x)^2(x^2-4xy+2y^2)}{8x^2(2y-x)^2}F(y)\,dy\\
&\left.+\int_0^x \frac{5y^2-12xy}{16x^2}F(y)\,dy + \int_x^Y \frac{3x^2-4xy-6y^2}{16y^2}F(y)\,dy\right).
\end{aligned} \tag{4.9}$$

Here $x = p^2$ and $y = q^2$, where q is an integration momentum, $N = 3$ for QCD. The last four terms in brackets represent diagrams with one usual gauge vertex (see the last three diagrams at Figure 4.1). These terms maintain the gauge invariance of results in this approximation. Note that one can additionally check the gauge invariance by introduction of longitudinal term $d_l\, k_\mu k_\nu/(k^2)^2$ in boson propagators to verify an independence of results on d_l in this approximation. Ghost contributions also give zero result in the present approximation due to vertex (4.6) being transversal:

$$\begin{aligned} p_\mu\, V(p,q,k)_{\mu\nu\rho} &= q_\nu\, V(p,q,k)_{\mu\nu\rho} = k_\rho\, V(p,q,k)_{\mu\nu\rho} = 0, \\ V(p,q,k)_{\mu\nu\rho} &= g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk) \\ &\quad + g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho. \end{aligned} \tag{4.10}$$

Gauge invariance might be also violated by terms arising from momentum dependence of form-factor F. However this problem does not arise in the approximation corresponding to equation (4.9) and becomes essential for taking into account of g^2 terms. In this case ghost contributions also do not cancel. The problem of the gauge invariance of the next approximations has to be considered in future studies.

We introduce in equation (4.9) an effective cut-off Y, which bounds a "low-momentum" region where our nonperturbative effects act and consider the equation at interval $[0,\ Y]$ under condition

$$F(Y) = 0. \tag{4.11}$$

The value of this parameter is to be defined in the process of a solution of the compensation equation. We shall solve equation (4.9) by iterations. That is we expand its terms being proportional to g in powers of x and take at first only a constant term. Thus we have

$$\begin{aligned} F_0(x) = &-\frac{G^2 N}{64\,\pi^2}\Bigg(\int_0^Y F_0(y)\, y dy - \frac{1}{12\,x^2}\int_0^x F_0(y)\, y^3 dy \\ &+ \frac{1}{6\,x}\int_0^x F_0(y)\, y^2 dy + \frac{x}{6}\int_x^Y F_0(y)\, dy - \frac{x^2}{12}\int_x^Y \frac{F_0(y)}{y}\, dy\Bigg) \\ &+ \frac{87\, G g N}{512\,\pi^2}\int_0^Y F_0(y)\, dy. \end{aligned} \tag{4.12}$$

Expression (4.12) again provides an equation of the type which were already studied in Chapters 2 and 3, where the way of obtaining solutions of equations analogous to (4.12) are described. Indeed, by successive differentiation of equation (4.12) we come to the Meijer differential equation

$$\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)F_0(x) + \frac{G^2 N x^2}{64\,\pi^2}\, F_0(x) = A, \tag{4.13}$$

$$A = 4\left(-\frac{G^2 N}{64\,\pi^2}\int_0^Y F_0(y)\, y dy + \frac{87\, G g N}{512\,\pi^2}\int_0^Y F_0(y)\, dy\right),$$

which solution again is expressed in terms of Meijer functions

$$
\begin{aligned}
F_0(x) &= C_1 G^{10}_{04}(z|1/2,1,-1/2,-1) + C_2 G^{10}_{04}(z|1,1/2,-1/2,-1)\\
&- \frac{GN}{128\pi^2}\, G^{31}_{15}(z\,|^{0}_{1,1/2,0,-1/2,-1}) \int_0^Y \left(Gy - \frac{87\,g}{8}\right) F_0(y)\,dy,\\
G^{31}_{15}(z\,|^{0}_{1,1/2,0,-1/2,-1}) &= \frac{1}{2z} - G^{30}_{04}(z\,|1,1/2,-1,-1/2),\\
z &= \frac{G^2 N x^2}{1024\pi^2}.
\end{aligned}
\tag{4.14}
$$

Constants C_1, C_2 are defined by the following boundary conditions

$$
\begin{aligned}
\left[2z^2 \frac{d^3 F_0(z)}{dz^3} + 9z \frac{d^2 F_0(z)}{dz^2} + \frac{dF_0(z)}{dz}\right]_{z=z_0} &= 0,\\
\left[2z^2 \frac{d^2 F_0(z)}{dz^2} + 5z \frac{dF_0(z)}{dz} + F_0(z)\right]_{z=z_0} &= 0,\\
z_0 &= \frac{G^2 N Y^2}{1024\pi^2}.
\end{aligned}
\tag{4.15}
$$

Conditions (4.11, 4.15) defines set of parameters

$$
z_0 = \infty, \quad C_1 = 0, \quad C_2 = 0. \tag{4.16}
$$

The normalization condition for form-factor $F(0) = 1$ here is

$$
-\frac{G^2 N}{64\pi^2} \int_0^\infty F_0(y)\,y\,dy + \frac{87\,GgN}{512\pi^2} \int_0^\infty F_0(y)\,dy = 1. \tag{4.17}
$$

However the first integral in (4.17) diverges due to asymptotic

$$
G^{31}_{15}\left(z\,|^{0}_{1,1/2,0,-1/2,-1}\right) \to \frac{1}{2z}, \quad z \to \infty,
$$

and we have no consistent solution. In view of this we consider the next approximation. We substitute solution (4.14) with account of (4.17) into terms of equation (4.9) being proportional to gauge constant g but the constant ones and calculate terms proportional to $\sqrt{z}$. Now we have bearing in mind the normalization condition

$$
\begin{aligned}
F(z) &= 1 + \frac{85\,g\sqrt{N}\sqrt{z}}{96\pi}\Bigg(\ln z + 4\gamma + 4\ln 2 - \frac{1975}{168}\\
&+ \frac{1}{2} G^{31}_{15}\left(z_0\,|^{0}_{0,0,1/2,-1,-1/2}\right)\Bigg) - \frac{2}{3z}\int_0^z F(t)\,t\,dt\\
&- \frac{2z}{3}\int_z^{z_0} F(t)\frac{dt}{t} + \frac{4}{3\sqrt{z}}\int_0^z F(t)\sqrt{t}\,dt + \frac{4\sqrt{z}}{3}\int_z^{z_0} F(t)\frac{dt}{\sqrt{t}},
\end{aligned}
\tag{4.18}
$$

where γ is the Euler constant. We look for a solution of (4.18) in the form

$$F(z) = \frac{1}{2} G_{15}^{31}\left(z\,|^{0}_{1,1/2,0,-1/2,-1}\right) - \frac{85 g\sqrt{N}}{128\pi} G_{15}^{31}\left(z|^{1/2}_{1,1/2,1/2,-1/2,-1}\right)$$
$$- C_1 G_{04}^{10}\,(z|1/2,\ 1, -1/2, -1) + C_2\, G_{04}^{10}\,(z|1,\ 1/2, -1/2, -1)\,. \tag{4.19}$$

We have also conditions

$$1 + 8\int_0^{z_0} F(z)\,dz = \frac{87\,g\,\sqrt{N}}{32\,\pi}\int_0^{z_0} F_0(z)\,\frac{dz}{\sqrt{z}}, \tag{4.20}$$

$$F(z_0) = 0, \tag{4.21}$$

and boundary conditions analogous to (4.15). The last condition (4.21) means smooth transition from the nontrivial solution to trivial one $G = 0$. Knowing form (4.19) of a solution we calculate both sides of relation (4.18) in two different points in interval $0 < z < z_0$ and having four equations for four parameters solve the set. With $N = 3$ we obtain the following solution, which we use to describe QCD case

$$g(z_0) = 3.8166, \quad z_0 = 0.009553,$$
$$C_1 = -5.19055, \quad C_2 = 5.46167. \tag{4.22}$$

We would draw attention to the fixed value of parameter z_0. The solution exists only for this value (4.22) and it plays the role of eigenvalue. As a matter of fact from the beginning the existence of such eigenvalue is by no means evident. This parameter z_0 defines scale appropriate to the solution. That is why we take value of running coupling g in solution (4.22) just at this point.

Note, that we use notations $F(z)$ and $F(x)$ for the form-factor, bearing in mind, that these two functions are connected by the substitution of variable according to (4.14).

Let us take three-loop expression for $\alpha_s(\mu^2)$ (1.69) and take for normalization in low energy region with $N_f = 3$ its value at mass of τ-lepton. We have

$$\alpha_s(M_\tau = 1777\,\text{MeV}) = 0.320 \pm 0.005. \tag{4.23}$$

From here we obtain

$$\Lambda_3 = (345 \pm 19)\,\text{MeV}. \tag{4.24}$$

We normalize the running coupling by condition

$$\alpha_s(x_0) = \frac{g(z_0)^2}{4\,\pi} = 1.15515, \tag{4.25}$$

where coupling constant g entering in expression (4.22) is just corresponding to this normalization point. Now from definition of z (4.14) and value z_0 (4.22) we have

$$G = \frac{1}{\Lambda_G^2}, \quad \Lambda_G = (264 \pm 7)\,\text{MeV}. \tag{4.26}$$

Typical range around 250 MeV is natural for strong interaction. It is also worth mentioning the value of the momentum which corresponds to boundary of nonperturbative region z_0. From equations (4.22, 4.26) we have for this momentum

$$p_0 = (630 \pm 18)\ \text{MeV}. \tag{4.27}$$

Nonperturbative boundary (4.27) seems also natural from phenomenological point of view.

We have to bear in mind, of course, that all these results are obtained under chosen approximation. Generally speaking, results could change with another choice of form of dependence on the three variables in expression (4.8). By considering examples, we have get convinced, that such change does not influence coefficient afore the logarithm in brackets in expression for inhomogeneous part of equation (4.18), while it may lead to some change in the constant term of the expression in brackets. It is important to understand how small changes in this term influence results. In view of this we consider additional term ϵ in the inhomogeneous part of (4.18). Thus we have the following modified expression

$$1 + \frac{85 g\sqrt{N}\sqrt{z}}{96\pi}\left(\ln z + 4\gamma + 4\ln 2 - \frac{1975}{168} + \frac{1}{2}\, G^{31}_{15}\left(z_0\,\Big|^{0}_{0,0,1/2,-1,-1/2}\right) + \epsilon\right). \tag{4.28}$$

Let us take for example value $\epsilon = 0.13$. In this case instead of (4.22) we have

$$\begin{aligned} g(z_0) &= 3.11587, \quad z_0 = 0.0153348,\\ C_1 &= -4.47289, \quad C_2 = 3.62922, \end{aligned} \tag{4.29}$$

that in the same way as for case $\epsilon = 0$ leads to the following parameters

$$\alpha_s(x_0) = 0.7726, \quad G = \frac{1}{\Lambda_G^2}, \quad \Lambda_G = (273.5 \pm 7.0)\ \text{MeV}. \tag{4.30}$$

Another example $\epsilon = 0.15$. In this case we have

$$\begin{aligned} g(z_0) &= 3.03685, \quad z_0 = 0.0163105,\\ C_1 &= -4.37005, \quad C_2 = 3.43372,\\ \alpha_s(x_0) &= 0.7339, \quad G = \frac{1}{\Lambda_G^2}, \quad \Lambda_G = (276.4 \pm 7.0)\ \text{MeV}. \end{aligned} \tag{4.31}$$

4.2 Running coupling

In the previous section N. N. Bogoliubov compensation principle was applied to studies of a spontaneous generation of an effective nonlocal interaction (4.2) in QCD.

It is of the utmost interest to study an influence of interaction (4.2) on the behavior of strong running coupling $\alpha_s(Q^2)$ in the region below z_0 i. e., $Q < p_0$ (4.27).

For the purpose we rely on considerations connected with the renormalization group approach [1] (for application to QCD see, e. g., [2]). We have the one loop perturbative expression for QCD β-function.

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(11 - \frac{2N_f}{3}\right), \tag{4.32}$$

We shall take additional contributions for small momentum $k^2 \to 0$ of our new interactions according to diagrams shown in Figure 4.2, that gives instead of (4.32)

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\left(11 - \frac{2N_f}{3}\right) - \frac{405\sqrt{3}\,g(z0)}{2\pi}\Phi(0)\right]. \tag{4.33}$$

Here we see a decisive difference in behavior of perturbative β (4.32), which acts at large momenta $p > p_0$ and nonperturbative one for small $p \simeq 0$ (4.33). The sign of β changes between these regions. So $\alpha_s(p^2)$ for $p^2 \to 0$ is also positive as well as for large p. The behavior in between can not be obtained by application of renormalization group. However, we may use the dependence of diagrams of Figure 4.2. It is described by function $\Phi(x)$:

$$\begin{aligned}
&\Phi(z) = \int_z^{z_{01}} \frac{x - 3z/4}{x - z/2} F(x)dx + \int_{3z/4}^{z} \frac{4(x - 3z/4)^2}{z(x - z/2)} F(x)dx, \quad z < z01,\\
&\Phi(z) = \int_{3z/4}^{z_{01}} \frac{4(x - 3z/4)^2}{z(x - z/2)} F(x)dx, \quad z_{01} < z < \frac{4z_{01}}{3},\\
&\Phi(z) = 0, \quad z > \frac{4z_{01}}{3}, \quad z_{01} = \sqrt{z_0}, \quad x = \frac{\sqrt{3}Gq^2}{32\pi}, \quad z = \frac{\sqrt{3}Gk^2}{32\pi}.
\end{aligned} \tag{4.34}$$

Thus in approximation using the one-loop expression corresponding to diagrams of Figure 4.2 with $N_f = 3$ we have

$$\alpha_s(x) = \frac{\alpha(\bar{x}_0)}{1 + \frac{\alpha_s(\bar{x}_0)}{6\pi}\left(\frac{27}{2} - \frac{405\sqrt{3}\,g(z0)}{2\pi}\Phi(x)\right)\ln\left(\frac{x}{\bar{x}_0}\right)}, \tag{4.35}$$

where $\Phi(x)$ is defined by the loop diagrams of Figure 4.2. Due to this vertex being gauge invariant, there is no contribution of ghost fields. With G defined by (4.26), $g(z_0)$ defined by (4.22) and $k^2 = Q^2$ we have the behavior of $\alpha_s(Q)$.

With fixed parameter ϵ in (4.28) we calculate the behavior of running coupling. Let us begin with initial case $\epsilon = 0$. We have value of $\alpha_s(Q)$ at the beginning point of nonperturbative contribution $\bar{z}_{01} = \frac{4}{3}z_{01}$, corresponding to $Q = 726$ MeV.

$$\alpha_s(\bar{z}_{01}) = 0.936. \tag{4.36}$$

The boundary of nonperturbative region $Q_0 = 726$ MeV seems quite reasonable.

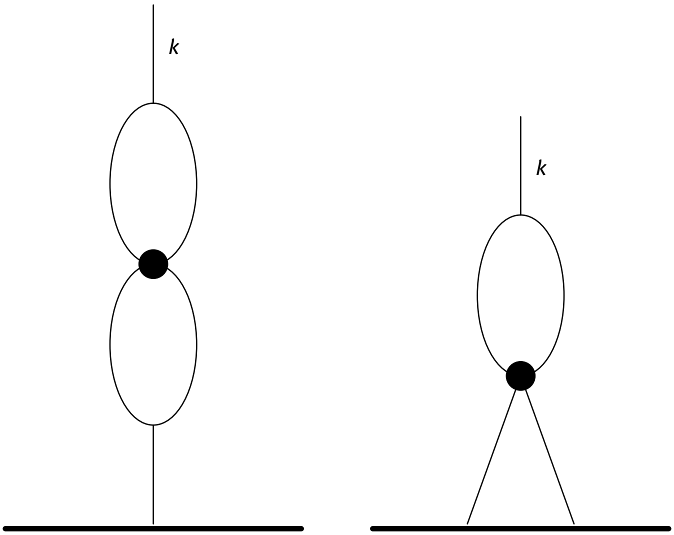


Fig. 4.2. Diagrams, describing the contribution of nonperturbative vertex (4.7), denoted by the black spot, to the running coupling $\alpha_s(k^2)$. Simple lines correspond to gluons and thick lines correspond to quarks.

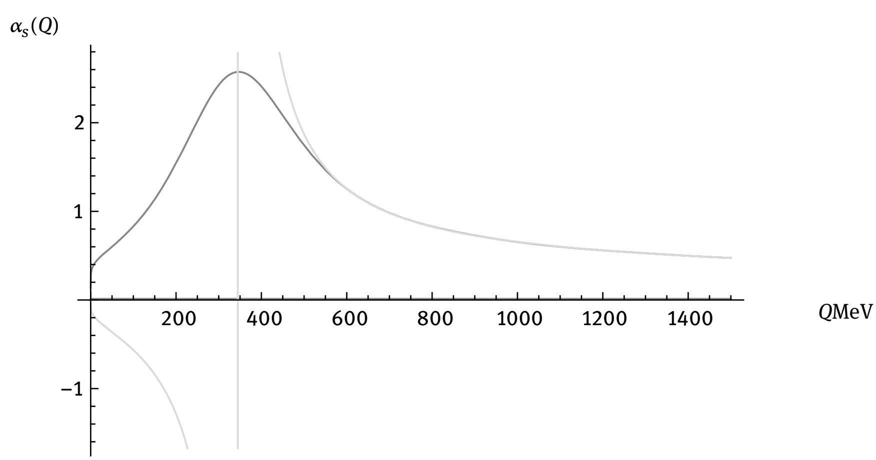


Fig. 4.3. Dependence of the running coupling α_s. The continuous line corresponds to α_s with nonperturbative contribution (4.35), the discontinuous one with a pole, denoted by the vertical line, corresponds to the usual perturbative one-loop expression.

It is interesting to study a dependence on value of ϵ. For example, for $\epsilon = 0.13$ we have the dependence presented in Figure 4.4. We see, that qualitatively the dependence is the same as in Figure 4.3, but the value of α_s at the maximum is lower. This, of course, leads to change of mean value of the coupling in the nonperturbative region. Remind,

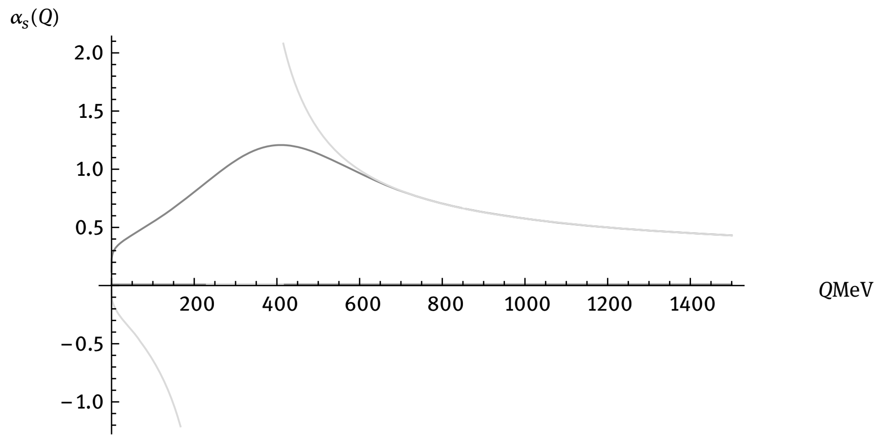


Fig. 4.4. Dependence of the running coupling α_s for $\epsilon = 0.13$. The continuous line corresponds to α_s with nonperturbative contribution (4.35), the discontinuous one with a pole corresponds to the usual perturbative one-loop expression.

that this parameter was essential for results of Nambu–Jona-Lasinio model, which was considered in Section 3.3.

We have three-loop expression for $\alpha_s(x)$ (1.69), which allows us to define value of α_s at $\bar{p}_0^2 = 4/3p_0^2$. According to previous results we normalize the running coupling by condition (4.36). The last values being presented correspond to central values of parameters defined above.

We would like to draw attention to the result, presented at Figure 4.3, which consists in absence of the Landau pole in expression (4.35). Remind, that in perturbative calculation up to four loops the singularity at the Landau pole point is always present. Only by taking into account of the nonperturbative effects we achieve elimination of this very unpleasant feature, which was seriously considered as a proof of the inconsistency of the quantum field theory [10, 11].

There is also a feature of expression (4.35), which deserves being mentioned. The limit of $\alpha_s(Q)$ for $Q \to 0$ is zero. It may be interesting for consideration of the color confinement conditions [79, 80]. Let us note, that similar behavior of the running coupling corresponds also to results, obtained by lattice calculations. For example, in the most recent lattice work [81] the dependence of the running coupling resembles that of Figure 4.4.

The average α_s in the nonperturbative region for $\epsilon = 0.13$

$$\bar{\alpha}_s = \frac{1}{Q_0}\int_0^{Q_0} \alpha_s(Q)\,dQ = 0.87, \quad Q_0 = \sqrt{\frac{128\pi z_{01}}{3\sqrt{3}\,G}}. \tag{4.37}$$

For $\epsilon = 0.15$ $\bar{\alpha}_s = 0.84$. These values, as well as value (4.36) of the coupling at the boundary of the nonperturbative region, agree with estimate (3.26), which give satisfactory description of nonperturbative parameters including quark condensate and the pion mass (see Section 3.3).

4.3 The gluon condensate

One of important nonperturbative parameters is the gluon condensate, that is the following vacuum average

$$V_2 = \left\langle \frac{g^2}{4\,\pi^2}\, F^a_{\mu\nu}\, F^a_{\mu\nu} \right\rangle. \tag{4.38}$$

Let us estimate this parameter in our approach. We apply our method to the first nonperturbative contributions, presented in Figure 4.5, which is proportional to $g\,G$. It is important to introduce Feynman rule for contribution of operator (4.38) in brackets. We denote it by skew cross in Figure 4.5

$$V_{FF}(\mu, \nu; p) = \imath\, \frac{g^2}{\pi^2}\, (g_{\mu\nu}\, p^2 - p_\mu\, p_\nu). \tag{4.39}$$

With distribution of integration momenta denoted in Figure 4.5 form-factor in both types of diagrams according to (4.8) has the same argument:

$$F\left(p^2 + \frac{3}{4} q^2\right). \tag{4.40}$$

It comes out, that the second and the third terms in the second row of Figure 4.5 are twice each of the previous terms. Thus the sum is equal to the result for the first diagram multiplied by 10.

We have after the Wick rotation

$$V_2 = \frac{10 \times 24\, g^3\, G}{(2\,\pi)^8\, \pi^2} \int F\left(p^2 + \frac{3}{4} q^2\right) \frac{12\,(p^2\, q^2 - pq^2)}{q^2 (p - q/2)^2 (p + q/2)^2}\, dp\, dq. \tag{4.41}$$

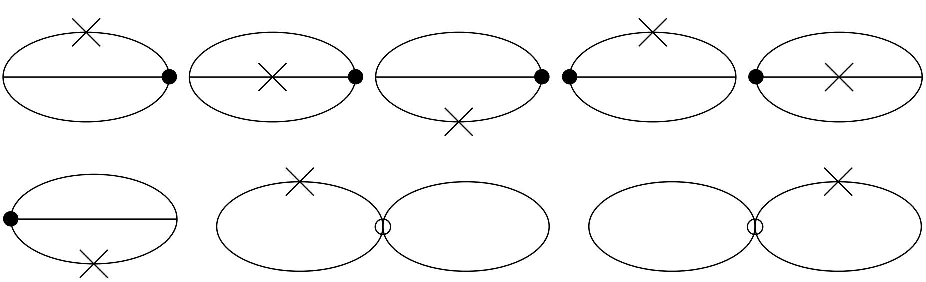


Fig. 4.5. Diagrams for calculation of the gluon condensate. Lines – gluons, black circle – triple vertex (5.4), empty circle – four gluon vertex (6.10) with corresponding form-factor and skew cross – vertex (4.39). Momenta directed to the right are $p - q/2, q, -p - q/2$ for bug-like diagrams and $p - q/2, p + q/2$ for ∞-like diagrams.

Using the following integral by angle θ

$$\int_0^{\pi} \frac{\sin^2(\theta)\, d\theta}{(p^2 + \frac{q^2}{4})^2 - (pq)^2} = \frac{\pi}{2(x + \frac{y}{4})} \left[\theta\left(x - \frac{y}{4}\right) \frac{1}{x} + \theta\left(\frac{y}{4} - x\right) \frac{4}{y} \right], \tag{4.42}$$
$$x = p^2, \quad y = q^2,$$

we obtain the following expression for quantity (4.41)

$$V_2 = \frac{5\, g^3\, 2^{11}}{G^2 \pi^3 \sqrt{3}} \int_0^{\sqrt{z_0}} F(t)\, I_t\, dt, \tag{4.43}$$

$$I_t = 12 \left(\int_0^{4t/3} \left(t - \frac{3y}{4}\right) dy \; - \; 4 \int_t^{4t/3} \frac{(t-y)^2 (t - 3y/4)}{(t - y/2) y} dy - \int_0^t \frac{(t-y)^2}{t - y/2} dy \right),$$
$$t = \frac{G\sqrt{3}}{2^5\, \pi} \left(x + \frac{3\, y}{4} \right).$$

We have already expressions (4.19, 4.22) for form-factor $F(z)$, $z = t^2$. So calculation here is direct and we obtain, using values for g (4.22) and the central value in definition of G (4.26)

$$V_2 = \frac{5\, g^3\, 2^{10}}{\pi^3 \sqrt{3}\, G^2} 12 \left(2 - 6 \ln \frac{4}{3} \right) \int_0^{z_0} F(z) \sqrt{z}\, dz = 0.0096 \,\text{GeV}^4, \tag{4.44}$$

Provided we take nonzero value for ϵ in expression (4.28) results for gluon condensate read

$$V_2 = 0.0120 \,\text{GeV}^4 \; (\epsilon = 0.13),$$
$$V_2 = 0.0128 \,\text{GeV}^4 \; (\epsilon = 0.15). \tag{4.45}$$

So in this approximation we have the nonzero nonperturbative parameter V_2. Its value agrees within accuracy of determination of this parameter with phenomenological values (1.83), $V_2 \simeq 0.012\,\text{GeV}^4$ [24], $V_2 \simeq 0.010\,\text{GeV}^4$ [82]. Values (4.44, 4.45) show variation in the range of an uncertainty of its phenomenological definition. Thus we can state, that our nonperturbative approach allows to calculate safely this important parameter.

Let us also estimate vacuum average V_3

$$V_3 = \langle g^3 f_{abc} F^a_{\mu\nu} F^b_{\nu\rho} F^c_{\rho\mu} \rangle. \tag{4.46}$$

Quite analogous calculations give, e. g., with $\epsilon = 0.13$

$$V_3 = \frac{g^3\, 2^{17}}{G^3} \left(2 - 6 \ln \frac{4}{3} \right) \int_0^{z_0} z F(z) dz = 0.00744 \,\text{GeV}^6. \tag{4.47}$$

This value also agrees estimates (1.83).

4.4 The glueball

The existence of anomalous interaction (4.2) makes possible to consider gluonic states. We shall consider scalar glueball X_0 state to get indications if value of the nonperturbative constant (4.26) may be used for adequate description of the nonperturbative effects of the strong interaction. For the purpose we use Bethe–Salpeter equation with the kernel corresponding to one-gluon exchange with our (point-like) anomalous three-gluon interaction (4.2). We take for vertex of X_0 interaction with two gluons in the following form

$$\frac{G_{gb}}{2} F^a_{\mu\nu} F^a_{\mu\nu} X_0 \Psi_{gb}(x), \quad x = p^2, \tag{4.48}$$

where $\Psi_{gb}(x)$ is a Bethe–Salpeter wave function. We have for the first approximation (zero momentum of X_0)

$$\Psi_{gb}(x) = -\frac{3G^2}{16\pi^2}\Big(\frac{1}{2x^2}\int_0^x y^3\Psi_{gb}(y)dy - \frac{1}{x}\int_0^x y^2\Psi_{gb}(y)dy \\ - 3\int_0^Y y\Psi_{gb}(y)dy - x\int_x^Y \Psi_{gb}(y)dy + \frac{x^2}{2}\int_x^Y \frac{\Psi_{gb}(y)}{y}dy\Big), \tag{4.49}$$

where we take again the upper limit Y of integration as in (4.9) due to form-factor of interaction (4.2) $F(x) = 0$ for $x \geq Y$. Again by successive differentiations we obtain from equation (4.49) the following differential equation:

$$\Big(z'\frac{d}{dz'}+1\Big)\Big(z'\frac{d}{dz'}+\frac{1}{2}\Big)\Big(z'\frac{d}{dz'}-\frac{1}{2}\Big)\Big(z'\frac{d}{dz'}-1\Big)\Psi_{gb}(z') = z'\Psi_{gb}(z') + \frac{C}{4}, \tag{4.50}$$

$$C = 4\int_0^{\bar{z}_0}\Psi_{gb}(t')dt', \quad z' = \frac{9\,G^2x^2}{128\pi^2}, \quad t' = \frac{9\,G^2y^2}{128\pi^2}.$$

Comparing variable z' in equation (4.50) with the initial variable z in Eq. (4.14) we see relation $z' = 24\,z$. This means also, that $\bar{z}_0 = 24z_0$, z_0 from solution (4.22). In new variables equation (4.49), in which we also have taken into account terms, proportional to gauge coupling g and mass of the bound state squared m^2, looks like

$$\Psi_{gb}(z') = 1 - \frac{2}{3\,z'}\int_0^{z'}\Psi_{gb}(t')t'dt' + \frac{4}{3\sqrt{z'}}\int_0^{z'}\Psi_{gb}(t')\sqrt{t'}dt' \\ + \frac{4\sqrt{z'}}{3}\int_{z'}^{\bar{z}_0}\frac{\Psi_{gb}(t')}{\sqrt{t'}}dt' - \frac{2z'}{3}\int_{z'}^{\bar{z}_0}\frac{\Psi_{gb}(t')}{t'}dt', \tag{4.51}$$

$$1 = 4\int_0^{\bar{z}_0}\Psi_{gb}(t')dt' + \Big(\kappa + \frac{3g\sqrt{2}}{2\pi}\Big)\int_0^{\bar{z}_0}\frac{\Psi_{gb}(t')}{\sqrt{t'}}dt'.$$

Here κ is connected with the bound state mass m in the following way:

$$\kappa = -\frac{3\,G\,M_{gb}^2}{8\sqrt{2}\,\pi}. \tag{4.52}$$

According to expression (4.50) we look for the solution of equation (4.49) in the following form

$$\begin{aligned}\Psi_{gb}(z') = {} & \frac{\pi}{2}\,G_{15}^{21}\left(z'|_{1,\,0,\,1/2,-1/2,-1}^{0}\right) + C_1\,G_{04}^{20}\left(z'|1,\ 1/2,-1/2,-1\right) \\ & + C_2\,G_{04}^{10}\left(-z'|1,\ 1/2,-1/2,-1\right).\end{aligned} \tag{4.53}$$

By substituting expression (4.53) into set of equations (4.51) and using the values of g and z_0 (4.22) we obtain unique solution for parameters

$$C_1 = 1.07899, \quad C_2 = -1.38099, \quad \kappa = -2.6415. \tag{4.54}$$

Now from values (4.26, 4.54), using relation (6.60), we have the lightest scalar glueball mass

$$M_{gb} = 1479 \pm 40\,\text{MeV}. \tag{4.55}$$

This value is quite natural, the more so, that the most serious candidate for being the lightest scalar glueball is the state $f_0(1500)$ (see recent review [83]) with mass 1507 ± 5 MeV, that evidently agrees our number (4.55).

Now we have to obtain the coupling constant of the scalar glueball entering in the expression of the effective interaction (4.48). For the purpose we use the normalization condition for Bethe–Salpeter wave function $\Psi(t)$.

$$1 = \frac{\sqrt{2}\,G_{gb}^2}{\pi\,G}\int_0^{\bar z_0}\frac{\Psi_{gb}(t')^2}{\sqrt{t'}}\,dt'. \tag{4.56}$$

Substituting into equation (4.56) solution (4.53, 4.54) and calculating the integral, we obtain

$$\begin{aligned}G_{gb}^2 &= \frac{\pi\,G}{\sqrt{2}\,I} = 1.825\,G, \\ I &= \int_0^{\bar z_0}\frac{\Psi_{gb}(t')^2}{\sqrt{t'}}\,dt' = 1.21732.\end{aligned} \tag{4.57}$$

From result (4.57) we have the following value of the glueball coupling:

$$G_{gb} = \frac{1}{190.337\,\text{MeV}} = \frac{5.254}{\text{GeV}}. \tag{4.58}$$

4.5 Conclusion

An existence of a nontrivial solution of a compensation equation is extremely restrictive. In the most cases such solutions do not exist at all. When we start from a renormalizable theory we have arbitrary value for its coupling constant. Provided there exists stable nontrivial solution of a compensation equation the coupling is fixed as well as the parameters of this nontrivial solution. Note, that application of the same approach to the electro-weak theory to be described below also leads to strong restrictions on parameters of the theory including the coupling constant.

We also may state, that in the case, discussed in the present chapter, just the nontrivial solution is the stable one, because the theory with the Landau pole is unstable.

We consider the results for the gluon condensate (4.44,4.45) and the glueball mass (4.55) as a confirmation of efficiency of our approach in application to nonperturbative contributions to QCD.

We consider the present results for low-momenta α_s to be encouraging and promising for further applications of the Bogoliubov compensation approach to principal problems of elementary particles physics. Let us emphasize, that elimination of the Landau pole, which is achieved due to existence of the effective anomalous three-gluon interaction, removes evident instability of the theory. This result may serve as an additional argument for stability of the solution being obtained here in comparison with the perturbative one. We shall consider problem of stability in more details in what follows.

In the next chapter we apply the method to light hadrons and their interactions. This problem is without doubt connected with nonperturbative effects. Let us watch, how compensation approach works in this field.

5 Nambu–Jona-Lasinio effective interaction

5.1 Introduction

The Nambu–Jona-Lasinio model, which we have described in Section 1.6.2, has manifested itself to be a good phenomenological tool for low-energy hadron physics. For details of the Nambu–Jona-Lasinio model see works [31, 33–37]. In the present chapter we consider the problem of a spontaneous generation of the Nambu–Jona-Lasinio interaction, following the compensation approach, introduced and elaborated in previous Chapters 3 and 4.

It is well-known, that the fundamental perturbative theory QCD is valid in the region of large q^2. In low-momenta region NJL model supplements the fundamental QCD. It is important, that common property of both theories consists in the chiral symmetry, which defines main features of low-energy hadron physics. However till now there was no direct derivation of NJL model from QCD. Therefore the problem to find a relation between parameters of NJL and those of QCD for a long time was quite actual. Some attempts in this direction were accompanied by inevitable introduction of additional parameters (see, *e.g.* [38, 84]).

We have already discussed main properties of the model in Section 1.6.2. In Section 3.3 we have considered application of the N. N. Bogoliubov compensation approach [42] to the problem of spontaneous generation of the NJL interaction. The results of Section 3.3 seems encouraging albeit the approximation being used there is rather rude and needs an introduction of cut-off Λ, which however is defined in the course of the solution of compensation equations.

In this chapter we use the method developed in Chapter 3 based on the N. N. Bogolubov compensation approach [42]. As a result we demonstrate the spontaneous generation of the Nambu–Jona-Lasinio effective interaction, which contains no additional parameters but the QCD ones. For the first time main parameters of low-energy hadron physics were calculated in this approach [77] in chiral limit that is for current mass of light quarks $m_0 = 0$. In this chapter we develop the approach for demonstration of spontaneous generation of the Nambu–Jona-Lasinio interaction and apply results for description of light spin zero and spin one mesons [78, 85].

5.2 Effective NJL interaction

Now we start with QCD Lagrangian with two light quarks (u and d) with number of colors $N = 3$

$$L = \sum_{k=1}^{2}\left(\frac{\imath}{2}\left(\bar{\psi}_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar{\psi}_k\gamma_\mu\psi_k\right) - m_0\bar{\psi}_k\psi_k + g_s\bar{\psi}_k\gamma_\mu t^a A^a_\mu\psi_k\right) - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu}\right), \quad (5.1)$$

where we use the standard QCD notations.

Let us assume that a nonlocal NJL interaction is spontaneously generated in this theory. We use Bogoliubov approach, described in Chapter 3 (see also [40, 42]) to check this assumption. In accordance to the approach, application of which to such problems are described in details in Section 3.5, we look for a nontrivial solution of a compensation equation, which is formulated on the basis of the Bogoliubov procedure add-subtract. Namely let us rewrite the initial expression (5.1) in the form

$$L = L_0 + L_{int},$$

$$\begin{aligned} L_0 = & \frac{\imath}{2}\left(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\right) - \frac{1}{4}F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} \\ & - m_0\bar\psi\,\psi + \frac{G_1}{2}\cdot\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\right) \\ & + \frac{G_2}{2}\cdot\left(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\right), \end{aligned} \tag{5.2}$$

$$\begin{aligned} L_{int} = & g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\right) \\ & - \frac{G_1}{2}\cdot\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\right) \\ & - \frac{G_2}{2}\cdot\left(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\right). \end{aligned} \tag{5.3}$$

Here ψ is the isotopic doublet of quark fields, color summation is performed inside of each fermion bilinear combination, $F_{0\,\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and notation $G_1/2\,\cdot\bar\psi\psi\bar\psi\psi$ corresponds to nonlocal vertex in the momentum space

$$\imath\,(2\pi)^4\,G_1\,F_1(p1,p2,p3,p4)\,\delta(p1+p2+p3+p4), \tag{5.4}$$

where $F_1(p1,p2,p3,p4)$ is a form-factor and $p1,\ p2,\ p3,\ p4$ are incoming momenta. In the same way we define vertices, containing Dirac and isotopic matrices. We comment the composition of the vector sector, which here contain only iso-vector terms, in what follows.

Let us consider expression (5.2) as the new free Lagrangian L_0, whereas expression (5.3) as the new interaction Lagrangian L_{int}. Then compensation conditions (see again Chapter 3) will consist in demand of full connected four-fermion vertices, following from Lagrangian L_0, to be zero. This demand gives a set of nonlinear equations for form-factors F_i.

These equations according to terminology of Chapter 3 are called compensation equations. In a study of these equations the existence of a perturbative trivial solution (in our case $G_i = 0$) is always evident, but a nonperturbative nontrivial solution may also exist. Just the quest of a nontrivial solution inspires the main interest in such problems. It is impossible to find an exact nontrivial solution in a realistic theory, therefore the goal of a study is a quest of an adequate approach, the first nonperturbative approximation of which describes the main features of the problem. Improvement of a precision of results is to be achieved by corrections to the initial first approximation.

Thus our task is to formulate the first approximation. Here the experience acquired in the course of considering of scalar model in Section 3.5 is useful. Now in view of obtaining the first approximation we would make the following assumptions.

(1) In compensation equations we restrict ourselves by terms with loop numbers 0, 1, 2. For one-loop case only trivial solution exists. Two-loop terms lead to integral equations, which may have nontrivial solutions. So the account of two-loop terms leads to the first nontrivial approximation.
(2) In compensation equations we perform a procedure of linearizing over form-factor, which leads to linear integral equations. It means that in loop terms only one vertex contains the form-factor, while other vertices are considered to be point-like. In diagram form equation for form-factor F_1 is presented in Figure 5.1.

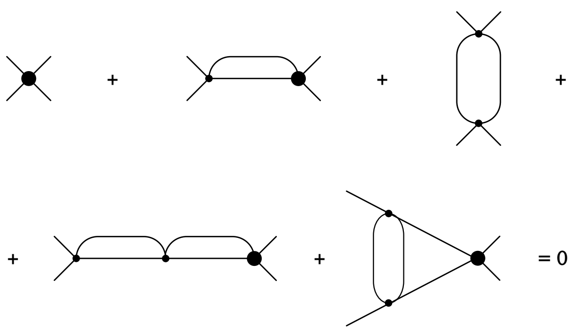


Fig. 5.1. Diagram corresponding to compensation equation in the Nambu–Jona–Lasinio nonlocal interaction.

An accuracy of this procedure was estimated in Section 3.8 (see also work [74]) to be of the order of magnitude of ten *per cent*.
(3) While evaluating diagrams with point-like vertices diverging integrals appear. Bearing in mind that as a result of the study we obtain form-factors decreasing at momentum infinity, we use an intermediate regularization by introducing ultraviolet cut-off Λ in the diverging integrals. It will be shown that results do not depend on the value of this cut-off.
(4) We use a special approximation for integrals, which is connected with transfer of a quark mass from its propagator to the lower limit of momentum integration. Effectively this leads to introduction of infra-red cut-off at the lower limit of integration by Euclidean momentum squared q^2 at value m^2. To justify this prescription let us consider a typical integral to be encountered here and perform simple evaluations. Functions which we use here depend on variable of the form $\alpha\, q^2$, where α is a parameter having $1/m^2$ dimension. Now we

have

$$\int_0^\infty \frac{F(\alpha\, q^2)\, dq^2}{(q^2+m^2)^k} = \int_{m^2}^\infty \frac{F(\alpha\,(q'^2-m^2))\, dq'^2}{(q'^2)^k}$$
$$= \int_{m^2}^\infty \frac{F(\alpha\, q'^2)\, dq'^2}{(q'^2)^k} - \alpha\, m^2 \int_{m^2}^\infty \frac{F'(\alpha\, q'^2)\, dq'^2}{(q'^2)^k} + \cdots\,, \tag{5.5}$$

Thus the introduction of infra-red cut-off at m^2 which actually consists in the following substitution corresponds to accuracy, which is defined by parameter $\alpha\, m^2$. As we shall see, this parameter for our solutions does not exceed order of magnitude of few *per cent*. We use this tool throughout the present book. In doing this we keep at numerators only the leading terms in m expansions because taking into account of the next terms evidently means supererogation of accuracy

$$\int_0^\infty \frac{F(\alpha\, q^2)\, dq^2}{(q^2+m^2)^k} \to \int_{m^2}^\infty \frac{F(\alpha\, q^2)\, dq^2}{(q^2)^k}. \tag{5.6}$$

(5) We shall take into account only the first two terms of the $1/N$ expansion. Neglected terms gives contribution, which values are defined by parameter $1/(4\,N)$. Here additional factor 4 in the denominator is connected with structure of NJL interaction in Lagrangian (5.3). Indeed a trace in color indices is always accompanied by a trace in spinor indices, which gives factor 4. Thus this approximation defines accuracy $\simeq$ 8 %.

Let us formulate compensation equations taking into account all the prescriptions being introduced. For free Lagrangian L_0 (5.2) full connected four-fermion vertices are to vanish. One can succeed in obtaining analytic solutions for the following set of momentum variables (see Figure 5.1): left-hand legs have momenta p and $-p$, and right-hand legs have zero momenta. In particular this kinematics suits for description of zero-mass bound states. The construction of expressions with an arbitrary set of momenta is the problem for the subsequent approximations. In the present chapter we shall use the next approximation for obtaining parameters of scalar and pseudo-scalar mesons.

Now following the rules being stated above we obtain the following equation for form-factor $F_1(p)$ in scalar channel

$$\begin{aligned} G_1 F_1(p^2) = {} & \frac{G_1^2 N \Lambda^2}{2\pi^2}\left(1 + \frac{1}{4N} - \frac{G_1 N}{2\pi^4}\left(1 + \frac{1}{2N}\right)\int \frac{F_1(q^2)\, dq}{q^2}\right) \\ & + \frac{3 G_1 G_2}{8\pi^2}\left(2\Lambda^2 + p^2 \ln\frac{p^2}{\Lambda^2} - \frac{3}{2}p^2 - \frac{m_0^4}{2\,p^2}\right) - \frac{(G_1^2 + 6 G_1 G_2) N}{32\,\pi^6} \\ & \times \int \left(2\Lambda^2 + (p-q)^2 \ln\frac{(p-q)^2}{\Lambda^2} - \frac{3(p-q)^2}{2} - \frac{m_0^4}{2(p-q)^2}\right)\frac{G_1 F_1(q^2) dq}{q^2}. \end{aligned} \tag{5.7}$$

Here integration is performed in the four-dimensional Euclidean momentum space with infra-red cut-off at m_0^2. One-loop expressions contains terms proportional to N and 1 while two-loop terms correspond to N^2 and N. The leading terms are the same for scalar and pseudo-scalar cases. We perform the study with the scalar channel, because it defines spontaneous breaking of the chiral symmetry. Equation (5.7) evidently has trivial solution $G_1 = 0$. Bearing in mind our goal to look for nontrivial solutions we divide the equation by G_1 and perform angular integration in four-dimensional Euclidean space. The necessary formulae for these calculation are presented in Section 5.8. From (5.7) we have

$$\begin{aligned}
F_1(x) = A &+ \frac{3G_2}{8\pi^2}\left(2\Lambda^2 + x\ln\frac{x}{\Lambda^2} - \frac{3}{2}x - \frac{\mu^2}{2x}\right)\\
&- \frac{(G_1^2+6G_1G_2)N}{32\pi^4}\left(\frac{1}{6x}\int_\mu^x (y^2-3\mu^2)F_1(y)dy\right.\\
&+ \frac{3}{2}\int_\mu^x yF_1(y)dy + \ln x\int_\mu^x yF_1(y)dy + x\ln x\int_\mu^x F_1(y)dy\\
&+ \int_x^\infty y\ln y F_1(y)dy + x\int_x^\infty \left(\ln y + \frac{3}{2}\right)F_1(y)dy\\
&+ \frac{x^2-3\mu^2}{6}\int_x^\infty \frac{F_1(y)}{y}\,dy + \left(2\Lambda^2 - \frac{3}{2}x\right)\int_\mu^\infty F_1(y)\,dy\\
&\left.- \frac{3}{2}\int_\mu^\infty yF_1(y)\,dy - \ln\,\Lambda^2\left(\int_\mu^\infty yF_1(y)\,dy + x\int_\mu^\infty F_1(y)\,dy\right)\right),\\
A &= \frac{G_1^2 N\Lambda^2}{2\pi^2}\left(1+\frac{1}{4N} - \frac{G_1 N}{2\pi^2}\left(1+\frac{1}{2N}\right)\int_\mu^\infty F_1(y)\,dy\right),\\
&\mu = m_0^2, \quad x = p^2, \quad y = q^2.
\end{aligned} \tag{5.8}$$

Equation (5.8) by a sequential six-fold differentiation reduces to the following differential equation

$$\begin{aligned}
&\frac{d^2}{dx^2}\left(x\frac{d^2}{dx^2}\left(x\frac{d^2}{dx^2}\left(xF_1(x)\right) + \frac{\beta m_0^4}{4}F_1(x)\right)\right) = \beta\frac{F_1(x)}{x}.\\
&\beta = \frac{(G_1^2+6\,G_1G_2)\,N}{16\,\pi^4},
\end{aligned} \tag{5.9}$$

with corresponding boundary conditions.

Equation (5.9) reduces to the Meijer equation. Namely with the simple substitution we have

$$\left(\left(z\frac{d}{dz}-b\right)\left(z\frac{d}{dz}-a\right)z\frac{d}{dz}\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-1\right)-z\right)F_1(z)=0,$$

$$z=\frac{\beta x^2}{2^6},\quad a=-\frac{1-\sqrt{1-64u_0}}{4},\quad b=-\frac{1+\sqrt{1-64u_0}}{4},\quad u_0=\frac{\beta m_0^4}{64}. \tag{5.10}$$

Boundary conditions for equation (5.10) are formulated in the same way as in previous cases. At first we have to choose solutions decreasing at infinity, that is combination of the following three solutions

$$F_1(z)=C_1\,G^{40}_{06}\left(z\,|1,\,\frac{1}{2},\,\frac{1}{2},\,0,\,a,\,b\right) \tag{5.11}$$
$$+C_2\,G^{40}_{06}\left(z\,|1,\,\frac{1}{2},\,b,\,a,\,\frac{1}{2},\,0\right)+C_3\,G^{40}_{06}\left(z\,|1,\,0,\,b,\,a,\,\frac{1}{2},\,\frac{1}{2}\right).$$

Constants C_i are defined by boundary conditions

$$\frac{3\,G_2}{8\pi^2}-\frac{\beta}{2}\int\limits_{m_0^2}^{\infty}F_1(y)\,dy=0,$$

$$\int\limits_{m_0^2}^{\infty}y\,F_1(y)\,dy=0,\qquad\int\limits_{m_0^2}^{\infty}y^2\,F_1(y)\,dy=0, \tag{5.12}$$

which one obtains from integral equation (5.8) by considering asymptotic behavior of integral terms at infinity. These conditions and condition $A=0$ as well provide cancelation of all terms in equation (5.8) being proportional to Λ^2 and $\ln\,\Lambda^2$. Thus the result does not depend on a value of parameter Λ. By solving linear set (5.12), in which solution (5.11) is substituted, we obtain the unique solution. Value of parameter u_0, which is connected with initial quark mass, and ratio of two constants G_i we obtain from conditions $F_1(\mu)=1$ and

$$A=\frac{G_1N\Lambda^2}{2\pi^2}\left(1+\frac{1}{4N}-\frac{G_1N}{2\pi^2}\left(1+\frac{1}{2N}\right)\int\limits_{m_0^2}^{\infty}F_1(y)\,dy\right)=$$

$$=\left(1+\frac{1}{4N}\right)\frac{G_1N\Lambda^2}{2\pi^2}\left(1-\frac{6G_2(4N+2)}{(G_1+6G_2)(4N+1)}\right)=0, \tag{5.13}$$

$$F_1(u_0)=1,\quad u_0=\frac{\beta\,m_0^4}{2^6}=\frac{N\,(G_1^2+6G_1G_2)\,m_0^4}{1024\,\pi^4}.$$

The last line here presents the obvious condition of normalization of a form-factor on the mass shell. Now relations (5.13) give for $N=3$ with the account of the first of conditions (5.12)

$$u_0 = 1.726 \cdot 10^{-8} \simeq 2 \cdot 10^{-8}, \quad G_1 = \frac{6}{13}\, G_2. \tag{5.14}$$

So G_1 and G_2 are both defined in terms of m_0. Thus we have the unique nontrivial solution of the compensation equation, which contains no additional parameters.

The form-factor now reads as (5.11) with

$$C_1 = 0.28323, \quad C_2 = -1.90129 \cdot 10^{-8}, \quad C_3 = -3.42638 \cdot 10^{-10}. \tag{5.15}$$

In what follows we use the notation $F_1(z)$ for expression (5.11), where z is always the dimensionless variable defined by expression (5.10). We have $F_1(u_0) = 1$ and $F_1(z)$ decreases with z increasing in the following way

$$F_1(z) \to \frac{D}{z^{\frac{1}{6}}} \exp\left(-3\left(1 - \imath\sqrt{3}\right) z^{\frac{1}{6}}\right) + h.c.,$$

where D is a complex constant. It is important, that the solution exists only for positive G_2 and due to (5.14) for positive G_1 as well.

At this point we would comment the problem of accuracy of our method of taking into account of quark mass m_0. A possible corrections being proportional to m_0^2 correspond to dimensionless variable (5.5) where $\alpha = \sqrt{\beta}/8$

$$\alpha\, m_0^2 = \sqrt{u_0} \simeq 10^{-4}, \tag{5.16}$$

and so they are not significant for definition of form-factor $F_1(z)$.

It is interesting to consider dependence of results, firstly for value m_0 on value of number of colors N_c. The set of equations (5.12) has unique solution for each value of N_c. First of all, let us consider limit $N_c \to \infty$. The solution here is

$$u_0 = 0, \quad C_2 = C_3 = 0, \quad C_1 = \frac{1}{2\sqrt{\pi}}$$
$$F_1(z) = \frac{1}{2\sqrt{\pi}}\, G^{40}_{06}\left(z\,|0, \frac{1}{2}, \frac{1}{2}, 1, -\frac{1}{2}, 0\right). \tag{5.17}$$

This result means, that for $N_c \to \infty$ we have the zero current mass for the light quark doublet, in other words we have the exact chiral symmetry. This result seems natural. We would emphasize, that it is obtained in the framework of the approach without any additional assumption. In fact this result qualitatively explains smallness of the light quarks' current mass m_0.

With N_c decreasing m_0 increases achieving value (5.14) for $N_c = 3$. Let us present values of u_0 for several values of N_c

$$N_c = 2 : \ u_0 = 1.065 \cdot 10^{-7}, \quad N_c = 4 : \ u_0 = 4.75 \cdot 10^{-9},$$
$$N_c = 5 : \ u_0 = 1.74 \cdot 10^{-9}, \quad N_c = 10 : \ u_0 = 8.00 \cdot 10^{-11}. \tag{5.18}$$

With one *per cent* accuracy we may approximate this results by the following simple dependence

$$u_0 \simeq \frac{A}{N_c^{\beta}}, \quad A = 2.3267\,10^{-6}, \quad \beta = 4.46366. \tag{5.19}$$

We would also comment the composition of the vector sector. For a nontrivial solution with $G_1 \neq 0$ we calculate one-loop terms giving contribution to equation for form-factor of vector terms. As a result of the first approximation we obtain just the isotopic vector terms, which are presented in expression (5.3).

Let us note, that at this stage we have two possibilities: trivial solution $G_i = 0$ and nontrivial one (5.11, 5.14, 5.15). We shall see below that a choice between the possibilities will be determined by the QCD interaction.

5.3 Scalar and pseudo-scalar states

Now with the nontrivial solution of the compensation equation we arrive at an effective theory in which there are already no undesirable four-fermion terms in free Lagrangian (5.2) while they are evidently present in interaction Lagrangian (5.3). Indeed four-fermion terms in these two parts of the full Lagrangian differ in sign and the existence of the nontrivial solution of compensation equation for Lagrangian (5.2) means nonexistence of the would be analogous equation, formulated for signs of four-fermion terms in interaction Lagrangian (5.3).[1]

So provided the nontrivial solution is realized the compensated terms go out from Lagrangian (5.2) and we obtain the following Lagrangian

$$\begin{aligned}
L = {} & \frac{\imath}{2}\left(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\right) - \frac{1}{4}\,F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} - m_0\bar\psi\,\psi \\
& + g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\right) \\
& - \frac{G_1}{2}\cdot\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\right) \\
& - \frac{G_2}{2}\cdot\left(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\right),
\end{aligned} \tag{5.20}$$

where G_1, G_2 are defined by relations (5.14) and form-factor F_1 is defined by equations (5.11, 5.15).

1 In other words the fact, that sum of a series $\sum G^n a_n = 0$ for some value G, by no means leads to a conclusion, that sum of the same series with $G \to -G$ vanishes as well.

Here we have to comment the meaning of the strong coupling constant g_s. We have already discussed in Sections 1.4, 4.2 running coupling $g_s^2/4\pi = \alpha_s(q^2)$ depending on the momentum variable. We need this coupling constant in the low-momenta region. However the perturbation theory in QCD does not work for small q^2. We assume that in this region $\alpha_s(q^2)$ may be approximated by its average value $\bar{\alpha}_s$. This assumption is very close to conception of a frozen strong coupling at low momenta [86]. The consideration of average low-momenta value of $\bar{\alpha}_s$ performed in Section 4.2 (see also [86–90]) leads to the definition of a possible range of values of $\bar{\alpha}_s$ from 0.7 up to 0.9. So in what follows we use constant $\bar{\alpha}_s$ which is assumed to fit this interval of possible values. Let us remind, that in Section 4.2 such average was calculated for several possible sets of initial parameters and these averages turn to agree with the possible range of variation of $\bar{\alpha}_s$

$$0.7 < \bar{\alpha}_s < 0.9, \tag{5.21}$$

in the nonperturbative region. In what follows in Chapter 5 we shall use notation $\alpha_s \equiv \bar{\alpha}_s$.

Thus, bound state problems in the present approach are formulated starting from Lagrangian (5.20).

Let us write down Bethe–Salpeter equation for a state in zero-spin (scalar and pseudo-scalar) channel in the same approximation as was used in equation (5.7). Let us begin with massless states. The definitions of momenta are the same as in equation (5.7)

$$\Psi(p^2) = \frac{G_1 N}{2\pi^4}\int\frac{\Psi(q^2)\,dq}{q^2} + \frac{(G_1^2 + 6G_1G_2)N}{32\pi^6} \times\int\left(2\Lambda^2 + (p-q)^2\ln\frac{(p-q)^2}{\Lambda^2} - \frac{3}{2}(p-q)^2 - \frac{m^4}{2\,(p-q)^2}\right)\frac{\Psi(q^2)\,dq}{q^2}. \tag{5.22}$$

Here m is a quark mass, which of course do not coincide with m_0. We define the value of m after considering the spontaneous breaking of the chiral symmetry.

After angular integrations we obtain the one-dimensional equation similar to equation (5.8)

$$\begin{aligned}\Psi(x) = {}& \frac{G_1N}{2\pi^2}\int_{m^2}^{\infty}\Psi(y)dy + \frac{(G_1^2 + 6G_1G_2)N}{32\pi^4}\Bigg(\frac{3}{2}\int_{m^2}^{x} y\Psi(y)dy + \frac{1}{6x}\int_{m^2}^{x}(y^2 - 3m^4)\Psi(y)dy \\ & + \ln x\int_{m^2}^{x} y\Psi(y)dy + x\ln x\int_{m^2}^{x}\Psi(y)dy + \int_{x}^{\infty} y\ln y\Psi(y)dy + x\int_{x}^{\infty}\left(\ln y + \frac{3}{2}\right)\Psi(y)dy \\ & + \frac{x^2 - 3m^4}{6}\int_{x}^{\infty}\frac{\Psi(y)}{y}\,dy + \left(2\bar{\Lambda}^2 - \frac{3}{2}x\right)\int_{m^2}^{\infty}\Psi(y)\,dy \\ & - \frac{3}{2}\int_{m^2}^{\infty} y\Psi(y)\,dy - \ln\Lambda^2\left(\int_{m^2}^{\infty} y\Psi(y)\,dy + x\int_{m^2}^{\infty}\Psi(y)\,dy\right)\Bigg),\end{aligned} \tag{5.23}$$

The corresponding differential equation for $\Psi(x)$ is almost the same, as the previous one (5.9) with one essential difference. Namely the sign before β is opposite.

$$\left(\left(z\frac{d}{dz}-\bar{b}\right)\left(z\frac{d}{dz}-\bar{a}\right)z\frac{d}{dz}\left(z\frac{d}{dz}-\frac{1}{2}\right)+\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-1\right)+z\right)\Psi(z)=0, \quad (5.24)$$

$$z=\frac{\beta x^2}{2^6}, \quad \bar{a}=\frac{-1+\sqrt{1+64u}}{4}, \quad \bar{b}=\frac{-1-\sqrt{1+64u}}{4}, \quad u=\frac{\beta m^4}{64}.$$

In this case we have the following solution decreasing at infinity

$$\Psi(z)=C_1^*G_{06}^{30}\left(z\,|1,\frac{1}{2},0,\frac{1}{2},\bar{a},\bar{b}\right)+C_2^*G_{06}^{30}\left(z\,|1,\frac{1}{2},\frac{1}{2},0,\bar{a},\bar{b}\right)$$
$$+C_3^*\,G_{06}^{30}\left(z\,|1,\bar{a},\bar{b},\frac{1}{2},\frac{1}{2},0\right)+C_4^*\,G_{06}^{30}\left(z\,|\frac{1}{2},\bar{a},\bar{b},1,\frac{1}{2},0\right). \quad (5.25)$$

Constants C_i^* are defined by the following conditions

$$\int_u^\infty\frac{\Psi(z)dz}{\sqrt{z}}=0, \quad \int_u^\infty\Psi(z)dz=0, \quad \int_u^\infty\sqrt{z}\Psi(z)dz=0, \quad \Psi(u)=1. \quad (5.26)$$

where $u=\beta m^4/2^6$. Let us remind, that boundary conditions (5.26) guarantee cancelation of terms in equation (5.22) containing cut-off Λ. Performing integrations in expressions (5.26), we have the following set of equations

$$C_1^*G_{06}^{30}\left(u|1,\frac{1}{2},0,\frac{1}{2},\bar{a},\bar{b}\right)+C_2^*\,G_{06}^{30}\left(u|1,\frac{1}{2},\frac{1}{2},0,\bar{a},\bar{b}\right)$$
$$+C_3^*G_{06}^{30}\left(u|1,\bar{a},\bar{b},\frac{1}{2},\frac{1}{2},0\right)+C_4^*G_{06}^{30}\left(u|\frac{1}{2},\bar{a},\bar{b},1,\frac{1}{2},0\right)=1,$$

$$-C_1^*G_{06}^{30}\left(u\,|\frac{3}{2},1,\frac{1}{2},0,\frac{1}{2}+\bar{a},\frac{1}{2}+\bar{b}\right)$$
$$+C_2^*\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\bar{a})\Gamma(\frac{1}{2}-\bar{b})}-G_{17}^{31}\left(u\,|^{1}_{\frac{3}{2},1,1,\frac{1}{2},0,\frac{1}{2}+\bar{a},\frac{1}{2}+\bar{b}}\right)\right]$$
$$-C_3^*\,G_{06}^{30}\left(u\,|\frac{3}{2},\frac{1}{2}+\bar{a},\frac{1}{2}+\bar{b},\frac{1}{2},1,0\right)-C_4^*\,G_{06}^{30}\left(u\,|1,\frac{1}{2}+\bar{a},\frac{1}{2}+\bar{b},\frac{3}{2},\frac{1}{2},0\right)=0,$$

$$C_1^*\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(-\frac{1}{2})\Gamma(-\bar{a})\Gamma(-\bar{b})}-G_{17}^{31}\left(u\,|^{1}_{2,3/2,1,3/2,0,1+\bar{a},1+\bar{b}}\right)\right]$$
$$-C_2^*G_{06}^{30}\left(u|2,\frac{3}{2},\frac{3}{2},0,1+\bar{a},1+\bar{b}\right)-C_3^*G_{06}^{30}\left(u|2,1+\bar{a},1+\bar{b},\frac{3}{2},\frac{3}{2},0\right)$$
$$-C_4^*\,G_{06}^{30}\left(u\,|\frac{3}{2},1+\bar{a},1+\bar{b},2,\frac{3}{2},0\right)=0, \quad (5.27)$$

$$C_2^*\left[\frac{\Gamma(\frac{5}{2})}{\Gamma(-\frac{1}{2})\Gamma(-\frac{1}{2}-\bar{a})\Gamma(-\frac{1}{2}-\bar{b})}-G_{17}^{31}\left(u|^{1}_{\frac{5}{2},2,2,\frac{3}{2},0,\frac{3}{2}+\bar{a},\frac{3}{2}+\bar{b}}\right)\right]$$
$$-C_1^*G_{17}^{31}\left(u\,|^{1}_{\frac{5}{2},2,\frac{3}{2},2,0,\frac{3}{2}+\bar{a},\frac{3}{2}+\bar{b}}\right)-C_3^*G_{17}^{31}\left(u\,|^{1}_{\frac{5}{2},\frac{3}{2}+\bar{a},\frac{3}{2}+\bar{b},2,2,\frac{3}{2},0}\right)$$
$$-C_4^*\,G_{17}^{31}\left(u\,|^{1}_{2,\frac{3}{2}+\bar{a},\frac{3}{2}+\bar{b},\frac{5}{2},2,\frac{3}{2},0}\right)=0.$$

For a given value of u these conditions (5.27) uniquely define four coefficients C_i^*. The result, that equation (5.22) has unique solution, which satisfies all boundary conditions, corresponds to existence of a zero-mass state in the same approximation as is used for compensation equation (5.7). This is quite natural due to Bogoliubov–Goldstone theorem [40, 43, 44].

However we have to take into account the QCD interaction as well as an interaction of these mesons (ϕ and π_a) with quarks. Indeed we have just shown the existence of this states and so the following effective meson-quark interaction is to exist

$$-g\left(\phi\,\bar{\psi}\,\psi + \imath\,\pi_a\,\bar{\psi}\,\gamma_5\,\tau_a\psi\right), \tag{5.28}$$

where g is defined by normalization condition of zero-spin states

$$\frac{g^2\,N}{4\,\pi^2}\,I_2 = 1, \quad I_2 = \int_{m^2}^{\infty} \frac{\Psi(p^2)^2\,dp^2}{p^2} = \int_{u}^{\infty} \frac{\Psi(z)^2\,dz}{2\,z}. \tag{5.29}$$

The form-factor of interaction (5.28) for our standard quark momenta prescription $(p, -p)$ is a Bethe–Salpeter wave function defined by eqs. (5.25, 5.27). The account of contributions of meson-quark interaction was considered in the framework of the Nambu–Jona-Lasinio model, e. g., in works [91, 92] and was shown to be corresponding to the next order of the $1/N_c$ expansion. In diagram representation the Bethe–Salpeter equation is shown in Figure 5.2.

Let us calculate a mass correction term due to these contributions. For the purpose let us take into account terms of the first order in P^2, where P is the momentum of a scalar (and pseudo-scalar) meson and one-loop terms being due to quark-gluon QCD interaction and quark-meson vertices. Note that for the last loops we use massless meson exchange. We define momenta of left-hand legs in Figure 5.2 to be $p + P/2$ and

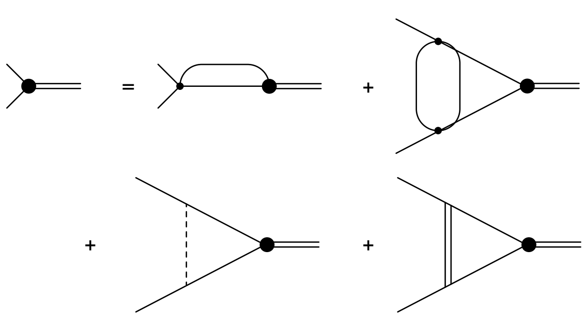


Fig. 5.2. Diagram corresponding to Bethe–Salpeter equation for the Nambu–Jona–Lasinio effective interaction.

$-p + P/2$ and obtain the following equation

$$
\begin{aligned}
\Psi_P(p^2) = {} & \frac{G_1 N}{2\pi^4} \int \frac{\Psi_P(q^2)dq}{q^2} \left(1 - \frac{3P^2}{4q^2} + \frac{(qP)^2}{(q^2)^2}\right) \\
& + \frac{(G_1^2 + 6G_1G_2)N}{32\pi^6} \int \left(1 - \frac{3P^2}{4q^2} + \frac{(qP)^2}{(q^2)^2}\right) \\
& \times \left(2\Lambda^2 + (p-q)^2 \ln \frac{(p-q)^2}{\Lambda^2} - \frac{3}{2}(p-q)^2 - \frac{m^4}{2(p-q)^2}\right) \\
& \times \frac{\Psi_P(q^2)\, dq}{q^2} + \left(\frac{g_s^2}{4\pi^4} + \frac{g^2}{8\pi^4}\right) \int \frac{\Psi_P(q^2)\, dq}{q^2(q-p)^2}.
\end{aligned}
\tag{5.30}
$$

In the course of the QCD term calculation we use the transverse Landau gauge [2].

Let us multiply equation (5.30) by $\Psi_P(p^2)/p^2$ at $P = 0$ and integrate by p. Due to equation (5.22) being satisfied we have

$$
-\frac{P^2}{2} \int \frac{\Psi(q^2)^2 dq}{(q^2)^2} + \left(\frac{g_s^2}{4\pi^4} + \frac{g^2}{8\pi^4}\right) \int \frac{\Psi(p^2)\, dp}{p^2} \int \frac{\Psi(q^2)\, dq}{q^2(q-p)^2} = 0. \tag{5.31}
$$

After an angular integration we obtain the following relation

$$
\begin{aligned}
\frac{P^2\pi^2}{2} I_2 &= \frac{2g_s^2 + g^2}{8} \int_{m^2}^{\infty} \Psi(x)dx \left(\frac{1}{x}\int_{m^2}^{x} \Psi(y)dy + \int_{x}^{\infty} \frac{\Psi(y)dy}{y}\right) \\
&= \frac{2g_s^2 + g^2}{2\sqrt{\beta}} \int_u^{\infty} \frac{\Psi(z)dz}{z} \int_u^{z} \frac{\Psi(t)dt}{\sqrt{t}} = \frac{(2g_s^2 + g^2)\, I_5}{2\sqrt{\beta}},
\end{aligned}
\tag{5.32}
$$

$$
z = \frac{\beta x^2}{64}, \quad t = \frac{\beta y^2}{64}.
$$

The integral inside of I_5 with account of boundary conditions (5.26) reads

$$
\begin{aligned}
\int_u^z \frac{\Psi(t)\, dt}{\sqrt{t}} = {} & C_1^* G_{06}^{30}\left(z \,\middle|\, \frac{1}{2}, 1, \frac{3}{2}, 0, \frac{1}{2} + \bar{a}, \frac{1}{2} + \bar{b}\right) \\
& - C_2^* G_{06}^{30}\left(z \,\middle|\, 0, 1, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} + \bar{a}, \frac{1}{2} + \bar{b}\right) \\
& + C_3^* G_{06}^{30}\left(z \,\middle|\, \frac{3}{2}, \frac{1}{2} + \bar{a}, \frac{1}{2} + \bar{b}, 0, \frac{1}{2}, 1\right) \\
& + C_4^* G_{06}^{30}\left(z \,\middle|\, 1, \frac{1}{2} + \bar{a}, \frac{1}{2} + \bar{b}, 0, \frac{1}{2}, \frac{3}{2}\right),
\end{aligned}
\tag{5.33}
$$

and after a substitution of relation (5.33) into integral I_5 it is to be calculated numerically. Note that while evaluating integral (5.33) we use the following relation, which

2 This gauge leads to an absence of renormalization of the both vertex and spinor field in the one loop approximation.

is formulated in Section 2.3 (see relations (2.150, 2.151)).

$$G^{31}_{17}\left(z|^{1}_{1,c,d,0,g,a,b}\right) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(1-g)\Gamma(1-a)\Gamma(1-b)} - G^{30}_{06}\left(z|\,0,c,d,g,a,b\right). \tag{5.34}$$

Integral I_5 turns to be positive, so the mass squared of scalar and pseudo-scalar mesons is shifted to negative value

$$m^2_\phi = -\left(\frac{\alpha_s}{\pi} + \frac{g^2}{8\pi^2}\right)\frac{8\,I_5}{\sqrt{\beta}I_2}. \tag{5.35}$$

This means that we obtain scalar and pseudo-scalar states with negative mass squared (tachyons).

5.4 Spontaneous breaking of the chiral symmetry

The negative value of m^2_ϕ (5.35) means instability of the vacuum. Therefore we have to consider an effective potential depending on scalar field ϕ. In doing this we need an expression for mass operator of the quark $\Sigma(p^2)$. The Schwinger–Dyson equation defining this function in our approximation reads as follows

$$\begin{aligned}\Sigma(p^2) = m_0 &+ \frac{G_1 N}{2\pi^4}\int\frac{\Sigma(q^2)\,dq}{q^2} + \frac{(G_1^2+6G_1G_2)N}{32\pi^6}\\ &\times\int\left(2\Lambda^2 + (p-q)^2\ln\frac{(p-q)^2}{\Lambda^2} - \frac{3}{2}(p-q)^2 - \frac{m^4}{2\,(p-q)^2}\right)\\ &\times\frac{\Sigma(q^2)\,dq}{q^2} + \left(\frac{g_s^2}{4\pi^4} + \frac{g^2}{8\pi^4}\right)\int\frac{\Sigma(q^2)\,dq}{q^2(q-p)^2}.\end{aligned} \tag{5.36}$$

The first approximation corresponds to $m_0 = g_s = g = 0$. Then equation (5.36) exactly coincides with equation (5.22) for Bethe–Salpeter wave function $\Psi(p^2)$ (5.25). Similar situation takes place in standard NJL model [37]. For nonzero m_0 we have without gluon and meson corrections

$$\Sigma(x) = m_0 + (m-m_0)\,\Psi(x), \quad \Sigma(-m^2) = m. \tag{5.37}$$

Emphasize that approximate solution (5.37) of equation (5.36) exists for any value of m. For definition of m one has to turn to the of the chiral symmetry.

Let us write down the effective potential which defines a possibility of the symmetry breaking. We look for terms proportional to ϕ^n for $n = 1, 2, 3, 4$. The term with $n = 2$ is evidently defined by (5.35). For terms with $n = 3, 4$ we take quark-loop diagrams with three and four scalar legs respectfully and as a result we have the following effective potential

$$V = C + m^4\Bigg(-\left(1-\left(\frac{u_0}{u}\right)^{1/4}\right)\left(\frac{1}{8\,\pi^2}+\frac{\alpha_s}{\pi\,g^2}\right)\frac{I_5\,\xi}{\sqrt{u}\,I_2} - \left(\frac{1}{8\,\pi^2}+\frac{\alpha_s}{\pi\,g^2}\right)\frac{I_5\,\xi^2}{2\,\sqrt{u}\,I_2} + \frac{3\,\xi^3}{2\,\pi^2}\left(\left(\frac{u_0}{u}\right)^{1/4} I_3 + \left(1-\left(\frac{u_0}{u}\right)^{1/4}\right) I_4\right) + \frac{3\,\xi^4}{8\,\pi^2}I_4\Bigg), \quad \xi = \frac{g\,\phi}{m}. \tag{5.38}$$

Here

$$I_3 = \int_u^\infty \frac{\Psi(z)^3\,dz}{2\,z}, \quad I_4 = \int_u^\infty \frac{\Psi(z)^4\,dz}{2\,z}. \tag{5.39}$$

The connection between terms with $n = 1$ and $n = 2$ is obtained from the fact, that the tadpole term due to expression (5.37) gives just the same contribution as the two-loop one up to factor $(m - m_0)/g$. The contribution to the tadpole term being proportional to m_0 is zero due to boundary conditions (5.26).

Effective potential is defined by relation (5.38) up to a constant term C, which does not influence the position of a minimum. However this constant term shifts the value of V that can be important for the problem of a stability of different solutions. We discuss this point below.

As for one-loop terms with $n \geq 5$, they all converge with point-like vertices. In this case they can be calculated and summed up to give the following additional term

$$\Delta V = m^4\left(\frac{1}{16\pi^2}(1-\xi)^4\ln|1-\xi| + \frac{\xi}{16\pi^2} - \frac{7\xi^2}{32\pi^2} + \frac{13\xi^3}{48\pi^2} - \frac{25\xi^4}{192\pi^2}\right), \tag{5.40}$$

which evidently does not destroy stability conditions and turns to influence results quite insignificantly. Thus we neglect it.

We look for a minimum of potential (5.38) that is for a solution of the following equation

$$\frac{\partial V}{\partial \xi} = 0. \tag{5.41}$$

Constituent quark mass is expressed through the vacuum expectation value of scalar field ϕ

$$m = m_0 + g\,\eta, \quad \eta = \langle\phi\rangle. \tag{5.42}$$

Bearing in mind definitions (5.13, 5.25) of parameters u_0 and u, we come to the conclusion, that the position of minimum ξ_0 has to be the following

$$\xi_0 = \left(1-\left(\frac{u_0}{u}\right)^{1/4}\right). \tag{5.43}$$

Thus from relations (5.29, 5.38, 5.41, 5.43) we obtain the following expression for α_s

$$\alpha_s = \frac{\pi\sqrt{u}}{I_5}\left(1-\left(\frac{u_0}{u}\right)^{\frac{1}{4}}\right)\left(3\left(\frac{u_0}{u}\right)^{\frac{1}{4}} I_3 + 4\left(1-\left(\frac{u_0}{u}\right)^{\frac{1}{4}}\right) I_4\right) - \frac{\pi}{6 I_2}. \tag{5.44}$$

Here all integrals are functions of u and so relation (5.44) defines function $\alpha_s(u)$.

Now it is the proper place to comment the problem of stability. From the very beginning we have two solutions: the trivial one $G_1 = G_2 = 0,\ m = m_0$ and the nontrivial one, which in details is presented above. Here we have to study the constant term in effective potential (5.38). It is connected with the following vacuum averages

$$C = \frac{1}{4}\,\langle F^a_{\mu\nu} F^a_{\mu\nu}\rangle + \Sigma_i\, m_i\,\langle \bar{\psi}_i\,\psi_i\rangle, \tag{5.45}$$

where the first term represent the gluon condensate. The relation of gluon and quark condensates to the problem of stability of the chiral symmetry breaking is considered in works [96, 97]. The first term is proportional to the gluon condensate V_2, which in our approach is calculated in Section 4.3. The light quark condensate $\langle \bar{q}q\rangle$ will be considered below.

Provided the nontrivial solution corresponds to the minimal negative value of effective potential (5.38) while the trivial solution corresponds to its value zero we are to conclude, that just the nontrivial solution is stable and thus the nontrivial solution is to describe the observable physical quantities. We discuss details of the problem after obtaining numerical results for parameters of the present approach.

We apply quark mass operator (5.37) to obtain also the expression for pion decay constant f_π. Considering one-loop quark diagram for decay amplitude of process $\pi^+ \to \mu^+\,\nu_\mu$ we have

$$\begin{aligned} f_\pi &= \frac{g\,N}{4\,\pi^2} \int\limits_{m^2}^{\infty} \Big((m - m_0)\,\Psi(y)^2 + m_0\,\Psi(y) \Big)\,\frac{dy}{y} \\ &= \frac{g\,N}{4\,\pi^2}\,\big((m - m_0)\,I_2 + m_0\,I_1 \big), \quad I_1 = \int\limits_{u}^{\infty} \frac{\Psi(z)\,dz}{2\,z}. \end{aligned} \tag{5.46}$$

Provided either $m_0 = 0$ or $I_2 = I_1$ we get with account of normalization condition (5.29) just the original Goldberger–Treiman relation $m = g\,f_\pi$. We use full relation (5.46). However let us note, that values of the two integrals are close $I_2 \simeq I_1$ and the simple original relation works with sufficient accuracy.

5.5 Pion mass and the quark condensate

In relations (5.38, 5.46) we have used approximation (5.37) for the quark mass operator. For calculation of the pion mass and the quark condensate we need the next approximation for mass operator. In view of this we reformulate equation (5.36) for the following function

$$\Phi(p^2) = \frac{\Sigma(p^2) - m_0}{m - m_0}, \tag{5.47}$$

and the first approximation for Φ is just Ψ. Then we introduce (5.47) into (5.36) to obtain

$$\Phi(p^2) = \frac{G_1\,N}{8\,\pi^4}\int \frac{((m-m_0)\,\Phi(q^2)+m_0)\,dq}{(m-m_0)\,q^2}\left(4+\frac{2g_s^2+g^2}{G_1N(q-p)^2}\right)$$
$$+\frac{(G_1^2+6G_1G_2)N}{32\,\pi^6}\int\left(2\Lambda^2+(p-q)^2\ln\frac{(p-q)^2}{\Lambda^2}-\frac{3}{2}(p-q)^2-\frac{m^4}{2\,(p-q)^2}\right)$$
$$\frac{((m-m_0)\,\Phi(q^2)+m_0)\,dq}{(m-m_0)\,q^2}. \tag{5.48}$$

Now we subtract equation (5.30) from equation (5.48) and obtain the following relation

$$D(p^2) = \frac{G_1\,N}{2\,\pi^4}\left(\int\frac{((m-m_0)\,D(q^2)+m_0)\,dq}{(m-m_0)\,q^2}\right.$$
$$\left.+\int\frac{\Psi(q^2)\,dq}{q^2}\left(\frac{3\,P^2}{4\,q^2}-\frac{(qP)^2}{(q^2)^2}\right)\right)+\frac{(G_1^2+6G_1G_2)N}{32\,\pi^6}$$
$$\times\left(\int\left(2\Lambda^2+(p-q)^2\ln\frac{(p-q)^2}{\Lambda^2}-\frac{3}{2}(p-q)^2-\frac{m^4}{2\,(p-q)^2}\right)\right.$$
$$\left.\times\left(\frac{(m-m_0)D(q^2)+m_0}{m-m_0}+\Psi(q^2)\left(\frac{3\,P^2}{4\,q^2}-\frac{(qP)^2}{(q^2)^2}\right)\right)\frac{dq}{q^2}\right.$$
$$\left.+\left[\frac{g_s^2}{4\pi^4}+\frac{g^2}{8\pi^4}\right]\right]\int\frac{((m-m_0)D(q^2)+m_0)dq}{(m-m_0)q^2(q-p)^2}. \tag{5.49}$$
$$D(p^2) = \Phi(p^2)-\Psi(p^2).$$

The analogous procedure is applied in the standard NJL model [37, 38] while proving the Gell-Mann–Oaks–Renner relation [46]. Then we again multiply (5.49) by $\Psi(p^2)/p^2$ and integrate over dp. Due to equation (5.30) be satisfied only terms being proportional either to $P^2 = -m_\pi^2$ or to m_0 do not cancel and finally we have the following relation for mass of the π-meson

$$m_\pi^2 = \frac{m^2\,m_0}{2\,\pi\,(m-m_0)\,I_2\,\sqrt{u}}\left(\alpha_s+\frac{g^2}{8\,\pi}\right)I_{\ln}, \tag{5.50}$$
$$I_{\ln} = -\int_u^\infty\frac{\ln z}{\sqrt{z}}\Psi(z)dz = C_1^*\,G_{06}^{30}\left(u\,|\frac{3}{2},\frac{1}{2},0,\frac{1}{2}+a,\frac{1}{2}+b,0\right)$$
$$-C_2^*\,G_{06}^{30}\left(u|\frac{3}{2},0,0,\frac{1}{2},\frac{1}{2}+a,\frac{1}{2}+b\right)$$
$$-C_3^*\,G_{06}^{30}\left(u|\frac{3}{2},\frac{1}{2}+a,\frac{1}{2}+b,\frac{1}{2},0,0\right)$$
$$+C_4^*\,G_{06}^{30}\left(u\,|0,\frac{1}{2}+a,\frac{1}{2}+b,\frac{3}{2},\frac{1}{2},0\right).$$

We see that pion mass squared is proportional to m_0 in accordance to the result of well-known work [46]. Note, that contributions being proportional to m_0 arising from the first two terms of equation (5.49) are summed to overall zero due to consequences of boundary conditions (5.27).

From equation (5.47) we obtain also the next approximation for Φ, which leads to a nonzero value of the quark condensate

$$\begin{aligned}\langle \bar{q}\, q\rangle &= -\frac{4\,N}{(2\,\pi)^4}\int \frac{\Sigma(q)-m_0}{q^2+m^2}\,dq \\ &= \frac{N(m-m_0)}{\pi^2\sqrt{\beta}}\left[\frac{\alpha_s}{\pi}+\frac{g^2}{8\pi^2}\right]\left[\int_u^\infty \frac{dz}{\sqrt{z}}\left[\frac{1}{\sqrt{z}}\int_u^z \frac{\Psi(t)dt}{\sqrt{t}}+\int_z^\infty \frac{\Psi(t)dt}{t}\right]\right. \\ &= -\frac{N(m-m_0)}{\pi^2\sqrt{\beta}}\left[\frac{\alpha_s}{\pi}+\frac{g^2}{8\pi^2}\right]\left[\left[\int_u^\infty \frac{\Psi(t)\ln t\,dt}{\sqrt{t}}+2\sqrt{u}\int_u^\infty \frac{\Psi(t)dt}{t}\right].\right.\end{aligned} \tag{5.51}$$

After evaluating the integrals we have

$$\begin{aligned}\langle \bar{q}q\rangle = &\left[\alpha_s+\frac{g^2}{8\pi}\right]\frac{3m^2(m-m_0)}{8\pi^3\sqrt{u}} \\ &\times\left[C_1^*\left(G_{06}^{30}\left(u\middle|\frac{3}{2},\frac{1}{2},0,0,\frac{1}{2}+a,\frac{1}{2}+b\right)\right.\right. \\ &\left.+2\,G_{06}^{30}\left(u\,\middle|1,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}+a,\frac{1}{2}+b\right)\right) \\ &-C_2^*\left(G_{06}^{30}\left(u\middle|\frac{3}{2},0,0,\frac{1}{2},\frac{1}{2}+a,\frac{1}{2}+b\right)\right. \\ &\left.+2G_{06}^{30}\left(u\middle|1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2}+a,\frac{1}{2}+b\right)\right) \\ &-C_3^*\left(G_{06}^{30}\left(u\middle|\frac{3}{2},\frac{1}{2}+a,\frac{1}{2}+b,\frac{1}{2},0,0\right)\right. \\ &\left.+2G_{06}^{30}\left(u\middle|\frac{1}{2},\frac{1}{2}+a,\frac{1}{2}+b,1,1,\frac{1}{2}\right)\right) \\ &+C_4^*\left(G_{06}^{30}\left(u\middle|\frac{1}{2}+a,\frac{1}{2}+b,0,\frac{3}{2},\frac{1}{2},0\right)\right. \\ &\left.\left.+2G_{06}^{30}\left(u\middle|1,\frac{1}{2}+a,\frac{1}{2}+b,1,\frac{1}{2},\frac{1}{2}\right)\right)\right].\end{aligned} \tag{5.52}$$

Scalar field ϕ corresponds to the σ-meson. To estimate mass of the σ-meson we use relation (3.33), which was already presented in Section 3.3. Thus in the first approximation we have

$$m_\sigma = \frac{g_s}{G_1}. \tag{5.53}$$

The $\sigma\,\pi\,\pi$ vertex gives according to triangle one-loop diagram the following coupling constant

$$g_{\sigma\pi\pi} = \frac{g^3\, N m}{4\,\pi^2} \int\limits_u^\infty \frac{\Psi(z)^3}{z}, \tag{5.54}$$

and the σ-meson width reads

$$\Gamma_\sigma = \frac{3\, g^2_{\sigma\pi\pi}}{16\,\pi\, m_\sigma^2} \sqrt{m_\sigma^2 - 4\, m_\pi^2}. \tag{5.55}$$

5.6 Numerical results and discussion

Now we have expressions for all quantities under study. Then we proceed as follows.

(1) We calculate function α_s (5.44) depending on parameter u (5.25) and get convinced, that the interesting range of α_s corresponds to u varying in the following region

$$0.0005 < u < 0.002. \tag{5.56}$$

In doing this we use parameter $u_0 = 1.725\,10^{-8}$ according to relation (5.14) and calculate constants C_i^*, $i = 1, 2, 3, 4$ from boundary conditions (5.27) thus defining $\Psi(z)$. Having $\Psi(z)$ we calculate integrals I_j, $j = 1, 2, 3, 4, 5$.

(2) We fix value $f_\pi = 92.4$ MeV.

(3) Then for given u in range (5.56) from (5.46) we obtain constituent quark mass m.

(4) Having m and α_s we calculate m_π from (5.50).
For u in range (5.56) m_π varies insignificantly between 133 MeV and 127 MeV with maximal value 134.8 MeV at $u = 0.0009$, that corresponds to $\alpha_s = 0.673$ and $m_0 = 20.27$ MeV. Considering this maximal value of m_π, we present a set of calculated parameters for these conditions including quark condensate (5.52) and parameters of the σ-meson (5.53), (5.55) as well

$$\begin{aligned} \alpha_s &= 0.673, & m_0 &= 20.3\,\text{MeV}, \\ m_\pi &= 134\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV}, \\ m_\sigma &= 771.0\,\text{MeV}, & \Gamma_\sigma &= 367.7\,\text{MeV}, \\ m &= 295\,\text{MeV}, & \langle \bar{q}\, q\rangle &= -\,(222\,\text{MeV})^3, \\ G_1 &= \frac{1}{(244.0\,\text{MeV})^2}, & g &= 3.16. \end{aligned} \tag{5.57}$$

However we have no strong arguments on behalf of value α_s (5.57). Thus it would be instructive to consider also other values of α_s, that means other values of parameter u.

Let us take $u = 0.001$. We have in this case

$$\begin{aligned} \alpha_s &= 0.700, & m_0 &= 19.1\,\text{MeV}, \\ m_\pi &= 132.0\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV}, \\ m_\sigma &= 767.2\,\text{MeV}, & \Gamma_\sigma &= 385.5\,\text{MeV}, \\ m &= 296.7\,\text{MeV}, & \langle \bar{q} q \rangle &= -(222.8\,\text{MeV})^3, \\ G_1 &= \frac{1}{(239.0\,\text{MeV})^2}, & g &= 3.21. \end{aligned} \tag{5.58}$$

Let us take $u = 0.0015$. Then we obtain in the same way

$$\begin{aligned} \alpha_s &= 0.818, & m_0 &= 18.0\,\text{MeV}, \\ m_\pi &= 128.9\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV}, \\ m_\sigma &= 754.8\,\text{MeV}, & \Gamma_\sigma &= 426.0\,\text{MeV}, \\ m &= 309.5\,\text{MeV}, & \langle \bar{q} q \rangle &= -(228.8\,\text{MeV})^3, \\ G_1 &= \frac{1}{(225.3\,\text{MeV})^2}, & g &= 3.35. \end{aligned} \tag{5.59}$$

For $u = 0.002$ we obtain in the same way

$$\begin{aligned} \alpha_s &= 0.910, & m_0 &= 17.3\,\text{MeV}, \\ m_\pi &= 127.3\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV}, \\ m_\sigma &= 745.2\,\text{MeV}, & \Gamma_\sigma &= 440.0\,\text{MeV}, \\ m &= 318.9\,\text{MeV}, & \langle \bar{q} q \rangle &= -(232.9\,\text{MeV})^3, \\ G_1 &= \frac{1}{(216.0\,\text{MeV})^2}, & g &= 3.45. \end{aligned} \tag{5.60}$$

Note, that for these calculations uncertainties due to our method of infrared cutoff are defined not by estimate (5.16) but by the following quantities

$$\sqrt{u} \simeq 3\,10^{-2}\,(u = 0.001), \quad \sqrt{u} \simeq 5\,10^{-2}\,(u = 0.002),$$

that is the accuracy of numbers (5.57–5.60) is not better than (3–5) %. There are also other sources of uncertainties and so we may estimate the overall accuracy to be of order of 10 %. The main contribution to this estimate is provided by the next orders of $1/N$ expansion, according to the discussion in the Section 5.2.

Bearing in mind the last remarks, we may consider the correspondence of our results for m_π, f_π and $\langle \bar{q} q \rangle$ to existing data being quite satisfactory. Indeed, in Section 4.2 of Chapter 4 we have calculated values of average running coupling $\bar{\alpha}_s$ in the nonperturbative region (4.37). It comes out to be approximately $0.8 - 0.9$ in dependence on the parameter ϵ. This corresponds to results (5.59, 5.60). Value for the constituent light quark mass is consistent, for example, with estimate (1.35), obtained from consideration of the nucleon magnetic moments in Section 1.2. The quark condensate also is close to its phenomenological value (1.83).

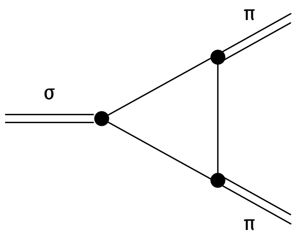


Fig. 5.3. Diagram corresponding to the $\sigma\pi\pi$ effective coupling constant. Simple lines represent quarks.

As for parameters of the σ-meson, experimental data according to [4] give a wide range for their possible values

$$\begin{aligned} 400\,\text{MeV} &< m_\sigma < 550\,\text{MeV}, \\ 400\,\text{MeV} &< \Gamma_\sigma < 700\,\text{MeV}. \end{aligned} \tag{5.61}$$

Let us note, that recent determinations [93, 94] of the σ-meson parameters give more definite results. They are respectfully the following

$$\begin{aligned} m_\sigma &= (470 \pm 30)\,\text{MeV}, \quad \Gamma_\sigma = (590 \pm 40)\,\text{MeV}, \\ m_\sigma &= (541 \pm 39)\,\text{MeV}, \quad \Gamma_\sigma = (504 \pm 80)\,\text{MeV}. \end{aligned} \tag{5.62}$$

There is also analysis of the $\pi - \pi$ data with light σ [95], which also agrees with σ parameters (5.61, 5.62). The data give values for m_σ essentially smaller, than that shown in (5.57, 5.58 5.59, 5.60) of the present calculations. There is a reliable argumentation, that the first approximation does not take into account effects of four quark admixture in the σ-meson [47]. The effective vertex $\sigma\pi\pi$ in the first approximation is described by diagram shown in Figure 5.3. According to the diagram we have the following expression for effective coupling $g_{\sigma\pi\pi}$

$$g_{\sigma\pi\pi} = \frac{N g^3 m}{4\pi^2} \int_u^\infty \frac{\Psi(t)^3}{t} dt. \tag{5.63}$$

Let us estimate contributions under discussion by taking into account of simple π-meson loop diagram, shown in Figure 5.4 This diagram give the following contribution to the mass squared of the σ-meson

$$\Delta(m_\sigma^2) = -\frac{3\,g_{\sigma\pi\pi}^2}{16\pi^2} \int_u^\infty \frac{\Psi(z)^2}{z} dz, \quad m_\sigma = \sqrt{m_{0\sigma}^2 + \Delta(m_\sigma^2)}. \tag{5.64}$$

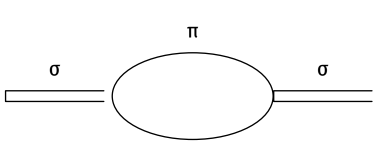


Fig. 5.4. Diagram corresponding to contribution of meson loop to the sigma meson mass.

Here we take for an effective vertex $\sigma\,\pi\,\pi$ just the Bethe–Salpeter wave function (5.25), which defines $\sigma\,\bar{q}\,q$ interaction. Thus the result of the calculation can be only qualitative and is to be understood as an estimate. However, it is important, that the sign of contribution (5.64) is obtained safely and the result of reducing the σ mass due to an account of four-quark contributions is quite reliable.

Let us also show expression for the σ meson width

$$\Gamma_\sigma = \frac{3\,g^2_{\sigma\pi\pi}\,\sqrt{m_\sigma^2 - 4\,m_\pi^2}}{16\,\pi\,m_\sigma^2}. \tag{5.65}$$

After application of corrections (5.64) we come to the modified table of results. In these calculations of the σ meson parameters we take for pion mass $m_\pi = 138\,\text{MeV}$, which corresponds to average of the charged and the neutral pion mass.

Let us take $u = 0.001$. We have in this case

$$\begin{aligned}
\alpha_s &= 0.700, & m_0 &= 19.1\,\text{MeV},\\
m_\pi &= 132.0\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV},\\
m_\sigma &= 458.9\,\text{MeV}, & \Gamma_\sigma &= 548.4\,\text{MeV},\\
m &= 296.7\,\text{MeV}, & \langle\bar{q}\,q\rangle &= -(222.8\,\text{MeV})^3,\\
G_1 &= \frac{1}{(239.0\,\text{MeV})^2}, & g &= 3.21.
\end{aligned} \tag{5.66}$$

Let us take $u = 0.0015$. Then we obtain in the same way

$$\begin{aligned}
\alpha_s &= 0.818, & m_0 &= 18.0\,\text{MeV},\\
m_\pi &= 128.9\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV},\\
m_\sigma &= 447.6\,\text{MeV}, & \Gamma_\sigma &= 601.8\,\text{MeV},\\
m &= 309.5\,\text{MeV}, & \langle\bar{q}\,q\rangle &= -(228.8\,\text{MeV})^3,\\
G_1 &= \frac{1}{(225.3\,\text{MeV})^2}, & g &= 3.35.
\end{aligned} \tag{5.67}$$

For $u = 0.002$ we obtain in the same way

$$\begin{aligned}
\alpha_s &= 0.910, & m_0 &= 17.3\,\text{MeV},\\
m_\pi &= 127.3\,\text{MeV}, & f_\pi &= 92.4\,\text{MeV},\\
m_\sigma &= 444.9\,\text{MeV}, & \Gamma_\sigma &= 630.1\,\text{MeV},\\
m &= 318.9\,\text{MeV}, & \langle\bar{q}\,q\rangle &= -(232.9\,\text{MeV})^3,\\
G_1 &= \frac{1}{(216.0\,\text{MeV})^2}, & g &= 3.45.
\end{aligned} \tag{5.68}$$

Now parameters of the σ meson are in agreement with experimental bounds (5.61, 5.62). For example, the sets of results for the three values u fit into error bars of the first result (5.62). We are to bear in mind our estimated accuracy $\simeq 10\,\%$.

Let us also draw attention to value of the quark condensate $\langle\bar{q}q\rangle$. The results for this quantity also agree within 10 % accuracy with phenomenological value (1.83): $-(230\,\text{MeV})^3$. The agreement is even better than in original Nambu–Jona-Lasinio model (see (1.107) in Section 1.6.2), where agreement was achieved with adjusting of four parameters.

Now let us comment the problem of stability. To estimate value of effective potential (5.38) we take numbers (5.57, 5.58, 5.59, 5.60) for α_s, m, $\langle\bar{q}\,q\rangle$ and use in expression (5.45) values for the gluon condensate (4.44, 4.45). In the point of a minimum in expression (5.38)

$$\xi_0 = 1 - \left(\frac{u_0}{u}\right)^{\frac{1}{4}}, \quad \xi = \frac{g\,\phi}{m}, \tag{5.69}$$

where ϕ is nonzero vacuum average of the scalar field. We have calculated mass m and quark condensate $\langle\bar{q}\,q\rangle$ for u and d quarks, while in (5.45) contribution of all quarks is implied. As a matter of fact, contribution of s-quark may be essential, and the heavier quarks may also give an additional increment. Just to get impression of the dependence of stability conditions on values of average $\bar{\alpha}_s$, we assume that the contribution of other quarks is equal to value $m\,\langle\bar{q}\,q\rangle$ for one light quark. So we substitute for quark condensate term in (5.45)

$$3\,m\,\langle\bar{q}\,q\rangle, \tag{5.70}$$

and now all parameters are in our disposal. Let us note, that similar arguments were expressed in works [96, 97].

Substituting calculated values into (5.38, 5.45) we perform direct calculations for three values of u. The results for vacuum density VD are presented in Table 5.1

We see from Table 5.1, that with $\bar{\alpha}_s$ increasing vacuum density VD decreases and at some value of the average coupling became negative. The value for absence of nonperturbative contributions, that is for the trivial solution, has to be zero. Thus we come to the important conclusion.

There exists some critical value $\bar{\alpha}_s^{\text{crit}}$, which corresponds to a phase transition. For $\bar{\alpha}_s < \bar{\alpha}_s^{\text{crit}}$ the perturbative phase is realized, in which vacuum averages V_2 and $\langle\bar{q}q\rangle$ are zero and the trivial solutions for compensation equations are accomplished. On the contrary, $\bar{\alpha}_s > \bar{\alpha}_s^{\text{crit}}$ corresponds to the nonperturbative phase, in which nontrivial solutions of compensation equations are realized and both anomalous three-gluon interaction (4.2, 4.6) and Nambu–Jona-Lasinio interaction (5.3, 5.4) are generated. Ac-

Table 5.1. Vacuum density VD in dependence of average $\bar{\alpha}_s$. Gluon condensate V_2 is defined in (4.38)

$\bar{\alpha}_s$	u	g	m	$-\langle\bar{q}q\rangle^{\frac{1}{3}}$ MeV	V_2 GeV^4	VD GeV^4
0.84	0.001608	3.375	311.8	229.8	0.0128	0.0012
0.87	0.001765	3.408	314.9	231.2	0.0120	0.0004
0.91	0.002	3.450	318.9	232.9	0.0108	−0.0014

cording to Table 5.1 under premise (5.70) this critical value is approximately

$$\bar{\alpha}_s^{\text{crit}} = 0.88. \tag{5.71}$$

We conclude, that the description of low momenta nonperturbative region comes out to be quite satisfactory in case we unite results of two Chapters 4 and 5. Thus we effectively describe this region with the only parameter, which can be chosen to be either Λ_{QCD} or f_π. Remind, that the interval of possible average values of the strong coupling agrees with calculations of previous Chapter 4.

To conclude we would like to emphasize that the present approach for the first time permits to determine parameters of effective interaction inherent to the Nambu–Jona-Lasinio model in terms of parameters of the fundamental QCD. The optimal value of $\alpha_s = 0.8$–0.9 in (5.67, 5.68) is quite reasonable from the point of view of the existing knowledge on its low-momenta behavior. As for value of current quark mass $m_0 \simeq 18\,\text{MeV}$, it seems to be rather larger than usual values $m_0(2\,\text{GeV}) \simeq 2$–$6\,\text{MeV}$(see Table 1.2). To comment the situation let us note, that firstly the low value of m_0 being mentioned corresponds to perturbative region and the problem how this running parameter varies while $\sqrt{q^2}$ moves to low energy region deserves a special study. In considering of the running m_0 we have to take into account the effective NJL interaction as well. Secondly, the lattice studies give as a rule rather high values for m_0, e. g., in work [98] values of m_0 correspond just to few tens of MeV. The smaller values are to be obtained in the continuous limit, which till now is performed only by an extrapolation procedure.

So we may state that the aim of the consideration is achieved. We have begun with the demonstration of the nontrivial solution of the compensation equation. The appearance of scalar and pseudo-scalar excitations (mesons) in the same approximation is a consequence of its existence. The account of QCD interaction and of meson-quark interaction leads to the shift of their masses squared to the negative region, i.e. to the appearance of tachyons, which are necessary for scalar condensate to arise. As a result we obtain the standard scheme leading to the spontaneous breaking of the chiral symmetry. Subsequent approximations of the approach are related to values of the quark condensate and of the pion mass.

We have shown that the application of the method of our compensation method, which is based on Bogoliubov compensation approach, to the low-energy region of hadron physics leads to quite reasonable results. Let us once more emphasize that we have no additional parameters but those entering in the low-energy QCD: α_s and m_0. Thus we derive the effective interaction of NJL type from the fundamental QCD. On this point we can do the more strong statement. Namely, the interval of possible values of $\alpha_s(z_0)$ was calculated in Chapter 4 (see (4.25, 4.30). The most valuable information, which is to be compared with initial parameters is obtained in Section 4.2. Indeed values for average $\bar{\alpha}_s$ in the nonperturbative region e. g., (4.37) are just fitting the interval $0.8 < \bar{\alpha}_s < 0.9$. Remind, that this interval also includes the estimated $\alpha_s^{\text{crit}} = 0.88$ (5.71). Thus, we may state, that all the results for description of low-momenta being presented till now in Chapters 4 and 5 are obtained with the only one initial parameter either Λ_{QCD} or f_π. This conclusion illustrates very important property of a theory with a spontaneously generated effective interaction. The conditions, imposed by compensation equations are very restrictive. In the most cases these equations have no nontrivial solution at all. But with existence of a nontrivial solution, for example, in the case, described in Chapters 4 and 5, the set of conditions defines almost all the parameters and functions, entering in the theory.

It is quite worth emphasizing, that the phenomenon of α_s^{crit} (5.71) is occurring only with combined action of both the three-gluon effective interaction and the Nambu–Jona-Lasinio effective interaction. Thus these two parts of the theory are connected by conditions of stability. So albeit we have considered conditions for spontaneous generation of these interactions separately, it seems that they may be realized only in a combination. It is important physical conclusion.

A development of the present approach in application to the hadron physics quite deserves attention. In particular it is advisable to apply the approach to calculation of parameters of vector mesons, e. g., of the ρ meson. We describe this application in the subsequent section.

5.7 Vector mesons

In previous sections effective nonlocal $SU(2) \times SU(2)$ NJL model was derived in the framework of the fundamental QCD. All the parameters of the model are expressed through QCD parameters: current light quark mass m_0 and average nonperturbative α_s. The results for scalar and pseudo-scalar mesons are in satisfactory agreement to existing data. In the present work the same model without introduction of any additional parameters is applied for a description of masses and strong decay widths of ρ- and a_1-mesons. The results for both scalar and vector sectors agree with data with only one adjusted parameter m_0, with account of average α_s, which range of variation is considered in the previous section.

We have shown above, that low-energy hadron physics is effectively described in the framework of the Nambu–Jona-Lasinio model, which is spontaneously generated with conventional QCD taken as an input.

In previous sections we have succeed in obtaining description of $SU(2) \times SU(2)$ NJL model using only QCD parameters. As a result a nonlocal version of NJL model was obtained with uniquely defined form-factor. Thus ultra-violet divergences disappear, therefore there is no need of introduction of parameter Λ. Constants G_1 and G_2 are expressed through m_0 and strong constant α_s in the nonperturbative region.

Remind, that application of these results to the sector of scalar and pseudo-scalar mesons leads to satisfactory description of π and σ masses, constant of weak pion decay f_π and of strong $\sigma \to \pi\pi$ decay. Emphasize, that only parameters m_0 and α_s were used.

It is worth noting, that in Chapter 4 estimate of average nonperturbative value α_s was obtained. The same Bogoliubov approach for a study of the effective nonlocal three-gluon interaction results in existence of the stable solution for the definite form of nonperturbative contributions to running coupling $\alpha_s(q^2)$. This corresponds to average value for the running coupling in the nonperturbative region α_s =0.7–0.9. Taking into account this result only one parameter m_0 remains in our disposal. Note, that previous results lead to a consistent value of the gluon condensate.

Here we use the nonlocal NJL model with the same parameters m_0 and α_s for calculation of masses and decay widths of vector and axial-vector mesons ρ and a_1. Remind that we introduce no new parameters at all.

5.7.1 Compensation equations for effective form-factors

In this section we apply previous results to calculation of parameters of vector mesons. In the same way as above we start from the standard Lagrangian of QCD (5.1) with two light quarks and number of colors $N = 3$

Let us rewrite the initial expression (5.1) in the form

$$L = \frac{\imath}{2}\left(\bar\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\mu\psi\right) - \frac{1}{4}F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} - m_0\bar\psi\,\psi \tag{5.72}$$

$$\begin{aligned}
&+\frac{G_1}{2}\cdot\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\right) + \frac{G_2}{2}\cdot\left(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi\right.\\
&\left.+\bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\right) + \frac{G_3}{2}\cdot\left(\bar\psi\gamma_\mu\psi\,\bar\psi\gamma_\mu\psi + \bar\psi\gamma_5\gamma_\mu\psi\bar\psi\gamma_5\gamma_\mu\psi\right)\\
&+g_s\,\bar\psi\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}\left(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}\right)\\
&-\frac{G_1}{2}\cdot\left(\bar\psi\tau^b\gamma_5\psi\,\bar\psi\tau^b\gamma_5\psi - \bar\psi\,\psi\,\bar\psi\,\psi\right)\\
&-\frac{G_2}{2}\cdot\left(\bar\psi\tau^b\gamma_\mu\psi\,\bar\psi\tau^b\gamma_\mu\psi + \bar\psi\tau^b\gamma_5\gamma_\mu\psi\bar\psi\tau^b\gamma_5\gamma_\mu\psi\right)\\
&-\frac{G_3}{2}\cdot\left(\bar\psi\gamma_\mu\psi\,\bar\psi\gamma_\mu\psi + \bar\psi\gamma_5\gamma_\mu\psi\bar\psi\gamma_5\gamma_\mu\psi\right).
\end{aligned}\tag{5.73}$$

Here ψ is isotopic doublet, color summation is performed inside each spinor bilinear combination, $F_{0\,\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and e.g. notation $G_2\cdot\bar\psi\tau^b\gamma_\mu\psi\bar\psi\tau^b\gamma_\mu\psi$ means nonlocal vertex in the momentum space

$$\imath\,(2\pi)^4\,G_2\,\tau^b\gamma_\mu \times \tau^b\gamma_\mu F_V(p1,p2,p3,p4)\,\delta(p1+p2+p3+p4), \tag{5.74}$$

and form-factor F_v depends on incoming momenta. The Lagrangian contains contribution of both G_2 and G_3 which are connected correspondingly to isovector and isoscalar terms. Here we consider compensation equation only for isovector four-fermion terms.

Now we consider the first three lines of the Lagrangian (5.72) as new free Lagrangian L_0, and the four last ones as interaction Lagrangian L_{int} (5.73). Then compensation conditions again will consist in demand of full connected four-fermion vertices, following from Lagrangian L_0, to be zero. This demand gives a set of nonlinear compensation equations for form-factors F_V.

Let us emphasize, that again the existence of a perturbative trivial solution (in our case $G_i = 0$) is evident, but a nonperturbative nontrivial solution may also exist. In the present problem as well as in previous ones we look for an adequate approach, the first nonperturbative approximation of which describes the main features of the problem. Improvement of a precision of results is to be achieved by corrections to the initial first approximation.

The definition of the approximation is already formulated in Section 5.2.

Note, that in case of vector vertices there are two Lorentz structures and thus we have generally speaking two form-factors instead of one in Section 5.2. However the corresponding set of equations has no explicit solution similar to that described by relations (5.11), (5.14), (5.15) and so we proceed in the following way. In our approximation we impose simplified kinematic condition that left-side legs of diagrams have momenta p and $-p$, while right-side ones have zero momenta. Now in addition to terms proportional to $\gamma_\varrho \times \gamma_\varrho$, which we are interested in, terms of the form $\hat{p} \times \hat{p}$ may be also present, Supposing, that the presence of a form-factor connected with the last structure gives small corrections we shall transform the initial equation (in diagram form see Figure 5.1) to the scalar one contracting it with projector of the form

$$\frac{1}{12}\left(\gamma_\varrho - \frac{\hat{p}\, p_\varrho}{p^2}\right). \tag{5.75}$$

In the process of the study we have considered also equations obtained with use of projectors of more general form, namely

$$\frac{1}{4\,(4-d)}\left(\gamma_\varrho - d\,\frac{\hat{p}\, p_\varrho}{p^2}\right). \tag{5.76}$$

It becomes clear, that for values d between 1 and 2 the corresponding solutions lead to spread of physical values under interest in the range of 5–7 %, that corresponds accuracy of the method as a whole. So we take the formulated projection procedure as a component of the first approximation.

Now the demand of compensation of full connected four-fermion vertices proportional to G_2 multiplied by the vector form-factor leads us to the following equation, which in diagram form has the same representation, shown in Figure 5.1, as the equation in the scalar case.

$$\begin{aligned} G_2\, F_V\left(p^2\right) &+ \frac{{G_2}^2}{\pi^2}\left(\frac{65}{72}\,p^2 - \frac{7}{12}\,p^2 \ln\left(\frac{p^2}{\Lambda^2}\right) - \frac{5}{4}\,\Lambda^2\right) \\ &+ \frac{G_3\, G_2}{2\,\pi^2}\left(-\frac{43}{72}\,p^2 + \frac{5}{12}\,p^2 \ln\left(\frac{p^2}{\Lambda^2}\right) + \frac{3}{4}\,\Lambda^2\right) \\ &+ \frac{{G_2}^2\, N}{32\,\pi^6}\int\limits_{m_0}^{\infty} F_V\left(k^2\right)\left({G_2}^2\, N\Lambda^2 - 4\,\pi^2\right) d^4k \end{aligned}$$

$$
\begin{aligned}
&+\frac{{G_1}^2}{\pi}\left(\frac{11}{288}p^2-\frac{1}{16}\Lambda^2-\frac{1}{48}p^2\ln\left(\frac{p^2}{\Lambda^2}\right)\right)\\
&+\frac{{G_2}^3N}{2\pi^6}\left(\frac{7}{36}\int_{m_0}^{\infty}\left(\frac{2(kp)^2}{p^2}+k^2\right)(p-k)^2\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\right.\\
&\quad+\int_{m_0}^{\infty}\left(-\frac{31}{108}\frac{(kp)^2}{p^2}-\frac{109}{864}k^2\right)(p-k)^2\,\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\\
&\quad+\int_{m_0}^{\infty}\left(\frac{1}{18}k^2p^2\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)+\frac{3}{16}\left(\frac{2(kp)^2}{p^2}+k^2\right)\Lambda^2\right.\\
&\quad\left.-\frac{5}{432}\left(-(kp)^2+3\,k^2p^2\right)\right)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}-\int_{m_0}^{\infty}\left(\frac{1}{48}\frac{\left((kp)^2-k^2p^2\right){m_0}^4}{(p-k)^4}\right.\\
&\quad\left.\left.-\frac{1}{96}\frac{{m_0}^4\left(7\,k^2p^2+8\,(kp)^2\right)}{(p-k)^2\,p^2}\right)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\right)+\frac{G_2{G_3}^2N}{2\pi^6}\\
&\times\left(\int_{m_0}^{\infty}\left(\frac{55}{576}\frac{(kp)^2}{p^2}+\frac{17}{2304}k^2\right)(p-k)^2\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\right.\\
&\quad+\int_{m_0}^{\infty}\left(\frac{17}{216}\frac{(kp)^2}{p^2}+\frac{1009}{13824}k^2\right)(p-k)^2\,\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\\
&\quad+\int_{m_0}^{\infty}\left(\frac{11}{384}k^2p^2\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)-\frac{49}{768}\left(\frac{2(kp)^2}{p^2}+k^2\right)\Lambda^2\right.\\
&\quad\left.+\frac{5}{6912}\left(31(kp)^2-33\,k^2p^2\right)\right)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\\
&\quad+\int_{m_0}^{\infty}\left(\frac{1}{288}\frac{\left((kp)^2-k^2p^2\right){m_0}^4}{(p-k)^4}+\frac{1}{384}\frac{{m_0}^4\left(7\,k^2p^2+8\,(kp)^2\right)}{(p-k)^2\,p^2}\right)\\
&\quad\left.\times\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\right)\\
&+\frac{G_2{G_1}^2N}{2\pi^6}\left(-\frac{1}{288}\int_{m_0}^{\infty}\left(\left(\frac{4(kp)^2}{p^2}-k^2\right)\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)\right.\right.\\
&\quad\left.-\frac{1}{1728}\left(\frac{32(kp)^2}{p^2}+k^2\right)\right)(p-k)^2\,\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\\
&\quad+\int_{m_0}^{\infty}\left(-\frac{1}{864}\left(-5+6\,\ln\left(\frac{(p-k)^2}{\Lambda^2}\right)\right)\left(k^2p^2-(kp)^2\right)\right.
\end{aligned}
\tag{5.77}
$$

$$
\begin{aligned}
&+\frac{1}{96}\left(\frac{2(kp)^2}{p^2}+k^2\right)\Lambda^2\bigg)\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\\
&+\int_{m_0}^{\infty}\left(-\frac{1}{192}\,\frac{m_0{}^4\left(5\,k^2p^2-2\,kp^2\right)}{(p-k)^2\,p^2}+1/48\,\frac{m_0{}^4\left(k^2p^2-kp^2\right)}{(p-k)^4}\right)\\
&\times\frac{F_V\left(k^2\right)d^4k}{(k^2)^2}\Bigg)=0.
\end{aligned}
$$

Here a conversion to Euclidean momentum space is performed, at one-loop level terms proportional to N and 1 are taken into account and for two loops respectively N^2 and N. The lower limit of integration is defined by current quark mass m_0 corresponding to value $u_0 = 1.78\,10^{-8}$, which is obtained in the course of consideration of scalar form-factor (see (5.14)). We also use relation

$$
G_1=\frac{6}{13}\,G_2, \tag{5.78}
$$

which is derived above. After dividing by G_2 that correspond to our intention to find a nontrivial solution we integrate by angular variables of four-dimensional space and we have as a result

$$
\begin{aligned}
F_V\,(x)\;&+\frac{N}{\pi^4}\Bigg(\Bigg(\Bigg(\left(\frac{1}{64}\,G_2{}^2-\frac{1}{256}\,G_3{}^2-\frac{1}{128}\,G_1{}^2\right)\int_{m_0}^{x}\frac{1}{x}\quad F_V\,(y)\,dy\\
&+\left(\frac{1}{96}\,G_2{}^2+\frac{1}{96}\,G_3\,G_2+\frac{1}{384}\,G_1{}^2\right)\int_{m_0}^{x}\frac{y}{x^2}F_V\,(y)\,dy\Bigg)m_0{}^4\\
&+\left(\frac{13}{96}\,G_2{}^2-\frac{5}{96}\,G_3\,G_2+\frac{1}{192}\,G_1{}^2+\frac{1}{384}\,G_3{}^2\right)\ln{(x)}\,x\int_{m_0}^{x}F_V\,(y)\,dy\\
&+\left(\frac{1}{8}\,G_2{}^2-\frac{7}{96}\,G_3\,G_2\right)\ln{(x)}\int_{m_0}^{x}y\,F_V\,(y)\,dy\\
&+\left(\frac{7}{32}\,G_2{}^2-\frac{29}{288}\,G_3\,G_2+\frac{1}{384}\,G_3{}^2+\frac{1}{192}\,G_1{}^2\right)\int_{m_0}^{x}y\,F_V\,(y)\,dy\\
&+\frac{1}{x}\left(-\frac{13}{1152}G_3G_2+\frac{1}{128}G_2{}^2-\frac{1}{1536}G_3{}^2-\frac{1}{768}G_1{}^2\right)\int_{m_0}^{x}y^2F_V\,(y)\,dy\\
&+\left(\frac{1}{11520}G_3{}^2-\frac{1}{1920}G_3G_2+\frac{7}{2880}G_2{}^2+\frac{1}{5760}G_1^2\right)\frac{1}{x^2}\int_{m_0}^{x}y^3F_V\,(y)\,dy\\
&+\Bigg(\left(\frac{7}{384}\,G_1^2+\frac{5}{192}\,G_2{}^2+\frac{1}{24}\,G_3\,G_2+\frac{1}{256}\,G_3{}^2\right)x\int_{x}^{\infty}\frac{1}{y^2}\,F_V\,(y)\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{3}{128}G_1{}^2-\frac{1}{128}G_3{}^2-\frac{1}{32}G_3G_2\right)\int_x^\infty\frac{1}{y}F_V(y)\,dy\Bigg)m_0{}^4\\
&+\left(\frac{1}{640}G_2{}^2-\frac{1}{3840}G_1{}^2-\frac{1}{7680}G_3{}^2-\frac{13}{5760}G_3G_2\right)x^3\int_x^\infty\frac{1}{y^2}F_V(y)\,dy\\
&+\left(\frac{5}{1152}G_3G_2+\frac{1}{384}G_1{}^2+\frac{1}{64}G_2{}^2+\frac{1}{768}G_3{}^2\right)x^2\int_x^\infty\frac{1}{y}F_V(y)\,dy\\
&+\left(\frac{13}{96}G_2{}^2-\frac{5}{96}G_3G_2+\frac{1}{192}G_1{}^2+\frac{1}{384}G_3{}^2\right)x\int_x^\infty\ln(y)F_V(y)\,dy\\
&+\left(\frac{61}{288}G_2{}^2+\frac{1}{1152}G_3{}^2-\frac{11}{96}G_3G_2+\frac{1}{576}G_1{}^2\right)x\int_x^\infty F_V(y)\,dy\\
&+\left(\frac{1}{8}G_2{}^2-\frac{7}{96}G_3G_2\right)\int_x^\infty\ln(y)\,y\,F_V(y)\,dy\Bigg)\\
&+\frac{N}{\pi^4}\Bigg(\Bigg(\left(\frac{1}{192}G_3G_2-\frac{1}{32}G_2{}^2\right)x\int_{m_0}^\infty\frac{1}{y^2}F_V(y)\,dy\\
&+\left(\frac{1}{64}G_3G_2-\frac{3}{32}G_2{}^2\right)\int_{m_0}^\infty\frac{1}{y}F_V(y)\,dy\Bigg)m_0{}^4\\
&+\left(\frac{5}{96}G_3G_2-\frac{13}{96}G_2{}^2-\frac{1}{192}G_1{}^2-\frac{1}{384}G_3{}^2\right)\ln\Lambda^2x\int_{m_0}^\infty F_V(y)\,dy\\
&+\left(\frac{43}{576}G_3G_2-\frac{119}{576}G_2{}^2-\frac{11}{2304}G_3{}^2-\frac{11}{1152}G_1{}^2\right)x\int_{m_0}^\infty F_V(y)\,dy\\
&+\left(\frac{7}{96}G_3G_2-\frac{1}{8}G_2{}^2\right)\ln\left(\Lambda^2\right)\int_{m_0}^\infty yF_V(y)\,dy\\
&+\left(\frac{53}{576}G_3G_2-\frac{1}{192}G_1{}^2-\frac{1}{384}G_3{}^2-\frac{19}{96}G_2{}^2\right)\int_{m_0}^\infty yF_V(y)\,dy\\
&+\left(-\frac{3}{32}G_3G_2+\frac{1}{128}G_3{}^2+\frac{9}{32}G_2{}^2+\frac{1}{64}G_1{}^2\right)\Lambda^2\int_{m_0}^\infty F_V(y)\,dy\Bigg)\\
&+\frac{G_2N\left(G_2N\Lambda^2-4\pi^2\right)}{16\pi^4}\int_{m_0}^\infty F_V(y)\,dy+\frac{G_2}{\pi^2}\\
&\times\left(\frac{65}{72}x-\frac{7}{12}x\ln\frac{x}{\Lambda^2}-\frac{5}{4}\Lambda^2\right)+\frac{G_3}{2\pi^2}\left(-\frac{43}{72}x+\frac{5}{12}x\ln\frac{x}{\Lambda^2}+\frac{3}{4}\Lambda^2\right)\\
&+\frac{G_1{}^2}{288G_2\pi^2}\left(11x-18\Lambda^2-6x\ln\frac{x}{\Lambda^2}\right)=0.
\end{aligned}
\tag{5.79}
$$

In view of looking for solutions of equation (5.79) we apply the differential operator

$$\frac{d^3}{dx^3}x\frac{d^2}{dx^2}x\frac{d^3}{dx^3}x^2,$$

to this equation. As a result we obtain a differential equation, which with account of the following substitution

$$z = \beta x^2, \quad \beta = \frac{1}{2^6}\frac{N\,G_2\,(12G_2 - 7G_3)}{24\,\pi^4}, \tag{5.80}$$

reduces to the following form

$$\begin{aligned}\left(z\frac{d}{dz}-b_1\right)\left(z\frac{d}{dz}-b_2\right)\left(z\frac{d}{dz}-b_3\right)\left(z\frac{d}{dz}-b_4\right)\left(z\frac{d}{dz}-b_5\right)&\\ \times\left(z\frac{d}{dz}-b_6\right)\times\left(z\frac{d}{dz}-b_7\right)\left(z\frac{d}{dz}-b_8\right)F_V(z)&\\ = z\left(z\frac{d}{dz}-a_1+1\right)\left(z\frac{d}{dz}-a_2+1\right)F_V(z),&\end{aligned} \tag{5.81}$$

i.e. it is Meijer equation of the eighth order. Solutions of the equation are represented in terms of the Meijer functions [71] with parameters b_i, a_i, which we can calculate provided G_3 is defined. We can naturally admit $G_3 = G_2$ following rules of NJL model (see [39] and Section 1.6.2). In what follows we present confirmation of this assumption. In this case we have

$$\begin{aligned}&b_1 := 1.5, \quad b_2 = 1, \quad b_3 = 0.499991384, \quad b_4 = .500008866,\\ &b_5 = -1.45597130\cdot 10^{-7}, \quad b_6 = 0, \quad b_7 = -0.50000003,\\ &b_8 = -1.0000001, \quad a_1 := -0.3944464, \quad a_2 := 1.9013991.\end{aligned} \tag{5.82}$$

Values of parameters are calculated with account of value m_0.

To obtain a solution of the integral equation we choose four linearly independent solutions of equation (5.81) decreasing at infinity and form the following linear combination with coefficients C_i

$$\begin{aligned}F_V(z) = C_1 G^{51}_{28}\left(z|^{a_1,a_2}_{b5,b4,b3,b2,b1,b8,b7,b6}\right) + C_2 G^{51}_{28}\left(z|^{a_1,a_2}_{b6,b5,b3,b2,b1,b8,b7,b4}\right)&\\ + C_3\, G^{51}_{28}\left(z|^{a_1,a_2}_{b7,b4,b3,b2,b1,b8,b6,b5}\right) + C_4\, G^{71}_{28}\left(z|^{a_1,a_2}_{b8,b6,b5,b4,b3,b2,b1,b7}\right).&\end{aligned} \tag{5.83}$$

Coefficients C_i are fixed by boundary conditions, which are obtained in the same way as above

$$\begin{aligned}&3\left(\frac{13}{96}{G_2}^2 + \frac{1}{192}{G_1}^2 + \frac{1}{384}{G_3}^2 - \frac{5}{96}G_2G_3\right)\frac{1}{\pi^4\sqrt{\beta}}\int_{m_0^2}^{\infty} F_V\,(y)\,dy\\ &\quad -\frac{7}{12}\frac{G_2}{\pi^2} - \frac{1}{48}\frac{{G_1}^2}{\pi^2 G_2} + \frac{5}{24}\frac{G_3}{\pi^2} = 0,\\ &\int_{m_0^2}^{\infty} y\,F_V\,(y)\,dy = 0, \quad \int_{m_0^2}^{\infty} y^2\,F_V\,(y)\,dy = 0, \quad \int_{m_0^2}^{\infty} y^3\,F_V\,(y)\,dy = 0.\end{aligned} \tag{5.84}$$

As a result we have

$$C_1 := 0.3330348455, \qquad C_2 := 6.254973002 \cdot 10^{-8},$$
$$C_3 := 3.452159489 \cdot 10^{-8}, \quad C_4 := 2.105889777 \cdot 10^{-15}. \tag{5.85}$$

Unlike of scalar case of Section 5.2 we here do not force the form-factor value at lower integration limit to be unity. Using this condition one might try to define ratio of G_2 and G_3. However assuming equality of these constants we avoid solution of additional complicated transcendental equation, but we acquire a criterion of self-consistency of our approach as a whole, because calculations show, that changing this ratio in reasonable range we have satisfactory results for values of the form-factor at the normalization point. In our case we have $F_V(u_0) = 0.96094$ and so we consider our assumption to be justified with reasonable accuracy. Admissible are values of ratio $\frac{G_2}{G_3} = \chi$ from 1 up to 1.2 as well. For the last value $F_V(u_0) = 1.098993576$. As a matter of fact to fix the ratio one should consider also equation for isoscalar vector terms. However this leads to a considerable complication of the procedure and so here we only noting, that preliminary estimates show that just for range $\chi = 1 - 1.2$ values of isoscalar vector form-factor differs from unity not more than by 10 %. So admitting $\chi = 1$ we formulate the ground approximation bearing in mind necessity of further corrections.

5.7.2 Wave functions of vector states

We have the nontrivial solution of the compensation equation and thus four-fermion terms are excluded from free Lagrangian. There is of course no compensation in interaction Lagrangian, which contains these terms with opposite sign. So we can study a problem of bound states with account of this four-fermion interaction. The Bethe–Salpeter equation for vector case in the same approximation as above (see Figure 5.2) has the following form. Remind that the first approximation corresponds to zero-mass states (in this approximation there is the same equation for vector and axial-vector).

$$\begin{aligned}
\Psi_V(y) = \frac{N}{\pi^4}\Bigg(m^4\Bigg(\left(\frac{3}{256}G_2{}^2 - \frac{G_1{}^2}{128}\right)\int_{m^2}^{x}\frac{1}{x}\;\Psi_V(y)\,dy \\
+\left(\frac{G_2{}^2}{48} + \frac{G_1{}^2}{384}\right)\int_{m^2}^{x}\frac{y}{x^2}\Psi_V(y)\,dy\Bigg) + \left(\frac{23}{11520}G_2{}^2 + \frac{G_1^2}{5760}\right) \\
\times\int_{m^2}^{x}\frac{y^3}{x^2}\Psi_V(y)\,dy + \left(\frac{11}{128}G_2{}^2 + \frac{G_1{}^2}{192}\right)\ln(x)\,x\int_{m^2}^{x}\Psi_V(y)\,dy \\
+\frac{5}{96}G_2{}^2\ln(x)\int_{m^2}^{x}y\Psi_V(y)\,dy + \left(\frac{139}{1152}G_2{}^2 + \frac{G_1{}^2}{192}\right)\int_{m^2}^{x}y\Psi_V(y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{19}{4608}G_2{}^2+\frac{G_1{}^2}{768}\right)\int\limits_{m^2}^{x}\frac{y^2}{x}\Psi_V(y)dy+\left(\left(\frac{7G_1^2}{384}+\frac{55}{768}G_2{}^2\right)\right.\\
&\times\int\limits_{x}^{\infty}\frac{x}{y^2}\Psi_V(y)\,dy-\left(\frac{3G_1{}^2}{128}+\frac{5G_2{}^2}{128}\right)\int\limits_{x}^{\infty}\frac{\Psi_V(y)}{y}dy\Bigg)m^4\\
&-\left(\frac{G_1{}^2}{3840}+\frac{19G_2{}^2}{23040}\right)\int\limits_{x}^{\infty}\frac{x^3}{y^2}\Psi_V(y)\,dy+\left(\frac{G_1{}^2}{384}+\frac{49}{2304}G_2{}^2\right)\\
&\times\int\limits_{x}^{\infty}\frac{x^2}{y}\Psi_V(y)\,dy+\left(\frac{11G_2{}^2}{128}+\frac{G_1{}^2}{192}\right)x\int\limits_{x}^{\infty}\ln(y)\,\Psi_V(y)\,dy\\
&+\left(\frac{113}{1152}G_2{}^2+\frac{G_1{}^2}{576}\right)x\int\limits_{x}^{\infty}\Psi_V(y)\,dy+\frac{5}{96}G_2{}^2\int\limits_{x}^{\infty}\ln(y)\,y\Psi_V(y)\,dy\Bigg) \qquad (5.86)\\
&+\frac{N}{\pi^4}\left(\left(-\frac{5}{192}G_2{}^2\int\limits_{m^2}^{\infty}\frac{x}{y^2}\Psi_V dy+\frac{5}{64}G_2{}^2\int\limits_{m^2}^{\infty}\frac{\Psi_V}{y}dy\right)m^4\right.\\
&-\left(\frac{11}{128}G_2{}^2+\frac{G_1{}^2}{192}\right)\ln\left(\Lambda^2\right)x\int\limits_{m^2}^{\infty}\Psi_V dy\\
&-\left(\frac{35}{256}G_2{}^2+\frac{11}{1152}G_1{}^2\right)x\int\limits_{m^2}^{\infty}\Psi_V dy-\frac{5}{96}G_2{}^2\ln\left(\Lambda^2\right)\int\limits_{m^2}^{\infty}y\Psi_V dy\\
&+\left(-\frac{G_1{}^2}{192}-\frac{125}{1152}G_2{}^2\right)\int\limits_{m^2}^{\infty}y\,\Psi_V\,dy+\left(\frac{25}{128}G_2{}^2+\frac{G_1{}^2}{64}\right)\Lambda^2\\
&\times\int\limits_{m^2}^{\infty}\Psi_V(y)\,dy\Bigg)+\left(\alpha_s-\frac{3}{8}\frac{g_V{}^2}{\pi}\right)\left(\frac{1}{9\pi}\int\limits_{m^2}^{x}\Psi_V(y)\left(\frac{15}{x}+\frac{2y}{x^2}\right)dy\right.\\
&\left.+\frac{1}{9\pi}\int\limits_{x}^{\infty}\Psi_V(y)\left(\frac{12}{y}+\frac{5x}{y^2}\right)dy\right).
\end{aligned}
$$

Here besides the same kernel as in equation (5.79) we take into account also one-gluon exchange and one-meson exchange with corresponding constants α_s and $g_V^2/4\pi$. Note that contributions of (pseudo-)scalar mesons here cancel. In equation (5.86) enters constituent mass m instead of current mass in equation (5.79). For parameter m we use results of previous results in Section 5.4 where it was obtained from stability condition for the effective potential. This procedure allows to define m corresponding to value of α_s. In the same way as in Section 5.6 we take values of $u = \beta m^4$, which correspond to values of α_s in the range under study. We perform calculations for $u = 0.001,\ 0.0015,\ 0.002$. Values of α_s, are presented in the summarizing table.

Differential equation now is the following

$$\left(z\frac{d}{dz}-b_1\right)\left(z\frac{d}{dz}-b_2\right)\left(z\frac{d}{dz}-b_3\right)\left(z\frac{d}{dz}-b_4\right)$$
$$\times\left(z\frac{d}{dz}-b_5\right)\left(z\frac{d}{dz}-b_6\right)\left(z\frac{d}{dz}-b_7\right)\left(z\frac{d}{dz}-b_8\right)\Psi_V(z)$$
$$=-z\left(z\frac{d}{dz}-a_1+1\right)\left(z\frac{d}{dz}-a_2+1\right)\Psi_V(z), \quad (5.87)$$

where

$$z=\beta x^2, \quad \beta=\frac{1}{2^6}\frac{N\,G_2\,(12\,G_2-7\,G_3)}{24\,\pi^4}, \quad \xi=\frac{G_1}{G_2},$$
$$a_1=\frac{1}{80}\frac{59\,\xi^2+6-\sqrt{8281\,\xi^4+708\,\xi^2+36}}{\xi^2},$$
$$a_2=\frac{1}{80}\frac{59\,\xi^2+6+\sqrt{8281\,\xi^4+708\,\xi^2+36}}{\xi^2}, \quad (5.88)$$

and coefficients b_i are roots of the following equation ($G_3=G_2$)

$$\frac{N\,m^4\,G_2^2}{\pi^4}\left(-\frac{2859\,b}{2704}-\frac{1217\,b^2}{676}+\frac{28195\,b^3}{2704}+\frac{435\,b^4}{338}\right.$$
$$\left.-\frac{11539\,b^5}{676}+\frac{1388\,b^6}{169}\right)+\left(\frac{\alpha_s}{\pi}-\frac{3\,g_V^2}{8\,\pi^2}\right)$$
$$\left(32\,b-\frac{592\,b^2}{3}+\frac{880\,b^3}{3}+\frac{320\,b^4}{3}-\frac{1216\,b^5}{3}+\frac{512\,b^6}{3}\right)$$
$$+48\,b^2-128\,b^3-176\,b^4+640\,b^5-512\,b^7+256\,b^8=0. \quad (5.89)$$

Solution of equation (5.87) decreasing at infinity has the following general form

$$\Psi_V(z)=C_1\,G^{4\,1}_{2\,8}\left(z\,|^{a_1,a_2}_{b1,b2,b3,b5,b4,b8,b7,b6}\right) \quad (5.90)$$
$$+C_2\,G^{4\,1}_{2\,8}\left(z\,|^{a_1,a_2}_{b1,b2,b3,b4,b5,b6,b7,b8}\right)+C_3\,G^{4\,1}_{2\,8}\left(z\,|^{a_1,a_2}_{b1,b2,b5,b6,b3,b4,b7,b8}\right)$$
$$+C_4\,G^{6\,1}_{2\,8}\left(z\,|^{a_1,a_2}_{b1,b2,b3,b4,b5,b7,b6,b8}\right)+C_5\,G^{6\,1}_{2\,8}\left(z\,|^{a_1,a_2}_{b1,b2,b3,b5,b6,b8,b7,b4}\right).$$

For $u=0.001$ with account of results (5.66) values of parameters b_i read

$$b_1=1.5, \quad b_2=1, \quad b_4=0.5, \quad b_3=0.6309939, \quad (5.91)$$
$$b_5=0.1329938, \quad b_6=0, \quad b_7=-0.7007615, \quad b_8=-1.0632261.$$

In the same way we obtain parameters for two other values of u:
$u=0.0015$ (5.67)

$$b_1=1.5, \quad b_2=1, \quad b_4=0.5, \quad b_3=0.6481721, \quad (5.92)$$
$$b_5=0.1365796, \quad b_6=0, \quad b_7=-0.7118443, \quad b_8=-1.0729073,$$

and $u = 0.002$ (5.68)

$$
\begin{aligned}
&b_1 = 1.5, && b_2 = 1, \quad b_4 = 0.5, && b_3 = 0.6625915, \\
&b_5 = 0.1384193, && b_6 = 0, \quad b_7 = -0.7197738, && b_8 = -1.0812370,
\end{aligned}
\tag{5.93}
$$

Parameters a_i are the same as before (5.82). Coefficients C_i are defined from the boundary conditions

$$
\begin{aligned}
&\Psi_V\left(m^2\right) = 1, \qquad \int_{m^2}^{\infty} \Psi_V(y)\,dy = 0, \qquad \int_{m^2}^{\infty} y\,\Psi_V(y)\,dy = 0, \\
&\int_{m^2}^{\infty} y^2\,\Psi_V(y)\,dy = 0, \qquad \int_{m^2}^{\infty} y^3\,\Psi_V(y)\,dy = 0,
\end{aligned}
\tag{5.94}
$$

and value g_V is given by the iterative procedure being defined by normalization condition in one-loop approximation

$$
\frac{N g_V^2}{12\pi^2} \int_{\tilde{u}}^{\infty} \frac{\Psi_V(z)^2}{z}\,dz = 1, \quad \tilde{u} = \frac{\beta}{\beta_0} u, \quad \beta_0 = \frac{(G_1^2 + 6\,G_1 G_2)\,N}{16\,\pi^4}. \tag{5.95}
$$

Ratio

$$
\frac{\beta}{\beta_0} = \frac{845}{754}, \tag{5.96}
$$

gives coefficients for transitions to variable $z \sim p^4$ respectfully for vector and scalar sectors. Expression for β_0 is obtained in Section 5.2.

For applications asymptotic of Ψ_V at infinity is essential. Considering equation (5.87) and its solution (5.90) we obtain with account of general expression (2.142)

$$
\begin{aligned}
&\Psi_V(z)_{z\to\infty} = C_V\, z^c\, \cos(6\,z^{\frac{1}{6}} + \phi_V), \qquad c = -\frac{5 + 2\sum_i a_i - 2\sum_i b_i}{12}, \\
&\sum_i a_i = 1.50695, \quad \sum_i b_i = 2, \quad c = -\,0.334492.
\end{aligned}
\tag{5.97}
$$

Now we proceed to calculation of observable parameters. Initial estimate of ρ-meson mass is given by expression (3.33), which for our case reads

$$
M_{0\,\rho} = \frac{g_V}{\sqrt{G_2}}, \tag{5.98}
$$

where G_2 is defined from relation (5.78) and values G_1 are calculated for chosen values u and are presented before in relations (5.57, 5.66, 5.67, 5.68). According to these considerations we present below the first approximation M_ρ^0 for the ρ-meson mass

In view of estimating of other important parameter of the ρ-meson, namely its width we have to obtain the coupling constant in vertex $\rho\,\pi\,\pi$ in accordance with the effective interaction

$$
L_{int} = g_{\rho\pi\pi}\,\epsilon^{abc}\,\rho_a^\mu\,\partial_\mu \pi^b\,\pi^c, \tag{5.99}
$$

Coupling constant $g_{\rho\pi\pi}$ of ρ-decay to two π-mesons we find with triangle diagram according to the following relation

$$g_{\rho\pi\pi} = {g_s}^2 \, g_V \, \frac{3}{4\,\pi^2} \int_u^\infty \frac{\Psi(z)^2 \;\; \Psi_V\left(\frac{\beta}{\beta_0}\, z\right)}{z} dz, \tag{5.100}$$

where $\Psi(z)$ is the Bethe–Salpeter wave function for scalar states and g_s is scalar meson coupling according to definition in Section 5.3.

The width of ρ is the following

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2 \left({M_\rho}^2 - 4\,{m_\pi}^2\right)^{3/2}}{24\,\pi {M_\rho}^2}. \tag{5.101}$$

This coupling constant in the first approximation is connected with quark loop, presented in Figure 5.5.

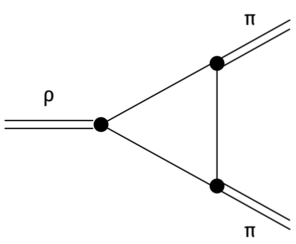


Fig. 5.5. Diagram corresponding to the quark loop, which defines $\rho\,\pi\,\pi$ effective coupling.

Now we calculate parameters of the corresponding wave functions and using relations (5.95, 5.96, 5.98, 5.100, 5.101). With parameters (5.66):

$$u = 0.001, \quad \alpha_s = 0.7, \quad G_1 = \frac{1}{(239.0\,\text{MeV})^2},$$

we have

$$\begin{aligned} C_1 &= 1.6997393, & C_2 &= 0.10299083, & C_3 &= 0.01072961, \\ C_4 &= 0.005643271, & C_5 &= -0.0001956055, & g_V &= 4.0046, \\ M_{0\,\rho} &= 650.2\,\text{MeV}, & g_{\rho\pi\pi} &= 5.65, & \Gamma_\rho &= 134.1\,\text{MeV}. \end{aligned} \tag{5.102}$$

With parameters (5.67):

$$u = 0.0015, \quad \alpha_s = 0.818, \quad G_1 = \frac{1}{(225.3\,\text{MeV})^2},$$

we have

$$\begin{aligned} C_1 &= 1.6910230, & C_2 &= 0.1290819, & C_3 &= 0.0157750, \\ C_4 &= 0.007378365, & C_5 &= -0.0003219670, & g_V &= 4.240, \\ M_{0\,\rho} &= 649.0\,\text{MeV}, & g_{\rho\pi\pi} &= 5.82, & \Gamma_\rho &= 142.3\,\text{MeV}. \end{aligned} \tag{5.103}$$

With parameters (5.68):

$$u = 0.002, \quad \alpha_s = 0.91, \quad G_1 = \frac{1}{(216.0\,\mathrm{MeV})^2},$$

we have

$$\begin{aligned} C_1 &= 1.6849163, & C_2 &= 0.1515008, & C_3 &= 0.0269341, \\ C_4 &= 0.00891981256, & C_5 &= -0.00045756, & g_v &= 4.415, \\ M_{0\,\rho} &= 647.8\,\mathrm{MeV}, & g_{\rho\pi\pi} &= 5.89, & \Gamma_\rho &= 146.2\,\mathrm{MeV}. \end{aligned} \tag{5.104}$$

Bearing in mind also results for other parameters (5.57, 5.58, 5.59, 5.60) we see quite an admissible agreement with data, especially for variant (5.60, 5.104). However, the ρ-meson mass is significantly smaller than its physical value 775 MeV. It may mean, that there are other contributions to the mass. Indeed, relation (5.98) take into account only two-quark composition of the ρ-meson. The four-quark contributions may be taken into account by considering a diagram with π-meson loop. We have already taken into account the π-meson loop contributions while considering parameters of the σ meson. The diagram corresponding to such contribution is presented in Figure 5.6. This contribution is described by the following expression in case of point-like vertices in effective Lagrangian (5.99)

$$\Delta(M_\rho^2) = \frac{g_{\rho\pi\pi}^2}{8\,\pi^2} \int dq^2. \tag{5.105}$$

Integral in (5.105) evidently diverges. However, we know the nature of effective interactions, which act only in a restricted region of the momentum space. In the Nambu–Jona-Lasinio theory, which consequences we now are studying this restriction is provided by form-factor $F(q^2)$ and corresponding Bethe–Salpeter wave functions. In the same way as in the case of the additional contribution to σ meson mass (5.64) in Section 5.6, we use for the purpose wave function Ψ_V, being obtained above. Now we have

$$\begin{aligned} \Delta(M_\rho^2) &= \frac{g_{\rho\pi\pi}^2}{8\,\pi^2} \int_{m^2}^{\infty} \Psi_V(q^2)^2\, dq^2 = \frac{g_{\rho\pi\pi}^2 \sqrt{2}}{\sqrt{21}\, G_1} I_V, \\ I_V &= \int_{u_0}^{\infty} \frac{\Psi_V(z)^2}{\sqrt{z}} dz. \end{aligned} \tag{5.106}$$

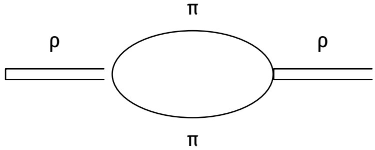


Fig. 5.6. The meson loop diagram, giving contribution to the ρ-meson mass.

Due to asymptotic (5.97) of Ψ_V integral I_V converges and we calculate the corrected mass of the ρ meson according to the following expression

$$M_\rho = \sqrt{M_{0\,\rho}^2 + \frac{g_{\rho\pi\pi}^2\,\sqrt{2}}{\sqrt{21}\,G_1}\,I_V}. \tag{5.107}$$

Thus we have the following change in results for values of u : 0.001, 0.0015, 0.002 respectively

$$M_\rho = 830.6\,\text{MeV}, \quad M_\rho = 820.2\,\text{MeV}, \quad M_\rho = 810.0\,\text{MeV}. \tag{5.108}$$

Thus ρ-meson mass is also consistent with experimental value $M_\rho = 775\,\text{MeV}$ (see Table 1.4) in the range of the anticipated accuracy $\simeq$ 10% for the range of values u being considered here.

As a matter of fact, the effective wave function $\Psi(\rho\,\pi\,\pi)$ has to be inserted in the integral (5.106). However, for the moment this wave function is not obtained yet, and we use instead of it wave function Ψ_V, which describes transition $\rho\,\bar{q}\,q$. While calculating the additional term for the σ mass, we have already mentioned, that the result is to be considered as an estimate. Nevertheless, in both cases results improve agreement with the real physical parameters. It is remarkable, that albeit the estimation of integrals is approximate, the signs of additional terms are defined exactly. In case of the σ meson it is negative and in case of the ρ meson it is positive, that just corresponds to an improvement of the agreement in both cases.

5.7.3 Results and discussion

In the last two Chapters 4, 5 we have considered strong interactions in the nonperturbative low momenta region. At first the three-gluon anomalous interaction was shown to be possibly spontaneously generated. It turns to be possible only provided the usual gauge coupling constant g be fixed at the boundary of the nonperturbative region p_0. The result is quite remarkable. Then we introduce dimensional coupling constant G, which is fixed by a value of the running coupling $\alpha_s(Q^2)$ in the definite point. We have taken for this point the mass of the τ lepton, due to better precision of $\alpha_s(Q^2)$ determination at this value of Q^2. For example, for $\epsilon = 0.13$ (see (4.30, 4.37)) we have

$$\alpha_s(z_0) = 0.773, \quad G = \frac{1}{(273.5\,\text{MeV})^2},$$
$$\bar{\alpha}_s = 0.87. \tag{5.109}$$

On the other hand, average value $\bar{\alpha}_s$ was one of the two initial parameters of the spontaneously generated Nambu–Jona-Lasinio interaction, which was in details studied in Chapter 5. It could not be a coincidence, that just value (5.109) corresponds to the region of better agreement for low-energy hadron physics, which manifests itself in studying of parameters of π, σ, ρ mesons and the gluon and the quark condensates (see relations (4.45, 5.59, 5.60)). This corresponds to region of $\bar{\alpha}_s$ values 0.8–0.9.

5.8 Necessary formulae

Let us present below the set of expressions for angle integrals in four-dimensional Euclid space, which was used in this chapter and will be used in what follows. Integrals involving logarithms may be found in textbook [75]. Remind, that $x = p^2$, $y = q^2$.

$$\int \frac{d^4 q\, F(q^2)}{(p-q)^2} = \pi^2 \int_0^\infty y dy F(y) \left(\vartheta(x-y)\frac{1}{x} + \vartheta(y-x)\frac{1}{y} \right),$$

$$\int \frac{d^4 q\, F(q^2)\,(pq)}{(p-q)^2} = \frac{\pi^2}{2} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\frac{y}{x} + \vartheta(y-x)\frac{x}{y} \right),$$

$$\int \frac{d^4 q\, F(q^2)\,(pq)^2}{(p-q)^2} = \frac{\pi^2}{4} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{y^2}{x} + y\right) + \vartheta(y-x)\left(x + \frac{x^2}{y}\right) \right),$$

$$\int \frac{d^4 q\, F(q^2)\,(pq)^3}{(p-q)^2} = \frac{\pi^2}{8} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{y^3}{x} + 2y^2\right) + \vartheta(y-x)\left(2x^2 + \frac{x^3}{y}\right) \right),$$

$$\int \frac{d^4 q F(q^2)\,(pq)^4}{(p-q)^2} = \frac{\pi^2}{16} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{y^4}{x} + 3y^3 + 2xy^2\right)\right.$$

$$\left. + \vartheta(y-x)\left(2x^2 y + 3x^3 + \frac{x^4}{y}\right) \right),$$

$$\int \frac{d^4 q F(q^2)(pq)^5}{(p-q)^2} = \frac{\pi^2}{32} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{y^5}{x} + 4y^4 + 5xy^3\right)\right.$$

$$\left. + \vartheta(y-x)\left(5x^3 y + 4x^4 + \frac{x^5}{y}\right) \right),$$

$$\int \frac{d^4 q\, F(q^2)(p, p-q)}{((p-q)^2)^2} = \pi^2 \int_0^\infty y dy F(y) \vartheta(x-y)\frac{1}{x},$$

$$\int \frac{d^4 q\, F(q^2)(q, q-p)}{((p-q)^2)^2} = \pi^2 \int_0^\infty y dy F(y) \vartheta(y-x)\frac{1}{y},$$

$$\int \frac{d^4 q\, F(q^2)(p, p-q)(pq)}{((p-q)^2)^2} = \frac{\pi^2}{4} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\frac{3y}{x} - \vartheta(y-x)\frac{x}{y} \right),$$

$$\int \frac{d^4 q\, F(q^2)(p, p-q)(pq)^2}{((p-q)^2)^2} = \frac{\pi^2}{4} \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{2y^2}{x} + y\right) - \vartheta(y-x)\frac{x^2}{y} \right),$$

$$\int d^4 q\, F(q^2) \ln{(p-q)^2} = \pi^2 \int_0^\infty y dy F(y) \left(\vartheta(x-y)\left(\frac{y}{2x} + \ln x\right) + \vartheta(y-x)\left(\ln y + \frac{x}{2y}\right) \right),$$

$$\int d^4 q\, F(q^2)(pq) \ln (p-q)^2$$
$$= \frac{\pi^2}{6} \int_0^\infty y dy F(y) \left(\vartheta(x-y) \left(\frac{y^2}{x} - 3y \right) + \vartheta(y-x) \left(-3x + \frac{x^2}{y} \right) \right),$$

$$\int d^4 q\, F(q^2)(pq)^2 \ln (p-q)^2$$
$$= \frac{\pi^2}{4} \int_0^\infty y dy F(y) \left[\vartheta(x-y) \left(\frac{y^3}{4x} + xy \ln x \right) + \vartheta(y-x) \left(xy \ln y + \frac{x^3}{4y} \right) \right],$$

$$\int d^4 q\, F(q^2)(pq)^3 \ln (p-q)^2 = \frac{\pi^2}{8} \int_0^\infty y dy F(y)$$
$$\times \left(\vartheta(x-y) \left(\frac{y^4}{5x} + \frac{y^3}{3} - 2xy^2 \right) + \vartheta(y-x) \left(-2x^2 y + \frac{x^3}{3} + \frac{x^4}{5y} \right) \right),$$

$$\int d^4 q\, F(q^2)(pq)^4 \ln (p-q)^2 = \frac{\pi^2}{8} \int_0^\infty y dy F(y) \left(\vartheta(x-y) \left(\frac{y^5}{12x} + \frac{y^4}{4} - \frac{xy^3}{4} + x^2 y^2 \ln x \right) \right.$$
$$\left. + \vartheta(y-x) \left(x^2 y^2 \ln y + \frac{x^5}{12y} + \frac{x^4}{4} + \frac{yx^3}{4} \right) \right),$$

$$\int d^4 q\, F(q^2)(pq)^5 \ln (p-q)^2 = \frac{\pi^2}{32} \int_0^\infty y dy F(y) \left(\vartheta(x-y) \left(\frac{y^6}{7x} + \frac{3y^5}{5} + \frac{xy^4}{3} - 5x^2 y^3 \right) \right.$$
$$\left. + \vartheta(y-x) \left(\frac{x^6}{7y} + \frac{3x^5}{5} + \frac{x^4 y}{3} - 5x^3 y^2 \right) \right).$$

6 Three-boson interaction

In previous chapters N. N. Bogoliubov compensation approach [40–42] was applied to studies of spontaneous generation of effective nonlocal interactions in QCD. Spontaneous generation of Nambu–Jona-Lasinio like interaction was studied in Chapter 5. We have achieved a description of the low energy hadron physics in terms of initial QCD parameters, which turns to be quite successful including values of parameters: average strong coupling in the nonperturbative region $\bar{\alpha}_s$, the gluon condensate V_2, the glueball mass M_{gb}, the pion mass m_π, the pion decay constant f_π, the σ meson mass m_σ, the quark condensate $\langle \bar{q}q \rangle$, the ρ meson mass and width M_ρ, Γ_ρ. The starting point of the application of the approach to QCD was the study of a possibility of a spontaneous generation of anomalous three-gluon effective interaction (4.2). The nontrivial solution of compensation equation (4.9) was obtained. Consequences of the solution was shown to be quite reasonable.

We may suppose, that such solutions may occur also in other nonabelian gauge theories. The most interesting such theory is the Standard Model electroweak theory EWT, which is briefly described in Section 2.1.

Let us consider the electroweak theory and consider a possibility of spontaneous generation of an anomalous three-boson interaction of the form

$$-\frac{G}{3!} \cdot \epsilon_{abc}\, W^a_{\mu\nu}\, W^b_{\nu\rho}\, W^c_{\rho\mu}. \tag{6.1}$$

In considering this possibility we follow the same approximation scheme, which was used in QCD.

The main principle of the approach is to check if an effective interaction could be generated in a chosen variant of a renormalizable theory. In view of this one performs "add and subtract" procedure for the effective interaction with a form-factor. Then one assumes the presence of the effective interaction in the interaction Lagrangian and the same term with the opposite sign is assigned to the newly defined free Lagrangian. This transformation of the initial Lagrangian is evidently identical. However such free Lagrangian contains completely improper term, corresponding to the effective interaction of the opposite sign. Then one has to formulate a compensation equation, which guarantees that this new free Lagrangian is a genuine free one, that is effects of the uncommon term sum up to zero. Provided a nontrivial solution of this equation exists, one can state the generation of the effective interaction to be possible. Now we apply this procedure to our problem. The presentation of this chapter is connected with results obtained in works [99–103].

In the present chapter we start with studying a possibility of a spontaneous generation of interaction (6.1).

6.1 Compensation equation for anomalous three-boson interaction

We start with the Lagrangian of the electro-weak interaction with 3 lepton ψ_k and color quark q_k doublets with gauge group $SU(2)$. That is we restrict the gauge sector to triplet of W^a_μ only. Thus we consider $U(1)$ abelian gauge field B to be decoupled, that means approximation $\sin^2\theta_W \ll 1$. Thus we suppose that neutral component

$$W^3 = \cos\theta_W\, Z + \sin\theta_W\, A, \tag{6.2}$$

after the procedure of the symmetry breaking has the same mass as the charged ones. Simply speaking we take $W^3 \equiv Z$. However in applications to real processes we shall use relation (6.2).

$$\begin{aligned} L = &\sum_{k=1}^{3}\left(\frac{\imath}{2}\left(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\right) + \frac{g}{2}\,\bar\psi_k\gamma_\mu\tau^a W^a_\mu\psi_k\right)\\ &+\sum_{k=1}^{3}\left(\frac{\imath}{2}\left(\bar q_k\gamma_\mu\partial_\mu q_k - \partial_\mu\bar q_k\gamma_\mu q_k\right) + \frac{g}{2}\,\bar q_k\gamma_\mu\tau^a W^a_\mu q_k\right)\\ &-\frac{1}{4}\left(W^a_{\mu\nu}W^a_{\mu\nu}\right),\quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc}W^b_\mu W^c_\nu. \end{aligned} \tag{6.3}$$

where we use the standard notations. In accordance to the compensation approach [40–42] in application to QFT, described in Chapter 3, we look for a nontrivial solution of a compensation equation, which is formulated on the basis of the Bogoliubov procedure add–subtract. Namely let us write down the initial expression (6.3) in the following form

$$L = L_0 + L_{int},$$

$$\begin{aligned} L_0 = = &\sum_{k=1}^{3}\left(\frac{\imath}{2}\left(\bar\psi_k\gamma_\mu\partial_\mu\psi_k - \partial_\mu\bar\psi_k\gamma_\mu\psi_k\right) - m_k\bar\psi_k\psi_k\right.\\ &\left.+\frac{\imath}{2}\left(\bar q_k\gamma_\mu\partial_\mu q_k - \partial_\mu\bar q_k\gamma_\mu q_k\right) - M_k\bar q_k q_k\right)\\ &-\frac{1}{4}\,W^a_{\mu\nu}W^a_{\mu\nu} + \frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}, \end{aligned} \tag{6.4}$$

$$\begin{aligned} L_{int} = &\frac{g}{2}\sum_{k=1}^{3}\left(\bar\psi_k\gamma_\mu\tau^a W^a_\mu\psi_k + \bar q_k\gamma_\mu\tau^a W^a_\mu q_k\right)\\ &-\frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}. \end{aligned} \tag{6.5}$$

Here isotopic summation is performed inside of each quark bi-linear combination, and notation $-\frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}$ means corresponding nonlocal vertex in the momentum space

$$\begin{aligned} &(2\pi)^4\,G\,\epsilon_{abc}\,(g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk)\\ &\quad + g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho)\,F(p,q,k)\,\delta(p+q+k) + \dots, \end{aligned} \tag{6.6}$$

where $F(p,q,k)$ is a form-factor and $p,\mu,a;\ \ q,\nu,b;\ \ k,\rho,c$ are respectfully incoming momenta, Lorentz indices and weak isotopic indices of W-bosons. We mean also that there are present four-boson, five-boson and six-boson vertices according to expression for $W^a_{\mu\nu}$ (6.3). Note, that inclusion of total W-boson term $W^a_{\mu\nu}\,W^a_{\mu\nu}$ in the new free Lagrangian (6.4) is performed in view of maintaining the gauge invariance of the approach.

Effective interaction (6.1) is usually called anomalous three-boson interaction and it is considered for long time on phenomenological grounds [104]. Note, that the first attempt to obtain the anomalous three-boson interaction in the framework of the Bogoliubov compensation approach was done in work [105]. Our interaction constant G is connected with conventional definitions in the following way

$$G = -\frac{g\,\lambda}{M_W^2}. \tag{6.7}$$

The current limitations for parameter λ read [106–108],

$$\lambda = -0.016^{+0.021}_{-0.023}, \qquad -\,0.059 < \lambda < 0.026\,(95\%\,C.L.). \tag{6.8}$$

$$\lambda_\gamma = -0.022\ \pm 0.019, \qquad -\,0.038 < \lambda < 0.030 \tag{6.9}$$

where the first result in the second row (6.9) is obtained recently by joint analysis of LEP data by the four experimental groups: ALEPH, DELPHI, L3, OPAL and the second one is obtained in recent LHC studies. Due to our approximation $\sin^2\theta_W \ll 1$ we use the same M_W for both charged $W^\pm$ and neutral W^0 bosons and assume no difference in anomalous interaction for Z and γ, i.e. $\lambda_Z = \lambda_\gamma = \lambda$.

Let us consider expression (6.4) as the new free Lagrangian L_0, whereas expression (6.5) as the new interaction Lagrangian L_{int}. It is important to note, that we put into the new free Lagrangian the full quadratic in W term including boson self-interaction, because we prefer to maintain gauge invariance of the approximation being used. Indeed, we shall use both four-fold term from the last term in (6.4) and triple one from the last but one term of (6.4). Then compensation conditions will consist in demand of full connected three-boson vertices of the structure (6.6), following from Lagrangian L_0, to be zero. This demand gives a nonlinear equation for form-factor F.

In such way we again come to a compensation equation. In a study of these equations it is always evident the existence of a perturbative trivial solution (in our case $G = 0$), but, in general, a nonperturbative nontrivial solution may also exist. Just the quest of a nontrivial solution inspires the main interest in such problems. One can not succeed in finding an exact nontrivial solution in a realistic theory, therefore the goal of a study is a quest of an adequate approach, the first nonperturbative approximation of which describes the main features of the problem. Improvement of a precision of results is to be achieved by corrections to the initial first approximation.

Thus our task is to formulate the first approximation. Here the previous studies, described in Chapters 3, 4, 5 could be helpful. Namely, in the first approximation we follow the same approximation which was formulated in Section 4.1. Let us recall the main points of the approximation in application to the electro-weak case.

(1) In compensation equation we restrict ourselves by terms with loop numbers 0, 1.
(2) We reduce thus obtained nonlinear compensation equation to a linear integral equation. It means that in loop terms only one vertex contains the form-factor, being defined above, while other vertices are considered to be point-like. In diagram form equation for form-factor F is the same as presented in Figure 4.1. Here four-leg vertex correspond to interaction of four bosons due to our effective three-field interaction. In our approximation we take here point-like vertex with interaction constant proportional to $g\,G$.
(3) We integrate by angular variables of the 4-dimensional Euclidean space. The necessary rules are presented in Section 5.8.

We have already mentioned, that results of application of the approximation agree with physical values with average accuracy $\simeq 10$–$15\,\%$. Thus we could hope for such accuracy in the present problem.

At first let us present the expression for four-boson vertex. The expression is similar to (4.7) and the difference is connected with the change of symmetry group $SU(3) \to SU(2)$.

$$
\begin{aligned}
\frac{V(p,m,\lambda;\ q,n,\sigma;\ k,r,\tau;\ l,s,\pi)}{(2\,\pi)^4} &= gG\Big(\epsilon^{amn}\epsilon^{ars}\Big(U(k,l;\sigma,\tau,\pi,\lambda)\\
&- U(k,l;\lambda,\tau,\pi,\sigma) - U(l,k;\sigma,\pi,\tau,\lambda) + U(l,k;\lambda,\pi,\tau,\sigma)\\
&+ U(p,q;\pi,\lambda,\sigma,\tau) - U(p,q;\tau,\lambda,\sigma,\pi) - U(q,p;\pi,\sigma,\lambda,\tau)\\
&+ U(q,p;\tau,\sigma,\lambda,\pi)\Big) - \epsilon^{arn}\,\epsilon^{ams}\Big(U(p,l;\sigma,\lambda,\pi,\tau)\\
&- U(l,p;\sigma,\pi,\lambda,\tau) - U(p,l;\tau,\lambda,\pi,\sigma) + U(l,p;\tau,\pi,\lambda,\sigma)\\
&+ U(k,q;\pi,\tau,\sigma,\lambda) - U(q,k;\pi,\sigma,\tau,\lambda) - U(k,q;\lambda,\tau,\sigma,\pi)\\
&+ U(q,k;\lambda,\sigma,\tau,\pi)\Big) + \epsilon^{asn}\,\epsilon^{amr}\Big(U(k,p;\sigma,\lambda,\tau,\pi) \qquad (6.10)\\
&- U(p,k;\sigma,\tau,\lambda,\pi) + U(p,k;\pi,\tau,\lambda,\sigma) - U(k,p;\pi,\lambda,\tau,\sigma)\\
&- U(l,q;\lambda,\pi,\sigma,\tau) + U(l,q;\tau,\pi,\sigma,\lambda) - U(q,l;\tau,\sigma,\pi,\lambda)\\
&+ U(q,l;\lambda,\sigma,\pi,\tau)\Big)\Big),\\
U(k,l;\sigma,\tau,\pi,\lambda) &= k_\sigma\, l_\tau\, g_{\pi\lambda} - k_\sigma\, l_\lambda\, g_{\pi\tau} + k_\pi\, l_\lambda\, g_{\sigma\tau}\\
&- (kl) g_{\sigma\tau} g_{\pi\lambda}\, F(k,l,-(k+l)).
\end{aligned}
$$

Here triad p, m, λ *etc* means correspondingly incoming momentum, isotopic index, Lorentz index of a boson.

Let us formulate compensation equations in this approximation. For free Lagrangian L_0 full connected three-boson vertices with Lorentz structure (6.6) are to vanish. Thus we come to the compensation equation, which in diagram form is exactly the same, as is presented in Figure 4.1. One can succeed in obtaining analytic solutions for the following set of momentum variables (see Figure 4.1): left-hand legs have momenta p and $-p$, and a right-hand leg has zero momenta. However in our approximation we need form-factor F also for nonzero values of this momentum. We have already define the first approximation for a dependence on three variables in the form (4.8). So here we also use the following simple dependence on all three variables

$$F(p_1, p_2, p_3) = F\left(\frac{p_1^2 + p_2^2 + p_3^2}{2}\right), \tag{6.11}$$

We consider the representation (6.11) to be the first approximation with a hope to consider corrections in forthcoming studies.

Now according to the rules being stated above we obtain the following equation for form-factor $F(x)$

$$\begin{aligned}
F(x) = &-\frac{G^2 N}{64\,\pi^2}\left(\int_0^Y F(y)\,y dy - \frac{1}{12\,x^2}\int_0^x F(y)\,y^3 dy\right.\\
&\left.+\frac{1}{6\,x}\int_0^x F(y)\,y^2 dy + \frac{x}{6}\int_x^Y F(y)\,dy - \frac{x^2}{12}\int_x^Y \frac{F(y)}{y}\,dy\right)\\
&+\frac{G\,g\,N}{16\,\pi^2}\int_0^Y F(y)\,dy + \frac{G\,g\,N}{24\,\pi^2}\left(\int_{3x/4}^x \frac{(3x-4y)^2(2y-3x)}{x^2(x-2y)}F(y)\,dy\right.\\
&\left.+\int_x^Y \frac{(5x-6y)}{(x-2y)}F(y)\,dy\right) + \frac{G\,g\,N}{32\pi^2}\left(\int_x^Y \frac{3(x^2-2y^2)}{8(2y-x)^2}\,F(y)\,dy\right.\\
&+\int_{3x/4}^x \frac{3(4y-3x)^2(x^2-4xy+2y^2)}{8x^2(2y-x)^2}\,F(y)\,dy\\
&\left.+\int_0^x \frac{5y^2-12xy}{16x^2}\,F(y)\,dy + \int_x^Y \frac{3x^2-4xy-6y^2}{16y^2}\,F(y)\,dy\right).
\end{aligned} \tag{6.12}$$

Here $x = p^2$ and $y = q^2$, where q is an integration momentum, $N = 2$. The last four terms in brackets represent diagrams with one usual gauge vertex (see three last diagrams at Figure 4.1). These terms maintain the gauge invariance of results in this approximation. Note that one can additionally check the gauge invariance by introduction of longitudinal term $d_l\,k_\mu k_\nu/(k^2)^2$ in boson propagators to verify independence of results on d_l in this approximation. Ghosts contributions also give zero result in the present approximation due to vertex (6.6) being transversal:

$$\begin{aligned}
p_\mu\,V(p,q,k)_{\mu\nu\rho} &= q_\nu\,V(p,q,k)_{\mu\nu\rho} = k_\rho\,V(p,q,k)_{\mu\nu\rho} = 0,\\
V(p,q,k)_{\mu\nu\rho} &= g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk)\\
&\quad + g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho.
\end{aligned} \tag{6.13}$$

Gauge invariance might be also violated by terms arising from momentum dependence of form-factor F. However this problem does not arise in the approximation corresponding to equation (6.12) and becomes essential for taking into account of g^2 terms. In this case ghost contributions also do not cancel. The problem of gauge invariance of the next approximations has to be considered in future studies.

We again introduce in equation (6.12) an effective cut-off Y, which bounds a "low-momentum" region where our nonperturbative effects act and consider the equation at interval $[0, Y]$ under condition

$$F(Y) = 0. \tag{6.14}$$

We shall solve equation (6.12) by iterations. That is we expand its terms being proportional to g in powers of x and take at first only constant term. Thus we have

$$\begin{aligned} F_0(x) = &-\frac{G^2 N}{64\pi^2}\Bigg(\int_0^Y F_0(y)\, y dy - \frac{1}{12 x^2}\int_0^x F_0(y)\, y^3 dy \\ &+ \frac{1}{6x}\int_0^x F_0(y)\, y^2 dy + \frac{x}{6}\int_x^Y F_0(y)\, dy - \frac{x^2}{12}\int_x^Y \frac{F_0(y)}{y}\, dy\Bigg) \\ &+ \frac{87\, G g N}{512\pi^2}\int_0^Y F_0(y)\, dy. \end{aligned} \tag{6.15}$$

Expression (6.15) provides an equation of the type which were repeatedly studied above starting from Chapter 2, where the way of obtaining solutions of equations analogous to (6.15) are described. Indeed, by successive differentiation of equation (6.15) we come to Meijer differential equation

$$\begin{aligned} &\left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+1\right)\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)F_0(x) + \frac{G^2 N x^2}{64\pi^2}F_0(x) \\ &= 4\left[-\frac{G^2 N}{64\pi^2}\int_0^Y F_0(y)\, y dy + \frac{87\, G g N}{512\pi^2}\int_0^Y F_0(y)\, dy\right], \end{aligned} \tag{6.16}$$

which solution looks like

$$\begin{aligned} F_0(z) = &\ C_1\, G^{10}_{04}(z\,|1/2,\ 1, -1/2, -1) \\ &+ C_2\, G^{10}_{04}(z\,|1,\ 1/2, -1/2, -1) \\ &- \frac{GN}{128\pi^2}\, G^{31}_{15}\left(z\,|^{0}_{1,1/2,0,-1/2,-1}\right)\int_0^Y \left(Gy - \frac{87 g}{8}\right) F_0(y)\, dy, \\ G^{31}_{15}&\left(z\,|^{0}_{1,1/2,0,-1/2,-1}\right) = \frac{1}{2z} - G^{30}_{04}\left(z\,|1,\ 1/2, -1, -1/2\right), \\ z &= \frac{G^2 N x^2}{1024\pi^2}. \end{aligned} \tag{6.17}$$

Constants C_1, C_2 are defined by the following boundary conditions

$$\left[2\,z^2\frac{d^3\,F_0(z)}{dz^3}+9\,z\,\frac{d^2\,F_0(z)}{dz^2}+\frac{d\,F_0(z)}{dz}\right]_{z=z_0}=0,\quad z_0=\frac{G^2\,N\,Y^2}{1024\,\pi^2},$$
$$\left[2\,z^2\,\frac{d^2\,F_0(z)}{dz^2}+5\,z\,\frac{d\,F_0(z)}{dz}+F_0(z)\right]_{z=z_0}=0. \tag{6.18}$$

Conditions (6.14, 6.18) defines set of parameters

$$z_0=\infty,\quad C_1=0\,,\quad C_2=0. \tag{6.19}$$

The normalization condition for form-factor $F(0)=1$ here is the following

$$-\frac{G^2\,N}{64\,\pi^2}\int_0^\infty F_0(y)\,y\,dy+\frac{87\,G\,g\,N}{512\,\pi^2}\int_0^\infty F_0(y)\,dy\;=\;1. \tag{6.20}$$

However the first integral in (6.20) diverges due to asymptotic

$$G_{15}^{31}\left(z\,|^{0}_{1,\,1/2,\,0,-1/2,-1}\right)\;\to\;\frac{1}{2\,z},\quad z\;\to\;\infty,$$

and we have no consistent solution. In view of this we consider the next approximation. We substitute solution (6.17) with account of (6.20) into terms of equation (6.12) being proportional to gauge constant g but the constant ones and calculate terms proportional to $\sqrt{z}$. Now we have bearing in mind the normalization condition

$$F(z)=1+\frac{85\,g\,\sqrt{N}\,\sqrt{z}}{96\,\pi}\left(\ln\,z+4\,\gamma+4\,\ln\,2+\frac{1}{2}\,G_{15}^{31}\left(z_0\,|^{0}_{0,0,1/2,-1,-1/2}\right)-\frac{3160}{357}\right)$$
$$+\frac{2}{3\,z}\int_0^z F(t)\,t\,dt-\frac{4}{3\,\sqrt{z}}\int_0^z F(t)\,\sqrt{t}\,dt-\frac{4\,\sqrt{z}}{3}\int_z^{z_0} F(t)\frac{dt}{\sqrt{t}}+\frac{2\,z}{3}\int_z^{z_0} F(t)\,\frac{dt}{t}, \tag{6.21}$$

where γ is the Euler constant. We look for solution of (6.21) in the form

$$F(z)=\frac{1}{2}G_{15}^{31}\left(z\,|^{0}_{1,\,1/2,0,-1/2,-1}\right)-\frac{85\,g\sqrt{N}}{128\,\pi}\,G_{15}^{31}\left(z\,|^{1/2}_{1,\,1/2,\,1/2,-1/2,-1}\right)$$
$$+\,C_1\,G_{04}^{10}\,(z\,|1/2,\;1,-1/2,-1)+\;C_2\,G_{04}^{10}\,(z\,|1,\;1/2,-1/2,-1)\,. \tag{6.22}$$

We have also conditions

$$1+8\int_0^{z_0} F(z)\,dz=\frac{87\,g\,\sqrt{N}}{32\,\pi}\int_0^{z_0} F_0(z)\,\frac{dz}{\sqrt{z}}, \tag{6.23}$$

$$F(z_0)\,=\,0, \tag{6.24}$$

and boundary conditions analogous to (6.18). The last condition (6.24) means smooth transition from the nontrivial solution to trivial one $G\;=\;0$ for $z\;>\;z_0$. Knowing form (6.22) of a solution we calculate both sides of relation (6.21) in two different points in interval $0 < z < z_0$ and having four equations for four parameters solve the set. With $N = 2$ we obtain the following solution, which we use to describe the electroweak case

$$\begin{aligned} g(z_0) &= 0.60366, \qquad z_0 = 9.61750, \\ C_1 &= -0.035096, \quad C_2 = -0.051104. \end{aligned} \tag{6.25}$$

We would draw attention to the fixed value of parameter z_0. The solution exists only for this value (6.25) and it plays the role of eigenvalue. As a matter of fact from the beginning the existence of such eigenvalue is by no means evident. This parameter z_0 defines scale appropriate to the solution. That is why we take value of running coupling g in solution (6.25) just at this point.

Let us remind, that the solution with smaller value of $z_0 = 0.009553$ and rather large $g(z_0) = 3.8166$, which with $N = 3$ corresponds to the strong interaction theory QCD, was considered above in Chapter 4.

We have one-loop three fermion generation expression for running electroweak coupling $\alpha_{ew}(p^2)$

$$\alpha_{ew}(x) = \frac{6\,\pi\,\alpha_{ew}(x_0)}{6\,\pi + 5\,\alpha_{ew}(x_0)\ln(x/x_0)}, \qquad x = p^2, \tag{6.26}$$

We normalize the running coupling by condition

$$\alpha_{ew}(x_0) = \frac{g(z_0)^2}{4\,\pi} = 0.0290, \tag{6.27}$$

where coupling constant g entering in expression (6.23) is just corresponding to this normalization point. Note that value (6.27) is not far from physical value of α_{ew} at the W-boson mass

$$\alpha_{ew}(M_W) = 0.0337. \tag{6.28}$$

To compare these values properly one needs a relation connecting G and M_W, which follows from expression for the running coupling (6.26). Let us remind that connection of variables z and $x = p^2$ is given in (6.17). Thus from value of α_{ew} in point z_0 we obtain an equation for variable $M_W^2\,G$. Solving this equation for set of parameters (6.25) and bearing in mind definition (6.7) we obtain corresponding value of parameter λ

$$|\lambda| = 2.88 \cdot 10^{-6}. \tag{6.29}$$

This result evidently does not contradict limitations (6.8, 6.9).

While considering the analogous interaction in QCD, we have studied the dependence of results on value of parameter ϵ, which was introduced to check stability of result in respect to small perturbations. Let as also take expression (4.28) for the inhomogeneous part of equation (6.21) and consider variation of its solution in respect to value of ϵ. Let us present several examples. For $\epsilon = 0.095$ we have parameters of the solution including also the corresponding value of λ

$$\begin{aligned} g(z_0) &= 0.6227, & z_0 &= 8.972601, \\ C_1 &= -0.0432122, & C_2 &= -0.0535126, \\ \lambda &= -0.00747. \end{aligned} \tag{6.30}$$

Note, that condition (6.26) defines only $|\lambda|$. However, bearing in mind, that our non-trivial solution exists for positive G and λ is defined by relation (6.7), we obtain the negative sign in the last line of (6.30).
For $\epsilon = 0.11$ we have

$$\begin{aligned} g(z_0) &= 0.6258, & z_0 &= 8.874286, \\ C_1 &= -0.0445799, & C_2 &= -0.0538548, \\ \lambda &= -0.025. \end{aligned} \tag{6.31}$$

For $\epsilon = 0.13$ we have

$$\begin{aligned} g(z_0) &= 0.6299, & z_0 &= 8.74465, \\ C_1 &= -0.0464409, & C_2 &= -0.0542931, \\ \lambda &= -0.12. \end{aligned} \tag{6.32}$$

We see, that solutions (6.30, 6.31) are consistent with restrictions (6.8, 6.9), while solution (6.32) already significantly contradicts these limitations. Thus limitations (6.8, 6.9), e. g., $-0.038 < \lambda < 0.030$, result in the following condition

$$\epsilon < 0.1154. \tag{6.33}$$

As a result of these considerations we see noticeable distinction in the two cases. In QCD variation of ϵ do not lead to crucial changes in physical parameters. For example, the gluon condensate for $0 < \epsilon < 0.15$ remains inside experimental uncertainties. However in the electroweak theory the dependence on ϵ is very sharp. As a matter of fact this difference is mostly due to smooth momentum dependence of α_{ew}, while in QCD running coupling at small momenta varies significantly. So, from the point of view of the effective interactions, the electroweak case at the present stage of the study is much less predictive, than QCD. It would be desirable to have some additional information, which could help to define G. Of course, direct searches for anomalous three-boson interaction (6.1) at the upgraded LHC could either give definite answer, in case of the discovery of the interaction, or provide more stringent limitations. However in the same way, as the triple gluon interaction (4.2) leads in Section 4.4 to glueball states, the interaction (6.1) may lead to states consisting of W bosons. Thus another way to check possibility of an existence of the anomalous interaction can be provided by searches for such W-balls. Let us consider this possibility in few subsequent sections.

Here we would present value for the analogue of the gluon condensate, which could be named the W condensate. Indeed, all calculations, which were performed in Section 4.3, could be applied to the electroweak case with obvious change of parameters and transmutation $SU(3) \to SU(2)$. Now, let us define the following quantity

$$V_2^{EW} = \left\langle \frac{g^2}{4\,\pi^2}\, W^a_{\mu\nu}\, W^a_{\mu\nu} \right\rangle. \tag{6.34}$$

Then we use the same diagrams as shown in Figure 4.5 and calculate value (6.34). In doing this we have to take into account the following considerations.

(1) The total convolution of structure constants for $SU(3)$ is equal

$$f^{abc}\, f^{abc} = 24, \tag{6.35}$$

while for $SU(2)$ the same quantity is equal

$$\epsilon^{abc}\, \epsilon^{abc} = 6. \tag{6.36}$$

Thus for the electroweak case one has to divide result (4.44) by 4, that is the ratio of (6.35) and (6.36).

(2) One has to multiply result (4.44) by

$$\frac{3\sqrt{3}}{2\sqrt{2}}, \tag{6.37}$$

due to definition of variable z, which is proportional to N.

After performing these simple substitutions we obtain the following expression for W condensate (6.34)

$$\begin{aligned} V_2^{EW} &= \frac{45\, g^3\, 2^9}{\pi^3\sqrt{2}\, G^2}\left(2 - 6\ln\frac{4}{3}\right)\int\limits_0^{z_0} F(z)\sqrt{z}\, dz \\ &= \frac{45\, g\, 2^9\, M_W^4}{\pi^3\sqrt{2}\,\lambda^2}\left(2 - 6\ln\frac{4}{3}\right)\int\limits_0^{z_0} F(z)\sqrt{z}\, dz. \end{aligned} \tag{6.38}$$

Here all parameters including those entering into definition of form-factor $F(z)$ (6.22) are defined in the present chapter. With different values of ϵ they are shown in (6.25, 6.29) for $\epsilon = 0$ and in (6.30, 6.31, 6.32) for $\epsilon \neq 0$.

For example, for $\epsilon = 0.11$ we have

$$V_2^{EW} = -\,2.55\ \mathrm{TeV}^4, \tag{6.39}$$

and for $\epsilon = 0.13$ we have

$$V_2^{EW} = -\,0.108\ \mathrm{TeV}^4. \tag{6.40}$$

It is interesting, that unlike the QCD case this vacuum average is negative. We see here also very sharp dependence on parameter ϵ.

6.2 Effective strong interaction in the weak gauge sector

In Section 6.1 the possibility of a spontaneous generation of effective nonlocal triple gauge invariant interaction (6.1) was demonstrated. Let us here remind the form of the interaction in view of further discussion of its properties

$$
\begin{aligned}
&-\frac{g\,\lambda}{3!\,M_W^2}\,F(p_i)\,\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu},\\
&W^3_{\mu\nu} = \cos\theta_W\,Z_{\mu\nu} + \sin\theta_W\,A_{\mu\nu},\\
&W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc}\,W^b_\mu W^c_\nu.
\end{aligned}
\tag{6.41}
$$

where $g(M_W) \simeq 0.65$ is the electro-weak coupling and uniquely defined form-factor $F(p_i)$ guarantees effective interaction (6.41) acting in a limited region of the momentum space. The accuracy of an approximate scheme being used, was estimated to be $\simeq$ 10 %. Experimental limitations for parameter of the anomalous interaction λ [104] are shown in (6.8, 6.9). Parameter λ is connected with coupling constant G by relation (6.7)

$$
G = -\frac{g\,\lambda}{M_W^2},
$$

Would-be existence of effective interaction (6.41) leads to important nonperturbative effects in the electro-weak interaction.

Interaction (6.41) increases with increasing momenta p. For estimation of an effective dimensionless coupling we choose symmetric momenta (p,q,k) in vertex corresponding to the interaction

$$
\begin{aligned}
&-(2\pi)^4\,\frac{g\,\lambda}{M_W^2}\,\epsilon_{abc}\,(g_{\mu\nu}(q_\rho pk - p_\rho qk)\\
&+ g_{\nu\rho}(k_\mu pq - q_\mu pk) + g_{\rho\mu}(p_\nu qk - k_\nu pq)\\
&+ q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho)\,F(p,q,k)\,\delta(p+q+k) + \dots,
\end{aligned}
\tag{6.42}
$$

where p, μ, a; q, ν, b; k, ρ, c are respectfully incoming momenta, Lorentz indices and weak isotopic indices of W-bosons. We mean also that there are present four-boson, five-boson and six-boson vertices according to expression for $W^a_{\mu\nu}$ (6.1). In what follows we shall use four boson vertex, which corresponds to the following interaction

$$
\Delta L = \frac{g\,G}{2}\epsilon_{abc}\,\epsilon_{aed}\,W^e_\mu\,W^d_\nu\,W^b_{\nu\rho}\,W^c_{\rho\mu}.
\tag{6.43}
$$

Explicit expression for the corresponding vertex is presented above (6.10). Form-factor $F(p,q,k)$ is obtained in Section 6.1 using the following approximate dependence on the three variables

$$
F(p,q,k) = F\left(\frac{p^2+q^2+k^2}{2}\right).
\tag{6.44}
$$

Symmetric condition means

$$
pq = pk = qk = \frac{p^2}{2} = \frac{q^2}{2} = \frac{k^2}{2} = \frac{x}{2}.
\tag{6.45}
$$

Interaction (6.41) increases with increasing momenta p and corresponds to effective dimensionless coupling being of the following order of magnitude

$$g_{eff} = \frac{|g\,\lambda|\,p^2}{2M_W^2}\,F\left(\frac{3\,p^2}{2}\right). \tag{6.46}$$

Form-factor $F(x)$ in Section 6.1 is expressed in terms of the Meijer functions

$$\begin{aligned} F(z) = {} & \frac{1}{2}\,G^{31}_{15}\left(z\,|^{0}_{1,\,1/2,\,0,-1/2,-1}\right) - \frac{85\,g_0\,\sqrt{2}}{128\,\pi}\,G^{31}_{15}\left(z\,|^{1/2}_{1,\,1/2,\,1/2,-1/2,-1}\right) \\ & + C_1\,G^{10}_{04}\,(z\,|1/2,\ 1,-1/2,-1) + C_2\,G^{10}_{04}\,(z\,|1,\ 1/2,-1/2,-1)\,. \end{aligned} \tag{6.47}$$

$$z = \frac{G^2\,(p^2)^2}{512\,\pi^2}.$$

$$g_0 = 0.6037, \quad C_1 = -0.0351, \quad C_2 = -0.0511, \tag{6.48}$$

where g_0 is value of the electro-weak running coupling at momentum p_0 corresponding to value of variable z

$$z_0 = 9.6175, \tag{6.49}$$

and

$$F(z) = 0, \quad z > z_0. \tag{6.50}$$

Thus running g_{eff} in dependence on variable $t = G\,p^2$ is the following

$$g_{eff}(t) = \frac{t}{2}\,F\left(\frac{9\,t^2}{2048\,\pi^2}\right), \quad t = G\,p^2. \tag{6.51}$$

Behavior of $g_{eff}(t)$ is presented in Figure 6.1. We see that for $t \simeq 22$ the coupling reaches maximal value $g_{eff} = 3.63$, that is corresponding effective α is the following

$$\alpha_{eff} = \frac{g_{eff}^2}{4\,\pi} = 1.049. \tag{6.52}$$

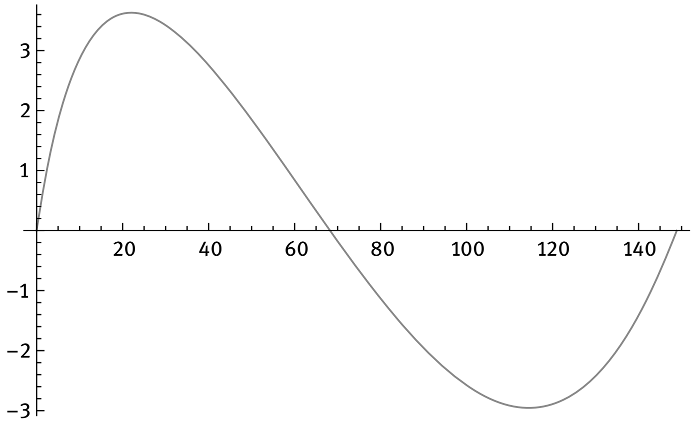


Fig. 6.1. Behavior of the effective coupling $g_{eff}(t)$, $t = G\,p^2$; $g_{eff}(t) = 0$ for $t > 148$.

Thus for sufficiently large momentum interaction (6.41) becomes strong and may lead to physical consequences analogous to that of the usual strong interaction (QCD). In particular bound states and resonances constituting of W-s – W-balls (W-hadrons) may appear.

6.3 Scalar bound state of two W-s

Let us consider a possibility of existence of bound state X of two W with mass M_s. For the beginning let us consider such state X with spin 0 and weak isotopic spin also 0. Then vertex of XWW interaction has the following form

$$\frac{G_X}{2}\, W^a_{\mu\nu}\, W^a_{\mu\nu}\, X\, \Psi_0, \tag{6.53}$$

where Ψ_0 is a Bethe–Salpeter wave function of the bound state. Again due to gauge invariance there is also three-boson term

$$-\, g\, G_X\, \epsilon_{abc}\, W^a_{0\,\mu\nu}\, W^b_\mu\, W^c_\nu\, X, \tag{6.54}$$

where $W_{0\,\mu\nu}$ is the gauge W field without nonlinear $[W_\mu, W_\nu]$ term. There are four-boson terms also, but we do not use them here. In what follows we use expressions (6.53, 6.54).

The main interactions forming the bound state are just nonperturbative interactions (6.41, 6.53). This means that we take into account exchange of vector boson W as well as of scalar bound state X itself. In diagram form the corresponding Bethe–Salpeter equation is presented in Figure 6.2. We expand the kernel of the equation in powers of M_W^2 and M_s^2 and obtain the following equation with introduction of more

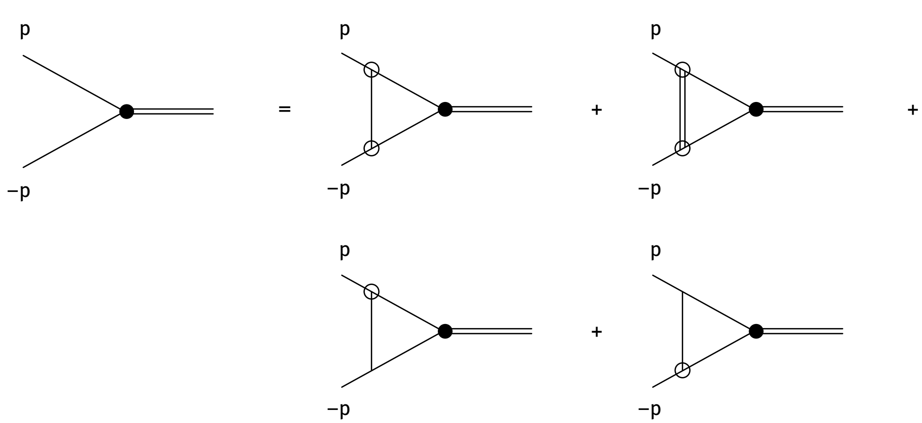


Fig. 6.2. Diagram representation of Bethe–Salpeter equation for W-W bound state. Black spot corresponds to *XWW* vertex (6.53) with BS wave function. Empty circles correspond to point-like anomalous three-gluon vertex (6.41), double circle – point-like XWW vertex (6.53). Simple point – usual gauge triple *W* interaction. A double line represents the bound state *X*, a simple line represents W.

suitable variable

$$z = \frac{G^2(p^2)^2}{64\pi^2}, \quad t = \frac{G^2(q^2)^2}{64\pi^2},$$

p is external momentum, q is the integration momentum.

$$\begin{aligned}
\Psi_0(z) = {} & 4\int_0^{z_0'} \Psi_0(t)dt - \frac{2}{3z}\int_0^z \Psi_0(t)tdt + \frac{4}{3\sqrt{z}}\int_0^z \Psi_0(t)\sqrt{t}\,dt \\
& + \frac{4\sqrt{z}}{3}\int_z^{z_0'} \frac{\Psi_0(t)}{\sqrt{t}}dt - \frac{2z}{3}\int_z^{z_0'} \frac{\Psi_0(t)}{t}dt + \mu\Bigg(-\frac{1}{z}\int_0^z \Psi_0(t)\sqrt{t}dt \\
& + \frac{2}{\sqrt{z}}\int_0^z \Psi_0(t)dt + 6\int_0^{z_0'} \frac{\Psi_0(t)}{\sqrt{t}}dt + 2\sqrt{z}\int_z^{z_0'} \frac{\Psi_0(t)}{t}dt - z\int_z^{z_0'} \frac{\Psi_0(t)}{t\sqrt{t}}dt\Bigg) \\
& - \mu_s\Bigg(\frac{1}{8z\sqrt{z}}\int_0^z \Psi_0(t)t\,dt - \frac{25}{64z}\int_0^z \Psi_0(t)\sqrt{t}\,dt \\
& + \frac{19}{64\sqrt{z}}\int_0^z \Psi_0(t)dt + \frac{11}{8}\int_0^z \frac{\Psi_0(t)}{\sqrt{t}}dt + \frac{19}{16}\int_z^{z_0'} \frac{\Psi_0(t)}{\sqrt{t}}dt \\
& + \frac{5\sqrt{z}}{16}\int_z^{z_0'} \frac{\Psi_0(t)}{t}dt - \frac{5z}{64}\int_z^{z_0'} \frac{\Psi_0(t)}{t\sqrt{t}}dt - \frac{z\sqrt{z}}{64}\int_z^{z_0'} \frac{\Psi_0(t)}{t^2}dt\Bigg) \\
& - \frac{\kappa}{12\pi}\Bigg(\frac{1}{2z}\int_0^z \Psi_0(t)\sqrt{t}\,dt + \frac{3}{2\sqrt{z}}\int_0^z \Psi_0(t)dt + \frac{3}{2}\int_z^{z_0'} \frac{\Psi_0(t)}{\sqrt{t}}dt + \frac{\sqrt{z}}{2}\int_z^{z_0'} \frac{\Psi_0(t)}{t}dt\Bigg) \\
& + \frac{g}{4\pi}\Bigg(-\frac{1}{z}\int_0^z \Psi_0(t)\sqrt{t}dt + \frac{3}{\sqrt{z}}\int_0^z \Psi_0(t)dt + 3\int_z^{z_0'} \frac{\Psi_0(t)}{\sqrt{t}}dt - \sqrt{z}\int_z^{z_0'} \frac{\Psi_0(t)}{t}dt\Bigg).
\end{aligned} \tag{6.55}$$

In equation (6.55) g is the electro-weak gauge coupling and the following notations are used

$$\mu = \frac{GM_W^2}{6\pi}, \quad \mu_s = \frac{GM_s^2}{6\pi}, \quad \kappa = \frac{G_X^2}{G}. \tag{6.56}$$

The first five terms in the rhs of equation (6.55) is the main (zero approximation) part. These terms and the terms being proportional to μ^2 and μ_s^2 are obtained from the main triangle diagram (the second one in the upper line of Figure 6.2) by expanding its expression in powers of $(M_W^2)^n$ and $(M_s^2)^n$. Then we take into account terms with n=0,1. Estimates show, that higher powers can be neglected. The term being proportional to κ, that is to G_X^2, corresponds to the third diagram in the upper line of Figure 6.2. Terms with gauge electro-weak coupling g enters due to diagrams of the second line of Figure 6.2. Upper limit z_0' is introduced as usually in our approach, according to which z_0' may be either ∞ or some finite quantity. That is z_0' is defined in a process of solving the problem. Physical meaning of this parameter corresponds to a definition of effective

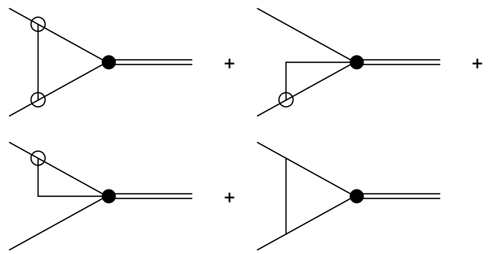
1 = 

Fig. 6.3. Diagram representation of normalization condition $\Psi_0(0) = 1$. Four leg vertex corresponds to interaction (6.54). All the external momenta are zero. Other notations are the same as in Figure 6.2

cut-off Λ', which bounds a low-momentum region, where the nonperturbative effects are significant. For form-factor of interaction (6.41) the upper limit z_0 is defined by results (6.25).

The Bethe–Salpeter wave function in the first approximation is normalized by condition $\Psi_0(0) = 1$, which corresponds to the following equality

$$4\int_0^{z_0'} \Psi_0(t)dt + \frac{2\sqrt{2}}{\pi}\int_0^{z_0} \frac{g\,F(t)}{\sqrt{t}}\,dt + \frac{3}{32\,\pi^2}\int_\mu^{z_0'} \frac{g^2\,\Psi_0(t)}{t}\,dt = 1, \tag{6.57}$$

where $F(t)$ and z_0 are defined by equations (6.22–6.25). In diagram form this condition is presented in Figure 6.3.

We have to take into account also the normalization condition for the Bethe–Salpeter wave function, which defines interaction constant G_X. This condition guarantees the proper form of the effective propagator for bound state X, as we have mentioned in Section 1.5. In diagram form it is presented in Figure 6.4. Here each diagram means a coefficient before external momentum squared p^2, that is for expression $\Phi(p^2, ...)$ we put

$$\frac{\partial}{\partial p^2}\Phi(p^2, ...)_{p^2=0}.$$

Diagrams in Figure 6.4 correspond to the following expression

$$\frac{\kappa}{8\,\pi}\left(9\,I_0 - \frac{25}{16\,\pi}\,D^2\right) = 1,$$
$$I_0 = \int_0^{z_0'} \frac{\Psi_1^2(z)\,dz}{\sqrt{z}}, \quad D = \int_0^{z_0'} \frac{\sqrt{g}\,\Psi_1(z)\,dz}{\sqrt{z}}. \tag{6.58}$$

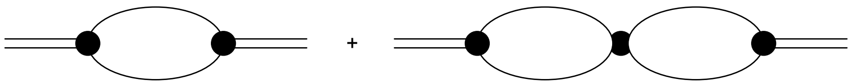
1 = 

Fig. 6.4. Diagrams for normalization condition of $X\,W\,W$-vertex. Four-leg vertex corresponds to vertex (6.10) being proportional to $g\,G$.

We shall solve equation (6.55) by iterations. We take as the first approximation for the problem the set of equations consisting of:

(1) the upper line of equation (6.55), that is (6.55) with $\mu = \mu_s = \kappa = g = 0$;
(2) condition $\Psi_0(0) = 1$ (6.57);
(3) normalization condition (6.58) for the BS wave function.

There are few solutions of set of equations (6.55, 6.57) but only one of them leads to positive M_s^2. It reads in terms of Meijer functions

$$\Psi_1(z) = \frac{\pi}{2}\, G_{15}^{21}\left(z|^{0}_{1,0,1/2,-1/2,-1}\right) + C_1\, G_{0}^{20}\left(z|_{1,1/2,-1/2,-1}\right) + C_2\, G_{0}^{10}\left(-z|_{1,1/2,-1/2,-1}\right), \tag{6.59}$$
$$z_0' = 44.151234, \quad C_1 = 3.05437, \quad C_2 = -0.0011964.$$

Now we use solution (6.59) and obtain parameter κ (6.55) with the aid of normalization condition for XWW coupling (6.58).

With Ψ_1 (6.59) we obtain from (6.58)

$$\kappa = 0.592411. \tag{6.60}$$

Then we multiply full equation (6.55) by $\Psi_1(z)$ from the right and integrate the result by z in interval $(0, z_0')$. It is easy to see by changing the order in double integrals, that all terms being of zero order in μ, μ_s, κ, g vanish, and we have the following equation

$$-\mu_s\left(\frac{3\,I_1}{64} - \frac{5\,I_2}{64} + \frac{95\,I_3}{64} + \frac{11\,I_4}{8} - \frac{I_5}{64}\right) + \\ + \mu\left(-I_1 + 3\,I_2 + 14\,I_3 + 6\,I_4\right) - \frac{\kappa}{12\,\pi}\left(I_2 + 3\,I_3\right) + \frac{3\,I_{g3} - I_{g2}}{4\,\pi} = 0, \tag{6.61}$$

where

$$\begin{aligned} I_1 &= \int_0^{z_0'} \frac{\Psi_1(z)\,dz}{z\sqrt{z}} \int_0^z \Psi_1(t) t\,dt, \quad I_2 = \int_0^{z_0'} \frac{\Psi_1(z)\,dz}{z} \int_0^z \Psi_1(t)\sqrt{t}\,dt, \\ I_3 &= \int_0^{z_0'} \frac{\Psi_1(z)\,dz}{\sqrt{z}} \int_0^z \Psi_1(t)\,dt, \quad I_4 = \int_0^{z_0'} \Psi_1(z)\,dz \int_0^z \frac{\Psi_1(t)\,dt}{\sqrt{t}}, \\ I_5 &= \int_0^{z_0'} \frac{\Psi_1(z)\,dz}{z^2} \int_0^z \Psi_1(t) t\sqrt{t}\,dt, \\ I_{g2} &= \int_0^{z_0'} \frac{g\,\Psi_1(z)\,dz}{z} \int_0^z \Psi_1(t)\sqrt{t}\,dt, \quad I_{g3} = \int_0^{z_0'} \frac{g\,\Psi_1(z)\,dz}{\sqrt{z}} \int_0^z \Psi_1(t)\,dt. \end{aligned} \tag{6.62}$$

Now we define running coupling g

$$g(z) = \frac{g(M_W)}{\sqrt{1 + \frac{5\,g^2(M_W)}{24\,\pi^2}\ln\left(1 + \frac{8\,\pi\sqrt{z}}{G\,M_W^2}\right)}}. \tag{6.63}$$

It enters in integrals (6.61, 6.62). The next question is if one can define possible mass M_s? For example, provided we choose $M_s = 125.5\,\text{GeV}$ that means

$$\mu_s = \mu\,\frac{125.5^2}{80.4^2}, \tag{6.64}$$

Then with this value of μ_s bearing in mind relations (6.56, 6.60) we have

$$G_X = 0.000668\,\text{GeV}^{-1}, \quad G = \frac{0.00484}{M_W^2}. \tag{6.65}$$

Result (6.65) means parameter of anomalous three-boson interaction (6.41) with account of relation (6.7)

$$\lambda = -\frac{G\,M_W^2}{g(M_W)} = -\,0.00744, \tag{6.66}$$

which doubtless agrees limitations (6.8). More than that, this value almost precisely coincides with result (6.30). With this result we have simple dependence of such resonance mass on value of λ

$$M_s = 125.5\,\sqrt{\frac{0.00744}{|\lambda|}}\;\text{GeV}. \tag{6.67}$$

One can also connect value M_s with parameter ϵ, which was discussed above. Provided parameter ϵ in sets of results (6.29)–(6.32) increases, the mass of the scalar state decreases, while for decreasing ϵ the mass increases up. To illustrate this dependence we present below estimates for the mass of the scalar W-ball for several values of ϵ

$$\begin{aligned}
\epsilon = 0 \;:\; M_s &\simeq 6400\,\text{GeV},\\
\epsilon = 0.11 \;:\; M_s &\simeq 68.5\,\text{GeV},\\
\epsilon = 0.13 \;:\; M_s &\simeq 31.2\text{GeV}.
\end{aligned} \tag{6.68}$$

Now there is a question, if a scalar W-ball really exists? For answering this question one has to look for either WW or ZZ resonances with sufficiently high masses. For example, one might ask, if this option could be applied to the newly discovered state [57, 58] with mass $125.7\,\text{GeV}$, which we have already mentioned in Section 2.1.2.

Let us consider this possibility in more details. Now we have scalar state X with coupling (6.53, 6.65). In calculations of decay parameters and cross-sections we use CompHEP package [109]. We use parameter G_X (6.65) being obtained above and $M_s = 125\,\text{GeV}$. Cross-section of X production at LHC reads

$$\begin{aligned}
\sigma(p + p \to X + ..)_{7\,\text{TeV}} &= 0.16\,pb,\\
\sigma(p + p \to X + ..)_{8\,\text{TeV}} &= 0.19\,pb.
\end{aligned} \tag{6.69}$$

Parameters of X-decay are the following

$$
\begin{gathered}
\Gamma_t(X) = 0.000502\,\text{GeV}\,, \\
BR(X \to \gamma\gamma) = 0.430, \quad BR(X \to \gamma Z) = 0.305, \\
BR(X \to 4\,l(\mu, e)) \;=\; 0.00092, \quad BR(X \to b\,\bar{b}) = 0.000024, \\
BR(X \to \gamma e^+ e^-) = 0.0231, \quad BR(X \to \gamma\mu^+\mu^-) = 0.016, \\
BR(X \to \gamma\tau^+\tau^-) = 0.0125, \quad BR(X \to \gamma u\bar{u}) = 0.0478, \\
BR(X \to \gamma c\bar{c}) = 0.0368, \quad BR(X \to \gamma d\bar{d}) = 0.0446, \\
BR(X \to \gamma s\bar{s})) = 0.0430, \quad BR(X \to \gamma b\bar{b}) = 0.0416.
\end{gathered} \tag{6.70}
$$

For decay $X \to b\bar{b}$ we calculate the evident triangle diagram and use $m_b(125\,\text{GeV}) \simeq 2.9$ GeV. Branching ratios for decays to other fermion pairs are even smaller.

Experimental data give in the region of the state the following results for $\sigma_{\gamma\gamma} = \sigma_X\, BR(X \to \gamma\gamma)$ [57, 58]. Let us draw attention to notation μ_x in the discussion of experimental data on 125.7 GeV state. This notation, which is now used everywhere means ratio of an experimental result for some quantity x to the value for the same quantity, calculated for the Standard Model Higgs particle with mass 125.7 GeV. Thus all parameters under a study are to equal to unity provided the state under discussion is the genuine Standard Model Higgs particle. Now for the decay channel to two photons we have

$$
\begin{aligned}
\mu_{\gamma\gamma} &= \frac{\sigma \times BR(X \to \gamma\gamma)_{exp}}{\sigma \times BR(X \to \gamma\gamma)_{SM}} = 1.3 \pm 0.4, \\
\mu_{\gamma\gamma} &= \frac{\sigma \times BR(X \to \gamma\gamma)_{exp}}{\sigma \times BR(X \to \gamma\gamma)_{SM}} = 1.6 \pm 0.4.
\end{aligned} \tag{6.71}
$$

Here $\sigma \times BR(H \to \gamma\gamma)_{SM} \;\simeq\; 0.04\,pb$ for $\sqrt{s} = 7$ TeV is the Standard Model value for the quantity under discussion, upper line corresponds to ATLAS data [57] and the lower line corresponds to CMS data [58]. Firstly both limitations are quite consistent. Secondly our value for the same quantity from (6.69, 6.70) reads

$$
\mu_{\gamma\gamma} = \frac{\sigma \times BR(X \to \gamma\gamma)_{calc}}{\sigma \times BR(X \to \gamma\gamma)_{SM}} = 1.6, \tag{6.72}
$$

that also agrees results (6.71), however it essentially exceeds the SM value. At this point it is advisable to discuss accuracy of our approximations.

The former experience concerning both applications to the QCD and the Nambu–Jona-Lasinio interaction in Chapters 4, 5 and to the electro-weak interaction in Chapter 6 shows that average accuracy of the method is around 10% in values of different parameters. So we may assume, that in the present estimations of coupling constant G_X we also have the same accuracy. For the cross-section this means possible deviation up to 15% of the calculated value. Thus we would change (6.72) to the following result

$$
\mu_{\gamma\gamma} = (1.6 \pm 0.24), \tag{6.73}
$$

In any case result (6.73) agrees (6.71).

There are also data for the 125.7 GeV state production in the kinematical region of vector boson fusion (VBF). Here there is a marked, but not decisive, deviation from the Standard Model Higgs option. With our calculations we obtain significant difference:

$$\mu_{\gamma\gamma}(VBF) = 3.0 \pm 0.3. \tag{6.74}$$

This could be compared with experimental value

$$\mu_{\gamma\gamma}(VBF) = 2.3 \pm 1.1.(CMS) \tag{6.75}$$

There are also indications for an excess around 125.7 GeV in four leptons states. With our numbers (6.69, 6.70) we have for decay $X \to l^+ l^+ l^- l^- (l = \mu, e)$:

$$\sigma \times BR \;=\; (0.00013 \pm 0.00003)\,pb. \tag{6.76}$$

This is approximately six times smaller the the Standard Model result. This result is to be compared to the latest data [110]

$$\mu(4\,l)_{CMS} = 0.93^{+0.26}_{-0.24}(stat)^{+0.13}_{-0.09}(syst). \tag{6.77}$$

Our estimation (6.76) already contradicts these data. Thus the interpretation of the 125.7 GeV state as a W-ball is not likely for the moment. The experimental data, which become more precise with accomplishing of the LHC studies, give the increasing confirmation for the 125.7 state being just the Higgs scalar particle. Although some admixture of other scalar states may be also considered.

So we state, that a check of possible existence of W-balls, which we discussed here needs a consideration of other possibilities. That means first of all, masses M_s being different from 125.7 GeV. Let us consider few possible examples. For example we would take two possible masses of a scalar W-ball

$$M_{s1} = 150\,\text{GeV}, \quad M_{s2} = 200\,\text{GeV}. \tag{6.78}$$

By performing the same operations as in the previous case, we come for the first mass in (6.78) the following value of coupling constant G_x

$$G_X = 0.000558\,\text{GeV}^{-1}. \tag{6.79}$$

With this value we have the following result for scalar state X production cross-section at $\sqrt{s} = 8$ TeV

$$\sigma_X = 0.0792\,pb. \tag{6.80}$$

The decay properties of the state is described by the following results

$$\Gamma_t(150) = 0.617\,\text{MeV}, \quad BR(X \to Z\gamma) \;=\; 0.635, \tag{6.81}$$
$$BR(X \to \gamma\gamma) = 0.364, \quad BR(X \to l^+l^-l^+l^-) = 0.00146.$$

For the second value of the mass in (6.78)we have the following value of coupling constant G_x

$$G_X = 0.000419\,\mathrm{GeV}^{-1} \quad (M_{s2} = 200\,\mathrm{GeV}). \tag{6.82}$$

With this value the result for scalar state X production cross-section at $\sqrt{s} = 8$ TeV looks like

$$\sigma_X = 0.0323\,pb. \tag{6.83}$$

And the decay properties of the state are the following

$$\begin{aligned} \Gamma_t(200) = 6.538\,\mathrm{MeV}, \quad BR(X \to Z\,\gamma) &= 0.184, \\ BR(X \to \gamma\,\gamma) = 0.364, \quad BR(X \to W^+\,W^-) &= 0.648, \\ BR(X \to Z\,Z) = 0.113, \quad BR(X \to l^+ l^- l^+ l^-) &= 0.0000457. \end{aligned} \tag{6.84}$$

We would emphasize importance of channel $X \to \gamma\, l^+ l^-$ for identification of a state as a W-ball. This channel may serve for an accurate test of our conjecture because the SM values for this channel are essentially smaller [111].

According to (6.80, 6.81) we have for the process at $\sqrt{s} = 8$ TeV with $M_s = 150$ GeV

$$\sigma(p\,p \to (X \to \gamma\, l^+\, l^-) + \ldots) = 0.0036\,pb, \tag{6.85}$$

that is quite a significant effect.

The same calculations give for $M_s = 200$ GeV

$$\sigma(p\,p \to (X \to \gamma\, l^+\, l^-) + \ldots) = 0.00043\,pb, \tag{6.86}$$

The main results of comparison with the would be option of $M_s = 125$ GeV are presented in the following Table 6.1. Here the signal-strength μ is a ratio of a quantity under consideration and of the same for the SM.

We have here comparison with data for both variants: the Standard Model Higgs and wouldbe WW state. As we have already mentioned, the last data indicate on behalf

Table 6.1. Comparison of experimental data to SM Higgs option and the W-hadrons option

process	μ_{exp}	μ_{calc} ($\boldsymbol{W - ball}$)
$H(X) \to \gamma\gamma$ ATLAS	1.4 ± 0.3	1.6
$H(X) \to \gamma\gamma$ CMS	1.6 ± 0.4	1.6
$H(X) \to \gamma\gamma_{VBF}$ CMS	2.3 ± 1.0	3.0
$H(X) \to 4\,l$ ATLAS	1.2 ± 0.6	$\simeq 0.15$
$H(X) \to 4\,l$ CMS	$0.93^{+0.26}_{-0.23}(st)^{+0.13}_{-0.09}(sys)$	$\simeq 0.15$
$H(X) \to b\bar{b}$ ATLAS	-0.4 ± 1.0	$\simeq 0$
$H(X) \to b\bar{b}$ CMS	1.0 ± 0.65	$\simeq 0$
$H(X) \to \tau\bar{\tau}$ ATLAS	$0.16^{+1.72}_{-1.84}$	$\simeq 0$
$H(X) \to \tau\bar{\tau}$ CMS	0.78 ± 0.27	$\simeq 0$
$H(X) \to \gamma\, l^+ l^-$	?	7.0 ± 1.4

of the Standard Model Higgs option [1](see also [112]). We would emphasize importance of channel $X \rightarrow \gamma l^+ l^-$ for identification of would W-balls, which may occur at other values of masses.

We would draw attention to the nonperturbative effects, which are decisive for a confirmation of the existence of anomalous three-boson interaction (6.1). Just W-balls in case of confirmation of their existence would follow from nonperturbative electroweak physics almost in the same way as the usual hadrons follow from nonperturbative effects in QCD. Let us note, that W-balls with higher spins are also of an interest. In particular, in work [102] the possible WW bound state with spin 1 is applied for an interpretation of TEVATRON data [113, 114].

6.4 Muon g-2

We have already mentioned, that measurements of anomalous magnetic moments of particles several times have been crucial points in verification of main theories of the Standard Model. This refers to a_e – anomalous magnetic moment of the electron in QED (Section 1.3). There was also one of the decisive contribution to establishing of the most important notion for strong interaction, namely of the color, which we have discussed in Section 1.2. Measurements of the anomalous magnetic moment of the muon $(g-2)_\mu$ (see the last publication [60] and the extensive review [115]) provides the only significant deviation of the experiment from predictions of the Standard Model. According to recent analysis of the problem [61–63] this deviation Δa_μ safely exceeds four s.d. and comprises the following values correspondingly

$$\begin{aligned} \Delta a_\mu &= (3.493 \pm 0.823)\, 10^{-9}, \\ \Delta a_\mu &= (3.935 \pm 0.523_{th} \pm 0.63_{ex})\, 10^{-9}. \end{aligned} \tag{6.87}$$

The most recent analysis of the problem [63] gives rather lower values of the difference, which with two different methods of accounting the hadron contribution are the following

$$\begin{aligned} \Delta a_\mu &= (2.87 \pm 0.80)\, 10^{-9}, \\ \Delta a_\mu &= (2.61 \pm 0.78)\, 10^{-9}. \end{aligned} \tag{6.88}$$

These results do not contradict the previous ones (6.87) and the discrepancy on the level of 3.5 s.d. persists.

It should be emphasized, that the deviation from the Standard Model calculations means the deviation from perturbative calculations in the electro-weak theory. However there quite may be nonperturbative contributions to physical quantities, just

1 Of course, one have to bear in mind also other options for interpretation of the effect.

those discussed above, in particular, the discrepancy might be due to the effect of a spontaneous generation of effective nonlocal interaction in the electro-weak theory, described in the the present chapter.

In the present section we apply the previous results to the problem of discrepancy Δa_μ. It will come clear, that the effect under discussion is quite natural in the theory with account of the spontaneous generation of an effective interaction in the conventional electro-weak theory, corresponding to expression (6.1).

Thus we start with the spontaneously generated anomalous three-boson interaction of the form

$$-\frac{G}{3!}\cdot\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}, \tag{6.89}$$

which corresponds to three-boson vertex (6.6)

$$\begin{aligned}&(2\pi)^4\,G\,\epsilon_{abc}\,(g_{\mu\nu}(q_\rho pk-p_\rho qk)+g_{\nu\rho}(k_\mu pq-q_\mu pk)\\&\quad+g_{\rho\mu}(p_\nu qk-k_\nu pq)+q_\mu k_\nu p_\rho-k_\mu p_\nu q_\rho)\\&\quad\times F(p,q,k)\,\delta(p+q+k)+\ldots,\end{aligned} \tag{6.90}$$

where $F(p,q,k)$ is a form-factor and $p,\mu,a;\;\; q,\nu,b;\;\; k,\rho,c$ are respectfully incoming momenta, Lorentz indices and weak isotopic indices of W-bosons. We mean also that there are present four-boson, five-boson and six-boson vertices according to the well-known nonlinear expression for $W^a_{\mu\nu}$. Note, that in the approximation used we maintain the gauge invariance of the approach.

In the course of the study the following simple dependence of form-factor F on all three variables was used

$$F(p_1,\,p_2,\,p_3)=F\left(\frac{p_1^2+p_2^2+p_3^2}{2}\right), \tag{6.91}$$

The expression for four-boson vertex is presented above (6.10), where triad p, m, λ *etc* means correspondingly momentum, isotopic index, Lorentz index of a boson; g is the usual gauge coupling constant of the electro-weak interaction. Note, that in vertex (6.10) only momenta of two legs are present in various combinations. Thus a leg here is either “momentum” one or “sterile” one. We shall use these notations in what follows while discussing distribution of momenta in diagrams.

Now in the way of studying the problem in this section we get convinced, that there exists a nontrivial solution, which is expressed in terms of Meijer functions.

The solution for the form-factor is unique and has the the following form for $0<z<z_0$

$$\begin{aligned}F(z)&=\frac{1}{2}\,G^{31}_{15}\Big(z\,|^{0}_{1,\,1/2,\,0,\,-1/2,\,-1}\Big)-\frac{85\,g\sqrt{2}}{512\,\pi}\,G^{31}_{15}\Big(z\,|^{1/2}_{1,\,1/2,\,1/2,\,-1/2,\,-1}\Big)+\\&\quad+\,C_1\,G^{10}_{04}\Big(z\,|1/2,\,1,-1/2,-1\Big)+C_2\,G^{10}_{04}\Big(z\,|1,\,1/2,-1/2,-1\Big).\end{aligned} \tag{6.92}$$

$$z=\frac{G^2\,x^2}{512\pi^2},\quad x=p^2.$$

For $z \geq z_0$ we have the trivial solution

$$F(z) = 0. \tag{6.93}$$

Parameters of solution (6.92) are the following

$$\begin{aligned} &g = g(z_0) = 0.60366, \quad z_0 = 9.61750, \\ &C_1 = -0.035096, \quad C_2 = -0.051104. \end{aligned} \tag{6.94}$$

We would draw attention to the fixed value of parameter z_0. The solution exists only for this value (6.94) and it plays the role of eigenvalue. As a matter of fact, from the beginning the existence of such eigenvalue is by no means evident. The definite value for $g(z_0)$ is also worth mentioning. Let us note, that $g(z_0)$ is the value of running gauge coupling g at the momentum Q_0, which is defined by relation $G^2\, Q_0^4 = 512\,\pi^2\, z_0$.

Remind, that an existence of a nontrivial solution of a compensation equation is extremely restrictive. In the most cases such solutions do not exist at all. When we start from a renormalizable theory we have arbitrary value for its coupling constant. Provided there exists nontrivial solution of a compensation equation the coupling is fixed as well as the parameters of this nontrivial solution.

Now let us consider a contribution of interaction (6.89) with a form-factor defined by relations (6.11, 6.92, 6.94) to the anomalous magnetic moment of the charged spin one half particle with mass m, for example, of the muon. The first approximation described by the simplest diagram presented in Figure 6.5 gives zero. This result is immediately connected with Lorentz structure of anomalous vertex (6.90).

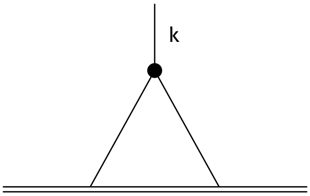


Fig. 6.5. One loop diagram for calculation of new contribution to the muon magnetic moment. Vertical line represents the photon, simple lines – W bosons, black spot – triple vertex (6.1) with corresponding form-factor. Double line represents the muon.

Thus to obtain nonzero contribution to a_μ we look for two-loop diagrams, which give contribution to three-boson Lorentz structure of the usual gauge vertex.

$$V_{p,\mu;q,\nu;k,\rho} = g_{\mu\nu}(q-p)_\rho + g_{\nu\rho}(k-q)_\mu + g_{\rho\mu}(p-k)_\nu. \tag{6.95}$$

Diagrams shown in Figure 6.6, in which four-boson vertex corresponds to terms in expression (6.10) with "momentum" legs entering to oval loops achieve this goal.

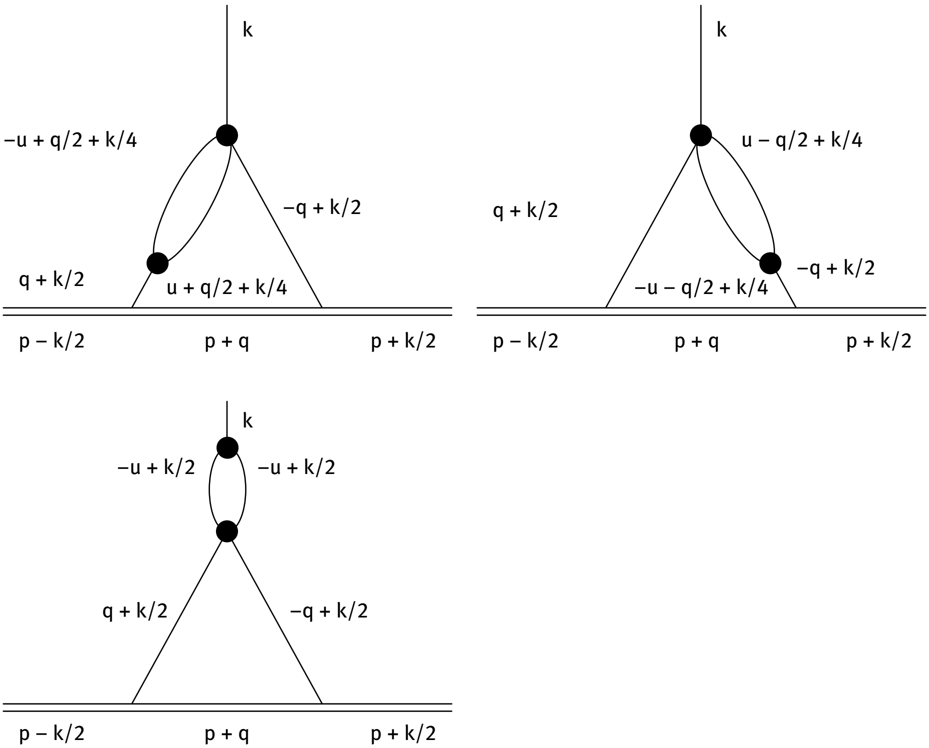


Fig. 6.6. Two loop diagrams for calculation of new contribution to the muon magnetic moment. Vertical line represents the photon, simple lines – W bosons, black spots – triple vertex (6.6) and four leg vertex (6.10) with corresponding form-factors. Double line represents the muon. The momenta are directed downwards and rightwards (the muon). The momentum of integration in the oval-like loops is denoted u.

The calculations are performed in the unitary gauge. The main contribution to the result is given by increasing terms in nominators in vector boson propagators, containing M_W^{-2}

$$\frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}}{q^2 - M_W^2},$$

while in denominators we put $M_W = 0$. The estimate of accuracy of this procedure will be given below.

In the course of calculations finite renormalization of the gauge coupling g is performed. In doing this we single out the constant contribution to the Lorentz structure (6.95) of triple boson vertices including oval-like loops. After this procedure only the two first diagrams in Figure 6.6 give contribution to the value of the magnetic moment. Remind, that the structure of the anomalous vertex (6.89) gives zero contribution.

We have for the main contribution to a magnetic moment according to diagrams Figure 6.6 the following Lorentz structure

$$\frac{\hat{p}\,\hat{k}\,\gamma_a}{4} - \frac{\gamma_a\,\hat{k}\,\hat{p}}{4} - \frac{1}{3}\hat{p}\,\gamma_a\,\hat{p} \;=\; \frac{m}{6}\,\frac{\hat{k}\,\gamma_a - \gamma_a\,\hat{k}}{2}, \tag{6.96}$$

which is multiplied by the two-loop integral, which we calculate in the Euclid four dimensional momentum space. The integration momentum inside an oval loop is u and inside a triangle loop is q. Denoting $u^2 = x$ and $q^2 = y$ we have from (6.91) combination $x + \frac{3y}{4}$ for arguments of both form-factors. Thus substitution

$$t \;=\; x + \frac{3y}{4}, \tag{6.97}$$

is quite natural. Using variables t (6.97) and y we have the following expression for coefficient before the magnetic moment structure with account of (6.96)

$$\frac{m\,e\,g^2 G^2}{12(16\pi^2)^2 M_W^2}\int_0^Y dt$$
$$F^2(t)\left(\int_0^t \frac{4\,t\,dy}{(6t-3y)} + \int_t^{4t/3} \frac{4\,t(16t^3 - 48t^2y + 48ty^2 - 15y^3)\,dy}{3(2t-y)y^3}\right), \tag{6.98}$$

where Y is defined by the relation

$$z_0 = \frac{G^2\,Y^2}{512\,\pi^2}.$$

From (6.98) with definitions of variable z and of the form-factor (6.92, 6.94) we obtain the following final result for the contribution to a_μ

$$\Delta a_\mu = \frac{g(z_0)^2\,m^2}{3\,\pi^2\,M_W^2}\left(20\,\ln\left[\frac{4}{3}\right] - \frac{13}{3}\right) \times \int_0^{z_0} F^2(z)\,dz = 2.775 \times 10^{-9}. \tag{6.99}$$

where for the numerical result we have used only values of the muon mass and the W-boson mass. All other parameters are defined by solution (6.92) with parameters (6.94). Let us draw attention to the disappearance of the effective interaction coupling constant G from expression (6.99). This is due to entering of factor G^2 into the denominator according to definition of variable z. Thus the main result does not depend on λ. This parameter influence only the next approximations. Let us estimate possible corrections due to $M_W \neq 0$. They are defined by the following parameter

$$\frac{\sqrt{2}g|\lambda|}{32\pi} = 0.0005, \tag{6.100}$$

with the maximal value of $|\lambda| = 0.059$ from restrictions (6.9). Thus this correction is negligible. The other correction may be connected with the value of gauge coupling

$g(z_0)$. It is taken from solution (6.25). However it is possible to calculate experimental value for this parameter. Let us start from the well-known expression for the running electro-weak coupling with the total number of flavors

$$g^2(Q^2) = \frac{g^2}{1 + \frac{5\,g^2}{24\pi^2}\,\ln\left[\frac{Q^2}{M_W^2}\right]}, \tag{6.101}$$

$$g = g(M_W^2) = 0.65, \quad Q^2(z_0) = \frac{32\,\pi\sqrt{z_0}\,M_W^2}{\sqrt{2}\,g\,|\lambda|}$$

Then with the same $|\lambda| = 0.059$ we obtain $g(z_0) = 0.626$ and with this value we have instead of result (6.99)

$$\Delta a_\mu = 2.987 \times 10^{-9}. \tag{6.102}$$

This value is few *per cent* larger than value (6.99). There may be also other corrections to result (6.99). The examples being studied earlier have given estimate for accuracy of the approximation $\simeq 10\%$ [74, 78]. Bearing in mind this estimate, the result for the nonperturbative contribution to a_μ is advertised to be the following

$$\Delta a_\mu = (2.78\, \pm 0.28) \times 10^{-9}. \tag{6.103}$$

The result, as well as values (6.99, 6.102), evidently agrees deviation (6.87) within error bars.

There are proposals to connect the Δa_μ effect with theories beyond the Standard Model, for example with effects of super-symmetric variants [116]. However with such proposals one inevitably introduces additional parameters to adjust the discrepancy. Here we have no adjusting parameters. Therefore the result (6.103) is to be considered as an evidence for confirmation of the Standard Model. What is necessary, is to learn how to calculate nonperturbative contributions. The method, which is used in the present calculations, gives quite an adequate result. Thus we consider the result to provide a convincing confirmation of the approach.

7 Possible four-fermion interaction of heavy quarks

7.1 Four-fermion interaction of heavy quarks

Let us remind that the adequate description of low-momenta region in QCD can be achieved by an introduction of the effective Nambu–Jona-Lasinio interaction [31, 32] (see recent review [39]). We have considered this problem in the previous chapters. In the framework of the compensation approach the spontaneous generation of NJL-type interaction was obtained in works [77, 78] and is described in Chapter 5. In these studies pions and other light hadrons are described as bound states of light quarks, which are formed due to the effective Nambu–Jona-Lasinio interaction with account of the QCD interaction.

In the present section we explore the analogous considerations and assume that scalar fields which substitute elementary Higgs fields might be formed by bound states of heavy quarks t, b. This possibility was proposed in works [117–119] and was considered in a number of publications (see, e.g. [120]). It comes clear, that estimates of mass of the t-quark in this model gives result which exceeds significantly its measured value. In the present consideration we obtain the four-fermion interaction in the framework of Bogoliubov compensation approach, while in the works on the model being cited, the interaction was postulated. In our approach parameters of the problem are defined by an unique solution of a set of equations quite analogously the Nambu–Jona-Lasinio case considered above. In particular we shall see that the t-quark mass is quite consistent with the current data.

We start with Lagrangian (6.3) in which both gauge bosons W and spinor particles (leptons and quarks) are massless. As the first stage we consider approximation in which only the most heavy particles acquire masses, namely W-s and the t-quark while all other ones remain massless. In view of this we introduce left doublet $\Psi_L = (1+\gamma_5)/2 \cdot (t,\, b)$ and right singlet $T_R = (1-\gamma_5)/2 \cdot t$. Then we study a possibility of a spontaneous generation of the following effective nonlocal four-fermion interaction

$$\begin{aligned} L_{ff} = {} & G_1\, \bar\Psi^\alpha_L\, T_{R\,\alpha}\, \bar T^\beta_R\, \Psi_{L\,\beta} + G_2\, \bar\Psi^\alpha_L\, T_{R\,\beta}\, \bar T^\beta_R\, \Psi_{L\,\alpha} \\ & + \frac{G_3}{2}\, \bar\Psi^\alpha_L\, \gamma_\mu\, \Psi_{L\,\alpha}\, \bar\Psi^\beta_L\, \gamma_\mu\, \Psi_{L\,\beta} + \frac{G_4}{2}\, \bar T^\alpha_R\, \gamma_\mu\, T_{R\,\alpha}\, \bar T^\beta_R\, \gamma_\mu\, T_{R\,\beta}. \end{aligned} \tag{7.1}$$

where α, β are color indices. We shall formulate and solve compensation equations for form-factors of the first two interactions, while consideration of the two last ones is postponed for the next approximations. However, coupling constants G_3, G_4 essentially influence the forthcoming results. Here we follow the procedure described in Chapter 5, which deal with four-fermion Nambu–Jona-Lasinio interaction.

In diagram form the compensation equation is shown at Figure 7.1. Form-factors F_i are introduced according to the following representation for four-quark vertices

$$G_1\, F_1(x)\, \delta^\alpha_\alpha\, \frac{1-\gamma_5}{2} \times \delta^\beta_\beta \frac{1-\gamma_5}{2} + G_2\, F_2(x)\, \delta^\alpha_\beta\, \frac{1-\gamma_5}{2} \times \delta^\beta_\alpha \frac{1-\gamma_5}{2}. \tag{7.2}$$

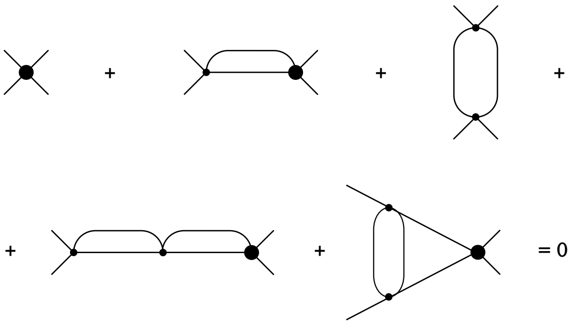


Fig. 7.1. The graphic representation of linear compensation equations (7.3), (7.4).

Following our method, formulated in the previous chapters, we come to the following compensation equations for form-factors $F_1(x)$ and $F_2(x)$, $x = p^2$, corresponding respectively to the first two terms in (7.1).

$$
\begin{aligned}
\Phi(x) = {} & \frac{\Lambda^2(N_c^2 G_1^2 + 2N_c G_1 G_2 + G_2^2)}{8\pi^2(N_c G_1 + G_2)}\left(1 - \frac{N_c G_1 + G_2}{8\pi^2} \times \int_0^{\bar{Y}} \Phi(y)dy\right) \\
& + \left(\Lambda^2 + \frac{x}{2}\ln\frac{x}{\Lambda^2} - \frac{3x}{4}\right) \times \frac{G_1^2 + G_2^2 + 2N_c G_1 G_2 + 2\bar{G}(N_c + 1)(G_1 + G_2)}{32\pi^2(N_c G_1 + G_2)} \\
& - \frac{G_1^2 + G_2^2 + 2N_c G_1 G_2 + 2\bar{G}(N_c + 1)(G_1 + G_2)}{2^9\pi^4} K \times \Phi,
\end{aligned} \tag{7.3}
$$

$$
\begin{aligned}
F_2(x) = {} & \frac{\Lambda^2 G_2}{8\pi^2}\left(1 - \frac{G_2}{8\pi^2}\int_0^{\bar{Y}} F_2(y)dy\right) \\
& + \left(\Lambda^2 + \frac{x}{2}\ln\frac{x}{\Lambda^2} - \frac{3x}{4}\right)\frac{G_1^2 + G_2^2 + 2\bar{G}(G_1 + G_2(N_c + 1))}{32\pi^2 G_2} \\
& - \frac{G_1^2 + G_2^2 + 2\bar{G}(G_1 + G_2(N_c + 1)}{2^9\pi^4} K \times F_2, \;\; \Phi(\bar{Y}) = F_2(\bar{Y}) = 0, \\
\Phi(x) = {} & \frac{N_c G_1 F_1 + G_2 F_2}{N_c G_1 + G_2}, \quad \bar{G} = \frac{G_3 + G_4}{2}, \quad x = p^2, \quad y = q^2.
\end{aligned} \tag{7.4}
$$

Here number $N_c = 3$ and a kernel term in equations is the following

$$
\begin{aligned}
K \times F = {} & (\Lambda^2 - x\ln\Lambda^2)\int_0^{\bar{Y}} F(y)dy - \ln\Lambda^2 \int_0^{\bar{Y}} F(y)ydy + \frac{1}{6x}\int_0^x F(y)y^2 dy \\
& + \ln x \int_0^x F(y)ydy + x\left(\ln x - \frac{3}{2}\right)\int_0^x F(y)dy + \int_x^{\bar{Y}} y\left(\ln y - \frac{3}{2}\right)F(y)dy \\
& + x\int_x^{\bar{Y}} \ln y F(y)dy + \frac{x^2}{6}\int_x^{\bar{Y}} \frac{F(y)}{y}dy.
\end{aligned} \tag{7.5}
$$

and Λ is an auxiliary cut-off, which disappears from all expressions with all conditions for solutions be fulfilled.

Introducing substitution

$$G_1 = \rho \bar{G}, \quad G_2 = \omega \bar{G}, \tag{7.6}$$

and comparing the two equations (7.3, 7.4) we get convinced, that both equations become being the same under the following condition

$$\rho = 0. \tag{7.7}$$

and we remain with one equation

$$\begin{aligned} F_2(z) = & \frac{\sqrt{\omega^2 + 8\omega}}{\omega} \sqrt{z}\,(\ln z - 3) \\ & - 16\Bigg[\frac{1}{6\sqrt{z}} \int_0^z F_2(t)\sqrt{t}\,dt + \frac{\ln z}{2} \int_0^z F_2(t)\,dt + \frac{\sqrt{z}(\ln z - 3)}{2} \int_0^z \frac{F_2(t)}{\sqrt{t}} dt \\ & + \frac{1}{2} \int_z^{\bar{z}_0} (\ln t - 3) F_2(t)\,dt + \frac{\sqrt{z}}{2} \int_z^{\bar{z}_0} \ln t \frac{F_2(t)}{\sqrt{t}}\,dt + \frac{z}{6} \int_z^{\bar{z}_0} \frac{F_2(t)}{t}\,dt\Bigg], \\ & z = \frac{(\omega^2 + 8\omega)\bar{G}^2 x^2}{2^{14}\pi^4}, \quad t = \frac{(\omega^2 + 8\omega)\bar{G}^2 y^2}{2^{14}\pi^4}, \quad \bar{z}_0 = \frac{(\omega^2 + 8\omega)\bar{G}^2 \bar{Y}^2}{2^{14}\pi^4}. \end{aligned} \tag{7.8}$$

Here we omit all terms containing auxiliary cut-off Λ due to their cancelation, which occurs here in the same way, as in Chapter 5.

Performing consecutive differentiations of equation (7.8) we obtain the following differential equation for F_2

$$\left(z\frac{d}{dz}+\frac{1}{2}\right)\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-\frac{1}{2}\right)\left(z\frac{d}{dz}-1\right) F_2(z) + z\,F_2(z) = 0, \tag{7.9}$$

Equation (7.9) is equivalent to integral equation (7.8) provided the following boundary conditions being fulfilled

$$\int_0^{\bar{z}_0} \frac{F_2(t)}{\sqrt{t}}\,dt = \frac{\sqrt{\omega^2 + 8\omega}}{8\,\omega}, \quad F_2(\bar{z}_0) = 0, \quad \int_0^{\bar{z}_0} F_2(t)\sqrt{t}\,dt = 0, \quad \int_0^{\bar{z}_0} F_2(t)\,dt = 0. \tag{7.10}$$

Note that just boundary conditions (7.10) lead also to cancelation of all terms containing Λ. Differential equation (7.9) is a Meijer equation and the only solution of the problem (7.9, 7.10) is the following (for a definition and properties of Meijer functions see Section 2.3)

$$F_2(z) = \frac{1}{2\sqrt{\pi}} G_{06}^{40}\left(z|0, \frac{1}{2}, \frac{1}{2}, 1, -\frac{1}{2}, 0\right), \quad \bar{z}_0 = \infty. \tag{7.11}$$

Here we also take into account condition $F_2(0) = 1$ that gives

$$\omega = \frac{8}{3}. \tag{7.12}$$

We would draw attention to the fact, that unique solution (7.11) exists only for infinite upper limit in integrals and solution (7.11) exponentially decreases at infinity.

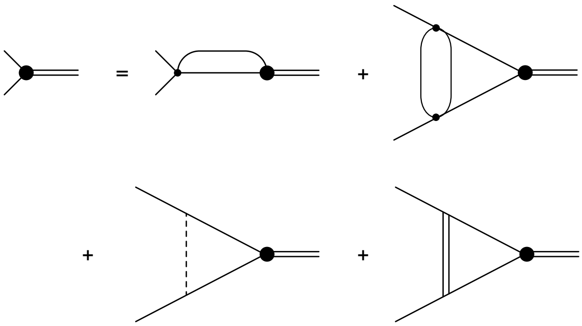


Fig. 7.2. The graphic representation of the Bethe–Salpeter equation for the bound state (8.41). The black spot represents the Bethe–Salpeter wave function, four-leg points represent interaction (7.1), dotted line is gluon, double line is the bound state itself.

7.2 Doublet bound state $\bar{\Psi}_L\, T_R$

Let us study a possibility of spin-zero doublet bound state $\bar{\Psi}_L\, T_R = \phi$, which, as we shall see, can be referred to a composite Higgs scalar. With account of interaction (7.1) using results of the previous section we have the following Bethe–Salpeter equation, in which we take into account the would-be t-quark mass (see Figure 7.2)

$$\Psi(x) = \frac{\bar{G}}{16\pi^2} \int \Psi(y)\, dy + \frac{G_2^2}{2^7\pi^4} K^* \times \Psi, \tag{7.13}$$

where in the first approximation we neglect contributions of diagrams of the second row in Figure 7.2. The modified integral operator K^* is defined in the same way as operator (7.5) with opposite sign, $\bar{Y} = \infty$, and lower limit of integration being changed for m^2, where m is of order of magnitude of the would-be t-quark mass (analogous equation see in Chapter 5).

Then we have again differential equation

$$\left(z\frac{d}{dz} - a_1\right)\left(z\frac{d}{dz} - a_2\right)\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz} - \frac{1}{2}\right)\left(z\frac{d}{dz} - \frac{1}{2}\right)\left(z\frac{d}{dz} - 1\right)\Psi(z) - z\,\Psi(z) = 0, \tag{7.14}$$

$$z = \frac{G_2^2 (p^2)^2}{2^{12}\pi^4}, \quad a_1 = -\frac{1 + \sqrt{1 + 64\mu}}{4}, \quad a_2 = -\frac{1 - \sqrt{1 + 64\mu}}{4}, \quad \mu = \frac{G_2^2 m^4}{2^{12}\pi^4}.$$

where the main difference with the compensation equation is the other sign of the last term, while variable z is just the same as in (7.9) with account of relation (7.12). Boundary conditions now are the following

$$\int_\mu^\infty \frac{\Psi(t)}{\sqrt{t}} dt = 0, \quad \int_\mu^\infty \Psi(t)\sqrt{t}\, dt = 0, \quad \int_\mu^\infty \Psi(t)\, dt = 0, \quad \Psi(\mu) = 1. \tag{7.15}$$

Solution of the problem is presented in the following form

$$\Psi(z) = C_1 G^{50}_{06}\left(z|a_1, a_2, \frac{1}{2}, \frac{1}{2}, 1, 0\right) + C_2 G^{30}_{06}\left(z|0, \frac{1}{2}, 1, a_1, a_2, \frac{1}{2}\right)$$
$$+ C_3\, G^{30}_{06}\left(z|\frac{1}{2}, \frac{1}{2}, 1, a_1, a_2, 0\right) + C_4\, G^{50}_{06}\left(z|a_1, a_2, 0, \frac{1}{2}, 1, \frac{1}{2}\right), \tag{7.16}$$

where C_i for given μ are uniquely defined by conditions (7.15).

We define interaction of the doublet ϕ, consisting of $\Psi_L\, T_R$, with heavy quarks

$$L_\phi = g_\phi(\phi^* \bar{\Psi}_L\, T_R + \phi\, \bar{T}_R \Psi_L), \tag{7.17}$$

where g_ϕ is the coupling constant of the new interaction to be defined by normalization condition of solution (7.16) of equation (7.14). Then we take into account the contribution of interaction of quarks with gluons and the exchange of ϕ as well (see Figure 7.2 the second row). Using standard perturbative method we obtain for the mass of the bound state under consideration the following expression in the same way as in Chapter 5, equations (5.30, 5.31, 5.35).

$$m_\phi^2 = -\frac{m_t^2\, I_5'}{\sqrt{\pi\,\mu}\, I_2'}, \quad I_2' = \int_\mu^\infty \frac{\Psi(z)^2\, dz}{z}, \tag{7.18}$$
$$I_5' = \int_\mu^\infty \frac{(16\,\pi\,\alpha_s(z) - g_\phi^2)\,\Psi(z)\, dz}{16\,\pi\, z} \int_\mu^z \frac{\Psi(t)\, dt}{\sqrt{t}}.$$

Here $\alpha_s(z)$ is the strong coupling with standard evolution, normalized at the t-quark mass, and we put $m = m_t$. Provided term with brackets inside integral I_5' being positive, bound state ϕ is a tachyon. Let us recall the well-known relation for t-quark mass, which is defined by nonzero vacuum average of $(\phi_2^* + \phi_2)/\sqrt{2}$. It reads

$$m_t = \frac{g_\phi\, \eta}{\sqrt{2}}, \tag{7.19}$$

where from phenomenology we know the value of the electro-weak scalar condensate $\eta = 246.2$ GeV. However in our approach there are additional contributions to this mass, which are due to diagram shown at Figure 7.3. That means that for experimental value of the t-quark mass we take the modified definition

$$m_t = \frac{g_\phi\, \eta}{\sqrt{2}} + \Delta M = \frac{g_\phi\, \eta}{f\,\sqrt{2}}. \tag{7.20}$$

According to these diagrams we have the following expression for ΔM

$$\Delta M = -\,4\, m_t \int_\mu^\infty \frac{F_2(z)\, dz}{\sqrt{z}} \int_\mu^\infty \frac{\alpha_s(z)\, F_2(z)\, dz}{2\,\pi\, z} - 4 \int_\mu^\infty \frac{m_t(z) F_2(z)\, dz}{\sqrt{z}}, \tag{7.21}$$
$$m_t(z) = m_t\left(1 + \frac{7\alpha_s(\mu)}{8\pi}\,\ln\frac{z}{\mu}\right)^{-\frac{4}{7}}.$$

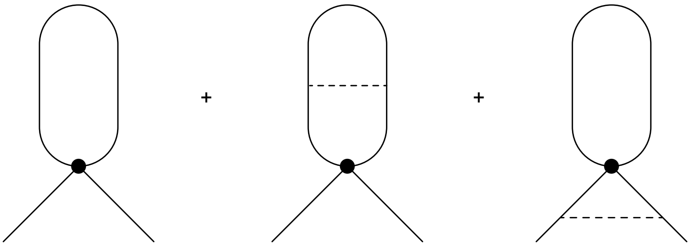


Fig. 7.3. Contributions to the t-quark mass. Dotted lines represent gluons.

Here the first term corresponds to gluon exchange between external legs and the second term corresponds to gluon exchanges inside the loop calculated with account of standard renormalization group mass evolution. Contributions of gluon exchanges from external legs to internal lines cancel. Now parameter f defined in (7.20) is the following

$$f = 1 + 4 \int_\mu^\infty \frac{F_2(z)dz}{\sqrt{z}} \int_\mu^\infty \frac{\alpha_s(z)F_2(z)\,dz}{2\,\pi\,z} + 4 \int_\mu^\infty \frac{m_t(z)F_2(z)\,dz}{m_t\,\sqrt{z}}. \tag{7.22}$$

Due to relation (7.10) factor f in (7.20, 7.22) is slightly larger than 2. For strong coupling $\alpha_s(z)$ we use the standard one-loop expression

$$\alpha_s(z) = \alpha_s(\mu)\left(1 + \frac{7\alpha_s(\mu)}{8\pi}\,\ln\frac{z}{\mu}\right)^{-1}. \quad \alpha_s(\mu) = 0.108, \tag{7.23}$$

where for strong coupling at the t-quark mass we take its value obtained by evolution expression (7.23) from its value at M_Z: $\alpha_s(M_Z) = 0.1184 \pm 0.0007$.

Let us consider the possibility when relation (7.18) leads to a tachyon state. In this case the neutral component of scalar ϕ consist of $\bar{t}_L\,t_R + \bar{t}_R\,t_L$ that is its vacuum average η corresponds to $\bar{t}\,t$ condensate [117–119]. For Higgs mechanism to be realized we need also four-fold interaction

$$L_{\phi 4} = \frac{\lambda}{2}\,(\phi^*\phi)^2. \tag{7.24}$$

Coupling constant in (7.24) is defined in terms of the following loop integral

$$\lambda = \frac{3\,g_\phi^4}{16\,\pi^2}\,I_4', \quad I_4' = \int_\mu^\infty \frac{\Psi(z)^4\,dz}{z}. \tag{7.25}$$

From well-known relations (see Section 2.1) $\eta^2 = -m_\phi^2/\lambda$ and the Higgs mass squared $M_H^2 = -\,2\,m_\phi^2$ we have

$$\eta^2 = \frac{16\pi\,m_t^2\,I_5'}{3\,g_\phi^4\,\sqrt{\mu}\,I_2'\,I_4'}, \quad M_H^2 = \frac{2\,m_t^2\,I_5'}{\pi\,\sqrt{\mu}\,I_2'}. \tag{7.26}$$

Fig. 7.4. Diagram representation of the normalization condition for bound state ϕ. Dotted lines represent gluons.

From (7.20) and (7.26) we have useful relation

$$2 = \frac{16\pi\, I_5'}{3\, g_\phi^2\, f^2\, \sqrt{\mu}\, I_2'\, I_4'}. \tag{7.27}$$

We obtain g_ϕ from a normalization condition, which is defined by diagrams of Figure 7.4

$$\frac{3g_\phi^2}{32\pi^2}\left(I_2' + \frac{\alpha_s(\mu)}{4\pi}\left((I_{22}')^2 + 2\, I_6'\right)\right) = 1, \tag{7.28}$$

$$I_{22}' = \int_\mu^\infty \frac{\Psi(t)\, dt}{t}, \quad I_6' = \int_\mu^\infty \frac{\Psi(z)\, dz}{z\sqrt{z}} \int_\mu^z \frac{\Psi(t)\, dt}{\sqrt{t}}.$$

Here we use strong coupling at the t-quark mass (7.23) and perform necessary calculations. In doing this we proceed in the following way: for six parameters μ, g_ϕ, η, m_t, M_H, f we have five relations (7.20, 7.26, 7.27, 7.28) and the well-known expression

$$M_W = \frac{g\,\eta}{2}, \tag{7.29}$$

where g is electroweak gauge interaction constant $g(M_W)$ at W mass. We obtain it by usual renormalization group evolution expression (6.26) from value g at z_0 (6.25). Let us remind that we consider M_W as an input. Thus for the moment we have two input parameters, which are safely known from the experiment

$$M_W = 80.4\,\text{GeV}, \quad \eta = 246.2\,\text{GeV}. \tag{7.30}$$

The last value corresponds to value of electro-weak coupling $g(M_W) = 0.653$. We would once more draw attention to results of compensation approach on unique definition of physical parameters. In particular, the value for electro-weak charge at point z_0 is consistent with this value. For explication of this point see relations (6.25, 6.29, 6.30, 6.31, 6.32).

Now we present thus obtained parameters

$$\begin{aligned} \mu &= 4.0675\, 10^{-12}, \quad f = 2.034, \quad g_\phi = 2.074, \\ m_t &= 177.0\,\text{GeV}, \quad M_H = 1803\,\text{GeV}. \end{aligned} \tag{7.31}$$

The most important result here is the t-quark mass, which is close to experimental value $M_t = 173.3 \pm 1.1$ GeV [4]. Really, the main difficulty of composite Higgs models [117–120] consists in too large m_t. Indeed the definition of g_ϕ in such models leads to $g_\phi \simeq 3$ and thus $m_t \simeq 500\,\text{GeV}$. We have all parameters, including important parameter f, being defined by self-consistent set of equations and the unique solution gives results (7.31), which for m_t is quite satisfactory.

However the mass of the Higgs scalar particle is quite large. Such large mass of H means, of course, very large width of H

$$\Gamma_H = 3784\,\text{GeV}, \quad BR(H \to W^+ W^-) = 51.4\%, \qquad (7.32)$$
$$BR(H \to Z Z) = 25.6\%, \quad BR(H \to \bar{t}\, t), = 23.0\%.$$

The large value for M_H seems to contradict to upper limit for this mass, which follows from considerations of the Landau pole in the $\lambda\phi^4$ theory. Emphasize, that this limit corresponds to the local theory and in our case of composite scalar fields is not relevant.

The most important argument against Higgs mass (7.31), is the LHC discovery of the state with mass 125.7 GeV, which properties are consistent with those of the Higgs particle [57, 58]. In case of a final confirmation of the Higgs mass in accordance to data [57, 58], results of this chapter are to be revised. Let us note, for example, that the coefficient in relation (7.18) for m_ϕ^2 is defined by difference $16\pi\alpha_s - g_\phi^2$, which contains two approximately equal terms. Under the approximation being used there, this difference turns to be positive, that corresponds to tachyon state $\bar{\Psi}_L\, \Psi_R$. Just this result leads to the interpretation of this state as corresponding to a constituent Higgs scalar field. It might be, that with an account of the next corrections the sign in relation (7.18) could change to the opposite. Then the state would be a usual $\bar{\Psi}_L\, \Psi_R$ resonance with mass of the order of magnitude $\sim$ TeV.

The most cardinal revision, while remaining in the framework of the compensation point of view, may consist in a conclusion, that one has to admit, that just trivial solutions of compensation equations (7.3, 7.4) are realized. Bearing in mind different possibilities, we nevertheless would discuss some problems, connected with a wouldbe existence of effective interaction (7.1).

As a matter of fact, predictions (7.31, 7.32) strictly speaking do not contradict experimental limitations yet. The possible effect would consist of registration of slight increasing of cross-sections $p + p \to W^+ + W^- + X$, $p + p \to Z + Z + X$, $p + p \to \bar{t} + t + X$ in region of invariant masses of two heavy particles $1\,\text{TeV} < M_{12} < 3\,\text{TeV}$.

7.3 Stability problem

In Chapter 5 we have considered conditions for stability of the nontrivial solution. We have became aware of the fact, that with variation of $\bar{\alpha}_s$ the vacuum energy density varies from positive (unstable) values to the negative ones, that corresponds to stability of a nontrivial solution. How things are in the electro-weak case?

Let us construct the vacuum energy density for this case. It consist of vacuum average of gauge boson term, vacuum average of the t-quark term (the $\bar{t}t$ condensate) and the scalar field term.

$$VD_{EW} = \frac{1}{4}\langle W^a_{\mu\nu} W^a_{\mu\nu}\rangle + m_t\langle \bar{t}t\rangle - \frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{7.33}$$

$AA.$

Here the W condensate is defined by relation (6.34) and the $\langle \bar{t}t\rangle$ condensate can be defined in exact analogy with $\langle \bar{q}q\rangle$ condensate in Chapter 5. Let us use relation (5.51) with $u = \mu \simeq 0$, $m_0 = 0$

$$\langle \bar{t}t\rangle = -\frac{3\,m_t}{\pi^2\sqrt{\beta}}\left[\frac{\alpha_s(m_t)}{\pi} + \frac{g_\phi^2}{8\pi^2}\right]\int_0^\infty \frac{\Psi(t)\,\ln t\,dt}{\sqrt{t}}, \tag{7.34}$$

$$\beta = \frac{G_2^2}{2^{12}\,\pi^4}. \tag{7.35}$$

where variable t is defined by expression (7.14).

Under the same conditions

$$\Psi(t) = \frac{\sqrt{\pi}}{2}\,G^{30}_{06}\left(t\,|0,\,\frac{1}{2},\,1,-\frac{1}{2},\,0,\,\frac{1}{2}\right), \tag{7.36}$$

and the integral in relation (7.35) can be easily evaluated by parts with the use of relation (2.152)

$$\int_0^\infty \frac{\Psi(t)\,\ln t\,dt}{\sqrt{t}} = -\frac{\sqrt{\pi}}{2}. \tag{7.37}$$

With this result we come to the final expression for the t-quark condensate

$$\langle \bar{t}t\rangle = \frac{3\cdot 2^4\,m_t\,\pi}{|G_2|}\left[\frac{\alpha_s(m_t)}{\pi} + \frac{g_\phi^2}{8\pi^2}\right] = \frac{3\,m_t^3}{4\pi\sqrt{\mu}}\left[\frac{\alpha_s(m_t)}{\pi} + \frac{g_\phi^2}{8\pi^2}\right]. \tag{7.38}$$

It is remarkable, that the situation here is quite contrary to the case of QCD. Indeed, in Chapters 4, 5 we have the gauge field (gluon) condensate being positive and the fermion (quark) condensate being negative. The two condensates almost cancel each other. In the electroweak case the gauge field (W) condensate (6.38) is negative (6.39, 6.40), while the t-quark condensate is positive on the contrary. We have already estimated the W condensate. The question is if these two condensates also close to canceling? So let us estimate expression (7.38). Using results (7.31, 7.23)

$$\mu = 4.0675\,10^{-12},\quad g_\phi = 2.074,\quad M_H = 1.803\,\text{TeV}$$
$$m_t = 0.177\,\text{TeV},\quad \alpha_s(m_t) = 0.108. \tag{7.39}$$

we obtain the following value for the t-quark condensate

$$m_t\langle \bar{t}t\rangle = 10.32\,\text{TeV}^4. \tag{7.40}$$

The coefficients in the scalar field part of expression (7.33) are defined by relations (7.18, 7.25). Then in the point of the minimum of expression (6.34) we have with account of definition (6.34)

$$VD_{EW} = \frac{\pi^2}{g^2} V_2^{EW} + m_t \langle \bar{t}t \rangle - \frac{M_H^2 \eta^2}{8}, \tag{7.41}$$

where V_2^{EW} is defined by expression (6.34). Substituting values (6.30, 6.31, 6.32, 7.39, 7.40) and $\eta = 0.2462$ TeV we obtain for values of $\epsilon = 0.095, 0.11, 0.13$ the following values of the density VD_{EW} correspondingly

$$\begin{aligned} VD_{EW} &= -692.8\,\mathrm{TeV}^4, \quad \epsilon = 0.095, \\ VD_{EW} &= -51.6\,\mathrm{TeV}^4, \quad \epsilon = 0.11, \\ VD_{EW} &= 7.68\,\mathrm{TeV}^4, \quad \epsilon = 0.13. \end{aligned} \tag{7.42}$$

We see, that the transition to positive values of the vacuum density occurs somewhere between the second and the third values of (7.42). The sharp dependence on the choice of parameter ϵ is connected with such dependence of the W condensate, as one readily sees e. g., from expressions (6.39, 6.40). So the point of the phase transition occurs just in this interval between $\epsilon = 0.11$ and $\epsilon = 0.13$, while our estimates of the preferable values of parameters corresponds to $\epsilon = 0.095$, that gives without doubts stability for the nontrivial solution.

7.4 Possible effects of the heavy quarks interaction

Interaction (7.1) of heavy quarks could manifest itself in physical effects. There is interesting data on decay $Z \to \bar{b}b$. Interaction (7.1) give contribution to this process due to diagram in Figure 7.5. From interaction (7.1) we see, that terms with couplings G_2 and G_3 give contribution to diagrams in Figure 7.5. The previous experience teaches to expect a significant contribution of these diagrams to the amplitude of the decay. However, as we see from Table 2.4 there is no significant deviations from the SM fit. Let us perform simple calculation of these diagrams, that gives the following contribution

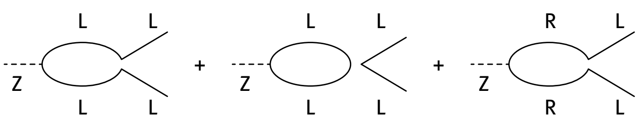


Fig. 7.5. Diagram describing the first contribution of four-quark interaction (7.1) to $z \to \bar{b}b$ decay. The left-side vertex corresponds to the SM $Z\bar{t}t$ and $Z\bar{b}b$ vertices (2.77). The four-fermion vertices with four L legs correspond to interaction with coupling G_3 and the mixed LR vertex corresponds to coupling G_2. The distribution of color indices is shown in (7.1).

to vertex $Z \to \bar{b}b$

$$\Delta V = \frac{g}{32\pi^2}\frac{1+\gamma_5}{2}\left(G_3\left(-\frac{1}{2}-\frac{2}{3}\sin^2\theta_W\right)+G_2\,\sin^2\theta_W\right)\int dq^2. \tag{7.43}$$

From definition (7.6) and result (7.12) we have

$$\Delta V = \frac{g}{32\pi^2}\frac{1+\gamma_5}{2}\left(G_3\left(-\frac{1}{2}-\frac{2}{3}\sin^2\theta_W\right)+\frac{8}{3}\,\bar{G}\,\sin^2\theta_W\right)\int dq^2. \tag{7.44}$$

Let us recall that $\bar{G}$ is the average of G_3 and G_4. For example, if we take G_3 to be equal to this average, we obtain

$$\Delta V = \frac{g}{32\pi^2}\frac{1+\gamma_5}{2}\,G_3\left(-\frac{1}{2}+2\,\sin^2\theta_W\right)\int dq^2. \tag{7.45}$$

Note, that the momentum integration is symbolically designated by $\int dq^2$. We have to bear in mind, that there should be a form-factor in the integral, that leads to the integral being inversely proportional to $G/16\pi^2$. In any case provided the expression in brackets in (7.45) is of order of unity, the contribution ΔV will contradict the satisfactory agreement of parameters of decay $Z \to \bar{b}b$ with the Standard Model fit of Table 2.4. However, expressions (7.44, 7.45) show an effective cancelation of two large contributions. In case $\sin^2\theta_W = 0.25$ we have zero for (7.45). The physical value is close to this number, namely, $\sin^2\theta_W = 0.2324 \pm 0.0012$ (see Table 2.4). Thus we conclude, that there is a mechanism of cancelation in the possible effect. The accuracy being achieved for the moment do not allow to make more definite assertion. However, one has to bear in mind, that just in this decay there are suspicious deviations from the Standard Model fit, especially in the forward-backward asymmetry F^b_{FB} (2.98), which comprises around 2.8 s.d.. It would be advisable to consider possible contributions of interaction (7.1) to this decay with better precision.

On the other hand, in case predictions of possible existence of interaction (7.1) will contradict future experimental results, it will not disprove the approach in a whole. In the framework of our reasoning it will mean, that set of equations (7.3, 7.4) leads to trivial solution, which is always possible. This does not exclude realization of nontrivial solutions in other cases.

To conclude we would emphasize, that albeit we discuss quite unusual effects, we do not deal with something beyond the Standard Model. We are just in the framework of the Standard Model. What makes difference with usual results is nonperturbative nontrivial solution of a compensation equation. There is of course also trivial perturbative solution. Which of the solutions is realized is to be defined by stability conditions. The problem of stability is extremely complicated and needs a special extensive study.

For the moment we can not exclude the possibility of realization of the trivial solution of set of compensation equations (7.3, 7.4). Then the conclusion of the very high mass of the Higgs scalar would be canceled.

With the present results we would draw attention to two important achievements provided by the nontrivial nonperturbative solution. The first one is unique determination of gauge electro-weak coupling constant $g(M_W)$ in close agreement with experimental value. The second result consists in calculation of the t-quark mass. At this point we would emphasize, that the existence of a nontrivial solution itself always leads to additional conditions for parameters of a problem under study. These two achievements strengthen the confidence in the correctness of applicability of the Bogoliubov compensation approach to the principal problems of elementary particles theory.

8 Overall conclusion

8.1 Short review of achievements of the compensation approach

In the book, which is proposed for the kind attention of a reader, the compensation conception is introduced and defended. It consists in the following premise.

We consider totality of renormalizable interactions of the Standard Model to be the genuine theory of the physical world. The excellent form of the perturbative calculations in the framework of this theory serves as strong support for the assertion.

However necessity of account of the nonperturbative effects in the framework of this genuine theory is evident. So the main premise of the book is the compensation approach, which is motivated by the successful application of the Bogoliubov compensation principle, first of all, to problems of statistical physics. That was famous superfluidity and superconductivity problems. The first one was solved by Bogoliubov himself, and his contribution to the solution of the second one is more than significant. The solutions of these problems were purely nonperturbative. The application of the same methods around fifty years ago had led to formulation of the conception of a spontaneous generation of masses. In the present book we undertake the following step and turn to the problem of a spontaneous generation of effective interactions. The realization of the goal needs a formulation of the tool. The proposed tool is not perfect, of course. When we perform the procedure of add–subtract, there is no approximation, but also there is not definite sense yet. Namely, how to formulate compensation condition, which exclude nondesirable interaction terms from the newly defined free Lagrangian? Strictly speaking it may be formulated only as a quantum functional condition, which is firstly unknown yet, and secondly even in case of successful formulation, problem of a solution is hopeless. Just this was said by N. N. Bogoliubov himself. So a formulation of an approximate procedure is inevitable, if one has intention to find out even something.

However we know, that an adequate approximation scheme for a functional formulation of a quantum field theory [1] is just the perturbation theory, i. e., the proper introduction of Feynman graphs. In an analogy to this well-known conclusion we formulate the first approximation also in terms of loop graphs with the number of loops being restricted by either one or two.

Thus there is an approximation. If it is adequate or not can not be said from the beginning. That is why we consider in the book a number of examples of an application of the method. It comes out, that these examples are instructive. What we are learning from these examples?

The first instructive example is the six-dimensional scalar model, which is considered in Sections 3.4–3.9. We start from a renormalizable $g\,\phi^3$ theory and ask, if there may be spontaneously generated the theory $G\phi^4$ with dimensional coupling constant $G \sim 1/M^2$, which is similar to the four-fermion coupling constant in the usual four-dimensional space. With this example we have succeeded to apply the compensation approach and to show how it works.

In Chapter 4 we consider the second example, being, as a matter of fact, the first real physical one. It starts with QCD initial theory, which contains the fundamental dimensional parameter Λ_{QCD}. The question, which we ask is if the anomalous three-gluon interaction with dimensional constant G, having dimension inverse mass squared, can exist? As the most important result we obtain the answer, that a nontrivial solution does exist and thus the new effective interaction can be spontaneously generated, provided initial gauge coupling constant g, defined on the boundary of the nonperturbative region Q_0 has uniquely defined value of the order of magnitude ~ 3. This result is of high importance. Firstly, it demonstrates the obvious fact, that as a rule, compensation equations have a trivial solution, that is we return to the initial perturbative problem without any change. However, there are sometimes special conditions, the fulfillment of which leads to a nontrivial solution. As we have got convinced, such conditions are imposed on parameters of the problem and not only on the parameters of the effective interaction, but also on the parameters of the initial renormalizable theory. This just occurs in QCD. So, when we have discussed status of the Standard Model we have recapitulated numerous parameters of the theory and have admit, that all these parameters are arbitrary. It is the general belief that the Standard Model exists for any value of a coupling constant, of a mass and of a mixing parameter. But already the example of Chapter 4 demonstrate, that if we choose theories, admitting the effective interactions with dimensional coupling constants, we have additional conditions imposed on the parameters of the initial theory. What may be the physical reason for choosing of such theories? It may be the reason of the stability. We have considered this problem in Chapter 4 and have mentioned that the theory with the spontaneously generated effective interaction (SGEI) differs from the same theory without (SGEI) by the absence of the Landau pole. While the Landau pole leads to instability of the theory, we may pretend, that just the theory, described in Chapter 4 is stable and thus is realized in the Nature. Let us emphasize once more, this theory can exist only for specified values of parameters. In particular, we obtain the value of average α_s in the nonperturbative region being ~ 0.8. As far as we know, there is no other possibility to achieve such result without additional assumptions.

In Chapter 5 we consider the spontaneous generation of the Nambu–Jona-Lasinio interaction and obtain the result, that when we start from QCD with two light quarks having initial parameters m_0 and average α_s, the interaction is completely defined and agreement with low momenta hadron physics (the most nonperturbative problem known) Starting from two initial parameters, e. g., $\bar{\alpha}_s$ and f_π we obtain a satisfactory description of low momenta strong interaction of mesons consisting of light quarks and gluons.

The important results of Chapter 6 refer to the electroweak interaction. As a matter of fact the compensation equation for three-boson effective interaction is in essence the same as for the three-gluon interaction in Chapter 4 (with change of $SU(3) \to$

$SU(2)$). It is of great interest, that this compensation equation has two and only two solutions, one of them suits for QCD and another one suits for EWT. Again we obtain unique value for coupling $g(z_0)$, which gives order of magnitude of the electroweak interaction

$$\frac{g(z_0)^2}{4\,\pi} \simeq 0.03,$$

that is close to the real physical value. I am forced to note once more, that there is no another method, which can determine the coupling, the more so, as in close agreement with its real value.

The existence of the electroweak anomalous three-boson effective interaction (6.1) leads to several consequences. The resonances or bound states consisting of $W\,W$ are predicted. Maybe, some signs of these states are already present in the data. It seems, that the particular importance provides result of Section 6.4. In this section we show, that the existence of the three-boson interaction (6.1) leads to additional contribution to the muon anomalous magnetic moment. The calculation with already defined parameters and the form-factor gives contribution, which removes the discrepancy in measurement of the muon $g-2$. The author is inclined to consider this result as a decisive confirmation of the approach. As a matter of fact, the problems with magnetic moments in many case play the decisive role in a confirmation of a theory. Let us remind, that in QED such role was plaid by calculation of anomalous magnetic moment of the electron a_e (1.55). In QCD understanding of magnetic moments (also anomalous) of the nucleons was crucial for establishing of the new quantum number color, as we have described in Section 1.2. The author express his conviction, that just the result of Section 6.4 on the muon anomalous magnetic moment may serve for justification of the nonperturbative approach with the spontaneously generated effective interactions.

Let us note, that nowadays the status of the discrepancy in the muon $g-2$ is not finally determined. The overall uncertainty of the effect, which is defined by uncertainties in both the experimental and the Standard Model theoretical results, comprises from 3 s. d. up to 4 s. d. depending on a method of a calculation of hadron contributions to a_μ. One may hope for a progress in a resolution of this problem in the near future. For example, the most recent publication [121], devoted to a calculation of the hadron contribution to the effect, supports the more pronounced discrepancy on the level of 4 s. d.. Provided the existence of the discrepancy being firmly established, our result becomes quite sound.

In application to the electroweak interaction of quarks we have considered only the heavy quarks (t, b) and the spontaneous generation of the t-quark mass only. In the present approximation all other quarks including b are massless. The real case needs consideration of all three quark and lepton generations. This means also introduction of mixing angles. The problem is which are conditions, imposed by the method, and on what extent mixing angles and fermion masses are defined. In any case the previous results give the reason to expect here an interesting continuation. Some examples of model results for mass ratios will be considered below.

When we consider the electroweak theory we restricted ourselves by heavy quarks only. The introduction of the real three generations of quarks needs also introduction of mixing angles. The evaluations being performed in Chapter 7 in this case would be much more complicated. As we have seen throughout the book, and in Chapter 7 in particular, the existence of a nontrivial solution impose strong restrictions on its parameters. One may expect the same consequences for parameters describing three generations of quarks. The model of Chapter 7 corresponds to only one massive quark, namely the t-quark, which mass was calculated to be of the correct size. Is it possible, that in a complicate case of three generations of quarks and leptons, one would succeed in obtaining satisfactory description of masses and mixing angles? The answer may be obtained only by trying to consider the problem. "The proof of pudding is eating".

It is worth mentioning, that in expressions for vacuum energy density (5.38, 5.45, 7.41) combinations of the gauge field condensate and of the quark condensate are present

$$VD_{\mathrm{QCD}} = \frac{1}{4}\langle F^a_{\mu\nu}\, F^a_{\mu\nu}\rangle + \sum_i\, m_i\langle \bar{q}_i q_i\rangle, \tag{8.1}$$

$$VD_{EW} = \frac{1}{4}\langle W^a_{\mu\nu}\, W^a_{\mu\nu}\rangle \;+\; m_t\langle \bar{t}t\rangle. \tag{8.2}$$

In both cases two terms of expressions have opposite signs. For QCD case (8.1) the first term is positive and the second one is negative, while for the electro-weak case *vice versa* the first term is negative and the second one is positive. In both cases calculations (see Table 5.1 and results (7.42)) shows presence of a critical point of a phase transition, in the vicinity of which total vacuum energy density is close to zero. In QCD it corresponds to value of average strong coupling

$$\bar{\alpha}_s \;\simeq\; 0.88, \tag{8.3}$$

which corresponds to value of parameter ϵ is according to result (4.37) somewhat less than $\epsilon = 0.13$. In the electroweak case according to (7.42) the point of the phase transition occurs to correspond to value of ϵ also slightly less than $\epsilon = 0.13$. This remarkable result one hardly can prescribe to simple coincidence.

The problem of vacuum energy density due to matter fields is very important for consideration of evolution of the Universe, because this density is directly connected with cosmological λ term. The problem is widely discussed starting from the well-known paper by S. Weinberg [124].

Indeed, the large value for the vacuum energy density means drastic change in the cosmological λ term and thus destroys any reliable model of the Universe evolution. However we need this quantity because it is the most important constituent of the Higgs mechanism. Thus we need almost total cancelation of the vacuum energy density. It may occur provided the real situation corresponds to close vicinity of the phase transition point, where the density is almost zero.

The result of approximate coincidence of phase transition points for the two main matter interactions, the quantum chromodynamics and the electroweak theory may be very important for this the most important ideological problem. We would emphasize, that the result of closeness of these points is achieved just in the theory including spontaneously generated effective interactions. For QCD we have interactions (4.2) and (5.2). For the electroweak theory we have analogous interactions (6.1) and (7.1).

The closeness of the two critical points in our approach is due to the most important feature of the compensation method. Compensation equations always have trivial solutions. And in most cases they do not have any other solution. An appearance of a nontrivial solution is quite a rare event. The existence of the nontrivial solution always impose strong restrictions on parameters of the problem. In this way we have obtained nontrivial solutions for spontaneous generation of effective anomalous three gauge boson interactions (4.2) and (6.1). In both case the existence of a solution is connected with strict fixation of the gauge coupling constant $g(z_0)$ on the boundary z_0 of the nonperturbative region. For the present general discussion it would be instructive to consider peculiarities of emerging of the solutions. Let us return to set of equations (4.18, 4.28), which contain the flexible parameter ϵ. Let us take number of colors $N = 3$. The set of equations for $\epsilon = 0$ has two solutions. The results of calculation are presented in Table 8.1

Table 8.1. Solutions of the compensation equation (6.1) in dependence on parameter ϵ for N=3

ϵ	z_{01}	$g_1(z_{01})$	z_{02}	$g_2(z_{02})$
−0.39	NO	∞	12.7165	0.4344
−0.38	0.00000245	213.46	12.627	0.4358
−0.37	0.00002376	68.7832	12.538	0.4372
−0.3	0.0006909	13.0498	11.931	0.4472
0.0	0.009553	3.8166	9.6175	0.4929
0.3	0.024342	2.5837	7.7065	0.5439
0.6	0.044331	2.1071	6.1246	0.6009
1.0	0.080961	1.7140	4.4249	0.6879
1.4	0.13537	1.5068	3.0873	0.7902
2.1	0.36996	1.2711	1.2753	1.0338
2.2	0.47729	1.2282	0.9963	1.0910
2.256	0.67041	1.1697	0.71186	1.1587
2.257	NO	NO	NO	NO

We would draw attention to several important points. Firstly, considering results, presented in Chapters 4 and 6 on properties of solutions of compensation equation (4.2, 6.1), which essentially are the same for both cases, we state, that for values of parameter ϵ under the study there are two and only two solutions. The first one corresponds to small boundary of the nonperturbative region z_0 and large coupling constant on

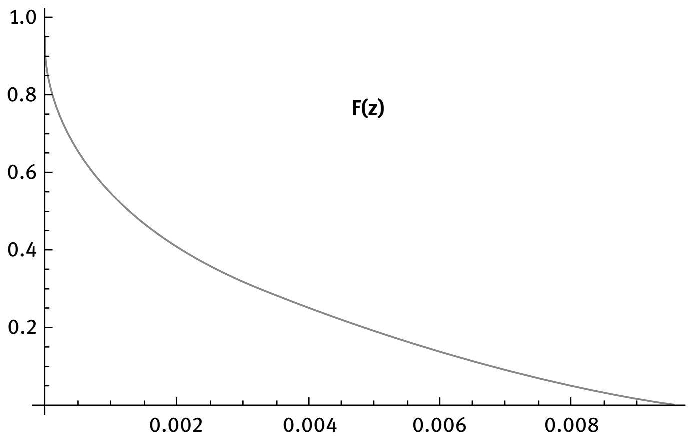


Fig. 8.1. The behavior of the formfactor for QCD.

this boundary $g(z_0)$. *Vise versa* the second one gives large z_0 and small $g(z_0)$. Results, presented in Table 8.1 for wider interval of parameter ϵ give more information. We see from the table, that two solutions exist only for some restricted interval of ϵ. With ϵ increasing the two solutions tend each to other. Somewhere at $\epsilon = \epsilon_0$ between $\epsilon = 2.256$ and $\epsilon = 2.257$ they coincide and for $\epsilon > \epsilon_0$ both solutions disappear. With ϵ decreasing the two solutions move away each from other and $g(z_0$ for the first one grows quickly and tends to infinity at value of ϵ between $\epsilon = -0.38$ and $\epsilon = -0.39$. Thus the first solution in this case also disappears.

We would emphasize, that situation, corresponding to calculated parameter $\epsilon = 0$ and close to this value positive and not large values of ϵ is quite remarkable. We have just two possible Yang–Mills gauge theories. One with large value of coupling g and

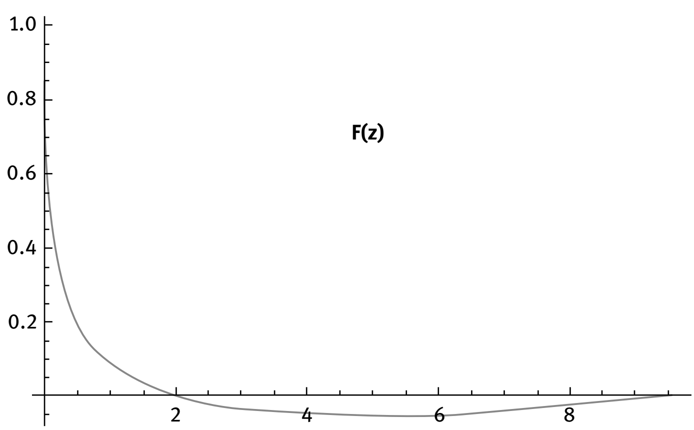


Fig. 8.2. The behavior of the formfactor for electroweak theory.

the other with smaller coupling. NOT MORE! And really we have in the Nature just the strong interaction QCD and the electroweak theory with moderate interaction. The first one has symmetry group $SU(3)$ and the second one has symmetry group $SU(2)$. We prescribe such groups to the obtained theories, however here we follow the ready knowledge. Are there real theoretical arguments for such prescription?

Two solutions are different. The main difference consists in number of zeros of function $F(z)$.

The first solution quoted in Table 8.1 corresponds to the QCD strong interaction with large value $g(z_0)$ and its form-factor F(z) does not change sign between zero and z_0. On the other hand, the second one with smaller value $g(z_0)$, which corresponds to the electroweak interaction has the form-factor with extra zero between zero and z_0. With parameter ϵ increasing values of $g(z_0)$ for two solutions tends each to other and do coincide at ϵ between 2.256 and 2.257. At the same point positions of zeroes z_0 of formfactors coincide also and for $\epsilon > 2.257$ both solutions disappear. When parameter ϵ decreases being negative, the QCD coupling increases very fast and at a point between -0.38 and -0.39 becomes infinite and the solution also disappears. We would show and discuss these properties to illustrate our general conclusion, that the existence of a nontrivial solution of a compensation equation is really a rare event and in the most cases there is only trivial solutions. Of course, parameter ϵ is by no means a flexible parameter. It is fixed in the framework of the approximation being used. Our main approximation means $\epsilon = 0$. However, we have tried other close values for ϵ mainly in view of a check of a stability of our results. We would emphasize the following conclusion based on application of these attempts. Firstly, as we have discussed just above, our value $\epsilon = 0$ hits in the narrow region of the existence of the nontrivial solution. Secondly, as we have shown in Chapter 4 the best agreement with data are achieved for small deviations from this calculated approximate value: $0.1 < \epsilon < 0.15$, that also corresponds to the stable existence area of the nontrivial solutions.

The existence of the electroweak anomalous three-boson effective interaction (6.1) leads to several consequences. The resonances or bound states consisting of $W\,W$ are predicted. Maybe, some signs of these states are already present in the data. It seems, that the particular importance provides result of Section 6.4. In this section we show, that the existence of the three-boson interaction (6.1) leads to additional contribution to the muon anomalous magnetic moment. The calculation with already defined parameters and the form-factor gives contribution, which removes the discrepancy in measurement of the muon $g-2$. The author is inclined to consider this result as a decisive confirmation of the approach. As a matter of fact, the problems with magnetic moments in many cases play the decisive role in a confirmation of a theory. Let us remind, that in QED such role was plaid by calculation of anomalous magnetic moment of the electron a_e (1.55). In QCD understanding of magnetic moments (also anomalous) of the nucleons was crucial for establishing of the new quantum number color, as we have described in Section 1.2. The author express his conviction, that just the result of Section 6.4 on the muon anomalous magnetic moment may serve for justification of the nonperturbative approach with the spontaneously generated effective interactions.

Of course, these results are obtained in the framework of the approximated calculations.

In application to the electroweak interaction of quarks we have considered only the heavy quarks (t, b) and the spontaneous generation of the t-quark mass only. In the present approximation all other quarks including b are massless. The real case need consideration of all three quark and lepton generations. This means also introduction of mixing angles. The problem is which are conditions, imposed by the method, and on what extent mixing angles and fermion masses are defined. In any case the previous results give reason to expect here interesting results.

We would also mention the result for current mass of light quarks. The result is essentially based on the extremely small value $u_0 \simeq 1.7 \cdot 10^{-8}$ (5.14), which necessarily follows from the unique solution for compensation equation (5.8). Such small number illustrates two important conclusion. Firstly, not all dimensionless parameters entering the description of the physics have to be of the order of magnitude of unity. The author encounters such assertion repeatedly, starting from the student time. However the example under discussion shows, that very small numbers may appear as solutions of equations, describing physical problems. In our case we deal with compensation equations, which have nontrivial solutions only provided numerous conditions being fulfilled. The appearance of such small numbers is by no means due the so called fine tuning. It follows just from the solution of the problem.

The compensation approach allows to fix at least few parameters of the Standard Model from those 25, being discussed in Section 2.2. We mean both coupling constants, the strong one and the electroweak one. We also calculate the t-quark mass in terms of the W boson mass. These results lessen number of parameters by 3. One dare to express well-founded hope, that subsequent studies will lead to achievement of a further information on these parameters. Of course it needs extensive efforts *e.g.* in introduction of nonzero masses of other quarks in compensation equations of type (7.8). In the next section we would try to consider simplified examples, which could show some features of possible results.

8.2 Examples of additional relations in the compensation approach

Following the approach used in Section 3.3 let us formulate the compensation equations for would-be four-fermion interaction of two types of quarks and two leptons, that is we consider one generation of fundamental fermions. For simplicity we call them “u”, “d”, “e” and “ν”, which in the standard way are represented by their left ψ_L and right ψ_R components. We admit initial masses for all participating fermions to be zero and we will look for possibility of them to acquire masses $m_i, i = 1, ...4$ respectively due to interaction with scalar Higgs-like composite field.

Then let us consider a possibility of spontaneous generation of the following interaction

$$\begin{aligned} L_{eff} = {} & G_1 \bar{u}_L\, u_R\, \bar{u}_R\, u_L + G_2 \bar{d}_L\, d_R\, \bar{d}_R\, d_L + G_4 \bar{e}_L\, e_R\, \bar{e}_R\, e_L \\ & + G_3 (\bar{u}_L\, u_R\, \bar{d}_R\, d_L + \bar{d}_L\, d_R\, \bar{u}_R\, u_L) + G_5 (\bar{u}_L\, u_R\, \bar{e}_R\, e_L + \bar{e}_L\, e_R\, \bar{u}_R\, u_L) \\ & + G_6 (\bar{e}_L\, e_R\, \bar{u}_R\, u_L + \bar{e}_R\, e_L\, \bar{u}_L\, u_R) + G_7\, \bar{\nu}_L\, \nu_R\, \bar{\nu}_R\, \nu_L \\ & + G_8 (\bar{\nu}_L\, \nu_R\, \bar{d}_R\, d_L + \bar{d}_L\, d_R\, \bar{\nu}_R\, \nu_L) + G_9 (\bar{\nu}_L\, \nu_R\, \bar{u}_R\, u_L + \bar{u}_L\, u_R\, \bar{\nu}_R\, \nu_L) \\ & + G_{10} (\bar{\nu}_L\, \nu_R\, \bar{e}_R\, e_L + \bar{e}_L\, e_R\, \bar{\nu}_R\, \nu_L). \end{aligned} \tag{8.4}$$

Here all coupling constants G_i have dimension of the inverse mass squared M^{-2}.

Now we would like to find out, if the four-fermion interaction (8.7) could be spontaneously generated. In doing this we again proceed with the add-subtract procedure

$$\begin{aligned} L &= L_0 + L_{int}, \\ L_0 &= \sum_{u,d} \bar{q}(x)(\imath\partial_\alpha\gamma_\alpha - m)q(x) + \sum_{e,\nu} \bar{l}(x)(\imath\partial_\alpha\gamma_\alpha - m)l(x) - L_{eff}, \\ L_{int} &= L_{0int} + L_{eff}. \end{aligned} \tag{8.5}$$

Then we have to compensate the undesirable term L_{eff} in the newly defined free Lagrangian. The relation, which serve to accomplish this goal, is called compensation equation. Necessarily we use approximate form of this equation. In diagram form the compensation equation for three fermions participating the interaction in one-loop approximation is presented in Figure 8.3.

Let us define effective cut-off Λ in integrals of equation (8.7). We shall see below, that Λ may be defined in the course of solution of compensation equations. With account of this definition we introduce the following dimensionless variables

$$\begin{aligned} & y_1 = \frac{G_1\,\Lambda^2}{8\,\pi^2}, \quad y_2 = \frac{G_2\,\Lambda^2}{8\,\pi^2}, \quad y_3 = \frac{G_3\,\Lambda^2}{8\,\pi^2}, \\ & z_1 = \frac{G_4\,\Lambda^2}{8\,\pi^2}, \quad z_2 = \frac{G_7\,\Lambda^2}{8\,\pi^2}, \quad z_3 = \frac{G_{10}\,\Lambda^2}{8\,\pi^2}, \\ & x_1 = \frac{G_5\,\Lambda^2}{8\,\pi^2}, \quad x_2 = \frac{G_9\,\Lambda^2}{8\,\pi^2}, \quad x_3 = \frac{G_6\,\Lambda^2}{8\,\pi^2}, \quad x_4 = \frac{G_8\,\Lambda^2}{8\,\pi^2}, \\ & \xi_1 = \frac{m_2}{m_1}, \qquad \xi_2 = \frac{m_3}{m_1}, \qquad \xi_3 = \frac{m_4}{m_1}. \end{aligned} \tag{8.6}$$

Then we consider scalar bound state consisting of all possible fermion-antifermion combinations $\bar{u}u$, $\bar{d}d$, $\bar{e}e$ and $\bar{\nu}\nu$. The corresponding set of Bethe–Salpeter equations is shown in Figure 8.4. In this way we come to the following set of ten compensation equations presented in Figure 8.3 and four Bethe–Salpeter equations shown in Figure 8.4. Let us note, that in Figure 8.4 we present also wouldbe contributions of gauge bosons exchanges, which in the present calculations are not taken into

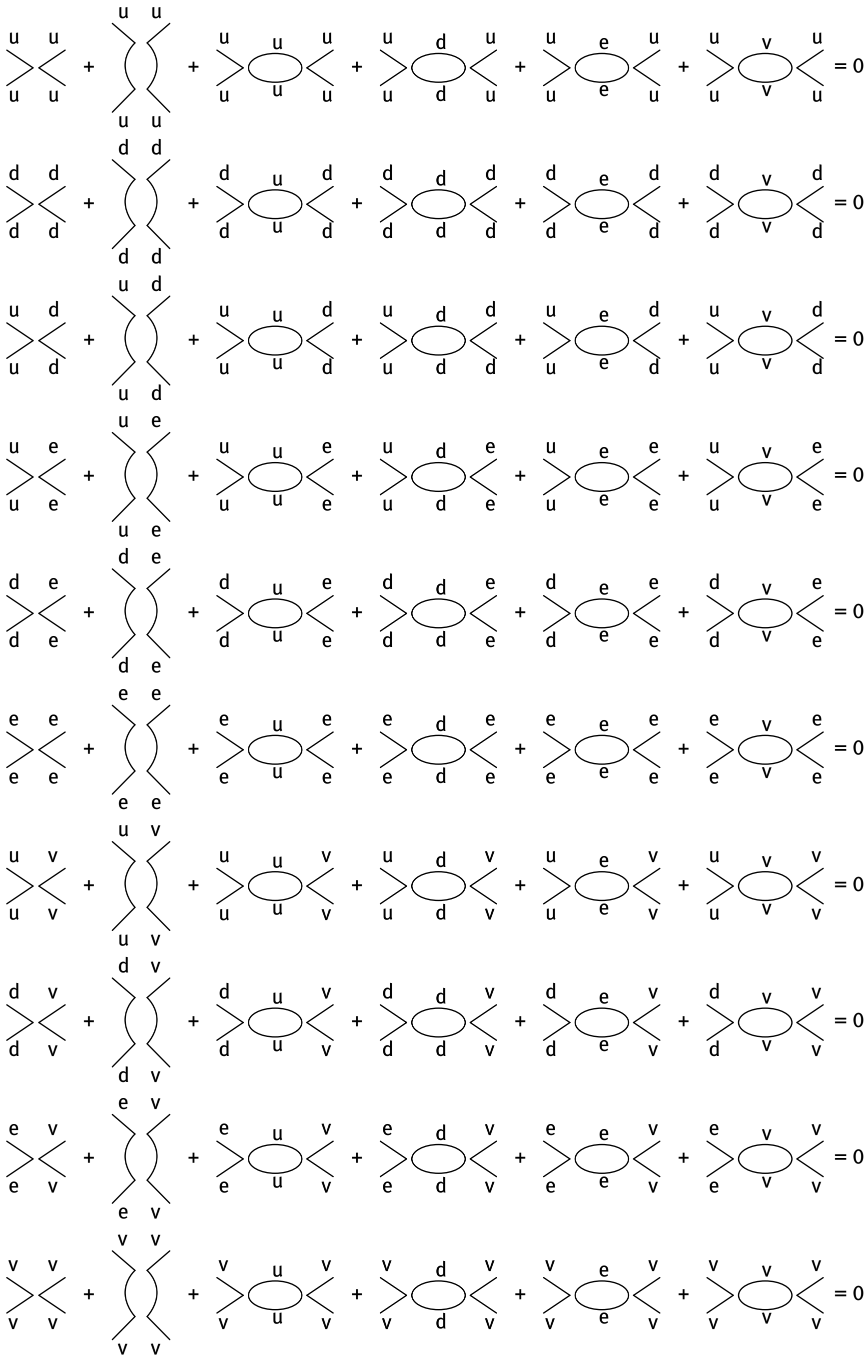


Fig. 8.3. Diagram representation of the compensation equation for spontaneous generation of interaction (8.7). Notations of quarks and lepton are shown by corresponding lines.

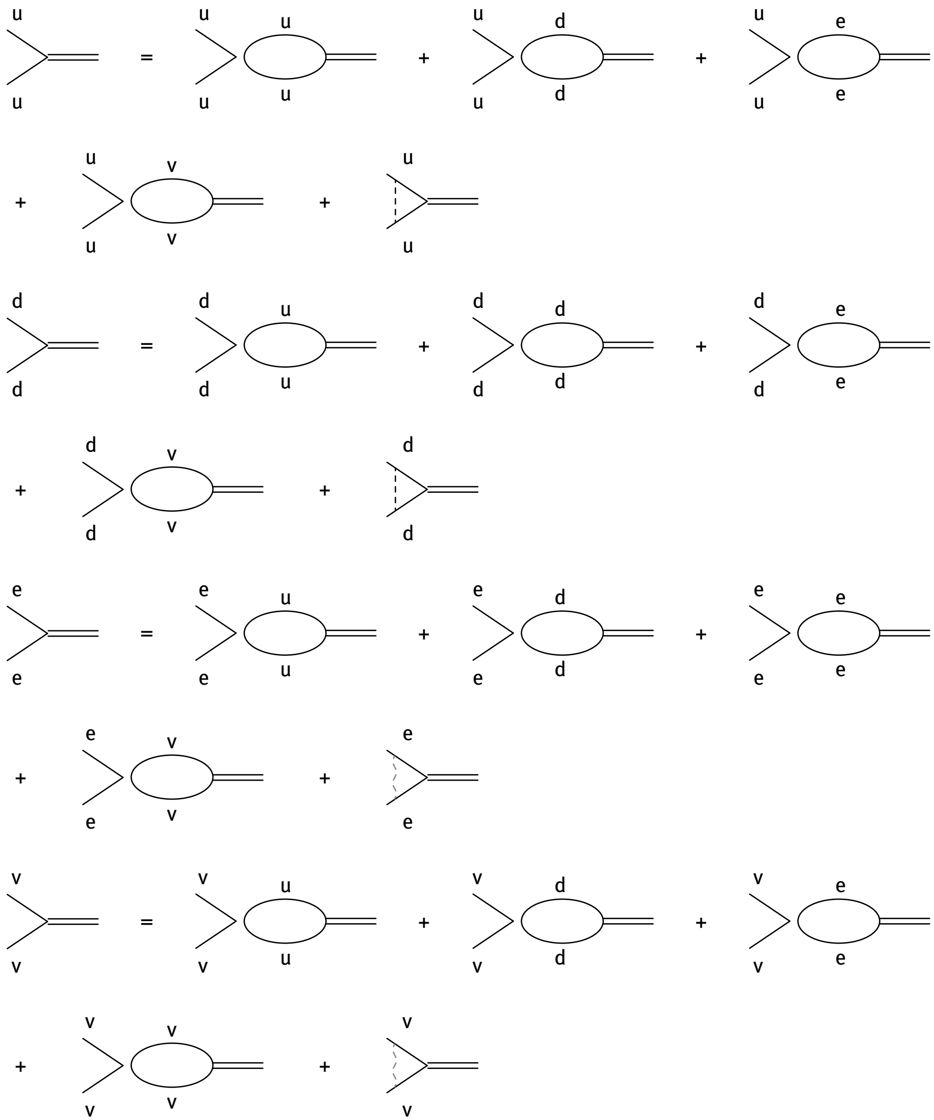


Fig. 8.4. Diagram representation of the Bethe–Salpeter equation for scalar bound state, included in set of equations (8.8). Notations of quarks and lepton are shown by corresponding lines. Contributions of gauge bosons exchanges (the last diagrams in each equation are not taken into account yet).

account. Note also, that terms with factor A arise from vertical diagrams in Figure 8.3. Let us remind, that the sign minus before linear terms in compensation equations is connected with opposite signs of terms corresponding to effective interactions in the new free Lagrangian and in the new interaction Lagrangian.

$$\begin{aligned}
-y_1 + A\,y_1^2 + 3(y_1^2 + y_3^2) + x_1^2 + x_2^2 &= 0,\\
-y_2 + A\,y_2^2\,\xi_1^2 + 3(y_2^2 + y_3^2) + x_3^2 + x_4^2 &= 0,\\
-y_3 + A\,y_3^2\,\xi_1 + 3\,y_3(y_1 + y_2) + x_1\,x_3 + x_2 x_4 &= 0,\\
-z_1 + A\,z_1^2\,\xi_2^2 + 3\,(x_1^2 + x_3^2) + z_1^2 + z_3^2 &= 0,\\
-z_2 + A\,z_2^2\,\xi_3^2 + 3\,(x_2^2 + x_4^2) + z_2^2 + z_3^2 &= 0,\\
-z_3 + A\,z_3^2\,\xi_2\,\xi_3 + 3\,(x_1\,x_2 + x_3\,x_4) + z_1 z_3 + z_2 z_3 &= 0,\\
-x_1 + A\,x_1^2\,\xi_2 + 3(x_1 y_1 + x_3 y_3) + x_1 z_1 + x_2 z_3 &= 0,\\
-x_2 + A\,x_2^2\,\xi_3 + 3(x_2 y_1 + x_3 y_3) + x_1 z_1 + x_2 z_3 &= 0,\\
-x_3 + A\,x_3^2\,\xi_1\,\xi_2 + 3(x_1 y_3 + x_4 y_3) + x_1 z_3 + x_2 z_2 &= 0,\\
-x_4 + A\,x_4^2\,\xi_1 \xi_3 + 3(x_2 y_3 + x_4 y_2) + x_3 z_3 + x_4 z_2 &= 0,\\
A = \frac{m_1^2}{4\,\Lambda^2}\,\ln\frac{\Lambda^2}{\bar m^2}&,
\end{aligned} \tag{8.7}$$

$$\begin{aligned}
\frac{1}{B} &= 3(y_1 + \xi_1\,y_3) + \xi_2 x_1 + \xi_3 x_2,\\
\frac{\xi_1}{B} &= 3(y_3 + \xi_1\,y_2) + \xi_2\,x_3 + \xi_3 x_4,\\
\frac{\xi_2}{B} &= 3(x_1 + \xi_1\,x_3) + \xi_2\,z_1 + \xi_3 z_3,\\
\frac{\xi_3}{B} &= 3(x_2 + \xi_1\,x_4) + \xi_2\,z_3 + \xi_3 z_2,\\
B &= 1 + \frac{m_0^2}{2\,\Lambda^2}\,\ln\frac{\Lambda^2}{\bar m^2},
\end{aligned} \tag{8.8}$$

where m_0 is the bound state mass and $\bar m$ is an average mass of participating fermions. Let us comment the appearance of mass parameters ξ_i in terms, corresponding to vertical diagrams in Figure 8.3. Due to the orthogonality of matrices

$$\frac{1+\gamma_5}{2},\quad \frac{1-\gamma_5}{2}, \tag{8.9}$$

terms containing $\hat q$ cancel and we are left only with mass terms in spinor propagators. Introduction of the average $\bar m$, instead of substituting in proper places different masses m_i, means of course an approximation. However due to logarithmic dependence on this parameter, this approximation seems to be reasonable. Factor A has to be very small and factor B has to be close to unity, because $\Lambda \gg m_i$. Ten equations (8.7) correspond to the set of compensation equations, while four equations (8.8) represent the Bethe–Salpeter equations. Let us remind, that after performing the compensation procedure, which means exclusion of four-fermion vertices in the newly defined free Lagrangian, we use the resulting coupling constants in the newly defined interaction Lagrangian with the opposite sign.

The appearance of ratios ξ_i in Bethe–Salpeter part (8.8) of the set presumably needs explanation. We assume, that the scalar composite state, which in our approach serves as a substitute of the elementary Higgs scalar, consists of all existing quark-antiquark and lepton-antilepton pairs $\bar{\psi}_L\,\psi_R$ (not only of heavy quarks $\bar{\Psi}_L\,\Psi_R$ as in work [100]). Then coupling of this scalar with different fermions will give their masses according to well known relation

$$g_a = \frac{g\,m_a}{\sqrt{2}\,M_W}. \tag{8.10}$$

On the other hand, Bethe–Salpeter wave functions are proportional to coupling constants g_a, where a is just the constituent particle. Thus we change a ratio of coupling constants by a ratio of corresponding masses ξ_i.

It seems advisable to refer here to the experience, acquired while considering the subsequent approximations in studies of the Nambu–Jona-Lasinio interaction. The first approximation, which was considered in Section 3.3 contains only horizontal diagrams. It leads to results, which reproduce the main features of the model. The next approximation, developed in Chapter 5, is of course more informative, but nevertheless the first one seems to be quite reasonable.

Now let us consider solutions of set (8.7, 8.8). First of all let us remind, that parameter A is very small, so we look for solutions, which are stable in the limit $A \to 0$. We also will consider only real solutions, because our variables just correspond to physical observable quantities. A number of such solutions is at least six. Namely, we have for $A = 0.0001$ these solutions

$$\begin{array}{llll} y_1 = 0.12500, & y_2 = y_1, & y_3 = -y_1, & \\ z_1 = y_1, & z_2 = y_1, & z_3 = -y_1, & \\ x_1 = y_1, & x_2 = -y_1, & x_3 = -y_1, & x_4 = y_1, \\ \xi_1 = -1, & \xi_2 = 1, & \xi_3 = -1, & B = 1.00001. \end{array} \tag{8.11}$$

$$\begin{array}{llll} y_1 = 0.12500, & y_2 = y_1, & y_3 = -y_1, & \\ z_1 = y_1, & z_2 = y_1, & z_3 = y_1, & \\ x_1 = y_1, & x_2 = y_1, & x_3 = -y_1, & x_4 = -y_1, \\ \xi_1 = -1, & \xi_2 = 1, & \xi_3 = 1, & B = 1.00001. \end{array} \tag{8.12}$$

$$\begin{aligned}
&y_1 = 0.24999, && y_2 = 0.33333, && y_3 = 0, \\
&z_1 = 0.24999, && z_2 = 0.56468, && z_3 = -0.38570, \\
&x_1 = -0.24999, && x_2 = x_3 = x_4 = 0, && \\
&\xi_1 = 0.86603, && \xi_2 = -1, && \xi_3 = 0, && B = 1.00003.
\end{aligned} \tag{8.13}$$

$$\begin{aligned}
&y_1 = 0.33332, && y_2 = 0.057288, && y_3 = 0, \\
&z_1 = 0.26344, && z_2 = 0.56470, && z_3 = -0.38570, \\
&x_1 = x_2 = 0, && x_3 = 0.12285, && x_4 = -0.17986, \\
&\xi_1 = \xi_2 = \xi_3 = 0, && B = 1.00003.
\end{aligned} \tag{8.14}$$

$$\begin{aligned}
&y_1 = 0.29077, && y_2 = 0.29077, && y_3 = -0.04256, \\
&z_1 = 0.25534, && z_2 = 0, && z_3 = 0, \\
&x_1 = 0.17801, && x_2 = x_4 = 0, && x_3 = 0.17801, \\
&\xi_1 = 1, && \xi_2 = 1.4344, && \xi_3 = 0, && B = 1.00003.
\end{aligned} \tag{8.15}$$

$$\begin{aligned}
&y_1 = 0.19313, && y_2 = 0.18758, && y_3 = 0.14295, \\
&z_1 = 0.857858, && z_2 = 0, && z_3 = 0, \\
&x_1 = -0.14116, && x_2 = x_4 = 0, && x_3 = 0.14393, \\
&\xi_1 = 1.069, && \xi_2 = 0.26728, && \xi_3 = 0, && B = 1.00002.
\end{aligned} \tag{8.16}$$

Of course, there is a temptation to confront these solutions with the existing generations of quarks and leptons. Let us note, that the fist three solutions (8.11, 8.12, 8.13) contain mass ratios ξ_i with negative signs, that is quite unnatural for fermions entering to one generation. Maybe the most suitable ones are the three last solutions (8.14, 8.15, 8.16). All these solutions have nonnegative parameters ξ_i and at least one lepton being massless, that might be a neutrino. The fourth solution (8.14) gives one (the first) fundamental fermion (quark) being much heavier, than three others, that reminds situation of the third generation with the very heavy t quark. The fifth solution (8.15) gives charged lepton mass approximately the same as those of quarks, that may hint the situation in the second generation with approximately equal masses of the muon and of the s-quark. The sixth solution (8.16) gives two different masses for the quark pair, while the wouldbe charged lepton has the mass approximately four times smaller than that of the first quark. This resembles situation for the first generation. Indeed, let us take for the electron mass its physical value $m_e = 0.51\,\mathrm{MeV}$ Then we have from (8.16)

$$m_e = 0.51\,\mathrm{MeV}, \quad m_u = \frac{m_e}{\xi_2} = 1.90\,\mathrm{MeV}, \quad m_d = \frac{m_e\,\xi_1}{\xi_2} = 2.04\,\mathrm{MeV}. \tag{8.17}$$

The wouldbe u-quark mass fits into error bars of its definition, while the wouldbe d-quark mass is rather lighter than its physical value [4]. Note, that in our estimates we have not taken into account the phenomenon of mixing of down quarks (d, s, b).

Of course, the similarity is rather reluctant and there is no overall explicit agreement with the real situation. Maybe one could move further with an application of a next approximation, which presumably needs a consideration of the Bethe–Salpeter equations with account of gauge interactions contributions, that is with account of a gluon exchange and of electroweak bosons exchanges. These exchanges are schematically drawn in Figure 8.4. The problem of an adequate formulation of the approximation needs a special investigation. Nevertheless, even a possibility to define ratios of the fundamental masses in the compensation approach is of a doubtless interest.

We would also draw attention to the important point, that for all solutions parameter B is close to unity, just as we have expected. With decreasing of parameter A, which is proportional to ratio squared of the mass of the first quark and cut-off Λ, parameter B tends to unity exactly.

It might be also worth mentioning, that we obtain three possible solutions for fermion generations, that just corresponds to the current knowledge.

The example being just considered shows possibility of definition of mass ratios in the compensation approach. There are also mixing angles in the Standard Model, e. g., the Weinberg angle θ_W in W^0, B mixing and down quarks mixing angles and a phase in the Cabibbo–Kobayashi–Maskawa matrix. The example being just considered shows possibility of definition of masses ratios in the compensation approach. There are also mixing angles in the Standard Model, e. g., the Weinberg angle θ_W in W^0, B mixing (2.42) and lower quarks mixing angles and a phase in the Cabibbo–Kobayashi–Maskawa matrix (2.3).

In view of this, let us consider another example, which deals with mixed states of s and d quarks, which mixing is described by the well-known Cabibbo relation [48]

$$q_1 = \cos\theta_c\, d + \sin\theta_c\, s, \;\; q_2 = -\sin\theta_c\, d + \cos\theta_c\, s. \tag{8.18}$$

where quarks q_i $(i = 1, 2)$ are lower members of quark doublets of the two first generations. Then in the test effective interaction we have to introduce terms with strangeness nonconservation. Thus let us take for the study the following four-fermion effective interaction of quarks q_i

$$\begin{aligned} L_{eff} = {} & G_2 \bar{q}_{1L} q_{1R}\, \bar{q}_{1R} q_{1L} + G_5 \bar{q}_{2L} q_{2R}\, \bar{q}_{2R} q_{2L} + G_{13}(\bar{q}_{1L} q_{2R}\, \bar{q}_{1R} q_{2L} + \bar{q}_{2L}\, q_{1R}\, \bar{q}_{2R}\, q_{1L}) \\ & + G_{10}(\bar{q}_{1L}\, q_{1R}\, \bar{q}_{2R}\, q_{2L} + \bar{q}_{2L}\, q_{2R}\, \bar{q}_{1R}\, q_{1L}) \\ & + G_{11}(\bar{q}_{2L}\, q_{1R}\, \bar{q}_{1R}\, q_{1L} + \bar{q}_{1L}\, q_{1R}\, \bar{q}_{2R}\, q_{1L} + \bar{q}_{1L}\, q_{1R}\, \bar{q}_{1R}\, q_{2L} + \bar{q}_{1L}\, q_{2R}\, \bar{q}_{1R}\, q_{1L}) \\ & + G_{12}(\bar{q}_{1L}\, q_{2R}\, \bar{q}_{2R}\, q_{2L} + \bar{q}_{2L}\, q_{2R}\, \bar{q}_{1R}\, q_{2L} + \bar{q}_{2L}\, q_{1R}\, \bar{q}_{2R}\, q_{2L} + \bar{q}_{2L}\, q_{2R}\, \bar{q}_{2R}\, q_{1L}), \end{aligned} \tag{8.19}$$

where as usually the color summation is meant inside each antiquark-quark term. Due to strangeness nonconservation the quadratic term in the corresponding Lagrangian contains also a mixing term

$$\Delta L = -m_1\, \bar{q}_1\, q_1 - m_1\, \xi\, \bar{q}_2\, q_2 - m_1\, \delta\, (\bar{q}_1\, q_2 + \bar{q}_2\, q_1). \tag{8.20}$$

Now the set of compensation equations, which is analogue of (8.7) with notations (8.6, 8.20) looks like

$$
\begin{aligned}
&- y_2 + 3\,(y_2^2 + y_{10}^2 + 2y_{11}^2) + A(y_2^2 + 2\,\xi\, y_{11}^2 + \xi^2 y_{13}^2 \\
&\qquad\qquad + 2\,\delta\, y_{11}(\delta\, y_{11} - 2\,y_2) + 2\,\delta\, y_{13}(\delta\, y_2 - 2\,y_{11})) = 0, \\
&- y_5 + 3\,(y_5^2 + y_{10}^2 + 2y_{12}^2) + A(y_{13}^2 + 2\,\xi\, y_{12}^2 + \xi^2 y_5^2 \\
&\qquad\qquad + 2\,\delta\, y_{12}(\delta\, y_{12} - 2\,y_5) + 2\,\delta\, y_{13}(\delta\, y_5 - 2\,y_{12})) = 0, \\
&- y_{13} + 3\,(y_{13}^2 + y_{11}^2 + y_{12}^2) + A(y_2\, y_{13} + 2\,\xi\, y_{11} y_{12} + \xi^2\, y_5 y_{13} \\
&\qquad + \delta^2 (y_2\, y_5 + y_{13}^2 + 2\,y_{11}\, y_{12}) - 2\,\delta(y_2\, y_{12} + y_{11} y_{13} + y_5\, y_{11} + y_{12}\, y_{13})) = 0, \\
&- y_{10} + 3\,(y_2\, y_{10} + y_5\, y_{10} + 2\,y_{11}\, y_{12}) + A(y_{11}^2 + \xi(y_{10}^2 + y_{13}^2) \\
&\qquad + \xi^2\, y_{12}^2 + 2\,\delta\, y_{12}(\delta\, y_{11} - 2\,y_{10}) + 2\,\delta\, y_{13}(\delta\, y_{10} - 2\,y_{11})) = 0, \\
&- y_{11} + 3\,(y_2\, y_{11} + y_{12}\, y_{10} + y_{11}\, y_{13}) + A(y_2\, y_{11} + \xi\, y_{11}(y_{10} + y_{13}) \\
&\qquad + \xi^2\, y_{12}\, y_{13} + \delta^2 (y_2\, y_{12} + y_{10}\, y_{11} + 2\,y_{11}\, y_{13}) \\
&\qquad\qquad - \delta(y_2\, y_{10} + y_2 y_{13} + y_{10}\, y_{13} + y_{13}^2 + 2\,y_{11}\, y_{12} + 2\,y_{11}^2)) = 0, \\
&- y_{12} + 3\,(y_5\, y_{12} + y_{11}\, y_{10} + y_{12}\, y_{13}) + A(y_{13}\, y_{11} + \xi\, y_{12}(y_{10} + y_{13}) \\
&\qquad + \xi^2\, y_{12}\, y_5 + \delta^2 (y_5\, y_{11} + y_{10}\, y_{12} + 2\,y_{12}\, y_{13}) \\
&\qquad\qquad - \delta(y_5\, y_{10} + y_5\, y_{13} + y_{10}\, y_{13} + y_{13}^2 + 2\,y_{12}^2 + 2\,y_{11}\, y_{12})) = 0.
\end{aligned}
\tag{8.21}
$$

The mixing angle (8.18) is defined by parameters ξ, δ according to the following relation

$$
\delta(1 - 2\,sin^2\phi_c) = -\ \sin\phi_c\,(1 - \xi)\,\sqrt{1 - sin^2\phi_c}. \tag{8.22}
$$

Masses of physical states d, s are the following

$$
m_d = \frac{m_1(\cos^2\phi_c - \xi\sin^2\phi_c)}{\cos^2\phi_c - \sin^2\phi_c}, \qquad m_s = \frac{m_1(\xi\cos^2\phi_c - \sin^2\phi_c)}{\cos^2\phi_c - \sin^2\phi_c}. \tag{8.23}
$$

The set of Bethe–Salpeter equations is now the following

$$
\begin{aligned}
\frac{1}{B} &= 3(y_2 + \xi\, y_{10} + 2\,\delta\, y_{11}), \\
\frac{\xi}{B} &= 3(\xi\, y_5 + y_{10} + 2\,\delta\, y_{12}), \\
\frac{\delta}{B} &= 3(y_{11} + \xi\, y_{12} + \delta\, y_{13}).
\end{aligned}
\tag{8.24}
$$

Remind, that parameters A and B are already defined in (8.7, 8.8). Now, there are numerous solutions of set of equations (8.21, 8.24). The most of them are complex and thus inadmissible from the physical point of view. There are also solutions with parameter B, which differs significantly from unity, that contradicts to definition (8.8) of this quantity. Among the real ones with $B \simeq 1$ there is a lot of solutions with either $\delta = 0$ or $\xi = 1$. According to relation (8.22) the first possibility corresponds to zero mixing $\sin\phi_c = 0$ and the second possibility corresponds to maximal mixing $\sin\phi_c = \pm\frac{1}{\sqrt{2}}$. There are few solutions with values of δ and ξ, which differ from $\delta = 0$ and $\xi = 1$. Let us show two examples of these solutions. The first one

$$\begin{aligned} &B = 1.00000333, && \xi = -1, && \delta = -2.538663 \cdot 10^{-7}, \\ &y_2 = 0.33333222, && y_5 = 0.33333222, && y_{13} = 1.07460479 \cdot 10^{-13}, \\ &y_{10} = 1.0736719 \cdot 10^{-14}, && y_{11} = -4.31859573 \cdot 10^{-8}, && y_{12} = 4.14358583 \cdot 10^{-8}. \end{aligned} \tag{8.25}$$

The second one

$$\begin{aligned} &B = 1.00000333, && \xi = 2.148573 \cdot 10^{-14}, && \delta = 1.4655801 \cdot 10^{-7}, \\ &y_2 = 0.33333222, && y_5 = 0.3333333 && y_{13} = 7.1707928 \cdot 10^{-15}, \\ &y_{10} = 5.05749407 \cdot 10^{-16}, && y_{11} = 4.885245 \cdot 10^{-8}, && y_{12} = -1.72524533 \cdot 10^{-9}. \end{aligned} \tag{8.26}$$

For solutions (8.25, 8.26) we have from relations (8.22, 8.23) respectively

$$\sin\phi_c = \pm 1.2693 \cdot 10^{-7}, \quad m_d = -m_s = m_1, \tag{8.27}$$

$$\begin{aligned} \sin\phi_c &= \pm 1.4658 \cdot 10^{-7}, \quad m_d = m_1, \\ m_s &= -3.64595 \cdot 10^{-23} m_1, \end{aligned} \tag{8.28}$$

We see here nonzero, but extremely small mixing angles. In the second case (8.28) we obtain also an extremely small second mass. These results can not be related to some real situation. However, they demonstrate, that physical parameters, namely mass ratios and mixing angles, could be, in principle, defined in the framework of the compensation approach. As a matter of fact, set of equations (8.21, 8.24) is oversimplified and do not take into account upper quarks (u and c). We have seen while considering one generation case above in this section, that an inclusion of all quarks and leptons leads to physically reasonable results on the one hand, however to a considerable complication of the problem on the other hand.

Provided one tries to introduce also other members of two generations into consideration, this lead to an essential increase in a number of equations of the set. The more so, as to approach the real situation one has to take into account three generations. Then the number of equations approaches a hundred. An analysis of solutions of such set of nonlinear equations needs a dedicated and extensive work. Examples, which were considered in the present section may just serve as a motivation for accomplishing this forthcoming work.

8.3 Weinberg mixing angle and the fine structure constant

To conclude the considerations of possibilities of determination of parameters of the Standard Model, let us demonstrate a simple model, which illustrates how the well-known Weinberg mixing angle could be defined. In previous chapters N. N. Bogoliubov compensation principle [41, 42] was applied to studies of a spontaneous generation of effective nonlocal interactions in renormalizable gauge theories.

In particular, Chapter 6 deals with an application of the approach to the electro-weak interaction and a possibility of spontaneous generation of effective anomalous three-boson interaction of the form

$$-\frac{G}{3!}\,F\,\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}, \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc} W^b_\mu W^c_\nu. \tag{8.29}$$

with uniquely defined form-factor $F(p_i)$, which guarantees effective interaction (8.53) acting in a limited region of the momentum space. It was done of course in the framework of an approximate scheme, which accuracy was estimated in Chapter 3 to be $\simeq$ (10–15)%. Would-be existence of effective interaction (8.29) leads to important non-perturbative effects in the electro-weak interaction. Remind, that our interaction constant G is connected with conventional definitions in the following way

$$G = -\frac{g\,\lambda}{M_W^2}, \tag{8.30}$$

where $g \simeq 0.65$ is the electro-weak coupling. The current limitations for parameter λ read [106]

$$\lambda = -\,0.016^{+0.021}_{-0.023}, \quad -\,0.059 < \lambda < 0.026\,(95\%\,C.L.). \tag{8.31}$$

Solutions of the compensation equation correspond to QCD with $g(z_0) = 3.8$ and for the electroweak interaction with [100, 103]

$$g(z_0) = 0.60366, \quad z_0 = 9.6175, \quad |\lambda| = 2.88 \cdot 10^{-6}. \tag{8.32}$$

Value of boundary momentum, that is effective cut-off Λ is defined by expression [100, 103]

$$\frac{2\,G^2\,\Lambda^4}{1024\,\pi^2} = \frac{2\,g^2\,\lambda^2\,\Lambda^4}{1024\,\pi^2\,M_W^4} = z_0. \tag{8.33}$$

As a rule the existence of a nontrivial solution of a compensation equation impose essential restrictions on parameters of a problem. Just the example of these restrictions is the definition of coupling constant $g(z_0)$ in (8.32). It is advisable to consider other possibilities for spontaneous generation of effective interactions and to find out, which restrictions on physical parameters may be imposed by an existence of nontrivial solutions.

To begin the considerations of the present section, let us demonstrate a simple model, which illustrates how the well-known Weinberg mixing angle could be defined.

Let us consider a possibility of a spontaneous generation of the following quartic effective interaction of electroweak gauge bosons

$$L_{eff} = G_1\, W^a_\mu W^d_\mu\, W^a_{\rho\sigma} W^d_{\rho\sigma} + G_2\, W^a_\mu W^a_\mu W^b_{\rho\sigma} W^b_{\rho\sigma} \\ + G_3\, W^a_\mu W^a_\mu B_{\rho\sigma} B_{\rho\sigma} + G_4\, Z_\mu Z_\mu W^b_{\rho\sigma} W^b_{\rho\sigma} + G_5\, Z_\mu Z_\mu B_{\rho\sigma} B_{\rho\sigma}. \tag{8.34}$$

where we maintain the residual gauge invariance for the electromagnetic field. Here indices a, d correspond to charged W-s, that is they take values 1, 2, while index b corresponds to three components of W defined by the initial formulation of the electroweak interaction. Let us remind the well-known relation, which connect fields W^0, B with physical fields of the Z boson and of the photon

$$W^0_\mu = \cos\,\theta_W\, Z_\mu + \sin\,\theta_W\, A_\mu, \quad B_\mu = -\,\sin\,\theta_W\, Z_\mu + \cos\,\theta_W\, A_\mu, \tag{8.35}$$

Interactions of type (8.34) were earlier introduced on phenomenological grounds in works [122, 123]. Let us introduce an effective cut-off Λ in the same way as we have done earlier while considering the spontaneous generation of effective interactions in QCD and in the electro-weak theory. Here we use for definition of Λ relation (8.33). Here we shall proceed just in the same way as earlier. Then let us consider a possibility of a spontaneous generation of interaction (8.34). Now we would like to find out, if interaction (8.34) could be spontaneously generated. In doing this we again proceed with the add-subtract procedure. Provided we start with usual form of the Lagrangian (6.3), which describes electro-weak gauge fields W^a and B

$$L = L_0 + L_{int},$$
$$L_0 = -\frac{1}{4}(W^a_{0\mu\nu}\, W^a_{0\mu\nu}) - \frac{1}{4}(B_{\mu\nu}\, B_{\mu\nu}), \tag{8.36}$$
$$L_{int} = -\frac{1}{4}(W^a_{\mu\nu}\, W^a_{\mu\nu} - W^a_{0\mu\nu}\, W^a_{0\mu\nu}). \tag{8.37}$$
$$W^a_{0\mu\nu} = \partial_\mu\, W^a_\nu - \partial_\nu\, W^a_\mu, \; B^a_{\mu\nu} = \partial_\mu\, B_\nu - \partial_\nu\, B_\mu.$$

and $W^a_{\mu\nu}$ is the well-known nonlinear Yang–Mills field of W-bosons. Then the we perform the add-subtract procedure of expression (8.34)

$$L = L'_0 + L'_{int},$$
$$L'_0 = L_0 - L_{eff}, \tag{8.38}$$
$$L'_{int} = L_{int} + L_{eff}. \tag{8.39}$$

Now let us formulate compensation equations. We are to demand, that considering the theory with Lagrangian L'_0 (8.38), all contributions to four-boson connected vertices, corresponding to interaction (8.34) are summed up to zero. That is the undesirable interaction part in the would-be free Lagrangian (8.38) is compensated. Then

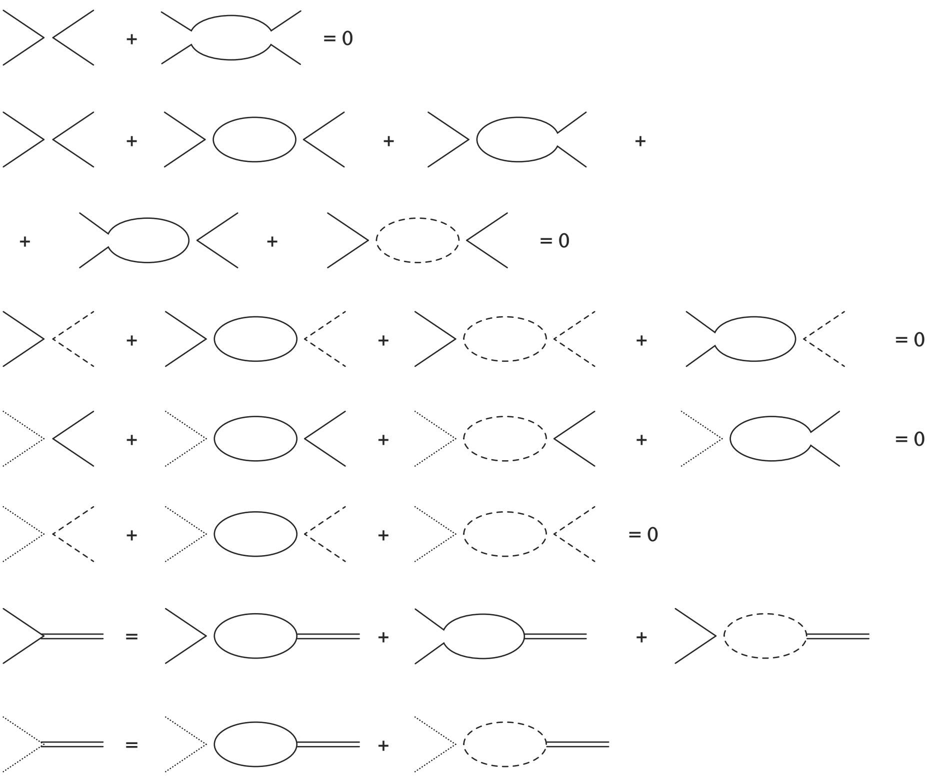


Fig. 8.5. Diagram representation of set (8.40) (the first five equations) and (8.41) (the two last ones). Simple line represent W-s, dotted lines represent B and lines, consisting of black spots, represent Z.

we are rested with interaction (8.34) only in the proper place (8.39) We have the following set of compensation equations

$$\begin{aligned}
&-x_1 + x_1^2 = 0,\\
&-x_2 + 2\,x_2^2 + 2\,x_1 x_2 + (1-a^2)\,x_3 x_4\, a^2\, x_2 x_4 = 0,\\
&-x_3 + x_1 x_3 + 2\,x_2 x_3 + a^2\, x_2\, x_5 + (1-a^2)\,x_3 x_5 = 0,\\
&-x_4 + x_1 x_4 + 2\,x_2 x_4 + a^2\, x_4 x_5 = 0,\\
&-x_5 + 2\,x_3 x_4 + a^2\, x_4\, x_5 + (1-a^2)\,x_5^2 = 0,\\
&x_i = \frac{3\,G_i\,\Lambda^2}{64\,\pi^2}.
\end{aligned} \tag{8.40}$$

Here $a = \cos\theta_W$. Factor 2 in several terms of equations here corresponds to sum by weak isotopic index $\delta^a_a = 2$, $a = 1, 2$. Then following the reasoning of the approach we assume, that the Higgs scalar corresponds to a bound state consisting of a complete set of fundamental particles. Note, that in Chapter 7 we have considered only the heaviest particle t quark as the main constituent of the Higgs scalar. We have to remark, that this assumption could be hardly consistent with value of the Higgs particle mass 125.7 GeV [57, 58]. Here we include the electro-weak bosons in consideration of the Higgs interaction. There are two Bethe–Salpeter equations for this problem, because constituents are either $W^a\,W^a$ or $Z\,Z$. In approximation of very large cut-off Λ these equations have the following form

$$x_1 + (2+a)\,x_2 + \frac{1-a^2}{a}\,x_3 + \beta = 1, \qquad (8.41)$$
$$(2+a)\,x_4 + \frac{1-a^2}{a}\,x_5 + \frac{\beta}{a} = \frac{1}{a},$$

where β describes additional contributions to equations. Now we look for solutions of set (8.40, 8.41) for variables x_i, a, β. We consider as physical solutions those having very small β. Of course, there is the trivial solution with $\beta = 1$: all $x_i = 0$. However there are also nontrivial solutions. Namely, there are the the following two ones with $x_1 = 1$

$$x_2 = 0, \quad x_3 = 0.729625, \quad x_4 = 0, \quad x_5 = 0, \qquad (8.42)$$
$$\beta_1 = 1, \quad \beta_2 = \frac{0.729625\,(a-1)}{a},$$

for any a, and the following three ones with $x_1 = 0$

$$x_2 = 0, \quad x_3 = 3.070337, \quad x_4 = 0, \quad x_5 = 3.61378,$$
$$a = 0.8504594, \quad \beta = -5.06\cdot 10^{-16}, \qquad (8.43)$$
$$x_2 = 0.48772, \quad x_3 = 0, \quad x_4 = 1.2654, \quad x_5 = 0,$$
$$a = 0.33801, \quad \beta = -1.2\cdot 10^{-5},$$
$$x_2 = 0.5, \quad x_3 = 1.09555, \quad x_4 = 0, \quad x_5 = 0,$$
$$a = -0.75556, \quad \beta = 1.$$

Very small $\beta \simeq -5\cdot 10^{-16}$ is appropriate only for the first solution of (8.43). (For solutions (8.42) smallness of β is achieved only for the second one with $a \to 1$, that is in an absence of the mixing.) Thus the only solution with physical meaning is just this one, which gives

$$\sin^2\theta_W = 1 - a^2 = 0.27672. \qquad (8.44)$$

This value corresponds to scale Λ (8.33), which corresponds to parameter z_0. At this scale the electroweak coupling according to (8.32)

$$\alpha_{ew}(z_0) = \frac{g(z_0)^2}{4\pi} = 0.028999 \qquad (8.45)$$

then electromagnetic coupling at the same scale is the following

$$\alpha(z_0) = \alpha_{ew}(z_0)\,\sin^2\theta_W(z_0) = 0.0080244. \qquad (8.46)$$

With the well-known evolution expression for electromagnetic coupling we have for six quark flavors ($\Lambda \gg M_W$)

$$\alpha(z_0) = \frac{\alpha(M_W)}{1 - \frac{5\alpha(M_W)}{6\pi}\ln\left[\frac{\Lambda^2}{M_W^2}\right]} = 0.0080244. \tag{8.47}$$

This gives with value Λ from expression (8.33) with account of (8.32) the following fine structure constant at scale of the W-boson mass

$$\alpha(M_W) = 0.00772. \tag{8.48}$$

to be compared with experimental value [4]

$$\alpha(M_Z) = 0.0077562 \pm 0.0000012. \tag{8.49}$$

Of course, set of equations (8.40, 8.41) is approximate. It quite may be, that with account of necessary corrections the agreement of the result with experimental number (8.49) will be not such indecently good.

The second solution gives mach larger value for $\sin^2\theta_W \simeq 0.89$. As a result this leads to $\alpha(M_W) \simeq 0.0235$, that is three times more, than (8.48, 8.49). Now we have one solution (8.48) being in agreement with actual physics and another one being in evident disagreement. Which one is to be used?

The answer is connected with the problem of a stability of solutions (8.43). The stability in the model is defined by sum of vacuum averages

$$\frac{1}{4}\langle W^a_{\mu\nu}\, W^a_{\mu\nu}\rangle + \frac{1}{4}\langle B_{\mu\nu}\, B_{\mu\nu}\rangle. \tag{8.50}$$

In Chapter 6 contribution (6.38) to the first term, which is due to effective interaction (8.29), is estimated. It was shown there, that the result is proportional to an integral involving the corresponding form-factor

$$\int_0^{z_0} F(z)\,\sqrt{z}\,dz, \tag{8.51}$$

which turns to be negative. Here we also consider effective interaction (8.29) in the framework of the electro-weak theory. Let us assume, that a still unknown form-factor of interaction (8.29) also leads to negative value of integral (8.51). Then in the same way as in Chapter 6 we conclude, that contribution of interaction (8.29) also leads to the negative result. In view of the result being proportional to coupling constants G_i, the value of (8.50) is negative with larger absolute value for larger values of G_i. Thus the first solution of (8.43) might be stable in comparison to the second one with $\beta = -1.2 \cdot 10^{-5}$. Thus result (8.48) may correspond to the most stable option.

The considerations of the present section can not be regarded as finally decisive and they are rather indications of how things might occur.

We would also draw attention to an appearance of very small numbers in solutions being considered. E. g., solution (8.43) contains parameter $\beta \simeq 5 \cdot 10^{-16}$. This might be useful in consideration of problems of hierarchy [67, 68].

8.4 Expectations

We have already emphasized above, that in case compensation equations have nontrivial solutions there are always a number of additional conditions, which give relations between physical parameters. Thus in case these solutions satisfy stability conditions and, as a result, realize in the Nature, these conditions are to be imposed on the real physics. We have demonstrated, that in case both of QCD and the electroweak theory nontrivial solutions with spontaneous generation of triple gauge boson interactions (4.2, 6.1) turn to be stable. One of the most important consequences of existence of these solutions are predictions of coupling constants of corresponding gauge theories. The conclusions may be formulated as follows:
there are two and only two Yang–Mills gauge vector theories, which differs by an intensity of interaction: the first option corresponds to interaction constant α_s being of order of unity at the boundary of the nonperturbative region (for $N_c = 3$ $\bar{\alpha}_s \simeq 0.9$), and the second option corresponds to interaction constant $\alpha_{EW} \simeq 0.03$ for $N_{EW} = 2$, that is surprisingly close to the physical value of the electroweak coupling. We have already calculated the t-quark mass in terms of the W mass (7.26, 7.29, 7.31). We have already mentioned, that with these results we shorten list of arbitrary constants of the Standard Model by number 3. What else?

In Section 8.2 we have presented examples, which demonstrate how compensation equations may define ratios of fundamental fermion masses. Provided one will succeed in considering analogous problem including all three generations of color quarks and leptons the important information may be achieved. In the would-be set of equations will be involved not only masses, but parameters of mixing as well. For the moment we could not safely judge on number of conditions, which would be provided by the set. Let us suppose that all ratios of quark and lepton masses will be defined by the set of compensation equations. Then from already calculated t-quark mass (7.31) we obtain also masses of the light quarks u, d. But these masses enter in QCD quantities and in the Nambu–Jona-Lasinio interaction, which is defined by QCD scale. Thus we obtain strict relation of the two scales, namely the QCD scale, e. g., f_π or Λ_{QCD}, with the electroweak scale, e. g., W mass.

We have demonstrated, that mixing angles may be also defined by the compensation equations. It is remarkable, that a possibility of a definition of the Weinberg mixing angle, which enters into important parameter $\sin^2\theta_W$, was also demonstrated. This result might be achieved by a consideration of an effective interactions of gauge bosons W^a and B (see Section 8.3). In case of a successful realization of this possibility, one could also define the value of the fine structure constant α, because gauge constant g of the electroweak interaction is already calculated in Section 6.1 and

$$\alpha = \sin^2\theta_W \frac{g^2}{4\pi}. \tag{8.52}$$

Value (8.48) for the fine structure constant being estimated in the previous section with satisfactory precision agrees its physical value. The author dares to express a hope, that a consecutive consideration of interactions of the Standard Model leads to definition of the totality of physical parameters, which is necessary to describe the totality of data in elementary particles physics. It could be achieved provided one could find the most stable solution of a total set of compensation equations and other necessary relations, e. g., Bethe–Salpeter equations for corresponding bound states, especially scalar ones, including the Higgs scalar. Of course, realization of the program needs extensive efforts and hard work. But the wouldbe result without doubt is worth trying.

It is also worth mentioning, that further efforts are necessary to develop the next approximations of the compensation approach. In the most cases we have worked with solutions of linear equations, which were obtained in the approximate scheme of linearization. The appropriate corrections to this scheme are needed.

In support of the approach under a discussion we would lay down just results of the present book. Let us emphasize the main points.

(1) The compensation approach leads to derivation of the well-known and very effective in the low momenta region of the strong interaction Nambu–Jona-Lasinio interaction, without introducing of any additional parameter but the fundamental QCD ones. In Chapter 5 and in Section 3.3 we have demonstrated, that the resulting scheme satisfactory describes data in the region of its applicability.
(2) The consideration of a possibility of spontaneous generation of anomalous three-gauge boson interactions in both QCD and EWT results in definition of values of gauge constants g in points of boundary of nonperturbative regions. Let us emphasize, that the values agree the nowadays knowledge on these fundamental quantities. As far as we know, there is no other approach, which could provide such result. The obtained results on the three-boson effective interaction allows to calculate nonperturbative quantities, e. g., the gluon condensate, in agreement with the phenomenology.
(3) We would also draw attention to a successful attempt of the calculation of the additional contribution to the anomalous magnetic moment of the muon in the framework of the electroweak theory with anomalous three-boson interaction (6.1). Let us remind, that Schwinger's calculation of the anomalous magnetic moment of the electron was in proper time one of the decisive arguments on behalf of quantum electrodynamics. Bearing in mind this history, the author dares to consider results of Section 6.4 as a strong indication for the verity of the compensation approach.

We have already mentioned a possible connection of the two fundamental scales: the QCD scale and the electroweak scale. However, there is one more fundamental scale, namely, the scale of gravitation, which is actually the Planck mass M_{Pl} (1.44). Let us ask the question, if it is possible to obtain a connection between the strong and the electroweak scales and the scale of the gravitation? From our point of view a possible answer could be achieved by consideration of the wouldbe effective interactions in the framework of the quantum gravitation theory. Of course, there is the essential difference between gauge theories of the Standard Model and the quantum gravitation. The last theory is nonrenormalizable. However, the cumulative experience, which partially is described in the book, may allow to hope for finding a possibility to deal with the problem of a spontaneous generation of effective interactions in this case as well. Just in view to support this assumption in the next section we consider an example, which demonstrates, how gravitation interaction may be connected with conventional particles' interactions described in Chapters 1, 2. Provided one could succeed in a definition of a connection of the scale of gravitation with the scales of the gauge theories, we might come to description of the Nature in terms of only one dimensional parameter e. g., the Planck mass. All other physical quantities then would be defined in terms of this fundamental M_{Pl}. In this case we might state, that fundamental physical parameters correspond to the most stable option , which just is realized in the Nature.

Of course, for the moment it is a dream. But nevertheless the goal is very attractive. Maybe, wouldbe realization of the dream might be achieved with the aid of notions and tools, described in the book, which is proposed to a kind attention of a reader.

Let us emphasize, that this possibility is alternative to the option of anthropic principle (see, e. g., [125]), which assumes multiplicity of Universes. The main foundation of this postulate is just a complete absence of any principle, which could fix values of parameters of the Standard Model. In Chapter 2 we have already noted, that the number N_{SM} of fundamental parameters of the Standard Model including those, which are related to neutrinos, comprises as many as 25. Then if each possible set of these parameters corresponds to a really existing Universe, then the power of set of the totality of Universes is

$$(continuum)^{N_{SM}}$$

On the other hand, the existence of a human being, who is capable to observe the Nature and to try to understand Its laws, is closely connected with actual values of the parameters of the Standard Model. The properties of nuclei and their isotopes are connected with parameters defining low-energy strong interaction, that is in our notations with $\bar{\alpha}_s$, m_u, m_d. The most important parameters, which define the rich variety of organic substances, which is inevitably necessary for the life generation and evolution, are just the fine structure constant α and the electron mass m_e.

Thus the anthropic principle assumes, that we live in the only Universe, which supplies conditions for an existence of a human being, that is in the Universe with such parameters α, $\bar{\alpha}_s$, m_u, m_d, m_e, which we consider now as real physical ones. All other Universes are principally unobservable.

The approach, which we have considered in the present book, provides a possibility to define at least some of these parameters. Indeed, in Chapter 4 we have obtained value $\bar{\alpha}_s$ (see, e. g., (4.37)) in agreement with its physical value. As for other parameters, in Section 8.2 we have discussed possibilities of their fixing in the framework of a spontaneous generation of effective interactions in the Standard Model. In Section 8.3 we have demonstrated a possibility to define the fine structure constant α. Relations, being obtained in Section 8.2 are not to be considered as satisfactory ones, but the examples, which give these results, may serve as leading indications for further more reliable studies. In case of successful realization of the program, we shall have a possibility to understand how values of the fundamental parameters are fixed. Then the conception of the uniqueness of the Universe might be established. That is, it might be, that the observable Universe corresponds to the most stable nontrivial solution of the Standard Model. The author does believe, that a possible way to this goal is connected with a phenomenon of a spontaneous generation of effective interactions in the framework of the Standard Model. Just this possibility we have considered in the present book.

We have already noted that further studies might lead to some connection of scales of the Standard Model with the scale of the gravity.

In view of considering of a possible connection of parameters of the Standard Model with the gravitation scale, in the following section we consider an example of a spontaneous generation of an anomalous three-graviton effective interaction, which is analogous to effective interactions (4.2, 6.1) in gauge theories of the Standard Model.

8.5 A possible effective interaction in the general relativity

Due to well-known problems of the dark matter and the dark energy numerous possibilities of modified gravity are considered (see, e.g. review [126] and recent work [127]). This approach assumes existence of new effective interactions of the gravitational field in addition to the fundamental Einstein–Hilbert Lagrangian. The main goal of the book is just to find out how such interactions can be generated. In view of the extreme interest of the problem of a modified gravity we would consider a possibility of a spontaneous generation of an effective interaction using the methods, being developed above. We would also follow the close analogy with effective interactions being studied in previous chapters.

In the present section we would show an example of how gravity interactions could be connected with interactions of the Standard Model.

Namely, we would discuss a possibility of anomalous gravitation interaction in terms of nonperturbative effects of the Einstein–Hilbert gravity. For the purpose we rely on the compensation approach, which is described in the book. In Chapters 3, 4, 5, 6, 7 this approach was applied to studies of a spontaneous generation of effective nonlocal interactions in renormalizable gauge theories. In particular, Chapter 6 deal with an application of the approach to the electro-weak interaction and a possibility of spontaneous generation of the effective anomalous three-boson interaction of the following form was demonstrated

$$\frac{g\,\lambda}{3!\,M_W^2}\,F\,\epsilon_{abc}\,W^a_{\mu\nu}\,W^b_{\nu\rho}\,W^c_{\rho\mu}, \tag{8.53}$$
$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc}\,W^b_\mu W^c_\nu.$$

where $g \simeq 0.65$ is the electro-weak coupling. Here $F(p_i)$ is a form-factor, which guarantees effective interaction (8.53) acting in a limited region of the momentum space. This form-factor is uniquely defined by the compensation equation of the Bogoliubov approach. We use an approximate scheme, which accuracy was estimated to be $\simeq$ (10–15) %. Up to this precision the approach gives unique results for physical parameters, so we have none adjusting parameter in the scheme. Would-be existence of effective interaction (8.53) leads to important nonperturbative effects in the electro-weak interaction. Its consequences were considered above. Note, that interaction (8.53) is extensively looked for experimentally and there exist experimental limitations for parameter λ (6.8).

We would take interaction (8.53) as a leading hint for choosing of an effective interaction in the gravity theory. Considering links between vector nonabelian gauge theories and the theory of the gravity one easily sees that gauge field $W^a_{\mu\nu}$ plays the same role as the Riemann curvature tensor $R^m_{n\,\mu\,\nu}$. Thus the anomalous interaction which is strictly analogous to interaction (8.53) is the following

$$\frac{G}{2!}\,F_G\,\sqrt{-g}\,\epsilon^{n_1 n_2 n_3 m_3}\,R_{m_1 n_1 \mu\nu}\,R_{m_2 n_2 \nu_1 \rho}\,R_{m_3 n_3 \rho_1 \mu_1} \times g^{m_1 m_2}\,g^{\nu\nu_1}\,g^{\rho\rho_1}\,g^{\mu\mu_1},$$
$$R_{mn\mu\nu} = g_{ms}\,R^s_{n\mu\nu}, \tag{8.54}$$
$$R^s_{n\mu\nu} = \frac{\partial\Gamma^s_{n\nu}}{\partial x_\mu} - \frac{\partial\Gamma^s_{n\mu}}{\partial x_\nu} + \Gamma^s_{r\mu}\,\Gamma^r_{n\nu} - \Gamma^s_{r\nu}\,\Gamma^r_{n\mu},$$
$$\Gamma^i_{kl} = \frac{1}{2}\,g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Here curvature tensor $R^s_{n\mu\nu}$ indeed plays a part of gauge field $W^a_{\mu\nu}$ and two indices s, n replace symmetry index a. The necessity of an introduction of absolute antisymmetric tensor $\epsilon^{n_1 n_2 n_3 m_3}$ is connected with the antisymmetry of the curvature tensor in respect to the last two indices. Here we have the direct analogy with the form of interaction (8.53). It is important to emphasize, that interaction (8.54) does not conserve parity. Coupling constant G has dimension M^{-5}. $F_G(q_i)$ in definition (8.54) is again some form-factor to be defined by a compensation equation. This equation in the first approximation according to the procedure of our approach corresponds to diagrams of Figure 8.6. The Lorentz structure of the anomalous three-graviton vertex is defined in Section 8.6 with the use of the FORM program. We also use the standard Feynman rules for the quantum gravitation [128, 129].

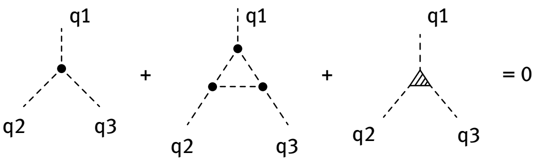


Fig. 8.6. Diagram representation of the compensation equation in the first approximation. Dotted lines correspond to gravitons, a black spot represents interaction (8.54), the striped triangle represents a contribution of the Standard Model diagrams.

Performing calculations using FORM we achieve the following integral equation with integrations in the four-dimensional Euclid momentum space

$$\begin{aligned} F(x) = F_{0G} + \frac{3G^2}{16\pi^2}\Bigg(&-\frac{1}{x^3}\int_0^x y^7 F(y)dy + \frac{4}{x^2}\int_0^x y^6 F(y)dy - \frac{3}{x}\int_0^x y^5 F(y)dy \\ &+ 8\int_0^x y^4 F(y)dy + 18\int_x^\infty y^4 F(y)dy - 25x\int_x^\infty y^3 F(y)dy \\ &+ 24x^2\int_x^\infty y^2 F(y)dy - 11x^3\int_x^\infty yF(y)dy + 2x^4\int_x^\infty F(y)dy\Bigg), \quad x = p^2. \end{aligned} \tag{8.55}$$

where F_{0G} means inhomogeneous part of the equation, which in Figure 8.6 is denoted by the striped triangle.

Assuming $F_{0G} = Const$, we obtain by successive differentiations of equation (8.55) the following linear differential equation for $F(x)$. Introducing new variable

$$z = \frac{81\, G^2\, x^5}{15625\, \pi^2}, \tag{8.56}$$

we have

$$\begin{aligned} &\Bigg[\left(z\frac{d}{dz}+\frac{3}{5}\right)\left(z\frac{d}{dz}+\frac{2}{5}\right)\left(z\frac{d}{dz}+\frac{1}{5}\right)\left(z\frac{d}{dz}\right) \\ &\quad\times\left(z\frac{d}{dz}-\frac{1}{5}\right)\left(z\frac{d}{dz}-\frac{2}{5}\right)\left(z\frac{d}{dz}-\frac{3}{5}\right)\left(z\frac{d}{dz}-\frac{4}{5}\right) \\ &\quad+z\left(z\frac{d}{dz}+\frac{14}{15}\right)\Bigg]F(z) = 0. \end{aligned} \tag{8.57}$$

Integral equation (8.55) is equivalent to differential equation (8.57) with boundary conditions. Taking into account these conditions we have the following solution, which we obtain in terms of Meijer functions in the same way as other solutions in the book

$$F(z) = C\, G^{50}_{18}\left(z\,\middle|^{1/15}_{0,\,1/5,\,2/5,\,3/5,\,4/5,\,-3/5\,-2/5,\,-1/5}\right). \tag{8.58}$$

Constant C is defined by normalization condition at $z = 0$: $F(0) = 1$, that gives

$$C = \frac{6\,\Gamma\left(\frac{1}{15}\right)}{125\,\Gamma\left(\frac{4}{5}\right)} = 0.5972001. \tag{8.59}$$

On the other hand, assuming $F_{0G} = 0$, we may calculate $F(0)$ from equations (8.55, 8.58), that gives

$$F(0) = \frac{18}{5}. \tag{8.60}$$

So there is evidently additional contribution to $F(0)$, that is

$$F_{0G} \neq 0. \tag{8.61}$$

This contribution might be given by diagrams including matter fields, for example, by those being presented in Figure 8.7. First of all we would draw attention to presence of Z exchange in Figure 8.7. The interaction of Z with neutrinos contains γ_5 matrix (see Feynman rules (2.77)) and so the Trace inevitably contains antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$, which is present in interaction (8.54). The vertex of a graviton interaction with a neutrino, as well as with any spinor field, is the following

$$V(\mu, \nu, p_1, p_2) = \imath\,\kappa\,(\gamma_\mu\,(p_1 + p_2)_\nu + \gamma_\nu\,(p_1 + p_2)_\mu), \tag{8.62}$$

where p_1 is the momentum of the incoming neutrino and p_2 is the same of the outcoming one.

We readily estimate, that this diagram gives the following contribution to the inhomogeneous part of the equation

$$F_{0G} = -\,C_G \frac{g^2 \kappa^3}{4(16\,\pi^2)^2 M_Z^2}\,\ln\frac{M_Z^2}{m_\nu^2}, \tag{8.63}$$

where κ is the usual gravitation coupling constant, g is the electroweak gauge constant, which is defined in Section 6.1, and C_G is a coefficient of order of unity. From the main equation (8.55) we have the following condition

$$F(0) + F_{0G} = 1. \tag{8.64}$$

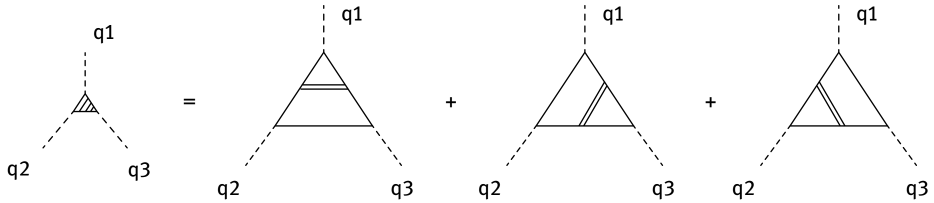


Fig. 8.7. Diagrams, describing the first approximation for the Standard Model contribution to three-graviton vertex (8.54). Simple lines correspond to matter fermions (neutrino *etc.*, double lines correspond to weak bosons Z, W.

Expression (8.63) has to be equal to

$$F_{0G} = 1 - F(0) = -3.0028. \tag{8.65}$$

Then with an account of number of neutrinos $N_\nu = 3$ and previous relations (8.55, 8.65) we obtain the following estimate for the coupling constant of the effective interaction (8.54) G. In doing this we have to bear in mind, that integral equation (8.55) is divided by coupling constant G due to the overall procedure for searches for nontrivial solutions of compensation equations. Thus we have

$$G \sim \frac{g^2\kappa^3}{4(16\pi^2)^2 M_Z^2} \ln \frac{M_Z^2}{m_\nu^2}. \tag{8.66}$$

As a matter of fact, for the moment we can not substitute a reliable value for the average neutrino mass m_ν in expression (8.66). We may safely assert, that it is not zero due to existence of the effect of neutrino oscillations. In any case it may not be more than $10^{-2}\,eV$ (see data (2.107) in Section 2.2). In view of this we have taken for the estimate just neutrinos, as particles having the smallest masses of all particles giving contribution to coupling constant G. It is evident, that massless particles, namely photons and gluons, do not give contribution due to parity conservation of their interactions. To obtain more definite connection between the two parameters G and κ one needs perform difficult calculations, which will be done elsewhere. However our estimate (8.66) allows us to consider effects of the interaction (8.54) and to conclude if it is advisable to continue studies in this direction.

With physical mass of Z and bearing in mind relation

$$\kappa = \frac{1}{M_{Pl}},$$

where Planck mass (1.44) is very large, we understand, that possible value (8.66) is essentially larger, than seemingly natural value, which one can estimate under premise, that only gravitational effects can define the quantity under the study

$$G_{Pl} \sim \kappa^5 = \frac{1}{M_{Pl}^5}. \tag{8.67}$$

The interaction (8.54) due to a presence of the antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ gives no contribution to spherically symmetric problems of gravitation (Schwartzschield solution, Friedmann solution *etc.*). However it could manifest itself in problems without spherical symmetry in a rotating system (e. g., spiral galaxy). The considerable enhancement of possible value (8.66) in comparison to natural value (8.67) by the following factor

$$\frac{G}{G_{Pl}} = \frac{g^2 M_{Pl}^2}{64\pi^2 M_Z^2} \simeq 10^{31}, \tag{8.68}$$

is quite remarkable and may lead to observable effects. Here we use also estimation of the logarithm in (8.66).

Let us note, that the propagator of a graviton is the following [128]

$$D(a,b,m,n,q) = \frac{g_{am}g_{bn} + g_{an}g_{bm} - g_{ab}g_{mn}}{\imath\,(2\pi)^4\,q^2}, \tag{8.69}$$

and the three-graviton vertex is presented in Section 8.6.

Thus we have a possible additional effective interaction which could be considered in the framework of modified gravitation. On the other hand the example shows, that spontaneous generation of effective interactions may occur in the gravitation theory as well.

Of course, for the moment we can not say, that the gravity scale M_{Pl} is somehow connected with scales of the Standard Model. However, this example could give hints on how this connection might be looked for.

8.6 Appendix

Here we present FORM program leading to the definition of vertex V corresponding to interaction (8.54). The explicit expression of V takes much more space than the program.

```
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,m,n,r,s,u,v,t,w;
G F(a,b,c,d) = (d_(a,c)*d_(b,d)+d_(a,d)*d_(b,c));
G FG(a,b,m,n)=d_(a,m)*d_(b,n)+d_(a,n)*d_(b,m)-d_(a,b)*d_(m,n);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,j,v,w,m,n,r,s,u,t;
G R(h,i,j,w,g,f)=F(i,w,g,f)*q(j)*q(h)-F(i,j,g,f)*q(w)*q(h)+
F(h,j,g,f)*q(w)*q(i)-F(h,w,g,f)*q(j)*q(i);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R11=R(b,i1,m,n,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
```

```
G R12=R(b,i2,n,r,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R13=R(a,i3,r,m,t,s);
id q=q3;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R1123=R11*R12*R13*e_(i1,i2,i3,a);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R21=R(b,i1,m,n,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R22=R(b,i2,n,r,t,s);
id q=q3;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R23=R(a,i3,r,m,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R2123=R21*R22*R23*e_(i1,i2,i3,a);
```

```
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R31=R(b,i1,m,n,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
S x,y;
G R32=R(b,i2,n,r,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R33=R(a,i3,r,m,t,s);
id q=q3;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R3123=R31*R32*R33*e_(i1,i2,i3,a);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R41=R(b,i1,m,n,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R42=R(b,i2,n,r,t,s);
id q=q3;
```

```
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R43=R(a,i3,r,m,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R4123=R41*R42*R43*e_(i1,i2,i3,a);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R51=R(b,i1,m,n,t,s);
id q=q3;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R52=R(b,i2,n,r,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R53=R(a,i3,r,m,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R5123=R51*R52*R53*e_(i1,i2,i3,a);
.sort
Print;
```

```
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R61=R(b,i1,m,n,t,s);
id q=q3;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R62=R(b,i2,n,r,u,v);
id q=q2;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R63=R(a,i3,r,m,g,f);
id q=q1;
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G R6123=R61*R62*R63*e_(i1,i2,i3,a);
.sort
Print;
.store
V k,p,q,q1,q2,q3;
I a,b,c,d,f,g,h,i,i1,i2,i3,j,v,w,m,n,r,s,u,t;
G V(g,f,u,v,t,s)=1/2*(R1123+R2123+R3123+R4123+R5123+R6123);
.sort
Print;
.store
.end
```

After calculation we obtain vertex $V(g,f,u,v,t,s)$, in which indices and momenta for three legs are the following (for each leg there are two indices)

$$g,\, f,\, q_1, \quad u,\, v,\, q_2, \quad t,\, s,\, q_3. \tag{8.70}$$

Just to illustrate the form of the total vertex, we can present the expression for the Lorentz structure of the three-gluon vertex (8.54) on the mass shell i. e., with the following conditions

$$q_1^2 = q_2^2 = q_3^2 = (q_1 q_2) = (q_1 q_3) = (q_2 q_3) = 0. \tag{8.71}$$

Remind, that the FORM notations are connected with those used throughout the book in the following way

$$p(a) = p_a, \quad d\_(v,s) = g_{vs}, \quad e\_(a,b,c,d) = \epsilon_{abcd}. \tag{8.72}$$

```
V(g,f,u,v,t,s) =
 2*e_(q1,q2,q3,f)*d_(v,s)*q1(u)*q2(t)*q3(g)
-2*e_(q1,q2,q3,f)*d_(v,s)*q1(t)*q2(g)*q3(u)
-2*e_(q1,q2,q3,f)*d_(v,t)*q1(s)*q2(g)*q3(u)
+2*e_(q1,q2,q3,f)*d_(v,t)*q1(u)*q2(s)*q3(g)
+2*e_(q1,q2,q3,f)*d_(s,u)*q1(v)*q2(t)*q3(g)
-2*e_(q1,q2,q3,f)*d_(s,u)*q1(t)*q2(g)*q3(v)
+2*e_(q1,q2,q3,f)*d_(u,t)*q1(v)*q2(s)*q3(g)
-2*e_(q1,q2,q3,f)*d_(u,t)*q1(s)*q2(g)*q3(v)
+2*e_(q1,q2,q3,g)*d_(v,s)*q1(u)*q2(t)*q3(f)
-2*e_(q1,q2,q3,g)*d_(v,s)*q1(t)*q2(f)*q3(u)
-2*e_(q1,q2,q3,g)*d_(v,t)*q1(s)*q2(f)*q3(u)
+2*e_(q1,q2,q3,g)*d_(v,t)*q1(u)*q2(s)*q3(f)
+2*e_(q1,q2,q3,g)*d_(s,u)*q1(v)*q2(t)*q3(f)
-2*e_(q1,q2,q3,g)*d_(s,u)*q1(t)*q2(f)*q3(v)
+2*e_(q1,q2,q3,g)*d_(u,t)*q1(v)*q2(s)*q3(f)
-2*e_(q1,q2,q3,g)*d_(u,t)*q1(s)*q2(f)*q3(v)
+2*e_(q1,q2,q3,v)*d_(f,s)*q1(u)*q2(t)*q3(g)
-2*e_(q1,q2,q3,v)*d_(f,s)*q1(t)*q2(g)*q3(u)
-2*e_(q1,q2,q3,v)*d_(f,t)*q1(s)*q2(g)*q3(u)
+2*e_(q1,q2,q3,v)*d_(f,t)*q1(u)*q2(s)*q3(g)
+2*e_(q1,q2,q3,v)*d_(g,s)*q1(u)*q2(t)*q3(f)
-2*e_(q1,q2,q3,v)*d_(g,s)*q1(t)*q2(f)*q3(u)
-2*e_(q1,q2,q3,v)*d_(g,t)*q1(s)*q2(f)*q3(u)
+2*e_(q1,q2,q3,v)*d_(g,t)*q1(u)*q2(s)*q3(f)
+2*e_(q1,q2,q3,s)*d_(f,v)*q1(u)*q2(t)*q3(g)
-2*e_(q1,q2,q3,s)*d_(f,v)*q1(t)*q2(g)*q3(u)
+2*e_(q1,q2,q3,s)*d_(f,u)*q1(v)*q2(t)*q3(g)
-2*e_(q1,q2,q3,s)*d_(f,u)*q1(t)*q2(g)*q3(v)
+2*e_(q1,q2,q3,s)*d_(g,v)*q1(u)*q2(t)*q3(f)
-2*e_(q1,q2,q3,s)*d_(g,v)*q1(t)*q2(f)*q3(u)
+2*e_(q1,q2,q3,s)*d_(g,u)*q1(v)*q2(t)*q3(f)
-2*e_(q1,q2,q3,s)*d_(g,u)*q1(t)*q2(f)*q3(v)
+2*e_(q1,q2,q3,u)*d_(f,s)*q1(v)*q2(t)*q3(g)
```

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-2*e_(q1,q2,q3,u)*d_(f,s)*q1(t)*q2(g)*q3(v)
+2*e_(q1,q2,q3,u)*d_(f,t)*q1(v)*q2(s)*q3(g)
-2*e_(q1,q2,q3,u)*d_(f,t)*q1(s)*q2(g)*q3(v)
+2*e_(q1,q2,q3,u)*d_(g,s)*q1(v)*q2(t)*q3(f)
-2*e_(q1,q2,q3,u)*d_(g,s)*q1(t)*q2(f)*q3(v)
+2*e_(q1,q2,q3,u)*d_(g,t)*q1(v)*q2(s)*q3(f)
-2*e_(q1,q2,q3,u)*d_(g,t)*q1(s)*q2(f)*q3(v)
-2*e_(q1,q2,q3,t)*d_(f,v)*q1(s)*q2(g)*q3(u)
+2*e_(q1,q2,q3,t)*d_(f,v)*q1(u)*q2(s)*q3(g)
+2*e_(q1,q2,q3,t)*d_(f,u)*q1(v)*q2(s)*q3(g)
-2*e_(q1,q2,q3,t)*d_(f,u)*q1(s)*q2(g)*q3(v)
-2*e_(q1,q2,q3,t)*d_(g,v)*q1(s)*q2(f)*q3(u)
+2*e_(q1,q2,q3,t)*d_(g,v)*q1(u)*q2(s)*q3(f)
+2*e_(q1,q2,q3,t)*d_(g,u)*q1(v)*q2(s)*q3(f)
-2*e_(q1,q2,q3,t)*d_(g,u)*q1(s)*q2(f)*q3(v)
+2*e_(q1,q2,f,s)*q1(v)*q2(t)*q3(g)*q3(u)
+2*e_(q1,q2,f,s)*q1(u)*q2(t)*q3(g)*q3(v)
-4*e_(q1,q2,f,s)*q1(t)*q2(g)*q3(v)*q3(u)
+2*e_(q1,q2,f,t)*q1(v)*q2(s)*q3(g)*q3(u)
-4*e_(q1,q2,f,t)*q1(s)*q2(g)*q3(v)*q3(u)
+2*e_(q1,q2,f,t)*q1(u)*q2(s)*q3(g)*q3(v)
+2*e_(q1,q2,g,s)*q1(v)*q2(t)*q3(f)*q3(u)
+2*e_(q1,q2,g,s)*q1(u)*q2(t)*q3(f)*q3(v)
-4*e_(q1,q2,g,s)*q1(t)*q2(f)*q3(v)*q3(u)
+2*e_(q1,q2,g,t)*q1(v)*q2(s)*q3(f)*q3(u)
-4*e_(q1,q2,g,t)*q1(s)*q2(f)*q3(v)*q3(u)
+2*e_(q1,q2,g,t)*q1(u)*q2(s)*q3(f)*q3(v)
+4*e_(q1,q2,v,s)*q1(u)*q2(t)*q3(f)*q3(g)
-2*e_(q1,q2,v,s)*q1(t)*q2(f)*q3(g)*q3(u)
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+4*e_(q1,q2,v,t)*q1(u)*q2(s)*q3(f)*q3(g)
-4*e_(q1,q2,s,u)*q1(v)*q2(t)*q3(f)*q3(g)
+2*e_(q1,q2,s,u)*q1(t)*q2(f)*q3(g)*q3(v)
+2*e_(q1,q2,s,u)*q1(t)*q2(g)*q3(f)*q3(v)
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-2*e_(q1,q2,u,t)*q1(s)*q2(f)*q3(g)*q3(v)
-2*e_(q1,q2,u,t)*q1(s)*q2(g)*q3(f)*q3(v)
+2*e_(q1,q3,f,v)*q1(s)*q2(g)*q2(t)*q3(u)
-4*e_(q1,q3,f,v)*q1(u)*q2(s)*q2(t)*q3(g)
+2*e_(q1,q3,f,v)*q1(t)*q2(g)*q2(s)*q3(u)
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-4*e_(q1,q3,f,u)*q1(v)*q2(s)*q2(t)*q3(g)
+2*e_(q1,q3,f,u)*q1(s)*q2(g)*q2(t)*q3(v)
+2*e_(q1,q3,f,u)*q1(t)*q2(g)*q2(s)*q3(v)
+2*e_(q1,q3,g,v)*q1(s)*q2(f)*q2(t)*q3(u)
-4*e_(q1,q3,g,v)*q1(u)*q2(s)*q2(t)*q3(f)
+2*e_(q1,q3,g,v)*q1(t)*q2(f)*q2(s)*q3(u)
-4*e_(q1,q3,g,u)*q1(v)*q2(s)*q2(t)*q3(f)
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+2*e_(q1,q3,u,t)*q1(v)*q2(f)*q2(s)*q3(g)
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-4*e_(q1,q3,u,t)*q1(s)*q2(f)*q2(g)*q3(v)
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-4*e_(q2,q3,f,s)*q1(v)*q1(u)*q2(t)*q3(g)
+2*e_(q2,q3,f,s)*q1(v)*q1(t)*q2(g)*q3(u)
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-2*e_(q2,q3,f,u)*q1(v)*q1(s)*q2(t)*q3(g)
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+4*e_(q2,q3,g,v)*q1(s)*q1(t)*q2(f)*q3(u)
-2*e_(q2,q3,g,v)*q1(u)*q1(t)*q2(s)*q3(f)
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+2*e_(q2,q3,g,s)*q1(v)*q1(t)*q2(f)*q3(u)
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-2*e_(q2,q3,g,u)*q1(v)*q1(s)*q2(t)*q3(f)
-2*e_(q2,q3,g,u)*q1(v)*q1(t)*q2(s)*q3(f)
+4*e_(q2,q3,g,u)*q1(s)*q1(t)*q2(f)*q3(v)
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+2*e_(q2,q3,g,t)*q1(v)*q1(s)*q2(f)*q3(u)
-4*e_(q2,q3,g,t)*q1(v)*q1(u)*q2(s)*q3(f)
+2*e_(q2,q3,g,t)*q1(s)*q1(u)*q2(f)*q3(v)
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Index

This monograph is devoted to the non-perturbative dynamics in the Standard Model (SM), the basic theory of all fundamental interactions in nature except gravity. The Standard Model is divided into two parts: the quantum chromodynamics (QCD) and the electro-weak theory (EWT) are well-defined renormalizable theories in which the perturbation theory is valid. However, for the adequate description of the real physics non-perturbative effects are inevitable. This book describes how these non-perturbative effects may be obtained in the framework of spontaneous generation of effective interactions. The well-known example of such effective interaction is provided by the famous Nambu-Jona-Lasinio effective interaction. Also a spontaneous generation of this interaction in the frame-work of QCD is described and applied to the method for other effective interactions in QCD and EWT. The method is based on N.N. Bogoliubov's conception of compensation equations. As a result we then describe the principal features of the Standard Model, e.g. Higgs sector, and significant non-perturbative effects including recent results obtained at LHC and TEVATRON.

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