Lecture 1

1.1 Introduction

This course has two aims.

Aim 1. To explain how Einstein's theory models gravitation as the curvature of space-time. This involves setting up some new mathematical machinery, notably tensor calculus. The mathematical background will be developed in parallel with the theory. In textbooks, it is often done the other way: mathematics first, theory after. But the textbooks do not have to be read in linear order. The approach here should make it clearer at each stage where we are going.

Aim 2. To treat the elementary theory of black holes.

The closest to a course text is Hughston and Tod: *An introduction to general relativity* (LMS student texts 5, CUP). For more advanced reading, R M Wald, *General relativity*, (Chicago) is strongly recommended. [Note: text in red is clickable]

General relativity is a theory of gravity. It grew from attempts to reconcile

a) Newton's theory of gravitation

b) special relativity

in much the same way (only more quickly) that special relativity grew out of Maxwell's theory and Galileo's principle of relativity. As in that case, the reconciliation involved rethinking our ideas about space and time—continuing the process begun with SR.

Special relativity teaches us that we must no longer think of space and time as separate. General relativity goes further: it allows space-time to be 'curved', and models gravity as the curvature. This leads to some interesting possibilities. For example, the universe might be spatially closed like the three-dimensional 'surface' of a sphere in four dimensions. For many years, there was some unease about the contrast between such far-reaching consequences of GR and the rather meagre set of observations through which it was verified—for example the fact that the orbit of Mercury rotates by $43''$ per century faster than it should in Newtonian theory and the fact that the positions of stars in the sky are slightly distorted by the 'bending of light' in the sun's gravitational field. However this position has changed completely in recent years with many new astronomical observations. Perhaps the most dramatic of these concern the binary pulsar PSR 1913+16: this consists of a neutron star, about 15 miles across, but with a mass of about 1.5 times that of the sun, orbiting another star once every 8 hours or so. Here one reverses the Mercury observation: one uses the rotation of the orbit to measure the masses, and then calculates the theoretical rate at which the orbital period should decrease as the two stars lose energy through gravitational radiation. The result, over 15 years, agrees with observation to within 0.5%.

It is worth keeping in mind this almost unbelievable correspondence between theory and observation as we go through what at times will seem some very technical and abstract mathematics. In spite of its mathematical abstraction, general relativity is now one of the most accurately confirmed theories of all time. An account of the current experimental and observational status of general relativity can be seen online in Clifford Wills' article in *Living Reviews*.

1.2 Newtonian gravity

The essential content of Newton's theory is contained in two equations. The first is

$$\nabla^2\phi = 4\pi G\rho, \tag{1}$$

where ϕ is the gravitational potential ρ is the matter density, and $G = 6.67 \times 10^{-11}$ in SI units (it has dimensions $L^3M^{-1}T^{-2}$). With appropriate boundary conditions, (1) determines the gravitational field generated by a source. The second is

$$\boldsymbol{g} = -\boldsymbol{\nabla}\phi. \tag{2}$$

This determines the force $m\boldsymbol{g}$ on a particle of mass m in a gravitational field with potential ϕ. The accuracy of the theory is remarkable: in the solar system, the only detectable discrepancy between the theoretical and actual motions of the planets is the orbit of Mercury, where it amounts to one part in 10^7.

The two equations contain the inverse-square law. By integrating over a region bounded by a surface S, and containing a total mass M, we obtain Gauss' law from the divergence theorem:

$$\int_S \boldsymbol{g}.\mathrm{d}\boldsymbol{S} = -4\pi GM. \tag{3}$$

If the field is spherically symmetric, for example if it is that *outside* a spherical star, then $\boldsymbol{g} = F(r)\hat{\boldsymbol{r}}$. By taking S to be a sphere of radius r, we obtain

$$F(r) = -\frac{GM}{r^2},$$

which is the inverse square law. The corresponding potential ϕ is not unique since we are free to add a constant. If we fix this by taking $\phi = 0$ at infinity, then

$$E = \tfrac{1}{2}mv^2 + m\phi = \tfrac{1}{2}mv^2 - \frac{GMm}{r}$$

is the total energy of a particle at of mass m and speed v.

One can see even in these very simple formulas some important issues.

Black holes. For the particle to escape to infinity, v must exceed the *escape velocity*

$$v = \sqrt{2GM/r}$$

(since E is conserved and v^2 must remain non-negative as $r \to \infty$). This is maximal at the star's surface, where r takes its lower possible value in the region outside the star. What if at this point we have $v = c$, the velocity of light? This will be the case if the radius of the star is $R = 2GM/c^2$, the so-called *Schwarzschild radius*. Then nothing can escape from the surface. But what if there is a mirror on the surface and we shine light down from infinity? It will be reflected at the surface and follow the same path back out again. Since orbits are reversible in Newtonian gravity, it will be reflected back to infinity. Clearly Newton does not provide a consistent picture of such a 'black hole'. One might not think this matters: for a star of the mass of the sun, the Schwarzschild radius is 3 km, which would make the star an apparently impossibly dense object with a density 10^{16} times that of water. However, such stars exist not much more massive than the sun. If one increases the mass to that of a galaxy, then the critical density goes down to that of air. So the issue must be addressed as a potentially real physical problem, and not simply as an absurd extrapolation.

Linearity. The equations of Newtonian gravity are *linear*: if you superimpose two mass distributions, then you need only add the corresponding gravitational fields. However, as bodies interact gravitationally, energy is transferred from their gravitational fields to the bodies themselves, and vice versa. Thus gravitational fields carry energy, and therefore by Einstein's principle $E = mc^2$, they have mass: that is gravitational fields should themselves generate gravitational fields. Thus a relativistic theory of gravity should be described by *nonlinear* equations.

1.3 Gravity and SR

(1) and (2) are very like the basic equations of electrostatics. This suggests that we should try to construct a relativistic theory of gravity in the same way as you did for electromagnetism. But nobody has succeeded in doing this in a way that works —i.e. that does not give incorrect predictions; and there is any case problems of principle about applying special relativity in a gravitational field.

Gravity and light.. The first problem is that light is affected by gravity, and this is incompatible with the basic notion in special relativity that light should travel in straight lines at the same constant speed in all inertial frames.

How do we know that light is affected by gravity? First because it is observed to be. For example, the path followed by light reaching the earth from a star in the direction of the sun is bent by the sun's gravitation field, as was first observed by Eddington's in 1919: he showed that, relative to their normal positions in the night sky, the positions of stars appeared to be displaced away from the centre of the sun when observed during a total eclipse. (The history of Eddington's observation is not quite as straightforward as it is sometimes presented. See, for example, Peter Coles' article Einstein, Eddington and the 1919 eclipse. More recent and dramatic evidence for the bending of light can be seen in the photographs taken by the Hubble telescope.)

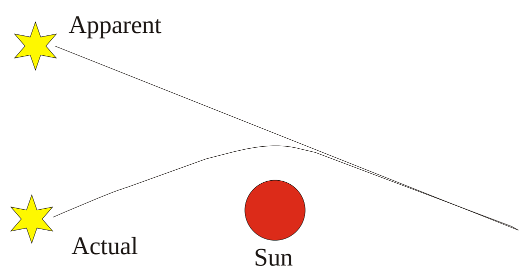


Figure 1: Bending of light

Second, there is Bondi's thought experiment: if photons were not affected by gravity, then one could in principle build a perpetual motion machine. Bondi imagined a machine consisting of a series of buckets attached to a conveyor belt. Each contains a single atom, with those on the right in an excited state and those on the left in a lower energy state. As these reach the bottom of the belt, the excited atoms emit light which is focused by two curved mirrors onto the atom at the top of the belt; the one at the bottom falls into the lower state and the one at the top is excited. Since $E = mc^2$, those on the right, which have more energy, should be heavier. The force of gravity should therefore keep the belt rotating. The way out is that photons lose energy as they climb up through the

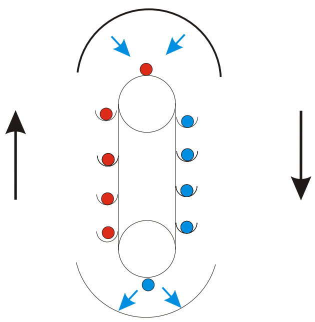


Figure 2: Bondi's *perpetuum mobile*

gravitational field. Since $E = \hbar\omega$, this means that they are red-shifted. Such a shift was measured directly by Pound and Rebka in 1959 by a remarkable experiment: over the 75ft height of the tower of the Jefferson building at Harvard, the red-shift is about 3 parts in 10^{14}.

The Pound-Rebka result is incompatible with SR, as illustrated in the space-time diagram. The blue lines are the top and bottom of the tower and the red lines are the worldlines of photons travelling up the tower. Since the top and bottom of the tower are at rest relative to each other, their worldlines in SR are parallel, which forces $\Delta t = \Delta t'$. So in a special-relativistic theory of gravity, there cannot be any gravitational red-shift.

Where are the inertial frames?. In special relativity, there is a preferred class of 'non-accelerating frames', called inertial frames. These are precisely the frames in which Newton's first law holds:

> In the absence of forces, particles move in straight lines at constant speed.

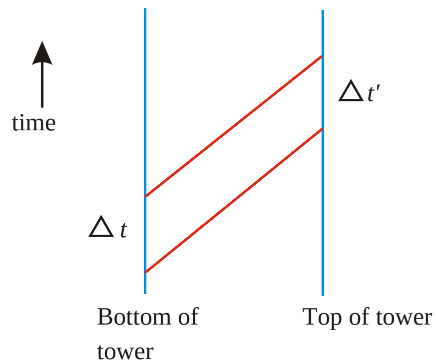


Figure 3: Pound and Rebka's measurement is incompatible with special relativity

How can we tell if we are in an inertial frame? At first glance the answer looks easy: we simply look to see if free particles travel in straight lines at constant speed relative to the frame. The problem is that gravity affects all matter equally, so there are no completely *free particles*. In the equation of motion,

$$m\ddot{\mathbf{r}} = -m\boldsymbol{\nabla}\phi \tag{4}$$

the two m's cancel, so all particles accelerate in the same way. This does not happen with other forces. In electromagnetism, there are (charged) particles that are affected by an electric field and others (uncharged) that are not; but in gravitation theory, there are no 'neutral' particles which we can think of as free of *all* forces.

The m on the LHS is *inertial mass*; that on the RHS is *gravitational mass* (the analogue of charge in electromagnetism). The exact equality of the two is the (weak) *principle of equivalence*. Galileo tested it by comparing the periods of pendula with weights made out of different substances. A celebrated 19$^{\text{th}}$ experiment by Eötvös checked very accurately (to one part in 10^9) the equality of the two ms in the gravitational force on a mass and the centrifugal force due to the Earth's rotation: these are the two components in the apparent gravitational field. More recent experiments—including lunar ranging measurements—have improvement the accuracy to one part in 10^{13}; and a planned space experiment (STEP) will test it to one part in 10^{18}.

We have good reason, therefore, to believe that the equality is exact. Einstein extrapolated it to the (*strong) equivalence principle*: there is no observable distinction between the effects of gravity and acceleration. In a closed box, there is no physical experiment

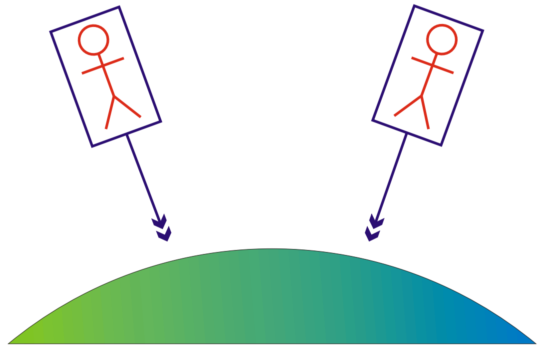


Figure 4: Relative acceleration in free fall

within the box that will reveal whether(i) the box is at rest on the earth's surface, or (ii) the box is accelerating at 1g in otherwise empty space. In both situations, those inside the box 'feel' a normal terrestrial gravitational field.

In a gravitational field, it is impossible to identify the global inertial frames of special relativity. But we can pick out *local* inertial frames in which gravity is turned off: these are the frames in free-fall. But if we can only work in frames in which gravity is turned off, then how can we observe gravity? The answer is: by shifting your point of view. Gravity manifests not as a 'force', but rather as the relative acceleration of nearby local inertial frames. The key difference between what happens near the earth and what happens in empty space is that (i) near the earth, two nearby 'boxes' in free fall (e.g. space stations) have a small relative acceleration because they are falling on paths that converge at the centre of the earth, while (ii) in empty space, they have no relative acceleration.

> **Starting point**. Special relativity holds over short distances and times in frames in free fall. Gravity is not a local force field, but shows up in the small relative acceleration between local inertial frames. In the presence of gravity, the transformation between local 'inertial' coordinates is not exactly linear.

The idea of *curvature* comes in here: the analogy is with map making. If one makes maps of the earth's surface by projecting onto a tangent plane from the centre of the earth, then overlapping maps will be slightly distorted relative to each other because of the curvature of the earth.

Lecture 2

2.1 Inertial coordinates

SR describes the relationship between physical observations made by different non-accelerating observers. Each observer labels events in space-time by four coordinates t, x, y, z. The coordinate systems of two observers are related by a Lorentz transformation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = L \begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} + T \tag{5}$$

where T is a column vector (which shifts the origin of the coordinates) and

$$L = \begin{pmatrix} L^0{}_0 & L^0{}_1 & L^0{}_2 & L^0{}_3 \\ L^1{}_0 & L^1{}_1 & L^1{}_2 & L^1{}_3 \\ L^2{}_0 & L^2{}_1 & L^2{}_2 & L^2{}_3 \\ L^3{}_0 & L^3{}_1 & L^3{}_2 & L^3{}_3 \end{pmatrix} \tag{6}$$

with $L^0{}_0 > 0$, $\det L = 1$, and $L^{-1} = gL^t g$, where g is the diagonal matrix with diagonal entries $1, -1, -1, -1$, (i.e. L is a proper orthochronous Lorentz transformation matrix). For example for a boost along the x-axis,

$$L = \begin{pmatrix} \gamma & \gamma u & 0 & 0 \\ \gamma u & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

where $\gamma = 1/\sqrt{1-u^2}$. Throughout this course, we shall take $c = 1$.

We call t, x, y, z an *inertial coordinate system* or an *inertial frame*.

To make the transition to GR, we need some new terminology and notation.

Inertial coordinates. These will be written $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. (It is important to keep track of the position of the indices: a lot of information will be stored by making a distinction between upper and lower indices). We can then write (5) in the compact form

$$x^a = \sum_{b=0}^{3} L^a{}_b \tilde{x}^b + T^a \quad (a = 0, 1, 2, 3). \tag{8}$$

(When you write the entries ${L^a}_b$ in L, make it clear that the upper index comes first: a labels the rows in L and b the columns). We note that

$$ {L^a}_b = \frac{\partial x^a}{\partial \tilde{x}^b} $$

and that

$$ {(L^{-1})^a}_b = \frac{\partial \tilde{x}^a}{\partial x^b} \,. $$

The summation and range convention. When an index is repeated in an expression (a *dummy* index), a sum over 0,1,2,3 is implied. An index that is not summed is a *free index*: any equation is understood to hold for all possible values of its free indices. To apply the conventions consistently, an index must never appear more than twice in any term in an expression, once as an upper index and once as a lower index.

The metric and the Kronecker delta. We define the quantities g_{ab}, g^{ab} and δ^a_b by

$$ g_{ab} = g^{ab} = \begin{cases} 1 & a = b = 0 \\ -1 & a = b \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \delta^a_b = \begin{cases} 1 & a = b \\ 0 & \text{otherwise} \end{cases} $$

(Note: later on, the 'metric coefficients' g_{ab} and g^{ab} will be more general functions of the space-time coordinates, but the Kronecker delta will always be defined in this way.)

Examples. (8) becomes

$$ x^a = {L^a}_b \tilde{x}^b + T^a. \tag{9} $$

(repetition of b implies summation over $0, 1, 2, 3$, while the range convention means that the equation is understood to hold as the free index a runs over the values $0, 1, 2, 3$). If two events have coordinates x^a and y^a in the first system and $\tilde{x}^a$ and $\tilde{y}^a$ in the second system, then

$$ x^a - y^a = {L^a}_b(\tilde{x}^b - \tilde{y}^b) = {L^a}_b \tilde{x}^b - {L^a}_b \tilde{y}^b \,. \tag{10} $$

(This illustrates that one must take care about what is meant by a 'term in an expression'. In principle, you should multiply out all the brackets before applying the rule—otherwise the three-fold repetition of b in the middle expression could cause confusion). The Lorentz condition $L^{\mathrm{t}} g L = g$ becomes

$$ {L^c}_a {L^d}_b g_{cd} = g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} = g_{ab} \,. $$

(Note that it does not matter in which order one writes the Ls and gs so long as the indices are 'wired up' correctly. In this equation a, b are free, while c, d are *dummy indices*—like dummy variables in an integral. The sum over c is the sum in the matrix product $L^{\mathrm{t}}g$, while the sum over d is the sum in the matrix product gL). Similarly, $L^{-1}g^{-1}(L^{\mathrm{t}})^{-1} = g^{-1}$ becomes

$$g^{cd}\frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial \tilde{x}^b}{\partial x^d} = g^{ab}\,. \tag{11}$$

If one combines two coordinate transformations

$$x^a = K^a{}_b\tilde{x}^b, \qquad \tilde{x}^a = L^a{}_b\hat{x}^b + T^a \tag{12}$$

then the result is

$$x^a = K^a{}_bL^b{}_c\hat{x}^c + K^a{}_bT^b\,. \tag{13}$$

(To avoid ambiguity, one changes the dummy index in the second equation before making the substitution. It is then clear that there are two sums, over $b = 0, 1, 2, 3$ and over $c = 0, 1, 2, 3$. If you did not do this, then you would end up with the ambiguous expression $K^a{}_bL^b{}_b$, which could mean $\sum_{b=0}^{3} K^a{}_bL^b{}_b$). A a final illustration, we note the identity

$$g_{ab}g^{bc} = \delta^c_a. \tag{14}$$

(This is the matrix identity $g^2 = 1$. Here c, a are the free indices and b is a dummy index).

2.2 4-vectors

A 4-vector has four components V^0, V^1, V^2, V^3, which transform under change of coordinates by

$$\begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = L \begin{pmatrix} \tilde{V}^0 \\ \tilde{V}^1 \\ \tilde{V}^2 \\ \tilde{V}^3 \end{pmatrix}, \tag{15}$$

That is $V^a = L^a{}_b\tilde{V}^b$. Note that $L^a{}_b = \partial x^a/\partial \tilde{x}^b$. The following definition will generalize easily to GR.

Definition 1
A 4-vector is an object with components V^a which transform by

$$V^a = \frac{\partial x^a}{\partial \tilde{x}^b}\tilde{V}^b$$

under change of inertial coordinates.

Example. The 4-velocity: if $x^a = x^a(s)$ is the worldline of a particle, parametrized by proper time s, then the 4-velocity has components $V^a = \mathrm{d}x^a/\mathrm{d}s$. Under coordinate change

$$V^a = \frac{\mathrm{d}x^a}{\mathrm{d}s} = \frac{\partial x^a}{\partial \tilde{x}^b}\frac{\mathrm{d}\tilde{x}^b}{\mathrm{d}s}, \tag{16}$$

so the 4-vector transformation rule is a consequence of the chain rule.

2.3 Tensors in Minkowski space

Two other objects have similar transformation rules.

(1) The *electromagnetic* (EM) *field*

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} \tag{17}$$

transforms by $F = L\tilde{F}L^t$. That is

$$F^{ab} = L^a{}_c L^b{}_d \tilde{F}^{cd} = \frac{\partial x^a}{\partial \tilde{x}^c}\frac{\partial x^b}{\partial \tilde{x}^d}\tilde{F}^{cd}. \tag{18}$$

(2) The *gradient covector* of a function $f(x^a)$ has components $\partial_a f$, where $\partial_a = \partial/\partial x^a$. These transform by the chain rule

$$\partial_a f = \frac{\partial \tilde{x}^b}{\partial x^a}\tilde{\partial}_b f. \tag{19}$$

Note that it is $\partial\tilde{x}/\partial x$ on the RHS, not $\partial x/\partial\tilde{x}$, so this is not the 4-vector transformation rule. Hence the term ‘covector’.

Definition 2
A tensor of type (p, q) is an object that assigns a set of components $T^{a\ldots b}{}_{c\ldots d}$ (p upper indices, q lower indices) to each inertial coordinate system, with the transformation rule under change of inertial coordinates

$$T^{a\ldots b}{}_{c\ldots d} = \frac{\partial x^a}{\partial \tilde{x}^e}\cdots\frac{\partial x^b}{\partial \tilde{x}^f}\frac{\partial \tilde{x}^h}{\partial x^c}\cdots\frac{\partial \tilde{x}^k}{\partial x^d}\tilde{T}^{e\ldots f}{}_{h\ldots k}.$$

A tensor can be defined at a single event, or along a curve, or on the whole of space-time, in which case the components are functions of the coordinates and we call T a tensor field. If $q = 0$ then there are only upper indices and the tensor is said to be contravariant; if $q = 0$, then there are only lower indices and the tensor is said to be covariant.

Examples. (1) A 4-vector V^a is a tensor of type $(1, 0)$. Also called vector or contravariant vector.

(2) The gradient covector $\partial_a f$ is a tensor of type $(0, 1)$. A tensor α_a of type $(0, 1)$ is generally called a covector or covariant vector.

(3)The Kronecker delta is a tensor of type $(1, 1)$ since

$$\delta^c_d \frac{\partial x^a}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^d}{\partial x^b} = \frac{\partial x^a}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^c}{\partial x^b} = \delta^a_b. \tag{20}$$

(4) The *contravariant metric* has components g^{ab} and is a tensor of type $(2, 0)$ (by 11). The *covariant metric* has components g_{ab} and is a tensor of type $(0, 2)$

Both the Kronecker delta and the metric are special in that they have the same components in every frame. For a general tensor, the components in different frames are not the same.

Lecture 3

3.1 Operations on tensors

Addition. S, T of the same type: $S + T$ has components

$$S^{a\ldots b}{}_{c\ldots d} + T^{a\ldots b}{}_{c\ldots d}.$$

Scalars. A scalar at an event is simply a number. A scalar field is a function on space-time. The value of a scalar is unchanged by coordinate transformations. We can multiply a tensor T by a scalar f to get a tensor of the same type with components $fT^{a\ldots b}{}_{c\ldots d}$.

The two operations make the space of tensors of type (p, q) into a vector space of dimension 4^{p+q}.

Tensor product. It S, T are tensors of types (p, q), (r, s), respectively, then the *tensor product* is the tensor of type $(p + r, q + s)$ with components $S^{a\ldots b}{}_{c\ldots d}T^{e\ldots f}{}_{g\ldots h}$. It is denoted by ST or $S \otimes T$.

Differentiation. If T is a tensor field of type (p, q), then ∇T is the tensor of type $(p, q+1)$ with components

$$\nabla_a T^{b\ldots c}{}_{d\ldots e} = \partial_a T^{b\ldots c}{}_{d\ldots e}, \quad \partial_a = \frac{\partial}{\partial x^a}.$$

Under change of *inertial* coordinates,

$$\begin{aligned}
\partial_a T^{b\ldots c}{}_{d\ldots e} &= \frac{\partial \tilde{x}^p}{\partial x^a}\tilde{\partial}_p\left[\frac{\partial x^b}{\partial \tilde{x}^q}\cdots\frac{\partial \tilde{x}^r}{\partial x^d}\tilde{T}^{q\ldots}{}_{r\ldots}\right] \\
&= \frac{\partial \tilde{x}^p}{\partial x^a}\frac{\partial x^b}{\partial \tilde{x}^q}\cdots\frac{\partial \tilde{x}^r}{\partial x^d}\tilde{\partial}_p\tilde{T}^{q\ldots}{}_{r\ldots},
\end{aligned}$$

which is the correct transformation rule. Note that we are still working in the context of special relativity: the calculation only works because $\partial x/\partial \tilde{x}$ is constant. We shall have to work harder to define differentiation in curved space-time.

Contraction. If T is of type $(p + 1, q + 1)$, then we can form a tensor S of type (p, q) by *contracting on the first upper index and first lower index* of T:

$$S^{b\ldots c}{}_{e\ldots f} = T^{ab\ldots c}{}_{ae\ldots f}$$

(sum over a). Under change of coordinates

$$\begin{aligned} S^{b\ldots c}{}_{e\ldots f} &= T^{ab\ldots c}{}_{ae\ldots f} \\ &= \frac{\partial x^a}{\partial \tilde{x}^p}\frac{\partial x^b}{\partial \tilde{x}^q}\cdots\frac{\partial x^c}{\partial \tilde{x}^r}\frac{\partial \tilde{x}^s}{\partial x^a}\frac{\partial \tilde{x}^t}{\partial x^e}\cdots\frac{\partial \tilde{x}^u}{\partial x^f}\tilde{T}^{pq\ldots r}{}_{st\ldots u} \\ &= \frac{\partial x^b}{\partial \tilde{x}^q}\cdots\frac{\partial x^c}{\partial \tilde{x}^r}\frac{\partial \tilde{x}^t}{\partial x^e}\cdots\frac{\partial \tilde{x}^u}{\partial x^f}\tilde{S}^{q\ldots r}{}_{t\ldots u} \end{aligned}$$

since

$$\frac{\partial x^a}{\partial \tilde{x}^p}\frac{\partial \tilde{x}^s}{\partial x^a} = \delta^s_p.$$

One can also contract on other pairs of indices (one upper and one lower).

Raising and lowering. If α is a covector and $U^a = g^{ab}\alpha_b$, then U is a 4-vector (tensor multiplication plus contraction). We write α^a for U^a and call the operation 'raising the index': raising the index changes the signs of the 1,2,3 components, but leaves the first component unchanged. The reverse operation is 'lowering the index': $V_a = g_{ab}V^b$. One similarly lowers and raises indices on tensors by taking the tensor product with the covariant or contravariant metric and contracting, e.g. $T^a{}_b = g_{bc}T^{ac}$. Warning: keep track of the order of the upper and lower indices: $T^a{}_b$ and $T_b{}^a$ are generally distinct, so don't write T^a_b.

Examples. (1) If f is scalar field, then $\nabla^a f$, where

$$(\nabla^a f) = (\partial_t f, -\partial_x f, -\partial_y f, -\partial_z f)$$

is a 4-vector field: it the 'gradient 4-vector'.

(2) If U and V are 4-vectors, then

$$g(U,V) = g_{ab}U^aV^b = U^aV_a = U_aV^a.$$

(3) Raising one index on g_{ab} or lowering one index on g^{ab} gives the Kronecker delta since $g^{ab}g_{bc} = \delta^a_c$.

(4) Suppose that $(S^a) = (1,0,0,0)$ and $(T^a) = (1,1,0,0)$. Then $S\otimes T$ and $T\otimes S$ have respective components

$$(S^aT^b) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (T^aS^b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $S\otimes T \neq T\otimes S$, but when written as matrices, as above, the components of $S\otimes T$ and $T\otimes S$ are related by transposition.

3.2 The energy-momentum tensor

Consider a cloud of particles ('dust'), in which the velocities of the individual particles vary smoothly from event to event and time to time. There is one worldline through each event and the 4-velocity is a 4-vector field. There are no external forces or interactions, so each particle moves in a straight line at constant speed.

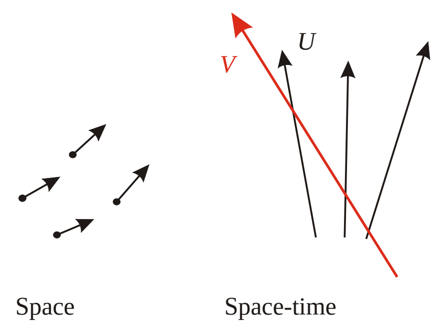


Figure 5: A 'dust' cloud

Problem What is the energy density seen by an observer with 4-velocity V? The answer depends on V because

(i) the energy of each individual particle depends on its velocity relative to the observer; and

(ii) moving volumes appear to contract.

Definition 3

The rest density ρ is a scalar. It is defined at an event A to be the rest-mass per unit volume measured in a frame in which the particles at A are at rest. If there are n particles per unit volume in this frame and each has rest mass m, then $\rho = nm$.

Consider the particles which occupy a unit volume at an event A in the rest frame of the particles at A. Suppose that in this frame the observer is moving along the negative x-axis with velocity v. To the observer, each particle at A appears to have velocity $(v, 0, 0)$ and to have energy $m\gamma(v) = m/\sqrt{1 - v^2}$. The particles appear to occupy a volume $1/\gamma(v) = \sqrt{1 - v^2}$. Therefore the observer measures the energy density to be $\gamma(v)^2\rho$.

Definition 4
The energy-momentum tensor of the dust cloud is the tensor field with components $T^{ab} = \rho U^a U^b$. It is a tensor of type $(2, 0)$ (scalar times the tensor product of two 4-vectors).

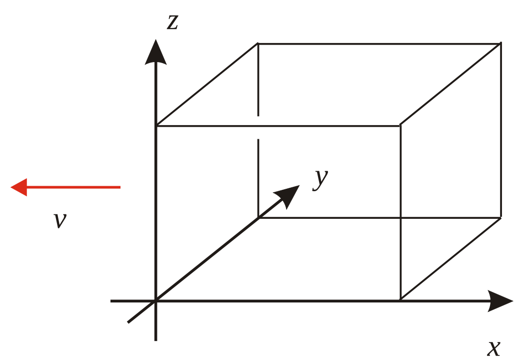


Figure 6: The transformation of density

Proposition 1
The energy density measured by an observer with 4-velocity V is $\rho_{\text{obs}} = T_{ab}V^aV^b$.

Proof. In the rest frame of the observer,

$$(V^a) = (1, 0, 0, 0) \qquad (U^a) = \gamma(v)(1, v, 0, 0).$$

Therefore $T_{ab}V^aV^b = \rho(U_aV^a)^2 = \rho\gamma(v)^2$. ∎

3.3 The continuity equation

In the rest frame of the particle P, the nearby particles all have small velocities and so the non-relativistic continuity equation holds

$$\frac{\partial \rho}{\partial t} + \rho \,\mathrm{div}\, \mathbf{u} = 0.$$

This expresses the conservation of mass and energy.

Proposition 2
$\nabla_a T^{ab} = 0$

Proof. We want to show that this holds at an event A. It is a tensor equation, and so it enough to show that it holds in one frame. Now

$$\nabla_a T^{ab} = \nabla_a(\rho U^a U^b) = \Big(U^a \nabla_a \rho + \rho(\nabla_a U^a)\Big) U^b + \rho U^a \nabla_a U^b.$$

Each particle moves in a straight line at constant speed. Therefore

$$0 = \frac{\mathrm{d}U^a}{\mathrm{d}s} = \frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\partial U^a}{\partial x^b} = U^b \nabla_b U^a.$$

In the rest frame of the particle at A,

$$U = \begin{cases} (1, \mathbf{0}) & \text{at } A \\ \gamma(u)(1, \mathbf{u}) & \text{near } A,\ u = |\mathbf{u}| << 1 \end{cases}$$

Therefore, at the event A,

$$\begin{aligned} U^a \nabla_a \rho &= \partial_t \rho \\ \nabla_a U^a &= \partial_t \gamma(u) + \partial_x(\gamma(u) u_1) + \partial_y(\gamma(u) u_2) + \partial_z(\gamma(u) u_3)\,, \end{aligned}$$

where $\partial_t = \partial/\partial t$ and so on. But since $\gamma(u) = 1/\sqrt{1-u^2}$, we have

$$\partial_t \gamma(u) = \gamma(u)^3 u \partial_t u = 0$$

at A, where $u = 0$. The other partial derivatives of $\gamma(u)$ vanish similarly. Therefore $\nabla_a U^a = \mathrm{div}\, \mathbf{u}$ at A and the result follows from the continuity equation. ∎

Lecture 4

4.1 Curved space-time

In Minkowski space, the components of the metric tensor are constant. In general relativity, they are functions of the space-time coordinates. The gravitational field is encoded in the second derivatives of the metric, which measure the *curvature* of space-time. Our starting point is the following set of guiding principles.

> GR1: SR holds over small distances and short times in frames in free fall, that is, in *local inertial frames*. In such frames we can set up *local inertial coordinates* and as in Minkowski space.

> GR2: Gravity appears as the relative acceleration of nearby local inertial frames.

We can only use inertial coordinates in a small neighbourhood of each event. If we want to describe what is happening in a larger region of space-time, we must use a general coordinate system x^a $(a = 0, 1, 2, 3)$ to label events.

> GR3: Principle of general covariance: the theory should be formulated in a way that does not give any special role to a particular coordinate system.

This generalizes the principle of relativity, which says the same for inertial coordinate systems in SR.

4.2 The metric

Let A be an event and let O be an observer in free fall at A. To O, SR must hold near A. Suppose that A sets up local inertial coordinates t, x, y, z in small region of space-time around A, with A as origin. Since O thinks SR holds locally, we have the following.

(1) The worldlines of free particles that pass through A satisfy

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = \frac{\mathrm{d}^2 z}{\mathrm{d}s^2} = 0$$

at $t = x = y = z = 0$, where s is proper time.

(2) Let B be an event near A with coordinates $\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, and put

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

Then, provided that we ignore third order terms, $\mathrm{d}s^2$ has the same interpretation as in SR. Therefore the following hold.

Timelike: If $\mathrm{d}s^2 > 0$, then $\mathrm{d}s$ is the time from A to B on a clock in free fall from A to B.

Null: If $\mathrm{d}s^2 = 0$, then A and B lie on the worldline of a photon.

Spacelike: If $\mathrm{d}s^2 < 0$, then $\mathrm{d}s^2 = -D^2$, where D is the distance from A to B measured in a frame in free fall in which A and B are simultaneous.

The only difference from SR is that in SR we would say that (1) held everywhere and that (2) was also true for widely separated events.

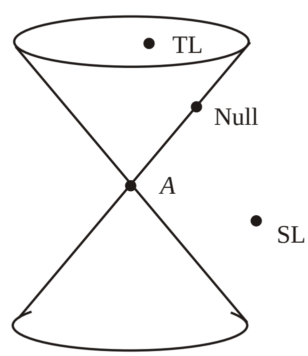


Figure 7: The displacement from A to B in the three cases

The coordinates t, x, y, z can only be used at A. Let us introduce a general coordinate system x^a on space-time, and translate these statements into the general system. Near A, t, x, y, z are some general functions of the x^as, so

$$\mathrm{d}t = \frac{\partial t}{\partial x^a}\mathrm{d}x^a + \text{second order terms in } \mathrm{d}x$$

and so on.

Therefore at A, ignoring third order terms in the $\mathrm{d}x^a$s,

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b,$$

where, in a generalization of our previous notation, the coefficients $g_{ab} = g_{ba}$ are functions of the coordinates constructed from the partial derivatives of the local inertial coordinates at an event with respect to the x^as. They are called the *metric coefficients*. For example, at A,

$$g_{01} = \frac{\partial t}{\partial x^0}\frac{\partial t}{\partial x^1} - \frac{\partial x}{\partial x^0}\frac{\partial x}{\partial x^1} - \frac{\partial y}{\partial x^0}\frac{\partial y}{\partial x^1} - \frac{\partial z}{\partial x^0}\frac{\partial z}{\partial x^1}.$$

We can make a similar transformation to local inertial coordinate systems near other events. Then $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$ everywhere, but the g_{ab}s now vary from event to event.

If we change from x^a to a new coordinate system $\tilde{x}^a$, then

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \left(g_{cd}\frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b}\right)\mathrm{d}\tilde{x}^a\mathrm{d}\tilde{x}^b.$$

In the new coordinate system, the metric coefficients are

$$\tilde{g}_{ab} = g_{cd}\frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b}$$

or in matrix notation

$$\tilde{g} = M^t g M \quad \text{where} \quad M = \left(\frac{\partial x^a}{\partial \tilde{x}^b}\right).$$

At any one event, we can reduce g to the the diagonal matrix with diagonal entries $1, -1, -1, -1$ by transforming to local interial coordinates at that event (but in general we cannot do this at all events simulataneously, as we shall see). Therefore the matrix g has one positive and three negative eigenvalues—which is usually expressed by saying that g has signature $+---$.

To summarize, in an arbitrary coordinates system, if $\mathrm{d}x^a$ is the coordinate separation between two nearby events A and B, then, to the second order in $\mathrm{d}x$

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

where the metric coefficients are evaluated at A and $\mathrm{d}s$ has the interpretation in (ds1)–(ds3).

The metric cooefficients g_{ab} have the following properties

MC1: they are smooth functions of the coordinates x^a;

MC2: they are symmetric $g_{ab} = g_{ba}$

MC3: they have signature $+---$ at every event

MC4: they transform under general coordinate transformations by

$$\tilde{g}_{ab} = g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b}.$$

Example. Suppose that $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2.$$

Then we can reduce $\mathrm{d}s^2$ to the form $\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ by the coordinate change $t = t$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. So this is just the metric of SR in a funny coordinate system (spherical polars). For a general metric, we cannot reduce to the Minkowski space form by a coordinate transformation. However, we can do it up to the second order at one event: the following proposition will allow us to recover local inertial coordinates from a general set of metric coefficients satisfying (MC1)–(MC4).

PROPOSITION 3
Let g_{ab} be a general set of metric coefficients such that (MC1)–(MC4) hold and let A be the event $x^a = 0$. Then there exists a coordinate system $\tilde{x}^a$ such that $\tilde{x}^a = 0$ and $\tilde{\partial}_c \tilde{g}_{ab} = 0$ at A.

Proof. Define new coordinates $\tilde{x}^a$ by $x^a = \tilde{x}^a - \frac{1}{2}\Gamma^a{}_{bc}\tilde{x}^b\tilde{x}^c$, where the $\Gamma^a{}_{bc}$s are constants such that $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$. Put $G_{ab} = g_{ab}(0)$ and $H_{cab} = \partial_c g_{ab}(0)$. Then, by Taylor's theorem,

$$g_{ab} = G_{ab} + x^c H_{cab} + O(2),$$

where '$O(2)$' denotes quadratic and higher order terms in the x^as. Hence

$$\begin{aligned} \tilde{g}_{ab} &= g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} \\ &= (G_{cd} + x^k H_{kcd})(\delta^c_a - \Gamma^c{}_{ae}\tilde{x}^e)(\delta^d_b - \Gamma^d{}_{bf}\tilde{x}^f) + O(2) \\ &= G_{ab} + \tilde{x}^c(H_{cab} - \Gamma_{abc} - \Gamma_{bac}) + O(2), \end{aligned}$$

where $\Gamma_{abc} = G_{ad}\Gamma^d{}_{bc}$; we have used $x^a = \tilde{x}^a + O(2)$, as well as changing the names of the dummy indices. We want to choose $\Gamma_{abc} = \Gamma_{acb}$ so that

$$H_{cab} = \Gamma_{abc} + \Gamma_{bac} \,.$$

By permuting the indices, we then also have

$$H_{bca} = \Gamma_{cab} + \Gamma_{acb}\, H_{abc} = \Gamma_{bca} + \Gamma_{cba} \,.$$

By adding the first two of these and subtracting the third, we obtain

$$\Gamma_{abc} = \tfrac{1}{2}(H_{cab} + H_{bca} - H_{abc}) \,,$$

and hence that

$$\Gamma^a{}_{bc} = \tfrac{1}{2}G^{ad}(H_{cdb} + H_{bcd} - H_{dbc}) \,,$$

where $G^{ab}G_{bc} = \delta^a_c$ (that is, (G^{ab}) is the inverse of the matrix G_{ab}.

Conversely, if we define $\Gamma^a{}_{bc}$ in this way, then we have

$$H_{cab} - \Gamma_{abc} - \Gamma_{bac} = H_{cab} - \tfrac{1}{2}(H_{cab} + H_{bca} - H_{abc} + H_{cba} + H_{acb} - H_{bac}) = 0,$$

since $H_{abc} = H_{acb}$. ∎

Note that

$$\Gamma^a{}_{bc} = \tfrac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \,.$$

evaluated at $x^a = 0$, where the g^{ab}s are the inverse (or *contravariant*) metric coefficients, defined by $g^{ab}g_{bc} = \delta^a_c$. The quantities Γ^a_{bc} are called the *Christoffel symbols*. We shall meet them again in the definition of the *Levi-Civita* connection.

Lecture 5

5.1 Existence of local inertial coordinates

We are trying to extract local inertial coordinates from a general set of metric coefficients g_{ab} satisfying (MC1)–(MC4). We have shown that by coordinate change, we can make $\partial_a g_{bc} = 0$ at the event $x^a = 0$, that is

$$g_{ab}(x) = g_{ab}(0) + O(2).$$

PROPOSITION 4
Let A be an event. Suppose that we have two coordinate systems x^a and $\tilde{x}^a$ such that at A, $x^a = \tilde{x}^a = 0$ and $\partial_a g_{bc} = \tilde{\partial}_a \tilde{g}_{bc} = 0$. Then there exist constants $M^a{}_b$ such that $x^a = M^a{}_b \tilde{x}^b + O(3)$ ('$O(3)$' means third order terms in x).

This says that the transformation is linear at A up to the second order in x: that is, the Taylor expansion about A of x^a in powers $\tilde{x}^a$ has no second order terms.

Proof. We have to show that at A, $\partial^2 x^a / \partial \tilde{x}^b \partial \tilde{x}^c = 0$. Now at all events,

$$\tilde{g}_{ab} = g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b}.$$

Therefore

$$\tilde{\partial}_e \tilde{g}_{ab} = \frac{\partial x^f}{\partial \tilde{x}^e} \partial_f g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} + g_{cd} \frac{\partial^2 x^c}{\partial \tilde{x}^a \partial \tilde{x}^e} \frac{\partial x^d}{\partial \tilde{x}^b} + g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial^2 x^d}{\partial \tilde{x}^b \partial \tilde{x}^e} \tag{21}$$

(note that the second two terms on the right-hand side differ by the interchange of a and b). Put

$$L_{abe} = g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial^2 x^d}{\partial \tilde{x}^b \partial \tilde{x}^e}\,.$$

Then $L_{abe} = L_{aeb}$ and (21) gives

$$\begin{aligned} L_{bae} + L_{abe} &= 0 \\ L_{eba} + L_{bea} &= 0 \\ L_{aeb} + L_{eab} &= 0\,. \end{aligned}$$

By adding the first and third, and subtracting the second, we obtain $L_{abe} = 0$. Hence

$$\frac{\partial^2 x^q}{\partial \tilde{x}^b \partial \tilde{x}^e} = \frac{\partial \tilde{x}^a}{\partial x^p} g^{pq} g_{cd} \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial^2 x^d}{\partial \tilde{x}^b \partial \tilde{x}^e} = \frac{\partial \tilde{x}^a}{\partial x^p} g^{pq} L_{abe} = 0\,,$$

which completes the proof. ∎

Proposition 5

Existence of local inertial coordinates for a general metric. Let $g_{ab}(x)$ be an set of metric coefficients satisfying (MC1)–(MC4) and let A be an event. Then there exists a coordinate system x^a such that $x^a = 0$ at A and

$$(g_{ab}(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + O(2).$$

This system is unique up to coordinate transformations $x^a \mapsto \tilde{x}^a$ where $x^a = L^a{}_b \tilde{x}^b + O(3)$ for some Lorentz transformation matrix $L = (L^a{}_b)$.

Proof. Choose an initial coordinate system such that $\partial_c g_{ab} = 0$ and $x^a = 0$ at A. Since g_{ab} has signature $+ - --$ we can find a matrix $M = (M^a{}_b)$ such that

$$M^t g M = (M^a{}_c g_{ab}(0) M^b{}_d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now make the linear coordinate change $x^a \mapsto M^a{}_b x^b$. The uniqueness statement follows from the previous proposition. ∎

The metric coefficients g_{ab} will describe a general gravitational field. The coordinates at A in the last proposition are interpreted as local inertial coordinates of an observer in free fall at A. For special metrics we can reduce g_{ab} to the diagonal form $\operatorname{diag}(1, -1, -1, -1)$ everywhere. We shall see that that is the case in which gravitational field vanishes, but for a general metric, such a coordinate transformation does not exist. To summarize:

(1) A gravitational field is described by a general set of metric coefficients satisfying (MC1)–(MC4), which encode the temporal and spatial separation of nearby events according to (ds1)–(ds3).

(2) The local inertial coordinates set up by an observer in free fall at an event A are the coordinates x^a such that at A, $x^a = 0$ and

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + O(2).$$

In local inertial coordinates, SR holds over small times and distances. It is conventional to write the metric in general coordinates in the form

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b\,.$$

For example, for Minkowski space in spherical polar coordinates

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\,.$$

5.2 Particle motion

In a local inertial coordinate system at an event A, $\partial_c g_{ab} = 0$ at A and the worldlines of free particles (particles in free fall) satisfy

$$\ddot{x}^a = 0 \quad \text{where} \quad \cdot = \frac{\mathrm{d}}{\mathrm{d}s}, \quad s = \text{proper time}$$

at the event A. What is the corresponding differential equation in a general coordinate system?

Our strategy is find a Lagrangian and use a result from classical mechanics about the transformation of Lagrange's equations under change of coordinates.

Invariance of Lagrange's equations. In classical mechanics, the equations of motion of a system with time-independent Lagrangian $L(q_a, \dot{q}_a)$ are Lagrange's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0\,,$$

where the q_as are generalized coordinates. The equations in a new coordinate system $\tilde{q}_a$ can be found by substituting

$$q_a = q_a(\tilde{q}), \qquad \dot{q}_a = \frac{\partial q_a}{\partial \tilde{q}_b}\dot{\tilde{q}}_b$$

into L and writing down Lagrange's equations in the new coordinates. This is the sense in which Lagrange's equations are invariant under coordinate transformations.

We shall take this result out of its original physical context and apply it as a statement about the way in which Lagrange's equations change under change of dependent variables. We shall put the space-time coordinates x^a in the role of the q_as and the proper time s in the role of time in classical mechanics. We are led to the following.

The geodesic hypothesis.. The worldlines of particles in free fall parametrized by proper time s satisfy the *geodesic equations*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$$

where $\cdot = \mathrm{d}/\mathrm{d}s$ and $L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$. Solution curves are called *geodesics*.

> Justification (1). In local inertial coordinates at an event, these reduce to $\ddot{x}^a = 0$ at the event.
>
> Justification (2). They are invariant: they take the same form in every coordinate system.

To understand (1), we write out the geodesic equations explicitly. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{ab}\dot{x}^b) - \tfrac{1}{2}(\partial_a g_{bc})\dot{x}^b\dot{x}^c = 0.$$

Hence

$$\begin{aligned} g_{ab}\ddot{x}^b + \tfrac{1}{2}\dot{x}^b\dot{x}^c(2\partial_c g_{ab} - \partial_a g_{bc}) &= 0 \\ \Rightarrow \quad \ddot{x}^a + \tfrac{1}{2}\dot{x}^b\dot{x}^c g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) &= 0 \\ \Rightarrow \quad \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c &= 0 \end{aligned}$$

where

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

These are the *Christoffel symbols* or *connection coefficients*. We shall see a lot of them. Since the Christoffel symbols vanish at an event A in local inertial coordinates at A, the equations reduce to $\ddot{x}^a = 0$ in these coordinates at A.

The second statement in justification follows from the general principles of Lagrangian mechanics and from the invariance of L: from (MC3),

$$\tilde{g}_{ab}\dot{\tilde{x}}^a\dot{\tilde{x}}^b = \tilde{g}_{ab}\frac{\partial \tilde{x}^a}{\partial x^c}\frac{\partial \tilde{x}^b}{\partial x^d}\dot{x}^c\dot{x}^d = g_{cd}\dot{x}^c\dot{x}^d\,.$$

Putting these two facts together, we have that the geodesic equations give the correct equations of motion at any chosen event in one particular coordinate system; and also that if they hold in one system, then they hold in every system. Hence they give the correct equations of motion everywhere in every coordinate system

It follows from the geodesic equations that L is constant. In fact, on the worldline of a particle in free fall, $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$, by definition, so $L = g_{ab}\dot{x}^a\dot{x}^b = 1$.

Example. For the Minkowski space metric written in spherical polar coordinates,

$$L = \tfrac{1}{2}(\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2)\,.$$

The geodesic equations are

$$\begin{aligned} \ddot{t} &= 0 & \ddot{\theta} + 2r^{-1}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\phi^2 &= 0, & \ddot{\phi} + 2r^{-1}\dot{r}\dot{\phi} + 2\cot\dot{\theta}\dot{\phi} &= 0\,. \end{aligned}$$

We can read off from these that, for example, $\Gamma^3_{13} = 1/r$ (we are labelling the coordinates so that $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$).

We can extend the discussion to photons, where the same argument leads to the *photon postulate* that the worldline of a photon is also given by the geodesic equations with $g_{ab}\dot{x}^a\dot{x}^b = 0$. The parameter s does not here have the interpretation of time: it is called an *affine parameter* and can be replaced by any linear function of s. Geodesics with $g_{ab}\dot{x}^a\dot{x}^b > 0$ are said to be *timelike*; those with $g_{ab}\dot{x}^a\dot{x}^b = 0$ are said to be *null*.

Lecture 6

6.1 Transformation of the Christoffel symbols

The Christoffel symbols are defined by

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{ba} - \partial_a g_{bc}).$$

If we make a coordinate change, then

$$\tilde{\Gamma}^a_{bc} = \tfrac{1}{2} \tilde{g}^{ad}(\tilde{\partial}_b \tilde{g}_{dc} + \tilde{\partial}_c \tilde{g}_{ba} - \tilde{\partial}_a \tilde{g}_{bc}).$$

How are Γ^a_{bc} and $\tilde{\Gamma}^a_{bc}$ related? We know that the geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b \dot{x}^c = 0 \tag{22}$$

transform to

$$\ddot{\tilde{x}}^a + \tilde{\Gamma}^a_{bc}\dot{\tilde{x}}^b \dot{\tilde{x}}^c = 0$$

because the Lagrangian they are derived from is invariant. Substitute

$$\dot{\tilde{x}}^a = \frac{\partial \tilde{x}^a}{\partial x^d}\dot{x}^d$$

into the second equation to get

$$\begin{aligned} 0 &= \frac{\partial \tilde{x}^a}{\partial x^d}\ddot{x}^d + \frac{\partial^2 \tilde{x}^a}{\partial x^d \partial x^e}\dot{x}^d \dot{x}^e + \tilde{\Gamma}^a_{ef}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c}\dot{x}^b \dot{x}^c \\ \Rightarrow \quad 0 &= \ddot{x}^p + \frac{\partial x^p}{\partial \tilde{x}^d}\left[\tilde{\Gamma}^d_{ef}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c} + \frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c}\right]\dot{x}^b \dot{x}^c\,, \end{aligned} \tag{23}$$

with the second line following form the first by multiplying by $\partial x^p / \partial \tilde{x}^a$ and summing over a. Hence since $\Gamma^a_{bc} = \Gamma^a_{cb}$, and since (23) and (22) are equivalent for all choices of free particle worldline,

$$\Gamma^a_{bc} = \frac{\partial x^a}{\partial \tilde{x}^d}\tilde{\Gamma}^d_{ef}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c} + \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c}$$

6.2 Manifolds

We now have half of general relativity.

(1) The gravitational field is encoded in the metric coefficients g_{ab}.

(2) Free particles move on timelike geodesics; photons move on null geodesics.

The picture is very like SR except that the light cones vary from event to event. So we know how gravity affects matter. We shall leave the other half—how matter generates gravity—until later. First we need some mathematical ideas to make precise our model of space-time.

What sort of object is space-time? We want to draw an analogy between the metric and the first fundamental form of a surface. A surface in space can be described parametrically by $\mathbf{r} = \mathbf{r}(u, v)$, where u, v are parameters. The distance $\mathrm{d}s$ between the nearby points (u, v) and $(u + \mathrm{d}u, v + \mathrm{d}v)$ is given by

$$\begin{aligned} \mathrm{d}s^2 &= \mathbf{r}_u.\mathbf{r}_u \mathrm{d}u^2 + 2\mathbf{r}_u.\mathbf{r}_v \mathrm{d}u\mathrm{d}v + \mathbf{r}_v.\mathbf{r}_v \mathrm{d}v^2 \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

This is the *first fundamental form* (never mind about the second). Like $\mathrm{d}s^2$ in space-time, it is a quadratic form in $(\mathrm{d}u, \mathrm{d}v)$, which measures the separation between two nearby points on the surface. The coefficients E, F, G are functions of the 'coordinates' u, v, like the metric coefficients g_{ab} in space-time. We note two points.

(1) In general, the metric cannot be reduced to the flat form $\mathrm{d}u^2 + \mathrm{d}v^2$ by changing the parameters (this is only possible if the surface has no *intrinsic* or *Gaussian curvature*).

(2) The surface may have nontrivial topology, in which case the parametrization cannot be used over the whole surface. In GR, similarly, we must allow for space-time to have a nontrivial topology: this plays an important part in the model space-times used in cosmology.

We want space-time to be a 'four-dimensional surface', but we don't want to have to think of it as embedded in a higher-dimensional flat space. Locally, the surface looks like a piece of 'curved Minkowski space' just as the surface locally looks a 'curved piece of the plane'. The appropriate mathematical object is the following.

Definition 5

A four-dimensional manifold is

(a) a connected Hausdorff topological space M, together with

(b) a collection of *charts* or *coordinate patches* (U, x^a), where $U \subset M$ is an open set and the x^as are functions $x^a : U \to \mathbb{R}$, $a = 0, 1, 2, 3$, such that the map

$$\mathbf{x} : U \to \mathbb{R}^4 : m \mapsto (x^0(m), x^1(m), x^2(m), x^3(m))$$

is a homeomorphism from U to an open subset $V \subset \mathbb{R}^4$.

Two conditions must hold (i) every point (='event') of M must lie in a coordinate patch; and (ii) if (U, x^a) and $(\tilde{U}, \tilde{x}^a)$ are charts, then the $\tilde{x}^a$s can be expressed as functions of the x^as on the intersection. We require that

$$(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \mapsto (x^0, x^1, x^2, x^3)$$

should be smooth (i.e. infinitely differentiable) and one-to-one, with

$$\det \left[\frac{\partial x^a}{\partial \tilde{x}^b} \right] \neq 0.$$

The same definition works, of course, in other dimensions.

6.3 Vectors and tensors

The various physical objects in space-time are represented by scalars (functions on space-time) or by scalars or tensors, which are objects which transform in simple ways under change of coordinates. The definition is the same as in SR, except that the coordinate changes are now general.

Definition 6

A tensor of type (p, q) is an object that assigns a set of components $T^{a\ldots b}{}_{c\ldots d}$ (p upper indices, q lower indices) to each local coordinate system, with the transformation rule under change of coordinates

$$T^{a\ldots b}{}_{c\ldots d} = \frac{\partial x^a}{\partial \tilde{x}^e} \cdots \frac{\partial x^b}{\partial \tilde{x}^f} \cdots \frac{\partial \tilde{x}^h}{\partial x^c} \cdots \frac{\partial \tilde{x}^k}{\partial x^d} \tilde{T}^{e\ldots f}{}_{h\ldots k}.$$

A tensor can be defined at a single event, or along a curve, or on the whole of space-time, in which case the components are functions of the coordinates and we call T a tensor field. If $q = 0$ then T ia a *contravariant* tensor; if $p = 0$, it is a *covariant* tensor.

An object that behaves as a tensor under change of local inertial coordinates at an event determines a tensor at the event under general coordinate transformations.

Note that since

$$\frac{\partial x^a}{\partial \tilde{x}^e}\frac{\partial \tilde{x}^e}{\partial x^b} = \delta^a_b\,, \tag{24}$$

one could equally well write the transformation law with all the tilde ($\tilde{}$) and un-tilde quantities interchanged.

Example. The metric g_{ab} is a tensor field of type $(0,2)$. It has the transformation law

$$g_{ab} = \tilde{g}_{cd}\frac{\partial \tilde{x}^c}{\partial x^a}\frac{\partial \tilde{x}^d}{\partial x^b}. \tag{25}$$

Example. The contravariant metric has components g^{ab}, where (g^{ab}) is the inverse matrix to g_{ab} (i.e. $g^{ab}g_{bc} = \delta^a_c$). It is a tensor of type $(2,0)$. This is proved from (25) by the following steps, which are well worth following carefully because they illustrate some basic techniques of index manipulation. The proof makes several uses of (24).

$$\begin{aligned}
\tilde{g}_{ab}\frac{\partial \tilde{x}^b}{\partial x^e} &= g_{ce}\frac{\partial x^c}{\partial \tilde{x}^a} && \text{(multiply both sides by } \frac{\partial \tilde{x}^b}{\partial x^e} \text{ and sum over } b\text{)}\\
\frac{\partial \tilde{x}^f}{\partial x^e}g^{eh} &= \tilde{g}^{af}\frac{\partial x^h}{\partial \tilde{x}^a} && \text{(multiply by } \tilde{g}^{af}g^{eh} \text{ and sum over } a, e\text{)}\\
g^{kh} &= \tilde{g}^{af}\frac{\partial x^h}{\partial \tilde{x}^a}\frac{\partial x^k}{\partial \tilde{x}^f} && \text{(multiply by } \frac{\partial x^k}{\partial \tilde{x}^f} \text{ and sum over } f\text{)}
\end{aligned}$$

Example. The gradient $\partial_a f$ of a scalar function is a covector field (tensor of type $(0,1)$).

Example. If $x^a = x^a(s)$ is the worldline of a particle parametrized by s, then $V^a = \mathrm{d}x^a/\mathrm{d}s$ is a 4-vector field along the worldline. If $g_{ab}V^aV^b = g_{ab}\dot{x}^a\dot{x}^b = 1$, then V is called the *4-velocity* and s is called the *proper time*.

We can carry out all the operations on tensors in exactly the same way as in SR with the exception of differentiation: the components $\partial_a T^{b\cdots}{}_{d\ldots}$ no longer obey the tensor transformation law. We shall come back to that.

Indices are raised and lowered by contracting with g^{ab} and g_{ab}, but this now involves more than just changing the signs of a few components. For example, if $T^{ab}{}_c$ is a tensor of type $(2,1)$, then the contraction $T^{ab}{}_b$ is a tensor of type $(1,0)$ (one free upper index a). If α_a is a covector then $g^{ab}\alpha_c$ is a a tensor of type $(2,1)$ and its contraction

$$\alpha^a = g^{ab}\alpha_b$$

is a vector. This is the operation of *raising the index*. One similarly lowers indices, for example by putting $X_a = g_{ab}X^b$. Raising followed by lowering returns to the starting point since

$$g_{ab}g^{bc} = \delta^c_a\,.$$

The exceptional operation, differentiation, is more subtle in a general space-time. In the next lecture, we start by considering how to define a derivative operation for vectors.

6.4 Summary of the mathematical formulation

Space-time is a four-dimensional manifold M with a metric tensor g_{ab} (symmetric, type (0,2), signature $+---$).

The points of M are the *events*. If $x^a = x^a(u)$ is the worldline of a particle, where u is a parameter, then

$$s = \int \sqrt{g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}u}\frac{\mathrm{d}x^b}{\mathrm{d}u}}\,\mathrm{d}u$$

is the proper time along the worldline: the time measured by a clock carried by the particle (this the *clock hypothesis*). The 4-vector V with components $V^a = \mathrm{d}x^a/\mathrm{d}s$ is the 4-velocity.

The metric determines the behaviour of free particles via the geodesic hypothesis

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}s^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = 0$$

where s is proper time.

If A is an event, then there exists a local coordinate system such that $x^a = 0$ at A and

$$(g_{ab}(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + O(2).$$

In these coordinates, $\Gamma^a_{bc} = 0$ at the origin (the event A). Such a coordinate system is interpreted as the local inertial coordinate system set up by an observer in free fall at A. We identify 4-vectors and tensors at A with vectors and tensors in SR by taking their components in local inertial coordinates.

The metric determines an inner product $g(X, Y) = X_aY^a$ on the space of 4-vectors at an event with signature $+---$ (that is, it is *not* positive definite). As in SR, we say that X is timelike if $X^aX_a > 0$, and so on.

Lecture 7

7.1 Differentiation of vectors

We need to be able to differentiate vectors and tensors—for example to be able to write down Einstein's equation's. But there is a problem—illustrated by the following examples.

Example. Let X^a be a vector field and put $\tilde{\partial}_b = \partial/\partial\tilde{x}^b$. Then

$$\begin{aligned}\tilde{\partial}_b \tilde{X}^a &= \frac{\partial x^c}{\partial \tilde{x}^b}\frac{\partial}{\partial x^c}\left(\frac{\partial \tilde{x}^a}{\partial x^d}X^d\right)\\ &= \frac{\partial x^c}{\partial \tilde{x}^b}\frac{\partial \tilde{x}^a}{\partial x^d}\partial_c X^d + \frac{\partial x^c}{\partial \tilde{x}^b}\frac{\partial^2 \tilde{x}^a}{\partial x^c \partial x^d}X^d\end{aligned}$$

The first term is what we want for a tensor transformation law. The second term is the problem (in SR, where the coordinate transformations are all linear, it vanishes automatically: the problem is that we must now consider general, nonlinear coordinate transformations).

Example. Let $x^a = x^a(s)$ be the worldline of a particle parametrized by proper time. Then

$$\frac{\mathrm{d}x^a}{\mathrm{d}s} = \frac{\partial x^a}{\partial \tilde{x}^b}\frac{\mathrm{d}\tilde{x}^b}{\mathrm{d}s}$$

which implies that the 4-velocity components transform in the right way. But

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}s^2} = \frac{\partial x^a}{\partial \tilde{x}^b}\frac{\mathrm{d}^2\tilde{x}^b}{\mathrm{d}s^2} + \frac{\partial^2 x^a}{\partial \tilde{x}^b \partial \tilde{x}^c}\frac{\mathrm{d}\tilde{x}^b}{\mathrm{d}s}\frac{\mathrm{d}\tilde{x}^c}{\mathrm{d}s}\,.$$

Again the second term is the obstruction to a nice transformation law: the obvious definition of 4-acceleration does not lead to a tensor.

There is however a way out that does lead to a tensor transformation law in both these cases. Recall that the Christoffel symbols

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{ba} - \partial_a g_{bc})$$

obey the transformation law

$$\Gamma^a_{bc} = \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c}\tilde{\Gamma}^d_{ef} + \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c} = \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c}\tilde{\Gamma}^d_{ef} - \frac{\partial^2 x^a}{\partial \tilde{x}^e \partial \tilde{x}^f}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\tilde{x}^f}{\partial x^c}\,.$$

The second term is exactly what we want to cancel the unwanted term in the first example. Therefore

$$\nabla_b X^a = \partial_b X^a + \Gamma^a_{bc} X^c$$

transforms as a tensor of type $(1,1)$. It is called the *covariant derivative* of X^a.

To show the transformation rule in detail,

$$\begin{aligned}\partial_a X^b + \Gamma^b_{cd} X^d &= \frac{\partial \tilde{x}^c}{\partial x^a}\frac{\partial}{\partial \tilde{x}^c}\left(\frac{\partial x^b}{\partial \tilde{x}^f}\tilde{X}^f\right) + \frac{\partial x^b}{\partial \tilde{x}^e}\frac{\partial \tilde{x}^f}{\partial x^c}\frac{\partial \tilde{x}^h}{\partial x^d}\tilde{\Gamma}^e_{fh} X^d - \frac{\partial^2 x^b}{\partial \tilde{x}^e \partial \tilde{x}^f}\frac{\partial \tilde{x}^e}{\partial x^c}\frac{\partial \tilde{x}^f}{\partial x^d} X^d \\ &= \frac{\partial \tilde{x}^c}{\partial x^a}\frac{\partial x^b}{\partial \tilde{x}^d}\left(\tilde{\partial}_c \tilde{X}^d + \tilde{\Gamma}^a_{de}\tilde{X}^e\right).\end{aligned}$$

In a coordinate system such that $\partial_a g_{bc} = 0$ at the event $x^a = 0$, we have $\Gamma^a_{bc} = 0$ at $x^a = 0$ and hence $\nabla_a X^b = \partial_a X^b$. Note that this holds only at the origin. We could have used this property to define the covariant derivative: that is, we could equally well have defined the covariant derivative by requiring that the value of $\nabla_a X^b$ at A should be the tensor that coincides with $\partial_a X^b$ in local inertial coordinates at A. Then the tensor transformation law would have enabled us to write down its components in a general coordinate system. It is a common technique to define a tensor by giving its components in a particular coordinate system and then to use the transformation law backwards.

7.2 Parallel transport

Consider two nearby events A and B with coordinates x^b and $x^b + \delta x^b$. To the first order in δx^b,

$$\delta x^b \nabla_b Y^a = \delta x^b \partial_b Y^a + \delta x^b \Gamma^a_{bc} Y^c = Y^a(x+\delta x) - \Big(Y^a(x) - \delta x^b \Gamma^a_{bc} Y^c(x)\Big).$$

Thus the covariant derivative compares $Y^a(x+\delta x)$ (the components of Y at B) with $Y^a(x) - \delta x^b \Gamma^a_{bc} Y^c$, which we think of as the result of displacing Y^a to the 'most nearly parallel vector at B'.

Definition 7

The vector at B with components $Y^a(A) - \delta x^c \Gamma^a_{bc} Y^c(A)$ is said to be obtained by *parallel transport* of Y^a from A.

In local inertial coordinates at A, $\Gamma = 0$ at A, the vector at B is the one with the same components (to the first order in δx). It makes more sense to express this in terms of parallel transport along a curve—we then don't have to worry about infinitesimals.

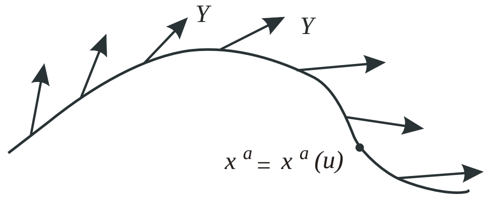


Figure 8: Parallel transport of Y along the curve $x^a = x^a(u)$.

Definition 8
A vector Y is a parallel transported (or propagated) along a curve $x^a = x^a(u)$ whenever

$$\frac{\mathrm{d}Y^a}{\mathrm{d}u} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}u}Y^c = 0.$$

This is a set of ordinary differential equations for the components Y^a as functions of the parameter u: it determines the Y^a in terms of their values at the initial point of the curve.

The tangent vector to a geodesic, which is a solution of

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0,$$

is parallel propagated along the geodesic: this is the sense in which geodesics are curves in a curved space-time which are 'as straight as possible'.

Parallel propagation around a closed curve need not bring you back to your starting point: this is a manifestation of curvature.

7.3 The wave equation

Suppose that u is a function on space-time. Then the partial derivatives $\partial_a u$ are the components of a covector, the gradient covector. We can define a vector field with components $\nabla^a u$ by putting

$$\nabla^a u = g^{ab}\partial_b u\,.$$

That is, by raising the index. This is the *gradient vector*. The *wave operator* sends u to

$$\nabla_a(\nabla^a u) = \partial_a(g^{ab}\partial_b u) + \Gamma^a_{ab} g^{bc}\partial_c u \,.$$

It is an exercise on problem sheet 4 to show that

$$\Gamma^b_{ab} = \tfrac{1}{2}\partial_a \log |g| \,,$$

where g is the determinant of the matrix (g_{ab})—it follows from the fact that if A is a square matrix depending on the coordinates x^a, then

$$\partial_a \log \det A = \operatorname{tr}(A^{-1}\partial_a A) \,.$$

This implies that

$$\Gamma^b_{ab} = \tfrac{1}{2} g^{bd}(\partial_b g_{ad} + \partial_a g_{bd} - \partial_d g_{ab}) = \tfrac{1}{2} g^{bd}\partial_a g_{bd} = \partial_a \log \sqrt{|g|} \,.$$

Thus the wave operator is also

$$u \mapsto \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^a}\left(\sqrt{|g|}\, g^{ab} \frac{\partial u}{\partial x^b}\right) \,.$$

It is a natural operator; that is, it is independent of the choice of coordinates.

Lecture 8

8.1 Covariant derivatives of tensors

We extend the defintion of ∇_a to scalars by putting $\nabla_a f = \partial_a$ and to covectors by

$$\nabla_a \alpha_b := \partial_a \alpha_b - \alpha_c \Gamma^c_{ab}\,.$$

By a similar argument to that used in the case of vectors, this transforms as a tensor of type $(0,2)$.

Exercise. Show that $\partial_a(\alpha_b Y^b) = (\nabla_a \alpha_b)Y^b + \alpha_b \nabla_a Y^b$. (Note that $\alpha_b Y^b$ is a scalar, so the gradient covector on the LHS is well defined.)

For a general tensor field, the covariant derivative has one gamma for each index. For example,

$$\nabla_a T^{bc}{}_d = \partial_a T^{bc}{}_d + \Gamma^b_{ae} T^{ec}{}_d + \Gamma^c_{ae} T^{be}{}_d - \Gamma^e_{ad} T^{bc}{}_e.$$

We also write $\nabla_a f$ for $\partial_a f$ when f is a scalar.

Properties of the covariant derivative.

(cd1) $\nabla_a (T^{\cdots}{}_{\ldots} + S^{\cdots}{}_{\ldots}) = \nabla_a T^{\cdots}{}_{\ldots} + \nabla_a S^{\cdots}{}_{\ldots}$.

(cd2) $\nabla_a (f T^{\cdots}{}_{\ldots}) = f \nabla_a T^{\cdots}{}_{\ldots} + (\nabla_a f) T^{\cdots}{}_{\ldots}$.

(cd3) $\nabla_a (T^{\cdots}{}_{\ldots} S^{\cdots}{}_{\ldots}) = \nabla_a (T^{\cdots}{}_{\ldots}) S^{\cdots}{}_{\ldots} + T^{\cdots}{}_{\ldots} \nabla_a (S^{\cdots}{}_{\ldots})$.

(cd4) $\nabla_a T^{bc}{}_b$ is the same whether the contraction is done before or after the differentiation.

(cd5) The covariant derivative of the Kronecker delta vanishes since

$$\nabla_a \delta^b_c = \partial_a \delta^b_c + \Gamma^b_{ad} \delta^d_c - \Gamma^d_{ac} \delta^b_d = 0.$$

(cd6) For a scalar f, but not for a general tensor,

$$\nabla_a \nabla_b f = \partial_a \partial_b f - \partial_c f \Gamma^c_{ab} = \nabla_b \nabla_a f.$$

(cd7) The covariant derivative of the metric tensor vanishes, since

$$\begin{aligned}
\nabla_a g_{bc} &= \partial_a g_{bc} - g_{dc} \Gamma^d_{ab} - g_{bd} \Gamma^d_{ac} \\
&= \partial_a g_{bc} - \tfrac{1}{2}\{\partial_a g_{cb} + \partial_b g_{ac} - \partial_c g_{ab}\} - \tfrac{1}{2}\{\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac}\} \\
&= 0
\end{aligned}$$

(cd8) $\nabla_a g^{bc} = 0$. This follows from (cd7) and

$$0 = \nabla_a(\delta^b_d) = \nabla_a(g^{bc}g_{cd}) = \nabla_a(g^{bc})g_{cd} + g^{bc}\nabla_a g_{cd} = 0.$$

Hence $\nabla_a g^{bc} = 0$.

It follows from (cd7) and (cd8) that raising and lowering can be interchanged with covariant differentiation. For example if X^a is a vector field, then $\nabla_a X_b$ is well-defined: it does not matter whether you lower the index on the X before or after the differentiation.

Example. *Maxwell's equations.* In a curved space-time and in the absence of sources, these are

$$\nabla_a F^{ab} = 0, \qquad \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$$

because these equations are covariant and reduce to the special relativity form in local inertial coordinates at a point. Gravity affects light through the Γs.

8.2 Connections

All that is needed to define the covariant derivative in a coordinate independent way is the transformation rule

$$\Gamma^a_{bc} = \frac{\partial x^a}{\partial \tilde{x}^d}\tilde{\Gamma}^d_{ef}\frac{\partial \tilde{x}^e}{\partial x^b}\frac{\partial \tilde{x}^f}{\partial x^c} + \frac{\partial x^a}{\partial \tilde{x}^d}\frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c}$$

A field of Γ^a_{bc}s with this transformation property is called a set of *connection coefficients*. The corresponding operator ∇ is called a connection—it connects the spaces of vectors and tensors at nearby events.

(cd1)-(cd5) hold for all connections; (cd6) holds only if the connection is torsion-free—that is $\Gamma^c_{ab} = \Gamma^c_{ba}$. (cd7) holds *in addition* only for

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{26}$$

This is the unique torsion-free connection for which the covariant derivative of the metric tensor vanishes (problem sheet). It is called the *Levi-Civita connection.*

Note that if Γ^a_{bc} is a set of connection coefficients and Q^a_{bc} is a tensor, then $\Gamma^a_{bc} + Q^a_{bc}$ is also a set of connection coefficients. All connections can be obtained in this way once once is given.

From now on ∇ will always be the Levi-Civita connection, defined by (26).

8.3 Curvature

In Minkowski space, there are global coordinate systems in which g_{ab} is constant: in such coordinates $\nabla_a = \partial_a$ and therefore $\nabla_a \nabla_b = \nabla_b \nabla_a$ (acting on vectors). So if in a general space-time, $\nabla_a \nabla_b \neq \nabla_b \nabla_a$ on vectors, then we know that the metric cannot be reduced to the SR form by a coordinate change.

PROPOSITION 6

For any metric g_{ab}, there is a tensor field $R_{abc}{}^d$ of type $(1,3)$ such that

$$\nabla_a \nabla_b X^d - \nabla_b \nabla_a X^d = R_{abc}{}^d X^c$$

for any 4-vector field X.

The tensor $R_{abc}{}^d$ is called the *Riemann tensor* or the *curvature tensor*.

Proof. From the definition of ∇,

$$\begin{aligned}
\nabla_a \nabla_b X^d &= \nabla_a (\partial_b X^d + \Gamma^d_{bc} X^c) \\
&= \partial_a \partial_b X^d + (\partial_a \Gamma^d_{bc}) X^c + \Gamma^d_{bc} \partial_a X^c \\
&\quad + \Gamma^d_{ae} (\partial_b X^e + \Gamma^e_{bc} X^c) - \Gamma^e_{ab} (\partial_e X^d + \Gamma^d_{ec} X^c)
\end{aligned}$$

Hence

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^d = (\partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} - \Gamma^d_{be} \Gamma^e_{ac} + \Gamma^d_{ae} \Gamma^e_{bc}) X^c.$$

We define the expression in brackets to be $R_{abc}{}^d$. We must show that it is a tensor. The direct method is horrible. We know, however, that the left-hand side is a tensor. Hence, if we change coordinates,

$$\begin{aligned}
\tilde{\nabla}_a \tilde{\nabla}_b \tilde{X}^d - \tilde{\nabla}_b \tilde{\nabla}_a \tilde{X}^d &= \frac{\partial x^p}{\partial \tilde{x}^a} \frac{\partial x^q}{\partial \tilde{x}^b} \frac{\partial \tilde{x}^d}{\partial x^s} R_{pqc}{}^s X^c \\
&= \frac{\partial x^p}{\partial \tilde{x}^a} \frac{\partial x^q}{\partial \tilde{x}^b} \frac{\partial x^r}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^d}{\partial x^s} R_{pqr}{}^s \tilde{X}^c. \qquad (27)
\end{aligned}$$

However, had we worked from the beginning in the tilde coordinates, we would have obtained

$$(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a) \tilde{X}^d = \tilde{R}_{abc}{}^d \tilde{X}^c, \qquad (28)$$

where $\tilde{R}_{abc}{}^d$ is defined in the same way as $R_{abc}{}^d$. Since (27) and (28) hold for any X, we deduce that

$$\tilde{R}_{abc}{}^d = \frac{\partial x^p}{\partial \tilde{x}^a} \frac{\partial x^q}{\partial \tilde{x}^b} \frac{\partial x^r}{\partial \tilde{x}^c} \frac{\partial \tilde{x}^d}{\partial x^s} R_{pqr}{}^s,$$

which is the tensor transformation law. ∎

Corollary 7

If there exists a vector field X such that $\nabla_a \nabla_b X^d \neq \nabla_b \nabla_a X^d$, there does not exist a coordinate system in which the metric coefficients are constant.

8.4 Symmetries of the Riemann tensor

In terms of the connection coefficients

$$R_{abc}{}^{d} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} - \Gamma^d_{be}\Gamma^e_{ac} + \Gamma^d_{ae}\Gamma^e_{bc}$$

Pick an event A and choose coordinates such that $\partial_a g_{bc} = 0$ at A. Then $\Gamma^a_{bc} = 0$ at A. We can drop all the first derivatives of g_{ab} and g^{ab}, but not the second derivatives. So, at the event A (but *not* elsewhere),

$$\begin{aligned} R_{abcd} &= g_{de}\partial_a(\Gamma^e_{bc}) - g_{de}\partial_b(\Gamma^e_{ac}) \\ &= \tfrac{1}{2}\partial_a(\partial_c g_{bd} + \partial_b g_{dc} - \partial_d g_{bc}) - \tfrac{1}{2}\partial_b(\partial_c g_{da} + \partial_a g_{dc} - \partial_d g_{ac}) \\ &= \tfrac{1}{2}[\partial_a\partial_c g_{bd} + \partial_b\partial_d g_{ac} - \partial_a\partial_d g_{bc} - \partial_b\partial_c g_{ad}] \end{aligned}$$

From this we deduce that

$$\begin{aligned} &(S1) \quad R_{abcd} = -R_{bacd} \\ &(S2) \quad R_{abcd} = R_{cdab} \\ &(S3) \quad R_{abcd} = -R_{abdc} \\ &(S4) \quad R_{abcd} + R_{bcad} + R_{cabd} = 0. \end{aligned}$$

Note that symmetries (S1), (S3) and (S4) imply (S2) (exercise).

Lecture 9

9.1 Bracket notation

For a general covariant tensor with p lower indices,

$$\begin{aligned} T_{[ab\ldots c]} &:= \frac{1}{p!}\sum_{\text{perms}} \text{sign}\,(\sigma) T_{\sigma(a)\sigma(b)\ldots\sigma(c)} \\ T_{(ab\ldots c)} &:= \frac{1}{p!}\sum_{\text{perms}} T_{\sigma(a)\sigma(b)\ldots\sigma(c)} \end{aligned}$$

where the sums are over the permutations σ of p objects, and $\text{sign}\,(\sigma)$ is 1 or -1 as σ is even or odd. For example,

$$\begin{aligned} T_{[ab]} &= \tfrac{1}{2}(T_{ab} - T_{ba}) \\ T_{(ab)} &= \tfrac{1}{2}(T_{ab} + T_{ba}) \\ T_{[abc]} &= \tfrac{1}{6}(T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{acb} - T_{cba}) \\ T_{(abc)} &= \tfrac{1}{6}(T_{abc} + T_{bca} + T_{cab} + T_{bac} + T_{acb} + T_{cba}) \end{aligned}$$

The same definitions apply to brackets on a subset of the indices and to brackets on upper indices. For example

$$T^{[ab](cd)} = \tfrac{1}{4}(T^{abcd} - T^{bacd} + T^{abdc} - T^{badc})\,.$$

There is a possibility of ambiguity over the order of the operations if two sets of brackets overlap (as, for example, in the expression $T_{[a(bc]d)}$); so overlaps are forbidden. Nested brackets, however, are unambiguous, although they can always be simplified since

$$T_{[\ldots(\ldots)\ldots]} = 0 = T_{(\ldots[\ldots]\ldots)}, \quad T_{[\ldots[\ldots]\ldots]} = T_{[\ldots]}, \quad T_{(\ldots(\ldots)\ldots)} = T_{(\ldots)}\,.$$

Examples. With this notation, the fourth symmetry of the Riemann tensor reads

$$R_{[abc]d} = 0\,.$$

Maxwell's equations (without sources) are

$$\nabla_a F^{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0\,.$$

The symmetries of the contravariant metric g^{ab} and of the alternating tensor ϵ_{abcd} (see problem sheet 1) can be expressed respectively as

$$g^{[ab]} = 0, \qquad \epsilon_{abcd} = \epsilon_{[abcd]}\,.$$

9.2 The Bianchi identity

Choose coordinates such that $\Gamma^a_{bc} = 0$ at an event. Then

$$\nabla_a R_{bcd}{}^e = \partial_a \partial_b \Gamma^e_{cd} - \partial_a \partial_c \Gamma^e_{bd} + \text{terms in } \Gamma\partial\Gamma \text{ and } \Gamma\Gamma\Gamma$$

Since the first term on the RHS is symmetric in ab and the second in ac,

$$\nabla_{[a} R_{bc]d}{}^e = 0$$

at the event in this coordinate system. However, this is a tensor equation, so it is valid in every coordinate system. It is called the *Bianchi identity*.

9.3 The operator D

Let ω be a the worldline of an observer (not necessarily in free fall). Let s be the proper time along ω and let $V^a = \mathrm{d}x^a/\mathrm{d}s$ be the 4-velocity of O: this is a vector defined at each point of ω.

We need a tool: a derivative operator that measures the rate of change of vectors and tensors as observed by O. Suppose that Y^a is a vector field. Then at each event on ω, we define DY^b by

$$DY^b = V^a \nabla_a Y^b = \frac{\mathrm{d}x^a}{\mathrm{d}s} \partial_a Y^b + \Gamma^b_{ac} V^a Y^c = \frac{\mathrm{d}Y^b}{\mathrm{d}s} + \Gamma^b_{ac} V^a Y^c.$$

The first equality makes it clear that DY^a is a well-defined vector at each point of ω. The last, that the values of DY^a along ω depend only on the values of $Y^a(s)$ along ω: so DY^a makes sense for vectors fields that are defined only along ω. Note that $DY^a = 0$ is the equation of parallel transport.

We similarly define D on tensor fields along ω by, for example,

$$DT^a_{\ b} = \frac{\mathrm{d}T^a_{\ b}}{\mathrm{d}s} + \Gamma^a_{cd} V^c T^d_{\ b} - \Gamma^d_{cb} V^c T^a_{\ d}.$$

If the observer is in free fall, then $D = \mathrm{d}/\mathrm{d}s$ in local inertial coordinates in which the observer is at rest at an event.

9.4 Geodesic deviation

The crucial property of curvature is that it measures the *relative acceleration of nearby particles in free fall*. That is, it directly encodes the gravitational field.

Imagine a cloud of particles in free fall. Let us suppose that an observer O is on one of the particles, and that this particle has worldline ω. Suppose that the observer looks at a nearby particle and measures its position in local inertial coordinates. In SR, it will move in a straight line at constant speed, and will have no acceleration. What happens in a gravitational field?

The 4-velocities of the particles form a vector field V^a. Since the individual particle worldlines are geodesic,

$$V^a \nabla_a V^b = DV^a = \frac{\mathrm{d}V^a}{\mathrm{d}s} + \Gamma^a_{bc} V^b V^c = 0.$$

Pick out a particle P near O, and at each event on ω, let Y^a be the 4-vector joining the event to a simultaneous event at P (since P is 'near' O, Y is small: we shall ignore second order terms in its components). In the LIC in which O is instantaneously at rest, Y has components $(0, \boldsymbol{y})$, where $\boldsymbol{y}$ is the position of P. If ω is given by $x^a = x^a(s)$ in general coordinates, P's worldline is $x^a(s) + Y^a(s) + O(2)$, where $O(2)$ denotes second-order and smaller terms in the components of $\boldsymbol{y}$. The parameter s here is the proper time along the worldline of O. The proper time separation $\mathrm{d}s$ between two nearby events $x^a(s)$ and $x^a(s+\delta s)$ on the worldline of O is the same to the second order in $\boldsymbol{y}$ as the proper time between the corresponding events, $x^a(s) + Y^a(s)$ and $x^a(s+\delta s) + Y^a(s+\delta s)$, on the worldline of P. To this approximation, therefore, s is also the proper time along P's worldline.

We note that Y^a is a vector field along ω such that $V^a Y_a = 0$ (because $Y = (0, \boldsymbol{y})$ and $V = (1, 0)$ in the LIC in which O is instantaneously at rest).

Since $DV^a = 0$, we have

$$0 = D(V_a Y^a) = V_a DY^a \quad \text{and} \quad 0 = D(V_a DY^a) = V_a D^2 Y^a.$$

In the local rest frame of O at an event on its worldline, the 4-velocity of O is $(1, \mathbf{0})$; and the vectors Y^a, DY^a and D^2Y^a are, respectively, of the form $(0, \boldsymbol{y})$, $(0, \boldsymbol{u})$ and $(0, \boldsymbol{a})$, where $\boldsymbol{u}$ is the relative velocity of P to O and $\boldsymbol{a}$ is the relative acceleration.

We are interested in the relative acceleration, and therefore in D^2Y^a. We want to express this in terms of the curvature. The key to this is the following result.

Proposition 8
$DY^a = Y^b\nabla_b V^a$.

Proof. We know that

$$V^a(P) = \frac{\mathrm{d}x^a}{\mathrm{d}s} + \frac{\mathrm{d}Y^a}{\mathrm{d}s} + O(2) = V^a(O) + \frac{\mathrm{d}Y^a}{\mathrm{d}s} + O(2)\,.$$

On the other hand, by expanding to the first order in ϵ,

$$V^a(P) = V^a(O) + Y^c\partial_c V^a + O(2)\,.$$

Therefore $\mathrm{d}Y^a/\mathrm{d}s = Y^c\partial_c V^a$. It follows that

$$DY^a = \frac{\mathrm{d}Y^a}{\mathrm{d}s} + \Gamma^a_{bc}V^bV^c = Y^c\partial_c V^a + \Gamma^a_{bc}V^bY^c = Y^b\nabla_b V^a\,,$$

which is the result we need. ∎

Now we can derive the *equation of geodesic deviation* (Jacobi equation), which is of central importance to the physical interpretation of curvature.

$$\begin{aligned} D^2Y^d &= D(Y^b\nabla_b V^d) \\ &= (DY^b)\nabla_b V^d + Y^b D(\nabla_b V^d) \\ &= (Y^a\nabla_a V^b)\nabla_b V^d + Y^b V^a\nabla_a\nabla_b V^d \\ &= Y^a(\nabla_a V^b)\nabla_b V^d + Y^b V^a\nabla_b\nabla_a V^d + R_{abc}{}^d V^aY^bV^c. \end{aligned}$$

But

$$V^a\nabla_b\nabla_a V^d = \nabla_b(V^a\nabla_a V^d) - (\nabla_b V^a)(\nabla_a V^d) = -(\nabla_b V^a)(\nabla_a V^d)$$

since $V^a\nabla_a V^d = 0$ by the geodesic equation. Therefore the first two terms in the last line cancel, and

$$D^2Y^d = R_{abc}{}^d V^aY^bV^c,$$

which is the geodesic deviation equation. It relates the relative acceleration of nearby particles in free fall to the curvature tensor.

Lecture 10

10.1 Tidal forces

The relative acceleration of two particles in free fall is determined by the equation of geodesic deviation

$$D^2Y^d = R_{abc}{}^d V^a V^c Y^b \,.$$

If you are travelling with one of the particles, with 4-velocity V, then the acceleration $\boldsymbol{a}$ of your neighbours is a linear function of their position. In local inertial coordinates in which you are (instantaneously) at rest, $V = (1, 0)$, $Y = (0, \boldsymbol{y})$, where $\boldsymbol{y}$ is the position vector of a nearby particle, and

$$\boldsymbol{a} = M\boldsymbol{y}$$

where M is the 3×3 matrix with entries

$$M_{ij} = R_{0i0}{}^j$$

$(i, j = 1, 2, 3)$. It is symmetric because of the symmteries of the Riemann tensor, and because the compoents are taken in local intertial coordinates.

What is the corresponding result in Newtonian gravity? Consider a cloud of particles in free fall. The acceleration of each particle is given by $\ddot{\boldsymbol{r}} = -\nabla\phi$, or in components, by $\ddot{r}_i = -\partial_i\phi$. Since the particle at O has acceleration $(-\partial_\phi)_O$ and the particle at P has acceleration $(-\partial_i\phi)_P$, the acceleration of that at P relative to that at O has components

$$a_i = (-\partial_i\phi)_P - (-\partial_i\phi)_O = -y_j\partial_j\partial_i\phi + O(2) \tag{29}$$

to the first order in $\boldsymbol{y}$, where $\boldsymbol{y}$ is the position vector of P relative to O, the second derivatives are evaluated at O, and there is a sum over $j = 1, 2, 3$.

Aside. Another way to read (29) is in terms of tidal forces. If O and P are joined by a light rod, and falling towards the moon with the rod pointing towards the moon, then then there is a tension force in the rod. This is true whichever particle is in the lead. If we think of O as the centre of the earth, and of P as a mass of water on the surface, then we see that this *tidal force* will push P away from the centre: this is true whether P is on the surface directly under the moon or on the opposite side of the earth. Thus the tidal force raises two 'humps' in the ocean, one under the moon and one on the other side. As the earth rotates, the humps move round, giving two high tides each day. We can trace the reason that there are two high tides to the symmetry of the second derivatives in $\partial_j\partial_i\phi$. Before Newton, even Galileo's explanation of this fact was confused (and erroneous).

When our observer in curved space-time looks at the nearby particles, therefore, he reckons that he is in a gravitational field with potential ϕ such that

$$M_{ij} = -\partial_i \partial_j \phi = R_{0i0}{}^{j}$$

at his location. Now in empty space, Poisson's equation reduces to $\nabla^2 \phi = 0$. That is $\partial_i \partial_i \phi = 0$ or, in other words, $\mathrm{tr} M = 0$. So in GR, we require $R_{abc}{}^{b} V^a V^c = 0$. Since this must hold for every 4-velocity V, we are led to *Einstein's vacuum field equation*

$$R_{ab} = 0\,, \tag{30}$$

where R_{ab} is the *Ricci tensor*, defined by $R_{ab} = R_{acb}{}^{c}$. The vacuum equation is in fact ten equations (for the ten independent components of the symmetric tensor R_{ab}) in ten unknowns (the ten independent components of the metric g_{ab}). The equations are nonlinear, as anticipated. The use of the summation convention makes them look very simple. Written out explicitly without this notation, each equation would contain 1248 terms. Not surprsingly, therefore, it is not easy to find solutions.

We shall see two justifications for the vacuum equation: (1) it reduces to the Newtonian equation in the weak field limit, and (2) it has a solution (the *Schwarzschild solution*) like $\phi = -GM/r$, which encodes the inverse-square law of gravity.

10.2 The weak field limit

We consider the case in which the gravitational field is *weak* and *slowly varying*. That is the metric is

$$w_{ab} = g_{ab} + \epsilon h_{ab}\,,$$

where $g_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski space metric, ϵ is a small parameter, and the coordinates are the Minkowski space inertial coordinates t, x, y, z. 'Weak' means that we ignore terms of order ϵ^2; 'slowly varying' means that $\partial_0 h_{ab} = O(\epsilon)$.

To find the vacuum equation in this case, we first have to find the contravariant metric w^{ab} (defined by $w^{ab} w_{bc} = \delta^a_c$). This is given by

$$w^{ab} = g^{ab} + O(\epsilon)\,,$$

where $g^{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the contravariant Minkowski space metric . There is an immediate possibility for confusion here since we are dealing simultaneously with two metrics, w and g, so we shall avoid raising and lowering indices.

The Christoffel symbols are given to the first order in ϵ by

$$\Gamma^a_{bc} = \tfrac{1}{2} w^{ad}(\partial_b w_{cd} + \partial_c w_{bd} - \partial_d w_{bc}) = \tfrac{1}{2}\epsilon g^{ad}(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) + O(\epsilon^2)$$

and therefore the Riemann tensor is

$$\begin{aligned} R_{abc}{}^d &= \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + O(\epsilon^2) \\ &= \tfrac{1}{2}\epsilon g^{de}(\partial_a\partial_c h_{be} + \partial_b \partial_e h_{ac} - \partial_a \partial_e h_{bc} - \partial_b \partial_c h_{ae}) + O(\epsilon^2)\,. \end{aligned} \tag{31}$$

Now consider the motion of a slow moving particle with worldline $x^a = x^a(t)$. Since the speed u is small, we have $\gamma(u) \sim 1$ and so we can identify the coordinate time t with the proper time along the wordline. The 4-velocity is

$$(V^a) = \left(\frac{\mathrm{d}x^a}{\mathrm{d}s}\right) = (1,0,0,0) + O(\epsilon), \quad s = t + O(\epsilon)\,,$$

and the geodesic equation is

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}s^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0\,.$$

To the first order, this is

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}s^2} + \Gamma^a_{00} = 0\,,$$

where

$$\Gamma^a_{00} = \begin{cases} O(\epsilon^2) & \text{when } a = 0 \\ \tfrac{1}{2}\epsilon\partial_a h_{00} + O(\epsilon^2) & \text{when } a \neq 0\,, \end{cases}$$

since $\partial_0 h_{ab} = O(\epsilon)$. When $a = 0$, this gives us $\mathrm{d}^2 t/\mathrm{d}s^2 = O(\epsilon^2)$. The other three components give the equation of motion

$$\ddot{\boldsymbol{r}} = -\tfrac{1}{2}\nabla(\epsilon h_{00}) + O(\epsilon^2)\,,$$

where the dot can be differentiation with respect to either t or s. Thus if we want to reduce GR to the Newtonian theory in this limit, then we must take $\phi = \frac{1}{2}\epsilon h_{00}$.

The derivatives of h_{ab} with respect to x^0 in (31) are all of order ϵ. So we have

$$R_{0b0}{}^d = \tfrac{1}{2}\epsilon g^{de}\partial_b\partial_e h_{00} + O(\epsilon^2)\,.$$

Therefore the $0,0$ component of the vacuum equation $R_{ab} = 0$ is

$$R_{0b0}{}^b = \tfrac{1}{2}\epsilon(-\partial_1^2 h_{00} - \partial_2^2 h_{00} - \partial_3^2 h_{00}) = -\tfrac{1}{2}\epsilon\nabla^2 h_{00} = 0\,,$$

that is, $\nabla^2\phi = 0$, which is the Laplace's equation (the vacuum equation in Newtonian theory). Thus in this limit, Einstein's theory reduces to Newton's. (What about the other nine components of the vacuum equation?).

10.3 The non-vacuum case

What happens when there is matter present? What is the analogue of Poisson's equation $\nabla^2\phi = 4\pi G\rho$?

We shall consider only the case in which the matter generating the gravitational field is a dust cloud. Then its energy density is encoded in the energy-momentum tensor $T^{ab} = \rho U^a U^b$ where U^a is the 4-velocity field of the dust and ρ is the energy (mass) density measured in the local rest frame. We know that $\partial_a T^{ab} = 0$ in local inertial coordinates since this *continuity equation* holds in SR. Therefore we have $\nabla_a T^{ab} = 0$ in general coordinates, since this is a tensor equation which reduces to the continuity equation in local inertial coordinates. This suggests that the GR field equation should equate R_{ab} to a constant multiple of T_{ab}. Unfortunately, this will not do because in general $\nabla_a R^{ab} \neq 0$. But there is a tensor closely related to the Ricci tensor which can be put on the LHS without contradiction: this is the *Einstein tensor*

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab}\,,$$

where $R = R_a{}^a$ is the *Ricci scalar* or *scalar curvature*.

PROPOSITION 9
For any space-time metric, $\nabla_a G^{ab} = 0$.

Proof. The Bianchi identity is

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0\ ,.$$

By contracting with $g^{ad}g^{ce}$, we obtain

$$0 = 2\nabla^a R_{ab} - \nabla_b R = 2\nabla^a (R_{ab} - \tfrac{1}{2}g_{ab}R) = 2\nabla^a G_{ab}\,,$$

which completes the proof. ∎

Our candidate for the field equation is $G_{ab} = \lambda\rho U_a U_b$, with λ constant. By contracting with g^{ab}, we obtain

$$R - 2R = \lambda T_a{}^a = \lambda\rho$$

since $g^{ab}g_{ab} = 4$ and $U^a U_a = 1$. So an equivalent form of the equation is

$$R_{ab} = \lambda(T_{ab} - \tfrac{1}{2}\rho g_{ab}).$$

Now in the coordinates we used in the weak field limit, $R_{00} = -\nabla^2\phi$, while $T_{00} = \rho$. Thus in this limit, we have $\nabla^2\phi = -\frac{1}{2}\lambda\rho$. To obtain the correct correspondence with the Newtonian theory, therefore, we must take $\lambda = -8\pi G$, which means that the field equation is

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -8\pi G T_{ab}\,.$$

This is in fact quite general: the tensor T_{ab} can be the energy-momentum tensor of any form of matter or energy.

In future, we shall always use units in which $G = 1 = c$ (note that given the unit of time (say the second), putting putting $c = 1$ fixes the unit of distance (the light-second) and putting $G = 1$ fixes the unit of mass.

Example. Calculate the conversion factors to SI units and take note that the units we shall use are not likely to find favour with engineers.

Lecture 11

11.1 Spherical symmetry

The next few sections will be of the form an extended worked example.

We want to find the gravitational field outside a spherical body of mass m, that is the solution of the vacuum equation analogous to the Newtonian potential $\phi = -Gm/r$ (for simplicity, we shall now choose units so that $G = 1$). By saying the solution has mass m, we mean that $g_{00} \sim 1 - 2m/r$ for large r, so that a long way from the body, where the field is weak, it looks like the field of a static spherically symmetric body of mass m: in operational terms, m is the mass measured by analysing orbits in the field of the body near infinity.

We want the metric to have the symmetries appropriate to a static spherical body. Now in spherical polar coordinates, the Minkowski space metric is

$$\mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,. \tag{32}$$

The expression in brackets is the metric (first fundamental form) on the unit sphere.

Our metric must reduce to (32) when $m = 0$ and in any case in the limit $r \to \infty$. This flat metric has the following features.

- No t-dependence.
- No $\mathrm{d}t\mathrm{d}r$, $\mathrm{d}t\mathrm{d}\phi$, or $\mathrm{d}t\mathrm{d}\theta$ terms: it is therefore *time reversible*—that is, invariant under $t \mapsto -t$.
- No $\mathrm{d}r\mathrm{d}\theta$ or $\mathrm{d}r\mathrm{d}\phi$ terms: thus at constant time, the radial vector is perpendicular to the surfaces of constant r.
- The metric on the surface $t = \text{const}$, $r = \text{const}$ is a constant multiple of the metric on the unit sphere.
- The coefficients of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ are independent of θ and ϕ.

The last three properties might be taken to characterize spherical symmetry. We shall assume that out metric has all these properties, and thus that it is of the form

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - C(r)r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,,$$

for some functions A, B, C of r. We can in fact take $C = 1$ without loss of generality by replacing r by $r\sqrt{C}$. Our task then is to solve the vacuum equation $R_{ab} = 0$, subject to the boundary conditions $A, B \to 1$ and $A = 1 - 2m/r + O(r^{-2})$ as $r \to \infty$.

11.2 The curvature tensor

We need to find R_{ab} in terms of A and B (with $C = 1$). We shall do this by a straightforward, but head-on method: a more elegant method can be found in Hughston and Tod, but it requires some understanding of differential forms.

The first step is to find the Christoffel symbols from the geodesic equations. These are the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$$

generated by the Lagrangian

$$L = \tfrac{1}{2}(A\dot{t}^2 - B\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2)\,,$$

with $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. The idea is to read off the Christoffel symbols by comparing the Lagrange equations with $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$.

The Lagrange equations are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big(A\dot{t}\Big) &= 0\\
\frac{\mathrm{d}}{\mathrm{d}s}\Big(-B\dot{r}\Big) - \tfrac{1}{2}A'\dot{t}^2 + \tfrac{1}{2}B'\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 &= 0\\
\frac{\mathrm{d}}{\mathrm{d}s}\Big(-r^2\dot{\theta}\Big) + r^2\sin\theta\cos\theta\,\dot{\phi}^2 &= 0\\
\frac{\mathrm{d}}{\mathrm{d}s}\Big(-r^2\sin^2\theta\dot{\phi}\Big) &= 0
\end{aligned}$$

where the *dot* denotes the derivative with respect to s. Equivalently

$$\begin{aligned}
\ddot{t} + A'A^{-1}\dot{t}\dot{r} &= 0\\
\ddot{r} + \tfrac{1}{2}A'B^{-1}\dot{t}^2 + \tfrac{1}{2}B'B^{-1}\dot{r}^2 - B^{-1}r\dot{\theta}^2 - B^{-1}r\sin^2\theta\,\dot{\phi}^2 &= 0\\
\ddot{\theta} + 2r^{-1}\dot{\theta}\dot{r} - \sin\theta\cos\theta\,\dot{\phi}^2 &= 0\\
\ddot{\phi} + 2r^{-1}\dot{\phi}\dot{r} + 2\cot\theta\,\dot{\theta}\dot{\phi} &= 0\,.
\end{aligned}$$

From these we read off the Christoffel symbols Γ^a_{bc} (note carefully the factors of $\frac{1}{2}$ when $b \neq c$: where do they come from?)

$(a = 0)$ $\quad \Gamma^0_{01} = \Gamma^0_{10} = A'/2A$

$$(a = 1) \qquad \Gamma^1_{00} = A'/2B, \quad \Gamma^1_{11} = B'/2B, \quad \Gamma^1_{22} = -r/B, \quad \Gamma^1_{33} = -r\sin^2\theta/B$$

$$(a = 2) \qquad \Gamma^2_{21} = \Gamma^2_{12} = r^{-1}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$(a = 3) \qquad \Gamma^3_{31} = \Gamma^3_{13} = r^{-1}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta\,.$$

All the others vanish. From the definition of the curvature tensor, we have

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} - \Gamma^d_{be}\Gamma^e_{ac} + \Gamma^d_{ae}\Gamma^e_{bc}\,.$$

The components R_{ac} of the Ricci tensor are then given by putting $b = d$ and summing. We find

$$\begin{aligned}
R_{232}{}^3 &= \partial_2\Gamma^3_{32} - \partial_3\Gamma^3_{22} - \Gamma^3_{3e}\Gamma^e_{22} + \Gamma^3_{2e}\Gamma^e_{32} \\
&= \partial_\theta(\cot\theta) + B^{-1} + \cot^2\theta \\
&= -1 + B^{-1} \\
R_{010}{}^1 &= -A''/2B + B'A'/4B^2 + A'^2/4BA \\
R_{020}{}^2 &= R_{030}{}^3 = -A'/2Br \\
R_{121}{}^2 &= R_{131}{}^3 = -B'/2Br \\
R_{101}{}^0 &= -BR_{010}{}^1/A \\
R_{303}{}^0 &= -r^2\sin^2\theta R_{030}{}^3/A = r\sin^2\theta A'/2BA \\
R_{313}{}^1 &= r^2\sin^2\theta R_{131}{}^3/B = -r\sin^2\theta B'/2B^2\,.
\end{aligned}$$

Finally, we obtain the vacuum equations in the form

$$R_{00} = R_{010}{}^1 + R_{020}{}^2 + R_{030}{}^3 = -A''/2B + B'A'/4B^2 + A'^2/4BA - A'/Br = 0 \tag{33}$$

together with

$$\begin{aligned}
R_{11} &= A''/2A - A'^2/4A^2 - B'A'/4BA - B'/Br = 0 && (34)\\
R_{22} &= R_{33}/\sin^2\theta = rA'/2BA - rB'/2B^2 + 1/B - 1 = 0\,. && (35)
\end{aligned}$$

All the other components of the Ricci tensor vanish identically (as can be seen by direct calculation or by using the fact that R_{ab} must have the same symmetries as the metric).

In all, we have three equations in the two unknowns A, B. Fortunately they are consistent.

If we take A times (33) and add A times (34), we get

$$AB' + BA' = 0\,,$$

and hence that AB is constant. Since we want $A, B \to 1$ as $r \to \infty$, we must therefore have $AB = 1$. By substituting into (35), we then get that $rA' + A = 1$ and hence that

$$A = \frac{1}{B} = 1 + \frac{k}{r}$$

for some constant k. But for large r, we want $A = 1 - 2m/r + O(r^2)$, so $k = -2m$, and the solution is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

This is the *Schwarzschild metric*. The method of derivation is notable only for the incentive it gives to find more subtle methods for tackling Einstein's equations.

Lecture 12

12.1 Stationary observers

An observer in a fixed location (relative to our coordinate system) has a wordline with constant r, θ, ϕ, and therefore has 4-velocity U with only the first component nonzero. Since $U^a U_a = 1$ and $U^0 > 0$, therefore, we must have

$$(U^a) = (1/\sqrt{1-2m/r}, 0, 0, 0)\,.$$

This worldline is *not* geodesic—as we know, for example, from the fact that an observer at rest on the earth's surface is accelerating relative to the local inertial frame. He interprets this acceleration as the 'force of gravity'.

In local inertial coordinates at an event on his worldline, the four-acceleration of the observer is $\alpha^a = \mathrm{d}U^a/\mathrm{d}s$; in arbitrary coordiantes, therefore, $\alpha^a = U^b \nabla_b U^a$. The acceleration actually felt by the observer is $\sqrt{-\alpha_a \alpha^a}$

By using the calculation of the Christoffel symbols in the last lecture, we have

$$(U^b \nabla_b U^a) = (U^0 \partial_0 U^a + \Gamma^a_{00} U^0 U^0) = (0, A'(U^0)^2/2B, 0, 0) = (0, \tfrac{1}{2}A', 0, 0)$$

where $A = 1/B = 1-2m/r$). Thus the 4-acceleration of the observer is $(\alpha^a) = (0, m/r^2, 0, 0)$, as one might expect by naive analogy with Newtonian theory. However, the acceleration felt by the observer is

$$g = \sqrt{-\alpha^a \alpha_a} = \frac{m}{r^2\sqrt{1-2m/r}} \tag{36}$$

Thus the 'force of gravity' is given by the same inverse-square law $g = m/r^2$ as in Newtonian theory for large r, but increases to infinity as r approaches the Schwarzschild radius $r = 2m$. We shall see later that $r = 2m$ is the 'event horizon' of a black hole. What we are observing is a consequence of the fact that inside a black hole, you would have to travel faster than light in order to stay in the same place (what would the Red Queen say?).

12.2 Potential energy

The t equation for the geodesic motion of a free particle is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0$$

since the Lagrangian is independent of t. Therefore $E = (1 - 2m/r)\dot{t}$ is constant along the particle worldline. What is the interpretation of this constant?

Suppose that the particle has 4-velocity V and unit mass. Then relative to our observer 'at rest' at some point in the particle's history, the particle has speed v given by

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}} = U^a V_a = g_{00} U^0 V^0 = \dot{t}\sqrt{1 - 2m/r}\,.$$

Therefore

$$E = \frac{\sqrt{1-2m/r}}{\sqrt{1-v^2}}\,.$$

For large r and small v, this is approximately

$$E = 1 + \tfrac{1}{2}v^2 - m/r + \text{smaller terms}\,.$$

Thus E is the sum of the rest energy (Mc^2 with $M = 1$ and $c = 1$), the kinetic energy $\frac{1}{2}v^2$ relative to the observer, and the Newtonian potential energy $-m/r$. Thus it is reasonable to interpret E as the *total energy* of the particle. We note that this is consistent with (36), which can be written $g = \partial_r \sqrt{1 - 2m/r}$, and which therefore also suggests that we should interpret $\sqrt{1 - 2m/r}$ as the potential energy of a unit mass particle at rest. Conservation of energy is a consequence of $\partial L/\partial t = 0$.

12.3 Photons

Geodesics in space-time are generated by the Lagrangian

$$L = \tfrac{1}{2} g_{ab}\dot{x}^a \dot{x}^b\,.$$

In the case of free particles, the dot denotes differentiation with respect to *proper time* s—the time measured by a clock carried by the particle. In the case of photons, the dot is differentiation with respect to an affine parameter, which is defined only up to a constant factor (and of course addition of a constant).

In Minkowski space, a photon worldline is a null line. The *frequency 4-vector* K is tangent to the worldline and encodes information about the frequency of the photon, as measured by a moving observer. If the observer has 4-velocity U, then the observed frequency is $\omega = U_a K^a$.

In special relativity, the frequency 4-vector is constant along the photon worldline. By our usual principle that special relativity should hold over short times and distances in local inertial coordinates, in general relativity, we must have that K is tangent to the photon worldline (now a null geodesic) and that $K^a \nabla_a K^b = 0$. But if we put $W^a = \mathrm{d}x^a/\mathrm{d}\sigma$, then the geodesic equation is $W^a \nabla_a W^b = 0$. So W is a constant multiple of K, and by rescaling σ, we can take $W = K$. Then the frequency 4-vector is given by $K^a = \mathrm{d}x^a/\mathrm{d}\sigma$.

12.4 Gravitational redshift

In the Schwarzschild metric,

$$L = \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right], \tag{37}$$

and $(1 - 2m/r)\dot{t}$ is constant along both timelike and null geodesics.

Consider two observers O_1 and O_2 at rest relative to the Schwarzschild coordinates at $r = r_1$ and $r = r_2$. If O_1 sends out a photon to O_2, and if the frequency measured by O_1 at transmission is ω_1, then what is the frequency at reception as measured by O_2?

Let the photon's worldline be $x^a = x^a(\sigma)$, where the affine parameter σ is chosen so that the frequency 4-vector is $K^a = \mathrm{d}x^a/\mathrm{d}\sigma$. Let ω denote frequency measured by a stationary observer at r. Then we have

$$\omega = U^a K_a = g_{00} U^0 K^0 = \dot{t}\sqrt{1 - 2m/r}, \qquad \cdot = \frac{\mathrm{d}}{\mathrm{d}\sigma}.$$

However $(1 - 2m/r)\dot{t}$ is constant along the worldline. Therefore $\omega\sqrt{1 - 2m/r}$ is also constant and so we have

$$\omega_2 = \omega_1\sqrt{\frac{1 - 2m/r_1}{1 - 2m/r_2}}.$$

This is the *gravitational redshift*. For large r_1, r_2, we have

$$\omega_2 \sim \omega_1(1 + m/r_2 - m/r_1),$$

so the change in freqeuncy is proportional to the difference gravitational potential between the two observers. This is precisely what is needed to avoid the paradox in the working of Bondi's perpetual motion machine.

Remark. The energy of a photon (relative to an observer) is $\hbar\omega$. The conservation law here can again be interpreted as 'conservation of energy'.

12.5 Killing vectors

A special role is played in these calculations by time symmetry: it is this that allows us to say what we mean by 'stationary' observers, and it is this that gives us the energy conservation.

More generally, if the metric coefficients g_{ab} are independent of one of the coordinates x^0, then $L = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b$ is independent of x^0, and so from Lagrange's equations

$$\frac{\partial L}{\partial \dot{x}^0} = g_{a0}\dot{x}^a$$

is constant along geodesics. But this quantity is equal to $T^a V_a$, where $V^a = \mathrm{d}x^a/\mathrm{d}s$ and T is the 4-vector field with components $(1, 0, 0, 0)$. We note that $T^a V_a$ is an invariant: it depends only on the 4-vectors V and T, and not on the choice of coordinates (although, of course, T will have components $(1, 0, 0, 0)$ only for special choices of coordinates).

Definition 9

A non-vanishing vector field T is said to be a *Killing vector field* or *Killing vector* whenever there exists a coordinate system in which T has components $(1, 0, 0, 0)$ and g_{ab} is independent of x^0.

We have just proved the following.

Proposition 10

If T is a Killing vector, then $T_a\dot{x}^a$ is constant along any geodesic.

Remark. For any non-vanishing vector field $T \neq 0$, there exists a coordinate system in which T has components $(1, 0, 0, 0)$.

How can we recognise a Killing vector and therefore derive a conserved quantity for free particle and photon orbits without making the transformation to the special coordinate system? To answer this, we look first at the defining property in the special coordinates. Here we have

$$0 = \partial_0 g_{ab} = T^c \partial_c g_{ab}\,.$$

But we also have

$$\begin{aligned} \nabla_a T_b &= \partial_a(g_{bc}T^c) - \tfrac{1}{2}T^c(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \\ \nabla_b T_a &= \partial_b(g_{ac}T^c) - \tfrac{1}{2}T^c(\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ba})\,. \end{aligned}$$

By adding, we get

$$\nabla_a T_b + \nabla_b T_a = T^c \partial_c g_{ab} = 0\,.$$

But the left-hand side is a tensor. Therefore it vanishes in one coordinate system if and only if it vanishes in every coordinate system. We have proved the following.

PROPOSITION 11

Let T^a be a non-vanishing vector field. If T^a is a Killing vector then $\nabla_a T_b + \nabla_b T_a = 0$ (in any coordinate system).

The converse is also true: this is proved by choosing a coordinate system in which the components are $(1,0,0,0)$, and by using the same calculation in reverse.

We can use (11) to prove Proposition (10) directly by starting from the geodesic condition in the form $V^a \nabla_a V_b = 0$, where $V^a = \dot{x}^a$. From this we get

$$(V^a T_a)^\bullet = V^a \nabla_a (V^b T_b) = V^a V^b \nabla_a T_b = \tfrac{1}{2} V^a V^b (\nabla_a T_b + \nabla_b T_a) = 0\,,$$

and hence that $V^a T_a$ is constant.

Exercise. The converse statement can be deduced from this: if $\dot{x}^a T_a$ is conserved along every geodesic, then T_a is a Killing vector).

We use Proposition (11) to extend the definition to arbitrary vector fields. We then have

DEFINITION 10

Alternative defintion: a vector field T^a is a *Killing vector* if $\nabla_a T_b + \nabla_b T_a = 0$.

Lecture 13

13.1 Massive particles

We shall now look at particle motion in the Schwarzschild background. Our main aim will be to derive corrections to Kepler's laws, so we shall think of the gravitational field as that of the sun. By a 'particle', we mean a very small body, such a planet, whose own gravitational field can be ignored.

The particle orbits are generated by the Lagrangian

$$L = \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right],$$

where the parameter is the proper time s and $\cdot = \mathrm{d}/\mathrm{d}s$. We shall assume that $r > 2m$, which means that we are looking at the external field of a spherical star rather than the field inside a black hole. Since L has no explicit dependence on t, ϕ, or s, we have three conservation laws.

$(\partial_t L = 0)$ $\quad E := (1 - 2m/r)\dot{t} = \text{constant};$

$(\partial_\phi L = 0)$ $\quad J := r^2 \sin^2\theta\, \dot{\phi} = \text{constant};$

$(\partial_s L = 0)$ $\quad L = \text{constant}.$

Since s is proper time, $g_{ab}\dot{x}^a\dot{x}^b = 1$, and therefore the third conservation law is $L = \frac{1}{2}$. We need one other equation to determine the orbits. We shall use the θ Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\dot{\theta}\right) - r^2 \sin\theta\cos\theta\,\dot{\phi}^2 = 0\,. \tag{38}$$

We could also write down the r equation, but it would contain no new information since we already have four equations for the four unknown coordinates t, r, θ, ϕ.

Because (38) is symmetric under $\theta \mapsto \pi/2 - \theta$, an orbit such that $\theta = \pi/2$, $\dot{\theta} = 0$ initially will have $\theta = \pi/2$ for all s. Because the field is spherically symmetric, we can understand all the orbits by studying only these equatorial orbits. So we shall put $\theta = \pi/2$. Then by combining the conservation laws, we have

$$1 = \frac{E^2}{1 - 2m/r} - \frac{\dot{r}^2}{1 - 2m/r} - \frac{J^2}{r^2}.$$

That is,

$$\dot{r}^2 = -\frac{J^2}{r^2}\left(1 - \frac{2m}{r}\right) + E^2 - \left(1 - \frac{2m}{r}\right).$$

This is a first-order differential equation for r as a function of proper time. As in Newtonian theory, the equation looks a bit simpler if we replace r by $u = m/r$ and use ϕ instead of s as the parameter. Now

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = -\frac{m}{r^2}\frac{\mathrm{d}r}{\mathrm{d}s}\Big/\frac{\mathrm{d}\phi}{\mathrm{d}s} = -\frac{m\dot{r}}{J}.$$

Hence when $J \neq 0$, that is when the orbit is not radial, we have

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{m^2E^2}{J^2} - u^2(1-2u) - \frac{m^2(1-2u)}{J^2}. \tag{39}$$

13.2 Comparison with the Newtonian theory

In Newton's theory, the particle (assumed to have unit mass) moves under the influence of the inverse-square law force m/r^2. The equatorial orbits are determined in plane polar coordinates r, ϕ by the conservation of the energy ε and the angular momentum J:

$$\varepsilon = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - m/r, \qquad J = r^2\dot{\phi}.$$

As before, we put $u = m/r$, $\mathrm{d}u/\mathrm{d}\phi = -m\dot{r}/J$. Then we have

$$\varepsilon = \frac{1}{2}\left(\frac{J^2}{m^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \frac{J^2u^2}{m^2}\right) - u.$$

To make comparison with Einstein's theory, we put $\beta = m/J$, $p = \mathrm{d}u/\mathrm{d}\phi$ and $k = \varepsilon m^2/J^2$ (Newton), $k = (E^2-1)m^2/2J^2$ (Einstein). Then we have (from above and from (39)):

Einstein $p^2 = 2\beta^2u + 2k - u^2 + 2u^3 =: f(u)$;

Newton $p^2 = 2\beta^2u + 2k - u^2 := g(u)$.

The only difference is the term $2u^3$, which is, of course, small when r is large.

The equations for the orbits can also be written in then second-order form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2\right) = \tfrac{1}{2}f'(u)$$

in Einstein's theory, or in the same way with $\frac{1}{2}g'(u)$ on the right in Newton's theory.

13.3 Newtonian orbits and the relativistic correction

We can see the effect of the extra term in one of the classical tests of general relativity, the perihelion advance (i.e. the fact that the point on each orbit where r is minimal—the perihelion—advances on each orbit in Einstein's theory, while in the Newtonian theory, it is always in the same place.[1]

In the Newtonian theory, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{m^2}{J^2}\,, \tag{40}$$

which implies that $u = \beta^2 + A\cos(\phi - \phi_0)$, for constant A, ϕ_0. By differentiating we get

$$p^2 = A^2 \sin^2(\phi - \phi_0) = A^2 - (u - \beta^2)^2\,.$$

Hence $A^2 = 2k + \beta^4$.

(1) If $k > 0$ then $|A| > \beta^2$ and $u = 0$ for some values of ϕ. In this case, the orbits are hyperbolic and the particle can escape to infinity.

(2) If $k < 0$ then $|A| < \beta^2$. In this case, u is bounded away from zero, and therefore $|r|$ is bounded and the orbits are elliptic. Special cases are the *circular orbits*, on which $u = u_0$ is constant, and so $\mathrm{d}u/\mathrm{d}\phi$ and $\mathrm{d}^2u/\mathrm{d}\phi^2$ vanish identically. These are therefore given by solving $g(u_0) = g'(u_0) = 0$ for u_0. The result is $u_0 = \beta^2$, where

$$g(\beta^2) = \beta^4 + 2k = 0\,.$$

That is, $k = -\frac{1}{2}\beta^4$.

We can rewrite (40) in the form

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + v = 0\,,$$

where $v = u - \beta^2$. This is the equation of simple harmonic motion with period 2π (the 'time' of course is not t or s, but the polar angle ϕ). Thus we can think of a general elliptic

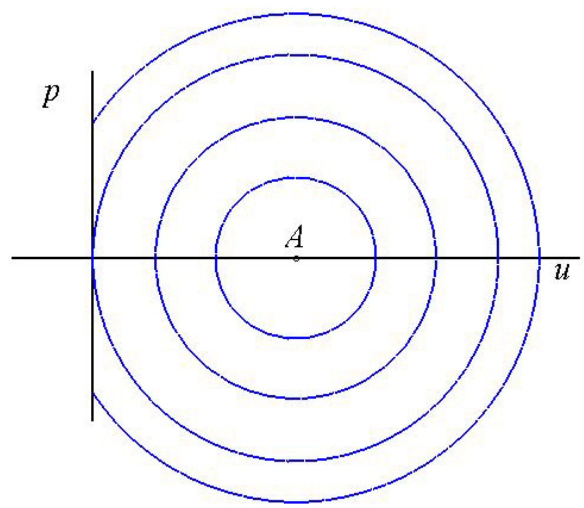
[1] In fact the perihelion of Mercury also advances in Newtonian theory because of interactions with other planets, notably Jupiter: the general relativistic effect is the additional advance that cannot be explained in this way. When it was first noted, before the discovery of general relativity, it was thought that it might be due to another planet orbiting the sun inside Mercury's orbit. This planet, which of course was never seen, was called *Vulcan*.

orbit as oscillating about circular orbit ($u = \beta^2$) with simple harmonic motion. The fact that in these oscillations the period of u as a function of ϕ is exactly 2π is what makes the elliptic orbits closed in Newtonian theory: each circuit of the origin adds 2π to ϕ and brings us back to the initial value of u. In particular, perihelion always occurs at the same value of ϕ: there is no perihelion advance in the two-body system.

One can gain some insight into the structure of the orbits in Newtonian theory by plotting the *phase portrait*, in which one represents by curves in the u, p-plane. If we fix β and plot the curves for varying values of k, the result is a set of concentric circles

$$p^2 + (u - \beta^2)^2 = 2k + \beta^4$$

centred on the circular orbit $u_0 = \beta^2$ (labelled A). The hyperbolic orbits are those that meet the p axis, the elliptic orbits are those that do not. The two families are separated by the parabolic orbit, which touches the p-axis at the origin.



13.4 The perihelion advance

In general relativity, there are also closed orbits $u = u_0$. The corresponding values of the constant β and E are found from $f'(u_0) = 0 = f(u_0)$.

Now consider an orbit $u = u_0 + v(\phi)$ which is almost circular, so that v is small. By substituting into the equation of motion, we obtain

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} = \tfrac{1}{2} f'(u) = \tfrac{1}{2} f'(u_0) + \tfrac{1}{2} v f''(u_0) + O(v^2)\,.$$

But $f'(u_0) = 0$, and $f''(u) = -2 + 12u$. Therefore v satisfies

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + (1 - 6u_0)v = 0$$

(ignoring the $O(v^2)$ error term). This is again the equation of simple harmonic motion (with ϕ as 'time'); so at least for orbits that are close to circular, we again have the picture that the particle's orbit oscillates about a circular orbit. However, not the 'period' is not 2π, but

$$\Phi = 2\pi/\sqrt{1 - 6u_0} \sim 2\pi + 6u_0\pi\,,$$

for small u_0, that is, large r_0. Thus if the particle starts at perihelion (r minimal, u maximal), then r returns to its initial value not after a whole rotation, but after ϕ has advanced through a further angle $6u_0\pi$. This is the *perihelion advance*.

For small u_0 (large r), we have $\Phi \sim 2\pi + 6u_0\pi$, so perihelion occurs at $\phi = 0, 2\pi + 6u_0\pi, \ldots$ (ignoring terms of order u_0^2). If we substitute $u_0 = m/r_0$ and put back in the constants (there is only one way to do this to get dimensions right), then the advance is

$$\frac{6Gm\pi}{r_0 c^2}$$

per revolution for an orbit of approximate radius r_0 (we are ignoring second order terms in $1/r_0$, as well as assuming that the orbit is 'nearly' circular).

In the case of Mercury, m, the mass of the sun, is 1.98×10^{30}. The radius is $r_0 = 5.79 \times 10^{10}$, and the constants are $G = 6.67 \times 10^{-11}$, and $c^2 = 9 \times 10^{16}$ (all in SI units). This gives the advance as around $40''$ per century (a more careful analysis gives $43''$). In the case of the binary pulsar, the advance is around 4^o per year.

Lecture 14

14.1 Circular orbits

The equatorial particle orbits in the Schwarzschild space-time are given by $p^2 = f(u)$, where

$$f(u) = 2\beta^2 u - u^2 + 2u^3 + 2k\,,$$

and $u = m/r$, $p = \mathrm{d}u/\mathrm{d}\phi$, $\beta = m/J$, $k = \frac{1}{2}(E^2 - 1)m^2/J^2$. In second order form they are solutions to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} = \tfrac{1}{2} f'(u)\,.$$

The *circular* orbits are those for which r and therefore also u are constant. They are given by $f(u) = 0$, $f'(u) = 0$. The second of these implies that

$$6u = 1 \pm \sqrt{1 - 12\beta^2},$$

which for small β gives $u = \beta^2$ and $u = \frac{1}{3} - \beta^2$ (ignoring terms of order β^4). The first root is the Newtonian circular orbit: this is still present in general relativity provided that the radius m/β^2 is large in comparison to m. The second is a new feature: it has radius close to $r = 3m$, which is only just above the Schwarzschild radius $r = 2m$, and of course it exists only if the the source of the gravitational field is all contained within the sphere $r = 3m$, so the metric still takes the Schwarzschild form at this radius. We shall see below that $r = 3m$ itself is a cicular photon orbit. A particle on the inner circular orbit has to be moving close to the velocity of light (relative to a stationary observer).

14.2 The phase portrait

We can understand more clearly the pattern of orbits by drawing the phase portrait in the u, p-plane for fixed β^2 and varying k (as we did above for the Newtonian theory). We first plot $2\beta^2 u - u^2 + 2u^3$ for different values of β^2:

$\beta^2 = 0$: (red curve).

$\beta^2 = 1/16$: the two roots of $2u^2 - u + 2\beta^2$ come into coincidence (blue curve).

$\beta^2 = 1/12$: the two roots of the u-derivative of $2\beta^2 u - u^2 + 2u^3$ come into coincidence (green curve).

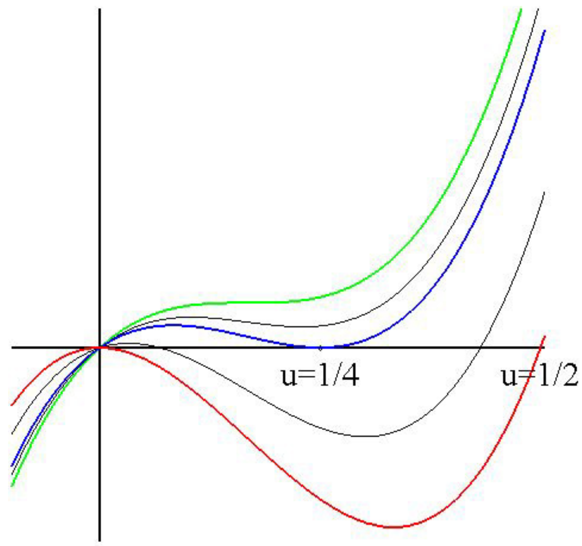
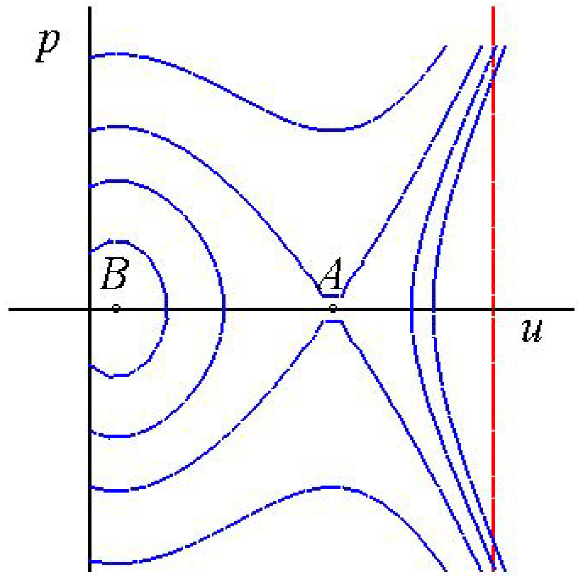


Figure 9: Plots of $2\beta^2 u - u^2 + 2u^3$.

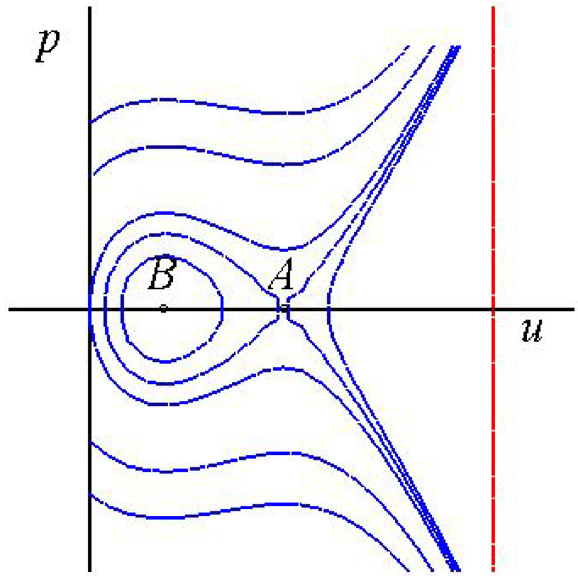
We shall consider the orbits only for $\frac{1}{2} > u > 0$, that is, for $r > 2m$ (if the vacuum region extnds that far, the portion of space-time in which $r < 2m$ is inside a black hole.

For different choices of β^2, the phase portraits are as follows. Note that for small u (large r), the portraits all correspond to the Newtonian one.

$0 < \beta^2 < 1/16$. In this case, there are two circular orbits, one stable (B), and the other unstable with $k > 0$ (A). A particle disturbed from the inner cicular orbit—the unstable one—can either spiral inwards or escape to infinity. The boundary $r = 2m$ is the red line.



$1/16 < \beta^2 < 1/12$. In this case, the inner unstable circular orbit A has $k < 0$: a particle disturbed from this orbit will not escape to infinity. As β^2 is increased, the two circular orbits move towards each other. They coincide when β^2 reaches $1/12$, at $r = 6m$.

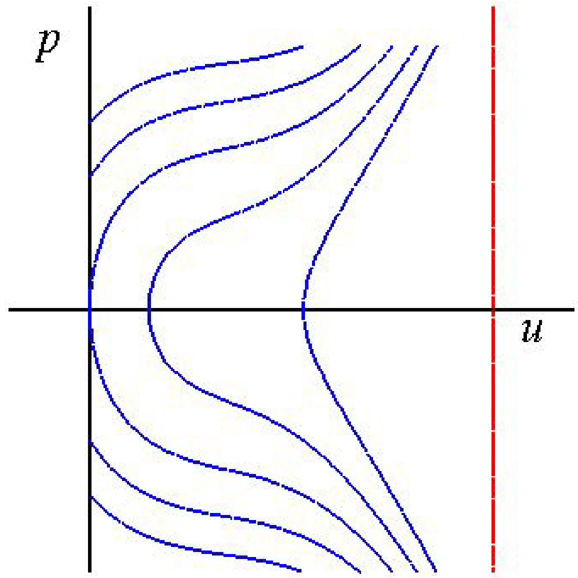


$\beta^2 > 1/12$. There are no closed orbits in this case: the angular momentum is too small. All orbits either escape to infinity or spiral inwards.
We note that in no case are there stable circular orbits with $r < 6m$: this is the minimum radius for a planetary orbit. For the sun, this minimum radius is 9 km. That is, it is well inside the sun: the limit is not relevant. In the case of black holes, the limit is important in the analysis of the infall of matter (usually from a companion star): it is matter falling in, and heating up as it does so, that is responsible for X-ray emissions from the neighbourhood of stellar mass black hole.

An interesting lesson to learn from the first case is that, contrary to popular belief, it is not easy to fall into a black hole. Suppose that the particle starts a long way from the Schrwarzschild radius ($r >> m$), and that the radial and tranverse components of its velocity relative to a stationary oserver are v_{rad} and v_{trans}, respectively. So long as these are small compared with 1 (the velocity of light), we have that

$$\beta^2 = \frac{m}{j} = \frac{m}{r}\frac{1}{v_{\mathrm{trans}}}$$

is small. The corresponding inner circular orbit is given approximately by $r = 3m$, and so $2k = 1/27$ on this orbit. It is clear from the first phase diagram, therefore, that for the



particle orbit to pass inside the black hole (i.e. to pass $u = \frac{1}{2}$), it must have

$$p^2 > \frac{1}{27}$$

at large r. But

$$p = -\frac{m\dot{r}}{r^2\dot{\phi}} = -\frac{m}{r}\frac{v_{\rm rad}}{v_{\rm trans}} .$$

If the particle is to fall into the black hole starting at large r, the angle α between its trajectory and the inward-point radial direction must satisfy

$$\tan\alpha = \frac{v_{\rm trans}}{v_{\rm rad}} < \frac{3\sqrt{3}m}{r} = \frac{3\sqrt{3}}{2}\frac{\text{Schwarzschild radius}}{\text{distance}} .$$

So, for example, if the black hole has the mass of the sun (with a Schwarzschild radius of 3km) and if initially r is the radius of the earth's orbit, then α has to lie in an impossibly small range: although the balck hole has a very strong gravitational field, it is very small, and is an almost impossible target to hit from any distance.

14.3 Photon orbits

Photon orbits are generated by the same Lagrangian

$$L = \frac{1}{2}\left[\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-2m/r} - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\right],$$

but now the dot is differentiation with respect to an affine parameter σ. Again we have three constants

Energy $E = (1-2m/r)\dot{t}$;

Angular momentum $J = r^2 \sin^2\theta\,\dot{\phi}$

Hamiltonian $L = 0$

only in this case, $h = 0$ since $g_{ab}\dot{x}^a\dot{x}^b$ vanishes because $\dot{x}^a$ is null. By proceeding in the same way as before, we obtain

$$p^2 = \alpha^2 + 2u^3 - u^2\,,$$

where $u = m/r$, $p = \mathrm{d}u/\mathrm{d}\phi$, and $\alpha^2 = m^2E^2/J^2$. We now have no sensible Newtonian model with which to make comparisons, but we note that without $2u^3$ term, the orbits would be given by

$$u = \alpha\cos(\phi - \phi_0), \qquad p = -\alpha\sin(\phi-\phi_0)\,.$$

That is,

$$r\cos(\phi - \phi_0) = m/\alpha\,, \tag{41}$$

which is the polar equation of a straight line. Thus the term $2u^3$ is the gravitational contribution: it is responsible for the 'bending of light by gravity'.

The phase portrait is found in the same way as before. First we plot $\alpha^2 + 2u^3 - u^2$ against u for different values of α, as in Figure 14.3

By taking the square root, we then get the phase portrait in Figure 14.3, with the different values of α labelling the different curves in the p, u-space. For large r (small u), the orbits look like hyperbolic Newtonian orbits: a photon travelling in from infinity will esacpe to infinity, but the trajectory will be deflected. As r decreases, the deflection increases, and the orbit can wind around the source many times. Ar $r = 3m$, it is possible for the photon to orbit in an unstable circular orbit.

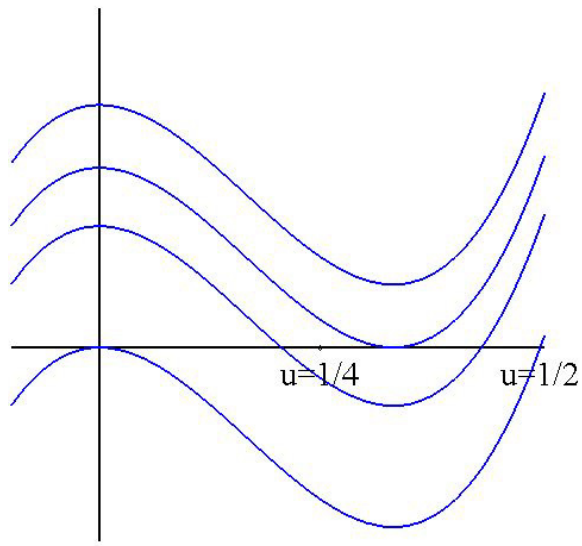


Figure 10: $\alpha^2 + 2u^3 - u^2$ against u for different values of α

14.4 The bending of light

Let us solve for the orbits on the assumption that r is large (and therefore $u^2 << u^3$). We shall look at the effect of the 'gravitational term' by putting $u = \alpha \cos\phi + v$, where α is small and $v = O(\alpha^2)$ (i.e. we are looking at a perturbation of (41) with $\phi_0 = 0$). By ignoring terms of order $v^2 = O(\alpha^4)$, we get

$$0 = u'^2 + u^2 - 2u^3 - \alpha^2 = -2\alpha v' \sin\phi + 2\alpha v \cos\phi - 2\alpha^3 \cos^3\phi + O(\alpha^4)$$

where the prime is differentiation with respect to ϕ. From this we obtain

$$\begin{aligned} & \sin\phi\, v' = v\cos\phi - \alpha^2\cos^3\phi \\ \Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{v}{\sin\phi}\right) = -\alpha^2\cos\phi\left(\frac{1}{\sin^2\phi} - 1\right) \\ \Rightarrow \qquad & \frac{v}{\sin\phi} = \frac{\alpha^2}{\sin\phi} + \alpha^2\sin\phi + K \\ \Rightarrow \qquad & v = \alpha^2(1 + \sin^2\phi)\,, \end{aligned}$$

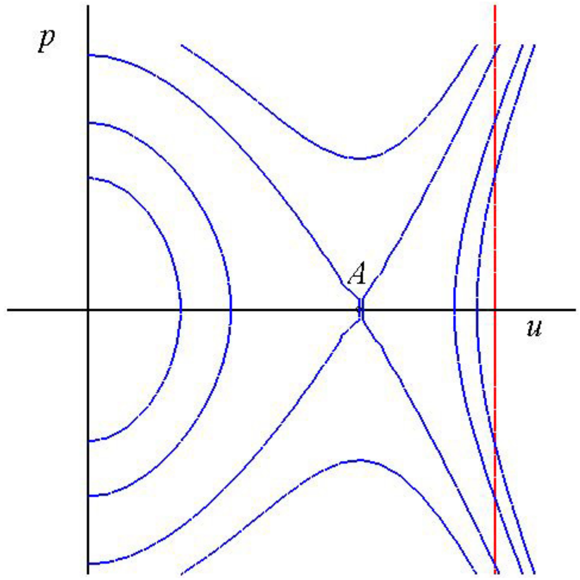


Figure 11: The phase portrait for photon orbits

since without loss of generality, we can set $K = 0$ by adjusting ϕ_0. Since the gravitational field is weak, we can interpret r, ϕ as plane polar coordinates in the equatorial plane. The unperturbed trajectory is the straight line $\alpha r \cos\phi = m$: a point on this goes to infinity as $\phi \to \pm\pi/2$. The perturbed trajectory, on the other hand, goes to infinity as $\phi \to \pm(\pi/2 + \gamma)$, where the angle γ is given by

$$0 = -\alpha \sin\gamma + \alpha^2(1 + \cos^2\gamma)\,,$$

which gives $\gamma = 2\alpha$ to the first order in α. The total deflection of the light ray is $2\gamma = 4\alpha = 4m/D$, where $D = m/\alpha$ is the value of r at the point of closest approach to the source of the unperturbed trajectory. In SI units, the deflection is

$$\frac{4mG}{Dc^2}.$$

For a light ray just grazing the surface of the sun, we have (in SI units) $D = 7 \times 10^8$ (the radius of the sun), $m = 2 \times 10^{30}$ (the mass of the sun), $c = 3 \times 10^8$, $G = 7 \times 10^{-11}$. The result is a deflection of 10^{-5} radians or $2''$. This is hard to see, of course, because of the difficulty in observing light that grazes the sun. Its effect can, however, be seen during a total eclipse, and was first observed by Eddington in 1918: the deflection causes the stars near the sun in sky to move outwards slightly from the centre of the sun, as compared with their their normal relative positions. Eddington compared photographs of the star field near the sun during a total eclipse with a photograph of the same star field when the sun was in a different position in the sky.

Lecture 15

15.1 The Schwarzschild radius

We shall now look more closely at what happens at the 'Schwarzschild radius', $r = 2m$. Clearly something goes wrong in the formula for the metric coefficients at that point: we shall see, however, that it is not the space-time geometry itself that is singular there, but simply the coordinates in which it is expressed. We shall see that the singular behaviour at $r = 2m$ (but not at $r = 0$) goes away when we make an appropriate change of coordinates.

Of course for a normal star the Schwarzschild radius is well inside the star itself: since it is not in the vacuum region, we would not expect $R_{ab} = 0$ to hold at $r = 2m$, and therefore the Schwarzschild solution would not be valid[2]. But, in the extreme case that all the body lies within its Schwarzschild radius, we have a (spherical) *black hole*.

In the case of the sun, for $r = 2m$ to be outside the sun (and therefore in the vacuum region), the sun would have to be compressed to a radius of 3km, which would imply almost unimaginable density. For a galaxy, however, the density at this critical compression is only that of air, and so it is not hard, at least in principle, to imagine a sufficiently advanced civilization directing the orbits of all the stars in galaxy so that all the matter ends up within the Schwarzschild radius. We must therefore take seriously the existence of black holes as a theoretical possibility.

15.2 Eddington–Finkelstein coordinates

The Schwarzschild metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\,.$$

We cannot simply ignore the part of space-time for which $r \leq 2m$ because an infalling observer will reach $r = 2m$ in finite proper time. The observer's worldline (if he falls radially, that is with constant θ and ϕ) is given by

$$E = (1 - 2m/r)\dot{t}, \qquad 1 = (1 - 2m/r)\dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r}\,,$$

[2]One can construct 'interior' Scwharzschild metrics with the energy-momentum tensor of various types of matter on the right-hand side of Einstein's equation; these are, of course, nonsingular at $r = 2m$.

where the parameter s is proper time. If we consider the special case $E = 1$, which arises when the observer falls from rest (with respect to the time Killing vector) at infinity, then we have $\dot{r}^2 = 2m/r$. For an ingoing observer, therefore,

$$\begin{aligned} \int \sqrt{r}\,\mathrm{d}r &= -\sqrt{2m}\int \mathrm{d}s \\ \Rightarrow \qquad \tfrac{2}{3}r^{3/2} &= \sqrt{2m}(k-s)\,; \qquad k = \text{constant}\,. \end{aligned}$$

from which we conclude that the proper time s taken to reach $r = 2m$ is finite. However the coordinate time taken, t, is infinite because

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(1 - \frac{2m}{r}\right)\frac{\sqrt{2m}}{\sqrt{r}}$$

and so

$$-\int \frac{r^{3/2}\,\mathrm{d}r}{r-2m} = \sqrt{2m}\int \mathrm{d}t\,.$$

But the intergal on the left-hand side diverges as $r \to 2m$.

To understand the space-time geometry of a black hole, we first look for a coordinate system in which the singularity at $r = 2m$ disappears. One can see what goes wrong with the given coordinates by looking at the null geodesics in the r, t-plane (the worldlines of photons travelling radially inwards or outwards). These are the curves given by

$$\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-2m/r} = 0\,,$$

from which we obtain

$$\int \mathrm{d}t = \pm\int \frac{\mathrm{d}r}{1-2m/r} = \pm\int\left(1 + \frac{2m}{r-2m}\right)\mathrm{d}r\,.$$

That is,

$$t \pm (r + 2m\log(r-2m)) = \text{constant}\,. \tag{42}$$

The radial null geodesics in the t, r-plane are shown in blue in Figure 15.2. The singular behaviour at $r = 2m$ (the red line) is associated with the fact that the curves 'bunch up' on the common asymptote at $r = 2m$. We also see that for large r, they look like the corresponding curves $r = \pm t$ in flat space-time. One should think of each curve as an

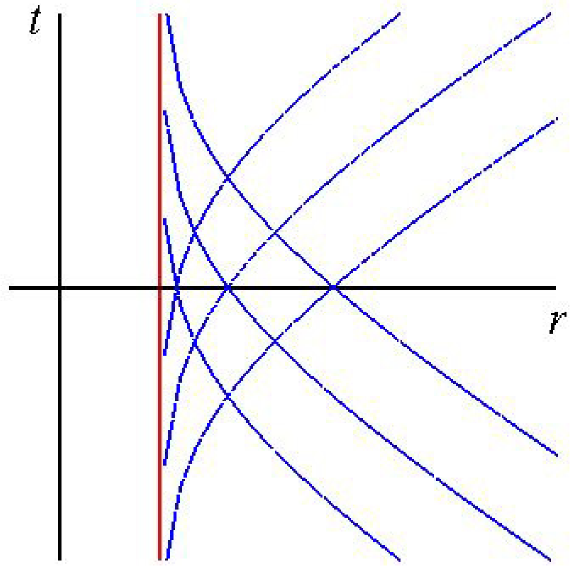


Figure 12: Radial null geodesics in the Schwarzschild metric

ingoing or outgoing spherical wave front: one can get a partial picture of this by rotating the about the t-axis to make the curves into surfaces of revolution. The are shown Figure 15.2, which is a space-time diagram with one spatial dimension suppressed. The blue surface is at $r = 2m$; the history of the outgoing wave-front is shown in red; and that of the ingoing one in green.

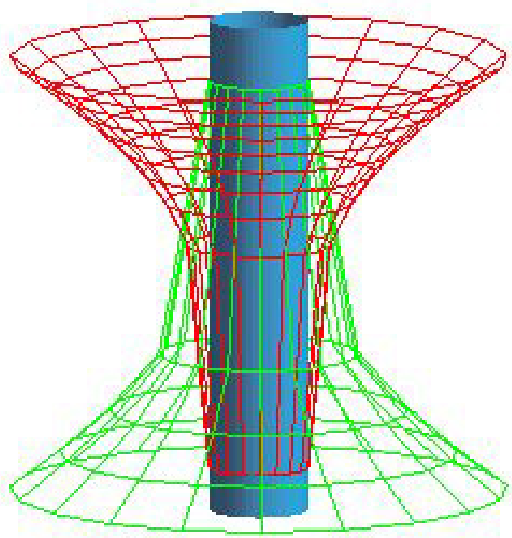


Figure 13: Ingoing and outgoing wavefronts in the Schwarzschild metric

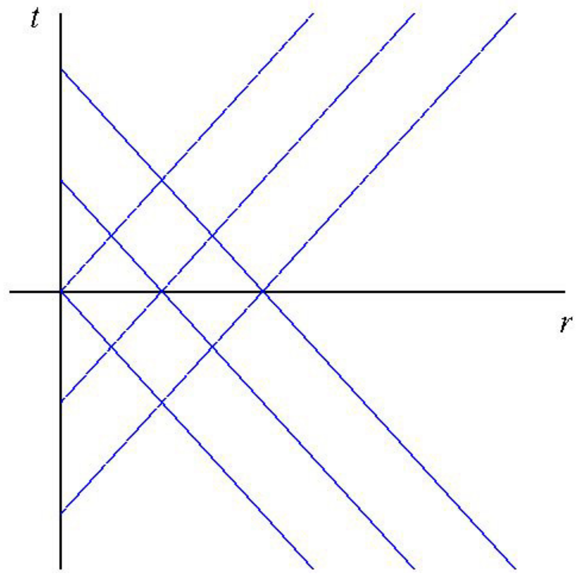


Figure 14: Radial null geodesics in Minkowski space

We should compare these pictures with the corresponding ones for Minkowski space. The corresponding radial null geodesics are the straight blue lines shown in Figure 15.2, and the corresponding in- and outgoing wave-fronts are the null cones of the points on the polar axis $r = 0$ (Figure 15.2).

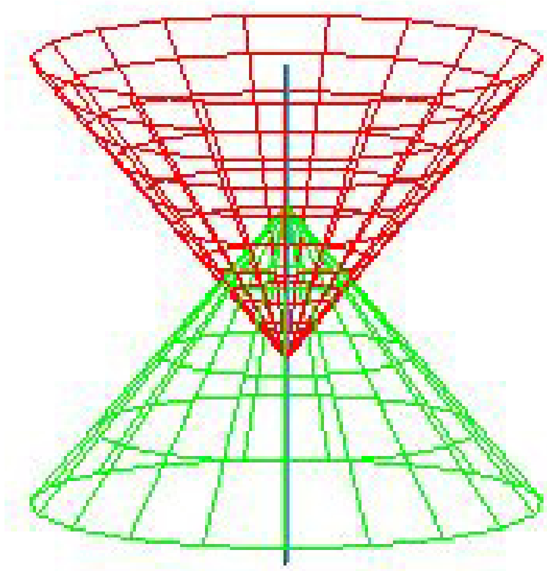


Figure 15: Ingoing and outgoing wavefronts in MInkowski space

In the Schwarzschild space-time, we can resolve the coordinate difficulties by 'compressing' the t coordinate as we approach $r = 2m$. Guided by (42), we make the transformation to coordinates v, r, θ, ϕ by putting

$$v = t + r + 2m\log(r - 2m)\,,$$

which gives

$$\mathrm{d}t = \mathrm{d}v - \frac{\mathrm{d}r}{1 - 2m/r}$$

and hence

$$\mathrm{d}s^2 = (1 - 2m/r)\,\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

The singular behaviour at $r = 2m$ has now disappeared. In the r, v-plane, the radial null geodesics are given by

$$v = \text{constant} \qquad \text{or} \qquad \left(1 - \frac{2m}{r}\right)\frac{\mathrm{d}v}{\mathrm{d}r} - 2 = 0\,.$$

The second equation can be integrated to give:

$$v = \int \frac{2r\,\mathrm{d}r}{r-2m} = 2r + 4m\,\log|r-2m| + \text{const.}$$

Thus the radial null geodesics are as shown in Figure 15.2.

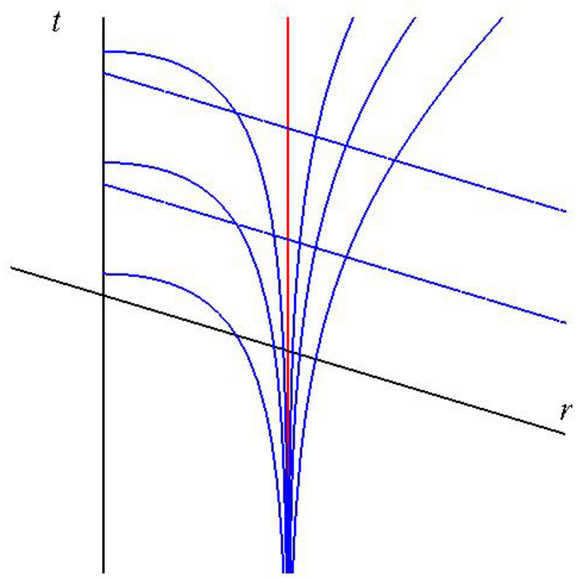


Figure 16: Radial null geodesics in Eddington-Finkelstein coordinates

We see from the from the fact that timelike curves must lie between the ingoing and outgoing null geodesics at every event that, although the space-time is nonsingular for $r < 2m$, it is not possible to escape to infinity. Hence the hypersurface $r = 2m$ is called the 'event horizon': it separates events of which observers outside can have knowledge from those inside of which they cannot. The events inside the event horizon are inside the 'black hole'. (The v-axis has been drawn at 45° to emphasize the fact that the lines of constant v are null.)

The histories of the ingoing and outgoing wave fronts outside the event horizon(the blue surface) are now as in Figure 15.2.

The singular behaviour of the the metric coefficients in the t, r coordinates is not a singularity of the space-time geometry (because it disappears in the v, r coordinates): it

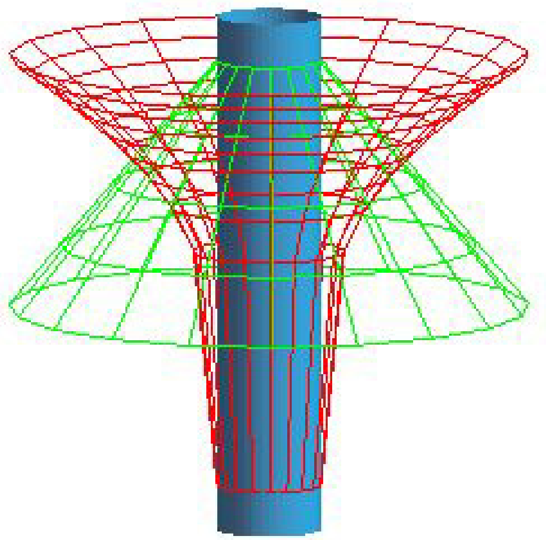


Figure 17: Ingoing and outgoing wavefronts in Eddington-Finkelstein coordinates

arises from the singular behaviour of the transformation from v, r to t, r coordinates at $r = 2m$—which shows itself in the picture in the fact that the curves of constant t are asymptotic to the line $r = 2m$. The transformation from v, r to t, r coordinates pushes the points of $(v, 2m)$ to $t = \infty$.

The Killing vector T with components $(1, 0, 0, 0)$ in the coordinates t, r, θ, ϕ has the same components in the new coordinate system, but we note that $T^a T_a = 1 - 2m/r$ both inside and outside the Schwarzschild radius. However

(i) for $r > 2m$, T is timelike and defines a standard of 'rest' (an observer whose 4-velocity is tangent to T is what we called a 'stationary observer');

(ii) for $r = 2m$, T is null: we can think of the event horizon as the history of a light wavefront 'at rest', hovering for ever between escaping to infinity and falling into the black hole;

(iii) for $r < 2m$, T is spacelike, and no observer can remain at rest.

The worldline of any observer inside the black hole must inevitably reach $r = 0$ in finite proper time (in fact, in a time of the same order of magnitude as light takes to travel a distance equal to the Schwarzschild radius). In fact $r = 0$ is a genuine singularity at which the tidal forces become infinite. One can see this from the fact the invariant $R_{abcd}R^{abcd}$ blows up like r^{-6}, and therefore there cannot exist a coordinate system in which the metric is well-behaved at $r = 0$. Thus once inside the black hole, an observer is not only unable to escape to infinity, but is equally unable to escape being crushed in the singularity in a very short time (the position is rather different if the black is rotating: see next term's lectures).

Lecture 16

16.1 Gravitational collapse

The Schwarzschild solution by itself does not provide a good model of a real black hole because it is a vacuum metric: there is no matter present to generate the gravitational field. In a real astrophysical situation one expects black holes to form from the collapse of stars after they have burnt up all their nuclear fuel: the collapse can form a white dwarf, which is supported against gravity by the 'electron degeneracy pressure'; however, above 1.4 times the mass of the sun, this pressure is insufficient, and collapse results in a neutron star—essentially a massive nucleus with an atomic number around 10^{58}. But again there is a limit to mass: above some critical mass (somewhere between 1.5 and 3 solar masses), no known physical process can prevent collapse to a black hole. Once the event horizon has formed, no conceivable process can prevent collapse to a singularity (the Penrose singularity theorem).

One can model the field of a spherically symmetric collapsing object by joining the Schwarzshild metric (to represent the field outside the body) to an interior metric (representing the field inside the collapsing star) across a spherically symmetric hypersurface represented by a timelike curve in the v, r-plane.

If we add one of the other coordinates by rotating about the line $r = 0$, then one obtains the following three-dimensional representation of the space-time:

16.2 Kruskal coordinates

It is interesting, however, to explore further the vacuum solution without joining on any interior solution. Here we look more closely at a curious feature of the Eddington-Finkelstein coordinates: they introduce time asymmetry that is not present in the original metric. That is, they do not treat the future and the past in an even-handed way. They adjoin the interior of a black hole to the exterior solution, or in the time-reversed form, a 'white hole'.

One can see what is going on here by transforming instead to *Kruskal coordinates*, in which both extensions can be made simultaneously. We start with the original form of the metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

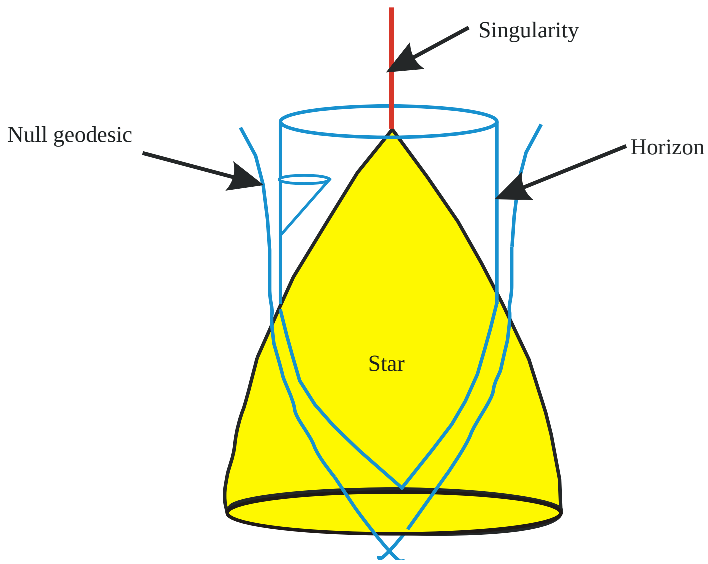


Figure 18: The collapse of a star to form a black hole

But now we transform to new coordinates U, V, θ, ϕ by putting

$$\frac{V}{U} = -\mathrm{e}^{t/2m}, \qquad UV = \mathrm{e}^{r/2m}(2m - r)\,.$$

That is, $V = \mathrm{e}^{v/4m}$, $U = -\mathrm{e}^{-u/4m}$, where

$$v = t + r + 2m\,\log(r - 2m), \qquad u = t - r - 2m\,\log(r - 2m)\,.$$

(Note that v is the Eddington-Finkelstein coordinate). Then

$$\mathrm{d}V = \frac{\mathrm{e}^{v/4m}}{4m}\left(\mathrm{d}t + \frac{\mathrm{d}r}{1 - 2m/r}\right), \quad \mathrm{d}U = \frac{\mathrm{e}^{-u/4m}}{4m}\left(\mathrm{d}t - \frac{\mathrm{d}r}{1 - 2m/r}\right)\,.$$

Hence

$$\mathrm{d}U\,\mathrm{d}V = \frac{r\,\mathrm{e}^{r/2m}}{16m^2}\left(1 - \frac{2m}{r}\right)\left(\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1 - 2m/r)^2}\right)\,.$$

Therefore in these new coordinates, the metric is

$$\mathrm{d}s^2 = 16m^2 r^{-1}\mathrm{e}^{-r/2m}\,\mathrm{d}U\mathrm{d}V - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

where r is defined as a function of U, V by $UV = \mathrm{e}^{r/2m}(2m - r)$.

To understand the geometry, let us look first at the corresponding transformation of Minkowski space. Here we start with

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

and make the coordinate change

$$U = -\mathrm{e}^{r-t}, \qquad V = \mathrm{e}^{t+r}\,.$$

Then the metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{-2r}\,\mathrm{d}U\,\mathrm{d}V - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \\ &= -\frac{\mathrm{d}U\,\mathrm{d}V}{UV} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \end{aligned}$$

If we suppress the angular coordinates, then the relationship between the two coordinate systems as shown in Figure 16.2. The U, V axes are null lines, and are therefore drawn at

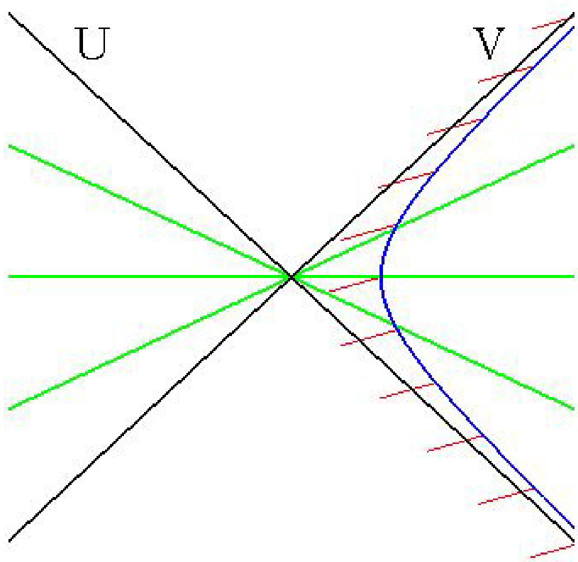


Figure 19: The U, V coordinates on Minkowski space

45^o to the horizontal, with time and the two coordinates U, V increasing up the page. The curves of constant t (green) are straight lines through the orgin; those of constant r are the hyperbolas $UV =$ constant, which have the U, V-axes as asymptotes. The transformation maps the whole of Minkowski space into the region

$$-UV > 1, \qquad U < 0, \qquad V > 0$$

in the U, V-plane. The blue hyperbola is the curve $UV = -1$; that is, $r = 0$.

In the Schwarzschild geometry, the picture is very similar, except that the metric continues in the U, V-plane to the region $UV < 2m$ (the boundary $UV = 2m$ is the image of the 'real' singularity at $r = 0$ in the r, t plane). Figure 16.2 again shows the U, V-plane,

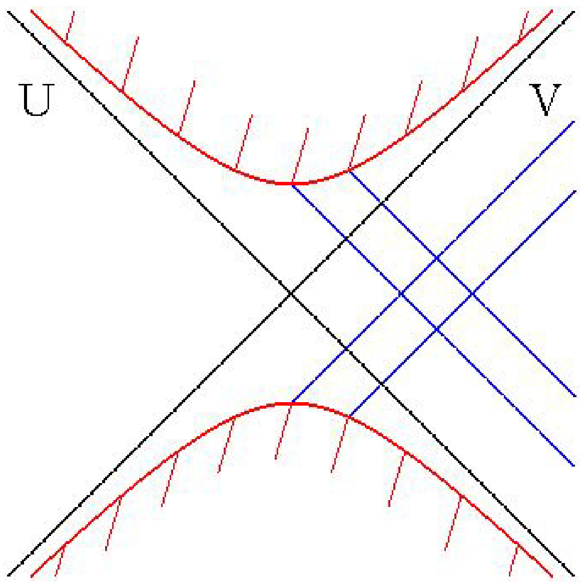


Figure 20: The Krusakal extension of the Schwarzschild geometry

with the axes drawn at 45^o to the horizontal. The blue lines are null. In this case, however, the metric is non-singular in the whole region bounded by the two branches of the hyperbola $UV = 1$ (on which $r = 0$): if we exclude the red-shaded region above and below these, then we have the *maximal analytic extension* of the Schwarzschild space-time, due to Kruskal.

If the black hole is formed by gravitational collapse, then we see only part of the diagram to the right: the rest must be replaced by a suitable interior metric. The portion covered by the Eddington-Finkelstein coordinates is the portion above the U-axis. The entire extended space-time contains both a black hole (the region $U > 0$, $V > 0$, which an observer can enter but not leave), and a 'white hole'—the time reverse of a black hole—namely the region $U < 0$ $V < 0$, which an observer can leave but not enter.

If we admit all values of U, V with $UV < 2m$, then we have the 'Kruskal extension' of the Schwarzschild space-time, which contains no matter: we can think of the 'm' in the metric as being entirely gravitational in origin, or perhaps we should think of it as the mass of the singularity at $r = 0$. Here we have no stellar boundary, and space-time looks like two 'external' regions, but they are joined together by a 'wormhole'.

The external regions are the two quadrants $V > 0 > U$ and $U > 0 > V$: for large $|UV|$, the metric looks in both like that of Minkowski space . We can see the way in they are connected by looking at the geometry of the spatial section $U = V$, on which r is given as a function of V by $V^2 = \mathrm{e}^{r/2m}(2m - r)$.

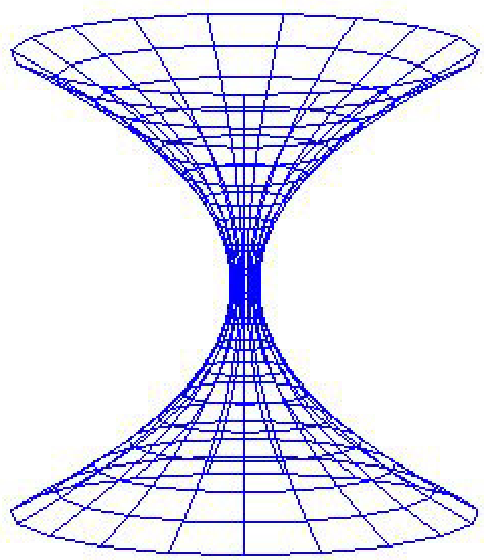


Figure 21: The spatial geometry at $t = 0$

On this r decreases to a minimum value of $2m$ and then increases again to infinity. If we put $\theta = \pi/2$ (so that we are looking at the 'equatorial plane'), then the metric is

$$\mathrm{d}s^2 = (1 - 2m/r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 = (1 + f'(r)^2)\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\,,$$

where $f = \sqrt{8m(r-2m)}$. This is the metric on a surface of revolution given by rotating the parabola $f = f(r)$ about the f-axis (Figure 16.2). Thus we can picture the hypersurface $U = V$ as two copies of Euclidean space (at large r), joined by a wormhole: to an observer in either of these external spaces, the geometry looks like that of a black hole. Of course one cannot actually travel through the wormhole: passing through $r = 2m$ takes one inside the event horizon, and inevitably into the singularity at $r = 0$.