



Analytical Mechanics for Relativity and Quantum Mechanics

Oliver Davis Johns

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for
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Oliver Davis Johns
San Francisco State University

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This book is dedicated to my parents,
Mary-Avolyn Davis Johns and Oliver Daniel Johns,
who showed me the larger world of the mind.
And to my wife, Lucy Halpern Johns,
whose love, and enthusiasm for science, made the book possible.

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PREFACE

The intended reader of this book is a graduate student beginning a doctoral program in physics or a closely related subject, who wants to understand the physical and mathematical foundations of analytical mechanics and the relation of classical mechanics to relativity and quantum theory.

The book's distinguishing feature is the introduction of extended Lagrangian and Hamiltonian methods that treat time as a *transformable coordinate*, rather than as the universal time parameter of traditional Newtonian physics. This extended theory is introduced in Part II, and is used for the more advanced topics such as covariant mechanics, Noether's theorem, canonical transformations, and Hamilton–Jacobi theory.

The obvious motivation for this extended approach is its consistency with special relativity. Since time is allowed to transform, the Lorentz transformation of special relativity becomes a canonical transformation. At the start of the twenty-first century, some hundred years after Einstein's 1905 papers, it is no longer acceptable to use the traditional definition of canonical transformation that excludes the Lorentz transformation. The book takes the position that special relativity is now a part of standard classical mechanics and should be treated integrally with the other, more traditional, topics. Chapters are included on special relativistic spacetime, fourvectors, and relativistic mechanics in fourvector notation. The extended Lagrangian and Hamiltonian methods are used to derive manifestly covariant forms of the Lagrange, Hamilton, and Hamilton–Jacobi equations.

In addition to its consistency with special relativity, the use of time as a coordinate has great value even in pre-relativistic physics. It could have been adopted in the nineteenth century, with mathematical elegance as the rationale. When an extended Lagrangian is used, the generalized energy theorem (sometimes called the Jacobi-integral theorem), becomes just another Lagrange equation. Noether's theorem, which normally requires a longer proof to deal with the intricacies of a varied time parameter, becomes a one-line corollary of Hamilton's principle. The use of extended phase space greatly simplifies the definition of canonical transformations. In the extended approach (but not in the traditional theory) a transformation is canonical if and only if it preserves the Hamilton equations. Canonical transformations can thus be characterized as the most general phase-space transformations under which the Hamilton equations are form invariant.

This is also a book for those who study analytical mechanics as a preliminary to a critical exploration of quantum mechanics. Comparisons to quantum mechanics appear throughout the text, and classical mechanics itself is presented in a way that will aid the reader in the study of quantum theory. A chapter is devoted to linear vector operators and dyadics, including a comparison to the bra-ket notation of quantum mechanics. Rotations are presented using an operator formalism similar to that used

in quantum theory, and the definition of the Euler angles follows the quantum mechanical convention. The extended Hamiltonian theory with time as a coordinate is compared to Dirac's formalism of primary phase-space constraints. The chapter on relativistic mechanics shows how to use covariant Hamiltonian theory to write the Klein–Gordon and Dirac wave functions. The chapter on Hamilton–Jacobi theory includes a discussion of the closely related Bohm hidden variable model of quantum mechanics.

The reader is assumed to be familiar with ordinary three-dimensional vectors, and to have studied undergraduate mechanics and linear algebra. Familiarity with the notation of modern differential geometry is not assumed. In order to appreciate the advance that the differential-geometric notation represents, a student should first acquire the background knowledge that was taken for granted by those who created it. The present book is designed to take the reader up to the point at which the methods of differential geometry should properly be introduced—before launching into phase-space flow, chaotic motion, and other topics where a geometric language is essential.

Each chapter in the text ends with a set of exercises, some of which extend the material in the chapter. The book attempts to maintain a level of mathematical rigor sufficient to allow the reader to see clearly the assumptions being made and their possible limitations. To assist the reader, arguments in the main body of the text frequently refer to the mathematical appendices, collected in Part III, that summarize various theorems that are essential for mechanics. I have found that even the most talented students sometimes lack an adequate mathematical background, particularly in linear algebra and many-variable calculus. The mathematical appendices are designed to refresh the reader's memory on these topics, and to give pointers to other texts where more information may be found.

This book can be used in the first year of a doctoral physics program to provide a necessary bridge from undergraduate mechanics to advanced relativity and quantum theory. Unfortunately, such bridge courses are sometimes dropped from the curriculum and replaced by a brief classical review in the graduate quantum course. The risk of this is that students may learn the recipes of quantum mechanics but lack knowledge of its classical roots. This seems particularly unwise at the moment, since several of the current problems in theoretical physics—the development of quantum information technology, and the problem of quantizing the gravitational field, to name two—require a fundamental rethinking of the quantum-classical connection. Since progress in physics depends on researchers who understand the foundations of theories and not just the techniques of their application, it is hoped that this text may encourage the retention or restoration of introductory graduate analytical mechanics courses.

Oliver Davis Johns
San Francisco, California
April 2005

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Readers are encouraged to send comments and corrections by electronic mail to ojohns@metacosmos.org. A web page with errata and addenda will be maintained at <http://www.metacosmos.org>.

–ODJ

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Part I

Introduction: The Traditional Theory

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BASIC DYNAMICS OF POINT PARTICLES AND COLLECTIONS

Modern mechanics begins with the publication in 1687 of Isaac Newton's *Principia*, an extension of the work of his predecessors, notably Galileo and Descartes, that allows him to explain mathematically what he calls the "System of the World": the motions of planets, moons, comets, tides. The three "Axioms, or Laws of Motion" in the *Principia* (Newton, 1729) are:

Law I: *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

Law II: *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

Law III: *To every Action there is always opposed an equal Reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

These axioms refer to the general behavior of a "body." It is clear from Newton's examples (projectiles, a top, planets, comets, a stone) in the same section that he intends these bodies to be macroscopic, ordinary objects.

But elsewhere Newton (1730) refers to the "particles of bodies" in ways that suggest an atomic theory in which the primitive, elementary objects are small, indestructible, "solid, massy, hard, impenetrable, movable particles." These are what we will call the *point particles* of Newtonian physics. Newton says of them that, "these primitive Particles being Solids, are incomparably harder than any porous Bodies compounded of them; even so very hard as never to wear or break in pieces; no ordinary Power being able to divide what God himself made one in the first Creation."

The present chapter will begin with the assumption that Newton's three axioms refer fundamentally to these point particles. After deriving the laws of momentum, angular momentum, and work–energy for point particles, we will show that, given certain plausible and universally accepted additional axioms, essentially the same laws can be proved to apply to macroscopic bodies, considered as collections of the elementary point particles.

1.1 Newton's Space and Time

Before discussing the laws of motion of point masses, we must consider the space and time in which that motion takes place. For Newton, space was logically and physically distinct from the masses that might occupy it. Space provided a static, absolute, and

independent reference with respect to which all particle positions and motions were to be measured. Space could be perceived by looking at the fixed stars which were presumed to be at rest relative to it. Newton also emphasized the ubiquity of space, comparing it to the *sensorium* of God.¹

Newton thought of time geometrically, comparing it to a mathematical point moving steadily along a straight line. As with space, the even flow of time was absolute and independent of objects. He writes in the *Principia*, “Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external.”²

In postulating an absolute space, Newton was breaking with Descartes, who held that the proper definition of motion was motion with respect to nearby objects. In the *Principia*, Newton uses the example of a spinning bucket filled with water to argue for absolute motion. If the bucket is suspended by a rope from a tree limb and then twisted, upon release the bucket will initially spin rapidly but the water will remain at rest. One observes that the surface of the water remains flat. Later, when the water has begun to rotate with the bucket, the surface of the water will now be concave, in response to the forces required to maintain its accelerated circular motion. If motion were to be measured with respect to proximate objects, one would expect the opposite observations. Initially, there is a large relative motion between the water and the proximate bucket, and later the two have nearly zero relative motion. So the Cartesian view would predict inertial effects initially, with the water surface becoming flat later, contrary to observation.

Newton realized that, as a practical matter, motion would often be measured by reference to objects rather than to absolute space directly. As we discuss in Section 14.1, the Galilean relativity principle states that Newton’s laws hold when position is measured with respect to *inertial systems* that are either at rest, or moving with constant velocity, relative to absolute space. But Newton considered these relative standards to be secondary, merely stand-ins for space.

Nearly the opposite view was held by Newton’s great opponent, Leibniz, who held that space is a “mere seeming thing” and that the only reality is the relation of objects. Their debate took the form of an exchange of letters, later published, between Leibniz and Clarke, Newton’s surrogate.³ Every student is urged to read them. The main difficulty for the modern reader is the abundance of theological arguments, mixed almost inextricably with the physical ones. One can appreciate the enormous progress that has been made since the seventeenth century in freeing physics from the constraints of theology. In the century after Newton and Leibniz, their two philosophical traditions continued to compete. But the success of the Newtonian method in explaining

¹Seventeenth century physiology held that the information from human sense organs is collected in a “sensorium” which the soul then views.

²Newton’s ideas about time were possibly influenced by those of his predecessor at Cambridge, Isaac Barrow. See Chapter 9 of Whitrow (1989).

³The correspondence is reprinted, with portions of Newton’s writings, in Alexander (1956).

experiments and phenomena led to its gradual ascendancy.⁴

Newton's space and time were challenged by Mach in the late nineteenth century. Mach argued, like Leibniz, that absolute space and time are illusory and that the only reality is the relation of objects.⁵ Mach also proposed that the inertia of a particle is related to the existence of other particles and presumably would vanish without them, an idea that Einstein referred to as *Mach's Principle*.

Einstein's special relativity unifies space and time. And in his general relativity the metric of the combined *spacetime* becomes dynamic rather than static and absolute. General relativity is Machian in the sense that the masses of the universe affect the local *curvature* of spacetime, but Newtonian in the sense that spacetime itself (now represented by the dynamic metric field) is something all pervasive that has definite properties even at points containing no masses.

For the remainder of Part I of the book, we will adopt the traditional Newtonian definition of space and time. In Part II, we will consider the modifications of Lagrangian and Hamiltonian mechanics that are needed to accommodate special relativity, in which space and time are combined and time becomes a transformable coordinate.

1.2 Single Point Particle

In this section, we assume the applicability of Newton's laws to point particles, and introduce the basic derived quantities: momentum, angular momentum, work, kinetic energy, and their relations.

An uncharged point particle is characterized completely by its mass m and its position \mathbf{r} relative to the origin of some inertial system of coordinates. The velocity $\mathbf{v} = d\mathbf{r}/dt$ and acceleration $\mathbf{a} = d\mathbf{v}/dt$ are derived by successive differentiation. Its momentum (which is what Newton called "motion" in his second law) is defined as

$$\mathbf{p} = m\mathbf{v} \tag{1.1}$$

Newton's second law then can be expressed as the law of momentum for point particles,

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} \tag{1.2}$$

Since the mass of a point particle is unchanging, this is equivalent to the more familiar $\mathbf{f} = m\mathbf{a}$. The requirement that the change of momentum is "in the direction of the right

⁴Leibnizian ideas continued to be influential, however. The great eighteenth century mathematician Euler, to whom our subject owes so much, published in 1768 a widely read book, *Letters Addressed to a German Princess*, in which he explained the science of his day to the lay person (Euler, 1823). He felt it necessary to devote some thirty pages of that book to refute Wolff, the chief proponent of Leibniz's philosophy. See also the detailed defense of Newton's ideas in Euler, L. (1748) "Reflexions sur l'Espace et le Temps," *Mémoires de l'Académie des Sciences de Berlin*, reprinted in Series III, Volume 2 of Euler (1911).

⁵See Mach (1907). Discussions of Mach's ideas are found in Rindler (1977, 2001) and Misner, Thorne and Wheeler (1973). A review of the history of spacetime theories from a Machian perspective is found in Barbour (1989, 2001). See also Barbour and Pfister (1995).

line" of the impressed force \mathbf{f} is guaranteed in modern notation by the use of vector quantities in the equations.

For the point particles, Newton's first law follows directly from eqn (1.2). When $\mathbf{f} = 0$, the time derivative of \mathbf{p} is zero and so \mathbf{p} is a constant vector. Note that eqn (1.2) is a *vector* relation. If, for example, the x -component of force f_x is zero, then the corresponding momentum component p_x will be constant regardless of what the other components may do.

The angular momentum \mathbf{j} of a point particle and the torque $\boldsymbol{\tau}$ acting on it are defined, respectively, as

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} \qquad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{f} \qquad (1.3)$$

It follows that the law of angular momentum for point particles is

$$\boldsymbol{\tau} = \frac{d\mathbf{j}}{dt} \qquad (1.4)$$

since

$$\frac{d\mathbf{j}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{f} = 0 + \boldsymbol{\tau} \qquad (1.5)$$

In a time dt the particle moves a vector distance $d\mathbf{r} = \mathbf{v} dt$. The work dW done by force \mathbf{f} in this time is defined as

$$dW = \mathbf{f} \cdot d\mathbf{r} \qquad (1.6)$$

This work is equal to the increment of the quantity $(1/2)mv^2$ since

$$dW = \mathbf{f} \cdot \mathbf{v} dt = \left(\frac{d(m\mathbf{v})}{dt} dt \right) \cdot \mathbf{v} = m (d\mathbf{v}) \cdot \mathbf{v} = d \left(\frac{1}{2}mv^2 \right) \qquad (1.7)$$

Taking a particle at rest to have zero kinetic energy, we define the kinetic energy T as

$$T = \frac{1}{2}mv^2 \qquad (1.8)$$

with the result that a work-energy theorem for point particles may be expressed as $dW = dT$ or

$$\mathbf{f} \cdot \mathbf{v} = \frac{dT}{dt} \qquad (1.9)$$

If the force \mathbf{f} is either zero or constantly perpendicular to \mathbf{v} (as is the case for purely magnetic forces on a charged particle, for example) then the left side of eqn (1.9) will vanish and the kinetic energy T will be constant.

1.3 Collective Variables

Now imagine a collection of N point particles labeled by index n , with masses m_1, m_2, \dots, m_N and positions $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$.

The other quantities defined in Section 1.2 will be indexed similarly, with $\mathbf{p}_n = m_n \mathbf{v}_n$, for example, referring to the momentum of the n th particle and \mathbf{f}_n denoting the

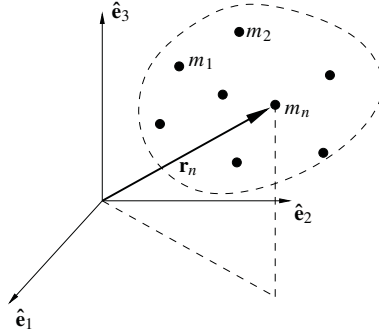


FIG. 1.1. A collection of point masses.

force acting on it. The total mass, momentum, force, angular momentum, torque, and kinetic energy of this collection may be defined by

$$M = \sum_{n=1}^N m_n \quad \mathbf{P} = \sum_{n=1}^N \mathbf{p}_n \quad \mathbf{F} = \sum_{n=1}^N \mathbf{f}_n \quad \mathbf{J} = \sum_{n=1}^N \mathbf{j}_n \quad \boldsymbol{\tau} = \sum_{n=1}^N \boldsymbol{\tau}_n \quad T = \sum_{n=1}^N T_n \quad (1.10)$$

Note that, in the cases of \mathbf{P} , \mathbf{F} , \mathbf{J} , and $\boldsymbol{\tau}$, these are *vector* sums. If a particular collection consisted of two identical particles moving at equal speeds in opposite directions, for example, \mathbf{P} would be zero.

In the following sections, we derive the equations of motion for these collective variables. All of the equations of Section 1.2 are assumed to hold individually for each particle in the collection, with the obvious addition of subscripts n to each quantity to label the particular particle being considered. For example, $\mathbf{v}_n = d\mathbf{r}_n/dt$, $\mathbf{a}_n = d\mathbf{v}_n/dt$, $\mathbf{p}_n = m_n\mathbf{v}_n$, $\mathbf{f}_n = d\mathbf{p}_n/dt$, $\mathbf{f}_n = m_n\mathbf{a}_n$, etc.

1.4 The Law of Momentum for Collections

We begin with the law of momentum. Differentiation of the sum for \mathbf{P} in eqn (1.10), using eqn (1.2) in the indexed form $d\mathbf{p}_n/dt = \mathbf{f}_n$, gives

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt} \sum_{n=1}^N \mathbf{p}_n = \sum_{n=1}^N \frac{d\mathbf{p}_n}{dt} = \sum_{n=1}^N \mathbf{f}_n = \mathbf{F} \quad (1.11)$$

The time rate of change of the total momentum is thus the total force.

But the force \mathbf{f}_n on the n th particle may be examined in more detail. Suppose that it can be written as the vector sum of an external force $\mathbf{f}_n^{(\text{ext})}$ coming from influences operating on the collection from outside it, and an internal force $\mathbf{f}_n^{(\text{int})}$ consisting of all forces that cannot be identified as external, such as forces on particle n coming from collision or other interaction with other particles in the collection. For example, if the collection were a globular cluster of stars (idealized here as point particles!) orbiting a galactic center, the external force on star n would be the gravitational attraction

from the galaxy, and the internal force would be the gravitational attraction of the other stars in the cluster. Thus

$$\mathbf{f}_n = \mathbf{f}_n^{(\text{ext})} + \mathbf{f}_n^{(\text{int})} \quad \text{and, correspondingly,} \quad \mathbf{F} = \mathbf{F}^{(\text{ext})} + \mathbf{F}^{(\text{int})} \quad (1.12)$$

where

$$\mathbf{F}^{(\text{ext})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \quad \text{and} \quad \mathbf{F}^{(\text{int})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{int})} \quad (1.13)$$

Axiom 1.4.1: The Law of Momentum

It is taken as an axiom in all branches of modern physics that, insofar as the action of outside influences can be represented by forces, the following Law of Momentum must hold:

$$\mathbf{F}^{(\text{ext})} = \frac{d\mathbf{P}}{dt} \quad (1.14)$$

It follows from this Law and eqn (1.11) that $\mathbf{F} = \mathbf{F}^{(\text{ext})}$ and hence $\mathbf{F}^{(\text{int})} = 0$. Identifying \mathbf{P} with Newton's "motion" of a body, and $\mathbf{F}^{(\text{ext})}$ with his "motive force impressed" on it, eqn (1.14) simply restates Newton's second law for bodies, now considered as collections of point particles.

An immediate consequence of the Law of Momentum is that the vanishing of $\mathbf{F}^{(\text{ext})}$ makes \mathbf{P} constant. We then say that \mathbf{P} is *conserved*. This rule of momentum conservation is generally believed to apply even for those situations that cannot be described correctly by the concept of force. This is the essential content of Newton's first law. The total momentum of an isolated body does not change.

1.5 The Law of Angular Momentum for Collections

The derivation of the Law of Angular Momentum is similar to the previous Section 1.4. Differentiation of the sum for \mathbf{J} in eqn (1.10), using eqn (1.4) in the indexed form $d\mathbf{j}_n/dt = \boldsymbol{\tau}_n$, gives

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt} \sum_{n=1}^N \mathbf{j}_n = \sum_{n=1}^N \frac{d\mathbf{j}_n}{dt} = \sum_{n=1}^N \boldsymbol{\tau}_n = \boldsymbol{\tau} \quad (1.15)$$

The time rate of change of the total angular momentum is thus the total torque.

Making the same division of forces into external and internal as was done in Section 1.4, we use the indexed form of eqn (1.3) to write the torque on particle n as the sum of external and internal torques,

$$\boldsymbol{\tau}_n = \mathbf{r}_n \times \mathbf{f}_n = \mathbf{r}_n \times \left(\mathbf{f}_n^{(\text{ext})} + \mathbf{f}_n^{(\text{int})} \right) = \boldsymbol{\tau}_n^{(\text{ext})} + \boldsymbol{\tau}_n^{(\text{int})} \quad (1.16)$$

where

$$\boldsymbol{\tau}_n^{(\text{ext})} = \mathbf{r}_n \times \mathbf{f}_n^{(\text{ext})} \quad \text{and} \quad \boldsymbol{\tau}_n^{(\text{int})} = \mathbf{r}_n \times \mathbf{f}_n^{(\text{int})} \quad (1.17)$$

Then, the total torque $\boldsymbol{\tau}$ defined in eqn (1.10) may then be written

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(\text{ext})} + \boldsymbol{\tau}^{(\text{int})} \quad (1.18)$$

where

$$\boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^N \boldsymbol{\tau}_n^{(\text{ext})} \quad \text{and} \quad \boldsymbol{\tau}^{(\text{int})} = \sum_{n=1}^N \boldsymbol{\tau}_n^{(\text{int})} \quad (1.19)$$

Axiom 1.5.1: The Law of Angular Momentum

It is taken as an axiom in all branches of modern physics that, insofar as the action of outside influences can be represented by forces, the following Law of Angular Momentum must hold:

$$\boldsymbol{\tau}^{(\text{ext})} = \frac{d\mathbf{J}}{dt} \quad (1.20)$$

It follows from this Law and eqn (1.15) that $\boldsymbol{\tau} = \boldsymbol{\tau}^{(\text{ext})}$ and hence $\boldsymbol{\tau}^{(\text{int})} = 0$. An immediate consequence of the Law of Angular Momentum is that the vanishing of $\boldsymbol{\tau}^{(\text{ext})}$ makes \mathbf{J} constant. We then say that \mathbf{J} is *conserved*. This rule of angular momentum conservation is generally believed to apply even for those situations that cannot be described correctly by the concept of force. The total angular momentum of an isolated body does not change.

It is important to notice that the Laws of Momentum and Angular Momentum are *vector* relations. For example, in eqn (1.14), if $F_y^{(\text{ext})} = 0$ then P_y is conserved regardless of the values of the other components of the total external force. A similar separation of components holds also in eqn (1.20).

1.6 “Derivations” of the Axioms

Although the Law of Momentum is an axiom, it can actually be “derived” if one accepts an outdated action-at-a-distance model of internal forces in which the force $\mathbf{f}_n^{(\text{int})}$ is taken as the instantaneous vector sum of forces on particle n coming from all of the other particles in the collection. Denote the force on particle n coming from particle n' as $\mathbf{f}_{nn'}$ and thus write

$$\mathbf{f}_n^{(\text{int})} = \sum_{\substack{n'=1 \\ n' \neq n}}^N \mathbf{f}_{nn'} \quad \text{and hence} \quad \mathbf{F}^{(\text{int})} = \sum_{n=1}^N \sum_{\substack{n'=1 \\ n' \neq n}}^N \mathbf{f}_{nn'} \quad (1.21)$$

In this model, Newton’s third law applied to the point particles implies that

$$\mathbf{f}_{nn'} = -\mathbf{f}_{n'n} \quad (1.22)$$

which makes the symmetric double sum in eqn (1.21) vanish identically. With $\mathbf{F}^{(\text{int})} = 0$, eqns (1.11, 1.12) then imply eqn (1.14), as was to be proved. Equation (1.22) is sometimes referred to as the *weak form of Newton’s third law*. We emphasize, however, that the Law of Momentum is more general than the action-at-a-distance model of the internal forces used in this derivation.

The Law of Angular Momentum is also an axiom but, just as in the case of linear momentum, it too can be “derived” from an outdated action-at-a-distance model of

internal forces. We again denote the force on particle n coming from particle n' as $\mathbf{f}_{nn'}$ and thus write

$$\boldsymbol{\tau}_n^{(\text{int})} = \mathbf{r}_n \times \mathbf{f}_n^{(\text{int})} = \sum_{\substack{n'=1 \\ n' \neq n}}^N \mathbf{r}_n \times \mathbf{f}_{nn'} \quad \text{and hence} \quad \boldsymbol{\tau}^{(\text{int})} = \sum_{n=1}^N \sum_{\substack{n'=1 \\ n' \neq n}}^N \mathbf{r}_n \times \mathbf{f}_{nn'} \quad (1.23)$$

It follows from eqn (1.22) that the second of eqn (1.23) may be rewritten as

$$\boldsymbol{\tau}^{(\text{int})} = \frac{1}{2} \sum_{n=1}^N \sum_{\substack{n'=1 \\ n' \neq n}}^N (\mathbf{r}_n - \mathbf{r}_{n'}) \times \mathbf{f}_{nn'} \quad (1.24)$$

If we now assume (which we did *not* need to assume in the linear momentum case) that the force $\mathbf{f}_{nn'}$ is *central*, that is parallel (or anti-parallel) to the line $(\mathbf{r}_n - \mathbf{r}_{n'})$ between particles n and n' , then it follows from the vanishing of the cross products that $\boldsymbol{\tau}^{(\text{int})}$ is zero, as was to be proved.

The addition of centrality to eqn (1.22) is sometimes called the *strong form of Newton's third law*. We emphasize that, as in the case of linear momentum, the Law of Angular Momentum is more general than the model of central, action-at-a-distance internal forces used in this last derivation.

For example, the laws of momentum and angular momentum can be applied correctly to the behavior of physical objects such as quartz spheres, whose internal structure requires modern solid-state physics for its description rather than Newtonian central forces between point masses. Yet, when there are identifiable external force fields acting, such as gravity for example, these objects will obey Axioms 1.4.1 and 1.5.1.

1.7 The Work–Energy Theorem for Collections

The work–energy theorem of eqn (1.9) can be extended to collections. Using the definition in eqn (1.10) together with the indexed form of eqn (1.8), the total kinetic energy is

$$T = \sum_{n=1}^N T_n = \frac{1}{2} \sum_{n=1}^N m_n v_n^2 \quad \text{with} \quad v_n^2 = \mathbf{v}_n \cdot \mathbf{v}_n \quad (1.25)$$

Then the time rate of change of T is equal to the rate at which work is done on all particles of the collection,

$$\frac{dT}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot \mathbf{v}_n \quad (1.26)$$

To prove this result, differentiate the sum for T in eqn (1.10), using eqn (1.9) in its indexed form $dT_n/dt = \mathbf{f}_n \cdot \mathbf{v}_n$ where $\mathbf{v}_n = d\mathbf{r}_n/dt$. Then

$$\frac{dT}{dt} = \frac{d}{dt} \sum_{n=1}^N T_n = \sum_{n=1}^N \frac{dT_n}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot \mathbf{v}_n \quad (1.27)$$

as was to be proved.

There is little benefit to introducing the separation of force \mathbf{f}_n into external and internal terms here, since the total kinetic energy T can be changed even when no external forces are present. For example, consider four identical particles initially at rest at the four corners of a plane square. If there is a gravitational internal force among those particles, they will begin to collapse toward the center of the square. Thus T will increase even though only internal forces are acting.

1.8 Potential and Total Energy for Collections

In some cases, there will exist a potential function $U = U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ from which all forces on all particles can be derived. Thus

$$\mathbf{f}_n = -\nabla_n U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = -\frac{\partial}{\partial \mathbf{r}_n} U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) \quad (1.28)$$

where⁶

$$\nabla_n = \frac{\partial}{\partial \mathbf{r}_n} = \sum_{i=1}^3 \hat{\mathbf{e}}_i \frac{\partial}{\partial x_{ni}} \quad (1.29)$$

and x_{ni} is the i^{th} coordinate of the n^{th} particle of the collection, that is, $\mathbf{r}_n = \sum_{i=1}^3 x_{ni} \hat{\mathbf{e}}_i$.

The total energy E is defined as $E = T + U$, where T is the total kinetic energy. Its rate of change is

$$\frac{dE}{dt} = \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)}{\partial t} \quad (1.30)$$

To see this, use the chain rule of partial differentiation and eqns (1.27, 1.28) to write

$$\frac{dT}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot \mathbf{v}_n = - \sum_{n=1}^N \mathbf{v}_n \cdot \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)}{\partial \mathbf{r}_n} = - \left(\frac{dU}{dt} - \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)}{\partial t} \right) \quad (1.31)$$

where the last equality implies eqn (1.30).

If the potential function $U = U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)$ happens not to depend explicitly on the time t , the partial derivative in eqn (1.30) will vanish and E will be a constant. The total energy of the collection is then said to be *conserved*.

1.9 The Center of Mass

All of the collective variables in eqn (1.10) are simple scalar or vector sums of individual quantities. The center of mass of the collection \mathbf{R} is only slightly more complicated. It is defined as the mass-weighted average position of the particles making up

⁶See Section A.11 for a discussion of the notation $\partial U / \partial \mathbf{r}_n$, including cautions about its proper use.

the collection,

$$\mathbf{R} = \frac{1}{M} \sum_{n=1}^N m_n \mathbf{r}_n \quad (1.32)$$

This \mathbf{R} can be used to define a new set of position vectors $\boldsymbol{\rho}_n$ for the point particles, called *relative position vectors*, that give the positions of masses relative to the center of mass, rather than relative to the origin of coordinates as the \mathbf{r}_n do.

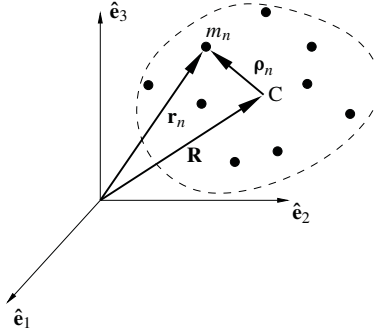


FIG. 1.2. Center of mass and relative position vectors. The center of mass is at C.

The definition is

$$\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R} \quad \text{or, equivalently,} \quad \mathbf{r}_n = \mathbf{R} + \boldsymbol{\rho}_n \quad (1.33)$$

The vector $\boldsymbol{\rho}_n$ can be thought of as the position of particle n as seen by an observer standing at the center of mass. The vectors $\boldsymbol{\rho}_n$ can be expanded in terms of Cartesian unit vectors $\hat{\mathbf{e}}_i$ as

$$\boldsymbol{\rho}_n = \sum_{i=1}^3 \rho_{ni} \hat{\mathbf{e}}_i \quad (1.34)$$

Component ρ_{ni} will be called the i th *relative coordinate* of particle n .

The velocity of the center of mass \mathbf{V} is obtained by differentiating eqn (1.32) with respect to the time,

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{1}{M} \sum_{n=1}^N m_n \mathbf{v}_n \quad (1.35)$$

Then, differentiation of eqn (1.33) yields

$$\dot{\boldsymbol{\rho}}_n = \mathbf{v}_n - \mathbf{V} \quad \text{or, equivalently,} \quad \mathbf{v}_n = \mathbf{V} + \dot{\boldsymbol{\rho}}_n \quad (1.36)$$

where the definition $\dot{\boldsymbol{\rho}}_n = d\boldsymbol{\rho}_n/dt$ is used. This quantity will be called the *relative velocity* of mass m_n . It may be thought of as the apparent velocity of m_n as seen by an observer riding on the center of mass. A particular $\dot{\boldsymbol{\rho}}_n$ may in some cases be nonzero even when $\mathbf{v}_n = 0$ and the mass m_n is at rest relative to absolute space, due

to the motion of the center of mass induced by motions of the other particles in the collection. Differentiating eqn (1.34) gives

$$\dot{\boldsymbol{\rho}}_n = \sum_{i=1}^3 \dot{\rho}_{ni} \hat{\mathbf{e}}_i \quad (1.37)$$

where the $\dot{\rho}_{ni}$ will be called the i th *relative velocity coordinate* of mass m_n .

An observer standing at the center of mass will calculate the center of mass to be at his feet, at zero distance from him, as is shown in the following lemma which will be used in the later proofs.

Lemma 1.9.1: Properties of Relative Vectors

A very useful property of vectors $\boldsymbol{\rho}_n$ and $\dot{\boldsymbol{\rho}}_n$ is

$$0 = \sum_{n=1}^N m_n \boldsymbol{\rho}_n \quad \text{and} \quad 0 = \sum_{n=1}^N m_n \dot{\boldsymbol{\rho}}_n \quad (1.38)$$

Proof: The proof of the first expression follows directly from the definitions in eqns (1.32, 1.33),

$$\sum_{n=1}^N m_n \boldsymbol{\rho}_n = \sum_{n=1}^N m_n (\mathbf{r}_n - \mathbf{R}) = \sum_{n=1}^N m_n \mathbf{r}_n - \sum_{n=1}^N m_n \mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0 \quad (1.39)$$

with the second expression following from time differentiation of the first one. □

1.10 Center of Mass and Momentum

Having defined the center of mass, we now can write various collective quantities in terms of the vectors \mathbf{R} , $\boldsymbol{\rho}$ and their derivatives. The total momentum \mathbf{P} introduced in eqn (1.10) can be expressed in terms of the total mass M and velocity of the center of mass \mathbf{V} by the remarkably simple equation

$$\mathbf{P} = M\mathbf{V} \quad (1.40)$$

To demonstrate this result, we use the second of eqn (1.36) to rewrite \mathbf{P} as

$$\mathbf{P} = \sum_{n=1}^N \mathbf{p}_n = \sum_{n=1}^N m_n \mathbf{v}_n = \sum_{n=1}^N m_n (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) = \sum_{n=1}^N m_n \mathbf{V} + \sum_{n=1}^N m_n \dot{\boldsymbol{\rho}}_n = M\mathbf{V} \quad (1.41)$$

where the Lemma 1.9.1 was used to get the last equality. The total momentum of a collection of particles is the same as would be produced by a single particle of mass M moving with the center of mass velocity \mathbf{V} .

The Law of Momentum in eqn (1.14) can then be written, using eqn (1.40) and the constancy of M , as

$$\mathbf{F}^{(\text{ext})} = \frac{d\mathbf{P}}{dt} = M\mathbf{A} \quad \text{where} \quad \mathbf{A} = \frac{d\mathbf{V}}{dt} \quad (1.42)$$

is the acceleration of the center of mass. Thus, beginning from the assumption that $\mathbf{f} = m\mathbf{a}$ for individual point particles, we have demonstrated that $\mathbf{F}^{(\text{ext})} = M\mathbf{A}$ for

composite bodies, provided that \mathbf{A} is defined precisely as the acceleration of the center of mass of the body. This last result is very close to Newton's original second law.

1.11 Center of Mass and Angular Momentum

The total angular momentum \mathbf{J} can also be rewritten in terms of center of mass and relative quantities. It is

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \quad (1.43)$$

where

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} \quad \text{and} \quad \mathbf{S} = \sum_{n=1}^N \boldsymbol{\rho}_n \times (m_n \dot{\boldsymbol{\rho}}_n) \quad (1.44)$$

will be referred to as the "orbital" and "spin" contributions to \mathbf{J} , respectively. Note that \mathbf{L} is just the angular momentum that would be produced by a single particle of mass M moving with the center of mass, and that \mathbf{S} is just the apparent angular momentum that would be calculated by an observer standing on the center of mass and using only quantities relative to herself.

To demonstrate this result, we begin with eqn (1.10) and the indexed form of eqn (1.3) to write

$$\mathbf{J} = \sum_{n=1}^N \mathbf{j}_n = \sum_{n=1}^N (\mathbf{r}_n \times \mathbf{p}_n) = \sum_{n=1}^N (\mathbf{r}_n \times m_n \mathbf{v}_n) = \sum_{n=1}^N m_n (\mathbf{r}_n \times \mathbf{v}_n) \quad (1.45)$$

Now we introduce the definitions in eqns (1.33, 1.36), and use the linearity of cross products to get

$$\begin{aligned} \mathbf{J} &= \sum_{n=1}^N m_n (\mathbf{R} + \boldsymbol{\rho}_n) \times (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) \\ &= \sum_{n=1}^N m_n \mathbf{R} \times \mathbf{V} + \sum_{n=1}^N m_n \mathbf{R} \times \dot{\boldsymbol{\rho}}_n + \sum_{n=1}^N m_n \boldsymbol{\rho}_n \times \mathbf{V} + \sum_{n=1}^N m_n \boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n \\ &= \left\{ \left(\sum_{n=1}^N m_n \right) \mathbf{R} \times \mathbf{V} \right\} + \left\{ \mathbf{R} \times \left(\sum_{n=1}^N m_n \dot{\boldsymbol{\rho}}_n \right) \right\} \\ &\quad + \left\{ \left(\sum_{n=1}^N m_n \boldsymbol{\rho}_n \right) \times \mathbf{V} \right\} + \left\{ \sum_{n=1}^N m_n \boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n \right\} \end{aligned} \quad (1.46)$$

where, in each term in curly brackets, quantities not depending on index n have been factored out of the sum. Lemma 1.9.1 now shows that the second and third terms vanish identically. The remaining two terms are identical to the \mathbf{L} and \mathbf{S} defined in eqn (1.44), as was to be proved.

1.12 Center of Mass and Torque

The Law of Angular Momentum, eqn (1.20), contains the total external torque $\boldsymbol{\tau}^{(\text{ext})}$. Using eqns (1.17, 1.18), it may be written

$$\boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^N \boldsymbol{\tau}_n^{(\text{ext})} = \sum_{n=1}^N \mathbf{r}_n \times \mathbf{f}_n^{(\text{ext})} \quad (1.47)$$

Substituting eqn (1.33) for \mathbf{r}_n then gives

$$\boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^N (\mathbf{R} + \boldsymbol{\rho}_n) \times \mathbf{f}_n^{(\text{ext})} = \mathbf{R} \times \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} + \sum_{n=1}^N \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} = \boldsymbol{\tau}_o^{(\text{ext})} + \boldsymbol{\tau}_s^{(\text{ext})} \quad (1.48)$$

where we have defined the “orbital” and “spin” external torques as

$$\boldsymbol{\tau}_o^{(\text{ext})} = \mathbf{R} \times \mathbf{F}^{(\text{ext})} \quad \text{and} \quad \boldsymbol{\tau}_s^{(\text{ext})} = \sum_{n=1}^N \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} \quad (1.49)$$

In a pattern that is becoming familiar, $\boldsymbol{\tau}_o^{(\text{ext})}$ is the torque that would result if the total external force on the collection acted on a particle at the center of mass, and $\boldsymbol{\tau}_s^{(\text{ext})}$ is the external torque on the collection that would be calculated by an observer standing at the center of mass and using $\boldsymbol{\rho}_n$ instead of \mathbf{r}_n as the moment arm.

1.13 Change of Angular Momentum

The Law of Angular Momentum in eqn (1.20) may now be broken down into separate parts, one for the orbital angular momentum \mathbf{L} and the other for the spin angular momentum \mathbf{S} . The rate of change of \mathbf{L} is equal to the orbital external torque,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_o^{(\text{ext})} \quad (1.50)$$

The demonstration is almost identical to that in Section 1.2 for the angular momentum of a single point particle,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} (\mathbf{R} \times \mathbf{P}) = \frac{d\mathbf{R}}{dt} \times \mathbf{P} + \mathbf{R} \times \frac{d\mathbf{P}}{dt} = \mathbf{V} \times M\mathbf{V} + \mathbf{R} \times \mathbf{F}^{(\text{ext})} = 0 + \boldsymbol{\tau}_o^{(\text{ext})} \quad (1.51)$$

where eqns (1.40, 1.49) and the Law of Momentum, eqn (1.14), have been used. The rate of change of \mathbf{S} is equal to the spin external torque,

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\tau}_s^{(\text{ext})} \quad (1.52)$$

The demonstration begins by using eqns (1.43, 1.48) to rewrite eqn (1.20) in the form

$$\boldsymbol{\tau}_o^{(\text{ext})} + \boldsymbol{\tau}_s^{(\text{ext})} = \frac{d\mathbf{L}}{dt} + \frac{d\mathbf{S}}{dt} \quad (1.53)$$

Equation (1.50) can then be used to cancel $d\mathbf{L}/dt$ with $\boldsymbol{\tau}_o^{(\text{ext})}$. Equating the remaining terms then gives eqn (1.52), as was to be shown.

Thus eqns (1.50, 1.52) give a separation of the Law of Angular Momentum into separate orbital and spin laws. The orbital angular momentum \mathbf{L} and the orbital torque $\boldsymbol{\tau}_o^{(\text{ext})}$ are exactly what would be produced if all of the mass of the collection were concentrated into a point particle at the center of mass. The evolution of the orbital angular momentum defined by eqn (1.50) is totally independent of the fact that the collection may or may not be spinning about the center of mass.

Equation (1.52), on the other hand, shows that the time evolution of the spin angular momentum \mathbf{S} is determined entirely by the external torque $\boldsymbol{\tau}_s^{(\text{ext})}$ measured by an observer standing at the center of mass, and is unaffected by the possible acceleration of the center of mass that may or may not be happening simultaneously.

1.14 Center of Mass and the Work–Energy Theorems

The total kinetic energy T may be expanded in the same way as the total angular momentum \mathbf{J} in Section 1.13. We may use $T_n = m_n v_n^2/2$ and $v_n^2 = \mathbf{v}_n \cdot \mathbf{v}_n$ to rewrite eqn (1.10), and then use eqn (1.36) to get

$$T = \sum_{n=1}^N T_n = \frac{1}{2} \sum_{n=1}^N m_n \mathbf{v}_n \cdot \mathbf{v}_n = \frac{1}{2} \sum_{n=1}^N m_n (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) \cdot (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) \quad (1.54)$$

Expanding the dot product and using Lemma 1.9.1 then gives

$$T = T_o + T_i \quad (1.55)$$

where

$$T_o = \frac{1}{2} M V^2 \quad \text{and} \quad T_i = \frac{1}{2} \sum_{n=1}^N m_n \|\dot{\boldsymbol{\rho}}_n\|^2 \quad (1.56)$$

are the orbital and internal kinetic energies, respectively. The time rate of change of the orbital kinetic energy is

$$\frac{dT_o}{dt} = \mathbf{F}^{(\text{ext})} \cdot \mathbf{V} \quad (1.57)$$

The demonstration uses eqn (1.40) and the Law of Momentum eqn (1.14),

$$\frac{dT_o}{dt} = \frac{d}{dt} \left(\frac{\mathbf{P} \cdot \mathbf{P}}{2M} \right) = \frac{\mathbf{P}}{M} \cdot \frac{d\mathbf{P}}{dt} = \mathbf{V} \cdot \mathbf{F}^{(\text{ext})} \quad (1.58)$$

as was to be shown.

The time rate of change of the internal kinetic energy T_i is

$$\frac{dT_i}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \quad (1.59)$$

The demonstration of eqn (1.59) is quite similar to that of eqn (1.52). We begin with

the collective work–energy theorem, eqn (1.27), rewritten using eqns (1.36, 1.55) as

$$\frac{dT_0}{dt} + \frac{dT_1}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) = \mathbf{V} \cdot \sum_{n=1}^N \mathbf{f}_n + \sum_{n=1}^N \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \quad (1.60)$$

The earlier result in Section 1.4 that $\mathbf{F} = \mathbf{F}^{(\text{ext})}$ then gives

$$\frac{dT_0}{dt} + \frac{dT_1}{dt} = \mathbf{V} \cdot \mathbf{F}^{(\text{ext})} + \sum_{n=1}^N \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \quad (1.61)$$

Using eqn (1.57) to cancel the first terms on each side gives eqn (1.59), as was to be shown. Note the *absence* of the superscript “(ext)” on \mathbf{f}_n in eqn (1.59). This is not a mistake! The internal kinetic energy T_1 can be changed by both external and internal forces, as we noted in Section 1.7.

1.15 Center of Mass as a Point Particle

It is remarkable that the center-of-mass motion of a body or other collection of point particles can be solved by imagining that the entire mass of the collection is a point particle at the center of mass \mathbf{R} with the entire external force $\mathbf{F}^{(\text{ext})}$ acting on that single point. The quantities and relations derived above,

$$\mathbf{P} = M\mathbf{V} \quad \mathbf{F}^{(\text{ext})} = \frac{d\mathbf{P}}{dt} \quad \mathbf{L} = \mathbf{R} \times \mathbf{P} \quad \frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_0^{(\text{ext})} = \mathbf{R} \times \mathbf{F}^{(\text{ext})} \quad (1.62)$$

and

$$T_0 = \frac{1}{2}MV^2 \quad \frac{dT_0}{dt} = \mathbf{F}^{(\text{ext})} \cdot \mathbf{V} \quad (1.63)$$

refer only to the total mass M , the center of mass \mathbf{R} , its derivative \mathbf{V} , and the total force $\mathbf{F}^{(\text{ext})}$. And yet these formulas replicate all of the results obtained in Section 1.2 for a single point particle.

If, as we have assumed, Newton’s laws apply fundamentally to Newtonian point particles, then these quantities and relations vindicate Newton’s application of them to “bodies” rather than point particles. A billiard ball (by which we mean the center of a billiard ball) moves according to the same laws as a single point particle of the same mass.

1.16 Special Results for Rigid Bodies

The results obtained up to this point apply to all collections, whether they be solid bodies or a diffuse gas of point particles. Now we consider special, idealized collections called *rigid bodies*. They are defined by the condition that the distance $\|\mathbf{r}_n - \mathbf{r}_{n'}\|$ between any two masses in the collection is constrained to be constant. In Chapter 8

on the kinematics of rigid-body motion, we will prove that this constraint implies the existence of a (generally time-varying) vector $\boldsymbol{\omega}$ and the relation given in eqn (8.93),

$$\dot{\boldsymbol{\rho}}_n = \boldsymbol{\omega} \times \boldsymbol{\rho}_n \quad (1.64)$$

between each relative velocity vector and the corresponding relative location vector. This relation has a number of interesting applications which we will discuss in later chapters. Here we point out one of them, the effect on eqn (1.59). Rewriting that equation and using eqn (1.64) gives

$$\frac{dT_I}{dt} = \sum_{n=1}^N \mathbf{f}_n \cdot \boldsymbol{\omega} \times \boldsymbol{\rho}_n = \boldsymbol{\omega} \cdot \sum_{n=1}^N \boldsymbol{\rho}_n \times \mathbf{f}_n = \boldsymbol{\omega} \cdot \sum_{n=1}^N (\mathbf{r}_n - \mathbf{R}) \times \mathbf{f}_n = \boldsymbol{\omega} \cdot (\boldsymbol{\tau} - \mathbf{R} \times \mathbf{F}) \quad (1.65)$$

where eqns (1.33, 1.10) have been used. But the Law of Momentum of Section 1.4 and the Law of Angular Momentum of Section 1.5 imply that

$$\mathbf{F} = \mathbf{F}^{(\text{ext})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^N \mathbf{r}_n \times \mathbf{f}_n^{(\text{ext})} \quad (1.66)$$

and hence that

$$\frac{dT_I}{dt} = \boldsymbol{\omega} \cdot (\boldsymbol{\tau}^{(\text{ext})} - \mathbf{R} \times \mathbf{F}^{(\text{ext})}) \quad (1.67)$$

depends only on the external forces $\mathbf{f}_n^{(\text{ext})}$. Thus, for rigid bodies and only for rigid bodies, we may add an “(ext)” to eqn (1.59) and write

$$\text{Rigid bodies only :} \quad \frac{dT_I}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \dot{\boldsymbol{\rho}}_n \quad (1.68)$$

It follows from eqns (1.55, 1.57, 1.68) that dT/dt for rigid bodies also depends only on external forces, and so we may write eqn (1.27) in the form

$$\text{Rigid bodies only :} \quad \frac{dT}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \mathbf{v}_n. \quad (1.69)$$

1.17 Exercises

Exercise 1.1 In spherical polar coordinates, the radius vector is $\mathbf{r} = r\hat{\mathbf{r}}$.

(a) Use the product and chain rules of differentiation, and the partial derivatives read from eqns (A.48 – A.51), to obtain the standard expression for $\mathbf{v} = d\mathbf{r}/dt$ as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}} \quad (1.70)$$

(b) By a similar process, derive the expression for $\mathbf{a} = d\mathbf{v}/dt$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}, \ddot{r}, \ddot{\theta}, \ddot{\phi}$.

Exercise 1.2 Derive the identities in eqns (A.69, A.71) and demonstrate that eqn (A.72) does follow from them.

Exercise 1.3 Consider a circular helix defined by

$$\mathbf{r} = a \cos\beta \hat{\mathbf{e}}_1 + a \sin\beta \hat{\mathbf{e}}_2 + c\beta \hat{\mathbf{e}}_3 \quad (1.71)$$

where a, c are given constants, and parameter β increases monotonically along the curve.

(a) Express the Serret–Frenet unit vectors $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$, the curvature ρ , and the torsion κ , in terms of $a, c, \beta, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$.

(b) Show that $\hat{\mathbf{n}}$ is always parallel to the x - y plane.

Exercise 1.4 In Section A.12 it is stated that the Serret–Frenet relations eqns (A.77, A.78, A.79) may be written as shown in eqn (A.80),

$$\frac{d\hat{\mathbf{t}}}{ds} = \omega \times \hat{\mathbf{t}} \quad \frac{d\hat{\mathbf{n}}}{ds} = \omega \times \hat{\mathbf{n}} \quad \frac{d\hat{\mathbf{b}}}{ds} = \omega \times \hat{\mathbf{b}} \quad (1.72)$$

where $\omega = \kappa \hat{\mathbf{t}} + \rho \hat{\mathbf{b}}$. Verify these formulas.

Exercise 1.5 A one tonne (1000 kg) spacecraft, in interstellar space far from large masses, explodes into three pieces. At the instant of the explosion, the spacecraft was at the origin of some inertial system of coordinates and had a velocity of 30 km/sec in the $+x$ direction relative to it. Precisely 10 sec after the explosion, two of the pieces are located simultaneously. They are a 300 kg piece at coordinates (400, 50, -20) km and a 500 kg piece at coordinates (240, 10, 32) km.

(a) Where was the third piece 10 sec after the explosion?

(b) Mission control wants to know where the missing piece will be 1 hour after the explosion. Give them a best estimate and an error circle. (Assume that the spacecraft had a largest dimension of 10 m, so that, at worst, a given piece might have come from a point 10 m from the center.)

(c) What if the spacecraft had been spinning end-over-end just before it exploded. Would the above answers change? At all? Appreciably? Explain.

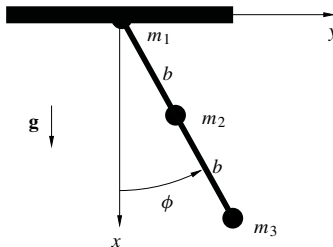


FIG. 1.3. Illustration for Exercise 1.6.

Exercise 1.6 Three equal point masses $m_1 = m_2 = m_3 = m$ are attached to a rigid, massless rod of total length $2b$. Masses #1 and #3 are at the ends of the rod and #2 is in the middle.

Mass m_1 is suspended from a frictionless pivot at the origin of an inertial coordinate system. Assume that the motion is constrained in a frictionless manner so that the masses all stay in the x - y plane. Let a uniform gravitational field $\mathbf{g} = g\hat{\mathbf{e}}_1$ act in the positive x -direction.

(a) Using plane polar coordinates, letting the r -direction be along the stick and letting ϕ be the angle between the stick and the x -axis, use the law of angular momentum to obtain $\ddot{\phi}$ and $\dot{\phi}^2$ as functions of ϕ .

(b) From the above, obtain $d^2\mathbf{r}_3/dt^2$ as a function of ϕ , $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$ and use

$$\mathbf{f}_3^{(\text{int})} = m_3 \frac{d^2\mathbf{r}_3}{dt^2} - m_3\mathbf{g} \quad (1.73)$$

to obtain the internal force $\mathbf{f}_3^{(\text{int})}$ on mass m_3 .

(c) If it is entirely due to central forces from m_1 and m_2 as is required by the “strong form” of the second law, then $\mathbf{f}_3^{(\text{int})}$ should be parallel to the stick. Is it? Explain.⁷

Exercise 1.7 Show clearly how eqns (1.55, 1.56) follow from eqn (1.54).

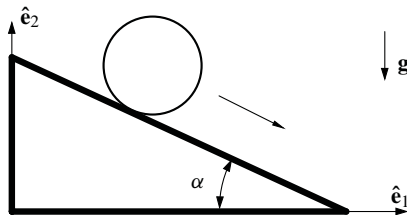


FIG. 1.4. Illustration for Exercise 1.8.

Exercise 1.8 A hollow, right-circular cylinder of mass M and radius a rolls without slipping straight down an inclined plane of angle α , starting from rest. Assume a uniform gravitational field $\mathbf{g} = -g\hat{\mathbf{e}}_2$ acting downwards.

(a) After the center of mass of the cylinder has fallen a distance h , what are the *vector* values of \mathbf{V} , \mathbf{P} , \mathbf{S} for the cylinder? [Note: This question should be answered *without* considering the details of the forces acting. Assume that rolling without slipping conserves energy.]

(b) Using your results in part (a), find the force $\mathbf{F}^{(\text{ext})}$ and spin torque $\boldsymbol{\tau}_s^{(\text{ext})}$ acting on the cylinder.

Exercise 1.9 Write out eqn (A.67) and verify that it does express the correct chain rule result for df/dt .

Exercise 1.10 If all external forces $\mathbf{f}_n^{(\text{ext})}$ on the point masses of a rigid body are derived from an *external* potential $U^{(\text{ext})}(\mathbf{r}_1, \dots, \mathbf{r}_D, t)$, show that the quantity $E = T + U^{(\text{ext})}$ obeys

$$\frac{dE}{dt} = \frac{\partial U^{(\text{ext})}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_D, t)}{\partial t} \quad (1.74)$$

⁷See Stadler, W. (1982) “Inadequacy of the Usual Newtonian Formulation for Certain Problems in Particle Mechanics,” *Am. J. Phys.* **50**, p. 595.

Exercise 1.11 Let a collection of point masses m_1, m_2, \dots, m_N move without interaction in a uniform, external gravitational field \mathbf{g} so that $\mathbf{f}_n = \mathbf{f}_n^{(\text{ext})} = m_n \mathbf{g}$.

(a) Demonstrate that a possible potential for this field is

$$U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = - \sum_{n=1}^N m_n \mathbf{r}_n \cdot \mathbf{g} \quad (1.75)$$

which may also be written as

$$U = -M\mathbf{R} \cdot \mathbf{g} \quad (1.76)$$

where M is the total mass of the collection, and \mathbf{R} is its center of mass.

(b) Express $\mathbf{F}^{(\text{ext})}$, $\boldsymbol{\tau}_o^{(\text{ext})}$, $\boldsymbol{\tau}_s^{(\text{ext})}$ in terms of M , \mathbf{g} , \mathbf{R} for this collection.

(c) Which of the following are conserved: E , \mathbf{P} , \mathbf{L} , \mathbf{S} , T_o , T_I ?

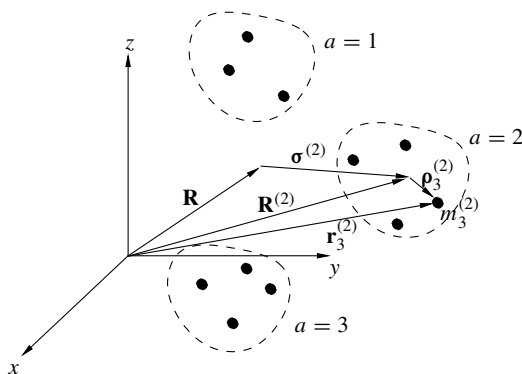


FIG. 1.5. Illustration for Exercise 1.12. Mass $m_3^{(2)}$ is the third mass in the second collection. Vector $\mathbf{R}^{(2)}$ is the center of mass of the second collection, and \mathbf{R} is the center of mass of the entire system.

Exercise 1.12 Suppose that a total collection is made up of C sub-collections, labeled by the index $a = 1, \dots, C$. The a th sub-collection has $N^{(a)}$ particles, mass $M^{(a)}$, momentum $\mathbf{P}^{(a)}$, center of mass $\mathbf{R}^{(a)}$, and center-of-mass velocity $\mathbf{V}^{(a)}$. (You might think of this as a globular cluster made up of stars. Each star is a sub-collection and the whole cluster is the total collection.)

(a) Demonstrate that the center of mass \mathbf{R} and momentum \mathbf{P} of the total collection may be written as

$$\mathbf{R} = \frac{1}{M} \sum_{a=1}^C M^{(a)} \mathbf{R}^{(a)} \quad \mathbf{P} = \sum_{a=1}^C \mathbf{P}^{(a)} \quad (1.77)$$

where

$$M = \sum_{a=1}^C M^{(a)} \quad \text{and} \quad \mathbf{P}^{(a)} = M^{(a)} \mathbf{V}^{(a)} \quad (1.78)$$

i.e. that the total center of mass and total momentum may be calculated by treating each sub-collection as a single particle with all of its mass at its center of mass.

(b) Let the n th mass of the a th sub-collection $m_n^{(a)}$ have location $\mathbf{r}_n^{(a)}$. Define $\boldsymbol{\sigma}^{(a)} = \mathbf{R}^{(a)} - \mathbf{R}$ and $\boldsymbol{\rho}_n^{(a)} = \mathbf{r}_n^{(a)} - \mathbf{R}^{(a)}$ so that

$$\mathbf{r}_n^{(a)} = \mathbf{R} + \boldsymbol{\sigma}^{(a)} + \boldsymbol{\rho}_n^{(a)} \quad (1.79)$$

Prove the identities

$$\sum_{n=1}^{N^{(a)}} m_n^{(a)} \boldsymbol{\rho}_n^{(a)} = 0 \quad \sum_{a=1}^C M^{(a)} \boldsymbol{\sigma}^{(a)} = 0 \quad (1.80)$$

and use them and their first time derivatives to demonstrate that the total angular momentum \mathbf{J} may be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{K} + \sum_{a=1}^C \mathbf{S}^{(a)} \quad (1.81)$$

where

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} \quad \mathbf{K} = \sum_{a=1}^C \boldsymbol{\sigma}^{(a)} \times M^{(a)} \dot{\boldsymbol{\sigma}}^{(a)} \quad \mathbf{S}^{(a)} = \sum_{n=1}^{N^{(a)}} \boldsymbol{\rho}_n^{(a)} \times m_n^{(a)} \dot{\boldsymbol{\rho}}_n^{(a)} \quad (1.82)$$

Note that \mathbf{K} is just the spin angular momentum that would result if each sub-collection were a point mass located at its center of mass. Then the sum over $\mathbf{S}^{(a)}$ adds the intrinsic spins of the sub-collections.

(c) Suppose that a system consists of a massless stick of length b with six point masses, each of mass m , held rigidly by a massless frame at the vertices of a plane hexagon centered on one end of the stick. Similarly, four point masses, each also of mass m , are arranged at the vertices of a plane square centered on the other end. How far from the first end is the center of mass of the whole system? Do you need to assume that the hexagon and the square are co-planar?

Exercise 1.13 Consider a system consisting of two point masses, m_1 at vector location \mathbf{r}_1 and m_2 at \mathbf{r}_2 , acted on only by internal forces \mathbf{f}_{12} and \mathbf{f}_{21} , respectively. Denote the vector from the first to the second mass by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. For this exercise, use the model in which the interaction between m_1 and m_2 is due entirely to these forces.

(a) Show that Axiom 1.4.1, implies that $\mathbf{f}_{21} + \mathbf{f}_{12} = 0$.

(b) Show that this and Axiom 1.5.1 imply that \mathbf{f}_{21} and \mathbf{f}_{12} must be parallel or anti-parallel to \mathbf{r} (i.e., be central forces).

(c) Prove that $d^2\mathbf{R}/dt^2 = 0$ and $\mu(d^2\mathbf{r}/dt^2) = \mathbf{f}_{21}$ where \mathbf{R} is the center of mass and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

(d) Show that a potential of the form $U(\mathbf{r}_1, \mathbf{r}_2) = U_0 f(r)$ where U_0 is a constant and $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ will produce forces $\mathbf{f}_{12} = -\partial U / \partial \mathbf{r}_1$ and $\mathbf{f}_{21} = -\partial U / \partial \mathbf{r}_2$ having the required properties.

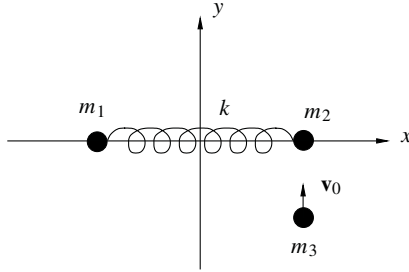


FIG. 1.6. Illustration for Exercise 1.14.

Exercise 1.14 Two masses m_1 and m_2 are connected by a massless spring of zero rest length, and force constant k . At time zero, the masses m_1 and m_2 lie at rest on the x axis at coordinates $(-a, 0, 0)$ and $(+a, 0, 0)$, respectively. Before time zero, a third mass m_3 is moving upwards with velocity $\mathbf{v}_0 = v_0 \hat{\mathbf{e}}_2$, x -coordinate a , and y -coordinate less than zero. At time zero, m_3 collides with, and sticks to, m_2 . Assume that the collision is impulsive, and is complete before m_1 or m_2 have changed position. Assume that the three masses are equal, with $m_1 = m_2 = m_3 = m$. Ignore gravity.

(a) Using the initial conditions of the problem to determine the constants of integration, write expressions for the center of mass vector \mathbf{R} and the relative position vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as functions of time for all $t > 0$.

(b) Write expressions for \mathbf{r}_1 and \mathbf{r}_2 , the vector locations of masses m_1 and m_2 , respectively, for all times $t > 0$.

(c) Show that mass m_1 has zero velocity at times $t_n = 2\pi n \sqrt{3m/2k}$, for $n = 0, 1, 2, \dots$ but that the masses never return to the x axis.

Exercise 1.15 Prove that the \mathbf{V}_\perp in eqn (A.4) can also be written as $\mathbf{V}_\perp = \hat{\mathbf{n}} \times (\mathbf{V} \times \hat{\mathbf{n}})$.

Exercise 1.16 Use eqn (A.61) to derive the related identity

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \quad (1.83)$$

and show that the triple cross product is not associative.

INTRODUCTION TO LAGRANGIAN MECHANICS

If modern mechanics began with Newton, modern *analytical* mechanics can be said to have begun with the work of the eighteenth century mathematicians who elaborated his ideas. Without changing Newton's fundamental principles, Euler, Laplace, and Lagrange developed elegant computational methods for the increasingly complex problems to which Newtonian mechanics was being applied.

The Lagrangian formulation of mechanics is, at first glance, merely an abstract way of writing Newton's second law. Someone approaching it for the first time will possibly find it ugly and counterintuitive. But the beauty of it is that, if ugly, it is terminally ugly. When simple Cartesian coordinates are replaced by the most general variables capable of describing the system adequately, the Lagrange equations do not change, do not become any more ugly than they were. The vector methods of Chapter 1 fail when a mechanical system is described by systems of coordinates much more general than the standard curvilinear ones. But such cases are treated easily by Lagrangian mechanics.

Another beauty of the Lagrangian method is that it frees us from the task of keeping track of the components of force vectors and the identities of the particles they act on. The whole of mechanics is reduced to an algebraic method. Lagrange himself was proud of the fact that his treatise on mechanics contained not a single figure.⁸

2.1 Configuration Space

In Chapter 1, the position of the n th point particle is given by the vector

$$\mathbf{r}_n = x_{n1}\hat{\mathbf{e}}_1 + x_{n2}\hat{\mathbf{e}}_2 + x_{n3}\hat{\mathbf{e}}_3 \quad (2.1)$$

where x_{n1} , x_{n2} , x_{n3} are its x , y , z coordinates, respectively. Lagrangian mechanics, however, uses what are called *generalized coordinates*, in which a particular coordinate is usually not tied to a particular particle. These generalized coordinates may be any set of independent variables capable of specifying the configuration of the system. Taken together, they define what is called *configuration space*.

For example, the simplest set of generalized coordinates is what we will call the *s-system*. Imagine all the Cartesian coordinates of N point masses listed in serial order,

⁸In the preface to his *Mécanique Analytique*, Lagrange wrote, "No diagrams are found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field." See Chapter 11 of Dugas (1955).

as in

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, \dots, x_{N1}, x_{N2}, x_{N3} \quad (2.2)$$

and define the corresponding s_i generalized coordinates as

$$s_1, s_2, s_3, s_4, s_5, s_6, s_7, \dots, s_{D-2}, s_{D-1}, s_D \quad (2.3)$$

where $D = 3N$ is called the number of *degrees of freedom* of the system. Thus $s_1 = x_{11}$, $s_2 = x_{12}$, $s_3 = x_{13}$, $s_4 = x_{21}$, \dots , $s_7 = x_{31}$, etc. For example, s_5 is the y -coordinate of the second particle.

Similarly, the force acting on the n th particle is

$$\mathbf{f}_n = f_{n1}\hat{\mathbf{e}}_1 + f_{n2}\hat{\mathbf{e}}_2 + f_{n3}\hat{\mathbf{e}}_3 \quad (2.4)$$

and we can define the *generalized forces* in the s -system, F_i , by the correspondence between the lists

$$f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, \dots, f_{N1}, f_{N2}, f_{N3} \quad (2.5)$$

and

$$F_1, F_2, F_3, F_4, F_5, F_6, F_7, \dots, F_{D-2}, F_{D-1}, F_D \quad (2.6)$$

Masses may also be relabeled by means of a correspondence between the lists

$$m_1, m_1, m_1, m_2, m_2, m_2, m_3, \dots, m_N, m_N, m_N \quad (2.7)$$

and

$$M_1, M_2, M_3, M_4, M_5, M_6, M_7, \dots, M_{D-2}, M_{D-1}, M_D \quad (2.8)$$

Note that $M_1 = M_2 = M_3 = m_1$, $M_4 = M_5 = M_6 = m_2$, etc.

With these definitions, Newton's second law can be written in either of two equivalent ways, the vector form from Chapter 1, or the equivalent form in the s -system,

$$\mathbf{f}_n = m_n \frac{d^2 \mathbf{r}_n}{dt^2} \quad \text{or} \quad F_i = M_i \frac{d^2 s_i}{dt^2} \quad (2.9)$$

where $n = 1, \dots, N$ and $i = 1, \dots, D$. The content of these two equations is identical, of course, but the second equation treats all coordinates equally, without reference to the particular particle that a coordinate belongs to.

Other physical quantities can be expressed in the s -system notation. For example, corresponding to the vector definition $\mathbf{p}_n = m_n \mathbf{v}_n$ for $n = 1, \dots, N$, the *generalized momentum* can be defined, for all $i = 1, \dots, D$, by

$$P_i = M_i \dot{s}_i \quad (2.10)$$

where $\dot{s}_i = ds_i/dt$ is called the *generalized velocity*. Then eqn (2.9) can be written in s -system notation as

$$F_i = \frac{dP_i}{dt} \quad (2.11)$$

for $i = 1, \dots, D$.

The kinetic energy defined in eqn (1.25) can also be written in the two equivalent ways, the first from Chapter 1, or the second using the s -system coordinates and masses,

$$T = \frac{1}{2} \sum_{n=1}^N m_n v_n^2 \quad \text{or} \quad T = \frac{1}{2} \sum_{i=1}^D M_i \dot{s}_i^2 \quad (2.12)$$

2.2 Newton's Second Law in Lagrangian Form

In Section 1.8 of Chapter 1, we noted that the total force on the n th particle can often be derived from a potential function $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$. Here, we are going to allow for the possibility that some, but perhaps not all, of the force on a particle can be derived from a potential so that

$$\mathbf{f}_n = -\nabla_n U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) + \mathbf{f}_n^{(\text{NP})} \quad (2.13)$$

where

$$\nabla_n = \frac{\partial}{\partial \mathbf{r}_n} = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_{n1}} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_{n2}} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_{n3}} \quad (2.14)$$

and superscript “NP” means that $\mathbf{f}_n^{(\text{NP})}$ is that part of the force that is Not derived from a Potential. Expressed in the s -system notation, eqn (2.13) becomes

$$F_i = -\frac{\partial}{\partial s_i} U(s_1, s_2, \dots, s_D, t) + F_i^{(\text{NP})} \quad (2.15)$$

where $i = 1, \dots, D$, and $U(s_1, \dots, s_D, t)$ is obtained by writing $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ out in terms of its Cartesian coordinates and then using the correspondence between eqns (2.2, 2.3) to translate to the s_i variables. Using eqns (2.11, 2.15), Newton's second law can now be written as

$$\frac{dP_i}{dt} = -\frac{\partial}{\partial s_i} U(s_1, s_2, \dots, s_D, t) + F_i^{(\text{NP})} \quad (2.16)$$

for $i = 1, \dots, D$.

To obtain the Lagrangian form of Newton's second law, define the Lagrangian $L = L(s, \dot{s}, t)$ as

$$L(s, \dot{s}, t) = T(\dot{s}) - U(s, t) \quad (2.17)$$

In expanded form, this is

$$L = L(s_1, s_2, \dots, s_D, \dot{s}_1, \dot{s}_2, \dots, \dot{s}_D, t) = \frac{1}{2} \sum_{j=1}^D M_j \dot{s}_j^2 - U(s_1, s_2, \dots, s_D, t) \quad (2.18)$$

Then it follows that

$$\frac{\partial}{\partial \dot{s}_i} L(s_1, s_2, \dots, s_D, \dot{s}_1, \dot{s}_2, \dots, \dot{s}_D, t) = M_i \dot{s}_i = P_i \quad (2.19)$$

and

$$\frac{\partial}{\partial s_i} L(s_1, s_2, \dots, s_D, \dot{s}_1, \dot{s}_2, \dots, \dot{s}_D, t) = -\frac{\partial}{\partial s_i} U(s_1, s_2, \dots, s_D, t) \quad (2.20)$$

so that eqn (2.16) may be rewritten as

$$\frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial s_i} = F_i^{(\text{NP})} \quad (2.21)$$

for $i = 1, \dots, D$. This is the Lagrangian form of Newton's second law, as expressed in the s -system of coordinates. Note that we have used the usual shorthand, abbreviating $L(s_1, \dots, s_D, \dot{s}_1, \dots, \dot{s}_D, t)$ to the shorter form $L(s, \dot{s}, t)$.

2.3 A Simple Example

Suppose one particle of mass m is acted on by a spherically symmetric, harmonic oscillator force attracting it to the origin. Then

$$L = \frac{1}{2} \left(M_1 \dot{s}_1^2 + M_2 \dot{s}_2^2 + M_3 \dot{s}_3^2 \right) - \frac{1}{2} k \left(s_1^2 + s_2^2 + s_3^2 \right) \quad (2.22)$$

But, in problems this simple, it is often clearer to replace s_1, s_2, s_3 by x, y, z , $\dot{s}_1, \dot{s}_2, \dot{s}_3$ by $\dot{x}, \dot{y}, \dot{z}$, and M_1, M_2, M_3 by m , giving

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - \frac{1}{2} k \left(x^2 + y^2 + z^2 \right) \quad (2.23)$$

We can use this more familiar notation while still thinking of the s -system in the back of our minds. Then eqn (2.21) becomes

$$\begin{aligned} \text{For } i = 1: & \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{x}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial x} = 0 & \quad \text{or} & \quad m\ddot{x} + kx = 0 \\ \text{For } i = 2: & \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{y}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial y} = 0 & \quad \text{or} & \quad m\ddot{y} + ky = 0 \\ \text{For } i = 3: & \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{z}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial z} = 0 & \quad \text{or} & \quad m\ddot{z} + kz = 0 \end{aligned} \quad (2.24)$$

which are the correct differential equations of motion for this problem.

2.4 Arbitrary Generalized Coordinates

The generalized coordinates of the s -system are only a trivial re-labelling of Cartesian coordinates. The real power of the Lagrangian method appears when we move to more general coordinate sets.

Let q_1, q_2, \dots, q_D be any set of D independent variables, which we will call the q -system, such that their values completely specify all of the s -system values, and vice

versa. We write each of the s_i , for $i = 1, \dots, D$, as a function of these q variables and possibly also the time t ,

$$s_i = s_i(q_1, q_2, \dots, q_D, t) \quad (2.25)$$

The only restriction placed on the set q_1, q_2, \dots, q_D is that eqn (2.25) must be invertible in an open neighborhood of every point of configuration space, so that we can write the inverse relations, for $k = 1, \dots, D$,

$$q_k = q_k(s_1, s_2, \dots, s_D, t) \quad (2.26)$$

As proved in Theorem D.24.1, the necessary and sufficient condition for this inversion is the Jacobian determinant condition

$$\left| \frac{\partial s(q, t)}{\partial q} \right| \neq 0 \quad (2.27)$$

where the $D \times D$ Jacobian matrix $(\partial s(q, t)/\partial q)$ is defined, for $i, k = 1, \dots, D$, by⁹

$$\left(\frac{\partial s(q, t)}{\partial q} \right)_{ik} = \frac{\partial s_i(q, t)}{\partial q_k} \quad (2.28)$$

Generalized coordinates q which obey eqn (2.27) at every point¹⁰ will be referred to as *good generalized coordinates*.

Note that we may define a matrix $(\partial q(s, t)/\partial s)$ by using eqn (2.26) to write its matrix elements, for $i, k = 1, \dots, D$, as

$$\left(\frac{\partial q(s, t)}{\partial s} \right)_{ki} = \frac{\partial q_k(s, t)}{\partial s_i} \quad (2.29)$$

It follows from eqn (2.27) and the discussion in Section D.25 of Appendix D that matrix $(\partial s(q, t)/\partial q)$ has eqn (2.29) as its inverse matrix

$$\left(\frac{\partial s(q, t)}{\partial q} \right)^{-1} = \left(\frac{\partial q(s, t)}{\partial s} \right) \quad \text{so that} \quad \sum_{k=1}^3 \frac{\partial s_i(q, t)}{\partial q_k} \frac{\partial q_k(s, t)}{\partial s_l} = \delta_{il} \quad (2.30)$$

In the next four sections, we derive some important relations between the s- and q-systems.¹¹ Then, in Section 2.9 we will prove the main result of this chapter: The Lagrange equations in a general q-system have the same form as that derived in eqn (2.21) for the s-system.

⁹Note that here, and throughout the chapter, we often use the shorthand notations $q = q_1, \dots, q_D$ and $s = s_1, \dots, s_D$ in which a single, unsubscripted letter stands for a set of variables.

¹⁰In practice, this condition may be violated in regions whose dimensionality is less than D . For example, in the transition to spherical polar coordinates, the condition is violated on the whole of the z -axis. Such regions may be excluded, and then approached as a limit.

¹¹Of course the q-system, being general, includes the s-system as a special case. But we will continue to refer to these two systems in this and the next few chapters to illustrate the methods of transformation between systems. The s-system is particularly important because of its close relation to Cartesian coordinates.

2.5 Generalized Velocities in the q-System

In Section 2.1 we defined $\dot{s}_i = ds_i/dt$ as the generalized velocities in the s-system. A similar definition, $\dot{q}_k = dq_k/dt$, is used for generalized velocities in the q-system. Since s_i in eqn (2.25) depends only on q, t , the relation between \dot{s} and \dot{q} takes a simple form.

Using the chain rule to differentiate eqn (2.25) with respect to the time allows $\dot{s}_i = ds_i/dt$ to be expanded as a function of q and its time derivatives. The expansion is

$$\dot{s}_i = \frac{ds_i(q, t)}{dt} = \sum_{k=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial s_i(q, t)}{\partial t} = \sum_{k=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \dot{q}_k + \frac{\partial s_i(q, t)}{\partial t} \quad (2.31)$$

for each $i = 1, \dots, D$, where the generalized velocities in the q-system are denoted in the last expression by $\dot{q}_k = dq_k/dt$. Inspection of eqn (2.31) shows that each \dot{s}_i depends on q, t through the dependency of the partial derivatives on these quantities, and on \dot{q} due to the \dot{q}_k factors in the sum. Thus

$$\dot{s}_i = \dot{s}_i(q, \dot{q}, t) = \dot{s}_i(q_1, q_2, \dots, q_D, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_D, t) \quad (2.32)$$

2.6 Generalized Forces in the q-System

Given the generalized force F_i in the s-system, the generalized force Q_k in the q-system is defined as

$$Q_k = \sum_{i=1}^D F_i \frac{\partial s_i(q, t)}{\partial q_k} \quad \text{with the inverse} \quad F_i = \sum_{k=1}^D Q_k \frac{\partial q_k(s, t)}{\partial s_i} \quad (2.33)$$

The reason for this definition will become apparent in Section 2.9.

Substituting eqn (2.15) into this equation gives

$$Q_k = - \sum_{i=1}^D \frac{\partial U(s, t)}{\partial s_i} \frac{\partial s_i(q, t)}{\partial q_k} + \sum_{i=1}^D F_i^{(\text{NP})} \frac{\partial s_i(q, t)}{\partial q_k} \quad (2.34)$$

If we consider the potential $U(q, t)$ in the q-system to be the same function as $U(s, t)$ but expressed in the q, t variable set, then substitution of eqn (2.25) into $U(s, t)$ gives

$$U = U(q, t) = U(s_1(q, t), s_2(q, t), \dots, s_D(q, t), t) \quad (2.35)$$

Thus the chain rule expansion of the compound function gives

$$\frac{\partial U(q, t)}{\partial q_k} = \sum_{i=1}^D \frac{\partial U(s, t)}{\partial s_i} \frac{\partial s_i(q, t)}{\partial q_k} \quad (2.36)$$

Equation (2.34) then becomes

$$Q_k = - \frac{\partial U(q, t)}{\partial q_k} + Q_k^{(\text{NP})} \quad (2.37)$$

where we have defined $Q_k^{(\text{NP})}$ to be the q-system generalized force corresponding to

$F_i^{(\text{NP})}$ according to the rule defined in eqn (2.33),

$$Q_k^{(\text{NP})} = \sum_{i=1}^D F_i^{(\text{NP})} \frac{\partial s_i(q, t)}{\partial q_k} \quad \text{with the inverse} \quad F_i^{(\text{NP})} = \sum_{k=1}^D Q_k^{(\text{NP})} \frac{\partial q_k(s, t)}{\partial s_i} \quad (2.38)$$

2.7 The Lagrangian Expressed in the q-System

We have defined the Lagrangian $L(s, \dot{s}, t)$ in the s -system in eqn (2.18) above. The Lagrangian in the q -system $L(q, \dot{q}, t)$ is defined to be the *same function*, but expressed in terms of the q, \dot{q}, t variable set.¹² Substituting eqns (2.25, 2.32) into $L(s, \dot{s}, t)$ gives the Lagrangian as a compound function of q, \dot{q}, t ,

$$L(q, \dot{q}, t) = L(s_1(q, t), s_2(q, t), \dots, s_D(q, t), \dot{s}_1(q, \dot{q}, t), \dot{s}_2(q, \dot{q}, t), \dots, \dot{s}_D(q, \dot{q}, t), t) \quad (2.39)$$

Equation (2.18) and the expansion in eqn (2.31) then give the Lagrangian in the q -system in an expanded form

$$\begin{aligned} L &= L(q, \dot{q}, t) = \frac{1}{2} \sum_{j=1}^D M_j (\dot{s}_j(q, \dot{q}, t))^2 - U(s_1(q, t), s_2(q, t), \dots, s_D(q, t), t) \\ &= \frac{1}{2} \sum_{j=1}^D M_j \left(\sum_{k=1}^D \frac{\partial s_j(q, t)}{\partial q_k} \dot{q}_k + \frac{\partial s_j(q, t)}{\partial t} \right) \left(\sum_{l=1}^D \frac{\partial s_j(q, t)}{\partial q_l} \dot{q}_l + \frac{\partial s_j(q, t)}{\partial t} \right) \\ &\quad - U(s_1(q, t), s_2(q, t), \dots, s_D(q, t), t) \end{aligned} \quad (2.40)$$

where each \dot{s}_j factor has been replaced by a separate sum. Exchanging the order of the finite sums and collecting terms then gives

$$L = L(q, \dot{q}, t) = T_2(q, \dot{q}, t) + T_1(q, \dot{q}, t) + T_0(q, t) - U(q, t) \quad (2.41)$$

where the kinetic energy is broken down into three terms

$$T(q, \dot{q}, t) = T_2(q, \dot{q}, t) + T_1(q, \dot{q}, t) + T_0(q, t) \quad (2.42)$$

where

$$T_2(q, \dot{q}, t) = \frac{1}{2} \sum_{k=1}^D \sum_{l=1}^D m_{kl}(q, t) \dot{q}_k \dot{q}_l \quad \text{with} \quad m_{kl}(q, t) = \sum_{j=1}^D M_j \frac{\partial s_j(q, t)}{\partial q_k} \frac{\partial s_j(q, t)}{\partial q_l} \quad (2.43)$$

is homogeneous of degree two in the set of variables $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_D$,

$$T_1(q, \dot{q}, t) = \sum_{k=1}^D n_k(q, t) \dot{q}_k \quad \text{with} \quad n_k(q, t) = \sum_{j=1}^D M_j \frac{\partial s_j(q, t)}{\partial q_k} \frac{\partial s_j(q, t)}{\partial t} \quad (2.44)$$

¹²We follow the physics custom which uses the same letter L in both the s and q systems, and considers $L(s, \dot{s}, t)$ and $L(q, \dot{q}, t)$ to be the same function expressed in different coordinates. See the discussion in Section D.5.

is homogeneous of degree one in the same variables, and

$$T_0(q, t) = \frac{1}{2} \sum_{j=1}^D M_j \left(\frac{\partial s_j(q, t)}{\partial t} \right)^2 \quad (2.45)$$

and

$$U(q, t) = U(s_1(q, t), s_2(q, t), \dots, s_D(q, t), t) \quad (2.46)$$

are independent of the generalized velocities \dot{q} .

2.8 Two Important Identities

The proof of the form invariance of the Lagrange equations in Section 2.9 requires the following Lemma. These two identities are formal consequences of eqn (2.31) and the properties of partial derivatives, and are true only because of the simple form of eqn (2.25) in which the $s_i = s_i(q, t)$ depend only on q and t .

Lemma 2.8.1: Identities in Configuration Space

It follows from the expansion in eqn (2.31) that the following two identities hold,

$$\frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial \dot{q}_k} = \frac{\partial s_i(q, t)}{\partial q_k} \quad \text{and} \quad \frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial s_i(q, t)}{\partial q_k} \right) \quad (2.47)$$

Proof: The first of these follows immediately from the fact that both $\partial s_i(q, t)/\partial q_k$ and $\partial s_i(q, t)/\partial t$ in eqn (2.31) are functions only of q, t , so that the explicit linear term in \dot{q}_k is the only place that the variables \dot{q} appear. The partial derivative of $\dot{s}_i(q, \dot{q}, t)$ with respect to \dot{q}_k is thus the coefficient of the \dot{q}_k in eqn (2.31), which proves the first of eqn (2.47).

The second identity in eqn (2.47) requires a somewhat longer proof. From eqn (2.31), the left side of this second equation may be written as

$$\frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial q_k} = \sum_{l=1}^D \frac{\partial}{\partial q_k} \left(\frac{\partial s_i(q, t)}{\partial q_l} \right) \dot{q}_l + \frac{\partial}{\partial q_k} \left(\frac{\partial s_i(q, t)}{\partial t} \right) \quad (2.48)$$

The right side of the second equation may be expanded by noting that, for any function $g(q, t)$,

$$\frac{dg(q, t)}{dt} = \sum_{l=1}^D \frac{\partial g(q, t)}{\partial q_l} \dot{q}_l + \frac{\partial g(q, t)}{\partial t} \quad (2.49)$$

Setting $g(q, t) = \partial s_i(q, t)/\partial q_k$ thus gives the right side of the second equation as

$$\frac{d}{dt} \left(\frac{\partial s_i(q, t)}{\partial q_k} \right) = \sum_{l=1}^D \frac{\partial}{\partial q_l} \left(\frac{\partial s_i(q, t)}{\partial q_k} \right) \dot{q}_l + \frac{\partial}{\partial t} \left(\frac{\partial s_i(q, t)}{\partial q_k} \right) \quad (2.50)$$

which is equal to eqn (2.48) when the order of partial derivatives is exchanged. \square

2.9 Invariance of the Lagrange Equations

We now come to the main theorem of this chapter: The Lagrange equations are form invariant under a change of generalized coordinates.

Theorem 2.9.1: Invariance of Lagrange Equations

Assume that a change of coordinates is made from the s -system to the q -system (assumed to be any good generalized coordinates), as defined by eqn (2.25). Define the Lagrangian function in the q -system by eqn (2.39), and the non-potential generalized force in the q -system by eqn (2.38). Then the Lagrange equations in the s -system,

$$\frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial s_i} = F_i^{(\text{NP})} \quad (2.51)$$

hold for all $i = 1, \dots, D$ if and only if the Lagrange equations in the q -system,

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = Q_k^{(\text{NP})} \quad (2.52)$$

hold for all $k = 1, \dots, D$.

Proof: We first prove that eqn (2.51) implies eqn (2.52). Multiplying both sides of eqn (2.51) by $\partial s_i(q, t)/\partial q_k$ and summing over i gives

$$\sum_{i=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \sum_{i=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \frac{\partial L(s, \dot{s}, t)}{\partial s_i} = \sum_{i=1}^D \frac{\partial s_i(q, t)}{\partial q_k} F_i^{(\text{NP})} \quad (2.53)$$

If f and g are any functions, it follows from the product rule for differentiation that $f(dg/dt) = d(fg)/dt - g(df/dt)$. Applying this rule with $f = \partial s_i(q, t)/\partial q_k$ and $g = \partial L(s, \dot{s}, t)/\partial \dot{s}_i$ allows the first term in eqn (2.53) to be rewritten as

$$\begin{aligned} & \sum_{i=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) \\ &= \sum_{i=1}^D \frac{d}{dt} \left(\frac{\partial s_i(q, t)}{\partial q_k} \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \frac{d}{dt} \left(\frac{\partial s_i(q, t)}{\partial q_k} \right) \\ &= \sum_{i=1}^D \frac{d}{dt} \left(\frac{\partial \dot{s}_i(q, t)}{\partial \dot{q}_k} \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial q_k} \end{aligned} \quad (2.54)$$

where the first and second identities in eqn (2.47) were used to rewrite the first and second terms on the right side of eqn (2.54), respectively.

Thus, rearranging terms slightly and using eqn (2.38) to replace the term on the

right by $Q_k^{(NP)}$, eqn (2.53) may be written as

$$\frac{d}{dt} \left(\sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \frac{\partial \dot{s}_i(q, t)}{\partial \dot{q}_k} \right) - \left(\sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial s_i} \frac{\partial s_i(q, t)}{\partial q_k} + \sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial q_k} \right) = Q_k^{(NP)} \quad (2.55)$$

But the first parenthesis in eqn (2.55) is the chain rule expansion of $\partial L(q, \dot{q}, t)/\partial \dot{q}_k$ where $L(q, \dot{q}, t)$ is the compound function defined in eqn (2.39). And the second parenthesis in eqn (2.55) is the chain rule expansion of $\partial L(q, \dot{q}, t)/\partial q_k$. Thus eqn (2.55) becomes

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = Q_k^{(NP)} \quad (2.56)$$

which is the same as eqn (2.52), as was to be proved.

To prove the converse, that eqn (2.52) implies eqn (2.51), we start from eqn (2.56) and reverse the chain of algebra to arrive at eqn (2.53). Multiplying that equation by $\partial q_k(s, t)/\partial s_j$, summing over $k = 1, \dots, D$, and using eqn (2.30) gives

$$\frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_j} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial s_j} = F_j^{(NP)} \quad (2.57)$$

which is identical to eqn (2.51), as was to be proved. \square

2.10 Relation Between Any Two Systems

The q-system above is taken to be *any* good system of generalized coordinates. If we imagine it and any other good system, which we may call the r-system, then it follows from what we've done above that the Lagrange equations in this r-system are equivalent to the Lagrange equations in the q-system. Both of them are equivalent to the s-system, hence they are equivalent to each other. But it may be useful to state explicitly the relations between the q- and the r-systems. We state these relations without proof, since their proof follows the pattern just established in going from the s-system to the q-system.

The transformation between the q- and r-systems is

$$q_k = q_k(r, t) \quad \text{and the inverse} \quad r_j = r_j(q, t) \quad (2.58)$$

Since both the q- and the r-systems are good generalized coordinates, the determinant conditions for transformations between them are

$$\left| \frac{\partial q(r, t)}{\partial r} \right| \neq 0 \quad \text{and} \quad \left| \frac{\partial r(q, t)}{\partial q} \right| \neq 0 \quad (2.59)$$

All generalized forces in the q-system Q_k are related to those in the r-system R_j by

$$R_j = \sum_{k=1}^D Q_k \frac{\partial q_k(r, t)}{\partial r_j} \quad \text{and the inverse} \quad Q_k = \sum_{j=1}^D R_j \frac{\partial r_j(q, t)}{\partial q_k} \quad (2.60)$$

The Lagrangian in the r-system is defined as the compound function obtained by substituting $q_k = q_k(r, t)$ and $\dot{q}_k = \dot{q}_k(r, \dot{r}, t)$ into $L(q, \dot{q}, t)$ as

$$L(r, \dot{r}, t) = L(q(r, t), \dot{q}(r, \dot{r}, t), t) \quad (2.61)$$

Then

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = Q_k^{(\text{NP})} \quad (2.62)$$

for all $k = 1, \dots, D$ if and only if

$$\frac{d}{dt} \left(\frac{\partial L(r, \dot{r}, t)}{\partial \dot{r}_j} \right) - \frac{\partial L(r, \dot{r}, t)}{\partial r_j} = R_j^{(\text{NP})} \quad (2.63)$$

for all $j = 1, \dots, D$.

2.11 More of the Simple Example

Suppose that the simple example of Section 2.3 is transformed to a q-system consisting of spherical polar coordinates. Choose $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$. Then for $i = 1, 2, 3$, respectively, the equations $s_i = s_i(q, t)$ in eqn (2.25) take the form

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta \quad (2.64)$$

and the equations $\dot{s}_i = \dot{s}_i(q_1, q_2, \dots, q_D, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_D, t)$ of eqn (2.32) are, again for $i = 1, 2, 3$, respectively,

$$\begin{aligned} \dot{x} &= \dot{r} \sin \theta \cos \phi + r \dot{\theta} \cos \theta \cos \phi - r \sin \theta \dot{\phi} \sin \phi \\ \dot{y} &= \dot{r} \sin \theta \sin \phi + r \dot{\theta} \cos \theta \sin \phi + r \sin \theta \dot{\phi} \cos \phi \\ \dot{z} &= \dot{r} \cos \theta - r \dot{\theta} \sin \theta \end{aligned} \quad (2.65)$$

Note that these equations are linear in the dotted variables, as advertised in eqn (2.31). Substituting eqns (2.64, 2.65) into the Lagrangian of eqn (2.23) following the recipe given in eqn (2.39), we obtain, after some simplification,

$$L = L(q, \dot{q}, t) = \frac{1}{2}m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{1}{2}kr^2 \quad (2.66)$$

The three Lagrange equations eqn (2.52) are then, for $k = 1, 2, 3$, respectively,

$$\begin{aligned} k = 1: \quad & \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{r}} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial r} = 0 \quad \text{or} \quad m\ddot{r} - mr\dot{\theta}^2 - mr \sin^2 \theta \dot{\phi}^2 + kr = 0 \\ k = 2: \quad & \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{\theta}} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial \theta} = 0 \quad \text{or} \quad \frac{d}{dt} (mr^2 \dot{\theta}) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \\ k = 3: \quad & \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{\phi}} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial \phi} = 0 \quad \text{or} \quad \frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = 0 \end{aligned} \quad (2.67)$$

which are the correct equations of motion in the q-system.

2.12 Generalized Momenta in the q-System

In eqn (2.19), the generalized momenta $P_i = M_i \dot{s}_i$ in the s-system were derived from partial differentiation of the Lagrangian, $P_i = \partial L(s, \dot{s}, t) / \partial \dot{s}_i$. The generalized momenta in the q-system are *defined* by a similar partial differentiation,

$$p_k = p_k(q, \dot{q}, t) = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad (2.68)$$

The expansion of the Lagrangian in eqns (2.41 – 2.46) shows that this p_k can be expanded as

$$p_k(q, \dot{q}, t) = \sum_{l=1}^D m_{kl}(q, t) \dot{q}_l + n_k(q, t) \quad (2.69)$$

A transformation law can be found between the generalized momenta in the s- and q-systems. Using eqns (2.39, 2.68) and the chain rule gives

$$p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} = \sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \frac{\partial \dot{s}_i(q, \dot{q}, t)}{\partial \dot{q}_k} = \sum_{i=1}^D P_i \frac{\partial s_i(q, t)}{\partial q_k} \quad (2.70)$$

where eqn (2.19), and the first of eqn (2.47) from Lemma 2.8.1, have been used in the final expression. Using eqn (2.30), the inverse relation can also be written

$$P_i = \sum_{k=1}^D p_k \frac{\partial q_k(s, t)}{\partial s_i} \quad (2.71)$$

The pair of quantities q_k, p_k are referred to as *conjugates*. The p_k is called the *conjugate momentum* of coordinate q_k , and the q_k is called the *conjugate coordinate* of momentum p_k . The same nomenclature is applied also to the pair s_i, P_i , and to similar pairs in any system of coordinates.

2.13 Ignorable Coordinates

The Lagrange equations in the general q-system, eqn (2.52), may be written in the form of two coupled equations,

$$\dot{p}_k = \frac{\partial L(q, \dot{q}, t)}{\partial q_k} + Q_k^{(\text{NP})} \quad \text{and} \quad p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad (2.72)$$

If $Q_k^{(\text{NP})} = 0$ and $\partial L(q, \dot{q}, t) / \partial q_k = 0$ for a particular k value, then we say that the variable q_k is *ignorable*. In this case, its conjugate momentum p_k is said to be *conserved*, which means that its time derivative vanishes and hence it is equal to a constant which may be taken to be its value at $t = 0$. If q_k is ignorable, then

$$p_k(t) = C_k = p_k(0) \quad (2.73)$$

For example, variable ϕ in Section 2.11 is ignorable.

2.14 Some Remarks About Units

Notice that the generalized coordinates in the s-system all have units of length. But the generalized coordinates in the q-system may have other units. For example, in the simple example in Section 2.11, the variables q_2 and q_3 are angles and hence unitless. However, there are certain products that will always have the same units, regardless of which system is used.

Using eqns (2.25, 2.70), with the notation δq_k for a differential at fixed time with $\delta t = 0$, the chain rule gives

$$\sum_{k=1}^D p_k \delta q_k = \sum_{k=1}^D \left(\sum_{i=1}^D P_i \frac{\partial s_i(q, t)}{\partial q_k} \right) \delta q_k = \sum_{i=1}^D P_i \left(\sum_{k=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \delta q_k \right) = \sum_{i=1}^D P_i \delta s_i \quad (2.74)$$

It follows that the units of each product $p_k q_k$ must be the same as the units of the products $P_i s_i$, which are ML^2/T , the units of what is called *action*. Thus, in the simple example of Section 2.11, the p_2 and p_3 generalized momenta are seen to be angular momenta, which have the same units as action.

Similarly, denoting differentials with time fixed by δq_k and δs_i , eqn (2.33) and the chain rule show that

$$\sum_{k=1}^D Q_k \delta q_k = \sum_{k=1}^D \left(\sum_{i=1}^D F_i \frac{\partial s_i(q, t)}{\partial q_k} \right) \delta q_k = \sum_{i=1}^D F_i \left(\sum_{k=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \delta q_k \right) = \sum_{i=1}^D F_i \delta s_i \quad (2.75)$$

It follows that each product $Q_k q_k$ must have the same units as the products $F_i s_i$, which are ML^2/T^2 , the units of work and energy. Thus, in the simple example of Section 2.11, the Q_2 and Q_3 generalized forces are torques, which have the same units as work.

The results in this section can be very useful, allowing a unit check of sorts to be performed even in complex Lagrangian systems for which the units of the q_k may be very strange.

2.15 The Generalized Energy Function

We have defined generalized coordinates, velocities, and momenta. We now define what may be thought of as a generalized energy. The *generalized energy function* (sometimes called the Jacobi-integral function) H_q in a general q-system is defined to be

$$H_q = H_q(q, \dot{q}, t) = \sum_{k=1}^D \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \dot{q}_k - L(q, \dot{q}, t) = \sum_{k=1}^D p_k(q, \dot{q}, t) \dot{q}_k - L(q, \dot{q}, t) \quad (2.76)$$

The generalized energy function in the s-system is defined similarly,

$$H_s = H_s(s, \dot{s}, t) = \sum_{i=1}^D \frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \dot{s}_i - L(s, \dot{s}, t) = \sum_{i=1}^D P_i(s, \dot{s}, t) \dot{s}_i - L(s, \dot{s}, t) \quad (2.77)$$

The subscripts on H_q and H_s are to emphasize that, unlike the Lagrangian function in the s - and q -systems, the H_q and H_s are not in general the same function. One cannot go from one to the other by simply making a coordinate substitution as we did for L .

Theorem 2.15.1: The Generalized Energy Theorem

The total time derivatives of the generalized energy functions are given by

$$\frac{dH_q}{dt} = \dot{H}_q = \sum_{k=1}^D Q_k^{(NP)} \dot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial t} \quad (2.78)$$

$$\frac{dH_s}{dt} = \dot{H}_s = \sum_{i=1}^D F_i^{(NP)} \dot{s}_i - \frac{\partial L(s, \dot{s}, t)}{\partial t} \quad (2.79)$$

Proof: The proof will be given for the q -system since the s -system proof is identical. From eqn (2.78),

$$\begin{aligned} \dot{H}_q &= \sum_{k=1}^D (\dot{p}_k \dot{q}_k + p_k \ddot{q}_k) - \frac{dL(q, \dot{q}, t)}{dt} \\ &= \sum_{k=1}^D \left(\dot{p}_k \dot{q}_k + p_k \ddot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} \dot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \ddot{q}_k \right) - \frac{\partial L(q, \dot{q}, t)}{\partial t} \end{aligned} \quad (2.80)$$

Using eqn (2.68) to cancel the \ddot{q}_k terms, and eqn (2.72) for \dot{p}_k , gives eqn (2.78) as was to be proved. \square

Equations (2.78, 2.79) are generalized work–energy theorems. If the non-potential forces vanish identically for all index values, and if the Lagrangian does not contain the letter t explicitly, then the generalized energy function will be conserved. For example, in the q -system $Q_k^{(NP)} = 0$ and $\partial L(q, \dot{q}, t)/\partial t = 0$ would imply that $\dot{H}_q = 0$, and hence that

$$H_q(q(t), \dot{q}(t), t) = C = H_q(q(0), \dot{q}(0), 0) \quad (2.81)$$

2.16 The Generalized Energy and the Total Energy

One can easily show using eqns (2.18, 2.77) that

$$H_s = \frac{1}{2} \sum_{j=1}^D M_j \dot{s}_j^2 + U(s_1, s_2, \dots, s_D, t) = T + U = E \quad (2.82)$$

where E is identical to the total energy defined in Section 1.8. So the s -system generalized energy function H_s is equal to the total energy.

The situation is different in the general q -system, however. Using eqns (2.41, 2.76),

$$H_q = \sum_{k=1}^D \dot{q}_k \frac{\partial T_2(q, \dot{q}, t)}{\partial \dot{q}_k} + \sum_{k=1}^D \dot{q}_k \frac{\partial T_1(q, \dot{q}, t)}{\partial \dot{q}_k} + \sum_{k=1}^D \dot{q}_k \frac{\partial T_0(q, \dot{q}, t)}{\partial \dot{q}_k} - L(q, \dot{q}, t) \quad (2.83)$$

Since functions T_l are homogeneous of degree l in the generalized velocities \dot{q}_k , the

Euler condition from Theorem D.31.1 shows that

$$\sum_{k=1}^D \dot{q}_k \frac{\partial T_2(q, \dot{q}, t)}{\partial \dot{q}_k} = 2T_2(q, \dot{q}, t) \quad \sum_{k=1}^D \dot{q}_k \frac{\partial T_1(q, \dot{q}, t)}{\partial \dot{q}_k} = T_1(q, \dot{q}, t)$$

$$\sum_{k=1}^D \dot{q}_k \frac{\partial T_0(q, \dot{q}, t)}{\partial \dot{q}_k} = 0 \quad (2.84)$$

Therefore

$$H_q = 2T_2 + T_1 - (T_2 + T_1 + T_0 - U) = T_2 - T_0 + U \quad (2.85)$$

and H_q is related to $H_s = T + U$ by

$$H_q = (T + U) - (T_1 + 2T_0) = H_s - (T_1 + 2T_0) = E - (T_1 + 2T_0) \quad (2.86)$$

which is not in general equal to the total energy E .

Examination of eqns (2.44, 2.45, 2.86) shows that the condition for H_q to equal $H_s = T + U$ is for the coordinate transformation equation not to contain the letter t explicitly. Then $\partial s_i(q, t)/\partial t = 0$, which in turn implies that both T_1 and T_0 are zero. Thus

$$\frac{\partial s_i(q, t)}{\partial t} = 0 \quad \text{implies that} \quad H_q = H_s = T + U \quad (2.87)$$

Note to the Reader: The condition for H_q to be conserved (which, in the absence of non-potential forces, is $\partial L(q, \dot{q}, t)/\partial t = 0$) is *independent* of the condition for $H_q = T + U$ (which is $\partial s_i(q, t)/\partial t = 0$). The H_q may be conserved even when the total energy E is not.

The generalized energy function is most useful in problem solutions when it is conserved. And if H_q is conserved, it usually makes little difference to the problem solution whether or not H_q equals $T + U$. For conservation implies the equation $H_q(q, \dot{q}, t) = C$, a first-order differential equation and a first integral of the equations of motion, regardless of the relation of H_q to the total energy.

2.17 Velocity Dependent Potentials

The problem of N charged particles in a given, externally applied electromagnetic field can also be reduced to Lagrangian form. We use the s-system of generalized coordinates, expressed in vector notation.

The Lorentz force acting on the n th particle is

$$\mathbf{f}_n = q_n^{(\text{ch})} \mathbf{E}(\mathbf{r}_n, t) + \frac{q_n^{(\text{ch})}}{c} \mathbf{v}_n \times \mathbf{B}(\mathbf{r}_n, t) \quad (2.88)$$

where $q_n^{(\text{ch})}$ is the charge of the particle, $\mathbf{E}(\mathbf{r}, t)$ is the electric field, $\mathbf{B}(\mathbf{r}, t)$ is the magnetic induction field, and

$$\mathbf{r}_n = x_{n1} \hat{\mathbf{e}}_1 + x_{n2} \hat{\mathbf{e}}_2 + x_{n3} \hat{\mathbf{e}}_3 \quad \text{and} \quad \mathbf{v}_n = \dot{x}_{n1} \hat{\mathbf{e}}_1 + \dot{x}_{n2} \hat{\mathbf{e}}_2 + \dot{x}_{n3} \hat{\mathbf{e}}_3 \quad (2.89)$$

are the particle's position and velocity, respectively.

Introducing the scalar potential $\Phi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$, the electric and magnetic induction fields at the particle location \mathbf{r}_n may be written

$$\mathbf{E}(\mathbf{r}_n, t) = -\frac{\partial\Phi(\mathbf{r}_n, t)}{\partial\mathbf{r}_n} - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r}_n, t)}{\partial t} \quad \text{and} \quad \mathbf{B}(\mathbf{r}_n, t) = \nabla_n \times \mathbf{A}(\mathbf{r}_n, t) \quad (2.90)$$

where the notation for the gradient vector operator

$$\frac{\partial}{\partial\mathbf{r}_n} = \nabla_n = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_{n1}} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_{n2}} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_{n3}} \quad (2.91)$$

has been introduced.¹³

Substituting eqn (2.90) into the Lorentz force eqn (2.88) gives

$$\begin{aligned} \mathbf{f}_n &= q_n^{(\text{ch})} \left(-\frac{\partial\Phi(\mathbf{r}_n, t)}{\partial\mathbf{r}_n} - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r}_n, t)}{\partial t} \right) + \frac{q_n^{(\text{ch})}}{c} \mathbf{v}_n \times (\nabla_n \times \mathbf{A}(\mathbf{r}_n, t)) \\ &= -\frac{\partial}{\partial\mathbf{r}_n} \left(q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) \right) - \frac{\partial}{\partial t} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) \\ &\quad + \frac{\partial}{\partial\mathbf{r}_n} \left(\mathbf{v}_n \cdot \frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) - \mathbf{v}_n \cdot \frac{\partial}{\partial\mathbf{r}_n} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) \end{aligned} \quad (2.92)$$

where the triple cross product has been expanded, using the usual Lagrangian list of variables $\mathbf{r}_n, \mathbf{v}_n, t$ to define the meaning of the partial differentials in the next-to-last term.

Noting that the total time derivative of $(q_n^{(\text{ch})} \mathbf{A}(\mathbf{r}_n, t)/c)$ can be written, using the chain rule, as

$$\frac{d}{dt} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) = \mathbf{v}_n \cdot \frac{\partial}{\partial\mathbf{r}_n} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) + \frac{\partial}{\partial t} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) \quad (2.93)$$

eqn (2.92) becomes

$$\mathbf{f}_n = -\frac{\partial}{\partial\mathbf{r}_n} \left(q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) - \mathbf{v}_n \cdot \frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) - \frac{d}{dt} \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) \quad (2.94)$$

Defining the velocity dependent potential $U^{(\text{vel})}$ by

$$U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t) = \sum_{n=1}^N \left(q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) - \mathbf{v}_n \cdot \frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \right) \quad (2.95)$$

gives

$$\frac{\partial U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t)}{\partial\mathbf{v}_n} = -\frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \quad (2.96)$$

¹³See Section A.11 for a discussion of this notation, and cautions for its proper use.

where the operator

$$\frac{\partial}{\partial \mathbf{v}_n} = \hat{\mathbf{e}}_1 \frac{\partial}{\partial \dot{x}_{n1}} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial \dot{x}_{n2}} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial \dot{x}_{n3}} \quad (2.97)$$

has been introduced and the identity eqn (A.71) of Section A.11 has been used. Thus, finally

$$\mathbf{f}_n = \frac{d}{dt} \left(\frac{\partial U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}_n} \right) - \frac{\partial U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}_n} \quad (2.98)$$

expresses the Lorentz force in Lagrangian form.

In the present vector notation, the total kinetic energy is given in the first of eqn (2.12). Again using the identities in Section A.11, the equations of motion for this problem can thus be written as the following sequence of equivalent expressions

$$\frac{d}{dt} (m_n \mathbf{v}_n) = \mathbf{f}_n \quad (2.99)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \mathbf{v}_n} \right) = \frac{d}{dt} \left(\frac{\partial U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}_n} \right) - \frac{\partial U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}_n} \quad (2.100)$$

and hence

$$\frac{d}{dt} \left(\frac{\partial L(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}_n} \right) - \frac{\partial L(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}_n} = 0 \quad (2.101)$$

where the Lagrangian function for velocity dependent potentials is defined as

$$L(\mathbf{r}, \mathbf{v}, t) = T(\mathbf{v}) - U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t) \quad (2.102)$$

Written out, the Lagrangian is thus

$$L = \frac{1}{2} \sum_{n=1}^N m_n v_n^2 - \sum_{n=1}^N q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) + \sum_{n=1}^N \mathbf{v}_n \cdot \frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \quad (2.103)$$

The generalized momenta of particles in an electromagnetic field are not simply the particle momenta $\mathbf{p}_n = m_n \mathbf{v}_n$. They are

$$\underline{\mathbf{p}}_n = \frac{\partial L(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}_n} = m_n \mathbf{v}_n + \frac{q_n^{(\text{ch})}}{c} \mathbf{A}(\mathbf{r}_n, t) \quad (2.104)$$

which might be considered as the vector sum of a particle momentum $\mathbf{p}_n = m_n \mathbf{v}_n$ and a field momentum $q_n^{(\text{ch})} \mathbf{A}(\mathbf{r}_n, t)/c$. It is this generalized momentum that is conserved when the coordinate \mathbf{r}_n is ignorable.

The generalized energy function can also be found,

$$H_s = \sum_{n=1}^N \mathbf{v}_n \cdot \underline{\mathbf{p}}_n - L = \frac{1}{2} \sum_{n=1}^N m_n v_n^2 + \sum_{n=1}^N q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) \quad (2.105)$$

Note that, even though we are in the s-system, the generalized energy function here is not equal to $T + U^{(\text{vel})}$ since the terms linear in the velocity have canceled. However, the generalized energy eqn (2.105) is equal to the total energy of the system of charges as it is usually defined in electrodynamics.

It seems surprising that a complicated velocity-dependent force like the Lorentz force of electrodynamics can be written in the Lagrangian form of eqn (2.98). Why do electrodynamics and Lagrangian mechanics fit together so neatly? We leave that question for the reader to ponder.

Other velocity-dependent potentials are possible. The general rule for their use follows the same pattern as the electromagnetic example. In the s -system with velocity dependent potential $U^{(\text{vel})} = U^{(\text{vel})}(s, \dot{s}, t)$, the generalized forces can be defined as

$$F_i = \frac{d}{dt} \left(\frac{\partial U^{(\text{vel})}(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \frac{\partial U^{(\text{vel})}(s, \dot{s}, t)}{\partial s_i} \quad (2.106)$$

and the Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{s}_i} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial s_i} = 0 \quad \text{where} \quad L(s, \dot{s}, t) = T(\dot{s}) - U^{(\text{vel})}(s, \dot{s}, t) \quad (2.107)$$

Using the general q -system, the velocity-dependent potential will be $U^{(\text{vel})}(q, \dot{q}, t)$, obtained as a compound function from $U^{(\text{vel})}(s, \dot{s}, t)$,

$$U^{(\text{vel})} = U^{(\text{vel})}(q, \dot{q}, t) = U^{(\text{vel})}(s(q, t), \dot{s}(q, \dot{q}, t), t) \quad (2.108)$$

The generalized forces are

$$Q_k = \frac{d}{dt} \left(\frac{\partial U^{(\text{vel})}(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial U^{(\text{vel})}(q, \dot{q}, t)}{\partial q_k} \quad (2.109)$$

and the Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad \text{where} \quad L(q, \dot{q}, t) = T(q, \dot{q}, t) - U^{(\text{vel})}(q, \dot{q}, t) \quad (2.110)$$

The zero on the right in eqns (2.107, 2.110) follows from the assumption that that no forces other than those produced by $U^{(\text{vel})}$ are present.

2.18 Exercises

Exercise 2.1

(a) Calculate the Jacobian matrix for $s_1 = x$, $s_2 = y$, $s_3 = z$ and $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$, the transformation from Cartesian to spherical-polar coordinates. Show that r , θ , ϕ are good generalized coordinates except on the z -axis.

(b) Work out in detail the derivation of eqn (2.66) from eqn (2.23).

Exercise 2.2

(a) Calculate the Jacobian matrix for $s_1 = x$, $s_2 = y$, $s_3 = z$ and $q_1 = \rho$, $q_2 = \phi$, and $q_3 = z$, the transformation from Cartesian to cylindrical-polar coordinates. Show that ρ , ϕ , z are good generalized coordinates except on the z -axis.

(b) Starting from the Lagrangian in eqn (2.23), work out in detail the transformation to the Lagrangian $L(\rho, \phi, z, \dot{\rho}, \dot{\phi}, \dot{z}, t)$ for this system.

Exercise 2.3 We are given a Lagrangian $L(q, \dot{q}, t)$. Assume that there are no non-potential forces. Let $f(q, t)$ be an arbitrary function of $q = q_1, q_2, \dots, q_N$ and possibly the time t . Show that if $q_k = q_k(t)$ are a solution of

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad (2.111)$$

then the *same* $q_k(t)$ are also a solution of

$$\frac{d}{dt} \left(\frac{\partial L'(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L'(q, \dot{q}, t)}{\partial q_k} = 0 \quad (2.112)$$

where

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \quad (2.113)$$

This problem shows that L and L' are equivalent Lagrangians. The same solution will be found no matter which one is used.

Exercise 2.4 Consider a collection that consists of just two masses, m_1 and m_2 . We can define the center of mass \mathbf{R} and the vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ as in Section 1.9. However, the components of these three vectors are *not* suitable generalized coordinates. For one thing, there are nine of them, whereas the number of degrees of freedom D is only six (the six components of \mathbf{r}_1 and \mathbf{r}_2). Suppose that we define a new vector \mathbf{r} by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and define $\mathbf{v} = d\mathbf{r}/dt$ as its time derivative. Also it will be useful to define a reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$.

(a) Write \mathbf{r}_1 and \mathbf{r}_2 in terms of \mathbf{R} and \mathbf{r} and the appropriate masses. Then show that the six components of \mathbf{R} and \mathbf{r} satisfy the Jacobian determinant condition and so *are* good generalized coordinates.

(b) Write \mathbf{P} , \mathbf{L} , \mathbf{S} , T_0 , and T_1 in terms of μ , M , \mathbf{R} , \mathbf{r} , \mathbf{V} , and \mathbf{v} only.

Exercise 2.5 Suppose that the two masses in Exercise 2.4 have a motion defined by a Lagrangian function

$$L = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - U(\mathbf{r}_2 - \mathbf{r}_1) \quad (2.114)$$

where $v_1 = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1}$ and $\mathbf{v}_1 = d\mathbf{r}_1/dt$, with similar definitions for the second mass.

(a) Rewrite the Lagrangian in terms of the variables \mathbf{r} , \mathbf{R} and their derivatives. Show that this Lagrangian can be written as the sum of two terms, one of which depends only on \mathbf{R} and its time derivative and the other only on \mathbf{r} and its time derivative. (Such Lagrangian systems are called *separable*.)

(b) Show that the three components of \mathbf{R} are ignorable coordinates, and that the total momentum of the system is conserved.

Exercise 2.6 A mass m is acted on by a force derived from the generalized potential

$$U^{(\text{vel})}(\mathbf{r}, \mathbf{v}, t) = U(r) + \boldsymbol{\sigma} \cdot \mathbf{L} \quad (2.115)$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \quad \boldsymbol{\sigma} = \sigma \hat{\mathbf{e}}_3 \quad \mathbf{L} = \mathbf{r} \times m\mathbf{v} \quad (2.116)$$

and \mathbf{r} and \mathbf{v} are the position and velocity of the mass relative to some inertial coordinate system.

(a) Express $U^{(\text{vel})}$ in Cartesian coordinates (the s-system) $x, y, z, \dot{x}, \dot{y}, \dot{z}$ and find the force \mathbf{F} (i.e. find the three components F_x, F_y, F_z).

(b) Now express $U^{(\text{vel})}$ in spherical polar coordinates (which we might call the q-system) $r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}$ and find the generalized force Q_k for $k = 1, 2, 3$.

(c) The force vector we found in part (a) can be re-expressed in terms of the spherical polar unit vectors as

$$\mathbf{F} = F_r \hat{\mathbf{r}} + F_\theta \hat{\boldsymbol{\theta}} + F_\phi \hat{\boldsymbol{\phi}} \quad (2.117)$$

where

$$F_r = \hat{\mathbf{r}} \cdot \mathbf{F} \quad F_\theta = \hat{\boldsymbol{\theta}} \cdot \mathbf{F} \quad F_\phi = \hat{\boldsymbol{\phi}} \cdot \mathbf{F} \quad (2.118)$$

Show that Q_r is equal to F_r .

(d) However, Q_ϕ is not equal to F_ϕ . Show that Q_ϕ is equal to the z -component of the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

(e) Verify that the units of $Q_\phi \delta\phi$ do obey the rule described in Section 2.14.

Exercise 2.7 Given an electric potential $\Phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$, the electric and magnetic induction fields can be expressed as

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (2.119)$$

We know that the \mathbf{E} and \mathbf{B} fields are left invariant by a gauge transformation of the potentials. That is, if

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \quad \Phi' = \Phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \quad (2.120)$$

and \mathbf{E}' and \mathbf{B}' are found using eqn (2.119) but with Φ and \mathbf{A} replaced by the primed potentials Φ' and \mathbf{A}' , then it can be shown that $\mathbf{E}' = \mathbf{E}$ and $\mathbf{B}' = \mathbf{B}$.

(a) For the case of a single particle of mass m and charge $q^{(\text{ch})}$, use eqn (2.103) to write out two Lagrangians, one using the original potentials and one using the primed potentials. Call them L and L' .

(b) Find

$$\underline{\mathbf{p}} = \frac{\partial L(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} \quad \text{and} \quad \underline{\mathbf{p}}' = \frac{\partial L'(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} \quad (2.121)$$

(c) Show that

$$L' = L + \frac{df(\mathbf{r}, t)}{dt} \quad (2.122)$$

and write $f(\mathbf{r}, t)$ in terms of $\chi(\mathbf{r}, t)$.

(d) If $\mathbf{r} = \mathbf{r}(t)$ is a solution to the Lagrange equations with L , is it also a solution to the Lagrange equations with L' ? Should it be? If it is, show why it is, and if not show why it is not.

Exercise 2.8 A single particle of mass m in one dimension has the Lagrangian in some q system of coordinates

$$L(q_1, \dot{q}_1, t) = \frac{1}{2} m \frac{a^2 \dot{q}_1^2}{q_1^4} - \frac{m\omega^2 a^2}{2 q_1^2} \quad (2.123)$$

where a and ω are given constants having appropriate units.

- (a) Find the generalized momentum p_1 and the generalized energy function $H_q(q, \dot{q}, t)$ for the q system. Is the generalized energy conserved?
- (b) Suppose that the q system coordinates are related to those of the s system by $q_1 = a/s_1$. Write the Lagrangian in the s -system, $L(s, \dot{s}, t)$.
- (c) Find the generalized momentum P_1 and the generalized energy function $H_s(s, \dot{s}, t)$ for the s -system. Is the generalized energy conserved?
- (d) Show that the momenta p_1 and P_1 are related as predicted by eqn (2.71).
- (e) When expressed in the same coordinate system, is H_s equal to H_q ? Why should it be?

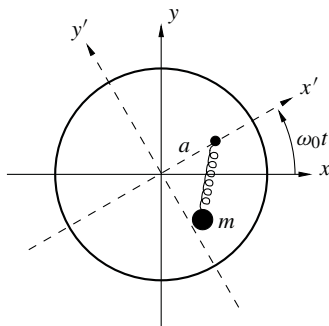


FIG. 2.1. Illustration for Exercise 2.9.

Exercise 2.9 A horizontal, circular table with a frictionless top surface is constrained to rotate about a vertical line through its center, with constant angular velocity ω_0 . A peg is driven into the table top at a distance a from the center of the circle. A mass m slides freely on the top surface of the table, connected to the peg by a massless spring of force constant k and zero rest length. Take the s -system to be an inertial system of Cartesian coordinates x, y with origin at the center of the table top, and the q -system to be rotating Cartesian coordinates x', y' defined so that $\hat{\mathbf{e}}'_1$ defines a line passing through the peg. Ignore the z -coordinate, and treat this problem as one with two degrees of freedom. The transformation between coordinates of the mass in the two systems is

$$x = x' \cos \omega_0 t - y' \sin \omega_0 t \quad y = x' \sin \omega_0 t + y' \cos \omega_0 t \quad (2.124)$$

- (a) Write $L(s, \dot{s}, t)$ in the s -system and $L(q, \dot{q}, t)$ in the q -system.
- (b) Write H_s in the s -system. Is it equal to $T + U$? Is it conserved?
- (c) Write H_q in the q -system. Is it equal to $T + U$? Is it conserved?

Exercise 2.10 A one-dimensional system has the Lagrangian

$$L(q, \dot{q}, t) = \frac{m}{2} \left(\dot{q}_1^2 \sin^2 \omega t + \omega^2 q_1^2 \cos^2 \omega t + \omega q_1 \dot{q}_1 \sin 2\omega t \right) - mgq_1 \sin \omega t \quad (2.125)$$

where $0 < t < \pi/\omega$.

- (a) Find the generalized energy function for the q -system, $H_q(q, \dot{q}, t)$. Is it conserved?
- (b) Make a change of generalized coordinates, with the new coordinate r_1 defined by $q_1 = r_1 / \sin \omega t$, as in Section 2.10. Write the Lagrangian in the r -system, $L(r, \dot{r}, t)$.
- (c) Find the generalized energy function for the r -system, $H_r(r, \dot{r}, t)$. Is it conserved?

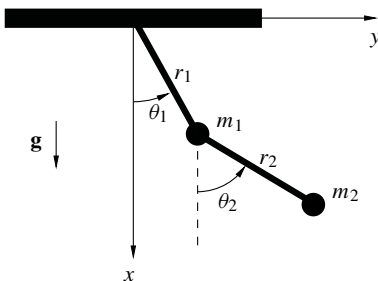


FIG. 2.2. Illustration for Exercise 2.11.

Exercise 2.11 Consider a plane double pendulum with rigid, massless, but possibly extensible sticks. It has a mass m_1 at coordinates x_1, y_1 and a mass m_2 at x_2, y_2 . Gravity $\mathbf{g} = g\hat{\mathbf{e}}_1$ acts downwards. Ignore the z coordinate in this problem, and assume that all pivots are frictionless. In the s -system $s = x_1, y_1, x_2, y_2$, the Lagrangian is

$$L(s, \dot{s}, t) = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) + m_1 g x_1 + m_2 g x_2 \quad (2.126)$$

- (a) Consider a change of generalized coordinates to the q -system $q = r_1, \theta_1, r_2, \theta_2$ shown in the diagram. Write the four transformation equations of the form $s_i = s_i(q, t)$ for $i = 1, \dots, 4$.
- (b) Calculate the Jacobian determinant $|\partial s / \partial q|$ for this transformation and find the conditions under which the q -system are good generalized coordinates.
- (c) Write the Lagrangian $L(q, \dot{q}, t)$ in the q -system.

Exercise 2.12 A point particle of mass m and charge q moves near a very long wire carrying a current I . Choose the $\hat{\mathbf{e}}_3$ axis along the wire in the direction of the current. In the region near the wire, the vector potential in terms of cylindrical polar coordinates ρ, ϕ, z is

$$\mathbf{A} = \frac{I}{2\pi c} \ln\left(\frac{\rho}{\rho_0}\right) \hat{\mathbf{z}} \quad (2.127)$$

where ρ_0 is some arbitrarily chosen ρ value. Assume the electric potential Φ to be zero.

- (a) Write the Lagrangian $L = L(\rho, \phi, z, \dot{\rho}, \dot{\phi}, \dot{z}, t)$ for the particle, using cylindrical polar coordinates.
- (b) Find the generalized momenta $p_\rho, p_\phi,$ and p_z .
- (c) Write the three Lagrange equations, and show that ϕ and z are *ignorable* coordinates.
- (d) Use the ϕ and z Lagrange equations to write expressions for $\dot{\phi}$ and \dot{z} as functions of ρ and integration constants.
- (e) Write the generalized energy function. Is it conserved? Use it to express $\dot{\rho}^2$ as a function only of ρ and some constants that can be determined at $t = 0$.

LAGRANGIAN THEORY OF CONSTRAINTS

One attractive feature of the Lagrangian method is the ease with which it solves so-called constraint problems. But, as the reader will see, applying the correct method for a particular problem can be something of an art. We present several different ways of solving such problems, with examples of each. With experience, the reader will become adept at choosing among them.

In the previous chapter, the generalized coordinates were assumed to be independent variables. But there are problems of interest in which these coordinates are not independent, but rather are forced into particular relations by what are called *constraints*. For example, the x, y, z coordinates of a point mass falling under gravity are independent. But if the mass is forced to slide on the surface of a plane, there would be a constraint in a form such as $\alpha x + \beta y + \gamma z - \Lambda = 0$ tying them together. The present chapter shows that such constraints can be incorporated into the Lagrangian method in a particularly convenient way. If the constraints are idealized (such as frictionless surfaces or perfectly rigid bodies), then the equations of motion can be solved without knowing the forces of constraint. Also, the number of degrees of freedom of the Lagrangian system can be reduced by one for each constraint applied.

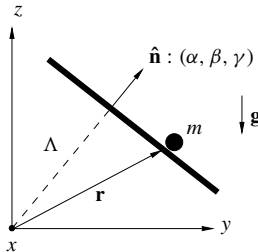


FIG. 3.1. Example of a holonomic constraint. The mass m is constrained to move on the surface of a plane defined by $\hat{\mathbf{n}} \cdot \mathbf{r} = \Lambda$. Constants α, β, γ are the components of a unit vector perpendicular to the plane, and Λ is the perpendicular distance from the plane to the origin.

3.1 Constraints Defined

The simplest class of constraints are those called *holonomic*. A constraint is holonomic if it can be represented by a single function of the generalized coordinates, equated to zero, as in

$$G_a(q, t) = 0 \quad (3.1)$$

for $a = 1, \dots, C$, where each a value is considered a separate constraint.

The relation among the coordinates may vary with time. For example, if the plane in the previous paragraph had a time-varying distance from the origin $\Lambda(t)$, the constraint, using the s -system, would be

$$G_1(s, t) = \alpha x + \beta y + \gamma z - \Lambda(t) = 0 \tag{3.2}$$

If the number of constraints C is greater than one, care must be taken to ensure that the constraint equations are functionally independent. Otherwise the actual number of constraints may be fewer than the number listed. As discussed in Theorem D.28.1, the condition for functional independence of the constraints is that the $C \times D$ matrix whose elements are

$$\left(\frac{\partial G}{\partial q} \right)_{ak} = \frac{\partial G_a(q, t)}{\partial q_k} \tag{3.3}$$

must have rank C . In other words, there must be a nonzero $C \times C$ determinant, called a critical minor,¹⁴ constructed by selecting the C rows, and C of the D columns, of eqn (3.3). We will assume throughout that all sets of constraints obey this condition.

3.2 Virtual Displacement

In the treatment of Lagrangian constraint problems, it is very convenient to define the new concept of *virtual displacement*.

Definition 3.2.1: Virtual Displacements Defined

A virtual displacement of a function $f = f(q, t)$ is its differential, but with the convention that the time t is held fixed so that $\delta t = 0$. These virtual displacements are denoted with a lowercase Greek δ to distinguish them from normal differentials.

The virtual displacement of a function $f(q, t)$ is then,

$$\delta f = \sum_{k=1}^D \frac{\partial f(q, t)}{\partial q_k} \delta q_k \tag{3.4}$$

and the virtual displacement of the constraint function $G_a(q, t)$ defined in Section 3.1 is

$$\delta G_a = \sum_{k=1}^D \frac{\partial G_a(q, t)}{\partial q_k} \delta q_k \tag{3.5}$$

These definitions also apply to the s -system. By the chain rule, virtual displacements in the s - and q -systems are related by

$$\delta s_i = \sum_{k=1}^D \frac{\partial s_i(q, t)}{\partial q_k} \delta q_k \quad \text{and the inverse relation} \quad \delta q_k = \sum_{i=1}^D \frac{\partial q_k(s, t)}{\partial s_i} \delta s_i \tag{3.6}$$

¹⁴See Section B.17 of Appendix B for a discussion of the rank of a matrix.

and the differential of G_a can be written equivalently as

$$\delta G_a = \sum_{i=1}^D \frac{\partial G_a(s, t)}{\partial s_i} \delta s_i \quad (3.7)$$

Definition 3.2.2: Virtual Displacements Re-defined

The definition of virtual displacement in Definition 3.2.1 is now extended to include the condition that the δq_k must be chosen so that, at each instant of time, and for all $a = 1, \dots, C$,

$$\delta G_a = 0 \quad (3.8)$$

Virtual displacements at a frozen instant of time must be such that the constraints are maintained, that both $G_a(q, t) = 0$ and $G_a(q + \delta q, t) = 0$. For example, if the only constraint is that a single mass must move on a flat, horizontal elevator floor located at $z = h(t)$, then the constraint equation is $G_1(s, t) = z - h(t) = 0$ and the only allowed nonzero virtual displacements are δx and δy . The constraint requires δz to equal zero. The virtual displacements are constrained to remain in the instantaneous surface of constraint as it is at time t , even though that surface may be moving as t evolves.

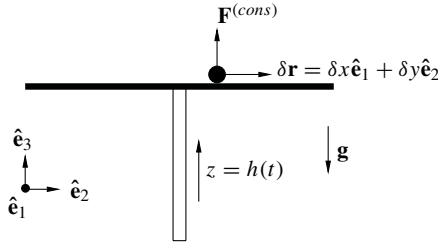


FIG. 3.2. A mass is constrained to slide without friction on the floor of an elevator which is moving upwards. The constraint is $z = h(t)$. The virtual displacement $\delta \mathbf{r}$ is parallel to the *instantaneous* position of the floor, even though the floor is moving. Thus $\delta z = 0$ as shown. Since the floor is frictionless, $\mathbf{F}^{(\text{cons})}$ is perpendicular to the floor and hence $\delta W^{(\text{cons})} = \mathbf{F}^{(\text{cons})} \cdot \delta \mathbf{r} = 0$.

3.3 Virtual Work

Generally, constraints are maintained by the actions of forces, like the force exerted on the mass by elevator floor in the previous example. We will denote these *forces of constraint* by $Q_k^{(\text{cons})}$ in the q -system or $F_i^{(\text{cons})}$ in the s -system. These forces of constraint (and indeed any generalized forces) in the two systems are related by the same transformation formulas as in Section 2.6,

$$Q_k^{(\text{cons})} = \sum_{i=1}^D F_i^{(\text{cons})} \frac{\partial s_i(q, t)}{\partial q_k} \quad \text{and} \quad F_i^{(\text{cons})} = \sum_{k=1}^D Q_k^{(\text{cons})} \frac{\partial q_k(s, t)}{\partial s_i} \quad (3.9)$$

The *virtual work of the forces of constraint* is defined as

$$\delta W^{(\text{cons})} = \sum_{i=1}^D F_i^{(\text{cons})} \delta s_i = \sum_{k=1}^D Q_k^{(\text{cons})} \delta q_k \quad (3.10)$$

where the second equality follows from eqns (3.6, 3.9).¹⁵

The problems that can be dealt with easily by Lagrangian theory of constraints are those in which, at least as an idealization, *the forces of constraint do no virtual work*. This portentous phrase means simply that

$$\delta W^{(\text{cons})} = 0 \quad (3.11)$$

for all allowed virtual displacements. For example, if the elevator floor in the previous section is made of frictionless ice, then the only constraint force will be a normal force $\mathbf{F}^{(\text{cons})} = F_z^{(\text{cons})} \hat{\mathbf{e}}_3$. Then, since $\delta z = 0$, the virtual displacement $\delta \mathbf{r} = \delta x \hat{\mathbf{e}}_1 + \delta y \hat{\mathbf{e}}_2$ will be perpendicular to the constraint force, leading at once to the conclusion that

$$\delta W^{(\text{cons})} = \sum_{i=1}^D F_i^{(\text{cons})} \delta s_i = \mathbf{F}^{(\text{cons})} \cdot \delta \mathbf{r} = 0 \quad (3.12)$$

A wide class of problems can be imagined in which masses slide without friction on various surfaces. For example, a coin sliding inside a spherical fish bowl made of frictionless ice, with the q-system taken to be spherical polar coordinates, would have $\delta r = 0$ and $Q_\theta^{(\text{cons})} = Q_\phi^{(\text{cons})} = 0$, leading again to $\delta W^{(\text{cons})} = 0$. A bead sliding on a frictionless wire of arbitrary shape would have a force of constraint perpendicular to the wire but virtual displacement only along it, again producing zero virtual work.

A less obvious example is that the cohesive forces binding the masses of an idealized, perfectly rigid body also do no virtual work. The proof of this statement must be deferred until the motion of rigid bodies is treated in later chapters. (It is proved in Theorem 8.13.1.) Systems of rigid rods linked by frictionless pivots and joints, such as the single or multiple pendulum, also have constraint forces that do no virtual work. Another important example is that the friction force acting when a wheel rolls without slipping on some surface does no virtual work, since the force acts at the contact point, which does not move in virtual displacements.

Virtual work is not the same as the real, physical work that may be done by the constraint forces. In the example of the elevator floor in Figure 3.2, if the elevator is moving upwards then the floor definitely will do real work on the mass as time evolves. But it will not do virtual work. The rule is that when the constraints are not time dependent, then the forces of constraint that do no virtual work will also not do real work. But when the constraints are time varying, with $\partial G_a(s, t)/\partial t \neq 0$, then forces of constraint that do no virtual work may still do real work.

¹⁵The use of the symbol $\delta W^{(\text{cons})}$ in eqn (3.10) is *not* meant to imply the existence of a work function $W^{(\text{cons})}(q, t)$ from which the generalized forces of constraint can be derived by partial differentiation. Such a function exists only in trivial cases. The forces of constraint take whatever values are necessary to maintain the constraints. In general, this means that they depend on the first and second time derivatives of the generalized coordinates, as well as the generalized coordinates and the time.

3.4 Form of the Forces of Constraint

The reason for the above definitions of virtual displacement and virtual work is to allow us to state the following theorem.

Theorem 3.4.1: Form of the Forces of Constraint

Given a system of constraints defined as in eqn (3.1), and virtual displacements obeying eqn (3.8), the virtual work of the forces of constraint vanishes,

$$\delta W^{(\text{cons})} = 0 \quad (3.13)$$

for all allowed virtual displacements if and only if there exist λ_a factors, called Lagrange multipliers, such that the constraint forces can be written in the following form

$$Q_k^{(\text{cons})} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \quad \text{or, equivalently,} \quad F_i^{(\text{cons})} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(s, t)}{\partial s_i} \quad (3.14)$$

where the λ_a factors in the first of eqn (3.14) are the same as those in the second.

Proof: The equivalence of the two equations in eqn (3.14) follows from eqn (3.9) and the chain rule.

We prove the theorem in the general q-system. The proof in the s-system is similar. First, we prove that eqn (3.14) implies eqn (3.13). The definition eqn (3.10) gives

$$\begin{aligned} \delta W^{(\text{cons})} &= \sum_{k=1}^D Q_k^{(\text{cons})} \delta q_k = \sum_{k=1}^D \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \delta q_k \\ &= \sum_{a=1}^C \lambda_a \sum_{k=1}^D \frac{\partial G_a(q, t)}{\partial q_k} \delta q_k = \sum_{a=1}^C \lambda_a \delta G_a \end{aligned} \quad (3.15)$$

where eqn (3.14) was used. But, by the definition of virtual displacement in Section 3.3, $\delta G_a = 0$ for all a . Hence $\delta W^{(\text{cons})} = 0$, as was to be proved.

The converse proof, that $\delta W^{(\text{cons})} = 0$ implies eqn (3.14), is a bit more involved. As a preliminary to the proof, note that the matrix eqn (3.3) discussed in Section 3.1 is assumed to have rank C . Since the order in which generalized coordinates are indexed is arbitrary, we may gain some clarity without loss of generality by assuming that its critical minor consists of its C rows and its last C columns, from $(D - C + 1)$ to D . We then denote¹⁶ by $q^{(f)}$ the set of free variables $q_1, \dots, q_{(D-C)}$ and by $q^{(b)}$ the set of bound variables $q_{(D-C+1)}, \dots, q_D$. The constraint conditions eqns (3.5, 3.8)

¹⁶The choice of bound and free variables here is not unique. In a given problem, there may be several critical minors of eqn (3.3) and hence several ways in which the free-bound division can be made.

imply

$$0 = \delta G_a = \sum_{k=1}^D g_{ak} \delta q_k = \sum_{k=1}^{D-C} g_{ak} \delta q_k^{(f)} + \sum_{l=D-C+1}^D g_{al} \delta q_l^{(b)} \quad (3.16)$$

for $a = 1, \dots, C$, where we have introduced the notation

$$g_{ak} = \frac{\partial G_a(q, t)}{\partial q_k} \quad (3.17)$$

and have separated the sums over the free (superscript (f) , index k) and bound (superscript (b) , index l) variables in the last form of the expression.

The $C \times C$ matrix $\mathbf{g}^{(b)}$ whose ai th matrix element is defined to be

$$g_{ai}^{(b)} = g_{a(D-C+i)} \quad (3.18)$$

is nonsingular by the assumption that the last C columns of eqn (3.3) are a critical minor of that matrix. Therefore, the inverse $\mathbf{g}^{(b)-1}$ exists and may be used to solve eqn (3.16) for the bound virtual displacements in terms of the free ones. Thus, for $i = 1, \dots, C$,

$$\delta q_{D-C+i}^{(b)} = - \sum_{a=1}^C \sum_{k=1}^{D-C} g_{ia}^{(b)-1} g_{ak} \delta q_k^{(f)} \quad (3.19)$$

With this relation now assumed, eqn (3.16) becomes an identity, satisfied regardless of the values we choose for the $\delta q^{(f)}$ displacements. Thus the $\delta q^{(f)}$ are not bound by the constraints and may be assigned any values, just as the name “free” suggests.

Now form an expression by multiplying eqn (3.8) by an unknown function λ_a and subtracting the sum over a from eqn (3.11). Since each constituent of this expression is zero by assumption, the expression also vanishes. Thus

$$\begin{aligned} 0 &= \delta W^{(\text{cons})} - \sum_{a=1}^C \lambda_a \delta G_a = \sum_{k=1}^D \left(Q_k^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{ak} \right) \delta q_k \\ &= \sum_{k=1}^{D-C} \left(Q_k^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{ak} \right) \delta q_k^{(f)} + \sum_{l=D-C+1}^D \left(Q_l^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{al} \right) \delta q_l^{(b)} \end{aligned} \quad (3.20)$$

where we have once again separated the sums over free and bound variables. The last sum in eqn (3.20) may be written

$$\sum_{l=D-C+1}^D \left(Q_l^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{al} \right) \delta q_l^{(b)} = \sum_{i=1}^C \left(Q_{D-C+i}^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{ai}^{(b)} \right) \delta q_{D-C+i}^{(b)} \quad (3.21)$$

The λ_a can now be chosen to be

$$\lambda_a = \sum_{i=1}^C Q_{D-C+i}^{(\text{cons})} g_{ia}^{(b)-1} \quad (3.22)$$

which makes

$$Q_{D-C+i}^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{ai}^{(b)} = 0 \quad \text{or equivalently} \quad Q_l^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{al} = 0 \quad (3.23)$$

identically for all $i = 1, \dots, C$, or equivalently for all $l = D - C + 1, \dots, D$.

Thus the choice in eqn (3.22) makes the last sum in eqn (3.20) zero. Equation (3.20) then reduces to

$$0 = \sum_{k=1}^{D-C} \left(Q_k^{(\text{cons})} - \sum_{a=1}^C \lambda_a g_{ak} \right) \delta q_k^{(f)} \quad (3.24)$$

Now invoke the independence of the free displacements to set the $\delta q_k^{(f)}$ nonzero one at a time, thus establishing that, for $k = 1, \dots, D - C$,

$$Q_k^{(\text{cons})} - \sum_{j=1}^C \lambda_j g_{jk} = 0 \quad (3.25)$$

Together with the second of eqn (3.23) for $l = D - C + 1, \dots, D$, this establishes eqn (3.14) for all k values, as was to be proved. \square

3.5 General Lagrange Equations with Constraints

There is a wide class of idealized systems in which it can be assumed that the only forces acting are either constraint forces or forces derived from a potential function. Such systems are sometimes called *monogenic*. For such systems, the only non-potential forces appearing are the constraint forces. Thus $Q_k^{(\text{NP})} = Q_k^{(\text{cons})}$ and the general Lagrange equations, eqn (2.52), become

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = Q_k^{(\text{cons})} \quad (3.26)$$

For forces of constraint that do no virtual work, Theorem 3.4.1 then allows us to write the Lagrange equations for the constrained motion in a form that can be solved without knowing the forces of constraint in advance. This result is one of the triumphs of the Lagrangian method.

Theorem 3.5.1: General Lagrange Equations with Constraints

If the only non-potential forces in a problem are the forces of constraint and if those forces of constraint do no virtual work, then the Lagrange equations become, for $k = 1, \dots, D$,

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \quad (3.27)$$

Together with the set of constraint equations

$$G_a(q, t) = 0 \quad (3.28)$$

for $a = 1, \dots, C$, these are $(D + C)$ equations in the $(D + C)$ variables $q_1, \dots, q_D, \lambda_1, \dots, \lambda_C$ and so may be solved for these variables.

Proof: Since the forces of constraint are assumed to do no virtual work, Theorem 3.4.1 applies and eqn (3.14) may be substituted into eqn (3.26) to give the desired result. \square

In applying these formulas, the partial derivatives in eqn (3.27) must be calculated first, and *then* the constraints eqn (3.28) applied to simplify the resulting differential equation. Note that applying the constraints before taking the partial derivatives in eqn (3.27) would in general lead to error.

A second triumph of the Lagrangian method is that, not only can the problem be solved without knowing the forces of constraint in advance, but also the same solution allows one to calculate what those constraint forces must have been.

Corollary 3.5.2: Calculation of Constraint Forces

After the problem is solved for the equations of motion $q_k = q_k(t)$ by use of Theorem 3.5.1, one can then calculate what the forces of constraint were.

Proof: The solution to eqns (3.27, 3.28) gives the Lagrange multipliers $\lambda_1, \dots, \lambda_C$ as well as the coordinates q_1, \dots, q_D . These λ_a values can then be inserted into eqn (3.14) to give the forces of constraint in the q- or s-systems. \square

The general Lagrange equations, eqn (3.27), have been given in the q-system. But equations of exactly the same form are true in any system of generalized coordinates. Just replace the letter q by s or r for the s- or r-systems, respectively.

3.6 An Alternate Notation for Holonomic Constraints

Some texts write eqn (3.27) in an alternate notation that the reader should be aware of.¹⁷ They define a new Lagrangian \tilde{L} that includes the constraint functions,

$$\tilde{L}(q, \dot{q}, t, \lambda) = L(q, \dot{q}, t) + \sum_{a=1}^C \lambda_a G_a(q, t) \quad (3.29)$$

Then eqn (3.27) can be written in the same form as the Lagrange equations without constraints. It becomes

$$\frac{d}{dt} \left(\frac{\partial \tilde{L}(q, \dot{q}, t, \lambda)}{\partial \dot{q}_k} \right) - \frac{\partial \tilde{L}(q, \dot{q}, t, \lambda)}{\partial q_k} = 0 \quad (3.30)$$

Unfortunately, most of these texts do not include the λ in the list of variables in $\tilde{L}(q, \dot{q}, t, \lambda)$, which leads the reader to wonder how to take partial derivatives of the λ_a . When encountering this notation, one should mentally add λ to the list of variables in $\tilde{L}(q, \dot{q}, t, \lambda)$ so that the λ_a are held constant when partials with respect to q_k and \dot{q}_k are taken.

¹⁷A notation similar to this one is also often used in the general calculus of variations.

3.7 Example of the General Method

Let us return to the simple example of Section 2.3 but now with the constraint

$$0 = G_1(s, t) = \alpha x + \beta y + \gamma z - \Lambda \quad (3.31)$$

discussed in the introduction to the present chapter. Applying eqn (3.27) and using the Lagrangian in eqn (2.23) gives

$$i = 1: \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{x}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial x} = \lambda_1 \alpha \quad \text{or} \quad m\ddot{x} + kx = \lambda_1 \alpha \quad (3.32)$$

$$i = 2: \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{y}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial y} = \lambda_1 \beta \quad \text{or} \quad m\ddot{y} + ky = \lambda_1 \beta \quad (3.33)$$

$$i = 3: \quad \frac{d}{dt} \left(\frac{\partial L(s, \dot{s}, t)}{\partial \dot{z}} \right) - \frac{\partial L(s, \dot{s}, t)}{\partial z} = \lambda_1 \gamma \quad \text{or} \quad m\ddot{z} + kz = \lambda_1 \gamma \quad (3.34)$$

which, together with the constraint equation eqn (3.31) can be solved for the four unknowns x, y, z, λ_1 .

3.8 Reduction of Degrees of Freedom

One of the benefits of the Lagrangian method is that holonomic constraints that do no virtual work may be used to reduce the number of degrees of freedom (i.e., the number of generalized coordinates) from D to $(D - C)$ where C is the number of independent constraints. After this reduction, the forces of constraint and the constrained variables both disappear from the calculation, leaving Lagrange equations that look like those of an unconstrained system of $(D - C)$ degrees of freedom.

This reduction theorem is based on the idea of a *reduced Lagrangian*. Using the division into free and bound variables from Theorem 3.4.1, we note that the constraint equations, eqn (3.1), may be written as

$$0 = G_a(q^{(f)}, q^{(b)}, t) \quad (3.35)$$

for $a = 1, \dots, C$, where we have written the dependency on the free and bound variables separately. As proved in Theorem D.26.1 of Appendix D, the nonsingularity of the matrix we have called $\mathbf{g}^{(b)}$ in Theorem 3.4.1 is a sufficient condition for eqn (3.35) to be solved for the bound variables. For $l = (D - C + 1), \dots, D$,

$$q_l^{(b)} = q_l^{(b)}(q^{(f)}, t) \quad (3.36)$$

Taking the time derivative of eqn (3.36) we also obtain the generalized velocities of the bound variables,

$$\dot{q}_l^{(b)} = \dot{q}_l^{(b)}(q^{(f)}, \dot{q}^{(f)}, t) \quad (3.37)$$

The reduced Lagrangian \bar{L} is defined as the original Lagrangian L with eqns (3.36, 3.37) substituted into it to eliminate the bound variables and their derivatives. Writ-

ing the original Lagrangian with its free and bound variables listed separately,

$$L = L(q, \dot{q}, t) = L(q^{(f)}, q^{(b)}, \dot{q}^{(f)}, \dot{q}^{(b)}, t) \quad (3.38)$$

the reduced Lagrangian is

$$\bar{L}(q^{(f)}, \dot{q}^{(f)}, t) = L(q^{(f)}, q^{(b)}(q^{(f)}, t), \dot{q}^{(f)}, \dot{q}^{(b)}(q^{(f)}, \dot{q}^{(f)}, t), t) \quad (3.39)$$

We may now state the reduction theorem.

Theorem 3.8.1: Reduced Lagrange Equations

If the forces of constraint do no virtual work, and if the constraints are holonomic and functionally independent, then the equations of motion of the system can be reduced to

$$\frac{d}{dt} \left(\frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k} \right) - \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial q_k} = 0 \quad (3.40)$$

for $k = 1, \dots, (D - C)$, where the reduced Lagrangian is defined by eqn (3.39). These are $(D - C)$ equations in $(D - C)$ unknowns and so may be solved for the free variables as functions of time. Thus a complete solution for the motion of the system is obtained.

Proof: The main burden of the proof is to justify the zero on the right side of eqn (3.40). Constraints are present, yet there are no Lagrange multiplier expressions on the right like the ones we saw in eqn (3.27).

The proof begins by a transformation to a new system of good generalized coordinates, similar to that discussed in Section 2.10. This new system, which we will call the special r-system, has its last C variables *defined to be equal* to the C constraint functions G_a . Thus, for $a = 1, \dots, C$,

$$r_{D-C+a} = G_a(q, t) = G_a(q^{(f)}, q^{(b)}, t) \quad (3.41)$$

The remaining $(D - C)$ variables of the r-system are set equal to free variables $q^{(f)}$. For $k = 1, \dots, (D - C)$,

$$r_k = q_k^{(f)} \quad (3.42)$$

This choice guarantees that the Jacobian determinant condition eqn (2.59) is satisfied and hence that the special r-system is a set of good generalized coordinates. For the second determinant in that equation will have the block form

$$\left| \frac{\partial r(q, t)}{\partial q} \right| = \left| \begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ \hline \left(\frac{\partial G(q, t)}{\partial q^{(f)}} \right) & \mathbf{g}^{(b)} \end{array} \right| = \left| \mathbf{g}^{(b)} \right| \quad (3.43)$$

where $\mathbf{g}^{(b)}$ is the matrix defined in eqns (3.17, 3.18). The determinant of this matrix is nonzero by the above assumption concerning the critical minor of eqn (3.3). Note that we denoted the $(D - C) \times (D - C)$ identity matrix by \mathbf{U} . It will be useful also to label free and bound r -variables as $r^{(f)} = r_1, \dots, r_{(D-C)}$ and $r^{(b)} = r_{(D-C+1)}, \dots, r_D$.

Due to the definition in eqn (3.41), in the r-system each constraint function depends only on a single $r^{(b)}$ coordinate,

$$G_a(r, t) = r_{(D-C+a)}^{(b)} \quad (3.44)$$

for $a = 1, \dots, C$. Thus the general Lagrange equations, eqn (3.27), now expressed in the special r-system, are

$$\frac{d}{dt} \left(\frac{\partial L(r, \dot{r}, t)}{\partial \dot{r}_k} \right) - \frac{\partial L(r, \dot{r}, t)}{\partial r_k} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(r, t)}{\partial r_k} = 0 \quad (3.45)$$

for $k = 1, \dots, (D - C)$, and

$$\frac{d}{dt} \left(\frac{\partial L(r, \dot{r}, t)}{\partial \dot{r}_l} \right) - \frac{\partial L(r, \dot{r}, t)}{\partial r_l} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(r, t)}{\partial r_l} = \lambda_{(l-D+C)} \quad (3.46)$$

for $l = (D - C + 1), \dots, D$, with the equation of constraint $0 = G_a(r, t)$ giving

$$r_l^{(b)} = 0 \quad (3.47)$$

for $l = (D - C + 1), \dots, D$. The zero on the right side of eqn (3.45) follows from eqn (3.44). In the r-system, the $G_a(r, t)$ constraint functions do not depend on the free variables and hence $\partial G_a(r, t)/\partial r_k = 0$ for $k = 1, \dots, (D - C)$.

Two immediate simplifications are possible now. First, we can drop eqn (3.46). As we will see, it is not needed to solve the problem. Second, we can note that the partials in eqn (3.45) are all with respect to the free variables. Thus eqn (3.47) setting the bound variables to zero can be applied in eqn (3.45) even before the partial derivatives are taken. For $k = 1, \dots, (D - C)$, we can write

$$\left. \frac{\partial L(r, \dot{r}, t)}{\partial r_k} \right|_{r^{(b)}, \dot{r}^{(b)}=0} = \frac{\partial}{\partial r_k} (L(r, \dot{r}, t)|_{r^{(b)}, \dot{r}^{(b)}=0}) \quad (3.48)$$

with a similar result for partials with respect to the free \dot{r}_k . If we define the *reduced* Lagrangian \bar{L} by

$$\bar{L}(r^{(f)}, \dot{r}^{(f)}, t) = L(r, \dot{r}, t)|_{r^{(b)}, \dot{r}^{(b)}=0} \quad (3.49)$$

eqn (3.45) can then be written, for $k = 1, \dots, (D - C)$, as

$$\frac{d}{dt} \left(\frac{\partial \bar{L}(r^{(f)}, \dot{r}^{(f)}, t)}{\partial \dot{r}_k} \right) - \frac{\partial \bar{L}(r^{(f)}, \dot{r}^{(f)}, t)}{\partial r_k} = 0 \quad (3.50)$$

But, except for the use of $r^{(f)}$ to denote the free variables rather than $q^{(f)}$, the reduced Lagrangian $\bar{L}(r^{(f)}, \dot{r}^{(f)}, t)$ in eqn (3.49) is identical¹⁸ to the reduced Lagrangian

¹⁸To obtain the Lagrangian $L(r, \dot{r}, t)$, the definitions $r^{(f)} = q^{(f)}$ and $r^{(b)} = G(q^{(f)}, q^{(b)}, t)$ in eqns (3.41, 3.42) must be inverted to give $q^{(f)} = r^{(f)}$ and $q^{(b)} = q^{(b)}(r^{(f)}, r^{(b)}, t)$. These functions and their derivatives are then substituted into $L(q, \dot{q}, t)$ to get $L(r, \dot{r}, t)$. Setting $r^{(b)} = 0$ and $\dot{r}^{(b)} = 0$ in $L(r, \dot{r}, t)$, as is done in eqn (3.49), then gives a result that becomes identical to eqn (3.39) when labeling of the free variables is changed from $r^{(f)}$ to the equivalent $q^{(f)}$.

$\bar{L}(q^{(f)}, \dot{q}^{(f)}, t)$ defined in eqn (3.39). And recall that eqn (3.42) makes $q_k = r_k$ for all $k = 1, \dots, (D - C)$. Thus eqn (3.50) may be rewritten as

$$\frac{d}{dt} \left(\frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k} \right) - \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial q_k} = 0 \quad (3.51)$$

for $k = 1, \dots, (D - C)$, as was to be proved. □

3.9 Example of a Reduction

Suppose that we have a system of one mass m moving under an acceleration of gravity $\mathbf{g} = -g\hat{\mathbf{e}}_3$. The Lagrangian in the s -system (with $s_1 = x, s_2 = y, s_3 = z$) is

$$L(s, \dot{s}, t) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \quad (3.52)$$

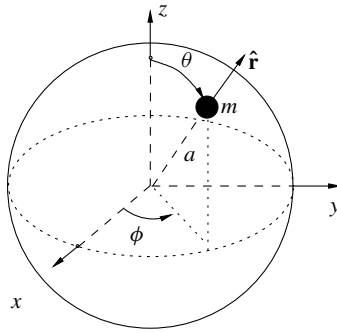


FIG. 3.3. Mass m is constrained to slide without friction on the surface of a sphere of radius a . Gravity is assumed to be acting downward, in the negative z direction.

Now suppose that the mass is constrained to move on the surface of a frictionless sphere of radius a by the constraint equation

$$0 = G_1(s, t) = \sqrt{x^2 + y^2 + z^2} - a \quad (3.53)$$

Assuming that we are interested only in motions above the x - y plane, we can solve eqn (3.53) for z giving

$$z = \sqrt{a^2 - x^2 - y^2} \quad (3.54)$$

and its derivative

$$\dot{z} = -\frac{x\dot{x} + y\dot{y}}{\sqrt{a^2 - x^2 - y^2}} \quad (3.55)$$

We define¹⁹ the set of free variables to be $s^{(f)} = x, y$ and the single bound variable to be $s^{(b)} = z$. Substituting eqns (3.54, 3.55) into eqn (3.52) gives the reduced

¹⁹Note again that there are often several possible ways of making the bound-free division. Here it is obvious from the symmetry of the problem that any one of x, y, z could be chosen to be the bound variable.

Lagrangian

$$\bar{L}(s^{(f)}, \dot{s}^{(f)}, t) = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + \frac{(x\dot{x} + y\dot{y})^2}{a^2 - x^2 - y^2} \right) - mg\sqrt{a^2 - x^2 - y^2} \quad (3.56)$$

from which we can derive the two Lagrange equations

$$i = 1: \quad \frac{d}{dt} \left(\frac{\partial \bar{L}(s^{(f)}, \dot{s}^{(f)}, t)}{\partial \dot{x}} \right) - \frac{\partial \bar{L}(s^{(f)}, \dot{s}^{(f)}, t)}{\partial x} = 0 \quad (3.57)$$

$$i = 2: \quad \frac{d}{dt} \left(\frac{\partial \bar{L}(s^{(f)}, \dot{s}^{(f)}, t)}{\partial \dot{y}} \right) - \frac{\partial \bar{L}(s^{(f)}, \dot{s}^{(f)}, t)}{\partial y} = 0 \quad (3.58)$$

and so solve the problem. The number of degrees of freedom has been reduced from $D = 3$ to $D - C = 3 - 1 = 2$.

3.10 Example of a Simpler Reduction Method

In some special cases, it may be possible to choose an initial q-system that matches the symmetries of the constraints. Then the calculation of the reduced Lagrangian becomes particularly simple.

Suppose that the initial q-system is chosen so that the equations of constraint depend *only* on the bound variables $q^{(b)}$. It follows that the constraint equations

$$0 = G_a(q, t) = G_a(q^{(b)}, t) \quad (3.59)$$

for $a = 1, \dots, C$, constitute C independent functions of the C variables $q^{(b)}$ and the time. Thus the solution for the bound variables in eqn (3.36) now gives these bound variables as functions of time alone, rather than as functions of the free variables and the time. Thus

$$q_l^{(b)} = q_l^{(b)}(t) \quad (3.60)$$

for $l = (D - C + 1), \dots, D$. The derivatives $\dot{q}_l^{(b)} = \dot{q}_l^{(b)}(t)$ may then be calculated from these equations, and will also be functions of time only. The calculation of the reduced Lagrangian is thus simplified.

For example, the constraint of Section 3.9 has spherical symmetry. If we choose a system of coordinates $q_1 = \theta$, $q_2 = \phi$, $q_3 = r$ where r, θ, ϕ are spherical polar coordinates, then the constraint equation depends only on q_3 . So we may define the free variables to be $q^{(f)} = \theta, \phi$ and the single bound variable to be $q_3^{(b)} = r$. When converted to this q-system, the constraint equation, eqn (3.53), becomes

$$G_1(q, t) = r - a \quad (3.61)$$

which depends only on the bound variable r and so can be solved immediately for $r = a$ and $\dot{r} = 0$.

In spherical polar coordinates, the full Lagrangian of eqn (3.52) becomes

$$L(q, \dot{q}, t) = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - mgr \cos \theta \quad (3.62)$$

Solving the constraint equation eqn (3.61) for $r = a$, $\dot{r} = 0$ and inserting these into eqn (3.62) gives the simple reduced Lagrangian

$$\bar{L}(q^{(f)}, \dot{q}^{(f)}, t) = \frac{m}{2} \left(a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2 \right) - mga \cos \theta \quad (3.63)$$

from which we derive the two reduced Lagrange equations

$$k = 1: \quad \frac{d}{dt} \left(\frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{\theta}} \right) - \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \theta} = 0 \quad (3.64)$$

$$k = 2: \quad \frac{d}{dt} \left(\frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{\phi}} \right) - \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \phi} = 0 \quad (3.65)$$

which may be used to derive the equations of motion.

3.11 Recovery of the Forces of Constraint

We have given several methods for finding the equations of motion of the system without knowing the forces of constraint. Let us suppose that one of them has been used, and that we now have the complete solution to the problem,

$$q_k = q_k(t) \quad \text{and} \quad \dot{q}_k = \dot{q}_k(t) \quad (3.66)$$

for $k = 1, \dots, D$. But suppose that we are curious, or otherwise need to know, the forces of constraint that must be acting to produce this motion. One method has already been given, in Corollary 3.5.2. Here we will treat this problem in a more general way which includes solution methods that do not produce the Lagrange multipliers λ_a directly.

Let us define A_k , for $k = 1, \dots, D$, to be those functions of time obtained by putting the solution, eqn (3.66), into the left side of eqn (3.26),

$$A_k = \left\{ \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} \right\} \bigg|_{\substack{q=q(t) \\ \dot{q}=\dot{q}(t)}} \quad (3.67)$$

Then eqn (3.26) gives the forces of constraint in the q -system directly as

$$Q_k^{(\text{cons})} = A_k \quad (3.68)$$

The forces of constraint in other systems can then be found from these by using the standard transformation formulas like eqn (3.9). One cautionary note: In evaluating the right side of eqn (3.67), it is essential to use the *full* Lagrangian, not some reduced form of it. Also, all indicated partial derivatives must be taken first, *before* the solutions $q = q(t)$ are introduced.

But suppose we want to find the forces of constraint in some other system, such as the s -system. Instead of using eqn (3.9) to convert $Q_k^{(\text{cons})}$ to the s -system, it is sometimes easier to find the Lagrange multipliers λ_a as an intermediate step. Then these same Lagrange multipliers can be used to find the forces of constraint in *any* system of coordinates by making use of equations like eqn (3.14).

Let us define

$$B_{ak} = \left. \frac{\partial G_a(q, t)}{\partial q_k} \right|_{q=q(t)} \quad (3.69)$$

Then, evaluating both sides of the general Lagrange equations in eqn (3.27) using the known solution from eqn (3.66), gives the set of linear equations for the Lagrange multipliers λ_a ,

$$A_k = \sum_{a=1}^C \lambda_a B_{ak} \quad (3.70)$$

where $k = 1, \dots, D$. These equations are redundant. Since the matrix \mathbf{B} has rank C by assumption, one can always select C of them to solve for the C Lagrange multipliers λ_a , using Cramer's rule or some other method. The forces of constraint in, for example, the s -system can then be found from eqn (3.14),

$$F_i^{(\text{cons})} = \sum_{a=1}^C \lambda_a \left. \frac{\partial G_a(s, t)}{\partial s_i} \right|_{s=s(t)} \quad (3.71)$$

for $i = 1, \dots, D$, where the partial derivatives on the right are evaluated using the known solution, eqn (3.66), now expressed in the s -system.

Although the formal description given here for finding the λ_a may seem complex, in practice it is often quite simple to apply, as will be seen in the next section.

3.12 Example of a Recovery

As an example, imagine that we need the forces of constraint exerted by the sphere in Section 3.10. In this example, $q_1^{(f)} = \theta$ and $q_2^{(f)} = \phi$ are the free variables, and $q_3^{(b)} = r$ is the bound variable, and there is only one constraint, $C = 1$. Also, that constraint $G_1(q, t) = r - a$ depends only on $q_3^{(b)} = r$. So there is only one nonzero matrix element,

$$B_{13} = \left. \frac{\partial G_1(q, t)}{\partial q_3} \right|_{q=q(t)} = 1 \quad (3.72)$$

and eqn (3.70) for λ_1 reduces to

$$A_3 = \lambda_1 B_{13} = \lambda_1 \quad (3.73)$$

It remains to evaluate A_3 . Using the *full* Lagrangian from eqn (3.62) gives

$$\begin{aligned}
 A_3 &= \left\{ \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{r}} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial r} \right\} \bigg|_{\substack{q=q(t) \\ \dot{q}=\dot{q}(t)}} \\
 &= \left\{ \frac{d}{dt} (m\dot{r}) - m \left(+r\dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 - g \cos \theta \right) \right\} \bigg|_{\substack{q=q(t) \\ \dot{q}=\dot{q}(t)}} \\
 &= -m \left(+a\dot{\theta}^2 + a \sin^2 \theta \dot{\phi}^2 - g \cos \theta \right) \bigg|_{\substack{q=q(t) \\ \dot{q}=\dot{q}(t)}} \quad (3.74)
 \end{aligned}$$

where the θ and ϕ in the final expression must be evaluated using the known solution previously obtained. Thus

$$\lambda_1 = -m \left(+a\dot{\theta}^2 + a \sin^2 \theta \dot{\phi}^2 - g \cos \theta \right) \bigg|_{\substack{q=q(t) \\ \dot{q}=\dot{q}(t)}} \quad (3.75)$$

Using this same λ_1 , the forces of constraint in the Cartesian s -system are then given by eqn (3.71) in the form, for $i = 1, 2, 3$,

$$F_i^{(\text{cons})} = \lambda_1 \frac{\partial G_1(s, t)}{\partial s_i} \bigg|_{s=s(t)} = \lambda_1 \frac{x_i}{\sqrt{x^2 + y^2 + z^2}} \bigg|_{s=s(t)} \quad (3.76)$$

where eqn (3.53) was used. In vector form,

$$\mathbf{F}^{(\text{cons})} = \lambda_1 \hat{\mathbf{r}} \quad (3.77)$$

which verifies the expected result that the force of constraint is entirely in the radial direction.

3.13 Generalized Energy Theorem with Constraints

The generalized energy function in a system with constraints is the same as that defined in Section 2.15. The generalized energy theorem is modified, however.

Theorem 3.13.1: Generalized Energy Theorem with Constraints

When the only non-potential forces are constraint forces that do no virtual work, the generalized energy theorem becomes

$$\dot{H}_q = -\frac{\partial L(q, \dot{q}, t)}{\partial t} - \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial t} \quad (3.78)$$

where H_q is the same generalized energy function as was defined earlier by eqn (2.76).

Proof: First note that, using the standard definition eqn (2.68) for the generalized momenta, the general Lagrange equations of eqn (3.27) may be written in the alternate form

$$\dot{p}_k = \frac{\partial L(q, \dot{q}, t)}{\partial q_k} + \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \quad (3.79)$$

The proof of eqn (3.78) is the same as the proof given in Theorem 2.15.1 up to and including eqn (2.80). The \ddot{q}_k terms cancel as before, but the use of eqn (3.79) for \dot{p}_k instead of eqn (2.72) leads, after some cancellation, to the expression

$$\begin{aligned} \dot{H}_q &= \sum_{k=1}^D \left(\sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \right) \dot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial t} \\ &= \sum_{a=1}^C \lambda_a \left(\sum_{k=1}^D \frac{\partial G_a(q, t)}{\partial q_k} \dot{q}_k \right) - \frac{\partial L(q, \dot{q}, t)}{\partial t} \\ &= \sum_{a=1}^C \lambda_a \left(\frac{dG_a(q, t)}{dt} - \frac{\partial G_a(q, t)}{\partial t} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial t} \end{aligned} \quad (3.80)$$

But eqn (3.1) implies that $dG_a(q, t)/dt = 0$, leading at once to eqn (3.78), as was to be proved. \square

It follows from Theorem 3.13.1 that, if both the Lagrangian $L(q, \dot{q}, t)$ and the constraint functions $G_a(q, t)$ in the q -system do not contain the letter t explicitly, the generalized energy function H_q will be a constant of the motion, equal to its initial value at $t = 0$.

The result in the s -system is similar, with a similar proof,²⁰

$$\dot{H}_s = -\frac{\partial L(s, \dot{s}, t)}{\partial t} - \sum_{a=1}^C \lambda_a \frac{\partial G_a(s, t)}{\partial t} \quad (3.81)$$

An alternate generalized energy theorem is also possible in systems in which holonomic constraints have been used to reduce the number of degrees of freedom from D to $(D - C)$. It begins with the reduced Lagrangian of eqn (3.39). Define a reduced, generalized energy function \bar{H}_q by

$$\bar{H}_q = \sum_{k=1}^{(D-C)} \dot{q}_k^{(f)} \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k} - \bar{L}(q^{(f)}, \dot{q}^{(f)}, t) \quad (3.82)$$

Then a proof almost identical to that in Theorem 2.15.1 shows that

$$\frac{d\bar{H}_q}{dt} = -\frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial t} \quad (3.83)$$

²⁰Equation (3.81) illustrates again that the forces of constraint may do real work even when they do no virtual work. In the s -system, the generalized energy function H_s will always equal the total energy. When the constraint is time varying so that $\partial G_a(s, t)/\partial t \neq 0$, the constraint forces are seen to contribute to the rate of change of H_s .

Thus, if the reduced Lagrangian \bar{L} does not contain the letter t explicitly, then the reduced generalized energy function \bar{H}_q will be a constant of the motion, equal to its initial value at $t = 0$.

3.14 Tractable Non-Holonomic Constraints

To be treated by the Lagrangian method, constraints must at least define a definite relation between displacements of the generalized coordinates. Other things that might be thought of as constraints, such as inequalities like $a_k \leq q_k \leq b_k$ defining walls of a room, cannot be treated by the methods described here.²¹

However, differential constraints of the form

$$0 = \sum_{k=1}^D g_{ak}(q, t) dq_k + g_{a0}(q, t) dt \quad (3.84)$$

or equivalently in the s -system

$$0 = \sum_{i=1}^D f_{ai}(s, t) ds_i + f_{a0}(s, t) dt \quad (3.85)$$

where $a = 1, \dots, C$, and the constraints are related by

$$g_{ak} = \sum_{i=1}^D f_{ai} \frac{\partial s_i(q, t)}{\partial q_k} \quad \text{and} \quad g_{a0} = f_{a0} + \sum_{i=1}^D f_{ai} \frac{\partial s_i(q, t)}{\partial t} \quad (3.86)$$

can be treated even though the differential expression in eqn (3.84) is not a perfect differential and hence cannot be integrated to give a holonomic constraint function $G_a(q, t)$. These will be called *tractable non-holonomic* constraints.²²

In the case of tractable but non-holonomic constraints, the allowed virtual displacements are defined to be those that satisfy an equation equivalent to eqns (eqn (3.5), eqn (3.8)). For $a = 1, \dots, C$, with $\partial G_a / \partial q_k$ replaced by g_{ak} ,

$$0 = \sum_{k=1}^D g_{ak} \delta q_k \quad (3.87)$$

Theorem 3.4.1 then can be generalized to say that $\delta W^{(\text{cons})} = 0$ for all allowed

²¹In Lagrangian mechanics, a ball confined to a box with perfectly elastic, rigid walls would be treated as a series of problems. Each problem would end when the ball hits a wall, the reflection conditions would be applied, and the next problem would begin with the resulting initial conditions.

²²The condition for differential expression eqn (3.84) to be a perfect differential which can be integrated to yield a potential function like $G_a(q, t)$ is given in Section D.20. Since each term of the homogeneous eqn (3.84) could be multiplied by an integrating function $u_a(q, t)$ without changing the implied relation between the differentials dq_k , the general condition for the integrability of eqn (3.84) for the a th constraint is that, for some nonzero integrating function $u_a(q, t)$, $\partial(u_a g_{ak}) / \partial q_l = \partial(u_a g_{al}) / \partial q_k$ for every pair of indices k, l . Also, $\partial(u_a g_{ak}) / \partial t = \partial(u_a g_{a0}) / \partial q_k$ must hold for every k value. If no such integrating function exists, then the constraint is non-holonomic.

virtual displacements if and only if the constraint forces have the following form

$$Q_k^{(\text{cons})} = \sum_{a=1}^C \lambda_a g_{ak}(q, t) \quad \text{or, equivalently,} \quad F_i^{(\text{cons})} = \sum_{a=1}^C \lambda_a f_{ai}(s, t) \quad (3.88)$$

If the constraint is holonomic, then $g_{ak} = \partial G_a(q, t)/\partial q_k$ and we recover eqn (3.14). But if the constraint is non-holonomic eqn (3.88) still applies, with the g_{ak} taken from eqn (3.84). The proof of this generalization is the same as that in Section 3.4. That proof used only virtual displacements, and the fact that g_{ak} was equal to $\partial G_a/\partial q_k$ played no essential role in it.

Thus, for the case of tractable but non-holonomic constraints, the general Lagrange equations, eqn (3.27), become

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = \sum_{a=1}^C \lambda_a g_{ak}(q, t) \quad (3.89)$$

with the constraint equation

$$0 = \sum_{k=1}^D g_{ak}(q, t) \dot{q}_k + g_{a0}(q, t) \quad (3.90)$$

where $k = 1, \dots, D$ and $a = 1, \dots, C$. These are $(D + C)$ differential equations for the $(D + C)$ unknown functions q, λ and therefore can be solved. Similar equations hold in the s -system, with f_{ai} in place of g_{ak} .

The generalized energy theorem, Theorem 3.13.1, becomes

$$\dot{H}_q = -\frac{\partial L(q, \dot{q}, t)}{\partial t} - \sum_{a=1}^C \lambda_a g_{a0} \quad (3.91)$$

or, in the s -system,

$$\dot{H}_s = -\frac{\partial L(s, \dot{s}, t)}{\partial t} - \sum_{a=1}^C \lambda_a f_{a0} \quad (3.92)$$

Some problems combine holonomic and non-holonomic constraints. In that case, the holonomic ones may be used to reduce the degrees of freedom of the system as outlined in Section 3.8. The non-holonomic ones may then be included by using the methods of the present section, but starting from the reduced Lagrangian.

3.15 Exercises

General note: These exercises are intended to help you master the Lagrangian theory of constraints. Therefore, they *must* be done using those methods, even if some of them are so simple that elementary approaches would also be possible.

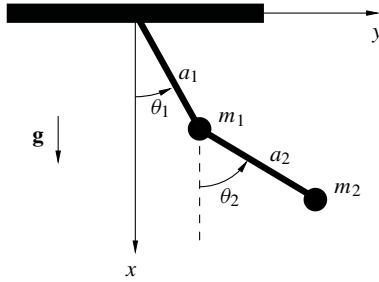


FIG. 3.4. Illustration for Exercise 3.1.

Exercise 3.1 Consider the plane double pendulum from Exercise 2.11.

- (a) The sticks of the pendulum are now constrained to have fixed lengths a_1 and a_2 . Write the two constraint functions $G_1(s, t)$ and $G_2(s, t)$ in terms of the s -system variables. Now express these same functions in terms of the q -system variables, as $G_1(q, t)$ and $G_2(q, t)$.
- (b) Use the full Lagrangians from Exercise 2.11, $L(s, \dot{s}, t)$ in the s -system and $L(q, \dot{q}, t)$ in the q -system, to write all four Lagrange equations, using the Lagrange multipliers λ_1 and λ_2 as appropriate. Do this in both the s - and the q -systems.
- (c) Taking θ_1 and θ_2 as your free variables, write the reduced Lagrangian in the reduced q -system $\bar{L}(q^{(f)}, \dot{q}^{(f)}, t)$ and the two Lagrange equations for the free variables in that system.
- (d) Suppose that you are able to solve the equations in part (c) for $\theta_1(t)$ and $\theta_2(t)$. State clearly, showing the exact formulas you would use, how you would calculate the Cartesian components of the force of constraint on each of the masses.

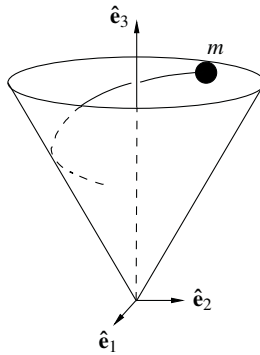


FIG. 3.5. Illustration for Exercise 3.2.

Exercise 3.2 A mass m slides on the inner surface of a conical hole in frictionless ice. The cone has half-angle α . Gravity acts downward, $\mathbf{g} = -g\hat{\mathbf{e}}_3$. At $t=0$, the mass has spherical polar coordinates $r_0 > 0$, $\dot{r}_0 = 0$, $\phi_0 = \pi$, $\dot{\phi}_0 > 0$. With the origin of coordinates at the vertex of the cone (bottom of the hole), the mass is constrained to have $\theta = \alpha$. The ice is fragile. Its surface can only provide a normal force less than F_{\max} . Find the radius r_b at which the mass breaks through the ice surface.

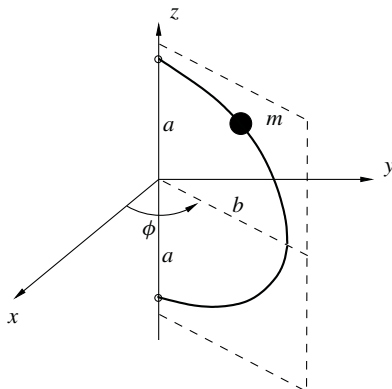


FIG. 3.6. Illustration for Exercises 3.3 and 3.4. The dotted rectangle represents an imaginary door swinging from hinges on the z -axis. The wire of the parabola is entirely in the plane of this door.

Exercise 3.3 A rigid wire of negligible mass is bent into the shape of a parabola and suspended from the z -axis by frictionless pivots at $z = \pm a$. The equation of the wire, in cylindrical polar coordinates, is $\rho = b(1 - z^2/a^2)$. A bead of mass m slides without friction on the wire. The acceleration of gravity is $\mathbf{g} = -g\hat{\mathbf{e}}_3$. Choose cylindrical polar coordinates as your generalized coordinates. Assume the initial conditions at $t = 0$ as follows: $0 < z_0 < a$, $\dot{z}_0 = 0$, $\phi_0 = 0$, $\dot{\phi}_0 > 0$.

- Write the full Lagrangian for this problem.
- Write the three Lagrange equations, using Lagrange multipliers as appropriate.
- Now, choosing your free variables to be z and ϕ , write the reduced Lagrangian \bar{L} and the two Lagrange equations derived from it.
- Use the result of (c) to write the reduced generalized energy \bar{H}_q . Is it conserved? If so why, if not why not?
- Use the results so far obtained to write expressions for ρ , ϕ , $\dot{\rho}$, $\dot{\phi}$, \dot{z} , $\ddot{\rho}$, $\ddot{\phi}$, \ddot{z} as functions of z only. [Note: These expressions, and the ones in the next part, may of course also depend on the initial values z_0 , $\dot{\phi}_0$ and on the parameters a , b , m , and g .]
- Find the Cartesian vector force of constraint exerted by the wire on the mass for $t \geq 0$, expressing it as a function of z only.

Exercise 3.4 This problem has the same geometry as Exercise 3.3, but now there is an additional constraint: $\phi = \omega_0 t$ where ω_0 is a given constant.

- Choosing z as your free coordinate, form the reduced Lagrangian and write the single Lagrange equation derived from it.
- Derive the reduced generalized energy function. Is it conserved? If so why, if not why not? [Note: This \bar{H}_q will not be the same as the one derived in Exercise 3.3.]
- What is the smallest value of ω_0 such that there will be at least one point (equilibrium point) such that if z_0 is set equal to that value with $\dot{z}_0 = 0$, the mass will remain at that height for all time?

Exercise 3.5 We use cylindrical polar coordinates in this problem. A roller-coaster car of mass m slides without friction on a track defined by the constraints

$$\rho = \rho_0 + a\phi \quad \text{and} \quad z = z_0 - b\phi \quad (3.93)$$

where $a, b > 0$. At $t=0$ the mass is at rest at

$$\mathbf{r}_0 = \rho_0 \hat{\mathbf{e}}_1 + z_0 \hat{\mathbf{e}}_3 \quad (3.94)$$

(a) Write the full Lagrangian $L(q, \dot{q}, t)$ using a q-system consisting of cylindrical polar coordinates ϕ, ρ, z .

(b) Write the three Lagrange equations in the q-system, putting in λ_1 and λ_2 correctly.

(c) Now use the constraints to eliminate $\rho, z, \dot{\rho}, \dot{z}$, leaving ϕ as your free variable. Write the

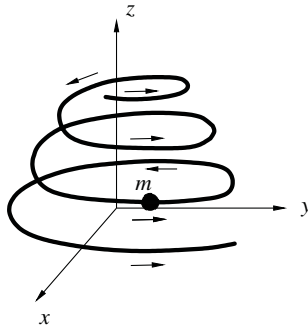


FIG. 3.7. Illustration for Exercise 3.5.

reduced Lagrangian $\bar{L} = \bar{L}(\phi, \dot{\phi}, t)$.

(d) Write the Lagrange equation using $\bar{L}(\phi, \dot{\phi}, t)$ and solve the resulting equation for $\ddot{\phi}$ as a function of ϕ and $\dot{\phi}$.

(e) Write the reduced generalized energy \bar{H}_q based on the reduced Lagrangian \bar{L} , and use it to derive an equation for $\dot{\phi}$ as a function of ϕ and an integration constant that you determine from the given initial conditions.

(f) From parts (d) and (e) you now have $\ddot{\phi}$ as a function of ϕ and $\dot{\phi}$, and also $\dot{\phi}$ as a function of ϕ . Thus you effectively have both $\dot{\phi}$ and $\ddot{\phi}$ as functions of ϕ only. Write an expression for the Cartesian vector force of constraint that the track exerts on the car, writing it as a function only of the given parameters and the variables $\phi, \dot{\phi}, \ddot{\phi}$. [This expression, of course, could now be used to write the force of constraint out as a function of ϕ only if you wished. But it is clearer just to leave the result as it is, and cite the results of parts (d) and (e) to anyone who wants it as a function of ϕ only. (e.g. a designer who needs to know how strong to make the track.)]

Exercise 3.6 Suppose a mass m_1 slides without friction on a horizontal table. There is a hole in the center of the table. A massless string runs along the table top from m_1 to the hole, through the hole, and down below the table, where it is attached to another mass m_2 . The origin of coordinates is at the center of the hole, with the $\hat{\mathbf{e}}_3$ axis pointing upwards. Gravity acts downwards. Consider the hole to have a size big enough to let the string through without

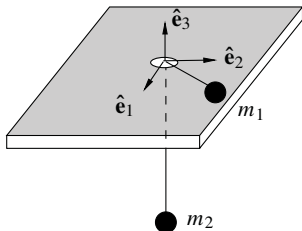


FIG. 3.8. Illustration for Exercise 3.6.

friction, but small enough to be neglected in our calculations.

(a) Using cylindrical polar coordinates for m_1 and Cartesian coordinates for m_2 , write the full Lagrangian for this two-mass system.

(b) Now apply the following constraints: Mass m_1 is always at the level of the table's surface. Mass m_2 is enclosed in a vertical plastic tube just large enough to hold it at $x_2 = y_2 = 0$ while exerting no friction forces on it. The string length is ℓ_0 and never changes. With these constraints, write the full Lagrange equations, including the Lagrange multipliers as required.

(c) Use the constraints to write a reduced Lagrangian with free coordinates ρ_1 and ϕ_1 , the cylindrical polar coordinates of the mass m_1 on the top of the table.

(d) Write the two reduced Lagrange equations and show that the one for ϕ_1 can be integrated immediately to give $\dot{\phi}_1$ as a function of ρ_1 and constants determined at $t = 0$. Assume that $\dot{\phi}_1(0) > 0$ at time zero. Use this result to write the other reduced Lagrange equation as an ordinary differential equation involving only ρ_1 and its derivatives.

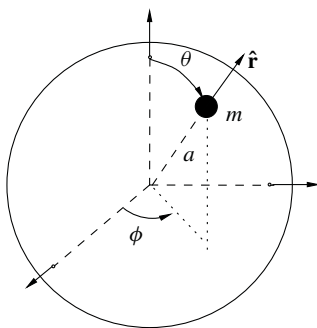


FIG. 3.9. Illustration for Exercise 3.7.

Exercise 3.7 Consider a single mass m to be sliding without friction on the outside surface of a sphere of radius a . Suppose that at time zero, it has spherical polar coordinates $\theta_0 > 0$, $\phi_0 = 0$ and generalized velocities $\dot{\theta}_0 = 0$ and $\dot{\phi}_0 > 0$.

(a) Using spherical polar coordinates, write both the full Lagrangian and the reduced Lagrangian for this problem.

(b) Write the Lagrange equations for both the full and the reduced Lagrangians.

(c) Use the reduced Lagrangian \bar{L} to write the reduced generalized energy \bar{H}_q .

(d) The mass will leave the surface of the sphere at the instant at which the normal force

of constraint becomes negative (the sphere cannot pull in on the mass, only push outwards). Write an expression for the angle θ_{\max} at which the mass leaves the sphere and the problem ends.

(e) Define the parameter γ by $a\dot{\phi}_0^2 = \gamma g$, where g is the acceleration of gravity, and express θ_{\max} as a function of a , g , γ , θ_0 only. In the case with $\gamma = 0$, find θ_{\max} in the limit $\theta_0 \rightarrow 0$.

(f) With $\theta_0 = 10^\circ$, find the numerical value of θ_{\max} for the case in which $\gamma = 0$. How big would γ have to be in order to reduce θ_{\max} to 45° ?

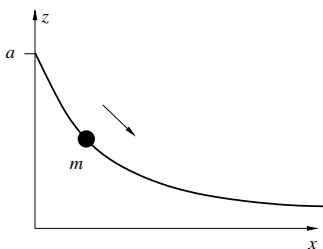


FIG. 3.10. Figure for Exercise 3.8.

Exercise 3.8 A bead of mass m slides without friction on a rigid wire that lies in the x - z plane and has the shape $z = ae^{-\gamma x}$, where γ is some given positive constant. Gravity acts downwards, with $\mathbf{g} = -g\hat{\mathbf{e}}_3$.

(a) Write the full Lagrangian for this problem, and write equations for the two holonomic constraints, first that the mass is confined to the plane $y = 0$, and second that it is confined to the surface $z = ae^{-\gamma x}$.

(b) Write the three Lagrange equations, introducing the Lagrange multipliers λ_1 and λ_2 as appropriate.

(c) Use the constraints to write a reduced Lagrangian $\bar{L}(x, \dot{x}, t)$, with x serving as the single free coordinate. Derive the reduced generalized energy from this reduced Lagrangian, and use it to find an expression for \dot{x}^2 as a function x . (Assume that the mass is released from rest at the point $x = 0$.) Also use the reduced Lagrange equation to find an expression for \ddot{x} as a function of x and \dot{x} .

(d) Write an expression for the Cartesian vector force of constraint $\mathbf{F}^{(\text{cons})}$ acting on the particle, expressing it as a function of x only. Check the limit of $\mathbf{F}^{(\text{cons})}$ as $x \rightarrow \infty$. Is it reasonable?

Exercise 3.9 A fixed, right circular cylinder (first cylinder) of radius a lies on its side, with its symmetry axis horizontal. A hollow right circular cylinder (second cylinder) of radius b and mass m , is free to roll without slipping on the first one. Assume that its symmetry axis remains aligned with that of the first cylinder. The full Lagrangian for the second cylinder's motion is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}mb^2\dot{\phi}^2 - mgr \cos \theta \quad (3.95)$$

where r is the distance between the axes of the two cylinders, and θ and ϕ are the angles shown in the figure. (This "full" Lagrangian is actually partially-reduced. Constraints not relevant to this exercise have already been applied.) Notice that ϕ is the angle between vertical

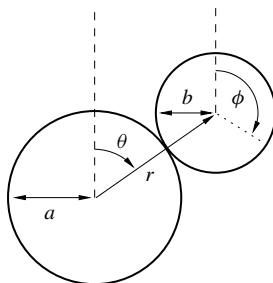


FIG. 3.11. Figure for Exercise 3.9. The upper cylinder rolls without slipping on the lower one.

and a mark inscribed on the face of the second cylinder.

(a) Write the two constraint functions, G_1 expressing the constraint that the second cylinder is in contact with the first one, and G_2 expressing the condition of rolling without slipping. Assume that $\phi = 0$ when $\theta = 0$.

(b) Write a (completely) reduced Lagrangian $\bar{L}(\theta, \dot{\theta}, t)$ and use the reduced generalized energy theorem to express $\dot{\theta}$ as a function of θ . Assume the second cylinder to be initially at rest, and at a very small distance to the right of $\theta = 0$.

(c) Use the full Lagrangian eqn (3.95) to write the three Lagrange equations, introducing Lagrange multipliers as appropriate. Find the generalized force of constraint $Q_r^{(cons)}$ for the r variable and use it to find the angle θ_c at which the rolling cylinder will lose contact with the fixed one.

INTRODUCTION TO HAMILTONIAN MECHANICS

The power of Lagrangian mechanics has caused generations of students to wonder why it is necessary, or even desirable, to recast mechanics in Hamiltonian form. The answer, which must be taken largely on faith at this point, is that the Hamiltonian formulation is a much better base from which to build more advanced methods. The Hamilton equations have an elegant symmetry that the Lagrange equations lack.

Another answer, not directly related to classical mechanics, is that the Hamiltonian function is used to write the Schroedinger equation of quantum mechanics, as discussed in Section 4.7.

4.1 Phase Space

The differences between the Lagrange and Hamilton equations result mainly from the different variable sets in which they act. The Lagrangian variable set²³ is the set of generalized coordinates and velocities $q, \dot{q} = q_1, \dots, q_D, \dot{q}_1, \dots, \dot{q}_D$ whereas the Hamiltonian set is the set of generalized coordinates and momenta $q, p = q_1, \dots, q_D, p_1, \dots, p_D$.

The q_k in the Hamiltonian set are the same as the, assumedly good, generalized coordinates used in Lagrangian mechanics. And the p_k are the same as the generalized momenta that were defined in Section 2.12 as functions of the Lagrangian variables and the time,

$$p_k = p_k(q, \dot{q}, t) = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad (4.1)$$

In Hamiltonian mechanics, the coordinates and momenta in set q, p are lumped together and considered to be coordinates of a $2D$ dimensional space called *phase space*. The variables $q_1, \dots, q_D, p_1, \dots, p_D$ are referred to collectively as *canonical coordinates of phase space*. The q_k is called the k th canonical coordinate, and p_k is called the k th canonical momentum. The pair q_k, p_k for the same k value are called canonical conjugates. Hamiltonian mechanics is essentially Newton's second law translated from Lagrangian form into a form appropriate for this phase space.

In order for the phase-space variables $q_1, \dots, q_D, p_1, \dots, p_D$ to be an adequate set

²³In the previous chapters, we have made a distinction between the s-system coordinates s_1, \dots, s_D , which were just re-labeled Cartesian coordinates, and the q-system coordinates q_1, \dots, q_D which are the most general good generalized coordinates. We now drop this distinction and use only the general set q_1, \dots, q_D . Of course, being general, these coordinates include the s-system as a special case.

of variables for mechanics, eqn (4.1) must be invertible to give inverse functions,

$$\dot{q}_k = \dot{q}_k(q, p, t) \quad (4.2)$$

for $k = 1, \dots, D$, from which the Lagrangian variables \dot{q} can be found. Then knowledge of the phase-space variables $q_1, \dots, q_D, p_1, \dots, p_D$ will allow one to determine the Lagrangian variables $q_1, \dots, q_D, \dot{q}_1, \dots, \dot{q}_D$, from which the position and velocity of each mass in the system can be found.

By the inverse function theorem, Theorem D.24.1, the condition for such an inversion is the Jacobian determinant condition

$$\left| \frac{\partial p}{\partial \dot{q}} \right| \neq 0 \quad (4.3)$$

involving the determinant of a matrix defined by

$$\left(\frac{\partial p}{\partial \dot{q}} \right)_{kl} = \frac{\partial p_k(q, \dot{q}, t)}{\partial \dot{q}_l} = \frac{\partial^2 L(q, \dot{q}, t)}{\partial \dot{q}_l \partial \dot{q}_k} \quad (4.4)$$

The inversion leading to eqn (4.2) is always possible, as proved in the following theorem.

Theorem 4.1.1: Inversion of Momenta

The matrix $(\partial p/\partial \dot{q})$ defined in eqn (4.4) is nonsingular and positive definite. It therefore satisfies the determinant condition in eqn (4.3), which allows $p_k = p_k(q, \dot{q}, t)$ to be solved for $\dot{q}_k = \dot{q}_k(q, p, t)$.

Proof: It follows from the expansion of the Lagrangian in Section 2.7 that the kl matrix element in eqn (4.4) is

$$\frac{\partial p_k(q, \dot{q}, t)}{\partial \dot{q}_l} = m_{kl}(q, t) = \sum_{j=1}^D M_j \frac{\partial s_j(q, t)}{\partial q_k} \frac{\partial s_j(q, t)}{\partial q_l} \quad (4.5)$$

where the s_j are the Cartesian components of the s -system, and the M_j are the masses of the point particles.

Defining a matrix \bar{M} by its matrix elements $\bar{M}_{ij} = M_i \delta_{ij}$, eqn (4.5) may be written as

$$\left(\frac{\partial p}{\partial \dot{q}} \right) = \left(\frac{\partial s}{\partial q} \right)^T \bar{M} \left(\frac{\partial s}{\partial q} \right) \quad (4.6)$$

Properties 5 and 10 of Section B.11 then give the determinant of $(\partial p/\partial \dot{q})$ as

$$\left| \frac{\partial p}{\partial \dot{q}} \right| = \left| \frac{\partial s}{\partial q} \right|^2 |\bar{M}| = \left| \frac{\partial s}{\partial q} \right|^2 M_1 M_2 \cdots M_D \quad (4.7)$$

The nonsingularity of the matrix $(\partial s/\partial q)$ appearing in eqn (4.6) was shown in Section 2.4 to be the condition for the q to be a good system of generalized coordinates, which we are assuming here. Thus the determinant $|\partial s/\partial q|$ is nonzero. Since all of the

particle masses M_i are positive quantities, it follows that $|\partial p/\partial \dot{q}| \neq 0$. Thus $(\partial p/\partial \dot{q})$ is nonsingular.

We now show that the real, symmetric matrix $(\partial p/\partial \dot{q})$ is positive definite. If $[x] \neq [0]$ is an arbitrary, non-null column vector, it follows from eqn (4.5) that

$$[x]^T \left(\frac{\partial p}{\partial \dot{q}} \right) [x] = \sum_{j=1}^D M_j y_j^2 \quad (4.8)$$

where

$$y_j = \sum_{k=1}^D \frac{\partial s_j(q, t)}{\partial q_k} x_k \quad \text{or, in matrix form,} \quad [y] = \left(\frac{\partial s}{\partial q} \right) [x] \quad (4.9)$$

Since the matrix $(\partial s/\partial q)$ is nonsingular by assumption, it follows from Corollary B.19.2 that the column vector $[y]$ must also be non-null.

Since all point masses M_j are positive, nonzero numbers, and since at least one of the y_j must be nonzero, the right side of eqn (4.8) must be positive and nonzero. Hence

$$[x]^T m [x] = [x]^T \left(\frac{\partial p}{\partial \dot{q}} \right) [x] > 0 \quad (4.10)$$

Using the definition in Section C.1, this implies that $(\partial p/\partial \dot{q})$ is a positive definite matrix. \square

The theorem just proved means that any physical quantity expressed in terms of Lagrangian variables can equally well be expressed in terms of phase-space ones by simple substitution. Assuming that $f = f(q, \dot{q}, t)$ is given, the same function in terms of phase-space variables is defined as the compound function

$$f = f(q, p, t) = f(q, \dot{q}(q, p, t), t) \quad (4.11)$$

where eqn (4.2) has been used.

Since the matrix $m_{kl} = \partial p_k/\partial \dot{q}_l$ has been proved nonsingular, the theory of linear equations can be used to solve for the \dot{q}_k explicitly. The definition of canonical momenta in eqn (2.69) can be written as the D linear equations, for $k = 1, \dots, D$,

$$\sum_{l=1}^D m_{kl}(q, t) \dot{q}_l = p_k(q, \dot{q}, t) - n_k(q, t) \quad (4.12)$$

for the D unknowns \dot{q}_l . They can be solved by calculating the inverse m^{-1} of the nonsingular matrix m and writing

$$\dot{q}_l(q, p, t) = \sum_{k=1}^D m_{lk}^{-1} \{p_k - n_k(q, t)\} \quad (4.13)$$

or, equivalently, by using Cramer's rule from Section B.16.

Lagrangian methods are sometimes applied to physical systems that are not obviously derived from Newton's laws for point masses. In those cases, the proof given above may not be relevant. But the inversion of eqn (4.1) may still be possible. Systems in which eqn (4.1) can be inverted to give eqn (4.2) will be referred to as *well-defined Lagrangian systems*.

4.2 Hamilton Equations

The transformation from Lagrange to Hamilton equations is a Legendre transformation, of the sort defined in Section D.30, which the reader should consult for details. In this transformation, the Lagrangian function $L(q, \dot{q}, t)$ of the Lagrangian variable set q, \dot{q}, t is to be replaced by the Hamiltonian function $H(q, p, t)$ of the phase-space variable set q, p, t . Thus $L \rightarrow H$ corresponds to $f \rightarrow g$, and there is an exchange of variables $\dot{q} \leftrightarrow p$ corresponding to the exchange of y and w . The correspondences between the present case and the general quantities defined in Section D.30 are: $L \leftrightarrow f$, $(q, t) \leftrightarrow x$, $\dot{q} \leftrightarrow y$, $H \leftrightarrow g$, $p \leftrightarrow w$, $\dot{p} \leftrightarrow u$.

The first step in the Legendre transformation, as in eqn (D.114), is to define the new function H , still expressed in terms of the old variables q, \dot{q}, t . In the present case, this first step has already been done, in Section 2.15 where the generalized energy function was defined as a function of the Lagrangian variables,

$$H = H(q, \dot{q}, t) = \sum_{k=1}^D \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \dot{q}_k - L(q, \dot{q}, t) = \sum_{k=1}^D p_k(q, \dot{q}, t) \dot{q}_k - L(q, \dot{q}, t) \quad (4.14)$$

As noted in Section D.30, to complete the Legendre transformation it is necessary to write eqn (4.14) in terms of the correct variable set $q_1, \dots, q_D, p_1, \dots, p_D$. This can always be done. Theorem 4.1.1 proved that the equations $p_k = p_k(q, \dot{q}, t)$ can always be inverted with to give $\dot{q}_k = \dot{q}_k(q, p, t)$. Thus, one simply substitutes this inverse equation into eqn (4.14) to write $H(q, p, t)$ as the compound function

$$H = H(q, p, t) = H(q, \dot{q}(q, p, t), t) \quad (4.15)$$

Note to the Reader: This step of writing H in terms of phase-space variables is essential to the Hamiltonian method. The Hamilton equations will not be true without it. To emphasize its importance, we reserve the name ‘‘Hamiltonian’’ for the expression $H(q, p, t)$ that results *after* this step is taken.

Thus, when written in terms of q, p, t , the generalized energy function $H(q, \dot{q}, t)$ becomes the Hamiltonian $H(q, p, t)$. They are the same function, but written in terms of different variables and called by different names.²⁴

²⁴See Section D.5 for a discussion of the physics convention for labeling the same function expressed in different variables.

Following the Legendre transformation pattern in Section D.30, the differential of the H in eqn (4.14) can be written as

$$\begin{aligned} dH &= \sum_{k=1}^D (p_k d\dot{q}_k + \dot{q}_k dp_k) - dL \\ &= \sum_{k=1}^D \left(p_k d\dot{q}_k + \dot{q}_k dp_k - \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} dq_k \right) - \frac{\partial L(q, \dot{q}, t)}{\partial t} dt \end{aligned} \quad (4.16)$$

Assuming for now that $Q_k^{(NP)} = 0$, the Lagrange equations, eqn (2.52),

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad (4.17)$$

can be written in a compact form using the definition of p_k from eqn (4.1),

$$\dot{p}_k = \frac{\partial L(q, \dot{q}, t)}{\partial q_k} \quad \text{where} \quad p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad (4.18)$$

When eqn (4.18) is substituted into eqn (4.16), the $d\dot{q}_k$ terms cancel and eqn (4.16) becomes

$$dH = \sum_{k=1}^D (\dot{q}_k dp_k - \dot{p}_k dq_k) + \dot{H} dt \quad (4.19)$$

where the generalized energy theorem $\dot{H} = -\partial L(q, \dot{q}, t)/\partial t$ from Section 2.15 has been used in the last term on the right.

The differential in eqn (4.19) may now be compared to the differential of the function $H(q, p, t)$ defined in eqn (4.15), which is

$$dH = \sum_{k=1}^D \left(\frac{\partial H(q, p, t)}{\partial p_k} dp_k + \frac{\partial H(q, p, t)}{\partial q_k} dq_k \right) + \frac{\partial H(q, p, t)}{\partial t} dt \quad (4.20)$$

In the Legendre transformation method, the differentials of the original variables $dq, d\dot{q}, dt$ are taken to be independent. Theorem D.18.5 and eqn (4.3) imply that set of differentials $dq_1, \dots, dq_D, dp_1, \dots, dp_D, dt$ are also independent. Hence, using Lemma D.18.3, the equality of the left sides of eqns (4.19, 4.20) implies equality of the corresponding coefficient of each differential term, and hence

$$\dot{q}_k = \frac{\partial H(q, p, t)}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H(q, p, t)}{\partial q_k} \quad \dot{H} = \frac{\partial H(q, p, t)}{\partial t} \quad (4.21)$$

for $k = 1, \dots, D$. The first two of these expressions are called the *Hamilton equations*.

The Hamilton equations are two sets of coupled first-order differential equations for the phase-space variables q_k, p_k . They are very nearly symmetric in these variables.

Except for the minus sign, the second of eqn (4.21) is just the first one with q_k and p_k exchanged between the partial and the time derivative.

Since the Hamilton equations have been derived from the Lagrange equations by a Legendre transformation, which is invertible by definition, it follows that the Hamilton equations hold if and only if the Lagrange equations hold. Thus both are equivalent to Newton's second law.

Second law \iff Lagrange equations \iff Hamilton equations

As can be seen from the properties of Legendre transformations, the first expression in eqn (4.21) simply restates eqn (4.2). The Lagrangian definition in eqn (4.1) gives $p_k(q, \dot{q}, t) = \partial L(q, \dot{q}, t)/\partial \dot{q}_k$ and the first Hamilton equations just give the inverse relation $\dot{q}_k(q, p, t) = \partial H(q, p, t)/\partial p_k$.

The second Hamilton equations in eqn (4.21), $\dot{p}_k(q, p, t) = -\partial H(q, p, t)/\partial q_k$, are in a sense the "real" equations of motion, analogous to the Lagrange equations $\dot{p}_k = \partial L(q, \dot{q}, t)/\partial q_k$.

The last of eqn (4.21) equates $\dot{H} = dH/dt$, the total time rate of change of the quantity H , to the partial derivative $\partial H(q, p, t)/\partial t$ of the function $H(q, p, t)$. It is the phase-space analog of the Lagrangian generalized energy theorem $\dot{H} = -\partial L/\partial t$.

4.3 An Example of the Hamilton Equations

As an example of the transition from Lagrange to Hamilton equations of motion, consider the system of a single particle in a central potential from Section 2.11. Using q_1, q_2, q_3 equal to polar coordinates r, θ, ϕ , the Lagrangian is

$$L = L(q, \dot{q}, t) = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{1}{2}kr^2 \quad (4.22)$$

and therefore the generalized momenta p_k for $k = 1, 2, 3$ are given by eqn (4.1) as

$$p_r = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{r}} = m\dot{r} \quad p_\theta = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad p_\phi = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi} \quad (4.23)$$

and the generalized energy function calculated from eqn (2.76) is

$$H = H(q, \dot{q}, t) = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{1}{2}kr^2 \quad (4.24)$$

Inverting eqn (4.23) to solve for the \dot{q}_k gives eqn (4.2) in the form

$$\dot{r} = \frac{p_r}{m} \quad \dot{\theta} = \frac{p_\theta}{mr^2} \quad \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta} \quad (4.25)$$

Substituting these into the generalized energy function, eqn (4.24), then gives the Hamiltonian as a function of the correct phase-space variables,

$$H = H(q, p, t) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + \frac{1}{2}kr^2 \quad (4.26)$$

As is always the case, the first set of Hamilton equations from eqn (4.21), $\dot{q}_k = \partial H(q, p, t)/\partial p_k$ for $k = 1, 2, 3$, simply repeats eqn (4.25),

$$\dot{r} = \frac{\partial H(q, p, t)}{\partial p_r} = \frac{p_r}{m} \quad \dot{\theta} = \frac{\partial H(q, p, t)}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \dot{\phi} = \frac{\partial H(q, p, t)}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta} \quad (4.27)$$

The next Hamilton equations, $\dot{p}_k = -\partial H(q, p, t)/\partial q_k$ for $k = 1, 2, 3$, are the real equations of motion. They are

$$\dot{p}_r = -\frac{\partial H(q, p, t)}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2 \theta} - kr \quad (4.28)$$

$$\dot{p}_\theta = -\frac{\partial H(q, p, t)}{\partial \theta} = \frac{p_\phi^2 \cos \theta}{mr^2 \sin^3 \theta} \quad \text{and} \quad \dot{p}_\phi = -\frac{\partial H(q, p, t)}{\partial \phi} = 0 \quad (4.29)$$

The last equation in eqn (4.21) is $\dot{H} = \partial H(q, p, t)/\partial t$, which here implies that $\dot{H} = 0$ and so $H = H(0)$, where constant $H(0)$ is determined from the value of H at time zero. This is the Hamiltonian analog of the generalized energy theorem. Thus

$$\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + \frac{1}{2}kr^2 = H(0) \quad (4.30)$$

where

$$H(0) = \frac{p_r^2(0)}{2m} + \frac{p_\theta^2(0)}{2mr^2(0)} + \frac{p_\phi^2(0)}{2mr^2(0) \sin^2 \theta(0)} + \frac{1}{2}kr^2(0) \quad (4.31)$$

and the needed values of the canonical momenta at time zero $p_k(0)$ can be determined from eqn (4.23).

As noted in Section 2.13, in this example the coordinate ϕ is ignorable. The last of eqn (4.29) implies that $p_\phi = a$ where $a = p_\phi(0)$ is some constant determined from the value of p_ϕ at time zero. The constant value can be substituted into eqn (4.30) to give the generalized energy theorem in an even simpler form, with both the coordinate ϕ and its conjugate momentum p_ϕ absent,

$$\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{a^2}{2mr^2 \sin^2 \theta} + \frac{1}{2}kr^2 = H(0) \quad (4.32)$$

4.4 Non-Potential and Constraint Forces

The derivation of the Hamilton equations in Section 4.2 has assumed that all forces are derived from the potential $U(q, t)$. However, if non-potential forces are present, possibly including suitable constraint forces that do no virtual work, the Hamilton equations can be generalized easily. In the step leading to eqn (4.19) above, one simply replaces the Lagrange equation $\dot{p}_k = \partial L(q, \dot{q}, t)/\partial q_k$ and the generalized energy

theorem $\dot{H} = -\partial L(q, \dot{q}, t)/\partial t$ by the more general expressions from eqns (2.52, 2.78),

$$\dot{p}_k = \frac{\partial L(q, \dot{q}, t)}{\partial q_k} + Q_k^{(\text{NP})} \quad \text{and} \quad \dot{H} = \sum_{k=1}^D Q_k^{(\text{NP})} \dot{q}_k - \frac{\partial L(q, \dot{q}, t)}{\partial t} \quad (4.33)$$

leading to the Hamilton equations

$$\dot{q}_k = \frac{\partial H(q, p, t)}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H(q, p, t)}{\partial q_k} + Q_k^{(\text{NP})} \quad (4.34)$$

$$\dot{H} = \frac{\partial H(q, p, t)}{\partial t} + \sum_{k=1}^D Q_k^{(\text{NP})} \dot{q}_k(q, p, t) \quad (4.35)$$

These are the general Hamilton equations in the presence of non-potential forces.

When the non-potential forces all come from suitable constraints, then, as proved in Theorem 3.4.1, $Q_k^{(\text{NP})} = Q_k^{(\text{cons})} = \sum_{a=1}^C \lambda_a \partial G_a(q, t)/\partial q_k$ and hence the Hamilton equations become

$$\dot{q}_k = \frac{\partial H(q, p, t)}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H(q, p, t)}{\partial q_k} + \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \quad (4.36)$$

$$\dot{H} = \frac{\partial H(q, p, t)}{\partial t} - \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial t} \quad (4.37)$$

This last equation follows from the same argument as was used in the proof of Theorem 3.13.1.

4.5 Reduced Hamiltonian

When the forces of constraint in a Lagrangian problem do no virtual work, and the C constraints are holonomic and independent, Section 3.8 showed how to use the constraints to reduce the number of degrees of freedom of the problem from D to $D - C$. The reduced Lagrangian $\bar{L}(q^{(f)}, \dot{q}^{(f)}, t)$ defined there can be used to define a reduced generalized energy function, as was done in eqn (3.82) of Theorem 3.13.1,

$$\begin{aligned} \bar{H} &= \bar{H}(q^{(f)}, \dot{q}^{(f)}, t) = \sum_{k=1}^{D-C} \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k^{(f)}} \dot{q}_k^{(f)} - \bar{L}(q^{(f)}, \dot{q}^{(f)}, t) \\ &= \sum_{k=1}^{D-C} \bar{p}_k^{(f)}(q^{(f)}, \dot{q}^{(f)}, t) \dot{q}_k^{(f)} - \bar{L}(q^{(f)}, \dot{q}^{(f)}, t) \end{aligned} \quad (4.38)$$

where we have defined, for $k = 1, \dots, D - C$,

$$\bar{p}_k^{(f)} = \bar{p}_k^{(f)}(q^{(f)}, \dot{q}^{(f)}, t) = \frac{\partial \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k^{(f)}} \quad (4.39)$$

Since all of the reduced Lagrange equations, eqn (3.40), have zeroes on their right hand side, the same Legendre transformation procedure used in Section 4.2 above

can be used to define a reduced Hamiltonian $\bar{H}(q^{(f)}, \bar{p}^{(f)}, t)$ and reduced Hamilton equations

$$\dot{q}_k^{(f)} = \frac{\partial \bar{H}(q^{(f)}, \bar{p}^{(f)}, t)}{\partial \bar{p}_k^{(f)}} \quad \dot{\bar{p}}_k^{(f)} = -\frac{\partial \bar{H}(q^{(f)}, \bar{p}^{(f)}, t)}{\partial q_k^{(f)}} \quad \frac{d\bar{H}}{dt} = \frac{\partial \bar{H}(q^{(f)}, \bar{p}^{(f)}, t)}{\partial t} \quad (4.40)$$

for $k = 1, \dots, (D - C)$.

The constrained variables have been eliminated from the problem. The whole Hamiltonian procedure is just as if the original Lagrangian problem had been free of constraints from the start.

However, the above derivation leading to eqn (4.40) will be correct only if the definition $\bar{p}_k^{(f)} = \bar{p}_k^{(f)}(q^{(f)}, \dot{q}^{(f)}, t)$ from eqn (4.39) can actually be inverted to give

$$\dot{q}_k^{(f)} = \dot{q}_k^{(f)}(q^{(f)}, \bar{p}^{(f)}, t) \quad (4.41)$$

This inversion allows one to make the usual substitution

$$\bar{H} = \bar{H}(q^{(f)}, \bar{p}^{(f)}, t) = \bar{H}(q^{(f)}, \dot{q}^{(f)}(q^{(f)}, \bar{p}^{(f)}, t), t) \quad (4.42)$$

to convert the reduced generalized energy function $\bar{H}(q^{(f)}, \dot{q}^{(f)}, t)$ to the reduced Hamiltonian $\bar{H}(q^{(f)}, \bar{p}^{(f)}, t)$.

Again using Theorem D.24.1, the condition for the inversion of eqn (4.39) to give eqn (4.41) is

$$\left| \frac{\partial \bar{p}^{(f)}}{\partial \dot{q}^{(f)}} \right| \neq 0 \quad (4.43)$$

where the matrix $(\partial \bar{p}^{(f)} / \partial \dot{q}^{(f)})$ is defined by

$$\left(\frac{\partial \bar{p}^{(f)}}{\partial \dot{q}^{(f)}} \right)_{kl} = \frac{\partial \bar{p}_k^{(f)}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_l^{(f)}} = \frac{\partial^2 \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k^{(f)} \partial \dot{q}_l^{(f)}} \quad (4.44)$$

The following theorem proves that this inversion can always be done.

Theorem 4.5.1: Inversion of Reduced Momenta

The matrix $(\partial \bar{p}^{(f)} / \partial \dot{q}^{(f)})$ defined in eqn (4.44) is positive definite and hence nonsingular. Thus the inversion condition eqn (4.43) is always satisfied.

Proof: When the constraints are holonomic and functionally independent, the bound variables can be written as functions of the free ones as in eqn(3.36), $q_l^{(b)} = q_l^{(b)}(q^{(f)}, t)$. Substituting this result and its derivatives into the expansion of the full Lagrangian L in Section 2.7 gives the reduced Lagrangian \bar{L} in the form

$$\bar{L} = \frac{1}{2} \sum_{k=1}^{D-C} \sum_{l=1}^{D-C} \bar{m}_{kl} \dot{q}_k^{(f)} \dot{q}_l^{(f)} + \sum_{k=1}^{D-C} \bar{n}_k \dot{q}_k^{(f)} + \bar{T}_0 - \bar{U} \quad (4.45)$$

where, for $k, l = 1, \dots, (D - C)$,

$$\bar{m}_{kl} = m_{kl} + \sum_{i=D-C+1}^D \sum_{j=D-C+1}^D s_{ki}^T m_{ij} s_{jl} + \sum_{j=D-C+1}^D m_{kj} s_{jl} + \sum_{i=D-C+1}^D s_{ki}^T m_{il} \quad (4.46)$$

where \mathbf{m} is the matrix proved positive definite in Theorem 4.1.1 and, with $k = (D - C + 1), \dots, D$ and $i = 1, \dots, (D - C)$,

$$s_{ki} = \frac{\partial q_k^{(b)}(q^{(f)}, t)}{\partial q_i^{(f)}} \quad (4.47)$$

Putting eqn (4.45) into eqn (4.44) gives

$$\left(\frac{\partial \bar{p}^{(f)}}{\partial \dot{q}^{(f)}} \right)_{kl} = \frac{\partial^2 \bar{L}(q^{(f)}, \dot{q}^{(f)}, t)}{\partial \dot{q}_k^{(f)} \partial \dot{q}_l^{(f)}} = \bar{m}_{kl} \quad (4.48)$$

To prove $\bar{\mathbf{m}} = (\partial \bar{p}^{(f)} / \partial \dot{q}^{(f)})$ positive definite, let $[x]$ be any arbitrary, real, non-null column vector of dimension $(D - C)$. Then define another column vector $[y]$ of dimension D as the compound matrix

$$[y] = \begin{pmatrix} [x] \\ s[x] \end{pmatrix} \quad (4.49)$$

It follows from the positive-definiteness of \mathbf{m} that

$$[x]^T \bar{\mathbf{m}} [x] = [y]^T \mathbf{m} [y] > 0 \quad (4.50)$$

which proves that $\bar{\mathbf{m}}$ is also positive definite. It follows from Lemma C.1.1 that matrix $\bar{\mathbf{m}} = (\partial \bar{p}^{(f)} / \partial \dot{q}^{(f)})$ is nonsingular. Hence that the inversion condition eqn (4.43) is satisfied. \square

4.6 Poisson Brackets

In Hamiltonian mechanics, all physical quantities are represented by phase-space functions like that in eqn (4.11). Assuming now that no constraints are present, the Hamilton equations, eqn (4.21), and the chain rule can be used to write the total time derivative of such a function f in a useful form

$$\begin{aligned} \dot{f} &= \frac{df}{dt} = \sum_{k=1}^D \left(\frac{\partial f(q, p, t)}{\partial q_k} \dot{q}_k + \frac{\partial f(q, p, t)}{\partial p_k} \dot{p}_k \right) + \frac{\partial f(q, p, t)}{\partial t} \\ &= \sum_{k=1}^D \left(\frac{\partial f(q, p, t)}{\partial q_k} \frac{\partial H(q, p, t)}{\partial p_k} - \frac{\partial f(q, p, t)}{\partial p_k} \frac{\partial H(q, p, t)}{\partial q_k} \right) + \frac{\partial f(q, p, t)}{\partial t} \end{aligned} \quad (4.51)$$

The sum in eqn (4.51) appears frequently enough to merit a special notation for it. It is called the *Poisson bracket* $[f, H]$ of the two phase-space functions $f(q, p, t)$ and

$H(q, p, t)$, so that eqn (4.51) becomes

$$\dot{f} = \frac{df}{dt} = [f, H] + \frac{\partial f(q, p, t)}{\partial t} \quad (4.52)$$

If $df/dt = 0$, then phase-space function f is called a *constant of the motion*. Equation (4.52) thus implies that a phase-space function that is not an explicit function of t will be a constant of the motion if and only if it has a vanishing Poisson bracket with the Hamiltonian.

The Poisson bracket $[f, g]$ can be defined more generally, for any two phase-space functions $f = f(q, p, t)$ and $g = g(q, p, t)$,

$$[f, g] = \sum_{k=1}^D \left(\frac{\partial f(q, p, t)}{\partial q_k} \frac{\partial g(q, p, t)}{\partial p_k} - \frac{\partial g(q, p, t)}{\partial q_k} \frac{\partial f(q, p, t)}{\partial p_k} \right) \quad (4.53)$$

Note that, since partial derivatives are functions of the same variable set as was the function differentiated, the $[f, g]$ is itself another phase-space function.

This definition implies some useful algebraic properties. First, by construction, the Poisson bracket is anti-symmetric in the exchange of the two functions so that, for any f and g ,

$$[g, f] = -[f, g] \quad \text{and hence} \quad [f, f] = 0 \quad (4.54)$$

Also, when $f(q, p, t)$, $g(q, p, t)$, and $h = h(q, p, t)$ are any phase-space functions, and α, β are numbers or otherwise not functions of q, p , the following identities can be proved,

$$[f, (\alpha g + \beta h)] = \alpha [f, g] + \beta [f, h] \quad (4.55)$$

$$[f, gh] = g[f, h] + [f, g]h \quad (4.56)$$

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0 \quad (4.57)$$

where, for example, $[f, [g, h]]$ denotes the Poisson bracket of function f with the function $[g, h]$ which was obtained by taking the Poisson bracket of g and h . The last of the three identities is called the *Jacobi identity*.

The algebra of Poisson brackets closely resembles that of the commutators of operators discussed in Section 7.1. This similarity is exploited in quantum mechanics. One path from classical to quantum mechanics is to write Poisson bracket relations and then replace the phase-space functions by quantum operators, as is discussed in Section 12.13.

Poisson brackets can be used to write the Hamilton equations in Poisson bracket form. Replacing $f(q, p, t)$ in eqn (4.51) by the single variables q_k, p_k, H in succession allows eqn (4.21) to be written in the form, for any $k = 1, \dots, D$,

$$\dot{q}_k = [q_k, H] \quad \dot{p}_k = [p_k, H] \quad \dot{H} = [H, H] + \frac{\partial H(q, p, t)}{\partial t} = \frac{\partial H(q, p, t)}{\partial t} \quad (4.58)$$

The following identities follow directly from the definition in eqn (4.53). If one puts $f(q, p, t)$ and $g(q, p, t)$ equal to any single canonical coordinate or momentum,

then, for any choices $k, l = 1, \dots, D$, it follows that

$$[q_k, q_l] = 0 \quad [q_k, p_l] = \delta_{kl} \quad [p_k, p_l] = 0 \quad (4.59)$$

where δ_{kl} is the Kroeneker delta function. These are called the *fundamental Poisson brackets*, and are analogous to similar operator equations in quantum mechanics.

Poisson brackets also play a crucial role in the definition of what are called Canonical Transformations of phase-space variables. But we will defer that discussion until the extended Lagrangian and Hamiltonian methods, with time as a coordinate, are introduced in Part II of the book.

4.7 The Schroedinger Equation

The Hamiltonian is an essential element in the derivation of the Schroedinger equation of quantum mechanics. We illustrate this transition from classical to quantum mechanics by using the example of a single particle of mass m moving in a potential $U(x, y, z, t)$. For such a system, the Lagrangian is

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z, t) \quad (4.60)$$

from which we derive the Hamiltonian

$$H = H(q, p, t) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + U(x, y, z, t) \quad (4.61)$$

The generalized coordinates here are just the Cartesian coordinates of the particle, $q_1 = x, q_2 = y, q_3 = z$.

The standard recipe for the transition to quantum mechanics is to make the substitutions

$$H \rightarrow i\hbar \frac{\partial}{\partial t} \quad p_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad p_y \rightarrow -i\hbar \frac{\partial}{\partial y} \quad p_z \rightarrow -i\hbar \frac{\partial}{\partial z} \quad (4.62)$$

in eqn (4.61), and then introduce a Schroedinger wave function $\psi(x, y, x, t)$ for the differential operators to operate on, leading to

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} \right) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi + \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial y} \right) \left(-i\hbar \frac{\partial}{\partial y} \right) \psi \\ &+ \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial z} \right) \left(-i\hbar \frac{\partial}{\partial z} \right) \psi + U\psi \end{aligned} \quad (4.63)$$

The products of operators are interpreted as repeated application, leading to second partial derivatives. For example,

$$\left(-i\hbar \frac{\partial}{\partial x} \right) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi = \left(-i\hbar \frac{\partial}{\partial x} \left(-i\hbar \frac{\partial \psi}{\partial x} \right) \right) = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \quad (4.64)$$

The result is the Schroedinger equation, the fundamental equation of nonrelativistic

quantum mechanics. It is usually written as

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{-\hbar^2}{2m} \nabla^2 \psi + U \psi \quad (4.65)$$

where the *Laplacian* operator ∇^2 is defined as

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla \cdot (\nabla \psi) \quad (4.66)$$

where ∇ is the gradient operator defined in eqn (A.66).

4.8 The Ehrenfest Theorem

The square of the absolute value of the Schrodinger wave function serves as a probability density in quantum theory. In one dimensional problems, for example, in the limit $dx \rightarrow 0$ the quantity $\mathcal{P}(x, t) dx$ is the probability that the particle will be found between x and $x + dx$, where $\mathcal{P}(x) = \psi^*(x, t) \psi(x, t)$. Instead of predicting the actual values of classical variables like position and momentum, quantum theory predicts a most likely value called the *expectation value*. The recipe for finding the expectation value is: (1) First one forms the classical phase-space function $f(x, y, z, p_x, p_y, p_z)$ representing the physical variable. (2) One then replaces the classical q, p values by quantum mechanical operators. In the position basis we are using here as an example, the operators representing positions x, y, z are just the coordinates themselves, but for the momenta the substitution in eqn (4.62) must be used. (3) The expectation value is then

$$\langle f \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^* f(x, y, z, -i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}) \psi dx dy dz \quad (4.67)$$

where we assume throughout that the Schrodinger wave function is normalized to give a probability of one that the particle will be found at some position,

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^* \psi dx dy dz \quad (4.68)$$

For example, the expectation of the z -component of angular momentum, $L_z = xp_y - yp_x$, is

$$\langle L_z \rangle = -i\hbar \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^* \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi dx dy dz \quad (4.69)$$

Quantum mechanics also predicts an RMS deviation from the expectation value. It is defined as

$$\Delta f = \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^* \left(f(x, y, z, -i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}) - \langle f \rangle \right)^2 \psi dx dy dz} \quad (4.70)$$

Quantum mechanics is what is called a *cover theory* for classical mechanics. This means that quantum mechanics is the more comprehensive theory and should predict

all of the classical results obtained in this book, in the limited domain, called the *classical limit*, in which classical mechanics is adequate. Roughly speaking, this classical limit is reached when one may (to some acceptable degree of approximation) ignore Δf and treat the expectation value $\langle f \rangle$ as if it were the actual value of a classical phase-space function $f(q, p)$. However, it is difficult to give a general prescription for this limit, and each case must be approached individually.

The Ehrenfest theorem shows that, with some limitations, the Hamilton equations of classical mechanics also hold in quantum mechanics.

Theorem 4.8.1: Ehrenfest Theorem

With a Hamiltonian of the general form given in eqn (4.61), the expectation values of position and momenta obey equations which resemble the classical Hamilton equations,

$$\frac{d}{dt} \langle x_i \rangle = \left\langle \frac{\partial H(x, y, z, p_x, p_y, p_z)}{\partial p_i} \right\rangle \quad \frac{d}{dt} \langle p_i \rangle = - \left\langle \frac{\partial H(x, y, z, p_x, p_y, p_z)}{\partial x_i} \right\rangle \quad (4.71)$$

where $i = 1, 2, 3$ and $x_1 = x, x_2 = y, p_1 = p_x$, etc. The expressions on the right, like $\langle \partial H(x, y, z, p_x, p_y, p_z) / \partial x \rangle$ for example, are obtained by first taking the partial derivative of the classical Hamiltonian, then making the substitutions from eqn (4.62), and finally placing the resulting expression into eqn (4.67) to obtain its expectation value.

Proof: In quantum texts, for example Chapter 6 of Shankar (1994), eqn (4.71) is proved to follow from the Schroedinger equation, eqn (4.65). This proof is general and is not restricted to the classical limit. \square

The Ehrenfest theorem does not allow us simply to replace the classical variables by their expectation values. For example, in general

$$\left\langle \frac{\partial H(x, y, z, p_x, p_y, p_z)}{\partial x} \right\rangle \neq \frac{\partial H(\langle x \rangle, \langle y \rangle, \langle z \rangle, \langle p_x \rangle, \langle p_y \rangle, \langle p_z \rangle)}{\partial \langle x \rangle} \quad (4.72)$$

and the classical limit still requires careful consideration.

4.9 Exercises

Exercise 4.1 This exercise is to emphasize the importance of writing the Hamiltonian in terms of the correct variable set q, p, t before the Hamilton equations are applied. It shows that partial derivatives depend not only on the variable differentiated with respect to (t here) but also on the list of variables to be held constant as the derivative is taken. Define

$$u(t) = a + \frac{1}{2}bt^2 \quad \text{and} \quad v(t) = \frac{1}{2}bt^2 + ct^5 \quad (4.73)$$

Also define

$$f(u, t) = u + \sin \omega t + ct^5 \quad \text{and} \quad f(v, t) = a + v + \sin \omega t \quad (4.74)$$

(a) By writing each out as a function of t , prove that the two compound functions are equal, $f(u, t) = f(v, t)$ for every value of t . (Note that we are following the physicist's convention

of using the same letter f for both compound functions, as discussed in Section D.5.)

(b) Calculate $\partial f(u, t)/\partial t$ and $\partial f(v, t)/\partial t$ and show, by considering each as a function of t , that they are *not* equal.

Exercise 4.2 Suppose that a one-dimensional system has Lagrangian

$$L(q, \dot{q}, t) = \frac{1}{2}m\dot{q}_1^2 \sin^2 \omega t + m\omega q_1 \dot{q}_1 \sin \omega t \cos \omega t + \frac{1}{2}m\omega^2 q_1^2 \cos^2 \omega t - mgq_1 \sin \omega t \quad (4.75)$$

(a) Find an expression for the canonical momentum p_1 and solve it for \dot{q}_1 .

(b) Find the generalized energy function $H(q, \dot{q}, t)$, and use the result of part (a) to write it in terms of the correct Hamiltonian variables to give the Hamiltonian $H(q, p, t)$.

(c) Write the two Hamilton equations. Verify that the one for \dot{q}_1 is consistent with your result from part (a).

(d) Use the Hamilton equation $\dot{H} = \partial H(q, p, t)/\partial t$ to test whether or not H is conserved.

Exercise 4.3 In part (e) of Exercise 3.1, you derived a reduced Lagrangian $\bar{L}(q^{(f)}, \dot{q}^{(f)}, t)$ for the plane double pendulum, using the free coordinates $q^{(f)} = \theta_1, \theta_2$.

(a) Find the generalized momenta conjugate to these free coordinates and invert them to solve for $\dot{\theta}_1$ and $\dot{\theta}_2$ as functions of the momenta.

(b) Write the reduced Hamiltonian $\bar{H}(q^{(f)}, p^{(f)}, t)$ for this problem.

Exercise 4.4 A system with two degrees of freedom has a Lagrangian

$$L(q, \dot{q}, t) = a\dot{q}_1^2 + 2b\dot{q}_1\dot{q}_2 + c\dot{q}_2^2 + f \quad (4.76)$$

where a, b, c, f are given functions of q_1, q_2, t .

(a) Find the two generalized momenta $p_k(q, \dot{q}, t)$ and, using Cramer's rule or otherwise, write expressions for the \dot{q}_k as functions of q, p, t .

(b) Write the generalized energy function $H(q, \dot{q}, t)$ and express it in terms of the proper phase-space coordinates q, p, t to form the Hamiltonian $H(q, p, t)$.

(c) Verify that the Hamilton equations for \dot{q}_k simply restate your result from part (a).

Exercise 4.5 Suppose that we have a Hamiltonian $H(q, p, t)$ and the usual Hamilton equations

$$\dot{q}_k = \frac{\partial H(q, p, t)}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H(q, p, t)}{\partial q_k} \quad \dot{H} = \frac{\partial H(q, p, t)}{\partial t} \quad (4.77)$$

We want to make a Legendre transformation from $H(q, p, t)$ back to $L(q, \dot{q}, t)$. Note that the variables being exchanged are $p \leftrightarrow \dot{q}$, and that this is the inverse of the Legendre transformation we used to get H in the first place.

(a) Write an expression for $L(q, p, t)$ in terms of $H(q, p, t)$ using the rules of the Legendre transformation as outlined in Section D.30.

(b) Assume that the first of eqn (4.77) can be inverted to give $p_k = p_k(q, \dot{q}, t)$ and show how this can be used to write L in terms of the correct Lagrangian variables $L = L(q, \dot{q}, t)$.

(c) Write the differential dL and use it to derive the three Lagrange equations

$$p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad \dot{p}_k = \frac{\partial L(q, \dot{q}, t)}{\partial q_k} \quad \dot{H} = -\frac{\partial L(q, \dot{q}, t)}{\partial t} \quad (4.78)$$

Exercise 4.6 We know from Exercise 2.3 that any solution of the Lagrange equations with Lagrangian $L(q, \dot{q}, t)$ is also a solution of the Lagrange equations with an equivalent Lagrangian $L'(q, \dot{q}, t)$ where

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{df(q, t)}{dt} = L(q, \dot{q}, t) + \sum_{k=1}^N \frac{\partial f(q, t)}{\partial q_k} \dot{q}_k + \frac{\partial f(q, t)}{\partial t} \quad (4.79)$$

(a) Let the generalized momenta for these two Lagrangians be denoted

$$p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \quad \text{and} \quad p'_k = \frac{\partial L'(q, \dot{q}, t)}{\partial \dot{q}_k} \quad (4.80)$$

Write an equation for p'_k as a function of p_k and the partial derivatives of f .

(b) Find the generalized energy function $H'(q, \dot{q}, t)$ corresponding to the Lagrangian L' . Write it in terms of $H(q, \dot{q}, t)$, the generalized energy function corresponding to Lagrangian L , and partial derivatives of f as needed.

(c) Now assume that the original generalized energy function $H(q, \dot{q}, t)$ can be converted to a Hamiltonian $H(q, p, t)$ in the usual way. Use that fact to write H' as a function of variables q, p, t .

(d) Solve your expression for p'_k in part (a) for $p_k = p_k(q, p', t)$, and use that solution to write H' in terms of its correct Hamiltonian variables q, p', t ,

$$H' = H'(q, p', t) = H'(q, p \rightarrow p(q, p', t), t) \quad (4.81)$$

(e) Assume that the Hamilton equations for the original Hamiltonian H hold, and prove that the Hamilton equations for H' are also true

$$\dot{q}_k = \frac{\partial H'(q, p', t)}{\partial p'_k} \quad \text{and} \quad \dot{p}'_k = -\frac{\partial H'(q, p', t)}{\partial q_k} \quad (4.82)$$

Exercise 4.7 Charged particles in an electromagnetic field were treated in Section 2.17.

(a) Show that the Hamiltonian derived from the generalized energy in eqn (2.105) is

$$H = \sum_{n=1}^N \left(\frac{(\underline{\mathbf{p}}_n - q_n^{(\text{ch})} \mathbf{A}(\mathbf{r}_n, t)/c) \cdot (\underline{\mathbf{p}}_n - q_n^{(\text{ch})} \mathbf{A}(\mathbf{r}_n, t)/c)}{2m_n} + q_n^{(\text{ch})} \Phi(\mathbf{r}_n, t) \right) \quad (4.83)$$

where $\underline{\mathbf{p}}_n$ is the canonical momentum defined in eqn (2.104).

(b) Show that the Hamilton equations may be written in vector form as

$$\mathbf{v}_n = \frac{\partial H}{\partial \underline{\mathbf{p}}_n} \quad \dot{\underline{\mathbf{p}}}_n = -\frac{\partial H}{\partial \mathbf{r}_n} \quad (4.84)$$

(c) Show that the first Hamilton equation simply restates eqn (2.104).

(d) Use the quantum substitution analogous to eqn (4.62) but with the quantum operators²⁵

²⁵Note that, when there is a difference, the quantum operators replace $\underline{\mathbf{p}}$ and not the particle momentum $\mathbf{p} = m\mathbf{v}$. See also the discussion in Section 12.13. Equation (4.86) correctly describes a nonrelativistic, spinless, charged particle in an external electromagnetic field. See, for example, page 387 of Shankar (1994).

replacing components of the *canonical* momentum $\underline{\mathbf{p}}$,

$$H \rightarrow i\hbar \frac{\partial}{\partial t} \quad \underline{p}_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad \underline{p}_y \rightarrow -i\hbar \frac{\partial}{\partial y} \quad \underline{p}_z \rightarrow -i\hbar \frac{\partial}{\partial z} \quad (4.85)$$

to show that the Schroedinger equation for a single particle of mass m and charge q moving in a given electromagnetic field is

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{(-i\hbar \nabla - q^{(\text{ch})} \mathbf{A}(\mathbf{r}, t)/c) \cdot (-i\hbar \nabla - q^{(\text{ch})} \mathbf{A}(\mathbf{r}, t)/c)}{2m} \Psi + q^{(\text{ch})} \Phi(\mathbf{r}, t) \Psi \quad (4.86)$$

Note that the ∇ in the first of the two $(-i\hbar \nabla - q^{(\text{ch})} \mathbf{A}(\mathbf{r}, t)/c)$ factors *does* operate on the $\mathbf{A}(\mathbf{r}, t)$ function in the second one, as well as on Ψ .

Exercise 4.8 In Exercise 2.9 you found the generalized energy functions for a mass on a rotating table in two different coordinate systems, one fixed and one rotating. You should use the generalized energy functions and canonical momenta from your previous work as the starting point of the present problem.

- (a) Find the Hamiltonians $H(q, p, t)$ and $H'(q', p', t)$ in these two systems.
 (b) Use the Hamilton equations,

$$\dot{H} = \frac{\partial H(q, p, t)}{\partial t} \quad \text{and} \quad \dot{H}' = \frac{\partial H'(q', p', t)}{\partial t} \quad (4.87)$$

to verify your earlier result that H is not conserved but H' is.

Exercise 4.9 Consider a system consisting of a single particle.

- (a) Using the phase-space variables $q, p = x, y, z, p_x, p_y, p_z$, prove that for any phase-space function $f(q, p)$,

$$[f(q, p), p_x] = \frac{\partial f}{\partial x} \quad [f(q, p), p_y] = \frac{\partial f}{\partial y} \quad [f(q, p), p_z] = \frac{\partial f}{\partial z} \quad (4.88)$$

- (b) The orbital angular momentum of a single mass is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Prove that

$$[L_z, x] = y \quad [L_z, y] = -x \quad [L_z, z] = 0 \quad (4.89)$$

and

$$[L_z, p_x] = p_y \quad [L_z, p_y] = -p_x \quad [L_z, p_z] = 0 \quad (4.90)$$

THE CALCULUS OF VARIATIONS

The calculus of variations is of enormous importance, not just in analytical mechanics, but in the whole of theoretical physics. The present chapter introduces it in the context of the finite-dimensional configuration spaces discussed in previous chapters. Mastery of this relatively simple form of the theory will provide the background required to study more advanced topics such as the variations of fields in the complex spaces of quantum field theory.

To understand what a variation is, imagine a curve drawn between two given points in a three-dimensional Cartesian space. Such a curve is often called a *path* between these points. Now imagine a line integral along that path, integrating some quantity of interest to us. For example, that quantity might be simply the increment of distance, so that the integral would give the total length of the path.

Now imagine several different paths between these same two end points. The integrals along these different paths would, in general, be different. The *calculus of variations* is concerned with the comparison of these line integrals along different paths. The difference between the integral along some chosen path and the integral of the same quantity along other paths is called the *variation* of that integral.

For example, if the integrated quantity is total length, we might want to find the shortest distance between the two points. Just as the minimum of an ordinary function happens at a point at which its first-order rate of change vanishes (vanishing first derivative), so the shortest path turns out to be the path whose length is, to first order, equal to the length of its near neighbors. It is called an *extremum path*.²⁶ The variation of the integral about that extremum path will thus vanish to first order.

The presentation of the calculus of variations in the present chapter uses what we call the *General Parametric Method*. In it, a path is specified parametrically, by letting each of its coordinates be a function of some monotonically varying, but initially unspecified, parameter β . This method contrasts with some other textbooks in which one of the coordinates is used as the parameter, and the other variables are made functions of it rather than of a general β . The two methods are compared in detail in Sections 5.14 and 5.15. The General Parametric approach used here has much to recommend it, and the reader is urged to adopt it.

²⁶In ordinary calculus, after finding a point where the first derivative vanishes, we must evaluate the second derivative to see if the point is a maximum, minimum, or point of inflection. A similar test would be required also in the calculus of variations. However, the first-order theory presented in this chapter is not capable of such a test, so we must accept the extremum determination and try to guess from context whether the extremum is indeed a maximum or minimum.

5.1 Paths in an N -Dimensional Space

We want a mathematical characterization of paths that can be generalized to spaces of more than three dimensions. In a three-dimensional Cartesian example we could imagine a path to be represented by a perspective drawing of it, but that will not be possible in spaces of higher dimension. So, even in the three-dimensional example, we will choose to represent a path by giving the three Cartesian coordinates as functions of a common parameter β , as $x = x(\beta)$, $y = y(\beta)$, and $z = z(\beta)$, and picturing it graphically as the three, separate graphs of these three functions.

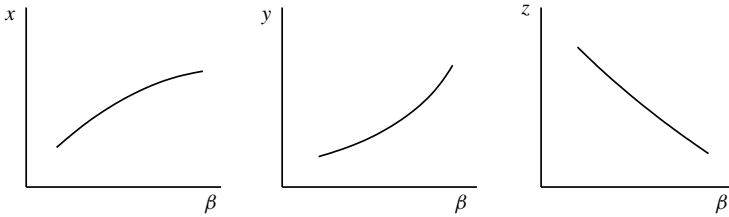


FIG. 5.1. A path in a three-dimensional space represented by three graphs.

This Cartesian example is now easily generalized to N -dimensional spaces. A path is characterized by making each of the coordinates x_k of such a space be a function of some parameter β that is unspecified except for the assumption that it increases monotonically as the represented point moves along the path. Thus, for $k = 1, \dots, N$,

$$x_k = x_k(\beta) \quad (5.1)$$

would be represented by N graphs, each one of a particular coordinate versus β . Together these N functions and their associated graphs represent a single path in the N -dimensional space, traced out as β advances.

The configuration spaces of mechanics described in Section 2.1, with x_k replaced by q_k , are one example of the kind of spaces in which the calculus of variations may be used. Chapter 6 is devoted to these applications. But the calculus of variations is more general than this particular application, and may also be used to solve problems that have nothing to do with mechanics.

Various paths will be given special names. First, imagine that some arbitrary path $x_k = x_k(\beta)$ has been chosen at the beginning of a calculation. This will be called the *chosen path* or the *unvaried path*. It will be considered to run between beginning and ending values of parameter β , denoted as β_1 and β_2 respectively, and to have the *end points* defined, for $k = 1, \dots, N$, by

$$x_k^{(1)} = x_k(\beta_1) \quad \text{and} \quad x_k^{(2)} = x_k(\beta_2) \quad (5.2)$$

After defining the chosen path, now consider another path, different from it but passing through the same end points. This will be called the *varied path*. A general way of writing such a varied path is to introduce a single scale parameter δa and a set

of N shape functions $\eta_k(\beta)$ so that the varied path $x_k(\beta, \delta a)$ is defined by its deviation from the chosen unvaried one. Thus, for $k = 1, \dots, N$,

$$x_k(\beta, \delta a) = x_k(\beta) + \delta a \eta_k(\beta) \quad (5.3)$$

The shape functions are finite, differentiable functions of β that are arbitrary except for the condition

$$\eta_k(\beta_1) = \eta_k(\beta_2) = 0 \quad (5.4)$$

which ensures that the varied and unvaried paths cross at the end points. Note that the scale parameter δa is a rough measure of the difference between the varied and unvaried paths, in that the two paths coalesce as δa goes to zero. This scale parameter is the same for all coordinates x_k and the same for the whole path; it is not a function of the index k or the parameter β .

5.2 Variations of Coordinates

The calculus of variations is based on comparisons of quantities evaluated on the varied path with the same quantities evaluated, at the same β value, on the unvaried path. The difference between such quantities evaluated on the two paths is called the *variation* of the quantity. For example, the coordinates themselves can be compared, leading to the variation $\delta x_k(\beta)$ defined, for all $k = 1, \dots, N$, by

$$\delta x_k(\beta) = x_k(\beta, \delta a) - x_k(\beta) = \delta a \eta_k(\beta) \quad (5.5)$$

Note that the comparison happens at fixed β , but that the variation $\delta x_k(\beta)$ is itself a function of β . For example, eqn (5.4) shows that it vanishes at the end points,

$$\delta x_k(\beta_1) = \delta x_k(\beta_2) = 0 \quad (5.6)$$

Another quantity to compare is the derivative of x_k with respect to β . On the unvaried path, this derivative is²⁷

$$\dot{x}_k(\beta) = \frac{dx_k(\beta)}{d\beta} \quad (5.7)$$

The derivative on the varied path is found by differentiating eqn (5.3), taking account of the fact that the scale parameter δa is not a function of β . It is

$$\frac{dx_k(\beta, \delta a)}{d\beta} = \frac{dx_k(\beta)}{d\beta} + \delta a \frac{d\eta_k}{d\beta} \quad (5.8)$$

or, in a simpler notation,

$$\dot{x}_k(\beta, \delta a) = \dot{x}_k(\beta) + \delta a \dot{\eta}_k(\beta) \quad (5.9)$$

²⁷Note that throughout this chapter we will denote total derivatives with respect to β by a dot placed above the differentiated quantity.

The variation $\delta\dot{x}_k(\beta)$ is found as the difference between eqns (5.8, 5.7),

$$\delta\dot{x}_k(\beta) = \frac{dx_k(\beta, \delta a)}{d\beta} - \frac{dx_k(\beta)}{d\beta} = \delta a \frac{d\eta_k(\beta)}{d\beta} = \delta a \dot{\eta}_k(\beta) \tag{5.10}$$

An important consequence of the definition eqn (5.10) is that

$$\delta\dot{x}_k(\beta) = \frac{d}{d\beta}\delta x_k(\beta) \tag{5.11}$$

Note that it follows from the definitions of this section that, for each value of β , the values on the varied path can be thought of as values on the chosen path plus a variation

$$x_k(\beta, \delta a) = x_k(\beta) + \delta x_k(\beta) \quad \text{and} \quad \dot{x}_k(\beta, \delta a) = \dot{x}_k(\beta) + \delta\dot{x}_k(\beta) \tag{5.12}$$

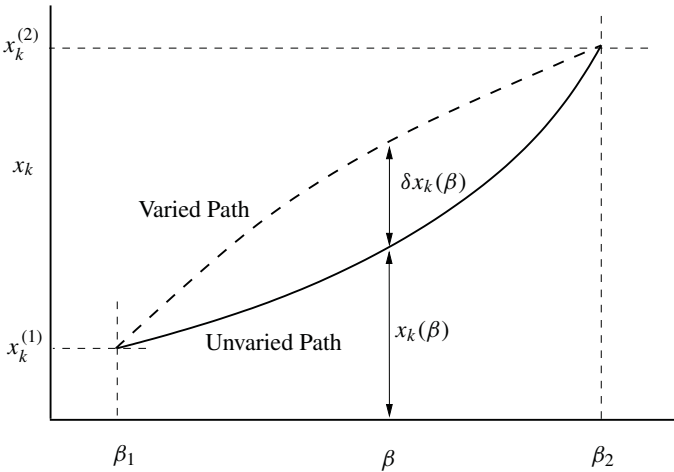


FIG. 5.2. Varied (dashed) and unvaried (solid) paths for a typical coordinate x_k .

5.3 Variations of Functions

The variation Δf of a function $f = f(x, \dot{x})$ is defined as the difference between its values on the varied and unvaried paths, again taken at the same value of β ,

$$\Delta f = f(x(\beta, \delta a), \dot{x}(\beta, \delta a)) - f(x(\beta), \dot{x}(\beta)) \tag{5.13}$$

The difference in eqn (5.13) may be expanded using a Taylor series, giving

$$\Delta f = \left(\frac{\partial f(x(\beta, h), \dot{x}(\beta, h))}{\partial h} \right)_{h=0} \delta a + \frac{1}{2} \left(\frac{\partial^2 f(x(\beta, h), \dot{x}(\beta, h))}{\partial h^2} \right)_{h=0} \delta a^2 + o(\delta a^2) \tag{5.14}$$

If Δf is calculated using only the first term on the right, the resulting quantity is called the *first-order variation* and is denoted δf . Thus

$$\delta f = \left(\frac{\partial f(x(\beta, h), \dot{x}(\beta, h))}{\partial h} \right)_{h=0} \delta a \quad (5.15)$$

Such first-order variations, which are the only ones used in the present text, are sufficient to determine extremum paths, but not to determine whether those paths are maxima, minima, or paths of inflection.

Note that the distinction between the variation Δf and the first-order variation δf was not needed in Section 5.2 when only the coordinates and their derivatives were being varied. The definition in eqn (5.3) contains δa only to the first power, and hence $\Delta x_k = \delta x_k$ with a similar result for the derivatives, $\Delta \dot{x}_k = \delta \dot{x}_k$.

One may expand the first partial derivative with respect to h in eqn (5.15) using eqns (5.3, 5.9) and the chain rule, giving

$$\begin{aligned} \delta f &= \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial x_k} \frac{\partial x_k(\beta, h)}{\partial h} + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \frac{\partial \dot{x}_k(\beta, h)}{\partial h} \right)_{h=0} \delta a \\ &= \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial x_k} \eta_k(\beta) \delta a + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \dot{\eta}_k(\beta) \delta a \right) \\ &= \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial x_k} \delta x_k(\beta) + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \delta \dot{x}_k(\beta) \right) \end{aligned} \quad (5.16)$$

where it is assumed that after the partials of f are taken, they are to be evaluated on the unvaried path with $h = 0$.

5.4 Variation of a Line Integral

The interesting applications of the calculus of variations involve variation of line integrals along paths. A line integral of a function $f(x, \dot{x})$ is taken along some line between end values β_1 and β_2 as

$$I = \int_{\beta_1}^{\beta_2} f(x(\beta), \dot{x}(\beta)) d\beta \quad (5.17)$$

When taken along the varied path, this integral I is a function of the scale parameter δa , and a functional of the chosen, unvaried path $x(\beta)$ and the shape function $\eta(\beta)$. It may be defined as

$$I(\delta a, [x], [\eta]) = \int_{\beta_1}^{\beta_2} f(x(\beta, \delta a), \dot{x}(\beta, \delta a)) d\beta \quad (5.18)$$

where the quantities in square brackets indicate functional dependence on the enclosed functions.²⁸ The line integral along the chosen or unvaried path is the same

²⁸A functional is a function of a function. The $[x]$ indicates that $I(\delta a, [x], [\eta])$ depends on the whole of the function $x(\beta)$ for all values $\beta_1 \leq \beta \leq \beta_2$.

integral, but with $\delta a = 0$,

$$I(0, [x], [\eta]) = \int_{\beta_1}^{\beta_2} f(x(\beta), \dot{x}(\beta)) d\beta \tag{5.19}$$

The variation of I is by definition the difference between these two integrals,

$$\begin{aligned} \Delta I &= I(\delta a, [x], [\eta]) - I(0, [x], [\eta]) \\ &= \int_{\beta_1}^{\beta_2} \{f(x(\beta, \delta a), \dot{x}(\beta, \delta a)) - f(x(\beta), \dot{x}(\beta))\} d\beta \\ &= \int_{\beta_1}^{\beta_2} \Delta f d\beta \end{aligned} \tag{5.20}$$

where Δf is the variation defined in eqn (5.13).

Since the scale parameter δa does not depend on β , inserting Δf from eqn (5.14) into eqn (5.20) gives

$$\begin{aligned} \Delta I &= \left\{ \int_{\beta_1}^{\beta_2} \left(\frac{\partial f(x(\beta, h), \dot{x}(\beta, h))}{\partial h} \right)_{h=0} d\beta \right\} \delta a \\ &\quad + \left\{ \frac{1}{2} \int_{\beta_1}^{\beta_2} \left(\frac{\partial^2 f(x(\beta, h), \dot{x}(\beta, h))}{\partial h^2} \right)_{h=0} d\beta \right\} \delta a^2 + o(\delta a^2) \end{aligned} \tag{5.21}$$

Using eqn (5.15), the first-order term in the variation ΔI may be written as

$$\delta I = \int_{\beta_1}^{\beta_2} \left(\frac{\partial f(x(\beta, h), \dot{x}(\beta, h))}{\partial h} \right)_{h=0} \delta a d\beta = \int_{\beta_1}^{\beta_2} \delta f d\beta \tag{5.22}$$

Substituting eqn (5.16) for δf gives

$$\delta I = \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial x_k} \delta x_k(\beta) + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \delta \dot{x}_k(\beta) \right) d\beta \tag{5.23}$$

It will be useful to modify eqn (5.23) slightly, using eqn (5.11) to do an integration by parts,

$$\begin{aligned} \delta I &= \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial x_k} \delta x_k(\beta) + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \frac{d}{d\beta} \delta x_k(\beta) \right) d\beta \\ &= \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left\{ \frac{\partial f(x, \dot{x})}{\partial x_k} \delta x_k(\beta) + \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \delta x_k(\beta) \right) - \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) \delta x_k(\beta) \right\} d\beta \end{aligned} \tag{5.24}$$

The perfect-differential term may be integrated immediately to give an integrand evaluated at the end points.

Thus, the first-order variation of the line integral reduces to the expression,

$$\delta I = \sum_{k=1}^N \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \delta x_k(\beta) \right) \Big|_{\beta_1}^{\beta_2} - \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \right\} \delta x_k(\beta) d\beta \quad (5.25)$$

5.5 Finding Extremum Paths

The typical use of the calculus of variations is to find paths that give extremum values to various line integrals. By definition, an extremum path is such that it, and all nearby paths that cross it at the end points, produce the same value for the line integral, to first order in δa . In other words, the chosen unvaried path is an extremum if the first-order variation vanishes, $\delta I = 0$, in analogy to the vanishing of the first derivative at the extremum points of functions in ordinary calculus.

The main theorem of the calculus of variations may now be stated.

Theorem 5.5.1: Euler–Lagrange Theorem

Assume a chosen unvaried path $x_k(\beta)$, varied paths $x_k(\beta, \delta a) = x_k(\beta) + \delta x_k(\beta)$ as defined in Sections 5.1 and 5.2, and a line integral

$$I = \int_{\beta_1}^{\beta_2} f(x, \dot{x}) d\beta \quad (5.26)$$

along those paths as specified in Section 5.4.

With the variations $\delta x_k(\beta)$ assumed arbitrary except for the condition that they vanish at the end points as stated in eqn (5.6), the unvaried path is an extremum path of this integral, with vanishing first-order variation $\delta I = 0$, if and only if the $x_k(\beta)$ of the unvaried path are a solution to the Euler–Lagrange differential equations²⁹

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} = 0 \quad (5.27)$$

for $k = 1, \dots, N$.

Proof: First, we assume eqn (5.27) and use eqn (5.25) to prove that $\delta I = 0$. Since eqn (5.27) holds for each value of k and β , the integrand of the second term on the right in eqn (5.25) vanishes identically and so the integral is zero. The first term also vanishes due to the assumed vanishing of the variations at the end points. Thus $\delta I = 0$ regardless of the $\delta x_k(\beta)$ used, as was to be proved.

The proof that $\delta I = 0$ implies eqn (5.27) also uses eqn (5.25). Assuming that $\delta I = 0$, and that the variations vanish at the end points, the first term on the right of

²⁹These equations are conventionally called the Euler–Lagrange equations, presumably to distinguish them from the Lagrange equations of mechanics, which have virtually the same form. One of the first uses of extremum principles was Fermat's Principle (see Section 5.6), but Euler made the first clear statement of the calculus of variations as a general computational method.

eqn (5.25) is zero, giving

$$0 = \delta I = \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \right\} \delta x_k(\beta) d\beta \quad (5.28)$$

Since the $\delta x_k(\beta)$ are arbitrary, they can be set *nonzero* one at a time. Suppose, for definiteness, we set all of them to zero except $\delta x_5(\beta)$. Then the sum in eqn (5.28) collapses to just the $k = 5$ term,

$$0 = \delta I = \int_{\beta_1}^{\beta_2} \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_5} \right) - \frac{\partial f(x, \dot{x})}{\partial x_5} \right\} \delta x_5(\beta) d\beta = \int_{\beta_1}^{\beta_2} \Gamma_5(\beta) \delta x_5(\beta) d\beta \quad (5.29)$$

where $\Gamma_5(\beta)$ stands for the quantity in the curly brackets. Now, exploiting the arbitrariness of $\delta x_5(\beta) = \eta_5(\beta) \delta a$, choose some arbitrary value $\beta_1 < \beta_0 < \beta_2$ and define $\eta_5(\beta)$ to be a continuous and continuously differentiable function which is zero except in a small range of β values $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$, and non-negative within that range. For example, one may choose $\eta_5(\beta) = \exp\{-1/\gamma^2\}$ where $\gamma = \sqrt{\varepsilon^2/(\beta - \beta_0)^2 - 1}$. Then, using the mean value theorem of the integral calculus to collapse the integral, eqn (5.29) reduces to

$$0 = \Gamma_5(\beta_0 + \theta_\varepsilon \varepsilon) C \delta a \quad (5.30)$$

where θ_ε is some number in the range $-1 \leq \theta_\varepsilon \leq 1$, and $C > 0$ is the integral of η_5 over its nonzero range. This implies that, for any nonzero ε value, $\Gamma_5(\beta_0 + \theta_\varepsilon \varepsilon)$ is zero for some θ_ε . Since the function f is assumed to be continuously differentiable, the function $\Gamma_5(\beta)$ is continuous. Taking the limit as $\varepsilon \rightarrow 0$ then gives $\Gamma_5(\beta_0) = \lim_{\varepsilon \rightarrow 0} \Gamma_5(\beta_0 + \theta_\varepsilon \varepsilon) = 0$, and so

$$0 = \Gamma_5(\beta_0) = \left. \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_5} \right) - \frac{\partial f(x, \dot{x})}{\partial x_5} \right\} \right|_{\beta=\beta_0} \quad (5.31)$$

But, since $k = 5$ and β_0 were arbitrarily chosen, any values may be chosen instead and so eqn (5.27) must be true for any k and β values, as was to be proved. When β_0 is one of the end values β_1 or β_2 , eqn (5.31) follows from its validity for interior values and the assumed continuity of Γ_5 . \square

Note that not every chosen unvaried path is an extremum path. It is quite possible to choose some unvaried path and define a varied path based on it, only to find that $\delta I \neq 0$. But if we choose an unvaried path that satisfies eqn (5.27), then we can be sure that it is an extremum path with $\delta I = 0$. Such a path can always be found, since the Euler–Lagrange equations in eqn (5.27) are N differential equations in N unknowns and so in principle can be solved exactly.

5.6 Example of an Extremum Path Calculation

In optics, Fermat’s Principle says that light rays always travel on paths that make the phase transit time T an extremum.³⁰ Denoting the phase velocity of light by v ,

³⁰Calling this Fermat’s principle of least times, as is often done, is inaccurate. It is actually Fermat’s principle of extremum times. For example, *all* rays going from an object point to a focus point through a

the time required for a wave crest to traverse a distance ds is $dT = ds/v = n ds/c$ where n is the index of refraction, and c is the vacuum speed of light. The quantity $n ds = dO$ is an increment of what is often called the Optical Path Length O . Thus, since $dT = dO/c$ it follows that $T = O/c$ and Fermat's principle may be restated by saying that the ray paths make an extremum of the integral defining the optical path length,

$$O = \int_1^2 n(x, y, z) ds \quad (5.32)$$

To rewrite eqn (5.32) in a form that can be treated by the calculus of variations, let x, y, z be functions of a monotonic parameter β so that, denoting derivatives with respect to β by a dot as usual,

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} d\beta \quad (5.33)$$

The line integral to be made an extremum becomes

$$O = \int_{\beta_1}^{\beta_2} n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} d\beta \quad (5.34)$$

With $x_1 = x$, $x_2 = y$, and $x_3 = z$ and with

$$f(x, \dot{x}) = n(x, y, z) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad (5.35)$$

Theorem 5.5.1 says that O will be an extremum along an unvaried path that is a solution to the three Euler-Lagrange equations, eqn (5.27), for $k = 1, 2, 3$,

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}} \right) - \frac{\partial f(x, \dot{x})}{\partial x} = 0 \quad (5.36)$$

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{y}} \right) - \frac{\partial f(x, \dot{x})}{\partial y} = 0 \quad (5.37)$$

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{z}} \right) - \frac{\partial f(x, \dot{x})}{\partial z} = 0 \quad (5.38)$$

Inserting eqn (5.35) into these equations gives the three equations

$$\frac{d}{d\beta} \left(\frac{n(x, y, z) \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \right) - \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \frac{\partial n(x, y, z)}{\partial x} = 0 \quad (5.39)$$

$$\frac{d}{d\beta} \left(\frac{n(x, y, z) \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \right) - \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \frac{\partial n(x, y, z)}{\partial y} = 0 \quad (5.40)$$

$$\frac{d}{d\beta} \left(\frac{n(x, y, z) \dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \right) - \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \frac{\partial n(x, y, z)}{\partial z} = 0 \quad (5.41)$$

perfect lens will have the same phase transit times.

The three eqns (5.39 – 5.41) are equivalent to the single vector equation

$$\frac{d}{d\beta} \left(\frac{n(x, y, z)}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{d\mathbf{r}}{d\beta} \right) - \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \nabla n(x, y, z) = 0 \quad (5.42)$$

which can be used to determine the extremum path.

Throughout the development so far, we have taken care not to specify the parameter β . It can be any quantity that increases monotonically along the unvaried path. But now, *after* all partial derivatives are taken and the final form of the Euler–Lagrange equation, eqn (5.42), has been found, we are at liberty to make a choice of β that will make its solution easier. One choice that is particularly appropriate to this problem is to choose β equal to the arc length s measured along the unvaried path starting at point 1.³¹ Then $\beta = s$ implies that

$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{d\beta} = \frac{ds}{ds} = 1 \quad (5.43)$$

on the unvaried path, so that eqn (5.42) simplifies to

$$\frac{d}{ds} (n\hat{\mathbf{t}}) - \nabla n = 0 \quad (5.44)$$

where $\hat{\mathbf{t}} = d\mathbf{r}/ds$ is the tangent unit vector defined in Section A.12. By inspection of this equation we can see that the path will curve in the direction of increasing index of refraction, giving, for example, a rough explanation of the desert mirages that occur when surface heat makes n smaller at the surface.

As a byproduct of this example, we can also prove that the extremum distance between two points is a straight line. If we set $n(x, y, z) = 1$ in the above problem, then the optical path length becomes just the geometrical path length, or distance. It follows that the extremum of geometrical path length is gotten by setting $n = 1$ in eqn (5.44), giving simply

$$\frac{d\hat{\mathbf{t}}}{ds} = 0 \quad (5.45)$$

But, as can be seen by reference to Section A.12, a path whose unit tangent vector is a constant is a straight line.

This Fermat's Principle example shows the utility of the General Parametric Method which leaves the parameter β unspecified until after all partial derivatives have been taken and the Euler–Lagrange equations obtained. Upon examination of eqn (5.42), it appeared that the choice $\beta = s$ allowed it to be simplified, and recast as a relation among Serret–Frenet vectors. In some other problem, examination of the Euler–Lagrange equations might suggest a different choice for β . (See, for example, Section 5.8.) By retaining β as an unspecified monotonic parameter until the end of calculations, one obtains the maximum flexibility in problem solution.

³¹It is important that this s be measured along the *unvaried* path. The arc lengths along the varied paths would depend on $\delta\alpha$, which would violate the condition that variations compare quantities at the same β value.

5.7 Invariance and Homogeneity

In Section 5.6, the integrand of the variational integral was the arc length ds weighted by a scalar function $n(x, y, z)$. This integral was translated into general parametric form by writing $ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} d\beta$.

In undergraduate texts, variational integrals are often presented in a form such as

$$I = \int g \left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx} \right) dx \quad (5.46)$$

in which the integration variable is one of the coordinates, here x for example, rather than a general parameter β . These integrals can always be recast into general parametric form by writing $dx = \dot{x} d\beta$, where $\dot{x} = dx/d\beta$ in the notation being used in this chapter. Then writing $dy/dx = \dot{y}/\dot{x}$ and $dz/dx = \dot{z}/\dot{x}$ gives

$$I = \int g \left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx} \right) dx = \int g \left(x, y, z, \frac{\dot{y}}{\dot{x}}, \frac{\dot{z}}{\dot{x}} \right) \dot{x} d\beta = \int f(x, y, z, \dot{x}, \dot{y}, \dot{z}) d\beta \quad (5.47)$$

with

$$f(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \dot{x} g \left(x, y, z, \frac{\dot{y}}{\dot{x}}, \frac{\dot{z}}{\dot{x}} \right) \quad (5.48)$$

The general parametric method can then be applied with $f(x, y, z, \dot{x}, \dot{y}, \dot{z})$ as the integrand.

Both the function in eqn (5.35) and the f in eqn (5.48) are seen to be homogeneous of degree one in the set of derivatives $\dot{x}, \dot{y}, \dot{z}$.³² This homogeneity is an essential element of the general parametric method. The integral I is equal to some physical or geometrical quantity that is to be extremized. The parameter β is just a dummy integration variable with no physical or geometrical significance. Its replacement by some other monotonic parameter $\theta = \theta(\beta)$ must not change the value of the integral I . Thus

$$I = \int_{\beta_1}^{\beta_2} f(x, \dot{x}) d\beta = \int_{\theta_1}^{\theta_2} f(x, x') d\theta = \int_{\beta_1}^{\beta_2} f \left(x, \left(\dot{x} \frac{d\beta}{d\theta} \right) \right) \frac{d\theta}{d\beta} d\beta \quad (5.49)$$

where $x'_k = dx_k/d\theta$ and hence $x'_k = \dot{x}_k(d\beta/d\theta)$, as has been indicated in the integrand of the last expression on the right. This equality holds for any values of the limits $x_k^{(1)} = x_k(\beta_1)$ and $x_k^{(2)} = x_k(\beta_2)$ and for any choice of path between them. It follows that the integrand $f(x, \dot{x})$ must satisfy the relation

$$f(x, \dot{x}) = f \left(x, \left(\dot{x} \frac{d\beta}{d\theta} \right) \right) \frac{d\theta}{d\beta} \quad (5.50)$$

The required invariance under a change of parameter thus implies the homogeneity of f , as stated in the following theorem.

³²Homogeneous functions are defined in Section D.31.

Theorem 5.7.1: Homogeneity

The value of the integral I is unchanged when parameter β is replaced by any other monotonic parameter $\theta = \theta(\beta)$ if and only if the integrand $f(x, \dot{x})$ is homogeneous of degree one in the set of derivatives $\dot{x} = \dot{x}_1, \dot{x}_2, \dots, \dot{x}_N$.

Proof: Equation (5.50) can be written as

$$f(x, (\dot{x}\lambda)) = \lambda f(x, \dot{x}) \quad (5.51)$$

where $\lambda = d\beta/d\theta$ is an arbitrary nonzero number. By Theorem D.31.1, this is the necessary and sufficient condition for $f(x, \dot{x})$ to be homogeneous of degree one in the set of derivatives \dot{x} . \square

It follows from the homogeneity of f that the Euler–Lagrange equations are also parameter independent.

Theorem 5.7.2: Invariance

If $\theta = \theta(\beta)$ is any monotonically varying parameter, then the $x_k(\beta)$ are a solution to the Euler–Lagrange equations with parameter β , as shown in eqn (5.27), if and only if the $x_k(\theta) = x_k(\beta(\theta))$ are a solution to the Euler–Lagrange equations with parameter θ ,

$$\frac{d}{d\theta} \left(\frac{\partial f(x, x')}{\partial x'_k} \right) - \frac{\partial f(x, x')}{\partial x_k} = 0 \quad (5.52)$$

for $k = 1, \dots, N$, where $x'_k = dx_k/d\theta$.

Proof: From Theorem 5.5.1, and a similar theorem with β replaced by θ , the Euler–Lagrange equations in β and θ hold if and only if

$$0 = \delta \int_{\beta_1}^{\beta_2} f(x, \dot{x}) d\beta \quad \text{and} \quad 0 = \delta \int_{\theta_2}^{\theta_1} f(x, x') d\theta \quad (5.53)$$

respectively. But, eqn (5.49) shows that the two integrals in eqn (5.53) are equal. Thus solution of the Euler–Lagrange equation in β implies the vanishing of both variations in eqn (5.53), which in turn implies the solution of the Euler–Lagrange equation in θ . The same argument holds with β and θ interchanged. \square

The homogeneity of the integrand $f(x, \dot{x})$ also has the consequence that the N Euler–Lagrange equations are redundant; only $(N - 1)$ of them are independent.

Theorem 5.7.3: Redundancy

The Euler–Lagrange equations are redundant. If some set of functions $x(\beta)$ satisfies the Euler–Lagrange equations in eqn (5.27) for $k = 0, 1, 2, \dots, (l - 1), (l + 1), \dots, N$, then the Euler–Lagrange equation for index l is also satisfied, except possibly at points where $\dot{x}_l = 0$.

Proof: From the Euler condition, Theorem D.31.1, the homogeneity of $f(x, \dot{x})$ proved in Theorem 5.7.1 implies that

$$0 = \sum_{k=1}^N \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \dot{x}_k - f(x, \dot{x}) \quad (5.54)$$

Differentiating this expression with respect to β and using the chain rule gives

$$\begin{aligned} 0 &= \sum_{k=1}^N \left\{ \dot{x}_k \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) + \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \ddot{x}_k - \frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \ddot{x}_k - \frac{\partial f(x, \dot{x})}{\partial x_k} \dot{x}_k \right\} \\ &= \sum_{k=1}^N \dot{x}_k \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \right\} \end{aligned} \quad (5.55)$$

Thus

$$\dot{x}_l \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_l} \right) - \frac{\partial f(x, \dot{x})}{\partial x_l} \right\} = - \sum_{k \neq l}^N \dot{x}_k \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \right\} \quad (5.56)$$

from which the theorem follows. \square

5.8 The Brachistochrone Problem

The general parametric method is particularly valuable when a proposed solution to the Euler–Lagrange equations is written in terms of some parameter that is not itself one of the variables of the problem. Then we can simplify the calculations by setting β equal to that parameter.

For example, the solution to the brachistochrone problem is known to be a cycloid, the locus of a point on the circumference of a circle that rolls without slipping on a line, usually written as

$$x = a(\theta - \sin \theta) \quad y = a(1 - \cos \theta) \quad z = 0 \quad (5.57)$$

where θ is the angle through which the circle has rolled. The Euler–Lagrange equations for this problem will be obtained below as usual, with β not yet specified. Then, after all partial derivatives have been taken, β can be set equal to θ to test whether or not eqn (5.57) is a solution. And, due to the redundancy noted in Theorem 5.7.3, only the two simplest of the three Euler–Lagrange equations will need to be tested.

The brachistochrone problem seeks the shape of a frictionless wire stretching between $(0, 0, 0)$ and $(x^{(2)}, y^{(2)}, 0)$ such that a bead of mass m sliding on the wire in a uniform gravitational field $\mathbf{g} = g\hat{\mathbf{e}}_2$ moves from the origin to the final point in minimum time T . By methods similar to those used in Section 5.6, and using energy

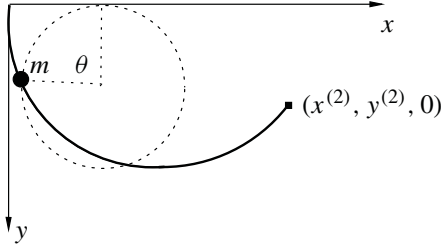


FIG. 5.3. Mass m slides on wire from origin to point $(x^{(2)}, y^{(2)}, 0)$ in minimum time.

conservation to get the speed of the bead v , the problem reduces to finding the extremum of the integral

$$I = T\sqrt{2g} = \sqrt{2g} \int \frac{ds}{v} = \int_{\beta_1}^{\beta_2} \sqrt{\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{y}} d\beta \tag{5.58}$$

With $x_1 = x, x_2 = y, x_3 = z$, and

$$f(x, \dot{x}) = \sqrt{\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{y}} \tag{5.59}$$

the Euler-Lagrange equations, eqn (5.27) for $k = 1, 2, 3$,

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} = 0 \tag{5.60}$$

reduce to the three equations

$$\frac{\dot{x}}{\sqrt{y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}} = C_1 \tag{5.61}$$

$$\frac{d}{d\beta} \left(\frac{\dot{y}}{\sqrt{y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}} \right) + \frac{1}{2} \sqrt{\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{y^3}} = 0 \quad \frac{\dot{z}}{\sqrt{y(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}} = C_2 \tag{5.62}$$

where C_1 and C_2 are integration constants.

The z -equation can be dealt with at once. Since the square root denominator is real and positive for the whole of the path, the second of eqn (5.62) implies that \dot{z} can never change sign. Thus, since β is monotonic, the function $z = z(\beta)$ can pass through both the initial and final points (both with $z = 0$) only if $z = 0$ for the whole path and $C_2 = 0$.

The other simple equation, eqn (5.61), can now be tested. Use the flexibility of the general parametric method to choose $\beta = \theta$, where θ is the cycloid parameter in the

proposed solution eqn (5.57). Then, with $\dot{x} = dx/d\theta$, etc., the left side of eqn (5.61) reduces to $1/\sqrt{2a}$ which is indeed a constant, and can be used to determine C_1 .

Since $\dot{y} = dy/d\theta \neq 0$ except at the isolated point $\theta = \pi$, Theorem 5.7.3 shows that the more complicated equation, the first of eqn (5.62), does not need to be tested. It is satisfied automatically due to the redundancy of the Euler–Lagrange equations. Thus eqn (5.57) does define the extremum path when the radius a is adjusted so that the cycloid curve passes through the final point $(x^{(2)}, y^{(2)}, 0)$.

5.9 Calculus of Variations with Constraints

Suppose now that we want to find the path that makes the integral in eqn (5.17) an extremum, but now subject to C holonomic constraints. These constraints are expressed by writing a functionally independent set of C functions of x and then requiring that the coordinates x_k for $k = 1, \dots, N$ and at each β value be such as to make these functions identically zero. Thus, for $a = 1, \dots, C$,

$$0 = G_a(x) \quad (5.63)$$

Using the definitions of unvaried path, varied path, and variation developed in Sections 5.2 and 5.3, these constraints are assumed to hold both on the unvaried path $0 = G_a(x(\beta))$, and on the varied path $0 = G_a(x(\beta, \delta a))$. It follows that $\Delta G_a = G_a(x(\beta, \delta a)) - G_a(x(\beta))$ is zero. Since the scale parameter δa is an arbitrary continuous parameter, it follows that the first-order variations δG_a are zero also. Thus, for $a = 1, \dots, C$,

$$\delta G_a = 0 \quad (5.64)$$

Theorem 5.9.1: Euler–Lagrange with Constraints

The integral

$$I = \int_{\beta_1}^{\beta_2} f(x, \dot{x}) d\beta \quad (5.65)$$

will be an extremum, with $\delta I = 0$ for variations that vanish at the end points but are otherwise arbitrary except for the constraints in eqn (5.63), if and only if there exist C functions λ_a such that, for $k = 1, \dots, N$,

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(x)}{\partial x_k} \quad (5.66)$$

Together, eqns (5.63, 5.66) constitute $N + C$ equations in the $N + C$ unknowns $x_0, \dots, x_N, \lambda_1, \dots, \lambda_C$ and so can be solved to find the extremum path. The functions λ_a are called³³ Lagrange multipliers.

³³In Chapter 2, the similarly denoted values λ_a were related to the forces of constraint. But the theory in the present chapter is more general. The Lagrange multipliers appear also in problems having nothing to do with forces or with the Lagrange equations of mechanics.

Proof: Using the definitions in Section 5.3 for variation of a function, the condition $\delta G_a = 0$ in eqn (5.64) may be expressed as a set of linear equations to be satisfied by the δx_k ,

$$0 = \delta G_a = \sum_{k=1}^N g_{ak} \delta x_k \tag{5.67}$$

for $a = 1, \dots, C$, where the $C \times N$ matrix \mathbf{g} is defined by

$$g_{ak} = \frac{\partial G_a(x)}{\partial x_k} \tag{5.68}$$

The condition for the functional independence of the constraints is that matrix \mathbf{g} must be of rank C and hence have a C -rowed critical minor. As was done in the proof of Theorem 3.4.1, we may reorder the coordinate indices so that this critical minor is formed from the C rows and the last C columns of \mathbf{g} . Then, the $C \times C$ matrix $\mathbf{g}^{(b)}$ defined by

$$g_{aj}^{(b)} = g_{a(N-C+j)} \tag{5.69}$$

will be nonsingular and have an inverse $\mathbf{g}^{(b)-1}$, since its determinant is the critical minor and hence is nonsingular by definition. Thus eqn (5.67) may be written as

$$0 = \delta G_a = \sum_{i=1}^{N-C} g_{ai} \delta x_i^{(f)} + \sum_{j=1}^C g_{aj}^{(b)} \delta x_{(N-C+j)}^{(b)} \tag{5.70}$$

which breaks the expression into two sums, first over what will be called the free coordinates, $x^{(f)} = x_1, \dots, x_{(N-C)}$ and then over the bound coordinates $x^{(b)} = x_{(N-C+1)}, \dots, x_N$. Then eqn (5.70) can be solved for the variations of the bound coordinates in terms of the variations of the free ones,

$$\delta x_{(N-C+j)}^{(b)} = - \sum_{a=1}^C \sum_{i=1}^{(N-C)} g_{ja}^{(b)-1} g_{ai} \delta x_i^{(f)} \tag{5.71}$$

Now to the main part of the proof. First we prove that, with the constraints, the extremum condition $\delta I = 0$ implies eqn (5.66). The δI here is the same as that derived earlier and given in eqn (5.25). Using the assumed vanishing of the variation at the end points to eliminate the integrated term, the assumed condition $\delta I = 0$ becomes

$$0 = \delta I = \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \left\{ \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \right\} \delta x_k d\beta = \int_{\beta_1}^{\beta_2} \sum_{k=1}^N \Gamma_k \delta x_k d\beta \tag{5.72}$$

where the notational definition

$$\Gamma_k = \frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}_k} \right) - \frac{\partial f(x, \dot{x})}{\partial x_k} \tag{5.73}$$

has been introduced. If the variations δx_k were all arbitrary and independent, as was assumed in Section 5.5, then eqn (5.72) would have the immediate consequence that

$\Gamma_k = 0$ for all k . But eqn (5.71) shows that the bound variations are not independent. Writing eqn (5.72) with separate sums over free and bound variables, substituting eqn (5.71) to eliminate the dependent variations, and reordering some finite sums, gives

$$\begin{aligned} 0 = \delta I &= \int_{\beta_1}^{\beta_2} \sum_{i=1}^{(N-C)} \Gamma_i \delta x_i^{(f)} d\beta + \int_{\beta_1}^{\beta_2} \sum_{j=1}^C \Gamma_{(N-C+j)} \delta x_{(N-C+j)}^{(b)} d\beta \\ &= \int_{\beta_1}^{\beta_2} \sum_{i=1}^{(N-C)} \left(\Gamma_i - \sum_{a=1}^C \sum_{j=1}^C \Gamma_{(N-C+j)} g_{ja}^{(b)-1} g_{ai} \right) \delta x_i^{(f)} d\beta \end{aligned} \quad (5.74)$$

With the definition

$$\lambda_a = \sum_{j=1}^C \Gamma_{(N-C+j)} g_{ja}^{(b)-1} \quad (5.75)$$

eqn (5.74) becomes

$$0 = \delta I = \int_{\beta_1}^{\beta_2} \sum_{i=1}^{(N-C)} \left(\Gamma_i - \sum_{a=1}^C \lambda_a g_{ai} \right) \delta x_i^{(f)} d\beta \quad (5.76)$$

But the variations of the free variables $\delta x_i^{(f)}$ for $i = 1, \dots, (N - C)$ are independent. The solution in eqn (5.71) satisfies the constraint equation, eqn (5.67), regardless of the choices of the $\delta x_i^{(f)}$. Thus an argument similar to that in Section 5.5, with $\delta x_i^{(f)}$ set nonzero one at a time in eqn (5.76), shows that $\delta I = 0$ implies

$$\Gamma_i - \sum_{a=1}^C \lambda_a g_{ai} = 0 \quad (5.77)$$

for all $i = 1, \dots, (N - C)$ and all values of β , which establishes eqn (5.66) for the free variables.

To see that the eqn (5.66) also hold for the bound variables, write an expression like eqn (5.77), but for the bound indices, and substitute eqn (5.75) for λ_a into it. Thus, for all $j = 1, \dots, C$,

$$\begin{aligned} \Gamma_{(N-C+j)} - \sum_{a=1}^C \lambda_a g_{a(N-C+j)} &= \Gamma_{(N-C+j)} - \sum_{a=1}^C \lambda_a g_{aj}^{(b)} \\ &= \Gamma_{(N-C+j)} - \sum_{a=1}^C \sum_{l=1}^C \Gamma_{(N-C+l)} g_{la}^{(b)-1} g_{aj}^{(b)} \\ &= \Gamma_{(N-C+j)} - \sum_{l=1}^C \Gamma_{(N-C+l)} \delta_{lj} = 0 \end{aligned} \quad (5.78)$$

Thus eqn (5.66) holds for all values $k = 1, \dots, N$, as was to be proved.

To prove the converse, that eqn (5.66) implies that $\delta I = 0$, note that after the constrained variations have been eliminated, the variation δI is equal to the right side of eqn (5.76). But eqn (5.66) implies that the integrand in this right side vanishes identically. Thus $\delta I = 0$, as was to be proved. \square

5.10 An Example with Constraints

Suppose that we want to find the extremum path between two points on the surface of a sphere. Such a path is called a *geodesic*. Using coordinates $x_1 = x, x_2 = y, \text{ and } x_3 = z$, the integral to be extremized is

$$S = \int_1^2 ds = \int_{\beta_1}^{\beta_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} d\beta \tag{5.79}$$

and the constraint is

$$0 = G_1(x) = \sqrt{x^2 + y^2 + z^2} - a \tag{5.80}$$

Using eqn (5.66) with

$$f(x, \dot{x}) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{5.81}$$

the constrained Euler–Lagrange equations for $k = 1, 2, 3$ are

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{x}} \right) - \frac{\partial f(x, \dot{x})}{\partial x} = \lambda_1 \frac{\partial G_1(x)}{\partial x} \tag{5.82}$$

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{y}} \right) - \frac{\partial f(x, \dot{x})}{\partial y} = \lambda_1 \frac{\partial G_1(x)}{\partial y} \tag{5.83}$$

$$\frac{d}{d\beta} \left(\frac{\partial f(x, \dot{x})}{\partial \dot{z}} \right) - \frac{\partial f(x, \dot{x})}{\partial z} = \lambda_1 \frac{\partial G_1(x)}{\partial z} \tag{5.84}$$

The three equations obtained by inserting eqns (5.80, 5.81) into these equations can be combined into a single vector equation

$$\frac{d}{d\beta} \left(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{d\mathbf{r}}{d\beta} \right) = \lambda_1 \frac{\mathbf{r}}{a} \tag{5.85}$$

where the constraint was used after the partials were taken to replace $\sqrt{x^2 + y^2 + z^2}$ by a .

If, as we did in Section 5.6, we now choose β to be the arc-length s measured along the unvaried path starting at point 1, eqn (5.85) can be simplified further to give

$$\frac{d\hat{\mathbf{t}}}{ds} = \lambda_1 \frac{\mathbf{r}}{a} \tag{5.86}$$

where $\hat{\mathbf{t}} = d\mathbf{r}/ds$. Using Serret–Frenet methods from Section A.12, along with eqn (5.80) in the vector form $\sqrt{\mathbf{r} \cdot \mathbf{r}} = a$, eqn (5.86) can be used to prove that the geodesic is a great circle, the intersection of the spherical surface with a plane passing through the center of the sphere.

5.11 Reduction of Degrees of Freedom

The method of Lagrange multipliers treated in Sections 5.9 and 5.10 has the advantage that it treats all the coordinates symmetrically, which avoids upsetting the natural symmetry of the problem. For example, this allowed the three Euler–Lagrange equations to be combined neatly into one vector expression, eqn (5.86).

However, in some problems the easiest method is simply to eliminate the constrained variables at the outset. Assume that the coordinates have been relabeled as was done in the proof in Section 5.9, with free coordinates $x^{(f)} = x_1, \dots, x_{(N-C)}$ and bound coordinates $x^{(b)} = x_{(N-C+1)}, \dots, x_N$. Then, by construction, $|\mathbf{g}^{(b)}| \neq 0$ where $\mathbf{g}^{(b)}$ is the matrix defined in eqn (5.69). But, from Theorem D.26.1, this is the necessary and sufficient condition for the constraint equations

$$0 = G_a(x) \quad (5.87)$$

for $a = 1, \dots, C$, to be solved for the bound variables in terms of the free ones, with the result for all $j = 1, \dots, C$,

$$x_{(N-C+j)}^{(b)} = x_{(N-C+j)}^{(b)}(x_1^{(f)}, \dots, x_{(N-C)}^{(f)}) = x_{(N-C+j)}^{(b)}(x^{(f)}) \quad (5.88)$$

These equations can then be differentiated using the chain rule to obtain

$$\dot{x}_{(N-C+j)}^{(b)} = \frac{dx_{(N-C+j)}^{(b)}}{d\beta} = \dot{x}_{(N-C+j)}^{(b)}(x^{(f)}, \dot{x}^{(f)}) \quad (5.89)$$

These expressions for the bound variables and their derivatives can then be substituted into the integral in eqn (5.65) to eliminate the bound variables, giving

$$I = \int_{\beta_1}^{\beta_2} \bar{f}(x^{(f)}, \dot{x}^{(f)}) d\beta \quad (5.90)$$

where

$$\bar{f}(x^{(f)}, \dot{x}^{(f)}) = f(x^{(f)}, x^{(b)}(x^{(f)}), \dot{x}^{(f)}, \dot{x}^{(b)}(x^{(f)}, \dot{x}^{(f)})) \quad (5.91)$$

is obtained by writing f with the free and bound variables listed separately as $f = f(x^{(f)}, x^{(b)}, \dot{x}^{(f)}, \dot{x}^{(b)})$ and then substituting eqns (5.88, 5.89) for the bound ones.

Now, eqn (5.90) can be taken as the start of a new problem with no constraints, which can be solved by the methods of Section 5.5.³⁴ Thus, the extremum condition is just eqn (5.27) with f replaced by \bar{f} and the number of variables reduced from N

³⁴The redundancy of the Euler–Lagrange equations proved in Theorem 5.7.3 will still apply to this new problem. If constraints have reduced the number of Euler–Lagrange equations from N to $N - C$, then under the conditions of that Theorem, satisfaction of $N - C - 1$ of them will imply satisfaction of the remaining one.

to $(N - C)$. Thus, for $k = 1, \dots, (N - C)$,

$$\frac{d}{d\beta} \left(\frac{\bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \dot{x}_k} \right) - \frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial x_k} = 0 \tag{5.92}$$

Whether this reduction method is superior to the Lagrange multiplier method of Section 5.9 depends somewhat on the choice of the original coordinates before the reduction is done. If original coordinates are chosen that reflect the symmetry of the constraints, the reduction method is often quite simple. In the next two sections we give two examples of the reduction method, the first with a nonoptimal choice of original coordinates, and the second with a better choice.

The use of holonomic constraints to reduce the number of dimensions of an extremum problem is quite straightforward. We have simply solved for the bound coordinates and eliminated them from the integral whose extremum path is to be found. This simplicity contrasts to the similar problem in Section 3.8, where we had the additional difficulty of accounting for the forces of constraint.

5.12 Example of a Reduction

Consider again the problem of finding the geodesics on a sphere of radius a , using the same Cartesian coordinates as in Section 5.10. Then, restricting our attention to paths entirely on the upper hemisphere, the constraint equation

$$0 = G_1(x) = \sqrt{x^2 + y^2 + z^2} - a \tag{5.93}$$

can be solved for the bound variable $x_3^{(b)} = z$ in terms of the free ones $x_1^{(f)} = x$ and $x_2^{(f)} = y$,

$$z = \sqrt{a^2 - x^2 - y^2} \tag{5.94}$$

Substituting this equation and its derivative into $f(x, \dot{x}) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ from eqn (5.81) gives

$$\bar{f}(x^{(f)}, \dot{x}^{(f)}) = \left(\dot{x}^2 + \dot{y}^2 + \frac{(x\dot{x} + y\dot{y})^2}{a^2 - x^2 - y^2} \right)^{1/2} \tag{5.95}$$

from which we obtain the two reduced Euler-Lagrange equations for the free variables with $k = 1, 2$

$$\frac{d}{d\beta} \left(\frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \dot{x}} \right) - \frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial x} = 0 \tag{5.96}$$

$$\frac{d}{d\beta} \left(\frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \dot{y}} \right) - \frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial y} = 0 \tag{5.97}$$

which may be solved for the extremum path.

5.13 Example of a Better Reduction

In Section 5.12, we used the constraint to eliminate the z variable. It is often much simpler to take the preliminary step of choosing a set of coordinates appropriate to the symmetries of the constraints, and *then* eliminate one of the variables.

Let us return again to the problem of the extremum path on the surface of a sphere, but now using the coordinates $x_1^{(f)} = \theta$, $x_2^{(f)} = \phi$, and $x_3^{(b)} = r$ where r, θ, ϕ are spherical polar coordinates. These coordinates are more appropriate for the spherical constraint. Then

$$S = \int_1^2 ds = \int_{\beta_1}^{\beta_2} \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2} d\beta \quad (5.98)$$

Due to the clever choice of coordinates, the constraint is reduced to a function of one variable only. Moreover, it can be solved to give the actual value of r , not just its expression in terms of the other variables,

$$0 = G_1(x) = r - a \quad (5.99)$$

Putting the constrained values $r = a$ and $\dot{r} = 0$ into

$$f(x, \dot{x}) = \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2} \quad (5.100)$$

gives the reduced function

$$\bar{f}(x^{(f)}, \dot{x}^{(f)}, \beta) = \sqrt{a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2} \quad (5.101)$$

and hence the two reduced Euler–Lagrange equations

$$\frac{d}{d\beta} \left(\frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \dot{\theta}} \right) - \frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \theta} = 0 \quad (5.102)$$

$$\frac{d}{d\beta} \left(\frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \dot{\phi}} \right) - \frac{\partial \bar{f}(x^{(f)}, \dot{x}^{(f)})}{\partial \phi} = 0 \quad (5.103)$$

which may be solved for the extremum path.

5.14 The Coordinate Parametric Method

In the *general parametric method* presented in this chapter, the integration parameter β in the line integrals is left unspecified until the end of the calculation, when it is selected to make the Euler–Lagrange equations as simple and transparent as possible. Some other textbooks, particularly undergraduate ones, instead choose a particular one of the variables as the integration parameter, and do so at the beginning of the calculation rather than at the end. Let us call this use of one of the coordinates as the integration parameter the *coordinate parametric method*.

Since the reader is likely to have studied the calculus of variations from those texts at some point, and since the traditional form of Hamilton's Principle presented in Chapter 6 closely resembles the coordinate parametric method, it will be useful to state and prove the Euler–Lagrange equations for that method.

Suppose that, after rearranging the coordinates if necessary, we denote the coordinate selected to be the integration parameter in the coordinate parametric method by x_1 . Derivatives of the other coordinates with respect to x_1 will be denoted by x'_k so that $x'_k = dx_k/dx_1$. The integral to be extremized in the coordinate parametric method may then be written as

$$I = \int_{x_1^{(1)}}^{x_1^{(2)}} g(x, x'_{[1]}) dx_1 \quad (5.104)$$

where x_2, \dots, x_N are the remaining variables and $x'_{[1]} = x'_2, \dots, x'_N$ are their derivatives with respect to x_1 . (Here x stands for all of the variables, x_1, \dots, x_N , and $x_{[1]}$ stands for all of the variables except x_1 .)

Theorem 5.14.1: Coordinate Euler–Lagrange Theorem

Assume that the variable x_1 chosen to be the integration parameter of the coordinate parametric method varies monotonically along the unvaried path. Then the first-order variation of eqn (5.104) vanishes, $\delta I = 0$, for arbitrary variations of the x_2, \dots, x_N variables with fixed end points (and no variation of x_1 itself), if and only if the unvaried path $x_k = x_k(x_1)$ is a solution to the Euler–Lagrange equations

$$\frac{d}{dx_1} \left(\frac{\partial g(x, x'_{[1]})}{\partial x'_k} \right) - \frac{\partial g(x, x'_{[1]})}{\partial x_k} = 0 \quad (5.105)$$

for $k = 2, \dots, N$.

Proof: The condition that x_1 must vary monotonically is essential. For if x_1 were to be constant along some region of the unvaried path while other coordinates varied, the derivatives $x'_k = dx_k/dx_1$ would be infinite and the method would fail. The present theorem can be proved by setting $\beta = x_1$ and $g = f$ in Theorem 5.5.1. The only difficulty is that β does not appear explicitly in $f(x, \dot{x})$, whereas x_1 does appear in $g(x, x'_{[1]})$. But a close inspection of the proof of Theorem 5.5.1 reveals that the presence of β in f would not invalidate the theorem. \square

The coordinate parametric method may also be used for problems with constraints.

Theorem 5.14.2: Coordinate Method with Constraints

Assume that the variable x_1 chosen to be the integration parameter of the coordinate parametric method varies monotonically along the unvaried path. Suppose that the variations are arbitrary except for the constraints, for $a = 1, \dots, C$,

$$G_a(x) = 0 \quad (5.106)$$

Then, again with fixed end points, the first-order variation of eqn (5.104) vanishes, $\delta I = 0$, if and only if the chosen unvaried path $x_k = x_k(x_1)$ is a solution to the Euler–Lagrange

equations

$$\frac{d}{dx_1} \left(\frac{\partial g(x, x'_{[1]})}{\partial x'_k} \right) - \frac{\partial g(x, x'_{[1]})}{\partial x_k} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(x)}{\partial x_k} \quad (5.107)$$

for $k = 2, \dots, N$.

Proof: As noted in the previous theorem, the condition that x_1 must vary monotonically is essential. To prove the present theorem, set $\beta = x_1$, $f = g$ in Theorem 5.9.1. The only difficulty is that x_1 appears explicitly in $g(x, x'_{[1]})$ and $G(x)$ but β does not appear explicitly in the $f(x, \dot{x})$ and $G(x)$ of Theorem 5.9.1. But examination will reveal that the proof of Theorem 5.9.1 remains valid even with an explicit dependence of these quantities on β . \square

One problem with the coordinate parametric method is that we have N coordinates x_1, \dots, x_N but only $N - 1$ Euler–Lagrange equations. Compared to the general parametric method in which there is an Euler–Lagrange equation for each coordinate, the Euler–Lagrange equation involving partial derivatives with respect to x_1 has gotten lost. This lost equation can be recovered by what is often called the *second form of the Euler–Lagrange equations*.

Theorem 5.14.3: Second Form of Euler–Lagrange Equations

The lost Euler–Lagrange equation in the coordinate parametric method may be recovered by defining the second form h as

$$h = \sum_{k=2}^N x'_k \frac{\partial g(x, x'_{[1]})}{\partial x'_k} - g(x, x'_{[1]}) \quad (5.108)$$

The lost Euler–Lagrange equation is then

$$\frac{dh}{dx_1} = -\frac{\partial g(x, x'_{[1]})}{\partial x_1} + \sum_{a=1}^C \lambda_a \frac{\partial G_a(x)}{\partial x_1} \quad (5.109)$$

In problems with no constraints, or in which the constrained variables have been eliminated, the last term on the right will be absent.

Proof: The proof closely parallels the proofs of the generalized energy theorems in Sections 2.15 and 3.13, with the substitutions $h \rightarrow H$, $x_1 \rightarrow t$, and $x_{[1]} \rightarrow q$, and will not be repeated here. \square

We note finally that any problem stated in the coordinate parametric form can be converted to general parametric form. Introducing the general monotonic parameter

β and writing $dx_1 = \dot{x}_1 d\beta$ and $x'_k = \dot{x}_k / \dot{x}_1$, the integral in eqn (5.104) may be written

$$I = \int_{x_1^{(1)}}^{x_1^{(2)}} g(x, x'_{[1]}) dx_1 = \int_{\beta_1}^{\beta_2} g\left(x, \left(\frac{\dot{x}_{[1]}}{\dot{x}_1}\right)\right) \dot{x}_1 d\beta = \int_{\beta_1}^{\beta_2} f(x, \dot{x}) d\beta \quad (5.110)$$

where

$$f(x, \dot{x}) = \dot{x}_1 g\left(x, \left(\frac{\dot{x}_{[1]}}{\dot{x}_1}\right)\right) \quad (5.111)$$

is the integrand for use in the general parametric method. There will now be N Euler–Lagrange equations, one for each coordinate. The lost equation that was recovered by the second form in Theorem 5.14.3 will be just another of the Euler–Lagrange equations of the general parametric method, and the second form will no longer be necessary.

5.15 Comparison of the Methods

Perhaps the clearest way to contrast the two methods is to re-do the example in Section 5.6, but now using the coordinate parametric method. Selecting x to be the integration parameter, the integral for the optical path length becomes

$$O = \int_{x^{(1)}}^{x^{(2)}} n(x, y, z) \sqrt{1 + y'^2 + z'^2} dx \quad (5.112)$$

where now $y' = dy/dx$ and $z' = dz/dx$. In terms of the definitions in Section 5.14, $x_1 = x$, $x_{[1]} = y, z$, and

$$g(x, x'_{[1]}) = g(x, y, z, y', z') = n(x, y, z) \sqrt{1 + y'^2 + z'^2} \quad (5.113)$$

Equation (5.105) of Theorem 5.14.1 gives the two Euler–Lagrange equations

$$\frac{d}{dx} \left(\frac{\partial g(x, y, z, y', z')}{\partial y'} \right) - \frac{\partial g(x, y, z, y', z')}{\partial y} = 0 \quad (5.114)$$

$$\frac{d}{dx} \left(\frac{\partial g(x, y, z, y', z')}{\partial z'} \right) - \frac{\partial g(x, y, z, y', z')}{\partial z} = 0 \quad (5.115)$$

Using eqn (5.113), these two equations reduce to

$$\frac{d}{dx} \left(\frac{y' n(x, y, z)}{\sqrt{1 + y'^2 + z'^2}} \right) - \sqrt{1 + y'^2 + z'^2} \frac{\partial n(x, y, z)}{\partial y} = 0 \quad (5.116)$$

$$\frac{d}{dx} \left(\frac{z' n(x, y, z)}{\sqrt{1 + y'^2 + z'^2}} \right) - \sqrt{1 + y'^2 + z'^2} \frac{\partial n(x, y, z)}{\partial z} = 0 \quad (5.117)$$

The lost third equation can be recovered by using the second form derived in Theorem 5.14.3. It is

$$h = y' \frac{\partial g(x, y, z, y', z')}{\partial y'} + z' \frac{\partial g(x, y, z, y', z')}{\partial z'} - g(x, y, z, y', z') = \frac{n(x, y, z)}{\sqrt{1 + y'^2 + z'^2}} \quad (5.118)$$

and eqn (5.109) becomes

$$\frac{d}{dx} \left(\frac{n(x, y, z)}{\sqrt{1 + y'^2 + z'^2}} \right) - \sqrt{1 + y'^2 + z'^2} \frac{\partial n(x, y, z)}{\partial x} = 0 \quad (5.119)$$

which is the lost equation, equivalent to eqn (5.39) of the general parametric method. Some extra work would now be required to cast these three equations into a simple vector form, as was done with the general parametric method in eqn (5.44) of Section 5.6

Many readers will have learned the calculus of variations using the coordinate parametric method. The present text is urging you to use the general parametric method instead. As you decide which method to adopt, in a particular problem or in your general perception of the calculus of variations, the following points should be considered:

1. As seen in the example in this section, the coordinate parametric method loses one Euler–Lagrange equation. Since Theorem 5.7.3 proves the Euler–Lagrange equations redundant, the solution path can still be found. But it is not always obvious at the start of a problem which one of the Euler–Lagrange equations one wishes to lose. It may turn out that the Euler–Lagrange equation lost as a result of your choice of the corresponding coordinate as integration parameter was actually the simplest one to solve.
2. The lost Euler–Lagrange equation in the coordinate method can always be recovered using the so-called second form of the Euler–Lagrange equations. But this requires more calculation. It seems preferable to use the general parametric method in which all of the available equations are present from the start. Rather than recovering information, it seems best not to lose it in the first place.
3. The coordinate method will fail when the coordinate chosen as the integration parameter happens to remain constant for a section of the path. But it is not always obvious in advance which coordinate can be trusted to vary monotonically. For example, if the problem were to find the geodesic on the surface of a paraboloid of revolution oriented with its symmetry axis along \hat{e}_3 , and if the cylindrical polar coordinate ϕ had been chosen as the integration parameter, it would be impossible to use the Euler–Lagrange equations of the coordinate parametric method to test whether the line $\phi = \text{constant}$ is a geodesic (which it is; see Exercise 5.3).
4. As seen in the example in this section, the premature choice of some variable like x as the integration parameter often destroys the symmetry of the Euler–Lagrange equations among the variables, and so makes it more difficult to put

the resulting differential equations into a simple form. The general parametric method, however, retains whatever symmetry the problem possesses.

5. Keeping β undetermined until after the partial derivatives are taken and the Euler–Lagrange equations are written out gives one maximum flexibility in solving the resulting differential equations. For example, we made different choices above: In Section 5.6, we chose $\beta = s$, the arc length along the solution path, and in Section 5.8, we chose $\beta = \theta$, the parameter of the cycloid solution.

5.16 Exercises

Exercise 5.1 Use the calculus of variations to solve the brachistochrone problem.

- (a) Verify the details of the example in Section 5.8, including the derivation of eqns (5.61, 5.62).
 (b) Carry out the demonstration that eqn (5.57) define the extremum path for the wire.

Exercise 5.2 Use the calculus of variations to find the extremum distance between two points on the surface of a sphere of radius a .

- (a) First do this problem entirely in Cartesian coordinates, with Lagrange multipliers as required. Show that the three eqns (5.82, 5.83, 5.84) really do lead to eqn (5.85). Show how the choice $\beta = s$ transforms eqn (5.85) into eqn (5.86). Use the Serret–Frenet methods of Section A.12 to prove that the extremum path is the intersection of the sphere’s surface with a plane passing through its center, i.e., a great circle.
 (b) Now do the problem again, but this time use spherical polar coordinates and a reduced \bar{f} as outlined in Section 5.13. Choosing the $\hat{\mathbf{e}}_3$ axis to pass through the initial point of your extremum line, show that this line is indeed a great circle.
 (c) Are the solutions to your differential equations necessarily the minimum distances between the two end points? Or could they be maximum distances?

Exercise 5.3 The general parametric method may be used to find geodesics on the surface of a paraboloid of revolution defined in terms of cylindrical coordinates ρ, ϕ, z by the equation $z = a\rho^2$.

- (a) Set up the integral to be minimized, using cylindrical polar coordinates.
 (b) Eliminate the z variable and write the reduced integrand $\bar{f}(\rho, \phi, \dot{\rho}, \dot{\phi})$ and the two associated Euler–Lagrange equations.
 (c) Consider the path: $\rho = \rho^{(1)}$ with ϕ varying, where $\rho^{(1)}$ is a constant. Show that this path satisfies the Euler–Lagrange equation for ϕ but not the one for ρ and hence is not a geodesic.
 (d) Explain how this result is consistent with the redundancy of the reduced Euler–Lagrange equations proved in Theorem 5.7.3. Why does satisfaction of the ϕ equation not imply satisfaction of the ρ equation?
 (e) Consider the path: $\phi = \phi^{(1)}$ with ρ varying, where $\phi^{(1)}$ is a constant. Show that this path is a geodesic.

Exercise 5.4

- (a) Using the result of Exercise 5.2, or otherwise, show that the shortest line of constant latitude on the surface of the Earth (horizontal line on a Mercator projection map) is generally *not* the shortest path between its two end points.
 (b) What line would be the exception to this rule?

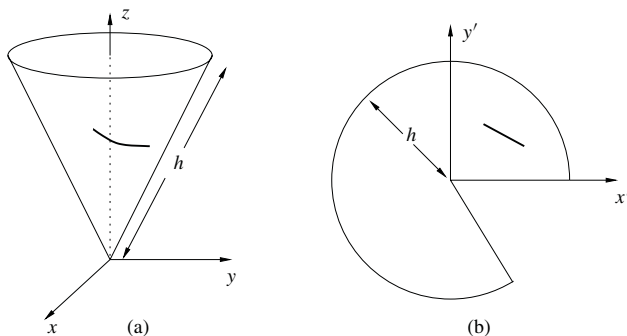


FIG. 5.4. Illustration for Exercise 5.5. The cone in (a) is cut along the x - z plane and flattened out as shown in (b). A straight line drawn on the flattened surface becomes a curve when the cone is reassembled.

Exercise 5.5 An inverted right-circular cone of half-angle α is placed with its apex at the origin of coordinates and its symmetry axis along \hat{e}_3 .

(a) Use the calculus of variations to find the two differential equations describing the extremum path between two general points on the surface of this cone. [Note: For example, you might use spherical polar coordinates with the constraint $\theta = \alpha$.]

(b) Suppose the cone to be cut along a line defined by the surface of the cone and the x - z plane. The cut cone is then flattened out and a straight line is drawn on the flattened surface. The cone is then reassembled. Use the Euler–Lagrange equations you found in part (a) to determine if the line you drew (now a curve, of course) is an extremum path on the surface of the cone.

Exercise 5.6 A right-circular cylinder of top radius a is oriented with its symmetry axis along \hat{e}_3 .

(a) Use the calculus of variations and cylindrical polar coordinates to find the two differential equations describing the extremum path between two general points on the surface of the cylinder.

(b) Choose β equal to s , the arc length along the curve, and solve for ϕ and z as functions of s .

(c) Suppose the cylinder surface to be cut along a line parallel to its symmetry axis, and the cut surface then flattened out onto a table. Draw a diagonal line on that flattened surface and then re-assemble the cylinder. Determine if the line you drew (now a curve, of course) is an extremum path on the surface of the cylinder.

Exercise 5.7

(a) Use the methods in Section 5.6 to show that the extremum (in this case, actually a minimum) distance between two points in a plane is a straight line.

(b) Surfaces that can be defined by the continuous motion of a rigid line are called *developable surfaces*. They have the property that, with suitable cuts, they can be flattened out onto a plane surface without stretching or tearing them. (The cone in Exercise 5.5 and the cylinder in Exercise 5.6 are examples.) Give an argument showing that all developable surfaces have the property that a straight line drawn on their flattened surfaces will be a geodesic

on the re-assembled curved surfaces. [Hint: Imagine the surface to have a regular arrangement of atoms, with separation ℓ in the surface that will not change when they are flattened or re-assembled.]

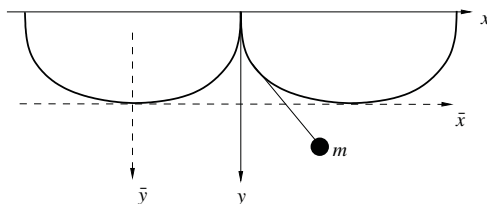


FIG. 5.5. Illustration for Exercise 5.8, Huygens' Isochronous Pendulum.

Exercise 5.8 Huygens' Isochronous Pendulum. A mass m hangs from a massless string of fixed length that swings between two metal sheaves bent into the shape of a cycloid whose formula is given in eqn (5.57), as shown. The straight part of the string is always tangent to the cycloid sheave at the point of last contact.

(a) If the string has length $\ell = 4a$, show that the path of the mass m is the same cycloid as eqn (5.57), but expressed in terms of displaced coordinates $\bar{x} = x + a\pi$ and $\bar{y} = y - 2a$. (The evolute of a cycloid is a cycloid.)

(b) Determine the period of oscillation of the mass, and show that it is independent of the amplitude of the pendulum's swing.

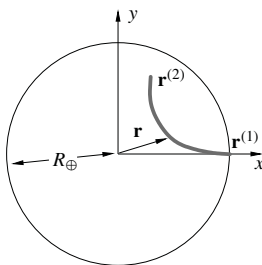


FIG. 5.6. Illustration for Exercise 5.9. The train enters the tunnel at $\mathbf{r}^{(1)} : (R_{\oplus}, 0, 0)$ and the tunnel ends at $\mathbf{r}^{(2)} : (x^{(2)}, y^{(2)}, 0)$.

Exercise 5.9 Suppose that a rail car moves without friction through a tunnel burrowed into the Earth. It starts from rest, and moves entirely under the influence of the nonuniform gravitational field inside the Earth, assumed here to be a sphere of uniform density with gravitational potential $\Phi = M_{\oplus}G (r^2 - 3R_{\oplus}^2) / 2R_{\oplus}^3$, where G is the gravitational constant, M_{\oplus} is the mass of the Earth, and R_{\oplus} is its radius. Ignore the rotation of the Earth. Assume that the tunnel lies entirely in the x - y plane where the origin of coordinates is at the center of the Earth and $\hat{\mathbf{e}}_1$ points directly toward the point of entry.

(a) Using the general parametric method, write the Euler-Lagrange equations for the path

that extremizes the transit time T from the entry point to the point $(x^{(2)}, y^{(2)}, 0)$.

(b) By choosing $\beta = \eta$ after the Euler–Lagrange equations are written, show that the solution to these equations is, for suitable choice of R, a , given by

$$x = (R - a) \cos \eta + a \cos \left(\frac{R - a}{a} \eta \right) \qquad y = (R - a) \sin \eta - a \sin \left(\frac{R - a}{a} \eta \right) \quad (5.120)$$

which are the equations of a *hypocycloid*, the line traced out by a point on the circumference of a circle of radius a that is rolling without slipping along the inside of a circle of radius $R > a$. The parameter η here is the plane-polar angle of the center of the rolling circle.

(c) Suppose that the far end of the extremum tunnel is back at the surface of the Earth. If D is the distance along the surface of the Earth between entry and exit points, what is the greatest depth reached by the tunnel? How long did the trip take?

(d) Now rewrite the Euler–Lagrange equations with the choice $\beta = s$, the arc length along the unvaried path. Write them as a single vector equation, using the notation of the Serret–Frenet theory of Section A.12. Taking \mathbf{r}_{\parallel} and \mathbf{r}_{\perp} to be the resolution of the radius vector \mathbf{r} into vectors parallel and perpendicular to the unit tangent vector $\hat{\mathbf{t}}$, write $d\hat{\mathbf{t}}/ds$ in terms of \mathbf{r}_{\perp} , R_{\oplus} , x , y , z only, where R_{\oplus} is the radius of the Earth.

6

HAMILTON'S PRINCIPLE

The general calculus of variations developed in Chapter 5 may be used to derive variational principles in mechanics. Two different, but closely related, variational principles are presented here: Hamilton's Principle and the phase-space Hamilton's Principle. One acts in the space of Lagrangian variables q, \dot{q}, t and the other in Hamiltonian phase space q, p, t .

Some authors believe that variational principles are the foundations of physics. For example, the classic analytical mechanics text of Landau and Lifshitz (1976) writes an action function on page two, and derives the whole of mechanics from it, including Newton's laws. Whether this is a fair judgement or not, it is certainly true that variational principles play a crucial role in quantum theory, general relativity, and theoretical physics in general.

6.1 Hamilton's Principle in Lagrangian Form

We now revert to the mechanics notation and denote dq_k/dt by \dot{q}_k . This is a change from the notation of Chapter 5 where $dx_k/d\beta$ was denoted by \dot{x}_k .

If we identify x_1 with the time t , identify $x_{[1]}$ with the Lagrangian generalized coordinates q , and restrict ourselves to cases in which no constraints are present, the unconstrained Euler–Lagrange equations of the coordinate parametric variational method in Section 5.14 become

$$\frac{d}{dt} \left(\frac{\partial g(t, q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial g(t, q, \dot{q})}{\partial q_k} = 0 \quad (6.1)$$

for $k = 1, \dots, D$. These equations are remarkably similar in form to the Lagrange equations of mechanics derived in Section 2.9 for the case with no constraints and all forces derived from a potential,

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad (6.2)$$

That similarity underlies Hamilton's Principle.³⁵

³⁵This similarity is so striking that it seems surprising that Hamilton's Principle was not stated clearly until the middle of the nineteenth century. One possible reason is the authority of Maupertuis, who insisted on theological grounds that the system trajectory must be a true *minimum* of some quantity, since a wise God would not waste means. See the discussion in Chapter 3 of Yourgrau and Mandelstam (1968).

Equation (6.1) is the condition for the extremum of the line integral of the coordinate parametric method, eqn (5.104). With the above substitutions, it becomes

$$I = \int_{t^{(1)}}^{t^{(2)}} g(t, q, \dot{q}) dt \quad (6.3)$$

Since eqns (6.1, 6.2) differ only by the appearance of either $g(t, q, \dot{q})$ or $L(q, \dot{q}, t)$ in them, this suggests that the Lagrange equations of mechanics can be derived from a variational principle that seeks to extremize the integral

$$I = \int_{t^{(1)}}^{t^{(2)}} L(q, \dot{q}, t) dt \quad (6.4)$$

In mechanics, this integral is called the *Action Integral*, or more simply, the *Action*. The *Hamilton's Principle* states that the natural path of system motion makes the action integral an extremum.

Theorem 6.1.1: Hamilton's Principle

With I defined as in eqn (6.4), and assuming variations that vanish at the end points, the first-order variation δI vanishes for arbitrary δq_k if and only if the $q_k(t)$ of the unvaried path are a solution to the Lagrange equations with $Q_k^{(NP)} = 0$. Thus the extremum condition $\delta I = 0$ holds if and only if, for all $k = 1, \dots, D$,

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad (6.5)$$

Proof: With the substitutions listed above, the integral in eqn (5.104) becomes identical to eqn (6.4). With those same substitutions, the Euler–Lagrange equations, eqn (5.105), become identical to eqn (6.5). Theorem 5.14.1 thus proves the present theorem. \square

The path in configuration space that is a solution to the Lagrange equations is often referred to as the *classical path*. This is the path of natural motion of a mechanical system as it responds to the forces included in the potential part of the Lagrangian. Thus we can say that $\delta I = 0$ for variations about a chosen unvaried path if and only if that chosen path is the classical path. Notice that many different unvaried paths could be chosen, but that the condition $\delta I = 0$ happens only for variations about the classical path. Fortunately, that classical path can be found by a procedure better than simple trial and error. It is found by solving the Lagrange equations.

6.2 Hamilton's Principle with Constraints

If we make the same substitutions as in Section 6.1, the constrained form of the coordinate parametric variational method derived in Theorem 5.14.2 implies a constrained form of Hamilton's Principle.

Theorem 6.2.1: Hamilton's Principle with Constraints

With I defined as in eqn (6.4), consider variations that vanish at the end points but are otherwise arbitrary, except for the C independent holonomic constraints given by eqn (3.1),

$$0 = G_a(q, t) \tag{6.6}$$

for $a = 1, \dots, C$. Then $\delta I = 0$ if and only if the $q_k(t)$ of the chosen path are a solution to the equations

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \tag{6.7}$$

for $k = 1, \dots, D$. Equation (6.7) is the correct equation of motion of the mechanical system if and only if the forces of constraint do no virtual work.

Proof: With the same substitutions as above in Section 6.1, Theorem 5.14.2 establishes that $\delta I = 0$ and the conditions in eqn (6.6) do imply eqn (6.7). And Theorem 3.5.1 establishes that eqn (6.7) is the correct equation of motion if and only if the forces of constraint do no virtual work. □

The quantities λ_a , which are called Lagrange multipliers in the calculus of variations, have a special interpretation in the mechanical problem. They are related to the forces of constraint and may be used, as in eqn (3.14), to derive those forces. Needless to say, this interpretation of the λ_a does not apply when the calculus of variations is used for nonmechanical problems.

6.3 Comments on Hamilton's Principle

We proved in Chapter 2 that the Lagrange equations hold if and only if each point mass of the mechanical system obeys Newton's second law. Thus the Lagrange equations are equivalent to the second law. In Theorem 6.1.1 we have proved that, when all forces are derived from a potential, $\delta I = 0$ if and only if the Lagrange equations are satisfied. Thus, the chain of logic has established, at least when no constraint or other non-potential forces are present, that Hamilton's Principle is equivalent to Newton's second law,

$$\text{Second Law} \iff \text{Lagrange Equations} \iff \text{Hamilton's Principle}$$

But the equivalence of Hamilton's Principle to Newton's second law is established only for the case when all forces are derived from a potential. If constraint forces are present in a mechanical system, this equivalence breaks down. Then Hamilton's Principle is equivalent to Newton's second law only in the idealized case in which the constraint forces do no virtual work. Hamilton's Principle always implies eqn (6.7), but that equation is incorrect when the constraint forces have friction and hence

do virtual work. If the forces of constraint do virtual work, the correct equations of motion would be something like

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = Q_k^{(\text{frict})} + \sum_{a=1}^C \lambda_a \frac{\partial G_a(q, t)}{\partial q_k} \quad (6.8)$$

where the $Q_k^{(\text{frict})}$ are the generalized forces of the friction.

Note to the Reader: If the forces of constraint in a mechanical system happen to *have* friction, and hence *do* virtual work, then eqn (6.7) will *not* be the correct equation of motion of the system. However, Hamilton's Principle in the form of the variational hypothesis in Theorem 6.2.1 would still imply eqn (6.7). Thus it is possible for an incorrect equation to be derived from a variational method.

Variational principles give an elegant way to express the results of mechanics. But one must realize that the calculus of variations is just a language, and like all languages can be used to make both true and false statements.

Hamilton's Principle in Section 6.1 is analogous to the form of the calculus of variations called the "coordinate parametric method," and described in Section 5.14. In that coordinate parametric method, some coordinate x_1 is prematurely removed from the list of varied coordinates and is made to play the role of integration variable. As seen in Section 6.1, the variable $t = x_1$ plays that role in Hamilton's Principle. As a result, Lagrangian mechanics does indeed require a "second form of the Euler-Lagrange equations" in analogy to that discussed in Theorem 5.14.3. That second form is just the generalized energy theorem $\dot{H} = -\partial L(q, \dot{q}, t) / \partial t$, which was derived in Section 2.15.

But Section 5.15 argued that the general parametric method is simpler and more complete than the coordinate parametric method. In the general parametric method, the coordinate t would be restored to its proper place as a generalized coordinate and the generalized energy theorem (the second form) would be restored to its proper place as just another Lagrange equation. The problem of restoring the apparently lost symmetry of Lagrangian mechanics, by treating t properly as a coordinate rather than as a parameter, is discussed in Part II of the book. Hamilton's Principle with time as a coordinate is treated in Chapter 13.

6.4 Phase-Space Hamilton's Principle

As stated in Chapter 4 on the Hamilton equations, the usefulness of phase space in more advanced analytical mechanics depends on the equal treatment of the canonical coordinates and momenta. To that end, we now use the calculus of variations to derive a phase-space form of Hamilton's Principle.

To begin, the action function defined in eqn (6.4) can be rewritten as a line integral involving the Hamiltonian. Solving eqn (4.14) for L , and introducing phase-space

variables, gives

$$I = \int_{t^{(1)}}^{t^{(2)}} L dt = \int_{t^{(1)}}^{t^{(2)}} \left(\sum_{k=1}^D p_k \dot{q}_k - H(q, p, t) \right) dt \tag{6.9}$$

The first-order variation of the line integral in eqn (6.9) may now be taken, using the definitions of variations of a function and an integral from Sections 5.3 and 5.4. The result will be

$$\begin{aligned} \delta I &= \int_{t^{(1)}}^{t^{(2)}} \left(\sum_{k=1}^D (p_k \delta \dot{q}_k + \dot{q}_k \delta p_k) - \delta H(q, p, t) \right) dt \\ &= \sum_{k=1}^D (p_k \delta q_k) \Big|_{t^{(1)}}^{t^{(2)}} + \int_{t^{(1)}}^{t^{(2)}} \sum_{k=1}^D \left\{ \left(\dot{q}_k - \frac{\partial H(q, p, t)}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H(q, p, t)}{\partial q_k} \right) \delta q_k \right\} dt \end{aligned} \tag{6.10}$$

where an integration by parts has been done.

Before proceeding to state a phase-space Hamilton's Principle, we must first discuss the meaning to be given to variations δp_k of the canonical momenta in eqn (6.10). In Lagrangian mechanics, the generalized momenta $p_k = p_k(q, \dot{q}, t)$ are functions of the Lagrangian variables q, \dot{q}, t and hence δp_k would be calculated using eqn (5.16). The result would be

$$\delta p_k = \sum_{k=1}^D \left(\frac{\partial p_k(q, \dot{q}, t)}{\partial q_k} \delta q_k + \frac{\partial p_k(q, \dot{q}, t)}{\partial \dot{q}_k} \delta \dot{q}_k \right) \tag{6.11}$$

which would make the variation δp_k depend on δq_k and its time derivative and hence not be an independent variation.

But we want a phase-space Hamilton's Principle that treats the coordinates and momenta equally. Thus both δq_k and δp_k should be treated as independent variations, unrelated to each other. Therefore, we temporarily forget both the equation $p_k = p_k(q, \dot{q}, t)$ and its inverse $\dot{q}_k = \dot{q}_k(q, p, t)$. (As we will see, these relations will be recovered at the end of the calculation.) Thus eqn (6.10) will be considered as an expression involving two equally unknown sets of functions q and p . Equation (6.11) will therefore no longer hold. The variations of q_k and p_k will now be defined by the two equations, both holding for $k = 1, \dots, D$,

$$q_k(t, \delta a) = q_k(t) + \delta a \eta_k(t) \tag{6.12}$$

$$p_k(t, \delta a) = p_k(t) + \delta a \chi_k(t) \tag{6.13}$$

where the shape functions η_k and χ_k are considered to be arbitrary and independent of one another. Since we wanted both q and p to be considered simply as coordinates of phase space, we have applied the definition of variation of coordinates from Section 5.2 to both q and p . Then, as in that section, the variations $\delta q_k = \delta a \eta_k$ and $\delta p_k = \delta a \chi_k$ will all be arbitrary and independent.

We may now state the phase-space form of Hamilton's Principle.

Theorem 6.4.1: Phase-Space Hamilton's Principle

With F defined to be the integrand of eqn (6.9),

$$F(q, p, \dot{q}, \dot{p}, t) = \sum_{k=1}^D p_k \dot{q}_k - H(q, p, t) \quad (6.14)$$

the action integral

$$I = \int_{t^{(1)}}^{t^{(2)}} L dt = \int_{t^{(1)}}^{t^{(2)}} F(q, p, \dot{q}, \dot{p}, t) dt \quad (6.15)$$

will be an extremum, $\delta I = 0$, for variations δq and δp that are arbitrary except for the requirement that they vanish at the end points $t^{(1)}$ and $t^{(2)}$, if and only if the Hamilton equations

$$\dot{q}_k = \frac{\partial H(q, p, t)}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H(q, p, t)}{\partial q_k} \quad (6.16)$$

hold on the unvaried path.

Proof: Since δq_k vanishes at $t^{(1)}$ and $t^{(2)}$ by assumption, the integrated term vanishes and eqn (6.10) becomes

$$\delta I = \int_{t^{(1)}}^{t^{(2)}} \sum_{k=1}^D \left\{ \left(\dot{q}_k - \frac{\partial H(q, p, t)}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H(q, p, t)}{\partial q_k} \right) \delta q_k \right\} dt \quad (6.17)$$

Since both δq and δp are now arbitrary and independent, they may be set nonzero one at a time. Hence $\delta I = 0$ if and only if 6.16 hold, as was to be proved.³⁶ \square

Notice that the first Hamilton equation gives $\dot{q}_k = \dot{q}_k(q, p, t)$ as an equation of motion. Thus the relation between p and \dot{q} is recovered. The difference between the Hamiltonian and Lagrangian approaches is that in Lagrangian theory the relation $\dot{q}_k = \dot{q}_k(q, p, t)$ is an identity, true both on the unvaried and on all varied paths. But in the phase-space form of Hamilton's Principle, that relation is an equation of motion that is true only on the classical path. It is part of the definition of the classical path.

6.5 Exercises

Exercise 6.1 This exercise gives an alternate proof of Theorem 6.4.1, using Theorem 5.14.1.

(a) With the substitutions $x_{[1]} = q_1, \dots, q_D, p_1, \dots, p_D$ and $x_1 = t$, show that the Euler–Lagrange equations in eqn (5.105) become, for $k = 1, \dots, D$,

$$\frac{d}{dt} \left(\frac{\partial F(q, p, \dot{q}, \dot{p}, t)}{\partial \dot{q}_k} \right) - \frac{\partial F(q, p, \dot{q}, \dot{p}, t)}{\partial q_k} = 0 \quad (6.18)$$

$$\frac{d}{dt} \left(\frac{\partial F(q, p, \dot{q}, \dot{p}, t)}{\partial \dot{p}_k} \right) - \frac{\partial F(q, p, \dot{q}, \dot{p}, t)}{\partial p_k} = 0 \quad (6.19)$$

(b) With F given by eqn (6.14), show that these Euler–Lagrange equations imply the Hamilton equations, eqn (6.16).

³⁶See the proof of the Euler–Lagrange theorem, Theorem 5.5.1, for more detail about setting arbitrary variations nonzero one at a time.

LINEAR OPERATORS AND DYADICS

Linear vector functions of vectors, and the related dyadic notation, are important in the study of rigid body motion and the covariant formulations of relativistic mechanics. In this chapter we introduce these topics and present methods which we will need later.

Linear vector functions of vectors have a rich structure, with up to nine independent parameters needed to characterize them, and vector outputs that need not even have the same directions as the vector inputs. The subject of linear vector operators merits a chapter to itself not only for its importance in analytical mechanics, but also because study of it will help the reader to master the operator formalism of quantum mechanics.

7.1 Definition of Operators

It seems easiest to write linear vector functions of vectors using the operator notation familiar from quantum mechanics, but perfectly applicable here as well. To say that some function maps vector \mathbf{A} into vector \mathbf{B} we could write $\mathbf{B} = \vec{f}(\mathbf{A})$, where \vec{f} denotes the vector function. It is easier and clearer to write instead $\mathbf{B} = \mathcal{F}\mathbf{A}$, with operator \mathcal{F} thought of as operating to the right on \mathbf{A} and converting it into \mathbf{B} . The linearity of \mathcal{F} is expressed by defining its operation on $\mathbf{A} = \alpha\mathbf{V} + \beta\mathbf{W}$, where α, β are scalars, to be

$$\mathcal{F}\mathbf{A} = \mathcal{F}(\alpha\mathbf{V} + \beta\mathbf{W}) = \alpha\mathcal{F}\mathbf{V} + \beta\mathcal{F}\mathbf{W} \quad (7.1)$$

Linearity says that the result of operating on sum \mathbf{A} is the same as operating on each of its terms and then doing the sum. Also, the scalar factors α, β may be applied either before or after operation with \mathcal{F} , giving the same result in either case. For example, $\mathcal{F}(\alpha\mathbf{V}) = \alpha\mathcal{F}\mathbf{V}$.

Since operators are defined by their action on vectors, two operators are equal, $\mathcal{A} = \mathcal{B}$, if and only if

$$\mathcal{A}\mathbf{V} = \mathcal{B}\mathbf{V} \quad (7.2)$$

for any arbitrary vector \mathbf{V} . For linear operators, this condition is equivalent to requiring only that $\mathcal{A}\mathbf{V}_k = \mathcal{B}\mathbf{V}_k$ for any three, non-coplanar vectors $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ since any arbitrary vector \mathbf{V} can be expressed as a sum of these three.

The null operator \mathcal{O} and the identity (or unity) operator \mathcal{U} are defined by $\mathcal{O}\mathbf{V} = 0$ and $\mathcal{U}\mathbf{V} = \mathbf{V}$ for any vector \mathbf{V} , where we adopt the usual convention of denoting the null vector $\mathbf{0}$ by the number 0. The null operator \mathcal{O} is also usually denoted by just the number 0. This notational sloppiness seems not to lead to problems in either case.

Thus expressions like $\mathcal{A} = 0$ are allowed, although $\mathcal{A} = 3$ would be nonsense unless intended to be an (even more sloppy) short form for $\mathcal{A} = 3\mathcal{U}$.

Linear operators can be added, subtracted, and multiplied by numbers. The definition is that

$$\mathcal{C} = \alpha\mathcal{A} + \beta\mathcal{B} \quad (7.3)$$

if and only if

$$\mathcal{C}\mathbf{V} = \alpha\mathcal{A}\mathbf{V} + \beta\mathcal{B}\mathbf{V} \quad (7.4)$$

for any arbitrary vector \mathbf{V} . It follows from the properties of vector addition that addition of operators is commutative and associative,

$$\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A} \quad \text{and} \quad (\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C}) \quad (7.5)$$

The multiplication of operators is defined to mean successive application. Thus

$$\mathcal{C} = \mathcal{A}\mathcal{B} \quad \text{if and only if} \quad \mathcal{C}\mathbf{V} = \mathcal{A}(\mathcal{B}\mathbf{V}) \quad (7.6)$$

for any vector \mathbf{V} . Operator \mathcal{B} acts on \mathbf{V} first, producing another vector $\mathcal{B}\mathbf{V}$. The operator \mathcal{A} then acts on that vector to produce the final result. Operator multiplication is associative,

$$(\mathcal{A}\mathcal{B})\mathcal{C} = \mathcal{A}(\mathcal{B}\mathcal{C}) = \mathcal{A}\mathcal{B}\mathcal{C} \quad (7.7)$$

since all three expressions acting on an arbitrary \mathbf{V} reduce to the same result $\mathcal{A}(\mathcal{B}(\mathcal{C}\mathbf{V}))$.

However, operator multiplication is in general not commutative. In general $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$. The *commutator* of the two operators is another operator $[\mathcal{A}, \mathcal{B}]_c$ defined by³⁷

$$[\mathcal{A}, \mathcal{B}]_c = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \quad (7.8)$$

If $[\mathcal{A}, \mathcal{B}]_c = 0$, where here we use the number 0 for the null operator as noted above, then the two operators are said to *commute*. The commutator is anti-symmetric in the exchange of its two operators, and hence any operator commutes with itself,

$$[\mathcal{B}, \mathcal{A}]_c = -[\mathcal{A}, \mathcal{B}]_c \quad \text{and} \quad [\mathcal{A}, \mathcal{A}]_c = 0 \quad (7.9)$$

The evaluation of commutators is aided by some easily proved algebraic rules. With scalars β, γ ,

$$[\mathcal{A}, (\beta\mathcal{B} + \gamma\mathcal{C})]_c = \beta[\mathcal{A}, \mathcal{B}]_c + \gamma[\mathcal{A}, \mathcal{C}]_c \quad (7.10)$$

$$[\mathcal{A}\mathcal{B}, \mathcal{C}]_c = \mathcal{A}[\mathcal{B}, \mathcal{C}]_c + [\mathcal{A}, \mathcal{C}]_c\mathcal{B} \quad (7.11)$$

$$[\mathcal{F}, [\mathcal{G}, \mathcal{H}]_c]_c + [\mathcal{H}, [\mathcal{F}\mathcal{G}]_c]_c + [\mathcal{G}, [\mathcal{H}, \mathcal{F}]_c]_c = 0 \quad (7.12)$$

For every operator \mathcal{A} there is another operator \mathcal{A}^T called its *transpose*, which is

³⁷The subscript c is to distinguish the commutator of two operators from the similarly denoted Poisson bracket defined in Section 4.6. The algebra of commutators resembles that of Poisson brackets, as may be seen by comparing the identities in eqns (7.10 – 7.12) with those in eqns (4.55 – 4.57). The algebraic similarity of commutators and Poisson brackets has important consequences in quantum mechanics, as discussed in Section 12.13.

defined by the condition that

$$(\mathcal{A}\mathbf{V}) \cdot \mathbf{W} = \mathbf{V} \cdot \mathcal{A}^T\mathbf{W} \tag{7.13}$$

holds for any arbitrary vectors \mathbf{V}, \mathbf{W} .

It follows from this definition that $(\mathcal{A}^T)^T = \mathcal{A}$ and that the transpose of a product of operators is the product of the transposes, but in reverse order. To establish this last result, note that eqns (7.6, 7.13) imply that

$$(\mathcal{A}\mathcal{B}\mathbf{V}) \cdot \mathbf{W} = (\mathcal{A}(\mathcal{B}\mathbf{V})) \cdot \mathbf{W} = (\mathcal{B}\mathbf{V}) \cdot \mathcal{A}^T\mathbf{W} = \mathbf{V} \cdot \mathcal{B}^T\mathcal{A}^T\mathbf{W} \tag{7.14}$$

and hence, again using definition eqn (7.13), that

$$(\mathcal{A}\mathcal{B})^T = \mathcal{B}^T\mathcal{A}^T \tag{7.15}$$

7.2 Operators and Matrices

We know that, once an orthonormal basis $\hat{\mathbf{e}}_i$ is chosen for a three-dimensional vector space, a one-to-one relation can be established between vectors and the 3×1 matrices, called column vectors, made up of the vector components in that basis,

$$\mathbf{V} \iff [V] \quad \text{where} \quad [V] = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad \text{with} \quad V_i = \hat{\mathbf{e}}_i \cdot \mathbf{V} \tag{7.16}$$

for $i = 1, 2, 3$.

The relation is one-to-one because not only does every vector determine the its components by the last of eqn (7.16), but also, given its components, any vector \mathbf{V} can be determined by writing it as

$$\mathbf{V} = \sum_{j=1}^3 V_j \hat{\mathbf{e}}_j \tag{7.17}$$

Thus two vectors are equal, with $\mathbf{V} = \mathbf{W}$, if and only if $[V] = [W]$, or in component form $V_i = W_i$ for all $i = 1, 2, 3$.

Operators are similar to vectors in that, once an orthonormal basis is chosen, each operator is associated uniquely with a matrix. But in the case of operators, the matrix is a 3×3 square matrix with nine components.

Definition 7.2.1: Matrix Elements

Assuming that a basis $\hat{\mathbf{e}}_i$ has been chosen, there is a one-to-one relation between an operator and its matrix in this basis given by the definition

$$\mathcal{F} \iff \mathbf{F} \quad \text{where} \quad \mathbf{F} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \quad \text{with} \quad F_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{F}\hat{\mathbf{e}}_j \tag{7.18}$$

for $i, j = 1, 2, 3$. The nine numbers F_{ij} are called the matrix elements of operator \mathcal{F} in the $\hat{\mathbf{e}}_i$ basis.

Just as its components in some basis determine a vector \mathbf{V} , so the matrix in some basis determines the operator. Imagine that the linear operator \mathcal{F} operates on a vector expanded as $\mathbf{V} = \sum_{j=1}^3 V_j \hat{\mathbf{e}}_j$. Denote the result by \mathbf{W} . Then

$$\mathbf{W} = \mathcal{F}\mathbf{V} = \mathcal{F} \left(\sum_{j=1}^3 V_j \hat{\mathbf{e}}_j \right) = \sum_{j=1}^3 V_j \mathcal{F}\hat{\mathbf{e}}_j \quad (7.19)$$

where the linearity of operators from eqn (7.1) was used to derive the last equality. Then the component W_i of vector \mathbf{W} is

$$W_i = (\hat{\mathbf{e}}_i \cdot \mathbf{W}) = \sum_{j=1}^3 V_j (\hat{\mathbf{e}}_i \cdot \mathcal{F}\hat{\mathbf{e}}_j) = \sum_{j=1}^3 F_{ij} V_j \quad (7.20)$$

where eqn (7.18) was used to get the matrix elements F_{ij} . Equation (7.20) can be written in matrix notation as

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad \text{or, more succinctly,} \quad [W] = \mathbf{F}[V] \quad (7.21)$$

where the 3×3 matrix is denoted by single letter \mathbf{F} .

Thus, given any vector \mathbf{V} , knowledge of the matrix elements F_{ij} will uniquely determine the vector \mathbf{W} . Since operators are defined by their action on vectors, this defines \mathcal{F} completely. Thus $\mathcal{A} = \mathcal{B}$ if and only if $\mathbf{A} = \mathbf{B}$, or in component form $A_{ij} = B_{ij}$ for all $i, j = 1, 2, 3$.

The matrices corresponding to the null and unit operators are easily found from eqn (7.18). They are the null and unity matrices, with $O_{ij} = 0$ and $U_{ij} = \delta_{ij}$, respectively, where δ_{ij} is the Kroeneker delta function. Thus

$$\mathcal{O} \iff \mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{U} \iff \mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.22)$$

The matrix \mathbf{A}^T corresponding to the transposed operator \mathcal{A}^T defined in eqn (7.13) has the matrix elements $A_{ij}^T = A_{ji}$ for all $i, j = 1, 2, 3$. To see this, replace \mathbf{V} by $\hat{\mathbf{e}}_i$ and \mathbf{W} by $\hat{\mathbf{e}}_j$ in eqn (7.13), and use eqn (7.18). The matrix element A_{ij}^T of matrix \mathbf{A}^T is thus

$$A_{ij}^T = \hat{\mathbf{e}}_i \cdot \mathcal{A}^T \hat{\mathbf{e}}_j = (\mathcal{A}\hat{\mathbf{e}}_i) \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_j \cdot (\mathcal{A}\hat{\mathbf{e}}_i) = A_{ji} \quad (7.23)$$

where the symmetry of dot products proved in Section A.2 has been used.

The result in eqn (7.23) corresponds exactly to the definition of the transpose of a matrix in Section B.2. Also, eqn (B.24) shows that

$$(\mathbf{A} \mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T \quad (7.24)$$

which is consistent with eqn (7.15) for operators.

7.3 Addition and Multiplication

As discussed in Section 7.1, operators can be added or multiplied. The matrices corresponding to the resulting operators are obtained by addition or multiplication of the associated matrices.

Let operators \mathcal{A} , \mathcal{B} have corresponding matrices \mathbf{A} , \mathbf{B} , respectively. Let $\mathcal{C} = \alpha\mathcal{A} + \beta\mathcal{B}$. Then using eqns (7.4, 7.18) gives the corresponding matrix \mathbf{C} as

$$C_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{C}\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i \cdot (\alpha\mathcal{A} + \beta\mathcal{B})\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i \cdot (\alpha\mathcal{A}\hat{\mathbf{e}}_j + \beta\mathcal{B}\hat{\mathbf{e}}_j) = \alpha A_{ij} + \beta B_{ij} \quad (7.25)$$

which may be written in matrix form as $\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B}$.

Let $\mathcal{D} = \mathcal{A}\mathcal{B}$. The corresponding matrix \mathbf{D} is

$$D_{ik} = \hat{\mathbf{e}}_i \cdot \mathcal{D}\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i \cdot (\mathcal{A}\mathcal{B}\hat{\mathbf{e}}_k) = \hat{\mathbf{e}}_i \cdot (\mathcal{A}(\mathcal{B}\hat{\mathbf{e}}_k)) \quad (7.26)$$

where eqn (7.6) has been used to get the last equality. But, like any vector, $\mathcal{B}\hat{\mathbf{e}}_k$ can be expanded in the $\hat{\mathbf{e}}_j$ basis as

$$\mathcal{B}\hat{\mathbf{e}}_k = \sum_{j=1}^3 \hat{\mathbf{e}}_j (\hat{\mathbf{e}}_j \cdot \mathcal{B}\hat{\mathbf{e}}_k) = \sum_{j=1}^3 \hat{\mathbf{e}}_j B_{jk} \quad (7.27)$$

Putting this result into eqn (7.26) then gives

$$D_{ik} = \hat{\mathbf{e}}_i \cdot \left(\mathcal{A} \left(\sum_{j=1}^3 \hat{\mathbf{e}}_j B_{jk} \right) \right) = \sum_{j=1}^3 (\hat{\mathbf{e}}_i \cdot \mathcal{A}\hat{\mathbf{e}}_j) B_{jk} = \sum_{j=1}^3 A_{ij} B_{jk} \quad (7.28)$$

Since the second index j of A_{ij} matches the first index of B_{jk} , eqn (7.28) is equivalent to the matrix multiplication $\mathbf{D} = \mathbf{A}\mathbf{B}$.

Equations (7.25, 7.28) may be summarized as the correspondences

$$\alpha\mathcal{A} + \beta\mathcal{B} \iff \alpha\mathbf{A} + \beta\mathbf{B} \quad (7.29)$$

$$\mathcal{A}\mathcal{B} \iff \mathbf{A}\mathbf{B} \quad (7.30)$$

7.4 Determinant, Trace, and Inverse

Given the basis $\hat{\mathbf{e}}_i$, eqn (7.18) defines the nine components F_{ij} of the matrix \mathbf{F} that corresponds uniquely to operator \mathcal{F} , in exactly the same sense that the last of eqn (7.16) defines the three components V_i of the column vector $[V]$ that corresponds uniquely to vector \mathbf{V} in that same basis.

If an alternate orthonormal basis $\hat{\mathbf{e}}'_i$ is chosen, assumed also to be right-handed with $\hat{\mathbf{e}}'_1 \times \hat{\mathbf{e}}'_2 = \hat{\mathbf{e}}'_3$, then vector \mathbf{V} and operator \mathcal{F} will also have a unique relation to column vector $[V']$ and matrix \mathbf{F}' in this alternate basis. The vector components and matrix elements in the alternate basis are given by the same formulas as in Section

7.2, but now using the primed basis vectors. Thus

$$V'_i = \hat{\mathbf{e}}'_i \cdot \mathbf{V} \quad \text{and} \quad F'_{ij} = \hat{\mathbf{e}}'_i \cdot \mathcal{F} \hat{\mathbf{e}}'_j \quad (7.31)$$

The unique relation between vectors and column vectors, and between operators and matrices, holds in either basis, and hence

$$[V'] \iff \mathbf{V} \iff [V] \quad \text{and} \quad \mathbf{F}' \iff \mathcal{F} \iff \mathbf{F} \quad (7.32)$$

This chain of unique correspondence has the consequence that any equation involving matrices and column vectors in basis $\hat{\mathbf{e}}_i$ will be true if and only if the same equation is true when primes are put on all the matrices and column vectors, indicating that they refer to the alternate basis $\hat{\mathbf{e}}'_i$.

In general, even for the same i, j indices, V'_i and F'_{ij} will be quite different from V_i and F_{ij} . However, there are certain quantities calculated from these numbers that have the same value no matter what basis is used. These are called *invariant* or *basis-independent* quantities. Two such quantities are the determinant and trace.

Lemma 7.4.1: Invariance of Determinant and Trace

If an operator \mathcal{F} has corresponding matrices \mathbf{F} and \mathbf{F}' in the two bases $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$, respectively, then

$$|\mathbf{F}| = |\mathbf{F}'| \quad \text{and} \quad \text{Tr } \mathbf{F} = \text{Tr } \mathbf{F}' \quad (7.33)$$

Proof: In Section 8.32 of Chapter 8, it will be proved that $\mathbf{F}' = \mathbf{R}^T \mathbf{F} \mathbf{R}$ where \mathbf{R} is the matrix, expressed in the $\hat{\mathbf{e}}_i$ basis, of a proper orthogonal operator \mathcal{R} defined by $\hat{\mathbf{e}}'_i = \mathcal{R} \hat{\mathbf{e}}_i$ for $i = 1, 2, 3$. This operator will be proved there to have the property that $\mathbf{R}^T \mathbf{R} = \mathbf{U} = \mathbf{R} \mathbf{R}^T$ and $|\mathbf{R}| = 1$. It follows, using Property 5 and Property 10 of Section B.11, that $|\mathbf{F}'| = |\mathbf{R}^T| |\mathbf{F}| |\mathbf{R}| = |\mathbf{F}|$ as was to be proved. Also, from eqn (B.33), $\text{Tr } \mathbf{F}' = \text{Tr} (\mathbf{R}^T \mathbf{F} \mathbf{R}) = \text{Tr} (\mathbf{R} \mathbf{R}^T \mathbf{F}) = \text{Tr } \mathbf{F}$, as was to be proved. \square

Since these quantities are basis independent, the determinant and trace of an operator may be defined by selecting *any* basis $\hat{\mathbf{e}}_i$, determining the matrix \mathbf{F} corresponding to \mathcal{F} in that basis, and setting

$$\det \mathcal{F} = |\mathbf{F}| \quad \text{and} \quad \text{Tr } \mathcal{F} = \text{Tr } \mathbf{F} \quad (7.34)$$

It follows from definition eqn (7.34) and the corresponding properties of matrices in Section B.11 and eqn (B.33) that the determinant and trace of operators have the properties

$$\det \mathcal{A}^T = \det \mathcal{A} \quad \det (\mathcal{A}\mathcal{B}) = \det \mathcal{A} \det \mathcal{B} \quad (7.35)$$

$$\text{Tr} (\mathcal{A}\mathcal{B}\mathcal{C}) = \text{Tr} (\mathcal{C}\mathcal{A}\mathcal{B}) = \text{Tr} (\mathcal{B}\mathcal{C}\mathcal{A}) \quad (7.36)$$

An operator \mathcal{F} may or may not have an inverse. If the inverse exists, it is denoted \mathcal{F}^{-1} and has the defining property that, for both right and left multiplication, the

product of \mathcal{F} with its inverse is the identity operator \mathcal{U} ,

$$\mathcal{F}^{-1}\mathcal{F} = \mathcal{U} = \mathcal{F}\mathcal{F}^{-1} \quad (7.37)$$

The inverse is unique. If two operators both are inverses of a given \mathcal{F} , then they can be shown to be identical to each other.

The necessary and sufficient condition for the inverse of an operator \mathcal{F}^{-1} to exist is that $\det \mathcal{F} \neq 0$. This result follows from the definition in eqn (7.34) and the similar property of matrices proved in Section B.14.

If $\mathcal{C} = \mathcal{A}\mathcal{B}$, it is easily verified that the inverse is $\mathcal{C}^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$, provided of course that the inverses of \mathcal{A} and \mathcal{B} exist. The inverse of a product is the product of the inverses, in reverse order.

7.5 Special Operators

If an operator \mathcal{S} is identical to its transpose, $\mathcal{S}^T = \mathcal{S}$, then $S_{ij}^T = S_{ji} = S_{ij}$ and we say that it (and its matrix) are *symmetric*.³⁸

An *anti-symmetric* (an alternate term is *skew-symmetric*) operator is in a sense the opposite of a symmetric one. Such an operator is equal to the negative of its transpose. If operator \mathcal{W} is anti-symmetric, then $\mathcal{W}^T = -\mathcal{W}$ and its matrix elements obey $W_{ij}^T = W_{ji} = -W_{ij}$.

The most general anti-symmetric operator has a matrix containing only three independent matrix elements,

$$\mathbf{W} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (7.38)$$

or equivalently

$$W_{ij} = \sum_{k=1}^3 \varepsilon_{ikj} \omega_k \quad (7.39)$$

for $i, j = 1, 2, 3$, where $\omega_1, \omega_2, \omega_3$ are three arbitrarily chosen numbers that together determine \mathbf{W} .

The operation of an anti-symmetric operator \mathcal{W} on a vector can be represented as a cross-product.

Lemma 7.5.1: Equivalent Cross-Product

If we define a vector $\boldsymbol{\omega}$ whose components are the same three numbers ω_i found in eqns (7.38, 7.39),

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{e}}_1 + \omega_2 \hat{\mathbf{e}}_2 + \omega_3 \hat{\mathbf{e}}_3 \quad (7.40)$$

then the action of operator \mathcal{W} on an arbitrary vector \mathbf{V} is the same as the cross product of vector $\boldsymbol{\omega}$ with that vector;

$$\mathcal{W}\mathbf{V} = \boldsymbol{\omega} \times \mathbf{V} \quad (7.41)$$

³⁸Matrix symmetries are treated in Section B.4.

Proof: Let $\mathbf{A} = \boldsymbol{\omega} \times \mathbf{V}$. Then

$$A_i = \hat{\mathbf{e}}_i \cdot \mathbf{A} = \hat{\mathbf{e}}_i \cdot \boldsymbol{\omega} \times \mathbf{V} = \sum_{j=1}^3 \sum_{k=1}^3 \omega_k V_j (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_j) = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ikj} \omega_k V_j = \sum_{j=1}^3 W_{ij} V_j \quad (7.42)$$

which is the component form of the matrix equation $[A] = \mathbf{W}[V]$. Since there is a one-to-one correspondence between operators and matrices, it follows that $\mathbf{A} = \mathcal{W}\mathbf{V}$, as was to be proved. \square

If the anti-symmetric operator is given initially, the components of $\boldsymbol{\omega}$ can be extracted from its matrix by

$$\omega_k = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ikj} W_{ij} \quad (7.43)$$

where identity eqn (A.65) has been used.

Another important special class of operators is orthogonal operators. An operator \mathcal{R} is orthogonal if it has an inverse \mathcal{R}^{-1} and its inverse is equal to its transpose,

$$\mathcal{R}^{-1} = \mathcal{R}^T \quad (7.44)$$

Thus the property of inverses in eqn (7.37) implies that

$$\mathcal{R}\mathcal{R}^T = \mathcal{U} = \mathcal{R}^T\mathcal{R} \quad (7.45)$$

for orthogonal operators. Orthogonal operators will be used to characterize rotations in Chapter 8.

7.6 Dyadics

There is yet another way of writing linear vector functions of vectors in common use: Dyadics. Those who have studied the Dirac notation in quantum mechanics should find them familiar. Those who have not can learn dyadics here and get a head start on mastering the Dirac notation.

We begin by defining a single-termed dyadic, or *dyad*, \mathbb{D} as a pair of vectors \mathbf{a} and \mathbf{b} written side-by-side with no operation between them such as dot or cross product,

$$\mathbb{D} = \mathbf{a}\mathbf{b} \quad (7.46)$$

This strange-looking object is intended to be an operator on vectors. But, unlike the operators defined above, it operates either to its right or to its left, and by means of dot products rather than directly. Thus, dotting \mathbb{D} to its right onto vector \mathbf{V} is defined to give

$$\mathbb{D} \cdot \mathbf{V} = \mathbf{a}(\mathbf{b} \cdot \mathbf{V}) \quad (7.47)$$

which is a vector parallel to \mathbf{a} . The dyad \mathbb{D} can also be dotted to the left on a vector \mathbf{V} to yield

$$\mathbf{V} \cdot \mathbb{D} = (\mathbf{V} \cdot \mathbf{a})\mathbf{b} \quad (7.48)$$

which is a vector parallel to \mathbf{b} . We see at once that left and right dotting will generally

give different output vectors, since \mathbf{a} need not be parallel to \mathbf{b} . By its definition in terms of the dot product, the dyadic operation is a linear function of vector \mathbf{V} .

We define a law of addition for dyads similar to that for operators above. Suppose that two dyads are $\mathbb{D}_1 = \mathbf{ab}$ and $\mathbb{D}_2 = \mathbf{cd}$. Then multiplication by scalars and addition as in

$$\mathbb{D} = \alpha\mathbb{D}_1 + \beta\mathbb{D}_2 \quad (7.49)$$

are defined by the rule that the operation of \mathbb{D} on any arbitrary vector \mathbf{V} is

$$\mathbb{D} \cdot \mathbf{V} = \alpha\mathbb{D}_1 \cdot \mathbf{V} + \beta\mathbb{D}_2 \cdot \mathbf{V} \quad (7.50)$$

A similar rule holds for left multiplication. The sum of one or more dyads is called a *dyadic*. The rule for the addition of two dyadics is the same as eqn (7.49) for dyads.

Now suppose that we have a linear operator like the \mathcal{F} discussed in Section 7.1. Since the matrix elements F_{ij} of this operator are just numbers like α and β in eqn (7.49), we can define a dyadic corresponding to operator \mathcal{F} by using this addition rule to write the nine-termed sum

$$\mathbb{F} = \sum_{i=1}^3 \sum_{j=1}^3 F_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \quad (7.51)$$

This dyadic is often denoted in equivalent ways, by using the freedom to write the numerical factor F_{ij} either before the pair of vectors, between the pair (as is often done in quantum mechanics), or after both of them, as in

$$\mathbb{F} = \sum_{i=1}^3 \sum_{j=1}^3 F_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i F_{ij} \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j F_{ij} \quad (7.52)$$

Conversely, if we are given a dyadic \mathbb{F} , the matrix elements F_{ij} in the $\hat{\mathbf{e}}_i$ basis can be determined by dotting from both sides with unit vectors, since

$$\hat{\mathbf{e}}_i \cdot \mathbb{F} \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i \cdot \left(\sum_{k=1}^3 \sum_{l=1}^3 \hat{\mathbf{e}}_k F_{kl} \hat{\mathbf{e}}_l \right) \cdot \hat{\mathbf{e}}_j = \sum_{k=1}^3 \sum_{l=1}^3 \delta_{ik} F_{kl} \delta_{lj} = F_{ij} \quad (7.53)$$

As an example of a case in which the dyadic is given initially and the matrix and operator derived from it, consider the dyad $\mathbb{D} = \mathbf{ab}$ in eqn (7.46). Then

$$D_{ij} = \hat{\mathbf{e}}_i \cdot \mathbb{D} \cdot \hat{\mathbf{e}}_j = (\hat{\mathbf{e}}_i \cdot \mathbf{a}) (\mathbf{b} \cdot \hat{\mathbf{e}}_j) = a_i b_j \quad (7.54)$$

The matrix element of this simple dyad is just the product of the components of the two vectors. General dyadics, of course, will not have matrices with this simple product form.

By its construction, the dyadic \mathbb{F} dotted onto any vector \mathbf{V} has the same effect as the operator \mathcal{F} acting on that same vector.

Lemma 7.6.1: Equivalence of Operator and Dyadic

If \mathbf{V} is any vector, then

$$\mathcal{F}\mathbf{V} = \mathbb{F} \cdot \mathbf{V} \quad (7.55)$$

Proof: Let $\mathbf{W} = \mathcal{F}\mathbf{V}$ define the vector \mathbf{W} . Then from eqn (7.20) $W_i = \sum_{j=1}^3 F_{ij} V_j$. The dyadic acting on \mathbf{V} gives the same vector \mathbf{W} ,

$$\mathbb{F} \cdot \mathbf{V} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i F_{ij} (\hat{\mathbf{e}}_j \cdot \mathbf{V}) = \sum_{i=1}^3 \hat{\mathbf{e}}_i \sum_{j=1}^3 F_{ij} V_j = \sum_{i=1}^3 \hat{\mathbf{e}}_i W_i = \mathbf{W} \quad (7.56)$$

which establishes eqn (7.55). \square

Like operators and matrices, dyadics can also be multiplied. The product

$$\mathbb{C} = \mathbb{A} \cdot \mathbb{B} \quad (7.57)$$

is defined by considering its operation on an arbitrary vector \mathbf{V} . The dyadic \mathbb{B} is first dotted with vector \mathbf{V} and the dyadic \mathbb{A} is then dotted onto the resulting vector,

$$\mathbb{C} \cdot \mathbf{V} = (\mathbb{A} \cdot \mathbb{B}) \cdot \mathbf{V} = \mathbb{A} \cdot (\mathbb{B} \cdot \mathbf{V}) \quad (7.58)$$

Thus, from Lemma 7.6.1,

$$\mathcal{A}\mathcal{B}\mathbf{V} = \mathbb{A} \cdot (\mathbb{B} \cdot \mathbf{V}) \quad (7.59)$$

for any vector \mathbf{V} .

Like operator multiplication in eqn (7.7), dyadic multiplication is associative by definition, since

$$(\mathbb{A} \cdot \mathbb{B}) \cdot \mathbb{C} = \mathbb{A} \cdot (\mathbb{B} \cdot \mathbb{C}) = \mathbb{A} \cdot \mathbb{B} \cdot \mathbb{C} \quad (7.60)$$

Just as for operators in Section 7.5, the determinant and trace of a dyadic are defined to be the determinant and trace of its associated matrix in some basis. The inverse dyadic is the dyadic constructed from the inverse matrix, and exists if and only if the dyadic has a nonzero determinant.

The transpose of a dyadic is constructed from the transposed matrix, using eqn (7.52). If a dyadic \mathbb{F} has a matrix \mathbb{F} then the transpose is defined as

$$\mathbb{F}^T = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i F_{ij}^T \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i F_{ji} \hat{\mathbf{e}}_j \quad (7.61)$$

It follows that left multiplication of \mathbb{F} by \mathbf{V} gives the same result as right multiplication of \mathbb{F}^T by the same vector. That is,

$$\mathbf{V} \cdot \mathbb{F} = \mathbb{F}^T \cdot \mathbf{V} \quad (7.62)$$

for any vector \mathbf{V} .

7.7 Resolution of Unity

Consider the identity operator \mathcal{U} which has $\mathcal{U}\mathbf{V} = \mathbf{V}$ for any vector \mathbf{V} . The dyadic form \mathbb{U} of this operator is of particular interest. From eqn (7.22),

$$\mathbb{U} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i \delta_{ij} \hat{\mathbf{e}}_j = \sum_{i=1}^3 \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \quad (7.63)$$

This dyadic is called a *resolution of unity* in basis $\hat{\mathbf{e}}_i$. Since $\mathcal{U}\mathbf{V} = \mathbf{V}$, it follows that $\mathbb{U} \cdot \mathbf{V} = \mathbf{V}$. The resolution of unity can be used as a convenient device to expand vectors and other operators in a basis. For example,

$$\begin{aligned} \mathbf{V} &= \mathbb{U} \cdot \mathbf{V} = (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3) \cdot \mathbf{V} \\ &= \hat{\mathbf{e}}_1 (\hat{\mathbf{e}}_1 \cdot \mathbf{V}) + \hat{\mathbf{e}}_2 (\hat{\mathbf{e}}_2 \cdot \mathbf{V}) + \hat{\mathbf{e}}_3 (\hat{\mathbf{e}}_3 \cdot \mathbf{V}) \end{aligned} \quad (7.64)$$

simply restates eqn (A.10). Any vector \mathbf{V} in any expression can always be replaced by either $\mathbb{U} \cdot \mathbf{V}$ or $\mathbf{V} \cdot \mathbb{U}$. The result is always to expand the expression in terms of components in the resolution's basis, in this case $\hat{\mathbf{e}}_i$.

7.8 Operators, Components, Matrices, and Dyadics

The equation $\mathbf{W} = \mathcal{F}\mathbf{V}$ can now be written in four equivalent ways: operator, component, matrix, and dyadic:

$$\mathbf{W} = \mathcal{F}\mathbf{V} \quad W_i = \sum_{j=1}^3 F_{ij} V_j \quad [W] = \mathbf{F} [V] \quad \mathbf{W} = \mathbb{F} \cdot \mathbf{V} \quad (7.65)$$

Each of the four expressions in eqn (7.65) is a different way of saying the same thing, and each of them implies the others. This, and the various other equivalences proved in the preceding sections of this chapter, can be summarized as a theorem, which we state here without further proof.

Theorem 7.8.1: Equivalence of Operators, Matrices, Dyadics

Any equation involving the addition, multiplication, transposition, and inversion of operators, and the action of operators on vectors, will be true if and only if the same equation is true with matrices or dyadics substituted for the operators. In the matrix case, of course, the vectors must also be replaced by column vectors.

As an example of this theorem, consider the following equivalent expressions,

$$\mathcal{A}\mathcal{B}^T\mathbf{V} + \alpha\mathcal{C}^{-1}\mathcal{D}\mathbf{W} = \mathbf{Y} \quad (7.66)$$

$$\sum_{j=1}^3 \sum_{k=1}^3 A_{ij} B_{jk}^T V_k + \alpha \sum_{j=1}^3 \sum_{k=1}^3 C_{ij}^{-1} D_{jk} W_k = Y_i \quad (7.67)$$

$$\mathbf{A} \mathbf{B}^T [V] + \alpha \mathbf{C}^{-1} \mathbf{D} [W] = [Y] \quad (7.68)$$

$$\mathbb{A} \cdot \mathbb{B}^T \cdot \mathbf{V} + \alpha \mathbb{C}^{-1} \cdot \mathbb{D} \cdot \mathbf{W} = \mathbf{Y} \quad (7.69)$$

Each of them is true if and only if the other three are true.

The theorem of this section is of great use. It means that operator equations can be proved by proving the equivalent component or matrix equations in some basis, and vice versa. Throughout the text, we will use this theorem to go back and forth between operator and matrix relations, often with little warning, assuming that the reader understands that they are equivalent.

For example, if $\mathcal{F}(\eta) = \mathcal{A}(\eta) \mathcal{B}(\eta)$ is a product of two operators, each of which is a function of some parameter η , then the product rule for differentiation,

$$\frac{d\mathcal{F}(\eta)}{d\eta} = \frac{d\mathcal{A}(\eta)}{d\eta} \mathcal{B}(\eta) + \mathcal{A}(\eta) \frac{d\mathcal{B}(\eta)}{d\eta} \quad (7.70)$$

follows from the usual product rule for differentiation of the component expansion

$$F_{ik}(\eta) = \sum_{j=1}^3 A_{ij}(\eta) B_{jk}(\eta) \quad (7.71)$$

where the matrix elements $A_{ij}(\eta)$, etc., are now just ordinary functions of η .

For the remainder of this chapter, we will exploit Theorem 7.8.1 to translate the properties of matrices summarized in Appendix B into operator and dyadic forms. Since the proofs are given in Appendix B, we will often simply state the results here and refer the reader to that Appendix for more information.

7.9 Complex Vectors and Operators

A real vector is one whose components in some Cartesian basis are *all* purely real numbers. (The Cartesian basis vectors themselves are always considered real in these determinations.) If at least one component is an imaginary or complex number, the vector is complex. A general complex vector \mathbf{V} may be written

$$\mathbf{V} = \mathbf{V}_R + i\mathbf{V}_I \quad (7.72)$$

where real vector \mathbf{V}_R collects all of the real parts of the components of \mathbf{V} and real vector \mathbf{V}_I collects all of the imaginary parts. For example, if $\mathbf{V} = 3\hat{\mathbf{e}}_1 + (2 - 4i)\hat{\mathbf{e}}_2 + 6i\hat{\mathbf{e}}_3$ then $\mathbf{V}_R = 3\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2$ and $\mathbf{V}_I = -4\hat{\mathbf{e}}_2 + 6\hat{\mathbf{e}}_3$.

Operators and dyadics can also be real or complex. The definition is similar to that for vectors. An operator \mathcal{F} and dyadic \mathbb{F} is real only if *all* of its matrix elements F_{ij} in some Cartesian basis are real numbers. If even one matrix element is imaginary or complex, the operator is complex. The transpose, and the definitions of symmetric, anti-symmetric, and orthogonal operators, must be generalized when complex operators are considered.

The complex conjugate of an operator can be defined as that operator all of whose matrix elements in some Cartesian basis $\hat{\mathbf{e}}_i$ are the complex conjugates of the original ones. If \mathcal{F} has matrix elements F_{ij} , then

$$\mathcal{F}^* \quad \text{has matrix elements} \quad F_{ij}^* \quad (7.73)$$

Thus an operator is real if and only if $\mathcal{F} = \mathcal{F}^*$. Otherwise, it is complex.

The generalization of transpose is *Hermitian conjugate*. The Hermitian conjugate of operator \mathcal{F} is denoted \mathcal{F}^\dagger and is defined in a way similar to eqn (7.13) for the transpose, by the condition that

$$(\mathcal{F}\mathbf{V})^* \cdot \mathbf{W} = \mathbf{V}^* \cdot (\mathcal{F}^\dagger \mathbf{W}) \tag{7.74}$$

for any vectors \mathbf{V}, \mathbf{W} . As was done for the transpose in eqn (7.23), basis vectors may be substituted for the vectors in eqn (7.74), giving the relation

$$F_{ij}^\dagger = F_{ji}^* \tag{7.75}$$

for the matrix elements F_{ij}^\dagger of matrix \mathbf{F}^\dagger .

Notice that the Hermitian conjugate can be considered as the combination of transpose and complex conjugate in either order,

$$\mathcal{F}^\dagger = (\mathcal{F}^T)^* = (\mathcal{F}^*)^T \tag{7.76}$$

as can be seen by considering the matrix elements of each expression.

If the operator is real, then the complex conjugations have no effect, and $\mathcal{F}^\dagger = \mathcal{F}^T$ holds. Thus the definition of Hermitian conjugate for possibly complex operators is a generalization of the definition of transpose for real ones.

The generalization of symmetric is *Hermitian*. If an operator \mathcal{H} is equal to its Hermitian conjugate, then it is Hermitian. Then $\mathcal{H}^\dagger = \mathcal{H}$ and hence $H_{ij}^\dagger = H_{ji}^* = H_{ij}$. For real operators, the complex conjugation would have no effect, and hence a real Hermitian operator is a real symmetric one. Thus the definition of Hermitian for possibly complex operators is a generalization of definition of symmetric for real ones. Similarly, anti-Hermitian operators can be defined that generalize anti-symmetric ones.

The generalization of orthogonal is *unitary*. An operator \mathcal{T} is unitary if it is non-singular and if its inverse is equal to its Hermitian conjugate,

$$\mathcal{T}^{-1} = \mathcal{T}^\dagger \tag{7.77}$$

with the consequence that

$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{U} = \mathcal{T}^\dagger\mathcal{T} \tag{7.78}$$

As seen above, for real operators there is no distinction between transpose and Hermitian conjugate. Hence a real unitary operator would be a real orthogonal one. Thus the definition of unitary for possibly complex operators is a generalization of the definition of orthogonal for real ones.

For complex operators, the determinants obey

$$\det \mathcal{F}^* = (\det \mathcal{F})^* \quad \text{and hence also} \quad \det \mathcal{F}^\dagger = (\det \mathcal{F})^* \tag{7.79}$$

Just as a complex vector can be written as the sum of its real and imaginary parts as in eqn (7.72), any complex operator can be written as the sum of two Hermitian

operators,

$$\mathcal{F} = \mathcal{F}_R + i\mathcal{F}_I \quad (7.80)$$

where the Hermitian operators \mathcal{F}_R and \mathcal{F}_I are

$$\mathcal{F}_R = \frac{1}{2}(\mathcal{F} + \mathcal{F}^\dagger) \quad \text{and} \quad \mathcal{F}_I = -\frac{i}{2}(\mathcal{F} - \mathcal{F}^\dagger) \quad (7.81)$$

7.10 Real and Complex Inner Products

Recall that, in the space of real vectors, the inner product of two vectors can be written in column vector and component forms, as

$$\mathbf{V} \cdot \mathbf{W} = \sum_{i=1}^3 V_i W_i = (V_1 \ V_2 \ V_3) \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = [\mathbf{V}]^T [\mathbf{W}] \quad (\text{real vectors}) \quad (7.82)$$

In a space of complex vectors, this definition of inner product must be modified. Orthogonality, norm, etc., for such a complex vector space are based on a generalized inner product consisting of the dot product *and* complex conjugation of the left-hand vector,

$$\mathbf{V}^* \cdot \mathbf{W} = \sum_{i=1}^3 V_i^* W_i = (V_1^* \ V_2^* \ V_3^*) \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = [\mathbf{V}]^\dagger [\mathbf{W}] \quad (\text{complex vectors}) \quad (7.83)$$

Note that the transpose of the column vector $[\mathbf{V}]^T$ in the real case becomes the Hermitian conjugate $[\mathbf{V}]^\dagger$ in the complex case.

The redefinition of inner product for complex vectors is necessary in order to preserve an important property that dot products have in real vector spaces: The norm of a vector must be non-negative and be zero only for the null vector. Thus we have, with the redefinition,

$$V^2 = \|\mathbf{V}\|^2 = \mathbf{V}^* \cdot \mathbf{V} = (\mathbf{V}_R - i\mathbf{V}_I) \cdot (\mathbf{V}_R + i\mathbf{V}_I) = \|\mathbf{V}_R\|^2 + \|\mathbf{V}_I\|^2 \quad (7.84)$$

which clearly has the desired non-negative property. The rule is that, when using complex vector spaces, one must always be sure that the left-hand vector in an inner product is complex conjugated before the dot product is taken.

7.11 Eigenvectors and Eigenvalues

An operator \mathcal{F} acts on some vectors (but not others) in a particularly simple way: it gives an output vector which is just the input vector multiplied by a numerical scale factor. Those vectors are called the *eigenvectors* of \mathcal{F} (“eigen” is German for “own”) and the scale factors (in general different for each eigenvector) are called *eigenvalues*. The equation

$$\mathcal{F}\mathbf{V}^{(k)} = \lambda_k \mathbf{V}^{(k)} \quad (7.85)$$

defines $\mathbf{V}^{(k)}$ to be an eigenvector, and λ_k to be the associated eigenvalue, of operator \mathcal{F} . The integer k labels the different eigenvalues and corresponding eigenvectors that

\mathcal{F} may have. The set of eigenvectors and eigenvalues of an operator may in many cases characterize it completely, as we will see, and so their determination is of particular importance.³⁹

We may rewrite this equation, and its equivalent matrix equation in some basis, in the forms

$$(\mathcal{F} - \lambda_k \mathcal{U}) \mathbf{V}^{(k)} = 0 \quad \text{and} \quad (\mathbf{F} - \lambda_k \mathbf{U}) [\mathbf{V}^{(k)}] = 0. \quad (7.86)$$

where $[\mathbf{V}^{(k)}]$ is the column vector of components $V_i^{(k)} = \hat{\mathbf{e}}_i \cdot \mathbf{V}^{(k)}$ of eigenvector $\mathbf{V}^{(k)}$ in the chosen basis. The matrix equation, and hence the operator equation also, has a solution other than the null vector if and only if

$$\det(\mathcal{F} - \lambda_k \mathcal{U}) = 0 \quad \text{with matrix equivalent} \quad |\mathbf{F} - \lambda_k \mathbf{U}| = 0. \quad (7.87)$$

This cubic equation has three eigenvalue solutions $\lambda_1, \lambda_2, \lambda_3$ which may in general be complex numbers. For each of those solutions, $(\mathbf{F} - \lambda_k \mathbf{U})$ has rank less than three, and so a non-null eigenvector solution to eqn (7.86) can be found.⁴⁰ These eigenvectors are usually *normalized* by dividing each one by its magnitude to produce a unit vector. The form of eqn (7.85) shows that these normalized vectors $\hat{\mathbf{V}}^{(k)} = \mathbf{V}^{(k)} / \|\mathbf{V}^{(k)}\|$ are still eigenvectors.

7.12 Eigenvectors of Real Symmetric Operator

Real symmetric operators \mathcal{S} , obeying $\mathcal{S}^T = \mathcal{S}$, are an important special case. We list here some properties of their eigenvalues and eigenvectors. The listed properties are proved in Section B.24.

1. The eigenvalues λ_k of real symmetric operators are all real.
2. Since all matrix elements S_{ij} are real numbers, the eigenvector solutions eqn (7.86) may be taken to be real vectors.
3. If two eigenvalues are different, $\lambda_k \neq \lambda_n$, then the corresponding eigenvectors are orthogonal, $\hat{\mathbf{V}}_k \cdot \hat{\mathbf{V}}_n = 0$.
4. Three orthogonal unit eigenvectors of \mathcal{S} can always be found. These three eigenvectors obey $\hat{\mathbf{V}}_k \cdot \hat{\mathbf{V}}_n = \delta_{kn}$ and are said to form a *complete orthonormal set*. The word “complete” is used here to indicate that these three eigenvectors could be used as an orthonormal basis in place of $\hat{\mathbf{e}}_i$ if desired.

7.13 Eigenvectors of Real Anti-Symmetric Operator

The eigenvalue problem for the real, anti-symmetric operators described in Section 7.5 is of particular importance in the study of rigid body rotations. Fortunately, the eigenvalues and eigenvectors of the most general anti-symmetric operator in three dimensions can be found in a standard form.

³⁹The reader should refer to Section B.23 for more detail about finding eigenvalues and eigenvectors.

⁴⁰See Section B.19.

Theorem 7.13.1: Eigenvectors of Anti-Symmetric Operators

If \mathcal{W} is a real anti-symmetric operator obeying $\mathcal{W}^T = -\mathcal{W}$ with eigenvector equation

$$\mathcal{W}\hat{\mathbf{V}}^{(k)} = \lambda_k \hat{\mathbf{V}}^{(k)} \tag{7.88}$$

then its eigenvalues and corresponding eigenvectors are

$$\lambda_1 = i\omega \quad \lambda_2 = -i\omega \quad \lambda_3 = 0 \tag{7.89}$$

and

$$\hat{\mathbf{V}}^{(1)} = (\hat{\mathbf{a}} - i\hat{\mathbf{b}})/\sqrt{2}, \quad \hat{\mathbf{V}}^{(2)} = (\hat{\mathbf{a}} + i\hat{\mathbf{b}})/\sqrt{2}, \quad \hat{\mathbf{V}}^{(3)} = \boldsymbol{\omega}/\omega \tag{7.90}$$

where $\boldsymbol{\omega}$ is the vector defined in eqn (7.40), $\omega = \|\boldsymbol{\omega}\|$ is its magnitude, $\hat{\mathbf{a}}$ is some real unit vector perpendicular to $\boldsymbol{\omega}$, and $\hat{\mathbf{b}} = (\boldsymbol{\omega}/\omega) \times \hat{\mathbf{a}}$ is also a real unit vector, perpendicular to both $\hat{\mathbf{a}}$ and $\boldsymbol{\omega}$.

Proof: Direct computation of eqn (7.87) using the matrix in eqn (7.38) shows the eigenvalues to be as stated. Lemma 7.5.1 showed that $\mathcal{W}\mathbf{V} = \boldsymbol{\omega} \times \mathbf{V}$ for any vector \mathbf{V} . Applying this result, one easily proves that $\mathcal{W}\hat{\mathbf{V}}^{(k)} = \boldsymbol{\omega} \times \hat{\mathbf{V}}^{(k)} = \lambda_k \hat{\mathbf{V}}^{(k)}$ for $k = 1, 2, 3$, as was to be proved. □

Note that the three eigenvalues of an anti-symmetric \mathcal{W} are always distinct. They would be equal (all zero) only in the case $\omega = 0$ which would imply a null operator. The three eigenvectors of \mathcal{W} are orthonormal using the extended definition of inner product appropriate for vectors with complex components discussed in Section 7.10 above. They are easily shown to obey $\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl}$.

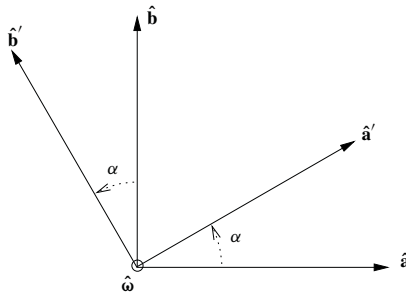


FIG. 7.1. Construction of eigenvectors of \mathcal{W}

One might think that the eigenvalue problem for \mathcal{W} is not really solved, due to the arbitrary choice of vector $\hat{\mathbf{a}}$. But in fact we have solved the problem as well as eigenvalue problems can ever be solved. To show this, we begin with a lemma.

Lemma 7.13.2: Underdetermination of Eigenvectors

Normalized eigenvectors are determined only up to an arbitrary phase factor $\exp(i\alpha_k)$,

where α_k are real numbers that may in general be different for the different eigenvectors. If an eigenvector problem is solved to give an orthonormal set of eigenvectors $\hat{\mathbf{V}}^{(k)}$, then

$$\hat{\mathbf{V}}^{(k)'} = \exp(i\alpha_k) \hat{\mathbf{V}}^{(k)} \tag{7.91}$$

are also an orthonormal solution to the same problem.

Proof: Equation (7.85) is homogeneous in the eigenvectors. Thus, when any normalized eigenvector $\hat{\mathbf{V}}^{(k)}$ is multiplied by a factor of the form $\exp(i\alpha_k)$, the $\exp(i\alpha_k)$ factors on left and right of eqn (7.85) will cancel and the result will still be an eigenvector. The resulting set of eigenvectors will also still be normalized and mutually orthogonal, since

$$\begin{aligned} \left(\hat{\mathbf{V}}^{(k)'}\right)^* \cdot \hat{\mathbf{V}}^{(l)'} &= \left(\exp i\alpha_k \hat{\mathbf{V}}^{(k)}\right)^* \cdot \left(\exp i\alpha_l \hat{\mathbf{V}}^{(l)}\right) = \exp(i\alpha_l - i\alpha_k) \hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} \\ &= \exp(i\alpha_l - i\alpha_k) \delta_{kl} = \delta_{kl} \end{aligned} \tag{7.92}$$

using of course the extended definition of dot product. □

It follows that the eigenvector equation eqn (7.88) can never determine $\hat{\mathbf{a}}$ completely. It can be any unit vector lying in the plane perpendicular to $\boldsymbol{\omega}$. To see this, use the real, orthogonal unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ defined above to derive the identities

$$\exp(i\alpha) (\hat{\mathbf{a}} - i\hat{\mathbf{b}}) = (\hat{\mathbf{a}}' - i\hat{\mathbf{b}}') \quad \text{and} \quad \exp(-i\alpha) (\hat{\mathbf{a}} + i\hat{\mathbf{b}}) = (\hat{\mathbf{a}}' + i\hat{\mathbf{b}}') \tag{7.93}$$

where

$$\hat{\mathbf{a}}' = \cos \alpha \hat{\mathbf{a}} + \sin \alpha \hat{\mathbf{b}} \quad \text{and} \quad \hat{\mathbf{b}}' = -\sin \alpha \hat{\mathbf{a}} + \cos \alpha \hat{\mathbf{b}} \tag{7.94}$$

are $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ rotated by the same angle α in the plane they define. Note that $\hat{\mathbf{b}}' = (\boldsymbol{\omega}/\omega) \times \hat{\mathbf{a}}'$ and $\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}}' = 0$ remain true. Thus the rotation of $\hat{\mathbf{a}}$ by angle α to produce some other vector $\hat{\mathbf{a}}'$ lying in the same plane leads to the new set of eigenvectors

$$\hat{\mathbf{V}}^{(1)'} = \exp(i\alpha) \hat{\mathbf{V}}^{(1)} \quad \hat{\mathbf{V}}^{(2)'} = \exp(-i\alpha) \hat{\mathbf{V}}^{(2)} \quad \hat{\mathbf{V}}^{(3)'} = \exp(0) \hat{\mathbf{V}}^{(1)} \tag{7.95}$$

By Lemma 7.13.2, the eigenvector equation cannot distinguish between the two sets and so cannot determine the vector $\hat{\mathbf{a}}$.

7.14 Normal Operators

The properties of normal operators given below are proved for normal matrices in the last three sections of Appendix B. Their correctness as operator relations is a consequence of the one-to-one correspondence between operators and matrices proved in Section 7.8. Note that the eigenvectors in Section B.26 and following sections are denoted by $[x_i^{(k)}]$ whereas we are using $[V_i^{(k)}]$ here.

An operator will have a complete orthonormal set of eigenvectors obeying

$$\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl} \quad (7.96)$$

if and only if it is a *normal operator*. An operator \mathcal{A} is called a normal operator if it commutes with its Hermitian conjugate,

$$\mathcal{A}^\dagger \mathcal{A} = \mathcal{A} \mathcal{A}^\dagger \quad \text{or, equivalently,} \quad [\mathcal{A}, \mathcal{A}^\dagger]_c = 0 \quad (7.97)$$

Most operators that might be used in mechanics are normal. Real symmetric, real anti-symmetric, real orthogonal, Hermitian, anti-Hermitian, and unitary operators are all normal operators.

To exploit the properties of normal operators, we define a linear operator \mathcal{D} to be the operator that converts each of the basis vectors $\hat{\mathbf{e}}_k$ into the corresponding unit eigenvector of the normal operator

$$\mathcal{D}\hat{\mathbf{e}}_k = \hat{\mathbf{V}}^{(k)} \quad (7.98)$$

for $k = 1, 2, 3$. Since the eigenvectors $\hat{\mathbf{V}}^{(k)}$ will in general not be real vectors, it follows that \mathcal{D} will not in general be a real operator. It follows from definition eqn (7.98) that the matrix elements of \mathcal{D} in the $\hat{\mathbf{e}}_i$ basis are

$$D_{ik} = \hat{\mathbf{e}}_i \cdot \mathcal{D}\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{V}}^{(k)} = V_i^{(k)} \quad (7.99)$$

so that matrix element D_{ik} is equal to the i th component of the k th eigenvector, and the matrix \mathbf{D} can be constructed by writing the components of the three normalized eigenvectors as its three columns, as in

$$\mathbf{D} = \begin{pmatrix} V_1^{(1)} & V_1^{(2)} & V_1^{(3)} \\ V_2^{(1)} & V_2^{(2)} & V_2^{(3)} \\ V_3^{(1)} & V_3^{(2)} & V_3^{(3)} \end{pmatrix} \quad (7.100)$$

The orthogonality condition eqn (7.96) may now be written out as

$$\delta_{kl} = \hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \sum_{i=1}^3 V_i^{(k)*} V_i^{(l)} = \sum_{i=1}^3 D_{ik}^* D_{il} = \sum_{i=1}^3 D_{ki}^\dagger D_{il} \quad (7.101)$$

where the definition of Hermitian conjugate in eqn (7.75) was used. Thus

$$U_{kl} = \delta_{kl} = \left(\mathbf{D}^\dagger \mathbf{D} \right)_{kl} \quad (7.102)$$

for $k, l = 1, 2, 3$, and hence

$$\mathbf{U} = \mathbf{D}^\dagger \mathbf{D} \quad (7.103)$$

As proved in Theorem B.22.2 this is sufficient to prove that \mathcal{D} is a unitary operator

obeying eqn (7.78)

$$\mathcal{D}^\dagger \mathcal{D} = \mathcal{U} = \mathcal{D} \mathcal{D}^\dagger \tag{7.104}$$

Let us now define the operator \mathcal{E} by the two equivalent formulae

$$\mathcal{E} = \mathcal{D}^\dagger \mathcal{A} \mathcal{D} \quad \text{and} \quad \mathcal{A} = \mathcal{D} \mathcal{E} \mathcal{D}^\dagger \tag{7.105}$$

By eqns (7.30, 7.99), the matrix elements of \mathcal{E} in the \hat{e}_k basis are

$$\begin{aligned} E_{kl} &= \left(\mathbf{D}^\dagger \mathbf{A} \mathbf{D} \right)_{kl} = \sum_{i=1}^3 \sum_{j=1}^3 D_{ki}^\dagger A_{ij} D_{jl} = \sum_{i=1}^3 V_i^{(k)*} \sum_{j=1}^3 A_{ij} V_j^{(l)} \\ &= \sum_{i=1}^3 V_i^{(k)*} \lambda_l V_i^{(l)} = \lambda_l \delta_{kl} \end{aligned} \tag{7.106}$$

where the component expansions of eqns (7.85, 7.96) have been used to write

$$\sum_{j=1}^3 A_{ij} V_j^{(l)} = \lambda_l V_i^{(l)} \quad \text{and} \quad \sum_{i=1}^3 V_i^{(k)*} V_i^{(l)} = \delta_{kl} \tag{7.107}$$

Thus the matrix of operator \mathcal{E} is

$$\mathbf{E} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{7.108}$$

a diagonal matrix with the eigenvectors as its diagonal elements. We say that the operator \mathcal{D} reduces \mathcal{A} to a diagonal operator \mathcal{E} .

7.15 Determinant and Trace of Normal Operator

For a normal operator, the determinant and trace defined in Section 7.4 can be written in terms of the eigenvalues of the operator. Taking the determinant of the second of eqn (7.105) and using eqn (7.35) gives

$$\det \mathcal{A} = \det \left(\mathcal{D} \mathcal{E} \mathcal{D}^\dagger \right) = \det \mathcal{D} \det \mathcal{E} \det \mathcal{D}^\dagger \tag{7.109}$$

Also, taking the determinant of eqn (7.103) gives

$$1 = \det \mathcal{U} = \det \left(\mathcal{D}^\dagger \mathcal{D} \right) = \det \mathcal{D}^\dagger \det \mathcal{D} \tag{7.110}$$

Thus, noting the diagonal form of \mathbf{E} in eqn (7.108), we obtain

$$\det \mathcal{A} = \det \mathcal{E} = \lambda_1 \lambda_2 \lambda_3 \tag{7.111}$$

Similarly, taking the trace of both sides of eqn (7.105) gives

$$\text{Tr} \mathcal{A} = \text{Tr} \left(\mathcal{D} \mathcal{E} \mathcal{D}^\dagger \right) = \text{Tr} \left(\mathcal{D}^\dagger \mathcal{D} \mathcal{E} \right) = \text{Tr} (\mathcal{U} \mathcal{E}) = \text{Tr} \mathcal{E} = \lambda_1 + \lambda_2 + \lambda_3 \tag{7.112}$$

where eqn (7.36) was used, and the final value of the trace was obtained by inspection of eqn (7.108).

7.16 Eigen-Dyadic Expansion of Normal Operator

Any linear operator has an equivalent dyadic as defined in Section 7.6. However, for normal operators, that dyadic can be expanded in a form that depends only on the eigenvectors and eigenvalues of the operator. We will call this the *eigen-dyadic expansion*. Since operators and dyadics are equivalent, normal operators are thus completely determined by their eigenvectors and eigenvalues. Expansion of this sort are used, for example, in the proof of the Euler Theorem in Chapter 8. But they are also important to the reader because of their frequent use in quantum theory.

Theorem 7.16.1: Eigen-Dyadic Expansion

If \mathcal{A} is a normal operator whose eigenvalues λ_k and orthonormal eigenvectors $\hat{\mathbf{V}}^{(k)}$ are known, then its dyadic \mathbb{A} can be expanded in eigen-dyadic form

$$\mathbb{A} = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \quad (7.113)$$

which expresses \mathbb{A} entirely in terms of the eigenvalues and eigenvectors of \mathcal{A} .

Proof: This result follows from the component expansion of the second of eqn (7.105), which is

$$A_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 D_{ik} E_{kl} D_{lj}^\dagger = \sum_{k=1}^3 \sum_{l=1}^3 V_i^{(k)} \lambda_l \delta_{kl} V_j^{(l)*} = \sum_{k=1}^3 V_i^{(k)} \lambda_k V_j^{(k)*} \quad (7.114)$$

where eqns (7.99, 7.106) have been used. Substituting that result into the definition of the dyadic \mathbb{A} in eqn (7.52) gives

$$\mathbb{A} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i A_{ij} \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \hat{\mathbf{e}}_i V_i^{(k)} \lambda_k V_j^{(k)*} \hat{\mathbf{e}}_j = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \quad (7.115)$$

which is eqn (7.113). □

Thus the expansion of $\mathbf{W} = \mathbb{A} \cdot \mathbf{V}$ for a general vector \mathbf{V} in eqn (7.56), can equally well be written as

$$\mathbf{W} = \mathbb{A} \cdot \mathbf{V} = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \cdot \mathbf{V} \quad (7.116)$$

Note that the right vector in eqn (7.113) is already complex conjugated and is simply dotted onto \mathbf{V} in eqn (7.116) without change.

As shown in Lemma 7.13.2, the eigenvectors are not uniquely determined. If each $\hat{\mathbf{V}}^{(k)}$ is multiplied by a factor $\exp(i\alpha_k)$, the result will be an orthonormal set of eigenvectors that are equivalent to the original ones. However, this indeterminacy does not affect the dyadic defined in eqn (7.113).

Lemma 7.16.2: Uniqueness of Eigen-Dyadic

The eigen-dyadic in eqn (7.113) is uniquely determined even though the eigenvectors are not.

Proof: Replacing $\hat{\mathbf{V}}^{(k)}$ by $\exp(i\alpha_k)\hat{\mathbf{V}}^{(k)}$ in eqn (7.113) gives

$$\begin{aligned} \sum_{k=1}^3 \left(\exp(i\alpha_k)\hat{\mathbf{V}}^{(k)} \right) \lambda_k \left(\exp(i\alpha_k)\hat{\mathbf{V}}^{(k)} \right)^* \\ = \sum_{k=1}^3 \exp(i\alpha_k - i\alpha_k)\hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \end{aligned} \tag{7.117}$$

which is identical to the original dyadic \mathbb{A} . □

The resolution of unity dyadic in eqn (7.63) of Section 7.7 can also be expanded in terms of the eigenvectors of a normal operator. The second equality in eqn (7.104) implies that

$$\delta_{ij} = \sum_{k=1}^3 D_{ik} D_{kj}^\dagger = \sum_{k=1}^3 V_i^{(k)} V_j^{(k)*} \tag{7.118}$$

It follows that

$$\mathbb{U} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i \delta_{ij} \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \hat{\mathbf{e}}_i V_i^{(k)} V_j^{(k)*} \hat{\mathbf{e}}_j = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \hat{\mathbf{V}}^{(k)*} \tag{7.119}$$

which is the required expansion. Note that the only difference between eqns (7.113, 7.119) is that the former multiplies the terms by the eigenvalues λ_k before adding them.

7.17 Functions of Normal Operators

The eigen-dyadic expansion eqn (7.113) can be used to define general functions of normal operators and dyadics.

Definition 7.17.1: Functions of Normal Operators

If a function $f(z)$ is well defined for all eigenvalues λ_k of a normal operator \mathcal{A} , then the dyadic function $\mathbb{F} = f(\mathbb{A})$ of the dyadic \mathbb{A} is defined by

$$\mathbb{F} = f(\mathbb{A}) = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} f(\lambda_k) \hat{\mathbf{V}}^{(k)*} \tag{7.120}$$

which is the same as eqn (7.113), but with $f(\lambda_k)$ replacing λ_k .

The operator function $\mathcal{F} = f(\mathcal{A})$ of the normal operator \mathcal{A} is then defined by the condition that its effect on any vector \mathbf{V} be the same as that of the dyadic: $\mathcal{F}\mathbf{V} = \mathbb{F} \cdot \mathbf{V}$.

This definition has the consequence that $\mathcal{F} = f(\mathcal{A})$ has the same eigenvectors as does \mathcal{A} , and eigenvalues $\gamma_k = f(\lambda_k)$,

$$\mathcal{F}\hat{\mathbf{V}}^{(k)} = \gamma_k \hat{\mathbf{V}}^{(k)} \quad \text{where} \quad \gamma_k = f(\lambda_k) \tag{7.121}$$

To see this, note that

$$\mathcal{F}\hat{\mathbf{V}}^{(k)} = \mathbb{F} \cdot \hat{\mathbf{V}}^{(k)} = \sum_{l=1}^3 \hat{\mathbf{V}}^{(l)} f(\lambda_l) \hat{\mathbf{V}}^{(l)*} \cdot \hat{\mathbf{V}}^{(k)} = \sum_{l=1}^3 \hat{\mathbf{V}}^{(l)} f(\lambda_l) \delta_{lk} = f(\lambda_k) \hat{\mathbf{V}}^{(k)} \tag{7.122}$$

This result is important because it proves that any well-defined function of a normal operator is also a normal operator, with the same orthonormal set of eigenvectors. Of course, the *eigenvalues* γ_k in general are different from the eigenvalues λ_k .

If the function f is a very simple one, like $f(z) = z^n$ where n is some positive integer then, as we would expect, \mathcal{F} is the product of \mathcal{A} with itself n times, as in

$$\mathcal{F} = \mathcal{A}^n = \overbrace{\mathcal{A} \cdots \mathcal{A}}^{n \text{ factors}} \tag{7.123}$$

For example, consider the case $n = 2$. Then eqn (7.113), and the orthogonality relation $\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl}$, give

$$\begin{aligned} \mathbb{A}^2 = \mathbb{A} \cdot \mathbb{A} &= \left(\sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \right) \cdot \left(\sum_{l=1}^3 \hat{\mathbf{V}}^{(l)} \lambda_l \hat{\mathbf{V}}^{(l)*} \right) \\ &= \sum_{k=1}^3 \sum_{l=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \delta_{kl} \lambda_l \hat{\mathbf{V}}^{(l)*} = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k^2 \hat{\mathbf{V}}^{(k)*} \end{aligned} \tag{7.124}$$

which is the definition in eqn (7.120) for this function. This result for $n = 2$ can be generalized in an obvious way to any positive integer n .

The definition eqn (7.120) is well defined even if the function $f(z)$ does not have a power series expansion. But if it does have one, the following theorem applies.

Theorem 7.17.2: Function as Power Series

If function $f(z)$ has a power series expansion

$$f = f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n \tag{7.125}$$

and all eigenvalues λ_k of \mathcal{A} lie in the circle of convergence of the power series, then the operator function $\mathcal{F} = f(\mathcal{A})$ in Definition 7.17.1 equals a convergent power series in operator \mathcal{A} ,

$$\mathcal{F} = f(\mathcal{A}) = a_0 \mathcal{U} + a_1 \mathcal{A} + a_2 \mathcal{A}^2 + \cdots \tag{7.126}$$

Proof: Using eqn (7.123) repeatedly gives

$$\begin{aligned} \mathbb{F} = f(\mathbb{A}) &= \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \left(a_0 + a_1 \lambda_k + a_2 \lambda_k^2 + \dots \right) \hat{\mathbf{V}}^{(k)*} & (7.127) \\ &= a_0 \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \hat{\mathbf{V}}^{(k)*} + a_1 \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} + a_2 \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k^2 \hat{\mathbf{V}}^{(k)*} + \dots \\ &= a_0 \mathcal{U} + a_1 \mathbb{A} + a_2 \mathbb{A}^2 + \dots \end{aligned}$$

which converges whenever all eigenvalues of \mathcal{A} lie in the circle of convergence of the power series. The equivalent operator equation is then the power series

$$\mathcal{F} = f(\mathcal{A}) = a_0 \mathcal{U} + a_1 \mathcal{A} + a_2 \mathcal{A}^2 + \dots \tag{7.128}$$

□

7.18 The Exponential Function

The exponential function is of particular importance in the treatment of rotation operators. The power series expansion of this function,

$$f(z) = \exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \tag{7.129}$$

converges for any z . Thus, for any normal operator \mathcal{A} , Theorem 7.17.2 shows that the function $\mathcal{F} = \exp(\mathcal{A})$ can be expanded in a power series as

$$\mathcal{F} = \exp(\mathcal{A}) = \mathcal{U} + \mathcal{A} + \frac{\mathcal{A}^2}{2!} + \frac{\mathcal{A}^3}{3!} + \dots \tag{7.130}$$

Then, if \mathcal{A} has eigenvectors $\hat{\mathbf{V}}^{(k)}$ and eigenvalues λ_k , the operator $\mathcal{F} = \exp(\mathcal{A})$ is also a normal operator with the same eigenvectors, and eigenvalues $\gamma_k = \exp(\lambda_k)$.

If θ is a scalar, and if \mathcal{A} does not depend on θ , then the function $f(z) = \exp(\theta z)$ produces the power series

$$\mathcal{F}(\theta) = \exp(\theta \mathcal{A}) = \mathcal{U} + \theta \mathcal{A} + \frac{\theta^2 \mathcal{A}^2}{2!} + \frac{\theta^3 \mathcal{A}^3}{3!} + \dots \tag{7.131}$$

Differentiating eqn (7.131) term-by-term gives

$$\frac{d}{d\theta} \exp(\theta \mathcal{A}) = \mathcal{A} + \theta \mathcal{A}^2 + \frac{\theta^2 \mathcal{A}^3}{2!} + \dots = \mathcal{A} \left(\mathcal{U} + \theta \mathcal{A} + \frac{\theta^2 \mathcal{A}^2}{2!} + \dots \right) = \mathcal{A} \exp(\theta \mathcal{A}) \tag{7.132}$$

which shows that $\mathcal{F}(\theta)$ defined in eqn (7.131) is a solution to the operator differential equation

$$\frac{d\mathcal{F}(\theta)}{d\theta} = \mathcal{A} \mathcal{F}(\theta) \tag{7.133}$$

where the initial condition $\mathcal{F}(0) = \mathcal{U}$ is assumed.

If two operators commute, then they can be manipulated in the same way as ordinary numbers. Thus it follows from the same proofs as are found in standard calculus books that if operators \mathcal{A} and \mathcal{B} commute, $[\mathcal{A}, \mathcal{B}]_c = 0$, the exponential functions will have the property that

$$\exp(\mathcal{A}) \exp(\mathcal{B}) = \exp(\mathcal{A} + \mathcal{B}) = \exp(\mathcal{B}) \exp(\mathcal{A}) \quad (7.134)$$

Since any operator \mathcal{A} commutes with $(-\mathcal{A})$, it follows that

$$\exp(\mathcal{A}) \exp(-\mathcal{A}) = \exp(-\mathcal{A}) \exp(\mathcal{A}) = \exp(0\mathcal{A}) = \mathcal{U} \quad (7.135)$$

Thus, whether \mathcal{A} is singular or nonsingular, $\mathcal{F} = \exp(\theta\mathcal{A})$ is always nonsingular, with the inverse $\mathcal{F}^{-1} = \exp(-\theta\mathcal{A})$.

7.19 The Dirac Notation

Quantum mechanics uses complex vectors and operators similar to those described in Sections 7.9 and 7.10. The main difference is that the quantum vectors may have infinite dimension.

Quantum mechanics also uses a different notation for complex vectors, called the Dirac notation.⁴¹ We have denoted a vector by \mathbf{V} where the use of bold-face type indicates that it is a vector, and the letter “ V ” is a label indicating which vector it is. The Dirac notation denotes a vector by what is called a *ket* $|V\rangle$ where the $| \rangle$ indicates that this is a vector, and the letter “ V ” is its label.

Inner products, which we write $\mathbf{V}^* \cdot \mathbf{W}$, are written by reversing the ket to form what is called a *bra* $\langle V|$, so that together the two parts of the inner product form a *bra-ket* $\langle V|W\rangle$. Note that the bar is not doubled in the inner product of a bra and a ket.

Operators are variously notated. One common notation, which we will adopt here, is to place a hat symbol over the operator. For example, an equation that we would write $\mathbf{W} = \mathcal{F}\mathbf{V}$ would be $|W\rangle = \widehat{\mathcal{F}}|V\rangle$ in Dirac notation.

The Dirac notation is essentially dyadic. The dyadic \mathbb{F} defined in eqn (7.52) is written in Dirac notation with the ket and bra vectors poised to make inner products to the left or the right. Thus, the dyadic associated with operator \mathcal{F} would be written

$$\widehat{\mathcal{F}} = \sum_{i=1}^3 \sum_{i'=1}^3 |e_i\rangle F_{ii'} \langle e_{i'}| \quad (7.136)$$

where the matrix element that we write $F_{ii'} = \hat{\mathbf{e}}_i \cdot \mathcal{F}\hat{\mathbf{e}}_{i'}$ is written as

$$F_{ii'} = \langle e_i | \widehat{\mathcal{F}} | e_{i'} \rangle \quad (7.137)$$

The kets $|e_i\rangle$ here are the Cartesian unit vectors that we denote by $\hat{\mathbf{e}}_i$. Notice that in quantum mechanics, the distinction between operator \mathcal{F} and associated dyadic \mathbb{F} is ignored. So, in eqn (7.136) the operator is considered to be *equal* to its dyadic.

⁴¹Most quantum texts treat the Dirac notation. For a definitive statement of it, see Dirac (1935).

Eigenvectors that are labeled by an index k are often denoted by kets using just that index as their label. Thus eqn (7.85) in the Dirac notation is

$$\hat{\mathcal{F}}|k\rangle = \lambda_k |k\rangle \quad (7.138)$$

where the eigenvector we denoted by $\hat{\mathbf{V}}^{(k)}$ is denoted simply by $|k\rangle$. This extreme freedom in choosing labels for bras and kets is one of the strengths of the Dirac notation. The orthogonality of the eigenvectors in eqn (7.96) becomes simply $\langle k|l\rangle = \delta_{kl}$.

The Eigen-Dyadic of a normal operator defined in eqn (7.113) is then written

$$\hat{\mathcal{F}} = \sum_{k=1}^3 |k\rangle \lambda_k \langle k| \quad (7.139)$$

and the resolution of unity from eqn (7.119) is written

$$\hat{\mathcal{U}} = \sum_{k=1}^3 |k\rangle \langle k| \quad (7.140)$$

In the Dirac notation, the definition of Hermitian conjugate is extended to apply also to bras and kets. Since, from eqn (7.83), the inner product is expressed in terms of component column vectors by using the Hermitian conjugate of $[V]$,

$$\langle V|W\rangle = [V]^\dagger[W] \quad (7.141)$$

the Dirac notation defines the bra $\langle V|$ as the Hermitian conjugate of the ket $|V\rangle$, as in $\langle V| = |V\rangle^\dagger$. The bras are considered to be a separate vector space, called the dual space, and expressions like $\langle V| + |W\rangle$ adding a bra and a ket make no sense and are forbidden.

The flexibility in labeling kets leads to certain limitations in the Dirac notation. If a ket is multiplied by a number, that number cannot be taken inside the ket. Thus $\alpha|\psi\rangle \neq |\alpha\psi\rangle$ since the expression on the right is nonsense, a label multiplied by a number. Also, in the case of the eigenkets $|k\rangle$ such a usage could lead to errors. Clearly, $3|1\rangle \neq |3\rangle$ since the eigenkets $|1\rangle$ and $|3\rangle$ are distinct members of an orthonormal set of eigenvectors.

7.20 Exercises

Exercise 7.1 A 3×3 real matrix \mathbf{R} can be thought of as three 3×1 column vectors,

$$\mathbf{R} = \left(\begin{array}{c} \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix} \quad \begin{bmatrix} R_{12} \\ R_{22} \\ R_{32} \end{bmatrix} \quad \begin{bmatrix} R_{13} \\ R_{23} \\ R_{33} \end{bmatrix} \end{array} \right) \quad (7.142)$$

(a) Using the formalism of sums and indices, write out the ij components of both sides of the equation

$$\mathbf{R}^T \mathbf{R} = \mathbf{U} \quad (7.143)$$

and show that it is true if and only if the three column vectors in eqn (7.142) are normalized and mutually orthogonal (i.e. an orthonormal set of vectors).

(b) Show that eqn (7.143) is true if and only if \mathbf{R}^{-1} exists and $\mathbf{R}^T = \mathbf{R}^{-1}$, and hence that a real matrix is orthogonal if and only if its column vectors are an orthonormal set.

Exercise 7.2 In the following, you may use the fact that whenever operators \mathcal{A} and \mathcal{B} commute, so that $[\mathcal{A}, \mathcal{B}]_c = 0$, then

$$e^{\mathcal{A}}e^{\mathcal{B}} = e^{(\mathcal{A}+\mathcal{B})} = e^{\mathcal{B}}e^{\mathcal{A}} \quad (7.144)$$

[Although it doesn't matter for this exercise, note that this equation is *not* true when they fail to commute.]

(a) Suppose that \mathcal{A} is a normal operator. Prove that \mathcal{B} defined by

$$\mathcal{B} = e^{\mathcal{A}} = \mathcal{U} + \mathcal{A} + \frac{1}{2!}\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^3 + \dots \quad (7.145)$$

is also a normal operator.

(b) Prove that if eqn (7.145) holds and \mathcal{A} is a normal operator, then

$$\det \mathcal{B} = e^{\text{Tr} \mathcal{A}} \quad (7.146)$$

(c) Let \mathcal{A} be a normal operator, which may or may not be singular. Prove that an operator \mathcal{B} defined from this \mathcal{A} by eqn (7.145) has an inverse given by

$$\mathcal{B}^{-1} = e^{-\mathcal{A}} \quad (7.147)$$

and hence is nonsingular.

(d) Prove that if \mathcal{A} is a real, anti-symmetric operator, then the \mathcal{B} defined in eqn (7.145) will be a real orthogonal operator. Find the value of its determinant $\det \mathcal{B}$.

(e) Use the power series expansion of the exponential to prove that

$$\mathcal{C}\mathcal{B}\mathcal{C}^{-1} = e^{\mathcal{C}\mathcal{A}\mathcal{C}^{-1}} \quad (7.148)$$

where \mathcal{C} is any nonsingular operator, and eqn (7.145) is assumed to hold.

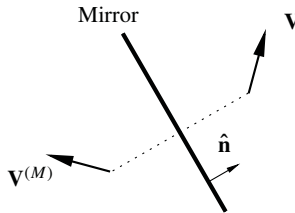


FIG. 7.2. Illustration for Exercise 7.3.

Exercise 7.3 Consider a plane mirror. Denote the unit vector normal to its surface and pointing out into the room by $\hat{\mathbf{n}}$.

(a) The operator \mathcal{M} converts a general vector \mathbf{V} in front of the mirror into its reflected image

$\mathbf{V}^{(M)} = \mathcal{M}\mathbf{V}$ behind the mirror. Find a general expression for its matrix elements M_{ij} in terms of the components n_i of vector $\hat{\mathbf{n}}$. [Hint: Write \mathbf{V} and $\mathbf{V}^{(M)}$ as sums of vectors parallel and perpendicular to \mathbf{n} using eqn (A.3).]

(b) Is \mathcal{M} an orthogonal operator? What is $\det \mathcal{M}$?

(c) Write the dyadic \mathbb{M} corresponding to operator \mathcal{M} , expressing it in terms of the unit dyadic and $\hat{\mathbf{n}}$.

Exercise 7.4 Refer to eqn (A.3) of Appendix A. Operators \mathcal{P}_{\parallel} and \mathcal{P}_{\perp} , which are called *projection operators*, are defined by

$$\mathbf{V}_{\parallel} = \mathcal{P}_{\parallel}\mathbf{V} \quad \mathbf{V}_{\perp} = \mathcal{P}_{\perp}\mathbf{V} \quad (7.149)$$

for any general vector \mathbf{V} .

(a) Find the matrices of these two operators, writing them in terms of the components of $\hat{\mathbf{n}}$.

(b) Prove that

$$(\mathcal{P}_{\parallel})^2 = \mathcal{P}_{\parallel} \quad (\mathcal{P}_{\perp})^2 = \mathcal{P}_{\perp} \quad \mathcal{P}_{\parallel}\mathcal{P}_{\perp} = 0 = \mathcal{P}_{\perp}\mathcal{P}_{\parallel} \quad \mathcal{P}_{\parallel} + \mathcal{P}_{\perp} = \mathcal{U} \quad (7.150)$$

(c) Are \mathcal{P}_{\parallel} and \mathcal{P}_{\perp} orthogonal operators? Do they have inverses?

(d) Write the projection dyadics \mathbb{P}_{\parallel} and \mathbb{P}_{\perp} corresponding to \mathcal{P}_{\parallel} and \mathcal{P}_{\perp} , respectively. Write them in terms of the unit dyadic and the vector $\hat{\mathbf{n}}$.

Exercise 7.5 In Section 7.5, the general anti-symmetric operator \mathcal{W} is defined in terms of a vector $\boldsymbol{\omega}$ by its action $\mathcal{W}\mathbf{V} = \boldsymbol{\omega} \times \mathbf{V}$ on any arbitrary vector \mathbf{V} .

(a) Use the eigenvalues and eigenvectors listed in eqns (7.89, 7.90) to verify that, for $k = 1, 2, 3$,

$$\mathcal{W}\hat{\mathbf{V}}^{(k)} = \lambda_k \hat{\mathbf{V}}^{(k)} \quad (7.151)$$

(b) Verify that these eigenvectors are orthogonal and normalized, using the extended definition of dot product appropriate for complex vectors,

$$\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl} \quad (7.152)$$

(c) Derive the identities given in eqns (7.93, 7.94). Use them to write out the alternate eigenvectors $\hat{\mathbf{V}}^{(k)'}$ in eqn (7.95) in terms of $\hat{\mathbf{a}}'$, $\hat{\mathbf{b}}'$, and $\boldsymbol{\omega}$.

Exercise 7.6 A complex, *spherical basis*, $\hat{\mathbf{e}}_m^{(\text{sp})}$ for $m = -1, 0, +1$, in a three-dimensional Cartesian space may be defined as

$$\hat{\mathbf{e}}_{+1}^{(\text{sp})} = \frac{1}{\sqrt{2}}(-\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2) \quad \hat{\mathbf{e}}_0^{(\text{sp})} = \hat{\mathbf{e}}_3 \quad \hat{\mathbf{e}}_{-1}^{(\text{sp})} = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2) \quad (7.153)$$

(a) Prove that these basis vectors are orthonormal, using the complex inner product defined in Section 7.10,

$$\hat{\mathbf{e}}_m^{(\text{sp})*} \cdot \hat{\mathbf{e}}_{m'}^{(\text{sp})} = \delta_{mm'} \quad (7.154)$$

(b) Show that the dyadic

$$\mathbb{U} = \hat{\mathbf{e}}_{+1}^{(\text{sp})}\hat{\mathbf{e}}_{+1}^{(\text{sp})*} + \hat{\mathbf{e}}_0^{(\text{sp})}\hat{\mathbf{e}}_0^{(\text{sp})*} + \hat{\mathbf{e}}_{-1}^{(\text{sp})}\hat{\mathbf{e}}_{-1}^{(\text{sp})*} \quad (7.155)$$

is equal to the resolution of unity dyadic \mathbb{U} defined in eqn (7.63).

(c) Use the resolution of unity in eqn (7.155) to prove that a general vector \mathbf{A} can be expanded as

$$\mathbf{A} = \sum_{m=+1,0,-1} A_m^{(\text{sp})} \hat{\mathbf{e}}_m^{(\text{sp})} \quad \text{where} \quad A_m^{(\text{sp})} = \hat{\mathbf{e}}_m^{(\text{sp})*} \cdot \mathbf{A} \quad (7.156)$$

[Recall that dyadics in complex spaces are written with the right-hand vector *already* complex conjugated so that no further complex conjugation is required when dotting them onto vectors.]

Exercise 7.7

(a) Apply the expansion in eqn (7.156) to the radius vector \mathbf{r} . Write the resulting $r_m^{(\text{sp})}$ components in terms of x, y, z .

(b) Write the components $r_m^{(\text{sp})}$ in terms of spherical polar coordinates, and demonstrate that, for $m = +1, 0, -1$,

$$r_m^{(\text{sp})} = \sqrt{\frac{4\pi}{3}} r Y_1^m(\theta, \phi) \quad (7.157)$$

where the $Y_\ell^m(\theta, \phi)$ are the standard spherical harmonics for $\ell = 1$ as listed, for example on page 337 of (Shankar, 1994).

Exercise 7.8

(a) Use the definition of components in eqn (7.156) with the standard Cartesian expansion $\mathbf{A} = \sum_{i=1}^3 A_i \hat{\mathbf{e}}_i$ to show that the spherical components can be written in terms of the Cartesian ones as

$$A_m^{(\text{sp})} = \sum_{i=1}^3 T_{mi} A_i \quad \text{or, equivalently,} \quad \begin{bmatrix} A_{+1}^{(\text{sp})} \\ A_0^{(\text{sp})} \\ A_{-1}^{(\text{sp})} \end{bmatrix} = \mathbf{T} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \quad (7.158)$$

where

$$T_{mi} = \hat{\mathbf{e}}_m^{(\text{sp})*} \cdot \hat{\mathbf{e}}_i \quad (7.159)$$

(b) Demonstrate that the matrix \mathbf{T} must be unitary, and check that the matrix you wrote is indeed unitary.

(c) Using the resolution of unity eqn (7.155) or otherwise, show that the equation $\mathbf{B} = \mathcal{F}\mathbf{A}$ can be written as the equivalent component equation

$$B_m^{(\text{sp})} = \sum_{m'=+1,0,-1} F_{mm'}^{(\text{sp})} A_{m'}^{(\text{sp})} \quad \text{where} \quad \mathbf{F}^{(\text{sp})} = \mathbf{T} \mathbf{F} \mathbf{T}^\dagger \quad (7.160)$$

gives the spherical matrix $\mathbf{F}^{(\text{sp})}$ in terms of the standard Cartesian matrix \mathbf{F} defined in eqn (7.18).

Exercise 7.9 An operator \mathcal{F} is defined in terms of the unit operator \mathcal{U} and a real, anti-symmetric operator \mathcal{W} by

$$\mathcal{F} = \mathcal{U} + \mathcal{W} \quad (7.161)$$

The operator \mathcal{W} is associated with a given vector $\boldsymbol{\omega}$, as described in Lemma 7.5.1.

(a) Show that \mathcal{F} is a normal operator.

(b) Show that $\hat{\mathbf{V}}^{(3)} = \hat{\boldsymbol{\omega}}$ is an eigenvector of \mathcal{F} .

(c) Find the eigenvalues of \mathcal{F} and hence prove that \mathcal{F} is nonsingular for any value of $\boldsymbol{\omega}$.

Exercise 7.10 Consider a real, anti-symmetric operator \mathcal{N} . Suppose that the associated vector discussed in Section 7.5 is the unit vector $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2) / \sqrt{5}$. (Thus $\mathcal{N}\mathbf{V} = \hat{\mathbf{n}} \times \mathbf{V}$ for any vector \mathbf{V} .)

- (a) Show that $\hat{\mathbf{a}} = \hat{\mathbf{e}}_3$ is a suitable choice of the unit vector $\hat{\mathbf{a}}$ discussed in Section 7.13. Use it to find three eigenvectors of \mathcal{N} . Show that they are orthonormal, using the complex definition of inner product from eqn (7.83).
- (b) Consider now an operator defined by $\mathcal{F} = \mathcal{U} + 2\mathcal{N} - 3\mathcal{N}^2$, where \mathcal{U} is the unit operator. Find eigenvectors and eigenvalues of \mathcal{F} .

KINEMATICS OF ROTATION

In this chapter, we develop the techniques needed to define the location and orientation of a moving rigid body. Roughly speaking, rotation can be defined as what a rigid body does. For example, imagine an artist's construction consisting of straight sticks of various lengths glued together at their ends to make a rigid structure. As you turn such a construction in your hands, or move it closer for a better look, you will notice that the lengths of the sticks, and the angles between them, do not change. Thinking of those sticks as vectors, their general motion can be described by a class of linear operators called rotation operators, which have the special property that they preserve all vector lengths and relative orientations.

8.1 Characterization of Rigid Bodies

The concept of a *rigid body* is an idealization, since all real objects have some degree of elasticity. However, the theory in the present and following chapters, based on this idealization, provides a good first approximation to the behavior of many real objects.

Definition 8.1.1: Definition of Rigid Body

A rigid body can be defined as a collection of point masses such that the distances between them do not change. If \mathbf{r}_l and \mathbf{r}_n are the locations of any two masses m_l and m_n in the body, relative to some inertial coordinate system, the body is rigid if and only if the distances d_{ln} defined for all l, n values by

$$\mathbf{r}_l - \mathbf{r}_n = \mathbf{d}_{ln} \quad \text{and} \quad \|\mathbf{d}_{ln}\| = d_{ln} \quad (8.1)$$

remain constant as the body moves.

We will refer to vectors \mathbf{d}_{ln} between masses m_l and m_n as *internal vectors*.

The above definition implies that the dot product of any two internal vectors is a constant, regardless of where the masses occur in the rigid body.

Lemma 8.1.2: Constancy of Dot Products

For any masses m_a, m_b, m_p, m_q of a rigid body,

$$\mathbf{d}_{ab} \cdot \mathbf{d}_{pq} = \text{constant} \quad (8.2)$$

Proof: The lemma is proved in two stages. First consider any three distinct masses m_a, m_b, m_p and the vectors between pairs of them. They form a triangle so that

$$\mathbf{d}_{ap} - \mathbf{d}_{bp} = \mathbf{d}_{ab} \quad (8.3)$$

As the rigid body moves, this triangle will remain anchored to the same three masses.

Now calculate the squared magnitude of the left and right sides of eqn (8.3),

$$\|\mathbf{d}_{ap} - \mathbf{d}_{bp}\|^2 = \|\mathbf{d}_{ab}\|^2 \quad \text{or} \quad d_{ap}^2 - 2(\mathbf{d}_{ap} \cdot \mathbf{d}_{bp}) + d_{bp}^2 = d_{ab}^2 \quad (8.4)$$

The constancy defined in eqn (8.1) then implies that the squared terms in eqn (8.4) are all constants, and hence that the dot product must also be constant. The dot product of vectors \mathbf{d}_{ap} and \mathbf{d}_{bp} , both of which start from mass m_p , must therefore also remain constant.

Now consider *any* two internal vectors of the rigid body. Call them \mathbf{d}_{ab} and \mathbf{d}_{pq} . Picking some other mass, which will be labeled with index c , we can write

$$\mathbf{d}_{ab} = \mathbf{d}_{ac} - \mathbf{d}_{bc} \quad \text{and} \quad \mathbf{d}_{pq} = \mathbf{d}_{pc} - \mathbf{d}_{qc} \quad (8.5)$$

Therefore the expression

$$\mathbf{d}_{ab} \cdot \mathbf{d}_{pq} = (\mathbf{d}_{ac} - \mathbf{d}_{bc}) \cdot (\mathbf{d}_{pc} - \mathbf{d}_{qc}) \quad (8.6)$$

contains only dot products of internal vectors originating at the common single mass m_c . But all such dot products have just been proved to be constant, so $\mathbf{d}_{ab} \cdot \mathbf{d}_{pq}$ must be a constant, which completes the proof of the lemma. \square

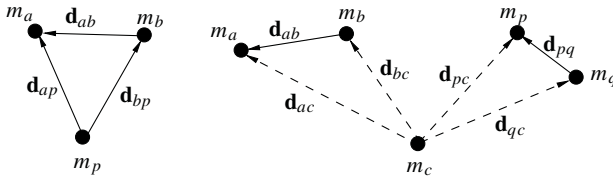


FIG. 8.1. Relations among internal vectors.

The dot product of two internal vectors is the product of their magnitudes, which are constant by the above definition, and the cosine of the angle between them. Thus the above definition and lemma also establish that the relative angle between any two internal vectors must remain constant as the body moves.

8.2 The Center of Mass of a Rigid Body

The center of mass \mathbf{R} of any collection of point masses, including that of a rigid body, is given in eqn (1.32) as

$$\mathbf{R} = \frac{1}{M} \sum_{n=1}^N m_n \mathbf{r}_n \quad (8.7)$$

The relative position vector $\boldsymbol{\rho}_n$ of mass m_n is then defined in eqn (1.33) by the equation

$$\mathbf{r}_n = \mathbf{R} + \boldsymbol{\rho}_n \quad (8.8)$$

The relative position vectors may be used to give an alternate characterization of a rigid body.

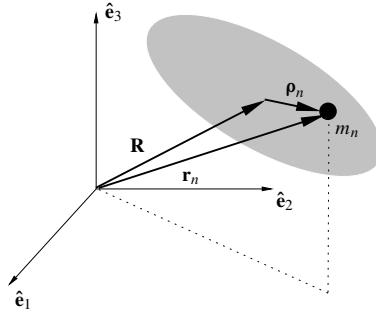


FIG. 8.2. Center of mass and relative vectors for a rigid body. A typical mass m_n of the body is shown.

Lemma 8.2.1: Constant Dot Product of Relative Position Vectors

A body is a rigid body if and only if, for all masses m_l and m_n ,

$$\boldsymbol{\rho}_l \cdot \boldsymbol{\rho}_n = \text{constant} \quad (8.9)$$

Proof: First show that eqn (8.9) implies the constancy of d_{ln} , and hence that the body is a rigid body according to Definition 8.1.1. Equation (8.8) shows that any internal vector may be written

$$\mathbf{d}_{ln} = \mathbf{r}_l - \mathbf{r}_n = \boldsymbol{\rho}_l - \boldsymbol{\rho}_n \quad (8.10)$$

since the \mathbf{R} terms cancel. Thus

$$d_{ln}^2 = \mathbf{d}_{ln} \cdot \mathbf{d}_{ln} = (\boldsymbol{\rho}_l - \boldsymbol{\rho}_n) \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}_n) \quad (8.11)$$

Equation (8.9) implies that all dot products in the expansion of the right side of eqn (8.11) are constant. Hence each d_{ln} is constant, as was to be proved.

Now prove the converse, that Definition 8.1.1 implies eqn (8.9). Use eqns (8.7, 8.8) to write, for any mass m_n ,

$$\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R} = \frac{1}{M} \sum_{p=1}^N m_p (\mathbf{r}_n - \mathbf{r}_p) = \frac{1}{M} \sum_{p=1}^N m_p \mathbf{d}_{np} \quad (8.12)$$

Using eqn (8.12), the expression $\boldsymbol{\rho}_l \cdot \boldsymbol{\rho}_n$ in eqn (8.9) becomes

$$\boldsymbol{\rho}_l \cdot \boldsymbol{\rho}_n = \frac{1}{M^2} \sum_{p=1}^N \sum_{q=1}^N m_p m_q \mathbf{d}_{lp} \cdot \mathbf{d}_{nq} \quad (8.13)$$

which contains only dot products of the form $\mathbf{d}_{lp} \cdot \mathbf{d}_{nq}$, all of which were proved constant by Lemma 8.1.2, which completes the proof. \square

In general, the center of mass will not be at the location of one of the point masses. In fact, for a hollow body like a basketball or a teacup, the center of mass may be at some distance from the masses. But eqn (8.9) with $l = n$ implies that the distance of the center of mass from any of the point masses is a constant. The center of mass moves rigidly with the body just as if it were one of the point masses.

8.3 General Definition of Rotation Operator

Rigid bodies have been defined by the condition that the dot product of any two relative position vectors $\rho_l \cdot \rho_n$ must remain constant as the body rotates. We now investigate a class of linear operators called *rotation operators* that preserve the inner product of *any* two vectors and are therefore appropriate for describing the rotation of rigid bodies. These rotation operators will be applied to the kinematics of rigid bodies in Section 8.9.

The first property of rotation operators is linearity. Let a general rotation operator \mathcal{R} transform a general vector \mathbf{V} into the vector $\mathbf{V}^{(R)}$,

$$\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V} \quad (8.14)$$

Since we want operators that reproduce the behavior of rigid bodies, the first requirement placed on this operator must be that a triangle of internal vectors such as eqn (8.3) must be transformed into the same triangle of transformed vectors with

$$\mathbf{d}_{ap}^{(R)} - \mathbf{d}_{bp}^{(R)} = \mathbf{d}_{ab}^{(R)} \quad (8.15)$$

or, introducing the operator,

$$\mathcal{R}\mathbf{d}_{ap} - \mathcal{R}\mathbf{d}_{bp} = \mathcal{R}\mathbf{d}_{ab} = \mathcal{R}(\mathbf{d}_{ap} - \mathbf{d}_{bp}) \quad (8.16)$$

This condition is satisfied by requiring \mathcal{R} to be a *linear* operator, as defined in eqn (7.1) of Chapter 7, so that for any vectors \mathbf{V} and \mathbf{W} ,

$$\mathcal{R}(\alpha\mathbf{V} + \beta\mathbf{W}) = \alpha\mathcal{R}\mathbf{V} + \beta\mathcal{R}\mathbf{W} \quad (8.17)$$

However, linearity alone is not sufficient. In addition to being linear, the rotation operator must satisfy the conditions of the following definition.

Definition 8.3.1: Rotation Operator Defined

A rotation operator, sometimes referred to as a rotation, is defined as a linear operator that also satisfies any one of the following three equivalent definitions. Each of the definitions implies the other two.

1. Given vectors \mathbf{V} and \mathbf{W} , define $\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V}$ and $\mathbf{W}^{(R)} = \mathcal{R}\mathbf{W}$. Then a linear operator \mathcal{R} is a rotation operator if and only if

$$\mathbf{V} \cdot \mathbf{W} = \mathbf{V}^{(R)} \cdot \mathbf{W}^{(R)} \quad (8.18)$$

is satisfied for any, arbitrary \mathbf{V} , \mathbf{W} .

2. The linear operator \mathcal{R} is a rotation operator if and only if there is some orthonormal triad of vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ obeying

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \quad (8.19)$$

such that $\hat{\mathbf{e}}_1^{(R)}, \hat{\mathbf{e}}_2^{(R)}, \hat{\mathbf{e}}_3^{(R)}$ is also an orthonormal triad of vectors, obeying

$$\hat{\mathbf{e}}_i^{(R)} \cdot \hat{\mathbf{e}}_j^{(R)} = \delta_{ij} \quad (8.20)$$

where $\hat{\mathbf{e}}_i^{(R)} = \mathcal{R}\hat{\mathbf{e}}_i$ for each index $i = 1, 2, 3$.

3. A linear operator \mathcal{R} is a rotation operator if and only if it possesses an inverse and its inverse is equal to its transpose,

$$\mathcal{R}^{-1} \text{ exists, and } \mathcal{R}^{-1} = \mathcal{R}^T \tag{8.21}$$

so that

$$\mathcal{R}^T \mathcal{R} = \mathcal{U} = \mathcal{R} \mathcal{R}^T \tag{8.22}$$

where \mathcal{U} is the unit operator. As discussed in Section 7.5, this is the definition of an orthogonal operator, so this definition requires \mathcal{R} to be a real, linear, orthogonal operator.

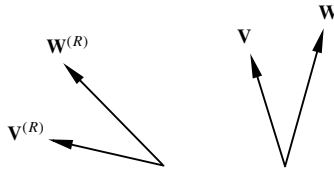


FIG. 8.3. The lengths of the rotated vectors $\mathbf{V}^{(R)}$ and $\mathbf{W}^{(R)}$ are the same as the original vectors. Also the angle between them is same as between the original ones.

Proof: (Proof of equivalence) We now prove that the condition in each of these definitions implies the condition in the following one, in the pattern $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. This implies that an operator \mathcal{R} satisfying any of the definitions will also satisfy the other two, and thus that the three definitions are equivalent.

Since the vectors \mathbf{V} and \mathbf{W} in Definition 1 are assumed arbitrary, they can be taken to be $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$. Thus the condition of Definition 1 implies that of Definition 2.

The condition in Definition 2 can be written, for $i, j = 1, 2, 3$,

$$\hat{\mathbf{e}}_i^{(R)} \cdot \hat{\mathbf{e}}_j^{(R)} = \mathcal{R} \hat{\mathbf{e}}_i \cdot \mathcal{R} \hat{\mathbf{e}}_j = \delta_{ij} \tag{8.23}$$

Using the definition of the identity and transpose operators from Section 7.1, this becomes

$$\hat{\mathbf{e}}_i \cdot \mathcal{R}^T \mathcal{R} \hat{\mathbf{e}}_j = \delta_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{U} \hat{\mathbf{e}}_j \quad \text{or} \quad (\mathcal{R}^T \mathcal{R})_{ij} = U_{ij} \tag{8.24}$$

Since all of the matrix elements of the operators $(\mathcal{R}^T \mathcal{R})$ and \mathcal{U} are equal, the operators are equal and⁴²

$$\mathcal{R}^T \mathcal{R} = \mathcal{U} \tag{8.25}$$

Taking the determinants of both sides of eqn (8.25) and using eqn (7.35), shows that

$$(\det \mathcal{R})^2 = (\det \mathcal{R}^T)(\det \mathcal{R}) = \det \mathcal{U} = 1 \tag{8.26}$$

with the result that

$$\det \mathcal{R} = \pm 1 \neq 0 \tag{8.27}$$

⁴²As proved for matrices in Theorem B.22.1, eqn (8.25) is actually a necessary and sufficient condition for \mathcal{R} to be an orthogonal operator. That proof is repeated here.

Thus \mathcal{R} is nonsingular, and, \mathcal{R}^{-1} exists. Using $\mathcal{U} = \mathcal{R}\mathcal{R}^{-1}$ and eqn (8.25) gives

$$\mathcal{R}^T = \mathcal{R}^T\mathcal{U} = \mathcal{R}^T\mathcal{R}\mathcal{R}^{-1} = \mathcal{U}\mathcal{R}^{-1} = \mathcal{R}^{-1} \quad (8.28)$$

which is the condition in Definition 3 and implies eqn (8.22).

Introducing the operator \mathcal{R} , the condition in Definition 1 can be written using the definition of the transpose operator in eqn (7.13),

$$\mathbf{V} \cdot \mathbf{W} = \mathbf{V}^{(R)} \cdot \mathbf{W}^{(R)} = (\mathcal{R}\mathbf{V}) \cdot (\mathcal{R}\mathbf{W}) = \mathbf{V} \cdot (\mathcal{R}^T\mathcal{R}\mathbf{W}) \quad (8.29)$$

Thus the orthogonality condition $\mathcal{R}^T\mathcal{R} = \mathcal{U}$ from Definition 3 implies the condition of Definition 1, completing the circle of inference. \square

8.4 Rotation Matrices

From the general discussion of linear operators in Section 7.2, we know that $\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V}$ implies and is implied by the equation

$$V_i^{(R)} = \sum_{j=1}^3 R_{ij} V_j \quad \text{where} \quad R_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{R}\hat{\mathbf{e}}_j \quad (8.30)$$

are the matrix elements of the matrix \mathbf{R} associated with the operator \mathcal{R} .

Equation (8.30) gives the components $V_i^{(R)}$ of $\mathbf{V}^{(R)}$ in the $\hat{\mathbf{e}}_i$ basis, in terms of the components V_j of the original vector \mathbf{V} in that same basis. It may also be written in matrix form, as

$$[V^{(R)}] = \mathbf{R}[V] \quad (8.31)$$

where $[V]$ is the column vector of components V_j and $[V^{(R)}]$ is the column vector of components $V_i^{(R)}$.

As an example of a rotation operator, consider the rotation denoted $\mathcal{R}[\theta\hat{\mathbf{e}}_3]$, a rotation by angle θ about the $\hat{\mathbf{e}}_3$ axis. The second of eqn (8.30) shows that the matrix element R_{ij} is the dot product of $\hat{\mathbf{e}}_i$ with the rotated image of $\hat{\mathbf{e}}_j$. Thus $R_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j^{(R)}$. Evaluating these dot products gives the matrix

$$\mathbf{R}[\theta\hat{\mathbf{e}}_3] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (8.32)$$

The reader should verify all of the matrix elements of eqn (8.32), and also check that this matrix, and hence the associated operator, are orthogonal and have determinant equal to plus one. A general prescription for deriving the operators and matrices for rotation about any axis will be given in Section 8.18.

8.5 Some Properties of Rotation Operators

By Definition 2 of Section 8.3, the rotated images of the basis vectors $\hat{\mathbf{e}}_i$ are also three, mutually orthogonal unit vectors and hence form a basis in the space. Like any vectors, these rotated basis vectors may be expanded in the original basis, as

$$\hat{\mathbf{e}}_i^{(R)} = \sum_{a=1}^3 \hat{\mathbf{e}}_a (\hat{\mathbf{e}}_a \cdot \hat{\mathbf{e}}_i^{(R)}) = \sum_{a=1}^3 \hat{\mathbf{e}}_a (\hat{\mathbf{e}}_a \cdot \mathcal{R}\hat{\mathbf{e}}_i) = \sum_{a=1}^3 \hat{\mathbf{e}}_a R_{ai} = \sum_{a=1}^3 R_{ia}^T \hat{\mathbf{e}}_a \quad (8.33)$$

Note that the basis vectors transform using the transposed matrix \mathbf{R}^T .

It is useful to define rotated versions of the Kroeneker delta function and the Levi-Civita function defined in Section A.5. It follows from Definition 1 of Section 8.3, that the rotated Kroeneker-delta function is the same as the original one,

$$\delta_{ij}^{(R)} = \hat{\mathbf{e}}_i^{(R)} \cdot \hat{\mathbf{e}}_j^{(R)} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \quad (8.34)$$

Using eqn (8.33), the rotated Levi-Civita function may be expanded as

$$\varepsilon_{ijk}^{(R)} = \hat{\mathbf{e}}_i^{(R)} \times \hat{\mathbf{e}}_j^{(R)} \cdot \hat{\mathbf{e}}_k^{(R)} = \sum_{a=1}^3 \sum_{b=1}^3 \sum_{c=1}^3 R_{ai} R_{bj} R_{ck} \hat{\mathbf{e}}_a \times \hat{\mathbf{e}}_b \cdot \hat{\mathbf{e}}_c = \sum_{a=1}^3 \sum_{b=1}^3 \sum_{c=1}^3 R_{ai} R_{bj} R_{ck} \varepsilon_{abc} \quad (8.35)$$

It follows from eqn (B.37) and the properties of ε_{abc} listed in Section A.5 that

$$\varepsilon_{123}^{(R)} = \sum_{a=1}^3 \sum_{b=1}^3 \sum_{c=1}^3 R_{a1} R_{b2} R_{c3} \varepsilon_{abc} = |\mathbf{R}| = \det \mathcal{R} \quad (8.36)$$

Since exchange of two indices of $\varepsilon_{ijk}^{(R)}$ implies the exchange of two corresponding indices of ε_{abc} in eqn (8.35), one obtains

$$\varepsilon_{ijk}^{(R)} = \hat{\mathbf{e}}_i^{(R)} \times \hat{\mathbf{e}}_j^{(R)} \cdot \hat{\mathbf{e}}_k^{(R)} = \det \mathcal{R} \varepsilon_{ijk} \quad (8.37)$$

8.6 Proper and Improper Rotation Operators

Equation (8.27) states that the determinant of a rotation operator must be either +1 or -1. Rotation operators with $\det \mathcal{R} = +1$ are called *proper rotation operators*, or *proper rotations*. Those with $\det \mathcal{R} = -1$ are called *improper rotation operators*. These operators are also referred to as proper or improper orthogonal operators.

For example, consider the identity operator \mathcal{U} . It is orthogonal, since $\mathcal{U}^T = \mathcal{U}$ and therefore $\mathcal{U}\mathcal{U}^T = \mathcal{U}^2 = \mathcal{U}$. The identity can be thought of as a degenerate proper rotation (by zero angle), since $\det \mathcal{U} = +1$.

But the total inversion operator $\mathcal{T} = -\mathcal{U}$, which converts every vector \mathbf{V} into $-\mathbf{V}$, is also orthogonal since $\mathcal{T}\mathcal{T}^T = \mathcal{U}^2 = \mathcal{U}$. But $\det \mathcal{T} = -1$ and so the total inversion operator is an improper rotation.

The distinction between proper and improper rotations is of no importance for dot products, since, by Definition 1 of Section 8.3,

$$\mathbf{V}^{(R)} \cdot \mathbf{W}^{(R)} = \mathbf{V} \cdot \mathbf{W} \quad (8.38)$$

in either case.

But cross products are sensitive to the distinction, as proved in the following theorem.

Theorem 8.6.1: Rotated Cross Products

With the definitions $\mathbf{A}^{(R)} = \mathcal{R}\mathbf{A}$, $\mathbf{B}^{(R)} = \mathcal{R}\mathbf{B}$, and $\mathbf{C}^{(R)} = \mathcal{R}\mathbf{C}$,

$$\mathbf{A} = \mathbf{B} \times \mathbf{C} \quad \text{implies} \quad \mathbf{A}^{(R)} = (\det \mathcal{R}) (\mathbf{B}^{(R)} \times \mathbf{C}^{(R)}) \quad (8.39)$$

Proof: Writing $\mathbf{A} = \sum_{k=1}^3 A_k \hat{\mathbf{e}}_k$, with a similar expansion for \mathbf{B} and \mathbf{C} , it follows from the linearity of operator \mathcal{R} that

$$\mathbf{A}^{(R)} = \mathcal{R}\mathbf{A} = \mathcal{R} \sum_{k=1}^3 A_k \hat{\mathbf{e}}_k = \sum_{k=1}^3 A_k \mathcal{R}\hat{\mathbf{e}}_k = \sum_{k=1}^3 A_k \hat{\mathbf{e}}_k^{(R)} \quad (8.40)$$

with a similar expressions for $\mathbf{B}^{(R)}$ and $\mathbf{C}^{(R)}$. Thus

$$\mathbf{B}^{(R)} \times \mathbf{C}^{(R)} = \sum_{i=1}^3 \sum_{j=1}^3 B_i C_j \hat{\mathbf{e}}_i^{(R)} \times \hat{\mathbf{e}}_j^{(R)} = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 B_i C_j (\hat{\mathbf{e}}_i^{(R)} \times \hat{\mathbf{e}}_j^{(R)} \cdot \hat{\mathbf{e}}_k^{(R)}) \hat{\mathbf{e}}_k^{(R)} \quad (8.41)$$

where the last expression expands the vector $\hat{\mathbf{e}}_i^{(R)} \times \hat{\mathbf{e}}_j^{(R)}$ in the rotated basis. Using eqns (8.37, 8.40), and the expansion of cross products from eqn (A.16), then gives

$$\mathbf{B}^{(R)} \times \mathbf{C}^{(R)} = (\det \mathcal{R}) \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} B_i C_j \hat{\mathbf{e}}_k^{(R)} = (\det \mathcal{R}) \sum_{k=1}^3 A_k \hat{\mathbf{e}}_k^{(R)} = (\det \mathcal{R}) \mathbf{A}^{(R)} \quad (8.42)$$

as was to be proved. □

We will be concerned almost entirely with proper rotations, for which $\det \mathcal{R} = +1$. Using eqn (8.39), we may now give a necessary and sufficient condition for \mathcal{R} to be a proper rotation operator.

Definition 8.6.2: Proper Rotation Operator

The linear operator \mathcal{R} is a proper rotation operator if and only if it satisfies Definition 2 of Section 8.3 as well as the condition that the original and the rotated basis vectors $\hat{\mathbf{e}}_i^{(R)} = \mathcal{R}\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$ both form right-handed systems, obeying

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \quad \text{and} \quad \hat{\mathbf{e}}_1^{(R)} \times \hat{\mathbf{e}}_2^{(R)} = \hat{\mathbf{e}}_3^{(R)} \quad (8.43)$$

8.7 The Rotation Group

As for any linear operator, the product of two rotation operators is defined to mean successive application. Thus, for any vector \mathbf{V} , the product $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$ implies that

$$\mathcal{R}\mathbf{V} = (\mathcal{R}_1\mathcal{R}_2)\mathbf{V} = \mathcal{R}_1(\mathcal{R}_2\mathbf{V}) \quad (8.44)$$

in which the right operator \mathcal{R}_2 is applied first to \mathbf{V} and the left operator \mathcal{R}_1 is then applied to the result.

A set of objects is said to form a *group* if a binary operation called *group multiplication* of the objects is defined and if a set of group axioms is satisfied. The common usage is to say that the objects form a group *under* that particular group multiplication. We show that proper rotations form a group under the operator multiplication defined in eqn (8.44).

1. The first axiom is closure. The group product of two objects must be an object in the same group. Thus, the product of two proper rotations must also be a proper rotation. If $\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$ and rotations \mathcal{R}_1 and \mathcal{R}_2 both satisfy Definition 3 of Section 8.3, then

$$\mathcal{R}^{-1} = (\mathcal{R}_1\mathcal{R}_2)^{-1} = \mathcal{R}_2^{-1}\mathcal{R}_1^{-1} = \mathcal{R}_2^T\mathcal{R}_1^T = (\mathcal{R}_1\mathcal{R}_2)^T = \mathcal{R}^T \quad (8.45)$$

shows that \mathcal{R} also satisfies the same definition and hence is also a rotation. Moreover, if \mathcal{R}_1 and \mathcal{R}_2 are proper rotations, then

$$\det \mathcal{R} = \det (\mathcal{R}_1\mathcal{R}_2) = \det \mathcal{R}_1 \det \mathcal{R}_2 = (+1)(+1) = +1 \quad (8.46)$$

shows that \mathcal{R} is also a proper rotation. Thus closure is proved.

2. There must be an identity in the group such that pre- or post-multiplication of any object by that identity does not change the object. We have previously seen that the identity, or unity, operator \mathcal{U} is a proper rotation operator.
3. Every object in the group must have an inverse in the group, such that pre- or post-multiplication of that object by its inverse yields the identity object. As noted in the proof of Definition 3 of Section 8.3, the inverse \mathcal{R}^{-1} of a proper rotation always exists. To see that the inverse is also a proper rotation, set $\mathcal{B} = \mathcal{R}^{-1} = \mathcal{R}^T$, where \mathcal{R} is a proper rotation operator. Then

$$\mathcal{B}^{-1} = (\mathcal{R}^{-1})^{-1} = \mathcal{R} = (\mathcal{R}^T)^T = \mathcal{B}^T \quad (8.47)$$

which shows that \mathcal{B} is a rotation operator. Also $1 = \det \mathcal{R} = \det \mathcal{B}^T = \det \mathcal{B}$ shows that \mathcal{B} is a proper rotation.

4. Group multiplication must be associative. Proper rotation operators obey $(\mathcal{R}_1\mathcal{R}_2)\mathcal{R}_3 = \mathcal{R}_1(\mathcal{R}_2\mathcal{R}_3)$ since, as discussed in Section 7.1, both sides are equal to $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$.

The group of proper rotations is designated $SO(3)$, which stands for the special (determinant equal to $+1$), orthogonal group in three dimensions.

If the product of a pair of elements gives a result independent of their order, the group is said to be *Abelian*. The rotation operators form a non-Abelian group. A finite rotation $\mathcal{R}_1\mathcal{R}_2$ will not usually give the same end result as a finite rotation $\mathcal{R}_2\mathcal{R}_1$.

For example, place a closed book on the table in front of you, as if preparing to open and read it. Rotate it by 90° about a vertical axis, and then by 90° about an axis running from your left to your right hands. Now replace the book in its original position and do the same two rotations in reverse order. You will see that the final orientation of the book is indeed different.

We say that finite rotations do not *commute*. Writing the *commutator* of \mathcal{R}_1 and \mathcal{R}_2 as

$$[\mathcal{R}_1, \mathcal{R}_2]_c = \mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2\mathcal{R}_1 \quad (8.48)$$

we express this result by saying that proper rotations have in general a nonzero commutator and so form a non-Abelian group.

8.8 Kinematics of a Rigid Body

Let a rigid body have a center of mass \mathbf{R} and relative position vectors $\boldsymbol{\rho}_n$. As the rigid body moves, both \mathbf{R} and the $\boldsymbol{\rho}_n$ will be functions of time. At $t=0$, the position of the mass m_n relative to the origin of some inertial coordinate system will be

$$\mathbf{r}_n(0) = \mathbf{R}(0) + \boldsymbol{\rho}_n(0) \quad (8.49)$$

and at time t the location will be

$$\mathbf{r}_n(t) = \mathbf{R}(t) + \boldsymbol{\rho}_n(t) \quad (8.50)$$

As proved in Lemma 8.2.1, the dot product of any pair of relative position vectors is constant (including that of a vector with itself, giving its magnitude squared). Hence, these dot products also will be the same at all times t ,

$$\boldsymbol{\rho}_l(t) \cdot \boldsymbol{\rho}_n(t) = \boldsymbol{\rho}_l(0) \cdot \boldsymbol{\rho}_n(0) \quad (8.51)$$

Thus the problem of parameterizing the orientation of a rigid body (by which we mean defining the location of all of its masses m_n once its center of mass is known) boils down to finding an expression for the evolution of vectors $\boldsymbol{\rho}_n(t)$ that obey eqn (8.51) at all times t .

The first step toward such a parameterization is to construct a system of coordinates tied to the rigid body. With the rigid body at its initial position and orientation at time zero, it is always possible to select three non-coplanar relative position vectors. For simplicity, suppose that these are the first three of them $\boldsymbol{\rho}_1(0)$, $\boldsymbol{\rho}_2(0)$, $\boldsymbol{\rho}_3(0)$. Now

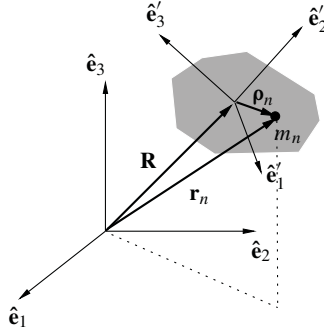


FIG. 8.4. The body system unit vectors $\hat{\mathbf{e}}'_i$ are rigidly fixed in the moving body. The relative position vector $\boldsymbol{\rho}_n$ is also fixed in the body. Hence its components in the body system are constants.

apply the Schmidt orthogonalization method⁴³ to these vectors to construct a right-handed, orthonormal set of unit vectors $\hat{\mathbf{e}}'_1(0), \hat{\mathbf{e}}'_2(0), \hat{\mathbf{e}}'_3(0)$. Thus, by construction,

$$\hat{\mathbf{e}}'_i(0) = \sum_{k=1}^3 \alpha_{ik} \boldsymbol{\rho}_k(0) \quad \text{and} \quad \delta_{ij} = \hat{\mathbf{e}}'_i(0) \cdot \hat{\mathbf{e}}'_j(0) = \sum_{k=1}^3 \sum_{l=1}^3 \alpha_{ik} \alpha_{jl} \boldsymbol{\rho}_k(0) \cdot \boldsymbol{\rho}_l(0) \quad (8.52)$$

where the α_{ik} are coefficients specified by the Schmidt method.

Now define vectors $\hat{\mathbf{e}}'_i(t)$ at time t by

$$\hat{\mathbf{e}}'_i(t) = \sum_{k=1}^3 \alpha_{ik} \boldsymbol{\rho}_k(t) \quad (8.53)$$

where the α_{ik} factors in eqn (8.53) are defined to be the same as those in eqn (8.52). It follows from eqns (8.51, 8.52) that

$$\begin{aligned} \hat{\mathbf{e}}'_i(t) \cdot \hat{\mathbf{e}}'_j(t) &= \sum_{k=1}^3 \sum_{l=1}^3 \alpha_{ik} \alpha_{jl} \boldsymbol{\rho}_k(t) \cdot \boldsymbol{\rho}_l(t) \\ &= \sum_{k=1}^3 \sum_{l=1}^3 \alpha_{ik} \alpha_{jl} \boldsymbol{\rho}_k(0) \cdot \boldsymbol{\rho}_l(0) = \hat{\mathbf{e}}'_i(0) \cdot \hat{\mathbf{e}}'_j(0) = \delta_{ij} \end{aligned} \quad (8.54)$$

which shows that the $\hat{\mathbf{e}}'_i(t)$ are also an orthonormal set of unit vectors for all t .

The coordinate system consisting of these three orthonormal vectors $\hat{\mathbf{e}}'_i(t)$, with its origin at the center of mass, will be called the *body system*. The relative position vectors can be expanded in this body system as

$$\boldsymbol{\rho}_n(t) = \sum_{i=1}^3 \rho'_{ni}(t) \hat{\mathbf{e}}'_i(t) \quad (8.55)$$

Equation (8.51) implies that the components $\rho'_{ni}(t)$ will be constants, always equal

⁴³See Section B.20.

to their values $\rho'_{ni}(0)$ at time zero,

$$\rho'_{ni}(t) = \hat{\mathbf{e}}'_i(t) \cdot \boldsymbol{\rho}_n(t) = \sum_{k=1}^3 \alpha_{ik} \boldsymbol{\rho}_k(t) \cdot \boldsymbol{\rho}_n(t) = \sum_{k=1}^3 \alpha_{ik} \boldsymbol{\rho}_k(0) \cdot \boldsymbol{\rho}_n(0) = \hat{\mathbf{e}}'_i(0) \cdot \boldsymbol{\rho}_n(0) = \rho'_{ni}(0) \quad (8.56)$$

Thus the angles between the various $\hat{\mathbf{e}}'_i(t)$ and $\boldsymbol{\rho}_n(t)$ will never change. The $\hat{\mathbf{e}}'_i(t)$ are rigidly connected to the body and turn with it as it moves.

8.9 Rotation Operators and Rigid Bodies

The time evolution of a rigid body can be systematized by defining a time-dependent rotation operator $\mathcal{R}(t)$ by the condition that it maps each $\hat{\mathbf{e}}'_i(0)$ of the body system at time zero into its value $\hat{\mathbf{e}}'_i(t)$ at time t , as proved in the following theorem.

Theorem 8.9.1 Define a time dependent operator $\mathcal{R}(t)$ by the condition that, for $i = 1, 2, 3$,

$$\hat{\mathbf{e}}'_i(t) = \mathcal{R}(t) \hat{\mathbf{e}}'_i(0) \quad (8.57)$$

It follows that $\mathcal{R}(t)$ is a proper rotation operator obeying

$$\mathcal{R}(t)^T \mathcal{R}(t) = \mathcal{U} = \mathcal{R}(t) \mathcal{R}(t)^T \quad (8.58)$$

and $\det \mathcal{R}(t) = +1$ for all time t .

It also follows that

$$\boldsymbol{\rho}_n(t) = \mathcal{R}(t) \boldsymbol{\rho}_n(0) \quad (8.59)$$

and that

$$\boldsymbol{\rho}_l(t) \cdot \boldsymbol{\rho}_n(t) = \boldsymbol{\rho}_l(0) \cdot \boldsymbol{\rho}_n(0) \quad (8.60)$$

as is required for rigid bodies.

Proof: Identify $\hat{\mathbf{e}}'_i(t)$ with the rotated basis vector $\hat{\mathbf{e}}_i^{(R)}$ in Definition 2 of Section 8.3. Since eqn (8.54) proved the $\hat{\mathbf{e}}'_i(t)$ to be an orthonormal system of basis vectors, it follows from Definition 2 that $\mathcal{R}(t)$ is a rotation operator. Hence, by equivalent Definition 3, it obeys eqn (8.58) at all times t .

A general relative position vector $\boldsymbol{\rho}_n(t)$ can be expanded in the body system $\hat{\mathbf{e}}'_i(t)$ basis as given in eqn (8.55),

$$\boldsymbol{\rho}_n(t) = \sum_{i=1}^3 \rho'_{ni}(t) \hat{\mathbf{e}}'_i(t) \quad (8.61)$$

The components in this expansion were shown in eqn (8.56) to be constants, with $\rho'_{ni}(t) = \rho'_{ni}(0)$. Thus, using the linearity of $\mathcal{R}(t)$,

$$\boldsymbol{\rho}_n(t) = \sum_{i=1}^3 \rho'_{ni}(0) \hat{\mathbf{e}}'_i(t) = \sum_{i=1}^3 \rho'_{ni}(0) \mathcal{R}(t) \hat{\mathbf{e}}'_i(0) = \mathcal{R}(t) \left(\sum_{i=1}^3 \rho'_{ni}(0) \hat{\mathbf{e}}'_i(0) \right) = \mathcal{R}(t) \boldsymbol{\rho}_n(0) \quad (8.62)$$

as was to be proved.

It then follows from the orthogonality of $\mathcal{R}(t)$ and the definition of transpose in eqn (7.13) that

$$\boldsymbol{\rho}_l(t) \cdot \boldsymbol{\rho}_n(t) = \mathcal{R}(t)\boldsymbol{\rho}_l(0) \cdot \mathcal{R}(t)\boldsymbol{\rho}_n(0) = \boldsymbol{\rho}_l(0) \cdot \mathcal{R}^T(t)\mathcal{R}(t)\boldsymbol{\rho}_n(0) = \boldsymbol{\rho}_l(0) \cdot \boldsymbol{\rho}_n(0) \quad (8.63)$$

which is eqn (8.60).

It follows from eqn (8.57) that at time zero, $\mathcal{R}(0) = \mathcal{U}$, which has determinant $+1$. Since the vectors $\boldsymbol{\rho}_n(t)$ of the rigid body, and hence the body system unit vectors $\hat{\mathbf{e}}'_i(t)$, are assumed to evolve continuously with time, the determinant cannot make a discontinuous jump to the only other possible value -1 . Thus $\det \mathcal{R}(t) = +1$, $\mathcal{R}(t)$ is a proper rotation, and the body system unit vectors $\hat{\mathbf{e}}'_i(t)$ remain a right-handed, orthonormal triad for all time t . \square

8.10 Differentiation of a Rotation Operator

We now have an operator $\mathcal{R}(t)$ that allows any vector of a rigid body at time t to be expressed in terms of that vector at time zero. But Lagrangian mechanics also needs expressions for the velocities of the point masses of the rigid body. To obtain these velocities, we now derive the time derivatives of the operator $\mathcal{R}(t)$ and of the vectors rotated by it.

Suppose that $\mathcal{R}(t)$ acts on an arbitrary constant vector \mathbf{V} to produce a time-varying rotated vector $\mathbf{V}^{(R)}(t)$ as in

$$\mathbf{V}^{(R)}(t) = \mathcal{R}(t)\mathbf{V} \quad (8.64)$$

Taking the derivative of eqn (8.64) gives

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \frac{d\mathcal{R}(t)}{dt}\mathbf{V} \quad (8.65)$$

since \mathbf{V} is a constant. The meaning of this last equation is perhaps made clearer if we express eqns (8.64, 8.65) in component form as

$$V_i^{(R)}(t) = \sum_{j=1}^3 R_{ij}(t)V_j \quad (8.66)$$

and

$$\frac{dV_i^{(R)}(t)}{dt} = \sum_{j=1}^3 \frac{dR_{ij}(t)}{dt}V_j \quad (8.67)$$

Comparison of eqns (8.65, 8.67) shows that $d\mathcal{R}(t)/dt$ is that operator such that *each* of its matrix elements is the time derivative of the corresponding matrix element

$R_{ij}(t)$. Written out, the matrix of $d\mathcal{R}(t)/dt$ is

$$\frac{d\mathbf{R}(t)}{dt} = \begin{pmatrix} \frac{dR_{11}}{dt} & \frac{dR_{12}}{dt} & \frac{dR_{13}}{dt} \\ \frac{dR_{21}}{dt} & \frac{dR_{22}}{dt} & \frac{dR_{23}}{dt} \\ \frac{dR_{31}}{dt} & \frac{dR_{32}}{dt} & \frac{dR_{33}}{dt} \end{pmatrix} \quad (8.68)$$

in which each element is differentiated.

Continuing, we use eqn (8.58) to write eqn (8.65) in the form

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \frac{d\mathcal{R}(t)}{dt}\mathcal{U}\mathbf{V} = \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T\mathcal{R}(t)\mathbf{V} = \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T\mathbf{V}^{(R)}(t) \quad (8.69)$$

where eqn (8.64) has been used to get the last equality. Defining the operator $\mathcal{W}(t)$ by

$$\mathcal{W}(t) = \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T \quad (8.70)$$

then gives

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \mathcal{W}(t)\mathbf{V}^{(R)}(t) \quad (8.71)$$

which expresses the time derivative in terms of the current value of $\mathbf{V}^{(R)}(t)$ at time t .

The real, time-varying operator $\mathcal{W}(t)$ defined by eqn (8.70) is anti-symmetric. From eqn (8.58) we have $\mathcal{U} = \mathcal{R}(t)\mathcal{R}(t)^T$. Differentiating both sides of this equation with respect to t using the product rule eqn (7.70) gives

$$\begin{aligned} 0 &= \frac{d\mathcal{U}}{dt} = \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T + \mathcal{R}(t)\frac{d\mathcal{R}(t)^T}{dt} \\ &= \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T + \mathcal{R}(t)\left(\frac{d\mathcal{R}(t)}{dt}\right)^T = \mathcal{W}(t) + \mathcal{W}(t)^T \end{aligned} \quad (8.72)$$

which implies the anti-symmetry

$$\mathcal{W}(t)^T = -\mathcal{W}(t) \quad (8.73)$$

In deriving eqn (8.72) we used the fact that taking the transpose of an operator and then differentiating it with respect to time produces the same result as doing the same two operations in reverse order. This operator identity follows from the same identity for matrices

$$\frac{d\mathbf{R}(t)^T}{dt} = \left(\frac{d\mathbf{R}(t)}{dt}\right)^T \quad (8.74)$$

which can be obtained by inspection of eqn (8.68).

In Section 7.5, we determined that the most general real, anti-symmetric operator acting on a vector is equivalent to a vector $\boldsymbol{\omega}$ acting by means of a cross product. Thus there is some vector $\boldsymbol{\omega}(t)$ such that

$$\mathcal{W}(t)\mathbf{A} = \boldsymbol{\omega}(t) \times \mathbf{A} \quad (8.75)$$

for any arbitrary vector \mathbf{A} , where vector \mathbf{A} itself may or may not be time-varying. Hence the time derivative in eqn (8.71) can be written

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \mathcal{W}(t)\mathbf{V}^{(R)}(t) = \boldsymbol{\omega}(t) \times \mathbf{V}^{(R)}(t) \quad (8.76)$$

where $\boldsymbol{\omega}(t)$ is in general time varying since $\mathcal{W}(t)$ is.

The vector $\boldsymbol{\omega}(t)$ in eqn (8.76) is called the *angular velocity vector* of the time-varying rotation. Expanding this vector in the fixed, inertial $\hat{\mathbf{e}}_i$ basis,

$$\boldsymbol{\omega}(t) = \omega_1(t)\hat{\mathbf{e}}_1 + \omega_2(t)\hat{\mathbf{e}}_2 + \omega_3(t)\hat{\mathbf{e}}_3 \quad (8.77)$$

the matrix of operator $\mathcal{W}(t)$ can be obtained from eqns (7.38, 7.39) with the time dependence added. The matrix elements are

$$W_{ij}(t) = \sum_{k=1}^3 \varepsilon_{ikj} \omega_k(t) \quad (8.78)$$

and, written out, the matrix is

$$\mathbf{W}(t) = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \quad (8.79)$$

An operator differential equation for $\mathcal{R}(t)$ can also be written. Multiply both sides of eqn (8.70) from the right by $\mathcal{R}(t)$ to get

$$\mathcal{W}(t)\mathcal{R}(t) = \frac{d\mathcal{R}(t)}{dt}\mathcal{R}(t)^T\mathcal{R}(t) = \frac{d\mathcal{R}(t)}{dt}\mathcal{U} = \frac{d\mathcal{R}(t)}{dt} \quad (8.80)$$

and hence the differential equation

$$\frac{d\mathcal{R}(t)}{dt} = \mathcal{W}(t)\mathcal{R}(t) \quad (8.81)$$

8.11 Meaning of the Angular Velocity Vector

First, it is useful to establish some notation for later use. The angular velocity vector $\boldsymbol{\omega}(t)$ has a magnitude $\omega(t)$ and an associated unit vector $\hat{\boldsymbol{\omega}}(t)$ which we will typically

denote as $\hat{\mathbf{n}}(t)$ in order to make it easier to distinguish from $\boldsymbol{\omega}(t)$ itself. Thus

$$\hat{\mathbf{n}}(t) = \hat{\boldsymbol{\omega}}(t) = \frac{\boldsymbol{\omega}(t)}{\omega(t)} \quad (8.82)$$

In component form, this equation is

$$n_i(t) = \frac{\omega_i(t)}{\omega(t)} \quad (8.83)$$

for $i = 1, 2, 3$, where the unit vector $\hat{\mathbf{n}}(t)$ has the expansion

$$\hat{\mathbf{n}}(t) = n_1(t) \hat{\mathbf{e}}_1 + n_2(t) \hat{\mathbf{e}}_2 + n_3(t) \hat{\mathbf{e}}_3 \quad (8.84)$$

Hence, the angular velocity may be written as a magnitude times a unit vector direction,

$$\boldsymbol{\omega}(t) = \omega(t) \hat{\mathbf{n}}(t) \quad \text{or in component form} \quad \omega_i(t) = \omega(t) n_i(t) \quad (8.85)$$

Dividing eqns (8.78, 8.79) by the magnitude $\omega(t)$ allows one to define a new operator $\mathcal{N}(t) = \mathcal{W}(t)/\omega(t)$ with matrix elements based on the axis unit vector $\hat{\mathbf{n}}(t)$. Thus

$$\frac{W_{ij}(t)}{\omega(t)} = N_{ij}(t) = \sum_{k=1}^3 \varepsilon_{ikj} n_k(t) \quad (8.86)$$

with matrix

$$\frac{\mathbf{W}(t)}{\omega(t)} = \mathbf{N}(t) = \begin{pmatrix} 0 & -n_3(t) & n_2(t) \\ n_3(t) & 0 & -n_1(t) \\ -n_2(t) & n_1(t) & 0 \end{pmatrix} \quad (8.87)$$

such that

$$\mathcal{W}(t) = \omega(t)\mathcal{N}(t) \quad \text{and} \quad \mathbf{W}(t) = \omega(t)\mathbf{N}(t) \quad (8.88)$$

Also, dividing both sides of eqn (8.75) by the magnitude $\omega(t)$ shows that the action of the operator $\mathcal{N}(t)$ is equivalent to a cross product with the unit vector $\hat{\mathbf{n}}(t)$ as in

$$\mathcal{N}(t)\mathbf{A} = \hat{\mathbf{n}}(t) \times \mathbf{A} \quad (8.89)$$

for an arbitrary vector \mathbf{A} .

With this notation established, now consider the angular velocity vector. Multiplying eqn (8.76) by dt gives the differential relation

$$d\mathbf{V}^{(R)}(t) = \mathcal{W}(t)dt \mathbf{V}^{(R)}(t) = \boldsymbol{\omega}(t) dt \times \mathbf{V}^{(R)}(t) \quad (8.90)$$

which we may rewrite as

$$d\mathbf{V}^{(R)}(t) = (\omega(t) dt) \mathcal{N}(t)\mathbf{V}^{(R)}(t) = (\omega(t) dt) \hat{\mathbf{n}}(t) \times \mathbf{V}^{(R)}(t) \quad (8.91)$$

For small enough dt , the differential $d\mathbf{V}^{(R)}(t)$ approximates the change in $\mathbf{V}^{(R)}(t)$ during that time interval. From the properties of cross products, this change is a vector

perpendicular to both $\hat{\mathbf{n}}(t)$ and $\mathbf{V}^{(R)}(t)$, with magnitude $(\omega(t) dt) V^{(R)}(t) \sin \theta_{\omega V}$ where $\theta_{\omega V}$ is the angle between the two vectors. Geometrically, this is a rotation of vector $\mathbf{V}^{(R)}(t)$ about an instantaneous axis whose direction is given by the unit vector $\hat{\mathbf{n}}(t)$, with a rotation angle $d\Phi$ defined as

$$d\Phi = \omega(t) dt \quad \text{so that} \quad d\Phi \hat{\mathbf{n}}(t) = \boldsymbol{\omega}(t) dt \tag{8.92}$$

The angular velocity vector $\boldsymbol{\omega}(t)$ thus has a magnitude $\omega(t)$ that gives the instantaneous rate of rotation $\omega(t) = d\Phi/dt$, and an associated unit vector $\hat{\boldsymbol{\omega}}(t) = \hat{\mathbf{n}}(t)$ that gives the instantaneous axis of rotation.

In general, both of these quantities will change with time. Thus, even if eqn (8.92) could be integrated to obtain some angle Φ , in general that angle would be meaningless since each of the increments $d\Phi$ takes place at a different time and hence about a different axis.

Note that vectors parallel to the instantaneous axis are not changed at all in time interval dt since the cross product of the two parallel vectors in eqn (8.90) will vanish.

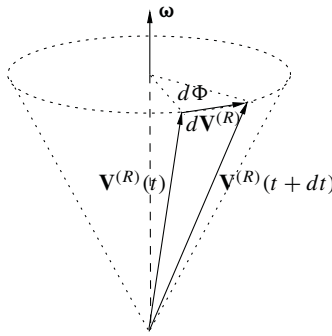


FIG. 8.5. Geometry of the angular velocity vector $\boldsymbol{\omega}$. The differential $d\mathbf{V}^{(R)}$ is seen to be perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{V}^{(R)}$, corresponding to the cross product in eqn (8.90).

8.12 Velocities of the Masses of a Rigid Body

The theory of Section 8.10 can be used to find the time derivative of the relative position vector $\boldsymbol{\rho}_n(t)$ discussed in Section 8.2.

From eqn (8.59) of Section 8.9, there is a time dependent rotation operator $\mathcal{R}(t)$ such that $\boldsymbol{\rho}_n(t) = \mathcal{R}(t)\boldsymbol{\rho}_n(0)$. Replacing $\mathbf{V}^{(R)}(t)$ by $\boldsymbol{\rho}_n(t)$ and \mathbf{V} by $\boldsymbol{\rho}_n(0)$ in eqn (8.64) allows eqn (8.76) to be written as

$$\frac{d\boldsymbol{\rho}_n(t)}{dt} = \boldsymbol{\omega}(t) \times \boldsymbol{\rho}_n(t) \tag{8.93}$$

This important formula was used in eqn (1.64) in Chapter 1, and will be used extensively in our discussion of rigid body dynamics.

The time derivative of eqn (8.8) then gives the velocity of mass m_n relative to the inertial origin as

$$\mathbf{v}_n = \frac{d\mathbf{r}_n(t)}{dt} = \frac{d\mathbf{R}(t)}{dt} + \frac{d\boldsymbol{\rho}_n(t)}{dt} = \frac{d\mathbf{R}(t)}{dt} + \boldsymbol{\omega}(t) \times \boldsymbol{\rho}_n(t) \quad (8.94)$$

It follows that the most general differential displacement of a rigid body in time dt can be described as a differential displacement $d\mathbf{R}$ of its center of mass, together with a rotation by an angle $d\Phi = \boldsymbol{\omega}(t) dt$ about an instantaneous axis $\hat{\mathbf{n}}(t)$ passing through the center of mass,

$$d\mathbf{r}_n = \mathbf{v}_n dt = \mathbf{V} dt + \boldsymbol{\omega}(t) dt \times \boldsymbol{\rho}_n(t) = d\mathbf{R} + d\Phi \hat{\mathbf{n}}(t) \times \boldsymbol{\rho}_n(t) \quad (8.95)$$

8.13 Savio's Theorem

In Section 3.3, we asserted that the cohesive forces holding a rigid body together do no virtual work. The results of Section 8.12 allow us to give a proof.

Before presenting the proof, we note that, although eqn (8.95) refers to a differential displacement in a time dt , it is actually more general. The parameter dt could be replaced by any parameter that varies monotonically as the body moves. Thus, the most general virtual displacement of a mass m_n of a rigid body, in the sense defined in Section 3.2, is given by

$$\delta\mathbf{r}_n = \delta\mathbf{R} + \delta\Phi \hat{\mathbf{n}} \times \boldsymbol{\rho}_n \quad (8.96)$$

where $\hat{\mathbf{n}}$ is some axis and $\delta\Phi$ is some angle. This is the most general virtual displacement that is consistent with the rigidity of the body.

Theorem 8.13.1: Savio's Theorem

If Axioms 1.4.1 and 1.5.1, the laws of linear and angular momentum,

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{(\text{ext})} \quad \text{and} \quad \frac{d\mathbf{J}}{dt} = \boldsymbol{\tau}^{(\text{ext})} \quad (8.97)$$

*are assumed to hold for a rigid body, considered as a collection of point masses, then the internal forces of cohesion will do no virtual work.*⁴⁴

Proof: For a rigid body, we identify as internal forces of constraint $\mathbf{f}_n^{(\text{int})}$ all those forces that are not explicitly external. Then, as discussed in Sections 1.4 and 1.5, eqn (8.97) implies that

$$\mathbf{F}^{(\text{int})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{int})} = 0 \quad \text{and} \quad \boldsymbol{\tau}^{(\text{int})} = \sum_{n=1}^N \mathbf{r}_n \times \mathbf{f}_n^{(\text{int})} = 0 \quad (8.98)$$

The virtual work done by these internal forces of constraint is defined in Section 3.3.

⁴⁴The theorem that the general laws of momentum are sufficient to establish the vanishing of rigid-body virtual work was derived by the late Mario Savio while he was a graduate student at San Francisco State University.

In vector form, it is

$$\delta W^{(\text{cons})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{int})} \cdot \delta \mathbf{r}_n \quad (8.99)$$

Using the virtual displacement from eqn (8.96) and the definition $\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R}$ from eqn (1.33) gives

$$\delta W^{(\text{cons})} = \sum_{n=1}^N \mathbf{f}_n^{(\text{int})} \cdot \delta \mathbf{R} + \sum_{n=1}^N \mathbf{f}_n^{(\text{int})} \cdot \delta \Phi \hat{\mathbf{n}} \times (\mathbf{r}_n - \mathbf{R}) \quad (8.100)$$

Factoring quantities that have no index n out of the sums, and rearranging a triple scalar product, this becomes

$$\delta W^{(\text{cons})} = \mathbf{F}^{(\text{int})} \cdot \delta \mathbf{R} + \boldsymbol{\tau}^{(\text{int})} \cdot \delta \Phi \hat{\mathbf{n}} - \mathbf{F}^{(\text{int})} \cdot \delta \Phi \hat{\mathbf{n}} \times \mathbf{R} \quad (8.101)$$

Equations (8.98) imply that each term on the right in eqn (8.101) is zero, and hence that $\delta W^{(\text{cons})} = 0$, as was to be proved. \square

8.14 Infinitesimal Rotation

Consider again the rotated vector $\mathbf{V}^{(R)}(t)$ in eqn (8.64) of Section 8.10. The difference $\Delta \mathbf{V}^{(R)}(t)$ between the vectors $\mathbf{V}^{(R)}(t + dt)$ and $\mathbf{V}^{(R)}(t)$ may be approximated by the differential $d\mathbf{V}^{(R)}(t)$ from eqn (8.91). The error of this approximation approaches zero in the limit as dt goes to zero. As discussed in Section D.12, the differential dt is *not* assumed to be a small quantity. But when it is large, the approximation of the difference $\Delta \mathbf{V}^{(R)}(t)$ by the differential $d\mathbf{V}^{(R)}(t)$ will in general be poor.

Thus we may use the definition of angle $d\Phi$ from eqn (8.92) to write

$$\Delta \mathbf{V}^{(R)}(t) = \mathbf{V}^{(R)}(t + dt) - \mathbf{V}^{(R)}(t) = d\mathbf{V}^{(R)}(t) + o(dt) = d\Phi \mathcal{N}(t) \mathbf{V}^{(R)}(t) + o(dt) \quad (8.102)$$

and hence⁴⁵

$$\begin{aligned} \mathbf{V}^{(R)}(t + dt) &= \mathbf{V}^{(R)}(t) + d\Phi \mathcal{N}(t) \mathbf{V}^{(R)}(t) + o(dt) \\ &= (\mathcal{U} + d\Phi \mathcal{N}(t)) \mathbf{V}^{(R)}(t) + o(dt) = \mathcal{R}_I[d\Phi \hat{\mathbf{n}}(t)] \mathbf{V}^{(R)}(t) + o(dt) \end{aligned} \quad (8.103)$$

The operator

$$\mathcal{R}_I[d\Phi \hat{\mathbf{n}}(t)] = \mathcal{U} + d\Phi \mathcal{N}(t) \quad (8.104)$$

defined in this equation will be referred to as an *infinitesimal rotation operator*. To order $o(dt)$ in the limit $dt \rightarrow 0$, it transforms $\mathbf{V}^{(R)}(t)$ into its value $\mathbf{V}^{(R)}(t + dt)$ at time $t + dt$. The notation $\mathcal{R}_I[d\Phi \hat{\mathbf{n}}(t)]$ should be read as the rotation by angle $d\Phi$ about instantaneous axis $\hat{\mathbf{n}}(t)$.

⁴⁵The symbol $o(dt)$ is discussed in Section D.11. Including it in an equation means that terms of *smaller* order than dt are being dropped. In the present context, this means that terms in dt^2 or higher powers are dropped since $\lim_{dt \rightarrow 0} (dt)^n / dt = 0$ for $n \geq 2$.

Note that the operator $\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)]$ is indeed a rotation when terms in dt^2 and higher powers are neglected, since it satisfies the orthogonality condition

$$\begin{aligned} \mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] \mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)]^T &= (\mathcal{U} + d\Phi\mathcal{N}(t)) (\mathcal{U} + d\Phi\mathcal{N}(t))^T \\ &= (\mathcal{U} + d\Phi\mathcal{N}(t)) (\mathcal{U} - d\Phi\mathcal{N}(t)) = \mathcal{U} + o(dt) \end{aligned} \quad (8.105)$$

due to the cancellation of the terms that are linear in $d\Phi$ and hence linear in dt .

8.15 Addition of Angular Velocities

In Section 8.10 and subsequently, we have referred to the angular velocity $\boldsymbol{\omega}(t)$ as a “vector.” However, vectors have more assumed properties than just the ability to be used in cross products as in eqn (8.76). For example, vectors can be added, and their sum is independent of the order of the addends. We now use the concept of infinitesimal rotation to understand the geometrical meaning of expressions like

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_a(t) + \boldsymbol{\omega}_b(t) = \boldsymbol{\omega}_b(t) + \boldsymbol{\omega}_a(t) \quad (8.106)$$

If the addends are assumed to be angular velocity vectors like the ones discussed above, we now show that the sum $\boldsymbol{\omega}(t)$ in eqn (8.106), in either order, is also a legitimate angular velocity vector corresponding to the same definite infinitesimal rotation.

If eqn (8.106) is assumed, then we can use eqn (8.92) to write

$$d\Phi\hat{\mathbf{n}}(t) = \boldsymbol{\omega}(t) dt = \boldsymbol{\omega}_a(t) dt + \boldsymbol{\omega}_b(t) dt = d\Phi_a\hat{\mathbf{n}}_a(t) + d\Phi_b\hat{\mathbf{n}}_b(t) \quad (8.107)$$

where $d\Phi_a = \omega_a(t) dt$ and $d\Phi_b = \omega_b(t) dt$ are the differential angles of the “a” and “b” rotations. In operator form, this is

$$d\Phi\mathcal{N}(t) = d\Phi_a\mathcal{N}_a(t) + d\Phi_b\mathcal{N}_b(t) \quad (8.108)$$

where operators $\mathcal{N}(t)$, $\mathcal{N}_a(t)$, and $\mathcal{N}_b(t)$ are related to the unit vectors $\hat{\mathbf{n}}(t)$, $\hat{\mathbf{n}}_a(t)$, and $\hat{\mathbf{n}}_b(t)$ as in eqn (8.89).

Then eqn (8.104) gives the infinitesimal rotation corresponding to vector $\boldsymbol{\omega}(t)$ as

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{U} + d\Phi\mathcal{N}(t) = \mathcal{U} + d\Phi_a\mathcal{N}_a(t) + d\Phi_b\mathcal{N}_b(t) \quad (8.109)$$

But when terms of order dt^2 are dropped, this can be written

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = (\mathcal{U} + d\Phi_a\mathcal{N}_a(t)) (\mathcal{U} + d\Phi_b\mathcal{N}_b(t)) + o(dt) \quad (8.110)$$

or

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{R}_I[d\Phi_a\hat{\mathbf{n}}_a(t)] \mathcal{R}_I[d\Phi_b\hat{\mathbf{n}}_b(t)] + o(dt) \quad (8.111)$$

The sum of two angular velocity vectors thus corresponds to a compound infinitesimal rotation consisting of two successive infinitesimal rotations, first the one produced by $\boldsymbol{\omega}_b dt$, followed by the one produced by $\boldsymbol{\omega}_a dt$.

But, again due to the neglect of terms of order dt^2 , we could just as well write the products in reverse order,

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = (\mathcal{U} + d\Phi_b\mathcal{N}_b(t))(\mathcal{U} + d\Phi_a\mathcal{N}_a(t)) + o(dt) \quad (8.112)$$

or

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{R}_I[d\Phi_b\hat{\mathbf{n}}_b(t)] \mathcal{R}_I[d\Phi_a\hat{\mathbf{n}}_a(t)] + o(dt) \quad (8.113)$$

So the sum of two angular velocity vectors corresponds also to a compound infinitesimal rotation consisting of two successive infinitesimal rotations in the opposite order, first the one produced by $\omega_a dt$, followed by a second one produced by $\omega_b dt$.

Thus, with the understanding that terms dt^2 and higher are to be dropped, the sum in eqn (8.106) corresponds to the product of the two infinitesimal rotations in either order,

$$\mathcal{R}_I[d\Phi_a\hat{\mathbf{n}}_a(t)] \mathcal{R}_I[d\Phi_b\hat{\mathbf{n}}_b(t)] = \mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{R}_I[d\Phi_b\hat{\mathbf{n}}_b(t)] \mathcal{R}_I[d\Phi_a\hat{\mathbf{n}}_a(t)] \quad (8.114)$$

The sum of two angular velocity vectors, in either order, corresponds to the same product of two infinitesimal rotations, since the order of their application makes no difference when terms containing dt^2 and higher powers are dropped.

Thus the vector $\omega(t)$ in eqn (8.106) is a legitimate angular velocity and corresponds to the same definite, unambiguous infinitesimal rotation regardless of the order of addition. Angular velocities like $\omega(t)$ are thus vectors and have the algebraic properties associated with them.

Note to the Reader: Equation (8.114) illustrates an important fact. Although *finite* rotations do not commute in general, *infinitesimal* rotations always commute.

8.16 Fundamental Generators of Rotations

Since angular velocities can be added, it is legitimate to consider the expansion of $\omega(t)$ into its three Cartesian components to represent the product of three infinitesimal rotations. These three rotations are now considered.

Let

$$\omega(t) = \omega_1(t)\hat{\mathbf{e}}_1 + \omega_2(t)\hat{\mathbf{e}}_2 + \omega_3(t)\hat{\mathbf{e}}_3 \quad (8.115)$$

as in eqn (8.77). The components $\omega_i(t)$ in this expansion can be used to define the new quantities $d\Phi_i$ by

$$d\Phi_i = \omega_i(t) dt = d\Phi n_i(t) \quad (8.116)$$

for $i = 1, 2, 3$, where the second equality follows from eqn (8.85) and the definitions $d\Phi = \omega(t) dt$ and $n_i(t) = \omega_i(t)/\omega(t)$. In vector form, the component definitions in eqn (8.116) are equivalent to

$$\omega(t)dt = d\Phi\hat{\mathbf{n}}(t) = d\Phi_1\hat{\mathbf{e}}_1 + d\Phi_2\hat{\mathbf{e}}_2 + d\Phi_3\hat{\mathbf{e}}_3 \quad (8.117)$$

Applying the argument that led from eqn (8.109) to eqn (8.111), it follows that

the infinitesimal rotation corresponding to $\boldsymbol{\omega}(t) dt$ can be written as

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{R}_I[d\Phi_1\hat{\mathbf{e}}_1] \mathcal{R}_I[d\Phi_2\hat{\mathbf{e}}_2] \mathcal{R}_I[d\Phi_3\hat{\mathbf{e}}_3] + o(dt) \quad (8.118)$$

where the order of the operators on the right makes no difference to the product, since by assumption terms containing dt^2 and higher powers are being dropped. The components of the angular velocity vector can be thought of as producing a product of three infinitesimal rotations with angles $d\Phi_i = \omega_i(t) dt$ about the corresponding coordinate axes $\hat{\mathbf{e}}_i$, with the order of these infinitesimal rotations having no effect on the final outcome.

Equation (8.104) shows that each of the three operators on the right in eqn (8.118) has the form, for $i = 1, 2, 3$,

$$\mathcal{R}_I[d\Phi_i\hat{\mathbf{e}}_i] = \mathcal{U} + d\Phi_i\mathcal{J}^{(i)} \quad (8.119)$$

where each $\mathcal{J}^{(i)}$ is the operator \mathcal{N} evaluated for the special case in which $\hat{\mathbf{n}} = \hat{\mathbf{e}}_i$. When terms in dt^2 and higher powers are dropped, eqn (8.118) can also be written as

$$\mathcal{R}_I[d\Phi\hat{\mathbf{n}}(t)] = \mathcal{U} + \sum_{i=1}^3 d\Phi_i\mathcal{J}^{(i)} + o(dt) \quad (8.120)$$

The operators $\mathcal{J}^{(i)}$ are called the *fundamental generators of infinitesimal rotations* or, more simply, the *infinitesimal generators*. The matrices corresponding to the $\mathcal{J}^{(i)}$ operators can be derived by setting $\hat{\mathbf{n}}$ to be the vectors with components (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively, in eqn (8.87). They are

$$\mathbf{J}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{J}^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{J}^{(3)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8.121)$$

The infinitesimal generators $\mathcal{J}^{(i)}$ do not commute. To see this, recall from eqn (8.89) that $\mathcal{N}\mathbf{V} = \hat{\mathbf{n}} \times \mathbf{V}$ and hence that the $\mathcal{J}^{(i)}$ obey

$$\mathcal{J}^{(1)}\mathbf{V} = \hat{\mathbf{e}}_1 \times \mathbf{V} \quad \mathcal{J}^{(2)}\mathbf{V} = \hat{\mathbf{e}}_2 \times \mathbf{V} \quad \mathcal{J}^{(3)}\mathbf{V} = \hat{\mathbf{e}}_3 \times \mathbf{V} \quad (8.122)$$

Thus, the rule for the expansion of triple cross products, together with the rule of composition of linear operators, give, for any vector \mathbf{V} ,

$$\mathcal{J}^{(i)}\mathcal{J}^{(j)}\mathbf{V} = \hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_j \times \mathbf{V}) = \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_j \cdot \mathbf{V}) - \mathbf{V} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \quad (8.123)$$

and

$$\mathcal{J}^{(j)}\mathcal{J}^{(i)}\mathbf{V} = \hat{\mathbf{e}}_j \times (\hat{\mathbf{e}}_i \times \mathbf{V}) = \hat{\mathbf{e}}_j (\hat{\mathbf{e}}_i \cdot \mathbf{V}) - \mathbf{V} (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i) \quad (8.124)$$

with the result that

$$(\mathcal{J}^{(i)}\mathcal{J}^{(j)} - \mathcal{J}^{(j)}\mathcal{J}^{(i)})\mathbf{V} = \hat{\mathbf{e}}_j (\hat{\mathbf{e}}_i \cdot \mathbf{V}) - \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_j \cdot \mathbf{V}) = (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \times \mathbf{V} \quad (8.125)$$

Then using the expansion of $(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j)$ from eqn (A.14) gives

$$\left(\mathcal{J}^{(i)} \mathcal{J}^{(j)} - \mathcal{J}^{(j)} \mathcal{J}^{(i)} \right) \mathbf{V} = \sum_{k=1}^3 \varepsilon_{ijk} \hat{\mathbf{e}}_k \times \mathbf{V} = \sum_{k=1}^3 \varepsilon_{ijk} \mathcal{J}^{(k)} \mathbf{V} \quad (8.126)$$

Since \mathbf{V} is any general vector, eqn (8.126) implies that the commutators of two fundamental generators are

$$\left[\mathcal{J}^{(i)}, \mathcal{J}^{(j)} \right]_c = \left(\mathcal{J}^{(i)} \mathcal{J}^{(j)} - \mathcal{J}^{(j)} \mathcal{J}^{(i)} \right) = \sum_{k=1}^3 \varepsilon_{ijk} \mathcal{J}^{(k)} \quad (8.127)$$

or, writing the three cases of interest explicitly,

$$\left[\mathcal{J}^{(1)}, \mathcal{J}^{(2)} \right]_c = \mathcal{J}^{(3)} \quad \left[\mathcal{J}^{(3)}, \mathcal{J}^{(1)} \right]_c = \mathcal{J}^{(2)} \quad \left[\mathcal{J}^{(2)}, \mathcal{J}^{(3)} \right]_c = \mathcal{J}^{(1)} \quad (8.128)$$

These fundamental commutation relations control the structure of rotations in three-dimensional Cartesian spaces. Relations eqn (8.127) define what is called the *Lie algebra* of the rotation group.

The commutations for the matrices $\mathbf{J}^{(i)}$ must be the same as for the operators. These commutation relations can be read from eqn (8.127), or can be derived directly from eqn (8.121).

8.17 Rotation with a Fixed Axis

A time dependent rotation operator in general has a time varying instantaneous axis of rotation $\hat{\mathbf{n}}(t)$. However, there is an important special case in which one assumes that the axis of rotation is constrained to be a constant independent of time, $\hat{\mathbf{n}}(t) = \hat{\mathbf{n}}$ for all t where $\hat{\mathbf{n}}$ here is assumed not to be time varying. In this special case, unlike the general case, the integral of the differential angle $d\Phi$ defined in eqn (8.92) *does* have a simple geometric significance. It is the accumulated angle of the fixed-axis rotation.

The operator for rotation by angle Φ about fixed axis $\hat{\mathbf{n}}$ can be found in closed form. It will be denoted $\mathcal{R}[\Phi\hat{\mathbf{n}}]$, with $\mathbf{R}[\Phi\hat{\mathbf{n}}]$ for the corresponding matrix, and will be referred to as a *fixed-axis rotation*.

The derivation of this operator begins with the differential equation, eqn (8.81). Making a change of variable from t to Φ , using the definitions

$$\Phi = \int_0^t d\Phi = \int_0^t \omega(t') dt' \quad \text{and} \quad \frac{d\Phi}{dt} = \omega(t) \quad (8.129)$$

derived from eqn (8.92), gives the differential equation, eqn (8.81), in the form

$$\frac{d\mathcal{R}(\Phi)}{d\Phi} = \frac{1}{\omega(t)} \mathcal{W}(t) \mathcal{R}(\Phi) = \mathcal{N}(t) \mathcal{R}(\Phi) \quad (8.130)$$

where eqn (8.88) was used to get the final equality.

However, the assumed constancy of the axis unit vector $\hat{\mathbf{n}}(t) = \hat{\mathbf{n}}$ implies that operator $\mathcal{N}(t)$, defined at the beginning of Section 8.11, is also constant in time. Thus $\mathcal{N}(t) = \mathcal{N}$ where \mathcal{N} is a constant operator with a constant matrix composed of the components of $\hat{\mathbf{n}}$,

$$\mathbf{N} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \quad (8.131)$$

where

$$\hat{\mathbf{n}} = n_1 \hat{\mathbf{e}}_1 + n_2 \hat{\mathbf{e}}_2 + n_3 \hat{\mathbf{e}}_3 \quad (8.132)$$

is the constant axis of rotation. Thus eqn (8.130) becomes

$$\frac{d\mathcal{R}(\Phi)}{d\Phi} = \mathcal{N}\mathcal{R}(\Phi) \quad (8.133)$$

The solution to differential equation, eqn (8.133), with a constant operator such as \mathcal{N} has already been discussed in Section 7.18. From eqn (7.131), it is

$$\mathcal{R}(\Phi) = \exp(\Phi\mathcal{N}) \quad \text{or, in our preferred notation,} \quad \mathcal{R}[\Phi\hat{\mathbf{n}}] = \exp(\Phi\mathcal{N}) \quad (8.134)$$

The exponential in eqn (8.134) can be written in a number of ways. A vector can be defined by $\Phi = \Phi\hat{\mathbf{n}}$ and a vector with operator components by

$$\vec{\mathcal{J}} = \sum_{k=1}^3 \hat{\mathbf{e}}_k \mathcal{J}^{(k)} \quad (8.135)$$

where the fundamental infinitesimal generators from Section 8.16 have been used. Using the matrices defined in eqn (8.121), the matrix in eqn (8.131) and hence the corresponding operator \mathcal{N} can be expanded as

$$\mathbf{N} = \sum_{k=1}^3 n_k \mathbf{J}^{(k)} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^3 n_k \mathcal{J}^{(k)} \quad (8.136)$$

The product $\Phi\mathcal{N}$ in eqn (8.134) can then be written as

$$\Phi\mathcal{N} = \sum_{k=1}^3 \Phi n_k \mathcal{J}^{(k)} = \Phi\hat{\mathbf{n}} \cdot \vec{\mathcal{J}} = \Phi \cdot \vec{\mathcal{J}} \quad (8.137)$$

which allows eqn (8.134) to be written as

$$\mathcal{R}[\Phi\hat{\mathbf{n}}] = \exp(\Phi\hat{\mathbf{n}} \cdot \vec{\mathcal{J}}) = \exp(\Phi \cdot \vec{\mathcal{J}}). \quad (8.138)$$

An important special case arises when the fixed axis is chosen to be one of the coordinate unit vectors. Rotation by Φ about a coordinate axis $\hat{\mathbf{e}}_k$ becomes

$$\mathcal{R}[\Phi\hat{\mathbf{e}}_k] = \exp(\Phi\hat{\mathbf{e}}_k \cdot \vec{\mathcal{J}}) = \exp(\Phi\mathcal{J}^{(k)}) \quad (8.139)$$

8.18 Expansion of Fixed-Axis Rotation

As discussed in Section 7.18, eqn (8.134) may be expanded in a power series,

$$\mathcal{R}[\Phi \hat{\mathbf{n}}] = \exp(\Phi \mathcal{N}) = \mathcal{U} + \Phi \mathcal{N} + \frac{(\Phi \mathcal{N})^2}{2!} + \frac{(\Phi \mathcal{N})^3}{3!} + \dots \quad (8.140)$$

This power series may be written as the sum of a finite number of terms.

Theorem 8.18.1: Expansion of Fixed-Axis Rotation

A finite rotation by angle Φ about a fixed axis $\hat{\mathbf{n}}$ may be written as

$$\mathcal{R}[\Phi \hat{\mathbf{n}}] = \exp(\Phi \mathcal{N}) = \mathcal{U} \cos \Phi + \mathcal{N} \sin \Phi + \mathcal{M} (1 - \cos \Phi) \quad (8.141)$$

with the corresponding matrix in the $\hat{\mathbf{e}}_i$ basis,

$$\mathbf{R}[\Phi \hat{\mathbf{n}}] = \exp(\Phi \mathbf{N}) = \mathbf{U} \cos \Phi + \mathbf{N} \sin \Phi + \mathbf{M} (1 - \cos \Phi) \quad (8.142)$$

where operator \mathcal{M} has a matrix \mathbf{M} with matrix elements $M_{ij} = n_i n_j$.

Proof: The evaluation of the power series in eqn (8.140) is facilitated by recursion relations for powers of \mathcal{N} . Direct matrix multiplication of eqn (8.131) by itself, using the fact that $n_1^2 + n_2^2 + n_3^2 = 1$ for unit vector $\hat{\mathbf{n}}$, yields

$$\mathbf{N}^2 = \mathbf{M} - \mathbf{U} \quad \text{where} \quad \mathbf{M} = \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 \end{pmatrix} \quad (8.143)$$

and \mathbf{U} is the identity matrix. Note that matrix \mathbf{M} has the form $M_{ij} = n_i n_j$.

The next power is then

$$\mathbf{N}^3 = \mathbf{N}^2 \mathbf{N} = (\mathbf{M} - \mathbf{U}) \mathbf{N} = \mathbf{M} \mathbf{N} - \mathbf{N} \quad (8.144)$$

But $\mathbf{M} \mathbf{N} = 0$, as can be seen from eqn (8.143) and the total skew-symmetry of ε_{ijk} ,

$$(\mathbf{M} \mathbf{N})_{ij} = \sum_{k=1}^3 M_{ik} N_{kj} = \sum_{k=1}^3 \sum_{l=1}^3 n_i n_k \varepsilon_{klj} n_l = n_i \sum_{k=1}^3 \sum_{l=1}^3 n_k n_l \varepsilon_{klj} = 0 \quad (8.145)$$

This result, together with eqn (8.144), gives $\mathbf{N}^3 = -\mathbf{N}$. Thus

$$\mathcal{N}^2 = \mathcal{M} - \mathcal{U} \quad \mathcal{N}^3 = -\mathcal{N} \quad \mathcal{N}^4 = \mathcal{N}^3 \mathcal{N} = -\mathcal{N}^2 = -(\mathcal{M} - \mathcal{U}) \quad (8.146)$$

and so on through a repeating sequence. Collecting coefficients of \mathcal{U} , \mathcal{N} , and \mathcal{M} in eqn (8.140) gives

$$\exp(\Phi \mathcal{N}) = \mathcal{U} \left(1 - \frac{\Phi^2}{2!} + \frac{\Phi^4}{4!} - \dots \right) + \mathcal{N} \left(\Phi - \frac{\Phi^3}{3!} + \dots \right) + \mathcal{M} \left(\frac{\Phi^2}{2!} - \frac{\Phi^4}{4!} + \dots \right) \quad (8.147)$$

Identifying the power series in Φ with the power series of trigonometric functions gives eqn (8.141), as was to be proved. \square

For example, setting $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3$ gives

$$\mathbf{R}[\Phi\hat{\mathbf{e}}_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos \Phi + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin \Phi + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (1 - \cos \Phi) \quad (8.148)$$

which reproduces eqn (8.32) derived earlier for this special case.

The trace of $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ is easily obtained as the sum of the traces of the terms of eqn (8.141). It is

$$\text{Tr}\mathcal{R}[\Phi\hat{\mathbf{n}}] = 2 \cos \Phi + 1 \quad (8.149)$$

Note that the dyadic form of operator \mathcal{M} has the form of a dyad $\mathbb{M} = \hat{\mathbf{n}}\hat{\mathbf{n}}$ with the consequence that $\mathcal{M}\mathbf{V} = \mathbb{M} \cdot \mathbf{V} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{V})$.

The result of the finite rotation of a general vector \mathbf{V} by angle Φ about a fixed axis $\hat{\mathbf{n}}$ can thus be written as

$$\mathbf{V}^{(R)} = \mathcal{R}[\Phi\hat{\mathbf{n}}]\mathbf{V} = \mathbf{V} \cos \Phi + \hat{\mathbf{n}} \times \mathbf{V} \sin \Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{V})(1 - \cos \Phi) \quad (8.150)$$

where eqn (8.89) has also been used, with $\hat{\mathbf{n}}$ constant.

The geometric interpretation of eqn (8.150) is immediate. Use eqn (A.3) to write the original vector as a sum of vectors parallel and perpendicular to $\hat{\mathbf{n}}$, as in $\mathbf{V} = \mathbf{V}_{\parallel} + \mathbf{V}_{\perp}$. Then eqn (8.150) can be written in the same form, as $\mathbf{V}^{(R)} = \mathbf{V}_{\parallel}^{(R)} + \mathbf{V}_{\perp}^{(R)}$, where

$$\mathbf{V}_{\parallel}^{(R)} = \mathbf{V}_{\parallel} \quad \text{and} \quad \mathbf{V}_{\perp}^{(R)} = (\mathbf{V}_{\perp} \cos \Phi + \hat{\mathbf{n}} \times \mathbf{V}_{\perp} \sin \Phi) \quad (8.151)$$

The original vector component \mathbf{V}_{\parallel} parallel to rotation axis $\hat{\mathbf{n}}$ is unchanged by the rotation, as one would expect. The vector $\mathbf{V}_{\perp}^{(R)}$ perpendicular to $\hat{\mathbf{n}}$ has the same magnitude V_{\perp} as the original perpendicular vector \mathbf{V}_{\perp} , but is rotated by angle Φ in the right-hand sense about axis $\hat{\mathbf{n}}$.

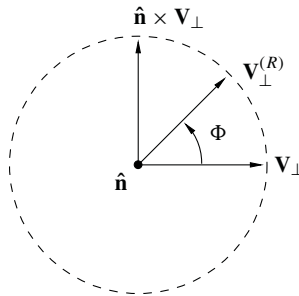


FIG. 8.6. Illustration of eqn (8.151). The component \mathbf{V}_{\perp} perpendicular to $\hat{\mathbf{n}}$ is rotated by angle Φ to give $\mathbf{V}_{\perp}^{(R)}$. The component parallel to $\hat{\mathbf{n}}$ is not changed.

Since rotation operators obey eqn (8.22), the inverse of a fixed-axis rotation is $\mathcal{R}[\Phi\hat{\mathbf{n}}]^{-1} = \mathcal{R}[\Phi\hat{\mathbf{n}}]^T$. The expansion eqn (8.140) gives

$$\mathcal{R}[\Phi\hat{\mathbf{n}}]^T = \{\exp(\Phi\mathcal{N})\}^T = \exp(\Phi\mathcal{N}^T) = \exp(-\Phi\mathcal{N}) = \mathcal{R}[-\Phi\hat{\mathbf{n}}] \quad (8.152)$$

since \mathcal{N} is anti-symmetric. Thus, as one would expect, the inverse is a rotation by the same angle about an oppositely directed axis,

$$\mathcal{R}[\Phi\hat{\mathbf{n}}]^{-1} = \mathcal{R}[-\Phi\hat{\mathbf{n}}] \quad (8.153)$$

8.19 Eigenvectors of the Fixed-Axis Rotation Operator

The eigenvectors and eigenvalues of the fixed-axis rotation operator $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ are easily derived. From eqn (8.134) we know that $\mathcal{R}[\Phi\hat{\mathbf{n}}] = \exp(\Phi\mathcal{N})$, where \mathcal{N} is a real anti-symmetric operator associated with unit vector $\hat{\mathbf{n}}$ by eqns (8.131, 8.132).

To begin, we solve the eigenvalue problem for \mathcal{N} by noting that this operator is identical to the \mathcal{W} treated in Section 7.13 except for the substitution of $\hat{\mathbf{n}}$ for $\boldsymbol{\omega}$. Setting $\omega = 1$ in eqn (7.89) since $\hat{\mathbf{n}}$ is a unit vector, gives the eigenvalues of operator \mathcal{N} as

$$\lambda_1^{(N)} = i \quad \lambda_2^{(N)} = -i \quad \lambda_3^{(N)} = 0 \quad (8.154)$$

with corresponding normalized eigenvectors

$$\hat{\mathbf{V}}^{(1)} = (\hat{\mathbf{a}} - i\hat{\mathbf{b}})/\sqrt{2}, \quad \hat{\mathbf{V}}^{(2)} = (\hat{\mathbf{a}} + i\hat{\mathbf{b}})/\sqrt{2}, \quad \text{and} \quad \hat{\mathbf{V}}^{(3)} = \hat{\mathbf{n}} \quad (8.155)$$

where $\hat{\mathbf{a}}$ is some real unit vector perpendicular to $\hat{\mathbf{n}}$ but otherwise arbitrary and $\hat{\mathbf{b}} = \hat{\mathbf{n}} \times \hat{\mathbf{a}}$ is also a real unit vector, perpendicular to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$.

By eqn (7.121) of Section 7.17, as discussed also in Section 7.18, the eigenvectors of $\mathcal{R}[\Phi\hat{\mathbf{n}}] = \exp(\theta\mathcal{N})$ are the same as those of \mathcal{N} , and the eigenvalues are exponential functions of those in eqn (8.154),

$$\lambda_1 = \exp(i\Phi) \quad \lambda_2 = \exp(-i\Phi) \quad \lambda_3 = \exp(0\Phi) = 1 \quad (8.156)$$

The dyadic $\mathbb{R}[\Phi\hat{\mathbf{n}}]$ corresponding to $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ can be obtained in eigen-dyadic form from eqn (7.113) of Theorem 7.16.1. It is

$$\mathbb{R}[\Phi\hat{\mathbf{n}}] = \sum_{k=1}^3 \mathbf{V}^{(k)} \lambda_k \mathbf{V}^{(k)*} \quad (8.157)$$

where the eigenvalues λ_k are from eqn (8.156).

The eigenvalue problem for $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ is now completely solved. As discussed in Section 7.13, making different choices of arbitrary unit vector $\hat{\mathbf{a}}$ is equivalent to multiplying the first two eigenvectors by $\exp(i\alpha)$ and $\exp(-i\alpha)$, respectively, where α is some real number. This is only a trivial change, since in any case eigenvectors are determined only up to a multiplicative constant of modulus unity. As proved in Lemma 7.16.2, such a multiplication also makes no change in the dyadic eqn (8.157), since the exponential factors cancel. Thus, in spite of its appearance, eqn (8.157) is in fact independent of the choice of $\hat{\mathbf{a}}$. If written out in terms of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{n}}$, eqn (8.157) will be seen to reduce to eqn (8.150) which depends only on Φ and $\hat{\mathbf{n}}$.

8.20 The Euler Theorem

We have now discussed two different types of rotations. The first, in Section 8.10, happens when a rigid body is rotated in a general way during a time t , first about one axis and then about another, and so on. The end product of all of this various motion is still a rotation, however. The operator $\mathcal{R}(t)$ at time t is an orthogonal operator.

The other type is what we have called a fixed-axis rotation, discussed in Section 8.17. In this case, the rigid body is rotated by an angle Φ about an axis that does not change, somewhat as if the rigid body were mounted on a lathe.

The Euler Theorem proves a result that may seem obvious: Any general rotation of the first type *could have been* accomplished by some fixed-axis rotation of the second type. This does not mean that it necessarily *was* accomplished by a fixed-axis rotation, only that it could have been. If one starts with some standard orientation of a rigid body at time zero and rotates it during time t in a general manner, the final orientation could as well have been produced by starting from the same standard orientation and rotating by some angle Φ about some fixed axis $\hat{\mathbf{n}}$.

Theorem 8.20.1: The Euler Theorem

For any general proper, orthogonal operator \mathcal{R} , there exist a fixed axis $\hat{\mathbf{n}}$ and an angle Φ in the range $0 \leq \Phi \leq \pi$ such that

$$\mathcal{R}[\Phi\hat{\mathbf{n}}] = \mathcal{R} \tag{8.158}$$

Proof: We show that the dyadic form of a general \mathcal{R} is identical to the dyadic form of some fixed-axis rotation $\mathcal{R}[\Phi\hat{\mathbf{n}}]$. Since two operators with identical dyadics are themselves identical, this will prove the theorem.

The first step is to find the eigenvalues of a general rotation \mathcal{R} . Use eqn (8.22) to write

$$(\mathcal{R} - \mathcal{U})\mathcal{R}^T = \mathcal{U} - \mathcal{R}^T = -(\mathcal{R} - \mathcal{U})^T \tag{8.159}$$

and then take determinants of both sides,

$$\det(\mathcal{R} - \mathcal{U}) \det \mathcal{R} = (-1)^3 \det(\mathcal{R} - \mathcal{U}) \tag{8.160}$$

where we used $\det \mathcal{R}^T = \det \mathcal{R}$, and $\det(\alpha\mathcal{R}) = \alpha^3 \det \mathcal{R}$ for three-dimensional operators. Since $\det \mathcal{R} = +1$ for proper orthogonal operators, the result is $\det(\mathcal{R} - \mathcal{U}) = 0$, which, according to eqn (7.86), shows that $+1$ is an eigenvalue of \mathcal{R} . Call this eigenvalue $\lambda_3 = 1$. To find the other two eigenvalues, we use eqns (7.111, 7.112) of Section 7.15 relating the determinant and trace to the eigenvalues of \mathcal{R} . Since the trace of \mathcal{R} is defined as the sum $\text{Tr}\mathcal{R} = (R_{11} + R_{22} + R_{33})$ which is a real number, the sum $(\lambda_1 + \lambda_2 + \lambda_3) = (\lambda_1 + \lambda_2 + 1)$ must be real, which implies the relation between imaginary parts $\Im(\lambda_2) = -\Im(\lambda_1)$. This, together with $1 = \det \mathcal{R} = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_2 (1)$, implies that

$$\lambda_1 = \exp(i\Phi) \quad \lambda_2 = \exp(-i\Phi) \quad \lambda_3 = 1 \tag{8.161}$$

where Φ is some real number. The value of this number can be found from

$$(R_{11} + R_{22} + R_{33}) = \text{Tr}\mathcal{R} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 \cos \Phi \tag{8.162}$$

where Φ can be restricted to the range $0 \leq \Phi \leq \pi$. The three eigenvalues of a general

proper orthogonal operator \mathcal{R} are thus determined uniquely.

The (real) eigenvector $\mathbf{V}^{(3)}$ corresponding to eigenvalue $+1$ is found by setting $\lambda_3 = 1$ in the eigenvector equation, eqn (7.86), and solving for the eigenvector. The equation to be solved is

$$(\mathcal{R} - \mathcal{U})\mathbf{V}^{(3)} = 0, \quad \text{or} \quad (\mathbf{R} - \mathbf{U})[V^{(3)}] = 0 \quad (8.163)$$

in matrix form.

It follows from eqn (8.163) in the form $\mathcal{R}\mathbf{V}^{(3)} = \mathbf{V}^{(3)}$ that the normalized eigenvector $\hat{\mathbf{n}} = \mathbf{V}^{(3)}/V^{(3)}$ is not changed by \mathcal{R} . Thus $\hat{\mathbf{n}}$ will be along the axis of rotation of \mathcal{R} . Only one rather trivial difficulty remains, the choice of direction for $\hat{\mathbf{n}}$. The eigenvector equation, eqn (8.163), only determines real unit vector $\hat{\mathbf{n}}$ up to a factor ± 1 . It is necessary to compare the action of \mathcal{R} on some vector not parallel to $\hat{\mathbf{n}}$. If that rotation is not in a right-handed sense about $\hat{\mathbf{n}}$, the direction of $\hat{\mathbf{n}}$ must be reversed. A unique axis direction and angle Φ are thus obtained, with positive angle Φ meaning rotation in a right-handed sense about $\hat{\mathbf{n}}$.

We now find the other two eigenvectors of \mathcal{R} . Since \mathcal{R} is a normal operator with three distinct eigenvalues, Lemma B.26.2 proves that it must possess three eigenvectors which are orthogonal in the extended sense $\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl}$. Setting $\hat{\mathbf{n}} = \hat{\mathbf{V}}^{(3)}$, and recalling that $\hat{\mathbf{n}}$ is real, the other two eigenvectors must be composed of real and imaginary parts, both of which are perpendicular to $\hat{\mathbf{n}}$. Setting $\hat{\mathbf{V}}^{(1)} = \mathbf{a} - i\mathbf{b}$ where \mathbf{a} and \mathbf{b} are unknown real vectors perpendicular to $\hat{\mathbf{n}}$, the first eigenvalue equation is

$$\mathcal{R}\hat{\mathbf{V}}^{(1)} = \lambda_1\hat{\mathbf{V}}^{(1)} \quad (8.164)$$

Since \mathcal{R} is a real operator and $\lambda_2 = \lambda_1^*$, the complex conjugate of eqn (8.164),

$$\mathcal{R}\hat{\mathbf{V}}^{(1)*} = \lambda_1^*\hat{\mathbf{V}}^{(1)*} = \lambda_2\hat{\mathbf{V}}^{(1)*} \quad (8.165)$$

implies that $\hat{\mathbf{V}}^{(2)} = \hat{\mathbf{V}}^{(1)*} = \mathbf{a} + i\mathbf{b}$ is the eigenvector corresponding to λ_2 .

The orthogonality of these two eigenvectors then implies the vanishing of the real and imaginary parts of the expression

$$0 = \hat{\mathbf{V}}^{(1)*} \cdot \hat{\mathbf{V}}^{(2)} = (\mathbf{a} + i\mathbf{b}) \cdot (\mathbf{a} + i\mathbf{b}) = (a^2 - b^2) + i(\mathbf{a} \cdot \mathbf{b}) \quad (8.166)$$

which requires that vectors \mathbf{a} and \mathbf{b} must be orthogonal and have the same magnitude. The vector \mathbf{b} must therefore be $\mathbf{b} = \hat{\mathbf{n}} \times \mathbf{a}$. (The other possible choice $\mathbf{b} = -\hat{\mathbf{n}} \times \mathbf{a}$ would have the effect of making positive Φ mean rotation about $\hat{\mathbf{n}}$ in the left-handed sense, whereas $\hat{\mathbf{n}}$ has already been chosen above so that \mathcal{R} produces rotation in the right-handed sense.) Normalizing the eigenvectors of \mathcal{R} using $\hat{\mathbf{V}}^{(k)*} \cdot \hat{\mathbf{V}}^{(k)} = 1$ for $k = 1, 2, 3$ shows that the eigenvectors of \mathcal{R} may be written as

$$\hat{\mathbf{V}}^{(1)} = (\hat{\mathbf{a}} - i\hat{\mathbf{b}})/\sqrt{2}, \quad \hat{\mathbf{V}}^{(2)} = (\hat{\mathbf{a}} + i\hat{\mathbf{b}})/\sqrt{2}, \quad \text{and} \quad \hat{\mathbf{V}}^{(3)} = \hat{\mathbf{n}} \quad (8.167)$$

where $\hat{\mathbf{a}}$ and hence $\hat{\mathbf{b}} = \hat{\mathbf{n}} \times \hat{\mathbf{a}}$ are unit vectors.

The dyadic form of normal operator \mathcal{R} is thus given by eqn (7.113) as

$$\mathbb{R} = \sum_{k=1}^3 \hat{\mathbf{V}}^{(k)} \lambda_k \hat{\mathbf{V}}^{(k)*} \quad (8.168)$$

where the eigenvalues are those in eqn (8.161) above, with the angle Φ found in eqn (8.162), and eigenvectors are those in eqn (8.167) with the axis $\hat{\mathbf{n}}$ found from eqn (8.163).

If we put the same angle Φ and same axis $\hat{\mathbf{n}}$ into eqn (8.157) of Section 8.19, we obtain a dyadic $\mathbb{R}[\Phi\hat{\mathbf{n}}]$ which is exactly the same as eqn (8.168), except for a possibly different choice of arbitrary unit vector \mathbf{a} . But, as discussed in Section 8.19, the dyadic is independent of the particular choice of \mathbf{a} . Different choices are equivalent to multiplying eigenvectors by phase factors of modulus unity that cancel from the dyadics. The dyadic of \mathcal{R} is thus identical to that of $\mathcal{R}[\Phi\hat{\mathbf{n}}]$. But two operators with identical dyadics are themselves identical. Hence $\mathcal{R}[\Phi\hat{\mathbf{n}}] = \mathcal{R}$, which proves the Euler Theorem. \square

8.21 Rotation of Operators

Suppose that $\mathbf{W} = \mathcal{F}\mathbf{V}$ where \mathcal{F} is a linear operator and \mathbf{V} a general vector. Suppose a rotation \mathcal{R} to act on both \mathbf{V} and \mathbf{W} giving $\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V}$ and $\mathbf{W}^{(R)} = \mathcal{R}\mathbf{W}$. Then we can find a linear operator $\mathcal{F}^{(R)}$ that will map $\mathbf{V}^{(R)}$ into $\mathbf{W}^{(R)}$ as in $\mathbf{W}^{(R)} = \mathcal{F}^{(R)}\mathbf{V}^{(R)}$. To do so, write

$$\mathbf{W}^{(R)} = \mathcal{R}\mathbf{W} = \mathcal{R}\mathcal{F}\mathbf{V} = \mathcal{R}\mathcal{F}\mathcal{R}^T\mathcal{R}\mathbf{V} = \mathcal{F}^{(R)}\mathbf{V}^{(R)} \quad (8.169)$$

which leads to the definition

$$\mathcal{F}^{(R)} = \mathcal{R}\mathcal{F}\mathcal{R}^T \quad (8.170)$$

We refer to $\mathcal{F}^{(R)}$ as a *rotated operator*, since its action on the rotated vectors mimics that of the original operator \mathcal{F} on the original vectors.

8.22 Rotation of the Fundamental Generators

The rotated operators of the fundamental infinitesimal generators $\mathcal{J}^{(k)}$ defined in Section 8.16 are of particular interest. They are

$$\mathcal{J}^{(k)(R)} = \mathcal{R}\mathcal{J}^{(k)}\mathcal{R}^T = \sum_{l=1}^3 \mathcal{J}^{(l)} R_{lk} = \sum_{l=1}^3 R_{kl}^T \mathcal{J}^{(l)} \quad (8.171)$$

where R_{lk} are the matrix elements of rotation \mathcal{R} .

To prove eqn (8.171), we let $\mathcal{J}^{(k)(R)} = \mathcal{R}\mathcal{J}^{(k)}\mathcal{R}^T$ act on a general vector \mathbf{V} . The result is

$$\mathcal{J}^{(k)(R)}\mathbf{V} = \mathcal{R}\mathcal{J}^{(k)}\mathcal{R}^T\mathbf{V} = \mathcal{R}(\hat{\mathbf{e}}_k \times (\mathcal{R}^T\mathbf{V})) \quad (8.172)$$

where eqn (8.122) was used. The invariance of cross products under proper rotation

from eqn (8.39) then gives

$$\mathcal{J}^{(k)(R)}\mathbf{V} = \mathcal{R}(\hat{\mathbf{e}}_k \times (\mathcal{R}^T\mathbf{V})) = (\mathcal{R}\hat{\mathbf{e}}_k) \times (\mathcal{R}\mathcal{R}^T\mathbf{V}) = (\mathcal{R}\hat{\mathbf{e}}_k) \times \mathbf{V} \quad (8.173)$$

Inserting a resolution of unity $\mathbb{U} = \sum_{l=1}^3 \hat{\mathbf{e}}_l \hat{\mathbf{e}}_l$ to expand $\mathcal{R}\hat{\mathbf{e}}_k$ as

$$\mathcal{R}\hat{\mathbf{e}}_k = \mathbb{U} \cdot (\mathcal{R}\hat{\mathbf{e}}_k) = \sum_{l=1}^3 \hat{\mathbf{e}}_l \hat{\mathbf{e}}_l \cdot \mathcal{R}\hat{\mathbf{e}}_k = \sum_{l=1}^3 \hat{\mathbf{e}}_l R_{lk} \quad (8.174)$$

eqn (8.173) becomes

$$\mathcal{J}^{(k)(R)}\mathbf{V} = (\mathcal{R}\hat{\mathbf{e}}_k) \times \mathbf{V} = \sum_{l=1}^3 R_{lk} \hat{\mathbf{e}}_l \times \mathbf{V} = \sum_{l=1}^3 R_{lk} \mathcal{J}^{(l)}\mathbf{V} \quad (8.175)$$

Since \mathbf{V} was an arbitrary vector we get finally the operator equality $\mathcal{J}^{(k)(R)} = \sum_{l=1}^3 \mathcal{J}^{(l)} R_{lk}$, which is the same as eqn (8.171).

Note the similarity between eqns (8.174, 8.171). The infinitesimal rotation generators $\mathcal{J}^{(k)}$ transform under rotation in the same way as the Cartesian basis vectors $\hat{\mathbf{e}}_k$ do.

8.23 Rotation of a Fixed-Axis Rotation

The rotation of a fixed-axis rotation operator may now be derived. Suppose the operator \mathcal{F} in eqn (8.170) to be a fixed-axis rotation $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ discussed in Section 8.17. Let this fixed-axis rotation map a general vector \mathbf{V} into another vector \mathbf{W} so that

$$\mathbf{W} = \mathcal{R}[\Phi\hat{\mathbf{n}}]\mathbf{V} \quad (8.176)$$

Now suppose some rotation \mathcal{R} (not usually the same as $\mathcal{R}[\Phi\hat{\mathbf{n}}]$) is applied to both \mathbf{V} and \mathbf{W} , to give $\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V}$ and $\mathbf{W}^{(R)} = \mathcal{R}\mathbf{W}$. We expect intuitively that a fixed-axis rotation by the same angle Φ but about a rotated axis $\hat{\mathbf{n}}^{(R)} = \mathcal{R}\hat{\mathbf{n}}$ should map $\mathbf{V}^{(R)}$ into $\mathbf{W}^{(R)}$ as in

$$\mathbf{W}^{(R)} = \mathcal{R}[\Phi\hat{\mathbf{n}}^{(R)}]\mathbf{V}^{(R)} \quad (8.177)$$

In effect, the original rotation should itself be rotated, its fixed axis changed from $\hat{\mathbf{n}}$ to $\hat{\mathbf{n}}^{(R)}$.

We now prove this important result formally.

Theorem 8.23.1: Rotation of Fixed-Axis Rotation

If $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ is a fixed-axis rotation operator, and if \mathcal{R} is some other rotation, then

$$\mathcal{R}[\Phi\hat{\mathbf{n}}]^{(R)} = \mathcal{R}\mathcal{R}[\Phi\hat{\mathbf{n}}]\mathcal{R}^T = \mathcal{R}[\Phi\hat{\mathbf{n}}^{(R)}] \quad (8.178)$$

where $\hat{\mathbf{n}}^{(R)} = \mathcal{R}\hat{\mathbf{n}}$.

Proof: The proof begins with eqn (8.140) which gives

$$\begin{aligned}\mathcal{R}[\Phi\hat{\mathbf{n}}]^{(R)} &= \mathcal{R}\mathcal{R}[\Phi\hat{\mathbf{n}}]\mathcal{R}^T = \mathcal{R}\exp(\Phi\mathcal{N})\mathcal{R}^T \\ &= \mathcal{R}\mathcal{U}\mathcal{R}^T + \mathcal{R}(\Phi\mathcal{N})\mathcal{R}^T + \mathcal{R}\frac{(\Phi\mathcal{N})^2}{2!}\mathcal{R}^T + \mathcal{R}\frac{(\Phi\mathcal{N})^3}{3!}\mathcal{R}^T + \dots\end{aligned}\quad (8.179)$$

where the linearity of \mathcal{R} has been used. Noting that

$$\mathcal{R}\mathcal{N}^k\mathcal{R}^T = \mathcal{R}\mathcal{N}\mathcal{N}\dots\mathcal{N}\mathcal{R}^T = \mathcal{R}\mathcal{N}\mathcal{R}^T\mathcal{R}\mathcal{N}\mathcal{R}^T\dots\mathcal{R}\mathcal{N}\mathcal{R}^T = (\mathcal{R}\mathcal{N}\mathcal{R}^T)^k \quad (8.180)$$

where unity operators in the form $\mathcal{U} = \mathcal{R}^T\mathcal{R}$ were inserted between \mathcal{N} factors, gives

$$\begin{aligned}\mathcal{R}[\Phi\hat{\mathbf{n}}]^{(R)} &= \mathcal{R}(\exp(\Phi\mathcal{N}))\mathcal{R}^T \\ &= \mathcal{R}\mathcal{U}\mathcal{R}^T + (\Phi\mathcal{R}\mathcal{N}\mathcal{R}^T) + \frac{(\Phi\mathcal{R}\mathcal{N}\mathcal{R}^T)^2}{2!} + \frac{(\Phi\mathcal{R}\mathcal{N}\mathcal{R}^T)^3}{3!} + \dots \\ &= \exp(\Phi\mathcal{R}\mathcal{N}\mathcal{R}^T)\end{aligned}\quad (8.181)$$

Now, by eqns (8.137, 8.171),

$$\mathcal{R}\mathcal{N}\mathcal{R}^T = \sum_{k=1}^3 n_k \mathcal{R}\mathcal{J}^{(k)}\mathcal{R}^T = \sum_{k=1}^3 n_k \sum_{l=1}^3 \mathcal{J}^{(l)} R_{lk} = \sum_{l=1}^3 n_l^{(R)} \mathcal{J}^{(l)} \quad (8.182)$$

where we have defined $n_l^{(R)} = \sum_{k=1}^3 R_{lk}n_k$ which, by eqn (8.30), is equivalent to $\hat{\mathbf{n}}^{(R)} = \mathcal{R}\hat{\mathbf{n}}$. Thus, again using eqn (8.137), eqn (8.181) becomes

$$\mathcal{R}[\Phi\hat{\mathbf{n}}]^{(R)} = \exp\left(\Phi \sum_{l=1}^3 n_l^{(R)} \mathcal{J}^{(l)}\right) = \mathcal{R}[\Phi\hat{\mathbf{n}}^{(R)}] \quad (8.183)$$

as was to be proved. A rotated fixed-axis rotation is indeed a rotation about a rotated fixed axis! \square

8.24 Parameterization of Rotation Operators

A general rotation \mathcal{R} would appear at first sight to require nine parameters, the nine matrix elements R_{ij} , to define it completely. But these nine matrix elements are not independent, being constrained by the six independent conditions coming from the orthogonality condition eqn (8.22). A general rotation can be completely defined by the values of only three independent parameters.

One obvious parameterization of \mathcal{R} would make use of the Euler Theorem of Section 8.20. As we saw there, any general \mathcal{R} determines the unique angle Φ and axis $\hat{\mathbf{n}}$ of an equivalent fixed-axis rotation $\mathcal{R} = \mathcal{R}[\Phi\hat{\mathbf{n}}]$. Since it is a unit vector, the fixed-axis $\hat{\mathbf{n}}$ can be parameterized by two numbers, its components in spherical-polar form, as in

$$n_1 = \sin\theta_n \cos\phi_n \quad n_2 = \sin\theta_n \sin\phi_n \quad n_3 = \cos\theta_n \quad (8.184)$$

where θ_n is the angle between $\hat{\mathbf{n}}$ and the $\hat{\mathbf{e}}_3$ axis, and ϕ_n is the azimuthal angle. Thus a general rotation \mathcal{R} can be uniquely parameterized by the three numbers Φ, θ_n, ϕ_n .

8.25 Differentiation of Parameterized Operator

When the rotation is varying with time, then the three parameters introduced in Section 8.24 will become time dependent also, $\Phi(t)$, $\theta_n(t)$, $\phi_n(t)$, which we may write as

$$\mathcal{R}(t) = \mathcal{R}[\Phi(t)\hat{\mathbf{n}}(t)] \quad (8.185)$$

At time t_1 , the rotation $\mathcal{R}(t_1)$ would be associated with the axis $\hat{\mathbf{n}}(t_1)$ and angle $\Phi(t_1)$ of the fixed-axis rotation that would carry the rigid body from some initial orientation to the rotated position at time t_1 . At a later time t_2 , the rotation $\mathcal{R}(t_2)$ would be associated with a different axis $\hat{\mathbf{n}}(t_2)$ and angle $\Phi(t_2)$ of a different fixed-axis rotation. Each rotation $\mathcal{R}[\Phi(t)\hat{\mathbf{n}}(t)]$ is about a fixed axis, but the required fixed axis is varying with time!

The angular velocity $\boldsymbol{\omega}(t)$ of such a time-varying rotation operator can be written in terms of the time derivatives of $\Phi(t)$ and $\hat{\mathbf{n}}(t)$.

Theorem 8.25.1: Angular Velocity of Parameterized Rotation

If we use the parameterization of eqn (8.185) to write a time-varying rotated vector as

$$\mathbf{V}^{(R)}(t) = \mathcal{R}[\Phi(t)\hat{\mathbf{n}}(t)]\mathbf{V} \quad (8.186)$$

then the time derivative can be written as in eqn (8.76),

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \boldsymbol{\omega}(t) \times \mathbf{V}^{(R)}(t) \quad (8.187)$$

where the angular velocity vector $\boldsymbol{\omega}$ can be expressed in terms of the parameters Φ , $\hat{\mathbf{n}}$ and their derivatives $\dot{\Phi} = d\Phi/dt$ and $d\hat{\mathbf{n}}/dt$ as

$$\boldsymbol{\omega}(t) = \dot{\Phi}\hat{\mathbf{n}} + \sin\Phi\frac{d\hat{\mathbf{n}}}{dt} + (1 - \cos\Phi)\hat{\mathbf{n}} \times \frac{d\hat{\mathbf{n}}}{dt} \quad (8.188)$$

Proof: To establish eqn (8.188), we begin by writing eqn (8.186) in the form

$$\mathbf{V}^{(R)}(t) = \mathcal{R}[\Phi(t)\hat{\mathbf{n}}(t)]\mathbf{V} = \left(\mathcal{U} + \sin\Phi\mathcal{N} + (1 - \cos\Phi)\mathcal{N}^2\right)\mathbf{V} \quad (8.189)$$

where the expansion of eqn (8.141) was used, with the first of eqn (8.146) used to substitute $\mathcal{M} = \mathcal{U} + \mathcal{N}^2$. Taking the time derivative of eqn (8.189) gives

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \left\{ \dot{\Phi} \left(\cos\Phi\mathcal{N} + \sin\Phi\mathcal{N}^2 \right) + \sin\Phi\frac{d\mathcal{N}}{dt} + (1 - \cos\Phi) \left(\frac{d\mathcal{N}}{dt}\mathcal{N} + \mathcal{N}\frac{d\mathcal{N}}{dt} \right) \right\} \mathbf{V} \quad (8.190)$$

From eqn (8.153), the inverse of eqn (8.189) is

$$\mathbf{V} = \mathcal{R}[-\Phi(t)\hat{\mathbf{n}}(t)]\mathbf{V}^{(R)}(t) = \left(\mathcal{U} - \sin\Phi\mathcal{N} + (1 - \cos\Phi)\mathcal{N}^2\right)\mathbf{V}^{(R)}(t) \quad (8.191)$$

Since $\hat{\mathbf{n}}$ is a unit vector with $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ it follows that $\hat{\mathbf{n}} \cdot (d\hat{\mathbf{n}}/dt) = 0$. Since, for any

arbitrary vector \mathbf{A} ,

$$\mathcal{N}\mathbf{A} = \hat{\mathbf{n}} \times \mathbf{A} \quad \text{and so} \quad \frac{d\mathcal{N}}{dt}\mathbf{A} = \frac{d\hat{\mathbf{n}}}{dt} \times \mathbf{A} \quad (8.192)$$

expanding the cross products using the rule of triple cross products gives

$$\mathcal{N}\frac{d\mathcal{N}}{dt}\mathcal{N}\mathbf{A} = 0 \quad \text{and hence} \quad \mathcal{N}\frac{d\mathcal{N}}{dt}\mathcal{N} = 0 \quad (8.193)$$

Substituting eqn (8.191) into eqn (8.190), and using eqn (8.193) as well as the identities in eqn (8.146) to simplify, gives

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \left\{ \dot{\Phi}\mathcal{N} + \sin\Phi \frac{d\mathcal{N}}{dt} + (1 - \cos\Phi) \left(\mathcal{N}\frac{d\mathcal{N}}{dt} - \frac{d\mathcal{N}}{dt}\mathcal{N} \right) \right\} \mathbf{V}^{(R)}(t) \quad (8.194)$$

Since, for any arbitrary vector \mathbf{A} ,

$$\left(\mathcal{N}\frac{d\mathcal{N}}{dt} - \frac{d\mathcal{N}}{dt}\mathcal{N} \right) \mathbf{A} = \left(\hat{\mathbf{n}} \times \frac{d\hat{\mathbf{n}}}{dt} \right) \times \mathbf{A} \quad (8.195)$$

eqn (8.194) is equivalent to

$$\frac{d\mathbf{V}^{(R)}(t)}{dt} = \left(\dot{\Phi}\hat{\mathbf{n}} + \sin\Phi \frac{d\hat{\mathbf{n}}}{dt} + (1 - \cos\Phi) \hat{\mathbf{n}} \times \frac{d\hat{\mathbf{n}}}{dt} \right) \times \mathbf{V}^{(R)}(t) \quad (8.196)$$

which completes the derivation of eqn (8.188). \square

An important consequence of eqn (8.188) is that if we happen to have a very simple time-dependent rotation, one with a time-varying $\Phi(t)$ but a fixed axis $\hat{\mathbf{n}}$ which does not vary with time, then

$$\frac{d\hat{\mathbf{n}}}{dt} = 0 \quad \text{and so} \quad \boldsymbol{\omega}(t) = \dot{\Phi}\hat{\mathbf{n}} \quad (8.197)$$

8.26 Euler Angles

For many problems, particularly in rigid body dynamics, the parameterization of a rotation by $\hat{\mathbf{n}}$ and Φ as in Section 8.24 is not the most convenient one. An alternate parameterization uses the three *Euler angles* α, β, γ defined by

$$\mathcal{R}[\alpha, \beta, \gamma] = \mathcal{R}[\alpha\hat{\mathbf{e}}_3] \mathcal{R}[\beta\hat{\mathbf{e}}_2] \mathcal{R}[\gamma\hat{\mathbf{e}}_3] \quad (8.198)$$

The definition in eqn (8.198) consists of three simple rotations about fixed coordinate axes: First by γ about the $\hat{\mathbf{e}}_3$ axis, then by β about the $\hat{\mathbf{e}}_2$ axis, then by α about the $\hat{\mathbf{e}}_3$ axis again. Since these are rotations by finite angles, their order is quite important, as we will see.

We prove the somewhat surprising fact that the product of these three rotations is capable of reproducing any rotation whatsoever.

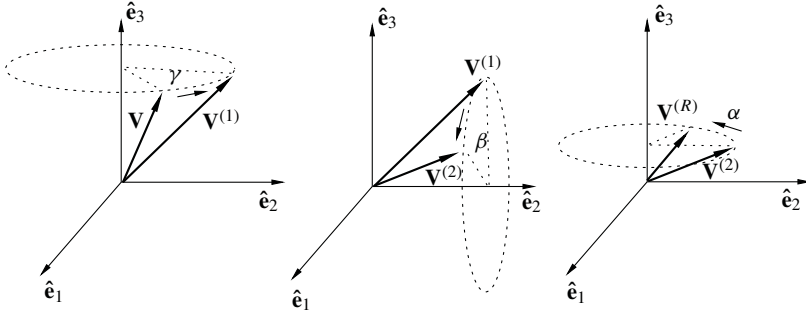


FIG. 8.7. Rotation of \mathbf{V} into $\mathbf{V}^{(R)}$ by the Euler angles. First, a rotation by γ about the $\hat{\mathbf{e}}_3$ axis rotates \mathbf{V} into $\mathbf{V}^{(1)}$. Then a rotation by β about the $\hat{\mathbf{e}}_2$ axis rotates $\mathbf{V}^{(1)}$ into $\mathbf{V}^{(2)}$. Finally a rotation by α about the $\hat{\mathbf{e}}_3$ axis rotates $\mathbf{V}^{(2)}$ into the final vector $\mathbf{V}^{(R)}$.

Theorem 8.26.1: Adequacy of Euler Angles

For any proper rotation \mathcal{R} , there are three angles α, β, γ in the ranges $-\pi < \alpha \leq \pi$, $0 \leq \beta \leq \pi$, $-\pi < \gamma \leq \pi$ such that

$$\mathcal{R}[\alpha, \beta, \gamma] = \mathcal{R} \tag{8.199}$$

Proof: We proved in Theorem 8.20.1 that for every proper rotation \mathcal{R} there are a unique axis $\hat{\mathbf{n}}$ and angle $0 \leq \Phi \leq \pi$ such that $\mathcal{R} = \mathcal{R}[\Phi\hat{\mathbf{n}}]$. So here we only need to prove that, given any fixed-axis rotation, there are three angles α, β, γ such that $\mathcal{R}[\alpha, \beta, \gamma] = \mathcal{R}[\Phi\hat{\mathbf{n}}]$. We begin by writing the matrices of each factor of eqn (8.198),

$$\begin{aligned} \mathbf{R}[\gamma\hat{\mathbf{e}}_3] &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{R}[\beta\hat{\mathbf{e}}_2] &= \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \\ \mathbf{R}[\alpha\hat{\mathbf{e}}_3] &= \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \end{aligned} \tag{8.200}$$

Multiplication of these three matrices then gives the matrix of $\mathcal{R}[\alpha, \beta, \gamma]$ as

$$\begin{aligned} \mathbf{R}[\alpha, \beta, \gamma] &= \mathbf{R}[\alpha\hat{\mathbf{e}}_3] \mathbf{R}[\beta\hat{\mathbf{e}}_2] \mathbf{R}[\gamma\hat{\mathbf{e}}_3] = \\ &\begin{pmatrix} (\cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma) & (-\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma) & \cos \alpha \sin \beta \\ (\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma) & (-\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma) & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix} \end{aligned} \tag{8.201}$$

We compare this matrix to eqn (8.142). Comparing the 33 elements of the two matrices gives

$$\cos \beta = n_3^2 + (1 - n_3^2) \cos \Phi \tag{8.202}$$

The components of unit vector $\hat{\mathbf{n}}$ obey $n_1^2 + n_2^2 + n_3^2 = 1$. It follows that $(1 - n_3^2) \geq 0$ and hence that, as Φ varies, the right side of eqn (8.202) has a maximum value of +1 and

a minimum value of $(-1 + 2n_3^2) \geq -1$. It follows that eqn (8.202) defines a unique β in the range $0 \leq \beta \leq \pi$.

Some special cases must now be considered. First, when $n_1 = n_2 = 0$ and hence $n_3^2 = 1$, the rotation is purely about the \hat{e}_3 axis. In this case, eqn (8.202) requires that $\beta = 0$. The other two angles are not separately determined, but may have any values such that $(\alpha + \gamma) = \Phi$.

Second, if $\Phi = 0$ the rotation is the trivial unity rotation, regardless of the value of $\hat{\mathbf{n}}$. Then eqn (8.202) requires that $\beta = 0$. The angles α and γ are again not separately determined but may have any values such that $(\alpha + \gamma) = 0$.

Having treated the cases $\Phi = 0$ and $n_3^2 = 1$ separately, we will henceforward assume that $\Phi > 0$ and $n_3^2 < 1$. With these assumptions, we cannot have $\beta = 0$, for that value would reduce eqn (8.202) to $(1 - n_3^2)(1 - \cos \Phi) = 0$, which would be impossible. The case $\beta = \pi$ is possible, however. Again using eqn (8.202), it can arise only when $n_3 = 0$ and $\Phi = \pi$. As can be seen by writing out $\mathbf{R}[\alpha, \pi, \gamma] = \mathbf{R}[\pi \hat{\mathbf{n}}]$ with $n_3 = 0$ assumed, the α and γ can then have any values such that $(\gamma - \alpha) = \theta$ where θ is some unique angle in the range $-\pi < \theta \leq \pi$ defined by the pair of equations $\sin \theta = 2n_1n_2$ and $\cos \theta = (n_2^2 - n_1^2)$.

The undetermination of α and γ for certain special values of Φ and $\hat{\mathbf{n}}$ is similar to the situation in spherical polar coordinates, where the polar angle ϕ is undetermined when $\theta = 0$. Here, as in the polar angle case, if Φ and $\hat{\mathbf{n}}$ are continuously differentiable functions of some parameter, the values of α and γ at the indeterminate points can be determined from the condition that α, β, γ also vary continuously.

We now find unique values of α and γ when $\Phi > 0$, $n_3^2 < 1$ and $\sin \beta > 0$. With β known from eqn (8.202), compare the 31 and 32 entries of the two sides of the matrix equation $\mathbf{R}[\alpha, \beta, \gamma] = \mathbf{R}[\Phi \hat{\mathbf{n}}]$ to obtain

$$\cos \gamma = \frac{n_2 \sin \Phi - n_3 n_1 (1 - \cos \Phi)}{\sin \beta} \quad (8.203)$$

$$\sin \gamma = \frac{n_1 \sin \Phi + n_3 n_2 (1 - \cos \Phi)}{\sin \beta} \quad (8.204)$$

Similarly, comparison of the 13 and 23 entries gives

$$\cos \alpha = \frac{n_2 \sin \Phi + n_1 n_3 (1 - \cos \Phi)}{\sin \beta} \quad (8.205)$$

$$\sin \alpha = \frac{-n_1 \sin \Phi + n_2 n_3 (1 - \cos \Phi)}{\sin \beta} \quad (8.206)$$

The sum of the squares of the right sides of eqn (8.203) and eqn (8.204) equals one. Thus a unique angle γ in the range $-\pi < \gamma \leq \pi$ is determined. Similarly, eqns (8.205, 8.206) determine a unique α in the range $-\pi < \alpha \leq \pi$.

The angles α, β, γ are now determined and five of the matrix elements matched. It remains to prove that the 11, 12, 21, and 22 elements of the two matrices are identical with these same choices of α, β, γ . This algebraic exercise is done in Exercise 8.12. Thus $\mathcal{R} = \mathcal{R}[\Phi \hat{\mathbf{n}}] = \mathcal{R}[\alpha, \beta, \gamma]$, as was to be proved. \square

The trace of $\mathcal{R}[\alpha, \beta, \gamma]$ is the sum of the diagonal terms of eqn (8.201). It can be simplified to

$$\text{Tr}\mathcal{R}[\alpha, \beta, \gamma] = \cos \beta + (1 + \cos \beta) \cos(\alpha + \gamma) \quad (8.207)$$

The definition eqn (8.198) rotates first by angle γ about $\hat{\mathbf{e}}_3$, then angle β about $\hat{\mathbf{e}}_2$, then by α about $\hat{\mathbf{e}}_3$ again. Repeated use of eqn (8.178) can be used to derive the following remarkable result.

Theorem 8.26.2: Euler Angles in Reverse Order

The rotation $\mathcal{R}[\alpha, \beta, \gamma]$ defined in eqn (8.198) can also be produced by three rotations that use the angles α, β, γ in reverse order; provided that each successive axis of rotation is changed to reflect the effect of rotations already performed,

$$\mathcal{R}[\alpha, \beta, \gamma] = \mathcal{R}[\alpha\hat{\mathbf{e}}_3] \mathcal{R}[\beta\hat{\mathbf{e}}_2] \mathcal{R}[\gamma\hat{\mathbf{e}}_3] = \mathcal{R}[\gamma\hat{\mathbf{e}}_3^{(R)}] \mathcal{R}[\beta\hat{\mathbf{y}}] \mathcal{R}[\alpha\hat{\mathbf{e}}_3] \quad (8.208)$$

where

$$\hat{\mathbf{e}}_3^{(R)} = \mathcal{R}[\beta\hat{\mathbf{y}}]\hat{\mathbf{e}}_3 = \mathcal{R}[\alpha, \beta, \gamma]\hat{\mathbf{e}}_3 \quad \text{and} \quad \hat{\mathbf{y}} = \mathcal{R}[\alpha\hat{\mathbf{e}}_3]\hat{\mathbf{e}}_2 \quad (8.209)$$

Proof: The first equality in eqn (8.208) simply repeats the definition eqn (8.198). The proof of the second equality is left as an exercise. \square

Note that the equivalence of the two forms in the first of eqn (8.209) follows from the fact that $\hat{\mathbf{e}}_3 = \mathcal{R}[\alpha\hat{\mathbf{e}}_3]\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_3^{(R)} = \mathcal{R}[\gamma\hat{\mathbf{e}}_3^{(R)}]\hat{\mathbf{e}}_3^{(R)}$. A unit vector is unchanged by a rotation of which it is the axis.

8.27 Fixed-Axis Rotation from Euler Angles

In Section 8.26 we began with a fixed axis rotation $\mathcal{R}[\Phi\hat{\mathbf{n}}]$ and derived the three Euler angles α, β , and γ . The inverse problem is also of interest.

We are given the three Euler angles α, β , and γ wish to find the equivalent finite rotation with

$$\mathcal{R}[\Phi\hat{\mathbf{n}}] = \mathcal{R}[\alpha, \beta, \gamma] \quad (8.210)$$

The angle is found by solving for $0 \leq \Phi \leq \pi$ in the expression

$$2 \cos \Phi + 1 = \cos \beta + (1 + \cos \beta) \cos(\alpha + \gamma) \quad (8.211)$$

that is found by equating the traces in eqns (8.149, 8.207).

The components of the axis come from a straightforward application of the eigenvector equation, eqn (8.163), using the matrix $\mathbf{R}[\alpha, \beta, \gamma]$ from eqn (8.201). The normalized eigenvector corresponding to $\lambda_3 = 1$ is $\hat{\mathbf{V}}^{(3)}$, which is the rotation axis. When $\beta = 0$ and $(\alpha + \gamma) = 0$, the rotation is the trivial identity transformation. In that case, the axis $\hat{\mathbf{n}}$ is undetermined since any vector is an eigenvector of \mathcal{U} with eigenvalue one. When $\beta = 0$ and $(\alpha + \gamma) \neq 0$, the components of this axis vector are $(0, 0, \pm 1)$ with the sign depending on the quadrant of $(\alpha + \gamma)$. When $\beta \neq 0$ but $(\alpha + \gamma) = 0$, the

components of the axis vector are $(-\sin \alpha, \cos \alpha, 0)$. When $\beta \neq 0$ and $(\alpha + \gamma) \neq 0$, the components of the (not yet normalized) axis vector are

$$V_1^{(3)} = (1 - \cos \beta) (\cos \alpha - \cos \gamma) \tag{8.212}$$

$$V_2^{(3)} = (1 - \cos \beta) (\sin \alpha + \sin \gamma) \tag{8.213}$$

$$V_3^{(3)} = \sin \beta (1 - \cos (\alpha + \gamma)) \tag{8.214}$$

The normalized rotation axis is then $\hat{\mathbf{n}} = \mathbf{V}^{(3)} / V^{(3)}$ where $\mathbf{V}^{(3)} = \sum_{i=1}^3 V_i^{(3)} \hat{\mathbf{e}}_i$ and $V^{(3)}$ is its magnitude. Just as in the Euler Theorem proof, the two sides of eqn (8.210) must be applied to some vector not parallel to $\hat{\mathbf{n}}$ and the results compared. If they fail to match, $\hat{\mathbf{n}}$ must be replaced by $-\hat{\mathbf{n}}$.

8.28 Time Derivative of a Product

It is useful to have general formulas for the time derivative of a rotation operator that is the product of time-dependent rotation operators

$$\mathcal{R}(t) = \mathcal{R}_a(t) \mathcal{R}_b(t) \tag{8.215}$$

Theorem 8.28.1: Angular Velocity of a Product

The angular velocity associated with the rotation $\mathcal{R}(t)$ defined in eqn (8.215) is

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_a(t) + \mathcal{R}_a(t) \boldsymbol{\omega}_b(t) \tag{8.216}$$

where $\boldsymbol{\omega}_a(t)$, $\boldsymbol{\omega}_b(t)$ are the angular velocities associated with $\mathcal{R}_a(t)$, $\mathcal{R}_b(t)$, respectively.

Proof: Using the product rule,

$$\frac{d\mathcal{R}(t)}{dt} = \frac{d\mathcal{R}_a(t)}{dt} \mathcal{R}_b(t) + \mathcal{R}_a(t) \frac{d\mathcal{R}_b(t)}{dt} \tag{8.217}$$

and hence, by eqn (8.70) of Section 8.10, the anti-symmetric operator associated with rotation $\mathcal{R}(t)$ is

$$\mathcal{W}(t) = \frac{d\mathcal{R}(t)}{dt} \mathcal{R}(t)^T = \frac{d\mathcal{R}_a(t)}{dt} \mathcal{R}_b(t) (\mathcal{R}_a(t) \mathcal{R}_b(t))^T + \mathcal{R}_a(t) \frac{d\mathcal{R}_b(t)}{dt} (\mathcal{R}_a(t) \mathcal{R}_b(t))^T \tag{8.218}$$

Since $(\mathcal{R}_a(t) \mathcal{R}_b(t))^T = \mathcal{R}_b^T(t) \mathcal{R}_a^T(t)$ and each operator is orthogonal, this expression reduces to

$$\mathcal{W}(t) = \frac{d\mathcal{R}_a(t)}{dt} \mathcal{R}_a^T(t) + \mathcal{R}_a(t) \frac{d\mathcal{R}_b(t)}{dt} \mathcal{R}_b^T(t) \mathcal{R}_a^T(t) = \mathcal{W}_a(t) + \mathcal{R}_a(t) \mathcal{W}_b(t) \mathcal{R}_a^T(t) \tag{8.219}$$

where $\mathcal{W}_a(t)$, $\mathcal{W}_b(t)$ are the anti-symmetric operators associated with rotations $\mathcal{R}_a(t)$,

$\mathcal{R}_b(t)$ respectively. Applying this to a general vector \mathbf{A} and using eqn (8.75),

$$\boldsymbol{\omega}(t) \times \mathbf{A} = \boldsymbol{\omega}_a(t) \times \mathbf{A} + \mathcal{R}_a(t) \left\{ \boldsymbol{\omega}_b(t) \times \left(\mathcal{R}_a^T(t) \mathbf{A} \right) \right\} \quad (8.220)$$

Then eqn (8.39) of Section 8.6 and the orthogonality of the operators gives

$$\begin{aligned} \boldsymbol{\omega}(t) \times \mathbf{A} &= \boldsymbol{\omega}_a(t) \times \mathbf{A} + \left\{ \mathcal{R}_a(t) \boldsymbol{\omega}_b(t) \times \left(\mathcal{R}_a^T(t) \mathbf{A} \right) \right\} \\ &= \boldsymbol{\omega}_a(t) \times \mathbf{A} + \left(\mathcal{R}_a(t) \boldsymbol{\omega}_b(t) \right) \times \mathbf{A} = \left(\boldsymbol{\omega}_a(t) + \mathcal{R}_a(t) \boldsymbol{\omega}_b(t) \right) \times \mathbf{A} \end{aligned} \quad (8.221)$$

Since vector \mathbf{A} is arbitrary, this implies eqn (8.216), as was to be proved. \square

The above theorem can be applied repeatedly to obtain the derivative of a product of any finite number of factors. Thus the angular velocity associated with

$$\mathcal{R}(t) = \mathcal{R}_a(t) \mathcal{R}_b(t) \mathcal{R}_c(t) \cdots \mathcal{R}_y(t) \mathcal{R}_z(t) \quad (8.222)$$

is

$$\begin{aligned} \boldsymbol{\omega}(t) &= \boldsymbol{\omega}_a(t) + \mathcal{R}_a(t) \boldsymbol{\omega}_b(t) \\ &\quad + \left(\mathcal{R}_a(t) \mathcal{R}_b(t) \right) \boldsymbol{\omega}_c(t) + \cdots + \left(\mathcal{R}_a(t) \mathcal{R}_b(t) \mathcal{R}_c(t) \cdots \mathcal{R}_y(t) \right) \boldsymbol{\omega}_z(t) \end{aligned} \quad (8.223)$$

in which each angular velocity is modified by all rotations that are applied after it.

8.29 Angular Velocity from Euler Angles

Time-dependent rotations can be parameterized by time-varying Euler angles. The angular velocity vector of the rotation can then be obtained as a function of the Euler angles and their first time derivatives.

Theorem 8.29.1: Angular Velocity from Euler Angles

Let a time-varying rotation be defined by

$$\mathcal{R}(t) = \mathcal{R}[\alpha(t), \beta(t), \gamma(t)] = \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \mathcal{R}[\beta(t) \hat{\mathbf{e}}_2] \mathcal{R}[\gamma(t) \hat{\mathbf{e}}_3] \quad (8.224)$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ give the three Euler angles as continuous, differentiable functions of the time. Then the vector $\boldsymbol{\omega}(t)$ associated with the time-dependent rotation in eqn (8.224) is

$$\boldsymbol{\omega}(t) = \dot{\alpha} \hat{\mathbf{e}}_3 + \dot{\beta} \hat{\mathbf{y}}(t) + \dot{\gamma} \hat{\mathbf{e}}_3^{(R)}(t) \quad (8.225)$$

where the dots represent time derivatives and $\hat{\mathbf{e}}_3^{(R)}(t) = \mathcal{R}[\alpha(t), \beta(t), \gamma(t)] \hat{\mathbf{e}}_3$ and $\hat{\mathbf{y}}(t) = \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \hat{\mathbf{e}}_2$ are the same vectors found in eqn (8.209) above.

Proof: Each of the products on the right side of eqn (8.224) is a rotation about a fixed coordinate axis. Hence, by eqn (8.197), the angular velocity vectors associated

with rotations $\mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3]$, $\mathcal{R}[\beta(t) \hat{\mathbf{e}}_2]$, $\mathcal{R}[\gamma(t) \hat{\mathbf{e}}_3]$ are $\boldsymbol{\omega}_\alpha(t) = \dot{\alpha} \hat{\mathbf{e}}_3$, $\boldsymbol{\omega}_\beta(t) = \dot{\beta} \hat{\mathbf{e}}_2$, and $\boldsymbol{\omega}_\gamma(t) = \dot{\gamma} \hat{\mathbf{e}}_3$, respectively. Then eqn (8.223) gives

$$\boldsymbol{\omega}(t) = \dot{\alpha} \hat{\mathbf{e}}_3 + \dot{\beta} \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \hat{\mathbf{e}}_2 + \dot{\gamma} \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \mathcal{R}[\beta(t) \hat{\mathbf{e}}_2] \hat{\mathbf{e}}_3 \quad (8.226)$$

Since rotation about a fixed axis does not change that axis vector, $\hat{\mathbf{e}}_3 = \mathcal{R}[\gamma(t) \hat{\mathbf{e}}_3] \hat{\mathbf{e}}_3$ and hence

$$\begin{aligned} \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \mathcal{R}[\beta(t) \hat{\mathbf{e}}_2] \hat{\mathbf{e}}_3 &= \mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \mathcal{R}[\beta(t) \hat{\mathbf{e}}_2] \mathcal{R}[\gamma(t) \hat{\mathbf{e}}_3] \hat{\mathbf{e}}_3 \\ &= \mathcal{R}[\alpha(t), \beta(t), \gamma(t)] \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3^{(R)}(t) \end{aligned} \quad (8.227)$$

Thus, using the definitions above, eqn (8.226) becomes identical to eqn (8.225), as was to be proved. \square

The expression for $\boldsymbol{\omega}$ in eqn (8.225) is not yet in a useful form. It needs to be expressed in terms of a single set of basis vectors. Denoting the components in the $\hat{\mathbf{e}}_k$ system by $\omega_k(t)$ gives

$$\boldsymbol{\omega}(t) = \omega_1(t) \hat{\mathbf{e}}_1(t) + \omega_2(t) \hat{\mathbf{e}}_2(t) + \omega_3(t) \hat{\mathbf{e}}_3(t) \quad (8.228)$$

where, using eqn (8.225), $\omega_k(t) = \hat{\mathbf{e}}_k \cdot \boldsymbol{\omega}(t)$ may be written as

$$\begin{aligned} \omega_k(t) &= \dot{\alpha} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_3 + \dot{\beta} \hat{\mathbf{e}}_k \cdot (\mathcal{R}[\alpha(t) \hat{\mathbf{e}}_3] \hat{\mathbf{e}}_2) + \dot{\gamma} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_3^{(R)}(t) \\ &= \dot{\alpha} \delta_{k3} + \dot{\beta} R_{k2}[\alpha(t) \hat{\mathbf{e}}_3] + \dot{\gamma} R_{k3}[\alpha(t), \beta(t), \gamma(t)] \end{aligned} \quad (8.229)$$

Writing the components out explicitly using eqns (8.200, 8.201) gives

$$\omega_1(t) = -\dot{\beta} \sin \alpha + \dot{\gamma} \cos \alpha \sin \beta \quad (8.230)$$

$$\omega_2(t) = \dot{\beta} \cos \alpha + \dot{\gamma} \sin \alpha \sin \beta \quad (8.231)$$

$$\omega_3(t) = \dot{\alpha} + \dot{\gamma} \cos \beta \quad (8.232)$$

where the Euler angles and their derivatives are all functions of time.

8.30 Active and Passive Rotations

We now return to the general treatment of the rotation operator. The rotation operator \mathcal{R} can be used either actively or passively. The active use, which is the only one we have discussed to this point, transforms each vector \mathbf{V} into a rotated vector $\mathbf{V}^{(R)} = \mathcal{R}\mathbf{V}$. Although we have not emphasized the point, it is implicit that the coordinate system unit vectors $\hat{\mathbf{e}}_i$ are not changed by this active rotation.

The passive use of \mathcal{R} makes the opposite choice. The unit vectors $\hat{\mathbf{e}}_i$ are rotated to form a new coordinate system, which we will denote $\hat{\mathbf{e}}'_i$ and define, for $i = 1, 2, 3$, as

$$\hat{\mathbf{e}}'_i = \hat{\mathbf{e}}_i^{(R)} = \mathcal{R}\hat{\mathbf{e}}_i \quad (8.233)$$

The vector \mathbf{V} , however, is *not changed* by passive rotations. In a sense, active rotations rotate the world while passive ones rotate the observer. Note that the same rotation operator \mathcal{R} is used in both cases, but used differently.

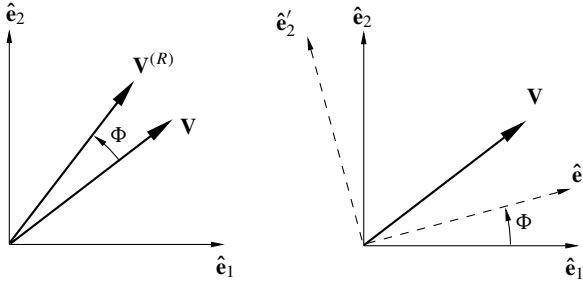


FIG. 8.8. On the left is an active rotation. The vector \mathbf{V} is rotated into the new vector $\mathbf{V}^{(R)}$ but the basis vectors $\hat{\mathbf{e}}_i$ do not change. On the right is a passive rotation. The basis vectors $\hat{\mathbf{e}}_i$ are rotated into new basis vectors $\hat{\mathbf{e}}'_i$ but the vector \mathbf{V} does not change.

We now consider passive rotations. We will refer to the original unit vectors $\hat{\mathbf{e}}_i$ as the old system or the original system, and often denote it by the letter o . The rotated unit vectors $\hat{\mathbf{e}}'_i$ will be called the new system or the rotated system, and will often be denoted by the letter o' .

We assume here and in the following sections that the $\hat{\mathbf{e}}_i$ are a right-handed orthonormal system and \mathcal{R} is a proper rotation operator. It follows from Definition 8.6.2 that the new system of unit vectors is orthonormal and right-handed,

$$\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \quad \text{and} \quad \hat{\mathbf{e}}'_1 \times \hat{\mathbf{e}}'_2 = \hat{\mathbf{e}}'_3 \tag{8.234}$$

The new unit vectors can be expanded in terms of the old ones by eqn (8.33) with the identification $\hat{\mathbf{e}}'_i = \hat{\mathbf{e}}_i^{(R)}$,

$$\hat{\mathbf{e}}'_i = \sum_{j=1}^3 \hat{\mathbf{e}}_j (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}'_i) = \sum_{j=1}^3 \hat{\mathbf{e}}_j (\hat{\mathbf{e}}_j \cdot \mathcal{R}\hat{\mathbf{e}}_i) = \sum_{j=1}^3 \hat{\mathbf{e}}_j R_{ji} = \sum_{j=1}^3 R_{ij}^T \hat{\mathbf{e}}_j \tag{8.235}$$

8.31 Passive Transformation of Vector Components

An important thing to notice is that, although the vectors \mathbf{V} do not change in passive rotations, their *components* do change. An unchanged vector \mathbf{V} can be expanded in either system

$$\sum_{i=1}^3 V_i \hat{\mathbf{e}}_i = \mathbf{V} = \sum_{i=1}^3 V'_i \hat{\mathbf{e}}'_i \tag{8.236}$$

where the components in the two systems are

$$V_i = \hat{\mathbf{e}}_i \cdot \mathbf{V} \quad \text{and} \quad V'_i = \hat{\mathbf{e}}'_i \cdot \mathbf{V} \tag{8.237}$$

The new components $V'_i = \mathbf{V} \cdot \hat{\mathbf{e}}'_i$ are different from the old ones $V_i = \mathbf{V} \cdot \hat{\mathbf{e}}_i$ even though \mathbf{V} is the same in both cases, because the unit vectors are different. We can

denote the two alternate expansions of the vector \mathbf{V} into components by the notation

$$\mathbf{V} : (V_1, V_2, V_3)_o \tag{8.238}$$

$$\mathbf{V} : (V'_1, V'_2, V'_3)_{o'} \tag{8.239}$$

We avoid using the equal sign here. A vector \mathbf{V} is *not* equal to its components, rather it is represented by its components in a particular reference system. As we see, the components in the o' system will be different from those in the o system, even though the vector \mathbf{V} is the same in both cases.

It follows from eqn (8.235) that the components of \mathbf{V} in the two systems are related by

$$V'_i = \hat{\mathbf{e}}'_i \cdot \mathbf{V} = \left(\sum_{j=1}^3 R_{ij}^T \hat{\mathbf{e}}_j \right) \cdot \mathbf{V} = \sum_{j=1}^3 R_{ij}^T V_j \tag{8.240}$$

This relation can also be written in matrix form. If we denote by $[V]$ the column vector of components in the o system, and by $[V']$ the column vector of components in the o' system, then eqn (8.240) can be written as

$$[V'] = \mathbf{R}^T [V] \tag{8.241}$$

8.32 Passive Transformation of Matrix Elements

Just as a vector \mathbf{V} has different components in the o and o' systems, an operator \mathcal{B} will also be represented by different matrix elements (which we might consider as the “components” of the operator) in the two systems. Since we are considering passive rotations now, the operator itself if not changed by the rotation, but its matrix elements are changed. In the two systems, we have

$$B_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{B} \hat{\mathbf{e}}_j \quad \text{and} \quad B'_{ij} = \hat{\mathbf{e}}'_i \cdot \mathcal{B} \hat{\mathbf{e}}'_j \tag{8.242}$$

which are related by

$$B'_{ij} = \hat{\mathbf{e}}'_i \cdot \mathcal{B} \hat{\mathbf{e}}'_j = \left(\sum_{k=1}^3 R_{ik}^T \hat{\mathbf{e}}_k \right) \cdot \mathcal{B} \left(\sum_{l=1}^3 R_{jl}^T \hat{\mathbf{e}}_l \right) = \sum_{k=1}^3 \sum_{l=1}^3 R_{ik}^T B_{kl} R_{lj} \tag{8.243}$$

which can be written as the matrix equation

$$\mathbf{B}' = \mathbf{R}^T \mathbf{B} \mathbf{R} \tag{8.244}$$

We say that the operator \mathcal{B} is represented by the matrix \mathbf{B} in the o system and by the matrix \mathbf{B}' in the o' system.

It follows from eqns (8.235, 8.243) that the dyadic associated with operator \mathcal{B} can be written out in either system, as

$$\sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i B_{ij} \hat{\mathbf{e}}_j = \mathbb{B} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}'_i B'_{ij} \hat{\mathbf{e}}'_j \tag{8.245}$$

Of particular interest is the unit operator \mathcal{U} . It follows from the orthogonality of \mathcal{R} and the transformation rule eqn (8.240) that its matrix elements are the same in any

system,

$$U_{ij} = \delta_{ij} = U'_{ij} \quad (8.246)$$

Thus the resolution-of-unity dyadic, the dyadic associated with \mathcal{U} , has exactly the same algebraic form in the two systems,

$$\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 = \mathbb{U} = \hat{\mathbf{e}}'_1 \hat{\mathbf{e}}'_1 + \hat{\mathbf{e}}'_2 \hat{\mathbf{e}}'_2 + \hat{\mathbf{e}}'_3 \hat{\mathbf{e}}'_3 \quad (8.247)$$

Multiplying a vector by the expansion of \mathbb{U} in the o (o') system will expand that vector in the o (o') system.

8.33 The Body Derivative

Let us now consider the case in which the o system is a fixed inertial system, but the rotation operator and the rotated system o' are both time varying. The basis vectors of the o' system will thus be functions of time. For $i = 1, 2, 3$,

$$\hat{\mathbf{e}}'_i(t) = \hat{\mathbf{e}}_i^{(R)}(t) = \mathcal{R}(t) \hat{\mathbf{e}}_i \quad (8.248)$$

From the consideration of time-dependent rotations in Section 8.10, we know that the time derivatives of the o' system basis vectors are

$$\frac{d\hat{\mathbf{e}}'_i(t)}{dt} = \boldsymbol{\omega}(t) \times \hat{\mathbf{e}}'_i(t) \quad (8.249)$$

where $\boldsymbol{\omega}(t)$ is the (generally time-dependent) angular velocity vector of the time varying rotation $\mathcal{R}(t)$.

Now consider the task of calculating the time-derivative of some vector \mathbf{V} . If we expand this vector in the o system, then the time derivative will be

$$\frac{d\mathbf{V}}{dt} = \frac{d}{dt} \sum_{i=1}^3 V_i \hat{\mathbf{e}}_i = \sum_{k=1}^3 \frac{dV_k}{dt} \hat{\mathbf{e}}_k \quad (8.250)$$

However, if we expand \mathbf{V} in the o' system, the same derivative will have a more complicated form, due to the time variation of the unit vectors. It is

$$\frac{d\mathbf{V}}{dt} = \frac{d}{dt} \sum_{i=1}^3 V'_i \hat{\mathbf{e}}'_i = \sum_{i=1}^3 \left(\frac{dV'_i}{dt} \hat{\mathbf{e}}'_i + V'_i \frac{d\hat{\mathbf{e}}'_i(t)}{dt} \right) = \sum_{i=1}^3 \left(\frac{dV'_i}{dt} \hat{\mathbf{e}}'_i + V'_i \boldsymbol{\omega}(t) \times \hat{\mathbf{e}}'_i(t) \right) \quad (8.251)$$

Collecting terms and using the linearity of cross products to factor $\boldsymbol{\omega}(t)$ out of the second term on the right gives

$$\frac{d\mathbf{V}}{dt} = \left\langle \frac{d\mathbf{V}}{dt} \right\rangle_b + \boldsymbol{\omega}(t) \times \mathbf{V} \quad (8.252)$$

where the first term on the right is the so-called *body derivative*, a vector defined as

$$\left\langle \frac{d\mathbf{V}}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{dV'_i}{dt} \hat{\mathbf{e}}'_i \quad (8.253)$$

To understand what the body derivative is, imagine an observer rotating with the o' system who is unaware that it and he are rotating. (We on the surface of the earth

are good examples.) If he is asked to calculate the time derivative of a vector, he will first express that vector in his o' reference system, and then calculate eqn (8.253), the body derivative. He thinks he is using eqn (8.250), but that is his error since his o' reference system is not, in fact, inertial. After he calculates the body derivative, we can correct his error by adding the term $\boldsymbol{\omega}(t) \times \mathbf{V}$.

So the recipe for getting the body derivative is: (1) Express the vector in the o' system, and then (2) take the time derivative as if the $\hat{\mathbf{e}}'_k$ basis vectors were constants. Note that, although this body derivative is calculated in a special way, nonetheless it is just an ordinary vector that can be expanded, if needed, in any coordinate system.

8.34 Passive Rotations and Rigid Bodies

We can identify the moving coordinate system $\hat{\mathbf{e}}'_i$ of passive rotations introduced in Section 8.30 with the similarly denoted coordinate system embedded in the moving rigid body in Section 8.9. The position and orientation of the rigid body at time t can be thought of as the position and orientation of this $\hat{\mathbf{e}}'_i$ system of coordinates, whose origin is at the center of mass of the body and whose orientation is given by

$$\hat{\mathbf{e}}'_i(t) = \mathcal{R}(t) \hat{\mathbf{e}}_i \tag{8.254}$$

derived from eqn (8.233). In this system of coordinates, the vectors $\boldsymbol{\rho}_n$ can be expressed as

$$\boldsymbol{\rho}_n(t) = \rho'_{n1}(t) \hat{\mathbf{e}}'_1(t) + \rho'_{n2}(t) \hat{\mathbf{e}}'_2(t) + \rho'_{n3}(t) \hat{\mathbf{e}}'_3(t) \tag{8.255}$$

where the components were shown in Section 8.8 to obey

$$\rho'_{ni}(t) = \hat{\mathbf{e}}'_i(t) \cdot \boldsymbol{\rho}_n(t) = \hat{\mathbf{e}}'_i(0) \cdot \boldsymbol{\rho}_n(0) = \rho'_{ni}(0) \tag{8.256}$$

and hence not vary with time.

Thus the time derivative of $\boldsymbol{\rho}_n(t)$, when expanded in terms of the body derivative and its correction becomes

$$\frac{d\boldsymbol{\rho}_n(t)}{dt} = \left\langle \frac{d\boldsymbol{\rho}_n(t)}{dt} \right\rangle_b + \boldsymbol{\omega}(t) \times \boldsymbol{\rho}_n(t) \tag{8.257}$$

where

$$\left\langle \frac{d\boldsymbol{\rho}_n(t)}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{d\rho'_{ni}(t)}{dt} \hat{\mathbf{e}}'_i = 0 \tag{8.258}$$

since eqn (8.256) implies that $d\rho'_{ni}(t)/dt = 0$. Thus

$$\dot{\boldsymbol{\rho}}_n(t) = \frac{d\boldsymbol{\rho}_n(t)}{dt} = \boldsymbol{\omega}(t) \times \boldsymbol{\rho}_n(t) \tag{8.259}$$

which reproduces eqn (8.93).

The coordinate system embedded in the rigid body, with unit vectors given by eqn (8.254) and origin at the center of mass of the body, will be used frequently in subsequent chapters. The constancy of the components $\rho'_{ni}(t) = \rho'_{ni}(0)$ in that system will lead to many simplifications.

8.35 Passive Use of Euler Angles

The time-dependent passive rotation in eqn (8.254) can be parameterized using time-dependent Euler angles, as developed in Section 8.26. Rather than simply using the standard definition in eqn (8.198), however, it is clearer to introduce the alternate form of the Euler angle operators in eqn (8.208) to write

$$\hat{e}'_i(t) = \mathcal{R}[\alpha(t), \beta(t), \gamma(t)] \hat{e}_i = \mathcal{R}[\gamma(t) \hat{e}'_3(t)] \mathcal{R}[\beta(t) \hat{y}(t)] \mathcal{R}[\alpha(t) \hat{e}_3] \hat{e}_i \quad (8.260)$$

where the definition $\hat{e}_3^{(R)}(t) = \hat{e}'_3(t)$ has been used, and where

$$\hat{e}'_3(t) = \mathcal{R}[\beta(t) \hat{y}(t)] \hat{e}_3 = \mathcal{R}[\alpha(t), \beta(t), \gamma(t)] \hat{e}_3 \quad \text{and} \quad \hat{y}(t) = \mathcal{R}[\alpha(t) \hat{e}_3] \hat{e}_2 \quad (8.261)$$

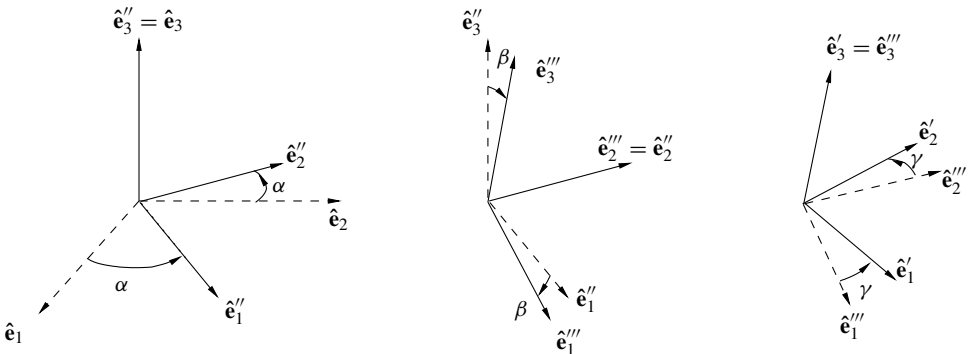


FIG. 8.9. Steps in the passive use of Euler angles. First a rotation by α about the \hat{e}_3 axis leads to \hat{e}''_i . Then (center figure) a rotation by β about the \hat{e}''_2 axis leads to \hat{e}'''_i . Finally, a rotation by γ about the \hat{e}'''_3 axis leads to the final orientation \hat{e}'_i .

Then the progression from \hat{e}_i to $\hat{e}'_i(t)$ can be decomposed into three easily visualized steps, the original triad is rotated by angle $\alpha(t)$ about the \hat{e}_3 axis by the operator $\mathcal{R}[\alpha(t) \hat{e}_3]$ to produce a triad that will be denoted $\hat{e}''_i(t)$. Thus

$$\hat{e}''_i(t) = \mathcal{R}[\alpha(t) \hat{e}_3] \hat{e}_i \quad (8.262)$$

for $i = 1, 2, 3$. The unit vector denoted $\hat{y}(t)$ in eqn (8.261) is seen to be the same as the vector $\hat{e}''_2(t)$ produced by this first rotation. It is the rotated y -axis. Note also that rotation about the z -axis does not change the z -axis, so $\hat{e}''_3 = \hat{e}_3$.

The triad $\hat{e}''_i(t)$ is then rotated by angle $\beta(t)$ about its own $\hat{e}''_2(t)$ -axis by the second rotation operator to act, $\mathcal{R}[\beta(t) \hat{y}(t)] = \mathcal{R}[\beta(t) \hat{e}''_2(t)]$. Call the resulting triad $\hat{e}'''_i(t)$. Thus, for $i = 1, 2, 3$,

$$\hat{e}'''_i(t) = \mathcal{R}[\beta(t) \hat{e}''_2(t)] \hat{e}''_i(t) \quad (8.263)$$

The new z -axis after this second rotation is $\hat{e}'''_3(t)$, which is in fact identical to the final z -axis $\hat{e}'_3(t)$. Also, since the y -axis is unchanged by a rotation about the y -axis, $\hat{e}'''_2(t) = \hat{e}''_2(t) = \hat{y}(t)$.

In the final step, the triad $\hat{\mathbf{e}}_i'''(t)$ is rotated by angle $\gamma(t)$ about its own $\hat{\mathbf{e}}_3'''(t)$ -axis by the operator $\mathcal{R}[\gamma(t)\hat{\mathbf{e}}_3'(t)] = \mathcal{R}[\gamma(t)\hat{\mathbf{e}}_3'''(t)]$ to produce the final triad $\hat{\mathbf{e}}_i'(t)$. For $i = 1, 2, 3$,

$$\hat{\mathbf{e}}_i'(t) = \mathcal{R}[\gamma(t)\hat{\mathbf{e}}_3'''(t)]\hat{\mathbf{e}}_i'''(t) \tag{8.264}$$

Note that rotation about the z -axis doesn't change the z -axis, and so $\hat{\mathbf{e}}_3'(t) = \hat{\mathbf{e}}_3'''(t)$, as was mentioned previously.

Thus a three-step process applied to the triad $\hat{\mathbf{e}}_i$, consisting of rotation about the original z -axis $\hat{\mathbf{e}}_3$ by α , rotation about the new y -axis $\hat{\mathbf{y}}(t)$ by β , and rotation about the even newer z -axis $\hat{\mathbf{e}}_3'(t)$ by γ , has led to the final triad $\hat{\mathbf{e}}_i'(t)$.

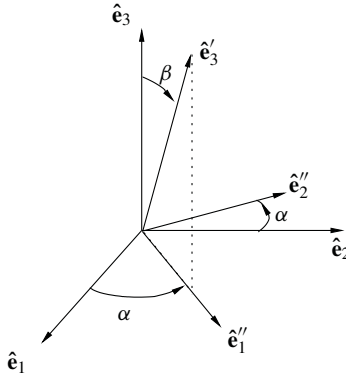


FIG. 8.10. Another view of the passive use of Euler angles. Not all unit vectors are shown, and the final rotation by γ is not shown. Note that $\hat{\mathbf{e}}_3'$ lies in the $\hat{\mathbf{e}}_3$ - $\hat{\mathbf{e}}_1''$ plane and has spherical polar angles α, β regardless of the value of γ .

The final z -axis $\hat{\mathbf{e}}_3'(t)$ will have spherical polar coordinates

$$1, \theta_{3'}, \phi_{3'} \quad \text{where} \quad \theta_{3'} = \beta \quad \text{and} \quad \phi_{3'} = \alpha \tag{8.265}$$

Hence, using either the definitions of spherical polar coordinates in Section A.8, or applying the matrix in eqn (8.201) to the column vector $(0, 0, 1)^T$ to obtain the components, the vector $\hat{\mathbf{e}}_3'(t)$ can be expressed in the $\hat{\mathbf{e}}_i$ coordinate system as

$$\hat{\mathbf{e}}_3'(t) = \sin \beta \cos \alpha \hat{\mathbf{e}}_1 + \sin \beta \sin \alpha \hat{\mathbf{e}}_2 + \cos \beta \hat{\mathbf{e}}_3 \tag{8.266}$$

Since parameterization of the body system by the three Euler angles will be used extensively in the following chapters, it will be useful to express the angular velocity vector in the body system. The general angular velocity is given by eqn (8.225) as

$$\boldsymbol{\omega}(t) = \dot{\alpha} \hat{\mathbf{e}}_3 + \dot{\beta} \hat{\mathbf{y}} + \dot{\gamma} \hat{\mathbf{e}}_3^{(R)} = \dot{\alpha} \hat{\mathbf{e}}_3 + \dot{\beta} \hat{\mathbf{e}}_2'''(t) + \dot{\gamma} \hat{\mathbf{e}}_3'(t) \tag{8.267}$$

where the last expression has been converted to the notation of the present section.

The expansion in the $\hat{\mathbf{e}}'_i(t)$ system is

$$\boldsymbol{\omega}(t) = \omega'_1(t) \hat{\mathbf{e}}'_1(t) + \omega'_2(t) \hat{\mathbf{e}}'_2(t) + \omega'_3(t) \hat{\mathbf{e}}'_3(t) \quad (8.268)$$

where, for $i = 1, 2, 3$,

$$\begin{aligned} \omega'_i(t) &= \hat{\mathbf{e}}'_i(t) \cdot \boldsymbol{\omega}(t) = \hat{\mathbf{e}}'_i(t) \cdot (\dot{\alpha} \hat{\mathbf{e}}_3 + \dot{\beta} \hat{\mathbf{e}}_2'''(t) + \dot{\gamma} \hat{\mathbf{e}}_3'(t)) \\ &= \dot{\alpha} R_{3i}[\alpha(t), \beta(t), \gamma(t)] + \dot{\beta} R_{2i}'''[\gamma(t) \hat{\mathbf{e}}_3'''] + \dot{\gamma} \delta_{i3} \end{aligned} \quad (8.269)$$

Noting that $R_{2i}'''[\gamma(t) \hat{\mathbf{e}}_3'''] = R_{2i}[\gamma(t) \hat{\mathbf{e}}_3]$, the matrices in eqn (8.200) through eqn (8.201) may be used to evaluate the needed matrix elements, giving finally

$$\omega'_1(t) = -\dot{\alpha} \sin \beta \cos \gamma + \dot{\beta} \sin \gamma \quad (8.270)$$

$$\omega'_2(t) = \dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma \quad (8.271)$$

$$\omega'_3(t) = \dot{\alpha} \cos \beta + \dot{\gamma} \quad (8.272)$$

and the Euler angles and their derivatives are all functions of time. These equations could also be derived, or checked as in Exercise 8.5, by applying eqn (8.240) directly to the inertial components of $\boldsymbol{\omega}(t)$ in eqns (8.230 – 8.232).

8.36 Exercises

Exercise 8.1 Consider a rotation $\mathcal{R}[\Phi \hat{\mathbf{n}}]$ with $\Phi = 30^\circ$ and a fixed axis $\hat{\mathbf{n}}$ that lies in the first octant and makes the same angle with each of the coordinate axes.

(a) Find numerical values for all nine components of the matrix $\mathbf{R}[\Phi \hat{\mathbf{n}}]$. [Note: It is much better to write the matrix elements in exact forms like, e.g., $\sqrt{3}/2$, rather than in terms of decimals.]

(b) Verify numerically that your matrix is a proper, orthogonal matrix.

(c) Check numerically, that $\text{Tr} \mathcal{R}[\Phi \hat{\mathbf{n}}] = 1 + 2 \cos \Phi$ for your matrix, as is required by eqn (8.149).

(d) Check numerically that $\mathbf{R}[\Phi \hat{\mathbf{n}}][n] = [n]$ where $[n]$ is the column vector of components of $\hat{\mathbf{n}}$. Why is this equation true?

Exercise 8.2 Consider a plane mirror. Denote the unit vector normal to its surface and pointing out into the room by $\hat{\mathbf{n}}$. Let an operator \mathcal{M} convert a general vector \mathbf{V} in front of the mirror into its reflected image $\mathbf{V}^{(M)} = \mathcal{M}\mathbf{V}$ behind the mirror. The matrix \mathbf{M} of this operator was found in Exercise 7.3.

(a) Consider the operator $\mathcal{R}[\pi \hat{\mathbf{n}}]$ that rotates vectors by 180° about the normal to the mirror. Find a general expression for its matrix elements $R_{ij}[\pi \hat{\mathbf{n}}]$ in terms of the components n_i of vector $\hat{\mathbf{n}}$.

(b) Show that $\mathbf{M} = \mathbf{T} \mathbf{R}[\pi \hat{\mathbf{n}}] = \mathbf{R}[\pi \hat{\mathbf{n}}] \mathbf{T}$ where \mathbf{T} is the matrix of the total inversion operator $\mathcal{T} = -\mathcal{U}$ discussed in Section 8.6. Mirror reflection is thus equivalent to total inversion followed or preceded by rotation by 180° about the normal to the surface of the mirror.

(c) Use the result (b) to argue that \mathcal{M} must be an improper rotation operator.

Exercise 8.3 Suppose that a rotation operator is defined by the Euler angles

$$\alpha = 45^\circ \quad \beta = 30^\circ \quad \gamma = -45^\circ \quad (8.273)$$

- (a) Write the numerical values of all nine matrix elements and the matrix $\mathbf{R}[\alpha, \beta, \gamma]$. [Note: It is much better to use exact forms such as, e.g., $\sqrt{2}/3$, rather than decimals. Please do it that way.]
 (b) Use the result of Exercise 7.1 to check that your matrix is orthogonal.
 (c) By the Euler Theorem, there must be some fixed-axis rotation such that, for some Φ in the range $0 \leq \Phi \leq \pi$ and some axis $\hat{\mathbf{n}}$

$$\mathbf{R}[\Phi\hat{\mathbf{n}}] = \mathbf{R}[\alpha, \beta, \gamma] \quad (8.274)$$

Find the numerical value of angle Φ by the condition that the traces of both sides of eqn (8.274) must be the same.

(d) The axis vector $\hat{\mathbf{n}}$ for this rotation has components $(-1, 1, 0)/\sqrt{2}$. Verify that $\mathbf{R}[\alpha, \beta, \gamma][n] = [n]$.

(e) Denoting $\hat{\mathbf{e}}_3^{(R)} = \mathcal{R}[\alpha, \beta, \gamma]\hat{\mathbf{e}}_3$, verify that $\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_3^{(R)} = \eta\hat{\mathbf{n}}$ where η is a positive number. Why is that so? What would it mean if η turned out to be a negative number?

Exercise 8.4 Use the results of the Theorem 8.23.1 repeatedly to prove the second equality in eqn (8.208) of Theorem 8.26.2.

Exercise 8.5 Use the passive transformation rule $[\omega'] = \mathbf{R}^T[\alpha, \beta, \gamma][\omega]$ from eqn (8.240), and the inertial system components ω_i of $\boldsymbol{\omega}$ from eqns (8.230 – 8.232), to obtain the components ω'_i of $\boldsymbol{\omega}$ in the body system stated in eqns (8.270 – 8.272).

Exercise 8.6 Use eqn (8.252) to show that the angular velocity in eqn (8.188) can also be written as

$$\boldsymbol{\omega} = \dot{\Phi}\hat{\mathbf{n}} + \sin\Phi \left\langle \frac{d\hat{\mathbf{n}}}{dt} \right\rangle_b - (1 - \cos\Phi)\hat{\mathbf{n}} \times \left\langle \frac{d\hat{\mathbf{n}}}{dt} \right\rangle_b \quad (8.275)$$

Exercise 8.7 Use eqns (8.104, 8.140) to show that fixed axis rotations and infinitesimal rotations are related by

$$\mathcal{R}[d\Phi\hat{\mathbf{n}}] = \mathcal{R}_I[d\Phi\hat{\mathbf{n}}] + o(d\Phi) \quad (8.276)$$

Exercise 8.8 This exercise refers to Sections 8.30 through 8.32.

(a) The rotation matrix has been defined throughout the chapter by its expression in the unrotated $\hat{\mathbf{e}}_i$ basis, $R_{ij} = \hat{\mathbf{e}}_i \cdot \mathcal{R}\hat{\mathbf{e}}_j$. Show that its expression in terms of the rotated basis, $R'_{ij} = \hat{\mathbf{e}}'_i \cdot \mathcal{R}\hat{\mathbf{e}}'_j$, obeys $R'_{ij} = R_{ij}$ and hence that \mathcal{R} has the same matrix in either basis.

(b) From eqn (8.70), one obtains the matrix equation $\mathbf{W} = (d\mathbf{R}/dt)\mathbf{R}^T$ where the matrices are expressed in the unrotated basis. Show that the matrix of the angular velocity operator \mathcal{W} in the rotated basis is

$$\mathbf{W}' = \mathbf{R}^T \frac{d\mathbf{R}}{dt} \quad (8.277)$$

(c) In eqn (8.78) the matrix elements of the angular velocity operator \mathcal{W} in the unrotated basis are written in terms of the components of the angular velocity vector $\boldsymbol{\omega}$ in that basis

as $W_{ij} = \sum_{k=1}^3 \varepsilon_{ikj} \omega_k$ where $\boldsymbol{\omega} = \sum_{k=1}^3 \omega_k \hat{\mathbf{e}}_k$. Show that the matrix elements of \mathcal{W} in the rotated basis have the same relation to the components ω'_k in that basis

$$W'_{ij} = \sum_{k=1}^3 \varepsilon_{ikj} \omega'_k \quad \text{where} \quad \boldsymbol{\omega} = \sum_{k=1}^3 \omega'_k \hat{\mathbf{e}}'_k \quad (8.278)$$

[Hint: Use $\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j \times \hat{\mathbf{e}}'_k = \varepsilon_{ijk} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k$ and the transformation rule eqn (8.235).]

Exercise 8.9 In eqn (7.160) of Exercise 7.8, the matrix representing an operator in the spherical basis was related to the standard Cartesian operator by the equation

$$\mathbf{F}^{(\text{sp})} = \mathbf{T} \mathbf{F} \mathbf{T}^\dagger \quad (8.279)$$

(a) Apply that transformation to the matrices defined in eqn (8.200) to find $\mathbf{R}^{(\text{sp})}[\gamma \hat{\mathbf{e}}_3]$, $\mathbf{R}^{(\text{sp})}[\beta \hat{\mathbf{e}}_2]$, and $\mathbf{R}^{(\text{sp})}[\alpha \hat{\mathbf{e}}_3]$.

(b) Check your work by comparing your $\mathbf{R}^{(\text{sp})}[\beta \hat{\mathbf{e}}_2]$ to the matrix with components $d_{mm'}^1(\beta)$ as listed in the page titled “Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions” in S. Eidelman et al. (2004) “Review of Particle Physics,” *Phys. Lett. B* **592**, 1. (That reference uses θ in place of β .)

(c) Equation (8.201) defines the matrix $\mathbf{R}[\alpha, \beta, \gamma]$ representing a general rotation parameterized in terms of Euler angles. Show that the matrix $\mathbf{R}^{(\text{sp})}[\alpha, \beta, \gamma]$ representing this rotation in the spherical basis can be written as

$$R_{mm'}^{(\text{sp})}[\alpha, \beta, \gamma] = e^{i\alpha m} d_{mm'}^1(\beta) e^{i\gamma m'} \quad (8.280)$$

This matrix, often denoted as $D_{mm'}^1[\alpha, \beta, \gamma]$, is used to represent rotations of state vectors of angular momentum $\ell = 1$ in quantum theory. See, for example, Chapter 12 of Shankar (1994).

Exercise 8.10 In eqn (8.138) it was shown that a rotation by angle Φ about a fixed axis $\hat{\mathbf{n}}$ can be written as

$$\mathcal{R}[\Phi \hat{\mathbf{n}}] = \exp\left(\Phi \hat{\mathbf{n}} \cdot \vec{\mathcal{J}}\right) \quad (8.281)$$

where $\vec{\mathcal{J}}$ is the vector with operator components defined in eqn (8.135).

(a) Prove that the commutation relations eqn (8.127) imply that, for two unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$,

$$\left[(\hat{\mathbf{n}}_1 \cdot \vec{\mathcal{J}}), (\hat{\mathbf{n}}_2 \cdot \vec{\mathcal{J}}) \right]_c = (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \vec{\mathcal{J}} \quad (8.282)$$

Exercise 8.11 In Sections 8.14 and 8.15 we demonstrated that infinitesimal rotations commute. This result can be expressed as

$$\left[\mathcal{R}[\varepsilon \hat{\mathbf{n}}_1], \mathcal{R}[\varepsilon \hat{\mathbf{n}}_2] \right]_c = o(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0 \quad (8.283)$$

where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are any two unit vectors. [Recall that the symbol $o(\varepsilon)$ means that the quantity is of *smaller* order than ε . See the definitions in Section D.11.]

(a) We now want to carry the calculation of the commutator to quadratic order. Use eqn (8.282) and

$$\mathcal{R}[\varepsilon \hat{\mathbf{n}}_1] = \mathcal{U} + \varepsilon (\hat{\mathbf{n}}_1 \cdot \vec{\mathcal{J}}) + \frac{\varepsilon^2}{2} (\hat{\mathbf{n}}_1 \cdot \vec{\mathcal{J}})^2 + o(\varepsilon^2) \quad (8.284)$$

together with a similar definition for $\hat{\mathbf{n}}_2$, where $\vec{\mathcal{J}}$ is defined in eqn (8.135), to prove that

$$[\mathcal{R}[\varepsilon \hat{\mathbf{n}}_1], \mathcal{R}[\varepsilon \hat{\mathbf{n}}_2]]_c = \varepsilon^2 (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \vec{\mathcal{J}} + o(\varepsilon^2) \quad (8.285)$$

(b) If we denote by $\Delta \mathbf{V}$ the *change* in vector \mathbf{V} due to rotation by angle ε about axis $\hat{\mathbf{n}}$, demonstrate that

$$\Delta \mathbf{V} = \mathbf{V}^{(R)} - \mathbf{V} = (\mathcal{R}[\varepsilon \hat{\mathbf{n}}] - \mathcal{U}) \mathbf{V} \quad (8.286)$$

(c) Now denote the cumulative change in \mathbf{V} resulting from successive rotations, first about axis $\hat{\mathbf{n}}_2$ and then about axis $\hat{\mathbf{n}}_1$, as

$$\Delta \mathbf{V}_{(1,2)} = (\mathcal{R}[\varepsilon \hat{\mathbf{n}}_1] \mathcal{R}[\varepsilon \hat{\mathbf{n}}_2] - \mathcal{U}) \mathbf{V} \quad (8.287)$$

and denote $\Delta \mathbf{V}_{(2,1)}$ as the change produced by the same two rotations but with the order reversed. Prove that the difference between these two changes is

$$\begin{aligned} \Delta \mathbf{V}_{(1,2)} - \Delta \mathbf{V}_{(2,1)} &= [\mathcal{R}[\varepsilon \hat{\mathbf{n}}_1], \mathcal{R}[\varepsilon \hat{\mathbf{n}}_2]]_c \mathbf{V} \\ &= (\mathcal{R}[\varepsilon^2 (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \vec{\mathcal{J}}] - \mathcal{U}) \mathbf{V} + o(\varepsilon^2) \end{aligned} \quad (8.288)$$

Thus the difference between the changes produced by pairs of rotations in opposite orders is, to second order in ε , equal to the change produced by a rotation about an axis parallel to $(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)$.

(d) Suppose that we first rotate successively about the x followed by the y axis. And then we start again and rotate successively about the y followed by the x axis. Show that the difference between the changes produced by two procedures is, to second order, equal to the change produced by a rotation about the z axis.

Exercise 8.12 Complete the proof of Theorem 8.26.1.

Exercise 8.13 Under the conditions $\beta \neq 0$ and $(\alpha + \gamma) \neq 0$ stated in Section 8.27, verify eqns (8.212 – 8.214).

Exercise 8.14 Find Euler angles α, β, γ for the following rotations:

(a) Rotation by $\pi/3$ radians about an axis $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2) / \sqrt{2}$.

(b) Rotation by π radians about an axis $\hat{\mathbf{n}} = ((\sqrt{3} - 1) \hat{\mathbf{e}}_1 + (\sqrt{3} + 1) \hat{\mathbf{e}}_2) / \sqrt{8}$.

(c) Rotation by $\pi/3$ radians about an axis $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3) / \sqrt{3}$.

Exercise 8.15 Use the methods of Section 8.18 to derive the matrix $\mathbf{R}[\beta \hat{\mathbf{e}}_2]$ in eqn (8.200).

ROTATIONAL DYNAMICS

The successful description of rigid-body motion is one of the triumphs of Newtonian mechanics. Having learned in the previous chapter how to specify the position and orientation of a rigid body, we now study its natural motion under impressed external forces and torques. The dynamical theorems of collective motion from Chapter 1 will be extended by use of the rotation operators whose properties were developed in Chapter 8.

9.1 Basic Facts of Rigid-Body Motion

The center of mass \mathbf{R} of a rigid body obeys the same formulas as those summarized in Section 1.15 of Chapter 1 for any collection of point masses,

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{(\text{ext})} \quad \text{where} \quad \mathbf{P} = M\mathbf{V} \quad \text{and} \quad \mathbf{V} = \frac{d\mathbf{R}}{dt} \quad (9.1)$$

The orbital angular momentum formulas are also the same,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_o^{(\text{ext})} \quad \text{where} \quad \mathbf{L} = \mathbf{R} \times \mathbf{P} \quad \text{and} \quad \boldsymbol{\tau}_o^{(\text{ext})} = \mathbf{R} \times \mathbf{F}^{(\text{ext})} \quad (9.2)$$

The spin angular momentum of a rigid body is the same as that defined in Section 1.11. It is

$$\mathbf{S} = \sum_{n=1}^N \boldsymbol{\rho}_n \times m_n \dot{\boldsymbol{\rho}}_n \quad (9.3)$$

and obeys the equation of motion derived in Section 1.13,

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\tau}_s^{(\text{ext})} \quad \text{where} \quad \boldsymbol{\tau}_s^{(\text{ext})} = \sum_{n=1}^N \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} \quad (9.4)$$

where $\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R}$ is the relative position vector defined in eqn (1.33).

The difference between a rigid body and a general collection of point masses is the relation

$$\dot{\boldsymbol{\rho}}_n = \boldsymbol{\omega} \times \boldsymbol{\rho}_n \quad (9.5)$$

that holds only for rigid bodies. This important formula for the time derivatives of the relative position vectors was initially stated in Section 8.12 and then re-derived in Section 8.34 using the concept of body derivative.

The application of eqn (9.5) allows the formulas for \mathbf{S} and its time derivative to be expressed in a very useful operator form, in which the properties of the rigid body

itself are contained in an operator \mathcal{I} called the inertia operator or inertia tensor. The exact form of this operator depends on the details of the experimental situation being treated. We begin with the case of a freely moving rigid body.

9.2 The Inertia Operator and the Spin

Consider a rigid body moving freely in empty space, for example a tumbling asteroid. Applying eqn (9.5) to the definition of \mathbf{S} in eqn (9.3) and expanding the triple vector product gives

$$\mathbf{S} = \sum_{n=1}^N m_n \boldsymbol{\rho}_n \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_n) = \sum_{n=1}^N m_n \{ (\boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_n) \boldsymbol{\omega} - \boldsymbol{\rho}_n (\boldsymbol{\rho}_n \cdot \boldsymbol{\omega}) \} \tag{9.6}$$

Section 8.34 describes the body system of coordinates $\hat{\mathbf{e}}'_i(t)$ that move with the rigid body. Expressed in terms of components in that system,

$$\mathbf{S} = \sum_{i=1}^3 S'_i \hat{\mathbf{e}}'_i \quad \boldsymbol{\rho}_n = \sum_{i=1}^3 \rho'_{ni} \hat{\mathbf{e}}'_i \quad \boldsymbol{\omega} = \sum_{i=1}^3 \omega'_i \hat{\mathbf{e}}'_i \tag{9.7}$$

and eqn (9.6) becomes

$$\begin{aligned} S'_i &= \sum_{n=1}^N m_n \left\{ (\rho'^2_{n1} + \rho'^2_{n2} + \rho'^2_{n3}) \omega'_i - \rho'_{ni} \sum_{j=1}^3 \rho'_{nj} \omega'_j \right\} \\ &= \sum_{j=1}^3 \sum_{n=1}^N m_n \left\{ (\rho'^2_{n1} + \rho'^2_{n2} + \rho'^2_{n3}) \delta_{ij} - \rho'_{ni} \rho'_{nj} \right\} \omega'_j \end{aligned} \tag{9.8}$$

where $\omega'_i = \sum_{j=1}^3 \delta_{ij} \omega'_j$ has been used.

Introducing the definition

$$I^{(\text{cm})'}_{ij} = \sum_{n=1}^N m_n \left\{ (\rho'^2_{n1} + \rho'^2_{n2} + \rho'^2_{n3}) \delta_{ij} - \rho'_{ni} \rho'_{nj} \right\} \tag{9.9}$$

allows eqn (9.8) to be written as

$$S'_i = \sum_{j=1}^3 I^{(\text{cm})'}_{ij} \omega'_j \tag{9.10}$$

The discussion of the equivalence of operators, matrices, and components in Section 7.8 can now be invoked to write

$$\mathbf{S} = \mathcal{I}^{(\text{cm})} \boldsymbol{\omega} \tag{9.11}$$

where $\mathcal{I}^{(\text{cm})}$ is that operator whose matrix elements in the body system are given by eqn (9.9). This operator will be called the *center-of-mass inertia operator*. Often it is also called the center-of-mass inertia tensor.

An important feature of this inertia operator is that its matrix elements in the body system are not time varying.

Lemma 9.2.1: Constancy of Matrix Elements

The body-system matrix elements

$$I_{ij}^{(\text{cm})'} = \hat{\mathbf{e}}_i' \cdot \mathcal{I}^{(\text{cm})} \hat{\mathbf{e}}_j' \quad (9.12)$$

of the operator $\mathcal{I}^{(\text{cm})}$ are constants, obeying

$$\frac{dI_{ij}^{(\text{cm})'}}{dt} = 0 \quad (9.13)$$

Proof: In Section 8.8, and again in Section 8.34, we saw that the components ρ'_{ni} of the relative position vectors in the body system are constants, with

$$\rho'_{ni}(t) = \rho'_{ni}(0) \quad \text{and hence} \quad \frac{d\rho'_{ni}}{dt} = 0 \quad (9.14)$$

But the expression for $I_{ij}^{(\text{cm})'}$ in eqn (9.9) contains only the components ρ'_{ni} , and hence $I_{ij}^{(\text{cm})'}$ must also be constant. \square

9.3 The Inertia Dyadic

Like any operator equation, eqn (9.11) can also be written in dyadic form. The last expression in eqn (9.6) can be written as

$$\begin{aligned} \mathbf{S} &= \sum_{n=1}^N m_n \{ (\boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_n) \boldsymbol{\omega} - \boldsymbol{\rho}_n (\boldsymbol{\rho}_n \cdot \boldsymbol{\omega}) \} \\ &= \sum_{n=1}^N m_n \{ (\boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_n) \mathbb{U} - \boldsymbol{\rho}_n \boldsymbol{\rho}_n \} \cdot \boldsymbol{\omega} = \mathbb{I}^{(\text{cm})} \cdot \boldsymbol{\omega} \end{aligned} \quad (9.15)$$

where the center-of-mass inertia dyadic is defined by

$$\mathbb{I}^{(\text{cm})} = \sum_{n=1}^N m_n \{ (\boldsymbol{\rho}_n \cdot \boldsymbol{\rho}_n) \mathbb{U} - \boldsymbol{\rho}_n \boldsymbol{\rho}_n \} \quad (9.16)$$

This same dyadic can also be derived from the component expression eqn (9.9) using the definition of a dyadic in terms of its matrix elements from eqn (7.52), applied here using basis vectors and matrix elements from the body system,

$$\mathbb{I}^{(\text{cm})} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}_i' I_{ij}^{(\text{cm})'} \hat{\mathbf{e}}_j' \quad (9.17)$$

An equivalent matrix equation can also be written. If we denote by $[S']$ and $[\omega']$ the column vectors of components of \mathbf{S} and $\boldsymbol{\omega}$ in the body system, then

$$[S'] = \mathbf{I}^{(\text{cm})'} [\omega'] \quad (9.18)$$

where the matrix elements of matrix $\mathbf{I}^{(\text{cm})'}$ are those given in eqn (9.9).

9.4 Kinetic Energy of a Rigid Body

The total kinetic energy of any collection, including a rigid body, is given in Section 1.14 as

$$T = T_0 + T_1 \quad \text{where} \quad T_0 = \frac{1}{2}MV^2 \quad \text{and} \quad T_1 = \frac{1}{2} \sum_{n=1}^N m_n \dot{\boldsymbol{\rho}}_n \cdot \dot{\boldsymbol{\rho}}_n \quad (9.19)$$

Using eqn (9.5), this last expression may be rewritten for a rigid body as

$$T_1 = \frac{1}{2} \sum_{n=1}^N m_n \dot{\boldsymbol{\rho}}_n \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_n) = \frac{1}{2} \sum_{n=1}^N m_n (\boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n) \cdot \boldsymbol{\omega} = \frac{1}{2} \mathbf{S} \cdot \boldsymbol{\omega} \quad (9.20)$$

where eqn (9.3) was used. Expanding \mathbf{S} using eqn (9.11) then gives

$$T_1 = \frac{1}{2} \boldsymbol{\omega} \cdot (\mathcal{I}^{(\text{cm})} \boldsymbol{\omega}) = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I_{ij}^{(\text{cm})'} \omega'_i \omega'_j \quad (9.21)$$

which expresses the internal kinetic energy in terms of the angular velocity and the inertia operator.

9.5 Meaning of the Inertia Operator

The diagonal matrix elements of the matrix $I^{(\text{cm})'}$ are called *moments of inertia*. For example, consider the element with $i = j = 3$,

$$I_{33}^{(\text{cm})'} = \sum_{n=1}^N m_n \left\{ (\rho'_{n1})^2 + \rho'_{n2})^2 + \rho'_{n3})^2 \right\} \delta_{33} - \rho'_{n3} \rho'_{n3} \left. \right\} = \sum_{n=1}^N m_n (\rho'_{n1})^2 + \rho'_{n2})^2 \quad (9.22)$$

which is the sum of each mass m_n multiplied by its perpendicular distance from a line parallel to $\hat{\mathbf{e}}'_3$ and passing through the center of mass.⁴⁶ The other two diagonal elements have similar expressions in terms of perpendicular distances from the other coordinate axes.

The off-diagonal elements of $I^{(\text{cm})'}$ are called *products of inertia*. For example, the element with $i = 1$ and $j = 3$ is

$$I_{13}^{(\text{cm})'} = \sum_{n=1}^N m_n \left\{ (\rho'_{n1})^2 + \rho'_{n2})^2 + \rho'_{n3})^2 \right\} \delta_{13} - \rho'_{n1} \rho'_{n3} \left. \right\} = - \sum_{n=1}^N m_n \rho'_{n1} \rho'_{n3} \quad (9.23)$$

The other off-diagonal elements are similar.

⁴⁶If we imagine the body system of coordinates to have its origin at the center of mass of the body, this is the distance of m_n from the $\hat{\mathbf{e}}'_3$ axis. It is the moment of inertia about that axis that would be measured by an observer standing at the center of mass.

9.6 Principal Axes

From eqn (9.9), we notice that, by construction, the inertia matrix is real and symmetric, with

$$I_{ji}^{(\text{cm})'} = I_{ij}^{(\text{cm})'} \quad (9.24)$$

As discussed in Section 7.12, any real symmetric operator in a three-dimensional space has three real, mutually orthogonal, and normalized eigenvectors $\mathbf{V}^{(k)}$ for $k = 1, 2, 3$ obeying the standard eigenvector equation

$$\mathcal{I}^{(\text{cm})} \mathbf{V}^{(k)} = \lambda_k \mathbf{V}^{(k)} \quad (9.25)$$

These three, orthonormal eigenvectors are called the *principal axes* of the rigid body. The corresponding real eigenvalues are the three solutions $\lambda_1, \lambda_2, \lambda_3$ of the cubic equation

$$\begin{vmatrix} (I_{11}^{(\text{cm})'} - \lambda) & I_{12}^{(\text{cm})'} & I_{13}^{(\text{cm})'} \\ I_{21}^{(\text{cm})'} & (I_{22}^{(\text{cm})'} - \lambda) & I_{23}^{(\text{cm})'} \\ I_{31}^{(\text{cm})'} & I_{32}^{(\text{cm})'} & (I_{33}^{(\text{cm})'} - \lambda) \end{vmatrix} = 0 \quad (9.26)$$

Since the matrix elements $I_{ji}^{(\text{cm})'}$ are all constant in time, the eigenvalues will also be constants.

The eigenvectors may be expanded in the body system as

$$\mathbf{V}^{(k)} = \sum_{i=1}^3 V_i^{(k)'} \hat{\mathbf{e}}_i' \quad (9.27)$$

where the components $V_i^{(k)'}$ of the k th eigenvector are found by solving the equation

$$\begin{pmatrix} (I_{11}^{(\text{cm})'} - \lambda_k) & I_{12}^{(\text{cm})'} & I_{13}^{(\text{cm})'} \\ I_{21}^{(\text{cm})'} & (I_{22}^{(\text{cm})'} - \lambda_k) & I_{23}^{(\text{cm})'} \\ I_{31}^{(\text{cm})'} & I_{32}^{(\text{cm})'} & (I_{33}^{(\text{cm})'} - \lambda_k) \end{pmatrix} \begin{pmatrix} V_1^{(k)'} \\ V_2^{(k)'} \\ V_3^{(k)'} \end{pmatrix} = 0 \quad (9.28)$$

and applying the normalization condition to obtain unit eigenvectors with $\hat{\mathbf{V}}^{(k)} \cdot \hat{\mathbf{V}}^{(k)} = 1$. Since the eigenvalues and matrix elements are all constant in time, the components $V_i^{(k)'}$ will also be constant. The three eigenvectors now form an orthonormal set, with

$$\mathcal{I}^{(\text{cm})} \hat{\mathbf{V}}^{(k)} = \lambda_k \hat{\mathbf{V}}^{(k)} \quad \text{and} \quad \hat{\mathbf{V}}^{(k)} \cdot \hat{\mathbf{V}}^{(l)} = \delta_{kl} \quad (9.29)$$

for all $k, l = 1, 2, 3$.

Now suppose that we choose a new set of basis vectors $\hat{\mathbf{e}}_i''$ equal to the eigenvectors just found,

$$\hat{\mathbf{e}}_i'' = \hat{\mathbf{V}}^{(i)} \quad (9.30)$$

for $i = 1, 2, 3$, where possibly the indices of the eigenvectors may need to interchanged, or one eigenvector replaced by its negative, to make sure that the $\hat{\mathbf{e}}_i''$ form

a right-handed set of basis vectors. Expressed in this new system, the inertia operator will have a matrix $I^{(cm)''}$ defined by its matrix elements

$$I_{ij}^{(cm)''} = \hat{\mathbf{e}}_i'' \cdot \left(\mathcal{I}^{(cm)} \hat{\mathbf{e}}_j'' \right) \quad (9.31)$$

But by eqns (9.25, 9.30), $\mathcal{I}^{(cm)} \hat{\mathbf{e}}_i'' = \lambda_i \hat{\mathbf{e}}_i''$ and hence, using the orthonormality of the new basis vectors,

$$I_{ij}^{(cm)''} = \hat{\mathbf{e}}_i'' \cdot \left(\lambda_j \hat{\mathbf{e}}_j'' \right) = \lambda_j \delta_{ij} \quad (9.32)$$

The eigenvalues will be denoted, for $j = 1, 2, 3$, by

$$\lambda_j = I_j^{(cm)''} \quad (9.33)$$

and will be called the *principal moments of inertia* of the rigid body. Thus, in the $\hat{\mathbf{e}}_i''$ system, the matrix corresponding to the inertia operator will be

$$I^{(cm)''} = \begin{pmatrix} I_1^{(cm)''} & 0 & 0 \\ 0 & I_2^{(cm)''} & 0 \\ 0 & 0 & I_3^{(cm)''} \end{pmatrix} \quad \text{with} \quad I_{ij}^{(cm)''} = I_j^{(cm)''} \delta_{ij} \quad (9.34)$$

Warning: Change of Notation

In subsequent work, unless explicitly stated otherwise, we will assume that any body system of coordinates used is *already* a principal axis system of the center-of-mass inertia operator. We assume that the task of finding principal axes, if necessary, has already been done. However, for notational simplicity, the double prime denoting the principal axis system above will be replaced by a single prime. The effect is that, dropping the double prime now, we will assume any body system to have a diagonal center-of-mass inertia matrix with

$$I^{(cm)'} = \begin{pmatrix} I_1^{(cm)'} & 0 & 0 \\ 0 & I_2^{(cm)'} & 0 \\ 0 & 0 & I_3^{(cm)'} \end{pmatrix} \quad \text{with} \quad I_{ij}^{(cm)'} = I_j^{(cm)'} \delta_{ij} \quad (9.35)$$

Thus all products of inertia such as in eqn (9.23) will vanish, and the principal moments of inertia will be given by eqn (9.9) with $i=j$,

$$I_i^{(cm)'} = \sum_{n=1}^N m_n \left\{ \left(\rho_{n1}'^2 + \rho_{n2}'^2 + \rho_{n3}'^2 \right) - \rho_{ni}'^2 \right\} \quad (9.36)$$

Use of the principal axis system leads to a considerable simplification. For example, eqn (9.10) becomes

$$S_i' = \sum_{j=1}^3 I_j^{(cm)'} \omega_j' = \sum_{j=1}^3 I_j^{(cm)'} \delta_{ij} \omega_j' = I_i^{(cm)'} \omega_i' \quad (9.37)$$

for each individual value $i = 1, 2, 3$, which says that each component of the spin is just the corresponding component of the angular velocity multiplied by the principal

moment of inertia,⁴⁷

$$S'_1 = I_1^{(\text{cm})'} \omega'_1 \quad S'_2 = I_2^{(\text{cm})'} \omega'_2 \quad S'_3 = I_3^{(\text{cm})'} \omega'_3 \quad (9.38)$$

The expression for the internal kinetic energy in eqn (9.21) also simplifies, to a single sum over the squares of the angular velocity components multiplied by the principal moments of inertia,

$$T_I = \frac{1}{2} \boldsymbol{\omega} \cdot (\mathcal{I}^{(\text{cm})} \boldsymbol{\omega}) = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I_{ij}^{(\text{cm})'} \omega'_i \omega'_j = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I_j^{(\text{cm})'} \delta_{ij} \omega'_i \omega'_j = \frac{1}{2} \sum_{i=1}^3 I_i^{(\text{cm})'} \omega_i'^2 \quad (9.39)$$

When expressed in terms of principal axis unit vectors, the center-of-mass inertia dyadic defined in eqn (9.17) also has a simple form. It becomes

$$\mathbb{I}^{(\text{cm})} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\mathbf{e}}'_i I_j^{(\text{cm})'} \delta_{ij} \hat{\mathbf{e}}'_j = \sum_{i=1}^3 \hat{\mathbf{e}}'_i I_i^{(\text{cm})'} \hat{\mathbf{e}}'_i \quad (9.40)$$

9.7 Guessing the Principal Axes

We know that any rigid body will have a system of principal axes. If necessary, we can choose three arbitrary body-fixed axes, calculate the inertia matrix, and then go through the procedure to find the principal axis eigenvectors. But in many situations of interest, the directions of the principal axes can be guessed (with certainty) from the symmetry of the rigid body. We give here several rules that can be used.

Lemma 9.7.1: The Plane-Figure Theorem

If the rigid body is flat and of negligible thickness (a plane figure), then the unit vector perpendicular to the plane will be a principal axis. Moreover, when the other two principal axes are found, the principal moments of inertia will obey the relation

$$I_3^{(\text{cm})'} = I_1^{(\text{cm})'} + I_2^{(\text{cm})'} \quad (9.41)$$

where we assume for definiteness that the perpendicular to the plane was chosen to be $\hat{\mathbf{e}}'_3$.

Proof: The proof begins by noting that all products of inertia involving the perpendicular direction will vanish. Assuming the perpendicular to be $\hat{\mathbf{e}}'_3$, eqn (9.9) gives, for $i = 1, 2$,

$$I_{i3}^{(\text{cm})'} = - \sum_{n=1}^N m_n \rho'_{ni} \rho'_{n3} \quad (9.42)$$

But $\rho'_{n3} = 0$ was assumed for all n values, hence $I_{13}^{(\text{cm})'} = I_{23}^{(\text{cm})'} = 0$ and the inertia

⁴⁷Readers accustomed to seeing the Einstein summation convention should note that no sum over i is intended or implied in eqn (9.37).

matrix has the form

$$\begin{pmatrix} I_{11}^{(cm)'} & I_{12}^{(cm)'} & 0 \\ I_{21}^{(cm)'} & I_{22}^{(cm)'} & 0 \\ 0 & 0 & I_{33}^{(cm)'} \end{pmatrix} \tag{9.43}$$

Thus the vector \hat{e}'_3 , which has components $(0, 0, 1)$, will be an eigenvector and hence a principal axis, as was to be proved.

The equality in eqn (9.41) can now be proved. With $\rho'_{n3} = 0$, the three principal moments of inertia in eqn (9.36) become

$$I_1^{(cm)'} = \sum_{n=1}^N m_n \rho'^2_{n2} \quad I_2^{(cm)'} = \sum_{n=1}^N m_n \rho'^2_{n1} \quad I_3^{(cm)'} = \sum_{n=1}^N m_n (\rho'^2_{n1} + \rho'^2_{n2}) \tag{9.44}$$

from which eqn (9.41) follows. □

Lemma 9.7.2: The Symmetry Rule

Suppose there is a symmetry plane passing through the center of mass of a rigid body such that, for each mass m_n on one side of the plane, there is a mirror-image mass $m_p = m_n$ on the other side. Then the perpendicular to the symmetry plane will be a principal axis.

Proof: The proof is similar to that of the Plane Figure Theorem. Assume \hat{e}'_3 chosen to be the perpendicular to the symmetry plane, so that the symmetry plane is the \hat{e}'_1 - \hat{e}'_2 plane. Then the sum in eqn (9.42) will vanish because each term of the form $m_n ab$ will be matched by a term $m_p a(-b)$ that cancels it. Thus, the inertia matrix will have the form shown in eqn (9.43) and so \hat{e}'_3 will once again be an eigenvector and hence a principal axis, as was to be proved. □

Lemma 9.7.3: Figures of Rotation

For any figure of rotation (such as might be turned on a lathe), the symmetry axis and any two unit vectors perpendicular to it, and to each other, will be the principal axes. Also, if we assume for definiteness (and according to the usual custom) that \hat{e}'_3 is along the symmetry axis, then

$$I_1^{(cm)'} = I_2^{(cm)'} \tag{9.45}$$

Proof: Note that any plane containing the symmetry axis of the figure will be a symmetry plane of the sort described in the Symmetry Rule above. Thus any unit vector perpendicular to the symmetry axis will be a principal axis. Choose two perpendicular vectors from this set. With these as two of the principal axes, the only remaining direction is the symmetry axis itself. Hence the symmetry axis must also be a principal axis, as was to be proved. Rotating the rigid body by 90° about the symmetry axis will move \hat{e}'_1 into \hat{e}'_2 but will not change the mass distribution. Hence eqn (9.45) follows. □

Lemma 9.7.4: Cuboids

For any cuboid (a body with six rectangular faces), the perpendiculars to the faces will be principal axes.

Proof: This rule follows from application of the symmetry rule with planes of symmetry parallel to the faces and cutting the cuboid into two equal parts. \square

For a continuous mass distribution with a mass density D , eqn (9.36) may be generalized to

$$I_i^{(\text{cm})'} = \int dm \left\{ (\rho_1')^2 + \rho_2'^2 + \rho_3'^2 \right\} - \rho_i'^2 \quad (9.46)$$

where $\boldsymbol{\rho} = \mathbf{r} - \mathbf{R}$ is the location of mass element $dm = D d^3\rho$ relative to the center of mass, and the integration is over the whole of the rigid body.

9.8 Time Evolution of the Spin

Assume now that the body axes $\hat{\mathbf{e}}'_i$ are the principal axes of the center-of-mass inertia operator $\mathcal{I}^{(\text{cm})}$. These $\hat{\mathbf{e}}'_i$ will be called the *principal axis system*. We continue the treatment of the tumbling asteroid introduced in Section 9.2 by considering the rate of change of its spin.

From eqn (9.38), in the principal axis system the spin takes the form

$$\mathbf{S} = \sum_{i=1}^3 I_i^{(\text{cm})'} \omega'_i \hat{\mathbf{e}}'_i \quad (9.47)$$

Its rate of change may be calculated using the body derivative introduced in Section 8.33,

$$\frac{d\mathbf{S}}{dt} = \left\langle \frac{d\mathbf{S}}{dt} \right\rangle_b + \boldsymbol{\omega} \times \mathbf{S} \quad (9.48)$$

where

$$\left\langle \frac{d\mathbf{S}}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{dS'_i}{dt} \hat{\mathbf{e}}'_i = \sum_{i=1}^3 I_i^{(\text{cm})'} \dot{\omega}'_i \hat{\mathbf{e}}'_i \quad (9.49)$$

where $\dot{\omega}'_i = d\omega'_i/dt$ and the constancy of $I_i^{(\text{cm})'}$ from Lemma 9.2.1 was used.

The equation of motion for \mathbf{S} from eqn (9.4) then becomes

$$\boldsymbol{\tau}_s^{(\text{ext})} = \frac{d\mathbf{S}}{dt} = \sum_{i=1}^3 I_i^{(\text{cm})'} \dot{\omega}'_i \hat{\mathbf{e}}'_i + \boldsymbol{\omega} \times \mathbf{S} \quad (9.50)$$

Expanding this equation in terms of components in the body system gives, for $i = 1, 2, 3$,

$$\tau'_{si} = I_i^{(\text{cm})'} \dot{\omega}'_i + \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \omega'_j S'_k = I_i^{(\text{cm})'} \dot{\omega}'_i + \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \omega'_j I_k^{(\text{cm})'} \omega'_k \quad (9.51)$$

where eqn (A.15) was used to expand the cross product and eqn (9.38) was used to get the second equality.

Equation (9.51) is often written in a slightly modified form,

$$I_i^{(\text{cm})'} \dot{\omega}'_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ikj} I_k^{(\text{cm})'} \omega'_k \omega'_j + \tau'_{si} \quad (9.52)$$

When each $i = 1, 2, 3$ component is written out, these equations have a symmetry which makes them easy to remember

$$I_1^{(\text{cm})'} \dot{\omega}'_1 = \omega'_2 \omega'_3 \left(I_2^{(\text{cm})'} - I_3^{(\text{cm})'} \right) + \tau'_{s1} \quad (9.53)$$

$$I_2^{(\text{cm})'} \dot{\omega}'_2 = \omega'_3 \omega'_1 \left(I_3^{(\text{cm})'} - I_1^{(\text{cm})'} \right) + \tau'_{s2} \quad (9.54)$$

$$I_3^{(\text{cm})'} \dot{\omega}'_3 = \omega'_1 \omega'_2 \left(I_1^{(\text{cm})'} - I_2^{(\text{cm})'} \right) + \tau'_{s3} \quad (9.55)$$

Each successive formula is gotten by a cyclic permutation of the integers 123 relative to the previous one.

Equations (9.53 – 9.55), like many others in this subject, are called the Euler equations. They give a set of coupled differential equations for the components ω'_i of the angular velocity vector relative to the body system of coordinates.

9.9 Torque-Free Motion of a Symmetric Body

Imagine now that the tumbling asteroid is replaced by a spaceship or other object (such as the Earth, or a football) with rotational symmetry about some axis. Taking the symmetry axis to be $\hat{\mathbf{e}}_3$ as is conventional, it follows from Lemma 9.7.3 that such objects have two equal principal moments of inertia $I_1^{(\text{cm})'} = I_2^{(\text{cm})'} \neq I_3^{(\text{cm})'}$. Bodies with $I_1^{(\text{cm})'} = I_2^{(\text{cm})'}$ will be referred to as *symmetric rigid bodies*.

Assume further that the symmetric body is moving with $\tau_s^{(\text{ext})} = 0$. The Euler equations of Section 9.8 can then be solved exactly for the angular velocity and spin as functions of time. Although the torque-free symmetric body is a simple case, the motion is surprisingly complicated.

We begin by assuming $I_1^{(\text{cm})'} = I_2^{(\text{cm})'}$ and writing the Euler equations eqns (9.53 – 9.55) as

$$I_1^{(\text{cm})'} \dot{\omega}'_1 = \omega'_2 \omega'_3 \left(I_1^{(\text{cm})'} - I_3^{(\text{cm})'} \right) \quad (9.56)$$

$$I_1^{(\text{cm})'} \dot{\omega}'_2 = \omega'_3 \omega'_1 \left(I_3^{(\text{cm})'} - I_1^{(\text{cm})'} \right) \quad (9.57)$$

$$I_3^{(\text{cm})'} \dot{\omega}'_3 = 0 \quad (9.58)$$

It follows at once from the third equation that ω'_3 is a constant equal to its value at time zero, $\omega'_3 = \omega'_{30}$. Then the other two equations can be rewritten as

$$\dot{\omega}'_1 = -\Omega_0 \omega'_2 \quad \dot{\omega}'_2 = \Omega_0 \omega'_1 \quad (9.59)$$

where the constant Ω_0 is defined by

$$\Omega_0 = \omega'_{30} \frac{\left(I_3^{(\text{cm})'} - I_1^{(\text{cm})'} \right)}{I_1^{(\text{cm})'}} \quad (9.60)$$

Throughout this and the following sections, we will assume for definiteness that the principal axis directions have been chosen so that $\omega'_{30} > 0$. Then $\Omega_0 > 0$ when $I_3^{(\text{cm})'} > I_1^{(\text{cm})'}$, as happens for oblate bodies like a thin circular disk, a thin square, or the Earth. But $\Omega_0 < 0$ when $I_3^{(\text{cm})'} < I_1^{(\text{cm})'}$, as happens for prolate bodies like a long rod, a long stick of square cross section, or an American or Rugby football.

The first of eqn (9.59) can be differentiated and the second substituted into it to give

$$\ddot{\omega}'_1 = -\Omega_0^2 \omega'_1 \quad (9.61)$$

which has the general solution

$$\omega'_1 = A \cos(\Omega_0 t + \delta) \quad (9.62)$$

where $A \geq 0$ and $-\pi < \delta \leq \pi$ are constants of integration to be determined at time zero. The other component is then

$$\omega'_2 = -\frac{\dot{\omega}'_1}{\Omega_0} = A \sin(\Omega_0 t + \delta) \quad (9.63)$$

The angular velocity vector is thus completely determined from its initial values. It is

$$\boldsymbol{\omega} = A \{ \cos(\Omega_0 t + \delta) \hat{\mathbf{e}}'_1 + \sin(\Omega_0 t + \delta) \hat{\mathbf{e}}'_2 \} + \omega'_{30} \hat{\mathbf{e}}'_3 = A \hat{\mathbf{n}}(t) + \omega'_{30} \hat{\mathbf{e}}'_3 \quad (9.64)$$

where the unit vector $\hat{\mathbf{n}}(t)$ is defined by

$$\hat{\mathbf{n}}(t) = \cos(\Omega_0 t + \delta) \hat{\mathbf{e}}'_1 + \sin(\Omega_0 t + \delta) \hat{\mathbf{e}}'_2 \quad (9.65)$$

As seen by an observer in the body system, the vector $\hat{\mathbf{n}}$ will rotate in a right-handed sense about the symmetry axis $\hat{\mathbf{e}}'_3$ when $I_3^{(\text{cm})'} > I_1^{(\text{cm})'}$ and in the opposite sense when $I_3^{(\text{cm})'} < I_1^{(\text{cm})'}$.

The spin angular momentum can also be written. Using eqn (9.38) and the assumed equality $I_1^{(\text{cm})'} = I_2^{(\text{cm})'}$, it is

$$\mathbf{S} = I_1^{(\text{cm})'} (\omega'_1 \hat{\mathbf{e}}'_1 + \omega'_2 \hat{\mathbf{e}}'_2) + I_3^{(\text{cm})'} \omega'_{30} \hat{\mathbf{e}}'_3 = I_1^{(\text{cm})'} A \hat{\mathbf{n}}(t) + I_3^{(\text{cm})'} \omega'_{30} \hat{\mathbf{e}}'_3 \quad (9.66)$$

It is seen that $\boldsymbol{\omega}$ and \mathbf{S} appear to rotate together about the symmetry axis $\hat{\mathbf{e}}'_3$. Their components perpendicular to the symmetry axis are both parallel to the same unit vector $\hat{\mathbf{n}}(t)$. The sign of A in eqn (9.62) has been chosen so that $\delta = 0$ will place both $\boldsymbol{\omega}$ and \mathbf{S} in the $\hat{\mathbf{e}}'_1$ - $\hat{\mathbf{e}}'_3$ plane at time zero.

The constant magnitude of \mathbf{S} is found from eqn (9.66) to be

$$S_0 = \|\mathbf{S}\| = \sqrt{\left(I_1^{(\text{cm})'} A\right)^2 + \left(I_3^{(\text{cm})'} \omega'_{30}\right)^2} \quad (9.67)$$

Since we know that a torque free rigid body has $d\mathbf{S}/dt = 0$, we know that \mathbf{S} must be a constant vector relative to inertial space, both in magnitude and direction. We exploit this constancy of \mathbf{S} by choosing the space-fixed, inertial coordinate system $\hat{\mathbf{e}}_k$ such that $\mathbf{S} = S_0 \hat{\mathbf{e}}_3$.

Since \mathbf{S} is an absolute constant, the time variation of the components of \mathbf{S} in eqn (9.66) must be due to the motion of the unit vectors $\hat{\mathbf{e}}'_i$, and hence of the rigid body in which they are embedded and whose orientation they define. Note that the solution with $A = 0$ is trivial, with \mathbf{S} , $\boldsymbol{\omega}$, and $\hat{\mathbf{e}}'_3$ all aligned and all constant in time. We will assume the interesting case $A > 0$ from now on.

The angle $\theta_{\omega 3'}$ between the vector $\boldsymbol{\omega}$ and the symmetry axis $\hat{\mathbf{e}}'_3$ can be determined from

$$\cos \theta_{\omega 3'} = \frac{\hat{\mathbf{e}}'_3 \cdot \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} = \frac{\omega'_{30}}{\sqrt{A^2 + \omega'^2_{30}}} \quad (9.68)$$

and is a constant. The angle $\theta_{33'}$ between the vector \mathbf{S} and the symmetry axis can be similarly determined from

$$\cos \theta_{33'} = \frac{\hat{\mathbf{e}}'_3 \cdot \mathbf{S}}{\|\mathbf{S}\|} = \frac{I_3^{(\text{cm})'} \omega'_{30}}{\sqrt{\left(I_1^{(\text{cm})'} A\right)^2 + \left(I_3^{(\text{cm})'} \omega'_{30}\right)^2}} \quad (9.69)$$

and is also a constant.

The assumptions that $\omega'_{30} > 0$ and $A > 0$ imply that $0 \leq \theta_{\omega 3'} \leq \pi/2$ and $0 \leq \theta_{33'} \leq \pi/2$. It follows from eqns (9.68, 9.69) that $I_3^{(\text{cm})'} > I_1^{(\text{cm})'}$, as for the Earth, implies that $\theta_{33'} < \theta_{\omega 3'}$. And $I_3^{(\text{cm})'} < I_1^{(\text{cm})'}$, as for a football, implies the opposite inequality $\theta_{33'} > \theta_{\omega 3'}$.

The motion of the torque-free symmetric body can be understood by a geometric construction. A space-fixed right circular cone, called the *space cone*, is drawn with its symmetry axis along \mathbf{S} and its surface defined by the path of $\boldsymbol{\omega}$. A body-fixed right circular cone called the *body cone*, is drawn with its symmetry axis along $\hat{\mathbf{e}}'_3$ and its surface defined by the path of $\boldsymbol{\omega}$ relative to the body system. The half angle of the body cone is thus $\theta_{\omega 3'}$. These two cones are placed so that $\boldsymbol{\omega}$ is always along their line of intersection, which makes the body cone roll on the space cone without slipping. The body cone carries the body system of coordinates with it as it rolls, and thus illustrates the motion of the body. The two cases, for $I_3^{(\text{cm})'} > I_1^{(\text{cm})'}$ and $I_3^{(\text{cm})'} < I_1^{(\text{cm})'}$, are shown in Figures 9.1 and 9.2. In cases like the Earth, the body cone encloses the space cone. In cases like the football, the body cone rolls on the outside of the space cone.

A great deal of qualitative information can be extracted from the results of this section. We now estimate some magnitudes of interest. For example, the Earth has a

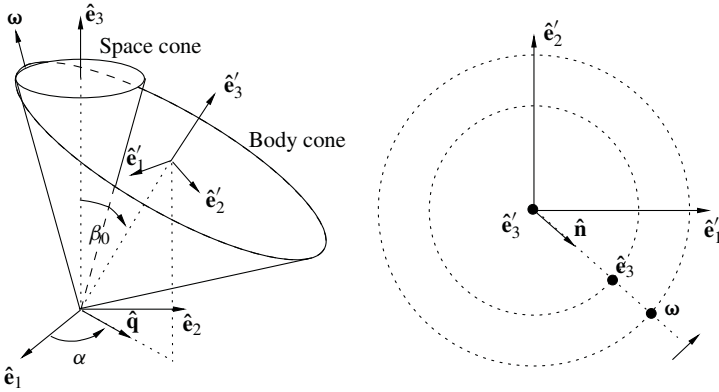


FIG. 9.1. For an oblate object like the Earth, $\Omega_0 > 0$. In the figure at the left, the body cone rolls without slipping on the space cone, carrying the \hat{e}'_i axes with it. The angular velocity ω is along the line of contact of the two cones, and the Euler angle α increases steadily. The figure on the right shows the motion from the viewpoint of an observer standing on the Earth at the north pole. The unit vector \hat{n} appears to move counter-clockwise, with the perpendicular components of \hat{e}_3 and ω lined up with it.

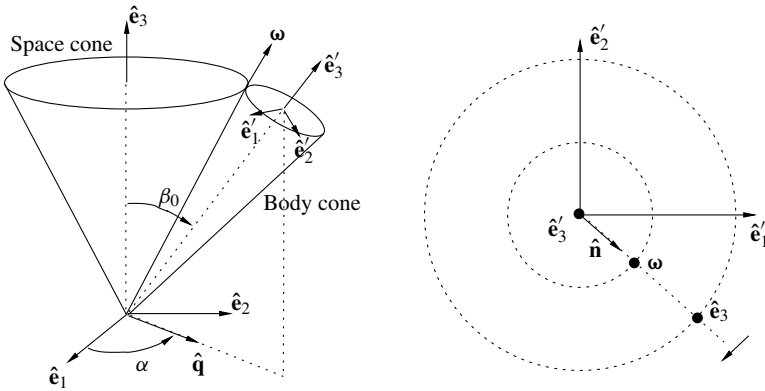


FIG. 9.2. For a prolate object such as a football, $\Omega_0 < 0$. In the figure on the left, the Euler angle α increases steadily as the body cone rolls without slipping on the space cone, carrying the \hat{e}'_i axes with it. The figure on the right shows the motion from the viewpoint of an observer riding on the nose of the football. The unit vector \hat{n} appears to move in a clockwise direction, with the perpendicular components of ω and \hat{e}_3 lined up with it.

small positive value of the ratio $(I_3^{(cm)'} - I_1^{(cm)'})/I_1^{(cm)'}$. To a body-system observer, the \hat{e}'_3 axis appears fixed and the vectors \mathbf{S} and ω (listed here in order of their angle from \hat{e}'_3) appear to rotate about it in a positive sense with an angular rate $\Omega_0 > 0$ that is slow compared to the total angular velocity $\|\omega\| \approx 2\pi/(1 \text{ day})$. The angle $\theta_{33'}$ is only slightly smaller than $\theta_{\omega 3'}$ and hence the space cone is small compared to the body cone, as is shown in Figure 9.1.

For another example, a thin rod has a ratio $(I_3^{(\text{cm})'} - I_1^{(\text{cm})}')/I_1^{(\text{cm})}'$ that is negative, and slightly greater than -1 . A body-system observer sees the symmetry axis $\hat{\mathbf{e}}_3'$ as fixed and the vectors $\boldsymbol{\omega}$ and \mathbf{S} (listed here in order of their angle from $\hat{\mathbf{e}}_3'$) appear to rotate about it in a negative sense, with an angular rate $\Omega_0 < 0$ that is nearly as large in magnitude as the total angular velocity $\|\boldsymbol{\omega}\|$. The angle $\theta_{33'}$ is considerably larger than $\theta_{\omega 3'}$ and hence the space cone is large compared to the body cone, as shown.

When tidal torques are ignored, the Earth is approximately a torque-free symmetric rigid body of the sort described here. If we assume an ideal case in which it is perfectly rigid and torque free, we can imagine the three vectors $\hat{\mathbf{e}}_3'$, $\boldsymbol{\omega}$, and \mathbf{S} to be drawn with a common origin at the center of the Earth and their lines extended out through the surface of the Earth. These lines would all pierce the snow at or near the north pole. The vector $\hat{\mathbf{e}}_3'$ defines the north pole, the geometric symmetry axis of the Earth, and would appear fixed to a polar observer standing in the snow. The trace of the other two vectors would appear to rotate in concentric circles around the north pole in a positive sense, with a common radius direction $\hat{\mathbf{n}}(t)$. They would make one complete circuit in a time $T_\Omega = 2\pi/\Omega_0$. This time can be calculated from the known oblateness of the Earth.

$$T_\Omega = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\omega_{30}} \frac{I_1^{(\text{cm})'}}{I_3^{(\text{cm})'} - I_1^{(\text{cm})}'} \cong \frac{I_1^{(\text{cm})}'}{I_3^{(\text{cm})'} - I_1^{(\text{cm})}'}} \text{ days} \cong 306 \text{ days} \quad (9.70)$$

The Earth's symmetry axis apparently does have a periodic variation, called the Chandler wobble, that can be associated with the effect calculated here. It has a small amplitude: The circles in the snow mentioned above would be of the order of 5 meters in radius. Also, it has a period of approximately 423 days and appears to be damped. Because of the damping, it is not simply a relic of the Earth's creation with some nonzero A value as the above analysis would suggest, but must be sustained by some present energy source not included in our analysis here.

9.10 Euler Angles of the Torque-Free Motion

The motion of the symmetric rigid body in Section 9.9 can also be described by establishing an inertial coordinate system with its $\hat{\mathbf{e}}_3$ axis along the fixed direction of the spin \mathbf{S} , and then using Euler angles α, β, γ to describe the orientation of the body-fixed $\hat{\mathbf{e}}_i'$ unit vectors relative to this inertial system. The motion of the body is then seen from the viewpoint of an inertial observer, perhaps someone watching a football pass as it spirals, or someone viewing the Earth's wobble from space.

In the present section, we assume the results derived in Section 9.9, but re-express them in terms of these Euler angles.

In terms of the Euler angles, the body symmetry axis $\hat{\mathbf{e}}_3'$ will have spherical polar coordinates $1, \beta, \alpha$ relative to inertial axes $\hat{\mathbf{e}}_k$, as was noted in eqn (8.265). The Euler angle β is the angle between $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_3'$ and therefore must be constant here, equal to the constant angle $0 \leq \theta_{33'} \leq \pi/2$ calculated in eqn (9.69). Thus $\beta = \beta_0 = \theta_{33'}$.

Equations (8.270 – 8.272) with $\beta = \beta_0$ and $\dot{\beta} = 0$ give the body system components of the angular velocity in terms of the Euler angles. The angular velocity is

$$\boldsymbol{\omega} = -\dot{\alpha} \sin \beta_0 \cos \gamma \hat{\mathbf{e}}'_1 + \dot{\alpha} \sin \beta_0 \sin \gamma \hat{\mathbf{e}}'_2 + (\dot{\alpha} \cos \beta_0 + \dot{\gamma}) \hat{\mathbf{e}}'_3 \quad (9.71)$$

Equating the three components of this vector to the components of $\boldsymbol{\omega}$ in eqn (9.64) gives

$$-\dot{\alpha} \sin \beta_0 \cos \gamma = A \cos (\Omega_0 t + \delta) \quad (9.72)$$

$$\dot{\alpha} \sin \beta_0 \sin \gamma = A \sin (\Omega_0 t + \delta) \quad (9.73)$$

$$\dot{\alpha} \cos \beta_0 + \dot{\gamma} = \omega'_{30} \quad (9.74)$$

It follows from eqns (9.72, 9.73) that $\dot{\alpha}$ and $\dot{\gamma}$ are constants with $\dot{\alpha} = \dot{\alpha}_0$ and $\dot{\gamma} = \dot{\gamma}_0$, and that $\dot{\gamma}_0 = -\Omega_0$. Then eqn (9.74) shows that $\dot{\alpha}_0$ must be a positive constant, $\dot{\alpha}_0 > 0$. With these conditions established, eqns (9.72 – 9.74) together imply that

$$\dot{\alpha}_0 \sin \beta_0 = A \quad \dot{\alpha}_0 \cos \beta_0 = \frac{I_3^{(cm)'} \omega'_{30}}{I_1^{(cm)'}} \quad \gamma = -\Omega_0 t - \delta + \pi \quad \alpha = \dot{\alpha}_0 t + \kappa \quad (9.75)$$

where the constant κ is determined from the initial conditions.

We now can express the vectors \mathbf{S} , $\hat{\mathbf{e}}'_3$, and $\boldsymbol{\omega}$ in the inertial system relative to which the Euler angles are defined. These expressions will be consistent with the geometrical constructions in Figures 9.1 and 9.2, and will show them from the inertial viewpoint.

The magnitude of the spin \mathbf{S} may be calculated in terms of Euler angles using eqns (9.67, 9.75). It is

$$S_0 = \|\mathbf{S}\| = I_1^{(cm)'} \dot{\alpha}_0 \quad (9.76)$$

Since the spin vector \mathbf{S} is constant for a torque-free body, and since the $\hat{\mathbf{e}}_3$ axis of the inertial system is defined to be along the direction of this vector, the spin vector will at all times be equal to $\hat{\mathbf{e}}_3$ times its magnitude, or

$$\mathbf{S} = I_1^{(cm)'} \dot{\alpha}_0 \hat{\mathbf{e}}_3 \quad (9.77)$$

The body symmetry vector $\hat{\mathbf{e}}'_3$ can be found from eqn (8.266). It is

$$\hat{\mathbf{e}}'_3 = \sin \beta_0 \hat{\mathbf{q}}(t) + \cos \beta_0 \hat{\mathbf{e}}_3 \quad (9.78)$$

where

$$\hat{\mathbf{q}}(t) = \cos(\dot{\alpha}_0 t + \kappa) \hat{\mathbf{e}}_1 + \sin(\dot{\alpha}_0 t + \kappa) \hat{\mathbf{e}}_2 \quad (9.79)$$

is a unit vector that rotates about $\hat{\mathbf{e}}_3$ in the positive sense. The expression for $\boldsymbol{\omega}$ in terms of inertial system unit vectors will be derived in Exercise 9.8.

To an inertial system observer viewing the Earth, the spin \mathbf{S} is constant and along the $\hat{\mathbf{e}}_3$ axis. The symmetry axis of the body $\hat{\mathbf{e}}'_3$ is at angle β_0 from the $\hat{\mathbf{e}}_3$ axis, and rotates about it in a positive sense, with a rate $\dot{\alpha}_0 > 0$ that is slightly larger than $\|\boldsymbol{\omega}\|$. The third Euler angle γ represents a rotation about the symmetry axis of the body

that is combined with the rotation already provided by α . It moves in a retrograde sense, with $\dot{\gamma}_0 < 0$ but small compared to $\|\boldsymbol{\omega}\|$. The vectors $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}'_3$ lie on opposite sides of $\hat{\mathbf{e}}_3$ and are both in the plane defined by $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{q}}(t)$.

To an inertial observer viewing the long rod or the football, the spin \mathbf{S} is constant and along the $\hat{\mathbf{e}}_3$ axis. The symmetry axis of the body $\hat{\mathbf{e}}'_3$ is at angle β_0 from the $\hat{\mathbf{e}}_3$ axis, and rotates about it in a positive sense, with a rate $\dot{\alpha}_0 > 0$ that is much smaller than $\|\boldsymbol{\omega}\|$. The third Euler angle γ moves in a positive sense, with $\dot{\gamma}_0 > 0$ and a magnitude only slightly smaller than $\|\boldsymbol{\omega}\|$. The vectors $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}'_3$ are on the same side of $\hat{\mathbf{e}}_3$ and are both in the plane defined by $\hat{\mathbf{e}}_3$ and $\hat{\mathbf{q}}(t)$.

9.11 Body with One Point Fixed

In Section 9.8, we considered a rigid body moving freely in empty space, like a tumbling asteroid. The motion of its center of mass was therefore governed by the same laws of motion as for any collection of masses, rigid or not.

We now consider another class of interesting cases, ones in which the rigid body is not floating freely but has one of its points constrained to be fixed. Examples are a top spinning with its point set into a depression that holds it fixed, a gyroscope with a point along its symmetry axis held fixed, etc.

Suppose that a point P of a rigid body is constrained to be at rest. Place an inertial coordinate system with its origin at that fixed point. Then the motion of the rigid body can be derived from the time evolution of the total angular momentum \mathbf{J} relative to this inertial system, as given in Axiom 1.5.1,

$$\frac{d\mathbf{J}}{dt} = \boldsymbol{\tau}^{(\text{ext})} \quad \text{where} \quad \mathbf{J} = \sum_{n=1}^N \mathbf{r}_n \times m_n \mathbf{v}_n = \mathbf{L} + \mathbf{S} \quad (9.80)$$

where \mathbf{L} and \mathbf{S} are defined in Section 1.11, and $\boldsymbol{\tau}^{(\text{ext})}$ is the total external torque relative to the origin of coordinates as defined in Section 1.5.

We want to find an operator \mathcal{I} that maps the angular velocity $\boldsymbol{\omega}$ into \mathbf{J} , similar to the operator $\mathcal{I}^{(\text{cm})}$ defined for the spin in Section 9.2. Since an operator expression for the spin, $\mathbf{S} = \mathcal{I}^{(\text{cm})} \boldsymbol{\omega}$, has already been derived in Section 9.2, an obvious approach is to find an operator expression for the orbital angular momentum \mathbf{L} and then to use $\mathbf{J} = \mathbf{L} + \mathbf{S}$ to find \mathbf{J} .

From eqns (9.1, 9.2),

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} \quad (9.81)$$

where $\mathbf{V} = d\mathbf{R}/dt$ is the velocity of the center of mass \mathbf{R} , and M is the total mass of the body. This is the same definition as for the tumbling asteroid, or for any collection of point masses. But, when one point of the body is fixed at the origin of coordinates, *both* ends of \mathbf{R} are now fixed relative to the rigid body, and so vector \mathbf{R} must move with the body.

To derive an expression for \mathbf{L} , let us begin by supposing that we have already found the principal axes of the rigid body relative to its center of mass, as discussed in Section 9.6. Then there is already a body-fixed coordinate system with principal

axis unit vectors $\hat{\mathbf{e}}'_i$ and its origin at the center of mass. The vector \mathbf{R} can be expanded in that body system as

$$\mathbf{R} = \sum_{i=1}^3 R'_i \hat{\mathbf{e}}'_i \quad \text{where} \quad R'_i = \hat{\mathbf{e}}'_i \cdot \mathbf{R} \quad (9.82)$$

Since both the unit vectors $\hat{\mathbf{e}}'_i$ and \mathbf{R} are now embedded in the same rigid body, the components R'_i will all be constants with $dR'_i/dt = 0$. Hence the body derivative of \mathbf{R} defined in Section 8.33 vanishes,

$$\left\langle \frac{d\mathbf{R}}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{dR'_i}{dt} \hat{\mathbf{e}}'_i = 0 \quad (9.83)$$

and the total time derivative reduces to

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \left\langle \frac{d\mathbf{R}}{dt} \right\rangle_b + \boldsymbol{\omega} \times \mathbf{R} = \boldsymbol{\omega} \times \mathbf{R} \quad (9.84)$$

The orbital angular momentum \mathbf{L} then becomes

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} = M\mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{R}) = M \left\{ R^2 \boldsymbol{\omega} - \mathbf{R}(\mathbf{R} \cdot \boldsymbol{\omega}) \right\} \quad (9.85)$$

Writing this equation out in terms of components in the body system gives

$$L'_i = M \left\{ \left(R_1'^2 + R_2'^2 + R_3'^2 \right) \omega'_i - R'_i \sum_{j=1}^3 R'_j \omega'_j \right\} = \sum_{j=1}^3 I_{ij}^{(\text{orb})'} \omega'_j \quad (9.86)$$

where the orbital inertia matrix in the body system is defined by

$$I_{ij}^{(\text{orb})'} = M \left\{ \left(R_1'^2 + R_2'^2 + R_3'^2 \right) \delta_{ij} - R'_i R'_j \right\} \quad (9.87)$$

Corresponding to the last expression in eqn (9.86), there is an operator equation

$$\mathbf{L} = \mathcal{I}^{(\text{orb})} \boldsymbol{\omega} \quad (9.88)$$

where $\mathcal{I}^{(\text{orb})}$ is the operator whose matrix elements in the body system are given by eqn (9.86). Notice that, since the R'_i are constants, as was discussed above, the matrix elements $I_{ij}^{(\text{orb})'}$ will also be constant in time and have zero time derivatives. The operator equation in eqn (9.88) may also be written as the equivalent dyadic equation

$$\mathbf{L} = \mathbb{I}^{(\text{orb})} \cdot \boldsymbol{\omega} \quad \text{where} \quad \mathbb{I}^{(\text{orb})} = M \left(R^2 \mathbb{U} - \mathbf{R}\mathbf{R} \right) \quad (9.89)$$

With the orbital angular momentum \mathbf{L} now determined, the total angular momen-

tum \mathbf{J} may be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathcal{I}^{(\text{orb})} \boldsymbol{\omega} + \mathcal{I}^{(\text{cm})} \boldsymbol{\omega} = \left(\mathcal{I}^{(\text{orb})} + \mathcal{I}^{(\text{cm})} \right) \boldsymbol{\omega} = \mathcal{I} \boldsymbol{\omega} \quad (9.90)$$

where the total inertia operator is defined by

$$\mathcal{I} = \mathcal{I}^{(\text{orb})} + \mathcal{I}^{(\text{cm})} \quad (9.91)$$

In terms of components in the body system, we have

$$J'_i = \sum_{j=1}^3 I'_{ij} \omega'_j \quad (9.92)$$

for $i = 1, 2, 3$, where

$$I'_{ij} = M \left\{ \left(R_1'^2 + R_2'^2 + R_3'^2 \right) \delta_{ij} - R'_i R'_j \right\} + I_j^{(\text{cm})'} \delta_{ij} \quad (9.93)$$

The delta function appears in the last term because the body system is assumed to be a principal axis system for the center of mass momentum operator $\mathcal{I}^{(\text{cm})}$. If the body system is not the center of mass principal axis system, then this term will be replaced by the non-diagonal matrix $I_j^{(\text{cm})'}$. Equation(9.93) will be referred to as the *translation of pivot theorem* since it expresses the inertia tensor about a fixed point displaced from the center of mass.

The dyadic equivalent to operator \mathcal{I} can also be written. It is

$$\mathbb{I} = \mathbb{I}^{(\text{orb})} + \mathbb{I}^{(\text{cm})} = M \left(R^2 \mathbb{U} - \mathbf{R}\mathbf{R} \right) + \mathbb{I}^{(\text{cm})} \quad (9.94)$$

where $\mathbb{I}^{(\text{cm})}$ is the dyadic expressed in the center of mass principal axis system by eqn (9.40).

In eqn (9.20), the internal kinetic energy T_I was given in terms of the spin and the angular velocity. The same can be done for the orbital kinetic energy T_O for rigid bodies moving with one point fixed. Starting with the definition in eqn (9.19), and using eqn (9.84),

$$T_O = \frac{1}{2} M V^2 = \frac{1}{2} M \mathbf{V} \cdot \mathbf{V} = \frac{1}{2} M \mathbf{V} \cdot \boldsymbol{\omega} \times \mathbf{R} = \frac{1}{2} \mathbf{R} \times M \mathbf{V} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} \quad (9.95)$$

Combining this result with eqn (9.20) then gives

$$T = T_O + T_I = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{S} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} = \frac{1}{2} \boldsymbol{\omega} \cdot (\mathcal{I} \boldsymbol{\omega}) \quad (9.96)$$

where eqn (9.90) was used. Expanding the last expression on the right in eqn (9.96) in the body basis gives

$$T = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I'_{ij} \omega'_i \omega'_j \quad (9.97)$$

which is the same as eqn (9.21) for the internal kinetic energy T_I , but with the center-of-mass inertia matrix $I_j^{(\text{cm})'}$ now replaced by the total inertia matrix I'_{ij} .

9.12 Preserving the Principal Axes

An unfortunate feature of eqn (9.93) is that, although the matrix $I^{(\text{cm})'}$ is diagonal by the assumption that the center of mass principal axis system is being used, the matrix I' may not be. Moving the reference point from the center of mass to fixed point P may introduce non-diagonal terms. If so, then the whole calculation of the principal axes will have to be done again.

However, there is a class of important special cases in which the center of mass principal axes are preserved. If the vector \mathbf{R} from the fixed point to the center of mass happens to lie along one of the $\hat{\mathbf{e}}'_i$ directions, then all products of inertia in eqn (9.87) will vanish. For example, suppose that $\mathbf{R} = R\hat{\mathbf{e}}'_3$. Then $R'_1 = R'_2 = 0$ and $R'_3 = R$ with the result that

$$I'_{ij} = MR^2 (\delta_{ij} - \delta_{i3}\delta_{j3}) + I_j^{(\text{cm})'} \delta_{ij} \quad (9.98)$$

Thus $i \neq j$ implies that $I'_{ij} = 0$ and the diagonal elements become

$$I'_{11} = MR^2 + I_1^{(\text{cm})'} \quad I'_{22} = MR^2 + I_2^{(\text{cm})'} \quad I'_{33} = I_3^{(\text{cm})'} \quad (9.99)$$

In general, when \mathbf{R} is along one of the center of mass principal axis $\hat{\mathbf{e}}'_k$ the principal axes of the problem are unchanged, the principal moments of inertia along the axes perpendicular to $\hat{\mathbf{e}}'_k$ have MR^2 added to them, and the principal moment of inertia along $\hat{\mathbf{e}}'_k$ itself is unchanged.

Assume now that we have preserved the center of mass principal axes, or otherwise found principal axes that make the total inertia operator \mathcal{I} diagonal, and are now using a principal axis system of the total inertia operator. The formulas for the total angular momentum and the total kinetic energy then become simpler, just as the formulas for the spin angular momentum and the internal kinetic energy did in Section 9.6.

The relation

$$\mathbf{J} = \mathcal{I}\boldsymbol{\omega} \quad (9.100)$$

is expressed in component form in eqn (9.92). If the body system is the principal axis system for \mathcal{I} then, for $i = 1, 2, 3$,

$$I'_{ij} = I'_i \delta_{ij} \quad \text{and hence} \quad J'_i = \sum_{j=1}^3 I'_j \delta_{ij} \omega'_j = I'_i \omega'_i \quad (9.101)$$

Just as in eqn (9.37), we emphasize that there is no sum implied in this last equation. Each component of J'_i is just the corresponding component of ω'_i multiplied by the principal moment of inertia I'_i ,

$$J'_1 = I'_1 \omega'_1 \quad J'_2 = I'_2 \omega'_2 \quad J'_3 = I'_3 \omega'_3 \quad (9.102)$$

In this same principal axis system, the total kinetic energy in eqn (9.97) simplifies to a single sum

$$T = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I'_j \delta_{ij} \omega'_i \omega'_j = \frac{1}{2} \sum_{i=1}^3 I'_i \omega'^2_i \quad (9.103)$$

9.13 Time Evolution with One Point Fixed

The time evolution of the spin was calculated in Section 9.8. The same methods used there for the spin equation of motion $d\mathbf{S}/dt = \boldsymbol{\tau}_s^{(\text{ext})}$ can also be used for the total angular momentum equation of motion $d\mathbf{J}/dt = \boldsymbol{\tau}^{(\text{ext})}$. Just replace S by J , and τ_s by τ , and $I^{(\text{cm})}$ by I throughout.

Assuming that we are now using body axes that are principal axes for the total inertia operator \mathcal{I} , the Euler equations analogous to eqns (9.53 – 9.55) are

$$I'_1 \dot{\omega}'_1 = \omega'_2 \omega'_3 (I'_2 - I'_3) + \tau'_1 \quad (9.104)$$

$$I'_2 \dot{\omega}'_2 = \omega'_3 \omega'_1 (I'_3 - I'_1) + \tau'_2 \quad (9.105)$$

$$I'_3 \dot{\omega}'_3 = \omega'_1 \omega'_2 (I'_1 - I'_2) + \tau'_3 \quad (9.106)$$

where the torque components are defined by $\tau'_i = \hat{\mathbf{e}}'_i \cdot \boldsymbol{\tau}^{(\text{ext})}$.

9.14 Body with One Point Fixed, Alternate Derivation

An operator expression for the total angular momentum \mathbf{J} of a rigid body moving with one point fixed can also be derived directly, without reference to \mathbf{L} and \mathbf{S} . The operator \mathcal{I} obtained will be the same as that derived in Section 9.11.

The basic definition of the total angular momentum of any collection, including a rigid body, is

$$\mathbf{J} = \sum_{n=1}^N \mathbf{r}_n \times m_n \mathbf{v}_n \quad (9.107)$$

where $\mathbf{v}_n = d\mathbf{r}_n/dt$ and \mathbf{r}_n is the vector from the fixed point P (which is taken as the origin of the inertial coordinate system) to the mass m_n .

Assume that some body-fixed system of coordinates has been defined. Since all of the vectors \mathbf{r}_n connect two points of the same rigid body, their components in this body system of coordinates must be constants, just as the components of \mathbf{R} were in Section 9.11. Thus the body derivatives vanish, $\langle d\mathbf{r}_n/dt \rangle_b = 0$, and

$$\mathbf{v}_n = d\mathbf{r}_n/dt = \boldsymbol{\omega} \times \mathbf{r}_n \quad (9.108)$$

Hence

$$\mathbf{J} = \sum_{n=1}^N m_n \mathbf{r}_n \times (\boldsymbol{\omega} \times \mathbf{r}_n) \quad (9.109)$$

Now using the same pattern found in Section 9.2, but with $\boldsymbol{\rho}_n$ replaced everywhere by \mathbf{r}_n , the components of \mathbf{J} in the body fixed system of coordinates can be reduced to

$$J'_i = \sum_{j=1}^3 I'_{ij} \omega'_j \quad (9.110)$$

where

$$I'_{ij} = \sum_{n=1}^N m_n \left\{ (r'_{n1}{}^2 + r'_{n2}{}^2 + r'_{n3}{}^2) \delta_{ij} - r'_{ni} r'_{nj} \right\} \quad (9.111)$$

Thus

$$\mathbf{J} = \mathcal{I} \boldsymbol{\omega} \quad (9.112)$$

where operator \mathcal{I} is the operator whose matrix elements in the body system are I'_{ij} . The dyadic form of \mathcal{I} is

$$\mathbb{I} = \sum_{n=1}^N m_n \{ (\mathbf{r}_n \cdot \mathbf{r}_n) \mathbb{U} - \mathbf{r}_n \mathbf{r}_n \} \quad (9.113)$$

The matrix I' defined in eqn (9.111) can be diagonalized to find a principal axis system for the total inertia tensor \mathcal{I} . The result will be the same (except for possible degeneracy of eigenvectors) as that obtained by the more indirect route taken in Sections 9.11 and 9.12.

9.15 Work–Energy Theorems

In Section 1.16 we showed that the rate of change of the total kinetic energy T of a rigid body can be written as

$$\frac{dT}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \mathbf{v}_n \quad (9.114)$$

Using eqn (9.108), in the case of a rigid body with one point fixed at the origin of an inertial coordinate system this result can be written as

$$\frac{dT}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \boldsymbol{\omega} \times \mathbf{r}_n = \left(\sum_{n=1}^N \mathbf{r}_n \times \mathbf{f}_n^{(\text{ext})} \right) \cdot \boldsymbol{\omega} = \boldsymbol{\tau}^{(\text{ext})} \cdot \boldsymbol{\omega} \quad (9.115)$$

where the definitions in eqns (1.17, 1.18) have been used. Thus an external torque that is always perpendicular to the angular velocity vector will do no work, and will not change the total kinetic energy of the rigid body.

A similar result holds for the internal kinetic energy T_I and the torque $\boldsymbol{\tau}_s^{(\text{ext})}$ defined in eqn (1.49). Starting again from the result in Section 1.16,

$$\frac{dT_I}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \dot{\boldsymbol{\rho}}_n \quad (9.116)$$

the rate of change of the internal kinetic energy can be written using eqn (9.5) as

$$\frac{dT_I}{dt} = \sum_{n=1}^N \mathbf{f}_n^{(\text{ext})} \cdot \boldsymbol{\omega} \times \boldsymbol{\rho}_n = \left(\sum_{n=1}^N \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} \right) \cdot \boldsymbol{\omega} = \boldsymbol{\tau}_s^{(\text{ext})} \cdot \boldsymbol{\omega} \quad (9.117)$$

9.16 Rotation with a Fixed Axis

There is a class of problems in which the rigid body is constrained even more severely than simply by having one point fixed. It might be constrained to rotate about a fixed axis, as on a lathe. Examples of this sort are often used in elementary textbooks to introduce students to “rotary motion.” However, it is instructive to see precisely how these elementary results fit into the general theory being presented here.

Imagine that a fixed axis passes through the rigid body and is rigidly connected to it. If the angle of rotation about the fixed axis is denoted Φ , then eqn (8.197) gives at once that

$$\boldsymbol{\omega} = \dot{\Phi} \hat{\mathbf{n}} \quad \text{or in component form} \quad \omega'_i = \dot{\Phi} n'_i \quad \text{for } i = 1, 2, 3 \quad (9.118)$$

where $\hat{\mathbf{n}}$ is a constant unit vector, along the fixed axis and pointing in the direction related to the positive direction of Φ by a right-hand rule. Taking the origin of an inertial coordinate system to be some point on the fixed axis, eqn (9.112) then gives the angular momentum as

$$\mathbf{J} = \mathcal{I} \boldsymbol{\omega} = \dot{\Phi} \mathcal{I} \hat{\mathbf{n}} \quad (9.119)$$

Note that \mathcal{I} is an *operator* and that in general \mathbf{J} will not point in the same direction as $\boldsymbol{\omega}$.

If, for example, a lathe is running and constraining $\dot{\Phi}$ to have a given value, or if Φ is otherwise known as a function of time, then $\boldsymbol{\omega}$ is known. The torques acting on the rigid body can be calculated by putting the known components of $\boldsymbol{\omega}$ from eqn (9.118) into the Euler equations, eqns (9.104 – 9.106), and solving for the torque components.

However, there is another class of problem in which the motor of the lathe is assumed to be disconnected, so that the rigid body moves freely about the fixed axis. An example might be a rear wheel of a front-wheel-drive automobile. The angle Φ then becomes a free dynamical variable. A differential equation for that variable can be derived that depends only on the component of torque parallel to the fixed axis $\hat{\mathbf{n}}$.

Dotting $\hat{\mathbf{n}}$ from the left onto both sides of eqn (9.80) gives

$$\frac{d}{dt} (\hat{\mathbf{n}} \cdot \mathbf{J}) = \hat{\mathbf{n}} \cdot \frac{d\mathbf{J}}{dt} = \hat{\mathbf{n}} \cdot \boldsymbol{\tau}^{(ext)} \quad (9.120)$$

where the constancy of $\hat{\mathbf{n}}$ allows it to be taken inside the time derivative. Then, introducing eqn (9.119) gives

$$\frac{d}{dt} (\hat{\mathbf{n}} \cdot \dot{\Phi} \mathcal{I} \hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\tau}^{(ext)} \quad \text{or} \quad I_n \ddot{\Phi} = \tau_n \quad (9.121)$$

where the definitions

$$I_n = \hat{\mathbf{n}} \cdot (\mathcal{I} \hat{\mathbf{n}}) \quad \text{and} \quad \tau_n = \hat{\mathbf{n}} \cdot \boldsymbol{\tau}^{(ext)} \quad (9.122)$$

have been introduced.

The I_n will be shown below to be the moment of inertia about the fixed rotation axis. It will be shown to be a constant, as has already been assumed in deriving eqn (9.121). The torque term τ_n is the component of the external torque parallel to the axis. In elementary textbooks, the expression $\tau_n = I_n \ddot{\Phi}$ from eqn (9.121) is sometimes referred to as the, “F equals MA of rotary motion.”

The quantity I_n is easiest to understand if the dyadic expression in eqn (9.103) is used,

$$\begin{aligned} I_n &= \hat{\mathbf{n}} \cdot (\mathcal{I}\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \mathbb{I} \cdot \hat{\mathbf{n}} \\ &= \sum_{n=1}^N m_n \{ (\mathbf{r}_n \cdot \mathbf{r}_n) \hat{\mathbf{n}} \cdot \mathbb{U} \cdot \hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \mathbf{r}_n) (\mathbf{r}_n \cdot \hat{\mathbf{n}}) \} \\ &= \sum_{n=1}^N m_n \{ (\mathbf{r}_n \cdot \mathbf{r}_n) - (\hat{\mathbf{n}} \cdot \mathbf{r}_n) (\mathbf{r}_n \cdot \hat{\mathbf{n}}) \} \end{aligned} \quad (9.123)$$

If we decompose each \mathbf{r}_n into a vector $\mathbf{r}_{n\parallel} = \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \mathbf{r}_n)$ parallel to $\hat{\mathbf{n}}$ and a vector $\mathbf{r}_{n\perp}$ perpendicular to $\hat{\mathbf{n}}$, in the manner described in Section A.2, the expression in the last of eqn (9.123) reduces to

$$I_n = \sum_{n=1}^N m_n \|\mathbf{r}_{n\perp}\|^2 \quad (9.124)$$

This expression is the sum of each mass multiplied by its perpendicular distance from the rotation axis, which is the definition of the moment of inertia about that axis. Since both the origin of coordinates on the axis, and the masses m_n are embedded rigidly in the same rigid body, all dot products in eqn (9.123) will be constants. Hence I_n is constant, as was asserted above.

In the special case that the fixed axis happens to pass through the center of mass of the rigid body, the above analysis will still hold, but with $\mathbf{J}, \mathcal{I}, \mathbb{I}, I, \mathbf{r}_n, \boldsymbol{\tau}^{(\text{ext})}$ replaced by $\mathbf{S}, \mathcal{I}^{(\text{cm})}, \mathbb{I}^{(\text{cm})}, I^{(\text{cm})}, \boldsymbol{\rho}_n, \boldsymbol{\tau}_s^{(\text{ext})}$, respectively.

Returning to the general case, there is an interesting relation between I_n and $I_n^{(\text{cm})}$, called the *parallel axis theorem*. Using the translation of pivot theorem from eqn (9.94),

$$I_n = \hat{\mathbf{n}} \cdot \mathbb{I} \cdot \hat{\mathbf{n}} = M \left\{ R^2 - (\hat{\mathbf{n}} \cdot \mathbf{R})^2 \right\} + \hat{\mathbf{n}} \cdot \mathbb{I}^{(\text{cm})} \cdot \hat{\mathbf{n}} = MR_{\perp}^2 + I_n^{(\text{cm})} \quad (9.125)$$

where $\mathbf{R} = \mathbf{R}_{\parallel} + \mathbf{R}_{\perp}$ decomposes \mathbf{R} into vectors parallel and perpendicular to $\hat{\mathbf{n}}$. The moment of inertia I_n about an axis $\hat{\mathbf{n}}$ is equal to the moment of inertia $I_n^{(\text{cm})}$ about a parallel axis passing through the center of mass, plus the total mass times the perpendicular distance between the two axes.

9.17 The Symmetric Top with One Point Fixed

Although the Euler equations eqns (9.104 – 9.106) are correct, they are less useful than they might be because the variables ω'_i in them are not good generalized coordinates in the Lagrangian sense described in Chapter 2. They are not even the time derivatives of good generalized coordinates.

However, a Lagrangian theory of rigid-body motion is possible, since, when used to specify the orientation of a body-fixed system of unit vectors, the Euler angles α , β , γ are good generalized coordinates. To demonstrate this from the Jacobian determinant condition of eqn (2.27) would be a daunting task indeed, since a rigid body with one point fixed has some 10^{25} degrees of freedom and $(10^{25} - 3)$ independent constraints. The “goodness” of the Euler angles must be established by going back to the property behind the Jacobian determinant condition: bi-uniqueness. The generalized coordinates are “good” if they, together with the constraints, uniquely determine the Cartesian coordinates of each of the point masses, and if conversely the Cartesian coordinates of all the point masses determine them uniquely. This bi-uniqueness condition is satisfied by the Euler angles.

Thus a reduced Lagrangian may be written using the usual formula $L = T - U$ where, after the constraints are applied, the (highly) reduced Lagrangian is

$$\bar{L}(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}, t) = T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) - U(\alpha, \beta, \gamma) \quad (9.126)$$

We wish to apply these Lagrangian methods to the motion of a symmetric top moving with a point on its symmetry axis fixed.

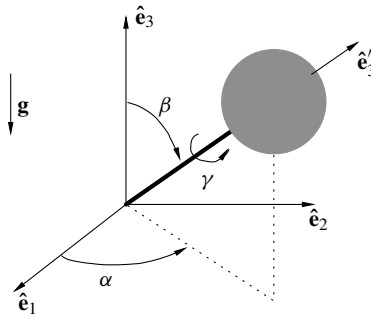


FIG. 9.3. A symmetric top spins with one point fixed, at the origin of an inertial coordinate system \hat{e}_i . The Euler angle γ represents the spin of the top about its symmetry axis \hat{e}'_3 .

We assume that the symmetric top is a body of revolution in the sense of Lemma 9.7.3 so that its symmetry axis, taken conventionally to be \hat{e}'_3 , and any two axes perpendicular to the symmetry axis, are principal axes of the center of mass inertia operator. Assume moreover that the top moves with a fixed point that is on the symmetry axis. Then, according to the analysis in Section 9.12, the center of mass principal axes will be preserved and will also be the principal axes of the total inertia tensor. Then the principal moments of inertia will obey $I'_1 = I'_2 \neq I'_3$.

The force of gravity is assumed to be acting in a downward direction on the top. An inertial coordinate system with its origin at the fixed point of the top is defined with its \hat{e}_3 axis upwards, so that $\mathbf{g} = -g\hat{e}_3$. As shown in Exercise 1.11 for a general collection, the potential energy of a rigid body in a uniform gravitational field is given by $U = -\mathbf{M}\mathbf{g} \cdot \mathbf{R}$ where \mathbf{R} points to its center of mass. Here, the vector \mathbf{R} is along the

$\hat{\mathbf{e}}'_3$ direction with $\mathbf{R} = R\hat{\mathbf{e}}'_3$, where R is the constant magnitude of \mathbf{R} . Thus

$$U(\alpha, \beta, \gamma) = -M(-g\hat{\mathbf{e}}_3) \cdot (R\hat{\mathbf{e}}'_3) = MgR\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_3 = MgR \cos \beta \quad (9.127)$$

since, as noted in Section 8.35, the spherical polar coordinates of $\hat{\mathbf{e}}'_3$ relative to the inertial system are $(1, \beta, \alpha)$.

The kinetic energy can be obtained from eqn (9.103). Setting $I'_1 = I'_2$ this equation reduces to

$$T = \frac{1}{2} \sum_{i=1}^3 I'_i \omega_i'^2 = \frac{1}{2} I'_1 (\omega_1'^2 + \omega_2'^2) + \frac{1}{2} I'_3 \omega_3'^2 \quad (9.128)$$

The angular velocity components in terms of the Euler angles and their derivatives are given in eqns (8.270 – 8.272). Substituting these and simplifying gives

$$T = \frac{1}{2} I'_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I'_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 \quad (9.129)$$

Thus the reduced Lagrangian is

$$\bar{L}(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}, t) = \frac{1}{2} I'_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I'_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - MgR \cos \beta \quad (9.130)$$

The variables α and γ are seen to be ignorable. So we deal with them first. The reduced Lagrange equation for γ is

$$\frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\gamma}} \right) - \frac{\partial \bar{L}}{\partial \gamma} = 0 \quad (9.131)$$

which simplifies to

$$(\dot{\alpha} \cos \beta + \dot{\gamma}) = \text{constant} \quad (9.132)$$

It will simplify later formulas if this constant is defined in terms of another constant A such that

$$\dot{\alpha} \cos \beta + \dot{\gamma} = \frac{I'_1 A}{I'_3} \quad (9.133)$$

Since eqn (8.272) gives $\omega'_3 = (\dot{\alpha} \cos \beta + \dot{\gamma})$, eqn (9.132) implies that ω'_3 is a constant, equal to its value at time zero, $\omega'_3 = \omega'_{30}$. The constant A in eqn (9.133) can be determined at time zero by the condition

$$A = \frac{I'_3 (\dot{\alpha}_0 \cos \beta_0 + \dot{\gamma}_0)}{I'_1} = \frac{I'_3 \omega'_{30}}{I'_1} \quad (9.134)$$

The reduced Lagrange equation for α is

$$\frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \bar{L}}{\partial \alpha} = 0 \quad (9.135)$$

which simplifies to

$$I_1' \dot{\alpha} \sin^2 \beta + I_3' (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta = \text{constant} \quad (9.136)$$

Again, this constant is defined in terms of another constant B such that

$$I_1' \dot{\alpha} \sin^2 \beta + I_3' (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta = I_1' B \quad (9.137)$$

Inserting eqn (9.133) and canceling the common I_1' factors, this becomes

$$\dot{\alpha} \sin^2 \beta + A \cos \beta = B \quad (9.138)$$

The constant B is thus determined from the conditions at time zero as

$$B = \dot{\alpha}_0 \sin^2 \beta_0 + A \cos \beta_0 \quad (9.139)$$

where A is determined from eqn (9.134).

The variable β is not ignorable. Its Lagrange equation involves a second time derivative of β and will be bypassed in favor of the generalized energy theorem that provides a first order differential equation for the same variable. The reduced generalized energy function defined in eqn (3.82) is

$$\begin{aligned} \bar{H} &= \dot{\alpha} \frac{\partial \bar{L}}{\partial \dot{\alpha}} + \dot{\beta} \frac{\partial \bar{L}}{\partial \dot{\beta}} + \dot{\gamma} \frac{\partial \bar{L}}{\partial \dot{\gamma}} - \bar{L} \\ &= \frac{1}{2} I_1' (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3' (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + MgR \cos \beta \end{aligned} \quad (9.140)$$

Since $\partial \bar{L}(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}, t)/\partial t = 0$ here, eqn (3.83) shows that the reduced generalized energy function is a constant equal to its value at time zero,

$$\begin{aligned} &\frac{1}{2} I_1' (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3' (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + MgR \cos \beta \\ &= \frac{1}{2} I_1' (\dot{\alpha}_0^2 \sin^2 \beta_0 + \dot{\beta}_0^2) + \frac{1}{2} I_3' (\dot{\alpha}_0 \cos \beta_0 + \dot{\gamma}_0)^2 + MgR \cos \beta_0 \end{aligned} \quad (9.141)$$

Due to the constancy of ω_3' noted in eqn (9.132), the terms involving I_3' cancel. Multiplying through by $2/I_1'$ then gives

$$\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2 + \frac{2MgR}{I_1'} \cos \beta = C \quad (9.142)$$

where C is a constant given in terms of conditions at time zero as

$$\dot{\alpha}_0^2 \sin^2 \beta_0 + \dot{\beta}_0^2 + \frac{2MgR}{I_1'} \cos \beta_0 = C \quad (9.143)$$

Equation (9.138) can be solved for $\dot{\alpha}$ as

$$\dot{\alpha} = \frac{B - A \cos \beta}{\sin^2 \beta} \quad (9.144)$$

and used to eliminate $\dot{\alpha}$ from eqn (9.142), giving

$$\frac{(B - A \cos \beta)^2}{\sin^2 \beta} + \dot{\beta}^2 + \frac{2MgR}{I_1'} \cos \beta = C \quad (9.145)$$

This equation can now be solved for $\dot{\beta}^2$

$$\dot{\beta}^2 = C - \frac{2MgR}{I_1'} \cos \beta - \frac{(B - A \cos \beta)^2}{\sin^2 \beta} \quad (9.146)$$

and simplified further by the usual substitution $u = \cos \beta$, which implies that $(1 - u^2) = \sin^2 \beta$ and $-\dot{u}/\sqrt{1 - u^2} = \dot{\beta}$, and gives

$$\dot{u}^2 = f(u) = \left(C - \frac{2MgR}{I_1'} u \right) (1 - u^2) - (B - Au)^2 \quad (9.147)$$

Equation (9.147) is a differential equation for the variable $u = \cos \beta$ and can be solved by writing

$$\frac{du}{dt} = \pm \sqrt{f(u)} \quad \text{and hence} \quad t = \int_0^t dt' = \pm \int_{u_0}^u \frac{du'}{\sqrt{f(u')}} = F(u, u_0) \quad (9.148)$$

and then inverting function F to get u as a function of time.

The general solution in eqn (9.148) involves elliptic integrals. For our purposes here, it will suffice to extract some generalities about the motion. The physical range of $u = \cos \beta$ is between -1 and $+1$. A plot of the cubic function $f(u)$ versus u will have $f(\pm\infty) = \pm\infty$ and $f(\pm 1) < 0$. Thus $f(u)$ must have one zero in the unphysical region $u > 1$. There must be two other zeroes $u_1 < u_2$ in the physical region, with $f(u) \geq 0$ for $u_1 \leq u \leq u_2$ since \dot{u}^2 in eqn (9.147) cannot be negative. The points u_1 and u_2 are called the turning points of the β motion. The β value will oscillate back and forth between these points. Note that smaller values of u correspond to the larger values of β and hence to lower positions of the top. Thus the top is lower at u_1 than it is at u_2 .

The oscillation of β is called the *nutation* of the top. While that nutation is in progress, the α variable is also changing with time. The change of α with time is called the *precession* of the top. Equation (9.144) gives

$$\dot{\alpha} = \frac{A}{1 - u^2} \left(\frac{B}{A} - u \right) \quad (9.149)$$

Thus the direction⁴⁸ of the α motion depends on the relative values of u and (B/A) . If $(B/A) > u_2$ then $\dot{\alpha} > 0$ always (direct precession). If $(B/A) < u_1$ then $\dot{\alpha} < 0$ always (retrograde precession). If $u_1 < (B/A) < u_2$ then $\dot{\alpha} < 0$ during the upper part of the nutation cycle and $\dot{\alpha} > 0$ for the lower part. The precession will then be stationary at the value $u_s = (B/A)$. The result of this last case will be a series of loops of the symmetry axis, one loop per nutation cycle.

⁴⁸We assume that the top is initially spun in a right-hand sense about axis \hat{e}_3' so that $A > 0$. If A were negative, the precession directions stated here would all be reversed.

9.18 The Initially Clamped Symmetric Top

There is one special case in which the analysis of Section 9.17 simplifies somewhat: The top whose symmetry axis is clamped at time zero so that $\dot{\alpha}_0 = \dot{\beta}_0 = 0$.⁴⁹ When the top is initially clamped, the various constants defined in Section 9.17 become

$$B = A \cos \beta_0 \quad C = \frac{2MgR}{I_1'} \cos \beta_0 \quad A = \frac{I_3' \dot{\gamma}_0}{I_1'} \quad (9.150)$$

and it is useful to define another constant

$$\psi_0 = \frac{I_1' A^2}{4MgR} = \frac{I_3'}{I_1'} \left(\frac{I_3' \dot{\gamma}_0^2 / 2}{2MgR} \right) \quad (9.151)$$

which is a rough measure of the speed of the top. Except for the factor (I_3'/I_1') , which is usually near unity, it is the ratio of the kinetic energy of the initial spin to the maximum range of the potential energy values. For a fast top, this parameter should therefore be very large.

For an initially clamped top with these parameters, eqn (9.147) becomes the product of a linear and a quadratic factor

$$\dot{u}^2 = f(u) = \frac{2MgR}{I_1'} (u_0 - u) g(u) \quad (9.152)$$

where $u_0 = \cos \beta_0$ and

$$g(u) = 1 - u^2 - 2\psi_0 (u_0 - u) \quad (9.153)$$

One zero of $f(u)$ is seen from eqn (9.152) to be the initial value, $u_2 = u_0$. The other turning point is that solution to the quadratic equation $g(u_1) = 0$ that lies in the physical range $-1 \leq u_1 \leq 1$. It is

$$u_1 = \psi_0 - \sqrt{\psi_0^2 - 2\psi_0 u_0 + 1} \quad (9.154)$$

Then, completing the square of the expression in the square root, the difference $u_0 - u_1$ may be written

$$u_0 - u_1 = |\psi_0 - u_0| \left(\sqrt{1 + \frac{1 - u_0^2}{(\psi_0 - u_0)^2}} - \frac{\psi_0 - u_0}{|\psi_0 - u_0|} \right) \quad (9.155)$$

This expression is seen to be essentially positive, which shows that $u_0 = u_2$ is the *upper* turning point. When released from its clamp, the top falls until it reaches u_1 at which point it turns in its β motion and returns to u_0 to begin another nutation cycle.

⁴⁹The third angle γ carries the spin of the top, and is not clamped. The $\dot{\gamma}_0$ is often very large, in fact.

The value of the precession rate $\dot{\alpha}$ is found from eqn (9.149). For the initially clamped top, this becomes

$$\dot{\alpha} = A \frac{u_0 - u}{1 - u^2} \quad (9.156)$$

Thus at the upper nutation turning point $u_2 = u_0$, the precession rate is zero. At the lower point u_1 , the precession rate is positive. The symmetry axis of the top therefore executes a series of cusps, stopping its precession at the upper turning point and maximizing it at the lower turning point. Equation (9.155) can be used to write the precession rate at u_1 as

$$\dot{\alpha}_1 = \frac{A |\psi_0 - u_0|}{1 - u_1^2} \left(\sqrt{1 + \frac{1 - u_0^2}{(\psi_0 - u_0)^2}} - \frac{\psi_0 - u_0}{|\psi_0 - u_0|} \right) \quad (9.157)$$

If a fast top is assumed, eqn (9.157) can be expanded in powers of the small, dimensionless quantity ψ_0^{-1} . If terms up to and including quadratic order in this quantity relative to unity are retained throughout, the approximate value is

$$\dot{\alpha}_1 = 2 \frac{MgR}{I_3' \dot{\gamma}_0} \left(1 - \frac{2u_0^2}{\psi_0^2} + \dots \right) \quad (9.158)$$

9.19 Approximate Treatment of the Symmetric Top

In beginning textbooks, the precession of a rapidly spinning top is treated approximately. The total angular momentum is assumed to be constant in magnitude and directed along the symmetry axis. Thus

$$\mathbf{J} \approx I_3' \dot{\gamma}_0 \hat{\mathbf{e}}_3' \quad (9.159)$$

In our exact treatment, this approximation is equivalent to assuming $\dot{\gamma}$ is a constant equal to its initial value $\dot{\gamma}_0$, and that $\dot{\gamma} \gg \dot{\alpha}, \dot{\beta}$. Then the gravitational torque is calculated from

$$\boldsymbol{\tau}^{(ext)} = \mathbf{R} \times (-Mg\hat{\mathbf{e}}_3) = MgR \sin \beta \hat{\mathbf{e}}_2''' \quad (9.160)$$

where $\hat{\mathbf{e}}_2'''$ is a unit vector lying in the $\hat{\mathbf{e}}_1$ - $\hat{\mathbf{e}}_2$ plane and making an angle α with the $\hat{\mathbf{e}}_2$ axis.⁵⁰ It is the result of rotating the $\hat{\mathbf{e}}_2$ axis by angle α in the right-hand sense about axis $\hat{\mathbf{e}}_3$.

The elementary treatments ignore nutation and assume $\beta = \beta_0$ for all time. Ignoring nutation, one can then calculate the time derivative of \mathbf{J} as

$$\frac{d\mathbf{J}}{dt} \cong I_3' \dot{\gamma}_0 \frac{d\hat{\mathbf{e}}_3'}{dt} \cong I_3' \dot{\gamma}_0 \sin \beta \dot{\alpha} \hat{\mathbf{e}}_2''' \quad (9.161)$$

where eqn (8.266) has been used. Equating eqns (9.160, 9.161), and canceling the

⁵⁰In Section 8.35, the $\hat{\mathbf{e}}_i'''$ unit vectors are the result of rotating the inertial system unit vectors $\hat{\mathbf{e}}_i$ by Euler angles α, β but not yet by γ . They are the next to last stage of the progression from the inertial to the rotated system.

common $\sin \beta$ factor then gives a constant precession rate

$$\dot{\alpha} \cong \frac{MgR}{I_3 \dot{\gamma}_0} \tag{9.162}$$

It is interesting to compare this approximate result with the values of $\dot{\alpha}$ obtained in Section 9.18 for an initially clamped top in the fast-top limit. The precession rate $\dot{\alpha}_1$ at the lower turning point is given by eqn (9.158). It is twice the approximate value in eqn (9.162). However, the precession rate at the upper turning point is zero, $\dot{\alpha}_2 = \dot{\alpha}_0 = 0$. So, in some rough sense, the elementary value might be thought of as an average between zero and a value twice too large. A careful treatment, however, would require a *time average* of $\dot{\alpha}$ over the nutation cycle, and not a simple average of its values at the turning points.

9.20 Inertial Forces

We put aside the dynamics of rigid bodies now and consider the problem of rotating, translating coordinate systems in general. The moving system considered now may or may not be the body system of a rigid body. An observer sitting on or in a rigid body and using its \hat{e}'_i system as his reference system (an astronaut riding on an asteroid) might be an example. Or, the moving system could be defined by the walls of a spacecraft which is accelerating and tumbling, or by the walls of a laboratory on the rotating earth.

An observer doing mechanics experiments in a laboratory that is translating and rotating with respect to inertial space will experience anomalies due to his non-inertial reference system. If the observer is unaware of the source of these anomalies, he may attribute them to forces acting on the masses in his experiments. These “forces” are called *inertial forces*, or sometimes *fictitious forces*.

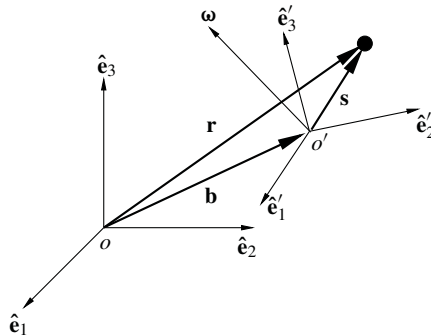


FIG. 9.4. A translating and rotating coordinate system o' has its origin at vector displacement \mathbf{b} relative to an inertial system o . A mass is located at \mathbf{r} relative to the inertial system and at \mathbf{s} relative to the moving system.

Let us suppose that an observer is using a reference system whose origin is located at $\mathbf{b}(t)$ relative to the origin of some inertial system, where $\mathbf{b}(t)$ is some general

function of time. And let his basis vectors $\hat{\mathbf{e}}'_i(t)$ be rotating such that their position at time t relative to some standard position at time zero is given by a rotation operator $\mathcal{R}(t)$ with an associated angular velocity vector $\boldsymbol{\omega}(t)$. The vector $\mathbf{b}(t)$ may point to the location of the center of mass of a rigid body as in the example of the astronaut, but it need not. It can be any displacement, just as $\mathcal{R}(t)$ can be any rotation.

The position of a mass m relative to the origin of the inertial system will be denoted by \mathbf{r} and the position of the same mass relative to the origin of the moving system by \mathbf{s} . Thus

$$\mathbf{r} = \mathbf{b} + \mathbf{s} \quad \mathbf{v} = \frac{d\mathbf{b}}{dt} + \mathbf{u} \quad \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{b}}{dt^2} + \frac{d^2\mathbf{s}}{dt^2} \quad (9.163)$$

where the inertial velocity and the velocity relative to the moving origin are denoted

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \mathbf{u} = \frac{d\mathbf{s}}{dt} \quad (9.164)$$

Now assume that the observer is not only measuring position relative to a moving origin, but is also expressing his vectors relative to moving unit vectors $\hat{\mathbf{e}}'_i$. Thus

$$\mathbf{s} = \sum_{i=1}^3 s'_i \hat{\mathbf{e}}'_i \quad \text{where} \quad s'_i = \hat{\mathbf{e}}'_i \cdot \mathbf{s} \quad (9.165)$$

and, using the body derivative developed in Section 8.33,

$$\mathbf{u} = \frac{d\mathbf{s}}{dt} = \left\langle \frac{d\mathbf{s}}{dt} \right\rangle_b + \boldsymbol{\omega} \times \mathbf{s} = \mathbf{u}_b + \boldsymbol{\omega} \times \mathbf{s} \quad (9.166)$$

where the body derivative will be denoted by \mathbf{u}_b . It can be expanded as

$$\mathbf{u}_b = \left\langle \frac{d\mathbf{s}}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{ds'_i}{dt} \hat{\mathbf{e}}'_i \quad (9.167)$$

and is the velocity that the observer actually measures. Note that \mathbf{u}_b is not only measured relative to a moving origin, but also is calculated as if the moving coordinate unit vectors were fixed.

The next time derivative may now be taken,

$$\frac{d^2\mathbf{s}}{dt^2} = \frac{d}{dt} \left(\frac{d\mathbf{s}}{dt} \right) = \frac{d}{dt} (\mathbf{u}_b + \boldsymbol{\omega} \times \mathbf{s}) = \frac{d\mathbf{u}_b}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{s} + \boldsymbol{\omega} \times \frac{d\mathbf{s}}{dt} \quad (9.168)$$

Each of the time derivatives in the last expression on the right may also be expanded using body derivatives. The first one becomes

$$\frac{d\mathbf{u}_b}{dt} = \left\langle \frac{d\mathbf{u}_b}{dt} \right\rangle_b + \boldsymbol{\omega} \times \mathbf{u}_b = \mathbf{a}_b + \boldsymbol{\omega} \times \mathbf{u}_b \quad (9.169)$$

where we have denoted

$$\mathbf{a}_b = \left\langle \frac{d\mathbf{u}_b}{dt} \right\rangle_b = \sum_{i=1}^3 \frac{d^2s'_i}{dt^2} \hat{\mathbf{e}}'_i \quad (9.170)$$

This is the acceleration that the observer would compute if he were operating in complete ignorance of the fact that both his origin and his basis vectors are moving.

Also

$$\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt} = \left\langle \frac{d\boldsymbol{\omega}}{dt} \right\rangle_b + \boldsymbol{\omega} \times \boldsymbol{\omega} = \left\langle \frac{d\boldsymbol{\omega}}{dt} \right\rangle_b \quad (9.171)$$

shows that there is no difference between the inertial and body derivative of the angular velocity vector itself.

Putting the results of eqns (9.166, 9.169) into eqn (9.168) then gives the second time derivative of \mathbf{s} entirely in terms of body derivatives

$$\begin{aligned} \frac{d^2\mathbf{s}}{dt^2} &= \mathbf{a}_b + \boldsymbol{\omega} \times \mathbf{u}_b + \dot{\boldsymbol{\omega}} \times \mathbf{s} + \boldsymbol{\omega} \times (\mathbf{u}_b + \boldsymbol{\omega} \times \mathbf{s}) \\ &= \mathbf{a}_b + \dot{\boldsymbol{\omega}} \times \mathbf{s} + 2\boldsymbol{\omega} \times \mathbf{u}_b + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) \end{aligned} \quad (9.172)$$

Now suppose that an experiment consists of observing the motion of a mass m acted on by net real force \mathbf{f} , which may be composed of contact forces, gravity, spring forces, electromagnetic forces, etc. Then, using eqn (9.163), Newton’s second law in the inertial system gives

$$\begin{aligned} \mathbf{f} &= m \frac{d^2\mathbf{r}}{dt^2} = m \frac{d^2\mathbf{b}}{dt^2} + m \frac{d^2\mathbf{s}}{dt^2} \\ &= m \frac{d^2\mathbf{b}}{dt^2} + m \{ \mathbf{a}_b + \dot{\boldsymbol{\omega}} \times \mathbf{s} + 2\boldsymbol{\omega} \times \mathbf{u}_b + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) \} \end{aligned} \quad (9.173)$$

The mass times acceleration measured by the observer in the moving system will thus be

$$\begin{aligned} m\mathbf{a}_b &= \mathbf{f} - m \frac{d^2\mathbf{b}}{dt^2} - m\dot{\boldsymbol{\omega}} \times \mathbf{s} - 2m\boldsymbol{\omega} \times \mathbf{u}_b - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) \\ &= \mathbf{f} + \mathbf{f}^{(\text{trans})} + \mathbf{f}^{(\text{ang})} + \mathbf{f}^{(\text{cor})} + \mathbf{f}^{(\text{cent})} \end{aligned} \quad (9.174)$$

where the inertial forces and their names are:

$$\mathbf{f}^{(\text{trans})} = -m \frac{d^2\mathbf{b}}{dt^2} \quad \text{Translation of origin force} \quad (9.175)$$

$$\mathbf{f}^{(\text{ang})} = -m\dot{\boldsymbol{\omega}} \times \mathbf{s} \quad \text{Change of angular velocity force} \quad (9.176)$$

$$\mathbf{f}^{(\text{cor})} = -2m\boldsymbol{\omega} \times \mathbf{u}_b \quad \text{Coriolis force} \quad (9.177)$$

$$\mathbf{f}^{(\text{cent})} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) \quad \text{Centrifugal force} \quad (9.178)$$

Notice that all of these inertial forces are proportional to the mass m of the particle. This proportionality comes from the fact that these “forces” are actually correction terms that appear when $m\mathbf{a}_b$ is used in place of $m\mathbf{a} = m(d^2\mathbf{r}/dt^2)$ in Newton’s second law.

A person driving a car that is rapidly accelerating forward will feel that the translation-of-origin inertial force is pushing her backwards into the car seat. Relative to inertial space, what is really happening is that the back of the seat of the car is pressing

forwards on her with a real force, to give her the acceleration she needs to keep up with the accelerating car.

A person sitting facing forward on the bench of a merry-go-round that is rapidly speeding up will feel that the change-of-angular-velocity force is pushing her backwards into the seat. The real, inertial effect is similar to the accelerating car.

Even when the angular acceleration is constant, a person on the merry-go-round will feel that a centrifugal force is pushing him outwards from the center so that he has to grab onto the pole to resist being thrown outwards. To see that the centrifugal force is outwards, use the expansion formula for triple cross products to write

$$\mathbf{f}^{(\text{cent})} = m\omega^2 \{ \mathbf{s} - \hat{\boldsymbol{\omega}} (\hat{\boldsymbol{\omega}} \cdot \mathbf{s}) \} = m\omega^2 \mathbf{s}_\perp \quad (9.179)$$

where \mathbf{s} has been decomposed as $\mathbf{s} = \mathbf{s}_\parallel + \mathbf{s}_\perp$, into vectors parallel and perpendicular to $\boldsymbol{\omega}$ using the method in Section A.2. What is really happening, relative to inertial space, is that in the absence of any forces, the rider would leave the merry-go-round and go off on a tangent line with a constant, straight-line velocity. His grabbing the pole provides the centripetal (inward) force that is required to keep him moving in a circle.

The Coriolis force is more subtle. It is a velocity-dependent inertial force that acts only on objects that are moving relative to the moving system, and always acts at right angles to \mathbf{u}_b . It can be understood by considering the merry-go-round once again. Suppose that a person riding on it throws a ball radially outwards relative to the merry-go-round system. The centrifugal inertial force will appear to accelerate the ball outwards, but will not change its apparent radial direction relative to the thrower. The Coriolis force, however, will appear to deflect the ball in a direction opposite to the direction of rotation of the merry-go-round. What is happening inertially is that the tangential velocity imparted to the ball as it is thrown is smaller than the tangential velocity of the region of the merry-go-round into which the ball flies. Thus the ball lags behind. The thrower attributes this lag to a Coriolis inertial force.

9.21 Laboratory on the Surface of the Earth

Assume that a coordinate system at the center of the earth with its unit vectors pointing toward fixed stars is approximately an inertial system. Consider a translating and rotating reference system with its origin on the surface of the Earth at latitude λ . For definiteness, assume that the moving system unit vectors are

$$\hat{\mathbf{e}}'_1 = \text{South} \quad \hat{\mathbf{e}}'_2 = \text{East} \quad \hat{\mathbf{e}}'_3 = \text{Up} \quad (9.180)$$

One immediate difference between this special case and the general theory of Section 9.20 is that the vector \mathbf{b} , which was there taken to be free to move in a general way, is now constrained to move with the Earth. Assuming the Earth to be

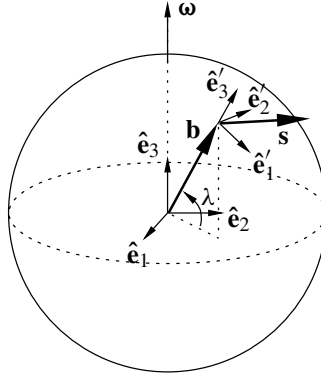


FIG. 9.5. The origin of the o' system is fixed to the surface of the Earth at north latitude λ . The o system at the center of the Earth is assumed to be inertial.

spherical, it is

$$\mathbf{b} = R_{\oplus} \hat{\mathbf{e}}'_3 \tag{9.181}$$

where R_{\oplus} is the radius of the Earth. Hence

$$\frac{d\mathbf{b}}{dt} = R_{\oplus} \frac{d\hat{\mathbf{e}}'_3}{dt} = R_{\oplus} \boldsymbol{\omega} \times \hat{\mathbf{e}}'_3 = \boldsymbol{\omega} \times \mathbf{b} \tag{9.182}$$

The angular velocity of the Earth can be found from eqn (8.197), assuming that the Earth rotates about the fixed axis $\hat{\mathbf{e}}_3$ with constant angular speed ω_0 ,

$$\boldsymbol{\omega} = \omega_0 \hat{\mathbf{e}}_3 = \omega_0 (-\cos \lambda \hat{\mathbf{e}}'_1 + \sin \lambda \hat{\mathbf{e}}'_3) \tag{9.183}$$

which gives

$$\frac{d\mathbf{b}}{dt} = \omega_0 R_{\oplus} \cos \lambda \hat{\mathbf{e}}'_2 \tag{9.184}$$

Thus

$$\frac{d^2\mathbf{b}}{dt^2} = \omega_0 R_{\oplus} \cos \lambda \frac{d\hat{\mathbf{e}}'_2}{dt} = \omega_0 R_{\oplus} \cos \lambda \boldsymbol{\omega} \times \hat{\mathbf{e}}'_2 = \boldsymbol{\omega} \times \frac{d\mathbf{b}}{dt} \tag{9.185}$$

Combining eqns (9.182, 9.185) then gives

$$\frac{d^2\mathbf{b}}{dt^2} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = -\omega_0^2 \{ \mathbf{b} - \hat{\boldsymbol{\omega}} (\hat{\boldsymbol{\omega}} \cdot \mathbf{b}) \} = -\omega_0^2 \mathbf{b}_{\perp} \tag{9.186}$$

where, again using the decomposition given in Section A.2, \mathbf{b}_{\perp} is the component of $\mathbf{b} = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}$ that points directly away from the symmetry axis of the earth $\hat{\boldsymbol{\omega}}$. Thus the translation of origin inertial force from eqn (9.175) is

$$\mathbf{f}^{(\text{trans})} = m\omega_0^2 \mathbf{b}_{\perp} = mR_{\oplus} \omega_0^2 \cos \lambda \hat{\mathbf{b}}_{\perp} \tag{9.187}$$

where $\hat{\mathbf{b}}_{\perp}$ is the unit vector formed from \mathbf{b}_{\perp} .

It is useful to write the real force \mathbf{f} as a vector sum of the gravitational force $m\mathbf{g}$ and all other forces $\mathbf{f}^{(\text{NG})}$ where the superscript stands for Not Gravitational. Then, assuming that the angular velocity of the Earth is approximately constant, the observer in the Earth laboratory will find that

$$m\mathbf{a}_b = \mathbf{f}^{(\text{NG})} + m \left(\mathbf{g} + R_{\oplus}\omega_0^2 \cos \lambda \hat{\mathbf{b}}_{\perp} \right) - 2m\boldsymbol{\omega} \times \mathbf{u}_b - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{s}) \quad (9.188)$$

The last term, the centrifugal inertial force from the rotation of the laboratory frame, is usually negligible and will be dropped. The term containing the gravitational force and the translational inertial force can be written as $m\mathbf{g}_e$, where

$$\mathbf{g}_e = \mathbf{g} + R_{\oplus}\omega_0^2 \cos \lambda \hat{\mathbf{b}}_{\perp} \quad (9.189)$$

is called the effective gravitational acceleration. It is the vector sum of the actual gravitational force of attraction towards the center of the Earth and a term pointing outwards from the Earth's rotation axis and due to the centripetal acceleration of the origin of coordinates as the Earth rotates. Then, finally, only the Coriolis inertial force remains, and

$$m\mathbf{a}_b = \mathbf{f}^{(\text{NG})} + m\mathbf{g}_e - 2m\boldsymbol{\omega} \times \mathbf{u}_b \quad (9.190)$$

An approximate calculation of the figure of the earth, called the *geoid*, can be made by initially assuming the Earth spherical as we have done and then calculating \mathbf{g}_e . To first order, the oceans of the Earth should have surfaces perpendicular to \mathbf{g}_e leading to a slight bulge at the equator, which is in fact observed.

Notice that, in the northern hemisphere, a wind from the north with $\mathbf{u}_b = u_b \hat{\mathbf{e}}'_1$ will be deflected to the west by the Coriolis force, while a wind from the south (opposite sign) will be deflected to the east. Similar deflections occur for east and west winds. This pattern is thought to be responsible for the weather pattern in which low pressure areas (with winds rushing in) in the northern hemisphere have winds circulating in a right-handed sense about the up axis. In the southern hemisphere, the direction of circulation is reversed.

The Coriolis force is usually negligible in laboratories on the surface of the Earth. However, if one designed a space station similar to the one in the film "2001 – A Space Odyssey" with rotation of a large toroidal ring providing an artificial gravity from the translation of origin inertial force, the Coriolis deflections could be troublesome in ordinary life. Exercise 9.5 considers such a space station.

9.22 Coriolis Force Calculations

The Coriolis inertial force is often small compared to other forces, such as gravity. This lends itself to an iterative approach.

A zeroth-order calculation is first done for the motion of the system with Coriolis forces ignored. The velocity \mathbf{u}_b is calculated from that zeroth-order result, and applied in eqn (9.177) to find a zeroth-order approximation to the Coriolis force $\mathbf{f}^{(\text{cor})}$. This approximate Coriolis force is then used to repeat the calculation for the motion of the

system, yielding a first-order approximation to the motion. This first-order approximation is often sufficient, at least for estimates. But, if necessary, the iteration can be repeated to second and higher orders.

For example, consider a projectile fired in a southerly direction from the surface of the Earth. With the Coriolis force ignored, the zeroth-order trajectory would be

$$\mathbf{s} = v_0 t \cos \alpha \hat{\mathbf{e}}'_1 + \left(v_0 t \sin \alpha - \frac{1}{2} g_e t^2 \right) \hat{\mathbf{e}}'_3 \quad (9.191)$$

where v_0 is the muzzle velocity of the cannon and α is its angle from horizontal. The zeroth-order body derivative is

$$\mathbf{u}_b = \left\langle \frac{d\mathbf{s}}{dt} \right\rangle_b = v_0 \cos \alpha \hat{\mathbf{e}}'_1 + (v_0 \sin \alpha - g_e t) \hat{\mathbf{e}}'_3 \quad (9.192)$$

which is used to write the zeroth-order Coriolis force as

$$\mathbf{f}^{(\text{cor})} = -2m\boldsymbol{\omega} \times \mathbf{u}_b = -2m\omega_0 \{v_0 \sin(\alpha + \lambda) - g_e t \cos \lambda\} \hat{\mathbf{e}}'_2 \quad (9.193)$$

After two integrations with the zeroth-order Coriolis force included, the first-order trajectory is found to be

$$\mathbf{s} = v_0 t \cos \alpha \hat{\mathbf{e}}'_1 - \omega_0 t^2 \left(v_0 \sin(\alpha + \lambda) - \frac{1}{3} g_e t \cos \lambda \right) \hat{\mathbf{e}}'_2 + \left(v_0 t \sin \alpha - \frac{1}{2} g_e t^2 \right) \hat{\mathbf{e}}'_3 \quad (9.194)$$

As shown in Exercise 9.11, assuming $\alpha = 45^\circ$ gives a first-order deflection at impact that is to the west in the whole of the northern hemisphere and in the southern hemisphere down to $\lambda \approx -20^\circ$.

9.23 The Magnetic – Coriolis Analogy

Suppose that we have a set of particles located at radius vectors $\mathbf{r}_n(t)$ relative to the origin of some inertial system of coordinates, where $n = 1, \dots, N$. Suppose that these particles have masses m_n and electrical charges $q_n^{(\text{ch})}$. If an external electric field $\mathbf{E}(\mathbf{r}, t)$ and a uniform and static external magnetic induction field $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0$ are present, these particles will be acted on by the Lorentz forces

$$\mathbf{f}_n = q_n^{(\text{ch})} \mathbf{E}(\mathbf{r}_n, t) + \frac{q_n^{(\text{ch})}}{c} \mathbf{v}_n \times \mathbf{B}_0 \quad (9.195)$$

where \mathbf{v}_n is the velocity of the n th particle relative to inertial space.

There is a striking analogy between the magnetic part of eqn (9.195) and the Coriolis force in a rotating coordinate system, given by eqn (9.177). Each involves the cross product of a uniform vector with a particle velocity. If we transform eqn (9.195) to a rotating coordinate system, this analogy allows us to use a suitably chosen $\boldsymbol{\omega}$ to cancel the magnetic part of the Lorentz force.

To exploit the analogy, consider that the motion of the particles under the force eqn (9.195) is now referred to a rotating system of coordinates. Assume that the results of Section 9.20 are applied with $\mathbf{b} = 0$ so that the inertial and rotating origins coincide, $\mathbf{v}_n = \mathbf{u}_n$, and $\mathbf{r}_n = \mathbf{s}_n$. The equations of motion in the rotating system are then

$$m_n(\mathbf{a}_n)_b = \mathbf{f}_n^{(\text{other})} + q_n^{(\text{ch})}\mathbf{E}(\mathbf{r}_n, t) + \frac{q_n^{(\text{ch})}}{c} \{(\mathbf{u}_n)_b + \boldsymbol{\omega} \times \mathbf{r}_n\} \times \mathbf{B}_0 - 2m_n\boldsymbol{\omega} \times (\mathbf{u}_n)_b - m_n\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_n) \quad (9.196)$$

which may be written as

$$m_n(\mathbf{a}_n)_b = \mathbf{f}_n^{(\text{other})} + q_n^{(\text{ch})}\mathbf{E}(\mathbf{r}_n, t) - \left(\frac{q_n^{(\text{ch})}}{c} \mathbf{B}_0 + 2m_n\boldsymbol{\omega} \right) \times (\mathbf{u}_n)_b - \frac{q_n^{(\text{ch})}}{c} \mathbf{B}_0 \times (\boldsymbol{\omega} \times \mathbf{r}_n) - m_n\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_n) \quad (9.197)$$

where $\mathbf{f}_n^{(\text{other})}$ represents forces other than electromagnetic, and $(\mathbf{u}_n)_b = \langle d\mathbf{r}_n/dt \rangle_b$ and $(\mathbf{a}_n)_b$ are the same body derivatives as defined in eqns (9.167, 9.170) but now applied to the n th mass.

If we assume that all of the particles have the same charge to mass ratio, $q_n^{(\text{ch})}/m_n = \chi$ independent of n , then the term in eqn (9.197) containing the body velocities $(\mathbf{u}_n)_b$ can be eliminated by choosing $\boldsymbol{\omega}$ equal to what we will call the *Larmour angular velocity*

$$\boldsymbol{\omega}_L = -\frac{\chi}{2c} \mathbf{B}_0 \quad (9.198)$$

Then the equation of motion in the rotating system becomes

$$m_n(\mathbf{a}_n)_b = \mathbf{f}_n^{(\text{other})} + q_n^{(\text{ch})}\mathbf{E}(\mathbf{r}_n, t) + m_n\boldsymbol{\omega}_L \times (\boldsymbol{\omega}_L \times \mathbf{r}_n) \quad (9.199)$$

Note that the centrifugal inertial force does not cancel. The combination of the magnetic and centrifugal forces gives an effective force that is centripetal (tending toward the center rather than away from it). In certain cases, this centripetal effective force will be small, which leads to the following result, which we state as a theorem.

Theorem 9.23.1: Larmour Theorem

A system of charged particles with a uniform charge to mass ratio χ is placed in a uniform, external magnetic field \mathbf{B}_0 . If one chooses the Larmour angular frequency as in eqn (9.198), and if the maximum centripetal force magnitude $\max_n(m_n\omega_L^2 r_n)$ is negligible compared to other forces, then the problem can be solved by solving the related problem

$$m_n(\mathbf{a}_n)_b = \mathbf{f}_n^{(\text{other})} + q_n^{(\text{ch})}\mathbf{E}(\mathbf{r}_n, t) \quad (9.200)$$

in the rotating system and then transforming the result back to the inertial system.

In other words, the motion with the magnetic field is approximately the same as the motion without it, but rotated by the Larmour angular velocity. Notice that, for positive charges, the direction of the Larmour rotation $\boldsymbol{\omega}_L$ is opposite to the direction of \mathbf{B}_0 .

9.24 Exercises

Exercise 9.1 Consider a flat (negligible thickness), uniform piece of rigid metal of mass m , cut in the shape of a 45° right triangle. Its center of mass is on the symmetry line from the 90° vertex and is one-third of the way up from the base.

- Guess the principal axis directions, and calculate the three principal moments of inertia relative to the center of mass.
- Check that your answers to part (a) obey the plane-figure theorem.

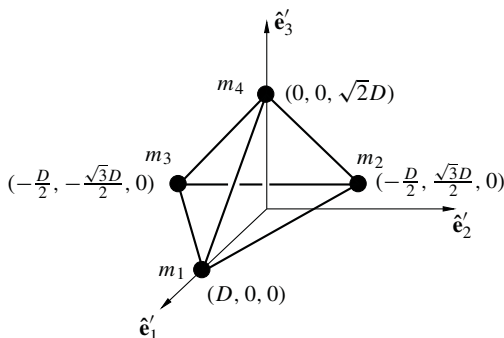


FIG. 9.6. Illustration for Exercise 9.2.

Exercise 9.2 Masses $m_1 = m_2 = m_3 = m_4 = m$ are located at the Cartesian coordinates shown. These masses are at the points of a regular tetrahedron. The four triangular faces are equilateral and identical.

- Use vector methods to check that distance $\overline{12}$ equals distance $\overline{23}$.
- Find the center of mass vector \mathbf{R} .
- Write out the four vectors $\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R}$ for $n = 1, 2, 3, 4$.
- The inertia operator is $\mathcal{I}^{(\text{cm})}$, with a matrix $I^{(\text{cm})'}$ in the $\hat{\mathbf{e}}'_i$ system defined by the matrix elements

$$I_{ij}^{(\text{cm})'} = \sum_{n=1}^4 m_n \left\{ \left(\rho_{n1}'^2 + \rho_{n2}'^2 + \rho_{n3}'^2 \right) \delta_{ij} - \rho_{ni}' \rho_{nj}' \right\} \quad (9.201)$$

Calculate the six independent moments and products of inertia in eqn (9.201) and write the matrix $I^{(\text{cm})'}$.

Exercise 9.3 A projectile is thrown vertically upward from the surface of the Earth. Its initial upward speed is v_0 . It reaches a maximum height, and then falls back to the ground.

- Calculate its first-order vector Coriolis deflection.
- The same projectile is now dropped from rest, its initial height being the same as the zeroth-order maximum height reached in part (a). Show that its first-order Coriolis deflection is in the opposite direction, and one-quarter as large, as that calculated in part (a).

Exercise 9.4 A square stick of mass m has square sides a and length b . With the origin of body-fixed coordinates at the center of mass, the principal axes of the stick are the symmetry

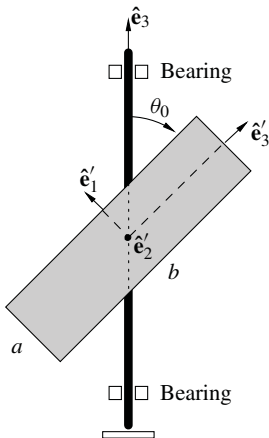


FIG. 9.7. Illustration for Exercise 9.4.

axes and the principal moments of inertia are $I_1^{(cm)'} = I_2^{(cm)'} = m(a^2 + b^2)/12$ and $I_3^{(cm)'} = ma^2/6$. A thin, massless rod is driven through the center of the stick, making an angle θ_0 with the stick's long axis. It is glued to the stick rigidly. The massless rod is suspended in frictionless bearings that hold it vertical (along the \hat{e}_3 space-fixed axis). Using an external motor, it is then rotated about that vertical axis with constant angular velocity $\omega_0 = \omega_0 \hat{e}_3$.

- (a) Write an expression for the angular velocity vector of the stick expressed in the body-fixed system \hat{e}_i' .
- (b) Write an expression for \mathbf{S} , the spin angular momentum vector of the stick, expressed in the body-fixed system.
- (c) Write an expression for the kinetic energy of the stick. How much work per second must the motor provide to keep the angular velocity constant?
- (d) Write an expression for the vector torque τ_s exerted on the system by the bearings and motor, also expressed in the body-fixed system.

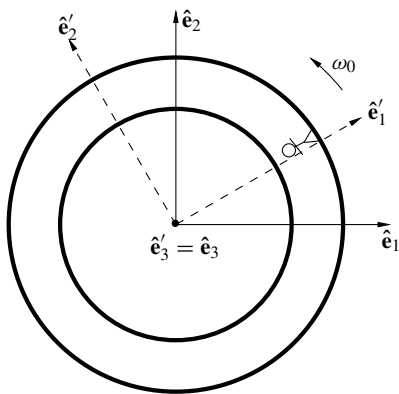


FIG. 9.8. Illustration for Exercise 9.5.

Exercise 9.5 Imagine a space station something like that in the film 2001: A wheel rotates about its symmetry axis with fixed angular velocity ω_0 , and with the living quarters on its outer rim. The centrifugal inertial force provides an effective “gravity” in which the astronauts live.

(a) If the outer rim of the wheel (at the feet of the astronaut pictured) has a radius of r_0 , what must the magnitude of the angular velocity ω_0 be in order to produce an effective gravity of g_e ? [Note: Assume that the living quarters are small compared to r_0 and so just calculate the effective gravity at that radius.]

(b) An astronaut drops a marble from a “height” h , so that it would land on her right toe in the absence of Coriolis forces. Write an expression for the (vector!) first-order displacement of the actual landing point from her right toe. [This expression must be in terms of the given parameters r_0, ω_0, h . For simplicity, take the right toe to be at moving-system coordinates $(r_0, 0, 0)$.]

(c) Evaluate the displacement found in part (b) numerically, with $r_0 = 500$ m, $g_e = 9.8$ m/s², $h = 180$ cm. By what distance will the marble miss the toe? Could you decrease this distance by changing ω_0 so that the effective gravity is smaller?

Exercise 9.6 The famous quarterback Joe Minnesota is drafted into the astronaut corps and sent into orbit to test Euler’s equations for a torque-free rigid body. He throws a pointed, near-ellipsoid of revolution [an American football] that has principal axes $\hat{\mathbf{e}}'_i$ and principal moments of inertia $I_1^{(\text{cm})'} = I_2^{(\text{cm})'} = \gamma I_0$, and $I_3^{(\text{cm})'} = I_0$ where $\gamma > 1$ and I_0 are given constants. Assume that no torques act on this football after it is thrown, and that it behaves as a rigid body.

(a) Suppose that at $t=0$ the initial components of the angular momentum vector $\boldsymbol{\omega}$ relative to the body-fixed principal-axis system are $\omega'_1 = a, \omega'_2 = 0, \omega'_3 = b$ where a, b are given, positive constants. What are the values of ω'_i ($i = 1, 2, 3$) for all future times?

(b) Consider the spin angular momentum vector \mathbf{S} . What are the values S'_i ($i = 1, 2, 3$), the components of the spin angular momentum relative to the body-fixed principal axes, for a general time $t > 0$?

(c) Suppose that *space-fixed* axes $\hat{\mathbf{e}}_i$ are chosen so that the initial spin angular momentum is along the $\hat{\mathbf{e}}_3$ axis, that is $\mathbf{S} = S_0 \hat{\mathbf{e}}_3$ at $t = 0$. What is the value of the constant S_0 ?

(d) What are the polar angles (call them α, β) of the symmetry axis of the football $\hat{\mathbf{e}}'_3$ relative to this space-fixed system? [For simplicity, assume $\alpha_0 = 0$ at $t=0$.]

Exercise 9.7 This exercise uses Lagrangian mechanics to derive the inertial forces discussed in Section 9.20. Suppose that we have a single particle in three dimensions with a Lagrangian

$$L(s, \dot{s}, t) = \frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) - U(s, t) \quad (9.202)$$

where the position and velocity are expressed in an inertial coordinate system as

$$\mathbf{r} = x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 + z \hat{\mathbf{e}}_3 \quad \text{and} \quad \mathbf{v} = \dot{x} \hat{\mathbf{e}}_1 + \dot{y} \hat{\mathbf{e}}_2 + \dot{z} \hat{\mathbf{e}}_3 \quad (9.203)$$

where the s-system coordinates are defined to be $s_1 = x, s_2 = y$, and $s_3 = z$.

(a) Now switch to another coordinate system with the same origin, but rotating with respect to the inertial one with a constant angular velocity $\boldsymbol{\omega}$. Call it the $\hat{\mathbf{e}}'_i$ system. In this system, the

position and body derivative will be

$$\mathbf{r} = x'\hat{\mathbf{e}}'_1 + y'\hat{\mathbf{e}}'_2 + z'\hat{\mathbf{e}}'_3 \quad \text{and} \quad \mathbf{u}_b = \dot{x}'\hat{\mathbf{e}}'_1 + \dot{y}'\hat{\mathbf{e}}'_2 + \dot{z}'\hat{\mathbf{e}}'_3 \quad (9.204)$$

where the q-system coordinates are defined to be $q_1 = x'$, $q_2 = y'$, and $q_3 = z'$. Show that eqn (9.202) becomes

$$L(q, \dot{q}, t) = \frac{1}{2}m(\mathbf{u}_b \cdot \mathbf{u}_b) + m\mathbf{u}_b \cdot \boldsymbol{\omega} \times \mathbf{r} + \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) - U(q, t) \quad (9.205)$$

(b) Show that the Lagrange equations in the q-system,

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q}, t)}{\partial q_k} = 0 \quad (9.206)$$

may be multiplied by the unit vectors $\hat{\mathbf{e}}'_k$ and added to obtain the equivalent vector expression,

$$\left\langle \frac{d}{dt} \left(\frac{\partial L(q, \dot{q}, t)}{\partial \mathbf{u}_b} \right) \right\rangle_b - \frac{\partial L(q, \dot{q}, t)}{\partial \mathbf{r}} = 0 \quad (9.207)$$

where the gradients are now to be interpreted in terms of the body system unit vectors. For example,

$$\frac{\partial L(q, \dot{q}, t)}{\partial \mathbf{r}} = \hat{\mathbf{e}}'_1 \frac{\partial L}{\partial x'} + \hat{\mathbf{e}}'_2 \frac{\partial L}{\partial y'} + \hat{\mathbf{e}}'_3 \frac{\partial L}{\partial z'} \quad (9.208)$$

(c) Use vector methods (i.e. do not write everything out in terms of components) to show that the equations of motion derived from eqn (9.207) are the same as eqn (9.174).

Exercise 9.8 This exercise continues the discussion in Section 9.10.

(a) Use the expressions in eqns (8.230 – 8.232) to write the angular velocity $\boldsymbol{\omega}$ of the torque-free symmetric body in terms of the Euler angles and unit vectors $\mathbf{q}(t)$ and $\hat{\mathbf{e}}_3$.

(b) Define a parameter $\Delta = (I_3^{(\text{cm})'} - I_1^{(\text{cm})'})/I_1^{(\text{cm})'}$. Show that $-1 \leq \Delta \leq 1$, with the negative extreme being a long rod (extreme prolate) and the positive extreme being a flat, symmetric disk (extreme oblate).

(c) Express $\boldsymbol{\omega}$ in terms of Δ , $\dot{\alpha}_0$, β_0 , $\hat{\mathbf{q}}$, $\hat{\mathbf{e}}_3$ only.

(d) Show that the half angle of the body cone $\theta_{3\omega}$ is given by

$$\cos \theta_{3\omega} = \frac{1 + \Delta \sin^2 \beta_0}{\sqrt{1 + \Delta(2 + \Delta) \sin^2 \beta_0}} \quad (9.209)$$

(e) Give arguments verifying the statements made in the last two paragraphs of Section 9.10.

Exercise 9.9 A thin, uniform disk of mass m and radius a has two extra point masses, each of mass κm , placed just inside its rim on diametrically opposite sides. The disk rolls without slipping down an inclined plane of angle η . Gravity acts downwards. The disk is supported by massless, frictionless shaft and side wheels which keep it perpendicular to the plane's surface. The center of mass of the disk is at $\mathbf{R} = X\hat{\mathbf{e}}_1 + Y\hat{\mathbf{e}}_2 + a\hat{\mathbf{e}}_3$.

(a) Show that the principal axes are the $\hat{\mathbf{e}}'_i$ shown in Figure 9.9, and calculate the principal

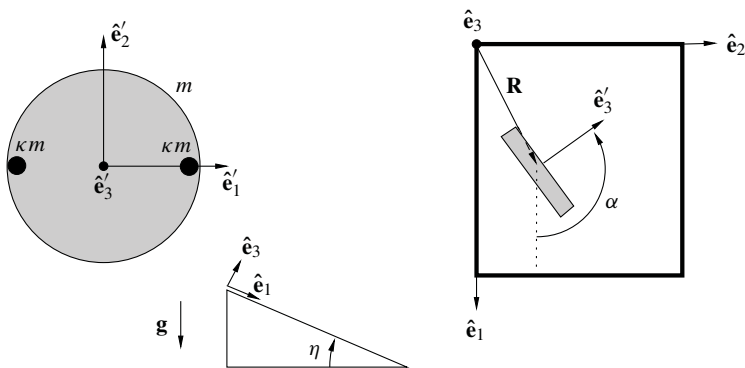


FIG. 9.9. Illustration for Exercise 9.9.

moments of inertia.

(b) Use Euler angles and X, Y, Z to write a reduced Lagrangian for this problem, including the holonomic constraints $\beta = \pi/2$ and $Z = a$.

(c) Write the Lagrange equations, including the tractable, non-holonomic constraints for rolling without slipping.

(d) Solve for the motion of the disk in the case $\kappa = 0$. Suppose the disk is at the origin at time zero with the values $\alpha_0 = \pi/2, \gamma_0 = 0, \dot{\gamma}_0 = 0, X_0 = Y_0 = 0$ but $\dot{\alpha}_0 > 0$. Find α, γ, X, Y as functions of time. Show that X returns to zero at times $t_n = \pi n / \dot{\alpha}_0$ where $n = 1, 2, 3, \dots$

Exercise 9.10 This exercise shows the connection between the inertia operator \mathcal{I} derived in Section 9.14 and the same operator $\mathcal{I} = \mathcal{I}^{(\text{orb})} + \mathcal{I}^{(\text{cm})}$ derived in the previous Sections 9.11 and 9.12. Substitute $\mathbf{r}_n = \mathbf{R} + \boldsymbol{\rho}_n$ into the dyadic expression eqn (9.113), and use Lemma 1.9.1 to obtain eqn (9.94), $\mathbb{I} = \mathbb{I}^{(\text{orb})} + \mathbb{I}^{(\text{cm})}$.

Exercise 9.11 This exercise continues the example in Section 9.22.

(a) Derive eqn (9.194) from eqn (9.193).

(b) Write an expression for the deflection $y'(t_f)$ of the projectile at the zeroth order time of impact $t_f = (2v_0 \sin \alpha) / g_e$.

(c) Assuming $\alpha = 45^\circ$, show that this deflection is to the west in the whole of the northern hemisphere and in the southern hemisphere for $\lambda \gtrsim -20^\circ$.

(d) If the zeroth order range of the projectile was 10 km and $\alpha = 45^\circ$, find the magnitude of the first order Coriolis deflection with $\lambda = 40^\circ$ in the northern hemisphere.

Exercise 9.12 Imagine a charged, symmetric rigid body consisting of particles of mass m_n and charge $q_n^{(\text{ch})}$. Let the center of mass of the body be at rest in an inertial system in which a uniform magnetic induction field \mathbf{B}_0 acts. Assume that the magnetic part of the Lorentz force eqn (9.195) is the only external force acting on the top.

(a) Using the vectors $\boldsymbol{\rho}_n$ and $\dot{\boldsymbol{\rho}}_n$ defined in Section 1.9 show that the external torque defined in Section 1.12 is

$$\boldsymbol{\tau}_s^{(\text{ext})} = \sum_{n=1}^N \frac{q_n^{(\text{ch})}}{c} \boldsymbol{\rho}_n \times (\dot{\boldsymbol{\rho}}_n \times \mathbf{B}_0) \tag{9.210}$$

(b) In electrostatics texts, this torque is written as

$$\boldsymbol{\tau}_s^{(\text{ext})} = \boldsymbol{\mu} \times \mathbf{B}_0 \quad \text{where} \quad \boldsymbol{\mu} = \sum_{n=1}^N \frac{q_n^{(\text{ch})}}{2c} (\boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n) \quad (9.211)$$

is the *magnetic moment* of the system. Show that the difference between eqns (9.210, 9.211) may be written as $\mathbb{D} \cdot \mathbf{B}_0$ where the dyadic is

$$\mathbb{D} = \frac{d}{dt} \left\{ \sum_{n=1}^N \frac{q_n^{(\text{ch})}}{2c} (\boldsymbol{\rho}_n \boldsymbol{\rho}_n - \rho_n^2 \mathbb{U}) \right\} \quad (9.212)$$

which will average to zero for periodic or bounded motions of the top.

(c) Accepting the form in eqn (9.211), and assuming that the masses have a uniform charge to mass ratio ($q_n^{(\text{ch})}/m_n = \chi$ not a function of n) show that $\boldsymbol{\mu} = (\chi/2c)\mathbf{S}$ where \mathbf{S} is the spin angular momentum defined in Section 1.11. The quantity $\chi/2c$ is called the *gyromagnetic ratio*.

(d) Use eqn (1.52), with the assumption that the body is rapidly spinning so that $\mathbf{S} = S\hat{\mathbf{e}}_3'$ remains true with constant spin magnitude S (the same approximation made in Section 9.19) to show that \mathbf{S} precesses with the Larmour angular velocity $\boldsymbol{\omega}_L$ derived in Section 9.23 and therefore verifies the Larmour theorem, Theorem 9.23.1.

(e) The electron has magnetic moment $\boldsymbol{\mu} = g(-e/2mc)\mathbf{S}$ where e is the absolute value of the charge of the electron, m is its mass, and the number g is predicted by the Dirac equation of quantum theory to be exactly 2 (radiative corrections raise it slightly). Show (using classical mechanics, although the same result can be obtained in quantum theory) that when placed in a uniform magnetic field, an electron precesses with an angular velocity $\boldsymbol{\omega}_e = (e/mc)\mathbf{B}_0$.

(f) Show that a classical model of the electron as spinning matter with a uniform charge to mass ratio is untenable because the Larmour theorem would predict a rate of precession disagreeing with experiment.

Exercise 9.13 A sphere of radius r_0 and a right, circular cylinder of radius a and height h both have the same mass M .

(a) The sphere and the cylinder have the same center-of-mass principal moments of inertia $I_k^{(\text{cm})'}$. Given r_0 , what must be the values of a and h ? What is the ratio a/h ?

(b) If the sphere is made from material of mass density ρ_s , what must be the mass density ρ_c of the cylinder? Show that your answer is independent of r_0 .

Exercise 9.14 A right circular cone of mass M , radius a , and height h has principal moments of inertia relative to a pivot at its point

$$I_1' = I_2' = \frac{3}{20}M(a^2 + 4h^2) \quad I_3' = \frac{3}{10}Ma^2 \quad (9.213)$$

Its center of mass is distance $3h/4$ from its point. Use the translation of pivot theorem, eqn (9.93), to find the center-of-mass principal moments of inertia $I_k^{(\text{cm})'}$ of the cone without doing any further integrations. (See also Section 9.12.)

Exercise 9.15

(a) Use the symmetry rule, Lemma 9.7.2, and the symmetry of the object to show that an equal-sided, regular tetrahedron has at least three non-coplanar eigenvector directions, each of which must have the same eigenvalue.

(b) Argue from the linearity of the eigenvalue equation $\mathcal{I}^{(\text{cm})}\mathbf{V}^{(k)} = \lambda_k\mathbf{V}^{(k)}$ that any vector is an eigenvector, with that same eigenvalue. And hence show that *any* three orthogonal unit vectors may be used as center-of-mass principal axes of the tetrahedron, and that the three corresponding principal moments of inertia will be equal.

(b) Show that this same result holds for any of the five regular Platonic solids: tetrahedron, cube, octahedron, icosahedron, dodecahedron.

Exercise 9.16 A right, circular cone of mass M , radius a and height h rolls without slipping on a horizontal table. The point of the cone is fixed to the table surface by a frictionless pivot. There is an inertial coordinate system $\hat{\mathbf{e}}_i$ with origin at the pivot, and $\hat{\mathbf{e}}_3$ perpendicularly upwards from the table surface. Gravity $\mathbf{g} = -g\hat{\mathbf{e}}_3$ acts downwards. Assume that the cone rolls steadily in the right-hand sense about $\hat{\mathbf{e}}_3$, making one circuit in T seconds.

(a) Use Euler angles and Section 8.29 to find the components ω_i of the angular velocity vector $\boldsymbol{\omega}$ relative to the inertial system.

(b) State why the angular velocity of the cone must always be parallel or anti-parallel to the line of contact between the cone and the table. Show that your $\boldsymbol{\omega}$ from (a) does indeed have this property.

(c) Use the results of Section 8.35 to find the components ω'_i of the angular velocity vector relative to the body principal axes of the cone. Use these in the Euler equations of motion from Section 9.13 to calculate the net torque that must be applied to the cone to sustain its motion.

(d) Demonstrate that this torque does not change the total kinetic energy of the cone, and hence that kinetic energy is a conserved quantity in this problem.

Exercise 9.17 A cube of mass M and side a is glued to a massless stick, with the stick perpendicular to a face of the cube. The distance from the far end of the stick to the center of mass of the cube is $R > a$.

(a) Suppose the far end of the stick to be fixed. What are the principal axes and principal moments of inertia relative to this fixed pivot?

(b) Now suppose that the cube is glued to the stick as before, but after being rotated by 37° relative to the stick. Assume the distance R is the same as before. How would the principal axes and principal moments of inertia differ from those of part (a)?

SMALL VIBRATIONS ABOUT EQUILIBRIUM

A number of interesting mechanical systems have one or more essentially stable equilibrium configurations. When disturbed slightly, they vibrate about equilibrium in characteristic patterns called normal modes. We present the Lagrangian theory of these small vibrations for the simple case of systems with a finite number of degrees of freedom.

The theory introduced here has wide application. For example, the normal mode oscillations of crystalline solids underlie both the overtone structure of a church bell and the definition of phonons in solid state physics. A similar formalism leads to photons as the quanta of modes of the electromagnetic field.

10.1 Equilibrium Defined

An *equilibrium point* in the configuration space of a mechanical system is a set of values $q_1^{(e)}, q_2^{(e)}, \dots, q_D^{(e)}$ for all of its generalized coordinates, such that the initial conditions $q_i(0) = q_i^{(e)}$ and $\dot{q}_i(0) = 0$ at time zero will make the system remain at rest at that point for all future times.

For example, a marble placed at rest at the bottom of a spherical bowl will remain there forever. A marble placed at rest, and very carefully, on the top of a sphere will also remain there so long as no forces other than gravity act, as will a marble placed on a flat, level table top.

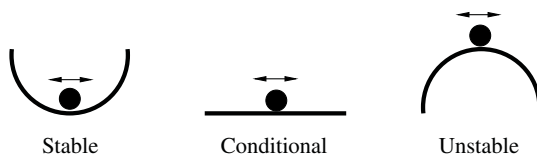


FIG. 10.1. Three categories of equilibrium points.

These examples illustrate the three types of equilibrium point. The first is called *stable*, since a small displacement of the marble would lead to a small vibration about the equilibrium point. The second is called *unstable*. Any disturbance of the marble will make it roll off the sphere with increasing speed. The third is called *conditional*. If the marble is displaced slightly to another point, but still placed at rest, it will stay at the new point and neither return to the first one nor move away. One important feature of the theory presented here is that it gives a systematic method for detecting stable, conditional, and unstable modes.

10.2 Finding Equilibrium Points

Assume now, as will be done throughout this chapter, that the q -system of coordinates is related to an inertial s -system by equations $s_i = s_i(q)$ that do not depend explicitly on the time, and that all forces are derived from a potential $V(q)$ which also does not depend explicitly on time. Also, we assume either that the system does not have any constraints, or that the constraints are holonomic, do not depend explicitly on time, and have been eliminated by the method of Section 3.8 to yield a reduced Lagrangian.

From Section 2.7, the Lagrangian of such a system is of the form

$$L = L(q, \dot{q}) = T_2 - U = \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D m_{ij}(q) \dot{q}_i \dot{q}_j - U(q) \quad (10.1)$$

and the Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial L(q, \dot{q})}{\partial q_k} = 0 \quad (10.2)$$

which can be written out as

$$\frac{d}{dt} \left(\sum_{j=1}^D m_{kj}(q) \dot{q}_j \right) - \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D \frac{\partial m_{ij}(q)}{\partial q_k} \dot{q}_i \dot{q}_j + \frac{\partial U(q)}{\partial q_k} = 0 \quad (10.3)$$

$$\sum_{j=1}^D \dot{m}_{kj}(q) \dot{q}_j + \sum_{j=1}^D m_{kj}(q) \ddot{q}_j - \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D \frac{\partial m_{ij}(q)}{\partial q_k} \dot{q}_i \dot{q}_j + \frac{\partial U(q)}{\partial q_k} = 0 \quad (10.4)$$

Theorem 4.1.1, and Theorem 4.5.1 for the reduced case, showed the matrix m to be nonsingular and hence to have an inverse. If all \dot{q}_k are set zero at a point, then the inverse matrix m^{-1} can be used to solve eqn (10.4) for the \ddot{q} at that same point.

$$\ddot{q}_j = - \sum_{k=1}^D m_{jk}^{-1} \frac{\partial U(q)}{\partial q_k} \quad (10.5)$$

It follows that all of the \ddot{q}_j will vanish, and the point will thus be an equilibrium point, if and only if all $\partial U(q)/\partial q_k$ are also zero at that point.

Thus $q^{(e)}$ is an equilibrium point if and only if, for all $i = 1, \dots, D$,

$$\left. \frac{\partial U(q)}{\partial q_i} \right|_{q=q^{(e)}} = 0 \quad (10.6)$$

These are D equations in D unknowns and hence can be solved for the $q_i^{(e)}$ for all $i = 1, \dots, D$. The point defined by eqn (10.6) satisfies the definition of equilibrium in Section 10.1 because the conditions $q_i(0) = q_i^{(e)}$ and $\dot{q}_i(0) = 0$ for all $i = 1, \dots, D$ imply that $\ddot{q}_i(0) = 0$ for all i values, with the result that the q never change.

10.3 Small Coordinates

Now assume that an equilibrium point has been found. We are interested in studying systems in which the generalized coordinates are only slightly displaced from their equilibrium values. Thus the quantities r_i defined, for all $i = 1, \dots, D$, by

$$r_i = q_i - q_i^{(e)} \quad \text{or, equivalently} \quad q_i = q_i^{(e)} + r_i \quad (10.7)$$

are assumed to be small, and are called the *small coordinates*. Their time derivatives $\dot{r}_i = \dot{q}_i$ are also assumed to be of the same order of smallness. For oscillatory solutions, this amounts to an assumption that all frequencies of vibration are finite.

It is clear by inspection that the r_i are a good system of generalized coordinates satisfying eqn (2.59). Thus, the Lagrangian in eqn (10.1) may be transformed to the r -system

$$L = L(r, \dot{r}) = L(q(r), \dot{q}(r, \dot{r})) = \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D m_{ij} (q^{(e)} + r) \dot{r}_i \dot{r}_j - U(q^{(e)} + r) \quad (10.8)$$

with the assurance that the Lagrange equations in the r -system will be valid.

Approximate equations of motion may be obtained by using the Taylor series to expand eqn (10.8) to second order in the small coordinates and their time derivatives

$$\begin{aligned} L(r, \dot{r}) = & L(r, \dot{r})|_{r, \dot{r}=0} + \sum_{i=1}^D \frac{\partial L(r, \dot{r})}{\partial r_i} \Big|_{r, \dot{r}=0} r_i + \sum_{i=1}^D \frac{\partial L(r, \dot{r})}{\partial \dot{r}_i} \Big|_{r, \dot{r}=0} \dot{r}_i \\ & + \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D \frac{\partial^2 L(r, \dot{r})}{\partial r_i \partial r_j} \Big|_{r, \dot{r}=0} r_i r_j + \sum_{i=1}^D \sum_{j=1}^D \frac{\partial^2 L(r, \dot{r})}{\partial r_i \partial \dot{r}_j} \Big|_{r, \dot{r}=0} r_i \dot{r}_j \\ & + \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D \frac{\partial^2 L(r, \dot{r})}{\partial \dot{r}_i \partial \dot{r}_j} \Big|_{r, \dot{r}=0} \dot{r}_i \dot{r}_j + o(h^2) \end{aligned} \quad (10.9)$$

where $h = \max_i \{r_i, \dot{r}_i\}$. Denoting the constant first term by

$$L_e = L(r, \dot{r})|_{r, \dot{r}=0} = -U(q^{(e)}) \quad (10.10)$$

and noting that the linear terms vanish identically due to the equilibrium condition eqn (10.6), the expansion in eqn (10.9) becomes

$$L(r, \dot{r}) = L_e + \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D (T_{ij} \dot{r}_i \dot{r}_j - V_{ij} r_i r_j) + o(h^2) \quad (10.11)$$

where the *constant* matrices \mathbf{T} and \mathbf{V} are defined by their matrix elements

$$T_{ij} = \frac{\partial^2 L(r, \dot{r})}{\partial \dot{r}_i \partial \dot{r}_j} \Big|_{r, \dot{r}=0} = m_{ij} (q^{(e)} + r) \Big|_{r=0} = m_{ij} (q^{(e)}) \quad (10.12)$$

$$V_{ij} = - \left. \frac{\partial^2 L(r, \dot{r})}{\partial r_i \partial r_j} \right|_{r, \dot{r}=0} = \left. \frac{\partial^2 U(q^{(e)} + r)}{\partial r_i \partial r_j} \right|_{r=0} = \left. \frac{\partial^2 U(q)}{\partial q_i \partial q_j} \right|_{q=q^{(e)}} \quad (10.13)$$

By construction, \mathbf{T} and \mathbf{V} are real, symmetric matrices.

The Lagrange equations in the r -system are

$$\frac{d}{dt} \left(\frac{\partial L(r, \dot{r})}{\partial \dot{r}_k} \right) - \frac{\partial L(r, \dot{r})}{\partial r_k} = 0 \quad (10.14)$$

Using these equations with the approximate Lagrangian of eqn (10.11) gives the approximate equations of motion, for all $i = 1, \dots, D$,

$$\sum_{j=1}^D (T_{ij} \ddot{r}_j + V_{ij} r_j) + o(h) = 0 \quad (10.15)$$

Since the Lagrangian was expanded up to and including quadratic order, these equations are correct to linear order in the small coordinates.

10.4 Normal Modes

The task of small vibrations theory is to solve the linearized, approximate equations of motion eqn (10.15) for the small coordinates r_i as functions of time. Equation (10.15) is a set of coupled, second-order differential equations. They can be decoupled by making yet another Lagrangian change of variables, from the r -system to a ρ -system called the system of *normal coordinates*. The variable ρ_k will be called the k th normal coordinate. The natural time evolution of the system produced by ρ_k will be referred to as motion in the k th *normal mode*.

The required transformation is linear. Introducing the constant coefficients C_{ik} the transformation is

$$r_i = \sum_{k=1}^D C_{ik} \rho_k \quad \text{and hence} \quad \dot{r}_i = \sum_{k=1}^D C_{ik} \dot{\rho}_k \quad (10.16)$$

Introducing these definitions into the Lagrangian of eqn (10.11), and dropping the constant L_e term, gives

$$\begin{aligned} L(\rho, \dot{\rho}) &= \frac{1}{2} \sum_{i=1}^D \sum_{j=1}^D \left\{ T_{ij} \left(\sum_{k=1}^D C_{ik} \dot{\rho}_k \right) \left(\sum_{l=1}^D C_{il} \dot{\rho}_l \right) - V_{ij} \left(\sum_{k=1}^D C_{ik} \rho_k \right) \left(\sum_{l=1}^D C_{il} \rho_l \right) \right\} \\ &= \frac{1}{2} \sum_{k=1}^D \sum_{l=1}^D \left\{ \left(\sum_{i=1}^D \sum_{j=1}^D C_{ik} T_{ij} C_{jl} \right) \dot{\rho}_k \dot{\rho}_l - \left(\sum_{i=1}^D \sum_{j=1}^D C_{ik} V_{ij} C_{jl} \right) \rho_k \rho_l \right\} \end{aligned} \quad (10.17)$$

where the $o(h^2)$ term will now be omitted, with the understanding that the equations are approximate to that order.

It is useful to write eqn (10.17) in matrix form. Define column vectors by

$$[r] = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_D \end{pmatrix} \quad \text{and} \quad [\rho] = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_D \end{pmatrix} \quad (10.18)$$

with the obvious extension to their time derivatives, and let the $D \times D$ matrices \mathbf{T} , \mathbf{V} , \mathbf{C} be defined by their sets of matrix elements T_{ij} , V_{ij} , C_{ij} , respectively. Then eqn (10.16) becomes

$$[r] = \mathbf{C}[\rho] \quad \text{and} \quad [\dot{r}] = \mathbf{C}[\dot{\rho}] \quad (10.19)$$

and the Lagrangian in eqn (10.17) can be written as

$$L(\rho, \dot{\rho}) = \frac{1}{2} \left([\dot{\rho}]^T \mathbf{C}^T \mathbf{T} \mathbf{C} [\dot{\rho}] - [\rho]^T \mathbf{C}^T \mathbf{V} \mathbf{C} [\rho] \right) \quad (10.20)$$

In Section 10.5, we will describe a method for finding a particular matrix \mathbf{C} that reduces \mathbf{T} to the unit matrix \mathbf{U} , and \mathbf{V} to a diagonal matrix \mathbf{F} with matrix elements $F_{ij} = \theta_i \delta_{ij}$.

$$\mathbf{C}^T \mathbf{T} \mathbf{C} = \mathbf{U} \quad \text{and} \quad \mathbf{C}^T \mathbf{V} \mathbf{C} = \mathbf{F} \quad (10.21)$$

Then eqn (10.20) becomes

$$L(\rho, \dot{\rho}) = \frac{1}{2} \left([\dot{\rho}]^T \mathbf{U} [\dot{\rho}] - [\rho]^T \mathbf{F} [\rho] \right) = \frac{1}{2} \sum_{k=1}^D \left(\dot{\rho}_k^2 - \theta_k \rho_k^2 \right) \quad (10.22)$$

The Lagrange equations in the ρ -system then give the equations of motion

$$\ddot{\rho}_k + \theta_k \rho_k = 0 \quad (10.23)$$

for $k = 1, \dots, D$. These equations have solutions that depend on the sign of the real constants θ_k . The most general solutions for the three possibilities can be written

$$\rho_k = \begin{cases} A_k \cos \omega_k t + B_k \sin \omega_k t & \text{for } \theta_k > 0 \quad \text{with } \omega_k = \sqrt{\theta_k} \\ A_k + B_k t & \text{for } \theta_k = 0 \\ A_k \cosh \gamma_k t + B_k \sinh \gamma_k t & \text{for } \theta_k < 0 \quad \text{with } \gamma_k = \sqrt{-\theta_k} \end{cases} \quad (10.24)$$

where the constants A_k , B_k are to be determined from the initial conditions at time zero. The meanings of these three solutions will be discussed in Section 10.6.

10.5 Generalized Eigenvalue Problem

The matrix \mathbf{C} that performs the reductions in eqn (10.21) can be found using the generalized eigenvalue methods from Appendix C.

With the substitutions of the real, symmetric small-vibration matrices \mathbf{V} and \mathbf{T} for the matrices \mathbf{A} and \mathbf{g} of that appendix, the generalized eigenvector equation in eqn (C.16) becomes

$$\mathbf{V}[z^{(k)}] = \theta_k \mathbf{T}[z^{(k)}] \quad (10.25)$$

The theory of Appendix C requires that \mathbf{T} , which is being substituted for \mathbf{g} there, must be a positive definite matrix. But that positive definiteness has already been proved earlier, in our discussion of Hamiltonian mechanics. It is proved in Theorem 4.1.1, and in Theorem 4.5.1 for the reduced case.

So the whole theory of Appendix C can now be applied to the present problem. The eigenvalues θ_k in eqn (10.25), all of which will be real, are the D solutions of eqn (C.18)

$$|\mathbf{V} - \theta \mathbf{T}| = 0 \quad (10.26)$$

and the corresponding eigenvectors $[z^{(k)}]$ are the solutions of eqn (C.17)

$$\{\mathbf{V} - \theta_k \mathbf{T}\}[z^{(k)}] = [0] \quad (10.27)$$

The generalized inner product defined in Section C.2 becomes

$$[x] \bullet [y] = [x]^T \mathbf{T}[y] = \sum_{i=1}^D \sum_{j=1}^D x_i T_{ij} y_j \quad (10.28)$$

The eigenvector solutions are then normalized by using

$$1 = [z^{(k)}] \bullet [z^{(k)}] = [z^{(k)}]^T \mathbf{T}[z^{(k)}] = \sum_{i=1}^D \sum_{j=1}^D z_i^{(k)} T_{ij} z_j^{(k)} \quad (10.29)$$

and, by Theorem C.3.2, D eigenvectors can always be found that are orthonormal in the generalized sense given in eqn (10.29)

$$[z^{(k)}] \bullet [z^{(l)}] = \delta_{kl} \quad (10.30)$$

Solution of the generalized eigenvalue problem produces the matrix \mathbf{C} defined in eqn (C.29) by

$$C_{ik} = z_i^{(k)} \quad (10.31)$$

Then, still with the substitutions of matrices \mathbf{V} and \mathbf{T} for the matrices \mathbf{A} and \mathbf{g} , eqn (C.32) proved in Theorem C.5.1 is identical to our desired result, eqn (10.21).

The reader may consult Appendix C for more detailed information about solution of the generalized eigenvalue problem.

10.6 Stability

At the beginning of this chapter, we spoke of stable, conditional, and unstable systems. Now, after seeing the general solution, we must speak more specifically of stable, conditional, and unstable *modes*. If $\theta_k > 0$, then mode k is stable and has an oscillatory time evolution. The conditional *zero frequency modes* with $\theta_k = 0$ will remain small provided that the B_k of that mode is zero. But when $\theta_k < 0$, the mode solution involves hyperbolic functions and will increase exponentially with time for any nonzero A_k or B_k . These are the unstable modes.

A system is unconditionally stable only if *all* of its modes are stable. Otherwise, there will be some initial conditions in which a conditional or unstable mode will cause the system to run away to infinity. One strength of the method presented here is that the stability of a complex system can be determined simply by solving eqn (10.26) for the eigenvalues. If all of them are positive, the system is stable.

If either of the coefficients A_k or B_k of mode k is nonzero, then we say that mode k is *excited*. Which modes are excited, and by how much, depends on the initial conditions at time zero. The determination of A_k and B_k from the initial conditions will now be considered.

10.7 Initial Conditions

Equation (10.16) gives the general solution for the time evolution of the small coordinates r_i in terms of the normal coordinates ρ_k given by eqn (10.24). We now show how to derive the general solution from the initial conditions at time zero.

As a preliminary to treating the initial conditions, we use the definition $C_{ik} = z_i^{(k)}$ from eqn (10.31) to write the solution eqn (10.16) as

$$r_i = \sum_{k=1}^D z_i^{(k)} \rho_k \quad (10.32)$$

where ρ_k is one of eqn (10.24). Introducing the column eigenvector $[z^{(k)}]$ defined by

$$[z^{(k)}] = \begin{pmatrix} z_1^{(k)} \\ z_2^{(k)} \\ \vdots \\ z_D^{(k)} \end{pmatrix} \quad (10.33)$$

and using eqn (10.18), eqn (10.32) may be written in matrix form as

$$[r] = \sum_{k=1}^D [z^{(k)}] \rho_k \quad (10.34)$$

This equation may be inverted, solved for ρ_k as functions of the r_i , by using the

generalized inner product defined in eqn (10.28),

$$[z^{(k)}] \bullet [r] = [z^{(k)}] \bullet \sum_{l=1}^D [z^{(l)}] \rho_l = \sum_{l=1}^D \delta_{kl} \rho_l = \rho_k \quad (10.35)$$

where the orthogonality relation eqn (10.30) was used.

The inverse eqn (10.35) is true at all times. But it is particularly useful to consider it at time zero. Then, using eqn (10.24),

$$A_k = \rho_k(0) = [z^{(k)}] \bullet [r(0)] \quad \text{and} \quad D_k = \dot{\rho}_k(0) = [z^{(k)}] \bullet [\dot{r}(0)] \quad (10.36)$$

where

$$D_k = \begin{cases} \omega_k B_k & \text{for } \theta_k > 0 \\ B_k & \text{for } \theta_k = 0 \\ \gamma_k B_k & \text{for } \theta_k < 0 \end{cases} \quad (10.37)$$

give the coefficients A_k and B_k in terms of the small coordinates $r_i(0)$ and velocities $\dot{r}_i(0)$ at time zero.

With the determination of the coefficients A_k and B_k , the problem is now completely solved. In matrix form, the solution is

$$[r] = \sum_{\theta_k > 0} [z^{(k)}] (A_k \cos \omega_k t + B_k \sin \omega_k t) + \sum_{\theta_k = 0} [z^{(k)}] (A_k + B_k t) + \sum_{\theta_k < 0} [z^{(k)}] (A_k \cosh \gamma_k t + B_k \sinh \gamma_k t) \quad (10.38)$$

10.8 The Energy of Small Vibrations

Since the chain of transformation equations from an inertial s -system to the final ρ -system of normal modes does not depend explicitly on the time, and since the constraints, if any, were also assumed not to be explicit time functions, it follows from the discussion in Section 2.16 that the generalized energy function in the ρ -system will be equal to the total energy. Using the Lagrangian $L(\rho, \dot{\rho})$ from eqn (10.22) in the definition of the generalized energy in eqn (2.76) gives at once that

$$H(\rho, \dot{\rho}) = V(q^{(e)}) + \frac{1}{2} \sum_{k=1}^D (\dot{\rho}_k^2 + \theta_k \rho_k^2) \quad (10.39)$$

where the L_e term from eqn (10.10) has now been restored. Using the solutions in eqn (10.24), the generalized energy function can be expanded as a sum over the stable, conditional, and unstable modes as

$$H(\rho, \dot{\rho}) = V(q^{(e)}) + \frac{1}{2} \sum_{\theta_k > 0} \omega_k^2 (A_k^2 + B_k^2) + \frac{1}{2} \sum_{\theta_k = 0} B_k^2 + \frac{1}{2} \sum_{\theta_k < 0} \gamma_k^2 (-A_k^2 + B_k^2) \quad (10.40)$$

Note that an initial displacement of an unstable mode may decrease the energy, as would be expected from the example of the sphere with a marble on top.

The Hamiltonian in the normal mode system may also be found, using the techniques of Chapter 4. Since $p_k = \partial L(\rho, \dot{\rho})/\partial \dot{\rho}_k = \dot{\rho}_k$, it is

$$H(\rho, p) = V(q^{(e)}) + \frac{1}{2} \sum_{k=1}^D (p_k^2 + \theta_k \rho_k^2) \quad (10.41)$$

Equation (10.41) is important in quantum mechanics. The classical normal mode analysis separates the stable modes (which are to be quantized) from the conditional and unstable ones. The coefficients ρ_k, p_k of the stable modes are replaced by quantum operators and eqn (10.41) becomes the Hamiltonian operator of a quantum system. Each mode is thus a quantum mechanical harmonic oscillator. In solids, the quanta of these mode oscillators are called phonons, in electrodynamics they are called photons. The zero-frequency conditional modes must be dealt with separately. Their quanta are sometimes referred to as Goldstone bosons.

10.9 Single Mode Excitations

Under some initial conditions, only one of the D possible modes will be excited. Assume the k th mode to be the one excited, so that $A_l = 0$ and $B_l = 0$ for all $l \neq k$. Then, taking the excited mode to be a stable one, the solution in eqn (10.38) will take the form

$$[r] = [z^{(k)}] (A_k \cos \omega_k t + B_k \sin \omega_k t) \quad (10.42)$$

where the sum has collapsed to the single index k of the excited mode. In component form, eqn (10.42) says that, for all $i = 1, \dots, D$,

$$r_i = z_i^{(k)} (A_k \cos \omega_k t + B_k \sin \omega_k t) \quad (10.43)$$

In a single mode excitation, all of the small coordinates r_i are seen to oscillate with the *same* frequency ω_k . Also, the ratios of the various small coordinates will be constant in time, and equal to the ratios of corresponding components of the k th eigenvector $[z^{(k)}]$. Thus, for all i, j values, and all times t ,

$$\frac{r_i(t)}{r_j(t)} = \frac{z_i^{(k)} (A_k \cos \omega_k t + B_k \sin \omega_k t)}{z_j^{(k)} (A_k \cos \omega_k t + B_k \sin \omega_k t)} = \frac{z_i^{(k)}}{z_j^{(k)}} \quad (10.44)$$

These ratios give what can be thought of as the *pattern* of the k th mode, how much and in what direction each coordinate moves relative to the others. Often, the frequency and pattern of a mode are all that are needed to answer a particular question about the system. In those cases, the step of normalizing the eigenvectors, as in eqn (10.29), is not necessary since the ratios in eqn (10.44) will be the same whether $[z^{(k)}]$ is normalized or not.

The initial conditions that will result in the single mode excitation of a particular mode k can be derived from eqn (10.36). Choose

$$r_i(0) = \alpha z_i^{(k)} \quad \text{and} \quad \dot{r}_i(0) = \beta z_i^{(k)} \quad (10.45)$$

for all i values, with α, β chosen to be small but arbitrary constants. Then, by construction,

$$[r(0)] = \alpha [z^{(k)}] \quad \text{and} \quad [\dot{r}(0)] = \beta [z^{(k)}] \quad (10.46)$$

and eqn (10.36) will give

$$A_l = [z^{(l)}] \bullet [r(0)] = [z^{(l)}] \bullet \alpha [z^{(k)}] = \alpha \delta_{lk} \quad (10.47)$$

$$B_l = \frac{1}{\omega_l} [z^{(l)}] \bullet [\dot{r}(0)] = \frac{1}{\omega_l} [z^{(l)}] \bullet \beta [z^{(k)}] = \frac{1}{\omega_k} \beta \delta_{lk} \quad (10.48)$$

which vanish for all $l \neq k$. The result is a single mode excitation of the k th mode with

$$r_i = z_i^{(k)} \left(\alpha \cos \omega_k t + \frac{\beta}{\omega_k} \sin \omega_k t \right) \quad (10.49)$$

as was desired.

Taking a case with all masses at rest at time zero as an example, a pure-mode excitation of the single mode k results when all of the masses of the system are displaced by small amounts, in exactly the pattern of the ratios given in eqn (10.44), held in those positions, and then released from rest. Then eqn (10.45) will hold for some α , and the β will be zero since all of the $\dot{r}_i(0)$ are assumed to vanish. Thus the single mode vibration of eqn (10.49) will be obtained, with $\beta = 0$.

10.10 A Simple Example

Let a rod be driven horizontally into a wall with two beads threaded onto it. Assume that there is no friction between the beads and the rod. The larger bead $m_1 = 2m$ is closer to the wall and is connected to the wall by a horizontal, massless spring of force constant $\gamma\kappa$ and rest length a . The γ here is a dimensionless constant that characterizes the relative strength of the two springs. Another massless, horizontal spring, of force constant κ and rest length a , connects m_1 to a smaller bead $m_2 = m$. Taking generalized coordinates q_1, q_2 to be the distances of the two beads from the wall, the reduced Lagrangian of the system is $L = T(\dot{q}) - U(q)$ where

$$T(\dot{q}) = \frac{1}{2} (m_1 \dot{q}_1^2 + m_2 \dot{q}_1^2) \quad (10.50)$$

$$U(q) = \frac{1}{2} \gamma \kappa (q_1 - a)^2 + \frac{1}{2} \kappa (q_2 - q_1 - a)^2 \quad (10.51)$$

Equation (10.6) becomes, for $i = 1, 2$,

$$0 = \left. \frac{\partial U(q)}{\partial q_1} \right|_{q=q^{(e)}} = \kappa \left\{ \gamma (q_1^{(e)} - a) - (q_2^{(e)} - q_1^{(e)} - a) \right\} \quad (10.52)$$

$$0 = \left. \frac{\partial U(q)}{\partial q_2} \right|_{q=q^{(e)}} = \kappa (q_2^{(e)} - q_1^{(e)} - a) \quad (10.53)$$

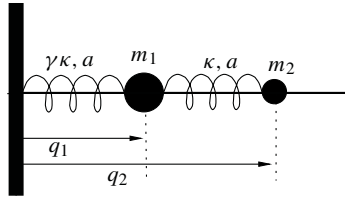


FIG. 10.2. Simple small-vibrations example.

The second of these equations implies that $q_2^{(e)} = q_1^{(e)} + a$. Putting that result in the first gives $q_1^{(e)} = a$. Thus the equilibrium point is determined by the two values

$$q_1^{(e)} = a \quad \text{and} \quad q_2^{(e)} = 2a \quad (10.54)$$

To determine the matrix \mathbf{T} , the first step is to find the required matrix elements $m_{ij}(q)$. Equating eqn (10.50) to the first term in eqn (10.1),

$$\frac{1}{2} (m_1 \dot{q}_1^2 + m_2 \dot{q}_1^2) = \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 m_{ij}(q) \dot{q}_i \dot{q}_j \quad (10.55)$$

from which one obtains $m_{ij}(q) = m_i \delta_{ij}$. Then, from eqn (10.12),

$$T_{ij} = m_{ij}(q^{(e)}) = m_i \delta_{ij} \quad (10.56)$$

and hence

$$\mathbf{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (10.57)$$

The step in eqn (10.56), evaluating $m_{ij}(q)$ at the equilibrium point $q^{(e)}$, is not really necessary in this simple example, but is included here because it *will* be necessary in more complicated cases.

The matrix \mathbf{V} is found from eqn (10.13). Thus, for $i, j = 1, 2$,

$$\begin{aligned} V_{11} &= \left. \frac{\partial^2 U(q)}{\partial q_1^2} \right|_{q=q^{(e)}} = (\gamma + 1)\kappa & V_{12} &= \left. \frac{\partial^2 U(q)}{\partial q_1 \partial q_2} \right|_{q=q^{(e)}} = -\kappa \\ V_{21} &= \left. \frac{\partial^2 U(q)}{\partial q_2 \partial q_1} \right|_{q=q^{(e)}} = -\kappa & V_{22} &= \left. \frac{\partial^2 U(q)}{\partial q_2^2} \right|_{q=q^{(e)}} = \kappa \end{aligned} \quad (10.58)$$

and hence

$$\mathbf{V} = \begin{pmatrix} (\gamma + 1)\kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix} \quad (10.59)$$

As in the determination of \mathbf{T} , the step of evaluating the second partial derivatives in eqn (10.58) at the equilibrium point $q^{(e)}$ is not necessary in this simple example, since

the second derivatives are already constants. But it is included here because it will be an essential step in more complicated examples.

Note also that the evaluation of V_{21} was not really necessary once V_{12} was found, since second partial derivatives give the same result in either order. In general problems, it will be sufficient to calculate the V_{ij} for the diagonal elements and those above the diagonal. Those below the diagonal are then determined by this symmetry of \mathbf{V} .

The eigenvalues of the generalized eigenvalue problem may now be found from eqn (10.26). First note that

$$\begin{aligned} \mathbf{V} - \theta \mathbf{T} &= \begin{pmatrix} (\gamma + 1)\kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix} - \theta \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \\ &= \kappa \left\{ \begin{pmatrix} (\gamma + 1) & -1 \\ -1 & 1 \end{pmatrix} - \frac{\theta m}{\kappa} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \kappa \begin{pmatrix} (\gamma + 1 - 2\phi) & -1 \\ -1 & (1 - \phi) \end{pmatrix} \end{aligned} \quad (10.60)$$

where we have now set $m_1 = 2m$ and $m_2 = m$ to their given values, and have defined a dimensionless quantity ϕ related to the eigenvalues θ by

$$\phi = \frac{\theta m}{\kappa} \quad \text{so that} \quad \theta = \frac{\kappa}{m} \phi \quad (10.61)$$

The eigenvalues are the two roots of the determinant equation

$$0 = |\mathbf{V} - \theta \mathbf{T}| = \kappa^2 \begin{vmatrix} (\gamma + 1 - 2\phi) & -1 \\ -1 & (1 - \phi) \end{vmatrix} \quad (10.62)$$

Since $\kappa \neq 0$ by assumption, the roots can be found by setting the determinant on the right equal to zero. The two roots are

$$\phi_1 = \frac{1}{4} \left(\gamma + 3 - \sqrt{\gamma^2 - 2\gamma + 9} \right) \quad \phi_2 = \frac{1}{4} \left(\gamma + 3 + \sqrt{\gamma^2 - 2\gamma + 9} \right) \quad (10.63)$$

Taking $\gamma = 1$ for simplicity from this point forward, the two roots become

$$\phi_1 = 1 - \frac{\sqrt{2}}{2} \quad \phi_2 = 1 + \frac{\sqrt{2}}{2} \quad (10.64)$$

and, from eqn (10.61), the actual eigenvalues are

$$\theta_1 = \frac{\kappa}{m} \left(1 - \frac{\sqrt{2}}{2} \right) \quad \theta_2 = \frac{\kappa}{m} \left(1 + \frac{\sqrt{2}}{2} \right) \quad (10.65)$$

The eigenvectors are now calculated using eqn (10.27), which may be written here, for $k = 1, 2$, as

$$\{ \mathbf{V} - \theta_k \mathbf{T} \} [z^{(k)}] = \kappa \begin{pmatrix} 2(1 - \phi_k) & -1 \\ -1 & (1 - \phi_k) \end{pmatrix} \begin{pmatrix} z_1^{(k)} \\ z_2^{(k)} \end{pmatrix} = 0 \quad (10.66)$$

where $\gamma = 1$ is still being assumed.

For $k = 1$, eqn (10.66) gives

$$\begin{pmatrix} \sqrt{2} & -1 \\ -1 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix} = 0 \quad (10.67)$$

which expands to the two equations

$$\sqrt{2}z_1^{(1)} - z_2^{(1)} = 0 \quad \text{and} \quad -z_1^{(1)} + \frac{\sqrt{2}}{2}z_2^{(1)} = 0 \quad (10.68)$$

The two equations are redundant. Solving either of them gives the ratio $z_2^{(1)}/z_1^{(1)} = \sqrt{2}$ and hence the eigenvector

$$[z^{(1)}] = N^{(1)} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad (10.69)$$

where the $N^{(k)}$ are normalization factors not yet determined.

A similar method for the mode $k = 2$ gives

$$[z^{(2)}] = N^{(2)} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad (10.70)$$

The reader should now verify that these two eigenvectors are orthogonal in the generalized sense, as required by eqn (10.30). They must satisfy

$$0 = [z^{(1)}] \bullet [z^{(2)}] = [z^{(1)}]^T \mathbb{T} [z^{(2)}] \quad (10.71)$$

using the generalized definition of inner product defined in eqn (10.28).

At this point in the calculation, we already know the frequency and pattern of each of the normal modes. In a pure mode-1 pattern, both masses would vibrate at the frequency

$$\omega_1 = \sqrt{\theta_1} = \sqrt{\frac{k}{m}} \left(1 - \frac{\sqrt{2}}{2} \right)^{1/2} \quad (10.72)$$

Equation (10.69) shows that mass m_2 would vibrate in phase with m_1 with an amplitude $\sqrt{2}$ times as great, since for all times, eqn (10.44) gives

$$\frac{r_2(t)}{r_1(t)} = \frac{z_2^{(1)}}{z_1^{(1)}} = \frac{\sqrt{2}}{1} \quad (10.73)$$

for a pure mode 1 excitation. A similar pattern results from a pure mode 2 excitation, but with $\sqrt{2}$ replaced by $-\sqrt{2}$.

As described in Section 10.9, inspection of eqn (10.69) shows that the system could be put into a pure mode-1 vibration by displacing mass m_1 by a small distance α from its equilibrium position, displacing mass m_2 by $\sqrt{2}\alpha$ in the same direction, and then releasing both masses from rest. The two masses would then continue to vibrate in the mode-1 pattern, both with frequency ω_1 .

In a pure mode-2 pattern, the higher-frequency mode here, both masses would vibrate at the frequency

$$\omega_2 = \sqrt{\theta_2} = \sqrt{\frac{k}{m} \left(1 + \frac{\sqrt{2}}{2} \right)^{1/2}} \tag{10.74}$$

eqn (10.70) shows that mass m_2 would vibrate exactly *out of phase* with m_1 with an amplitude $\sqrt{2}$ times as great. The system can be put into a pure mode-2 pattern by following a similar prescription to that just described for mode-1. Inspection of eqn (10.70) shows that a pure mode-2 vibration will be produced if one displaces mass m_1 by α and mass m_2 by $-\sqrt{2}\alpha$ in the *opposite* direction.

To find the small vibrations of the system for arbitrary initial conditions, it is necessary to find the normalization constants $N^{(k)}$ for the two modes, using eqn (10.29) with $k = 1, 2$. That calculation yields

$$N^{(1)} = N^{(2)} = \frac{1}{2\sqrt{m}} \tag{10.75}$$

The equality of these two factors is because of the simplicity of this example. In more complex problems, the normalizing factors will not generally be the same for the different modes.

As an example of a solution involving a particular initial condition, suppose that at time zero the mass m_2 is displaced by a small distance α while the mass m_1 is held fixed at its equilibrium position and not allowed to move. Then both masses are released from rest. Thus, the column vectors of initial conditions to be used in eqn (10.36) are

$$[r(0)] = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \quad \text{and} \quad [\dot{r}(0)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10.76}$$

with the results that $B_1 = B_2 = 0$ and

$$A_1 = [z^{(1)}] \bullet [r(0)] = N^{(1)} (1 \ \sqrt{2}) \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \frac{\alpha\sqrt{2m}}{2} \tag{10.77}$$

$$A_2 = [z^{(2)}] \bullet [r(0)] = N^{(2)} (1 \ -\sqrt{2}) \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = -\frac{\alpha\sqrt{2m}}{2} \tag{10.78}$$

The general solution from eqn (10.38) is then

$$\begin{aligned} [r] &= [z^{(1)}]A_1 \cos \omega_1 t + [z^{(2)}]A_2 \cos \omega_2 t \\ &= \frac{\alpha\sqrt{2}}{4} \left\{ \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos \omega_1 t - \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos \omega_2 t \right\} \end{aligned} \tag{10.79}$$

In terms of components, this is

$$r_1 = \frac{\alpha\sqrt{2}}{4} (\cos \omega_1 t - \cos \omega_2 t) \tag{10.80}$$

$$r_2 = \frac{\alpha}{2} (\cos \omega_1 t + \cos \omega_2 t) \tag{10.81}$$

For this initial condition, both normal modes are excited simultaneously. The resulting

motion can look quite complicated. Only two different frequencies of vibration are present, but neither periodicity may be obvious to the untrained eye. Certainly, pure mode vibrations are prettier and less interesting.

10.11 Zero-Frequency Modes

In the simple example of Section 10.10, the parameter γ that gives the strength of the first spring relative to the second one was chosen to be one. Here, let us consider a different choice. What if the first spring is decreased in strength until $\gamma = 0$. Then the pair of masses would be in a sense floating freely, disconnected from the wall.

The first thing one would notice about such a problem is that the two eqns (10.52, 10.53) would become redundant. They would both determine that $q_2^{(e)} = q_1^{(e)} + a$, but they would not determine the value of $q_1^{(e)}$. Physically, it is obvious why this is so. If the two masses are placed at rest anywhere on the rod, but separated by distance a equal to the rest length of the remaining spring, they will stay there forever. Any arbitrary $q_1^{(e)}$ choice then leads to a valid equilibrium point.

If we set $\gamma = 0$ in eqn (10.63), the two eigenvalues become

$$\theta_1 = 0 \quad \text{and} \quad \theta_2 = \frac{3k}{2m} \quad (10.82)$$

and the eigenvectors can be calculated by the same technique as in Section 10.10. They are

$$[z^{(1)}] = N^{(1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad [z^{(2)}] = N^{(2)} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (10.83)$$

where of course the $N^{(k)}$ normalization factors will be found to be different from those in Section 10.10.

The pattern of the zero-frequency mode is seen from the first of eqn (10.83) to be simply a translation of both masses without changing the distance between them. The pattern of mode-2 can be shown to be a vibration with the center of mass of the two masses fixed. The small vibrations method has automatically separated the motion into the same collective and internal form that was seen in Chapter 1.

The general solution from eqn (10.38) will now be

$$[r] = [z^{(1)}] (A_1 + B_1 t) + [z^{(2)}] (A_2 \cos \omega_2 t + B_2 \sin \omega_2 t) \quad (10.84)$$

where $\omega_2 = \sqrt{3k/2m}$ and where the A_k, B_k will be determined using the same methods as in Section 10.10, but now using the eigenvectors from eqn (10.83).

It is characteristic of systems with zero-frequency modes that the equilibrium points turn out to be under determined, as in this example. The condition for the equilibrium point to be uniquely determined is that the matrix \mathbf{V} must be nonsingular. But if the problem has n zero-frequency modes, that matrix will have rank $(D - n)$, and n of the components $q_i^{(e)}$ will have to be chosen arbitrarily. In the present example, one can verify that the choice $\gamma = 0$ does make the matrix \mathbf{V} in eqn (10.59) become singular.

Notice that the zero-frequency mode with $B_1 \neq 0$ in eqn (10.84) would lead to displacements of the r_i that become infinitely large. This would violate the condition that the r_i must be small. But the small vibrations theory can still give useful results, even in the presence of zero-frequency modes, provided that the B_k are all zero. For example, if all masses are taken to be at rest at time zero, then eqns (10.36, 10.37) imply that $B_k = 0$ for all k values.

10.12 Exercises

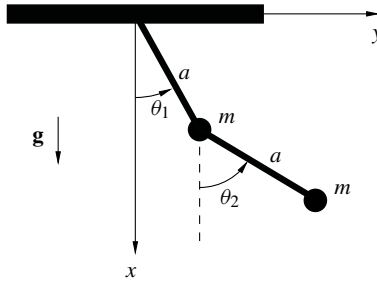


FIG. 10.3. Illustration for Exercise 10.1.

Exercise 10.1 Consider the double pendulum from Exercise 3.1. Assume that the sticks both have length a and the masses have the same value m .

- Write the reduced Lagrangian with free variables $q_1 = \theta_1$ and $q_2 = \theta_2$.
- Find values $q_k^{(e)}$ that give a stable equilibrium point.
- Find the constant matrices \mathbf{T} and \mathbf{V} used to write the Lagrangian of small vibrations in terms of small displacements.
- Find the normal frequencies and the associated column vectors $[z^{(1)}]$ and $[z^{(2)}]$. Draw diagrams showing the pattern of vibration for each of the two normal modes.
- Suppose that mass m_2 is displaced by small angle α at time zero while mass m_1 is held at its equilibrium point. The masses are released from rest. Show that there is no time t at which the masses will have these same positions again while at rest.

Exercise 10.2 Suppose that two equal masses $m_1 = m_2 = m$, where m is a given constant, are constrained to move along the centerline of a narrow, frictionless, horizontal, circular-toroidal tube of given constant radius R_0 . The two masses are connected by elastic cords that are also constrained to lie along the centerline of the tube. Assume that the toroidal tube is very narrow, so that the masses and cords are all at radius R_0 . Assume that there is no friction between the cords and the walls of the tube, and that the cords behave as ideal springs of zero rest length. That is, assume that the tension τ in each cord is $\tau = ks$ where k is the force constant of that cord and s is its length. Assume that the two cords have force constants $k_1 = 3k$ and $k_2 = k$, where k is a given constant.

- Find the equilibrium angles $\theta_1^{(0)}$ and $\theta_2^{(0)}$.
- Find the constant matrices \mathbf{T} and \mathbf{V} used to write the Lagrangian of small vibrations in

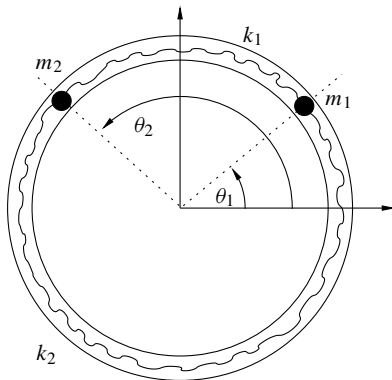


FIG. 10.4. Illustration for Exercise 10.2.

terms of small displacements.

- (c) Find the frequencies of the normal modes of small vibration about equilibrium.
- (d) For each frequency, find the un-normalized column vector that represents the pattern of vibration in that mode. Show by a diagram and a sentence or two exactly what the pattern of each mode is.

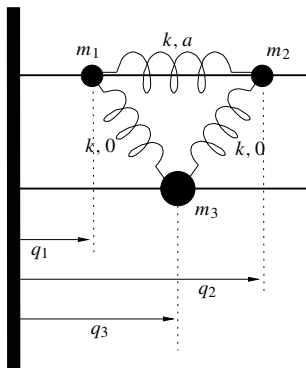


FIG. 10.5. Illustration for Exercise 10.3.

Exercise 10.3 Three beads, connected by massless springs, slide without friction on two horizontal, rigid wires whose vertical separation is D . Ignore gravity, which plays no role here. The two upper beads have $m_1 = m_2 = m$ and the lower bead has mass $m_3 = 2m$, where m is some given constant. All springs have force constant k . The two diagonal springs have zero rest length, while the horizontal one has rest length a . Assume that the springs remain in straight lines between the masses.

- (a) Taking the distances q_1, q_2, q_3 from the wall as your generalized coordinates, write the Lagrangian for this problem.
- (b) Find the equilibrium values $q_1^{(e)}, q_2^{(e)}, q_3^{(e)}$ for the coordinates. You may take $q_1^{(e)}$ as arbitrary.

trary, and derive the other two in terms of it.

(c) Write the matrices \mathbf{T} and \mathbf{V} for the small vibrations problem.

(d) Find the frequencies ω_k for the three normal modes.

(e) Solve for the eigenvectors $[z^{(k)}]$ corresponding to these normal modes. Find the normalization constants for these eigenvectors, and check that they are orthonormal in the sense $[z^{(k)}] \bullet [z^{(l)}] = \delta_{kl}$.

(f) Suppose that at time zero all masses are at rest. Masses m_1 and m_3 are at their equilibrium positions, but m_2 is displaced by a small amount α from its equilibrium position. Find the constants $A_1, A_2, A_3, B_1, B_2, B_3$ and write an expression for each small coordinate r_1, r_2, r_3 as a function of time for all $t > 0$.

Exercise 10.4 Suppose I have a small vibrations problem whose eigenvectors are the column vectors $[z^{(k)}]$. I want to excite *only* the l th mode so that all the masses and objects vibrate with the *single* frequency ω_l . How can I do that? Explain your answer using the formalism developed in the text.

Exercise 10.5 Show that a pure mode vibration with the pattern given by the second of eqn (10.83) would produce a motion with the center of mass at rest.

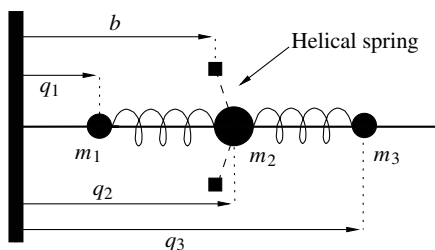


FIG. 10.6. Figure for Exercise 10.6.

Exercise 10.6 Three beads of mass $m_1 = m_3 = m$ and $m_2 = 2m$ slide without friction on a horizontal wire. They are connected by two massless springs of force constant k and rest length a . A helical spring of force constant γk and zero rest length is attached to the middle mass. The distances of the masses from the wall are q_1, q_2 , and q_3 as shown. The center point of the helical spring is distance b from the wall.

(a) Taking q_1, q_2 , and q_3 as your generalized coordinates, write the Lagrangian for this system.

(b) Find the equilibrium values $q_1^{(e)}, q_2^{(e)}, q_3^{(e)}$ for the three masses.

(c) Find the \mathbf{T} and \mathbf{V} matrices and use them to solve for the frequencies of the normal modes.

(d) Find the un-normalized eigenvectors of the normal modes. Find the frequencies and un-normalized eigenvectors for the specific case when $\gamma = 3$. Draw a rough diagram showing the pattern of each pure mode in this case.

(e) Show that in the limit $\gamma \rightarrow 0$ one of the normal mode frequencies goes to zero. Show that the pattern for that mode becomes a rigid-body translation of the system, with the inter-mass distances fixed. Show that in the $\gamma \rightarrow 0$ limit each of the other two modes leaves the center of mass at rest.

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Part II

Mechanics with Time as a Coordinate

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LAGRANGIAN MECHANICS WITH TIME AS A COORDINATE

In traditional analytical mechanics, it is assumed that the system traces out a path (one-dimensional curved line) in a D dimensional configuration space that is defined by writing each of the generalized coordinates as a function of the absolute, Newtonian time. Thus, for all $k = 1, \dots, D$, one writes the equation of motion as $q_k = q_k(t)$. This is the approach that was used throughout Part I of the book when traditional Lagrangian and Hamiltonian mechanics were introduced.

But more advanced topics, such as canonical transformations and Hamilton–Jacobi theory, are simplified if Lagrangian and Hamiltonian theory are recast in what we will call an *extended* form. The time becomes the zeroth generalized coordinate $q_0 = t$, and its conjugate momentum p_0 becomes the negative of the traditional generalized energy function H . There are then $(D + 1)$ *extended Lagrange equations*, the extra Lagrange equation being equivalent to the traditional generalized energy theorem.

The extended theory thus combines the traditional Lagrange equations and generalized energy theorem into a single set of equations, and restores the symmetry of the mathematical system in the sense discussed in Chapter 5. The traditional Lagrangian methods are analogous to the “coordinate parametric method” in the calculus of variations, described in Section 5.14. The extended Lagrangian theory is analogous to the recommended “general parametric method” presented from the beginning of Chapter 5. The generalized energy theorem of traditional Lagrangian theory, which had been a separate equation analogous to the “second form” of the Euler–Lagrange equations derived in Theorem 5.14.3, gets restored to its proper place as just another of the extended Lagrange equations, which now form a complete set of equations appropriate to the problem.

Besides the simplifications mentioned above, there is another motivation for use of the extended theory. It is now some hundred years since Einstein’s 1905 relativity papers. Special relativity is now an accepted part of classical mechanics. The use of time as a coordinate is essential so that, for example, the Lorentz transformation will not be excluded when we define canonical transformations. The extended Lagrangian theory of the present chapter will be essential in Chapter 16 where covariant Lagrangian mechanics is discussed.

However, consistency with special relativity is only one argument for use of the extended theory. It could as well have been introduced in the nineteenth century, before relativity, motivated by the mathematical elegance it brings to the treatment of canonical transformations and other advanced topics.

11.1 Time as a Coordinate

Time is to be removed from its role as a universal background parameter, and elevated to the status of a coordinate with $q_0 = t$. When time is a coordinate, the path of the system must be specified by introducing a new background parameter, which will be denoted by β . This parameter is not specified initially, except for the condition that it must be monotonically varying along the path of the system. Since time always advances, this condition is equivalent to requiring that $d\beta/dt$ is always finite and nonzero. Thus, the system traces out a path in an augmented ($D + 1$) dimensional configuration space defined, for all $k = 0, \dots, D$, by writing the equations of motion as $q_k = q_k(\beta)$, including $q_0 = q_0(\beta)$ which is the same as $t = t(\beta)$.

The idea of the extended method is that the monotonic parameter β is not to be specified until the end of a calculation, after all partial derivatives have been taken and the final differential equations of motion have been derived. Then it can be chosen at will, using whatever definition will make those differential equations simple. This eventual choice of β will depend on the nature of the mechanical system being studied. For example, one possible choice is just to set β equal to the Newtonian time t at the end. But, in special relativity, another possible (and covariant) choice is to set β equal to the proper time along the world line of some particle.

11.2 A Change of Notation

From this point forward in the book, we will make a notational change and use the dot over a quantity to denote its total derivative with respect to β rather than t , so that $\dot{q}_k = dq_k/d\beta$, including the case $k = 0$ for which $\dot{i} = dt/d\beta = dq_0/d\beta = \dot{q}_0$. We will continue to refer to \dot{q}_k as a generalized velocity, just as we did in the traditional Lagrangian theory, even though (as there) its units may not always be distance divided by time. Derivatives with respect to time, when needed, will either be written out explicitly or denoted by a prime. For example, the chain rule for total derivatives gives the relations

$$q'_k = \frac{dq_k}{dt} = \frac{\dot{q}_k}{\dot{i}} = \frac{\dot{q}_k}{\dot{q}_0} \quad (11.1)$$

In this and subsequent chapters, the notation of using a single unsubscripted variable to stand for a whole set of variables will be modified to include q_0 in the sets. Thus $q = q_0, q_1, q_2, \dots, q_D$ is a set of ($D + 1$) coordinates. And $\dot{q} = \dot{q}_0, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_D$ is the set of ($D + 1$) generalized velocities.

Another new notation will be $q_{[k]}$ to denote all of the ($D + 1$) variables *except* q_k . This same notation will also be applied to other sets, such as generalized velocities or momenta. Thus, for example,

$$\dot{q}_{[3]} = \dot{q}_0, \dot{q}_1, \dot{q}_2, \dot{q}_4, \dot{q}_5, \dots, \dot{q}_D \quad \text{and} \quad q_{[0]} = q_1, q_2, q_3, q_4, q_5, \dots, q_D \quad (11.2)$$

This notation will make it easier to compare the extended Lagrangian function to the traditional one.⁵¹

⁵¹The notations introduced here are similar to those used for the calculus of variations in Chapter 5.

11.3 Extended Lagrangian

The traditional Lagrangian defined in Chapter 2 can be written in the new notation as

$$L = L(q_{[0]}, q'_{[0]}, t) \quad (11.3)$$

where $q_{[0]}$ is the set defined in eqn (11.2), and $q'_{[0]}$ denotes the set of traditional generalized velocities

$$q'_{[0]} = q'_1, q'_2, \dots, q'_D = \frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots, \frac{dq_D}{dt} \quad (11.4)$$

Motivation for selecting the form of the extended Lagrangian can be found in the action integral of Hamilton's Principle. The definition in eqn (6.4) can be rewritten using β as the integration parameter instead of t . Introducing the notation of the present chapter, the action integral becomes

$$I = \int_{t_1}^{t_2} L(q_{[0]}, q'_{[0]}, t) dt = \int_{\beta_1}^{\beta_2} L\left(q_{[0]}, \frac{\dot{q}_{[0]}}{i}, t\right) i d\beta \quad (11.5)$$

where eqn (11.1) was used to replace $q'_{[0]}$ by the set

$$q'_{[0]} = \frac{\dot{q}_{[0]}}{i} = \frac{\dot{q}_1}{i}, \frac{\dot{q}_2}{i}, \dots, \frac{\dot{q}_D}{i} \quad (11.6)$$

where now $\dot{q}_k = dq_k/d\beta$. Equation (11.5) suggests the following definition.

Definition 11.3.1: Extended Lagrangian

Starting from the traditional Lagrangian in eqn (11.3), the extended Lagrangian \mathcal{L} is defined by

$$\mathcal{L}(q, \dot{q}) = iL(q_{[0]}, q'_{[0]}, t) = iL\left(q_{[0]}, \frac{\dot{q}_{[0]}}{i}, t\right) = \dot{q}_0 L\left(q_{[0]}, \frac{\dot{q}_{[0]}}{\dot{q}_0}, q_0\right) \quad (11.7)$$

where the last expression in eqn (11.7) introduces the definition $q_0 = t$.

This same definition can also be motivated by examining the prescription in eqn (5.111) for converting from the "coordinate parametric" variational method to the "general parametric" one in the calculus of variations. The identifications $g \rightarrow L$, $x_1 \rightarrow t$, $x_{[1]} \rightarrow q$, $f \rightarrow \mathcal{L}$ lead at once to eqn (11.7).

An important property of $\mathcal{L}(q, \dot{q})$ is that, by construction, it is homogeneous of degree one in the set of generalized velocities \dot{q} . We state this as a lemma for future reference.

Lemma 11.3.2: Homogeneity of Extended Lagrangian

The extended Lagrangian $\mathcal{L}(q, \dot{q})$ is homogeneous of degree one in the set of generalized velocities \dot{q} .

Proof: Let us denote by $\lambda\dot{q}$ the set $\lambda\dot{q}_0, \lambda\dot{q}_1, \dots, \lambda\dot{q}_D$ in which each generalized velocity is multiplied by the same nonzero number λ . It follows from the definition in eqn (11.7) that

$$\mathcal{L}(q, \lambda\dot{q}) = \lambda\dot{q}_0 L\left(q_{[0]}, \frac{\lambda\dot{q}_{[0]}}{\lambda\dot{q}_0}, q_0\right) = \lambda\mathcal{L}(q, \dot{q}) \quad (11.8)$$

since the λ factors cancel from the terms inside L . Using the definition in Section D.31, this implies that $\mathcal{L}(q, \dot{q})$ is homogeneous of degree one in the set $\dot{q} = \dot{q}_0, \dot{q}_1, \dots, \dot{q}_D$. \square

11.4 Extended Momenta

The generalized momenta in the extended theory are defined as

$$p_k = p_k(q, \dot{q}) = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \quad (11.9)$$

for $k = 0, \dots, D$. This definition is similar to eqn (2.68) for the traditional Lagrangian theory, except that \dot{q}_k now denotes differentiation with respect to β , and an additional momentum p_0 has been added.

For $k \neq 0$, the generalized momenta defined in eqn (11.9) are the *same functions* as the traditional momenta defined in eqn (2.68), simply re-expressed in terms of the new variables q, \dot{q} . Applying the definition in eqn (11.9) to eqn (11.7) gives, for $k \neq 0$,

$$\begin{aligned} p_k(q, \dot{q}) &= \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} = \dot{q}_0 \sum_{l=1}^D \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_l} \frac{\partial q'_l}{\partial \dot{q}_k} \\ &= \dot{q}_0 \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \frac{1}{\dot{q}_0} = \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \end{aligned} \quad (11.10)$$

where eqn (11.1) has been used to write $\partial q'_l / \partial \dot{q}_k = \delta_{kl}(1/\dot{q}_0)$. The last expression on the right of eqn (11.10) is exactly the traditional momentum defined in eqn (2.68). In the notation of the present chapter, that equation is

$$p_k(q_{[0]}, q'_{[0]}, t) = \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \quad (11.11)$$

Thus, for $k \neq 0$,

$$\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} = p_k(q, \dot{q}) = p_k(q_{[0]}, q'_{[0]}, t) = \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \quad (11.12)$$

as was asserted.

For $k = 0$, the momentum defined by eqn (11.9) is the negative of the traditional generalized energy function H defined in Section 2.15. Applying the definition in eqn

(11.9) to eqn (11.7) and using eqn (11.11) gives

$$\begin{aligned}
 p_0(q, \dot{q}) &= \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_0} = L(q_{[0]}, q'_{[0]}, t) + \dot{q}_0 \sum_{l=1}^D \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_l} \frac{\partial q'_l}{\partial \dot{q}_0} \\
 &= L(q_{[0]}, q'_{[0]}, t) - \dot{q}_0 \sum_{l=1}^D p_l(q_{[0]}, q'_{[0]}, t) \frac{\dot{q}_l}{\dot{q}_0^2} \\
 &= L(q_{[0]}, q'_{[0]}, t) - \sum_{l=1}^D p_l(q_{[0]}, q'_{[0]}, t) q'_l = -H(q_{[0]}, q'_{[0]}, t) \quad (11.13)
 \end{aligned}$$

where $H(q_{[0]}, q'_{[0]}, t)$ is the traditional generalized energy function defined in Section 2.15, rewritten in the notation of the present chapter. Thus

$$\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_0} = p_0(q, \dot{q}) = -H(q_{[0]}, q'_{[0]}, t) \quad (11.14)$$

as was asserted.

We conclude this section with two more lemmas of importance.

Lemma 11.4.1: Homogeneity of Momenta

The generalized momenta $p_k(q, \dot{q})$ are all homogeneous of degree zero in the generalized velocities.

Proof: Since Lemma 11.3.2 proved that $\mathcal{L}(q, \dot{q})$ is homogeneous of degree one in the generalized velocities, it follows from Theorem D.32.1 that the partial derivatives in eqn (11.9) must be homogeneous of degree zero. Thus it must be true that, using the same notation $\lambda \dot{q}$ as used in the proof of Lemma 11.3.2,

$$p_k(q, \lambda \dot{q}) = \lambda^0 p_k(q, \dot{q}) = p_k(q, \dot{q}) \quad (11.15)$$

This fact can also be seen by inspection of eqns (11.12, 11.14), noting that the expressions on the right side of these equations contain generalized velocities only as ratios like $q'_k = \dot{q}_k/\dot{q}_0$ from which the λ would cancel. □

Lemma 11.4.2: Extended Lagrangian and Momenta

The extended Lagrangian can be written in terms of the generalized velocities and momenta as

$$\mathcal{L}(q, \dot{q}) = \sum_{k=0}^D p_k(q, \dot{q}) \dot{q}_k \quad (11.16)$$

Proof: Since Lemma 11.3.2 proved that $\mathcal{L}(q, \dot{q})$ is homogeneous of degree one in the \dot{q} , the Euler Condition of Theorem D.31.1 applied to the present case implies that

$$\mathcal{L}(q, \dot{q}) = \sum_{k=0}^D \dot{q}_k \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \quad (11.17)$$

The theorem then follows from the definition in eqn (11.9). □

11.5 Extended Lagrange Equations

The extended Lagrange equations combine the traditional Lagrange equations and generalized energy theorem into one set of equations. We first state the extended Lagrange equations, and then prove their equivalence to the traditional formulas.

Definition 11.5.1: Extended Lagrange Equations

The extended Lagrange equations are, for $k = 0, \dots, D$,

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = Q_k^{(\text{NP})} \quad (11.18)$$

where

$$Q_k^{(\text{NP})} = i Q_k^{(\text{NP})} \quad Q_0^{(\text{NP})} = - \sum_{l=1}^D Q_l^{(\text{NP})} \dot{q}_l \quad (11.19)$$

for $k = 1, \dots, D$, where $Q_k^{(\text{NP})}$ are the non-potential generalized forces introduced in eqn (2.37).

When there are no non-potential forces, then the extended Lagrange equations in eqn (11.18) of course reduce to the standard form

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = 0 \quad (11.20)$$

The equivalence of eqn (11.18) to the traditional Lagrange equations and generalized energy theorem is proved in the following theorem.

Theorem 11.5.2: Equivalence of Extended Lagrange Equations

The extended Lagrange equations for $k \neq 0$ are equivalent to the traditional Lagrange equations, eqn (2.52). The extended Lagrange equation for $k = 0$ is equivalent to the generalized energy theorem, eqn (2.78).

Proof: First consider the case $k \neq 0$. The traditional Lagrange equation from Section 2.9 can be written in our present notation as

$$\frac{d}{dt} \left(\frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \right) - \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q_k} = Q_k^{(\text{NP})} \quad (11.21)$$

Multiplying through by $i = dt/d\beta$ and using eqn (11.19) then gives

$$\frac{d}{d\beta} \left(\frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} \right) - i \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q_k} = Q_k^{(\text{NP})} \quad (11.22)$$

But, from eqn (11.12)

$$\frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial q'_k} = p_k(q_{[0]}, q'_{[0]}, t) = p_k(q, \dot{q}) = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \quad (11.23)$$

Thus definition eqn (11.7) may be used to write eqn (11.22) as

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \mathcal{Q}_k^{(\text{NP})} \tag{11.24}$$

which is eqn (11.18) for the case $k \neq 0$, as was to be proved.

For the case $k = 0$, the generalized energy theorem, eqn (2.78) of Theorem 2.15.1, can be written in our present notation as

$$\frac{dH(q_{[0]}, q'_{[0]}, t)}{dt} = -\frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial t} + \sum_{l=1}^D \mathcal{Q}_l^{(\text{NP})} q'_l \tag{11.25}$$

Multiplying through by $i = dt/d\beta$ and using eqn (11.19) then gives

$$\frac{dH(q_{[0]}, q'_{[0]}, t)}{d\beta} = -i \frac{\partial L(q_{[0]}, q'_{[0]}, t)}{\partial t} - \mathcal{Q}_0^{(\text{NP})} \tag{11.26}$$

But eqn (11.14) gives

$$-H(q_{[0]}, q'_{[0]}, t) = p_0(q, \dot{q}) = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_0} \tag{11.27}$$

Thus, using definition eqn (11.7) and multiplying through by minus one, equation eqn (11.26) becomes

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_0} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_0} = \mathcal{Q}_0^{(\text{NP})} \tag{11.28}$$

which is eqn (11.18) for the case $k = 0$, as was to be proved. □

11.6 A Simple Example

The transition from traditional to extended Lagrange theory can be illustrated starting from the Lagrangian in eqn (2.66). In our current notation, it is

$$L(q_{[0]}, q'_{[0]}, t) = \frac{1}{2}m \left(r'^2 + r^2 \theta'^2 + r^2 \sin^2 \theta \phi'^2 \right) - \frac{1}{2}kr^2 \tag{11.29}$$

Following the recipe in eqn (11.7), each time derivative $q'_k = dq_k/dt$ is to be replaced by \dot{q}_k/i and the resulting equation is to be multiplied by i . Thus

$$\begin{aligned} \mathcal{L}(q, \dot{q}) &= i \left\{ \frac{1}{2}m \left(\left(\frac{\dot{r}}{i} \right)^2 + r^2 \left(\frac{\dot{\theta}}{i} \right)^2 + r^2 \sin^2 \theta \left(\frac{\dot{\phi}}{i} \right)^2 \right) - \frac{1}{2}kr^2 \right\} \\ &= \frac{m}{2i} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{i}{2}kr^2 \end{aligned} \tag{11.30}$$

The generalized momenta are defined in eqn (11.9). In this example, they are

$$\begin{aligned} \text{For } k = 0 \quad p_0 &= \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{t}} = -\frac{m}{2\dot{t}^2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - \frac{1}{2} k r^2 \\ &= -\left\{ \frac{1}{2} m \left(r'^2 + r^2 \theta'^2 + r^2 \sin^2 \theta \phi'^2 \right) + \frac{1}{2} k r^2 \right\} \end{aligned} \quad (11.31)$$

$$\text{For } k = 1 \quad p_r = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{r}} = m \frac{\dot{r}}{\dot{t}} = m r' \quad (11.32)$$

$$\text{For } k = 2 \quad p_\theta = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\theta}} = m \frac{r^2 \dot{\theta}}{\dot{t}} = m r^2 \theta' \quad (11.33)$$

$$\text{For } k = 3 \quad p_\phi = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\phi}} = m \frac{r^2 \sin^2 \theta \dot{\phi}}{\dot{t}} = m r^2 \sin^2 \theta \phi' \quad (11.34)$$

where eqn (11.1) has been used to write the last expression on the right in each case. Note that, for $k \neq 0$, these generalized momenta are indeed just the same as the traditional ones that would be derived from eqn (2.66) using the traditional definition in eqn (2.68). And for $k = 0$, the p_0 is just the negative of the traditional generalized energy function H that would be derived from eqn (2.76) for this example.

The extended Lagrange equations, eqn (11.18), are

$$\begin{aligned} \text{For } k = 0 \quad \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{t}} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial t} &= 0 \\ \text{or} \quad \frac{d}{d\beta} \left\{ \frac{m}{2\dot{t}^2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{1}{2} k r^2 \right\} &= 0 \end{aligned} \quad (11.35)$$

$$\begin{aligned} \text{For } k = 1 \quad \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial r} &= 0 \\ \text{or} \quad \frac{d}{d\beta} \left(\frac{m \dot{r}}{\dot{t}} \right) - \frac{m r \dot{\theta}^2}{\dot{t}} - \frac{m r \sin^2 \theta \dot{\phi}^2}{\dot{t}} + i k r &= 0 \end{aligned} \quad (11.36)$$

$$\begin{aligned} \text{For } k = 2 \quad \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \theta} &= 0 \\ \text{or} \quad \frac{d}{d\beta} \left(\frac{m r^2 \dot{\theta}}{\dot{t}} \right) - \frac{m r^2 \sin \theta \cos \theta \dot{\phi}}{\dot{t}} &= 0 \end{aligned} \quad (11.37)$$

$$\text{For } k = 3 \quad \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \phi} = 0 \quad \text{or} \quad \frac{d}{d\beta} \left(\frac{m r^2 \sin^2 \theta \dot{\phi}}{\dot{t}} \right) = 0 \quad (11.38)$$

By repeated use of eqn (11.1), the equations with $k = 1, 2, 3$ can be shown to be identical to the corresponding eqn (2.67). The equation with $k = 0$ is identical to the generalized energy theorem eqn (2.78) applied to this example. Note that both t and ϕ are ignorable coordinates, in the sense described in Section 2.13. Thus their conjugate momenta p_0 and p_ϕ , respectively, are constants of the motion. In the case of p_0 , this implies the constancy of the generalized energy function $H = -p_0$.

11.7 Invariance Under Change of Parameter

The choice of parameter β might seem to introduce an arbitrary element into mechanics. The whole development above could as well be done with some other monotonic parameter, say θ . But, although the generalized velocities and the extended Lagrangian would then be different, the resulting Lagrange equations would have the same form as the ones derived above and would be equivalent to them. This equivalence of form and content is often expressed by saying that the extended Lagrange equations are *form invariant* under a change of parameter. So the arbitrariness is only apparent.

Suppose that an alternate monotonic parameter $\theta = \theta(\beta)$ is introduced, with $d\theta/d\beta \neq 0$. Denoting $\tilde{q}_k = dq_k/d\theta$, the generalized velocities with the two different parameter choices are related by

$$\dot{q}_k = \tilde{q}_k \frac{d\theta}{d\beta} \quad \text{or, equivalently,} \quad \tilde{q}_k = \dot{q}_k \frac{d\beta}{d\theta} \quad (11.39)$$

Even though the generalized velocities and the Lagrangians are different, the generalized momenta p_k defined in eqn (11.9) will be the same no matter what parameter is chosen. Since, by Lemma 11.4.1, the generalized momenta are homogeneous of degree zero in the generalized velocities, the replacement of β by θ will not change them. From eqn (11.15) with $\lambda = d\theta/d\beta$,

$$p_k(q, \dot{q}) = p_k \left(q, \tilde{q} \frac{d\theta}{d\beta} \right) = \left(\frac{d\theta}{d\beta} \right)^0 p_k(q, \tilde{q}) = p_k(q, \tilde{q}) \quad (11.40)$$

This equality of these two momenta can also be derived from the work in Section 11.4, which proves that both of these momenta are equal to the traditional momenta and generalized energy functions, which are obviously independent of the choice of β or θ .

The Lagrange equations with parameter θ are

$$\frac{d}{d\theta} \left(\frac{\partial \tilde{\mathcal{L}}(q, \tilde{q})}{\partial \tilde{q}_k} \right) - \frac{\partial \tilde{\mathcal{L}}(q, \tilde{q})}{\partial q_k} = \tilde{\mathcal{Q}}_k^{(NP)} \quad (11.41)$$

where, using eqn (11.7), the Lagrangian $\tilde{\mathcal{L}}$ in terms of parameter θ is now defined in the same way as \mathcal{L} was in terms of β ,

$$\tilde{\mathcal{L}}(q, \tilde{q}) = \tilde{t}L(q_{[0]}, q'_{[0]}, t) = \tilde{t}L \left(q_{[0]}, \frac{\tilde{q}_{[0]}}{\tilde{t}}, t \right) = \tilde{t}L \left(q_{[0]}, \frac{\tilde{q}_{[0]}}{\tilde{q}_0}, q_0 \right) \quad (11.42)$$

where we used $\tilde{q}_k/\tilde{q}_0 = q'_k$ to get the second equality.

The Lagrange equations in eqn (11.41) are equivalent to the ones in eqn (11.18) in the following sense.

Theorem 11.7.1: Invariance Under Parameter Change

Functions $q_k(\theta)$ are a solution to eqn (11.41) if and only if $q_k(\beta) = q_k(\theta(\beta))$ are a solution to eqn (11.18).

Proof: Using the definitions in eqns (11.7, 11.9), the Lagrange equations, eqn (11.41), may be written as

$$\frac{dp_k(q, \tilde{q})}{d\theta} - \tilde{t} \frac{\partial L(q_{[0]}, q'_{[0]}, q_0)}{\partial q_k} = \tilde{Q}_k^{(\text{NP})} \quad (11.43)$$

Multiplying through by $d\theta/d\beta$ and using eqn (11.40), this becomes

$$\frac{dp_k(q, \dot{q})}{d\beta} - i \frac{\partial L(q_{[0]}, q'_{[0]}, q_0)}{\partial q_k} = Q_k^{(\text{NP})} \quad (11.44)$$

which can be rewritten as

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = Q_k^{(\text{NP})} \quad (11.45)$$

which is eqn (11.18). Thus, $q_k = q_k(\theta)$ will reduce the left side of eqn (11.41) to zero if and only if $q_k = q_k(\beta) = q_k(\theta(\beta))$ reduces the left side of eqn (11.45) to zero, as was to be proved. \square

11.8 Change of Generalized Coordinates

The extended Lagrange equations are also form invariant under coordinate transformations with β unchanged, including a larger class of Lagrangian coordinate transformations than those considered in Chapter 2. The advantage of the extended theory is that, since time is now a coordinate, it is allowed to transform in the same way as the other generalized coordinates. The form invariance of the extended Lagrange equations is stated in the following theorem.

Theorem 11.8.1: Invariance Under Coordinate Change

Let new generalized coordinates r_0, r_1, \dots, r_D be defined, for all $k = 0, \dots, D$, by

$$q_k = q_k(r_0, r_1, \dots, r_D) \quad (11.46)$$

where the following $(D+1) \times (D+1)$ Jacobian determinant condition is assumed to hold

$$\left| \frac{\partial q}{\partial r} \right| \neq 0 \quad \text{where} \quad \left(\frac{\partial q}{\partial r} \right)_{kl} = \frac{\partial q_k(r)}{\partial r_l} \quad (11.47)$$

for all $k, l = 0, \dots, D$. Let the extended Lagrangian in the r -system be the same function as $\mathcal{L}(q, \dot{q})$, but expressed in terms of the r -system coordinates and generalized velocities

$$\mathcal{L}(r, \dot{r}) = \mathcal{L}(q(r), \dot{q}(r, \dot{r})) \quad (11.48)$$

Also let the generalized non-potential forces in the r -system be defined by

$$\mathcal{R}_l^{(\text{NP})} = \sum_{k=0}^D Q_k^{(\text{NP})} \frac{\partial q_k(r)}{\partial r_l} \quad (11.49)$$

Then the Lagrange equations in the q -system

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \mathcal{Q}_k^{(NP)} \tag{11.50}$$

hold if and only if equations of the same form hold in the r -system

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(t, \dot{r})}{\partial \dot{r}_k} \right) - \frac{\partial \mathcal{L}(t, \dot{r})}{\partial r_k} = \mathcal{R}_k^{(NP)} \tag{11.51}$$

Proof: The proof of this theorem is very similar to the proofs of the invariance of the traditional Lagrange equations in Sections 2.9 and 2.10, but with the addition of coordinate q_0 and the substitution of β for t as the parameter. It will not be repeated here. □

The traditional Lagrangian coordinate transformations treated in Chapter 2 are included in the extended transformation theory as a special case. If $q_0 = r_0$ is assumed, then the Jacobian condition eqn (11.47) becomes identical to the condition in eqn (2.59).

The most important example of the new extended transformation theory is of course the Lorentz transformation of special relativity. If we define $q_0 = t$, $q_1 = x$, $q_2 = y$, $q_3 = z$ and $r_0 = t'$, $r_1 = x'$, $r_2 = y'$, $r_3 = z'$, then the standard Lorentz transformation

$$t = \Gamma \left(t' + \frac{V}{c^2} x' \right) \quad x = \Gamma (V t' + x') \quad y = y' \quad z = z' \tag{11.52}$$

satisfies eqn (11.47) and is acceptable in the extended theory. This transformation would not be possible in the traditional Lagrangian mechanics. The traditional theory is tied to the Newtonian conception of absolute time, in which time is an invariant parameter that is not allowed to transform.

11.9 Redundancy of the Extended Lagrange Equations

The $(D + 1)$ extended Lagrange equations are redundant, as is proved in the lemma below. With this lemma, the extended Lagrangian theory gives the conditions for the correctness of a custom of long standing in Lagrangian mechanics.

For example, in the problem of the symmetric top in Section 9.17, we solved two of the traditional Lagrange equations, the ones for Euler angles α and γ . We then moved to the generalized energy theorem to get a third equation. (In the extended Lagrangian theory this would be the extended Lagrange equation with $k = 0$.) We then solved for the motion of the top without even writing the Lagrange equation for the Euler angle β .

One might wonder if this omitted equation is actually satisfied by the solution we found. The answer is given by Lemma 11.9.1. The Euler angle β has a nonzero time derivative at all except isolated turning points of the motion. Thus the Lagrange equation for that angle is satisfied automatically and we were justified in not checking it.

Lemma 11.9.1: Redundancy of Extended Lagrange Equations

If some solution $q = q(\beta)$ satisfies all of the extended Lagrange equations, eqn (11.18), except the one with index l , and if $\dot{q}_l \neq 0$, then the extended Lagrange equation for index l is satisfied automatically.

Proof: The proof closely parallels the proof of Theorem 5.7.3 in the calculus of variations. The homogeneity of \mathcal{L} proved in Lemma 11.3.2 implies that

$$0 = \sum_{k=0}^D \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \dot{q}_k - \mathcal{L}(q, \dot{q}) \quad (11.53)$$

Taking the total derivative of both sides with respect to β then gives

$$\begin{aligned} 0 &= \sum_{k=0}^D \left\{ \dot{q}_k \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) + \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \ddot{q}_k - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \ddot{q}_k - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} \dot{q}_k \right\} \\ &= \sum_{k=0}^D \dot{q}_k \left\{ \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} \right\} \end{aligned} \quad (11.54)$$

Thus, assuming the extended Lagrange equations satisfied for $k \neq l$,

$$\dot{q}_l \left\{ \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_l} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_l} \right\} = - \sum_{\substack{k=0 \\ (k \neq l)}}^D \dot{q}_k Q_k^{(\text{NP})} = \dot{q}_l Q_l^{(\text{NP})} \quad (11.55)$$

where the definitions in eqn (11.19) have been used to get the last equality. Since $\dot{q}_l \neq 0$ by assumption, it can be cancelled, leading to the extended Lagrange equation for the index l , as was to be proved. \square

11.10 Forces of Constraint

Extended Lagrangian mechanics is most useful as a tool for advanced theoretical development. It is not intended for the mundane task of calculating the equations of motion in constraint problems. However, it is interesting to see how easily the Lagrangian theory of constraints can be translated into the extended form in which time is a coordinate.

The definition of a holonomic constraint in eqn (3.1) of Chapter 3 becomes, in our current notation,

$$0 = G_a(q_{[0]}, t) = G_a(q) \quad (11.56)$$

for $a = 1, \dots, C$. Theorem 3.4.1 demonstrated that when the only non-potential forces are constraint forces that do no virtual work, in the traditional sense defined by the vanishing of the $\delta W^{(\text{cons})}$ defined in eqn (3.10), then these forces can be written

in the form, for $k = 1, \dots, D$,

$$Q_k^{(\text{NP})} = Q_k^{(\text{cons})} = \sum_{a=1}^C \lambda_a \frac{\partial G_a(q_{[0]}, t)}{\partial q_k} \quad (11.57)$$

This result can be used to derive a similar expression for the extended generalized forces $Q_k^{(\text{NP})} = Q_k^{(\text{cons})}$ defined in eqn (11.19).

Theorem 11.10.1: Extended Forces of Constraint

If the only non-potential forces are constraint forces that do no virtual work in the traditional sense used in Chapter 3, then the non-potential forces of the extended Lagrangian theory are, for all $k = 0, \dots, D$, given by

$$Q_k^{(\text{NP})} = Q_k^{(\text{cons})} = \sum_{a=1}^C \tilde{\lambda}_a \frac{\partial G_a(q)}{\partial q_k} \quad (11.58)$$

where $\tilde{\lambda}_a = i\lambda_a$.

Proof: For $k \neq 0$, putting eqn (11.57) into the definitions in eqn (11.19) gives

$$Q_k^{(\text{NP})} = Q_k^{(\text{cons})} = iQ_k^{(\text{cons})} = \sum_{a=1}^C i\lambda_a \frac{\partial G_a(q_{[0]}, t)}{\partial q_k} \quad (11.59)$$

as was to be proved.

Following the same procedure for $k = 0$ gives

$$Q_0^{(\text{NP})} = Q_0^{(\text{cons})} = - \sum_{l=1}^D Q_l^{(\text{cons})} \dot{q}_l = - \sum_{a=1}^C \lambda_a \sum_{l=1}^D \frac{\partial G_a(q_{[0]}, t)}{\partial q_l} \dot{q}_l = \sum_{a=1}^C i\lambda_a \frac{\partial G_a(q_{[0]}, t)}{\partial t} \quad (11.60)$$

as was to be proved. The last equality in eqn (11.60) was obtained by using eqn (11.56) to write⁵²

$$0 = \frac{dG_a(q_{[0]}, t)}{d\beta} = \sum_{l=1}^D \frac{\partial G_a(q_{[0]}, t)}{\partial q_l} \dot{q}_l + \frac{\partial G_a(q_{[0]}, t)}{\partial t} \dot{t} \quad (11.61)$$

□

Equation (11.58) can now be used in eqn (11.18) to obtain the extended Lagrange equations in the presence of constraints. When the only non-potential forces are constraint forces that do no virtual work in the traditional sense, the extended Lagrange equations become

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \sum_{a=1}^C \tilde{\lambda}_a \frac{\partial G_a(q)}{\partial q_k} \quad (11.62)$$

for $k = 0, \dots, D$. Together with eqn (11.56), these are $(D + C + 1)$ equations in the $(D + C + 1)$ unknowns $q_0, q_1, \dots, q_D, \tilde{\lambda}_1, \dots, \tilde{\lambda}_C$ and so can be solved in principle.

⁵²For more detail, see the proof of the generalized energy theorem with constraints in Theorem 3.13.1.

Equation (11.62) is very similar in form to the traditional Lagrange equations with constraints given in eqn (3.27). The main difference is that now the generalized energy theorem of eqn (3.78) is included as one of the extended Lagrange equations. It is eqn (11.62) with $k = 0$.

The traditional definition of virtual displacements and virtual work from Sections 3.2 and 3.3 can also be translated into extended form. When time is a coordinate, the definition of a virtual displacement must be generalized to allow nonzero time displacements. The virtual displacements δq_k are now taken to be differentials of the $(D + 1)$ generalized coordinates that are arbitrary and independent except for the condition $\delta G_a(q) = 0$ which ensures that the constraints are not violated. This condition can be written as

$$0 = \delta G_a(q) = \sum_{k=0}^D \frac{\partial G_a(q)}{\partial q_k} \delta q_k = \sum_{k=1}^D \frac{\partial G_a(q)}{\partial q_k} \left(\delta q_k - \frac{dq_k}{dt} \delta t \right) \quad (11.63)$$

To obtain the last equality, the $\delta q_0 = \delta t$ term has been removed from the sum and written explicitly using eqn (11.61).

The notation δ instead of d for the differentials in virtual displacements is to remind us that they are not to be taken as the actual motion of the system produced by differential change $d\beta$. They are assumed to be arbitrary and independent, except for the constraints, even though each of the q_k is actually a function of β .

The virtual work of constraint forces when time is a coordinate can be defined as

$$\delta W^{(\text{cons})} = \sum_{k=0}^D Q_k^{(\text{cons})} \delta q_k \quad (11.64)$$

where the quantities $Q_k^{(\text{cons})}$ are the same as those defined in eqn (11.19), except with the superscript (NP) replaced by (cons) since the only non-potential forces acting are the constraint forces.

The definition of virtual work with time as a coordinate in eqn (11.64) is equivalent to the traditional one defined in Section 3.3. To see this, put the definitions in eqn (11.19) into eqn (11.64), again with the superscript (NP) replaced by (cons)

$$\begin{aligned} \delta W^{(\text{cons})} &= \sum_{k=1}^D Q_k^{(\text{cons})} \delta q_k + Q_0 \delta q_0 \\ &= \sum_{k=1}^D i Q_k^{(\text{cons})} \delta q_k - \sum_{l=1}^D Q_l^{(\text{cons})} \dot{q}_l \delta q_0 = i \sum_{k=1}^D Q_k^{(\text{cons})} \left(\delta q_k - \frac{dq_k}{dt} \delta t \right) \end{aligned} \quad (11.65)$$

The traditional virtual displacement at fixed time used in Chapter 3 should be identified with the expression $(\delta q_k - (dq_k/dt)\delta t)$ in the present section. If we denote the differentials at fixed time from Chapter 3 by $\delta_t q_k$ to distinguish them from the δq_k

being used here, then setting

$$\delta_t q_k = \left(\delta q_k - \frac{dq_k}{dt} \delta t \right) \tag{11.66}$$

for all $k = 1, \dots, D$ makes eqn (11.63) become identical to the traditional expression in eqn (3.5). Also, eqn (11.65) and the traditional expression eqn (3.10) become identical except for a i factor, which is of no importance since it is never zero and the only use of $\delta W^{(\text{cons})}$ is to be set equal to zero. Thus the two definitions of virtual work are equivalent, as was asserted.

The interpretation of eqn (11.66) is that, when time is a coordinate and so $\delta t \neq 0$, the traditional definition of virtual displacement is obtained by subtracting off the effect of δt to obtain a displacement $\delta_t q_k$ at fixed time. The extended definition in eqn (11.64) is equivalent to the traditional definition of $\delta W^{(\text{cons})}$ because it does that subtraction automatically.

For extended Lagrangian theory, a theorem very similar to the traditional Theorem 3.4.1 can be proved. The following theorem reaches the same conclusion as Theorem 11.10.1 above, but does so with no dependence on traditional Lagrangian theory of Chapter 3.

Theorem 11.10.2: Form of Forces of Constraint

Given the constraints defined by eqns (11.56, 11.63), the virtual work defined in eqn (11.64) is zero, $\delta W^{(\text{cons})} = 0$, if and only if, for all $k = 0, \dots, D$, the generalized forces of constraint have the form

$$Q_k^{(\text{cons})} = \sum_{a=1}^C \tilde{\lambda}_a \frac{\partial G_a(q)}{\partial q_k} \tag{11.67}$$

Proof: The proof is very similar to that in Theorem 3.4.1 and will not be repeated here. □

11.11 Reduced Lagrangians with Time as a Coordinate

Just as was proved for traditional Lagrangian theory in Theorem 3.8.1, holonomic constraints that do no virtual work can be used to reduce the number of degrees of freedom of extended Lagrangian systems. There are two ways that this can be done.

First, one might reduce the degrees of freedom using the traditional methods of Section 3.8 to obtain a reduced traditional Lagrangian \bar{L} . Then the transition to the extended theory in Definition 11.3.1 can be done starting from the traditional reduced \bar{L} Lagrangian instead of the full Lagrangian. Then the extended Lagrangian theory goes forward just as if the eliminated degrees of freedom had never existed.

Alternately, the transition to the extended theory can be done first, exactly as stated in Definition 11.3.1, starting from the full traditional Lagrangian L to obtain the full extended Lagrangian \mathcal{L} . Then the constraints can be used to reduce the number of degrees of freedom in the context of the extended Lagrangian theory.

This second method is the subject of the following theorem, which is analogous to Theorem 3.8.1 for the traditional Lagrangian theory.

Theorem 11.11.1: Reduced Lagrangians With Time as a Coordinate

In the theory with time as a coordinate, a reduced Lagrangian $\tilde{\mathcal{L}}(q^{(f)}, \dot{q}^{(f)})$ can be obtained by solving the constraint equations $G_a(q) = 0$ from eqn (11.56) for the bound variables as functions of the free ones, thus obtaining $q_l^{(b)} = q_l^{(b)}(q^{(f)})$ and from it $\dot{q}_l^{(b)} = \dot{q}_l^{(b)}(q^{(f)}, \dot{q}^{(f)})$, and then making the substitution

$$\tilde{\mathcal{L}}(q^{(f)}, \dot{q}^{(f)}) = \mathcal{L}\left(q^{(f)}, q^{(b)}(q^{(f)}), \dot{q}^{(f)}, \dot{q}^{(b)}(q^{(f)}, \dot{q}^{(f)})\right) \quad (11.68)$$

This reduced Lagrangian will be homogeneous of degree one in the reduced set of generalized velocities $\dot{q}^{(f)}$. If the forces of constraint do no virtual work, the extended Lagrange equations in the reduced variable set will then be

$$\frac{d}{d\beta} \left(\frac{\partial \tilde{\mathcal{L}}(q^{(f)}, \dot{q}^{(f)})}{\partial \dot{q}_k} \right) - \frac{\partial \tilde{\mathcal{L}}(q^{(f)}, \dot{q}^{(f)})}{\partial q_k} = 0 \quad (11.69)$$

for all of the $(D + 1 - C)$ index values k corresponding to the free variables.

Proof: The proof of this theorem is the same as that in Theorem 3.8.1, with the substitution of $\tilde{\mathcal{L}}$ for $\bar{\mathcal{L}}$ and the addition of the variables q_0 and \dot{q}_0 .

The homogeneity of $\tilde{\mathcal{L}}$ follows from fact that, in the extended Lagrangian theory with time as a coordinate, the derivatives of the bound variables have the simple linear form

$$\dot{q}_l^{(b)} = \sum_{\text{free } k} \frac{\partial q_l^{(b)}(q^{(f)})}{\partial q_k^{(f)}} \dot{q}_k^{(f)} \quad (11.70)$$

which preserves the homogeneity when the substitution in eqn (11.68) is made. \square

One interesting property of the method of Theorem 11.11.1 is that it is possible in some systems for the time q_0 to be chosen as one of the bound variables. Then the reduced Lagrangian $\tilde{\mathcal{L}}$ will not contain the time or its derivative. Such a case is examined in Exercise 11.6.

Note that the homogeneity of the reduced Lagrangian $\tilde{\mathcal{L}}$ proved in Theorem 11.11.1 means that Lemmas 11.3.2, 11.4.1, 11.4.2, and 11.9.1 that were proved for the full Lagrangian \mathcal{L} also apply to $\tilde{\mathcal{L}}$.

11.12 Exercises

Exercise 11.1 Write out the proof of Theorem 11.8.1 in detail.

Exercise 11.2 In Section 2.10 it was stated that the traditional transformation between any two systems of good generalized coordinates had to obey the $D \times D$ determinant condition $|\partial q(r_1, \dots, r_D, t)/\partial r| \neq 0$.

(a) Show that this condition, plus the identity transformation for the time $r_0 = q_0 = t$ assumed by the traditional theory, together imply the condition eqn (11.47) required for coordinate transformations in the extended Lagrangian theory.

(b) Explain why the extended transformation theory therefore includes the traditional transformation theory as a special case, but is more general.

Exercise 11.3 Demonstrate that the standard Lorentz transformation of eqn (11.52) satisfies the condition in eqn (11.47) and therefore is a legitimate extended Lagrangian transformation.

Exercise 11.4 A harmonic oscillator in three dimension has the traditional Lagrangian

$$L(q_{[0]}, q'_{[0]}, q_0) = \frac{1}{2}m(x'^2 + y'^2 + z'^2) - \frac{1}{2}k(x^2 + y^2 + z^2) \quad (11.71)$$

where $x' = dx/dt$, etc.

- Write the extended Lagrangian $\mathcal{L}(q, \dot{q})$ for this system.
- Write the generalized momentum p_0 and show that it is the negative of the traditional generalized energy function $H(q_{[0]}, q'_{[0]}, t)$.
- Use the extended Lagrange equation for $k = 0$ to show that the time is an ignorable coordinate, and that the energy of the system is a conserved quantity.
- Find the generalized momenta p_k for $k = 0, \dots, 3$ and use them to verify eqn (11.16) for this system.

Exercise 11.5

- Starting from the traditional Lagrangian for a system of charged particles in a given electromagnetic field, as shown in eqn (2.103), write the extended Lagrangian \mathcal{L} .
- Show that $\underline{p}_0 = \partial\mathcal{L}/\partial\dot{t}$ is the negative of the generalized energy function eqn (2.105).

Exercise 11.6 Suppose a single mass m is constrained to lie on the frictionless floor of a rising elevator, so that its z -coordinate obeys the constraint $z = at^2$. Assume that gravity acts downwards in the negative z -direction.

- Write the full traditional and extended Lagrangians for this problem.
- Select $q_0 = t$ to be the bound coordinate and x, y, z to be the free ones. Write the reduced Lagrangian $\tilde{\mathcal{L}}$ as derived in Section 11.11.
- Use this $\tilde{\mathcal{L}}$ to write the three extended Lagrange equations. Show that the x and y Lagrange equations are simple, and reduce dx/dt and dy/dt to constants. Argue that the third Lagrange equation is redundant and is automatically satisfied whenever the first two are.

Exercise 11.7 In the proof of Lemma 11.9.1, use eqn (11.19) to derive the last equality in eqn (11.55).

Exercise 11.8 In Exercise 2.7 we considered two traditional Lagrangians L and L' related by a gauge transformation. (Note that the prime here does not mean differentiation with respect to time.)

- Write the extended Lagrangians for these two cases, and show that

$$\mathcal{L}' = \mathcal{L} + \frac{q^{(\text{ch})}}{c} \frac{d\chi(q)}{d\beta} \quad (11.72)$$

- Derive the canonical momenta $\underline{p}_0, \underline{\mathbf{p}}$ and $\underline{p}'_0, \underline{\mathbf{p}}'$ from the two extended Lagrangians in eqn (11.72).

(c) Use the result of (b) to show that

$$\underline{p}'_0 = \underline{p}_0 + \frac{q^{(\text{ch})}}{c} \frac{\partial \chi}{\partial t} \quad \underline{\mathbf{p}}' = \underline{\mathbf{p}} + \frac{q^{(\text{ch})}}{c} \frac{\partial \chi}{\partial \mathbf{r}} \quad (11.73)$$

(d) Show that

$$\left(\underline{p}_0 + q^{(\text{ch})} \Phi \right) = \left(\underline{p}'_0 + q^{(\text{ch})} \Phi' \right) \quad \text{and} \quad \left(\underline{\mathbf{p}} - q^{(\text{ch})} \mathbf{A}/c \right) = \left(\underline{\mathbf{p}}' - q^{(\text{ch})} \mathbf{A}'/c \right) \quad (11.74)$$

and hence that these expressions are gauge invariant. Why must an expression like

$\left(\underline{\mathbf{p}} - q^{(\text{ch})} \mathbf{A}/c \right)$ be independent of the gauge of the electromagnetic field?

(e) Show that any solution $q_k = q_k(\beta)$ that solves the extended Lagrange equations with \mathcal{L} also solves the extended Lagrange equations with \mathcal{L}' .

HAMILTONIAN MECHANICS WITH TIME AS A COORDINATE

Chapter 11 used the traditional Lagrange equations of Chapter 2 as the basis for an extended Lagrangian theory in which time is treated as a coordinate. This extended theory combined the Lagrange equations and the generalized energy theorem into one set of equations.

In the present chapter, we will do the same with the traditional Hamilton equations of Chapter 4. The traditional Hamilton equations, including the Hamiltonian form of the generalized energy theorem, will be combined into one set of extended Hamilton equations in which time is treated as a coordinate.

The extended Hamilton theory developed in this chapter is of fundamental importance for the more advanced topics in mechanics. It is used in Chapter 16 to write the relativistically covariant Hamiltonian, which is then used to derive the Klein–Gordon equation of relativistic quantum mechanics. And the extended Hamilton equations provide the basis for our discussion of canonical transformations in Chapter 17.

12.1 Extended Phase Space

The objective of extended Hamiltonian theory is to write the equations of motion in terms of an extended set of phase-space variables that includes the new coordinate $q_0 = t$ and its conjugate momentum p_0 defined in Section 11.4. When the new coordinates are included, the phase space becomes $(2D + 2)$ -dimensional, with the canonical coordinates

$$q, p = q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D \quad (12.1)$$

Solutions to the traditional Hamilton equations of Chapter 4 gave the equations of motion of the system as a trajectory through the traditional phase space, in the form $q_k = q_k(t)$ and $p_k = p_k(t)$ for $k = 1, \dots, D$. The equations of motion of extended Hamiltonian theory will give the phase-space trajectory as functions of the new parameter β introduced in Section 11.1, $q_k = q_k(\beta)$ and $p_k = p_k(\beta)$ for $k = 0, \dots, D$, where the phase space now includes the two new coordinates q_0 and p_0 .

12.2 Dependency Relation

From the viewpoint of extended Lagrangian mechanics, the $(2D + 2)$ variables q, p of the extended phase space defined above are not independent. They have a single dependency relation among them, in the sense treated in Section D.29. We begin by examining this dependency, since the function that describes it will also play the role of the extended Hamiltonian in our extended theory.

In extended Lagrangian theory, the generalized momenta are those defined in eqns (11.9, 11.14)

$$p_i(q, \dot{q}) = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_i} \quad p_0(q, \dot{q}) = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_0} = -H(q_{[0]}, q'_{[0]}, t) \quad (12.2)$$

for $i = 1, \dots, D$, where $q'_k = dq_k/dt = \dot{q}_k/\dot{q}_0$ and the traditional generalized energy function $H(q_{[0]}, q'_{[0]}, t)$ has been written in the notation of Section 11.2, which we will continue to use in the present chapter.

In Theorem 4.1.1 it was proved that the D -rowed square matrix defined, for $i, j = 1, \dots, D$, by

$$\left(\frac{\partial p_{[0]}}{\partial q'_{[0]}} \right)_{ij} = \frac{\partial p_i(q_{[0]}, q'_{[0]}, t)}{\partial q'_j} \quad (12.3)$$

was nonsingular, and hence that the equations $p_k = p_k(q_{[0]}, q'_{[0]}, t)$ could be inverted to give $q'_k = q'_k(q_{[0]}, p_{[0]}, t)$. The traditional generalized energy function was then rewritten in terms of the traditional phase-space variables $q_{[0]}, p_{[0]}, t$, and was thereafter referred to as the Hamiltonian.

This change of variables does not change the function, however. The traditional Hamilton is the same function as the generalized energy, and is defined as the compound function

$$H = H(q_{[0]}, p_{[0]}, t) = H(q_{[0]}, q'_{[0]}(q_{[0]}, p_{[0]}, t), t) \quad (12.4)$$

Thus the second of eqn (12.2) can be written as the identity

$$p_0(q, \dot{q}) = -H(q_{[0]}, p_{[0]}(q, \dot{q}), q_0) \quad (12.5)$$

where t has now been replaced by q_0 .

As is customary (see Section D.29), we express this dependency among the extended phase-space variables q, p by defining a dependency function $\mathcal{K}(q, p)$ and setting it equal to zero.

Thus, we write the dependency relation as

$$\mathcal{K}(q, p) = 0 \quad \text{where} \quad \mathcal{K}(q, p) = p_0 + H(q_{[0]}, p_{[0]}, q_0) \quad (12.6)$$

12.3 Only One Dependency Relation

It is important, particularly when we want to derive the extended Hamilton equations from Hamilton's Principle in Chapter 13, to establish that there is only *one* dependency relation among the extended phase-space variables, as is proved in the following lemma and theorem.

Lemma 12.3.1: Rank of Transformation Matrix

The $(D + 1)$ -rowed square matrix defined by

$$\left(\frac{\partial p}{\partial \dot{q}}\right)_{ij} = \frac{\partial p_i(q, p)}{\partial \dot{q}_j} = \frac{\partial^2 \mathcal{L}(q, p)}{\partial \dot{q}_j \partial \dot{q}_i} \tag{12.7}$$

for $i, j = 0, \dots, D$, is singular and has rank D :

$$\left|\frac{\partial p}{\partial \dot{q}}\right| = 0 \quad \text{and} \quad \text{Rank}\left(\frac{\partial p}{\partial \dot{q}}\right) = D \tag{12.8}$$

Proof: First, to prove the singularity. The homogeneity of the p_k proved in Lemma 11.4.1 and the Euler Condition of Theorem D.31.1 together imply that

$$0 = \sum_{l=0}^D \dot{q}_l \frac{p_k(q, \dot{q})}{\partial \dot{q}_l} = \sum_{l=0}^D \left(\frac{\partial p}{\partial \dot{q}}\right)_{kl} \dot{q}_l \tag{12.9}$$

where the last expression is written in the form of a matrix multiplication. But at least one of the \dot{q}_i is nonzero, since it is always true that $\dot{q}_0 = (dt/d\beta) \neq 0$. Thus Corollary B.19.2 implies that the matrix $(\partial p/\partial \dot{q})$ is singular, as was to be proved.

The proof that $(\partial p/\partial \dot{q})$ has rank D then consists of demonstrating that, though singular, it has a nonzero D -rowed critical minor. The matrix $(\partial p/\partial \dot{q})$ may be written out as

$$\left(\frac{\partial p}{\partial \dot{q}}\right) = \begin{pmatrix} \frac{\partial p_0}{\partial \dot{q}_0} & \frac{\partial p_0}{\partial \dot{q}_1} & \dots & \frac{\partial p_0}{\partial \dot{q}_D} \\ \frac{\partial p_1}{\partial \dot{q}_0} & \frac{\partial p_1}{\partial \dot{q}_1} & \dots & \frac{\partial p_1}{\partial \dot{q}_D} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial p_D}{\partial \dot{q}_0} & \frac{\partial p_D}{\partial \dot{q}_1} & \dots & \frac{\partial p_D}{\partial \dot{q}_D} \end{pmatrix} \tag{12.10}$$

Removing the first row and the first column of this matrix leaves a D -rowed square matrix consisting of all rows and columns with nonzero indices. Section 11.4 demonstrated that, for $k \neq 0$, the extended momenta p_k in this matrix are identical to the traditional ones defined in Section 2.12. Hence, except for nonzero i factors on each matrix element, this matrix is the same as the matrix written above in eqn (12.3) and proved nonsingular in Theorem 4.1.1. Thus this D -rowed matrix is nonsingular and its determinant constitutes a D -rowed critical minor for the matrix in eqn (12.10), as was to be proved. □

Theorem 12.3.2: One Dependency Relation

Considered as functions of the extended Lagrangian variables $q_0, q_1, \dots, q_D, \dot{q}_0, \dot{q}_1, \dots, \dot{q}_D$ the phase-space coordinates listed in eqn (12.1) have exactly one dependency relation of the sort defined in Section D.29.

Proof: Apply Theorem D.29.1 to the present problem, with $M = N = (2D + 2)$, the functions f_k replaced by the set q, p , and the variables x_i by the set q, \dot{q} . The number of dependency relations then depends on the rank of the matrix $(\partial f/\partial x)$, which here has the block form

$$\begin{pmatrix} \mathbf{U} & \mathbf{0} \\ \left(\frac{\partial p}{\partial q}\right) & \left(\frac{\partial p}{\partial \dot{q}}\right) \end{pmatrix} \quad (12.11)$$

in which the matrix is written as four $(D + 1) \times (D + 1)$ blocks. The \mathbf{U} is the unit matrix, $\mathbf{0}$ is the null matrix, $(\partial p/\partial q)$ is a matrix defined similarly to eqn (12.7) but using q rather than \dot{q} , and the lower-right block $(\partial p/\partial \dot{q})$ is the matrix written in eqn (12.10) above.

It follows from the discussion of the determinants of partitioned matrices in Section B.15 that the determinant of the matrix in eqn (12.11) is the same as the determinant of its lower, right-hand block $(\partial p/\partial \dot{q})$, which determinant was proved zero in Lemma 12.3.1. Also, it follows from that same lemma that the matrix that would remain when the p_0 row and the \dot{q}_0 column were removed from eqn (12.11) would be nonsingular. Thus eqn (12.11) is a $(2D + 2)$ -rowed square matrix of rank $(2D + 1)$.

If the rank of the matrix eqn (12.11) is r , then Theorem D.29.1 proves that the number of dependency relations among the q, p will be $M - r = 2D + 2 - r$. Since we have shown that the rank of this matrix is $r = 2D + 1$, one less than the dimension of the matrix, it follows that $M - r = 1$ and that there is one and only one dependency relation, as was to be proved. \square

12.4 From Traditional to Extended Hamiltonian Mechanics

The principal objective of extended Hamiltonian theory is to write the equations of motion of the system in a form that treats all of the variables of phase space on an equal footing. This was also the objective of the traditional Hamiltonian theory developed in Chapter 4. The difference is that the extended phase space includes a new pair of canonically conjugate coordinates, the time q_0 and its conjugate momentum p_0 .

In the extended Hamiltonian theory, the momenta p_k (including p_0), which are derived quantities in extended Lagrangian theory, are now to be considered as *independent* coordinates in phase space. This freeing of the momenta in Hamiltonian mechanics means that the Lagrangian identities, eqn (12.2), no longer apply in Hamiltonian theory. The same relations will be recovered, however, not as identities but as consequences of the Hamilton equations of motion at the end of the calculation.

In particular, it is important to note that the dependency relation eqn (12.6) is no longer to be assumed in advance. In Lagrangian mechanics, $\mathcal{K}(q, p) = 0$ is an identity. But in Hamiltonian mechanics, $\mathcal{K}(q, p)$ is to be treated as just another phase-space function whose value is determined by the Hamilton equations of motion.

In Chapter 11, the extended Lagrangian theory was obtained by converting the traditional Lagrange equations to a parametric form. Similarly, extended Hamiltonian mechanics is obtained by converting the traditional Hamilton equations to a parametric form that uses derivatives with respect to an arbitrary monotonic parameter β rather than with respect to the time as in traditional theory, and that includes the Hamiltonian form of the generalized energy theorem as one of the extended Hamilton equations. This conversion is most transparently accomplished if we define the dependency function $\mathcal{K}(q, p)$ itself to be the extended Hamiltonian function and write the extended Hamilton equations in terms of it.

Definition 12.4.1: Standard Extended Hamiltonian

We define the standard extended Hamiltonian to be the function that appeared in the dependency relation of eqn (12.6).

$$\mathcal{K}(q, p) = p_0 + H(q_{[0]}, p_{[0]}, q_0) \quad (12.12)$$

Just as for the extended Lagrangian, this definition is not unique. Several different forms are possible, but the simplest one is this standard function. We next use this extended Hamiltonian to state the extended Hamilton equations.⁵³

Definition 12.4.2: Extended Hamilton Equations

With the extended Hamiltonian $\mathcal{K}(q, p)$ defined in eqn (12.12), the extended Hamilton equations are

$$\dot{q}_k = \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}(q, p)}{\partial q_k} \quad (12.13)$$

for all $k = 0, \dots, D$, where the dot denotes differentiation with respect to parameter β .

An immediate consequence of the extended Hamilton equations is that

$$\frac{d\mathcal{K}}{d\beta} = \sum_{k=0}^D \left(\frac{\partial \mathcal{K}(q, p)}{\partial q_k} \dot{q}_k + \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \dot{p}_k \right) = \sum_{k=0}^D (-\dot{p}_k \dot{q}_k + \dot{q}_k \dot{p}_k) = 0 \quad (12.14)$$

Thus the relation $\mathcal{K} = 0$, which was an identity in Lagrangian mechanics, is obtained as a consequence of the Hamilton equations of motion, provided that we choose \mathcal{K} to be zero at some initial value of β .⁵⁴ We will assume throughout that this initial condition has been specified. Thus the relation

$$\mathcal{K}(q, p) = 0 \quad \text{or} \quad p_0 = -H(q_{[0]}, p_{[0]}(q, \dot{q}), q_0) \quad (12.15)$$

is recovered, not as an identity but as a consequence of the Hamilton equations of motion. Note that the condition $\mathcal{K}(q, p) = 0$ must not be applied until *after* all partial derivatives in eqn (12.13) have been taken. It may then be applied to simplify our equations if necessary.

⁵³What we call the extended Hamilton equations are referred to by Lanczos (1970) as “the parametric form of the canonical equations.” See also Rund (1966).

⁵⁴Choosing some nonzero constant for \mathcal{K} at time zero would be equivalent to a trivial redefinition of the traditional Hamiltonian. The value zero is the simplest choice and will be adopted here.

12.5 Equivalence to Traditional Hamilton Equations

We now prove that these extended Hamilton equations are equivalent to a combination of the traditional Hamilton equations, and the traditional Hamiltonian form of the generalized energy theorem.

Theorem 12.5.1: Equivalence of Extended Hamilton Equations

Let the extended Hamiltonian $\mathcal{K}(q, p)$ be defined as in eqn (12.12). Then the extended Hamilton equations, eqn (12.13), when $k = 0$ are equivalent to the traditional Hamiltonian form of the generalized energy theorem. And the extended Hamilton equations, eqn (12.13), when $k \neq 0$ are equivalent to the traditional Hamilton equations.

Proof: Expressed in our present notation, the traditional Hamilton equations in eqn (4.21) are

$$q'_k = \frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial p_k} \quad p'_k = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_k} \quad (12.16)$$

and the traditional Hamiltonian form of the generalized energy theorem, the third of eqn (4.21), is

$$\frac{dH}{dt} = H' = \frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_0} \quad (12.17)$$

where, as throughout this chapter, the prime indicates differentiation with respect to the time.

First consider the $k = 0$ case. Then, eqn (12.13) becomes

$$\dot{q}_0 = 1 \quad \dot{p}_0 = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_0} \quad (12.18)$$

The derivative $\dot{q}_0 = dt/d\beta$, which here has the value one, can never be zero, since both the time t and parameter β vary monotonically along the system path in configuration space. Combining the two equations in eqn (12.18) and writing t for q_0 therefore gives

$$\frac{dp_0}{dt} = \frac{\dot{p}_0}{\dot{q}_0} = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial t} \quad (12.19)$$

But, from eqn (12.15), $p_0 = -H(q_{[0]}, p_{[0]}, q_0)$. Hence eqn (12.19) becomes

$$-\frac{dH}{dt} = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_0} \quad (12.20)$$

which is eqn (12.17), as was to be proved.

When $k \neq 0$, eqn (12.13) become

$$\dot{q}_k = \frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_k} \quad (12.21)$$

Again using the first of eqn (12.18), these become

$$\frac{dq_k}{dt} = \frac{\dot{q}_k}{\dot{q}_0} = \frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial p_k} \quad \text{and} \quad \frac{dp_k}{dt} = \frac{\dot{p}_k}{\dot{q}_0} = -\frac{\partial H(q_{[0]}, p_{[0]}, q_0)}{\partial q_k} \quad (12.22)$$

which are eqn (12.16), as was to be proved. \square

The privileged treatment of $q_0 = t$ in the first of eqn (12.18) is due to the simple form chosen for the extended Hamiltonian eqn (12.12). As will be shown in Section 12.8, other choices of \mathcal{K} are possible that treat some other coordinate q_l similarly, or that have no particular coordinate with a unit derivative.

12.6 Example of Extended Hamilton Equations

Consider the simple example of a three dimensional harmonic oscillator in Section 2.3. In the notation of the present chapter, the traditional Lagrangian function is

$$L\left(q_{[0]}, \frac{dq_{[0]}}{dt}, q_0\right) = \frac{1}{2}m\left(x'^2 + y'^2 + z'^2\right) - \frac{1}{2}k\left(x^2 + y^2 + z^2\right) \quad (12.23)$$

where we have written $q_0, q_1, q_2, q_3 = t, x, y, z$ for clarity, and we continue to use the notation $x' = dx/dt$, etc. The extended Lagrangian is defined in eqn (11.7). It is

$$\mathcal{L}(q, \dot{q}) = \frac{m}{2\dot{t}}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{\dot{t}}{2}k\left(x^2 + y^2 + z^2\right) \quad (12.24)$$

Using the methods of Chapter 4, the traditional Hamiltonian function is found to be

$$H(q_{[0]}, p_{[0]}, q_0) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) \quad (12.25)$$

The extended Hamiltonian is then defined in eqn (12.12) to be

$$\mathcal{K}(q, p) = p_0 + \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(x^2 + y^2 + z^2\right) \quad (12.26)$$

and the extended Hamilton equations are

$$i = \frac{\partial \mathcal{K}(q, p)}{\partial p_0} \quad \dot{x} = \frac{\partial \mathcal{K}(q, p)}{\partial p_x} \quad \dot{y} = \frac{\partial \mathcal{K}(q, p)}{\partial p_y} \quad \dot{z} = \frac{\partial \mathcal{K}(q, p)}{\partial p_z} \quad (12.27)$$

and

$$\dot{p}_0 = -\frac{\partial \mathcal{K}(q, p)}{\partial t} \quad \dot{p}_x = -\frac{\partial \mathcal{K}(q, p)}{\partial x} \quad \dot{p}_y = -\frac{\partial \mathcal{K}(q, p)}{\partial y} \quad \dot{p}_z = -\frac{\partial \mathcal{K}(q, p)}{\partial z} \quad (12.28)$$

12.7 Equivalent Extended Hamiltonians

We have introduced an arbitrary monotonic parameter β and used it to write the extended Hamilton equations (including the Hamiltonian form of the generalized energy theorem) in a parametric form. The solutions to the extended Hamilton equations, eqn (12.13), will be referred to as *phase-space trajectories*.

The parameter β may be any quantity that is assumed to vary monotonically along the system path. This parameter independence results from the fact that the right-hand sides of the extended Hamilton equations, eqn (12.13), do not contain β explicitly. Sets of coupled differential equations of this type can be reduced to a smaller set with parameter β eliminated, as is discussed in detail in Section D.36. For example, if we know that $\dot{q}_l \neq 0$ in some region, then we can write a set of $2N + 1$ equations by dividing each of eqn (12.13) by \dot{q}_l , giving, for all $j = 0, \dots, D$, $j \neq l$, and $k = 0, \dots, D$,

$$\frac{dq_j}{dq_l} = \frac{\dot{q}_j}{\dot{q}_l} = \frac{\partial \mathcal{K}(q, p) / \partial p_j}{\partial \mathcal{K}(q, p) / \partial p_l} = A_j(q, p) \quad \frac{dp_k}{dq_l} = \frac{\dot{p}_k}{\dot{q}_l} = -\frac{\partial \mathcal{K}(q, p) / \partial q_k}{\partial \mathcal{K}(q, p) / \partial p_l} = B_k(q, p) \quad (12.29)$$

The number of equations has been reduced by one, and the parameter β is gone. The independent variable is now q_l . Theorem D.36.2 demonstrates that, given initial values $q_j^{(0)}$ and $p_k^{(0)}$ at some value $q_l^{(0)}$ of the new independent variable, these equations have a *unique* solution. For all $j, k = 0, \dots, D$, $j \neq l$, we have

$$q_j = q_j(q_l) \quad \text{and} \quad p_k = p_k(q_l) \quad (12.30)$$

which define a *unique relation* among the phase-space variables q, p .

This unique relation eqn (12.30) among the $2N + 2$ phase-space variables *defines* the *phase-space trajectory* of the system. Such a trajectory determines completely the physical behavior of the system, and hence constitutes a unique solution to the mechanics problem. It does not matter physically how q_j , for example, is related to the parameter β . Only the relations *among* the phase-space variables q, p themselves are physically meaningful. Since the Hamilton equations exist only to predict system behavior, two different Hamiltonians that imply the same phase-space trajectory will be taken to be equivalent.

Note to the Reader: Two extended Hamiltonians and sets of extended Hamilton equations are *equivalent* if and only if they produce the same phase-space trajectory and hence the same system behavior.

Theorem D.36.2 (which the reader is advised to study) also shows that we may, if it is convenient in a particular context, reintroduce the parameter β into eqn (12.30) and so write them in parametric form $q_k = q_k(\beta)$ and $p_k = p_k(\beta)$, for all $k = 0, \dots, D$. However, it will still be true that the phase-space trajectory depends only on $D + 1$ arbitrary initial values, not on $D + 2$ as the parametric form might suggest.

In Section 11.7, we showed that the extended Lagrange equations are invariant under a change of parameter from β to any other monotonic parameter θ . In the

extended Hamiltonian theory, such a change of parameter would have the effect of introducing a uniform multiplicative factor $d\theta/d\beta$ into eqn (12.13) since, for example, $\dot{q}_k = (dq_k/d\theta)(d\theta/d\beta)$. These factors would cancel from eqn (12.29) and hence the phase-space trajectory defined would be the same no matter which parameter was used. In the extended Hamiltonian theory, invariance under change of parameter means that the same phase-space trajectory is obtained no matter what the monotonic parameter β may be.

12.8 Alternate Hamiltonians

Theorem 12.5.1 used the standard Hamiltonian $\mathcal{K}(q, p)$ defined in eqn (12.12) to prove the extended Hamilton equations equivalent to the traditional results. It might seem that nothing has been accomplished except the introduction of a parameter β which has just been shown to be removable. But the point of introducing β and writing eqn (12.13) has been to obtain equations that treat all of the $2N + 2$ phase-space coordinates q, p on an equal footing. The standard Hamiltonian eqn (12.12) leads to $\dot{q}_0 = 1$ and hence treats time in a privileged way. But there are other Hamiltonians that are equivalent to the standard one in the sense defined in Section 12.7, that do not have this simple relation between t and β . Two cases of interest are proved in the following lemmas.

Lemma 12.8.1: Multiplication by Nonzero Function

If an alternate extended Hamiltonian is defined by $\mathcal{K}_a(q, p) = g(q, p)\mathcal{K}(q, p)$, where $g(q, p)$ is any phase-space function that is known to be nonzero, then $\mathcal{K}(q, p) = 0$ from eqn (12.15) holds if and only if $\mathcal{K}_a(q, p) = 0$ holds. The extended Hamilton equations with $\mathcal{K}_a(q, p)$ are equivalent to those with $\mathcal{K}(q, p)$.

Proof: The extended Hamilton equations with the alternate Hamiltonian $\mathcal{K}_a(q, p) = g(q, p)\mathcal{K}(q, p)$ are

$$\dot{q}_k = \frac{\partial \mathcal{K}_a(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}_a(q, p)}{\partial q_k} \tag{12.31}$$

We show that these equations are equivalent to the standard extended Hamilton equations, eqn (12.13).

Consider the first of eqn (12.31) for a particular k value. The partial derivative is

$$\frac{\partial \mathcal{K}_a(q, p)}{\partial p_k} = g(q, p) \frac{\partial \mathcal{K}(q, p)}{\partial p_k} + \frac{\partial g(q, p)}{\partial p_k} \mathcal{K}(q, p) \tag{12.32}$$

But now, after the partial derivatives are all taken, we can use eqn (12.15) to write $\mathcal{K}(q, p) = 0$. Thus eqn (12.32) reduces to

$$\frac{\partial \mathcal{K}_a(q, p)}{\partial p_k} = g(q, p) \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \tag{12.33}$$

Similarly,

$$\frac{\partial \mathcal{K}_a(q, p)}{\partial q_k} = g(q, p) \frac{\partial \mathcal{K}(q, p)}{\partial q_k}$$

If we now choose some coordinate with $\dot{q}_l \neq 0$ and form the ratios in eqn (12.29),

the nonzero $g(q, p)$ factors will cancel. Hence the functions $A_j(q, p)$ and $B_k(q, p)$ in eqn (12.29) will be the same whether \mathcal{K} or \mathcal{K}_a is used for the extended Hamiltonian. The two extended Hamiltonians will produce identical phase-space trajectories and therefore are equivalent. \square

We now present the second way in which an alternate extended Hamiltonian can be formed.

Lemma 12.8.2: Solution for an Alternate Momentum

Suppose that there is a range of β values for which some particular generalized velocity \dot{q}_l is nonzero. Then, the dependency relation $\mathcal{K}(q, p) = 0$ can be solved for $p_l = p_l(q, p_{[l]})$ and rewritten as

$$\mathcal{K}_b(q, p) = 0 \quad \text{where} \quad \mathcal{K}_b(q, p) = p_l - p_l(q, p_{[l]}) \quad (12.34)$$

and $\mathcal{K}(q, p) = 0$ if and only if $\mathcal{K}_b(q, p) = 0$. The extended Hamilton equations with \mathcal{K}_b are equivalent to the standard equations in eqn (12.13).

Proof: First, we prove that the dependency relation can be written in the form eqn (12.34). Since $\dot{q}_l \neq 0$ by assumption, eqn (12.13) gives

$$\frac{\partial \mathcal{K}(q, p)}{\partial p_l} = \dot{q}_l \neq 0 \quad (12.35)$$

By the implicit function theorem, Theorem D.26.1 with the identifications $N = 1$, $f \rightarrow \mathcal{K}$, $y_1 \rightarrow p_l$, and $x \rightarrow q, p_{[l]}$, this is the necessary and sufficient condition for the equation $\mathcal{K}(q, p) = 0$ to be solved for p_l giving $p_l = p_l(q, p_{[l]})$.

The extended Hamilton equations with \mathcal{K}_b are

$$\dot{q}_k = \frac{\partial \mathcal{K}_b(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}_b(q, p)}{\partial q_k} \quad (12.36)$$

To show them equivalent to the standard extended Hamilton equations of eqn (12.13), we make use of eqn (D.103), with the same identifications as above, to write for all $k = 0, \dots, D$

$$\frac{\partial \mathcal{K}_b}{\partial q_k} = -\frac{\partial p_l(q, p_{[l]})}{\partial q_k} = \left(\frac{\partial \mathcal{K}}{\partial p_l} \right)^{-1} \frac{\partial \mathcal{K}}{\partial q_k} \quad (12.37)$$

and for all $k \neq l$

$$\frac{\partial \mathcal{K}_b}{\partial p_k} = -\frac{\partial p_l(q, p_{[l]})}{\partial p_k} = \left(\frac{\partial \mathcal{K}}{\partial p_l} \right)^{-1} \frac{\partial \mathcal{K}}{\partial p_k} \quad (12.38)$$

Also

$$\frac{\partial \mathcal{K}_b}{\partial p_l} = 1 = \left(\frac{\partial \mathcal{K}}{\partial p_l} \right)^{-1} \frac{\partial \mathcal{K}}{\partial p_l}$$

If we now form the ratios in eqn (12.29), the nonzero $(\partial \mathcal{K} / \partial p_l)^{-1}$ factors will cancel. Hence the functions $A_j(q, p)$ and $B_k(q, p)$ will be the same whether \mathcal{K} or \mathcal{K}_b is used for the extended Hamiltonian. The two extended Hamiltonians will produce identical phase-space trajectories and therefore are equivalent. \square

12.9 Alternate Traditional Hamiltonians

The traditional Hamilton equations treat the time t in a special way. It is an interesting curiosity that Hamilton equations of the traditional form can also be derived that treat some other variable specially.

For ranges of β with $\dot{q}_l \neq 0$, an alternate traditional Hamiltonian $H^{(l)}$ can be defined by

$$H^{(l)}(q_{[l]}, p_{[l]}, q_l) = -p_l(q, p_{[l]}) \tag{12.39}$$

where $p_l(q, p_{[l]})$ is the function defined in Lemma 12.8.2. Then the alternate extended Hamiltonian \mathcal{K}_b may also be written in a form parallel to eqn (12.12)

$$\mathcal{K}_b(q, p) = p_l + H^{(l)}(q_{[l]}, p_{[l]}, q_l) \tag{12.40}$$

The traditional Hamiltonian defined in eqn (12.39) can be used to write the *traditional* Hamilton equations in an alternate form. They are, for $i = 0, \dots, (l - 1), (l + 1), \dots, D$,

$$\frac{dq_i}{dq_l} = \frac{\partial H^{(l)}(q_{[l]}, p_{[l]}, q_l)}{\partial p_i} \qquad \frac{dp_i}{dq_l} = -\frac{\partial H^{(l)}(q_{[l]}, p_{[l]}, q_l)}{\partial q_i} \tag{12.41}$$

These equations can be proved by starting from eqn (12.36), and using the same chain of logic used in Theorem 12.5.1, but now with index 0 replaced by index l . Such alternate traditional Hamiltonians are discussed, for example, in Corben and Stehle (1960).

12.10 Not a Legendre Transformation

It might seem that Hamilton equations with time as a coordinate could have been derived by the same technique as was applied to derive the traditional Hamiltonian H in Chapter 4. Since we already have an extended Lagrangian $\mathcal{L}(q, \dot{q})$ from Chapter 11, we might try to make a Legendre transformation to an extended Hamiltonian \mathcal{H} by the rule

$$\mathcal{H}(q, \dot{q}) = \sum_{k=0}^D \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \dot{q}_k - \mathcal{L}(q, \dot{q}) = \sum_{k=0}^D p_k(q, \dot{q}) \dot{q}_k - \mathcal{L}(q, \dot{q}) \tag{12.42}$$

where the definition of p_k from eqn (11.9) has been introduced. We would then express $\mathcal{H}(q, \dot{q})$ in terms of the correct variable set as $\mathcal{H}(q, p)$, leading to Hamilton equations

$$\dot{q}_k = \frac{\partial \mathcal{H}(q, p)}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial \mathcal{H}(q, p)}{\partial q_k} \tag{12.43}$$

However, this procedure fails on two counts.

First, as was proved in Lemma 11.4.2, the fact that $\mathcal{L}(q, \dot{q})$ is homogeneous of degree one⁵⁵ in the generalized velocities implies that the $\mathcal{H}(q, \dot{q})$ in eqn (12.42) is *identically* zero. We emphasize that $\sum_{k=0}^D p_k(q, \dot{q})\dot{q}_k$ is *exactly* the same function of the variables q, \dot{q} as $\mathcal{L}(q, \dot{q})$ is. Thus, not only is $\mathcal{H}(q, \dot{q})$ equal to zero, but all of its partial derivatives with respect to q_k and \dot{q}_k are also zero. We can denote this by writing $\mathcal{H} \equiv 0$. Thus the Legendre transformation method fails at its first step. The function $\mathcal{H}(q, \dot{q})$ cannot be written.

Second, even if a function $\mathcal{H}(q, \dot{q})$ could be found, we could not carry out the next step in the Legendre transformation by writing it in terms of q, p and thus transforming it into a correct Hamiltonian $\mathcal{H}(q, p)$. To make that change of variables, the equations defining the momenta in terms of the coordinates and velocities $p_k = p_k(q, \dot{q})$ would have to be solved for the \dot{q}_k giving $\dot{q}_k = \dot{q}_k(q, p)$, which could then be substituted into $\mathcal{H}(q, \dot{q})$ to give the Hamiltonian as $\mathcal{H}(q, p) = \mathcal{H}(q, \dot{q}(q, p))$. But solving for the \dot{q} is not possible. The necessary and sufficient condition for it is that the $(D + 1)$ -rowed square matrix $(\partial p/\partial \dot{q})$ defined in eqn (12.7) must be nonsingular. But that matrix was proved singular in Lemma 12.3.1.

Thus the extended Hamiltonian theory cannot be obtained by a Legendre transformation from the extended Lagrangian theory. Rather, we have used the traditional Hamiltonian function $H(q_{[0]}, q'_{[0]}, t)$ to define an extended Hamiltonian $\mathcal{K}(q, p)$ directly, as in eqn (12.12). The extended Hamilton equations eqn (12.13) then determine the phase-space trajectory of the system. The Lagrangian identity between p_0 and $-H(q_{[0]}, q'_{[0]}, t)$ is not assumed in the extended Hamiltonian theory. It is, as it were, “forgotten”. Thus, unlike $\mathcal{H}(q, \dot{q}) \equiv 0$, the extended Hamiltonian $\mathcal{K}(q, p)$ is *not* identically zero. It has nonzero partial derivatives and is *set* equal to zero at the *end* of the calculation as a consequence of the Hamilton equations of motion.⁵⁶

12.11 Dirac’s Theory of Phase-Space Constraints

A theory of primary phase-space constraints developed by Dirac is closely related to our extended Hamiltonian theory. Since the reader may already have encountered Dirac’s ideas, or will in the future, it will be useful to discuss his formalism here.

Dirac believed that the route from classical to quantum mechanics required a Hamiltonian that could be used to derive something like a Schroedinger equation, much as is done in our Section 4.7 or in his derivation of the Dirac equation to be described in Section 16.15. The identical vanishing of the $\mathcal{H}(q, \dot{q})$ defined in eqn (12.42) was therefore a problem. Dirac (1964) addressed this problem by creating what he

⁵⁵As was discussed in Section 5.15 in the context of the calculus of variations, the homogeneity of $\mathcal{L}(q, \dot{q})$ is a consequence of the completeness of the set of extended Lagrange equations. No equations have been left out that could be recovered by a “second form” of the Euler–Lagrange equations. Thus, in the Lagrangian context, $\mathcal{H}(q, \dot{q})$ vanishes because it is not needed. The reader might look again at the example in Section 5.6. If she tries to define a “second form” of the Euler–Lagrange equations in that example using the prescription $h = \dot{x}(\partial f/\partial \dot{x}) + \dot{y}(\partial f/\partial \dot{y}) + \dot{z}(\partial f/\partial \dot{z}) - f$ she will find the result to be identically zero, just as in eqn (12.42), and for the same reason.

⁵⁶In Dirac (1964), the same distinction is made. The \mathcal{H} is said to be *strongly zero*, while \mathcal{K} is said to be *weakly zero*.

called a “generalized Hamiltonian formalism” with modified Hamilton equations, and phase-space constraints in the form $\phi_m(q, p) = 0$. Dirac’s formalism is intended to include complex problems such as Hamiltonian theory in the curved spaces of general relativity. We will present only enough of it to understand its relation to the extended Hamiltonian theory that we have developed in this chapter.

Let us suppose for a moment that we have a Hamiltonian function $\mathcal{H}(q, p)$, such as the one proved not to exist in Section 12.10, as well as N phase-space constraints $\phi_m(q, p) = 0$. Dirac uses Hamilton’s Principle and the calculus of variations to derive the equations

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k} + \sum_{m=1}^N \lambda_m \frac{\partial \phi_m}{\partial p_k} \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k} - \sum_{m=1}^N \lambda_m \frac{\partial \phi_m}{\partial q_k} \quad (12.44)$$

Except for our assumption that $\mathcal{H} \equiv 0$, his derivation is similar to the one we give in Section 13.4. He rewrites these Hamilton equations by defining an effective Hamiltonian

$$\mathcal{H}_{\text{eff}}(q, p, \lambda) = \mathcal{H}(q, p) + \sum_{m=1}^N \lambda_m \phi_m(q, p) \quad (12.45)$$

in which the constraints and their Lagrange multipliers have been incorporated into \mathcal{H}_{eff} .⁵⁷ The result is that eqn (12.44) now take the standard Hamiltonian form⁵⁸

$$\dot{q}_k = \frac{\partial \mathcal{H}_{\text{eff}}}{\partial p_k} \quad \dot{p}_k = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial q_k} \quad (12.46)$$

But, of course, \mathcal{H} does vanish identically, as discussed in Section 12.10. Thus we are left with only the constraint functions

$$\mathcal{H}_{\text{eff}}(q, p, \lambda) = \sum_{m=1}^N \lambda_m \phi_m \quad (12.47)$$

Dirac makes a distinction between primary and secondary constraints, and between first-class and second-class constraint functions. He says that the primary constraints, “are consequences merely of the equations that define the momentum variables.” It is clear that our function $\mathcal{K}(q, p)$ defined in eqn (12.12) is exactly the sort of primary constraint that Dirac’s intends, since it derives from the Lagrangian definition of p_0 . Dirac defines secondary constraints as those that emerge from a consistency condition among the primary ones. He defines a first-class constraint function as one whose Poisson bracket with all of the other constraint functions is itself weakly zero. Otherwise the function is second-class.

⁵⁷This incorporation procedure is the same as is often done in Lagrangian mechanics and the calculus of variations, as discussed in Section 3.6.

⁵⁸Dirac actually writes the effective Hamilton equations in an Poisson bracket form, similar to our eqn (12.60). But the two ways of writing the Hamilton equations are equivalent.

The reader is referred to the cited reference for details of these distinctions. For our purposes here, they are unnecessary since we have proved in Section 12.3 that there is one *and only one* dependency relation in our theory. Thus $\mathcal{K}(q, p) = 0$ is a primary, first-class constraint, and the only one present.

It follows that, setting $\phi_1 = \mathcal{K}$ for the single constraint, Dirac's eqn (12.47) becomes simply

$$\mathcal{H}_{\text{eff}}(q, p, \lambda) = \lambda_1 \mathcal{K}(q, p) \quad (12.48)$$

and his effective Hamilton equations eqn (12.46) reduce to

$$\dot{q}_k = \lambda_1 \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\lambda_1 \frac{\partial \mathcal{K}(q, p)}{\partial q_k} \quad (12.49)$$

The multiplier λ_1 cannot be zero, since that would imply that $\dot{t} = 0$ in contradiction to the monotonic variation of both t and β . A nonzero λ_1 cancels from eqn (12.29) and so eqn (12.49) are equivalent to the standard extended Hamilton equations eqn (12.13). Dirac's result is therefore equivalent to our extended Hamilton equations, eqn (12.13).

It seems, at least in the context of the extended Hamiltonian theory presented in this chapter, that the principal difference between Dirac's approach and the one we have used here is that we refer to \mathcal{K} as a dependency function and an extended Hamiltonian, while Dirac would refer to the same \mathcal{K} as a primary, first-class constraint that appears in an effective Hamiltonian. The resulting equations of motion are equivalent in either case.

By basing his treatment on the calculus of variations, Dirac also assumes implicitly that any equation of motion derived from Hamilton's Principle, using a consistent Lagrangian and well-defined constraints, must be true. But we see in Theorems 6.2.1 and 13.1.2 that there are cases in which a physically incorrect equation can be derived from Hamilton's Principle with constraints. Thus, while it is interesting to derive the extended Hamilton equations from the calculus of variations, as Dirac has done and as we do in Theorem 13.4.1, it is important that we have proved them to be correct in Theorem 12.5.1 using a proof that is entirely independent of the calculus of variations.

12.12 Poisson Brackets with Time as a Coordinate

In extended Hamiltonian mechanics, all quantities of physical interest are assumed to be expressed as functions of the phase-space variables. For example,

$$f = f(q, p) = f(q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D) \quad (12.50)$$

The extended Poisson bracket of two such functions can be defined in the same way as in Section 4.6, but now with the additional variables q_0 and p_0 .

Definition 12.12.1: Extended Poisson Brackets

In the theory with time as a coordinate, the extended Poisson bracket of two phase-space

functions $f(q, p)$ and $g(q, p)$ is another phase-space function defined by

$$[f, g] = \sum_{k=0}^D \left(\frac{\partial f(q, p)}{\partial q_k} \frac{\partial g(q, p)}{\partial p_k} - \frac{\partial g(q, p)}{\partial q_k} \frac{\partial f(q, p)}{\partial p_k} \right) \quad (12.51)$$

The properties in eqn (4.54) through eqn (4.57) follow from the anti-symmetry of the Poisson bracket, and so hold also for this extended definition. They are, with f, g , and h any phase-space functions,

$$[f, g] = -[g, f] \quad [f, f] = 0 \quad (12.52)$$

$$[f, (\alpha g + \beta h)] = \alpha [f, g] + \beta [f, h] \quad [f, gh] = g[f, h] + [f, g]h \quad (12.53)$$

and the Jacobi identity

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0 \quad (12.54)$$

The derivative of a function f with respect to β can be found from its Poisson bracket. If $f = f(q, p)$ is some phase-space function and $\mathcal{K}(q, p)$ is the extended Hamiltonian, then

$$\dot{f} = [f, \mathcal{K}] \quad (12.55)$$

To see this, apply the chain rule,

$$\dot{f} = \frac{df}{d\beta} = \sum_{k=0}^D \left(\frac{\partial f(q, p)}{\partial q_k} \dot{q}_k + \frac{\partial f(q, p)}{\partial p_k} \dot{p}_k \right) \quad (12.56)$$

and then the extended Hamilton equations, eqn (12.13) to get

$$\dot{f} = \sum_{k=0}^D \left(\frac{\partial f(q, p)}{\partial q_k} \frac{\partial \mathcal{K}(q, p)}{\partial p_k} - \frac{\partial f(q, p)}{\partial p_k} \frac{\partial \mathcal{K}(q, p)}{\partial q_k} \right) = [f, \mathcal{K}] \quad (12.57)$$

as was asserted.

It follows from eqn (12.55) that quantity f is a *conserved quantity* or *constant of the motion*, defined as a quantity with $\dot{f} = 0$, if and only if it has a vanishing Poisson bracket with the extended Hamiltonian function \mathcal{K} . Equation (12.55) does not require the additional $\partial f(q, p)/\partial t$ term that was needed in eqn (4.52) using the traditional Hamiltonian. This is because any time dependence of f is already included by making the time be a phase-space variable q_0 .

The Poisson bracket technique for calculating \dot{f} also allows new constants of the motion to be found.

Theorem 12.12.2: Poisson’s Theorem

If phase-space functions $f(q, p)$ and $g(q, p)$ are constants of the motion, then the phase-space function $[f, g]$ is also a constant of the motion.

Proof: The proof is left to the reader as Exercise 12.3. □

The derivative of f with respect to some coordinate with $\dot{q}_l \neq 0$ can also be found. For example, to calculate df/dt we can use the same equation eqn (12.55) applied to the simple phase-space function $t = q_0$. Then

$$\dot{t} = [q_0, \mathcal{K}] = \frac{\partial \mathcal{K}(q, p)}{\partial p_0} \quad (12.58)$$

and so

$$\frac{df}{dt} = \frac{\dot{f}}{\dot{t}} = \frac{[f, \mathcal{K}]}{[q_0, \mathcal{K}]} \quad (12.59)$$

As in the traditional case, the extended Hamilton equations eqn (12.13) can be written as Poisson bracket expressions. They are

$$\dot{q}_k = [q_k, \mathcal{K}] \quad \text{and} \quad \dot{p}_k = [p_k, \mathcal{K}] \quad (12.60)$$

And an extended set of fundamental Poisson brackets can also be defined. They are, for all $k, l = 0, \dots, D$,

$$[q_k, q_l] = 0 \quad [q_k, p_l] = \delta_{kl} \quad [p_k, p_l] = 0 \quad (12.61)$$

Phase-space functions in extended Hamiltonian theory can be written as functions of the entire set of independent phase-space coordinates $q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D$. We might be concerned that the same physical quantity could be written in two different ways, for example as $f = f(g(p_0), q, p)$, or $f = f(g(-H), q, p)$ where $H = H(q_{[0]}, p_{[0]}, q_0)$, which might lead to different values for \dot{f} in eqn (12.55). But Exercise 12.2 shows that the same value of \dot{f} is obtained no matter which way f is written.

12.13 Poisson Brackets and Quantum Commutators

In classical mechanics, the introduction of phase space and the Hamiltonian formalism can be thought of simply as a mathematical technique that re-expresses Newton's second law in a particularly elegant manner, but that adds nothing to its physical basis. The situation is different in quantum mechanics, however. The *canonical momenta*⁵⁹ of phase space determine the wave length of a quantum wave, which in turn governs the pattern of wave interference phenomena. Thus phase space is given a new physical significance beyond that found in classical theory.

The fundamental difference between classical and quantum mechanics is that the latter is a *wave* theory, with particle interference and superposition phenomena that are unknown in the classical mechanics of particles. The connection between the

⁵⁹Note that, as discussed initially in Section 2.17, when a magnetic field acts on charged particles, the canonical momentum \mathbf{p} is different from the particle momentum \mathbf{p} . In those cases, it is the *canonical* momentum that governs interference phenomena and appears in the quantum uncertainty principle.

particle variables of classical mechanics and wave variables of quantum mechanics was originally stated in the early wave mechanics by the deBroglie relations

$$E = \hbar\omega \quad p_x = \hbar k_x \quad p_y = \hbar k_y \quad p_z = \hbar k_z \quad (12.62)$$

between the classical variables E , \mathbf{p} and the angular frequency ω and wave vector \mathbf{k} of a matter wave. (The vector \mathbf{k} points in the direction of wave travel and its magnitude k is related to the wave length λ of the wave by $k = 2\pi/\lambda$.) A plane wave in wave mechanics is then of the form

$$\psi_0 = a \exp \{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\} = a \exp \{(i/\hbar)(\mathbf{p} \cdot \mathbf{r} - Et)\} \quad (12.63)$$

The motivation for the definition $\hat{\mathbf{p}} = -i\hbar(\partial/\partial\mathbf{r})$, used for example in eqn (4.67) and Exercise 12.1, is then that the plane wave eqn (12.63) is an eigenvector of this operator

$$\hat{\mathbf{p}}\psi_0 = -i\hbar \frac{\partial}{\partial\mathbf{r}}\psi_0 = \hbar\mathbf{k}\psi_0 = \mathbf{p}\psi_0 \quad (12.64)$$

where we have introduced the common notation of using the classical variable with a hat over it to represent the corresponding quantum operator.⁶⁰ Using this same notation, the position operators in Schroedinger theory are identical to the classical variables, hence $\hat{x} = x$, etc. In the extended Hamiltonian theory, we can also introduce the operator $\hat{t} = t$, and the zeroth momentum operator $\hat{p}_0 = -i\hbar\partial/\partial t$ that obeys the eigenvalue equation

$$\hat{p}_0\psi_0 = -i\hbar \frac{\partial}{\partial t}\psi_0 = -\hbar\omega\psi_0 = -E\psi_0 \quad (12.65)$$

corresponding to the definition $p_0 = -H$.

It follows that the commutators of the quantum operators have an algebraic structure that closely resembles the fundamental Poisson brackets of the classical variables in eqn (12.61). For any wave function ψ , the commutators of the quantum operators acting on ψ are, for $k, l = 0, 1, 2, 3$,⁶¹

$$[\hat{q}_k, \hat{q}_l]_c \psi = 0 \quad [\hat{q}_k, \hat{p}_l]_c \psi = i\hbar\delta_{kl}\psi \quad [\hat{p}_k, \hat{p}_l]_c \psi = 0 \quad (12.66)$$

where we denote $\hat{q}_0 = \hat{t}$, $\hat{q}_1 = \hat{x}$, etc. Since the wave function ψ is arbitrary, these imply the operator equations

$$[\hat{q}_k, \hat{q}_l]_c = 0 \quad [\hat{q}_k, \hat{p}_l]_c = i\hbar\delta_{kl} \quad [\hat{p}_k, \hat{p}_l]_c = 0 \quad (12.67)$$

which have the same structure as the Poisson bracket relations eqn (12.61) except for the addition of the reduced Planck constant \hbar as a scale factor with units of action, and the addition of i due to the complex space used for quantum mechanical operators.

⁶⁰A wide hat will be used for quantum operators, to distinguish them from Cartesian unit vectors that are denoted by a narrow hat. Thus $\hat{\mathbf{p}}$ is a unit vector, and $\hat{\mathbf{p}}$ is a quantum operator.

⁶¹Recall that, we use a subscript c on the bracket symbol “ $]$ ” to denote a commutator of two operators. Thus $[\hat{q}_k, \hat{p}_l]_c$ is the commutator of two operators, while $[q_k, p_l]$ as in eqn (12.61) is the Poisson bracket of two phase-space functions.

This analogy between Poisson brackets and quantum commutators can also be used to state the Ehrenfest theorem of Section 4.8 in a form analogous to eqn (4.58). Assuming the simple form for the traditional Hamiltonian given in eqn (4.61), it follows from the general rules of commutators described in Section 7.1 that, for $i = 1, 2, 3$, the Ehrenfest relations in eqn (4.71) may be written as

$$i\hbar \frac{d}{dt} \langle \hat{q}_i \rangle = \langle [\hat{q}_i, \hat{H}]_c \rangle \quad i\hbar \frac{d}{dt} \langle \hat{p}_i \rangle = \langle [\hat{p}_i, \hat{H}]_c \rangle \quad (12.68)$$

where \hat{H} is the traditional quantum Hamiltonian operator defined as

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + U(\hat{x}, \hat{y}, \hat{z}, t) \quad (12.69)$$

Again, except for the addition of the $i\hbar$ factors and the necessity to take expectation values, these have the same structure as the Poisson bracket form of the traditional Hamilton equations in eqn (4.58).

The transition from classical to quantum mechanics by the replacement of phase-space variables q, p by quantum operators \hat{q}, \hat{p} leads at once to the Schroedinger equation, as seen, for example, in Exercise 12.1. However, despite the above noted motivations for it, this transition can appear to be more of a recipe than a fundamentally motivated theory. It is justified ultimately by its great success as a predictor of experimental results. The close analogies between classical phase-space variables and quantum operators, such as the analogy between Poisson brackets and quantum commutators presented in this section, suggest that any future theory that seeks to understand *why* particles sometimes behave as waves must also confront the fundamental origin of particle mass and momentum.

12.14 Exercises

Exercise 12.1 Suppose we have a traditional Lagrangian for a single particle of mass m

$$L = \frac{m}{2} (x'^2 + y'^2 + z'^2) - U(x, y, z, t) \quad (12.70)$$

where $x' = dx/dt$, etc.

(a) Write the traditional Hamiltonian $H(q_{[0]}, p_{[0]}, t)$, the extended Lagrangian $\mathcal{L}(q, \dot{q})$, and the standard form of the extended Hamiltonian $\mathcal{K}(q, p)$ for this case.

(b) Show that the Schroedinger equation derived in Section 4.7 can also be derived by putting the quantum substitutions

$$p_0 \rightarrow -i\hbar \frac{\partial}{\partial t} \quad p_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad p_y \rightarrow -i\hbar \frac{\partial}{\partial y} \quad p_z \rightarrow -i\hbar \frac{\partial}{\partial z} \quad (12.71)$$

into the dependency relation $\mathcal{K}(q, p) = 0$.

Exercise 12.2

(a) If f is defined as the compound function $f = f(g(q, p), q, p)$, show that

$$[f, \mathcal{K}] = [f, \mathcal{K}]_g + \frac{\partial f(g, q, p)}{\partial g} [g, \mathcal{K}] \quad (12.72)$$

where $[f, \mathcal{K}]_g$ is the Poisson bracket evaluated as if g were a constant.

(b) Define f_1 and f_2 as two phase-space functions that differ only in the dependency of g on either p_0 or $-H(q, p_{[0]})$,

$$f_1 = f(g(p_0), q, p) \quad f_2 = f(g(-H(q, p_{[0]})), q, p) \quad (12.73)$$

Show that the application of eqn (12.15) *after* all partial derivatives are taken gives

$$f_1 = f_2 \quad \text{and} \quad [f_1, \mathcal{K}] = [f_2, \mathcal{K}] \quad (12.74)$$

(c) Explain why this proves the assertion made in the last paragraph of Section 12.12.

Exercise 12.3

(a) Use the Jacobi identity, eqn (12.54) to prove Theorem 12.12.2.

(b) Consider the case of a single mass with phase-space variables $q, p = t, x, y, z, p_0, p_x, p_y, p_z$. The vector angular momentum of the particle is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Derive the Poisson bracket relations

$$[L_x, L_y] = L_z \quad [L_z, L_x] = L_y \quad [L_y, L_z] = L_x \quad (12.75)$$

(c) Make a short argument supporting the following proposition if true, or opposing it if false: Any single particle system in which any two of L_x, L_y , and L_z are conserved must also conserve the third one.

Exercise 12.4 Consider a system of a single particle of mass m with traditional Lagrangian

$$L(q_{[0]}, \dot{q}_{[0]}, t) = \frac{1}{2}m(x'^2 + y'^2 + z'^2) - \frac{1}{2}k(x^2 + y^2) \quad (12.76)$$

where $x' = dx/dt$, etc.

(a) Write the traditional Hamiltonian $H(q_{[0]}, p_{[0]}, t)$, the extended Lagrangian $\mathcal{L}(q, \dot{q})$, and the standard form of the extended Hamiltonian $\mathcal{K}(q, p)$ for this system.

(b) Use the extended Hamilton equations to demonstrate that L_z is conserved for this system, but L_x and L_y are not. Does this contradict the result of Exercise 12.3?

Exercise 12.5 Consider again the case of a single mass with phase-space variables $q, p = t, x, y, z, p_0, p_x, p_y, p_z$ treated in Exercise 12.3

(a) State why no system of canonical phase-space coordinates q, p can contain both L_x and L_y .

(b) Show that $[L^2, L_k] = 0$ where $L^2 = \mathbf{L} \cdot \mathbf{L}$ and $k = x, y, z$. State why a set of canonical phase-space coordinates could contain both L^2 and L_z .

Exercise 12.6 A single projectile of mass m moving in two dimensions has the traditional Lagrangian

$$L = \frac{1}{2}m \left(x'^2 + z'^2 \right) - mgz \quad (12.77)$$

where $x' = dx/dt$, etc.

(a) Write the traditional Hamiltonian $H(q_{[0]}, p_{[0]}, t)$, the extended Lagrangian $\mathcal{L}(q, \dot{q})$, and the standard extended Hamiltonian $\mathcal{K}(q, p)$ for this system.

(b) Assuming that $\dot{x}(t=0) > 0$ at time zero, and hence for all future times, write the alternate extended Hamiltonian \mathcal{K}_b in eqn (12.34) with the momentum p_x playing the role of p_t . Show that the extended Hamilton equations with \mathcal{K}_b imply that $\dot{x} = 1$.

(c) Derive an alternate traditional Hamiltonian $H^{(l)}(t, z, p_0, p_z, x)$ as described in Section 12.9. Use it to write out the alternate traditional Hamilton equations eqn (12.41). Show from these equations that p_0 and p_x are conserved quantities, with $dp_0/dx = 0$ and $dp_x/dx = 0$.

Exercise 12.7 Consider now a single particle of mass m and charge $q^{(\text{ch})}$ moving in a given electromagnetic field.

(a) Starting with the traditional Lagrangian eqn (2.103) with $N=1$, write the traditional Hamiltonian $H(q_{[0]}, \underline{p}_{[0]}, t)$, the extended Lagrangian \mathcal{L} , and the standard form of the extended Hamiltonian \mathcal{K} for this case.

(b) Show that the only difference between this \mathcal{K} and that of a free particle is that p_0 is replaced by $(\underline{p}_0 + q^{(\text{ch})}\Phi)$ and \mathbf{p} is replaced by $(\underline{\mathbf{p}} - q^{(\text{ch})}\mathbf{A}/c)$. In quantum texts this is often referred to as “the gauge-invariant substitution”. (See Exercise 11.8 for proof of its gauge invariance.)

(c) Show that putting the quantum substitutions

$$\underline{p}_0 \rightarrow -i\hbar \frac{\partial}{\partial t} \quad \underline{p}_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad \underline{p}_y \rightarrow -i\hbar \frac{\partial}{\partial y} \quad \underline{p}_z \rightarrow -i\hbar \frac{\partial}{\partial z} \quad (12.78)$$

into \mathcal{K} leads to the same Schroedinger equation as was derived in Exercise 4.7.

Exercise 12.8 Show that the definitions $\hat{p}_0 = -i\hbar \partial/\partial t$ and $\hat{\mathbf{p}} = -i\hbar (\partial/\partial \mathbf{r})$ imply eqn (12.66).

HAMILTON'S PRINCIPLE AND NOETHER'S THEOREM

Hamilton's Principle has already been treated in the context of traditional Lagrangian and Hamiltonian mechanics. The reader should review Chapter 6 since many of the ideas there also apply when time is a coordinate.

In this chapter, we present extended forms of Hamilton's Principle and the phase space Hamilton's principle based on the extended Lagrangian and Hamiltonian methods developed in Chapters 11 and 12.

We also present Noether's theorem, a method for using symmetries of the extended Lagrangian to identify quantities that are conserved during the motion of the system. Noether's theorem is a powerful technique for discovering conserved quantities in complex Lagrangian systems. We present the basics of the method in the simple context of Lagrangian systems with a finite number of degrees of freedom.

13.1 Extended Hamilton's Principle

The extended action function is defined as

$$I = I(\delta a, [q], [\eta]) = \int_{\beta_1}^{\beta_2} \mathcal{L}(q, \dot{q}) d\beta \quad (13.1)$$

Putting eqn (11.7) into this definition gives

$$\int_{\beta_1}^{\beta_2} \mathcal{L}(q, \dot{q}) d\beta = \int_{\beta_1}^{\beta_2} L(q_{[0]}, q'_{[0]}, t) i d\beta = \int_{t^{(1)}}^{t^{(2)}} L(q_{[0]}, q'_{[0]}, t) dt \quad (13.2)$$

The last expression in this equation is the same as the traditional action function in eqn (6.4), but now expressed in the notation introduced in Section 11.2 in which $q'_k = dq_k/dt = \dot{q}_k/\dot{q}_0$. The only difference between the extended action function and the traditional one is that eqn (13.1) is written in a form that uses β as the integration parameter and hence allows q_0 to be varied along with the other generalized coordinates.

Both the traditional and the extended Hamilton's Principles are an application of the calculus of variations to mechanics. The traditional Hamilton's Principle used the coordinate parametric method of Section 5.14. The extended Hamilton's Principle uses the general parametric method presented in the body of Chapter 5.

The general parametric method in the calculus of variations in Chapter 5 will be applied in the present chapter with the variable set x_1, \dots, x_N used there replaced

by the set q_0, \dots, q_D . Thus there are now $N = D + 1$ independent variables. Equation (5.3) defining the varied path in the calculus of variations now includes a variable with $k = 0$, giving

$$q_0(\beta, \delta a) = q_0(\beta) + \eta_0(\beta)\delta a \quad \text{or, equivalently,} \quad t(\beta, \delta a) = t(\beta) + \eta_0(\beta)\delta a \quad (13.3)$$

In the traditional theory, time is not varied and there is no function η_0 . We can now state the extended Hamilton's Principle, as the following theorem.

Theorem 13.1.1: Extended Hamilton's Principle

With the action integral I defined as in eqn (13.1), and assuming variations that vanish at the end points, the first-order variation δI vanishes for arbitrary δq_k if and only if the $q_k(\beta)$ of the chosen path are a solution to the Lagrange equations, eqn (11.18), with $\mathcal{Q}_k^{(\text{NP})} = 0$. That is, $\delta I = 0$ if and only if, for $k = 0, \dots, D$,

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = 0 \quad (13.4)$$

on the unvaried path.

Proof: Replacing x by q , and replacing the $f(x, \dot{x})$ by $\mathcal{L}(q, \dot{q})$, Theorem 5.5.1 proves the present theorem. \square

The extended Hamilton's Principle can also be applied in an obvious way to systems with holonomic constraints. We state the relevant theorem.

Theorem 13.1.2: Extended Hamilton's Principle with Constraints

With I defined as in eqn (13.1), and variations that vanish at the end points but are otherwise arbitrary except for the holonomic constraints given by eqn (11.56)

$$0 = G_a(q) \quad (13.5)$$

for $a = 1, \dots, C$, then the first-order variation δI about a chosen unvaried path vanishes, $\delta I = 0$, if and only if the $q_k(\beta)$ of the chosen path are a solution to the Lagrange equations in eqn (11.62)

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} = \sum_{a=1}^C \tilde{\lambda}_a \frac{\partial G_a(q)}{\partial q_k} \quad (13.6)$$

for $k = 0, \dots, D$. Equations (13.5, 13.6) are $D + C + 1$ equations in the $D + C + 1$ unknowns $q_0, q_1, \dots, q_D, \tilde{\lambda}_1, \dots, \tilde{\lambda}_C$ and can be solved for them.

Equations (13.6) are the correct equations of motion if and only if the forces of constraint do no virtual work.

Proof: This theorem follows from Theorem 5.9.1 with the replacement of variables x by q , the substitution of \mathcal{L} for f and $G_a(q)$ for $G_a(x)$. Theorem 11.10.2 proves that the right side of eqn (13.6) is correct if and only if the forces of constraint do no virtual work. \square

Note to the Reader: It is important to realize that the correctness of eqn (13.6) in mechanics depends on the condition that the forces of constraint do no virtual work. If the virtual work is nonzero, then eqn (13.6) will be false. But $\delta I = 0$ and eqn (13.5) will still imply eqn (13.6). Thus it is possible for Hamilton's Principle to imply a false equation. The reader might look at a similar note following Theorem 13.1.2 for more information.

13.2 Noether's Theorem

In previous uses of the action function, eqn (13.1), the variations δq_k have been assumed to be arbitrary except possibly for some constraints. Then $\delta I = 0$ has been proved equivalent to the condition that the unvaried path satisfies the Lagrange equations, and hence is the classical path of system motion.

But another use of eqn (13.1) begins by *setting* the unvaried path to be the classical path and also *setting* the variations δq_k to be specific functions, at least one of which must be nonzero at the end points. These δq_k are chosen so that the integral $I(\delta a, [q], [\eta])$ on the varied path has the same value as the integral $I(0, [q], [\eta])$ on the unvaried path to first order, so that $\delta I = 0$. Such a choice of the δq_k reveals what is called a *symmetry* of the Lagrangian system. The variational calculus can then be used to derive a constant of the motion associated with that symmetry. The method is described in the following theorem.⁶²

Theorem 13.2.1: Noether's Theorem

Taking the unvaried path to be the classical path, select particular functions $\eta_k(\beta)$ defined in Section 5.2 so that, to first order in the scale parameter δa ,

$$I(\delta a, [q], [\eta]) = I(0, [q], [\eta]) + o(\delta a) \quad (13.7)$$

and therefore $\delta I = 0$. At least one of the selected $\eta_k(\beta)$ must be nonzero at the end points, so that the $\delta q_k = \eta_k(\beta)\delta a$ do not all vanish there. Then the expression

$$\sum_{k=0}^D p_k(q, \dot{q})\delta q_k = \delta a \sum_{k=0}^D p_k(q, \dot{q})\eta_k(\beta) \quad (13.8)$$

with the selected functions $\eta_k(\beta)$, will be a constant of the motion.

⁶²E. Noether (1918) "Invariante Variationsprobleme," *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse*, 235. English translation, M. A. Tavel (1971) *Transport Theory and Statistical Physics*, 1(3), 183.

Noether's original paper contained two theorems. The first theorem (the *global* theorem) assumed that the δq_k are obtained from transformations, such as rotations for example, that form a group parameterized by a set of constant coefficients. The second theorem (the *local* theorem) allowed transformations whose groups were parameterized by non-constant functions. It has been suggested that Noether's global theorem and her local theorem are of independent importance, and could with justice be referred to as Noether's "first" and "second" theorems. See K. Brading and H.R. Brown, "Symmetries and Noether's Theorems" in Brading and Castellani (2003).

The theorem we call "Noether's Theorem" corresponds to the first, or global, theorem. The redundancy of the extended Lagrange equations, of which we give a direct proof in Lemma 11.9.1, is used by Noether as an example consequence of her second theorem.

Proof: Making the identifications $x \rightarrow q$, $f \rightarrow \mathcal{L}$, the integral eqn (5.17) becomes identical to eqn (13.1). The development in Section 5.4 can then be applied. Of particular interest is eqn (5.25), which now becomes

$$\delta I = \sum_{k=0}^D \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \delta q_k(\beta) \right) \Big|_{\beta_1}^{\beta_2} - \int_{\beta_1}^{\beta_2} \sum_{k=0}^D \left\{ \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} \right\} \delta q_k(\beta) d\beta \quad (13.9)$$

Since the unvaried path was chosen to be the classical path, the Lagrange equation, eqn (13.4), is satisfied on the unvaried path. Thus the integrand is identically zero and eqn (13.9) reduces to

$$\begin{aligned} \delta I &= \sum_{k=0}^D \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \delta q_k(\beta) \right) \Big|_{\beta_1}^{\beta_2} \\ &= \delta a \sum_{k=0}^D p_k(q(\beta_2), \dot{q}(\beta_2)) \eta_k(\beta_2) - \delta a \sum_{k=0}^D p_k(q(\beta_1), \dot{q}(\beta_1)) \eta_k(\beta_1) \end{aligned} \quad (13.10)$$

Since the end points β_1, β_2 are arbitrary and δa is nonzero, the assumed choice of the $\delta q_k = \eta_k(\beta) \delta a$ to make $\delta I = 0$ implies that the expression in eqn (13.8) has the same value at any two β values, as was to be proved. \square

A theorem similar to Theorem 13.2.1 can also be proved in traditional Lagrangian mechanics. But, due to the complexity of varying the time t when time is also being used as an integration variable, the proof of the theorem can be several pages long. The same result is obtained here, with much less effort, by using the extended Lagrangian approach in which the variation of $q_0 = t$ is no different from the variation of any other coordinate. The proof of Noether's Theorem just given required only one short paragraph, and is an almost trivial corollary of the extended Hamilton's Principle.

13.3 Examples of Noether's Theorem

Noether's Theorem is rather abstract, so we present several examples. First, consider the Lagrangian in eqn (11.30). Then eqn (13.1) becomes

$$I(\delta a, [q], [\eta]) = \int_{\beta_1}^{\beta_2} \left\{ \frac{m}{2i} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{i}{2} k r^2 \right\} d\beta \quad (13.11)$$

Since the integrand in eqn (13.11) does not contain the letter t explicitly, we can select variations $\eta_k = \delta_{k0} C_0$ where C_0 is some constant. Thus $\eta_k = 0$ for $k \neq 0$. Since C_0 is assumed to be a constant, the variations for $k = 0$ are $\delta t = C_0 \delta a$, and $\delta \dot{t} = \delta a \dot{\eta}_0 = 0$. Then eqn (13.7) will hold because the selected variation will change neither the integrand nor the range of integration. It follows from Noether's Theorem that

$$\sum_{k=0}^D p_k(q, \dot{q}) \eta_k(\beta) = p_0(q, \dot{q}) C_0 \quad (13.12)$$

is a constant of the motion. Thus the momentum $p_0(q, \dot{q})$ is conserved. Examination of eqn (11.31) shows that p_0 is the negative of the traditional generalized energy

function, so we have proved the conservation of that function. Thus symmetry under uniform time translation at fixed values of the other coordinates implies conservation of energy.

For another example, consider the same Lagrangian and note that eqn (13.11) also does not contain the variable ϕ explicitly. Therefore, we can choose $\eta_k = \delta_{k3}C_3$ where C_3 is some constant. Then eqn (13.7) will again be satisfied, and Nother's theorem will predict the constant of the motion

$$\sum_{k=0}^D p_k(q, \dot{q})\eta_k(\beta) = p_3(q, \dot{q})C_3 \tag{13.13}$$

Thus the momentum $p_3 = p_\phi$ is a conserved quantity. Symmetry under rotation about the z -axis implies conservation of p_ϕ , which is equal to the z -component of the angular momentum.

Of course, both p_0 and p_ϕ had already been recognized as constants of the motion in Section 11.6, since the corresponding variables t and ϕ are ignorable. A more interesting example is to treat the same problem as in Section 11.6, but using Cartesian coordinates. Then it will not be so obvious that there is a constant of the motion like p_ϕ .

In Cartesian coordinates, the extended Lagrangian becomes, with $q_0 = t$, $q_1 = x$, $q_2 = y$, and $q_3 = z$,

$$\mathcal{L}(q, \dot{q}) = \frac{m}{2i} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{i}{2}k (x^2 + y^2 + z^2) \tag{13.14}$$

and eqn (13.1) is thus

$$I(\delta a, [q], [\eta]) = \int_{\beta_1}^{\beta_2} \left\{ \frac{m}{2i} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{i}{2}k (x^2 + y^2 + z^2) \right\} d\beta \tag{13.15}$$

The obvious spherical symmetry of this integrand suggests that it is unchanged by a rotation. Consideration of eqn (8.32), the matrix for rotation about the z -axis, for a small angle $\theta = \delta a$ suggests that we might choose $\eta_0 = 0$, $\eta_1 = -y$, $\eta_2 = x$, and $\eta_3 = 0$. Then, to first order in δa ,

$$(x^{(R)} - x) = \delta x = -y\delta a \quad \text{and} \quad (y^{(R)} - y) = \delta y = x\delta a \tag{13.16}$$

agrees with the rotation produced by eqn (8.32), since the cosine function becomes just the number 1 and $\sin(\delta a) = \delta a$ to this order.

The reader can easily verify, either by direct computation or by noticing that eqn (13.14) can be written as

$$\mathcal{L}(q, \dot{q}) = \frac{m}{2i} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) - \frac{i}{2}k (\mathbf{r} \cdot \mathbf{r}) \tag{13.17}$$

which contains dot products invariant under rotations, that to first order in δa the integrand in eqn (13.15) is unchanged by this choice of the η_k , and that eqn (13.7)

is therefore satisfied. Then Noether's theorem implies that there is a constant of the motion given by

$$\sum_{k=0}^D p_k(q, \dot{q}) \eta_k(\beta) = -p_x y + p_y x \quad (13.18)$$

which is the z -component of the angular momentum, the same quantity as the p_ϕ proved constant earlier using spherical polar coordinates.

The appropriate choice of the η_k to make $\delta I = 0$ reveals a symmetry of the Lagrangian system. The value of the Noether theorem is that it allows symmetries to be translated into conserved quantities. The last example was the most interesting because rotational symmetry produced a conserved quantity even though none of x , y , z was an ignorable coordinate.

13.4 Hamilton's Principle in an Extended Phase Space

In Section 6.4, a phase-space form of Hamilton's Principle was developed in the context of traditional Hamiltonian mechanics. The reader should refer to that section for background. A similar phase-space Hamilton's Principle will be presented here, but using the extended theory in which time is a coordinate.

In the extended theory, the phase-space action function is defined by substituting eqn (11.16) into the definition in eqn (13.1). It is

$$I = \int_{\beta_1}^{\beta_2} \mathcal{L} d\beta = \int_{\beta_1}^{\beta_2} \sum_{k=0}^D p_k \dot{q}_k d\beta \quad (13.19)$$

Here, as in Section 6.4, the idea is to write a variational principle that varies all of the canonical coordinates q and p of phase space independently. In the extended phase space, this set includes the two new variables q_0 and p_0 .

The variational calculus of Chapter 5 will be applied with $N = (2D + 2)$, and the variables x_1, \dots, x_N used there defined to be the whole of the set $q_0, \dots, q_D, p_0, \dots, p_D$ listed in eqn (12.1). The function $f(x, \dot{x})$ introduced in Section 5.3 will then be identified with the integrand $F(q, p, \dot{q}, \dot{p})$ in eqn (13.19), which will be considered to be a function of the variables q , p and their derivatives

$$F(q, p, \dot{q}, \dot{p}) = \sum_{k=0}^D p_k \dot{q}_k \quad (13.20)$$

Just as in the traditional theory of Section 6.4, the definition eqn (11.9) relating the canonical momenta to the coordinates and velocities is to be forgotten here. The momenta p_k are taken to be independent of the coordinates. For example, the definition of coordinate variation in eqn (5.5) becomes the two equations

$$\delta q_k = \delta a \theta_k(\beta) \quad \text{and} \quad \delta p_k = \delta a \chi_k(\beta) \quad (13.21)$$

which hold for all $k = 0, \dots, D$, where we have replaced the shape function η by the two independent sets of shape functions θ and χ .

The phase space Hamilton's Principle also requires application of the dependency relation $\mathcal{K}(q, p) = 0$ from eqn (12.6). From the viewpoint of the calculus of variations, this dependency relation is a constraint on the variations δq and δp . However, this constraint is not of the same sort as was discussed in Chapter 3. Those constraints were enforced by forces of constraint in particular mechanical systems. The constraint $\mathcal{K}(q, p) = 0$ is unrelated to forces. It is a kinematic rather than a dynamic constraint.

We now prove the extended form of the phase-space Hamilton's Principle.

Theorem 13.4.1: Extended Hamilton's Principle in Phase Space

Given the phase space action integral

$$I = \int_{\beta_1}^{\beta_2} \mathcal{L} d\beta = \int_{\beta_1}^{\beta_2} \sum_{k=0}^D p_k \dot{q}_k d\beta = \int_{\beta_1}^{\beta_2} F(q, p, \dot{q}, \dot{p}) d\beta \tag{13.22}$$

with $F(q, p, \dot{q}, \dot{p})$ defined in eqn (13.20), the first order variation δI will be zero for all δq and δp that vanish at the end points but are otherwise arbitrary and independent except for the single constraint

$$0 = \mathcal{K}(q, p) = p_0 + H(q_{[0]}, p_{[0]}, q_0) \tag{13.23}$$

if and only if the extended Hamilton equations, eqn (12.13),

$$\dot{q}_k = \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}(q, p)}{\partial q_k} \tag{13.24}$$

are satisfied on the unvaried path.

Proof: With the identifications $x \rightarrow q, p, f(x, \dot{x},) \rightarrow F(q, p, \dot{q}, \dot{p}), G_1(x) \rightarrow \mathcal{K}(q, p)$, and setting $C = 1$, the premises in Theorem 5.9.1 become identical to those of the present theorem. With the same substitutions, the Euler-Lagrange equations eqn (5.66) become the two equations

$$\frac{d}{d\beta} \left(\frac{\partial F(q, p, \dot{q}, \dot{p})}{\partial \dot{q}_k} \right) - \frac{\partial F(q, p, \dot{q}, \dot{p})}{\partial q_k} = \lambda_1 \frac{\partial \mathcal{K}(q, p)}{\partial q_k} \tag{13.25}$$

$$\frac{d}{d\beta} \left(\frac{\partial F(q, p, \dot{q}, \dot{p})}{\partial \dot{p}_k} \right) - \frac{\partial F(q, p, \dot{q}, \dot{p})}{\partial p_k} = \lambda_1 \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \tag{13.26}$$

which are to hold for all $k = 0, \dots, D$. Evaluating the left-hand sides gives

$$\dot{q}_k = (-\lambda_1) \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -(-\lambda_1) \frac{\partial \mathcal{K}(q, p)}{\partial q_k} \tag{13.27}$$

The Lagrange multiplier λ_1 is not determined. But it is known that it must be nonzero, because a zero value would imply that $0 = \dot{q}_0 = \dot{t}$, in contradiction to the monotonic variation of both β and t along any system path. Thus the multiplier cancels from eqn (12.29) that determines system trajectories. The same trajectory is obtained no matter what nonzero value of λ_1 is used. Thus eqn (13.27) are equivalent to the standard Hamilton equations, as was to be proved. □

13.5 Exercises

Exercise 13.1

(a) Apply Noether's theorem to the extended Lagrangian \mathcal{L} in eqn (13.14), with the variational choice

$$\delta \mathbf{r} = (\mathcal{R}[\delta \Phi \hat{\mathbf{n}}] - \mathcal{U}) \mathbf{r} = \delta \Phi \hat{\mathbf{n}} \times \mathbf{r} \quad (13.28)$$

where $\hat{\mathbf{n}}$ is an arbitrary unit vector. Thus prove that the angular momentum vector \mathbf{J} is a constant of the motion for this system.

(b) Would the same be true for

$$\mathcal{L} = \frac{m}{2i} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{iq^{(\text{ch})} Q^{(\text{ch})}}{4\pi \sqrt{x^2 + y^2 + z^2}} \quad (13.29)$$

where $q^{(\text{ch})}$ is the charge of a particle moving in the electric field of another charge $Q^{(\text{ch})}$ fixed at the origin of coordinates?

Exercise 13.2 Suppose that a system of N point masses has a potential function

$$U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{n=1}^N \sum_{n'=1}^{n-1} f(\mathbf{r}_n - \mathbf{r}_{n'}) \quad (13.30)$$

Use Noether's theorem to prove that the total momentum vector \mathbf{P} is a constant of the motion.

Exercise 13.3 Show that eqns (13.25, 13.26) reduce to eqn (13.27), as asserted in Theorem 13.4.1.

RELATIVITY AND SPACETIME

When it was proposed at the start of the twentieth century, special relativity was revolutionary and controversial. Now, some hundred years later, the consensus in physics is that, at least in local spaces where the effects of curvature are negligible, standard special relativity will reliably predict the outcome of any experiment to which it is applied. The Lorentz transformation is considered a symmetry of nature, and new theories are crafted with Lorentz invariance as a necessary feature.

In the early twenty-first century, the remaining task is to better understand the *implications* of the special and general theories of relativity, to incorporate them fully into our theoretical understanding. In particular, the task of unifying relativity and quantum theory is incomplete, and a quantum theory of gravity remains elusive. This book attempts to prepare the reader for this process of incorporation by at least presenting elementary Lagrangian and Hamiltonian mechanics, and the theory of canonical transformations, in a way that uses time as a coordinate and so does not exclude special relativity from the outset.

In the present chapter, we introduce the ideas of special relativity with the assumption that it is an established theory whose experimental efficacy is unquestioned. We attempt to help the reader understand that theory more deeply, to see what the relativistic effects may be telling us about the world and the nature of spacetime. Those who are familiar with elementary special relativity from earlier study may find Section 14.5, which analyzes relativity from a surveyor's viewpoint, to be an interesting counterpoint to their previous reading.

14.1 Galilean Relativity

The relativity principle of Newtonian physics underlies all discussions of relativity, so we begin with it. This relativity principle is commonly called Galilean relativity, although the same ideas are found in the *Principia*, and in works before Galileo. Certainly, Galileo gave a detailed and poetic statement of it in a book of wide influence, the *Dialog Concerning the Two Chief World Systems* (Galilei, 1632).

Galileo uses the example of passengers in a ship, below decks, with no view outside. Galileo's protagonist Salviati asks his listeners to imagine that "there are with you some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it." The behaviors of all these things are to be observed with the ship standing still in harbor. Salviati then tells his audience to "have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that.

You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.”

Although Galileo made enormous discoveries, such as that the trajectory of a projectile can be considered as simultaneous independent horizontal and vertical motions, his physics was still tied to the Earth. He believed that an initially horizontal *unforced* motion would continue in a circular path parallel to the surface of the Earth forever.⁶³ So, when Galileo speaks of a ship at rest or in uniform motion, he means with reference to the Earth.

Newton’s physics was free of such earthly constraints. He held that an unforced motion would continue in a straight line with constant speed, forever. In the Scholium of the Principia, Newton uses the same ship analogy as used by Galileo. But the reference for determining rest and constant velocity is plainly stated to be absolute space rather than the Earth.

The Galilean relativity principle, modified to refer to Newton’s absolute space, can be translated into a modern idiom as follows.

Definition 14.1.1: Galilean Relativity Principle

Consider a closed room with walls sufficient to prevent the detection of signals from outside it. No experiment performed entirely inside such a room can detect whether the room is at rest or moving with constant velocity relative to absolute space.

In Newtonian physics, both the room and the objects in it are moving in an external absolute space. Distances measured relative to this space are compounded, using what we now refer to as the rules of vector addition. Using Figure 14.1, and assuming for simplicity that the two coordinate systems coincide at time zero and are oriented with their relative velocity along both x -axes, the relations between measured distances and times are

$$t = t' \quad x = Vt' + x' \quad y = y' \quad z = z' \quad (14.1)$$

where $\mathbf{V} = V\hat{\mathbf{e}}_1 = V\hat{\mathbf{e}}'_1$ is the constant velocity of the S' system relative to the S system.

Equation (14.1) is called the *Galilean coordinate transformation*. Time transforms identically, $t = t'$, because in Newtonian physics time is taken to be an absolute quantity. Taking differentials and dividing by $dt = dt'$ leads at once to the Galilean velocity transformation formula. Denoting $v_x = dx/dt$, $v'_x = dx'/dt'$, etc., it is

$$v_x = V + v'_x \quad v_y = v'_y \quad v_z = v'_z \quad (14.2)$$

Since V is constant, another time derivative then shows that acceleration is invariant. With $a_x = dv_x/dt$, $a'_x = dv'_x/dt'$, etc., the transformation of acceleration is

$$a_x = a'_x \quad a_y = a'_y \quad a_z = a'_z \quad (14.3)$$

Hence, in the Newtonian model of point masses acted on by forces, the Galilean relativity principle holds if the forces and masses are also invariant. If $\mathbf{f} = \mathbf{f}'$ and

⁶³See the discussion in Chapter IV of Koyré (1957).

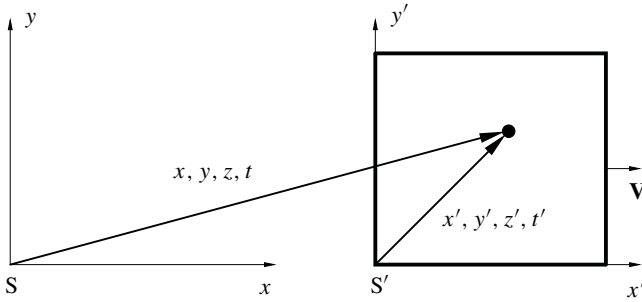


FIG. 14.1. A closed room moves with constant velocity V relative to a fixed coordinate system S . The S and S' axes coincide at time zero.

$m = m'$, then eqn (14.3) implies that the equation of motion relative to the body-fixed coordinates, $\mathbf{f}' = m'\mathbf{a}'$, is the same whether the room is at rest with $V = 0$ or moving at constant-velocity.

In Newtonian physics, coordinate systems moving uniformly with respect to absolute space are called *inertial reference systems*. The second law $\mathbf{f} = m\mathbf{a}$ holds when distances are measured with respect to any inertial system. And the Galilean relativity principle says that no local experiment can distinguish between different inertial systems.

14.2 Conflict with the Aether

The nineteenth century discovery that light is a transverse electromagnetic wave suggested that some medium, which came to be called the *aether*, must exist in which that wave propagates. Waves, after all, are not things; they are states of collective motion of something else. For example, a wave in the ocean is not itself a physical thing but rather a propagating collective motion of the surface water. And the transversality of the light waves suggested strongly that the medium had to have a rigid structure, be something like a crystal of solid material. And yet, this medium of propagation was invisible, unobserved except for the existence of the waves. One nineteenth century observer⁶⁴ commented wryly that, “For more than two generations the main, if not the only, function of the word ‘aether’ has been to furnish a nominative case to the verb ‘to undulate’.”

There is an obvious conflict between the aether and the Galilean principle of relativity: An observer in a closed room can do experiments with light waves. It was assumed in the nineteenth century that matter moved through the aether with no resistance and that the aether was all pervasive, filling all of the Newtonian absolute space and presumably at rest relative to it. So, no matter how closed the room was in Definition 14.1.1, it would be impossible to exclude the aether. Moreover, the fact

⁶⁴From Lord Salisbury’s presidential address to the British Association in 1894 as quoted in the Encyclopedia Britannica, 1911 edition, under entry AETHER.

that an observer in the room could even *do* experiments with light proved that aether must be present in the room to act as the propagation medium.

But measurement of the speed of light within a room would allow an observer there to measure his own velocity relative to the aether and therefore assumedly with respect to absolute space. This is a direct violation of the Galilean relativity principle, which asserts that V cannot be detected by experiments done entirely inside a closed room. Using eqn (14.2) and the assumption that all light propagates with a speed c relative to an aether that is at rest in absolute space, the speed of light relative to the room should be $c - V$ in the forward direction of the room's motion, and $c + V$ in the backward direction. The analogy is to a helicopter (the closed room) flying over a lake (the aether). If water waves on the lake's surface have speed c relative to the lake and the helicopter has speed V relative to the lake, then the waves will appear to the pilot to move at speed $c - V$ in the direction of the helicopter's motion as the helicopter partially overtakes them. Thus measurement of the speed of light in different directions inside the closed room could allow V to be determined, which would violate Definition 14.1.1.

At the end of the nineteenth century, it became technically possible to detect the speed of a room on the Earth relative to the aether, using the Michelson–Morley interferometer. The experiment failed to detect this motion, even though the aether theory predicted an effect well within the detection limits. This null result was a severe crisis for the aether theory.

14.3 Einsteinian Relativity

The conflict between the Galilean relativity principle and electrodynamics is not limited to light waves in an aether. Also, the Maxwell equations governing the basic dynamics of electric and magnetic fields are not form invariant under the Galilean transformation, eqn (14.1). Lorentz⁶⁵ used the requirement that the Maxwell equations must be form invariant under transformations from the aether system to other inertial systems to derive what is now called the *Lorentz transformation*. Under the same conditions as eqn (14.1), the transformation equations derived by Lorentz are

$$ct = \Gamma(ct' + Bx') \quad x = \Gamma(Bct' + x') \quad y = y' \quad z = z' \quad (14.4)$$

where c is the speed of light relative to the aether, and $B = V/c$, $\Gamma = (1 - B^2)^{-1/2}$ are unitless constants derived from the constant relative speed V of the two systems. But, since he shared the common view that the Newtonian time was absolute, Lorentz referred to the transformed time t' in the moving system as “local time” as opposed to t which he called the “true time.”

Einstein's great contribution was to extend the Newtonian concept of time. He realized that the transformation of Lorentz could be re-derived from the Galilean

⁶⁵H.A. Lorentz (1904) “Electromagnetic Phenomena in a System Moving with any Velocity Less than that of Light,” *Proceedings of the Academy of Sciences of Amsterdam*, 6. English translation reprinted in Einstein, Lorentz, Minkowski and Weyl (1923).

relativity principle provided that: (1) The time t' is not less “true” than t , but is the time actually measured by suitably synchronized physical clocks at rest in the moving system, and (2) The Galilean relativity principle is assumed to include the assertion that the speed of light in any direction has the *same* value c when measured in *either* the S or the S' system.

After an introduction describing the synchronization of clocks, Einstein's first 1905 paper states two axioms.⁶⁶ They are:

1. The Principle of Relativity: *The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion.*
2. The Principle of the Constancy of the Velocity of Light: *Any ray of light moves in the “stationary” system of coordinates with the determined velocity c , whether the ray be emitted by a stationary or a moving body.*

The first axiom is very close to being the Galilean relativity principle, simply translated into coordinate language.⁶⁷ It differs from Definition 14.1.1 principally in its careful avoidance of the idea of an absolute space.

Einstein's second axiom is curious. An aether theorist of Einstein's day would have had no difficulty in accepting it, for the aether theory says that the velocity of light is determined relative to the aether, and is not affected by the speed of the source of the disturbance. The analogy is for the pilot of the helicopter mentioned above to throw a stone in the lake. The speed of the water ripples (the light) relative to the lake (the aether) does not depend on the velocity of the stone (the light source).

It is only in connection with a *strict* interpretation of Axiom 1 that the Principle of the Constancy of the Velocity of Light has relevance. Ten paragraphs into his article, Einstein states that relevance, “light (as required by the constancy of the velocity of light, in combination with the principle of relativity) is also propagated with velocity c when measured in the moving system.”

To carry out the derivation of the Lorentz transformation, Einstein makes some plausible auxiliary assumptions. He uses the same simplified geometry as in Figure 14.1, and assumes that the three moving coordinate axes (which are perpendicular when viewed in the S' system by construction) appear perpendicular, and parallel to the S system axes, when viewed from the stationary S system. He uses a particular method for synchronizing co-moving clocks, and also assumes that the transformation equations must be linear. A number of good treatments of the derivation of the Lorentz transformation from Einstein's axioms can be found in the literature, so it will not be repeated here. However, probably the clearest and most careful derivation is

⁶⁶A. Einstein (1905) “On the Electrodynamics of Moving Bodies” *Annalen der Physik*, 17. Translated in Einstein, Lorentz, Minkowski and Weyl (1923).

⁶⁷In A. Einstein (1916) “The Foundation of the General Theory of Relativity,” *Annalen der Physik*, 49, Einstein says that this postulate, “is also satisfied by the mechanics of Galileo and Newton.”

that given by Einstein himself, in his original paper. The reader is urged to study at least the first four sections of Einstein's 1905 article.

One important point about Einstein's approach is that he presents special relativity as what is called a *Principle Theory*. He begins with axioms rather than experiments or deductions from other theories. With such a theory, experimental verification is crucial. And special relativity has been abundantly verified. But there is then a lingering question as to *why* those axioms are true, and what physical effects underlie them.

14.4 What Is a Coordinate System?

The Lorentz transformation is a transformation between coordinate systems. But the interesting question is how even *one* of these coordinate systems can be set up. The nineteenth century image of a coordinate system as three perpendicular sticks, with an observer at their intersection holding a clock, is not adequate. Among other problems, the finite speed of light would lead to a time delay between a distant event⁶⁸ and its observation by the observer at the origin.

To avoid these systematic time-delay errors, it is necessary to have many stationary observers (human or robot), each equipped with a synchronized clock, placed throughout the region of observation. Each observer makes observations only in his immediate *local* neighborhood, whose size is small enough to make time delays negligible. These clocks and observers form an *observer team*, which replaces the single observer of nineteenth century physics.⁶⁹

Aside from the nontrivial problem of synchronizing all of the clocks, it is important to realize the essential level of abstraction that is introduced by the use of an observer team instead of a single observer. Because observations are made by a team, the combined result of their observations is never what is actually seen by any single observer. The observers in the team make only local observations, which consist of the x , y , z address of the observer and the time t of the observed local event as measured on that observer's clock. These are the unprocessed raw data.

After the experiment is over, all of these scattered observations are reported back to some central data collector, say the observer at the origin. She may use the collected data to construct something like a motion picture or video animation, possibly a stereographic one, consisting of a sequence of frames, each frame labeled with a different value of *reported* time. She places in the animation frame labeled with a particular value of t only those reported events whose local observer assigned that particular time to them. She then projects these frames on a screen in a time sequence, perhaps in slow motion to help the viewer. Viewing this motion picture is the closest that we can come to "seeing" in a coherent way what happened in the experiment.

⁶⁸An *event* in relativity theory is a point in both space and time. It can be thought of as the very brief flash of a very small flash bulb. Analysis of experiments in relativity is aided by breaking them down into a succession of events.

⁶⁹Other schemes to eliminate the time-delay problem can be imagined. But they will be equivalent to the one presented here, which has the virtue that it illustrates clearly the issues raised.

When pondering the seemingly strange features of special relativity, the reader should remember that our truest picture of spatiotemporal reality is this artificially constructed composite, assembled after the fact from the separate observations of a team of observers.

In the following section, we consider the physical steps required to set up a coordinate system. The process is presented as a survey, a series of experiments. The end result is a team of observers, each with a definite address and a synchronized clock, ready to make observations.

14.5 A Survey of Spacetime

One way to understand better what a coordinate system is, and what lies behind the success of special relativity, is to imagine a coordinate system as the end result of a series of experiments—a survey of spacetime. Let us imagine that a group of twenty-third century physicists and engineers, carrying instrumentation sufficient to detect the very small corrections due to relativistic effects at small speeds, is sent to an empty region of space far from significant masses, where special relativity should apply. Imagine that they conduct a survey in which they confront the *effects* predicted by Einstein's special relativity theory, and are forced to work around them to establish a coordinate system.

In describing this imaginary task, we assume the experimental *efficacy* of standard special relativity, that the *effects* predicted by it will certainly be observed experimentally. Experimental results that the survey group will certainly observe due to their being predicted by standard special relativity⁷⁰ will be marked with the acronym "SR". As discussed in the introduction to this chapter, there is currently no doubt that the *predictions* of standard special relativity are correct. So, in this imaginary exercise, rather than starting with the principle of relativity and deriving the relativistic effects such as time dilation and Lorentz contraction, we reverse the order. We begin with the effects and show how they force a survey team to arrive at a coordination of spacetime which has the principle of relativity as one of its features.

A possible source of caution for any approach of this sort is that rods and clocks—taken to be primitive things in the survey described—are anything but simple objects. Any treatment of spacetime using rods and clocks faces this essential circularity: Space and time cannot be defined without rods and clocks, but the theory of rods and clocks (electromagnetism and quantum theory) cannot even be begun until space and time are defined. We can argue only that the general success of physics as a theoretical and experimental enterprise suggests that this circularity is benign and does not lead to paradox.

⁷⁰Standard special relativity is taken here to include the axiom that (when possible inertial forces are taken account of) sufficiently local and brief measurements by a moving observer will yield values identical to those that would be obtained in an inertial coordinate system relative to which the observer is instantaneously at rest.

14.5.1 Space Frame

The survey group's first task is to establish a framework with respect to which position and velocity can be defined. Imagine the group to be equipped with measuring rods which are assembled into a cubical grid with equal cells of some standard length and with right-angle corners defined by, for example, use of 3-4-5 right triangles. Construction of such a grid is treated by a number of authors, all of whom seem to feel it to be straightforward, so we will not dwell on this stage here.⁷¹ Assume the grid to be arbitrarily oriented but non-accelerating and non-rotating, as controlled by placing small test masses at rest at three or more non-linear points and observing them not to drift relative to the grid. Imagine also that the survey group has chosen an initial state of motion for the grid such that the local flux distribution of the three-degree cosmic radiation field will appear to have no low-order spherical harmonics, indicating that the inertial grid is co-moving with the local expansion of the universe. Choosing some arbitrary vertex in the grid as the origin point $(0, 0, 0)$, any other point may be given an address (x, y, z) by counting along three perpendicular directions from the origin and setting, for example, $x = n_x \ell_0$ where ℓ_0 is the standard length of rods in the grid, and n_x is the number of y - z planes pierced. Finer position gradations can be made by interpolation between vertices, using any signal whose speed is independent of position and direction (as in the acoustic spark chamber, for example). We assume also that the three coordinate directions are chosen to make a right-handed coordinate system.

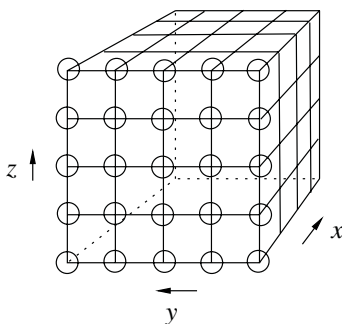


FIG. 14.2. Cubical grid used by the S group to set spatial locations. The circles represent the clocks (only those on the y - z plane are shown) that will be placed at each vertex.

Suppose that the length $\Delta \ell$ of a rod is determined by laying it out along one of the coordinate directions and counting the number of standard rods along its length. It will be observed (SR) that the Pythagorean rule

$$\Delta \ell = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \quad (14.5)$$

holds when this same rod is placed at rest between any two points in the grid regard-

⁷¹For example, see Bridgman (1962) or Taylor and Wheeler (1992).

less of their location (homogeneity) and relative orientation (isotropy) where, e.g., Δx is the difference in the x-values of the ends of the rod. Homogeneity in time will also be observed, so that the $\Delta \ell$ for a static stick arrangement agrees with eqn (14.5) no matter *when* it is measured.

14.5.2 *Clocks*

Now the survey group introduces standard clocks. Clocks for our purposes are physical devices that tick regularly so that the passage of time can be measured by $t = n_t T_0$ where T_0 is the standard tick time and n_t is the number of ticks since some arbitrary zero of time for that particular clock.

The accuracy of standard clocks of a given type could be estimated by placing a group of identical clocks side-by-side and comparing the time interval they record between a pair of local events. The variance of these measurements could be taken to be a measure of the intrinsic accuracy of clocks of that type. Prospective new generations of clocks are taken to be successful if they pass this mutual consistency test to some new level of accuracy, and also if they agree with the previous generation's clocks to within the accuracy of that previous generation.

A clock that measures time by exploiting a natural cyclic process of some sort, with care taken to isolate the process and avoid interfering with its regularity, is taken to be a good clock. It would, of course, always be possible to make a bad clock from a good one by, for example, adding or subtracting a bit from its counter after every n counts. If all clocks were submitted to this same "cooking" (where n might even be chosen differently at different spatial locations relative to the cubical grid, or at different times if a common zero of time were defined in some way), then they all would still pass the above accuracy test when placed together. But they would not be good clocks for our purposes. The statements we make about the behavior of standard clocks below refer to good clocks.

14.5.3 *Round-Trip Light Speed*

Let the survey group place a good, standard clock at any vertex A of the grid, and send a light pulse from that point to some other vertex B, from which it is reflected back to A. The total round-trip time Δt of the light pulse can be used to define the *round-trip speed* of light c as

$$c = \frac{2\Delta \ell}{\Delta t} \quad (14.6)$$

where $\Delta \ell$ is the Pythagorean distance between the clock and the reflecting vertex. This speed will be observed (SR) to be the same regardless of the location of the clock and the other vertex. Homogeneity in time will also be observed, the same value c being obtained no matter *when* the measurement is begun. Notice that only one clock was used in the light speed determination, so only the average round-trip light speed is measured directly.

14.5.4 *The One-Way Speed of Light: Unknown but Static*

The formula eqn (14.6) makes sense because we believe that a light pulse actually propagates from A to B and back during the time interval Δt . Although (since we do not yet have synchronized clocks at A and B) we cannot yet measure the time Δt_{ab} that the light pulse takes to go from A to B, nor the time Δt_{ba} that it takes to return from B to A, we can assume that any future determination of those quantities must have the property $\Delta t = \Delta t_{ab} + \Delta t_{ba}$ necessary for consistency with eqn (14.6).

This consistency condition, together with the isotropy and homogeneity (both spatial and temporal) observed for the two-way speed defined in eqn (14.6), can be used by the survey group to establish that both Δt_{ab} and Δt_{ba} , and hence the corresponding speeds, $c_{ab} = \Delta \ell / \Delta t_{ab}$ and $c_{ba} = \Delta \ell / \Delta t_{ba}$, must be *static* quantities. That is, even though we do not yet have a method of determining Δt_{ab} , we know that its eventual value must be found to be the same regardless of *when* the initial pulse leaves A. And a similar statement can be made for Δt_{ba} .

To derive the static nature of these one-way times, suppose that the survey group places a clock at A and another one at B. These clocks are not synchronized. A pulse of light is emitted from A, is reflected from B, is re-reflected from A, and so on. Due to the homogeneity, isotropy, and static nature of the round-trip speed of light in eqn (14.6), we know that the round trip times for all ABA paths (measured by the clock at A) and also for all BAB paths (measured by the clock at B) will be the same: $\Delta t = 2\Delta \ell / c$. If we denote the initial departure from A by index 1, the first reflection at B by index 2, the next reflection at A by index 3, etc., we have

$$2\Delta \ell / c = \Delta t_{ab}^{12} + \Delta t_{ba}^{23} = \Delta t_{ab}^{34} + \Delta t_{ba}^{45} = \dots \quad (14.7)$$

for all round trips beginning and ending at A, and

$$2\Delta \ell / c = \Delta t_{ba}^{23} + \Delta t_{ab}^{34} = \Delta t_{ba}^{45} + \Delta t_{ab}^{56} = \dots \quad (14.8)$$

for all round trips beginning and ending at B. Combining these equations we obtain

$$\Delta t_{ab}^{12} = \Delta t_{ab}^{34} = \Delta t_{ab}^{56} = \dots \quad \text{and} \quad \Delta t_{ba}^{23} = \Delta t_{ba}^{45} = \Delta t_{ba}^{67} = \dots \quad (14.9)$$

which demonstrates that Δt_{ab} is always the same, no matter how long the experiment is run. The same is true for Δt_{ba} . Thus these are static quantities.

14.5.5 *Standard Clocks at Rest but Separated*

Imagine that the survey group tests a collection of standard clocks to make sure that they tick at the same rate when placed together, for example by comparing their recorded time intervals between a pair of local events. Now the clocks are dispersed and placed at rest at the various vertices of the cubic grid. The survey group can determine that they still tick at the same equal rate as they did before dispersal. Note that these clocks were not synchronized initially, and are not assumed to be synchronized after their final placement.

To establish the common ticking rate of a clock at A with one at B, imagine a pulse of light to be emitted from A at time t_{a1} and another one at a later time t_{a2} , both measured by the stationary clock at A which thus records the time interval $(t_{a2} - t_{a1})$ between these emissions.

Using the static, one-way transit time Δt_{ab} derived in Section 14.5.4, the pulses arrive at B separated by the time interval $(t_{b2} - t_{b1}) = (t_{a2} + \Delta t_{ab2}) - (t_{a1} + \Delta t_{ab1}) = (t_{a2} - t_{a1})$, where the static nature of Δt_{ab} was used to cancel Δt_{ab2} and Δt_{ab1} . Thus if the clock at B records times t_{b1} and t_{b2} for the arrival of the two pulses there, the group will observe (SR) that $(t_{b2} - t_{b1})$ agrees with the quantity $(t_{a2} - t_{a1})$. This establishes that the clock at B is *ticking at the same rate* as the one at A, even though the two clocks are not synchronized.

Note that these standard clocks do not yet define an extended time-measurement system, since they are not yet synchronized. But synchronization is just the setting of the zero point of the clock's counter. Even without synchronization, the group has established a distributed set of clocks that are known to tick at the same rate and so define a common measure of time *interval* at any fixed vertex throughout the grid.

14.5.6 Proper Velocity

If some method of clock synchronization were already in place, the group could define the velocity of a moving object by observing a distance traveled relative to the grid and dividing that number by the elapsed time measured from a pair of clocks at the beginning and end of the travel.

But such a measurement would require the use of at least two observers, and synchronization of two separated clocks has not yet been accomplished. So, the survey group reverts to another velocity definition that will prove useful, the *proper velocity*. An observer riding on or with a moving object carries a clock. As he moves, he looks at the cubical grid and counts the number of sticks of the grid that he passes during an elapsed time $\Delta\tau$ measured *on the moving clock*. This interval $\Delta\tau$ is called the “self” or “proper” time interval, since it is measured on *on that single, moving clock* carried by the observer. Then the x -component of the proper velocity, for example, is defined as $u_x = \Delta x / \Delta\tau$, where Δx is the number of y - z planes pierced by the motion times their standard separation ℓ_0 and $\Delta\tau$ is the elapsed time measured by the moving clock. (We assume that, by using finer gradations of the grid, this quantity can be measured to any desired accuracy and treated as a continuous variable in the usual way.)

The proper speed is defined as $u = \Delta\ell / \Delta\tau$. The Pythagorean theorem of eqn (14.5) then gives

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} \quad (14.10)$$

The survey group will observe (SR) that the proper speed defined in this way can range from zero to infinity. (In the Minkowskian language to be introduced in Chapter 15, proper velocity is the spatial part of the velocity fourvector.)

14.5.7 *Discovery of Time Dilation*

Temporarily synchronizing two standard clocks at rest at the same location, and then moving one of them around a closed circuit and back to the original location, the survey group will discover that the two clocks have lost their synchronization. Turning this observation into a controlled experiment, they place a standard clock at rest at point A of the grid, and let another clock move around some closed (but not necessarily circular) path, returning periodically to A. They let the moving clock maintain a constant proper speed u . The time between successive crossings of point A is Δt_{aa} according to the fixed clock at A, and $\Delta \tau_{aa}$ according to the moving clock. They find (SR) that these quantities are always related by the formula

$$\Delta t_{aa} = \Delta \tau_{aa} \sqrt{1 + u^2/c^2} \quad (14.11)$$

where c is the round-trip speed of light defined above.

The question now arises as to the source of this time difference. Is its motion distorting the moving clock, or is the motion of the moving clock distorting the fixed one? The group answers this question by doing two experiments at the same time, with two moving clocks whose different trajectories both pass through A, each experiment using the same fixed clock at A. If the presence of a moving clock upsets the running of the stationary one, then a second moving clock in the same vicinity should upset the relation eqn (14.11) for the first one. But the presence of the second moving clock has no such effect (SR). Also, the group may observe, using the techniques defined above, that the stationary clock at A continues to tick at the same rate as all the other stationary clocks distributed throughout the grid. The conclusion is that the moving clock is the one affected by the motion, and not the stationary one.

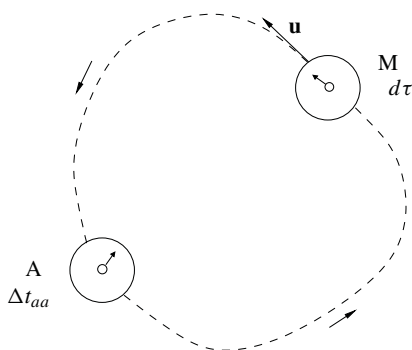


FIG. 14.3. Moving clock M moves on a closed path starting and ending at fixed clock A. The time for one cycle is Δt_{aa} on the fixed clock. Clock M records interval $d\tau$ while its proper speed is u .

The next question is whether the moving clock accumulates its distortion continuously during its motion. To test the hypothesis that it is continuously distorted, the group does a series of experiments in which the moving clock has varying proper

speed. The result in all cases is correctly reproduced by (SR)

$$\Delta t_{aa} = \oint_A^A \sqrt{1 + u^2/c^2} d\tau \quad (14.12)$$

where $d\tau$ is a differential time interval on the moving clock measured at an instant at which the proper speed is u . The conclusion is that time intervals $d\tau$ measured on a moving clock are *at each instant* distorted by the current value of the factor $\sqrt{1 + u^2/c^2}$ and hence eqn (14.12) implies the *differential* relation

$$dt = d\tau \sqrt{1 + u^2/c^2} \quad (14.13)$$

between the proper time interval $d\tau$ actually measured on a moving clock, and what we may call a *corrected or undistorted time* interval dt .

After this correction, a clock moving in any closed path starting and ending at a stationary clock at A will agree with that stationary clock as to the time elapsed during the transit.

$$\Delta t_{aa} = \oint_A^A dt = \oint_A^A \sqrt{1 + u^2/c^2} d\tau \quad (14.14)$$

The corrected time interval dt in eqn (14.13) removes the distortion of the moving clock coming from its motion and reduces it to consistency with the stationary ones. The survey group has thus discovered how to measure the standard time interval dt using moving clocks.

Some caution is required here, however. To derive a differential relation such as eqn (14.13) from a closed integral relation such as eqn (14.12), one must eliminate the possibility of an integrable contribution to the right-hand side of eqn (14.13) of the form $d\Phi$ where $\Phi(x, y, z)$ is some single-valued function of position in the grid. Such a contribution would integrate out of eqn (14.12), and could not be detected by any of the closed-path experiments.

The survey group may reasonably use a symmetry argument to eliminate this possibility. The survey is taking place far from any significant gravitating objects, in a cubical grid which is in free fall. The survey group has even taken the precaution of choosing a state of motion which makes the local flux of the three-degree cosmic radiation field appear to be maximally isotropic on the sky. Any maximum or minimum of function $\Phi(x, y, z)$ would select a particular place in the grid as special, in contradiction to the apparent homogeneity of the local space. And any nonzero gradient of function $\Phi(x, y, z)$ would define a particular spatial direction, in contradiction to the apparent isotropy. The survey group can find no physical source of such a violation of homogeneity and isotropy and so will conclude that any such function $\Phi(x, y, z)$ would have to be a constant, and hence that $d\Phi$ would be zero.

Still, the presumed total homogeneity and isotropy of the local environment remain conjectures by the survey group.

14.5.8 Corrected Velocity from Corrected Time Interval

A moving clock was used above to define its own proper velocity $\mathbf{u} = d\mathbf{r}/d\tau$, with components u_x, u_y, u_z . The corrected time interval of eqn (14.13) can then be used by the survey group to define what we will call a *corrected velocity* \mathbf{v} .

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\tau} \frac{d\tau}{dt} = \frac{\mathbf{u}}{\sqrt{1 + u^2/c^2}} \quad (14.15)$$

and a corresponding corrected speed $v = u/\sqrt{1 + u^2/c^2}$. These two speeds are thus related by

$$\sqrt{1 + u^2/c^2} \sqrt{1 - v^2/c^2} = 1 \quad (14.16)$$

so that eqn (14.15) may also be written in inverse form as

$$\mathbf{u} = \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma \mathbf{v} \quad (14.17)$$

where the notation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{1 + u^2/c^2} \quad (14.18)$$

has been introduced.⁷² As the proper speed ranges from zero to infinity, the corrected speed ranges from zero to c , the round-trip speed of light.

Note that the survey group has derived the corrected velocity \mathbf{v} *without* making any use of synchronized clocks. All measurements up to this point have been quite independent of clock synchronization and have depended only on the experimentally determined *rates of ticking* of clocks in various states of motion.

14.5.9 Discovery of Lorentz Contraction

The survey group may suspect that, since clocks are distorted by motion, possibly measuring sticks such as those used in the grid might also be affected. Possible distortion of dimensions perpendicular to the motion is easily tested. Imagine the group to make a square hole in a metal plate, and a metal cube that just passes through the hole at perpendicular incidence and near zero speed. If they now pass the cube through the hole at higher and higher proper velocities, they will see no change in the tolerance of its passage through the hole. They conclude that dimensions perpendicular to a velocity are unchanged by it.

The dimension along the motion is more difficult to measure. If there were a system of synchronized clocks available, which at this point there is not, the group could simply observe the leading and trailing ends of the moving cube at some single synchronized time. In the absence of such a system of synchronized clocks, the survey must resort to a more indirect method.

⁷²Note that this γ is *not* the same quantity as the constant Γ appearing in the Lorentz transformation eqn (14.13). The Γ is a function of V , the constant relative velocity of the S and S' systems. But γ is a variable quantity that changes as the clock's velocity changes.

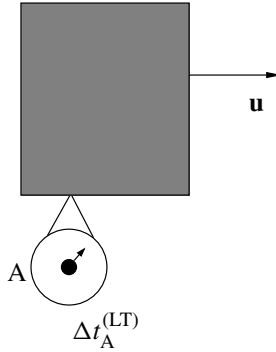


FIG. 14.4. A cube of rest dimension L_0 moves past a fixed clock A. The cube has proper speed u as measured by a clock riding on the cube (not shown). The clock A records a time interval $\Delta t_A^{(LT)}$ required for the cube to pass over it.

Imagine the moving cube to be carried, at constant proper velocity perpendicular to one face, past a fixed clock at A. The time interval taken by the cube to pass A is measured by the clock fixed at A to be $\Delta t_A^{(LT)}$, the time interval between passage of the Leading and Trailing faces. In addition to this fixed clock, now imagine another, separate and not synchronized, clock to be *attached to* the moving cube. The proper speed of the cube u can be measured using that attached clock in the manner described above. The group then uses eqn (14.15) to remove the error caused by the motion of the attached clock and so calculate the corrected speed of the moving cube v . The length of the cube, in the dimension along the line of its velocity, is then defined to be its corrected velocity multiplied by the time interval $\Delta t_A^{(LT)}$ required for it to pass by the fixed clock at A.

$$L = v \Delta t_A^{(LT)} \tag{14.19}$$

If the sides of the cube measured at rest are L_0 , it will be observed (SR) that

$$L = \frac{L_0}{\sqrt{1 + u^2/c^2}} = L_0 \sqrt{1 - v^2/c^2} \tag{14.20}$$

Note that the clock at A requires no correction, since it is a standard clock at rest in the grid.

14.5.10 Synchronization of Distant Clocks

The survey group has now dealt with two of the major impediments to its survey: Time dilation and Lorentz contraction. And it has done this without as yet having a synchronized system of clocks at different locations in the cubical grid. We now come to clock synchronization as the last step.

First, we note that the survey group already has a system of clocks distributed throughout the grid, ones that they know to be ticking at the same standard rate. In a sense, dt is already defined. All that is required to finish the job is to find some

systematic way of setting the zero of time for these stationary standard clocks. The most natural way is to make use of eqn (14.13), the general formula for correcting the time interval of a clock moving with the self-measured proper speed u .

To synchronize a clock at B with one at A, the procedure is as follows: Carry a standard clock M from A to B. At the instant of M's departure, reset M to the value t_{a0} read currently on the clock at A. Record M's proper speed u during the whole trip, continuously calculating the accumulated corrected time

$$\Delta t_{ab} = \int_A^B dt = \int_A^B \sqrt{1 + u^2/c^2} d\tau \quad (14.21)$$

where $d\tau$ is the small time increment measured by the moving clock M and u is its instantaneous proper speed. At the instant when the clock M reaches B, the stationary clock at B is reset to the time $t_b = t_{a0} + \Delta t_{ab}$. This procedure has the merit that slow clock transport is not required, which might appeal to the survey group on the grounds of efficiency.

14.5.11 Internal Consistency of Synchronization

Using the above synchronization method, every stationary clock will be found to be already synchronized with itself when M is carried around a closed path beginning and ending at that clock. For if the clock M is carried to B and then returned to A, possibly by a different return path, the total elapsed corrected time will be as given earlier by eqn (14.14)

$$\Delta t_{aa} = \oint_A^A \sqrt{1 + u^2/c^2} d\tau = \int_A^B \sqrt{1 + u^2/c^2} d\tau + \int_B^A \sqrt{1 + u^2/c^2} d\tau \quad (14.22)$$

and the clock at A at that instant will thus be found to agree already with the value $t_a = t_{a0} + \Delta t_{aa}$ that would be calculated to synchronize it.

Since $t_{a0} + \Delta t_{aa} = t_{a0} + \Delta t_{ab} + \Delta t_{ba} = t_b + \Delta t_{ba}$, this return of clock M back to A establishes the reflexivity and path independence of the synchronization procedure. For if clock B is synchronized with clock A using the outgoing path, then it is also true that clock A will be found to be already synchronized with clock B using any of many possible return paths, which may be different in both position and proper-speed profile. Thus not only does "B synchronized with A" imply "A synchronized with B," proving reflexivity, but also the synchronization of A with B is found true using any return path, proving path independence.

Transitivity also follows. If clock B is synchronized with clock A, and clock C is synchronized with clock B, then clock C will be found to be synchronized with clock A.

14.5.12 Two Limiting Cases

The above general method of clock synchronization has two limiting cases of interest.

Slowly-Moving Clock In the first limiting case, the transported clock M is moved very slowly, with $u \ll c$. Then eqn (14.21) may be expanded as

$$\Delta t_{ab} = \int_A^B \left(1 + \frac{u^2}{2c^2} + \dots \right) d\tau = \Delta\tau_{ab} + \int_A^B \left(\frac{u^2}{2c^2} + \dots \right) \frac{d\ell}{u} \tag{14.23}$$

where the relation $d\tau = d\ell/u$, which follows from the definition $u = d\ell/d\tau$, has been used in the last integral. If the total path length between A and B through which clock M is carried is ℓ , and the maximum proper speed is u_{\max} , then in the $u_{\max}/c \rightarrow 0$ limit the last term in eqn (14.23) is of order $(u_{\max}/c)(\ell/c)$. Thus $\Delta t_{ab} = \Delta\tau_{ab}$ plus a term which can be made as small as needed by a suitably small choice of u_{\max} .

This slow-clock method of synchronization has the advantage that no calculations are needed to get Δt_{ab} since it can just be read off the moving clock directly: $\Delta t_{ab} = \Delta\tau_{ab}$ to whatever accuracy desired. When clock M arrives at B, the stationary clock there should just be set to $t_{a0} + \Delta\tau_{ab}$. The disadvantage of this method is, of course, that the synchronization process might be very slow if high accuracy is needed.

Rapidly Moving Clock In the second limit, the clock M is transported with a very large proper speed $u \gg c$. Then eqn (14.21) may be expanded

$$\Delta t_{ab} = \int_A^B \left(1 + \frac{c^2}{2u^2} + \dots \right) \frac{u}{c} d\tau = \frac{\ell}{c} + \int_A^B \left(\frac{c^2}{2u^2} + \dots \right) \frac{d\ell}{c} \tag{14.24}$$

where the relation $d\ell = u d\tau$, which follows from the definition $u = d\ell/d\tau$, has been used, and ℓ is the total length of the path through which M is carried. If we denote the minimum proper speed by u_{\min} , then in the $c/u_{\min} \rightarrow 0$ limit, the last term in eqn (14.24) is of order $(c/u_{\min})(\ell/u_{\min})$, which can be made as small as needed by a suitably large choice of u_{\min} .

Upon arrival of the clock M at point B, the clock there should be set to $t_{a0} + \ell/c$. With this method of synchronization, it is not even necessary to read the time on the moving clock. Only the path length ℓ and the round-trip speed of light c are needed.

14.5.13 The One-way Speed of Light

The survey group can now determine the one-way speed of light between points A and B by transporting a moving clock M in a straight line between these points at very high proper speed u . They will find (SR) that a light pulse leaving A at the same instant will arrive at B at almost the same instant as the moving clock, with the agreement getting better the higher u is. (Of course u can never be infinite, only approach it as a limit.) Thus, according to eqn (14.24), the synchronized clocks at A and B will record a limiting common travel time $\Delta t_{ab} = \ell/c$ for both M and the light pulse, where now ℓ is the Pythagorean distance between A and B. It follows that the one-way speed of the light pulse is the same as the two-way speed, $c_{ab} = \ell/\Delta t_{ab} = c$. A similar argument applies between any two clocks. So the one-way speed of light between B and A is also $c_{ba} = c$.

This result can be used as an alternate method of clock synchronization, one that gives the same results as those in the previous subsections, but may be simpler to

implement: The time-signal method. Simply send a light pulse from A at t_{a0} and, at the instant of its arrival at B, set the clock at B to the value $t_b = t_{a0} + \ell/c$, where ℓ is the Pythagorean distance between the two clocks.

The clock synchronization method used by Einstein in his 1905 paper is closely related to this time-signal method. It follows from the time signal method that when a light pulse leaves A at t_{a1} , is reflected from B at t_b and returns to A at time t_{a2} , the three times are related by

$$t_b = \frac{1}{2}(t_{a1} + t_{a2}) \quad (14.25)$$

which was used by Einstein as a definition of t_b .

14.5.14 Lorentz Contraction by Another Method

Now that the clocks at each vertex of the cubic grid are synchronized, the survey group can do an alternate experiment to determine the Lorentz contraction of a cube that is moving with constant proper velocity perpendicular to one face. Two observers fixed in the grid observe the location of a corner of the leading and trailing faces of the cube at the same instant of time, as measured on their two synchronized clocks. Calculating the distance between the two points using the Pythagorean theorem, they will find (SR) that the measurement agrees with eqn (14.20).

14.5.15 Coordinate Time and Velocity

In Section 14.5.7, the “corrected” time interval dt between two events on a moving object was written in terms of the proper time interval $d\tau$ between those two events as read by a clock moving with the object. This was done *before* the stationary clocks at the vertices of the grid were synchronized. But, since this corrected time interval has now been used to *determine* the synchronization of the stationary grid clocks, it follows that the same time interval dt would also be measured by these vertex clocks. (Of course, two of them would be needed since two events on a moving object take place at different points of the grid.) We will now refer to time read on the synchronized, stationary vertex clocks as the *coordinate time* in the S system. The “corrected” time intervals and coordinate time intervals are therefore identical, and in later chapters will usually be referred to simply as intervals of coordinate time.

The synchronized clocks at the vertices of the grid may now be used to define what will be called the *coordinate velocity* of a moving object. Consider two events on the moving object. They will be separated in the grid by a vector displacement $d\mathbf{r}$ and by a time interval dt measured by two different but synchronized stationary clocks. The coordinate velocity is then defined by $\mathbf{v} = d\mathbf{r}/dt$. Since the corrected and coordinate time intervals are equal, the “corrected” velocity \mathbf{v} of a moving object defined in Section 14.5.8 will be the same as this coordinate velocity. Thus, in later chapters, the “corrected” velocity will usually be referred to as the coordinate velocity.

It follows that the earlier formulas, relating proper time interval and velocity to corrected time interval and velocity, also apply to coordinate time interval and coor-

dinate velocity. Thus, for example,

$$dt = \gamma d\tau \quad \text{and} \quad \mathbf{v} = \gamma \mathbf{u} \quad \text{where} \quad \gamma = \sqrt{1 + u^2/c^2} = 1/\sqrt{1 - v^2/c^2} \quad (14.26)$$

relate the proper time interval $d\tau$ and velocity \mathbf{u} to the coordinate time interval dt and velocity \mathbf{v} .

14.5.16 *Universality*

The survey group will also have observed another important phenomenon that deserves to be emphasized: universality. It will be found (SR) that *any* sort of good clock, carried along with a standard moving clock M and initially synchronized with M at the start of the travel, will remain synchronized with M to within its own accuracy. Quartz watches, radioactive decays, elementary particle lifetimes, even presumably human hearts, will all be observed to participate equally in the time-dilation effects.

Universality will also be observed (SR) in Lorentz contraction. Equal-sized, rigid cubes of diverse materials placed side-by-side will be observed to remain congruent when they move with the same constant proper velocity.

The group has now finished its survey. The cubical grid and observers holding synchronized clocks at each vertex form an observer team (in the sense described in Section 14.4) adequate for determination of the t, x, y, z coordinates of any event. The survey described here has not been based on axioms or other preconceptions about the nature of the surveyed spacetime. Special relativistic effects such as time dilation and Lorentz contractions have been encountered as surprising complications that had to be worked around, and the final coordinate system (observer team) has incorporated these relativistic effects in a conservative way. The resulting coordinate system is identical to one that could have been derived from Einstein's axioms.

14.6 The Lorentz Transformation

As discussed in Section 14.4, the important and difficult problem was to create even one coordinate system, which we now assume has been done as described in Section 14.5 or in some equivalent way. If we consider that a second survey group establishes another coordinate system using exactly the same process as in Section 14.5, we will find that the Lorentz transformation between two systems is already implicit in the results of the first survey.

Suppose a second survey to be done. The second survey group uses sets of standard rods and clocks identical to the first one. They use exactly the same sequence of experiments as the first survey did, with identical interpretations of them. The only difference is that we will assume the second survey to have slightly mismeasured the isotropy of the three-degree background radiation, and hence to have built a cubical grid that is moving uniformly relative to the first one. However, we will assume that the second survey group is unaware of this error, and will do its survey using *exactly* the same assumptions as used by the first survey.

The original survey will be referred to as S , and the second survey as S' . To simplify the algebra, we make the usual assumption that the origin of the second grid has

coordinate velocity $\mathbf{V} = V\hat{\mathbf{e}}_1$ as measured by the S system, that the second survey group has accidentally chosen its own x' -axis along this same direction, and that its y' -axis is such that points on the y' -axis have $x = Vt$ and $z = 0$ for all time t . We also assume that the moving-system origin is accidentally chosen so that it passes the origin of the rest system at rest-system time $t = 0$, and that both S and S' are right-handed systems.

As the S' group goes through the steps listed in Section 14.5, we can use the already established S coordinate system to verify that the S' group will obtain exactly the same experimental outcomes as the S group did. We give a brief summary of this process.

14.6.1 Second System – Space Frame

Following the procedure detailed in Section 14.5.1, the S' group assigns the address (x', y', z') where

$$x' = n'_x \ell_0 \quad y' = n'_y \ell_0 \quad z' = n'_z \ell_0 \quad (14.27)$$

to a vertex that is (n'_x, n'_y, n'_z) counts from an S' -system origin. The lengths ℓ_0 are those values stamped on the sides of the standard rods, just as they were for the original survey.

According to eqn (14.20) with $v = V$, the S group will observe the standard rods along the x' -axis of the moving system to be contracted to $\ell = \sqrt{1 - (V/c)^2} \ell_0$ while those along the y' - and z' -axes will have their uncontracted length ℓ_0 . The S group might question the use of the marked length ℓ_0 in the first of eqn (14.27), since rods in the x' direction are actually contracted. But S will understand this S' coordinate definition, since the universality noted in Section 14.5.16 implies that everything S' might compare to these rods is itself contracted by the same factor and so the contraction is invisible to S' . Equation (14.27) is in fact the first notable example of the S' group's insistence on using *exactly* the same methods as S used.

Given eqn (14.27), vertex (x', y', z') will, at S time t , be assigned the S-system address (x, y, z) where $y = y'$ and $z = z'$, but

$$x = Vt + n'_x \ell = Vt + n'_x \sqrt{1 - (V/c)^2} \ell_0 = Vt + \sqrt{1 - (V/c)^2} x' \quad (14.28)$$

Thus the S group will obtain the last three Lorentz transformation equations

$$x' = \Gamma(x - Vt) \quad y' = y \quad z' = z \quad (14.29)$$

where

$$\Gamma = \frac{1}{\sqrt{1 - (V/c)^2}} \quad (14.30)$$

The S group will also verify the S' group's discovery of the Pythagorean theorem. The universality of Lorentz contraction implies that the x' dimension of both the measured rod and the rods of the S' grid will be shortened by the same factor. Thus if the experiment were prepared first with S' at rest relative to S (in which state we

already know the Pythagorean theorem to hold), and then accelerated gently to its final situation while the S' group slept, the numbers $\Delta n'_x$, $\Delta n'_y$, $\Delta n'_z$ would not change and the S' group would observe eqn (14.5) to remain true.

14.6.2 Second System – Clocks and Two-Way Light Speed

The S' group now introduces standard clocks and carries out the experiments described in Sections 14.5.2 through 14.5.5. Using the established S coordinate system, we can verify that the S' group will obtain exactly the same results as the S group did.

For example, S will verify that all clocks at rest in the S' system tick at the same rate, since those clocks all have the same speed V and thus obey

$$dt = \frac{dt'|_{x'y'z'}}{\sqrt{1 - V^2/c^2}} \quad (14.31)$$

where $dt'|_{x'y'z'}$ is a time interval measured on a clock at rest in S' at address x' , y' , z' .

Also, the experiment in which the S' group measures the two-way speed of light to be static, homogeneous, and isotropic according to the formula

$$c' = \frac{2\Delta\ell'}{\Delta t'|_{x'y'z'}} \quad (14.32)$$

will be verified by the S system, which will view it as what is called a *light clock*. If a light signal moves from a clock A' to a clock B' and back, both clocks being at rest in S' , the S system will calculate the total transit time to be

$$\Delta t = \frac{2\Delta\ell'}{c\sqrt{1 - V^2/c^2}} \quad (14.33)$$

where the Lorentz contraction formula eqn (14.20) has been used, along with the Pythagorean theorem in the S' system

$$\Delta\ell' = \sqrt{(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2} \quad (14.34)$$

Using eqn (14.31) to write $\Delta t'|_{x'y'z'} = \sqrt{1 - V^2/c^2}\Delta t$ and putting this result into eqn (14.32) then gives

$$c' = \frac{2\Delta\ell'}{\Delta t\sqrt{1 - V^2/c^2}} = c \quad (14.35)$$

where eqn (14.33) was used to get the final equality. This verifies the S' result that c' is homogeneous and isotropic and also demonstrates that $c' = c$. The two-way speeds of light in the two systems are numerically identical. Henceforward, we will denote this common value as c , with no primes.

14.6.3 Second System – Proper Velocity

Following the path of the first survey in Section 14.5.6, the second survey group now uses the proper time $\Delta\tau$ read from a moving clock to define the proper velocity. The x' -component of the proper velocity, for example, is defined as $u'_x = \Delta x' / \Delta\tau$, where $\Delta x'$ is the number of y' - z' planes pierced by the motion times their standard separation, and $\Delta\tau$ is the elapsed time measured by the moving clock. The proper speed is defined as $u' = \Delta\ell' / \Delta\tau$, and the Pythagorean theorem in the S' system then gives

$$u' = \sqrt{u'^2_x + u'^2_y + u'^2_z} \quad (14.36)$$

The S group can use results already obtained in the S' survey to derive a transformation law between proper velocities in the two systems. Dividing the differentials of eqn (14.29) by the proper time interval $d\tau$ and using $dt/d\tau = \sqrt{1 + u^2/c^2}$ from eqn (14.26), gives

$$u'_x = \Gamma \left(u_x - V\sqrt{1 + u^2/c^2} \right) \quad u'_y = u_y \quad u'_z = u_z \quad (14.37)$$

Defining $\gamma = \sqrt{1 + u^2/c^2}$ and $\gamma' = \sqrt{1 + u'^2/c^2}$, eqn (14.37) implies the following relation that will be of use later

$$\gamma = \Gamma \left(\gamma' + \frac{Vu'_x}{c^2} \right) \quad (14.38)$$

It follows from these equations that $u/c \rightarrow \infty$ if and only if $u'/c \rightarrow \infty$. Since a light pulse was demonstrated in the S system to have an infinite u value, it follows that the same light pulse viewed from S' has an infinite u' value also.

14.6.4 Second System – Time Dilation

Still following the example of the first survey, the S' group observes that a clock moving in a closed circuit beginning and ending at the same point A' of the S' system obeys

$$\Delta t'_{a'a'} = \oint_{A'} \sqrt{1 + u'^2/c^2} d\tau \quad (14.39)$$

where $\Delta t'_{a'a'}$ is the time interval of one cycle as measured by a clock fixed at A' and $d\tau$ is the proper time interval of the moving clock at the instant when its proper speed is u' . The S' group then follows the same line of reasoning as outlined in Section 14.5.7 of the S group's survey. They conclude that the moving clock is distorted by its motion, and that the time interval dt' corrected for this distortion is

$$dt' = \sqrt{1 + u'^2/c^2} d\tau \quad (14.40)$$

As did the S group, the S' group use symmetry arguments to discount the possibility that the true relation might be

$$dt' = \sqrt{1 + u'^2/c^2} d\tau + d\Phi' \quad (14.41)$$

where $\Phi'(x', y', z')$ is some single-valued function of position in the S' system that integrates out of eqn (14.39). Like the S group, they argue from the apparent isotropy and homogeneity of the local space—plus their precaution (failed, as we know) of choosing a state of motion from which the three-degree radiation field appears isotropic—that no such $\Phi'(x', y', z')$ term is present in eqn (14.41).

The S group verifies eqn (14.39) but not eqn (14.40). Multiplying eqn (14.38) through by $d\tau$ and using the definition $u'_x = dx'/d\tau$ gives

$$dt = \gamma d\tau = \Gamma \left(\gamma' d\tau + \frac{V dx'}{c^2} \right) \tag{14.42}$$

where the relation $dt = \gamma d\tau$ from eqn (14.26) has also been used. Using eqn (14.42), together with $dt'|_{x'y'z'} = \sqrt{1 - V^2/c^2} dt$ from eqn (14.31), the S group derives the relation

$$dt'|_{x'y'z'} = \sqrt{1 + u'^2/c^2} d\tau + \frac{V dx'}{c^2} \tag{14.43}$$

which illustrates that the correction rule eqn (14.40) does *not* bring the proper time interval $d\tau$ into agreement with the time interval $dt'|_{x'y'z'}$ shown on the clocks fixed in S' . (Note that the S system time interval dt is used in intermediate steps in this derivation, but is absent from the final result. Equation (14.43) represents the time interval that elapses on a clock fixed in S' while a moving clock measures time interval $d\tau$ and moves an S' -distance dx' .) According to the S group, the S' group is not in an isotropic space but in one marked by a velocity \mathbf{V} relative to the truly isotropic S system. The function Φ' thus was incorrectly assumed to be a constant. Its actual form is found by the S group to be

$$\Phi' = \frac{Vx'}{c^2} \tag{14.44}$$

Nonetheless, we are here considering the case in which the S' survey group is ignorant of its error and persists in using the definition of dt' given in eqn (14.40). The S' group is assumed to do *exactly* as the S group did, including the using of eqn (14.40) exactly as S used eqn (14.13).

14.6.5 Second System – Velocity and Lorentz Contraction

Having adopted the measurement of dt' from eqn (14.40), the S' group continues the program followed by the S group in Sections 14.5.8 and 14.5.9. They define a corrected velocity

$$\mathbf{v}' = \frac{d\mathbf{r}'}{d\tau} \frac{d\tau}{dt'} = \frac{\mathbf{u}'}{\sqrt{1 + u'^2/c^2}} \tag{14.45}$$

and a corresponding corrected speed $v' = u'/\sqrt{1 + u'^2/c^2}$ such that

$$\mathbf{u}' = \frac{\mathbf{v}'}{\sqrt{1 - v'^2/c^2}} = \gamma' \mathbf{v}' \tag{14.46}$$

where

$$\gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}} = \sqrt{1 + u'^2/c^2} \quad (14.47)$$

The S' group then uses this corrected velocity to measure the length of a cube, in the dimension oriented along the line of its velocity, to be

$$L' = v' \Delta t'_{A'} = \frac{L'_0}{\sqrt{1 + u'^2/c^2}} = L'_0 \sqrt{1 - v'^2/c^2} \quad (14.48)$$

where L'_0 is the side of the cube when at rest in S' .

The S system will verify these experimental results to be correct *provided that* one accepts the strange (to the S group) definition of time interval dt' being used by the S' group.

14.6.6 Second System – Clock Synchronization

The second survey group now follows the S procedures of Sections 14.5.10 through 14.5.13 to synchronize standard clocks fixed at the vertices of the S' grid. A clock at B' is synchronized with one at A' by carrying a moving clock M from A' to B' . Clock M is first synchronized with the clock at A' . If it leaves A' at time t'_{a0} , then at the instant of arrival at B' the clock there is reset to $t'_b = t'_{a0} + \Delta t'_{ab}$ where

$$\Delta t'_{ab} = \int_{A'}^{B'} dt' = \int_{A'}^{B'} \sqrt{1 + u'^2/c^2} d\tau \quad (14.49)$$

where $d\tau$ is the small time increment measured by the moving clock and u' is its instantaneous proper speed.

The S' group will find the same consequences (internal consistency, limiting cases, one-way speeds of light) as found earlier by the S group. In particular, the one-way speed of light between two clocks A' and B' fixed in S' will be found to be

$$c'_{ab} = c = c'_{ba} \quad (14.50)$$

just as for the S system. These results will be the same as for the S system because they follow directly from eqn (14.49).

14.6.7 Second System – Emergence of the Lorentz Transformation

The S group observes that the S' synchronization process implies the standard Lorentz transformation between the two systems, again provided one follows the S' group's use of the definition of dt' in eqn (14.40). Using this definition, eqn (14.42) can be written as

$$dt = \Gamma \left(dt' + \frac{V dx'}{c^2} \right) \quad (14.51)$$

If a clock M is carried from the origin of S' to the point x', y', z' , starting at the instant when the origins of S and S' cross (an event taken to be at the arbitrary zero of time in both systems), then according to eqn (14.49) the clock at x', y', z' will be reset

to $t' = \int_{o'}^{x'y'z'} dt'$ at the instant of M's arrival. Integrating each term of eqn (14.51) between these same limits thus gives

$$t = \Gamma \left(t' + \frac{Vx'}{c^2} \right) \tag{14.52}$$

Substituting this result into eqn (14.29) and solving for x, y, z then gives the transformation between the coordinates of the two surveys

$$\begin{aligned} ct &= \Gamma (ct' + Bx') \\ x &= \Gamma (Bct' + x') \\ y &= y' \\ z &= z' \end{aligned} \tag{14.53}$$

where $B = V/c$ and $\Gamma = \sqrt{1 - B^2}$. We will refer to this as the *standard Lorentz transformation*. It holds when the axes of the two systems are aligned as described at the beginning of Section 14.6.

14.6.8 Second System – Coordinate Time and Velocity

The discussion of coordinate time and velocity in Section 14.5.11 applies also to the S' system developed by the second survey. The time measured by the synchronized clocks placed at the vertices of the S' grid will be referred to as the *coordinate time* in the S' system. Since the “corrected” time interval dt' obtained in Section 14.6.4 has been used to *determine* the synchronization of clocks at the vertices of the S' grid, it follows that two events on an object moving relative to the S' grid will have a corrected time difference equal to their coordinate time difference.

The *coordinate velocity* of an object moving relative to the S' system can also be defined, by considering two events on the object. They will occur at two points of the S' grid with vector displacement $d\mathbf{r}'$ and time interval dt' as measured by two different but synchronized vertex clocks. The coordinate velocity in the S' system is then defined as $\mathbf{v}' = d\mathbf{r}'/dt'$. As in the S system, this velocity is identical to the corrected velocity defined in Section 14.6.5.

It follows that the equations relating the proper time interval and proper velocity to the corrected time interval and corrected velocity apply also to the coordinate time interval and coordinate velocity. Thus, like eqn (14.26) in the S system, we have

$$dt' = \gamma' d\tau \quad \text{and} \quad \mathbf{v}' = \gamma' \mathbf{u}' \quad \text{where} \quad \gamma' = \sqrt{1 + u'^2/c^2} = 1/\sqrt{1 - v'^2/c^2} \tag{14.54}$$

relating the proper time interval $d\tau$ and proper velocity \mathbf{u}' to the coordinate time interval dt' and coordinate velocity \mathbf{v}' in the S' system.

14.7 The Principle of Relativity

Since the *effects predicted* by standard special relativity were assumed in the first survey, Section 14.5, and since the coordinate system produced by that survey was then

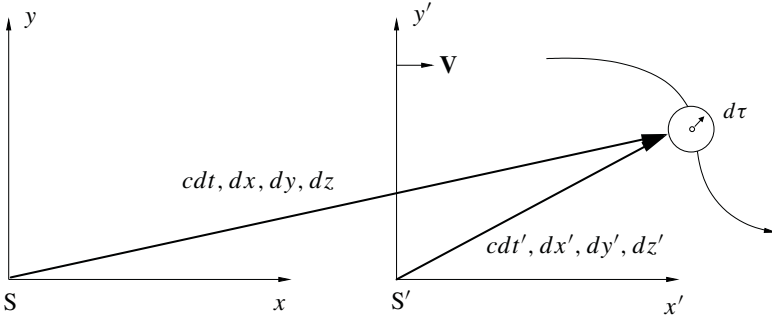


FIG. 14.5. Two flashes of a strobe light on a moving clock are separated by proper time $d\tau$ on the moving clock, by $dt = \gamma d\tau$ on the S system clocks, and by $dt' = \gamma' d\tau$ on the S' system clocks.

used to predict the effects encountered by the second survey, it is not too surprising that the standard Lorentz transformation in eqn (14.53) has emerged at the end of the process. The benefit of considering the Lorentz transformation as the end product of experimental surveys is that we get some insight into *how* it emerged.

The two survey groups have followed exactly the same experimental procedures to establish systems S and S'. One crucial common procedure was the assertion by each group that it was the one at rest in a locally homogeneous, isotropic space. Hence each group used a symmetry argument to infer a differential relation, eqn (14.13) for the S survey and eqn (14.40) for the S' survey, from an experimentally measured integral one. The surprising result is that, in both cases, this *assertion* of homogeneity and isotropy suffices to produce a system that is indeed homogeneous and isotropic in all of its internally measured properties like speeds of light, etc.

Moreover, these two systems seem truly indistinguishable. Inverting the set of equations, eqn (14.53), using Cramer's rule for linear equations, yields an inverse transformation which differs only by the exchange of primed and unprimed coordinates and the replacement of V by $-V$ (and hence B by $-B$),

$$\begin{aligned}
 ct' &= \Gamma (ct - Bx) \\
 x' &= \Gamma (-Bct + x) \\
 y' &= y \\
 z' &= z
 \end{aligned}
 \tag{14.55}$$

There is nothing internal to the two systems, nor in the transformation between them, to prevent us from viewing S' as the rest system, and S as a system moving along its negative x' axis direction with speed V in the opposite direction.

Thus Einstein's two principles are recovered, at least as far as the outcomes of the two surveys are concerned. Each group may assert that *it* is at rest in an isotropic, homogeneous space and the other group therefore cannot be. But there is no local *experimental* way to decide between these assertions.

14.8 Lorentzian Relativity

As we noted above, Lorentz derived his transformation from the requirement that the Maxwell equations must have the same form in both the S and S' systems. After Einstein's 1905 paper, Lorentz also abandoned the absolute Newtonian time but tried to hold on to the concept of an aether.⁷³ His idea was that the relativistic effects encountered by the S survey group were due to physical interactions of clocks and cubes with the background aether. The details of the S survey, in particular the fact that the time-dilation factor $\sqrt{1 + u^2/c^2}$ is the same as the Lorentz contraction factor, are then responsible for the principle of relativity and the equivalence of the S and S' systems.

Lorentz thus views time dilation of clocks moving relative to the aether to be due to a physical action of the aether on a moving clock. His approach makes relativity a *physical* theory (albeit with mysterious and unknown mechanisms of interaction), rather than just a statement of principles.

However, Lorentz's view did not prevail. By the 1930s the consensus was that the aether was an unnecessary and unobservable intellectual construct. The problem it was meant to solve, the provision of a medium within which light waves could propagate, was resolved by assuming that the electric and magnetic fields themselves were "things" on the same footing as real particles. Light was then viewed as a wave resulting from the interaction and mutual reinforcement of these real field objects. The aether was relegated to the dustbin of history, with other failed concepts such as phlogiston or N-rays. In Einstein's magisterial phrase, from his first 1905 paper, "the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest."

However, while the relativistic revolution was destroying the concept of the aether, another revolution was underway that offers some support to the Lorentzian program: quantum mechanics. The quantum theory of electrons and photons makes essential use of the concept of a *vacuum state*. This is the state of nature when everything removable is removed, and when all quantum fields are in their ground states. Due to the uncertainty principle, the quantum vacuum state has nonzero energy. At present, calculations of the energy density of the vacuum state give infinite answers. But its physical reality is demonstrated experimentally by the Casimir effect, in which finite *changes* in the vacuum energy can be observed. Also, each species of elementary particle in the standard model gives rise to what are called vacuum fluctuations in which particles are spontaneously created and destroyed. It seems, in sum, that empty space has become quite a busy place, with much in it that is undeniably physically real.

The physicality of the vacuum state of quantum mechanics opens the possibility that the Lorentzian approach to relativity might be reconsidered, with the aether replaced by a physical substance that we might call by a more neutral term, the *physical vacuum*. It might be that the relativistic effects are due to interactions of clocks and cubes with the same physical vacuum that is also the arena for the vacuum state of

⁷³See, for example, the last chapter of Lorentz (1952).

quantum mechanics.

14.9 Mechanism and Relativity

There is a tradition in physics that goes back to Descartes' arguments with the Aristotelian physics that preceded him. Called the *mechanist* tradition, it holds that physical objects have only a small number of intrinsic properties and that these properties can be described geometrically or mathematically. This was in opposition to the Aristotelian propensity to treat each new phenomenon as the emergence of another as yet unobserved (or *occult*, as in hidden) property of the objects. Descartes' mechanics also had the idea of proximate interaction. Things interact by collision, or by contact. Thus Leibniz ridiculed Newton's idea of a gravitational attraction acting between distant masses, calling it "occult".

A modern statement of the mechanist tradition might be the assumption that physical effects have physical causes, and that those physical causes must act locally. Thus the field picture of electrodynamics states that, rather than two electric charges acting on each other at a distance, the charges produce an electric field that fills the space between them and acts locally on each charge. The idea of field propagation at less than the speed of light, combined with local interactions, is sometimes referred to as *Einstein locality*.

We might ask whether the mechanist tradition is consistent with special relativity. In particular, is there an explanation for the time dilation of a moving clock that is local and involves the interaction of physical objects?

To explore this question, let us return to a very early stage in the first survey, when we had only a cubical grid, a standard clock A at some fixed vertex of that grid and a standard clock M free to move about in the grid. If M is carried around some closed path starting and ending at A, the elapsed time Δt_{aa} measured on the stationary clock and the elapsed time $\Delta \tau_{aa}$ measured on the moving clock are related by eqn (14.12), so that

$$\Delta t_{aa} = \oint_A^A \sqrt{1 + u^2/c^2} d\tau \quad \text{and} \quad \Delta \tau_{aa} = \oint_A^A d\tau \quad (14.56)$$

We note that this equation summarizes an *experimental* result that does not depend on such choices as the definition of coordinate time made later in the S survey. There is little room for doubt that the difference between Δt_{aa} and $\Delta \tau_{aa} = \oint_A^A d\tau$, the cycle time measured on the moving clock M, is a real effect that is independent of any elements that could be considered arbitrary. We argued in Section 14.5.7 that this difference must be due to a distortion of the running of clock M, and not of clock A. Let us now seek a physical cause of this distortion.

First consider the hypothesis that M is distorted by its motion relative to the other physical objects of the experiment, the rods and the fixed clock at A. This can be disproved by setting up multiple sets of physical grids and multiple clocks at various points in them or even moving freely in random ways. According to standard special relativity (SR), the distortion of M relative to the original grid and the original clock A will not be affected. Equation (14.56) will be unchanged. If a physical interaction

between M and the grid or between M and A were the cause of M's distortion, we would expect that similar physical interactions with the other grids and clocks would also distort M. But no such extra distortion will be observed.

Possibly the distortion of M is due to the forces that must be applied to it to make it move in a non-inertial path? Those forces are indeed physical and are applied locally to M. However, we can set up an experiment in which M travels out from A on a straight, inertial path with no forces acting on it, then circles around on the arc of some circle and returns to A again along another straight, inertial path. Then M will experience forces only during the circular part of its path. On its outward leg, M may not have experienced any acceleration since its distant past. Or, if M is constructed, for example, from a swarm of decaying muons, it may never have experienced an acceleration at all, having been emitted with its present momentum value. Yet these inertial portions of M's path contribute their fair share to the integral in eqn (14.56), regardless. The hypothesis that forces are the source of the distortion of M does not work out in detail. One can even imagine an experiment with no forces on the clocks at all, with an inertial, force-free outbound clock transferring its elapsed time count locally to an inertial, force-free inbound one as they pass each other. The conclusion is that the distortion of M is not due to whatever agent it is that produces the accelerations necessary to maintain its motion.

Mechanism requires that this physical effect, the dilation of the moving clock, must be due to some local interaction between the clock and some other *physical* thing. The mechanist tradition in physics will allow occult phenomena, those that *just happen*, as a temporary expedient but is not satisfied until physical causes are found for them. The question is: What is the physical thing with which a dilated clock is interacting?

One response has been to consider the metric field $g_{\mu\nu}$ of general relativity to be a physical thing, an idea akin to the reification of the electric field. The metric field is present in all of space and thus could interact locally with moving clocks. As mentioned in Section 14.8, another possibility would be to revive Lorentzian relativity, but with the aether replaced by a physical vacuum filling all space. The time dilation of clocks would be a consequence of the local interaction between the clocks and that physical vacuum. Then the metric field of general relativity would not be considered a thing in itself, but rather a way of representing local properties of the physical vacuum.

14.10 Exercises

The questions in these exercises should be answered using the standard Lorentz transformation eqn (14.53) and its inverse eqn (14.55), together with the axiom noted in the footnote on page 319.

Exercise 14.1

- (a) Suppose that two events take place at the same S' location but different times. Show that the time intervals between these events in the S and S' systems are related by $dt = \Gamma dt'$. What is the proper time interval $d\tau$ between these events?
- (b) Suppose that two events (event A at the end nearest to the origin, and event B at the other

end) take place at the ends of a stick of length $\Delta\ell'$ which is at rest in S' and aligned parallel to the x' -axis. Suppose that these two events are simultaneous as seen by the S system. Find the distance $x'_b - x'_a$ and the time interval $t'_b - t'_a$ between them as seen by the S' system.

(c) State why that the distance $(x_b - x_a)$ is a good measure of the length of the stick ℓ as seen by the S system. Show that it obeys the standard Lorentz contraction formula $\Delta\ell = \sqrt{1 - V^2/c^2}\Delta\ell'$.

Exercise 14.2

(a) Show that eqns (14.15, 14.17) are equivalent, and that either of them implies eqn (14.16).

(b) Use standard special relativity to predict the experimental result eqn (14.12).

Exercise 14.3 Show, using the standard Lorentz transformation, that the definition of the length of a moving object in eqn (14.19) implies the experimental result eqn (14.20).

Exercise 14.4 Suppose that a light clock is set up, consisting of a stick of length $\Delta\ell'$ at rest in the S' system but not parallel to any of the axes. There are mirrors at each end of the stick and one cycle of the light clock is the round-trip time of a light pulse that is reflected back and forth between them. Calculate the round trip cycle time of the light clock Δt as seen by the S system, and thus verify eqn (14.33).

Exercise 14.5 Prove that eqn (14.37) implies eqn (14.38).

Exercise 14.6 Use Cramer's rule (not assuming the principle of relativity, which is what we're seeking to prove) to show that eqn (14.53) implies the inverse eqn (14.55).

Exercise 14.7 Discuss the following questions.

(a) If time dilation is due to an interaction of clocks with a physical vacuum, then why is time dilation universal? Why does that dilation have exactly the same magnitude for all sorts of moving clocks, with presumably different interaction modes?

(b) If there is no physical vacuum, then what is the physical mechanism responsible for time dilation? Is one needed?

FOURVECTORS AND OPERATORS

The previous Chapter 14 discussed some of the philosophical questions raised by special relativity. The present chapter develops techniques that allow relativistically covariant calculations to be done in an elegant manner.

We introduce what will be called *fourvectors*. These are analogous to the familiar vectors in three-dimensional Cartesian space (which will now be referred to as *threevectors*) used throughout the earlier chapters of the text. The difference is that, in addition to the three spatial components, fourvectors will have an additional zeroth component associated with time. This additional component allows us to deal with the fact that the Lorentz transformation of special relativity transforms time as well as spatial coordinates.

The theory of fourvectors and operators is presented using an *invariant* notation. Rather than considering a fourvector to be only a set of components in some basis, we will consider it to be something that models some physical property of the experiment under study and is independent of the choice of coordinate system.

The reader is already familiar with this sort of notation because, for example in discussing rotations in Chapter 8, we have used threewector equations wherever possible, rather than writing out component equations in a particular basis. We are here extending to relativity (with fourvectors) the same techniques that made the theory of rotations (with threevectors) tractable. The reader is urged to think of fourvector equations in this invariant way.

The present chapter can only give the barest introduction to the vast subject of fourvectors and tensors in special and general relativity. We introduce the subject here in the simple context of special relativity, and present enough of it to allow the reader to understand the special relativistic generalization of mechanics in Chapter 16.

15.1 Fourvectors

Sections 8.30 and 8.31 treated passive rotations in ordinary three-dimensional Cartesian spaces. We saw there that any threewector, for example the differential spatial displacement threewector $d\mathbf{r}$ connecting two spatially separated points, can be expanded in either an original o system or a rotated o' system, according to the rule

$$dx\hat{\mathbf{e}}_1 + dy\hat{\mathbf{e}}_2 + dz\hat{\mathbf{e}}_3 = d\mathbf{r} = dx'\hat{\mathbf{e}}'_1 + dy'\hat{\mathbf{e}}'_2 + dz'\hat{\mathbf{e}}'_3 \quad (15.1)$$

where the differential components and basis vectors in the two systems are related by the orthogonal transformations eqns (8.235, 8.240).

From eqn (8.234), the basis threevectors obey

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \quad \text{and} \quad \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}'_j = \delta_{ij} \quad (15.2)$$

It follows that the square of the Pythagorean distance $d\ell$ between the two points is given by a form invariant⁷⁴ expression $(d\ell)^2 = d\mathbf{r} \cdot d\mathbf{r}$ where

$$(dx)^2 + (dy)^2 + (dz)^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx')^2 + (dy')^2 + (dz')^2 \quad (15.3)$$

Minkowski⁷⁵ noted that a similar formalism can be introduced in special relativity by the addition of a *time* coordinate to the original three coordinates of Newtonian space. We can define components, basis fourvectors, and an inner product of pairs of fourvectors—leading to an invariant quantity analogous to eqn (15.3).

The differential displacement between two *events* in special relativity can be represented by its coordinates cdt, dx, dy, dz in an S system and its corresponding coordinates cdt', dx', dy', dz' in the S' system (assumed, at least in this introductory section, to have the same standard configuration as that described in Section 14.6). Taking the differentials of the standard Lorentz transformation, eqn (14.53), the transformation between the coordinate differentials is found to be

$$\begin{aligned} cdt &= \Gamma \left(cdt' + \frac{V}{c} dx' \right) \\ dx &= \Gamma \left(\frac{V}{c} cdt' + dx' \right) \\ dy &= dy' \\ dz &= dz' \end{aligned} \quad (15.4)$$

As the reader can verify, the following quadratic expression is form invariant under this transformation of coordinates

$$-(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \quad (15.5)$$

Acting by analogy with Cartesian threevectors, we seek to define a displacement fourvector $d\mathbf{r}$ in relativity theory such that its inner product with itself $d\mathbf{r} \cdot d\mathbf{r}$ will reproduce the form invariant quantity in eqn (15.5). Following the pattern of eqn (15.1), we introduce basis fourvectors $\hat{\mathbf{e}}_\mu$ and write⁷⁶

$$d\mathbf{r} = dx^0 \hat{\mathbf{e}}_0 + dx^1 \hat{\mathbf{e}}_1 + dx^2 \hat{\mathbf{e}}_2 + dx^3 \hat{\mathbf{e}}_3 \quad (15.6)$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. This fourvector $d\mathbf{r}$ models the relativistic

⁷⁴Recall that a quantity or expression is called *invariant* if it has the same numerical value in two different coordinate systems. A *form* (short for algebraic form) is an algebraic expression written in terms of coordinates. Thus an expression that has both the same algebraic *form* and the same numerical *value* when expressed in two different coordinate systems is called an *invariant form*, or a *form invariant* expression. Such expressions are also often referred to by the shorter term *invariants*.

⁷⁵H. Minkowski (1908) "Space and Time," Address at the 80th assembly of German Natural Scientists and Physicians. English translation in Einstein, Lorentz, Minkowski and Weyl (1923).

⁷⁶We follow the established convention of using superscripts for the indices of fourvector components and subscripts for the indices of basis fourvectors. Note also that fourvectors will be set in **bold sans serif** type to distinguish them from threevectors that are set in **bold serif**.

interval between two events. It is an invariant object of the sort mentioned in the introduction to this chapter.

To see what the basis fourvectors mean, we consider an example from three-dimensional space and then the analog in relativity. We know that in the threevector case, the meaning of the basis vector $\hat{\mathbf{e}}_1$ is that a threevector displacement $d\mathbf{r} = dx\hat{\mathbf{e}}_1$ connects two points with the same y and z coordinates but differing x . Analogously, the meaning of the basis fourvector $\hat{\mathbf{e}}_1$ is that a fourvector displacement $d\mathbf{r} = dx^1\hat{\mathbf{e}}_1$ connects two *events* that have the same y , z , and t coordinates. Hence the two events connected by $dx^1\hat{\mathbf{e}}_1$ are also *simultaneous* in the S system.

Basis fourvectors can also be introduced in the S' system. These are defined by the requirement that the fourvector displacement $d\mathbf{r}$ defined in eqn (15.6) can also be written as an expansion in terms of the S' basis fourvectors

$$d\mathbf{r} = dx'^0\hat{\mathbf{e}}'_0 + dx'^1\hat{\mathbf{e}}'_1 + dx'^2\hat{\mathbf{e}}'_2 + dx'^3\hat{\mathbf{e}}'_3 \quad (15.7)$$

Thus, in analogy to eqn (15.1), we now have

$$dx^0\hat{\mathbf{e}}_0 + dx^1\hat{\mathbf{e}}_1 + dx^2\hat{\mathbf{e}}_2 + dx^3\hat{\mathbf{e}}_3 = d\mathbf{r} = dx'^0\hat{\mathbf{e}}'_0 + dx'^1\hat{\mathbf{e}}'_1 + dx'^2\hat{\mathbf{e}}'_2 + dx'^3\hat{\mathbf{e}}'_3 \quad (15.8)$$

For rotations of threevectors, the basis vectors appearing in eqn (15.1) were required to transform by an orthogonal matrix. In the fourvector case, the transformation law for the basis vectors can be obtained by substituting the inverse of eqn (15.4) into eqn (15.8) and using the fact that the differentials dx^μ are independent quantities that can be set nonzero one at a time. The result is

$$\begin{aligned} \hat{\mathbf{e}}_0 &= \Gamma \left(\hat{\mathbf{e}}'_0 - \frac{V}{c} \hat{\mathbf{e}}'_1 \right) \\ \hat{\mathbf{e}}_1 &= \Gamma \left(-\frac{V}{c} \hat{\mathbf{e}}'_0 + \hat{\mathbf{e}}'_1 \right) \\ \hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}'_3 \end{aligned} \quad (15.9)$$

Just as was done for threevectors, we can interpret the expansions in eqn (15.8) by saying that the same fourvector displacement $d\mathbf{r}$ is *represented* by its components in either the S or the S' systems

$$d\mathbf{r} : (cdt, dx, dy, dz)_S \quad d\mathbf{r} : (cdt', dx', dy', dz')_{S'} \quad (15.10)$$

We make the distinction of using the symbol “:” rather than “=” in eqn (15.10). As discussed in Section A.5 for the threevector case, it is not correct to set the fourvector $d\mathbf{r}$ *equal* to its components. The *same* fourvector $d\mathbf{r}$ is represented by *different* sets of components in the two systems. The two sets are related, of course, by the Lorentz transformation in eqn (15.4).

15.2 Inner Product

The *inner product* (also called the *dot product*) in the space of fourvectors can now be defined. But because of the minus sign on the time entry in eqn (15.5), an inner product that will reproduce that expression cannot have the standard Cartesian form. We adopt the definition that the basis fourvectors have inner products given, for all $\mu, \nu = 0, 1, 2, 3$, by

$$\hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu = g_{\mu\nu} \quad (15.11)$$

where $g_{\mu\nu}$ are the matrix elements of a four-rowed diagonal matrix \mathbf{g} defined as

$$\mathbf{g} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \quad (15.12)$$

It follows from eqns (15.9, 15.11) that basis vectors in the S' system have the same inner product

$$\hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu = g_{\mu\nu} \quad (15.13)$$

with the *same matrix* \mathbf{g} in both eqns (15.11 and 15.13). This \mathbf{g} is the matrix of what is called the *Minkowski metric*. We also define the inner product to be a linear operation, just it was in Cartesian three space.

With the fourvector $d\mathbf{r}$ defined as in eqn (15.6), and the inner product of basis fourvectors as defined in eqn (15.11), we may now use the assumed linearity of dot products to write

$$\begin{aligned} d\mathbf{r} \cdot d\mathbf{r} &= \left(\sum_{\mu=0}^3 dx^\mu \hat{\mathbf{e}}_\mu \right) \cdot \left(\sum_{\nu=0}^3 dx^\nu \hat{\mathbf{e}}_\nu \right) = \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx^\mu dx^\nu \hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu \\ &= \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx^\mu dx^\nu g_{\mu\nu} = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \end{aligned} \quad (15.14)$$

with a similar expression, but with primed components and basis vectors substituted for the unprimed ones, when $d\mathbf{r}$ is expanded in the S' system

$$\begin{aligned} d\mathbf{r} \cdot d\mathbf{r} &= \left(\sum_{\mu=0}^3 dx'^\mu \hat{\mathbf{e}}'_\mu \right) \cdot \left(\sum_{\nu=0}^3 dx'^\nu \hat{\mathbf{e}}'_\nu \right) = \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx'^\mu dx'^\nu \hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu \\ &= \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx'^\mu dx'^\nu g_{\mu\nu} = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \end{aligned} \quad (15.15)$$

Note the *absence* of a prime on $g_{\mu\nu}$ in eqn (15.15). This matrix is the same in both systems.

The form invariant expression in eqn (15.5) may now be written using the fourvector formalism, as the dot product of two displacement fourvectors expanded indifferently in either the S or the S' coordinate system

$$-(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = d\mathbf{r} \cdot d\mathbf{r} = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \quad (15.16)$$

Thus, following Minkowski, we have successfully defined a system of fourvector displacements $d\mathbf{r}$ whose dot product reproduces the Lorentz invariant expression eqn (15.5).

15.3 Choice of Metric

The choice of Minkowski matrix in eqn (15.12) has a certain degree of arbitrariness. We could as well have used its negative, with diagonal elements $(1, -1, -1, -1)$. Unfortunately, about half of the physics community uses our choice, and the other half uses the other.⁷⁷ And once a researcher adopts one or the other convention, it is difficult to change because a large number of signs, and imaginary versus real quantities in quantum field theory, depend on the choice.

A choice must be made, however, and for an analytical mechanics book like this one that choice is strongly influenced by the desire to have the threevector parts of fourvectors behave algebraically just as threevectors have behaved in the earlier chapters. With the metric in eqn (15.12), the spatial basis fourvectors with $i, j = 1, 2, 3$ obey $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ which is the same as eqn (15.2). Thus, we may write the fourvector $d\mathbf{r}$ as

$$d\mathbf{r} = dx^0 \hat{\mathbf{e}}_0 + d\mathbf{r} \quad (15.17)$$

where the spatial part of $d\mathbf{r}$ is written as the threevector

$$d\mathbf{r} = dx \hat{\mathbf{e}}_1 + dy \hat{\mathbf{e}}_2 + dz \hat{\mathbf{e}}_3 \quad (15.18)$$

with the assurance that $d\mathbf{r} \cdot d\mathbf{r} \geq 0$, as has been true for threevectors throughout the discussion in earlier chapters.⁷⁸ This allows us to make use of our earlier treatments of threevectors and operators with no changes, with just the replacement of the basis vectors $\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$ by the corresponding basis fourvectors $\hat{\mathbf{e}}_i$.

15.4 Relativistic Interval

We saw in the case of Cartesian threevectors that the quantity $d\mathbf{r} \cdot d\mathbf{r}$ has an invariant geometric meaning: It is the square of the spatial Pythagorean distance between the

⁷⁷For example Misner, Thorne and Wheeler (1973) and Weinberg (1995) use our convention, while Bjorken and Drell (1964) and Jackson (1975) use the other.

⁷⁸The spatial, threevector part of the fourvector $d\mathbf{r}$ will be denoted by the same typeface $d\mathbf{r}$ as we have used throughout the text for Cartesian threevectors. Strictly speaking, eqn (15.18) is both a threevector (since it is invariant under spatial rotations at fixed time) and a fourvector (since it is composed of unit fourvectors). Note that fourvectors like $d\mathbf{r}$ are invariant objects under Lorentz transformations, but the spatial parts $d\mathbf{r}$ are not. If we transform to a coordinate system S' , the same fourvector as in eqn (15.17) becomes $d\mathbf{r} = dx'^0 \hat{\mathbf{e}}'_0 + d\mathbf{r}'$, with both $dx'^0 \neq dx^0$ and $d\mathbf{r}' \neq d\mathbf{r}$.

two points that are connected by $d\mathbf{r}$. We now investigate the meaning of the analogous fourvector dot product $d\mathbf{r} \cdot d\mathbf{r}$ in special relativity.

The invariant expression in eqn (15.16) can be positive or negative. We consider first the cases in which it is negative. If the two events connected by $d\mathbf{r}$ are like the successive flashes of a quickly flashing strobe light moving at less than the speed of light (imagine successive flashes of the navigation lights of a passing rocket ship), then eqn (15.16) can be written in terms of the coordinate velocity of the moving object, \mathbf{v} in the S system and \mathbf{v}' in the S' system. Factoring cdt from the left side of eqn (15.16) and cdt' from the right side, and using $dx/dt = v_x$, etc., one obtains

$$-\left(1 - \frac{v^2}{c^2}\right) c^2 (dt)^2 = d\mathbf{r} \cdot d\mathbf{r} = -\left(1 - \frac{v'^2}{c^2}\right) c^2 (dt')^2 \quad (15.19)$$

Then either eqn (14.26) in the S system or eqn (14.54) in the S' system can be used to write

$$d\mathbf{r} \cdot d\mathbf{r} = -c^2 (d\tau)^2 \quad (15.20)$$

where $d\tau$ is the proper time interval between events as read on a clock being carried along with the strobing object, as defined in Section 14.5 and shown in Figure 14.5. This proper time interval is a property of the moving clock and is independent of any particular choice of coordinate system. Thus $d\tau$ is an invariant quantity, as it must be since it is written here as the dot product of two fourvectors.

Two events like the successive flashes of a moving strobe light just considered are called *timelike separated*. The fourvector interval between such events has a negative dot product with itself, and its value is related to the proper time interval measured on a clock moving with the strobing object. From eqn (15.20),

$$d\tau = \sqrt{-\frac{d\mathbf{r} \cdot d\mathbf{r}}{c^2}} \quad \text{for timelike separation} \quad (15.21)$$

In the opposite case, if the $d\mathbf{r}$ connecting two events has $d\mathbf{r} \cdot d\mathbf{r} > 0$, then the two events are called *spacelike separated*. Then, in analogy to the proper time for timelike separated events, an invariant distance $d\ell$ between the two events can be defined by

$$d\ell = \sqrt{d\mathbf{r} \cdot d\mathbf{r}} \quad \text{for spacelike separation} \quad (15.22)$$

The intermediate case is when $d\mathbf{r}$ is a null vector, with $d\mathbf{r} \cdot d\mathbf{r} = 0$. The two events in this case are called *lightlike separated*. As can be seen by considering the $v \rightarrow c$ limit of eqn (15.19), such a $d\mathbf{r}$ connects successive events occurring on something moving with the speed of light. As we saw in Section 14.5, this is the ultimate attainable speed, corresponding to proper speed u equal to infinity. Two successive events on the wave front of a light wave would have a null displacement vector

$$d\mathbf{r} \cdot d\mathbf{r} = 0 \quad \text{for lightlike separation} \quad (15.23)$$

Exercise 15.6 shows that for any two timelike separated events, there is some coordinate system relative to which the two events appear to be at the same spatial point. Exercise 15.7 shows that for any two spacelike separated events, there is some coordinate system relative to which the two events are simultaneous.

15.5 Spacetime Diagram

It has become customary to plot events in special relativity on what is called a *spacetime diagram*, in which one or two spatial coordinates (but not more due to the limitations of our primate sense apparatus) are plotted horizontally and $x^0 = ct$ is plotted vertically. An event is a mathematical point in such a diagram.

If we plot the successive strobe events of the passing rocket ship discussed in Section 15.4, we find that these events can be connected by a continuous line, called the *world line* of the strobe light. These world lines play the same role in relativity theory that particle trajectories as a function of time play in Newtonian physics. They give a complete description of the locus of the moving object at various epochs.

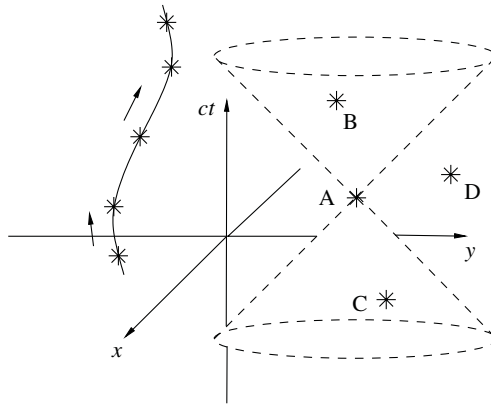


FIG. 15.1. Spacetime diagram showing, on the left, the world line drawn through successive flashes of a strobe light. The event A on the right is the origin of a light cone, shown as dashed lines. Event B is in the forward light cone of A, event C is in the backward light cone of A. Event D is outside the light cone of A and is not causally connected to it.

The spacetime diagram can also be used to define what is called a forward or backward *light cone* associated with a given event. As shown in the Figure 15.1, event B is in the forward light cone of event A if the displacement fourvector $d\mathbf{r}_{AB}$ connecting these two events obeys $d\mathbf{r}_{AB} \cdot d\mathbf{r}_{AB} < 0$ and $t_B > t_A$. That is, it consists of all of the events B later than event A that are timelike separated from event A. In that case, a physical object moving at less than the speed of light can actually be present at both event A and event B. (Note that an object can reach any location B from location A. The question is, can it get there in time?) Note that the world line of an object that is present at event A in a spacetime diagram will always remain within the forward light cone of event A.

The backwards light cone of event A is all of those events C such that A is in the forward light cone of C. This is the same as saying that $d\mathbf{r}_{CA} \cdot d\mathbf{r}_{CA} < 0$ and $t_C < t_A$.

Exercise 15.6 shows that any event B in the forward light cone of event A viewed in a spacetime diagram of the S system will still be in the forward light cone of A when viewed in any S' system connected to it by a proper Lorentz transformation. Thus the

temporal order of timelike separated events is an invariant. Exercise 15.7 shows that the same is not true for spacelike separated events. For them, even if $t_B > t_A$, there will be some system S' in which $t'_B = t'_A$ and another system S'' in which $t''_B < t''_A$ with the temporal order reversed.

Since no signal can propagate at greater than light speed, it is also true that the forward light cone of event A contains all of the events B that A can possibly influence. And that the backward light cone of event A contains all of those events C that can possibly influence A. We say that events within the light cones of A are *causally connected* to A, while those outside cannot be. Two events A and D that are spacelike separated are outside each other's light cones and cannot be causally connected.

15.6 General Fourvectors

We have so far treated only one fourvector, the differential displacement between two events $d\mathbf{r}$. But, just as in ordinary Cartesian three space where force, velocity, acceleration are all threevectors, there are many other fourvectors. The displacement $d\mathbf{r}$ is in a sense the *template* fourvector that defines the class.

Suppose we define a mathematical object \mathbf{A} in fourvector notation by specifying a set of four components A^0, A^1, A^2, A^3 in the S system and writing⁷⁹

$$\mathbf{A} = A^0 \hat{\mathbf{e}}_0 + A^1 \hat{\mathbf{e}}_1 + A^2 \hat{\mathbf{e}}_2 + A^3 \hat{\mathbf{e}}_3 \quad (15.24)$$

Then \mathbf{A} will actually be a fourvector provided that its components in the S and S' systems can be shown to be related by the same Lorentz transformation as eqn (15.4) (note that we are still assuming the standard configuration here)

$$\begin{aligned} A^0 &= \Gamma \left(A'^0 + \frac{V}{c} A'^1 \right) \\ A^1 &= \Gamma \left(\frac{V}{c} A'^0 + A'^1 \right) \\ A^2 &= A'^2 \\ A^3 &= A'^3 \end{aligned} \quad (15.25)$$

Just as in the case of $d\mathbf{r}$ in Section 15.1, this transformation law insures that the same fourvector \mathbf{A} can be expressed in the S' system using the primed coordinates and basis fourvectors, as

$$\mathbf{A} = A'^0 \hat{\mathbf{e}}'_0 + A'^1 \hat{\mathbf{e}}'_1 + A'^2 \hat{\mathbf{e}}'_2 + A'^3 \hat{\mathbf{e}}'_3 \quad (15.26)$$

For example, we can define a *spacetime position* or *radius* fourvector \mathbf{r} as the displacement vector between the origin of coordinates and an event with coordinates

⁷⁹Recall that we denote fourvectors by bold sans serif type, and matrices by normal sans serif type. Thus \mathbf{A} is a fourvector, and A is a matrix.

$(ct, x, y, z)_S$ in the S system and $(ct', x', y', z')_{S'}$ in the S' system. It can be written as

$$ct \hat{\mathbf{e}}_0 + x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 + z \hat{\mathbf{e}}_3 = ct \hat{\mathbf{e}}_0 + \mathbf{r} = \mathbf{r} = ct' \hat{\mathbf{e}}'_0 + \mathbf{r}' = ct' \hat{\mathbf{e}}'_0 + x' \hat{\mathbf{e}}'_1 + y' \hat{\mathbf{e}}'_2 + z' \hat{\mathbf{e}}'_3 \quad (15.27)$$

where

$$\mathbf{r} = x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 + z \hat{\mathbf{e}}_3 \quad \text{and} \quad \mathbf{r}' = x' \hat{\mathbf{e}}'_1 + y' \hat{\mathbf{e}}'_2 + z' \hat{\mathbf{e}}'_3 \quad (15.28)$$

are the radius threectors, the spatial parts of \mathbf{r} , in the two systems. The fact that \mathbf{r} is a fourvector follows directly from comparison of eqn (15.25) with the Lorentz transformation in eqn (14.53).

If \mathbf{A} and \mathbf{B} are any fourvectors, it follows from the transformation equations eqn (15.25) and the same reasoning that led to eqn (15.16), that the quantity $\mathbf{A} \cdot \mathbf{B}$ is an invariant form, with the expansion in the S and S' systems given by

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \left(\sum_{\mu=0}^3 A^\mu \hat{\mathbf{e}}_\mu \right) \cdot \left(\sum_{\nu=0}^3 B^\nu \hat{\mathbf{e}}_\nu \right) = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \\ &= \left(\sum_{\mu=0}^3 A'^\mu \hat{\mathbf{e}}'_\mu \right) \cdot \left(\sum_{\nu=0}^3 A'^\nu \hat{\mathbf{e}}'_\nu \right) = -A'^0 B'^0 + A'^1 B'^1 + A'^2 B'^2 + A'^3 B'^3 \end{aligned} \quad (15.29)$$

Like the displacement fourvector $d\mathbf{r}$, general fourvectors can be timelike, spacelike, or lightlike, depending on the value and sign of $\mathbf{A} \cdot \mathbf{A}$. A timelike fourvector \mathbf{A} can also be forward timelike ($\mathbf{A} \cdot \mathbf{A} < 0$ and $A^0 > 0$) or backward timelike ($\mathbf{A} \cdot \mathbf{A} < 0$ and $A^0 < 0$).

15.7 Construction of New Fourvectors

An interesting way to form new fourvectors is by construction from previous fourvectors and invariant quantities. For example, the quantity $d\tau$ defined in eqn (15.21) is an invariant. Dividing each term in the expansion of $d\mathbf{r}$ in eqn (15.8) through by the same quantity $d\tau$ gives

$$\frac{dx^0}{d\tau} \hat{\mathbf{e}}_0 + \frac{dx^1}{d\tau} \hat{\mathbf{e}}_1 + \frac{dx^2}{d\tau} \hat{\mathbf{e}}_2 + \frac{dx^3}{d\tau} \hat{\mathbf{e}}_3 = \frac{d\mathbf{r}}{d\tau} = \frac{dx'^0}{d\tau} \hat{\mathbf{e}}'_0 + \frac{dx'^1}{d\tau} \hat{\mathbf{e}}'_1 + \frac{dx'^2}{d\tau} \hat{\mathbf{e}}'_2 + \frac{dx'^3}{d\tau} \hat{\mathbf{e}}'_3 \quad (15.30)$$

Thus we can define a new fourvector $\mathbf{u} = d\mathbf{r}/d\tau$ by giving its components in the S system as

$$u^0 = \frac{cdt}{d\tau} \quad u^1 = \frac{dx}{d\tau} \quad u^2 = \frac{dy}{d\tau} \quad u^3 = \frac{dz}{d\tau} \quad (15.31)$$

with the same definitions in the S' system

$$u'^0 = \frac{cdt'}{d\tau} \quad u'^1 = \frac{dx'}{d\tau} \quad u'^2 = \frac{dy'}{d\tau} \quad u'^3 = \frac{dz'}{d\tau} \quad (15.32)$$

(Note the *lack* of primes on the $d\tau$ and c . They are *invariants* with the same value for any coordinate system.)

With these definitions, the transformation equation for the components of \mathbf{u} can be obtained by dividing each of eqn (15.4) through by $d\tau$ to obtain

$$\begin{aligned}\frac{cdt}{d\tau} &= \Gamma \left(\frac{cdt'}{d\tau} + \frac{V}{c} \frac{dx'}{d\tau} \right) \\ \frac{dx}{d\tau} &= \Gamma \left(\frac{V}{c} \frac{cdt'}{d\tau} + \frac{dx'}{d\tau} \right) \\ \frac{dy}{d\tau} &= \frac{dy'}{d\tau} \\ \frac{dz}{d\tau} &= \frac{dz'}{d\tau}\end{aligned}\tag{15.33}$$

Comparison of these transformation equations with eqn (15.25), with u substituted for A , proves that the components of \mathbf{u} do transform correctly and that \mathbf{u} is indeed a fourvector.

The fourvector \mathbf{u} is called the *fourvector velocity*. In the S system it can be written as

$$\mathbf{u} = u^0 \hat{\mathbf{e}}_0 + u^1 \hat{\mathbf{e}}_1 + u^2 \hat{\mathbf{e}}_2 + u^3 \hat{\mathbf{e}}_3 = u^0 \hat{\mathbf{e}}_0 + \mathbf{u}\tag{15.34}$$

where the spatial part of \mathbf{u} is the threevector

$$\mathbf{u} = u^1 \hat{\mathbf{e}}_1 + u^2 \hat{\mathbf{e}}_2 + u^3 \hat{\mathbf{e}}_3\tag{15.35}$$

This \mathbf{u} is the same as the proper velocity threevector that was used extensively in the survey of spacetime in Section 14.5. Using the definition of coordinate velocity \mathbf{v} from eqn (14.26), the fourvector \mathbf{u} can also be written as

$$\mathbf{u} = \gamma c \hat{\mathbf{e}}_0 + \gamma \mathbf{v}\tag{15.36}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2} = \sqrt{1 + u^2/c^2}$.

This process can be repeated, taking differentials of u^μ and dividing again by $d\tau$ to construct what is called the fourvector acceleration \mathbf{w} , defined by its components in the S system as

$$w^0 = \frac{du^0}{d\tau} = \frac{cd^2t}{d\tau^2} \quad w^1 = \frac{du^1}{d\tau} = \frac{d^2x}{d\tau^2} \quad w^2 = \frac{du^2}{d\tau} = \frac{d^2y}{d\tau^2} \quad w^3 = \frac{du^3}{d\tau} = \frac{d^2z}{d\tau^2}\tag{15.37}$$

and similar definitions for primed components in the S' system. Taking second derivatives in eqn (15.33), the components of this fourvector acceleration are easily shown to satisfy the transformation condition eqn (15.25).

We will return to the fourvector velocity and fourvector acceleration in Chapter 16 when the covariant form of mechanics is developed.

15.8 Covariant and Contravariant Components

We have so far written all fourvector components with a superscript index and all basis fourvectors with a subscript index. Quantities with superscripts are called *contravariant* and those with subscripts *covariant*.

This introduction of upper and lower indices is needed because the presence of matrix \mathbf{g} in the definition of the inner product of two fourvectors introduces a complication into relativity theory that was absent in ordinary threevector algebra. A method has evolved for dealing with this complication by defining *two sets* of S system components for every fourvector: its contravariant components (the one we have used above) and its covariant components. (There are also two sets in the S' system, and indeed in any coordinate system.)

However, the manipulation of these contravariant and covariant indices can itself prove a challenge. One author refers to “index gymnastics.” We introduce the methods here even though they are scarcely needed in the examples treated in this book. It is hoped that they will be easier to understand in the present context, and that familiarity with them will prove useful when the reader moves on to more advanced topics such as general relativity.

The general fourvector has been written with contravariant components A^μ . We define a set of covariant components A_μ for the same fourvector \mathbf{A} by a rule, called “lowering the index”

$$A_\mu = \sum_{\nu=0}^3 g_{\mu\nu} A^\nu \tag{15.38}$$

The inner products like $\mathbf{A} \cdot \mathbf{A}$ can then be written in an alternate form involving one covariant and one contravariant component and only one sum

$$\mathbf{A} \cdot \mathbf{A} = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} A^\mu A^\nu = \sum_{\mu=0}^3 A_\mu A^\mu \tag{15.39}$$

Often, the last expression on the right of eqn (15.39) is written without an explicit summation sign, using what is called the *Einstein summation convention*: Any term that contains the same Greek index (here μ) on a contravariant and a covariant object is automatically to be summed over that index from 0 to 3. In our treatment of fourvectors here, we will write our sums using this Einstein index convention, but will continue to write the summation signs explicitly, as we have done throughout the text.

Indices can also be raised. The matrix \mathbf{g} has an inverse, \mathbf{g}^{-1} . In the simple Minkowski space of special relativity, \mathbf{g} is in fact its own inverse since $(\mathbf{g})^2 = \mathbf{U}$ where \mathbf{U} is the four-rowed identity matrix with diagonal elements (1, 1, 1, 1). In general relativity, the inverse of \mathbf{g} will still exist, but will not have such a simple form.

We define a contravariant metric with $g^{\mu\nu}$ defined as the $\mu\nu$ element of the inverse matrix \mathbf{g}^{-1} . It follows that $\mathbf{g}^{-1} \mathbf{g} = \mathbf{U} = \mathbf{g} \mathbf{g}^{-1}$, or in component form,

$$\sum_{\alpha=0}^3 g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu = \sum_{\alpha=0}^3 g_{\nu\alpha} g^{\alpha\mu} \tag{15.40}$$

The indexed quantity δ_ν^μ is defined to have the value one whenever $\mu = \nu$ and the value zero otherwise. It is the four-dimensional generalization of the Kroeneker delta function defined in Section A.5.

Using this inverse $g^{\mu\nu}$, the covariant components A_μ defined in eqn (15.38) can be raised again, using the rule called “raising the index”

$$A^\mu = \sum_{\nu=0}^3 g^{\mu\nu} A_\nu \quad (15.41)$$

The reader can show that this operation recovers the original contravariant components.

In special relativity, the Minkowski matrix \mathbf{g} is diagonal. In general relativity the matrix \mathbf{g} will not in general be diagonal but will still be symmetric, with $g_{\nu\mu} = g_{\mu\nu}$ and $g^{\nu\mu} = g^{\mu\nu}$. So it makes no difference which index of \mathbf{g} is summed. And the dot product of two fourvectors will be independent of their order. For example,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu = g_{\mu\nu} = g_{\nu\mu} = \hat{\mathbf{e}}_\nu \cdot \hat{\mathbf{e}}_\mu \quad (15.42)$$

Basis fourvectors can also have their indices raised and lowered, by the rules⁸⁰

$$\hat{\mathbf{e}}^\mu = \sum_{\nu=0}^3 g^{\mu\nu} \hat{\mathbf{e}}_\nu \quad \hat{\mathbf{e}}_\mu = \sum_{\nu=0}^3 g_{\mu\nu} \hat{\mathbf{e}}^\nu \quad (15.43)$$

A general fourvector \mathbf{A} can thus be expanded in a basis in two equivalent ways

$$\sum_{\mu=0}^3 A_\mu \hat{\mathbf{e}}^\mu = \mathbf{A} = \sum_{\mu=0}^3 A^\mu \hat{\mathbf{e}}_\mu \quad (15.44)$$

Dot products with contravariant (covariant) basis fourvectors can be used to recover the contravariant (covariant) components of fourvectors, since

$$\hat{\mathbf{e}}^\mu \cdot \hat{\mathbf{e}}_\nu = \delta_\nu^\mu \quad \text{and hence} \quad A^\mu = \hat{\mathbf{e}}^\mu \cdot \mathbf{A} \quad \text{and} \quad A_\mu = \hat{\mathbf{e}}_\mu \cdot \mathbf{A} \quad (15.45)$$

All of the formulas in this section apply equally well in the S' system. One simply places primes on the components like A'^μ and A'_μ and on the basis vectors like $\hat{\mathbf{e}}'^\mu$ and $\hat{\mathbf{e}}'_\mu$ (but not on the Kroeneker delta or the metric, as noted above).

⁸⁰In advanced treatments of general relativity, in which it is often useful to work without introducing an explicit metric, a *dual* vector space can be defined, consisting of what are called *oneforms*. These oneforms act on fourvectors to produce a number, which is called their *contraction*. A common notation, for example in Misner, Thorne and Wheeler (1973), for the contraction of a oneform σ with a fourvector \mathbf{A} is $\langle \sigma, \mathbf{A} \rangle$. If a metric is now introduced, which allows dot products to be defined, then for every oneform σ there is a uniquely associated fourvector \mathbf{v} defined by $\langle \sigma, \mathbf{A} \rangle = \mathbf{v} \cdot \mathbf{A}$ for any \mathbf{A} . Basis oneforms ω^μ in the dual space are also introduced, which contract with fourvectors according to $\langle \omega^\mu, \mathbf{A} \rangle = A^\mu$. Comparison of this result with the second equation in eqn (15.45) shows that the basis oneform ω^μ is uniquely associated with the contravariant basis fourvector $\hat{\mathbf{e}}^\mu$ since $\langle \omega^\mu, \mathbf{A} \rangle = \hat{\mathbf{e}}^\mu \cdot \mathbf{A}$. The contravariant basis vector $\hat{\mathbf{e}}^\mu$ is indeed a fourvector and not a oneform, but it plays the same algebraic role as ω^μ .

15.9 General Lorentz Transformations

In the space of Cartesian threevectors, we saw in Section 8.3 that a general rotation operator can be defined as one that preserves the value of three-dimensional dot products, as seen, for example, in the invariant expression eqn (15.3). In an analogous way, a general Lorentz transformation is defined as one that preserves dot products of fourvectors and hence leads to the invariant expression eqn (15.16).

A general linear transformation between differential quantities dx^μ representing the interval $d\mathbf{r}$ in the S system and dx'^μ representing it in the S' system will be denoted as

$$dx^\mu = \sum_{\alpha=0}^3 M^\mu_\alpha dx'^\alpha \quad \text{or} \quad [dx] = \mathbf{M}[dx'] \tag{15.46}$$

where the second expression is in matrix form, with the $\mu\alpha$ matrix element of matrix \mathbf{M} denoted by M^μ_α , and $[dx]$ denoting a four rowed column vector consisting of the four components of $d\mathbf{r}$.

Definition 15.9.1: General Lorentz Transformation

A general Lorentz transformation is a linear coordinate transformation, written as in eqn (15.46), that obeys the condition

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 M^\mu_\alpha M^\nu_\beta g_{\mu\nu} = g_{\alpha\beta} \quad \text{or, in equivalent matrix form,} \quad \mathbf{M}^T \mathbf{g} \mathbf{M} = \mathbf{g} \tag{15.47}$$

Lemma 15.9.2: Invariance of Inner Product

Definition 15.9.1 implies and is implied by the invariance of $d\mathbf{r} \cdot d\mathbf{r}$, where this inner product is expanded in the S and S' systems as

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 dx^\mu dx^\nu g_{\mu\nu} = d\mathbf{r} \cdot d\mathbf{r} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx'^\mu dx'^\nu g_{\mu\nu} \tag{15.48}$$

Proof: Using eqn (15.46), and changing the dummy indices of the second expression, eqn (15.48) becomes

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M^\mu_\alpha M^\nu_\beta dx'^\alpha dx'^\beta g_{\mu\nu} = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 g_{\alpha\beta} dx'^\alpha dx'^\beta \tag{15.49}$$

or, in matrix form,

$$[dx']^T \mathbf{M}^T \mathbf{g} \mathbf{M} [dx'] = [dx']^T \mathbf{g} [dx'] \tag{15.50}$$

Since the differentials dx'^μ are arbitrary continuous variables, and both \mathbf{g} and $\mathbf{M}^T \mathbf{g} \mathbf{M}$ are symmetric matrices, this condition holds if and only if $\mathbf{M}^T \mathbf{g} \mathbf{M} = \mathbf{g}$, which is the same as eqn (15.47). Thus, $d\mathbf{r} \cdot d\mathbf{r}$ is invariant if and only if Definition 15.9.1 is satisfied. □

It follows from eqn (15.47) that the Lorentz transformation matrix \mathbf{M} has a non-zero determinant and hence an inverse \mathbf{M}^{-1} . To avoid confusion between the “-1” superscript and contravariant indices, we will denote \mathbf{M}^{-1} as $\underline{\mathbf{M}}$. Thus, defining $\underline{M}^{\mu}_{\nu}$ to be the $\mu\nu$ matrix element of $\underline{\mathbf{M}}$,

$$\sum_{\alpha=0}^3 \underline{M}^{\mu}_{\alpha} M^{\alpha}_{\nu} = \delta^{\mu}_{\nu} = \sum_{\alpha=0}^3 M^{\mu}_{\alpha} \underline{M}^{\alpha}_{\nu} \quad \text{or} \quad \underline{\mathbf{M}} \mathbf{M} = \mathbf{U} = \mathbf{M} \underline{\mathbf{M}} \quad (15.51)$$

which can be used to write relations inverse to eqn (15.46)

$$dx'^{\mu} = \sum_{\nu=0}^3 \underline{M}^{\mu}_{\nu} dx^{\nu} \quad \text{or} \quad [dx'] = \underline{\mathbf{M}}[dx] \quad (15.52)$$

The transformation rule for fourvector components in eqn (15.46) implies a transformation rule for the basis fourvectors. The covariant basis fourvectors in the S and S' systems must be related in a way that makes the expansion of $d\mathbf{r}$ in the two systems equivalent, such that

$$\sum_{\mu=0}^3 dx^{\mu} \hat{\mathbf{e}}_{\mu} = d\mathbf{r} = \sum_{\mu=0}^3 dx'^{\mu} \hat{\mathbf{e}}'_{\mu} \quad (15.53)$$

This equation is true for general dx^{μ} if and only if the following transformation equations hold for the covariant basis fourvectors

$$\hat{\mathbf{e}}_{\mu} = \sum_{\nu=0}^3 \underline{M}^{\nu}_{\mu} \hat{\mathbf{e}}'_{\nu} \quad \hat{\mathbf{e}}'_{\mu} = \sum_{\nu=0}^3 M^{\nu}_{\mu} \hat{\mathbf{e}}_{\nu} \quad (15.54)$$

Since the transformation coefficients M^{μ}_{α} in eqn (15.46) are constants, the transformation rules for the radius fourvector \mathbf{r} defined in eqn (15.27) are the same for the differential fourvector displacement $d\mathbf{r}$. Thus we can write the transformation rules for the components x^{μ} of the radius fourvector \mathbf{r} by simply replacing dx^{μ} by x^{μ} , etc., in the formulas of this section.

Note that the transformation rules in this section are generalizations of those derived earlier. Equations (15.4, 15.9) assumed the standard configuration for the S and S' systems, with the x -axes aligned, etc. The equations of this section are more general, and apply to any Lorentz transformation, not just the standard one.

15.10 Transformation of Components

By the definition of fourvectors, the transformation rules derived in Section 15.9 for the template fourvector $d\mathbf{r}$ must apply also to any fourvector \mathbf{A} . Thus \mathbf{A} is a fourvector if and only if

$$A^{\mu} = \sum_{\alpha=0}^3 M^{\mu}_{\alpha} A'^{\alpha} \quad \text{and} \quad A'^{\mu} = \sum_{\alpha=0}^3 \underline{M}^{\mu}_{\alpha} A^{\alpha} \quad (15.55)$$

These equations are to be considered as the generalization of the transformation rules in eqn (15.25), which assumed the special case of standard Lorentz transformations.

If we apply the results from eqns (15.38, 15.41) for the raising and lowering of indices to eqn (15.55), we obtain a transformation rule for quantities with covariant indices. Applying $g_{\mu\nu}$ and $g^{\mu\nu}$ gives

$$A_\mu = \sum_{\alpha=0}^3 g_{\mu\alpha} A^\alpha = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 \sum_{\nu=0}^3 g_{\mu\alpha} M^\alpha_\beta g^{\beta\nu} A'_\nu \quad (15.56)$$

Equation (15.47) implies that

$$\underline{\mathbf{M}} = \mathbf{g}^{-1} \mathbf{M}^T \mathbf{g} \quad (15.57)$$

or, in component form,

$$\underline{M}^{\nu}_\mu = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 g^{\nu\beta} M^\alpha_\beta g_{\alpha\mu} = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 g_{\mu\alpha} M^\alpha_\beta g^{\beta\nu} \quad (15.58)$$

where the order of terms has been changed to get the last equality. (Note that the symmetry of the \mathbf{g} allowed us also to exchange the indices on $g^{\beta\nu}$ and $g_{\mu\alpha}$.) Thus the transformation of covariant indices in eqn (15.56) can be written as

$$A_\mu = \sum_{\nu=0}^3 \underline{M}^{\nu}_\mu A'_\nu \quad \text{and the inverse} \quad A'_\mu = \sum_{\nu=0}^3 M^\nu_\mu A_\nu \quad (15.59)$$

which shows that covariant components transform by the same rule as the covariant basis fourvectors in eqn (15.54).

In a similar way, we can show that the contravariant basis fourvectors transform by the same rule as contravariant components in eqn (15.55).

$$\hat{\mathbf{e}}^\mu = \sum_{\nu=0}^3 M^\mu_\nu \hat{\mathbf{e}}'^\nu \quad \hat{\mathbf{e}}'^\mu = \sum_{\nu=0}^3 \underline{M}^\mu_\nu \hat{\mathbf{e}}^\nu \quad (15.60)$$

It follows from the transformation rules in this section that any sum involving the product of a covariant and a contravariant quantity will automatically be form invariant under Lorentz transformations. For example, expressions such as

$$\sum_{\mu=0}^3 A^\mu B_\mu = \mathbf{A} \cdot \mathbf{B} = \sum_{\mu=0}^3 A'^\mu B'_\mu \quad \text{and} \quad \sum_{\mu=0}^3 dx_\mu \hat{\mathbf{e}}^\mu = d\mathbf{r} = \sum_{\mu=0}^3 dx'_\mu \hat{\mathbf{e}}'^\mu \quad (15.61)$$

are form invariant. This is the main utility of the formalism of upper and lower indices.

15.11 Examples of Lorentz Transformations

We now give some examples of Lorentz transformations satisfying Definition 15.9.1.

15.11.1 The Standard Lorentz Transformation

The special case in which the coordinate axes of the S and S' systems are parallel, and the relative velocity is along the x -axis of each of them, is described in Section 14.6. This is referred to as the *standard* Lorentz transformation. Inspection of eqn (15.4) (or, equivalently, of eqn (14.53) for the radius fourvector \mathbf{r}) shows that the matrix M_{st} for this special case has the form

$$M_{st} = \begin{pmatrix} \Gamma & \Gamma B & 0 & 0 \\ \Gamma B & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } \Gamma = 1/\sqrt{1 - B^2} \quad \text{and } B = V/c \quad (15.62)$$

where V is the constant relative velocity of the two systems. From either eqn (15.57) or Cramer's rule, the inverse matrix is found to be

$$\underline{M}_{st} = \begin{pmatrix} \Gamma & -\Gamma B & 0 & 0 \\ -\Gamma B & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15.63)$$

Note that in this simple case the matrix M_{st} is symmetric, with $M_{st}^T = M_{st}$. General Lorentz transformation matrices are not symmetric, however, as the next example shows.

15.11.2 Rotations at Fixed Time

Threevector rotations of coordinates of the sort described in Section 8.3, with the extra condition that time is not transformed, are a special case of Lorentz transformations. For them, the matrix M_{rot} has the form

$$M_{rot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} \quad (15.64)$$

where the 3×3 sub-matrix \mathbf{R} is the orthogonal transformation matrix of a rotation in three-dimensional space, as discussed in Section 8.3. The reader can verify that, due to the orthogonality condition $\mathbf{R}^T \mathbf{R} = \mathbf{U} = \mathbf{R} \mathbf{R}^T$, this matrix M_{rot} does satisfy the condition Definition 15.9.1 and is therefore a Lorentz transformation.

15.11.3 General Boosts

If one begins with the standard Lorentz transformation and then applies the *same* proper rotation matrix \mathbf{R} to *both* the S and S' coordinate systems, the result is what

is called a boost in a general threevector direction \mathbf{V} . Boosts are sometimes called Lorentz transformations without rotation, but a more accurate term would be Lorentz transformations with identical rotation in both systems.

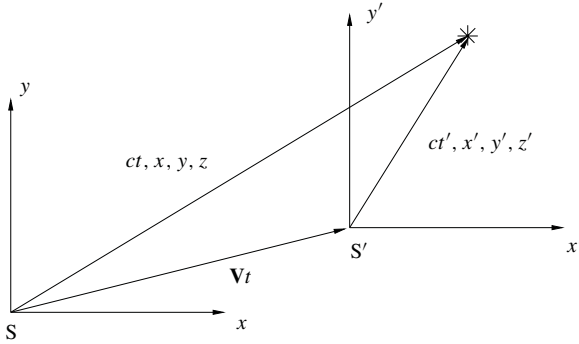


FIG. 15.2. Geometry of a boost in a general direction \mathbf{V} . The origins of S and S' coincide at $t = 0 = t'$. The diagram is somewhat schematic since the S' axes, which would appear to be perpendicular as seen by S' , would not appear perpendicular as seen by S, due to Lorentz contraction.

To derive the matrix M_{bst} for such a boost, we begin by writing the standard Lorentz transformation eqn (14.53) in threevector form as⁸¹

$$\begin{aligned}
 ct &= \Gamma \left\{ ct' + B \left(\hat{\mathbf{V}}' \cdot \mathbf{r}' \right) \right\} \\
 \mathbf{r} &\sim \left\{ \mathbf{r}' - \left(\hat{\mathbf{V}}' \cdot \mathbf{r}' \right) \hat{\mathbf{V}}' \right\} + \Gamma \left\{ Bct' \hat{\mathbf{V}}' + \left(\hat{\mathbf{V}}' \cdot \mathbf{r}' \right) \hat{\mathbf{V}}' \right\}
 \end{aligned}
 \tag{15.65}$$

where $\hat{\mathbf{V}}' = \hat{\mathbf{e}}'_1$, $\mathbf{r} = x \hat{\mathbf{e}}_1 + y \hat{\mathbf{e}}_2 + z \hat{\mathbf{e}}_3$, and $\mathbf{r}' = x' \hat{\mathbf{e}}'_1 + y' \hat{\mathbf{e}}'_2 + z' \hat{\mathbf{e}}'_3$.

Now assume that the S and S' coordinate systems are rotated by the *same* proper rotation matrix \mathbf{R} . Equations (15.65) are expressed as threevector relations, and hence are form invariant under such rotations. But the unit threevector $\hat{\mathbf{V}}'$ will now have different components. It will now be $\hat{\mathbf{V}}' = \alpha_1 \hat{\mathbf{e}}'_1 + \alpha_2 \hat{\mathbf{e}}'_2 + \alpha_3 \hat{\mathbf{e}}'_3$, where $(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1$. The resulting transformation of coordinates can then be expanded from eqn (15.65) as

$$\begin{aligned}
 ct &= \Gamma \left\{ ct' + B \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \right\} \\
 x &= \left\{ x' - \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_1 \right\} + \Gamma \left\{ Bct' \alpha_1 + \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_1 \right\} \\
 y &= \left\{ y' - \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_2 \right\} + \Gamma \left\{ Bct' \alpha_2 + \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_2 \right\} \\
 z &= \left\{ z' - \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_3 \right\} + \Gamma \left\{ Bct' \alpha_3 + \left(\alpha_1 x' + \alpha_2 y' + \alpha_3 z' \right) \alpha_3 \right\}
 \end{aligned}
 \tag{15.66}$$

⁸¹The notation $\mathbf{a} \sim \mathbf{b}'$ used here means that the threevector \mathbf{a} has the *same components* in the S system as the threevector \mathbf{b}' has in the S' system. We do not use an equal sign because the basis fourvectors in the two systems are not equal. For example, a displacement $\ell \hat{\mathbf{e}}_1$ would connect two events simultaneous in S. Those same two events would not be simultaneous in S' and hence could not be connected by any displacement of the form $\ell' \hat{\mathbf{e}}'_1$. Thus $\ell \hat{\mathbf{e}}_1 \sim \ell' \hat{\mathbf{e}}'_1$ but $\ell \hat{\mathbf{e}}_1 \neq \ell' \hat{\mathbf{e}}'_1$.

The matrix of a general boost can be read from eqn (15.66). It is

$$\mathbf{M}_{\text{bst}} = \begin{pmatrix} \Gamma & \Gamma B\alpha_1 & \Gamma B\alpha_2 & \Gamma B\alpha_3 \\ \Gamma B\alpha_1 & N_{11} & N_{12} & N_{13} \\ \Gamma B\alpha_2 & N_{21} & N_{22} & N_{23} \\ \Gamma B\alpha_3 & N_{31} & N_{32} & N_{33} \end{pmatrix} \quad \text{where} \quad N_{ij} = \delta_{ij} + (\Gamma - 1) \alpha_i \alpha_j \quad (15.67)$$

with the inverse matrix is given (using the same N_{ij} elements) by

$$\underline{\mathbf{M}}_{\text{bst}} = \begin{pmatrix} \Gamma & -\Gamma B\alpha_1 & -\Gamma B\alpha_2 & -\Gamma B\alpha_3 \\ -\Gamma B\alpha_1 & N_{11} & N_{12} & N_{13} \\ -\Gamma B\alpha_2 & N_{21} & N_{22} & N_{23} \\ -\Gamma B\alpha_3 & N_{31} & N_{32} & N_{33} \end{pmatrix} \quad (15.68)$$

It follows (see Exercise 15.11) that a point at rest in the S' system will appear to the S system to be moving with a threevector velocity $\mathbf{V} = V(\alpha_1 \hat{\mathbf{e}}_1 + \alpha_2 \hat{\mathbf{e}}_2 + \alpha_3 \hat{\mathbf{e}}_3)$, and that a point at rest in the S system will appear to the S' system to be moving with threevector velocity $(-\mathbf{V}')$ where $\mathbf{V}' = V(\alpha_1 \hat{\mathbf{e}}'_1 + \alpha_2 \hat{\mathbf{e}}'_2 + \alpha_3 \hat{\mathbf{e}}'_3)$. Thus $\mathbf{V} \sim \mathbf{V}'$. The threevector \mathbf{r}_\perp perpendicular to \mathbf{V} and the threevector \mathbf{r}'_\perp perpendicular to \mathbf{V}' will have the same components in the two systems. Thus $\mathbf{r}_\perp \sim \mathbf{r}'_\perp$.

Boosts are also called *velocity transformations*. They are completely specified once the threevector $\mathbf{V} = V(\alpha_1 \hat{\mathbf{e}}_1 + \alpha_2 \hat{\mathbf{e}}_2 + \alpha_3 \hat{\mathbf{e}}_3)$ is chosen.

15.12 Gradient Fourvector

A function $f(ct, x, y, z)$ has a differential given by the chain rule

$$df = \frac{\partial f}{\partial(ct)} d(ct) + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad (15.69)$$

A gradient fourvector ∂f is defined by the condition that this differential be reproduced by

$$df = d\mathbf{r} \cdot \partial f = d\mathbf{r} \cdot \frac{\partial f}{\partial \mathbf{r}} \quad \text{where} \quad d\mathbf{r} = \sum_{\mu=0}^3 dx^\mu \hat{\mathbf{e}}_\mu \quad (15.70)$$

The notation $\partial f = \partial f / \partial \mathbf{r}$ is similar to that discussed in Section A.11, and should be used with the same cautions.

Inspection of eqn (15.69) shows that a gradient fourvector satisfying eqn (15.70) is written most simply by using contravariant basis vectors, and covariant components defined as $\partial_\mu f = \partial f / \partial x^\mu$

$$\partial f = \frac{\partial f}{\partial \mathbf{r}} = \hat{\mathbf{e}}^0 \frac{\partial f}{\partial(ct)} + \hat{\mathbf{e}}^1 \frac{\partial f}{\partial x} + \hat{\mathbf{e}}^2 \frac{\partial f}{\partial y} + \hat{\mathbf{e}}^3 \frac{\partial f}{\partial z} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \frac{\partial f}{\partial x^\mu} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \partial_\mu f \quad (15.71)$$

It follows from this definition that

$$\begin{aligned} d\mathbf{r} \cdot \partial f &= \left(\sum_{\mu=0}^3 dx^\mu \hat{\mathbf{e}}_\mu \right) \cdot \left(\sum_{\nu=0}^3 \hat{\mathbf{e}}^\nu \frac{\partial f}{\partial x^\nu} \right) = \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx^\mu \frac{\partial f}{\partial x^\nu} \hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}^\nu \\ &= \sum_{\mu=0}^3 \sum_{\nu=0}^3 dx^\mu \frac{\partial f}{\partial x^\nu} \delta_\mu^\nu = \sum_{\mu=0}^3 dx^\mu \frac{\partial f}{\partial x^\mu} = df \end{aligned} \quad (15.72)$$

as required.

Like any fourvector, the same gradient fourvector can also be written using contravariant components and covariant basis vectors, as

$$\partial f = \frac{\partial f}{\partial \mathbf{r}} = \sum_{\mu=0}^3 \hat{\mathbf{e}}_\mu \partial^\mu f \quad \text{where} \quad \partial^\mu f = \sum_{\nu=0}^3 g^{\mu\nu} \frac{\partial f}{\partial x^\nu} = \frac{\partial f}{\partial x_\mu} \quad (15.73)$$

Thus the gradient can also be written as

$$\partial f = \frac{\partial f}{\partial \mathbf{r}} = -\hat{\mathbf{e}}_0 \frac{\partial f}{\partial(ct)} + \hat{\mathbf{e}}_1 \frac{\partial f}{\partial x} + \hat{\mathbf{e}}_2 \frac{\partial f}{\partial y} + \hat{\mathbf{e}}_3 \frac{\partial f}{\partial z} = -\hat{\mathbf{e}}_0 \frac{\partial f}{\partial(ct)} + \frac{\partial f}{\partial \mathbf{r}} \quad (15.74)$$

To prove that the quantity ∂f truly is a fourvector, we must check its transformation properties. If a function is defined in the S and S' systems by $f(ct, x, y, z) = f(ct', x', y', z')$ then the chain rule gives

$$\partial'_\mu f = \frac{\partial f}{\partial x'^\mu} = \sum_{\nu=0}^3 \frac{\partial f}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = \sum_{\nu=0}^3 \partial_\nu f M^\nu_\mu \quad (15.75)$$

which matches the second of eqn (15.59) for the inverse transformation of covariant components. Thus ∂f is a fourvector.

15.13 Manifest Covariance

If an expression relating physical variables can be written entirely in terms of fourvectors and invariants, it is said to be *manifestly covariant*. The exclusive use of manifestly covariant expressions in physics guarantees the consistency of the resulting theory with special relativity. If a manifestly covariant expression is correct when written out in terms of coordinates in some system, say the S' system, then it is guaranteed to be correct when written out in any coordinate system.

For example, consider the fourvector expression $\mathbf{K} = m\mathbf{w}$, where m is an invariant quantity. In the S' system, it takes the form $K'^\nu = mw'^\nu$. Assume that we know this expression to be correct. Since K'^ν and w'^ν are components of a fourvector, we may apply the same fourvector transformation from eqn (15.55) to both sides of this

expression to obtain the second equality

$$\sum_{\nu=0}^3 M^{\mu}_{\nu} K'^{\nu} = m \sum_{\nu=0}^3 M^{\mu}_{\nu} w'^{\nu} \quad \text{and hence} \quad K^{\mu} = m w^{\mu} \quad (15.76)$$

which shows that the same expression is also correct in the S system. This example illustrates the general rule stated in the following note.

Note to the Reader: If a manifestly covariant expression is true in one coordinate system, then it is automatically true in all coordinate systems obtained by proper Lorentz transformations. Thus, use of manifestly covariant expressions guarantees consistency with special relativity.

The elegance of the fourvector formalism, however, should not seduce us into believing that an expression must be true just because it is written in manifestly covariant form. A manifestly covariant expression might be false in all systems! For example, assuming the above expression to be true, we could also write the manifestly covariant expression $\mathbf{K} = m\beta (\mathbf{w} \cdot \mathbf{w}) \mathbf{w}$ where β is assumed to be some invariant constant. But this expression would be false. The fourvector notation is just an elegant language, and like any language can be used to make both true and false statements.

15.14 Formal Covariance

In Section 15.6 we saw how new fourvectors like \mathbf{u} can be derived by combining old fourvectors and invariant quantities like $d\tau$. But another way to make fourvectors is, as it were, by fiat.

We could, for example, define a new fourvector $\hat{\mathbf{n}}$ by giving its components in the S' system as $\hat{\mathbf{n}} : (1, 0, 0, 0)_{S'}$. Then we could use the Lorentz transformation rule eqn (15.55) to *define* the components of $\hat{\mathbf{n}}$ in some other system. Thus, its components in the S system would be $\hat{\mathbf{n}} : (M^0_0, M^1_0, M^2_0, M^3_0)_S$ where the M^{μ}_{ν} are the elements of the transformation matrix between the two systems.

We can then use this fourvector in a covariant expression, for example $\mathbf{r} \cdot \hat{\mathbf{n}} = \theta$ where θ is some invariant quantity. Such an expression uses the formalism of covariance (fourvectors and scalars) but violates the spirit of the principle of relativity. The vector $\hat{\mathbf{n}}$ by definition has a particular form in a particular coordinate system S' . Its components in another system cannot be found unless we know the transformation from our current system to that special system S' . Such equations are called *formally covariant*. They use the covariant formalism to express an essentially nonrelativistic idea since they contain elements (here $\hat{\mathbf{n}}$) that single out one particular coordinate system for special treatment.

15.15 The Lorentz Group

We saw in Section 8.7 that threevector rotations form a group. We now show that the same is true for Lorentz transformations.

As for rotations, the product of two Lorentz transformations is defined to mean successive transformation. Thus, if M denotes a Lorentz transformation and N denotes another, their product $C = MN$ is defined to mean the application of N and then the application of M to the result. Taking N^μ_ν to be the transformation matrix from S' to S'' and M^μ_ν to be the transformation matrix from S to S' , the transformation $C = MN$ from S to S'' has the matrix C where

$$C^\mu_\nu = \sum_{\alpha=0}^3 M^\mu_\alpha N^\alpha_\nu \quad \text{or, in matrix form,} \quad C = M N \quad (15.77)$$

A set of objects, often called a set of group *elements*, is said to form a *group* if an ordered binary operation called *group multiplication* of the elements is defined, and the standard group axioms listed below are satisfied. The common usage is to say that the objects form a group *under* that particular group multiplication. We show that Lorentz transformations form a group, under group multiplication defined as successive transformation.

1. The first axiom is closure. The group product of two elements must be an element in the same group. The product of two Lorentz transformations must also be a Lorentz transformation. Closure is proved by showing if M and N obey eqn (15.47) of Definition 15.9.1 then C does also, as follows from the matrix relation

$$C^T g C = N^T M^T g M N = N^T g N = g \quad (15.78)$$

2. There must be an identity element U in the group such that pre- or post-multiplication of any object by that identity does not change the object, $UM = M = MU$. The identity transformation, with matrix U where $U^\mu_\nu = \delta^\mu_\nu$ satisfies this condition. It also satisfies definition eqn (15.47) since $U^T g U = g$ is trivially true. Thus the identity is a member of the group, as required.
3. Every object M in the group must have an inverse \underline{M} in the group, such that pre- or post-multiplication of that object by its inverse yields the identity object, $\underline{M}M = U = M\underline{M}$. The inverse transformation \underline{M} defined in Section 15.9 satisfies this condition, as shown by eqn (15.51). The inverse is also a member of the group, as is demonstrated by multiplying both sides of eqn (15.47) from the left by \underline{M}^T and from the right by \underline{M} which gives

$$g = \underline{M}^T M^T g M \underline{M} = \underline{M}^T g \underline{M} \quad (15.79)$$

This proves that \underline{M} also satisfies eqn (15.47), and is hence also a Lorentz transformation and a member of the group.

4. Group multiplication must be associative. As for rotations, the associativity of group multiplication is a trivial consequence of its definition.

If the product of a pair of elements gives a result independent of their order, the group is said to be *Abelian*. Like rotations, Lorentz transformations form a non-Abelian group. In general $MN \neq NM$.

We have already made use of the fact that Lorentz transformations form a group under successive application, in Section 15.11.3 when we multiplied two rotations and a standard Lorentz transformation to form a general boost. Due to the group property of Lorentz transformations, we know that this boost is also a Lorentz transformation.

15.16 Proper Lorentz Transformations and the Little Group

The following are Lorentz transformations and hence members of the Lorentz group: time reversal with $M_{tr} = \mathbf{g}$; parity or spatial inversion with $M_{par} = -\mathbf{g}$; and total inversion with $M_{inv} = -\mathbf{U}$. Together with the identity \mathbf{U} , these elements define a four-element, abelian subgroup called the *Little Group*.

A subgroup is a subset of group elements such that there are closure, an identity, and inverses *within the subgroup*. The reader can easily verify that the product of any pair of these four elements yields another one of the four, proving subgroup closure. The identity element U is a member of the subgroup by definition. And each element of the subgroup is its own inverse. Also the reader can verify that the product of any pair of the four elements is the same regardless of the order in which they are multiplied, verifying that the subgroup is abelian.

If a Lorentz transformation has matrix \mathbf{M} with $M^0_0 \geq 1$ and determinant $|\mathbf{M}| = +1$ then it will be called a *proper Lorentz transformation*. Such transformations, (1) preserve the direction of time and, (2) preserve the right-handed sense of the spatial basis fourvectors $\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$. Every Lorentz transformation can be written as the product of a proper Lorentz transformation and a member of the Little Group.

Note that the standard Lorentz transformation and the general boost in Section 15.11 are proper Lorentz transformations. The rotation at fixed time is a proper Lorentz transformation if the 3×3 sub-matrix \mathbf{R} in eqn (15.64) is a proper rotation matrix in the sense described in Section 8.6.

Proper Lorentz transformations form a subgroup of the Lorentz group. Subgroup closure is proved in Exercise 15.13, which establishes that the product of any two proper Lorentz transformations is a proper Lorentz transformation. The identity U is a proper Lorentz transformation, and hence the subgroup contains an identity. And it follows from eqn (15.57) that the inverse of a proper Lorentz transformation will also be a proper Lorentz transformation. Thus all the group axioms are satisfied and proper Lorentz transformations form a subgroup. The subgroup of proper Lorentz transformations is said to be continuously connected to the identity U since, by Theorem 15.17.2, one can begin with any proper Lorentz transformation and pass continuously to the limits $\Phi \rightarrow 0$ and $V \rightarrow 0$, arriving at the identity transformation U .

15.17 Parameterization

The most general proper Lorentz transformation can be written as the product of a boost and a rotation. To demonstrate how this is done, we first need a preliminary result.

Lemma 15.17.1: A Relation Among Matrix Elements

Let M be the matrix of any Lorentz transformation. Then the matrix elements of M obey the following two identities

$$\left(M^0_0\right)^2 - \sum_{i=1}^3 \left(M^i_0\right)^2 = 1 = \left(M^0_0\right)^2 - \sum_{i=1}^3 \left(M^0_i\right)^2 \tag{15.80}$$

The proof of this lemma is in Exercise 15.9. We now state the main theorem.⁸²

Theorem 15.17.2: Lorentz Transformation as Product of Boost and Rotation

Let M be the matrix of any proper Lorentz transformation. Then there are a unique boost M_{bst} and a unique proper rotation M_{rot} such that

$$M = M_{\text{bst}} M_{\text{rot}} \tag{15.81}$$

It follows that a general Lorentz transformation can be parameterized uniquely by six numbers: The three components of the threevector $\Phi \hat{\mathbf{n}}$ (from the Euler theorem of Section 8.20) that specify the proper rotation M_{rot} and the three components of the threevector velocity $\mathbf{V} = V \hat{\mathbf{V}}$ which are shown in Section 15.11.3 to specify the boost M_{bst} uniquely.

Proof: According to the transformation law eqn (15.55) applied to fourvector \mathbf{r} , a point at rest at the origin of the S' system with coordinates $(ct', 0, 0, 0)_{S'}$ will be represented in the S system by the four coordinates $(M^0_0 ct', M^1_0 ct', M^2_0 ct', M^3_0 ct')_S$ and will therefore appear to have threevector velocity

$$\mathbf{V} = \sum_{i=1}^3 V_i \hat{\mathbf{e}}_i \quad \text{with components} \quad V_i = c M^i_0 / M^0_0 \tag{15.82}$$

It follows from Lemma 15.17.1 that this vector has magnitude $V = \sqrt{\mathbf{V} \cdot \mathbf{V}} \leq c$. Define M_{bst} to be the boost determined by this velocity \mathbf{V} .

Apply the inverse of this boost, $\underline{M}_{\text{bst}}$, to M to get a new matrix X defined by

$$X = \underline{M}_{\text{bst}} M \tag{15.83}$$

Performing the matrix multiplication shows that $X^0_0 = 1$. The matrix X is a product of Lorentz transformations and hence, by the group properties in Section 15.15, is a Lorentz transformation. Thus Lemma 15.17.1 applies and can be used to show that $X^i_0 = X^0_i = 0$ for $i = 1, 2, 3$. Thus X is a Lorentz transformation of the type given in Section 15.11.2, a rotation at fixed time. Since $X^0_0 = 1$ and $|X| = |\underline{M}_{\text{bst}}| |M| = 1$, X is also a proper rotation. Defining $M_{\text{rot}} = X$ and multiplying eqn (15.83) from the left by M_{bst} then gives eqn (15.81), which proves the theorem.

To prove the uniqueness of this factorization, suppose that some other boost $M_{\text{bst}}^{(*)}$ and rotation $M_{\text{rot}}^{(*)}$ can be found so that $M = M_{\text{bst}}^{(*)} M_{\text{rot}}^{(*)}$. It follows from this equation

⁸²This theorem and its proof are adapted from Wichmann (1974).

and eqn (15.81) that $\underline{\mathbf{M}}_{\text{bst}} \mathbf{M}_{\text{bst}}^{(*)} = \underline{\mathbf{M}}_{\text{rot}} \mathbf{M}_{\text{rot}}^{(*)}$. The right side of this equation is the product of two proper rotations, and hence a proper rotation. Thus the matrix product on the left must also be a proper rotation and hence have $\left(\underline{\mathbf{M}}_{\text{bst}} \mathbf{M}_{\text{bst}}^{(*)}\right)_0^0 = 1$. Carrying out the matrix multiplication, this condition implies

$$1 = \frac{1 - (\mathbf{V} \cdot \mathbf{V}^{(*)})/c^2}{\sqrt{1 - V^2/c^2} \sqrt{1 - V^{(*)2}/c^2}} \quad (15.84)$$

Since $(\mathbf{V} \cdot \mathbf{V}^{(*)}) \leq V V^{(*)}$, this equation can be satisfied if and only if $\mathbf{V} = \mathbf{V}^{(*)}$. Since a boost is completely determined by the choice of \mathbf{V} , this proves that the two boosts are the same and hence proves uniqueness. \square

15.18 Fourvector Operators

In Chapter 7, we developed the formalism of linear operators that map threevectors into other threevectors. Operators in the space of fourvectors can be defined similarly. Let

$$\mathbf{C} = \mathcal{F} \mathbf{A} \quad (15.85)$$

map fourvector \mathbf{A} into another fourvector \mathbf{C} . We assume this operation to be linear, so that

$$\mathcal{F}(\alpha \mathbf{A} + \beta \mathbf{B}) = \alpha \mathcal{F} \mathbf{A} + \beta \mathcal{F} \mathbf{B} \quad (15.86)$$

Using this linearity, as well as the assumed linearity of dot products, it follows that eqn (15.85) can be written out in terms of components in a particular coordinate system, say the S system, by writing

$$C^\mu = \hat{\mathbf{e}}^\mu \cdot \mathbf{C} = \hat{\mathbf{e}}^\mu \cdot \left\{ \mathcal{F} \left(\sum_{\nu=0}^3 A^\nu \hat{\mathbf{e}}_\nu \right) \right\} = \sum_{\nu=0}^3 F^\mu{}_\nu A^\nu \quad (15.87)$$

where

$$F^\mu{}_\nu = \hat{\mathbf{e}}^\mu \cdot (\mathcal{F} \hat{\mathbf{e}}_\nu) \quad (15.88)$$

are the matrix elements of the matrix \mathbf{F} that represents \mathcal{F} in the S coordinate system. This result can also be written as the matrix equation $[C] = \mathbf{F}[A]$.

Operators like \mathcal{F} are a particular example of a general class of mathematical objects called *tensors*. Since the matrix elements of operators have two indices, they are called tensors of rank two. Tensors of other ranks can also be defined. Fourvectors can be considered as tensors of rank one, for example. Tensors are an indispensable tool in general relativity, but their development here would take us too far afield. The reader can consult a number of excellent texts, particularly Misner, Thorne and Wheeler (1973), which uses an invariant approach similar to the one we have used.

Like fourvectors, the operator \mathcal{F} is considered to be an invariant object that models some physical property that is independent of our particular choice of coordinate

system. The same operator \mathcal{F} can thus be represented by its matrix elements in any coordinate system. In the S' system, \mathcal{F} is represented by

$$F'^{\mu}_{\nu} = \hat{\mathbf{e}}'^{\mu} \cdot (\mathcal{F} \hat{\mathbf{e}}'_{\nu}) \quad (15.89)$$

Using eqn (15.54), the relation between the matrix elements of \mathcal{F} in the S and S' coordinate systems can be written as

$$F^{\mu}_{\nu} = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M^{\mu}_{\alpha} F'^{\alpha}_{\beta} \underline{M}^{\beta}_{\nu} \quad \text{and} \quad F'^{\mu}_{\nu} = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 \underline{M}^{\mu}_{\alpha} F^{\alpha}_{\beta} M^{\beta}_{\nu} \quad (15.90)$$

in which the upper index transforms as a contravariant and the lower index as a covariant quantity.

The indices of F^{μ}_{ν} can be raised and lowered in the usual manner, giving, for example, equivalent components of \mathcal{F} like

$$F_{\mu\nu} = \sum_{\alpha=0}^3 g_{\mu\alpha} F^{\alpha}_{\nu} \quad \text{or} \quad F^{\mu\nu} = \sum_{\alpha=0}^3 F^{\mu}_{\alpha} g^{\alpha\nu} \quad (15.91)$$

The invariant trace and determinant of fourvector operators can also be defined. The trace is

$$\text{Tr } \mathbf{F} = \sum_{\mu=0}^3 F^{\mu}_{\mu} = \text{Tr } \mathcal{F} = \sum_{\mu=0}^3 F'^{\mu}_{\mu} = \text{Tr } \mathbf{F}' \quad (15.92)$$

and the determinant is

$$|\mathbf{F}| = \det \mathcal{F} = |\mathbf{F}'| \quad (15.93)$$

The invariance of these quantities follows from the matrix form of eqn (15.90),

$$\mathbf{F} = \mathbf{M} \mathbf{F}' \underline{\mathbf{M}} \quad \text{and} \quad \mathbf{F}' = \underline{\mathbf{M}} \mathbf{F} \mathbf{M} \quad (15.94)$$

together with $\underline{\mathbf{M}} \mathbf{M} = \mathbf{U}$ from eqn (15.51).

15.19 Fourvector Dyadics

The dyadic formalism of Section 7.6 can also be extended to the space of fourvectors. We present the formalism very briefly here, assuming that the reader has already understood the more extensive treatment of dyadics in the earlier section.

We define a dyadic \mathbb{F} associated with the \mathcal{F} by

$$\mathbb{F} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}_{\mu} F^{\mu}_{\nu} \hat{\mathbf{e}}^{\nu} \quad (15.95)$$

Then the action of \mathcal{F} on fourvector \mathbf{A} in eqn (15.87) can also be written as a dyadic

dot product

$$\mathcal{F}\mathbf{A} = \mathbb{F} \cdot \mathbf{A} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}_{\mu} F^{\mu}_{\nu} \hat{\mathbf{e}}^{\nu} \cdot \mathbf{A} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}_{\mu} F^{\mu}_{\nu} A^{\nu} = \sum_{\mu=0}^3 \hat{\mathbf{e}}_{\mu} C^{\mu} = \mathbf{C} \quad (15.96)$$

This result is analogous to the action of dyadics in the space of threevectors presented in Section 7.6. The matrix associated with a dyadic can be recovered by the rule

$$F^{\mu}_{\nu} = \hat{\mathbf{e}}^{\mu} \cdot \mathbb{F} \cdot \hat{\mathbf{e}}_{\nu} \quad (15.97)$$

which can be derived from eqn (15.95) by use of eqn (15.45).

Dyadics, like operators, are invariant objects. The *same* dyadic \mathbb{F} can also be expanded in the S' system as

$$\mathbb{F} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}'_{\mu} F'^{\mu}_{\nu} \hat{\mathbf{e}}'^{\nu} \quad \text{where} \quad F'^{\mu}_{\nu} = \hat{\mathbf{e}}'^{\mu} \cdot \mathbb{F} \cdot \hat{\mathbf{e}}'_{\nu} \quad (15.98)$$

Thus, as was noted for threevectors and operators in Chapter 7, we have four equivalent ways of presenting the same information:

$$\mathbf{C} = \mathcal{F}\mathbf{A} \quad \mathbf{C} = \mathbb{F} \cdot \mathbf{A} \quad C^{\mu} = \sum_{\nu=0}^3 F^{\mu}_{\nu} A^{\nu} \quad [C] = F[A] \quad (15.99)$$

The latter two expressions involve components in a particular basis, here the S system. They would take the same form (with primes added) in the S' system

$$C'^{\mu} = \sum_{\nu=0}^3 F'^{\mu}_{\nu} A'^{\nu} \quad [C'] = F'[A'] \quad (15.100)$$

15.20 Wedge Products

One particular dyadic of importance is what is called the *wedge product* \mathbb{W} of two fourvectors \mathbf{A} and \mathbf{B} . It is denoted as $\mathbf{A} \wedge \mathbf{B}$ and defined as the difference of two dyads

$$\mathbb{W} = \mathbf{A} \wedge \mathbf{B} = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} \quad (15.101)$$

It follows from eqn (15.97) that the matrix of the wedge product in the S system is

$$W^{\mu}_{\nu} = A^{\mu} B_{\nu} - B^{\mu} A_{\nu} \quad (15.102)$$

which of course can have its indices raised or lowered as needed. By construction, the wedge product has the properties

$$\mathbf{A} \wedge \mathbf{B} = -\mathbf{B} \wedge \mathbf{A} \quad \mathbf{A} \wedge \mathbf{A} = 0 \quad \mathbf{C} \wedge (\alpha \mathbf{A} + \beta \mathbf{B}) = \alpha \mathbf{C} \wedge \mathbf{A} + \beta \mathbf{C} \wedge \mathbf{B} \quad (15.103)$$

The wedge product of two fourvectors is analogous to the cross product of two threevectors (See Exercise 15.14). In three-dimensional Cartesian spaces, the cross product of two threevectors is a unique threevector orthogonal to the plane they define. But in four dimensions, the space perpendicular to the plane defined by two fourvectors is itself two dimensional. Hence, it is not possible to represent a cross product by a single fourvector.

15.21 Scalar, Fourvector, and Operator Fields

A scalar field is a function which assigns a particular number to each event. Since events can be specified in an invariant way by the radius fourvector \mathbf{r} , a scalar field can be written as $f = f(\mathbf{r})$. This function can be written out in terms of components in different coordinate systems. In the S and S' systems, for example,

$$f(ct, x, y, z) = f(\mathbf{r}) = f(ct', x', y', z') \tag{15.104}$$

where we have followed the convention discussed in Section D.5, using the same letter f to represent the function expressed in different coordinate systems. If the functional form of $f(ct, x, y, z)$ is known, and if f is known to be a scalar field, then eqn (15.104) and the Lorentz transformation eqn (14.53) for the components of fourvector \mathbf{r} can be used to calculate $f(ct', x', y', z')$.

Fourvector fields are functions that assign a particular fourvector to each event. They can be written as $\mathbf{A}(\mathbf{r})$. Thus, in the S and S' systems,

$$\sum_{\mu=0}^3 A^\mu(ct, x, y, z) \hat{\mathbf{e}}_\mu = \mathbf{A}(\mathbf{r}) = \sum_{\mu=0}^3 A'^\mu(ct', x', y', z') \hat{\mathbf{e}}'_\mu \tag{15.105}$$

where the components are related by eqn (15.55)

$$A^\mu(ct, x, y, z) = \sum_{\nu=0}^3 M^\mu_\nu A'^\nu(ct', x', y', z') \tag{15.106}$$

Field operators $\mathcal{F}(\mathbf{r})$ and dyadic fields $\mathbb{F}(\mathbf{r})$ are also useful. They can be written in dyadic form in the S and S' systems as

$$\sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}_\mu F^\mu_\nu(ct, x, y, z) \hat{\mathbf{e}}^\nu = \mathbb{F}(\mathbf{r}) = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}'_\mu F'^\mu_\nu(ct', x', y', z') \hat{\mathbf{e}}'^\nu \tag{15.107}$$

where the matrix elements transform according to the rule derived from eqn (15.90)

$$F^\mu_\nu(ct, x, y, z) = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M^\mu_\alpha F'^\alpha_\beta(ct', x', y', z') \underline{M}^\beta_\nu \tag{15.108}$$

Note that transforming a field is a two-step process. Using fourvector fields as an example, first one evaluates the right side of eqn (15.106). The resulting expression is $A^\mu(ct', x', y', z')$, a component in the S system as desired, but is still expressed in terms of S' system coordinates. The second step is to substitute the inverse Lorentz transformation eqn (14.55) into this expression to write it in terms of the correct S coordinates, as $A^\mu(ct, x, y, z)$. Transformation of field operators follows a similar two-step process.

15.22 Manifestly Covariant Form of Maxwell's Equations

In the following Chapter 16, we will consider the relativistic generalization of mechanics, including manifestly covariant Lagrangian and Hamiltonian methods. In that chapter, we will use the example of a charged point mass moving in an external, given electromagnetic field. The fourvector formulation of electromagnetism to be used there will be outlined here as an example, to help the reader see how a manifestly covariant theory can be developed. Since the Lorentz transformation was initially derived by Lorentz as the transformation theory leaving Maxwell's equations form invariant, we expect that a fourvector formulation of Maxwell's equations should be possible.

We begin with the threevector electrical current density

$$\mathbf{J}(ct, x, y, z) = \rho(ct, x, y, z) \mathbf{v}(ct, x, y, z) \quad (15.109)$$

derived from the charge density $\rho(ct, x, y, z)$ and the flow velocity field of that charge $\mathbf{v}(ct, x, y, z)$. These quantities obey the differential equation of charge conservation⁸³

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (15.110)$$

To put this expression in manifestly covariant form, we note that an observer moving with the flow of charge can draw an averaging sphere of radius ε containing charge $q_\varepsilon^{(\text{ch})}$ and thus define a comoving charge density $\rho_{(0)} = q_\varepsilon^{(\text{ch})} / (4\pi\varepsilon^3/3)$. It is taken as an axiom that the electric charge $q_\varepsilon^{(\text{ch})}$ has the same value in any reference system. An inertial observer team, relative to which the comoving observer has coordinate velocity \mathbf{v} , will thus observe a charge density modified only by the Lorentz contraction of the averaging sphere

$$\rho = \rho_{(0)}\gamma \quad \text{where} \quad \gamma = 1/\sqrt{1 - v^2/c^2} \quad (15.111)$$

leading to an observed current density $\mathbf{J} = \rho\mathbf{v} = \rho_{(0)}\gamma\mathbf{v}$. Comparing these results to the definition of the fourvector velocity \mathbf{u} in Section 15.6, we see that the charge density and the current density threevector can be combined to form a fourvector defined by

$$\mathbf{J} = c\rho\hat{\mathbf{e}}_0 + \mathbf{J} = \rho_{(0)}\mathbf{u} \quad (15.112)$$

This \mathbf{J} is a fourvector since \mathbf{u} has already been shown to be a fourvector in Section 15.6, and because $\rho_{(0)}$ (sometimes called the proper density) is independent of any particular observer team and hence is an invariant, much like the time $d\tau$ measured on a comoving clock was found to be an invariant. With the fourvector \mathbf{J} , the conservation of electric charge in eqn (15.110) can be expressed in manifestly covariant

⁸³The current density and equation of charge conservation are derived, for example, in Wangsness (1986).

form as the vanishing of a fourvector divergence defined by

$$\partial \cdot \mathbf{J} = \sum_{\mu=0}^3 \partial_{\mu} J^{\mu} = \frac{\partial(c\rho)}{\partial ct} + \nabla \cdot \mathbf{J} = 0 \quad (15.113)$$

The Maxwell equations in threevector form are⁸⁴

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (15.114)$$

$$\nabla \cdot \mathbf{E} = \rho \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c} \mathbf{J} \quad (15.115)$$

where we assume that the electric field $\mathbf{E}(ct, x, y, z)$ and magnetic induction field $\mathbf{B}(ct, x, y, z)$ are in a vacuum, except possibly for the presence of charge density ρ and current density \mathbf{J} .

The two homogeneous Maxwell equations, eqn (15.114), are turned into identities by the introduction of a potential $\Phi(ct, x, y, z)$ and a threevector potential $\mathbf{A}(ct, x, y, z)$. The electric and magnetic threevector fields in terms of these potentials are

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (15.116)$$

Substituting these definitions into eqn (15.115), and assuming the Lorentz gauge condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \quad (15.117)$$

the two inhomogeneous Maxwell equations become

$$\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \rho \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{1}{c} \mathbf{J} \quad (15.118)$$

The differential operators on the left sides of eqn (15.118) can be written as the divergence of a gradient fourvector. Taking the first of these equations as an example, we can write

$$\square^2 \Phi = \partial \cdot (\partial \Phi) = \left(\sum_{\mu=0}^3 \partial_{\mu} \hat{\mathbf{e}}^{\mu} \right) \cdot \left(\sum_{\nu=0}^3 \partial^{\nu} \Phi \hat{\mathbf{e}}_{\nu} \right) = -\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi \quad (15.119)$$

where the notation $\square^2 \Phi$, usually referred to as the *d'Alembertian* operator \square^2 acting on Φ , has been introduced for this expression.

⁸⁴We use Heaviside-Lorentz electrostatic units. The fields \mathbf{E} and \mathbf{B} have the same units, and factors of 4π do not appear in Maxwell's equations.

We now attempt to construct a fourvector potential \mathbf{A} by enumerating its components in the inertial system as

$$\mathbf{A} = \Phi \hat{\mathbf{e}}_0 + \mathbf{A} = \Phi \hat{\mathbf{e}}_0 + A^1 \hat{\mathbf{e}}_1 + A^2 \hat{\mathbf{e}}_2 + A^3 \hat{\mathbf{e}}_3 \quad (15.120)$$

The Lorentz gauge condition eqn (15.117) can then be written in manifestly covariant form as the vanishing of the divergence of \mathbf{A}

$$\partial \cdot \mathbf{A} = 0 \quad (15.121)$$

And the two eqn (15.118) become the single fourvector equation

$$-\square^2 \mathbf{A} = \frac{1}{c} \mathbf{J} \quad (15.122)$$

As can be verified by examination of eqn (15.119), the operator \square^2 is a form invariant differential operator. Also, \mathbf{J} was shown above to be a fourvector, due to the assumed invariance of electric charge. Hence, every equation of which \mathbf{A} is a solution is itself in a manifestly covariant form. It follows that a fourvector solution to these equations can be found. We conclude, therefore, that \mathbf{A} is a fourvector field.

The electromagnetic fields themselves cannot be cast into a simple fourvector form. Rather, one defines what is called the electromagnetic field tensor \mathcal{F} defined by writing its associated dyadic \mathbb{F} as a wedge product

$$\mathbb{F} = \partial \wedge \mathbf{A} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \hat{\mathbf{e}}_{\mu} F^{\mu}_{\nu} \hat{\mathbf{e}}^{\nu} \quad (15.123)$$

where, modifying the standard notation eqn (15.102) slightly to place the differential operator ∂_{ν} to the left of A^{μ} in the second term, the matrix of \mathbb{F} and \mathcal{F} has matrix elements

$$F^{\mu}_{\nu} = \partial^{\mu} A_{\nu} - \partial_{\nu} A^{\mu} \quad (15.124)$$

Evaluating these matrix elements using eqn (15.116), matrix \mathbb{F} is

$$\mathbb{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \quad (15.125)$$

The Lorentz threevector force \mathbf{f} acting on a point charge $q^{(\text{ch})}$ in a given electromagnetic field is

$$\mathbf{f} = q^{(\text{ch})} \mathbf{E} + \frac{q^{(\text{ch})}}{c} \mathbf{v} \times \mathbf{B} \quad (15.126)$$

Like the threevector velocity \mathbf{v} , the threevector force \mathbf{f} cannot be used directly as the spatial part of a fourvector. But, also like the velocity, there is a fourvector that is

closely related to it. A fourvector \mathbf{K} can be defined by dotting the electromagnetic field dyadic \mathbb{F} onto the fourvector velocity \mathbf{u} . It is

$$\mathbf{K} = \frac{q^{(\text{ch})}}{c} \mathbb{F} \cdot \mathbf{u} = K^0 \hat{\mathbf{e}}_0 + \gamma \mathbf{f} \quad \text{where} \quad K^0 = \frac{\gamma}{c} \mathbf{f} \cdot \mathbf{v} \quad (15.127)$$

and \mathbf{f} is the Lorentz threevector force from eqn (15.126). The reader can verify the equalities in eqn (15.127) by actually carrying out the indicated matrix multiplication using eqn (15.125). The fourvector \mathbf{K} is called the *Minkowski force*.

15.23 Exercises

Exercise 15.1 Use eqn (15.4) to prove the form invariance in eqn (15.5).

Exercise 15.2 Given fourvectors \mathbf{a} and \mathbf{b} that are both forward timelike,

- Prove that $\mathbf{a} \cdot \mathbf{b} < 0$.
- Prove that $\mathbf{a} + \mathbf{b}$ is also forward timelike.
- Prove that the same results remain true when the word “forward” is replaced by the word “backward” throughout.

Exercise 15.3 Suppose that fourvector \mathbf{a} is timelike, and $\mathbf{a} \cdot \mathbf{b} = 0$.

- Prove that fourvector \mathbf{b} is spacelike.
- Argue that, since basis fourvector $\hat{\mathbf{e}}_0$ is timelike, the basis vectors $\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$ must be spacelike. Show that they are.
- The converse to (a) is not true. A fourvector orthogonal to a spacelike vector is not necessarily timelike. Give an example that demonstrates this.

Exercise 15.4 Suppose that $\mathbf{a} = a^0 \hat{\mathbf{e}}_0 + \mathbf{a}$ is forward timelike.

- Find the \mathbf{V} defining a boost to a system S' in which $\mathbf{a}' = 0$.
- Show that the time unit vector of S' is given by $\hat{\mathbf{e}}_0' = \mathbf{a} / \sqrt{-\mathbf{a} \cdot \mathbf{a}}$.

Exercise 15.5 Suppose that $\mathbf{b} = b^0 \hat{\mathbf{e}}_0 + \mathbf{b}$ is spacelike, with $b^0 > 0$.

- Find a \mathbf{V} defining a boost to a system S' in which $b'^0 = 0$.
- Find a Lorentz boost to a system S' in which $b'^0 < 0$.

Exercise 15.6 Suppose that two events A and B are connected by $d\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$. Assume that $d\mathbf{r}$ is forward timelike, so that B is within the forward light cone of A and $t_B > t_A$.

- If S' is any coordinate system obtained from S by a proper Lorentz transformation, show that the temporal order of events A and B is still $t'_B > t'_A$.
- Show that a boost from S can always be found to a system S' in which A and B are at the same spatial location.

Exercise 15.7 Suppose that two events A and B have spacelike separation $d\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$ with $t_B > t_A$.

- Show that a boost from S can always be found to a system S' in which A and B are simultaneous.
- Show that there is a boost to a system S' in which the temporal order of the spacelike-separated events is reversed, with $t'_B < t'_A$.

Exercise 15.8 Prove that eqn (15.53) is true for general dx^μ if and only if eqn (15.54) holds.

Exercise 15.9 Use $M^T \mathbf{g} M = \mathbf{g}$ from eqn (15.47) of Definition 15.9.1 to prove Lemma 15.17.1.

Exercise 15.10 Prove that M_{rot} defined in eqn (15.64) is a Lorentz transformation satisfying Definition 15.9.1.

Exercise 15.11 Consider a boost from S to S' parameterized by the threevector \mathbf{V} , as discussed in Section 15.11.3.

- (a) Show that a point at rest in S' has threevector velocity \mathbf{V} as seen from S .
 (b) Show that a point at rest in S has a threevector velocity $-\mathbf{V}'$ as seen from S' , where $\mathbf{V} \sim \mathbf{V}'$.

Exercise 15.12

- (a) Prove that any Lorentz transformation matrix M can be written as $M = Z M_{\text{prop}}$ where M_{prop} is a proper Lorentz transformation and Z is one of the U , M_{tr} , M_{par} , M_{inv} defined in Section 15.16.
 (b) Show from (a) that any Lorentz transformation is continuously connected to some member of the Little Group.

Exercise 15.13 In Section 15.16 we defined a proper Lorentz transformation as one with $M^0_0 \geq 1$ and $|M| = 1$.

- (a) Use Lemma 15.17.1 and the Schwartz inequality $\mathbf{A} \cdot \mathbf{B} \leq AB$ to prove that the product of two proper Lorentz transformations is a proper Lorentz transformation.
 (b) Show that the proper Lorentz transformations form a subgroup of the Lorentz group.

Exercise 15.14 This exercise shows that a wedge product of two fourvectors includes the cross product of their threevector spatial parts. Suppose that $\mathbf{A} = A^0 \hat{\mathbf{e}}_0 + \mathbf{A}$ and $\mathbf{B} = B^0 \hat{\mathbf{e}}_0 + \mathbf{B}$. Define the threevectors $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ and $\mathbf{D} = A^0 \mathbf{B} - B^0 \mathbf{A}$.

- (a) Show from eqn (15.102) that $\mathbb{W} = \mathbf{A} \wedge \mathbf{B}$ has the matrix

$$\mathbb{W} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ D_x & 0 & C_z & -C_y \\ D_y & -C_z & 0 & C_x \\ D_z & C_y & -C_x & 0 \end{pmatrix} \quad (15.128)$$

- (b) If $\mathbf{Y} = \mathbb{W} \cdot \mathbf{X}$ where $\mathbf{X} = X^0 \hat{\mathbf{e}}_0 + \mathbf{X}$, show that $\mathbf{Y} = Y^0 \hat{\mathbf{e}}_0 + \mathbf{Y}$ where $Y^0 = \mathbf{D} \cdot \mathbf{X}$ and $\mathbf{Y} = \mathbf{D} X^0 - \mathbf{C} \times \mathbf{X}$.

Exercise 15.15

- (a) Use eqn (15.94) to prove the invariance of $\text{Tr} \mathcal{F}$ and $\det \mathcal{F}$ stated in eqns (15.92, 15.93).
 (b) Use eqns (15.93, 15.125) to prove that $(\mathbf{E} \cdot \mathbf{B})^2 = (\mathbf{E}' \cdot \mathbf{B}')^2$.
 (c) Use eqn (15.92) and the square of eqn (15.125) to prove that $E^2 - B^2 = E'^2 - B'^2$.

Exercise 15.16

- (a) Assuming a standard Lorentz transformation, use eqns (15.90, 15.125) to derive the trans-

formation rules for the electric and magnetic fields:

$$E_x = E'_x \quad E_y = \Gamma \left(E'_y + \frac{V}{c} B'_z \right) \quad E_z = \Gamma \left(E'_z - \frac{V}{c} B'_y \right) \quad (15.129)$$

$$B_x = B'_x \quad B_y = \Gamma \left(B'_y - \frac{V}{c} E'_z \right) \quad B_z = \Gamma \left(B'_z + \frac{V}{c} E'_y \right) \quad (15.130)$$

(b) Show that the results in (a) can be written for general boosts as

$$\mathbf{E}_{\parallel} \sim \mathbf{E}'_{\parallel} \quad \mathbf{E}_{\perp} \sim \Gamma \left(\mathbf{E}'_{\perp} - \frac{1}{c} \mathbf{V}' \times \mathbf{B}'_{\perp} \right) \quad (15.131)$$

$$\mathbf{B}_{\parallel} \sim \mathbf{B}'_{\parallel} \quad \mathbf{B}_{\perp} \sim \Gamma \left(\mathbf{B}'_{\perp} + \frac{1}{c} \mathbf{V}' \times \mathbf{E}'_{\perp} \right) \quad (15.132)$$

where \mathbf{E}_{\parallel} and \mathbf{E}_{\perp} are the components of \mathbf{E} parallel and perpendicular to \mathbf{V} , respectively, with similar definitions for the other fields.

RELATIVISTIC MECHANICS

It was apparent from its beginning that special relativity, developed as the invariance theory of electrodynamics, would require a modification of Newton's laws of motion. This chapter discusses that modified theory. The relativistically modified mechanics is presented and then recast into a fourvector form that demonstrates its consistency with special relativity. Traditional Lagrangian and Hamiltonian mechanics can incorporate these modifications. But the transition to a manifestly covariant Lagrangian and Hamiltonian mechanics requires use of the extended Lagrangian and Hamiltonian methods of Chapters 11 and 12.

16.1 Modification of Newton's Laws

In the same paper in which the Lorentz transformation is presented, Lorentz also derived a modified form for the momentum of a moving electron. The Newtonian form of the second law, $\mathbf{f} = d\mathbf{p}/dt$ with $\mathbf{p} = m\mathbf{v}$ was to be replaced by⁸⁵

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} \quad \text{with} \quad \mathbf{p} = m\gamma\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad (16.1)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the same time-dilation and Lorentz contraction factor as was defined in eqn (14.26) and m is a constant, independent of the velocity of the particle. Lorentz also noted that eqn (16.1) is equivalent to

$$\mathbf{f} = m_{\parallel}\mathbf{a}_{\parallel} + m_{\perp}\mathbf{a}_{\perp} \quad (16.2)$$

where \mathbf{a}_{\parallel} is the component of \mathbf{a} parallel to \mathbf{v} , and \mathbf{a}_{\perp} is the component of \mathbf{a} perpendicular to \mathbf{v} . The so-called longitudinal and transverse masses are, respectively,

$$m_{\parallel} = \gamma^3 m \quad \text{and} \quad m_{\perp} = \gamma m \quad (16.3)$$

At the end of his first 1905 paper,⁸⁶ Einstein also derived a modified form for Newton's second law. It is similar to that of Lorentz, but differs in defining the transverse mass as $m_{\perp} = \gamma^2 m$.

⁸⁵Lorentz, H.A. (1904) "Electromagnetic Phenomena in a System Moving with any Velocity Less than that of Light," *Proceedings of the Academy of Sciences of Amsterdam*, 6. English translation in Einstein, Lorentz, Minkowski and Weyl (1923). On page 23 the electron is modeled as a sphere of charge of radius R with electromagnetic mass $e^2/6\pi c^2 R$ and momentum $\mathbf{G} = (e^2/6\pi c^2 R)\beta\mathbf{v}$ where his β is the same as our γ . We suppress a factor ℓ which Lorentz sets to one at the end of his development.

⁸⁶Einstein, A. (1905) *On the Electrodynamics of Moving Bodies*, *Annalen der Physik*, 17. English translation in Einstein, Lorentz, Minkowski and Weyl (1923).

In our development here, we will take eqn (16.1) to be an established fact of nature. It applies, for example, when \mathbf{f} is the Lorentz force on a point particle of charge $q^{(\text{ch})}$ and mass m , leading to

$$q^{(\text{ch})}\mathbf{E} + \frac{q^{(\text{ch})}}{c}\mathbf{v} \times \mathbf{B} = \mathbf{f} = \frac{d}{dt} \left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) \quad (16.4)$$

This is one of the most thoroughly tested formulas in physics. It underlies the calculation of particle trajectories in all modern particle accelerators, for example, and hence is tested at a range of speeds up to nearly the speed of light.⁸⁷

In the same 1905 paper, Einstein also noted that eqn (16.2) would require a modification of the work–energy theorem. In our derivation of the Newtonian work–energy theorem in Section 1.2, the rate of doing work was shown to be equal to the rate of change of the Newtonian kinetic energy

$$\mathbf{f} \cdot \mathbf{v} = \frac{d(m\mathbf{v})}{dt} \cdot \mathbf{v} = \frac{dT}{dt} \quad \text{where} \quad T = \frac{1}{2}mv^2 \quad (16.5)$$

If, following Einstein, one retains $\mathbf{f} \cdot \mathbf{v}$ as the definition of the rate of doing work but adopts the modified definition of momentum in eqn (16.1), or equivalently uses eqn (16.2), then the rate of doing work is equal to the time derivative of a new quantity E , called the *relativistic energy*.

$$\mathbf{f} \cdot \mathbf{v} = \frac{d(\gamma m\mathbf{v})}{dt} \cdot \mathbf{v} = \frac{dE}{dt} \quad \text{where} \quad E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad (16.6)$$

Making the reasonable definition that kinetic energy T should be zero when the particle is at rest, the modified kinetic energy can be defined as

$$T = E - E_0 \quad \text{where} \quad E_0 = mc^2 \quad (16.7)$$

is a constant called the *rest energy*. Thus $\mathbf{f} \cdot \mathbf{v} = dT/dt$ still holds, but with a modified definition of T .

The mass m is also sometimes called the *rest mass* of the particle. This usage dates to a time when, as in eqn (16.3) for example, it was common to speak of masses that varied with the velocity of the particle. The modern convention, which we follow here, is to use only the single constant mass m , referred to simply as “the mass” of the particle.

Exercise 16.4 shows that the \mathbf{p} in eqn (16.1) and the T in eqn (16.7) reduce to the Newtonian expressions in the limit $v/c \rightarrow 0$.

⁸⁷A common joke is that eqn (16.4) is also politically invariant, since the ring of the Large Hadron Collider at the CERN laboratory crosses the border between France and Switzerland.

16.2 The Momentum Fourvector

After introducing the fourvector notation, Minkowski⁸⁸ also showed how to use it to express the modified relativistic mechanics. The threevector momentum \mathbf{p} defined in eqn (16.1) and the relativistic energy E defined in eqn (16.6) can be combined into a fourvector momentum $\mathbf{p} = m\mathbf{u}$.

Since the fourvector velocity \mathbf{u} was shown in eqn (15.36) to be

$$\mathbf{u} = \frac{d\mathbf{r}}{d\tau} = \gamma c \hat{\mathbf{e}}_0 + \gamma \mathbf{v} \quad (16.8)$$

we can write

$$\mathbf{p} = m\mathbf{u} = m\gamma c \hat{\mathbf{e}}_0 + m\gamma \mathbf{v} = \frac{E}{c} \hat{\mathbf{e}}_0 + \mathbf{p} \quad (16.9)$$

where $p^0 = E/c$ with E defined in eqn (16.6), and the spatial part \mathbf{p} is the same threevector momentum as defined in eqn (16.1). Adding the axiom that the mass m must be a Lorentz invariant makes this \mathbf{p} the product of an invariant and a fourvector and hence a fourvector.

Masses are never negative. So, for particles of nonzero mass, both the fourvector velocity \mathbf{u} and the fourvector momentum \mathbf{p} are forward timelike vectors. Dotting \mathbf{u} with itself gives

$$\mathbf{u} \cdot \mathbf{u} = \frac{d\mathbf{r}}{d\tau} \cdot \frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r} \cdot d\mathbf{r}}{(d\tau)^2} = -c^2 \quad (16.10)$$

where eqn (15.21) was used to get the last equality. Thus

$$\mathbf{p} \cdot \mathbf{p} = -m^2 c^2 \quad (16.11)$$

which may be used to get the special-relativistic relation between momentum and relativistic energy

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad (16.12)$$

where here $p = \sqrt{\mathbf{p} \cdot \mathbf{p}}$ is the magnitude of the threevector momentum \mathbf{p} .

16.3 Fourvector Form of Newton's Second Law

Lorentz's modified form of Newton's second law in eqn (16.1) is not yet in fourvector form. It is a threevector equation, and involves a derivative with respect to coordinate time. However, if we multiply both sides of it by γ we obtain an expression that is the derivative of the spatial part of fourvector \mathbf{p} defined in eqn (16.9) with respect to invariant proper time $d\tau$

$$\gamma \mathbf{f} = \gamma \frac{d\mathbf{p}}{dt} = \frac{d\mathbf{p}}{d\tau} \quad (16.13)$$

where $\gamma = dt/d\tau$ was used.

⁸⁸H. Minkowski (1908) *Space and Time*, Address at the 80th assembly of German Natural Scientists and Physicians. English translation in Einstein, Lorentz, Minkowski and Weyl (1923).

In a similar way, Einstein's modification of the work energy theorem in eqn (16.6) can be multiplied through by γ/c and written as a derivative with respect to proper time

$$\frac{\gamma}{c} (\mathbf{f} \cdot \mathbf{v}) = \frac{\gamma}{c} \frac{dE}{dt} = \frac{d(E/c)}{d\tau} \quad (16.14)$$

If we now define the fourvector Minkowski force \mathbf{K} by

$$\mathbf{K} = \frac{\gamma}{c} (\mathbf{f} \cdot \mathbf{v}) \hat{\mathbf{e}}_0 + \gamma \mathbf{f} \quad (16.15)$$

we see that eqns (16.13, 16.14) can be combined with the definition of \mathbf{p} in eqn (16.9) to write the manifestly covariant form of Newton's second law

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} \quad (16.16)$$

Since m is assumed to be constant along the world line of a particle, eqn (16.16) can also be written as

$$\mathbf{K} = m \frac{d\mathbf{u}}{d\tau} = m\mathbf{w} \quad (16.17)$$

where \mathbf{w} is the fourvector acceleration defined in eqn (15.37).

The definition in eqn (16.15) implies certain transformation rules for \mathbf{f} that may not be satisfied for all forces that have been used in Newtonian physics. If not, those forces would not be consistent with special relativity. The \mathbf{K} would not actually be a fourvector, despite the form in which it is written, and eqn (16.13) would be false in some coordinate systems.

But when \mathbf{f} is the Lorentz force of electrodynamics given in eqns (15.126, 16.4), it follows from eqn (15.127) that

$$\mathbf{K} = \gamma \frac{\mathbf{f} \cdot \mathbf{v}}{c} \hat{\mathbf{e}}_0 + \gamma \mathbf{f} = \frac{q^{(ch)}}{c} \mathbb{F} \cdot \mathbf{u} \quad (16.18)$$

which displays \mathbf{K} as an invariant $q^{(ch)}$ times the dot product of a fourvector dyadic \mathbb{F} and a fourvector \mathbf{u} , and therefore certainly a fourvector. So we have at least one fourvector Minkowski force, that derived from the Lorentz force of electrodynamics.

One interesting feature of covariant mechanics is that the fourvector acceleration has a zero dot product with the fourvector velocity. The vanishing of this dot product does not imply that their spatial parts are orthogonal, however, due to the presence of the time parts of both fourvectors. To see the vanishing dot product, we use eqn (16.10) to write

$$\mathbf{w} \cdot \mathbf{u} = \frac{d\mathbf{u}}{d\tau} \cdot \mathbf{u} = \frac{1}{2} \frac{d}{d\tau} (\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2} \frac{d}{d\tau} (-c^2) = 0 \quad (16.19)$$

It follows from this result and eqn (16.17) that

$$\mathbf{K} \cdot \mathbf{u} = 0 \quad (16.20)$$

When written out, this identity simply restates the modified work-energy theorem eqn (16.6).

16.4 Conservation of Fourvector Momentum

The identification of the relativistic energy E with the zeroth component of the fourvector momentum $\mathbf{p} = (E/c)\hat{\mathbf{e}}_0 + \mathbf{p}$ suggests that both E and \mathbf{p} are conserved in particle collisions and emissions. This suggestion is born out by experiment, and has become one of the foundation axioms of special relativity.

One consequence of the conservation of fourvector momentum, however, is that the mass m is not generally a conserved quantity, a marked departure from Newtonian theory. In a second paper⁸⁹ in 1905, Einstein considered the example of a particle at rest emitting equal light pulses in two opposite directions (see Exercise 16.6). If the total energy of the emitted light is L , he concluded that, “If a body gives off energy L in the form of radiation, its mass diminishes by L/c^2 .”

Einstein’s conclusion applies also to energy given off as the kinetic energy of massive particles. For example, if a stationary nucleus of mass m_0 decays into an alpha particle with mass m_α and a residual nucleus of mass m_1 , then conservation of relativistic energy, with the kinetic energies defined using eqn (16.7), gives

$$m_0c^2 = E_0 = E_\alpha + E_1 = m_\alpha c^2 + T_\alpha + m_1c^2 + T_1 \quad (16.21)$$

The total kinetic energy of the decay products is thus equal to c^2 times the mass deficit

$$T_\alpha + T_1 = (m_0 - m_\alpha - m_1)c^2 \quad (16.22)$$

The general result $T_{\text{total}} = |\Delta m|c^2$ for fission and fusion reactions is the basis of the atomic and hydrogen bombs. In a more benign application, the alpha decay of plutonium-238 provided power for the Cassini probe that reached Saturn in 2004, as estimated in Exercise 16.13.

16.5 Particles of Zero Mass

The relation, eqn (16.12), between relativistic energy and momentum simplifies in the extreme relativistic limit $p \gg mc$. Then

$$E = pc\sqrt{1 + \left(\frac{mc}{p}\right)^2} \rightarrow pc \quad \text{and} \quad T = E - mc^2 = pc\left(\sqrt{1 + (mc/p)^2} - \frac{mc}{p}\right) \rightarrow pc \quad (16.23)$$

The relativistic energy is dominated by the kinetic energy of the particle and the distinction between relativistic and kinetic energy disappears. It also follows from $pc/E = v/c$ (see Exercise 16.4) that $v/c \rightarrow 1$ in this limit.

There is a class of elementary particles for which the relation $E = pc$ holds exactly, and for low as well as high energies. For example, the deBroglie relations of quantum theory give energy $E = \hbar\omega$ and momentum $p = \hbar k$ for the photon, the quantum of light, where the angular frequency ω and wave vector \mathbf{k} are those of a light wave

⁸⁹Einstein, A. (1905) “Does the Inertia of a Body Depend Upon its Energy-Content?” *Annalen der Physik*, 17. English translation in Einstein, Lorentz, Minkowski and Weyl (1923).

and hence obey $\omega/k = c$. It follows that $E = pc$, a relation that can also be derived from classical electrodynamics by considering the Poynting vector and the momentum density of a light wave.

Such particles can be brought into the relativistic formalism by setting $m = 0$ in eqn (16.12). They are called *zero-mass particles*. Particles of zero mass are assumed to have $m = 0$ and speeds *always equal* to the speed of light. They are, in a sense, like massive particles always *at* their extreme relativistic limit. Hence, the definitions in eqns (16.1, 16.6) are indeterminate and cannot be used. The fourvector momentum of zero mass particles is found by specifying \mathbf{p} and using $E = pc$ to write

$$\mathbf{p} = \frac{E}{c} \hat{\mathbf{e}}_0 + \mathbf{p} = p \hat{\mathbf{e}}_0 + \mathbf{p} \tag{16.24}$$

where $p = \|\mathbf{p}\|$. The momentum of a zero-mass particle is therefore lightlike, with $\mathbf{p} \cdot \mathbf{p} = 0$.

The fourvector momentum of photons, or other particles of zero mass, is included in the conservation of momentum. For example, if the decaying nucleus in Section 16.4 also produced a gamma ray (an energetic photon), as it often does, then eqn (16.22) would be modified only by the addition of a term $T_{\text{phot}} = E_{\text{phot}} = p_{\text{phot}}c$ to its left-hand side.

16.6 Traditional Lagrangian

The traditional Lagrangian theory of Chapter 2 can be made consistent with the modified mechanics of Section 16.1. Our original derivation of Lagrangian mechanics in Section 2.2 was based on the identity

$$\frac{\partial T}{\partial \mathbf{v}} = m\mathbf{v} = \mathbf{p} \quad \text{where} \quad T = \frac{1}{2}mv^2 \tag{16.25}$$

Then the Newtonian form of the second law could be written in Lagrangian form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \mathbf{v}} \right) = \mathbf{f} \tag{16.26}$$

Introducing a potential with $\mathbf{f} = -\partial U/\partial \mathbf{r}$ and defining $L = T - U$ then led to the standard Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{r}} = 0 \tag{16.27}$$

A similar path can be followed in the relativistic case. We note that the modified form of the momentum in eqn (16.1) is obtained when the T of eqn (16.25) is replaced by $-mc^2\sqrt{1 - v^2/c^2}$ so that

$$\frac{\partial}{\partial \mathbf{v}} \left(-mc^2\sqrt{1 - v^2/c^2} \right) = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \mathbf{p} \tag{16.28}$$

The modified Newton's second law in eqn (16.1) then can be written in Lagrangian

form as

$$\frac{d}{dt} \left\{ \frac{\partial}{\partial \mathbf{v}} \left(-mc^2 \sqrt{1 - v^2/c^2} \right) \right\} = \mathbf{f} \quad (16.29)$$

Again introducing the potential, and defining

$$L = -mc^2 \sqrt{1 - v^2/c^2} - U \quad (16.30)$$

leads to the same standard Lagrange equation, eqn (16.27).⁹⁰

We will develop covariant Lagrangian and Hamiltonian mechanics here using the slightly more complicated problem of a point mass m with charge $q^{(\text{ch})}$ moving in a given, external electromagnetic field. We assume that the electromagnetic field contributions of $q^{(\text{ch})}$ itself can be ignored. The potential U in eqn (16.30) will be replaced by the velocity-dependent potential $U^{(\text{vel})}$ derived in Section 2.17, and the Lagrangian will be

$$L = -mc^2 \sqrt{1 - v^2/c^2} - U^{(\text{vel})} = -mc^2 \sqrt{1 - v^2/c^2} - q^{(\text{ch})} \Phi(\mathbf{r}, t) + \frac{q^{(\text{ch})}}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t) \quad (16.31)$$

Since the Lorentz force is shown in Section 2.17 to be

$$\mathbf{f} = \frac{d}{dt} \left(\frac{\partial U^{(\text{vel})}}{\partial \mathbf{v}} \right) - \left(\frac{\partial U^{(\text{vel})}}{\partial \mathbf{r}} \right) \quad (16.32)$$

the relativistic equation of motion eqn (16.29) will still be in the standard Lagrangian form, eqn (16.27).

With this Lagrangian, the generalized threevector *canonical* momentum is

$$\underline{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + \frac{q^{(\text{ch})}}{c} \mathbf{A} = \mathbf{p} + \frac{q^{(\text{ch})}}{c} \mathbf{A} \quad (16.33)$$

where \mathbf{p} is the same modified momentum defined in eqn (16.1). As in the Newtonian case in eqn (2.104), the canonical momentum threevector $\underline{\mathbf{p}}$ is seen to be the vector sum of a particle term \mathbf{p} and a field term $(q^{(\text{ch})}/c)\mathbf{A}$. The difference is that the particle momentum, which was $m\mathbf{v}$ in the Newtonian case, is now a threevector $\mathbf{p} = \gamma m\mathbf{v}$ that agrees with the modified definition of momentum in eqn (16.1).

The generalized energy H can be derived. Using the definition in eqn (2.76), it is

$$H = \underline{\mathbf{p}} \cdot \mathbf{v} - L = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + q^{(\text{ch})} \Phi = E + q^{(\text{ch})} \Phi \quad (16.34)$$

The particle term, which was $mv^2/2$ in the Newtonian case in eqn (2.105), is replaced by a term E that agrees with the modified definition of particle energy in eqn (16.6).

⁹⁰Notice that here, for the first time, the Lagrangian does not have the form $T - U$. Lagrangians can take any form that makes the Lagrange equations reproduce the correct equations of motion.

16.7 Traditional Hamiltonian

The generalized energy can be converted to the Hamiltonian by the techniques in Chapter 4. Using eqn (16.12) (which applies to the particle momentum \mathbf{p} rather than the canonical momentum $\underline{\mathbf{p}}$) to write the Hamiltonian in terms of the correct phase space variables gives

$$H = E + q^{(\text{ch})}\Phi = \sqrt{p^2c^2 + m^2c^4} + q^{(\text{ch})}\Phi = \sqrt{\|\underline{\mathbf{p}}c - q^{(\text{ch})}\mathbf{A}\|^2 + m^2c^4} + q^{(\text{ch})}\Phi \tag{16.35}$$

where eqn (16.33) has been solved for $\mathbf{p} = \underline{\mathbf{p}} - (q^{(\text{ch})}/c)\mathbf{A}$ and substituted into eqn (16.35) to obtain the final equality. The Hamilton equations are then

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \underline{\mathbf{p}}} \quad \text{and} \quad \frac{d\underline{\mathbf{p}}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} \tag{16.36}$$

16.8 Invariant Lagrangian

The traditional Lagrange and Hamilton equations just derived are covariant in the sense that they reproduce the relativistically modified equations of motion. But it is advantageous to write Lagrangian and Hamiltonian mechanics in a manifestly covariant form in which only invariants and fourvectors appear in the equations. The consistency with special relativity is then apparent by inspection.

The first step to a manifestly covariant Lagrangian mechanics is to convert the traditional Lagrangian of eqn (16.31) to the extended form, using the techniques of Chapter 11. If, as there, we let all of the generalized coordinates be functions of some general, monotonically increasing parameter β , and define $\dot{\mathbf{r}} = d\mathbf{r}/d\beta$, etc., then the extended Lagrangian \mathcal{L} is defined in eqn (11.7) as

$$\begin{aligned} \mathcal{L} = iL &= -mc^2\sqrt{i^2 - (iv)^2/c^2} - q^{(\text{ch})}i\Phi + \frac{q^{(\text{ch})}}{c}(iv) \cdot \mathbf{A} \\ &= -mc\sqrt{(ci)^2 - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})} - \frac{q^{(\text{ch})}}{c}(ci)\Phi + \frac{q^{(\text{ch})}}{c}\dot{\mathbf{r}} \cdot \mathbf{A} \end{aligned} \tag{16.37}$$

where the last expression has been rewritten using $iv = (dt/d\beta)(d\mathbf{r}/dt) = \dot{\mathbf{r}}$. We assume that, at the end of the calculation, the parameter β will be chosen to be an invariant quantity, as can always be done.

The radius fourvector defining events on the world line of the particle is

$$\mathbf{r} = \sum_{\mu=0}^3 x^\mu \hat{\mathbf{e}}_\mu = ct\hat{\mathbf{e}}_0 + \mathbf{r}$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. Taking its derivative with respect to the invariant β gives another fourvector

$$\dot{\mathbf{r}} = c\dot{t}\hat{\mathbf{e}}_0 + \dot{\mathbf{r}} \tag{16.38}$$

Using this fourvector and recalling from eqn (15.120) that $\mathbf{A} = \Phi\hat{\mathbf{e}}_0 + \mathbf{A}$ is also a

fourvector, the extended Lagrangian, eqn (16.37), can be written as⁹¹

$$\begin{aligned}\mathcal{L} &= -mc \sqrt{-\sum_{\mu=0}^3 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + \frac{q^{(\text{ch})}}{c} \sum_{\mu=0}^3 \dot{x}_\mu A^\mu \\ &= -mc \sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} + \frac{q^{(\text{ch})}}{c} \dot{\mathbf{r}} \cdot \mathbf{A}\end{aligned}\quad (16.39)$$

Since this expression involves only dot products of fourvectors, and the invariants m , c , and $q^{(\text{ch})}$, the extended Lagrangian \mathcal{L} is an invariant under Lorentz transformations, as desired.

16.9 Manifestly Covariant Lagrange Equations

We have obtained an extended Lagrangian that is manifestly form invariant under Lorentz transformations. The next step is to write the extended Lagrange equations themselves.

From eqn (11.18) the extended Lagrange equations can be written as

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}}{\partial \dot{i}} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{r}} = 0 \quad (16.40)$$

In Section 15.12 we defined the fourvector gradient operator ∂ as

$$\partial = \frac{\partial}{\partial \mathbf{r}} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \frac{\partial}{\partial x^\mu} = \hat{\mathbf{e}}^0 \frac{\partial}{\partial ct} + \frac{\partial}{\partial \mathbf{r}} \quad (16.41)$$

We now define another fourvector gradient, this time with respect to the dotted components of $\dot{\mathbf{r}}$. Define

$$\frac{\partial}{\partial \dot{\mathbf{r}}} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \frac{\partial}{\partial \dot{x}^\mu} = \hat{\mathbf{e}}^0 \frac{\partial}{\partial c\dot{i}} + \frac{\partial}{\partial \dot{\mathbf{r}}} \quad (16.42)$$

The transformation properties of this fourvector follow from those of $\dot{\mathbf{r}}$.

Multiplying the first of eqn (16.40) by $\hat{\mathbf{e}}^0/c$ and adding it to the second, the four Lagrange equations can now be combined into the following single fourvector Lagrange equation

$$\frac{d}{d\beta} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{r}} = 0 \quad (16.43)$$

The parameter β and the extended Lagrangian \mathcal{L} are invariants, and the gradient

⁹¹In Chapter 5 on the calculus of variations, and also in Chapter 11 on extended Lagrangian theory, we made a point of requiring that the parameter β *must remain unspecified* until the end of the calculation, *after* all partial derivatives have been taken. A premature choice of β violates the spirit of the method and can lead to error. Manifestly covariant Lagrangian theory is a prime example of the problems that could result. If we prematurely took β equal to the proper time τ measured along the world line of the particle, then eqn (16.39) would collapse to $\mathcal{L} = -mc^2 + (q^{(\text{ch})}/c)\mathbf{u} \cdot \mathbf{A}$ which would lead to nonsense results. This error is analogous to prematurely taking β equal to the arc length s in a calculation like that in Section 5.6, which also would lead to nonsense.

operators are fourvectors. Thus this equation exhibits Lagrange’s equations in a manifestly covariant form.

Since we are using the extended Lagrangian \mathcal{L} , we are now free to transform the time. (As described in Section 11.8, the Lagrangian \mathcal{L} is then simply rewritten in terms of the new coordinates.) In some situations, it will be useful to transform to a new set of generalized coordinates $q^{(\text{rel})} = ct, x, y, z$ which we refer to as the “relativistic coordinates.” The original canonical coordinates $q^{(\text{sta})} = t, x, y, z$ will be referred to as “standard coordinates.” The advantage is that the relativistic coordinates will then be identical to the contravariant components x^μ of the position fourvector $\mathbf{r} = \sum_{\mu=0}^3 x^\mu \hat{\mathbf{e}}_\mu$.⁹² Of course, the only difference is that $q_0^{(\text{rel})} = cq_0^{(\text{sta})}$ and the zeroth canonical momenta are related by $\underline{p}_0^{(\text{rel})} = \partial\mathcal{L}/\partial(ct) = (1/c)\partial\mathcal{L}/\partial t = \underline{p}_0^{(\text{sta})}/c$. We may sometimes use the relativistic coordinates without writing the explicit superscripts (rel) or (sta). It will be clear, either from context or from a specific statement, which system of coordinates is being used.

16.10 Momentum Fourvectors and Canonical Momenta

From the definition in eqn (11.9), and using eqn (16.37), the canonical momentum conjugate to the coordinate t in standard coordinates is $\underline{p}_0^{(\text{sta})}$ where

$$\underline{p}_0^{(\text{sta})} = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -\left(\frac{mc^2}{\sqrt{1-v^2/c^2}} + q^{(\text{ch})}\Phi\right) = -(E + q^{(\text{ch})}\Phi) \tag{16.44}$$

The other three components of canonical momentum, for $i = 1, 2, 3$, are the same in both the (rel) or (sta) coordinates. They are

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{mv_i}{\sqrt{1-v^2/c^2}} + \frac{q^{(\text{ch})}}{c}A_i = p_i + \frac{q^{(\text{ch})}}{c}A_i \tag{16.45}$$

If we multiply eqn (16.44) by $\hat{\mathbf{e}}^0/c$ and eqn (16.45) by $\hat{\mathbf{e}}^i$ and add them, we obtain

$$\underline{\mathbf{p}} = (\underline{p}_0^{(\text{sta})}/c)\hat{\mathbf{e}}^0 + \sum_{i=1}^3 \underline{p}_i \hat{\mathbf{e}}^i = \frac{-1}{c}(E + q^{(\text{ch})}\Phi)\hat{\mathbf{e}}^0 + \sum_{i=1}^3 \left(p_i + \frac{q^{(\text{ch})}}{c}A_i\right)\hat{\mathbf{e}}^i \tag{16.46}$$

Since the co- and contravariant quantities are related by $\hat{\mathbf{e}}^0 = -\hat{\mathbf{e}}_0$ and $A^i = A_i$ for $i \neq 0$, etc., eqn (16.46) may be rewritten as

$$\underline{\mathbf{p}} = \frac{1}{c}(E + q^{(\text{ch})}\Phi)\hat{\mathbf{e}}_0 + \sum_{i=1}^3 \left(p^i + \frac{q^{(\text{ch})}}{c}A^i\right)\hat{\mathbf{e}}_i = \mathbf{p} + \frac{q^{(\text{ch})}}{c}\mathbf{A} \tag{16.47}$$

where we have introduced the Minkowski momentum fourvector \mathbf{p} defined in eqn (16.9) and the fourvector \mathbf{A} defined in eqn (15.120). Since it is the sum of two

⁹²In Lagrangian mechanics, the different generalized coordinates (and generalized momenta) often have different units. However, all components of a *fourvector* must have the same units. These unit distinctions become less important when, as is often done in relativistic quantum theory, one uses a unit system in which the numerical value of the speed of light is unity. However, in an introductory treatment it seems best to retain the distinctions and use “normal” units.

fourvectors, and since c and $q^{(\text{ch})}$ are invariants, $\underline{\mathbf{p}}$ will also be a fourvector. It will be referred to as the *canonical momentum fourvector*. Thus the canonical momentum fourvector $\underline{\mathbf{p}}$ is the sum of a particle term \mathbf{p} and a field term $(q^{(\text{ch})}\mathbf{A}/c)$, where the particle term \mathbf{p} agrees with the Minkowski momentum definition in eqn (16.9).

This same canonical momentum fourvector $\underline{\mathbf{p}}$ defined in eqn (16.47) can also be obtained directly from the extended Lagrangian, eqn (16.39), without using explicit coordinates, as

$$\underline{\mathbf{p}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{mc\dot{\mathbf{r}}}{\sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} + \frac{q^{(\text{ch})}}{c}\mathbf{A} = m\mathbf{u} + \frac{q^{(\text{ch})}}{c}\mathbf{A} = \mathbf{p} + \frac{q^{(\text{ch})}}{c}\mathbf{A} \quad (16.48)$$

where eqn (16.142) was used. Notice that the canonical momentum fourvector $\underline{\mathbf{p}}$ is independent of the choice of β , as was proved in general in Section 11.7.

If we were to use the relativistic coordinates introduced at the end of Section 16.9, eqn (16.46) would become

$$\underline{\mathbf{p}} = p_0^{(\text{rel})}\hat{\mathbf{e}}^0 + \sum_{i=1}^3 p_i \hat{\mathbf{e}}^i \quad (16.49)$$

which shows another advantage of the relativistic coordinates. When they are used, the canonical momenta $p_{-\mu}^{(\text{rel})} = \partial \mathcal{L} / \partial x^\mu$ are identical to the covariant components of the canonical momentum fourvector.

16.11 Extended Hamiltonian

In Section 12.10 we discussed the impossibility of obtaining an extended Hamiltonian function by the traditional Legendre transformation method used in Chapter 4. The extended Lagrangian \mathcal{L} is homogeneous of degree one in the generalized coordinates \dot{q}_k and hence the prospective Hamiltonian function \mathcal{H} defined in eqn (12.42) is identically zero.

The same thing happens in the covariant theory being considered here. If we try to define \mathcal{H} by the usual Legendre transformation rule

$$\mathcal{H}(\mathbf{r}, \dot{\mathbf{r}}) = \dot{\mathbf{r}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} - \mathcal{L} \quad (16.50)$$

the function \mathcal{H} will be identically zero since

$$\dot{\mathbf{r}} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \dot{\mathbf{r}} \cdot \underline{\mathbf{p}} = \frac{mc\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{\sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} + \frac{q^{(\text{ch})}}{c}\dot{\mathbf{r}} \cdot \mathbf{A} = \mathcal{L} \quad (16.51)$$

This failure of the traditional Legendre transformation has caused some authors to reject the invariant Lagrangian \mathcal{L} and to try to modify it in various ways. But, as is explained in detail in Section 12.10, the extended Hamiltonian theory is *not* obtained from a Legendre transformation. The extended theory uses the relation between the

zero component of canonical momentum \underline{p}_0 and the traditional Hamiltonian H to write a dependency relation involving a dependency function \mathcal{K} that also serves as the extended Hamiltonian function.

Using the standard definition of extended Hamiltonian from eqn (12.12), and continuing to use the standard coordinates $q^{(\text{sta})} = t, x, y, z$,

$$\mathcal{K} = \underline{p}_0 + H = \underline{p}_0 + \sqrt{\|\underline{\mathbf{p}}c - q^{(\text{ch})}\mathbf{A}\|^2 + m^2c^4} + q^{(\text{ch})}\Phi \quad (16.52)$$

where the traditional Hamiltonian from eqn (16.35) has been introduced. In terms of this extended Hamiltonian, the Hamilton equations are those given in eqn (12.13)

$$i = \frac{\partial \mathcal{K}}{\partial \underline{p}_0} \quad \dot{\mathbf{r}} = \frac{\partial \mathcal{K}}{\partial \underline{\mathbf{p}}} \quad \dot{\underline{p}}_0 = -\frac{\partial \mathcal{K}}{\partial t} \quad \dot{\underline{\mathbf{p}}} = -\frac{\partial \mathcal{K}}{\partial \mathbf{r}} \quad (16.53)$$

16.12 Invariant Hamiltonian

Our objective is to write the Hamilton equations in a manifestly covariant manner, involving only fourvectors and invariants. The presence of the square root in eqn (16.52) makes this difficult to do. The most direct route is to use an alternate Hamiltonian \mathcal{K}_a as defined in Section 12.8 to rationalize the square root.

To define an alternate Hamiltonian, we first must find a nonzero phase space function $g(q, \underline{p})$, as demonstrated in Lemma 12.8.1. Consider the candidate function

$$\begin{aligned} g(q, \underline{p}) &= \frac{1}{c^2} \left\{ -\left(\underline{p}_0 + q^{(\text{ch})}\Phi\right) + \sqrt{\|\underline{\mathbf{p}}c - q^{(\text{ch})}\mathbf{A}\|^2 + m^2c^4} \right\} \\ &= \frac{1}{c^2} \left(E + \sqrt{p^2c^2 + m^2c^4} \right) = \frac{2E}{c^2} \end{aligned} \quad (16.54)$$

where eqn (16.44) was used. Since the relativistic energy E of a massive particle is positive and never zero, this g will be suitable. Then, the alternate Hamiltonian \mathcal{K}_a is defined as

$$\begin{aligned} \mathcal{K}_a &= g(q, \underline{p})\mathcal{K} = \frac{1}{c^2} \left\{ -\left(\underline{p}_0 + q^{(\text{ch})}\Phi\right) + \sqrt{\|\underline{\mathbf{p}}c - q^{(\text{ch})}\mathbf{A}\|^2 + m^2c^4} \right\} \\ &\quad \times \left\{ \left(\underline{p}_0 + q^{(\text{ch})}\Phi\right) + \sqrt{\|\underline{\mathbf{p}}c - q^{(\text{ch})}\mathbf{A}\|^2 + m^2c^4} \right\} \\ &= -\left(\frac{-p^0}{c} + \frac{q^{(\text{ch})}}{c}\Phi\right)^2 + \left\| \underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c}\mathbf{A} \right\|^2 + m^2c^2 \end{aligned} \quad (16.55)$$

where $\underline{p}_0 = -p^0$ was used. Since \underline{p}_0 here is the standard canonical momentum $\underline{p}_0^{(\text{sta})}$, the canonical momentum fourvector defined in Section 16.10 may be used to write

eqn (16.55) as

$$\mathcal{K}_a = \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) \cdot \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) + m^2 c^2 \quad (16.56)$$

which is composed only of invariants m , c , $q^{(\text{ch})}$ and the dot product of two fourvectors. Thus \mathcal{K}_a is a Lorentz invariant quantity, as desired.

16.13 Manifestly Covariant Hamilton Equations

Lemma 12.8.1 shows that the extended Hamilton equations with alternate Hamiltonian \mathcal{K}_a are equivalent to those with \mathcal{K} in eqn (16.53). The Hamilton equations retain the form given in eqn (16.53), with only the substitution of \mathcal{K}_a for \mathcal{K} . Thus we have the extended Hamilton equations

$$\dot{t} = \frac{\partial \mathcal{K}_a}{\partial \underline{p}_0^{(\text{sta})}} \quad \dot{\mathbf{r}} = \frac{\partial \mathcal{K}_a}{\partial \underline{\mathbf{p}}} \quad \dot{\underline{p}}_0^{(\text{sta})} = -\frac{\partial \mathcal{K}_a}{\partial t} \quad \dot{\underline{\mathbf{p}}} = -\frac{\partial \mathcal{K}_a}{\partial \mathbf{r}} \quad (16.57)$$

where we have used the superscript (sta) to emphasize that the standard coordinates $q = t, x, y, z$ are being used. Since the zeroth component of the canonical momentum fourvector $\underline{\mathbf{p}}$ is $\underline{p}_0 = \underline{p}_0^{(\text{rel})} = \underline{p}_0^{(\text{sta})}/c$, and since the zeroth component of the position fourvector \mathbf{r} is $x^0 = ct$, eqn (16.57) may be written in terms of fourvector components as

$$\dot{x}^0 = \frac{\partial \mathcal{K}_a}{\partial \underline{p}_0} \quad \dot{\mathbf{r}} = \frac{\partial \mathcal{K}_a}{\partial \underline{\mathbf{p}}} \quad \dot{\underline{p}}_0 = -\frac{\partial \mathcal{K}_a}{\partial x^0} \quad \dot{\underline{\mathbf{p}}} = -\frac{\partial \mathcal{K}_a}{\partial \mathbf{r}} \quad (16.58)$$

We now define a new fourvector $\dot{\underline{\mathbf{p}}} = \sum_{\mu=0}^3 \dot{p}_\mu \hat{\mathbf{e}}^\mu$ analogous to $\dot{\mathbf{r}}$ in eqn (16.38). Also we define a new gradient fourvector operator similar to $\partial/\partial \mathbf{r}$ but taking derivatives with respect to components of $\underline{\mathbf{p}}$

$$\frac{\partial}{\partial \underline{\mathbf{p}}} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \frac{\partial}{\partial \underline{p}^\mu} = \sum_{\mu=0}^3 \hat{\mathbf{e}}^\mu \frac{\partial}{\partial \underline{p}_\mu} \quad (16.59)$$

These definitions allow us to write eqn (16.58) in a manifestly covariant form, as desired

$$\dot{\mathbf{r}} = \frac{\partial \mathcal{K}_a}{\partial \underline{\mathbf{p}}} \quad \dot{\underline{\mathbf{p}}} = -\frac{\partial \mathcal{K}_a}{\partial \mathbf{r}} \quad (16.60)$$

It is interesting to examine the first of eqn (16.58). Using eqn (16.56), it is

$$\frac{d(ct)}{d\beta} = \frac{\partial \mathcal{K}_a}{\partial \underline{p}_0} = 2 \left(\underline{p}^0 - \frac{q^{(\text{ch})}}{c} \Phi \right) = \frac{2E}{c} = 2\gamma mc = 2mc \frac{dt}{d\tau} \quad (16.61)$$

where we have used $E = \gamma mc^2$ from eqn (16.6) and $dt/d\tau = \gamma$ from eqn (14.26). The nonzero differential dt may be cancelled, giving

$$\frac{d\tau}{d\beta} = 2m \quad \text{and hence} \quad \tau = \tau_0 + 2m\beta \quad (16.62)$$

where we have made the usual choice of taking $\beta = 0$ at the initial point of the trajectory, at which the proper time is set to an arbitrary value τ_0 . Equation (16.62)

suggests that the parameter β can be taken to be an invariant quantity like τ and m are. If, as will be done in Chapter 17, the Lorentz transformation is considered to be a canonical transformation (see Exercise 17.2), then Theorem 17.12.1 shows β to be an invariant by definition since that parameter is not transformed in canonical transformations.

16.14 The Klein–Gordon Equation

One of the recipes for passing from classical to quantum mechanics is the replacement of the generalized energy and the canonical⁹³ momenta of a particle by differential operators, as in

$$\underline{p}_0^{(\text{sta})} \rightarrow -i\hbar \frac{\partial}{\partial t} \quad \underline{p}_x \rightarrow -i\hbar \frac{\partial}{\partial x} \quad \underline{p}_y \rightarrow -i\hbar \frac{\partial}{\partial y} \quad \underline{p}_z \rightarrow -i\hbar \frac{\partial}{\partial z} \quad (16.63)$$

Since, as seen in Section 16.10, the canonical momentum fourvector and the gradient fourvector are

$$\underline{\mathbf{p}} = -(\underline{p}_0^{(\text{sta})}/c) \hat{\mathbf{e}}_0 + \underline{\mathbf{p}} \quad \text{and} \quad \frac{\partial}{\partial \mathbf{r}} = -\hat{\mathbf{e}}_0 \frac{\partial}{\partial ct} + \frac{\partial}{\partial \mathbf{r}} \quad (16.64)$$

the four substitutions in eqn (16.63) can be expressed as a single fourvector substitution

$$\underline{\mathbf{p}} \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{r}} \quad (16.65)$$

One can attempt to write a traditional Schroedinger equation by making these substitutions in the traditional Hamiltonian, eqn (16.35), giving

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= H \left(q, \underline{p} \rightarrow -i\hbar \frac{\partial}{\partial q} \right) \psi \\ &= \left\{ \sqrt{\left(-i\hbar c \frac{\partial}{\partial \mathbf{r}} - q^{(\text{ch})} \mathbf{A} \right) \cdot \left(-i\hbar c \frac{\partial}{\partial \mathbf{r}} - q^{(\text{ch})} \mathbf{A} \right) + m^2 c^4 + q^{(\text{ch})} \Phi} \right\} \psi \end{aligned} \quad (16.66)$$

However, this differential equation involves the square root of a differential operator and is difficult to apply.⁹⁴

In Section 4.7, the Schroedinger equation is written by equating $i\hbar \partial \psi / \partial t$ to the traditional Hamiltonian. Equation (16.66) uses that same method. But that is not the only way to obtain a quantum mechanical wave equation. It is not necessary to use the traditional Hamiltonian, or to have a Hamiltonian like the \mathcal{H} that was proved identically zero in Section 12.10. To obtain a quantum wave equation, *it is sufficient to have a dependency relation among the phase space variables* into which the substitutions

⁹³It is important to realize that it is the components of the *canonical* momentum $\underline{\mathbf{p}}$ that are replaced by quantum operators. Thus the uncertainty principle of quantum mechanics refers to the canonical and not the particle momenta. Of course, for a free particle these distinctions disappear.

⁹⁴Dirac proposed a method for linearizing the square root, leading to the Dirac equation to be discussed in Section 16.15.

eqn (16.63) can be made. The result will be a manifestly covariant wave equation. We now use this method to derive the Klein–Gordon equation.

The invariant Hamiltonian \mathcal{K}_a in eqn (16.56) defines a dependency relation in the extended phase space. It is written by equating \mathcal{K}_a to zero, as in

$$\mathcal{K}_a(q, \underline{p}) = \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) \cdot \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) + m^2 c^2 = 0 \quad (16.67)$$

Substituting the quantum differential operator eqn (16.65) for the canonical fourvector momentum $\underline{\mathbf{p}}$ in this dependency relation, and introducing a wave function Ψ , leads at once to the Klein–Gordon equation

$$\left\{ \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) + m^2 c^2 \right\} \Psi = 0 \quad (16.68)$$

which is a manifestly covariant differential equation for Ψ .

For a free particle with no electromagnetic fields acting, the Klein–Gordon equation reduces to

$$\left\{ \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) + m^2 c^2 \right\} \Psi = 0 \quad \text{or} \quad \left(-\square^2 + \frac{m^2 c^2}{\hbar^2} \right) \Psi = 0 \quad (16.69)$$

where \square^2 is the same d'Alembertian operator as is defined in eqn (15.119). The quantity \hbar/mc is often called the reduced Compton wavelength.

The Klein–Gordon equation in the form presented here can be applied to charged bose particles of zero spin. For details and for other related equations, the reader should consult Bjorken and Drell (1964) and Weinberg (1995).

16.15 The Dirac Equation

Another relativistic wave equation, this one useful for Fermi particles of half-integer spin such as the electron, was developed by Dirac. His original motivation was to avoid a difficulty with probability conservation by writing a first-order differential equation that would be equivalent to the second-order Klein–Gordon equation. Dirac's method (Dirac, 1935) began with the substitution of the quantum operators eqn (16.63) into the non-covariant Hamiltonian for a free particle, eqn (16.35), giving

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{\left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) + m^2 c^4} \psi \quad (16.70)$$

The square root on the right was then linearized by the introduction of matrix coefficients, and the resulting equation was recast into a manifestly covariant form and shown to imply the Klein–Gordon equation.

Since we already have the free particle Klein–Gordon equation in manifestly covariant form in eqn (16.69), we can obtain Dirac's result by a direct route. A set

of matrices can be found, usually called the *gamma matrices*, which obey the anti-commutation relations

$$\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \mathbf{U} \tag{16.71}$$

where \mathbf{U} is the unit matrix. We then write a formal fourvector with matrix coefficients as

$$\boldsymbol{\gamma} = \gamma^0 \hat{\mathbf{e}}_0 + \gamma^1 \hat{\mathbf{e}}_1 + \gamma^2 \hat{\mathbf{e}}_2 + \gamma^3 \hat{\mathbf{e}}_3 \tag{16.72}$$

From eqn (16.71), the gamma matrices obey $(\gamma^\mu)^2 = g^{\mu\mu} \mathbf{U}$, and $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$ for $\mu \neq \nu$. Taking the determinants of both sides of these expressions shows that the gamma matrices must have even dimension. The smallest dimension for which suitable gamma matrices can be found is four, which we now assume.

Using these gamma matrices, the Klein–Gordon equation, eqn (16.69), can be factored as

$$\left\{ \boldsymbol{\gamma} \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) + imc \mathbf{U} \right\} \left\{ \boldsymbol{\gamma} \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) - imc \mathbf{U} \right\} [\Psi] = 0 \tag{16.73}$$

where the wave function $[\Psi]$ is now a column vector with four rows. The free particle Dirac equation is the right-hand factor of eqn (16.73) equated to zero

$$\left\{ \boldsymbol{\gamma} \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) - imc \mathbf{U} \right\} [\Psi] = 0 \quad \text{or, equivalently,} \quad \left(\boldsymbol{\gamma} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{mc}{\hbar} \mathbf{U} \right) [\Psi] = 0 \tag{16.74}$$

To see that the factorization in eqn (16.73) is equivalent to the Klein–Gordon equation, use eqn (16.71) to write

$$\left(\boldsymbol{\gamma} \cdot \frac{\partial}{\partial \mathbf{r}} \right)^2 = \left(\sum_{\mu=0}^3 \gamma^\mu \partial_\mu \right) \left(\sum_{\nu=0}^3 \gamma^\nu \partial_\nu \right) = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g^{\mu\nu} \partial_\mu \partial_\nu \mathbf{U} = \mathbf{U} \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} \tag{16.75}$$

Using this result to multiply out the factors, eqn (16.73) becomes

$$\left\{ \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} \right) + m^2 c^2 \right\} [\Psi] = 0 \tag{16.76}$$

It follows from this equation that if $[\Psi]$ satisfies the Dirac equation, eqn (16.74), then each of its four components must satisfy the Klein–Gordon equation.

To establish the manifest covariance of the Dirac equation, we must first consider the transformation properties of $[\Psi]$ and the gamma matrices. A 4×4 , nonsingular transformation matrix $\mathbf{D}(M)$ can be found that depends on the Lorentz transformation M from the S to S' systems and satisfies the conditions

$$\gamma^\mu = \sum_{\nu=0}^3 M^\mu_\nu \mathbf{D} \gamma^\nu \mathbf{D}^{-1} \tag{16.77}$$

and $\mathbf{D}(MN) = \mathbf{D}(M) \mathbf{D}(N)$ where M, N are two Lorentz transformations. The matrices \mathbf{D} are said to form a *representation* of the Lorentz group. The transformation

rule for the Dirac wave function is $[\Psi] = D[\Psi']$. Using eqn (15.59) to transform the covariant components of $\partial/\partial\mathbf{r}$, these definitions imply that,

$$\left(\sum_{\mu=0}^3 \gamma^\mu \frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar} \mathbf{U} \right) [\Psi] = 0 \quad \text{if and only if} \quad \left(\sum_{\mu=0}^3 \gamma^\mu \frac{\partial}{\partial x'^\mu} + \frac{mc}{\hbar} \mathbf{U} \right) [\Psi'] = 0 \quad (16.78)$$

which establishes the form invariance of the Dirac equation.

The Dirac wave equation has four components, which are used to represent two states of electron spin for each of two signs of the electron energy. Since the Klein–Gordon equation, eqn (16.68), can be obtained from its free particle form eqn (16.69) by the replacement of $(-i\hbar\partial/\partial\mathbf{r})$ by $(-i\hbar\partial/\partial\mathbf{r} - q^{(\text{ch})}\mathbf{A}/c)$, Dirac conjectured that the correct wave equation for an electron in an electromagnetic field would follow from the same substitution in eqn (16.74), giving

$$\left\{ \boldsymbol{\gamma} \cdot \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) - imc \mathbf{U} \right\} [\Psi] = 0 \quad (16.79)$$

Positive energy solutions of this equation reproduce the energy spectrum of the hydrogen atom. For details, the reader is referred to Bethe and Jackiw (1968), Bjorken and Drell (1964), and Weinberg (1995).

16.16 The Manifestly Covariant N -Body Problem

The reader will be disappointed to learn that relativistic mechanics does not have a theory of collective motion that is as elegant and complete as the one presented in Chapter 1 for Newtonian mechanics. Even the definition of the total fourvector momentum \mathbf{P} of a system of point particles presents difficulties.

We will present the relativistic theory of collective motion using particles that interact only at what Synge (1955) calls “point catastrophes.” A system of masses moves independently except for point collisions among them. We assume that these collisions are impulsive and conserve all components of the fourvector momentum. This model, though obviously limited in its physical reality, has the advantage of presenting the important issues very clearly. A more physical model would require fields, and would require a level of mathematical complexity that would obscure the points we need to make.

We give only the outlines of the covariant theory of collective motion here. The reader is referred to Synge (1955), and particularly to Møller (1972), for more detail.

16.16.1 Problems With the Total Momentum

As can be seen from Figure 16.1, each of the particles m_n , where $n = 1, \dots, N$, has its own world line between collisions. We want to give a manifestly covariant definition of the total momentum of this collection. The immediate problem is to devise some coordinate-system-independent way of deciding *when* the momentum values should be collected and added to give the sum \mathbf{P} .

It seems that the momenta should be collected *simultaneously*. But simultaneity is relative, and so we have to ask which coordinate system should be used to define it.⁹⁵ Choice of one particular system will lead to a theory that selects that system for special treatment and hence is only formally covariant.

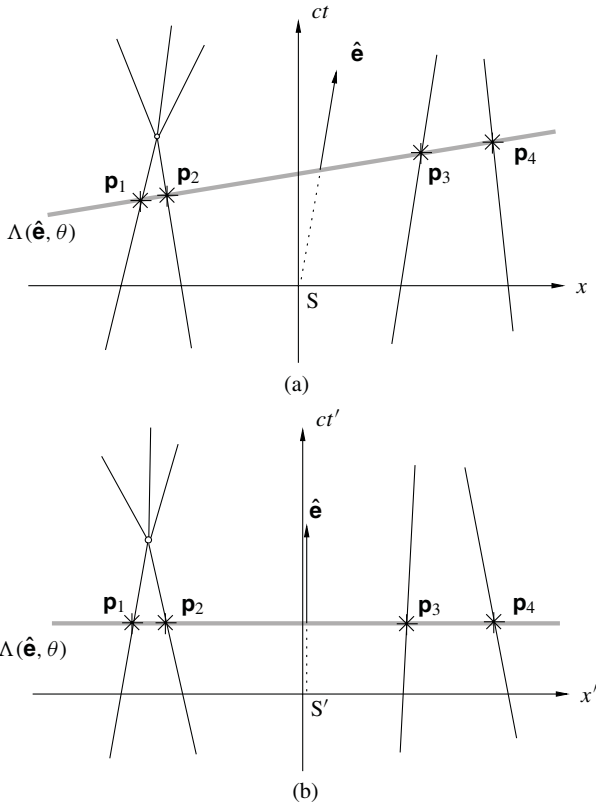


FIG. 16.1. The momenta \mathbf{p}_n are collected at events at which the hyperplane of simultaneity $\Lambda(\hat{\mathbf{e}}, \theta)$ (grey line) crosses the world lines of the four particles. A collision event with two particles entering and three emerging is shown on the left. Figure (a) is in a general system S . Figure (b) is in the system S' in which $\hat{\mathbf{e}}'_0 = \hat{\mathbf{e}}$. In (b), the coordinate time of the S' system is equal to θ and the four events are simultaneous, with $t'_n = t'$.

The problem of simultaneity is addressed by defining what we call a three dimensional *hyperplane of simultaneity* $\Lambda(\hat{\mathbf{e}}, \theta)$ cutting across the world lines of the particles. Events \mathbf{r} are in this hyperplane if they satisfy the equation

$$-\mathbf{r} \cdot \hat{\mathbf{e}} = c\theta \tag{16.80}$$

where $\hat{\mathbf{e}}$ is an arbitrary, forward timelike, unit fourvector that defines the orientation of the hyperplane, and θ will be called the *epoch parameter*. We will define the notation

⁹⁵A similar point is made in Section 5.7 of Rindler (1977).

$\mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}, \theta)$ to mean that $-\mathbf{r}_n \cdot \hat{\mathbf{e}} = c\theta$ so that event \mathbf{r}_n on the world line of mass m_n is in the hyperplane of simultaneity.

Since the orientation unit fourvector $\hat{\mathbf{e}}$ is forward timelike, Exercise 15.4 shows that a coordinate system S' can always be found in which $\hat{\mathbf{e}} : (1, 0, 0, 0)_{S'}$. Thus, the time unit vector of S' will be equal to the chosen orientation unit vector, $\hat{\mathbf{e}}'_0 = \hat{\mathbf{e}}$. It follows that events \mathbf{r}_n will be in the hyperplane if and only if they are simultaneous in S' and have $t'_n = t'$, the coordinate time of the S' system.

To understand the epoch parameter θ , consider a clock moving along a world line parallel to $\hat{\mathbf{e}}$ and passing through the origin (the dotted line segment in Figure 16.1). Let that world line intersect the hyperplane at event \mathbf{r} . Then θ is the proper time read on that clock, integrated from the origin event to \mathbf{r} . That clock will be at rest at the spatial origin of the S' system and will measure the coordinate time t' of that system.

The considerations above may be summarized as follows: When the relativity of simultaneity is taken into account, the total momentum is not a simple fourvector, but rather must be defined as a fourvector function of an arbitrary orientation unit fourvector $\hat{\mathbf{e}}$ and an invariant epoch parameter θ ,

$$\mathbf{P}(\hat{\mathbf{e}}, \theta) = \sum_{\substack{n=1 \\ \mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}, \theta)}}^N \mathbf{p}_n \quad (16.81)$$

The choice of $\hat{\mathbf{e}}$ determines the system S' in which the simultaneous collection of momenta is to be performed. In the system S' with $\hat{\mathbf{e}}'_0 = \hat{\mathbf{e}}$, the components of \mathbf{P} will be

$$P'^{\mu} = \sum_{\substack{n=1 \\ t'_n=t'=\theta}}^N p_n'^{\mu} \quad (16.82)$$

The fourvector expanded as $\mathbf{P} = P'^0 \hat{\mathbf{e}}'_0 + \mathbf{P}'$ for one choice of $\hat{\mathbf{e}}$ will in general not be the same as the fourvector similarly expanded with another choice. Any theory using such a total momentum will require a specific choice of $\hat{\mathbf{e}}$ and hence will be only formally covariant.

16.16.2 Isolated Systems

Fortunately, there is a wide class of situations in which the total momentum simplifies. If we assume, as he does, that the collection of N particles in Synge's point catastrophe model forms an *isolated system*, with no forces or other sources of momentum change acting on the system from outside it, then the fourvector momentum \mathbf{P} will be constant and independent of both $\hat{\mathbf{e}}$ and θ .

To see the constancy of \mathbf{P} for isolated systems, consider that if no outside influences are present, then the only source of momentum change for a particle is collision with some other particle. Thus the particles are force-free and have constant fourvector momenta between collisions. Since, in addition, the total fourvector momentum of the colliding particles coming into any vertex is by assumption equal to the total

fourvector momentum of the particles leaving it, the same fourvector \mathbf{P} will be obtained no matter when the plane of simultaneity intersects the particle world lines. In fact, the number of particles N need not be a constant. It makes no difference if two particles enter a collision and three exit, so long as the total fourvector momentum is conserved at the vertex. For isolated systems, the total momentum \mathbf{P} is independent of both the orientation and the location of the plane of simultaneity, and hence is not a function of $\hat{\mathbf{e}}$ or of θ .

Thus, for isolated systems, the components of the constant fourvector \mathbf{P} can be written using simultaneous collection in any frame. The same fourvector will result regardless of the choice. It follows that these components will satisfy the standard fourvector transformation rule from eqn (15.55)

$$P^\mu = \sum_{\nu=0}^3 M^\mu{}_\nu P'^\nu \tag{16.83}$$

For the rest of this section, we assume that the collection is an isolated system.

16.16.3 Rest Frame of the Collection

The constant fourvector momentum \mathbf{P} is the sum of forward timelike fourvectors \mathbf{p}_n . Exercise 15.2 shows that \mathbf{P} is also forward timelike. Thus a positive, constant, invariant quantity M can be defined by writing

$$M^2 c^2 = -\mathbf{P} \cdot \mathbf{P} \tag{16.84}$$

By analogy to eqn (16.11), this M will be called the *invariant mass of the collection*.

Since \mathbf{P} is timelike, Exercise 15.4 shows that there will always be some coordinate system in which the spatial part of \mathbf{P} vanishes. Let us call that system the *rest frame of the collection*, and denote it as S'' . If (see Exercise 15.4) we define a forward timelike unit fourvector by $\hat{\mathbf{e}}_P = \mathbf{P}/Mc$, then this rest frame S'' will have a time basis vector given by $\hat{\mathbf{e}}''_0 = \hat{\mathbf{e}}_P$.

In the S'' frame, $\mathbf{P}'' = 0$ and hence

$$\mathbf{P} = \frac{E''}{c} \hat{\mathbf{e}}''_0 \quad \text{where} \quad E'' = \sum_{\substack{n=1 \\ t''_n=t''}}^{N(t'')} E''_n \tag{16.85}$$

Also, the invariant mass M is related to the total energy in the S'' system by $E'' = Mc^2$. The vanishing of \mathbf{P}'' implies that the threevector momenta sum vectorially to zero in the S'' system

$$0 = \mathbf{P}'' = \sum_{\substack{n=1 \\ t''_n=t''}}^{N(t'')} \mathbf{p}''_n \tag{16.86}$$

This S'' system will be useful when we define the center of mass/energy of the collection.

16.16.4 Manifestly Covariant Center of Mass/Energy

The center of mass of a collection was defined in Chapter 1 as the mass-weighted average point. In Newtonian physics, this is

$$\mathbf{R} = \frac{1}{M} \sum_{n=1}^N m_n \mathbf{r}_n \quad (16.87)$$

An attempt to define the same sort of quantity in the relativistic case faces two immediate problems: The first is the simultaneity problem already encountered in Section 16.16.1. The second is that, as discussed in Section 16.4, the masses m_n of the particles are not conserved in collisions. If two particles collide, the particles emerging from the collision may have a total mass different from the total mass of the original two. Thus, if center of mass is defined using mass as the weighting factor, there will be impulsive, discontinuous changes in its location during particle collisions, leading to faster-than-light jumps.

To avoid discontinuous jumps in the center of mass location, we must define it using a weighting factor that is conserved in collisions. The most natural such quantity is the relativistic energy. For this reason, we will refer to the fourvector *center of mass/energy* rather than simply the center of mass as in Newtonian physics.

A manifestly covariant definition of the center of mass/energy can be written as

$$\mathbf{R} = \frac{1}{M^2 c^2} \sum_{\substack{n=1 \\ \mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}_P, \theta)}}^{N(\theta)} (-\mathbf{P} \cdot \mathbf{p}_n) \mathbf{r}_n \quad (16.88)$$

Notice that we have used the unit fourvector $\hat{\mathbf{e}}_P = \mathbf{P}/Mc$ defined in Section 16.16.3 to define the orientation of the plane of simultaneity. Thus \mathbf{R} is a fourvector function of the total momentum fourvector \mathbf{P} and the invariant epoch parameter θ . By using the previously determined total momentum fourvector \mathbf{P} to define $\hat{\mathbf{e}}_P$ and hence the orientation of the plane of simultaneity, we have avoided writing a merely formally covariant expression. The \mathbf{P} and hence $\hat{\mathbf{e}}_P$ represent invariant physical properties of the isolated collective system and are not merely chosen arbitrarily.

The fourvector definition eqn (16.88) takes its simplest form when written in terms of components in the rest system of the collection S'' . Then, as the reader can verify from eqn (16.85), it becomes

$$\mathbf{R} = \frac{1}{E''} \sum_{\substack{n=1 \\ t''_n = t''}}^{N(t'')} E''_n \mathbf{r}_n \quad (16.89)$$

The “center of mass” is really a center of relativistic energy, hence our reference to it as the center of mass/energy. It follows from eqn (16.89) that the world line of the

center of mass/energy is

$$\mathbf{R} = ct'' \hat{\mathbf{e}}_0'' + \mathbf{R}'' \quad \text{where} \quad \mathbf{R}'' = \frac{1}{E''} \sum_{\substack{n=1 \\ t_n''=t''}}^{N(t'')} E_n'' \mathbf{r}_n'' \quad (16.90)$$

16.16.5 Velocity of the Center of Mass/Energy

We expect that the center of mass/energy should be at rest in the rest system of the collection. We now show that this is the case. In the S'' system, $\hat{\mathbf{e}}_P = \hat{\mathbf{e}}_0''$, and the epoch parameter θ is equal to the coordinate time t'' . Thus we may write

$$\frac{d\mathbf{R}''}{d\theta} = \frac{d\mathbf{R}''}{dt''} = \frac{1}{E''} \frac{d}{dt''} \left(\sum_{\substack{n=1 \\ t_n''=t''}}^{N(t'')} E_n'' \mathbf{r}_n'' \right) \quad (16.91)$$

If the plane of simultaneity (which is the plane of constant t'' in this system) passes the vertex of a particle collision, the sum in brackets will not change. The colliding particles are all at the same spatial location, and the sum of the energies out equals the sum in. So, we need only consider regions between collisions. In these regions, the E_n'' are constants, and so

$$\frac{d\mathbf{R}''}{d\theta} = \frac{1}{E''} \sum_{\substack{n=1 \\ t_n''=t''}}^{N(t'')} E_n'' \frac{d\mathbf{r}_n''}{dt''} = \frac{1}{E''} \sum_{\substack{n=1 \\ t_n''=t''}}^{N(t'')} \gamma_n'' m_n c^2 \mathbf{v}_n'' = \frac{c^2}{E''} \sum_{\substack{n=1 \\ t_n''=t''}}^{N(t'')} \mathbf{p}_n'' = 0 \quad (16.92)$$

where eqn(16.86) was used to get the last equality. In this expression $\gamma_n'' = \sqrt{1 - (v_n'')^2 / c^2}$ is the time dilation factor of the n^{th} mass, \mathbf{v}_n'' is its coordinate velocity, and the definitions $E_n'' = \gamma_n'' m_n c^2$ and $\mathbf{p}_n'' = \gamma_n'' m_n \mathbf{v}_n''$ from Section 16.1 were used. Thus, since $d\mathbf{R}''/d\theta = 0$, we have from eqn (16.90) that

$$\frac{d\mathbf{R}}{d\theta} = \frac{d(ct'' \hat{\mathbf{e}}_0'')}{dt''} = c \hat{\mathbf{e}}_0'' = \frac{\mathbf{P}}{M} \quad \text{and so} \quad \mathbf{P} = M \frac{d\mathbf{R}}{d\theta} \quad (16.93)$$

The center of mass/energy is at rest in the S'' system and hence its world line is parallel to the $\hat{\mathbf{e}}_0''$ unit vector. The proper time interval $d\tau_R = \sqrt{-d\mathbf{R} \cdot d\mathbf{R}}/c$ along this world line is equal to dt'' . Thus $dt'' = d\theta = d\tau_R$ and we may write eqn (16.93) as a manifestly covariant equation

$$\mathbf{P} = M \mathbf{U} \quad \text{where} \quad \mathbf{U} = \frac{d\mathbf{R}}{d\tau_R} \quad (16.94)$$

The \mathbf{U} is the fourvector velocity of the center of mass/energy. This formula reproduces for collections the definition of the fourvector momentum of a single point particle in eqn (16.9). It is analogous to eqn (1.40) in the Newtonian theory of collective motion.

16.16.6 Angular Momenta

In Minkowski space, angular momenta are represented as dyadic wedge products. Thus, the total angular momentum \mathbb{J} of the collection can be written as

$$\mathbb{J} = \sum_{\substack{n=1 \\ \mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}_P, \theta)}}^{N(\theta)} \mathbf{r}_n \wedge \mathbf{p}_n \quad (16.95)$$

In analogy with the Newtonian case, we can also define orbital and spin angular momenta by

$$\mathbb{L} = \sum_{\substack{n=1 \\ \mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}_P, \theta)}}^{N(\theta)} \mathbf{R} \wedge \mathbf{p}_n = \mathbf{R} \wedge \mathbf{P} \quad (16.96)$$

$$\mathbb{S} = \sum_{\substack{n=1 \\ \mathbf{r}_n \subset \Lambda(\hat{\mathbf{e}}_P, \theta)}}^{N(\theta)} (\mathbf{r}_n - \mathbf{R}) \wedge \mathbf{p}_n \quad (16.97)$$

so that, by construction

$$\mathbb{J} = \mathbb{L} + \mathbb{S} \quad (16.98)$$

This relation is the analog of eqn (1.43) in Newtonian physics.

In the rest frame of the collection, $\mathbf{P} = (E''/c)\hat{\mathbf{e}}_0''$, $\mathbf{R} = ct''\hat{\mathbf{e}}_0'' + \mathbf{R}''$, and $\mathbf{r}_n = ct''\hat{\mathbf{e}}_0'' + \mathbf{r}_n''$ where, as shown in Section 16.16.3, the spatial part \mathbf{R}'' is a constant. Using these values, and Exercise 15.14, the matrices of these dyadics in this system can be written as

$$\mathbb{L}'' = -\frac{E''}{c} \begin{pmatrix} 0 & R''^1 & R''^2 & R''^3 \\ R''^1 & 0 & 0 & 0 \\ R''^2 & 0 & 0 & 0 \\ R''^3 & 0 & 0 & 0 \end{pmatrix} \quad \mathbb{S}'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_3'' & -S_2'' \\ 0 & -S_3'' & 0 & S_1'' \\ 0 & S_2'' & -S_1'' & 0 \end{pmatrix} \quad (16.99)$$

where the S_i'' are components of the threector \mathbf{S}'' defined by

$$\mathbf{S}'' = \sum_{\substack{n=1 \\ i_n''=i''}}^{N(\theta)} (\mathbf{r}_n'' - \mathbf{R}'') \times \mathbf{p}_n'' \quad (16.100)$$

Equation (16.98) then gives

$$\mathbb{J}'' = \begin{pmatrix} 0 & -(E''/c)R''^1 & -(E''/c)R''^2 & -(E''/c)R''^3 \\ -(E''/c)R''^1 & 0 & S_3'' & -S_2'' \\ -(E''/c)R''^2 & -S_3'' & 0 & S_1'' \\ -(E''/c)R''^3 & S_2'' & -S_1'' & 0 \end{pmatrix} \quad (16.101)$$

It follows from these matrices that $\mathbb{S} \cdot \mathbf{P} = 0$ since, in the S'' system, this product

becomes the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_3'' & -S_2'' \\ 0 & -S_3'' & 0 & S_1'' \\ 0 & S_2'' & -S_1'' & 0 \end{pmatrix} \begin{pmatrix} E''/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \tag{16.102}$$

Some authors use the condition $\mathbb{S} \cdot \mathbf{P} = 0$ together with eqn (16.97) to *define* the center of mass/energy \mathbf{R} . Our definition is equivalent to theirs.

Since we are assuming an isolated system, we expect that the angular momenta should be constants. For \mathbf{L}'' , this constancy follows immediately from the fact that \mathbf{R}'' and E'' are constants. For \mathbf{S}'' , the constancy can be seen by differentiating the three-vector \mathbf{S}'' in eqn (16.100) with respect to t'' . The passage of the plane of simultaneity across vertices of particle collision will not change \mathbf{S}'' , by the same argument as used above in deriving eqn (16.92). Between collisions, $d\mathbf{p}_n''/dt'' = 0$ and hence

$$\frac{d\mathbf{S}''}{dt''} = \sum_{\substack{n=1 \\ t_n''=t''}}^{N(\theta)} \left\{ \mathbf{v}_n'' \times \mathbf{p}_n'' + (\mathbf{r}_n'' - \mathbf{R}'') \times \frac{d\mathbf{p}_n''}{dt''} \right\} = \sum_{\substack{n=1 \\ t_n''=t''}}^{N(\theta)} (\mathbf{v}_n'' \times m_n \gamma'' \mathbf{v}_n'') = 0 \tag{16.103}$$

It follows that \mathbf{S}'' is a constant threevector, and hence by eqn (16.99) that \mathbb{S}'' is a constant matrix. The constancy of \mathbf{J}'' then follows from eqn (16.98).

Since $d\tau_R = dt''$ in the rest frame of the collection, these matrix results imply the manifestly covariant expressions

$$\frac{d\mathbb{L}}{d\tau_R} = 0 \qquad \frac{d\mathbb{S}}{d\tau_R} = 0 \qquad \frac{d\mathbf{J}}{d\tau_R} = 0 \tag{16.104}$$

The angular momenta of an isolated system are conserved, as was expected.

16.17 Covariant Serret–Frenet Theory

The derivation of the Serret–Frenet formulae for a curve in three dimensions, given in Section A.12, generalizes easily to world lines considered as curves in Minkowski space. The three orthonormal Serret–Frenet vectors of Section A.12 generalize to four orthonormal (in the sense of the Minkowski metric) Serret–Frenet fourvectors defined along the world line.⁹⁶

Suppose that a timelike world line is specified by writing t, x, y, z as functions of some monotonic parameter β

$$\mathbf{r}(\beta) = ct(\beta) \hat{\mathbf{e}}_0 + x(\beta) \hat{\mathbf{e}}_1 + y(\beta) \hat{\mathbf{e}}_2 + z(\beta) \hat{\mathbf{e}}_3 \tag{16.105}$$

For the purposes of deriving the Serret–Frenet vectors, the arc length ds along this world line can be defined as $cd\tau$ where τ is the proper time measured by a clock

⁹⁶Our treatment follows that in Section 2.7 of Synge and Schild (1969).

carried along the line starting from some specified zero point. Then

$$ds = cd\tau = \sqrt{-d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} d\beta \quad \text{and} \quad \dot{s} = \sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \quad (16.106)$$

It follows that

$$\hat{\mathbf{f}}_0 = \frac{d\mathbf{r}}{ds} = \frac{\dot{\mathbf{r}}}{\dot{s}} = \frac{\dot{\mathbf{r}}}{\sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \quad (16.107)$$

is a unit vector. We assume that both β and s increase in the direction of increasing t so that $\hat{\mathbf{f}}_0$ is a forward timelike unit vector obeying $\hat{\mathbf{f}}_0 \cdot \hat{\mathbf{f}}_0 = -1$. By construction, $\hat{\mathbf{f}}_0$ is tangent to the world line and points in the direction of increasing t . It is called the *unit tangent fourvector*.

Just as in the three dimensional case, we next differentiate the unit tangent vector, obtaining $d\hat{\mathbf{f}}_0/ds$. Differentiating the expression $\hat{\mathbf{f}}_0 \cdot \hat{\mathbf{f}}_0 = -1$ with respect to s proves that $(d\hat{\mathbf{f}}_0/ds) \cdot \hat{\mathbf{f}}_0 = 0$, from which it follows (Exercise 15.3) that $d\hat{\mathbf{f}}_0/ds$ is spacelike. The first curvature ρ_1 and the Serret–Frenet vector $\hat{\mathbf{f}}_1$ are then defined by

$$\rho_1 = \sqrt{\frac{d\hat{\mathbf{f}}_0}{ds} \cdot \frac{d\hat{\mathbf{f}}_0}{ds}} \quad \text{and} \quad \frac{d\hat{\mathbf{f}}_0}{ds} = \rho_1 \hat{\mathbf{f}}_1 \quad (16.108)$$

Thus $\hat{\mathbf{f}}_1$ is a spacelike unit vector obeying $\hat{\mathbf{f}}_1 \cdot \hat{\mathbf{f}}_1 = +1$.

Differentiating the expressions $\hat{\mathbf{f}}_1 \cdot \hat{\mathbf{f}}_1 = +1$ and $\hat{\mathbf{f}}_1 \cdot \hat{\mathbf{f}}_0 = 0$ with respect to s shows that $d\hat{\mathbf{f}}_1/ds$ has zero dot product with $\hat{\mathbf{f}}_1$, and obeys $(d\hat{\mathbf{f}}_1/ds) \cdot \hat{\mathbf{f}}_0 = -\rho_1$. It follows that the fourvector $(d\hat{\mathbf{f}}_1/ds) - \rho_1 \hat{\mathbf{f}}_0$ has a zero dot product with $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_0$, and is spacelike. The second curvature ρ_2 and the second spacelike Serret–Frenet vector $\hat{\mathbf{f}}_2$ are then defined by

$$\rho_2 = \sqrt{\left(\frac{d\hat{\mathbf{f}}_1}{ds} - \rho_1 \hat{\mathbf{f}}_0\right) \cdot \left(\frac{d\hat{\mathbf{f}}_1}{ds} - \rho_1 \hat{\mathbf{f}}_0\right)} \quad \text{and} \quad \frac{d\hat{\mathbf{f}}_1}{ds} = \rho_1 \hat{\mathbf{f}}_0 + \rho_2 \hat{\mathbf{f}}_2 \quad (16.109)$$

Continuing, we now consider $d\hat{\mathbf{f}}_2/ds$. Differentiating the expressions $\hat{\mathbf{f}}_2 \cdot \hat{\mathbf{f}}_2 = +1$, $\hat{\mathbf{f}}_2 \cdot \hat{\mathbf{f}}_1 = 0$, and $\hat{\mathbf{f}}_2 \cdot \hat{\mathbf{f}}_0 = 0$ with respect to s shows that $d\hat{\mathbf{f}}_2/ds$ has a zero dot product with $\hat{\mathbf{f}}_2$ and $\hat{\mathbf{f}}_0$, but obeys $(d\hat{\mathbf{f}}_2/ds) \cdot \hat{\mathbf{f}}_1 = -\rho_2$. Thus the fourvector $(d\hat{\mathbf{f}}_2/ds) + \rho_2 \hat{\mathbf{f}}_1$ is spacelike and has zero dot product with $\hat{\mathbf{f}}_2$, $\hat{\mathbf{f}}_1$, and $\hat{\mathbf{f}}_0$. The third curvature ρ_3 and third spacelike Serret–Frenet vector $\hat{\mathbf{f}}_3$ may then be defined by

$$\rho_3 = \pm \sqrt{\left(\frac{d\hat{\mathbf{f}}_2}{ds} + \rho_2 \hat{\mathbf{f}}_1\right) \cdot \left(\frac{d\hat{\mathbf{f}}_2}{ds} + \rho_2 \hat{\mathbf{f}}_1\right)} \quad \text{and} \quad \frac{d\hat{\mathbf{f}}_2}{ds} = -\rho_2 \hat{\mathbf{f}}_1 + \rho_3 \hat{\mathbf{f}}_3 \quad (16.110)$$

The curvatures ρ_1 and ρ_2 are taken to be non-negative. The sign of the curvature ρ_3 (which determines the direction of $\hat{\mathbf{f}}_3$) must be chosen to make the spacelike triad $\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3$ right handed.

The final step is to take the derivative $d\hat{\mathbf{f}}_3/ds$. Differentiating $\hat{\mathbf{f}}_3 \cdot \hat{\mathbf{f}}_3 = +1$ and $\hat{\mathbf{f}}_3 \cdot \hat{\mathbf{f}}_\mu = 0$ for $\mu = 0, 1, 2$ with respect to s then shows that $\left((d\hat{\mathbf{f}}_3/ds) + \rho_3 \hat{\mathbf{f}}_2 \right)$ is a spacelike vector having zero dot products with each of the four unit vectors $\hat{\mathbf{f}}_\mu$ for $\mu = 0, 1, 2, 3$. It follows that this vector must be null, leading to the final result

$$\frac{d\hat{\mathbf{f}}_3}{ds} = -\rho_3 \hat{\mathbf{f}}_2 \tag{16.111}$$

By construction, the Serret–Frenet vectors are orthonormal (in the sense of the Minkowski metric) and obey

$$\hat{\mathbf{f}}_\mu \cdot \hat{\mathbf{f}}_\nu = g_{\mu\nu} \tag{16.112}$$

for $\mu, \nu = 0, 1, 2, 3$, where $g_{\mu\nu}$ is the Minkowski metric defined in eqn (15.12).

The derivatives in eqns (16.108 – 16.111) may be summarized by the single dyadic equation

$$\frac{d\hat{\mathbf{f}}_\mu}{ds} = \mathbb{F} \cdot \hat{\mathbf{f}}_\mu \tag{16.113}$$

where

$$\mathbb{F} = \rho_1 (\hat{\mathbf{f}}_0 \wedge \hat{\mathbf{f}}_1) - \rho_2 (\hat{\mathbf{f}}_1 \wedge \hat{\mathbf{f}}_2) - \rho_3 (\hat{\mathbf{f}}_2 \wedge \hat{\mathbf{f}}_3) \tag{16.114}$$

Using the relation between dyadics and matrices described in Section 15.19 gives eqn (16.113) in the alternate matrix form

$$\frac{d\hat{\mathbf{f}}_\mu}{ds} = \sum_{\nu=0}^3 \hat{\mathbf{f}}_\nu F^\nu{}_\mu \quad \text{where} \quad F^\nu{}_\mu = \begin{pmatrix} 0 & \rho_1 & 0 & 0 \\ \rho_1 & 0 & -\rho_2 & 0 \\ 0 & \rho_2 & 0 & -\rho_3 \\ 0 & 0 & \rho_3 & 0 \end{pmatrix}_{\nu\mu} \tag{16.115}$$

16.18 Fermi–Walker Transport

The quadrad of unit vectors $\hat{\mathbf{f}}_\mu$ can be thought of as defining the walls of a small room that is being carried along the world line. The room has fourvector velocity $\mathbf{u} = c\hat{\mathbf{f}}_0$ defined by the unit tangent to the world line. In general, it may be accelerating and also rotating. We say that the room is being *Serret–Frenet transported* along the world line as β increases.

But we need a covariant definition of a different small room, one transported along the same world line but *without* the rotation. Such a room is said to be *Fermi–Walker transported* along the world line. The non-rotating room may be defined by a quadrad of unit fourvectors $\hat{\mathbf{g}}_\mu$ that obey the same transportation law as eqn (16.113) but with the ρ_2 and ρ_3 terms omitted. This quadrad is defined by

$$\frac{d\hat{\mathbf{g}}_\mu}{ds} = \rho_1 (\hat{\mathbf{f}}_0 \hat{\mathbf{f}}_1 - \hat{\mathbf{f}}_1 \hat{\mathbf{f}}_0) \cdot \hat{\mathbf{g}}_\mu \quad \text{and} \quad \hat{\mathbf{g}}_0 = \hat{\mathbf{f}}_0 = \frac{\mathbf{u}}{c} \tag{16.116}$$

with the initial condition that $\hat{\mathbf{g}}_\mu(\beta=0) = \hat{\mathbf{f}}_\mu(\beta=0)$. It is often seen in the literature in an equivalent form (Exercise 16.17) where \mathbf{u} and \mathbf{w} are the fourvector velocity and

acceleration, respectively,

$$\frac{d\hat{\mathbf{g}}_\mu}{d\tau} = \frac{1}{c^2} \{ \mathbf{u} (\mathbf{w} \cdot \hat{\mathbf{g}}_\mu) - \mathbf{w} (\mathbf{u} \cdot \hat{\mathbf{g}}_\mu) \} \tag{16.117}$$

Fermi–Walker transport can also be written as a dyadic multiplication

$$\frac{d\hat{\mathbf{g}}_\mu}{ds} = \mathbb{G} \cdot \hat{\mathbf{g}}_\mu \quad \text{where} \quad \mathbb{G} = \rho_1 \hat{\mathbf{f}}_0 \wedge \hat{\mathbf{f}}_1 = \frac{1}{c^3} \mathbf{u} \wedge \mathbf{w} \tag{16.118}$$

The Fermi–Walker transportation law, eqn (16.116), produces only those changes in the quadrad $\hat{\mathbf{g}}_\mu$ necessary to, (1) keep $\hat{\mathbf{g}}_0$ equal to \mathbf{u}/c so that the transported room shares the velocity of the world line at each β value, and (2) keep the quadrad orthonormal. Thus, for all β it remains true (Exercise 16.17) that

$$\hat{\mathbf{g}}_\mu \cdot \hat{\mathbf{g}}_\nu = g_{\mu\nu} \tag{16.119}$$

It is assumed in the special theory of relativity that a small room being Fermi–Walker transported along some world line is undergoing pure translation with no rotation. A small gyroscope with its axis aligned with one of the spacelike vectors $\hat{\mathbf{g}}_i$, and all acceleration applied at its center of mass so no torques act on it, will remain aligned with this same $\hat{\mathbf{g}}_i$ axis for all time.⁹⁷

For any β value, the orthonormal spacelike Serret–Frenet vectors $\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3$ span the same three-dimensional subspace as the orthonormal spacelike Fermi–Walker vectors $\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3$. It follows that there must be an orthogonal rotation matrix \mathbf{R} connecting them, as in the two equivalent expressions

$$\hat{\mathbf{f}}_i = \sum_{j=1}^3 \hat{\mathbf{g}}_j R_{ji} \quad \text{and} \quad \hat{\mathbf{g}}_j = \sum_{i=1}^3 \hat{\mathbf{f}}_i R_{ij}^T \tag{16.120}$$

The matrix \mathbf{R} represents the rotation of the $\hat{\mathbf{f}}_i$ basis vectors as seen in a room defined by the $\hat{\mathbf{g}}_j$.

To obtain the apparent angular velocity of this rotation, differentiate the first of eqn (16.120) and use eqns (16.113, 16.118), and the second of eqn (16.120) to write

$$\mathbb{F} \cdot \hat{\mathbf{f}}_i = \frac{d\hat{\mathbf{f}}_i}{ds} = \sum_{j=1}^3 \left(\frac{d\hat{\mathbf{g}}_j}{ds} R_{ji} + \hat{\mathbf{g}}_j \frac{dR_{ji}}{ds} \right) = \sum_{j=1}^3 (\mathbb{G} \cdot \hat{\mathbf{g}}_j) R_{ji} + \sum_{j=1}^3 \sum_{k=1}^3 \hat{\mathbf{f}}_k R_{kj}^T \frac{dR_{ji}}{ds} \tag{16.121}$$

The angular velocity matrix of the “rotated” basis $\hat{\mathbf{f}}_i$ relative to the “fixed” basis $\hat{\mathbf{g}}_j$ can

⁹⁷See Chapter 6 of Misner, Thorne and Wheeler (1973). The plausible assumptions must be made that the center of mass of a gyroscope is at its center and that its spin can be represented by a threevector \mathbf{S} , even though the definitions of center of mass and spin in Sections 16.16.4 and 16.16.6 are established only for isolated and hence non-accelerated systems.

be expressed in the $\hat{\mathbf{f}}_i$ system as (see Exercise 8.8)

$$\Omega'_{ij} = \left(\mathbf{R}^T \frac{d\mathbf{R}}{d\tau} \right)_{ij} = \sum_{k=1}^3 \varepsilon_{ikj} \omega'_k \tag{16.122}$$

Using the linearity of dyadic multiplication, eqn (16.121) then becomes

$$(\mathbb{F} - \mathbb{G}) \cdot \hat{\mathbf{f}}_i = \frac{1}{c} \sum_{k=1}^3 \hat{\mathbf{f}}_k \Omega'_{ki} = \frac{1}{c} \boldsymbol{\omega}_{\text{SF}} \times \hat{\mathbf{f}}_i \tag{16.123}$$

from which one obtains the angular velocity vector of the Serret–Frenet system as seen from the Fermi–Walker system

$$\boldsymbol{\omega}_{\text{SF}} = \sum_{i=1}^3 \omega'_i \hat{\mathbf{f}}_i = c \left(\rho_3 \hat{\mathbf{f}}_1 + \rho_2 \hat{\mathbf{f}}_3 \right) \tag{16.124}$$

The rate of change of the Serret–Frenet vectors with respect to proper time τ , as seen from a room defined by the non-rotating Fermi–Walker vectors, is then given by the threevector cross product

$$\left. \frac{d\hat{\mathbf{f}}_i}{d\tau} \right|_{\hat{\mathbf{g}}} = \boldsymbol{\omega}_{\text{SF}} \times \hat{\mathbf{f}}_i \tag{16.125}$$

where $i = 1, 2, 3$, and the subscript $\hat{\mathbf{g}}$ indicates that the derivatives are taken as if the Fermi–Walker vectors $\hat{\mathbf{g}}_\mu$ were constants.

The Serret–Frenet basis fourvectors are easy to calculate since they are obtained by a simple process of repeated differentiation along a world line. And the second of eqn (16.120) can then be used to write the Fermi–Walker basis vectors $\hat{\mathbf{g}}_j$. In some simple cases, this allows the Fermi–Walker basis fourvectors to be found explicitly, as is done in the next section. But, even when R_{ij} cannot be found in closed form, the angular velocity $\boldsymbol{\omega}_{\text{SF}}$ in eqn (16.124) can still provide useful information.

16.19 Example of Fermi–Walker Transport

The Serret–Frenet vectors can be used in some cases to calculate the Fermi–Walker basis vectors $\hat{\mathbf{g}}_\mu$ explicitly, as we now illustrate.

Suppose that a point is moving in a circle of radius a in the x - y plane of some inertial reference system, with fixed angular velocity ω . Taking the parameter β to be the coordinate time t , the world line can be viewed as a cylindrical spiral in the ct, x, y spacetime diagram

$$\mathbf{r}(t) = ct \hat{\mathbf{e}}_0 + a \cos(\omega t) \hat{\mathbf{e}}_1 + a \sin(\omega t) \hat{\mathbf{e}}_2 \tag{16.126}$$

The threevector velocity has constant magnitude $v = \omega a$, and the arc length is $s = ct = ct/\gamma$ where the Lorentz factor is $\gamma = (1 - (\omega a/c)^2)^{-1/2}$. The Serret–Frenet

vectors for this world line (Exercise 16.16) are $\hat{\mathbf{f}}_3 = \hat{\mathbf{e}}_3$ and

$$\hat{\mathbf{f}}_0 = \gamma \left(\hat{\mathbf{e}}_0 - \frac{a\omega}{c} \sin(\omega t) \hat{\mathbf{e}}_1 + \frac{a\omega}{c} \cos(\omega t) \hat{\mathbf{e}}_2 \right) \quad (16.127)$$

$$\hat{\mathbf{f}}_1 = - \left(\cos(\omega t) \hat{\mathbf{e}}_1 + \sin(\omega t) \hat{\mathbf{e}}_2 \right) \quad (16.128)$$

$$\hat{\mathbf{f}}_2 = \gamma \left(-\frac{a\omega}{c} \hat{\mathbf{e}}_0 + \sin(\omega t) \hat{\mathbf{e}}_1 - \cos(\omega t) \hat{\mathbf{e}}_2 \right) \quad (16.129)$$

and the curvatures are

$$\rho_1 = \frac{\gamma^2 a \omega^2}{c^2} \quad \rho_2 = \frac{\gamma^2 \omega}{c} \quad \rho_3 = 0 \quad (16.130)$$

The Fermi–Walker vectors can be calculated by noticing that, since $\rho_3=0$, and ρ_2 is constant, eqn (16.124) reduces to a rotation at a fixed rate about a fixed axis, $\boldsymbol{\omega}_{\text{SF}} = c\rho_2\hat{\mathbf{f}}_3 = c\rho_2\hat{\mathbf{g}}_3$. The rotation \mathbf{R} defined by eqn (16.120) thus can be obtained by integration, as in Section 8.18. It is $\mathbf{R} = \mathbf{R}[\phi\hat{\mathbf{g}}_3]$ where the angle is $\phi = c\rho_2\tau = \gamma\omega t$. The second of eqn (16.120) then reduces to

$$\hat{\mathbf{g}}_1 = \cos(\gamma\omega t) \hat{\mathbf{f}}_1 - \sin(\gamma\omega t) \hat{\mathbf{f}}_2 \quad \hat{\mathbf{g}}_2 = \sin(\gamma\omega t) \hat{\mathbf{f}}_1 + \cos(\gamma\omega t) \hat{\mathbf{f}}_2 \quad \hat{\mathbf{g}}_3 = \hat{\mathbf{f}}_3 \quad (16.131)$$

Putting the Serret–Frenet vectors from eqns (16.128, 16.129) into eqn (16.131), and recalling that $\hat{\mathbf{g}}_0 = \hat{\mathbf{f}}_0$ by definition, gives the Fermi–Walker vectors in terms of the inertial unit vectors. They are $\hat{\mathbf{g}}_3 = \hat{\mathbf{f}}_3 = \hat{\mathbf{e}}_3$ and

$$\hat{\mathbf{g}}_0 = \gamma \left(\hat{\mathbf{e}}_0 - \frac{a\omega}{c} \sin(\omega t) \hat{\mathbf{e}}_1 + \frac{a\omega}{c} \cos(\omega t) \hat{\mathbf{e}}_2 \right) \quad (16.132)$$

$$\begin{aligned} \hat{\mathbf{g}}_1 = \frac{\gamma a \omega}{c} \sin(\gamma\omega t) \hat{\mathbf{e}}_0 - \left(\cos(\gamma\omega t) \cos(\omega t) + \gamma \sin(\gamma\omega t) \sin(\omega t) \right) \hat{\mathbf{e}}_1 \\ - \left(\cos(\gamma\omega t) \sin(\omega t) - \gamma \sin(\gamma\omega t) \cos(\omega t) \right) \hat{\mathbf{e}}_2 \end{aligned} \quad (16.133)$$

$$\begin{aligned} \hat{\mathbf{g}}_2 = -\frac{\gamma a \omega}{c} \cos(\gamma\omega t) \hat{\mathbf{e}}_0 - \left(\sin(\gamma\omega t) \cos(\omega t) - \gamma \cos(\gamma\omega t) \sin(\omega t) \right) \hat{\mathbf{e}}_1 \\ - \left(\sin(\gamma\omega t) \sin(\omega t) + \gamma \cos(\gamma\omega t) \cos(\omega t) \right) \hat{\mathbf{e}}_2 \end{aligned} \quad (16.134)$$

If one assumes $v \ll c$ and expands these equations (Exercise 16.18) in powers of the small quantity $(\gamma - 1) \approx v^2/2c^2$, the leading term in $\hat{\mathbf{g}}_1$ will be

$$\hat{\mathbf{g}}_1 \approx -\cos(\omega_{\text{T}}t) \hat{\mathbf{e}}_1 + \sin(\omega_{\text{T}}t) \hat{\mathbf{e}}_2 \quad (16.135)$$

where

$$\omega_{\text{T}} = (\gamma - 1) \omega \approx a^2 \omega^3 / 2c^2 \quad (16.136)$$

A small, torque-free gyroscope transported along the spiral world line will remain aligned, for example, with the non-rotating $\hat{\mathbf{g}}_1$ unit vector. It will thus appear to the inertial system to be rotating in a negative sense about the $\hat{\mathbf{e}}_3$ axis with small, constant angular velocity ω_{T} . This retrograde rotation is called *Thomas precession*. When

applied to the small “gyroscope” consisting of the spin of the electron in the hydrogen atom, it reduces by a factor of two the effective spin-orbit coupling between the spin and orbital angular momenta.⁹⁸ (See Exercise 16.19 for a discussion of this effect in the context of the Bohr model.)

16.20 Exercises

Exercise 16.1 Prove that eqn (16.1) implies eqn (16.2), where $\mathbf{a}_{\parallel} = \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{a})$ and $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}$ are the components of \mathbf{a} parallel and perpendicular, respectively, to the current velocity threevector \mathbf{v} .

Exercise 16.2 Prove that

$$\frac{d}{dt} \left(\frac{mc^2}{\sqrt{1 - v^2/c^2}} \right) = \mathbf{v} \cdot \frac{d(\gamma m \mathbf{v})}{dt} \quad (16.137)$$

as is asserted in eqn (16.6).

Exercise 16.3 Use eqn (16.11) and the definitions of E and \mathbf{p} to derive eqn (16.12).

Exercise 16.4 The following exercise considers the Newtonian limits of relativistic quantities. Take $\delta > 0$ to be a small positive number. The meaning of the order symbol o is discussed in Section D.11.

(a) Show that, as $(v/c) \rightarrow 0$,

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = m\mathbf{v} \left(1 + \frac{v^2}{2c^2} + o\left((v/c)^{(4-\delta)}\right) \right) \quad (16.138)$$

and hence

$$\frac{\mathbf{p}}{mc} = \frac{\mathbf{v}}{c} \left(1 + \frac{v^2}{2c^2} + o\left((v/c)^{(4-\delta)}\right) \right) \quad (16.139)$$

(b) Show that, as $(v/c) \rightarrow 0$,

$$T = E - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \frac{1}{2}mv^2 \left(1 + \frac{3}{4} \left(\frac{v}{c} \right)^2 + o\left((v/c)^{(4-\delta)}\right) \right) \quad (16.140)$$

$$T = E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2 = \frac{p^2}{2m} \left(1 - \frac{1}{4} \left(\frac{p}{mc} \right)^2 + o\left((v/c)^{(4-\delta)}\right) \right) \quad (16.141)$$

(c) Show that $(\mathbf{v}/c) = (\mathbf{p}c/E)$ exactly, for all v/c .

⁹⁸This factor of two had to be put in by hand in the nonrelativistic treatment of the hydrogen atom that preceded the development of the Dirac equation. It is discussed in many introductory quantum texts. See, for example, Section 17.3 of Shankar (1994).

Exercise 16.5

(a) Prove the fourvector equivalents of the identities proved for threevectors in Section A.11. Prove that

$$\frac{\partial}{\partial \mathbf{r}} (\mathbf{r} \cdot \mathbf{C}) = \mathbf{C} \quad \text{and} \quad \frac{\partial}{\partial \dot{\mathbf{r}}} (\dot{\mathbf{r}} \cdot \mathbf{C}) = \mathbf{C} \quad (16.142)$$

(b) Show that

$$\frac{\partial}{\partial \dot{\mathbf{r}}} \left(\sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \right) = \frac{\dot{\mathbf{r}}}{\sqrt{-\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \quad \text{and} \quad \frac{\partial}{\partial \dot{\mathbf{r}}} \left(\frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A} \right) = \frac{q}{c} \mathbf{A} \quad (16.143)$$

which were used in eqn (16.48).

Exercise 16.6

(a) Use conservation of fourvector momentum to show that an isolated electron cannot emit a single photon.

(b) An excited atom at rest in S emits two photons of energies E_1 and E_2 . Denote the mass of the atom before the emissions by m_0 and the mass after by m . Show that, in the case in which the photons emerge in exactly opposite directions, the atom's mass is decreased by the ratio

$$\frac{m}{m_0} = \sqrt{\left(1 - \frac{2E_1}{m_0 c^2}\right) \left(1 - \frac{2E_2}{m_0 c^2}\right)} \quad (16.144)$$

(c) Show that if it also happens that $E_1 = E_2$, then the change in the atom's mass $\Delta m = m - m_0$ is given by

$$\Delta m = -\frac{E_1 + E_2}{c^2} \quad (16.145)$$

Exercise 16.7 Use eqn (16.37) and the definitions of E and \mathbf{p} to verify the results stated in eqns (16.44, 16.45).

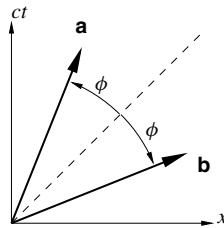


FIG. 16.2. Figure for Exercise 16.8. The fourvectors \mathbf{a} and \mathbf{b} have zero relativistic dot product. The dotted line is the projection of the light cone onto the x - ct plane.

Exercise 16.8 Suppose that $\mathbf{a} = a^0 \hat{\mathbf{e}}_0 + a^1 \hat{\mathbf{e}}_1$ is forward timelike, and that $\mathbf{b} = b^0 \hat{\mathbf{e}}_0 + b^1 \hat{\mathbf{e}}_1$ has $\mathbf{b} \cdot \mathbf{a} = 0$.

(a) Show that, when drawn on a spacetime diagram, \mathbf{a} makes an angle ϕ with the forward light cone where

$$\tan \phi = \frac{a^0 - a^1}{a^0 + a^1} \quad (16.146)$$

(b) Show that \mathbf{b} makes the same angle ϕ with the forward light cone, but on the opposite side.

- (c) Plot **a** and **b** schematically on a spacetime diagram for the case $a^1 < 0$. (Figure 16.2 shows the case $a^1 > 0$.)
- (d) Show that $0 < \phi < \pi/2$ with the end values approached as **a** approaches the lightlike limits ($|a^1|/a^0 \rightarrow 1$).

Exercise 16.9 Figure 16.1 shows the hyperplane of simultaneity defined in Section 16.16.1 when $\hat{\mathbf{e}} : (e^0, e^1, 0, 0)_S$.

- (a) Show that two events in the hyperplane have a separation $d\mathbf{r}$ that has a zero dot product with $\hat{\mathbf{e}}$. Show, using Exercise 16.8, that the plane and the orientation vector $\hat{\mathbf{e}}$ make equal angles above and below the light cone, as shown.
- (b) Show that if a clock has a world line parallel to **e** and passing through the origin event, then θ is the proper time measured on that clock as it moves from the origin event to **r**.
- (c) Show that if S' is a system in which $\hat{\mathbf{e}} : (1, 0, 0, 0)_{S'}$ then the coordinate time t' in system S' is equal to θ .

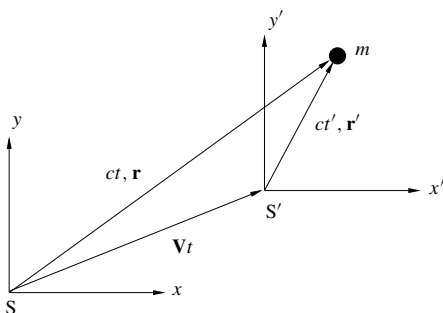


FIG. 16.3. Figure for Exercise 16.10. The particle m has coordinate velocity $\mathbf{v} = d\mathbf{r}/dt$ as seen from S and $\mathbf{v}' = d\mathbf{r}'/dt'$ as seen from S' . The systems S and S' are connected by a boost of threector velocity \mathbf{V} .

Exercise 16.10 Suppose that a particle has threector velocity \mathbf{v}' as seen from the S' system and \mathbf{v} as seen from S . Assume S and S' related by a general boost parameterized by the threector velocity \mathbf{V} between the systems.

- (a) Show that

$$\mathbf{v}_{\parallel} \sim \frac{\mathbf{V}' + \mathbf{v}'_{\parallel}}{(1 + Vv'_{\parallel}/c^2)} \quad \mathbf{v}_{\perp} \sim \frac{\mathbf{v}'_{\perp}}{\Gamma(1 + Vv'_{\parallel}/c^2)} \quad (16.147)$$

where $\mathbf{v}_{\parallel} = \hat{\mathbf{V}}(\hat{\mathbf{V}} \cdot \mathbf{v})$ is the component of \mathbf{v} parallel to \mathbf{V} and $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$ is the component perpendicular to \mathbf{V} , with similar definitions in the S' system.

Exercise 16.11

- (a) Show that when a boost has threector parameter $\mathbf{V} : (V, 0, 0)_S$, it reduces to the standard Lorentz transformation defined in Section 15.11.1.
- (b) Show that for a standard Lorentz transformation, the velocity addition in Exercise 16.10

reduces to

$$v_x = \frac{V + v'_x}{(1 + Vv'_x/c^2)} \quad v_y = \frac{v'_y}{\Gamma(1 + Vv'_x/c^2)} \quad v_z = \frac{v'_z}{\Gamma(1 + Vv'_x/c^2)} \quad (16.148)$$

(c) Suppose that the particle is a photon with $v' = c$ and $\mathbf{v}' = c(\cos \phi' \hat{\mathbf{e}}_1 + \sin \phi' \hat{\mathbf{e}}_2)$. Show that $v = c$ and $\mathbf{v} = c(\cos \phi \hat{\mathbf{e}}_1 + \sin \phi \hat{\mathbf{e}}_2)$ where

$$\cos \phi = \frac{\cos \phi' + V/c}{1 + V \cos \phi'/c} \quad \text{and} \quad \sin \phi = \frac{\sin \phi' / \Gamma}{1 + V \cos \phi'/c} \quad (16.149)$$

(d) If $\phi' = \pi/2$, show that $\cos \phi = V/c$ and $\sin \phi = \sqrt{1 - V^2/c^2}$.

Exercise 16.12 Referring to Exercise 16.10, we make the following definitions: $V/c = \tanh H$, $v_{\parallel}/c = \tanh \eta$, and $v'_{\parallel}/c = \tanh \eta'$. The quantities H, η, η' are called *rapidities* or *rapidity measures*.

(a) Show that the matrix eqn (15.62) of the standard Lorentz transformation can be written as

$$\mathbf{M}_{st} = \begin{pmatrix} \cosh H & \sinh H & 0 & 0 \\ \sinh H & \cosh H & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (16.150)$$

(b) Show that when a general boost is performed as in eqn (16.147), the rapidities add, with $\eta = H + \eta'$.

(c) Show that the velocity transformation in eqn (16.147) can be written as

$$\mathbf{v}_{\parallel} = c \hat{\mathbf{V}} \tanh \eta = c \hat{\mathbf{V}} \tanh (H + \eta') \quad \mathbf{v}_{\perp} \sim \mathbf{v}'_{\perp} \left(\frac{1}{\cosh H + \sinh H \tanh \eta'} \right) \quad (16.151)$$

Exercise 16.13 The power source of the Cassini spacecraft uses small cylinders of ^{238}Pu (in the form of plutonium dioxide) to produce heat. The half life of ^{238}Pu is 87.7 years, decaying predominantly by alpha decay to ^{234}U . Taking the masses to be $m_{238} = 238.049560$ amu, $m_{\alpha} = 4.002603$ amu, and $m_{234} = 234.040952$ amu, calculate how many grams of ^{238}Pu are required to produce 1 watt of power.

Exercise 16.14 This exercise treats what is called *hyperbolic motion*. Suppose that, at any instant, a point mass m has threevector acceleration $\mathbf{a}' = g \hat{\mathbf{e}}_1$ in a system S' in which m is instantaneously at rest, where g is a given constant.

(a) Derive the following values for the given fourvector dot products,

$$\mathbf{w} \cdot \mathbf{w} = g^2 \quad \mathbf{w} \cdot \mathbf{u} = 0 \quad \mathbf{u} \cdot \mathbf{u} = -c^2 \quad (16.152)$$

where \mathbf{u} is the velocity fourvector, and \mathbf{w} is the acceleration fourvector.

(b) Consider a fixed system S relative to which the mass m has threevector velocity $\mathbf{v} = v_1 \hat{\mathbf{e}}_1$.

Assume that S and S' are related by a standard Lorentz transformation. Prove that, relative to this S system, the mass has velocity fourvector components

$$u^0 = c \cosh\left(\frac{g\tau}{c}\right) \quad u^1 = c \sinh\left(\frac{g\tau}{c}\right) \quad u^2 = u^3 = 0 \quad (16.153)$$

where initial condition $u^1 = 0$ at $\tau = 0$ is assumed and τ is the proper time along the world line of the mass.

(c) Integrate again, assuming that $t = 0$ and $x = x_0$ at $\tau = 0$, to obtain the following coordinates for the mass as seen from the S system

$$x = x_0 + \frac{c^2}{g} \left\{ \cosh\left(\frac{g\tau}{c}\right) - 1 \right\} \quad ct = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c}\right) \quad (16.154)$$

(d) Prove that the threevector velocity relative to the S system is $\mathbf{v} = \hat{\mathbf{e}}_1 c \tanh(g\tau/c)$ and that the dilation factor $\gamma = dt/d\tau$ is $\gamma = \cosh(g\tau/c)$.

(e) Show that, still relative to the S system, the position of the mass can be expressed as a function of the coordinate time t as follows,

$$x = x_0 + \frac{c^2}{g} \left(\sqrt{1 + \left(\frac{gt}{c}\right)^2} - 1 \right) \quad (16.155)$$

(f) Show that the threevector force \mathbf{f} on the mass, as measured relative to the S system, must be $\mathbf{f} = mg\hat{\mathbf{e}}_1$, and that the threevector velocity is $\mathbf{v} = (gt/\gamma)\hat{\mathbf{e}}_1$ where the time dilation factor is $\gamma = \sqrt{1 + (gt/c)^2}$.

Exercise 16.15 Show that $[\Psi] = \mathbf{D}[\Psi']$ and eqn (16.77) imply the form invariance of the Dirac equation shown in eqn (16.78).

Exercise 16.16

(a) Apply the method of Section 16.17 to the example world line in eqn (16.126). Show that your results agree with eqns (16.127 – 16.130).

(b) Verify that your vectors $\hat{\mathbf{f}}_\mu$ obey eqn (16.112).

Exercise 16.17

(a) Use the definitions in Section 15.7 to show that the Fermi–Walker transport law eqn (16.116) can be written as eqn (16.117).

(b) Show that the definition $\hat{\mathbf{g}}_0 = \hat{\mathbf{f}}_0$ gives a $d\hat{\mathbf{g}}_0/ds$ that is a solution to the Fermi–Walker transport law in eqn (16.116).

(c) Show that the dyadic \mathbb{G} defined in eqn (16.118) has the property that $\mathbf{b} \cdot \mathbb{G} = -\mathbb{G} \cdot \mathbf{b}$ for any fourvector \mathbf{b} . Show that $d(\hat{\mathbf{g}}_\mu \cdot \hat{\mathbf{g}}_\nu)/ds = 0$ follows. Show that the assumed initial condition that the $\hat{\mathbf{g}}_\mu$ are equal to the $\hat{\mathbf{f}}_\mu$ at $\beta=0$ then implies that $\hat{\mathbf{g}}_\mu \cdot \hat{\mathbf{g}}_\nu = g_{\mu\nu}$ for all β .

Exercise 16.18

(a) The quantity $\Delta = \gamma - 1$ will be small when the velocity along a world line is small compared to the speed of light. Substitute $\gamma = \Delta + 1$ into eqns (16.132 – 16.134) and simplify the resulting equations (now depending on Δ and not on γ).

(b) Show that, considering Δ to be a small quantity, the leading term in the expansion of $\hat{\mathbf{g}}_1$ in powers of Δ is that given in eqn (16.135).

Exercise 16.19 In its simplest form, the Bohr model of the hydrogen atom consists of an electron of mass m and charge $-e$ moving in a plane, circular orbit around a fixed proton of charge e . The n th Bohr orbit is defined by its orbital angular momentum $L_n = n\hbar$. One assumes that, except for this quantization of angular momentum, the orbits are classically determined.

(a) Show that, in the Newtonian limit, the radius, speed, and total energy of the n th Bohr orbit are: $r_n = (\hbar/\alpha mc)n^2$, $v_n = \alpha c/n$, and $E_n = -\alpha^2 mc^2/2n^2$ where $\alpha = e^2/4\pi\hbar c \approx (1/137)$ is the Sommerfeld fine structure constant.

(b) Let an observer ride on the electron, in a small laboratory that is Fermi–Walker transported along its world line. Use Exercise 15.16 to find the magnetic induction field $\mathbf{B}_0 = B_0\hat{\mathbf{g}}_3$ (in the instantaneous rest frame of the electron) that results from the electron’s motion through the electric field of the proton. Assuming small velocities and keeping only leading order terms in v_n/c , use this B_0 and the results of Exercise 9.12 to derive a rate of precession for the electron spin in its own rest frame $\boldsymbol{\omega}_e = (\alpha^4 mc^2 \hat{\mathbf{g}}_3) / (\hbar n^5)$. This precession results from the interaction (called the *spin-orbit interaction*) between the electron’s intrinsic magnetic moment and the local \mathbf{B}_0 coming from its orbital motion.

(c) Show that for every n value, the Thomas angular velocity calculated approximately in eqn (16.136) is retrograde from the direction of $\boldsymbol{\omega}_e$, and has magnitude $\omega_T = \omega_e/2$. The *effective* rate of precession of the electron spin as seen by inertial observers thus will be $\boldsymbol{\omega}_{\text{eff}} = \omega_{\text{eff}}\hat{\mathbf{e}}_3$ where $\omega_{\text{eff}} = \omega_e - \omega_T = \omega_e/2$. The quantum mechanical treatment of the hydrogen atom, before the Dirac equation was developed, required a spin-orbit interaction term to be added to the Schroedinger equation in order to predict the fine structure correctly. But, just as the Bohr model predicts, the spin-orbit term had to be reduced by a factor of two to get experimental agreement. The factor of two is called the *Thomas factor*.

(d) Should we be surprised, or should we not be surprised, that the Thomas precession ω_T and the electromagnetic precession ω_e of the electron spin in each orbit of the Bohr atom differ by exactly a factor of two?⁹⁹

Exercise 16.20 Suppose that \mathbf{a} is an arbitrary fourvector and that $\Phi = \mathbf{a} \cdot \mathbf{b}$ is known to be an invariant, where $\mathbf{b} = b^0\hat{\mathbf{e}}_0 + \mathbf{b}$ is a set of quantities written in fourvector form.

(a) Demonstrate that \mathbf{b} is indeed a fourvector.

(b) The phase of a plane wave may be written as $\Phi = \mathbf{k} \cdot \mathbf{r} - \omega t$. Assume that the phase of a wave is a quantity obtained by counting wave crests and hence is a relativistic invariant. Since $\mathbf{r} = ct\hat{\mathbf{e}}_0 + \mathbf{r}$ is a fourvector, show that $\mathbf{k} = (\omega/c)\hat{\mathbf{e}}_0 + \mathbf{k}$ must also be a fourvector.

(c) Show that the deBroglie relations in eqn (12.62) may be written as $\mathbf{p} = \hbar\mathbf{k}$.

⁹⁹The Bohr model correctly predicts the Rydberg energy $\alpha^2 mc^2/2$ and the Thomas factor. However, in spite of spawning corporate logos with electrons in orbit around nuclei, it is plainly an incorrect model: (1) The lowest state $n = 1$ has angular momentum $L_1 = \hbar$, whereas the Schroedinger theory predicts $L = 0$ for the ground state. (2) Motion in a plane violates the uncertainty principle since it would require $\Delta z \Delta p_z = 0$.

CANONICAL TRANSFORMATIONS

The extended Lagrange equations were shown in Section 11.8 to be form invariant under coordinate transformations in which the new generalized coordinates, including the new time coordinate, are functions of the old coordinates and time. This was a generalization of the traditional Lagrangian invariance proof of Section 2.10 which did not allow the time to transform along with the other coordinates.

We now consider a broader class of transformations, called *canonical transformations*, which transform the whole of the extended phase space in a more general way. Each new canonical coordinate or momentum is allowed to be a function of *all* of the previous phase-space coordinates, including the previous canonical momenta. Thus the new position and time variables may depend, through their dependence on the momenta, on the old velocities as well as the old positions.

The Lagrange equations will not in general be form invariant under such transformations. But canonical transformations will preserve the form of the extended Hamilton equations developed in Chapter 12. In fact, as will be proved later in this chapter, canonical transformations are the most general phase-space transformations that preserve the extended Hamilton equations.

There are several equivalent definitions of canonical transformations. We present three of them here, the Poisson Bracket Condition, the Direct Condition, and the Lagrange Bracket Condition. And each of these three conditions has two forms, a long one which is written out in terms of partial derivatives, and a symplectic one consisting of a single matrix equation. We begin with the long form of the Lagrange Bracket Condition, and then, after introducing some necessary notation, derive the long and symplectic forms of all three.

Since we are now operating in an extended phase space in which the time is the transformable canonical coordinate q_0 , the definition of canonical transformation includes the Lorentz transformation of special relativity.

17.1 Definition of Canonical Transformations

We begin with the $(2D + 2)$ phase-space canonical coordinates listed in eqn (12.1)

$$q, p = q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D \quad (17.1)$$

and then transform to a new set of $(2D + 2)$ coordinates

$$Q, P = Q_0, Q_1, \dots, Q_D, P_0, P_1, \dots, P_D \quad (17.2)$$

using transformation equations of the form

$$Q_k = Q_k(q, p) = Q_k(q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D) \quad (17.3)$$

$$P_k = P_k(q, p) = P_k(q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D) \quad (17.4)$$

for $k = 0, \dots, D$.

In order to be a canonical transformation, the transformation equations must satisfy certain conditions. The motivation for these conditions will become clear as the chapter progresses. We begin with the most common, and in general the most useful, condition and then prove the other conditions equivalent to this one.

Definition 17.1.1: Poisson Bracket Condition – Long Form

The transformation $q, p \rightarrow Q, P$ defined by eqns (17.3, 17.4) is a canonical transformation if and only if the three equations

$$[Q_k, Q_l]_{qp} = 0 \quad [Q_k, P_l]_{qp} = \delta_{kl} \quad [P_k, P_l]_{qp} = 0 \quad (17.5)$$

are satisfied for all $k, l = 0, \dots, D$, where the Poisson brackets in eqn (17.5) are those defined in eqn (12.51) with f and g replaced by the various Q, P coordinates.

For example, the middle equation in eqn (17.5) can be written out as

$$[Q_k, P_l]_{qp} = \sum_{k'=0}^D \left(\frac{\partial Q_k(q, p)}{\partial q_{k'}} \frac{\partial P_l(q, p)}{\partial p_{k'}} - \frac{\partial P_l(q, p)}{\partial q_{k'}} \frac{\partial Q_k(q, p)}{\partial p_{k'}} \right) = \delta_{kl} \quad (17.6)$$

We attach the subscript qp here to emphasize that these brackets are to be evaluated using partial derivatives with respect to the q, p system of canonical coordinates.

17.2 Example of a Canonical Transformation

The special case of a transformation in which the Q_k depend only on the q_k is called a canonical transformation of the Lagrangian type. Suppose that such a transformation also has a simple linear form, with

$$Q_i = \sum_{k=0}^D a_{ik} q_k \quad P_i = \sum_{k=0}^D b_{ik} p_k \quad (17.7)$$

for $i = 0, \dots, D$, where \mathbf{a} and \mathbf{b} are constant matrices. Then, as done in Exercise 17.1, eqn (17.5) can be used to show that this transformation is canonical if and only if $\mathbf{a} \mathbf{b}^T = \mathbf{U}$. The Lorentz transformation is such a special case, and is treated in Exercise 17.2.

17.3 Symplectic Coordinates

Before continuing with canonical transformations, we must now digress to introduce some necessary notation. The development of canonical transformations will be simplified by the adoption of a single symbol for all of the canonical coordinates in eqn

(17.1). Define the coordinates γ_i for $i = 0, \dots, (2D + 1)$ by

$$\gamma_k = q_k \quad \text{and} \quad \gamma_{D+1+k} = p_k \tag{17.8}$$

for $k = 0, \dots, D$. Hence

$$\gamma_0, \gamma_1, \dots, \gamma_D, \gamma_{D+1}, \gamma_{D+2}, \dots, \gamma_{2D+1} = q_0, q_1, \dots, q_D, p_0, p_1, \dots, p_D \tag{17.9}$$

or, in a shortened form, $\gamma = q, p$. These γ_i will be referred to as the *symplectic coordinates* of phase space.

A similar definition is made for the Q, P coordinates.

$$\Gamma_k = Q_k \quad \text{and} \quad \Gamma_{D+1+k} = P_k \tag{17.10}$$

for $k = 0, \dots, D$, so that

$$\Gamma_0, \Gamma_1, \dots, \Gamma_D, \Gamma_{D+1}, \Gamma_{D+2}, \dots, \Gamma_{2D+1} = Q_0, Q_1, \dots, Q_D, P_0, P_1, \dots, P_D \tag{17.11}$$

or $\Gamma = Q, P$.

The transformation equations, eqns (17.3, 17.4), can then be expressed as a single equation. For all $i = 0, \dots, (2D + 1)$,

$$\Gamma_i = \Gamma_i(\gamma) = \Gamma_i(\gamma_0, \gamma_1, \dots, \gamma_D, \gamma_{D+1}, \gamma_{D+2}, \dots, \gamma_{2D+1}) \tag{17.12}$$

and we can then define a $(2D + 2) \times (2D + 2)$ Jacobian matrix of this transformation \mathbf{J} by writing its matrix elements as

$$J_{ij} = \frac{\partial \Gamma_i(\gamma)}{\partial \gamma_j} \tag{17.13}$$

for all $i, j = 0, \dots, (2D + 1)$. By construction, this matrix will have the form of four $(D + 1) \times (D + 1)$ blocks

$$\mathbf{J} = \begin{pmatrix} \left(\frac{\partial Q}{\partial q} \right) & \left(\frac{\partial Q}{\partial p} \right) \\ \left(\frac{\partial P}{\partial q} \right) & \left(\frac{\partial P}{\partial p} \right) \end{pmatrix} \tag{17.14}$$

where the four block matrices are defined, for all $k, l = 0, \dots, D$, by

$$\begin{aligned} \left(\frac{\partial Q}{\partial q} \right)_{kl} &= \frac{\partial Q_k(q, p)}{\partial q_l} & \left(\frac{\partial Q}{\partial p} \right)_{kl} &= \frac{\partial Q_k(q, p)}{\partial p_l} \\ \left(\frac{\partial P}{\partial q} \right)_{kl} &= \frac{\partial P_k(q, p)}{\partial q_l} & \left(\frac{\partial P}{\partial p} \right)_{kl} &= \frac{\partial P_k(q, p)}{\partial p_l} \end{aligned} \tag{17.15}$$

Derivatives with respect to the general parameter β will be denoted with a dot above the symbol, as was done in Chapters 11 and 12, and will be referred to as

generalized velocities of symplectic space. Thus $\dot{\gamma}_i = d\gamma_i/d\beta$. It will be useful to define column vectors of these derivatives as

$$[\dot{\gamma}] = \begin{pmatrix} \dot{\gamma}_0 \\ \vdots \\ \dot{\gamma}_D \\ \dot{\gamma}_{D+1} \\ \vdots \\ \dot{\gamma}_{2D+1} \end{pmatrix} = \begin{pmatrix} [\dot{q}] \\ [\dot{p}] \end{pmatrix} \quad (17.16)$$

where the last column vector expresses the $(2D+2)$ -rowed column vector $[\dot{\gamma}]$ in block form, in terms of the two $(D+1)$ -rowed column vectors

$$[\dot{q}] = \begin{pmatrix} \dot{q}_0 \\ \vdots \\ \dot{q}_D \end{pmatrix} \quad \text{and} \quad [\dot{p}] = \begin{pmatrix} \dot{p}_0 \\ \vdots \\ \dot{p}_D \end{pmatrix} \quad (17.17)$$

Functions of phase-space variables can also be rewritten as functions of the γ , with $f = f(\gamma) = f(q, p)$ obtained by simple substitution of the definition eqn (17.8) into $f(q, p)$. Then gradient column vectors can be defined as

$$\left[\frac{\partial f}{\partial \gamma} \right] = \begin{pmatrix} \frac{\partial f(\gamma)}{\partial \gamma_0} \\ \vdots \\ \frac{\partial f(\gamma)}{\partial \gamma_D} \\ \frac{\partial f(\gamma)}{\partial \gamma_{D+1}} \\ \vdots \\ \frac{\partial f(\gamma)}{\partial \gamma_{2D+1}} \end{pmatrix} = \begin{pmatrix} \left[\frac{\partial f}{\partial q} \right] \\ \left[\frac{\partial f}{\partial p} \right] \end{pmatrix} \quad (17.18)$$

where the last column vector expresses the $(2D+2)$ -rowed column vector in block form, in terms of the two $(D+1)$ -rowed column vectors

$$\left[\frac{\partial f}{\partial q} \right] = \begin{pmatrix} \frac{\partial f(q, p)}{\partial q_0} \\ \vdots \\ \frac{\partial f(q, p)}{\partial q_D} \end{pmatrix} \quad \text{and} \quad \left[\frac{\partial f}{\partial p} \right] = \begin{pmatrix} \frac{\partial f(q, p)}{\partial p_0} \\ \vdots \\ \frac{\partial f(q, p)}{\partial p_D} \end{pmatrix} \quad (17.19)$$

Definitions like eqns (17.16 – 17.19) will also be made for the Q, P variables, so that

$$[\dot{\Gamma}] = \begin{pmatrix} \dot{\Gamma}_0 \\ \vdots \\ \dot{\Gamma}_D \\ \dot{\Gamma}_{D+1} \\ \vdots \\ \dot{\Gamma}_{2D+1} \end{pmatrix} = \begin{pmatrix} [\dot{Q}] \\ [\dot{P}] \end{pmatrix} \tag{17.20}$$

and

$$\left[\frac{\partial f}{\partial \Gamma} \right] = \begin{pmatrix} \frac{\partial f(\Gamma)}{\partial \Gamma_0} \\ \vdots \\ \frac{\partial f(\Gamma)}{\partial \Gamma_D} \\ \frac{\partial f(\Gamma)}{\partial \Gamma_{D+1}} \\ \vdots \\ \frac{\partial f(\Gamma)}{\partial \Gamma_{2D+1}} \end{pmatrix} = \begin{pmatrix} \left[\frac{\partial f}{\partial Q} \right] \\ \left[\frac{\partial f}{\partial P} \right] \end{pmatrix} \tag{17.21}$$

with definitions of the constituent blocks similar to those in eqns (17.17, 17.19).

A phase-space function f is assumed to be represented by $f(q, p)$ or $f(Q, P)$ in the q, p or Q, P systems of coordinates, respectively. These representations are related by the equations¹⁰⁰

$$f(Q, P) = f(q(Q, P), p(Q, P)) \quad f(q, p) = f(Q(q, p), P(q, p)) \tag{17.22}$$

or, in symplectic notation

$$\begin{aligned} f(\Gamma) &= f(\gamma(\Gamma)) = f(\gamma_0(\Gamma), \dots, \gamma_{2D+1}(\Gamma)) \\ f(\gamma) &= f(\Gamma(\gamma)) = f(\Gamma_0(\gamma), \dots, \Gamma_{2D+1}(\gamma)) \end{aligned} \tag{17.23}$$

Thus, for example, $f(\gamma)$ can be obtained by starting with $f(\Gamma)$ and then simply substituting $\Gamma_i = \Gamma_i(\gamma)$ from eqn (17.12) into it. As discussed in Section D.5, the function $f(Q, P) = f(q, p)$, or $f(\Gamma) = f(\gamma)$ in symplectic notation, is considered to be the same function, just expressed in the two different coordinate systems.

The main use of this symplectic notation is to write the transformations of various quantities in a simple matrix form, as for example in the following lemma.

¹⁰⁰It is proved in Section 17.7 that, for canonical transformations, the equations $\Gamma_i = \Gamma_i(\gamma)$ can always be inverted to give $\gamma_i = \gamma_i(\Gamma)$.

Lemma 17.3.1: Transformation of Symplectic Quantities

Given the definitions in eqns (17.8 – 17.21), it follows that

$$[\dot{\Gamma}] = \mathbf{J} [\dot{\gamma}] \quad \text{and} \quad \left[\frac{\partial f}{\partial \gamma} \right] = \mathbf{J}^T \left[\frac{\partial f}{\partial \Gamma} \right] \quad (17.24)$$

Proof: The chain rule and eqn (17.13) give

$$\frac{d\Gamma_i}{d\beta} = \sum_{j=0}^{2D+1} \frac{\partial \Gamma_i(\gamma)}{\partial \gamma_j} \frac{d\gamma_j}{d\beta} = \sum_{j=0}^{2D+1} J_{ij} \frac{d\gamma_j}{d\beta} \quad (17.25)$$

which is the first of eqn (17.24) in component form, as was to be proved.

The chain rule and eqn (17.13) also give

$$\frac{\partial f(\gamma)}{\partial \gamma_i} = \sum_{j=0}^{2D+1} \frac{\partial f(\Gamma)}{\partial \Gamma_j} \frac{\partial \Gamma_j(\gamma)}{\partial \gamma_i} = \sum_{j=0}^{2D+1} J_{ij}^T \frac{\partial f(\Gamma)}{\partial \Gamma_j} \quad (17.26)$$

which is the second of eqn (17.24) in component form, as was to be proved. \square

17.4 Symplectic Matrix

The extended Hamilton equations and the definition of Poisson brackets can be written in useful matrix forms if we use the symplectic coordinates defined in Section 17.4 and a matrix \mathbf{s} , which will be referred to as the *symplectic matrix* and is defined as follows.

Definition 17.4.1: Symplectic Matrix

The $(2D + 2) \times (2D + 2)$ symplectic matrix \mathbf{s} is defined as the block matrix¹⁰¹

$$\mathbf{s} = \begin{pmatrix} \mathbf{0} & \mathbf{U} \\ -\mathbf{U} & \mathbf{0} \end{pmatrix} \quad (17.27)$$

where $\mathbf{0}$ is the $(D + 1) \times (D + 1)$ null matrix (all zeroes) and \mathbf{U} is the $(D + 1) \times (D + 1)$ unit matrix.

For example, when $D = 1$, the \mathbf{s} is

$$\mathbf{s} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (17.28)$$

This matrix has a number of interesting properties.

¹⁰¹If we extend the definition of the Kronecker delta function to include positive, negative, and zero integers, then the matrix elements of \mathbf{s} can be written as $s_{ij} = \delta_{i(j-D-1)} - \delta_{i(j+D+1)}$ for all $i, j = 0, \dots, (2D + 1)$.

Lemma 17.4.2: Properties of Symplectic Matrix

The symplectic matrix \mathbf{s} is skew-symmetric and orthogonal. Its determinant is $+1$.

Proof: By construction \mathbf{s} has the property $\mathbf{s}^T = -\mathbf{s}$ and hence is skew-symmetric. Direct multiplication of \mathbf{s} by itself shows that $\mathbf{s} \mathbf{s} = -\mathbf{U}$ where \mathbf{U} now is the $(2D + 2) \times (2D + 2)$ unit matrix. Thus

$$\mathbf{s}^T \mathbf{s} = (-\mathbf{s}) \mathbf{s} = \mathbf{U} = \mathbf{s} (-\mathbf{s}) = \mathbf{s} \mathbf{s}^T \quad (17.29)$$

which shows that \mathbf{s} is orthogonal, as was to be proved. To evaluate the determinant, note that $(D + 1)$ exchanges of rows (0 with $D + 1$, then 1 with $D + 2$, and so on) will reduce $|\mathbf{s}|$ to the determinant

$$|\mathbf{s}| = (-1)^{(D+1)} \begin{vmatrix} -\mathbf{U} & \mathbf{0} \\ \mathbf{0} & +\mathbf{U} \end{vmatrix} = (-1)^{(D+1)} (-1)^{(D+1)} = +1 \quad (17.30)$$

as was to be proved. □

17.5 Standard Equations in Symplectic Form

The motivation for the introduction of symplectic coordinates will now become clearer. Among other uses, they allow the Hamilton equations, and the Poisson bracket of two functions, to be written as simple matrix expressions.

The extended Hamilton equations are given in eqn (12.13). They are

$$\dot{q}_k = \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}(q, p)}{\partial q_k} \quad (17.31)$$

for all $k = 0, \dots, D$. These equations can be written in symplectic form.

Lemma 17.5.1: Hamilton Equations in Symplectic Form

Both of the extended Hamilton equations, eqn (17.31), can be written in symplectic form as the single equation

$$[\dot{\gamma}] = \mathbf{s} \left[\frac{\partial \mathcal{K}}{\partial \gamma} \right] \quad (17.32)$$

Proof: Expanded in block form using eqns (17.16, 17.18, 17.27), eqn (17.32) becomes

$$\begin{pmatrix} [\dot{q}] \\ [\dot{p}] \end{pmatrix} = \begin{pmatrix} \mathbf{0} & +\mathbf{U} \\ -\mathbf{U} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \left[\frac{\partial \mathcal{K}}{\partial q} \right] \\ \left[\frac{\partial \mathcal{K}}{\partial p} \right] \end{pmatrix} = \begin{pmatrix} + \left[\frac{\partial \mathcal{K}}{\partial p} \right] \\ - \left[\frac{\partial \mathcal{K}}{\partial q} \right] \end{pmatrix} \quad (17.33)$$

Equating each matrix element on both sides of eqn (17.33) reproduces eqn (17.31), as was to be proved. □

The definition of the extended Poisson bracket of two phase-space functions is given in eqn (12.51). It is

$$[f, g]_{qp} = \sum_{k=0}^D \left(\frac{\partial f(q, p)}{\partial q_k} \frac{\partial g(q, p)}{\partial p_k} - \frac{\partial g(q, p)}{\partial q_k} \frac{\partial f(q, p)}{\partial p_k} \right) \quad (17.34)$$

where we have added the subscript qp here to emphasize that the partial derivatives are to be taken with respect to the variables of the q, p system. This quantity can also be written in symplectic notation.

Lemma 17.5.2: Poisson Brackets in Symplectic Form

The extended Poisson bracket defined in eqn (17.34) can be written as

$$[f, g]_{qp} = \left[\frac{\partial f}{\partial \gamma} \right]^T \mathbf{s} \left[\frac{\partial g}{\partial \gamma} \right] \quad (17.35)$$

Proof: When expanded in block form, using eqns (17.18, 17.27), eqn (17.35) becomes

$$\begin{aligned} \left(\left[\frac{\partial f}{\partial q} \right]^T \left[\frac{\partial f}{\partial p} \right]^T \right) \begin{pmatrix} \mathbf{0} + \mathbf{U} \\ -\mathbf{U} \quad \mathbf{0} \end{pmatrix} \begin{pmatrix} \left[\frac{\partial g}{\partial q} \right] \\ \left[\frac{\partial g}{\partial p} \right] \end{pmatrix} &= \left(\left[\frac{\partial f}{\partial q} \right]^T \left[\frac{\partial f}{\partial p} \right]^T \right) \begin{pmatrix} + \left[\frac{\partial g}{\partial p} \right] \\ - \left[\frac{\partial g}{\partial q} \right] \end{pmatrix} \\ &= \left[\frac{\partial f}{\partial q} \right]^T \left[\frac{\partial g}{\partial p} \right] - \left[\frac{\partial f}{\partial p} \right]^T \left[\frac{\partial g}{\partial q} \right] \end{aligned} \quad (17.36)$$

When written out in terms of components using eqn (17.19), this becomes eqn (17.34), as was to be proved. \square

17.6 Poisson Bracket Condition

The symplectic notation will allow easy matrix proofs of the equivalence of the various conditions for a transformation to be canonical. This avoids what would otherwise be pages of formulas involving sums of partial derivatives. To begin, we restate the Poisson Bracket Condition from Section 17.1 in symplectic form.

Lemma 17.6.1: Poisson Bracket Condition – Short Form

The Poisson Bracket Condition for the transformation¹⁰² to be canonical is equivalent to the single equation

$$\mathbf{J} \mathbf{s} \mathbf{J}^T = \mathbf{s} \quad (17.37)$$

Thus the transformation is canonical if and only if this equation is satisfied.

¹⁰²For the remainder of this chapter, the transformation defined in eqns (17.3, 17.4), or alternately in eqn (17.12), will be referred to simply as “the transformation.”

Proof: The three equations of the Poisson bracket condition in eqn (17.5) can be combined into the single equation

$$[\Gamma_i, \Gamma_j]_{qp} = s_{ij} \tag{17.38}$$

for all $i, j = 0, \dots, (2D + 1)$, where the definitions in eqn (17.11) have been used. Using Lemma 17.5.2 with f, g replaced by Γ_i, Γ_j , this may be written as

$$\left[\frac{\partial \Gamma_i}{\partial \gamma} \right]^T \mathbf{s} \left[\frac{\partial \Gamma_j}{\partial \gamma} \right] = \sum_{k=0}^{(2D+1)} \sum_{l=0}^{(2D+1)} \frac{\partial \Gamma_i(\gamma)}{\partial \gamma_k} s_{kl} \frac{\partial \Gamma_j(\gamma)}{\partial \gamma_l} = s_{ij} \tag{17.39}$$

Introducing the definition of the Jacobian matrix from eqn (17.13) and using $J_{jl} = J_{lj}^T$, this becomes

$$\sum_{k=0}^{(2D+1)} \sum_{l=0}^{(2D+1)} J_{ik} s_{kl} J_{lj}^T = s_{ij} \tag{17.40}$$

which is just eqn (17.37) written out in terms of components. □

17.7 Inversion of Canonical Transformations

The matrix form of the Poisson bracket condition allows us to prove that canonical transformations are invertible.

Lemma 17.7.1: Nonsingularity of Jacobian Matrix

The Jacobian matrix of a canonical transformation is nonsingular. Its determinant is ± 1 . Thus eqns (17.3, 17.4) can be inverted, giving

$$q_k = q_k(Q, P) = q_k(Q_0, Q_1, \dots, Q_D, P_0, P_1, \dots, P_D) \tag{17.41}$$

$$p_k = p_k(Q, P) = p_k(Q_0, Q_1, \dots, Q_D, P_0, P_1, \dots, P_D) \tag{17.42}$$

or, in the equivalent symplectic form, eqn (17.12) can be inverted to give

$$\gamma_i = \gamma_i(\Gamma) = \gamma_i(\Gamma_0, \Gamma_1, \dots, \Gamma_D, \Gamma_{D+1}, \Gamma_{D+2}, \dots, \Gamma_{2D+1}) \tag{17.43}$$

Proof: Taking the determinant of both sides of eqn (17.37) and using the result $|\mathbf{s}| = 1$ from Lemma 17.4.2,

$$|\mathbf{J}| |\mathbf{s}| |\mathbf{J}^T| = |\mathbf{s}| \quad \text{and hence} \quad |\mathbf{J}|^2 = 1 \tag{17.44}$$

with the result $|\mathbf{J}| = \pm 1$, as was to be proved. It follows from the inverse function theorem, Theorem D.24.1, that eqn (17.12) can be inverted to give eqn (17.43), as was to be proved. □

Since the Jacobian matrix \mathbf{J} is nonsingular, it has an inverse \mathbf{J}^{-1} . The matrix elements of that inverse matrix are therefore

$$J_{ij}^{-1} = \frac{\partial \gamma_i(\Gamma)}{\partial \Gamma_j} \quad (17.45)$$

and it can be written in a block form similar to that in eqn (17.14) for the original transformation,

$$\mathbf{J}^{-1} = \begin{pmatrix} \left(\frac{\partial q}{\partial Q} \right) & \left(\frac{\partial q}{\partial P} \right) \\ \left(\frac{\partial p}{\partial Q} \right) & \left(\frac{\partial p}{\partial P} \right) \end{pmatrix} \quad (17.46)$$

where the four block matrices are defined, for all $k, l = 0, \dots, D$, by

$$\begin{aligned} \left(\frac{\partial q}{\partial Q} \right)_{kl} &= \frac{\partial q_k(Q, P)}{\partial Q_l} & \left(\frac{\partial q}{\partial P} \right)_{kl} &= \frac{\partial q_k(Q, P)}{\partial P_l} \\ \left(\frac{\partial p}{\partial Q} \right)_{kl} &= \frac{\partial p_k(Q, P)}{\partial Q_l} & \left(\frac{\partial p}{\partial P} \right)_{kl} &= \frac{\partial p_k(Q, P)}{\partial P_l} \end{aligned} \quad (17.47)$$

The nonsingularity of the Jacobian matrix \mathbf{J} has an important consequence. Since both t and the parameter β vary monotonically, the time parameter always has $\dot{t} \neq 0$ along any system trajectory. But after a canonical transformation, the new variables Q, P may have no easily identifiable connection to the monotonically varying time coordinate t . However, we do know that at least one of them must have a nonzero derivative at any system point.

Lemma 17.7.2: Non-Vanishing Derivatives of Q and P

After the most general canonical transformation, there will always be at least one member of the set Q_k, P_k for $k = 0, \dots, D$ which has a nonzero derivative with respect to parameter β .

Proof: Since \mathbf{J} is nonsingular and has a inverse \mathbf{J}^{-1} , the first of eqn (17.24) may be used to write

$$[\dot{\gamma}] = \mathbf{J}^{-1}[\dot{\Gamma}] \quad (17.48)$$

If all of the quantities \dot{Q}_k, \dot{P}_k were zero, then $[\dot{\Gamma}]$ would be the null column vector. Then eqn (17.48) would imply that $[\dot{\gamma}] = [0]$ in contradiction to $\dot{q}_0 = \dot{t} \neq 0$. Thus at least one of the \dot{Q}_k, \dot{P}_k must be nonzero. \square

17.8 Direct Condition

The inverse \mathbf{J}^{-1} that was proved to exist in Lemma 17.7.1 figures in the second of the conditions for a transformation to be canonical, the *Direct Condition*.

Lemma 17.8.1: Direct Condition – Short Form

The transformation is canonical if and only if \mathbf{J}^{-1} exists and

$$\mathbf{J}^{-1} = -\mathbf{s} \mathbf{J}^T \mathbf{s} \quad (17.49)$$

Proof: First prove that eqn (17.49) implies eqn (17.37). Multiplying both sides of eqn (17.49) from the left by J and from the right by s gives

$$J J^{-1} s = -J s J^T s \tag{17.50}$$

Since $J J^{-1} = U$ and $s^2 = -U$ from Lemma 17.4.2, eqn (17.50) reduces to eqn (17.37), as was to be proved.

Conversely, assume that eqn (17.37) holds. Then Lemma 17.7.1 has shown that J^{-1} exists. Multiply eqn (17.37) from the left by $s J^{-1}$ to give

$$s J^{-1} J s J^T = s J^{-1} s \quad \text{and hence} \quad -J^T = s J^{-1} s$$

Multiplying both sides from the left and right by s and once again using $s^2 = -U$ then gives eqn (17.49), as was to be proved. \square

The Direct Condition also has a long form. We derive it here from the symplectic short form just presented.

Lemma 17.8.2: Direct Condition – Long Form

The transformation is canonical if and only if it is invertible, so that eqns (17.41, 17.42) can be written, and the following four equalities hold for all $k, l = 0, \dots, D$:

$$\frac{\partial q_k(Q, P)}{\partial Q_l} = \frac{\partial P_l(q, p)}{\partial p_k} \qquad \frac{\partial q_k(Q, P)}{\partial P_l} = -\frac{\partial Q_l(q, p)}{\partial p_k} \tag{17.51}$$

$$\frac{\partial p_k(Q, P)}{\partial Q_l} = -\frac{\partial P_l(q, p)}{\partial q_k} \qquad \frac{\partial p_k(Q, P)}{\partial P_l} = \frac{\partial Q_l(q, p)}{\partial q_k} \tag{17.52}$$

Proof: Using eqns (17.14, 17.27, 17.46), eqn (17.49) may be written as

$$\begin{aligned} \begin{pmatrix} \left(\frac{\partial q}{\partial Q} \right) & \left(\frac{\partial q}{\partial P} \right) \\ \left(\frac{\partial p}{\partial Q} \right) & \left(\frac{\partial p}{\partial P} \right) \end{pmatrix} &= - \begin{pmatrix} 0 & +U \\ -U & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial Q}{\partial q} \right)^T & \left(\frac{\partial P}{\partial q} \right)^T \\ \left(\frac{\partial Q}{\partial p} \right)^T & \left(\frac{\partial P}{\partial p} \right)^T \end{pmatrix} \begin{pmatrix} 0 & +U \\ -U & 0 \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{\partial P}{\partial p} \right)^T & -\left(\frac{\partial Q}{\partial p} \right)^T \\ -\left(\frac{\partial P}{\partial q} \right)^T & \left(\frac{\partial Q}{\partial q} \right)^T \end{pmatrix} \end{aligned} \tag{17.53}$$

Equating the upper left blocks in the first and last matrices in this expression gives

$$\left(\frac{\partial q}{\partial Q} \right) = \left(\frac{\partial P}{\partial p} \right)^T \tag{17.54}$$

which, when written out, is the first of eqn (17.51). The other three equations in eqns (17.51, 17.52) are obtained by equating the other corresponding blocks, as was to be proved. \square

As written, the left sides of eqns (17.51, 17.52) are functions of Q, P and the right sides of q, p . Thus, eqns (17.3, 17.4) or eqns (17.41, 17.42) must be used to write both sides as functions of the same variable set before the equalities can be checked.

17.9 Lagrange Bracket Condition

The third condition is the Lagrange Bracket Condition. We prove its symplectic form first.

Lemma 17.9.1: Lagrange Bracket Condition – Short Form

The transformation is canonical if and only if

$$\mathbf{J}^T \mathbf{s} \mathbf{J} = \mathbf{s} \quad (17.55)$$

Proof: We prove that this equation implies and is implied by eqn (17.49). First assume that eqn (17.49) holds. Then multiplying both sides of eqn (17.49) from the left by \mathbf{s} and from the right by \mathbf{J} gives

$$\mathbf{s} \mathbf{J}^{-1} \mathbf{J} = -\mathbf{s}^2 \mathbf{J}^T \mathbf{s} \mathbf{J} \quad (17.56)$$

Since $\mathbf{J} \mathbf{J}^{-1} = \mathbf{U}$ and $\mathbf{s}^2 = -\mathbf{U}$ from Lemma 17.4.2, the eqn (17.56) reduces to eqn (17.55) as was to be proved.

Conversely, assume that eqn (17.55) holds. Then an argument similar to that in the proof of Lemma 17.7.1 shows that \mathbf{J} has an inverse. Multiply eqn (17.55) from the left by \mathbf{s} and from the right by \mathbf{J}^{-1} to obtain

$$\mathbf{s} \mathbf{J}^T \mathbf{s} \mathbf{J} \mathbf{J}^{-1} = \mathbf{s}^2 \mathbf{J}^{-1} \quad (17.57)$$

which reduces similarly to eqn (17.49), as was to be proved. \square

The long form of the Lagrange Bracket Condition can also be obtained.

Lemma 17.9.2: Lagrange Bracket Condition – Long Form

The transformation is canonical if and only if

$$\{\gamma_i, \gamma_j\}_{Q,P} = s_{ij} \quad (17.58)$$

for all $i, j = 0, \dots, (2D + 1)$ where the expressions in eqn (17.58) are called Lagrange Brackets and are defined by

$$\{\gamma_i, \gamma_j\}_{Q,P} = \sum_{k=0}^D \left(\frac{\partial Q_k(q, p)}{\partial \gamma_i} \frac{\partial P_k(q, p)}{\partial \gamma_j} - \frac{\partial Q_k(q, p)}{\partial \gamma_j} \frac{\partial P_k(q, p)}{\partial \gamma_i} \right) \quad (17.59)$$

In the q, p notation, eqn (17.58) becomes

$$\{q_k, q_l\}_{Q,P} = 0 \quad \{q_k, p_l\}_{Q,P} = \delta_{kl} \quad \{p_k, p_l\}_{Q,P} = 0 \quad (17.60)$$

for all $k, l = 0, \dots, D$.

Proof: When written out in terms of components, using the definition in eqn (17.13), eqn (17.55) becomes

$$\sum_{k=0}^{(2D+1)} \sum_{l=0}^{(2D+1)} \frac{\partial \Gamma_k(\gamma)}{\partial \gamma_i} s_{kl} \frac{\partial \Gamma_l(\gamma)}{\partial \gamma_j} = s_{ij} \tag{17.61}$$

Performing the sum over l using the definitions in eqns (17.27, 17.59), this becomes eqn (17.58). □

Corollary 17.9.3: Lagrange and Poisson Brackets

The Poisson and Lagrange brackets of canonical transformations obey the identity

$$\sum_{k=0}^{(2D+1)} \{ \gamma_k, \gamma_i \}_{Q,P} [\Gamma_k, \Gamma_j]_{qp} = \delta_{ij} \tag{17.62}$$

for all $i, j = 0, \dots, (2D + 1)$.

Proof: Using eqns (17.38, 17.58), the left side of eqn (17.62) becomes

$$\sum_{k=0}^{(2D+1)} s_{ki} s_{kj} = \sum_{k=0}^{(2D+1)} s_{ik}^T s_{kj} = \left(\mathbf{s}^T \mathbf{s} \right)_{ij} = \delta_{ij} \tag{17.63}$$

where the last equality follows from the orthogonality of \mathbf{s} proved in Lemma 17.4.2. □

17.10 The Canonical Group

In Section 8.7 we discussed the concept of groups of transformations and showed that rotations form a group. We now apply the same ideas to canonical transformations.

The first requirement is to define what is meant by group *multiplication*. Just as for rotation operators and Lorentz transformations, we define multiplication to mean repeated application. If A and B are canonical transformations, then a multiplication $C = AB$ will mean that B is applied and then A is applied to the result. Thus if $B : q, p \rightarrow Q, P$ is the transformation from q, p to Q, P and $A : Q, P \rightarrow X, Y$ is the transformation from Q, P to X, Y , then $C : q, p \rightarrow X, Y$ will be the transformation from q, p to X, Y that is the cumulative effect of these two successive transformations. Canonical transformations form a group under this form of group multiplication because they satisfy the same axioms as those listed in Section 8.7.

1. The first property of groups is closure. If A and B are canonical, then $C = AB$ must also be canonical. To see that this is true, consider the Jacobian matrix defined as in eqn (17.13) for each of the transformations. Let Θ_i be the symplectic coordinate of the system with coordinates X_k and momenta Y_k , defined

similarly to the definition of Γ_i in eqn (17.10). Then, using the chain rule,

$$\frac{\partial \Theta_i(\gamma)}{\partial \gamma_j} = \sum_{k=0}^{(2D+1)} \frac{\partial \Theta_i(\Gamma)}{\partial \Gamma_k} \frac{\partial \Gamma_k(\gamma)}{\partial \gamma_j} \quad \text{or} \quad \mathbf{J}^{(C)} = \mathbf{J}^{(A)} \mathbf{J}^{(B)} \quad (17.64)$$

in matrix form. Then the short form of the Poisson bracket condition in eqn (17.37) is satisfied for transformation C since

$$\mathbf{J}^{(C)} \mathbf{s} \left(\mathbf{J}^{(C)} \right)^T = \mathbf{J}^{(A)} \mathbf{J}^{(B)} \mathbf{s} \left(\mathbf{J}^{(B)} \right)^T \left(\mathbf{J}^{(A)} \right)^T = \mathbf{J}^{(A)} \mathbf{s} \left(\mathbf{J}^{(A)} \right)^T = \mathbf{s} \quad (17.65)$$

where we have used the assumption that both $\mathbf{J}^{(A)}$ and $\mathbf{J}^{(B)}$ satisfy this condition. Thus the product of two canonical transformations is a canonical transformation and the group property of closure is established.

2. Continuing as in Section 8.7, the next property is the existence of an identity. In the case of canonical transformations, the identity is the transformation $Q_k = q_k$ and $P_k = p_k$ whose Jacobian matrix is just the identity matrix $\mathbf{J} = \mathbf{U}$. This matrix satisfies the Poisson bracket condition, eqn (17.37), since $\mathbf{U} \mathbf{s} \mathbf{U}^T = \mathbf{s}$ follows from the definition of \mathbf{U} . Thus the identity transformation is canonical, which establishes the existence of an identity for the group.
3. Every member of a group, that is every canonical transformation in our case, must have an inverse that is also a member of the group. We have seen in Section 17.7 that canonical transformations are invertible and so have an inverse. We must now show that this inverse is canonical. Suppose that a transformation is canonical with a Jacobian matrix \mathbf{J} satisfying the Poisson bracket condition eqn (17.37). Taking the inverse of both sides of eqn (17.37) gives

$$\left(\mathbf{J}^T \right)^{-1} \mathbf{s}^{-1} \mathbf{J}^{-1} = \mathbf{s}^{-1} \quad \text{or} \quad \left(\mathbf{J}^{-1} \right)^T \mathbf{s} \left(\mathbf{J}^{-1} \right) = \mathbf{s} \quad (17.66)$$

where we used the result from Section 17.4 that $\mathbf{s}^{-1} = -\mathbf{s}$. Thus the inverse transformation, with Jacobian transformation matrix \mathbf{J}^{-1} satisfies the Lagrange bracket condition in eqn (17.55) and therefore is canonical and a member of the group.

4. The fourth required property is associativity. As in the case of rotations, it follows at once from our definition of multiplication as successive application.

17.11 Form Invariance of Poisson Brackets

We are familiar with the fact that the dot product of two vectors in a three-dimensional Cartesian space is unchanged by rotation of the coordinates of that space. In phase space, the Poisson bracket of two phase-space functions has a similar property. In fact, canonical transformations can be defined as the most general phase-space transformations that leave the extended Poisson brackets invariant.

To examine this invariance, we must first define the Poisson bracket in terms of the transformed variables Q, P .

Definition 17.11.1: Poisson Bracket in the Q,P System

The definition of the Poisson bracket of two phase-space functions $f(Q, P)$ and $g(Q, P)$ in the transformed Q, P system of coordinates uses the same form as eqn (17.34), but with Q, P substituted for q, p . It is

$$[f, g]_{QP} = \sum_{k=0}^D \left(\frac{\partial f(Q, P)}{\partial Q_k} \frac{\partial g(Q, P)}{\partial P_k} - \frac{\partial g(Q, P)}{\partial Q_k} \frac{\partial f(Q, P)}{\partial P_k} \right) \tag{17.67}$$

where the subscript QP indicates that the partials here are taken with respect to the transformed variable set.

The definition eqn (17.67) can also be written in symplectic form. Using the definition in eqn (17.21), the Poisson bracket in the Q, P system can be written as

$$[f, g]_{QP} = \left[\frac{\partial f}{\partial \Gamma} \right]^T \mathbf{s} \left[\frac{\partial g}{\partial \Gamma} \right] \tag{17.68}$$

The proof is the same as in Lemma 17.5.2, and will not be repeated here.

The expressions in eqns (17.67, 17.68) have the same form as eqns (17.34, 17.35), respectively. The only difference is the substitution of Q, P for q, p . The following theorem shows that they also have the same value. Such expressions, in which the same algebraic form in two systems yields the same value, are called *form invariant* expressions.

Theorem 17.11.2: Form Invariance of Poisson Brackets

The Poisson bracket of any two phase-space functions is form invariant

$$[f, g]_{QP} = [f, g]_{q,p} \tag{17.69}$$

if and only if the transformation $q, p \rightarrow Q, P$ is canonical.

Proof: First, assume eqn (17.69) and prove the transformation canonical. Since eqn (17.69) is assumed to hold for any phase-space functions, choose $f = Q_i$ and $g = Q_j$. Then

$$0 = [Q_i, Q_j]_{QP} = [Q_i, Q_j]_{qp} \tag{17.70}$$

where the first equality follows trivially from eqn (17.67) when the partial derivatives of the Q_i, Q_j are taken with respect to the Q, P system. This equation is identical to the first of eqn (17.5), the Poisson bracket condition for the transformation to be canonical. The other two equations in eqn (17.5) are proved similarly. Thus the transformation is canonical, as was to be proved.

Conversely, assume that the transformation is canonical and prove that eqn (17.69) holds. Substituting the transformation equations, eqn (17.24), the symplectic form in eqn (17.35) becomes

$$[f, g]_{qp} = \left[\frac{\partial f}{\partial \gamma} \right]^T \mathbf{s} \left[\frac{\partial g}{\partial \gamma} \right] = \left(\mathbf{J}^T \left[\frac{\partial f}{\partial \Gamma} \right] \right)^T \mathbf{s} \left(\mathbf{J}^T \left[\frac{\partial g}{\partial \Gamma} \right] \right) = \left[\frac{\partial f}{\partial \Gamma} \right] \mathbf{J} \mathbf{s} \mathbf{J}^T \left[\frac{\partial g}{\partial \Gamma} \right] \tag{17.71}$$

Since the transformation is assumed canonical, eqn (17.37) holds. Substituting this

result and using eqn (17.68) then gives

$$[f, g]_{qp} = \left[\frac{\partial f}{\partial \Gamma} \right]^T \mathfrak{s} \left[\frac{\partial g}{\partial \Gamma} \right] = [f, g]_{QP} \quad (17.72)$$

as was to be proved. \square

17.12 Form Invariance of the Hamilton Equations

We come now to the most important property of canonical transformations: They preserve the form of the extended Hamilton equations.

The present section proves two related theorems. First, we prove that a canonical transformation from a q, p system in which the extended Hamilton equations hold produces a Q, P system in which extended Hamilton equations of the same form hold, and with the same extended Hamiltonian \mathcal{K} . And, second, we prove that the equivalence of the Hamilton equations in the two systems is a sufficient condition for the transformation between them to be canonical.

Together, these two theorems imply that a transformation is canonical *if and only if* it preserves the form of the extended Hamilton equations. This result is an example of the superiority of the extended approach to Hamiltonian theory. It allows us to state that canonical transformations in the extended phase space are *the most general* phase-space transformations that preserve the extended Hamilton equations. A similar result could not be proved in the traditional theory, since multiplication of the time by a constant and division of the traditional Hamiltonian by that same constant preserves the traditional Hamilton equations (see Exercise 17.3). But such a transformation is not canonical in the traditional theory because it modifies time. In the extended theory, time is allowed to transform and such problem cases are included in the general definition of canonical transformations.

Before presenting the two theorems, however, it will be useful to emphasize the relation between canonical transformations and the extended Hamiltonian theory. Canonical transformations are nonsingular, invertible transformations between two independent sets of Hamiltonian variables $q, p \rightarrow Q, P$. They treat all of these variables equally. But, as noted in Section 12.4, the q, p are subject to a condition, $\mathcal{K}(q, p) = 0$, that is to be applied at the end of the calculation, after the partial derivatives of the Hamilton equations have been taken. When a canonical transformation is done, the new variables simply *inherit* that condition. It becomes $\mathcal{K}(Q, P) = 0$, where $\mathcal{K}(Q, P)$ is the *same function* (in the sense discussed in Section D.5) as $\mathcal{K}(q, p)$, but expressed in the new variable set.

Theorem 17.12.1: Canonical Transformation Implies Invariance

If the extended Hamilton equations from eqn (17.31)

$$\dot{q}_k = \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial \mathcal{K}(q, p)}{\partial q_k} \quad (17.73)$$

hold in the q, p system, and if the transformation $q, p \rightarrow Q, P$ is canonical, then the

extended Hamilton equations

$$\dot{Q}_k = \frac{\partial \mathcal{K}(Q, P)}{\partial P_k} \quad \text{and} \quad \dot{P}_k = -\frac{\partial \mathcal{K}(Q, P)}{\partial Q_k} \quad (17.74)$$

hold in the Q, P system, where the extended Hamiltonian $\mathcal{K}(Q, P)$ is the same function as the $\mathcal{K}(q, p)$ used there, simply re-expressed in terms of the Q, P coordinates by making the substitution

$$\mathcal{K}(Q, P) = \mathcal{K}(q(Q, P), p(Q, P)) \quad (17.75)$$

The parameter β is not transformed. It is the same in eqns (17.73, 17.74).

Proof: Since Lemma 17.7.1 proved the Jacobian transformation matrix \mathbf{J} to be nonsingular, canonical transformations are invertible and the substitution in eqn (17.75) can always be done.

We have shown in Lemma 17.5.1 that eqn (17.73) is equivalent to the symplectic form of the Hamilton equations in eqn (17.32). Similarly, both of the extended Hamilton equations, eqn (17.74), can be written in symplectic form as the single equation

$$[\dot{\Gamma}] = \mathbf{s} \begin{bmatrix} \partial \mathcal{K} \\ \partial \Gamma \end{bmatrix} \quad (17.76)$$

So we only need to prove the equivalence of the two symplectic forms, that

$$[\dot{\gamma}] = \mathbf{s} \begin{bmatrix} \partial \mathcal{K} \\ \partial \gamma \end{bmatrix} \quad \text{is equivalent to} \quad [\dot{\Gamma}] = \mathbf{s} \begin{bmatrix} \partial \mathcal{K} \\ \partial \Gamma \end{bmatrix} \quad (17.77)$$

The inverse of the nonsingular matrix \mathbf{J} may be used to write the transformation rules in eqn (17.24) as

$$[\dot{\gamma}] = \mathbf{J}^{-1} [\dot{\Gamma}] \quad \text{and} \quad \begin{bmatrix} \partial \mathcal{K} \\ \partial \gamma \end{bmatrix} = \mathbf{J}^T \begin{bmatrix} \partial \mathcal{K} \\ \partial \Gamma \end{bmatrix} \quad (17.78)$$

Applying these rules to the first of eqn (17.77) gives

$$\mathbf{J}^{-1} [\dot{\Gamma}] = \mathbf{s} \mathbf{J}^T \begin{bmatrix} \partial \mathcal{K} \\ \partial \Gamma \end{bmatrix} \quad (17.79)$$

Multiplying both sides by \mathbf{J} then gives

$$[\dot{\Gamma}] = \mathbf{J} \mathbf{s} \mathbf{J}^T \begin{bmatrix} \partial \mathcal{K} \\ \partial \Gamma \end{bmatrix} \quad (17.80)$$

Substituting the symplectic form of the Poisson Bracket Condition from eqn (17.37) then gives the second of eqn (17.77), as was to be proved. \square

The converse proof states that any transformation that preserves the form of the extended Hamilton equations must be canonical.

Theorem 17.12.2: Invariance Implies Canonical Transformation

If a transformation is invertible so that $\mathcal{K}(Q, P)$ can be defined by eqn (17.75), and if eqn (17.73) are equivalent to eqn (17.74) for any extended Hamiltonian and any choice of initial conditions, then the transformation $q, p \rightarrow Q, P$ is canonical.

Proof: As in Theorem 17.12.1, we will work entirely with the symplectic forms of the Hamilton equations.

By assumption, the first and second equations in eqn (17.77) are both true. Assuming the transformation invertible, and using the same transformations as in the proof of Theorem 17.12.1, the first of eqn (17.77) can be written as eqn (17.80). Subtracting the second of eqn (17.77) from this equation then gives

$$0 = \left(\mathbf{J} \mathbf{s} \mathbf{J}^T - \mathbf{s} \right) \left[\frac{\partial \mathcal{K}}{\partial \Gamma} \right] = \left(\mathbf{J} \mathbf{s} \mathbf{J}^T - \mathbf{s} \right) \left(\mathbf{J}^{-1} \right)^T \left[\frac{\partial \mathcal{K}}{\partial \gamma} \right] \quad (17.81)$$

The second equality used eqn (17.24) and the assumption that the transformation is invertible and hence has a nonsingular Jacobian matrix \mathbf{J} .

Assuming an extended Hamiltonian of the standard form in eqn (12.12), the column vector $[\partial \mathcal{K} / \partial \gamma]$ has an $i = (D + 1)$ component that is always equal to one. But the components from 0 to D are partial derivatives with respect to the q_k , which are shown by eqn (17.73) to be related to the \dot{p}_k . The potential and initial conditions can be chosen so that these quantities are nonzero one at a time. Also, the components from $i = (D + 2)$ to $(2D + 1)$ are partial derivatives with respect to the p_k for $k \neq 0$, and produce the generalized velocities \dot{q}_k . By suitable choice of the initial conditions, these values can also be set to be nonzero one at a time. Thus, eqn (17.81) holds for all of a linearly independent set of column vectors $[\partial \mathcal{K} / \partial \gamma]$. Since any column vector can be expanded in such a set, it follows that

$$0 = \left(\mathbf{J} \mathbf{s} \mathbf{J}^T - \mathbf{s} \right) \left(\mathbf{J}^{-1} \right)^T \quad (17.82)$$

Applying \mathbf{J}^T from the right on both sides then gives eqn (17.37), the symplectic Poisson Bracket Condition for the transformation to be canonical, as was to be proved.

It was shown in Section 12.8 that the extended Hamiltonian is not unique. Lemma 12.8.1 showed that it can be multiplied by any non-vanishing function $g(q, p)$. However, such a modification of \mathcal{K} will not affect the proof given here, since the function g will cancel from eqn (17.81). The proof here also applies with the modification in Lemma 12.8.2, with the nonzero \dot{q}_0 replaced by the assumedly nonzero \dot{q}_l described there. \square

17.13 Traditional Canonical Transformations

In many textbooks, the definition of canonical transformation is still the traditional one based on the Newtonian absolute time. The traditional definition implicitly includes the condition $Q_0 = q_0 = t$ which requires that time not transform. Thus the Lorentz transformation of special relativity is not considered canonical. The extended

definition presented here corrects this omission by allowing the time to transform along with the other coordinates of extended phase space. And the extended definition also includes the traditional one as a special case.

The traditional form of canonical transformations defines the transformation, for $k = 1, \dots, D$, as

$$T = Q_0 = q_0 = t \quad Q_k = Q_k(q_{[0]}, p_{[0]}, t) \quad P_k = P_k(q_{[0]}, p_{[0]}, t) \quad (17.83)$$

Note that the new coordinates and moment are allowed to depend on the time $t = q_0$ but not explicitly on the original Hamiltonian $-p_0$. The necessary conditions for the transformation to be canonical in the traditional sense are then, for all $k, l = 1, \dots, D$,

$$[Q_k, Q_l]^{(\text{trad})} = 0 \quad [Q_k, P_l]^{(\text{trad})} = \delta_{kl} \quad [P_k, P_l]^{(\text{trad})} = 0 \quad (17.84)$$

where the superscript (trad) denotes the traditional definition of Poisson bracket in the q, p system as given in eqn (4.53).

The traditional theory is rendered unnecessarily complicated by its privileged treatment of the time. Equations (17.83) do *not* completely specify the transformation. The traditional Hamiltonian function *does* transform, but that transformation is not included in eqn (17.83). If we denote the traditional Hamiltonian in the q, p system by $h(q_{[0]}, p_{[0]}, t)$ and the traditional Hamiltonian in the Q, P system by $H(Q_{[0]}, P_{[0]}, T)$, then the implicit transformation equation of the traditional theory is

$$H = h + g(q_{[0]}, p_{[0]}, t) \quad \text{or} \quad P_0 = p_0 - g(q, p_{[0]}) \quad (17.85)$$

where the function g must be a solution to the equations, for $k = 1, \dots, D$,

$$\frac{\partial Q_k(q_{[0]}, p_{[0]}, t)}{\partial t} = [Q_k, g]^{(\text{trad})} \quad \frac{\partial P_k(q_{[0]}, p_{[0]}, t)}{\partial t} = [P_k, g]^{(\text{trad})} \quad (17.86)$$

The implicit transformation eqn (17.85) is required in the traditional theory in order to secure the correctness of the Hamilton equations in both the $q_{[0]}, p_{[0]}$ system using h and the $Q_{[0]}, P_{[0]}$ system using H .

The following theorem proves the traditional theory to be a special, restricted case of the extended definition of canonical transformation in the present chapter.

Theorem 17.13.1: Restricted Canonical Transformations

A traditional canonical transformation defined by eqns (17.83, 17.85, 17.86) and satisfying the traditional condition eqn (17.84) is canonical in the extended sense defined by the extended Poisson bracket condition eqn (17.5).

Proof: In the following, we assume $Q_0 = q_0$, eqns (17.83, 17.85, 17.86), and the traditional conditions eqn (17.84). These are shown to imply each of the extended Poisson bracket conditions eqn (17.5).

When $k \neq 0$, the condition $Q_0 = q_0$ implies that $[Q_0, Q_k]_{qp} = \partial Q_k / \partial p_0$ and $[Q_0, P_k]_{qp} = \partial P_k / \partial p_0$. Thus the extended Poisson bracket conditions $[Q_0, Q_k]_{qp} = 0$ and $[Q_0, P_k]_{qp} = 0$ are satisfied because the transformation has the form in eqn (17.83) in which Q and P do not depend explicitly on p_0 .

The condition $Q_0 = q_0$ implies that $[Q_0, P_0]_{qp} = \partial P_0 / \partial p_0$. Thus the extended Poisson bracket condition $[Q_0, P_0]_{qp} = 1$ is satisfied because the function g defined in eqn (17.85) is of the general form $g = g(q, p_{[0]})$ and hence has $\partial g / \partial p_0 = 0$.

When $k \neq 0$, eqn (17.85) implies that $[Q_k, P_0]_{qp} = (\partial Q_k / \partial q_0 - [Q_k, g]^{(\text{trad})})$ and $[P_k, P_0]_{qp} = (\partial P_k / \partial q_0 - [P_k, g]^{(\text{trad})})$. Thus eqn (17.86) implies that the extended Poisson bracket conditions $[Q_k, P_0]_{qp} = 0$ and $[P_k, P_0]_{qp} = 0$ are satisfied.

When $k, l \neq 0$, then $[Q_k, Q_l]_{qp} = [Q_k, Q_l]^{(\text{trad})} = 0$, $[Q_k, P_l]_{qp} = [Q_k, P_l]^{(\text{trad})} = \delta_{kl}$, and $[P_k, P_l]_{qp} = [P_k, P_l]^{(\text{trad})} = 0$, which completes the proof. \square

The extended definition of canonical transformation used in this book thus includes the traditional definition as a special case. In addition to its inclusion of the Lorentz transformation, the extended definition also has the advantage of simplicity. Unlike the traditional definition, the extended definition has no implicit, unstated conditions like eqns (17.85, 17.86) to complicate it. Treating time as a transformable coordinate allows all the required conditions for the transformation to be canonical to be included in the necessary and sufficient conditions eqn (17.5).

17.14 Exercises

Exercise 17.1 Consider a canonical transformation with constant coefficients, defined by

$$Q_i = \sum_{k=0}^D a_{ik} q_k \quad \text{or, in matrix form,} \quad [Q] = \mathbf{a}[q] \quad (17.87)$$

$$P_i = \sum_{k=0}^D b_{ik} p_k \quad \text{or, in matrix form,} \quad [P] = \mathbf{b}[p] \quad (17.88)$$

for $i = 0, \dots, D$, where the elements a_{ik} and b_{ik} of the matrices \mathbf{a} and \mathbf{b} are all constants.

- If the matrix \mathbf{a} is given, find a general expression for the matrix \mathbf{b} .
- Canonical transformations of this type are of particular importance in quantum theory because the quantum substitution $p_k \rightarrow -i\hbar\partial/\partial q_k$ implies and is implied by the substitution $P_k \rightarrow -i\hbar\partial/\partial Q_k$. Prove this implication.
- If the matrix \mathbf{a} is

$$\mathbf{a} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} \quad (17.89)$$

find the numerical value of each element of the matrix \mathbf{b} .

Exercise 17.2 Consider a Lorentz transformation $q, p \rightarrow Q, P$. The canonical coordinates q are x^0, x^1, x^2, x^3 where the x^μ are the *contravariant* components of \mathbf{r} . The canonical momenta p are $\underline{p}_0, \underline{p}_1, \underline{p}_2, \underline{p}_3$ where the \underline{p}_μ are the *covariant* components of the canonical momentum fourvector $\underline{\mathbf{p}}$ defined in Section 16.10. Similarly identifying the Q, P with the primed fourvector components, x'^μ and \underline{p}'_μ , use the transformation rules eqns (15.55, 15.59) to prove that the Lorentz transformation is canonical. (Note that in this exercise we are using the “relativistic coordinates” discussed at the end of Section 16.9.)

Exercise 17.3 Suppose a system to have a traditional Hamiltonian $h(q_{[0]}, p_{[0]}, t)$. We make a transformation to new coordinates $Q_i = q_i$ and $P_i = p_i$, for $i = 1, \dots, D$, but $T = at$ and $H = h/a$ where a is some constant scaling parameter.

(a) Show that this scaling transformation preserves the form of the traditional Hamilton equations, i.e., that, for $i = 1, \dots, D$

$$\frac{dQ_i}{dT} = \frac{\partial H(Q_{[0]}, P_{[0]}, T)}{\partial P_i} \quad \text{and} \quad \frac{dP_i}{dT} = -\frac{\partial H(Q_{[0]}, P_{[0]}, T)}{\partial Q_i} \quad (17.90)$$

(b) This scaling transformation is not canonical in traditional theory since it transforms the time. However, show that it is canonical with the extended definition of canonical transformation used in this chapter. Show that $Q_0 = aq_0$ and $P_0 = p_0/a$, with the other coordinates and momenta transforming identically, is a canonical transformation.

Exercise 17.4 A transformation $q, p \rightarrow Q, P$ is defined, for all $i = 0, \dots, D$, by

$$Q_i = q_i \cos \theta_i + ap_i \sin \theta_i \quad (17.91)$$

$$P_i = -\frac{q_i}{a} \sin \theta_i + p_i \cos \theta_i \quad (17.92)$$

where $\theta_0, \theta_1, \theta_2, \dots, \theta_D$ are independent, constant parameters, and a is a given constant having appropriate units. (Note that no summation convention is being used here. For example, Q_2 and P_2 depend only on q_2, p_2 , and the constant parameter θ_2 .) Use the Poisson-bracket condition to show that this transformation is canonical for any set of θ_i values.

Exercise 17.5 Referring to Exercises 2.4 and 2.5, define the q variables to be $t, x_1, y_1, z_1, x_2, y_2, z_2$, the p variables to be $p_0, p_{1x}, p_{1y}, p_{1z}, p_{2x}, p_{2y}, p_{2z}$ the Q variables to be Q_0, x, y, z, X, Y, Z and the P variables to be $P_0, p_x, p_y, p_z, P_x, P_y, P_z$, where $\mathbf{p} = \mu(d\mathbf{r}/dt)$ and $\mathbf{P} = M(d\mathbf{R}/dt)$.

(a) With the assumption $P_0 = p_0$ and $Q_0 = q_0$, prove that the transformation from q, p to Q, P is canonical.

(b) Show that this is a canonical transformation of the type discussed in Exercise 17.1, and that the standard quantum substitutions

$$\mathbf{p}_1 \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{r}_1} \quad \text{and} \quad \mathbf{p}_2 \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{r}_2} \quad (17.93)$$

in the q, p system imply the quantum substitutions

$$\mathbf{p} \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{r}} \quad \text{and} \quad \mathbf{P} \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{R}} \quad (17.94)$$

in the Q, P system.

Exercise 17.6 Suppose that a system of two masses as described in Exercise 17.5 has a traditional Hamiltonian

$$H(q_{[0]}, p_{[0]}, t) = \frac{\mathbf{p}_1 \cdot \mathbf{p}_1}{2m_1} + \frac{\mathbf{p}_2 \cdot \mathbf{p}_2}{2m_2} + U(\mathbf{r}) \quad (17.95)$$

in the q, p system, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

(a) Write the extended Hamiltonian $\mathcal{K}(q, p)$ in the q, p system, and use the transformation equations to express the same extended Hamiltonian in the Q, P system as $\mathcal{K}(Q, P)$.

(b) Show that $\mathcal{K}(Q, P)$ can be written as $\mathcal{K} = P_0 + H_R(\mathbf{R}, \mathbf{P}) + H_r(\mathbf{r}, \mathbf{p})$ where H_R depends only on the variables \mathbf{R}, \mathbf{P} and the H_r depends only on the variables \mathbf{r}, \mathbf{p} . Such systems are called *separable*.

(b) Use the extended Hamilton equations to show that P_0 and \mathbf{P} are conserved. Is H_r a conserved quantity?

(c) Use $\mathcal{K}(Q, P)$ to write the Schroedinger equation in the Q, P system, making use of the quantum substitutions in eqn (17.94) and $P_0 \rightarrow -i\hbar(\partial/\partial Q_0) = -i\hbar(\partial/\partial t)$.

Exercise 17.7 Section 2.14 demonstrated that each product $q_k p_k$ for $k = 1, \dots, D$ has units of action.

(a) Demonstrate from the definition in Section 11.4 that the product $q_0 p_0$ also has units of action.

(b) Using Definition 17.1.1 or otherwise, demonstrate that, for all $k = 0, \dots, D$, each product $Q_k P_k$ after any canonical transformation will have the units of action.

Exercise 17.8 Consider the following four transformations

$$\begin{aligned} \#1 : \quad Q_0 &= -p_0 & Q_1 &= p_1 + aq_1 & P_0 &= bq_0 & P_1 &= cp_1 \\ \#2 : \quad Q_0 &= q_0 & Q_1 &= -p_1/a & P_0 &= bp_0 & P_1 &= -p_1 + cq_1 \\ \#3 : \quad Q_0 &= -p_0 & Q_1 &= q_1/a & P_0 &= bq_0 & P_1 &= -q_1 + cp_1 \\ \#4 : \quad Q_0 &= q_0 & Q_1 &= q_1 - ap_1 & P_0 &= bp_0 & P_1 &= cq_1 \end{aligned} \quad (17.96)$$

In each case, assume that a is given and find the b and c that will make the transformation canonical. (Note that the a, b, c may be different in each case.)

Exercise 17.9 Consider a transformation of phase-space variables defined by

$$Q_0 = q_0 \quad Q_1 = p_1 \quad Q_2 = q_2 \quad P_0 = p_0 \quad P_1 = (-q_1 - 3q_2) \quad P_2 = (-3p_1 + p_2) \quad (17.97)$$

(a) Using any method you want, verify that this transformation is canonical.

(b) Write out the 6×6 Jacobian matrix \mathbf{J} defined in eqn (17.14).

Exercise 17.10 A system with one degree of freedom has a traditional Hamiltonian

$$H(q_{[0]}, p_{[0]}, t) = \frac{a^2}{2mp_1^2} + \frac{m\omega^2 q_1^2 p_1^4}{2a^2} \quad (17.98)$$

(a) Write the extended Hamiltonian $\mathcal{K}(q, p)$ for this system.

(b) Consider the transformation $q, p \rightarrow Q, P$ defined as

$$Q_0 = q_0 = t \quad Q_1 = bq_1 p_1^2 \quad P_0 = p_0 \quad P_1 = -\frac{a}{p_1} \quad (17.99)$$

Assuming the constant a given, for what value of b is this transformation canonical?

(c) Use this transformation with your chosen value of b to write the extended Hamiltonian in terms of the Q, P variables, as in eqn (17.75).

(d) Write the extended Hamilton equations in the Q, P system and solve them for the system trajectory.

Exercise 17.11 Consider the transformation

$$\begin{aligned} Q_0 &= bq_0^2 & Q_1 &= q_1 & Q_2 &= aq_0 + q_2 \\ P_0 &= \frac{p_0 - ap_2}{2bq_0} & P_1 &= p_1 & P_2 &= p_2 + c(aq_0 + q_2)^2 \end{aligned} \quad (17.100)$$

For what values of the constants a, b, c is this a canonical transformation?

Exercise 17.12 Consider a canonical transformation that has $Q_0 = q_0$.

(a) Show that it must have $\partial P_0/\partial p_0 = 1$, as well as $\partial Q_i/\partial p_0 = 0$ and $\partial P_i/\partial p_0 = 0$ for all $i \neq 0$.

(b) Show that if either $\partial Q_i/\partial q_0 \neq 0$ or $\partial P_i/\partial q_0 \neq 0$ for any $i \neq 0$, then at least one of $\partial P_0/\partial q_i$ and $\partial P_0/\partial p_i$ must be nonzero for some $i \neq 0$.

GENERATING FUNCTIONS

In Chapter 17, several conditions were given that could be used for testing transformations to see whether they were canonical or not. But no methods, other than trial and error, were given for actually creating the canonical transformations to be tested. The present chapter supplies methods for creating transformations that will automatically be canonical.

Canonical transformations can be created by first choosing what are called *generating functions*. Using one of these generating functions in the formalism to be described will *generate* a transformation that will be canonical by construction. The generating functions can be quite general, leading to a wide selection of possible canonical transformations.

And not only does every generating function lead to a canonical transformation, the converse is also true. Given any canonical transformation, a generating function can always be found that will generate it.

18.1 Proto-Generating Functions

Before treating actual generating functions, we must first relate canonical transformations to the existence of an intermediate function $f(q, p)$ that will be called a *proto-generating function*.

Theorem 18.1.1: Existence of Proto-Generating Function

A transformation $q, p \rightarrow Q, P$ is canonical if and only if there exists a phase-space function $f(q, p)$ such that

$$df(q, p) = \sum_{k=0}^D p_k dq_k - \sum_{k=0}^D P_k(q, p) dQ_k(q, p) \quad (18.1)$$

In this expression, both sides are considered as functions of the q, p variable set. Thus the theorem is equivalent to the assertion that the right side of eqn (18.1) is a perfect differential when expanded in that set of variables.

Proof: First, we define a matrix \mathbf{t} in a manner analogous to the definition of \mathbf{s} in eqn (17.27),

$$\mathbf{t} = \begin{pmatrix} 0 & 0 \\ \mathbf{U} & 0 \end{pmatrix} \quad (18.2)$$

so that $\mathbf{s} = \mathbf{t}^T - \mathbf{t}$. Using the elements of this matrix, the right side of eqn (18.1)

may be written as

$$\sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} (\gamma_i t_{ij} d\gamma_j - \Gamma_i(\gamma) t_{ij} d\Gamma_j(\gamma)) \tag{18.3}$$

where the definitions $q, p = \gamma$ and $Q, P = \Gamma$ have been adopted from Section 17.3. The chain rule and the definition in eqn (17.13) give

$$d\Gamma_j(\gamma) = \sum_{k=0}^{2D+1} \frac{\partial \Gamma_j(\gamma)}{\partial \gamma_k} d\gamma_k = \sum_{k=0}^{2D+1} J_{jk} d\gamma_k \tag{18.4}$$

and hence, when expanded completely into differentials in the q, p system, eqn (18.3) becomes

$$\sum_{k=0}^{2D+1} \left(\sum_{i=0}^{2D+1} \gamma_i t_{ik} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} \Gamma_i(\gamma) t_{ij} J_{jk} \right) d\gamma_k = \sum_{k=0}^{2D+1} A_k d\gamma_k \tag{18.5}$$

where the definition

$$A_k = \sum_{i=0}^{2D+1} \gamma_i t_{ik} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} \Gamma_i(\gamma) t_{ij} J_{jk} \tag{18.6}$$

has been introduced. Applying Theorem D.20.1, this expression is a perfect differential and a potential function $f(q, p) = f(\gamma)$ exists satisfying $df = \sum_{k=0}^{2D+1} A_k d\gamma_k$, if and only if, for all $k, l = 0, \dots, (2D + 1)$,

$$\frac{\partial A_k(\gamma)}{\partial \gamma_l} = \frac{\partial A_l(\gamma)}{\partial \gamma_k} \tag{18.7}$$

When written out, eqn (18.7) becomes

$$\begin{aligned} t_{lk} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} \left(\frac{\partial \Gamma_i(\gamma)}{\partial \gamma_l} t_{ij} J_{jk} + \Gamma_i(\gamma) t_{ij} \frac{\partial J_{jk}}{\partial \gamma_l} \right) \\ = t_{kl} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} \left(\frac{\partial \Gamma_i(\gamma)}{\partial \gamma_k} t_{ij} J_{jl} + \Gamma_i(\gamma) t_{ij} \frac{\partial J_{jl}}{\partial \gamma_k} \right) \end{aligned} \tag{18.8}$$

Again using eqn (17.13), and cancelling the terms in $\partial J_{jk} / \partial \gamma_l = \partial J_{jl} / \partial \gamma_k$, this becomes

$$t_{lk} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} J_{il} t_{ij} J_{jk} = t_{kl} - \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} J_{ik} t_{ij} J_{jl} \tag{18.9}$$

which in matrix form is

$$\mathbf{t}^T - (\mathbf{J}^T \mathbf{t} \mathbf{J})^T = \mathbf{t} - (\mathbf{J}^T \mathbf{t} \mathbf{J}) \tag{18.10}$$

which reduces to

$$\mathbf{J}^T (\mathbf{t}^T - \mathbf{t}) \mathbf{J} = (\mathbf{t}^T - \mathbf{t}) \quad (18.11)$$

Since $\mathbf{s} = \mathbf{t}^T - \mathbf{t}$, this becomes

$$\mathbf{J}^T \mathbf{s} \mathbf{J} = \mathbf{s} \quad (18.12)$$

which is the Lagrange Bracket Condition from the transformation to be canonical, as proved in Lemma 17.9.1. Thus the proto-generating function $f(q, p)$ exists if and only if the transformation is canonical, as was to be proved.¹⁰³ \square

18.2 Generating Functions of the F_1 Type

The proto-generating function $f(q, p)$ is defined as a function of the q, p set of phase-space coordinates. However, generating functions themselves are always defined as functions of one category of variable from the q, p set and one from the Q, P set. The simplest case is a function of the form $F_1(q, Q)$, which depends on an old and a new coordinate. Aside from the requirement that it be continuously differentiable, the only restriction placed on this function is that the matrix \mathbf{m} defined by

$$m_{ij} = \frac{\partial^2 F_1(q, Q)}{\partial q_i \partial Q_j} \quad (18.13)$$

for all $i, j = 0, \dots, D$ must be nonsingular, with $|\mathbf{m}| \neq 0$.

A transformation $q, p \rightarrow Q, P$ is derived from $F_1(q, Q)$ by beginning with the following rules, which must hold for all $k = 0, \dots, D$,

$$p_k = p_k(q, Q) = \frac{\partial F_1(q, Q)}{\partial q_k} \quad P_k = P_k(q, Q) = -\frac{\partial F_1(q, Q)}{\partial Q_k} \quad (18.14)$$

These will be referred to as the “ F_1 rules.”

The derivation is a two step process. To begin, the first of eqn (18.14) is solved for Q_k as a function of q, p . By Theorem D.24.1, the inverse function theorem, the condition that $p_k = p_k(q, Q)$ can indeed be solved for Q_k is that the matrix whose ij th element is $\partial p_i(q, Q)/\partial Q_j$ must be nonsingular. But this matrix is seen to be identical to the matrix \mathbf{m} defined in eqn (18.13), and hence is nonsingular by assumption. Thus $Q_k = Q_k(q, p)$ can always be found.

Then this result can be substituted into the second of eqn (18.14) to give

$$P_k = P_k(q, p) = P_k(q, Q(q, p)) \quad (18.15)$$

as a compound function. Thus a complete transformation is constructed, having the same form as that described in eqns (17.3, 17.4).

The inverse transformation can also be derived from the same generating function. The second of eqn (18.14) is solved for $q_k = q_k(Q, P)$. Again using the inverse function theorem, this can be done if and only if the matrix whose elements are

¹⁰³This proof is adapted from Volume II of Desloges (1982).

$\partial P_i(q, Q)/\partial q_j$ is nonsingular. But that matrix is $-m^T$ and hence is nonsingular by assumption, so $q_k = q_k(Q, P)$ can always be found. This result is then substituted into the first of eqn (18.14) to obtain

$$p_k = p_k(Q, P) = p_k(q(Q, P), Q) \tag{18.16}$$

as a compound function. The completed transformation has the same form as the inverse transformation described in eqns (17.41, 17.42).

Thus both the transformation and its inverse are well defined. We now prove that the transformation is canonical.

Lemma 18.2.1: Canonical Transformation From F_1

The transformation and inverse defined by an F_1 generating function and eqn (18.14) are canonical.

Proof: We prove the transformation canonical. Its inverse will also be canonical, as demonstrated in Section 17.10 (item 3).

We prove the transformation canonical by demonstrating the existence of a correct proto-generating function. Define

$$f(q, p) = F_1(q, Q(q, p)) \tag{18.17}$$

Then

$$df = dF_1 = \sum_{k=0}^D \frac{\partial F_1(q, Q)}{\partial q_k} dq_k + \sum_{k=0}^D \frac{\partial F_1(q, Q)}{\partial Q_k} dQ_k = \sum_{k=0}^D p_k dq_k - \sum_{k=0}^D P_k dQ_k \tag{18.18}$$

where the F_1 rules in eqn (18.14) were used to get the last equality. Equation (18.18) has the same form as eqn (18.1), and hence Theorem 18.1.1 proves that the transformation is canonical. □

In spite of the generality of F_1 itself, the canonical transformations generated by F_1 functions are not completely general, as seen in the following lemma.

Lemma 18.2.2: Limitation of F_1 Transformations

Any canonical transformation generated by an F_1 function must have $|\partial Q(q, p)/\partial p| \neq 0$ where $(\partial Q(q, p)/\partial p)$ is the matrix whose elements are $(\partial Q(q, p)/\partial p)_{ij} = \partial Q_i(q, p)/\partial p_j$.

Proof: We have shown above that the matrix $(\partial^2 F_1(q, Q)/\partial q \partial Q) = (\partial p(q, Q)\partial Q)$ is nonsingular. This allowed $p_k = p_k(q, Q)$ to be solved for $Q_k = Q_k(q, p)$. We can now apply the consequence of the inverse function theorem in eqn (D.94) to obtain $(\partial Q(q, p)/\partial p) = (\partial p(q, Q)\partial Q)^{-1}$. Thus $(\partial Q(q, p)/\partial p)$ is the inverse of a nonsingular matrix and hence is nonsingular, as was to be proved. □

18.3 Generating Functions of the F_2 Type

Another important case is a generating function of the form $F_2 = F_2(q, P)$, which depends on the old coordinates and the new momenta. It too can be a general continuously differentiable function, being restricted only by the condition that the matrix n defined by

$$n_{ij} = \frac{\partial^2 F_2(q, P)}{\partial q_i \partial P_j} \quad (18.19)$$

must be nonsingular, with $|n| \neq 0$. The transformation is derived from the “ F_2 rules” which are

$$p_k = p_k(q, P) = \frac{\partial F_2(q, P)}{\partial q_k} \quad Q_k = Q_k(q, P) = \frac{\partial F_2(q, P)}{\partial P_k} \quad (18.20)$$

The first of eqn (18.20) is solved for $P_k = P_k(q, p)$. By Theorem D.24.1, this can be done if the matrix whose elements are $\partial p_i(q, P)/\partial P_j$ is nonsingular. But this matrix is the same as n defined in eqn (18.19) and hence is nonsingular by assumption. Thus the solution for $P_k = P_k(q, p)$ is always possible. This result is then substituted into the second of eqn (18.20) to give

$$Q_k = Q_k(q, p) = Q_k(q, P(q, p)) \quad (18.21)$$

as a compound function. Thus a complete transformation is defined, having the same form as eqns (17.3, 17.4).

As in the F_1 case, the inverse transformation can also be derived directly from the F_2 generating function. The method is similar. The second of eqn (18.20) is solved for $q_k = q_k(Q, P)$. By the inverse function theorem, this can always be done provided that the matrix whose elements are $\partial Q_i(q, P)/\partial q_j$ is nonsingular. But this matrix is seen to be the same as n^T , which is nonsingular by assumption. Thus the solution for $q_k = q_k(Q, P)$ is always possible. The rest of the inverse transformation then is obtained by substituting this result into the first of eqn (18.20) to get

$$p_k = p_k(Q, P) = p_k(q(Q, P), P) \quad (18.22)$$

as a compound function. The completed transformation has the same form as the inverse transformation eqns (17.41, 17.42).

The proof that the transformations generated by F_2 are canonical is only slightly different from Lemma 18.2.1.

Lemma 18.3.1: Canonical Transformation From F_2

The transformation and inverse defined by the F_2 generating function and eqn (18.20) are canonical.

Proof: We prove the transformation canonical. Its inverse then will also be canonical, as demonstrated in Section 17.10 (item 3),

Define

$$f(q, p) = F_2(q, P(q, p)) - \sum_{k=0}^D Q_k(q, p)P_k(q, p) \tag{18.23}$$

It follows that

$$\begin{aligned} df &= \sum_{k=0}^D \frac{\partial F_2(q, P)}{\partial q_k} dq_k + \sum_{k=0}^D \frac{\partial F_2(q, P)}{\partial P_k} dP_k - \left(\sum_{k=0}^D Q_k dq_k + \sum_{k=0}^D P_k dQ_k \right) \\ &= \sum_{k=0}^D p_k dq_k - \sum_{k=0}^D P_k dQ_k \end{aligned} \tag{18.24}$$

where the F_2 rules eqn (18.20) and some cancellation of terms were used to get the last equality. Equation (18.24) has the same form as eqn (18.1). Therefore a proto-generating function exists with and, by Theorem 18.1.1, the transformation is canonical. □

As for the F_1 case, the canonical transformations generated by F_2 are not completely general.

Lemma 18.3.2: Limitation of F_2 Transformations

Any canonical transformation generated by an F_2 function must have $|\partial P(q, p)/\partial p| \neq 0$ where $(\partial P(q, p)/\partial p)$ is the matrix whose elements are $(\partial P(q, p)/\partial p)_{ij} = \partial P_i(q, p)/\partial p_j$.

Proof: The proof is similar to that of Lemma 18.2.2. □

18.4 Examples of Generating Functions

We start with two very simple examples of the use of the F_1 and F_2 generating functions. Suppose that an F_1 is defined as

$$F_1(q, Q) = \sum_{k=0}^D q_k Q_k \tag{18.25}$$

The matrix m in eqn (18.13) is then just the unit matrix and so is nonsingular as required. The F_1 rules in eqn (18.14) give, for all $k = 0, \dots, D$, that

$$p_k = Q_k \qquad P_k = -q_k \tag{18.26}$$

Thus the new momenta are the negative of the old coordinates, and the new coordinates are the old momenta. The reader can verify that this transformation, though it may seem bizarre, does preserve the form of the extended Hamilton equations.

A second simple example uses an F_2 given by

$$F_2(q, P) = \sum_{k=0}^D q_k P_k \tag{18.27}$$

The matrix n in eqn (18.19) is just the unit matrix and so is nonsingular. The F_2 rules

give, for all $k = 0, \dots, D$,

$$p_k = P_k \quad Q_k = q_k \quad (18.28)$$

This is just the identity transformation, with the new coordinates and momenta equal to the old ones.

A more interesting example is

$$F_2(q, P) = \sum_{k=0}^D g_k(q) P_k \quad (18.29)$$

Now the matrix n is equal to a matrix with elements $\partial g_i(q)/\partial q_j$, which will therefore be assumed to be nonsingular. Applying the F_2 rules gives

$$p_k = \sum_{l=0}^D \frac{\partial g_l(q)}{\partial q_k} P_l \quad Q_k = g_k(q) \quad (18.30)$$

This is seen to be a transformation of the Lagrangian type, in which the new coordinates depend only on the old coordinates and not on the old momenta. Since the matrix $(\partial g/\partial q)$ is nonsingular by assumption, the first of eqn (18.30) can be inverted to give

$$Q_k = Q_k(q) \quad P_k = \sum_{l=0}^D \frac{\partial q_l(Q)}{\partial Q_k} p_l \quad (18.31)$$

where we have now replaced g_k by Q_k , which is equal to it. In the second of eqn (18.31), one will of course take the partial derivatives and then express the result in terms of the q, p variable set to obtain $P_k = P_k(q, p)$.

The transformation in eqn (18.31) is exactly what would be predicted by the extended Lagrangian theory of Chapter 11. If we have an extended Lagrangian $\mathcal{L}(q, \dot{q})$ and define a Lagrangian in the Q system by the usual rule¹⁰⁴

$$\mathcal{L}(Q, \dot{Q}) = \mathcal{L}(q(Q), \dot{q}(Q\dot{Q})) \quad (18.32)$$

then, according to Lagrangian theory, the canonical momenta in the two systems will be related by

$$P_k = \frac{\partial \mathcal{L}(Q, \dot{Q})}{\partial \dot{Q}_k} = \sum_{l=0}^D \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_l} \frac{\partial \dot{q}_l(Q, \dot{Q})}{\partial \dot{Q}_k} = \sum_{l=0}^D p_l \frac{\partial q_l(Q)}{\partial Q_k} \quad (18.33)$$

where Lemma 2.8.1 was used to get the final equality. Equation (18.33) agrees exactly with the second of eqn (18.31).

In Lagrangian theory, the transformation of the momenta was implicit. The theory of canonical transformations makes that transformation explicit, but the result is the

¹⁰⁴The Lagrangian transformation here is the same as in Section 11.8 with the notational substitution $r \rightarrow Q$.

same in this case. It follows that Lagrangian transformations of the type discussed in Section 11.8 are a special case of the more general canonical transformations. Note that, although the Q_k depend only on the q variables in Lagrangian transformations, the P_k in general do depend on both the q and p variables.

18.5 Other Simple Generating Functions

There are two other simple generating functions of some utility found in the literature.¹⁰⁵ Their definitions and use follow the same pattern as for the F_1 and F_2 so the relevant formulas will simply be stated here without detailed proof.¹⁰⁶

The generating function $\tilde{F}_1(p, P)$ must have $|\partial^2 \tilde{F}_1(p, P)/\partial p \partial P| \neq 0$. The rules for this function are

$$q_k(p, P) = \frac{\partial \tilde{F}_1(p, P)}{\partial p_k} \quad Q_k(p, P) = -\frac{\partial \tilde{F}_1(p, P)}{\partial P_k} \quad (18.34)$$

It can be used to define the proto-generating function

$$f(q, p) = -\tilde{F}_1(p, P(q, p)) + \sum_{k=0}^D q_k p_k - \sum_{k=0}^D Q_k(q, p) P_k(q, p) \quad (18.35)$$

The canonical transformations generated by the \tilde{F}_1 must have $|\partial P(q, p)/\partial q| \neq 0$.

The generating function $\tilde{F}_2(p, Q)$ must have $|\partial^2 \tilde{F}_2(p, Q)/\partial p \partial Q| \neq 0$. The rules for this function are

$$q_k(p, Q) = \frac{\partial \tilde{F}_2(p, Q)}{\partial p_k} \quad P_k(p, Q) = \frac{\partial \tilde{F}_2(p, Q)}{\partial Q_k} \quad (18.36)$$

It can be used to define the proto-generating function

$$f(q, p) = -\tilde{F}_2(p, Q(q, p)) + \sum_{k=0}^D q_k p_k \quad (18.37)$$

The canonical transformations generated by the \tilde{F}_2 must have $|\partial Q(q, p)/\partial q| \neq 0$.

¹⁰⁵One author commented wryly that, "The number of generating functions in the literature ranges from four to 4^N " (Arnold, 1978). We describe four simple types of generating functions, the F_1 , F_2 , \tilde{F}_1 , and \tilde{F}_2 . But, as will be shown in Section 18.9, all canonical transformations can be generated by one or more of the 2^N different *mixed* generating functions F , where $N = D + 1$ is the number of different values of the binary number α defined in eqn (18.38).

¹⁰⁶The numbering of the simple generating functions is arbitrary. Our numbering of the F_1 and F_2 functions agrees with that in Arnold (1978) and Goldstein, Poole and Safko (2002). Our \tilde{F}_1 and \tilde{F}_2 functions are related to the negatives of Goldstein's F_4 and F_3 , respectively. (Of course, these authors use the traditional definitions in which time does not transform.)

18.6 Mixed Generating Functions

Canonical transformations can also be produced by generating functions that are a particular mixture of the F_1 and F_2 types. They will be referred to as *mixed generating functions*. They are important because canonical transformations exist that do not have generating functions of the F_1 , F_2 , \tilde{F}_1 , or \tilde{F}_2 type. But, as we prove in Section 18.9 below, any canonical transformation will have at least one mixed generating function that generates it.

Mixed generating functions can be characterized by first selecting an arbitrary binary number α consisting of $(D + 1)$ binary digits,

$$\alpha = \alpha_D \cdots \alpha_1 \alpha_0 \quad (18.38)$$

Each of the α_k digits is either a one or a zero. These digits are assumed independent of each other. Thus $2^{(D+1)}$ different numbers α are possible, each producing a different canonical transformation. Also define $\bar{\alpha}_k = 1 - \alpha_k$ to be the complement of the digit α_k . It is zero when α_k is one, and one when α_k is zero. Thus

$$\alpha_k^2 = \alpha_k \quad \bar{\alpha}_k^2 = \bar{\alpha}_k \quad \alpha_k \bar{\alpha}_k = 0 \quad \text{and} \quad \alpha_k + \bar{\alpha}_k = 1 \quad (18.39)$$

for all k values.

After choosing α , we define variables X_k and Y_k by the rule, for all $k = 0, \dots, D$,

$$X_k = \bar{\alpha}_k Q_k - \alpha_k P_k \quad \text{and} \quad Y_k = \alpha_k Q_k + \bar{\alpha}_k P_k \quad (18.40)$$

so that $\alpha_k = 0$ makes $X_k, Y_k = Q_k, P_k$ and $\alpha_k = 1$ makes $X_k, Y_k = -P_k, Q_k$.

A mixed generating function $F(q, Y)$ can be any differentiable function of q, Y , except for the condition that the matrix \mathbf{o} whose ij th matrix element is

$$o_{ij} = \frac{\partial^2 F(q, Y)}{\partial q_i \partial Y_j} \quad (18.41)$$

must be nonsingular with $|\mathbf{o}| \neq 0$. A transformation can then be defined by what will be called the “ F rules” which are

$$p_k = p_k(q, Y) = \frac{\partial F(q, Y)}{\partial q_k} \quad X_k = X_k(q, Y) = \frac{\partial F(q, Y)}{\partial Y_k} \quad (18.42)$$

Note that these rules resemble the F_1 rules when $\alpha_k = 1$ and the F_2 rules when $\alpha_k = 0$. The F_1 rules in eqn (18.14) would result from the choice $\alpha = 11 \cdots 1$ for the binary number α defined in eqn (18.38), so that $\alpha_k = 1$ for all k . The F_2 rules in eqn (18.20) would result from the choice $\alpha = 00 \cdots 0$ so that $\alpha_k = 0$ for all k . These are the simplest choices, and show that the F_1 and F_2 generating functions above are just special cases of the mixed ones being discussed here. In general mixed generating functions, however, some of the binary digits α_k can be zero and others one.

A transformation can be obtained from the F rules by solving the first of eqn (18.42) for $Y_k = Y_k(q, p)$. By Theorem D.24.1, the inverse function theorem, this is

possible if and only if the matrix whose ij th matrix element is $\partial p_i(q, Y)/\partial Y_j$ is non-singular. But this matrix is the same as \mathbf{o} defined in eqn (18.41) and so is nonsingular by assumption. The solution for $Y_k = Y_k(q, p)$ is therefore always possible. Then this result is substituted into the second of eqn (18.42) to obtain

$$X_k = X_k(q, p) = X_k(q, Y(q, p)) \tag{18.43}$$

as a compound function. One thus obtains transformation equations in the form

$$X_k = X_k(q, p) \qquad Y_k = Y_k(q, p) \tag{18.44}$$

that can be translated back into equations for Q, P as functions of q, p by use of the inverses of eqn (18.40), which are

$$Q_k = \bar{\alpha}_k X_k + \alpha_k Y_k \qquad P_k = -\alpha_k X_k + \bar{\alpha}_k Y_k \tag{18.45}$$

Thus a well-defined transformation of coordinates is obtained, of the same standard form as eqns (17.3, 17.4).

The inverse transformation can also be obtained, again following a pattern similar to that used for F_1 and F_2 . The second of eqn (18.42) is solved for $q_k = q_k(X, Y)$. Again using the inverse function theorem, this solution is possible if the matrix whose ij th matrix element is $\partial X_i(q, Y)/\partial q_j$ is nonsingular. But this matrix is equal to \mathbf{o}^T , which is nonsingular by assumption, and so the solution is always possible. This equation is then substituted into the first of eqn (18.42) to obtain

$$p_k = p_k(X, Y) = p_k(q(X, Y), Y) \tag{18.46}$$

as a compound function. One thus obtains transformation equations in the form

$$q_k = q_k(X, Y) \qquad p_k = p_k(X, Y) \tag{18.47}$$

which can then be translated into functions of Q, P by use of eqn (18.40).

We now show that the transformation and its inverse are canonical.

Lemma 18.6.1: Canonical Transformation From F

For every choice of binary number α , the transformation and inverse obtained by use of an F generating function and eqn (18.42) are canonical.

Proof: We prove the transformation canonical. Its inverse then will also be canonical, as demonstrated in Section 17.10 (item 3).

Use the F generating function to define the proto-generating function

$$f(q, p) = F(q, Y(q, p)) - \sum_{k=0}^D \bar{\alpha}_k X_k(q, p) Y_k(q, p) \tag{18.48}$$

It follows that that

$$\begin{aligned}
 df &= \sum_{k=0}^D \frac{\partial F(q, Y)}{\partial q_k} dq_k + \sum_{k=0}^D \frac{\partial F(q, Y)}{\partial Y_k} dY_k - \left(\sum_{k=0}^D \bar{\alpha}_k Y_k dX_k + \sum_{k=0}^D \bar{\alpha}_k X_k dY_k \right) \\
 &= \sum_{k=0}^D p_k dq_k - \sum_{k=0}^D P_k dQ_k
 \end{aligned}
 \tag{18.49}$$

where the F rules in eqn (18.42) and the definitions in eqn (18.40) were used to get the last equality. Equation (18.49) agrees with eqn (18.1), and hence Theorem 18.1.1 shows that the transformation is canonical, as was to be proved. \square

An alternate way of writing mixed generating functions simply writes F out as a function of q and a mixture of Q and P variables. Suppose that the k values having $\alpha_k = 1$ are k_0, k_1, \dots, k_M and those having $\alpha_k = 0$ are $k_{(M+1)}, k_{(M+2)}, \dots, k_D$. Then $F(q, Y)$ becomes

$$F = F(q_0, \dots, q_D, Q_{k_0}, \dots, Q_{k_M}, P_{k_{(M+1)}}, \dots, P_{k_D})
 \tag{18.50}$$

The F_1 rules are applied for k values with Q dependencies, and the F_2 rules for those with P dependencies. Thus, the F -rules become

$$p_i = \frac{\partial F}{\partial q_i} \quad P_{k_a} = -\frac{\partial F}{\partial Q_{k_a}} \quad Q_{k_b} = \frac{\partial F}{\partial P_{k_b}}
 \tag{18.51}$$

where $i = 0, \dots, D, a = 0, \dots, M, b = (M + 1), \dots, D$, and the F is that function defined in eqn (18.50). The advantage of this notation is that the number α is not needed to specify the function. However, in writing a new generating function, the α notation is useful at the start, for example to test whether \mathfrak{o} is or is not singular.

18.7 Example of a Mixed Generating Function

As an example, suppose that $D = 3$ and that we want to exchange the role of coordinates and momenta for the first two variables but make no change in the last two. Then we can choose $\alpha = 0011$ and

$$F = \sum_{k=0}^3 q_k Y_k
 \tag{18.52}$$

The matrix \mathfrak{o} defined in eqn (18.41) will then be the unit matrix and hence nonsingular as required. The F rules in eqn (18.42) will give

$$p_k = Y_k \quad X_k = q_k
 \tag{18.53}$$

With the chosen α , these equations can be written out using eqn (18.40) with $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$, giving

$$\begin{aligned}
 p_0 = Q_0 & \quad -P_0 = q_0 & p_1 = Q_1 & \quad -P_1 = q_1 \\
 p_2 = P_2 & \quad Q_2 = q_2 & p_3 = P_3 & \quad Q_3 = q_3
 \end{aligned}
 \tag{18.54}$$

as desired.

Using the alternate notation of eqn (18.50), the generating function in eqn (18.52) becomes

$$F = q_0 Q_0 + q_1 Q_1 + q_2 P_2 + q_3 P_3 \tag{18.55}$$

from which eqn (18.54) can be derived by use of eqn (18.51).

18.8 Finding Simple Generating Functions

A chosen generating function can be used to produce a canonical transformation. However, what if we already have the canonical transformation and want to find a generating function that would produce it? This converse problem is now treated. Every canonical transformation has some generating function that generates it.

Given any canonical transformation, the first step in finding a generating function is to find the proto-generating function $f(q, p)$ defined in Theorem 18.1.1. As proved there, such a function can always be found. A method for doing so is described in Theorem D.20.1. We assume now that $f(q, p)$ has been found.

Now we must examine the Jacobian matrix J for the given canonical transformation, considering it in the block form given in eqn (17.14),

$$J = \begin{pmatrix} \left(\frac{\partial Q}{\partial q} \right) & \left(\frac{\partial Q}{\partial p} \right) \\ \left(\frac{\partial P}{\partial q} \right) & \left(\frac{\partial P}{\partial p} \right) \end{pmatrix} \tag{18.56}$$

The next step depends on the singularity or nonsingularity of the blocks of this Jacobian matrix. We consider the four simple cases in which one or more of the $F_1, F_2, \tilde{F}_1, \tilde{F}_2$ generating functions are adequate.

First the F_1 case. If the block $(\partial Q/\partial p)$ has a nonzero determinant, then, by Theorem D.24.1, the transformation equation $Q_k = Q_k(q, p)$ can be solved for $p_k = p_k(q, Q)$. Putting this result into $f(q, p)$ then gives the desired generating function as a compound function

$$F_1(q, Q) = f(q, p(q, Q)) \tag{18.57}$$

The differential of this F_1 is found from eqn (18.1). It is

$$dF_1 = df = \sum_{k=0}^D p_k dq_k - \sum_{k=0}^D P_k dQ_k \tag{18.58}$$

Thus this function obeys the correct F_1 rules in eqn (18.14).

It also passes the test that the matrix m defined in eqn (18.13) must be nonsingular. After an exchange of order of the partial derivatives, that matrix can be written as

$$m_{ij} = \frac{\partial^2 F_1(q, Q)}{\partial Q_j \partial q_i} = \frac{\partial p_i(q, Q)}{\partial Q_j} = \left(\frac{\partial Q}{\partial p} \right)^{-1}_{ij} \tag{18.59}$$

which is nonsingular by the assumption that $(\partial Q/\partial p)$ is.

This result may be combined with Lemma 18.2.2 to obtain the following necessary and sufficient condition: There is an F_1 generating function which generates the original transformation if and only if $(\partial Q/\partial p)$ is nonsingular.

The other cases are similar. We will quote only the necessary and sufficient condition for each of them, and the method of constructing the generating function from $f(q, p)$.

The F_2 case: If the block $(\partial P/\partial p)$ of the matrix \mathbf{J} is nonsingular, then the transformation equation $P_k = P_k(q, p)$ can be solved for $p_k = p_k(q, P)$. The generating function $F_2(q, P)$ is then obtained by putting this result into the expression for F_2 derived from eqn (18.23),

$$F_2(q, P) = f(q, p(q, P)) + \sum_{k=0}^D Q_k(q, p(q, P)) P_k \quad (18.60)$$

An F_2 function can be used if and only if the block $(\partial P/\partial p)$ is nonsingular.

The \tilde{F}_1 case: If the block $(\partial P/\partial q)$ of the matrix \mathbf{J} is nonsingular, then the transformation equation $P_k = P_k(q, p)$ can be solved for $q_k = q_k(p, P)$. The generating function $\tilde{F}_1(p, P)$ is then obtained by putting this result into the expression for \tilde{F}_1 derived from eqn (18.35),

$$\tilde{F}_1(p, P) = -f(q(p, P), p) + \sum_{k=0}^D q_k(p, P) p_k - \sum_{k=0}^D Q_k(q(p, P), p) P_k \quad (18.61)$$

An \tilde{F}_1 function can be used if and only if the block $(\partial P/\partial q)$ is nonsingular.

The \tilde{F}_2 case: If the block $(\partial Q/\partial q)$ of the matrix \mathbf{J} is nonsingular, then the transformation equation $Q_k = Q_k(q, p)$ can be solved for $q_k = q_k(p, Q)$. The generating function $\tilde{F}_2(p, Q)$ is then obtained by putting this result into the expression for \tilde{F}_2 derived from eqn (18.37),

$$\tilde{F}_2(p, Q) = -f(q(p, Q), p) + \sum_{k=0}^D q_k(p, Q) p_k \quad (18.62)$$

An \tilde{F}_2 function can be used if and only if the block $(\partial Q/\partial q)$ is nonsingular.

18.9 Finding Mixed Generating Functions

In Section 18.8 we found a simple generating function when any block of eqn (18.56) is nonsingular.¹⁰⁷ Unfortunately, it can happen that all four of the blocks of \mathbf{J} are singular, so that no simple generating function can be used. However, regardless of the singularity or nonsingularity of any of the blocks of eqn (18.56), there will always

¹⁰⁷Note that these nonsingularity conditions need not be exclusive. Canonical transformations often can be generated by more than one of the simple generating functions. However, there are cases in which one and only one of the simple forms can be used (see Exercise 18.9).

be a mixed generating function, of the sort described in Section 18.6, that generates the original canonical transformation.

Assume as before that a proto-generating function $f(q, p)$ has been found. To find the mixed generating function, write variables X_k and Y_k defined in eqn (18.40) as functions of the original variables q, p ,

$$X_k = X_k(q, p) = \bar{\alpha}_k Q_k(q, p) - \alpha_k P_k(q, p) \quad Y_k = Y_k(q, p) = \alpha_k Q_k(q, p) + \bar{\alpha}_k P_k(q, p) \tag{18.63}$$

and choose (by inspection or by trial and error) the arbitrary binary number α defined in eqn (18.38) such that the matrix $(\partial Y/\partial p)$ whose elements are

$$\left(\frac{\partial Y}{\partial p}\right)_{ij} = \frac{\partial Y_i(q, p)}{\partial p_j} \tag{18.64}$$

is nonsingular.¹⁰⁸

It follows from the nonsingularity of the matrix in eqn (18.64) that $Y_k = Y_k(q, p)$ from eqn (18.63) can be solved for $p_k = p_k(q, Y)$. The generating function then is obtained by solving eqn (18.48) for F and substituting $p_k = p_k(q, Y)$ into the resulting expression,

$$F(q, Y) = f(q, p(q, Y)) + \sum_{k=0}^D \bar{\alpha}_k X_k(q, p(q, Y)) Y_k \tag{18.65}$$

By using eqn (18.1) for df , and then substituting eqn (18.45) and using eqn (18.39), the differential of this function is found to be

$$dF = df + \sum_{k=0}^D (\bar{\alpha}_k X_k dY_k + \bar{\alpha}_k Y_k dX_k) = \sum_{k=0}^D p_k dq_k + \sum_{k=0}^D X_k dY_k \tag{18.66}$$

Thus the function obeys the correct F rules in eqn (18.42). It also passes the test that the matrix \mathfrak{o} defined in eqn (18.41) must be nonsingular. After an exchange of orders of partial differentiation, that matrix becomes

$$o_{ij} = \frac{\partial^2 F(q, Y)}{\partial Y_j \partial q_i} = \frac{\partial p_i(q, Y)}{\partial Y_j} = \left(\frac{\partial Y}{\partial p}\right)_{ij}^{-1} \tag{18.67}$$

Due to the nonsingularity of the matrix in eqn (18.64), this matrix \mathfrak{o} is nonsingular, as required. Thus F is a legitimate generating function and, using the methods discussed in Section 18.6 for mixed generating functions, the original canonical transformation can be obtained.

Note to the Reader: A transformation can be generated by a mixed generating function if and only if the transformation is canonical.

¹⁰⁸The proof that such an α always can be found requires an excursion into the algebra of symplectic spaces. It is given in Theorem E.10.1.

18.10 Finding Mixed Generating Functions—An Example

Suppose we have a system with $D = 2$ degrees of freedom and a canonical transformation defined as

$$Q_0 = q_0 \quad P_0 = p_0 \quad Q_1 = p_1 \quad P_1 = (-q_1 - 3q_2) \quad Q_2 = q_2 \quad P_2 = (-3p_1 + p_2) \quad (18.68)$$

For this transformation, the Jacobian matrix J written in the block form of eqn (18.56) is

$$J = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 1 \end{pmatrix} \end{pmatrix} \quad (18.69)$$

where the four blocks have been indicated. Using eqn (17.37), or some other test, the reader may verify (as already done in Exercise 17.9) that the transformation is indeed canonical.

The first step to finding a generating function is to find the proto-generating function $f(q, p)$ whose existence is proved by Theorem 18.1.1. It must be a solution to the equation that results when eqn (18.68) are substituted into eqn (18.1),

$$\begin{aligned} df &= \sum_{k=0}^2 p_k dq_k - \sum_{k=0}^2 P_k dQ_k \\ &= \{p_0 dq_0 + p_1 dq_1 + p_2 dq_2\} - \{p_0 dq_0 - (q_1 + 3q_2) dp_1 + (-3p_1 + p_2) dq_2\} \\ &= p_1 dq_1 + 3p_1 dq_2 + (q_1 + 3q_2) dp_1 \end{aligned} \quad (18.70)$$

Using the method outlined in the proof of Theorem D.20.1,

$$\begin{aligned} f &= C + \int_{(0,0,0)}^{(q_1,0,0)} p_1 dq_1 + \int_{(q_1,0,0)}^{(q_1,q_2,0)} 3p_1 dq_2 + \int_{(q_1,q_2,0)}^{(q_1,q_2,p_1)} (q_1 + 3q_2) dp_1 \\ &= C + 0 + 0 + (q_1 + 3q_2) p_1 \end{aligned} \quad (18.71)$$

Note that integrations along the dq_0, dp_0, dp_2 directions have been omitted since the coefficients of these differentials are zero. Thus, dropping the unnecessary constant C , the proto-generating function is

$$f(q, p) = (q_1 + 3q_2) p_1 \quad (18.72)$$

Now, if the upper right block ($\partial Q/\partial p$) of matrix J were nonsingular, we could express eqn (18.72) as a function of q, Q and thus define an F_1 function like eqn (18.57). Also, if the lower right block ($\partial P/\partial p$) were nonsingular, we could express eqn (18.72) as a function of q, P and use it to define an F_2 function like eqn (18.60),

and so on. But in this example, all of the blocks of J are singular and therefore a mixed generating function must be sought.

Since the first and last variables have $Q_k = q_k$ while the middle one has $Q_1 = p_1$, a good guess is that an $\alpha = 010$ might work, which will be of F_2 type for $k = 0, 2$ and of F_1 type for $k = 1$. With that choice, eqn (18.63) becomes

$$\begin{aligned} X_0 = Q_0 = q_0 & & X_1 = -P_1 = (q_1 + 3q_2) & & X_2 = Q_2 = q_2 \\ Y_0 = P_0 = p_0 & & Y_1 = Q_1 = p_1 & & Y_2 = P_2 = -3p_1 + p_2 \end{aligned} \quad (18.73)$$

where the second equality in each expression has been obtained by using eqn (18.68). The matrix $(\partial Y/\partial p)$ is therefore

$$\left(\frac{\partial Y}{\partial p}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \quad (18.74)$$

which has determinant equal to +1 and hence is nonsingular. Using the values of α_k and its complement $\bar{\alpha}_k$ from the choice $\alpha = 010$, the mixed generating function from eqn (18.65) is then

$$\begin{aligned} F(q, Y) &= f(q, p(q, Y)) + X_0(q, p(q, Y))Y_0 + X_2(q, p(q, Y))Y_2 \\ &= q_0Y_0 + (q_1 + 3q_2)Y_1 + q_2Y_2 \end{aligned} \quad (18.75)$$

The original canonical transformation can be derived from this function, using the chosen $\alpha = 010$ and the methods of Lemma 18.6.1, as the reader may verify.

In the alternate notation discussed at the end of Theorem D.20.1, the mixed generating function eqn (18.75) becomes

$$F = q_0P_0 + (q_1 + 3q_2)Q_1 + q_2P_2 \quad (18.76)$$

So the original canonical transformation can also be derived by using this function and eqn (18.51), as may also be verified.

18.11 Traditional Generating Functions

The theory of canonical transformations was originally developed in the nineteenth century, before relativity had introduced the possibility that time might be something other than an absolute, un-transformable parameter. Thus the traditional canonical transformation theory does not allow time to transform, and the traditional generating functions make the same assumption.

The canonical transformations and generating functions found in most current textbooks are still the traditional versions. Since the reader will certainly have encountered these books, it will be useful to compare our treatment of canonical transformations and generating functions to the traditional one. The traditional generating functions and canonical transformations can be derived as a special case of

our methods—the case in which time is not allowed to transform and the restriction $Q_0 = q_0$ is applied.

First, consider the case of F_2 generating functions. A canonical transformation with the restriction $Q_0 = q_0$ assumed can be derived from

$$F_2(q, P) = q_0 P_0 + \bar{F}_2(q, P_{[0]}) \quad (18.77)$$

where $\bar{F}_2(q, P_{[0]})$ is the traditional generating function, and where we are continuing to use the notation, introduced in Section 11.2, in which $P_{[0]} = P_1, P_2, \dots, P_D$ denotes all of the P_k variables except P_0 . The F_2 rules in eqn (18.20) then give, for $k = 0$,

$$p_0 = P_0 + \frac{\partial \bar{F}_2(q, P_{[0]})}{\partial q_0} \quad Q_0 = q_0 \quad (18.78)$$

and, for $k = 1, \dots, D$,

$$p_k = \frac{\partial \bar{F}_2(q, P_{[0]})}{\partial q_k} \quad Q_k = \frac{\partial \bar{F}_2(q, P_{[0]})}{\partial P_k} \quad (18.79)$$

In Section 12.2, it was shown that the traditional Hamiltonian is the negative of the zeroth momentum variable. If we follow the notation of the present chapter in which old variables are lower case and new ones upper case, we can write this as

$$p_0 = -h(q_{[0]}, p_{[0]}, q_0) \quad (18.80)$$

and the same relation in the Q, P system as

$$P_0 = -H(Q_{[0]}, P_{[0]}, Q_0) \quad (18.81)$$

Equation (18.78) then becomes

$$H = h + \frac{\partial \bar{F}_2(q, P_{[0]})}{\partial q_0} \quad Q_0 = q_0 \quad (18.82)$$

With the replacement of q_0 by t , eqns (18.79, 18.82) correspond to the traditional formulas found in other texts.¹⁰⁹

In the case of general mixed generating functions, the pattern is similar. We can define canonical transformations with $Q_0 = q_0$ by restricting the choice of α in eqn (18.38) to binary numbers with $\alpha_0 = 0$ and hence a zero in the first digit, as in $\alpha = \alpha_D \cdots \alpha_1 0$, and writing the generating function

$$F(q, Y) = q_0 Y_0 + \bar{F}(q, Y_{[0]}) \quad (18.83)$$

Then the F rules in eqn (18.42) give, for $k = 0$,

$$p_0 = P_0 + \frac{\partial \bar{F}(q, Y_{[0]})}{\partial q_0} \quad Q_0 = q_0 \quad (18.84)$$

¹⁰⁹For example, see Chapter 9 of Goldstein, Poole and Safko (2002). They use K for the new Hamiltonian and H for the old one, and, of course, have no bar over the F_2 .

and, for $k = 1, \dots, D$,

$$p_k = \frac{\partial \bar{F}(q, Y_{[0]})}{\partial q_k} \quad X_k = \frac{\partial \bar{F}(q, Y_{[0]})}{\partial Y_k} \quad (18.85)$$

As before, eqn (18.84) may be written

$$H = h + \frac{\partial \bar{F}(q, Y_{[0]})}{\partial q_0} \quad (18.86)$$

Then eqns (18.85, 18.86) are the same as the traditional form¹¹⁰ of the mixed generating functions.

The traditional \bar{F}_1 generating functions are related to mixed generating functions with the special choice $\alpha = 1 \cdots 10$. In the alternative notation of eqns (18.50, 18.51), the mixed generating function

$$F(q, P_0, Q_{[0]}) = q_0 P_0 + \bar{F}_1(q, Q_{[0]}) \quad (18.87)$$

will lead to $Q_0 = q_0$ with the other quantities transforming by the traditional \bar{F}_1 rules

$$Q_0 = q_0 \quad H = h + \frac{\partial \bar{F}_1(q, Q_{[0]})}{\partial q_0} \quad (18.88)$$

$$p_k = \frac{\partial \bar{F}_1(q, Q_{[0]})}{\partial q_k} \quad P_k = -\frac{\partial \bar{F}_1(q, Q_{[0]})}{\partial Q_k} \quad (18.89)$$

Traditional forms related to our \tilde{F}_1 and \tilde{F}_2 functions can be treated similarly.

18.12 Standard Form of Extended Hamiltonian Recovered

In Section 12.2, we began by writing a phase-space dependency function in the form $\mathcal{K}(q, p) = 0$ where

$$\mathcal{K}(q, p) = p_0 + H(q, p_{[0]}) \quad (18.90)$$

This same function was used as the standard extended Hamiltonian. In Section 12.8, we showed how to write other, extended Hamiltonians equivalent to this standard one.

Now imagine that a general canonical transformation has been done. We have seen in Section 17.12 that the extended Hamiltonian is invariant under canonical transformations. The extended Hamiltonian in the new system is the same function, and is obtained simply by substitution to write it in terms of the new Q, P variables

$$\mathcal{K}(Q, P) = \mathcal{K}(q(Q, P), p(Q, P)) \quad (18.91)$$

It follows from the canonical transformation procedure that, even if we begin with the standard form in eqn (18.90), after a canonical transformation the extended Hamiltonian $\mathcal{K}(Q, P)$ will not in general have the same standard form.

¹¹⁰See Chapter 77 of Desloges (1982). Goldstein, Poole and Safko (2002) mention mixed generating functions but give few details.

However, it is always possible to write an extended Hamiltonian in the Q, P variable set that is equivalent to eqn (18.91) but that recovers the standard form. To see this, we note that Lemma 17.7.2 proves that at least one of the derivatives \dot{Q}_k, \dot{P}_k must be nonzero. And Exercises 18.4 and 18.5 demonstrate that the variables Q, P may be relabeled (and the exchange $Q_l, P_l \rightarrow P_l, -Q_l$ made for particular l values if necessary) by use of trivial canonical transformations. It is therefore always possible to relabel and exchange the variables in such a way that $\dot{Q}_0 \neq 0$ for the coordinate labeled with the zero index.

Assume now that the relabelling and exchanges have been done if necessary so that $\dot{Q}_0 \neq 0$. One of the extended Hamilton equations is then

$$\frac{\partial \mathcal{K}(Q, P)}{\partial P_0} = \dot{Q}_0 \neq 0 \quad (18.92)$$

which (by the implicit function theorem, Theorem D.26.1) is the condition for the dependency relation $\mathcal{K}(Q, P) = 0$ to be solved for P_0 and hence written in the standard form

$$\mathcal{K}^{(\text{st})}(Q, P) = P_0 + H^{(\text{st})}(Q, P_{[0]}) = 0 \quad (18.93)$$

The particular function $H^{(\text{st})}$ obtained will depend on the details of the process. The equivalence of the extended Hamilton equations with $\mathcal{K}^{(\text{st})}(Q, P)$ to those with $\mathcal{K}(Q, P)$ follows from the same argument used in Lemma 12.8.2. Thus the standard form is recovered.

The results of this section have the important consequence that proofs in canonical transformation theory may assume the standard form eqn (18.93) for the extended Hamiltonian, and the condition $\dot{Q}_0 \neq 0$, with no loss of generality. The variable Q_0 will not in general be related to the time variable t , but the standard form is recovered nonetheless.

18.13 Differential Canonical Transformations

Differential canonical transformations (sometimes called infinitesimal canonical transformations) are those that differ from the identity transformation only by a differential scale factor δa that can approach zero as a limit. (This use of δa is similar to that in the calculus of variations in Section 5.1.) Since eqn (18.27) generates the identity, we may write the generating function of differential canonical transformations in the general form

$$F_2(q, P) = \sum_{k=0}^D q_k P_k + \delta a G(q, P) \quad (18.94)$$

where $G(q, P)$ is an arbitrary function of its stated variables, called the *generator of differential canonical transformations*. Applying the F_2 rules from eqn (18.20) gives

$$p_k = P_k + \delta a \frac{\partial G(q, P)}{\partial q_k} \quad Q_k = q_k + \delta a \frac{\partial G(q, P)}{\partial P_k} \quad (18.95)$$

Since the partial derivatives in each of eqn (18.95) are already multiplied by the small quantity δa , in the limit $\delta a \rightarrow 0$ we may write the differential canonical transforma-

tion as

$$Q_k = q_k + \delta a \frac{\partial G(q, p)}{\partial p_k} + o(\delta a) \quad P_k = p_k - \delta a \frac{\partial G(q, p)}{\partial q_k} + o(\delta a) \quad (18.96)$$

18.14 Active Canonical Transformations

Section 8.30 discussed active and passive rotations. In summary, active rotations change the system orientation while passive rotations only change the variables in terms of which an unchanged system orientation is expressed. A similar pattern holds for canonical transformations.

All canonical transformations treated up to now have been passive. Physical quantities are represented by phase-space functions $f(q, p)$ whose values are not changed by passive canonical transformations. They are simply re-expressed in terms of new variables, according to the standard rule

$$f(Q, P) = f(q(Q, P), p(Q, P)) \quad (18.97)$$

The extended Hamiltonian function $\mathcal{K}(q, p)$ follows this same rule, as seen in eqn (18.91).

We now consider active use of the same canonical transformations. In active canonical transformations, the transformation equations

$$Q_k = Q_k(q, p) \quad P_k = P_k(q, p) \quad (18.98)$$

are assumed to define new *values* for the variables q, p , without changing the coordinate system in which the phase-space functions are expressed. The values of the coordinates q, p are changed, with changes Δq_k and Δp_k defined by

$$\Delta q_k = Q_k(q, p) - q_k \quad \Delta p_k = P_k(q, p) - p_k \quad (18.99)$$

In the general case, this definition of active canonical transformations might not prove useful because, for example, the canonical transformation might be one that exchanges the role of coordinates and momenta. But it is very useful for the differential canonical transformations defined in Section 18.13 where it leads to $\Delta q_k = \delta q_k + o(\delta a)$ and $\Delta p_k = \delta p_k + o(\delta a)$ where the differential changes in the coordinates q, p are given by¹¹¹

$$\delta q_k = \delta a \frac{\partial G(q, p)}{\partial p_k} \quad \delta p_k = -\delta a \frac{\partial G(q, p)}{\partial q_k} \quad (18.100)$$

An immediate consequence is that the change Δf in the *value* of a phase-space function $f(q, p)$ is given by $\Delta f = \delta f + o(\delta a)$ where the differential change δf is

$$\delta f = \delta a \sum_{k=0}^D \left(\frac{\partial f}{\partial q_k} \delta q_k + \frac{\partial f}{\partial p_k} \delta p_k \right) = \delta a \sum_{k=0}^D \left(\frac{\partial f}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial G}{\partial q_k} \right) = \delta a [f, G] \quad (18.101)$$

where the last expression is the Poisson bracket as defined in Definition 12.12.1.

¹¹¹The general relation between differences like Δq_k and differentials like δq_k is discussed in Section D.12.

For an example, consider a system consisting of a single particle in three dimensions using Cartesian coordinates, and a generating function equal to the z -component of angular momentum,

$$G = xp_y - yp_x \quad (18.102)$$

In this case, eqn (18.100) gives

$$\delta x = -\delta a y \quad \delta y = \delta a x \quad \delta p_x = -\delta a p_y \quad \delta p_y = \delta a p_x \quad (18.103)$$

which are the same changes as are produced by an active rotation by differential angle δa about the z -axis (see Exercise 18.7). Thus the z -component of angular momentum generates rotations about the z -axis.

18.15 Phase-Space Analog of Noether Theorem

Noether's theorem, discussed in Section 13.2, allows symmetries of an extended Lagrangian system to be used to discover conserved quantities. The definition of active differential canonical transformations in Section 18.14 allows us to extend the same idea to phase space and Hamiltonian mechanics.

We define a *symmetry* in Hamiltonian mechanics to be an active differential canonical transformation that does not change the differential *value* of the extended Hamiltonian $\mathcal{K}(q, p)$, considered as a phase-space function. Applying eqn (18.101) to the extended Hamiltonian $\mathcal{K}(q, p)$ gives the differential change in its value due to a transformation generated by G as

$$\delta\mathcal{K} = \delta a [\mathcal{K}, G] \quad (18.104)$$

If $\delta\mathcal{K} = 0$ for a particular generator G , then we say that G generates a symmetry of the extended Hamiltonian system.

But $G(q, p)$ is just an ordinary phase-space function and so represents some property of the physical system, leading to the following theorem.

Theorem 18.15.1: Phase-Space Symmetry Theorem

If G generates a symmetry of the system, then G is a constant of the motion (i.e. is conserved) and the physical quantity it represents will remain constant as the system evolves.

Proof: We have proved in Section 12.12 that the rate of change of a phase-space function is given by its Poisson bracket with the extended Hamiltonian. Thus, substituting G for f in eqn (12.55) gives

$$\dot{G} = [G, \mathcal{K}] = -[\mathcal{K}, G] \quad (18.105)$$

where the anti-symmetry of the Poisson bracket has been used. If G generates a symmetry of the system, then eqn (18.104) and $\delta\mathcal{K} = 0$ together imply that $[\mathcal{K}, G] = 0$ and hence that $\dot{G} = 0$. Thus the generator G of a symmetry of the system will be a conserved quantity. \square

As an example of the phase-space symmetry theorem, consider a system of one particle in a Cartesian space with extended Hamiltonian

$$\mathcal{K} = p_0 + \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2}k(x^2 + y^2 + z^2) \tag{18.106}$$

Since this equation can also be written as $\mathcal{K} = p_0 + (\mathbf{p} \cdot \mathbf{p}) / 2m + k(\mathbf{r} \cdot \mathbf{r}) / 2$, which involves only dot products of threevectors, the rotation generated by eqn (18.102) will not change the value of \mathcal{K} , and hence that $\delta\mathcal{K} = 0$ for this canonical transformation. It follows that $G = xp_y - yp_x$ will be a constant of the motion, and that the z -component of angular momentum will be a conserved quantity.

18.16 Liouville Theorem

An important example of the active differential canonical transformations developed in Section 18.14 uses the extended Hamiltonian itself as the generator. If $G = \mathcal{K}$ then eqn (18.101) becomes

$$\delta f = \delta a [f, \mathcal{K}] = \delta a \frac{df}{d\beta} \tag{18.107}$$

where eqn (12.55) has been used. Thus \mathcal{K} generates the natural evolution of the system. In the $\delta a \rightarrow 0$ limit, the value $f + \delta f$ is the value of physical quantity f at the epoch $\beta + \delta a$. This evolution can also be seen from eqn (18.100), which becomes

$$\delta q_k = \delta a \frac{\partial \mathcal{K}(q, p)}{\partial p_k} = \delta a \dot{q}_k \qquad \delta p_k = -\delta a \frac{\partial \mathcal{K}(q, p)}{\partial q_k} = \delta a \dot{p}_k \tag{18.108}$$

Thus the natural evolution of the system point in phase space can be reproduced as an active differential canonical transformation generated by the extended Hamiltonian \mathcal{K} .

Consider now an integral defining the “volume” \mathcal{V} of phase space contained within a region defined by some boundary,

$$\mathcal{V} = \int dq_0 dq_1 \dots dq_D dp_0 dp_1 \dots dp_D \tag{18.109}$$

This phase-space volume will remain constant as the system evolves.

Theorem 18.16.1: Liouville Theorem

If a volume in phase space \mathcal{V} is defined as in eqn (18.109), then as the system evolves the boundaries of the region will change but the integral over the evolving region will be constant.

Proof: The natural evolution of the system has been shown to be an active differential canonical transformation. Lemma D.22.4 with $f = 1$ implies that the integral in eqn (18.109) transforms as

$$\int dQ_0 dQ_1 \dots dQ_D dP_0 dP_1 \dots dP_D = \int dq_0 dq_1 \dots dq_D dp_0 dp_1 \dots dp_D \left| \frac{\partial(Q, P)}{\partial(q, p)} \right| \tag{18.110}$$

when a canonical transformation changes the variables from q, p to Q, P . But the Jacobian in eqn (18.110) is the same as the determinant of the matrix \mathbf{J} , which was

shown in eqn (17.44) to have absolute value unity

$$\frac{\partial(Q, P)}{\partial(q, p)} = |J| = \pm 1 \quad (18.111)$$

Therefore the two integrals in eqn (18.110) are equal. In the active canonical transformation, however, the variables Q, P are just the values of q, p at epoch $\beta + \delta a$ and hence their integral in eqn (18.110) is the same as the integral in eqn (18.109) at $\beta + \delta a$. The equality of these two integrals proves that \mathcal{V} is a constant. \square

This theorem has an important consequence when one considers what may be called *Hamiltonian flow* problems. These consider a large set of identical systems, all of which obey the same extended Hamilton equations but generally with different initial conditions. Each system is represented by a point in phase space, which we may call a *system point*. Since any point in extended phase space has a unique trajectory passing through it, these system points will evolve along unique and non-intersecting trajectories. Therefore, if one defines a region in phase space by defining a boundary, the number of system points inside that boundary will be constant. (If a point reached a boundary, it could never cross it.) Thus a density function of system points per unit phase-space volume, $\rho = N/\mathcal{V}$, will have constant numerator and denominator and thus be a constant of the motion.

18.17 Exercises

Exercise 18.1

- (a) Demonstrate that each term of the $F_1, F_2, \tilde{F}_1, \tilde{F}_2$, and F generating functions must have the units of action.
 (b) Show that the product $X_k Y_k$ of the variables defined in eqn (18.40) will have units of action regardless of the choice of the binary number α .

Exercise 18.2

- (a) Write an F_2 generating function that will generate the general Lorentz transformation treated in Exercise 17.2.
 (b) Demonstrate that this F_2 does indeed generate both the Lorentz transformation and its inverse.

Exercise 18.3

Consider the canonical transformation in Exercise 17.4.

- (a) Find the proto-generating function $f(q, p)$ which obeys eqn (18.1),

$$df = \sum_{k=1}^D p_k \delta q_k - \sum_{k=1}^D P_k dQ \quad (18.112)$$

[Hint: $f(q, p) = \sum_{k=1}^D f_k(q_k, p_k)$ where each $f_k(q_k, p_k)$ is found by integrating $df_k = p_k dq_k - P_k dQ_k$ in the q_k, p_k plane.]

- (b) Use $f(q, p)$ to derive an $F_1(q, Q)$ generating function. What are the conditions on the θ_i for this generating function to be valid? Use it to re-derive the canonical transformation in Exercise 17.4.

(c) Use $f(q, p)$ to derive an $F_2(q, P)$ generating function. What are the conditions on the θ_i for this generating function to be valid? Use it to re-derive the canonical transformation in Exercise 17.4.

Exercise 18.4 Let k_0, k_1, \dots, k_D be any rearrangement of the integers $0, 1, 2, \dots, D$.

(a) Show that

$$F_2(q, P) = \sum_{i=0}^D q_{k_i} P_i \quad (18.113)$$

satisfies $|\mathbf{n}| \neq 0$ for the matrix defined in eqn (18.19), and hence is a legitimate F_2 generating function.

(b) Find the canonical transformation generated by this generating function, and hence show that any rearrangement of the variables q_k is a canonical transformation *provided that the same* rearrangement is applied to the p_k variables.

Exercise 18.5 Let $\alpha = \alpha_D \alpha_{D-1} \dots \alpha_0$ be the binary number as discussed in Section 18.6.

(a) Show that the mixed generating function

$$F(q, Y) = \sum_{k=0}^D q_k Y_k \quad (18.114)$$

has the effect of making the exchange $Q_k = p_k$ and $P_k = -q_k$ for those index values k with $\alpha_k = 1$ while acting as the identity with $Q_k = q_k$ and $P_k = p_k$ for those k with $\alpha_k = 0$.

Exercise 18.6 Section 18.4 gave the example of the generating function $F_2(q, P) = \sum_{k=0}^D g_k(q) P_k$ that generated transformations of the Lagrangian type, defined as transformations with $Q_k = Q_k(q)$ only. Now consider a generating function

$$F_2(q, P) = \sum_{k=0}^D q_k f_k(P) \quad (18.115)$$

(a) What condition must be imposed on the matrix with elements $\partial f_k(P)/\partial P_l$ in order for this to be a legitimate generating function?

(b) Show that in this transformation, the new momenta P_k are functions only of the old momenta p .

(c) Is this transformation of the Lagrangian type? If so why, if not why not?

Exercise 18.7

(a) Verify that eqn (18.103) follow from generating function eqn (18.102).

(b) Show that the relations in eqn (18.103) are the same as would result from active rotation by angle δa about the z-axis. That is, show that

$$\mathbf{r} + \delta \mathbf{r} = \mathcal{R}[\delta a \hat{\mathbf{e}}_3] \mathbf{r} + o(\delta a) \quad \mathbf{p} + \delta \mathbf{p} = \mathcal{R}[\delta a \hat{\mathbf{e}}_3] \mathbf{p} + o(\delta a) \quad (18.116)$$

where $\mathcal{R}[\delta a \hat{\mathbf{e}}_3]$ is the fixed-axis rotation operator defined in Section 8.17 and 8.18.

Exercise 18.8 The Hamilton-Jacobi theory to be presented in Chapter 19 allows us to find generating functions that simplify particular Hamiltonians. The harmonic oscillator in one dimension has an extended Hamiltonian in the q, p system

$$\mathcal{K}(q, p) = p_0 + \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1 \quad (18.117)$$

Sections 19.8 and 19.9 provide the generating function (see Exercise 19.7)

$$F_2(q, P) = q_0 (P_0 - P_1) + \frac{m\omega q_1}{2} \sqrt{\frac{2P_1}{m\omega^2} - q_1^2} + \frac{P_1}{\omega} \arcsin \left(q_1 \sqrt{\frac{m\omega^2}{2P_1}} \right) \quad (18.118)$$

(a) Find the transformed momenta $P_0(q, p)$ and $P_1(q, p)$, and demonstrate that, when expressed in the Q, P system, the extended Hamiltonian becomes simply $\mathcal{K}(Q, P) = P_0$.

(b) Show that all of the coordinates and momenta of the Q, P system are constants of the motion, except for $Q_0 = q_0 = t$.

(c) Use the inverse relation $q_1(Q, P)$ derived from the generating function to give a general solution for q_1 as a function of time and suitable constants to be determined at time zero.

Exercise 18.9 Consider the following four canonical transformations

$$\begin{aligned} \#1 : \quad Q_0 &= -p_0 & Q_1 &= p_1 + aq_1 & P_0 &= q_0 & P_1 &= p_1/a \\ \#2 : \quad Q_0 &= q_0 & Q_1 &= -p_1/a & P_0 &= p_0 & P_1 &= -p_1 + aq_1 \\ \tilde{\#1} : \quad Q_0 &= -p_0 & Q_1 &= q_1/a & P_0 &= q_0 & P_1 &= -q_1 + ap_1 \\ \tilde{\#2} : \quad Q_0 &= q_0 & Q_1 &= q_1 - ap_1 & P_0 &= p_0 & P_1 &= q_1/a \end{aligned} \quad (18.119)$$

(a) Write out the Jacobi matrix J in the form given in eqn (18.56) for each of these transformations.

(b) Verify that, of the four simple generating functions, #1 can only be generated by an F_1 , #2 only by an F_2 , #1 only by an \tilde{F}_1 , and #2 only by an \tilde{F}_2 .

Exercise 18.10 Consider the canonical transformation $\tilde{\#2}$ in eqn (18.119).

(a) Find the proto-generating function $f(q, p)$ and use it to write a generating function $\tilde{F}_2(p, Q)$. Verify your work by re-deriving eqn (18.119) from your generating function.

(b) Using the same $f(q, p)$ as in part (a), choose an appropriate binary number $\alpha = \alpha_1\alpha_0$ and write a mixed generating function $F(q, Y)$ for this canonical transformation. Again, verify your work by re-deriving $\tilde{\#2}$ in eqn (18.119) from your generating function.

Exercise 18.11 Consider the gauge transformation treated in Exercises 2.7, 11.8, and 12.7. Let the $\underline{p}_0, \mathbf{p}$ before the gauge transformation be the p variables, and the $\underline{p}'_0, \mathbf{p}'$ after the gauge transformation be the transformed P variables, where $q = t, x, y, z$ and the \underline{p}_0 and \underline{p}'_0 are the “standard” momenta discussed at the end of Section 16.9. Assume that a gauge transformation makes no change in the coordinates, so that $Q_k = q_k$.

(a) Use the Poisson bracket conditions to prove that this transformation is canonical.

(b) Find an F_2 generating function, and demonstrate that it does generate the transformation.

(c) Use the standard Hamiltonian $\mathcal{K}(q, p)$ from Exercise 12.7 and the simple substitution

rule eqn (17.75) to write the extended Hamiltonian $\mathcal{K}(Q, P)$. Put the $\mathcal{K}(Q, P)$ into a form that contains the gauge-transformed fields Φ' and \mathbf{A}' but not the gauge function χ .

(d) The form invariance of the traditional Hamilton equations under gauge transformations was derived with some labor in Exercise 4.6. Show that the form invariance of the extended Hamilton equations under gauge transformations follows at once from Theorem 17.12.1.

Exercise 18.12 Consider a system consisting of a single mass with phase-space variables $q, p = t, x, y, z, p_0, p_x, p_y, p_z$. Let $\hat{\mathbf{n}}$ be an arbitrary, constant unit vector.

(a) Show that the differential generating function $G = \hat{\mathbf{n}} \cdot \mathbf{p}$ generates a translation $\delta \mathbf{r} = \delta a \hat{\mathbf{n}}$ with no change in \mathbf{p} .

(b) Show that the differential generating function $G = -\hat{\mathbf{n}} \cdot \mathbf{r}$ generates a change of momentum $\delta \mathbf{p} = \delta a \hat{\mathbf{n}}$ with no change in position.

Exercise 18.13 Consider a system consisting of a single mass with phase-space variables $q, p = t, x, y, z, p_0, p_x, p_y, p_z$. Let the generating function be $G = \hat{\mathbf{n}} \cdot \mathbf{L}$ where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum vector and $\hat{\mathbf{n}}$ is an arbitrary, constant unit vector.

(a) Show that the differential changes of position and momentum are

$$\delta \mathbf{r} = \delta a \hat{\mathbf{n}} \times \mathbf{r} \quad \delta \mathbf{p} = \delta a \hat{\mathbf{n}} \times \mathbf{p} \quad (18.120)$$

(b) Show that the generating function $G = \hat{\mathbf{n}} \cdot \hat{\mathbf{L}}$ generates a differential rotation of \mathbf{r} and \mathbf{p} by angle δa about axis $\hat{\mathbf{n}}$.

Exercise 18.14 In Section 17.10 we showed that canonical transformations form a group, with group multiplication defined as successive transformation. We now show that this group is not *Abelian*. (The product of two canonical transformations *does* depend on their order.) Suppose that we make two successive active, differential canonical transformations, first with generator G_2 and then with generator G_1 , as in

$$q, p \xrightarrow{G_2} \bar{Q}, \bar{P} \xrightarrow{G_1} Q, P \quad (18.121)$$

(a) Define $\Delta q_k^{(2)} = \bar{Q}_k - q_k$, with similar definitions for the p variables. Use eqn (18.95) to prove that, when terms up to quadratic order in δa are retained,

$$\begin{aligned} \Delta q_k^{(2)} &= \delta a \frac{\partial G_2(q, \bar{P})}{\partial \bar{P}_k} = \delta a \frac{\partial G_2(q, p)}{\partial p_k} - (\delta a)^2 \sum_{l=0}^D \frac{\partial^2 G_2(q, p)}{\partial p_k \partial p_l} \frac{\partial G_2(q, p)}{\partial q_l} + o((\delta a)^2) \\ \Delta p_k^{(2)} &= -\delta a \frac{\partial G_2(q, \bar{P})}{\partial q_k} = -\delta a \frac{\partial G_2(q, p)}{\partial q_k} + (\delta a)^2 \sum_{l=0}^D \frac{\partial^2 G_2(q, p)}{\partial q_k \partial p_l} \frac{\partial G_2(q, p)}{\partial q_l} + o((\delta a)^2) \end{aligned} \quad (18.122)$$

(Recall that $o((\delta a)^2)$ means that the dropped terms are of order *smaller* than $(\delta a)^2$ as $\delta a \rightarrow 0$. See the definitions in Section D.11.)

(b) Define $\Delta q_k^{(1,2)} = Q_k - q_k$ where Q_k is the final value in eqn (18.121). Define $\Delta q_k^{(2,1)}$ similarly, but with the order of application of G_1 and G_2 reversed. Make similar definitions

for the p variables. Show that

$$\begin{aligned}\Delta q_k^{(1,2)} - \Delta q_k^{(2,1)} &= (\delta a)^2 \frac{\partial}{\partial p_k} [G_1(q, p), G_2(q, p)] + o((\delta a)^2) \\ \Delta p_k^{(1,2)} - \Delta p_k^{(2,1)} &= -(\delta a)^2 \frac{\partial}{\partial q_k} [G_1(q, p), G_2(q, p)] + o((\delta a)^2)\end{aligned}\quad (18.123)$$

Thus two successive canonical transformations produce a different result when their order of application is reversed.

(c) Compare eqn (18.123) with eqn (18.100). Show that, not only do the canonical transformations in reversed order produce different changes, but (when the calculation is carried to second order) the difference itself is a differential canonical transformation generated by a generating function $G = [G_1, G_2]$ equal to the Poisson bracket of G_1 and G_2 .

(d) Write a short paragraph justifying the following statement (or demolishing it if you disagree): Since L_x generates rotations about the x -axis, L_y generates rotations about the y -axis, and L_z generates rotations about the z -axis, the Poisson bracket of L_x and L_y *must be* L_z (as is shown by direct calculation in eqn (12.75)), due to the structure of the rotation group as demonstrated in Exercise 8.11.

Exercise 18.15

(a) Show that the transformation of coordinates in eqn (18.40) and the inverse relation in eqn (18.45) may be written in matrix form as

$$\begin{pmatrix} [X] \\ [Y] \end{pmatrix} = \mathbf{A} \begin{pmatrix} [Q] \\ [P] \end{pmatrix} \quad \begin{pmatrix} [Q] \\ [P] \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} [X] \\ [Y] \end{pmatrix}\quad (18.124)$$

where \mathbf{A} is the matrix defined in eqn (E.27).

(b) Verify that matrix \mathbf{A} is both canonical and orthogonal, i.e., that it satisfies both $\mathbf{s} = \mathbf{A}^T \mathbf{s} \mathbf{A}$ and $\mathbf{A}^T = \mathbf{A}^{-1}$.

Exercise 18.16 It is asserted that the canonical transformation in Exercise 17.11 can be derived from the generating function

$$F_2(q, P) = bq_0^2 P_0 + q_1 P_1 + (aq_0 + q_2) P_2 - \frac{c}{3} (aq_0 + q_2)^3 \quad (18.125)$$

(a) Verify that this function does generate eqn (17.100).

(b) Use the F_2 from eqn (18.125) to derive the inverse canonical transformation, giving q, p as functions of Q, P .

(c) The “direct conditions” for a transformation to be canonical (see Lemma 17.8.2) assert, among other relations, that $\partial P_2(q, p) / \partial q_0 = -\partial p_0(Q, P) / \partial Q_2$ when both sides are written in terms of some common sets of coordinates. Verify that this relation is true for the present transformation.

HAMILTON–JACOBI THEORY

The Hamilton–Jacobi theory is the apotheosis of Lagrangian and Hamiltonian mechanics: Action functions encode *all* of the possible trajectories of a mechanical system satisfying certain criteria. These action functions are the solutions of a nonlinear, first-order partial differential equation, called the Hamilton–Jacobi equation. The characteristic equations of this differential equation are the extended Hamilton equations defined in Chapter 12. Solution of a class of mechanics problems is thus reduced to the solution of a single partial differential equation.

Aside from its use as a problem solving tool, the Hamilton–Jacobi theory has particular importance because of its close relation to the Schroedinger formulation of quantum mechanics. We discuss that connection, and the Bohm hidden variable model and Feynman path integral method that are derived from it.

19.1 Definition of the Action

The trajectory of a mechanical system in configuration space¹¹² is typically specified by giving an initial value $q^{(1)} = q_0^{(1)}, q_1^{(1)}, \dots, q_D^{(1)}$ for all of the generalized coordinates, together with the set of initial generalized velocities $\dot{q}^{(1)}$. However, except in special cases that can be dealt with by partitioning the configuration space, a trajectory can also be specified by giving initial and final values of q . We assume here that there is a unique trajectory $q = q(\beta)$, called the classical path, that solves the Lagrange equations and passes through the points and times $q^{(1)}$ and $q^{(2)}$.

The first Hamilton–Jacobi action function $S_1(q^{(1)}, q^{(2)})$ is defined as the line integral of the extended Lagrangian along the unique classical path between the specified endpoints,

$$S_1(q^{(2)}, q^{(1)}) = \int_{q^{(1)}}^{q^{(2)}} \mathcal{L}(q, \dot{q}) d\beta \quad (19.1)$$

Although this integral resembles that used to define the action function of Hamilton’s Principle $I(\delta a, [q], [\eta])$ in eqn (13.1), there is an important difference. The Hamilton’s Principle action was an integral over varied and unvaried paths, taking different values as the path was varied. The Hamilton–Jacobi action function in eqn (19.1) is by definition *always* taken over the unique classical path (path that is a solution of the Lagrange equations) linking the end points.

¹¹²In this chapter, we continue to use the extended Lagrangian and Hamiltonian methods in which the zeroth generalized coordinate is the time, and the zeroth component of the momentum is the negative of the traditional Hamiltonian function. We also continue to use the notation q for the set of all generalized coordinates and $q_{[0]}$ for the set that includes all except $q_0 = t$.

Using the definition of \mathcal{L} from eqn (11.7), the action S_1 may be written equivalently in terms of the traditional Lagrangian function, as

$$S_1(q^{(2)}, q^{(1)}) = \int_{q^{(1)}}^{q^{(2)}} L\left(q_{[0]}, \frac{dq_{[0]}}{dt}, t\right) i d\beta = \int_{q_{[0]}^{(1)}, t^{(1)}}^{q_{[0]}^{(2)}, t^{(2)}} L\left(q_{[0]}, \frac{dq_{[0]}}{dt}, t\right) dt \quad (19.2)$$

where the integral is taken along the assumedly unique classical path connecting the end points and times. This traditional form is sometimes useful in model problems.

19.2 Momenta from the S_1 Action Function

Using the expression for \mathcal{L} from Lemma 11.4.2, the action can also be written as

$$S_1(q^{(2)}, q^{(1)}) = \int_{q^{(1)}}^{q^{(2)}} \sum_{k=0}^D p_k(q, \dot{q}) \dot{q}_k d\beta = \int_{q^{(1)}}^{q^{(2)}} \sum_{k=0}^D p_k(q, \dot{q}) dq_k \quad (19.3)$$

It might seem obvious from this expression that the partial derivatives of S_1 with respect to the coordinate q_k would give the momentum p_k . However, care is required because the definition of S_1 includes the requirement that the integral must be along the classical path between its endpoints. The partial derivative with respect to $q_k^{(2)}$ is

$$\begin{aligned} \frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_k^{(2)}} &= \\ \lim_{\delta q_k^{(2)} \rightarrow 0} &\left\{ \frac{S_1(q_0^{(2)}, \dots, q_k^{(2)} + \delta q_k^{(2)}, \dots, q_D^{(2)}, q^{(1)}) - S_1(q_0^{(2)}, \dots, q_k^{(2)}, \dots, q_D^{(2)}, q^{(1)})}{\delta q_k^{(2)}} \right\} \\ &= \lim_{\delta q_k^{(2)} \rightarrow 0} \frac{1}{\delta q_k^{(2)}} \left(\int_{q_k^{(1)}}^{q_k^{(1)} + \delta q_k^{(2)}} \mathcal{L}(q, \dot{q}) d\beta - \int_{q_k^{(1)}}^{q_k^{(2)}} \mathcal{L}(q, \dot{q}) d\beta \right) \quad (19.4) \end{aligned}$$

The first integral is along the classical path from $q^{(1)}$ to $q^{(2)} + \delta q^{(2)}$, which is *not* the same as the classical path between $q^{(1)}$ and $q^{(2)}$ used in the second integral.

The difference of integrals in eqn (19.4) can be evaluated by using the technique used in the proof of Noether’s theorem in Section 13.2. The classical path from $q^{(1)}$ to $q^{(2)} + \delta q^{(2)}$ can be considered a “varied” path, in the sense of the calculus of variations, relative to the “unvaried” classical path from $q^{(1)}$ to $q^{(2)}$. The first-order difference of the integrals is then the same as the first-order variation of the integral δI under this change of path. The results of Section 5.4 can then be applied, with the result

$$\delta I = \left(\sum_{k=0}^D \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \delta q_k(\beta) \right) \Big|_{\beta_1}^{\beta_2} - \int_{\beta_1}^{\beta_2} \sum_{k=0}^D \left\{ \frac{d}{d\beta} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_k} \right\} \delta q_k(\beta) d\beta \quad (19.5)$$

Since the Lagrange equations hold on the unvaried classical path, the integral term is identically zero. Since only the k th coordinate of the upper limit is varied here,

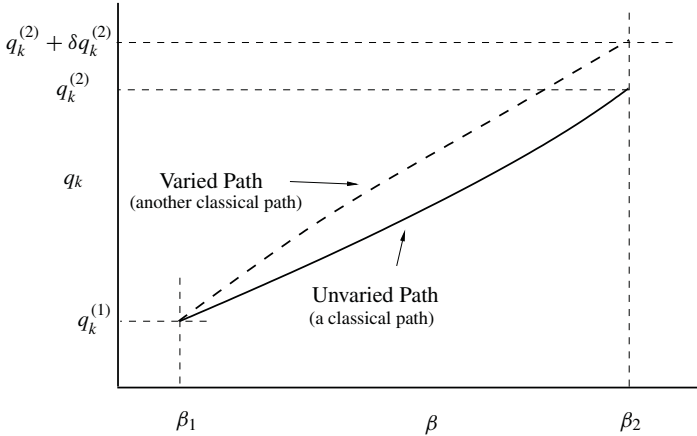


FIG. 19.1. Use of the calculus of variations to calculate $\partial S_1/\partial q_k$. The varied path is another classical path, but with an end point incremented by $\delta q_k^{(2)}$.

evaluation of the integrated term gives

$$\begin{aligned} \frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_k^{(2)}} &= \lim_{\delta q_k^{(2)} \rightarrow 0} \frac{\delta I}{\delta q_k^{(2)}} = \lim_{\delta q_k^{(2)} \rightarrow 0} \frac{1}{\delta q_k^{(2)}} \left(\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_k} \delta q_k(\beta) \right) \Big|_{\beta_1}^{\beta_2} \\ &= p_k(q^{(2)}, \dot{q}^{(2)}(q^{(2)}, q^{(1)})) = p_k^{(2)}(q^{(2)}, q^{(1)}) \end{aligned} \quad (19.6)$$

where we have made use of the fact that $\dot{q}^{(2)}$ depends on the path by which the upper end point was reached, and hence is a function of variables $q^{(2)}, q^{(1)}$ that determine that path.

A similar argument applies when the lower end point is varied. In summary, the partial derivatives for any $k = 0, 1, \dots, D$ are

$$p_k^{(2)}(q^{(2)}, q^{(1)}) = \frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_k^{(2)}} \quad \text{and} \quad p_k^{(1)}(q^{(2)}, q^{(1)}) = \frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_k^{(1)}} \quad (19.7)$$

The first of these momenta, for example, may be read as, “the k th momentum at the point and time $q^{(2)}$ of that particular motion that reached point and time $q^{(2)}$ from a beginning at point and time $q^{(1)}$.” Even in those cases where q_k is an ignorable coordinate so that p_k is constant along any particular classical path, these $p_k^{(2)}(q^{(2)}, q^{(1)})$ will still vary as $q^{(1)}$ and $q^{(2)}$, and hence the chosen classical path, are changed. The momenta $p_k^{(1)}(q^{(2)}, q^{(1)})$ have a similar meaning.

The momenta with $k = 0$ deserve special attention. Since the momenta in eqn

(19.6) are derived from the Lagrangian function, the identity eqn (12.5) holds and

$$\frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_0^{(2)}} = p_0^{(2)}(q^{(2)}, q^{(1)}) = -H(q_{[0]}^{(2)}, p_{[0]}^{(2)}(q^{(2)}, q^{(1)}), q_0^{(2)}) \quad (19.8)$$

$$\frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_0^{(1)}} = -p_0^{(1)}(q^{(2)}, q^{(1)}) = H(q_{[0]}^{(1)}, p_{[0]}^{(1)}(q^{(2)}, q^{(1)}), q_0^{(1)}) \quad (19.9)$$

For example, the first of these can be read as, “the negative of the value of the traditional Hamiltonian function at $q^{(2)}$ for that particular motion that reached $q^{(2)}$ from a beginning at $q^{(1)}$.” Like the other momenta, the traditional Hamiltonian function will vary with $q^{(1)}$ and $q^{(2)}$ even in those cases in which it is conserved along any particular system path.

19.3 The S_2 Action Function

An action function that depends on the initial momenta $p_{[0]}^{(1)}$ rather than the initial positions $q_{[0]}^{(1)}$ can be derived by a Legendre transformation that exchanges the role of these two quantities. Following the pattern in Section D.30, with $f \rightarrow S_1$, $g \rightarrow -S_2$, $x \rightarrow q_{[0]}$, $u \rightarrow -p_{[0]}$, we define

$$S_2(q^{(2)}, q^{(1)}) = S_1(q^{(2)}, q^{(1)}) - \sum_{i=1}^D q_i^{(1)} \frac{\partial S_1(q^{(2)}, q^{(1)})}{\partial q_i^{(1)}} = S_1 + \sum_{i=1}^D q_i^{(1)} p_i^{(1)} \quad (19.10)$$

Since a classical path also may be specified by its initial momenta and final position, the equations $p_i^{(1)} = p_i^{(1)}(q^{(2)}, q^{(1)})$ for $i = 1, \dots, D$ can be inverted to give $q_i^{(1)} = q_i^{(1)}(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)})$. This equation is substituted into eqn (19.10) to give the compound function,

$$S_2(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)}) = S_2(q^{(2)}, q_{[0]}^{(1)}(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)}), q_0^{(1)}) \quad (19.11)$$

The rules of the Legendre transformation then give, for $k = 0, 1, \dots, D$, and $i = 1, \dots, D$,

$$p_k^{(2)}(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)}) = \frac{\partial S_2(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)})}{\partial q_k^{(2)}} \quad (19.12)$$

$$p_0^{(1)}(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)}) = -\frac{\partial S_2(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)})}{\partial q_0^{(1)}} \quad (19.13)$$

$$q_i^{(1)}(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)}) = \frac{\partial S_2(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)})}{\partial p_i^{(1)}} \quad (19.14)$$

Equation (19.12) may be read as, “the momentum at point and time $q^{(2)}$ of that particular motion that reached $q^{(2)}$ from a beginning at time $q_0^{(1)}$ with initial momenta

$p_{[0]}^{(1)}$." Equation (19.14) gives, "the place $q_{[0]}^{(1)}$ that the motion must have started from at time $q_0^{(1)}$ in order that the initial momentum $p_{[0]}^{(1)}$ would carry it to $q_{[0]}^{(2)}$ at time $q_0^{(2)}$."

For the case $k = 0$, the momenta become

$$\frac{\partial S_2 \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right)}{\partial q_0^{(2)}} = p_0^{(2)} \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right) = -H \left(q_{[0]}^{(2)}, p_{[0]}^{(2)} \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right), q_0^{(2)} \right) \tag{19.15}$$

$$\frac{\partial S_2 \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right)}{\partial q_0^{(1)}} = -p_0^{(1)} \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right) = H \left(q_{[0]}^{(1)} \left(q^{(2)}, p_{[0]}^{(1)}, q_0^{(1)} \right), p_{[0]}^{(1)}, q_0^{(1)} \right) \tag{19.16}$$

with similar meanings.

19.4 Example of S_1 and S_2 Action Functions

The abstractions of the previous sections can be illustrated by the simple example of a mass m moving in three dimensions with no applied forces. Then we know that the momenta between starting point $\mathbf{r}^{(1)}, t^{(1)}$ and ending point $\mathbf{r}^{(2)}, t^{(2)}$ will have the constant values,

$$p_0 = -\frac{m \|\mathbf{r}^{(2)} - \mathbf{r}^{(1)}\|^2}{2(t^{(2)} - t^{(1)})^2} \quad \mathbf{p} = m \left(\frac{\mathbf{r}^{(2)} - \mathbf{r}^{(1)}}{t^{(2)} - t^{(1)}} \right) \tag{19.17}$$

Hence eqn (19.3) can be used to give

$$S_1 \left(q^{(2)}, q^{(1)} \right) = S_1 \left(\mathbf{r}^{(2)}, t^{(2)}, \mathbf{r}^{(1)}, t^{(1)} \right) = \int_{(1)}^{(2)} (p_0 dt + \mathbf{p} \cdot d\mathbf{r}) = \frac{m \|\mathbf{r}^{(2)} - \mathbf{r}^{(1)}\|^2}{2(t^{(2)} - t^{(1)})} \tag{19.18}$$

where we use the notation $q_0 = t, q_1 = x, q_2 = y, q_3 = z$ and $\mathbf{r} = x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 + z\hat{\mathbf{e}}_3$.

One can then verify, for example, that the first of eqn (19.7), written in vector form as

$$\mathbf{p}^{(2)} = \frac{\partial S_1}{\partial \mathbf{r}^{(2)}} = m \left(\frac{\mathbf{r}^{(2)} - \mathbf{r}^{(1)}}{t^{(2)} - t^{(1)}} \right) \tag{19.19}$$

is indeed the momentum at point and time 2 of that particular motion that reached that point from point and time 1.

The S_2 function can also be derived for this same example. Using the definition eqn(19.10) and then substituting the correct variables as indicated in eqn(19.11) gives

$$S_2 \left(\mathbf{r}^{(2)}, t^{(2)}, \mathbf{p}^{(1)}, t^{(1)} \right) = S_1 + \mathbf{r}^{(1)} \cdot \mathbf{p}^{(1)} = \mathbf{p}^{(1)} \cdot \mathbf{r}^{(2)} - \frac{\|\mathbf{p}^{(1)}\|^2}{2m} \left(t^{(2)} - t^{(1)} \right) \tag{19.20}$$

Then

$$\mathbf{p}^{(2)} = \frac{\partial S_2}{\partial \mathbf{r}^{(2)}} = \mathbf{p}^{(1)} \quad (19.21)$$

which is correct since the momentum is constant along the classical path of a free particle. The second of eqn (19.14) becomes

$$\mathbf{r}^{(1)} = \frac{\partial S_2}{\partial \mathbf{p}^{(1)}} = \mathbf{r}^{(2)} - \frac{\mathbf{p}^{(1)}}{m} (t^{(2)} - t^{(1)}) \quad (19.22)$$

which is correct since $\mathbf{p}^{(1)}/m$ is the constant velocity of the particle.

19.5 The Hamilton–Jacobi Equation

A phase space dependency function $\mathcal{K}(q, p)$ was introduced in Chapter 12. This same function serves as the extended Hamiltonian, as is shown in Section 12.4. The natural motion of the mechanical system in both the extended Lagrangian and Hamiltonian theories is such that $\mathcal{K}(q, p) = 0$ remains true on all trajectories. Hence, the momenta $p_k^{(2)}$ defined in either eqn (19.7) or eqn (19.12) must obey

$$\mathcal{K}(q^{(2)}, p^{(2)}) = 0 \quad (19.23)$$

Substituting the definitions of $p_k^{(2)}$ from these equations thus gives the first-order, partial differential equation

$$\mathcal{K}\left(q^{(2)}, \frac{\partial S_l}{\partial q^{(2)}}\right) = 0 \quad (19.24)$$

where l is either 1 or 2, depending on the S_l chosen. For example, using the standard definition of \mathcal{K} from eqn (12.12) gives

$$\frac{\partial S_l}{\partial t^{(2)}} + H\left(q_{[0]}^{(2)}, \frac{\partial S_l}{\partial q_{[0]}^{(2)}}, t^{(2)}\right) = 0 \quad (19.25)$$

We will now change notation in this chapter and drop the superscript “(2)” on the upper limits of the Hamilton–Jacobi action. Thus we will write the action functions as $S_1(q, q^{(1)})$ or $S_2(q, p_{[0]}^{(1)}, q_0^{(1)})$ with the understanding that the q is actually what we have called $q^{(2)}$, the upper limit of a classical trajectory from $q^{(1)}$. Then eqns (19.24, 19.25) may be written in a slightly less cluttered form as

$$\mathcal{K}\left(q, \frac{\partial S}{\partial q}\right) = 0 \quad \text{or} \quad \frac{\partial S}{\partial t} + H\left(q_{[0]}, \frac{\partial S}{\partial q_{[0]}}, t\right) = 0 \quad (19.26)$$

which is now considered as a partial differential equation for an unknown function $S(q)$ which may or may not be one of the S_l considered above. This is the *Hamilton–Jacobi equation*.

The Hamilton–Jacobi equation has been derived from two of its solutions, S_1 and S_2 . But now that we have a partial differential equation we can find other solutions. The question of interest to us will be the relation of these more general solutions to the problems of mechanics.

19.6 Hamilton's Characteristic Equations

The Hamilton–Jacobi equation is a first-order partial differential equation, since only first partial derivatives appear in it. It is non-linear, since the traditional Hamiltonian contains a term quadratic in the momenta p , which leads to terms such as $(\partial S/\partial q_k)^2$. For example, using the standard form of the extended Hamiltonian for a single particle moving under the influence of a potential U and restricting ourselves for clarity to systems of one degree of freedom (with $D = 1$), the Hamilton–Jacobi equation is

$$0 = \mathcal{K}(q, p) = \mathcal{K}\left(q, \frac{\partial S}{\partial q}\right) = \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x}\right)^2 + U(x, t) \tag{19.27}$$

A solution to the Hamilton–Jacobi equation $S = \Phi(t, x)$ can be thought of as defining an *integral surface* $0 = F(t, x, S) = \Phi(t, x) - S$ in a $(D + 2)$ -dimensional space whose coordinates are t, x, S . That integral surface can be found by first finding a system of curves, called *characteristic curves*, that lie entirely in it. A unique characteristic curve passes through each point t, x, S of the integral surface, and the system of these curves can be used to define the integral surface (see Figure 19.2).

These characteristic curves can be found by solving a set of ordinary, first-order, coupled differential equations called the *characteristic equations*, which we now derive. Denoting $\partial\Phi/\partial q_k = p_k$, the condition that $\mathcal{K}(q, p)$ must remain zero along any characteristic curve leads to the condition

$$0 = \delta\mathcal{K} = p_0\delta t + p_1\delta x - \delta S \tag{19.28}$$

where $\delta\mathbf{r} : (\delta t, \delta x, \delta S)$ can be visualized as a differential displacement vector along the characteristic curve starting from t, x, S , and $\mathbf{n} : (p_0, p_1, -1)$ can be thought of as a normal vector to the integral surface at that point. Thus the characteristic curve element $\delta\mathbf{r}$ obeys a Cartesian orthogonality condition $\delta\mathbf{r} \cdot \mathbf{n} = 0$. It follows from eqn (19.28) that $\delta\mathbf{r}$ lies in a plane that is perpendicular to \mathbf{n} and contains the point t, x, S .

The components of the normal vector \mathbf{n} must be determined from the Hamilton–Jacobi equation $\mathcal{K}(q, p) = 0$. However, that determination is incomplete. Assuming for definiteness that $\partial\mathcal{K}/\partial p_0 \neq 0$, and using the implicit function theorem in Section D.26, we can solve the Hamilton–Jacobi equation for $p_0 = p_0(t, x, p_1)$. But the value of p_1 is not determined. We say that, with p_1 as the free parameter and t, x, S fixed, there is a *one-parameter family* of possible normal vectors $\mathbf{n} : (p_0(t, x, p_1), p_1, -1)$ and hence a one-parameter family of possible planes.

Using the elements of the theory of surfaces outlined in Section D.37, we can now find the *envelope* of the one-parameter family of planes parameterized by p_1 . The *curve of intersection* of successive planes as p_1 varies is found by writing eqn (19.28) for $p_1 + dp_1$ and $p_1 - dp_1$ as $dp_1 \rightarrow 0$, as described in Section D.37. The curve of intersection is the solution of eqn (19.28) together with

$$0 = \frac{\partial p_0(t, x, p_1)}{\partial p_1} \delta t + \delta x = -\frac{\partial\mathcal{K}/\partial p_1}{\partial\mathcal{K}/\partial p_0} \delta t + \delta x \tag{19.29}$$

where eqn (D.103) was used to get the last equality. The envelope is the surface swept out by the curve of intersection as the parameter p_1 varies. Since the generating

surfaces are differently-oriented planes all of which pass through t, x, S , the envelope is a cone with its vertex at that point. It is called a *Monge cone*.

The characteristic curve $\delta\mathbf{r}$ through t, x, S is defined to be along one of the curves of intersection just found. Since $\delta\mathbf{r}$ also satisfies eqn (19.28), it will also lie in the integral surface. This combination of conditions means that the Monge cone must touch the integral surface, with $\delta\mathbf{r}$ lying along the line of contact. Defining a parameter β that is to vary along the characteristic curve by $\delta\beta = \delta t / (\partial\mathcal{K}/\partial p_0)$, eqn (19.29) becomes the first two characteristic equations

$$\frac{\delta t}{\delta\beta} = \frac{\partial\mathcal{K}(q, p)}{\partial p_0} \qquad \frac{\delta x}{\delta\beta} = \frac{\partial\mathcal{K}(q, p)}{\partial p_1} \qquad (19.30)$$

To complete the specification of the characteristic curves, it is necessary to find expressions for δp_0 and δp_1 . Since t and x are the independent variables, the Hamilton–Jacobi equation $\mathcal{K}(q, p) = 0$ continues to hold when these variables are independently incremented. Hence, incrementing t at fixed x gives

$$0 = \frac{\partial\mathcal{K}}{\partial t} + \frac{\partial\mathcal{K}}{\partial p_0} \frac{\partial p_0(q)}{\partial t} + \frac{\partial\mathcal{K}}{\partial p_1} \frac{\partial p_1(q)}{\partial t} = \frac{\partial\mathcal{K}}{\partial t} + \frac{\delta t}{\delta\beta} \frac{\partial p_0(q)}{\partial t} + \frac{\delta x}{\delta\beta} \frac{\partial p_0(q)}{\partial x} = \frac{\partial\mathcal{K}}{\partial t} + \frac{\delta p_0}{\delta\beta} \qquad (19.31)$$

where eqn (19.30) and the fact that $\partial p_1(q)/\partial t = \partial p_0(q)/\partial x$ were used to get the next-to-last equality. A similar equation is obtained when x is incremented at fixed t , giving

$$0 = \frac{\partial\mathcal{K}}{\partial x} + \frac{\partial\mathcal{K}}{\partial p_0} \frac{\partial p_0(q)}{\partial x} + \frac{\partial\mathcal{K}}{\partial p_1} \frac{\partial p_1(q)}{\partial x} = \frac{\partial\mathcal{K}}{\partial x} + \frac{\delta t}{\delta\beta} \frac{\partial p_1(q)}{\partial t} + \frac{\delta x}{\delta\beta} \frac{\partial p_1(q)}{\partial x} = \frac{\partial\mathcal{K}}{\partial x} + \frac{\delta p_1}{\delta\beta} \qquad (19.32)$$

Taking the limit $\delta\beta \rightarrow 0$, and assembling eqns (19.30 – 19.32), we obtain the set of characteristic equations

$$\dot{t} = \frac{\partial\mathcal{K}}{\partial p_0} \qquad \dot{x} = \frac{\partial\mathcal{K}}{\partial p_1} \qquad \dot{p}_0 = -\frac{\partial\mathcal{K}}{\partial t} \qquad \dot{p}_1 = -\frac{\partial\mathcal{K}}{\partial x} \qquad (19.33)$$

We recognize these characteristic equations as identical to the extended Hamilton equations of Definition 12.4.2.¹¹³ Because of this identity, the Hamilton equations are frequently referred to as *Hamilton’s characteristic equations*. As demonstrated in Section D.36, these equations have a solution that is unique once initial values of q and p are specified. Thus there is one and only one characteristic curve passing through each point of the integral surface.¹¹⁴ In Section 19.10, we give an example

¹¹³A fifth characteristic equation is $\dot{S} = p_0(\partial\mathcal{K}/\partial p_0) + p_1(\partial\mathcal{K}/\partial p_1)$, obtained by putting eqn (19.33) into eqn (19.28). However, since the unknown S does not appear explicitly in $\mathcal{K}(q, p)$, eqn (19.33) can always be solved independently, without using this equation for \dot{S} . It will be used, however, when the solution for S is desired, as in the Cauchy problem in Section 19.10. From Lemma 11.4.2, the characteristic equation for \dot{S} can also be written as $\dot{S} = p_0\dot{t} + p_1\dot{x} = \mathcal{L}$, which is consistent with the definition of S_1 in eqn (19.1).

¹¹⁴More detail about first-order partial differential equations, and proofs of the assertions made in this chapter, can be found in Courant and Hilbert (1962). That text, though definitive, is not particularly accessible. The reader might wish to begin by studying our Sections D.36 and D.37, and then an introductory treatment such as Chapter 10 of Ford (1955).

in which the Hamilton–Jacobi equation is solved by use of Hamilton’s characteristic equations.

However, one great contribution of Hamilton and Jacobi was to realize that this traditional order of solution can be reversed. Instead of solving the characteristic equations in order to find an integral of the Hamilton–Jacobi equation, one can solve the Hamilton–Jacobi equation directly (using methods to be discussed presently) and use it to find the characteristic curves, that is, to solve the Hamilton equations. This provides us with another tool for solving these equations and thus finding the phase-space trajectories of physical systems.

19.7 Complete Integrals

First-order partial differential equations of the form $\mathcal{K}(q, p, S) = 0$, where $p_i = \partial S/\partial q_i$ and at least one of the partial derivatives $\partial\mathcal{K}/\partial p_k$ is assumed to be nonzero, have solutions depending on $(D + 1)$ arbitrary constants of integration a_0, a_1, \dots, a_D . These are called *complete integrals*. In Section 19.8 we give a method for obtaining them. In this section, we show that complete integrals can be used to solve the extended Hamilton equations and hence determine the phase-space trajectories of physical systems.

Complete integrals may be written as

$$S = \Phi(q_0, q_1, \dots, q_D, a_0, a_1, \dots, a_D) = \Phi(q, a) \tag{19.34}$$

To be a legitimate complete integral, eqn (19.34) must satisfy a consistency condition. Denoting $\partial\Phi/\partial a_i = \Phi_{a_i}$ and $\partial^2\Phi/\partial a_i\partial q_j = \Phi_{a_i q_j}$, the $(D + 1) \times (D + 2)$ matrix

$$\begin{pmatrix} \Phi_{a_0} & \Phi_{a_0 q_0} & \cdots & \Phi_{a_0 q_D} \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{a_D} & \Phi_{a_D q_0} & \cdots & \Phi_{a_D q_D} \end{pmatrix} \tag{19.35}$$

must have rank $(D + 1)$. This is the condition for the equations $p_i = \Phi_{q_i}$ and eqn (19.34) to be solvable for $a_i(q, p, S)$, which can be then substituted back into eqn (19.34) to form a well-defined differential equation (not unique) of which eqn (19.34) is the solution. It also ensures that the constants a_0, a_1, \dots, a_D are truly independent in the sense that they cannot be combined into a smaller number of constants yielding the same function Φ .

The Hamilton–Jacobi equation is a special case, because it is of the form $\mathcal{K}(q, p) = 0$ and therefore contains the unknown function S *only* in the form of its partial derivatives $p_i = \partial S/\partial q_i$. Therefore, in the Hamilton–Jacobi case, one of the constants of integration is always additive. If S is a solution, so is $S + \alpha$ for any constant α . Thus, complete integrals of the Hamilton–Jacobi equation are of the form

$$S = \phi(q, a_{[0]}) + a_0 \tag{19.36}$$

where we use the notation $a_{[0]}$ to denote all of the constants except the additive one a_0 . It follows that eqn (19.35) has rank $(D + 1)$, and the complete integral is therefore

legitimate, if and only if the $D \times (D + 1)$ matrix

$$\mathbf{M} = \begin{pmatrix} \phi_{a_1 q_0} & \cdots & \phi_{a_1 q_D} \\ \vdots & \vdots & \vdots \\ \phi_{a_D q_0} & \cdots & \phi_{a_D q_D} \end{pmatrix} \quad (19.37)$$

has rank D . We assume now that, using the methods of Section 19.8 or otherwise, a complete integral of the Hamilton–Jacobi equation has been found satisfying this condition.

A complete integral of the Hamilton–Jacobi equation can be used to define trajectories in phase space which are solutions to the extended Hamilton equations. One introduces a set of new constants b_1, \dots, b_D and writes

$$b_j = \frac{\partial \phi(q, a_{[0]})}{\partial a_j} \quad p_i = \frac{\partial \phi(q, a_{[0]})}{\partial q_i} \quad (19.38)$$

where $i = 0, \dots, D$, $j = 1, \dots, D$ and the second equation simply repeats the definition of p_i . Let us suppose that one of the D -rowed critical minors of eqn (19.37) is the one obtained by expunging the column with derivatives with respect to q_l . In general, l may have any value. In the following, for simplicity we will assume that the coordinates and momenta have been relabeled if necessary so that the index l can be chosen to be the index 0. (This can always be done without loss of generality, as is shown in Section 18.12.)

Separating out the dependency on this q_0 , the first of eqn (19.38) may be written as

$$b_j = \frac{\partial \phi(q, a_{[0]})}{\partial a_j} = b_j(q_0, q_{[0]}, a_{[0]}) \quad (19.39)$$

The $D \times D$ matrix with elements $\partial b_j / \partial q_k = \partial^2 \phi / \partial a_j \partial q_k$ for $j, k = 1, \dots, D$ is a D -rowed critical minor of \mathbf{M} and therefore is nonsingular. By the inverse function theorem (Theorem D.24.1), this is the necessary and sufficient condition for eqn (19.39) to be solved for the $q_{[0]}$ giving, for $k = 1, \dots, D$,

$$q_k = q_k(q_0, b, a_{[0]}) \quad (19.40)$$

Writing the second of eqn (19.38) as $p_i = p_i(q_0, q_{[0]}, a_{[0]})$ and substituting eqn (19.40) into that expression, gives, for $i = 0, \dots, D$,

$$p_i = p_i(q_0, q_{[0]}(q_0, b, a_{[0]}), a_{[0]}) = p_i(q_0, b, a_{[0]}) \quad (19.41)$$

Thus we have all q, p except q_0 expressed as functions of q_0 , which defines a definite trajectory in phase space.

We now can prove that the trajectory defined by eqns (19.40, 19.41) satisfies Hamilton's characteristic equations.

Theorem 19.7.1: Solution of Hamilton Equations

The trajectory defined by eqns (19.40, 19.41) was extracted from complete integral eqn (19.36). This trajectory is a solution of the Hamilton equations, eqn (12.13).

Proof: Since the b_j in eqn (19.39) are constants, and since $\mathcal{K}(q, \partial\phi(q, a_{[0]})/\partial q) = 0$ for any values of $a_{[0]}$,

$$0 = \frac{db_j}{d\beta} = \sum_{i=0}^D M_{ji} \dot{q}_i \quad \text{and} \quad 0 = \frac{\partial \mathcal{K}}{\partial a_j} = \sum_{i=0}^D M_{ji} \frac{\partial \mathcal{K}(q, p)}{\partial p_i} \quad (19.42)$$

where it is assumed that the second of eqn (19.38) is substituted for p after the partial is taken. The matrix M with matrix elements $M_{ji} = \partial^2\phi(q, a_{[0]})/\partial a_j \partial q_i$ from eqn (19.37) has been used. This matrix is a $D \times (D + 1)$ matrix of rank D and hence, as discussed in Section B.19, these two homogeneous linear equations have solutions that are unique up to undetermined multipliers λ_1 and λ_2 . For $i = 0, \dots, D$, these solutions may be written as

$$\dot{q}_i = \lambda_1 c_i(q, a_{[0]}) \quad \text{and} \quad \frac{\partial \mathcal{K}(q, p)}{\partial p_i} = \lambda_2 c_i(q, a_{[0]}) \quad (19.43)$$

where the c_i are functions of the matrix elements of M and hence of $q, a_{[0]}$.

By assumption, at least one of the $\partial \mathcal{K}(q, p)/\partial p_i$ is nonzero. Since by Lemma 17.7.2 at least one of the set \dot{q}_i, \dot{p}_i must be nonzero in any system, it follows from eqn (19.40) that \dot{q}_0 must also be nonzero. Hence both of the λ_1 and λ_2 multipliers are nonzero, with the result that, for $i = 0, \dots, D$,

$$\dot{q}_i = \lambda \frac{\partial \mathcal{K}(q, p)}{\partial p_i} \quad (19.44)$$

where $\lambda = \lambda_1/\lambda_2$ is also nonzero.

Since the S in eqn (19.36) is by assumption an integral of the differential equation, the relation $\mathcal{K}(q, \partial\phi(q, a_{[0]})/\partial q) = 0$ holds for any values of the q . Thus, for $i = 0, \dots, D$,

$$0 = \frac{\partial \mathcal{K}(q, p)}{\partial q_i} + \sum_{k=0}^D \frac{\partial^2 \phi(q, a_{[0]})}{\partial q_k \partial q_i} \frac{\partial \mathcal{K}(q, p)}{\partial p_k} \quad (19.45)$$

where again it is assumed that the second of eqn (19.38) is substituted for p after the partial is taken. Equations (19.44, 19.45) may now be used to evaluate the derivative of the second of eqn (19.38), giving

$$\dot{p}_i = \sum_{k=0}^D \frac{\partial^2 \phi(q, a_{[0]})}{\partial q_k \partial q_i} \dot{q}_k = \sum_{k=0}^D \frac{\partial^2 \phi(q, a_{[0]})}{\partial q_k \partial q_i} \lambda \frac{\partial \mathcal{K}(q, p)}{\partial p_k} = -\lambda \frac{\partial \mathcal{K}(q, p)}{\partial q_i} \quad (19.46)$$

In summary, we have obtained the equations

$$\dot{q}_i = \lambda \frac{\partial \mathcal{K}(q, p)}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\lambda \frac{\partial \mathcal{K}(q, p)}{\partial q_i} \quad (19.47)$$

As is discussed in Section 12.7, for nonzero λ these equations are equivalent to the

extended Hamilton equations eqn (12.13),

$$\dot{q}_i = \frac{\partial \mathcal{K}(q, p)}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{K}(q, p)}{\partial q_i} \quad (19.48)$$

in the sense that any trajectory that solves one also solves the other. Thus the trajectory defined by eqns (19.40, 19.41) is a solution to the Hamilton equations. \square

19.8 Separation of Variables

The use of complete integrals to solve the Hamilton equations is of practical value only if some independent method is available for finding these complete integrals in the first place. For example, the S_1 and S_2 functions of Section 19.4 are complete integrals, but we had to know the trajectory already in order to set up the integral to calculate them. So they would be of little practical value for problem solving.

A standard method for finding complete integrals is the *separation of variables*. It is probably best explained by giving an example. Suppose that a single mass m moves in three dimensions under a uniform gravitational field $\mathbf{g} = -g\hat{\mathbf{e}}_3$. The extended Hamiltonian for this problem is, using $q_0, q_1, q_2, q_3 = t, x, y, z$,

$$\mathcal{K} = p_0 + \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz \quad (19.49)$$

and the Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right\} + mgz = 0 \quad (19.50)$$

We separate variables by using the trial solution

$$S(t, x, y, z) = T(t) + X(x) + Y(y) + Z(z) \quad (19.51)$$

which allows eqn (19.50) to be written as

$$\left\{ \frac{dT}{dt} \right\} + \left\{ \frac{1}{2m} \left(\frac{dX}{dx} \right)^2 \right\} + \left\{ \frac{1}{2m} \left(\frac{dY}{dy} \right)^2 \right\} + \left\{ \frac{1}{2m} \left(\frac{dZ}{dz} \right)^2 + mgz \right\} = 0 \quad (19.52)$$

Each of the terms in curly brackets in eqn (19.52) depends on only a single variable. Since the variables t, x, y, z are independent, we may hold t, x, y constant while z is allowed to vary. But, as z varies, the last expression in curly brackets must remain constant because of the constraint imposed by eqn (19.52) with constant values for the other variables. A similar argument can be applied to each of the terms in curly brackets, with the result that

$$\left\{ \frac{dT}{dt} \right\} = a_4 \quad \left\{ \frac{1}{2m} \left(\frac{dX}{dx} \right)^2 \right\} = a_1 \quad \left\{ \frac{1}{2m} \left(\frac{dY}{dy} \right)^2 \right\} = a_2 \quad \left\{ \frac{1}{2m} \left(\frac{dZ}{dz} \right)^2 + mgz \right\} = a_3 \quad (19.53)$$

where the constants must satisfy the relation $a_1 + a_2 + a_3 + a_4 = 0$.

Performing integrations of the ordinary differential equations in eqn (19.53) and putting the results into eqn (19.51) gives the complete integral $S(q, a) = a_0 + \phi(q, a_{[0]})$, where

$$\phi(q, a_{[0]}) = -(a_1 + a_2 + a_3)t + \varepsilon_1 \sqrt{2ma_1} x + \varepsilon_2 \sqrt{2ma_2} y + \varepsilon_3 \sqrt{\frac{8}{9mg^2}} (a_3 - mgz)^{3/2} \tag{19.54}$$

and the ε_i are factors of ± 1 coming from arbitrary choices of the signs of square roots in the integration.

Since the method of separation of variables has given us a complete integral, we may now use the methods of Section 19.7 to solve for the system trajectory. The first of eqn (19.38) gives

$$b_1 = -t + \varepsilon_1 \sqrt{\frac{m}{2a_1}} x \quad b_2 = -t + \varepsilon_2 \sqrt{\frac{m}{2a_2}} y \quad b_3 = -t - \varepsilon_3 \sqrt{\frac{2}{mg^2}} (a_3 - mgz)^{1/2} \tag{19.55}$$

from which one may calculate the trajectory. In particular, the equation for z is $z = z_0 + v_{30}t - gt^2/2$ where

$$v_{30} = -gb_3 \quad \text{and} \quad z_0 = \frac{a_3}{mg} - \frac{gb_3^2}{2} = \frac{1}{mg} \left(a_3 - \frac{mv_{30}^2}{2} \right) \tag{19.56}$$

19.9 Canonical Transformations

The extraction of a phase-space trajectory from a complete integral in Section 19.7 can also be viewed as a canonical transformation. In Courant and Hilbert (1962), this is referred to as a “new method” for proving that the trajectory defined by eqns (19.40, 19.41) is a solution of the Hamilton equations.

The new method makes a canonical transformation from the variable set q, p to a new set Q, P all but one of which will be constant. We assume that the phase-space variables have been relabeled if necessary so that $\dot{q}_0 \neq 0$. Also, for simplicity and without loss of generality, we assume that the extended Hamiltonian is in standard form. (See Section 18.12.)

Define an F_2 generating function of the sort described in Section 18.3 by making the substitution $a_j \rightarrow P_j$ in the function $\phi(q, a_{[0]})$ and writing

$$F_2(q, P) = q_0 P_0 + \phi(q, P_{[0]}) \tag{19.57}$$

Since matrix M from eqn (19.37) has rank D , it follows that the matrix $(\partial^2 F_2 / \partial q \partial P)$ defined in eqn (18.19) will be nonsingular, which is the condition for F_2 to be a legitimate generating function. The F_2 rules in eqn (18.20) then become, for $j = 1, \dots, D$,

$$p_0 = \frac{\partial F_2}{\partial q_0} = P_0 + \frac{\partial \phi(q, P_{[0]})}{\partial q_0} \quad Q_0 = \frac{\partial F_2}{\partial P_0} = q_0 \tag{19.58}$$

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial \phi(q, P_{[0]})}{\partial q_j} \quad Q_j = \frac{\partial F_2}{\partial P_j} = \frac{\partial \phi(q, P_{[0]})}{\partial P_j} \tag{19.59}$$

Since the complete integral is a solution of the Hamilton–Jacobi equation by assump-

tion, we have

$$\mathcal{K}\left(q, \frac{\partial\phi(q, P_{[0]})}{\partial q}\right) = \frac{\partial\phi(q, P_{[0]})}{\partial q_0} + H\left(q, \frac{\partial\phi(q, P_{[0]})}{\partial q_{[0]}}\right) = 0 \quad (19.60)$$

Thus the first of eqn (19.58) can be written as

$$P_0 = p_0 + H\left(q, \frac{\partial\phi(q, P_{[0]})}{\partial q_{[0]}}\right) = p_0 + H(q, p_{[0]}) = \mathcal{K}(q, p) \quad (19.61)$$

where the first of eqn (19.59) was used. As explained in Theorem 17.12.1, in canonical transformations the extended Hamiltonian in the new variables Q, P is always the same function as $\mathcal{K}(q, p)$, simply expressed in the new variable set by direct substitution. Hence eqn (19.61) implies

$$\mathcal{K}(Q, P) = \mathcal{K}(q(Q, P), p(Q, P)) = P_0 \quad (19.62)$$

As proved in Theorem 17.12.1, the Hamilton equations are form invariant under canonical transformations. Thus the Hamilton characteristic equations are satisfied in the original q, p system if and only if they hold in the Q, P system with the extended Hamiltonian $\mathcal{K}(Q, P) = P_0$. With that extended Hamiltonian, they are

$$\dot{Q}_0 = \frac{\partial\mathcal{K}(Q, P)}{\partial P_0} = 1 \quad \dot{Q}_j = \frac{\partial\mathcal{K}(Q, P)}{\partial P_j} = 0 \quad \dot{P}_i = -\frac{\partial\mathcal{K}(Q, P)}{\partial Q_i} = 0 \quad (19.63)$$

for $i = 0, \dots, D$ and $j = 1, \dots, D$.

We have already defined the $a_j = P_j$ as constants, and may now define the $Q_j = b_j$ to be constants also. Then eqn (19.63) hold by construction and hence, by form invariance, the Hamilton equations hold also in the original q, p system, which is what we wished to establish.

The method is essentially to make a canonical transformation from the variables q, p to new variables that are constants, except for $Q_0 = q_0$. In the traditional presentation of this method, in Courant and Hilbert (1962) for example, the time q_0 is not considered a variable. The traditional Hamiltonian in the transformed system vanishes and all of the remaining coordinates $Q_{[0]}, P$ are constant. In the extended theory, however, Lemma 17.7.2 has shown that in any system of canonical coordinates, at least one of the Q, P must have a non-vanishing derivative. Since we are assuming that the variable labeled q_0 is the non-constant one, and are hence using the standard form of the extended Hamiltonian, that single non-constant coordinate is $Q_0 = q_0$.

Since we have made the substitutions $P_j \rightarrow a_j, Q_j \rightarrow b_j, Q_0 = q_0$, the second F_2 transformation rule in eqn (19.59) may be written, for $j = 1, \dots, D$, as

$$b_j = \frac{\partial F_2}{\partial a_j} = \frac{\partial\phi(q, a_{[0]})}{\partial a_j} \quad (19.64)$$

which is identical to the first of eqn (19.38). Due to these same substitutions, the first

F_2 transformation rule in eqn (19.59) may be written as

$$p_0 = \frac{\partial F_2}{\partial q_0} = P_0 + \frac{\partial \phi(q, a_{[0]})}{\partial q_0} \quad p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial \phi(q, a_{[0]})}{\partial q_j} \quad (19.65)$$

Since, after all partials are taken, we can set $P_0 = \mathcal{K}(Q, P) = 0$, these two equations together are identical to the second of eqn (19.38). Thus the trajectory derived from the F_2 generating function will be identical to the trajectory derived earlier from eqn (19.38) directly.

19.10 General Integrals

The complete integrals described so far in this chapter are the main tools for actual problem solution using Hamilton–Jacobi methods. However, the Hamilton–Jacobi equation has other solutions that are of great theoretical interest. In particular, the general integrals are the ones most closely related to the Schrodinger wave function of quantum mechanics.

General integrals are those solutions of the Hamilton–Jacobi equation that do not depend explicitly on a set of integration constants. The most interesting ones for our purposes are those that solve what is called a *Cauchy problem*, in which one specifies the value of S on some chosen initial surface. If the initial surface is chosen properly, the general integral is then uniquely determined.

As we did for separation of variables, it is probably most useful to introduce general integrals by means of a simple example that can be generalized by the reader. We consider the same projectile problem as in Section 19.8, but now restricted to one dimension. Calling the vertical dimension x , and denoting $q_0 = t$ and $q_1 = x$, the extended Hamiltonian and resulting Hamilton–Jacobi equation are

$$\mathcal{K}(q, p) = p_0 + \frac{p_1^2}{2m} + mgx \quad \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + mgx = 0 \quad (19.66)$$

Suppose that we want a general integral that solves the Hamilton–Jacobi equation and passes through the initial surface C_1 (now a one-dimensional surface, or curve, since $D = 1$) denoted by introducing a new monotonic parameter τ and writing

$$t(\tau) = 0 \quad x(\tau) = \tau \quad S(\tau) = \alpha\tau \quad (19.67)$$

Because of the simplicity of this example, we can plot the initial curve C_1 and the eventual integral surface in a three-dimensional diagram with axes t, x, S as shown in Figure 19.2.

The general integral sought can be written as $S = \Phi(t, x)$. It will be useful to think of it in implicit form as $F(t, x, S) = \Phi(t, x) - S = 0$, which can be viewed geometrically as a surface in t, x, S space called an *integral surface*. (See Section D.37 for a discussion of surfaces in three dimensions.) The quantities $p_0 = \partial F / \partial t$, $p_1 = \partial F / \partial x$, and $-1 = \partial F / \partial S$ are then components of a “vector” in this space that is

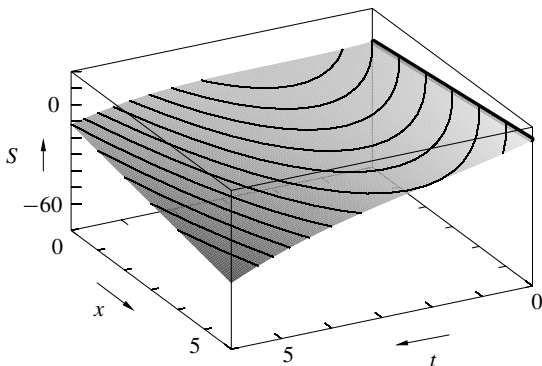


FIG. 19.2. General integral surface S for simple projectile. The initial curve is the line at the right edge of the surface. The characteristic curves (solutions to Hamilton equations) spread out from that line and lie in the integral surface. The time t is in seconds, the distance x is in units of $g(1 \text{ sec})^2$, the action S is in units of $mg^2(1 \text{ sec})^3$. The parameter α defining the initial curve in eqn (19.67) is chosen arbitrarily to be $\alpha = 2mg(1 \text{ sec})$.

normal to the integral surface. (If we imagine a small plane tangent to the surface at each point, then this vector is the normal to that tangent plane.)

The τ is some parameter that varies monotonically along the initial curve. Although it is convenient to express the result parametrically as we have done, clearly eqn (19.67) is stating that S has the value αx along the x -axis at $t = 0$. But one should not conclude from this example that the “initial” curve must necessarily be at time zero. Any initial curve that is not itself a characteristic curve may be used.

The first step toward the general integral is to determine initial values $p_0(\tau)$ and $p_1(\tau)$ at each τ along the initial curve C_1 . These quantities must satisfy the equation

$$0 = \frac{dF}{d\tau} = p_0(\tau) \frac{dt}{d\tau} + p_1(\tau) \frac{dx}{d\tau} - \frac{dS(\tau)}{d\tau} \quad (19.68)$$

and the Hamilton–Jacobi equation

$$p_0(\tau) + \frac{p_1^2(\tau)}{2m} + mgx(\tau) = 0 \quad (19.69)$$

Equation (19.68) says that the $p_0(\tau)$ and $p_1(\tau)$ must be chosen such that the initial curve $(dt/d\tau, dx/d\tau, dS/d\tau)$ lies in the integral surface $F = 0$. Equation (19.69) says that these same momenta must satisfy the Hamilton–Jacobi equation along C_1 . In our example, eqn (19.68) becomes $p_1(\tau) - \alpha = 0$. Substituting this result into the Hamilton–Jacobi equation gives $p_0(\tau) + \alpha^2/2m + mgx(\tau) = 0$ with the result that the momenta at each τ of the initial curve are determined to be

$$p_0(\tau) = - \left(\frac{\alpha^2}{2m} + mg\tau \right) \quad \text{and} \quad p_1(\tau) = \alpha \quad (19.70)$$

For any τ value, eqns (19.67, 19.70) together determine what may be called an *initial patch*, a small plane containing a small piece of the initial curve and having a normal

$(p_0(\tau), p_1(\tau), -1)$ that satisfies the Hamilton–Jacobi equation at that point. The set of all such patches as τ varies along C_1 is called an *initial strip*. This initial strip is the launching pad for our general integral.

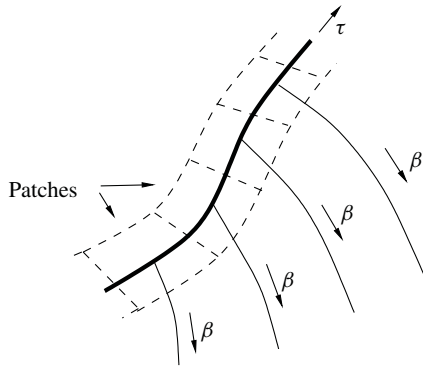


FIG. 19.3. As β increases, the characteristic curves branch out from points on the initial curve C_1 (heavy line parameterized by τ). These characteristic curves are in, and define, the integral surface. The patches along the initial curve are also shown. Together they make the initial strip.

The next step in finding the general integral is to write the *characteristic equations* of the problem and solve them. The characteristic differential equations are *defined* by the condition that their solutions, called *characteristic curves*, always lie in integral surfaces. We want to define a *family* of characteristic curves, with each curve in the family starting at some particular point τ on the initial curve C_1 . As can be seen from Figure 19.3, as parameter τ varies this family of characteristic curves will sweep out a surface in the space t, x, S . Since characteristic curves always lie in integral surfaces by definition, the surface swept out will define the general integral that we are seeking.

The characteristic equations of the Hamilton–Jacobi equation are the extended Hamilton equations, eqn (12.13), plus the additional equation mentioned in the footnote on page 468. In the present example, they reduce to

$$\dot{t} = \frac{\partial \mathcal{K}}{\partial p_0} = 1 \quad \dot{x} = \frac{\partial \mathcal{K}}{\partial p_1} = \frac{p_1}{m} \quad \dot{p}_0 = -\frac{\partial \mathcal{K}}{\partial t} = 0 \quad \dot{p}_1 = -\frac{\partial \mathcal{K}}{\partial x} = -mg \quad (19.71)$$

together with

$$\dot{S} = \sum_{k=0}^D p_k \frac{\partial \mathcal{K}}{\partial p_k} = p_0 + \frac{p_1^2}{m} \quad (19.72)$$

where in all cases the dot indicates differentiation with respect to some parameter β that varies monotonically along the characteristic curves. Performing the integrations using the initial conditions from the initial patches defined in eqns (19.67, 19.70), we obtain a family of characteristic curves, with each one beginning at a particular point

τ of the initial curve C_1 and moving out from there as β advances,

$$\begin{aligned} t(\tau, \beta) &= \beta & x(\tau, \beta) &= \tau + \frac{\alpha}{m}\beta - \frac{1}{2}g\beta^2 \\ p_0(\tau, \beta) &= -\left(\frac{\alpha^2}{2m} + mg\tau\right) & p_1(\tau, \beta) &= \alpha - mg\beta \end{aligned} \quad (19.73)$$

$$S(\tau, \beta) = (\alpha - mg\beta)\tau + \frac{\alpha^2}{2m}\beta - \alpha g\beta^2 + \frac{1}{3}mg^2\beta^3 \quad (19.74)$$

Note that the derivatives of these quantities with respect to β satisfy the characteristic equations, eqns (19.71, 19.72), and that their values when $\beta = 0$ reduce to the values on the initial curve C_1 as given by eqns (19.67, 19.70).

The $S(\tau, \beta)$ in eqn (19.74) is our general integral. However, the coordinates τ, β are inconvenient. We would like to have S as a function of t and x . To accomplish this change of variable, the equations

$$t(\tau, \beta) = \beta \quad x(\tau, \beta) = \tau + \frac{\alpha}{m}\beta - \frac{1}{2}g\beta^2 \quad (19.75)$$

from eqn (19.73) can be solved for $\tau(t, x)$ and $\beta(t, x)$. By the inverse function theorem (Theorem D.24.1), the condition for this inversion to be possible in some neighborhood of C_1 is that the determinant

$$\Delta = \begin{vmatrix} \frac{\partial t(\tau, \beta)}{\partial \tau} & \frac{\partial t(\tau, \beta)}{\partial \beta} \\ \frac{\partial x(\tau, \beta)}{\partial \tau} & \frac{\partial x(\tau, \beta)}{\partial \beta} \end{vmatrix} = \begin{vmatrix} \frac{\partial t(\tau, \beta)}{\partial \tau} & \frac{\partial \mathcal{K}}{\partial p_0} \\ \frac{\partial x(\tau, \beta)}{\partial \tau} & \frac{\partial \mathcal{K}}{\partial p_1} \end{vmatrix} \quad (19.76)$$

must be nonzero in the limit $\beta \rightarrow 0$. In our example, $\Delta = -1$ and the inversion gives

$$\beta(t, x) = t \quad \tau(t, x) = x - \frac{\alpha t}{m} + \frac{1}{2}gt^2 \quad (19.77)$$

Substitution of this result into eqn (19.74) then gives the general integral in its final form, which is the shaded surface shown in Figure 19.2,

$$S(t, x) = \alpha x - \left(mgx + \frac{\alpha^2}{2m}\right)t + \frac{\alpha}{2}gt^2 - \frac{mg^2}{6}t^3 \quad (19.78)$$

The reader can verify that this expression solves the Hamilton–Jacobi equation for all t, x values, and that as $t \rightarrow 0$ it reduces to value αx prescribed in eqn (19.67) on the initial curve. Thus a unique solution to the Cauchy problem has been found.

One feature of integrals to the Hamilton–Jacobi equation is that when the traditional Hamiltonian has the simple form eqn (19.27), the projections of the characteristic curves onto three-dimensional Cartesian space x, y, z will be lines perpendicular

to surfaces of constant S . This is because $\dot{x} = p_1/m$, etc., and hence one has the proportionality

$$\dot{x} : \dot{y} : \dot{z} = p_1 : p_2 : p_3 = \frac{\partial S}{\partial x} : \frac{\partial S}{\partial y} : \frac{\partial S}{\partial z} \tag{19.79}$$

This perpendicularity is not visible in Figure 19.2, however, since $D = 1$ and only the x -axis is present.¹¹⁵

The simple example given here can be generalized in a straightforward way to $D \geq 2$. The geometrical ideas become more difficult to visualize, however. It is useful to do the manipulations required while keeping in mind the analogy to the simple case $D = 1$. When $D \geq 2$, the initial curve becomes a D -dimensional initial surface, in a $D + 2$ -dimensional space with coordinates q_0, q_1, \dots, q_D, S . The initial surface is parameterized by D parameters τ_1, \dots, τ_D . The generalizations of eqns (19.68, 19.69) are

$$\sum_{i=0}^D p_i(\tau) \frac{dq_i(\tau)}{d\tau_j} - \frac{dS(\tau)}{d\tau_j} = 0 \qquad p_0(\tau) + H(q(\tau), p_{[0]}(\tau)) = 0 \tag{19.80}$$

for all $j = 1, \dots, D$, where τ stands for the set of parameters τ_1, \dots, τ_D .

The characteristic equations are the extended Hamilton equations, eqn (12.13), plus the extra equation

$$\dot{S} = \sum_{i=0}^D p_i \frac{\partial \mathcal{K}(q, p)}{\partial p_i} \tag{19.81}$$

They are to be solved for a D -parameter family of characteristic curves of the form

$$q_i = q_i(\tau_1, \dots, \tau_D, \beta) \qquad p_i = p_i(\tau_1, \dots, \tau_D, \beta) \qquad S = S(\tau_1, \dots, \tau_D, \beta) \tag{19.82}$$

where each characteristic curve begins on the initial surface at a point with parameters τ_1, \dots, τ_D . If the determinant

$$\Delta = \begin{vmatrix} \frac{\partial q_0}{\partial \tau_1} & \dots & \frac{\partial q_0}{\partial \tau_D} & \frac{\partial q_0}{\partial \beta} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial q_{D-1}}{\partial \tau_1} & \dots & \frac{\partial q_{D-1}}{\partial \tau_D} & \frac{\partial q_{D-1}}{\partial \beta} \\ \frac{\partial q_D}{\partial \tau_1} & \dots & \frac{\partial q_D}{\partial \tau_D} & \frac{\partial q_D}{\partial \beta} \end{vmatrix} \tag{19.83}$$

is nonzero (otherwise, the choice of initial surface should be modified until it is) then

¹¹⁵The relation in eqn (19.79) is not valid for charged particles in the presence of a magnetic induction field.

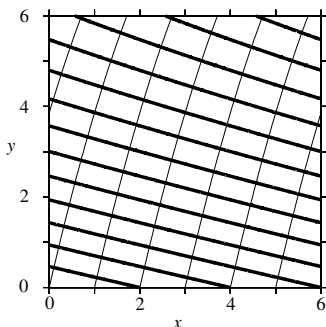


FIG. 19.4. Contour plot of the mono-energetic integral for the simple projectile in two dimensions. The heavy lines are contours of constant W . The lighter lines are projections onto the x - y plane of the characteristic curves (solutions to the Hamilton equations). The initial curve is the x -axis with $W = \alpha x$, as shown in eqn 19.87. The x and y coordinates are in units of $g(1 \text{ sec})^2$. The constants are chosen to be $\alpha = mg(1 \text{ sec})$ and $E = 10mg^2(1 \text{ sec})^2$

the first of eqn (19.82) can be solved for

$$\tau_j = \tau_j(q_0, \dots, q_D) \quad \beta = \beta(q_0, \dots, q_D) \quad (19.84)$$

which can be substituted into the last of eqn (19.82) to give the general integral

$$S = S(q_0, \dots, q_D) = S(\tau_1(q_0, \dots, q_D), \dots, \tau_D(q_0, \dots, q_D), \beta(q_0, \dots, q_D)) \quad (19.85)$$

19.11 Mono-Energetic Integrals

In the special case in which the traditional Hamiltonian does not depend explicitly on the time, there is yet another interesting class of integrals of the Hamilton–Jacobi equation. These are integrals that have only a trivial time dependence, and that depend on a non-additive constant, the energy E . We will call these *mono-energetic integrals*. They are like the general integrals in the sense that they do not depend on a complete set of integration constants, as the complete integrals of Section 19.7 do. And yet they are not completely general, since the characteristic curves that generate them are assumed all to have the same energy.

When the traditional Hamiltonian is not an explicit function of time, then the energy (more precisely the value of the traditional Hamiltonian, but we will continue here to use the term energy) is conserved along *any one* of the characteristic curves (phase-space trajectories) of any integral. But this does *not* mean that all of the characteristic curves in a truly general integral surface necessarily have the same energy. The energy will generally be different from curve to curve, as in Figure 19.2 for example.¹¹⁶ But *all* characteristic curves in the mono-energetic integrals *must* have the same energy E , by definition.

¹¹⁶For an even simpler example, consider the use of the S_1 action function in Section 19.2. We emphasized there that, despite its conservation along *any particular* curve, the traditional Hamiltonian $H = -\partial S_1 / \partial t^{(2)}$ defined in eqn (19.8) will be different if different trajectories are chosen (e.g. different endpoints $q^{(2)}$ while keeping the same $t^{(2)}$ value).

The mono-energetic integral is obtained by beginning the process of separation of variables described in Section 19.8 but only carrying it one step. If the traditional Hamiltonian does not depend on the time explicitly, then the partial separation $S = T(t) + W(q_{[0]})$ reduces the Hamilton–Jacobi equation, eqn (19.26), to

$$S = -Et + W(q_{[0]}, E) \quad \text{and} \quad E = H \left(q_{[0]}, \frac{\partial W(q_{[0]}, E)}{\partial q_{[0]}} \right) \quad (19.86)$$

where a constant energy value E is to be selected arbitrarily. A dependency of W on the constant E has been introduced to make clear that different E choices will lead to different solutions.

The function $W(q_{[0]}, E)$ is called *Hamilton’s characteristic function*, and the second equation in eqn (19.86) is sometimes called the time-independent Hamilton–Jacobi equation. We will refer to it as the mono-energetic Hamilton–Jacobi equation.

As an example, using a technique analogous to that in Section 19.10, a mono-energetic integral can be found for the simple projectile problem in two dimensions ($D = 2$ with coordinates x, y and gravity acting in the negative y direction) with the Cauchy initial curve

$$x(\tau) = \tau \quad y(\tau) = 0 \quad W(\tau) = \alpha\tau \quad (19.87)$$

The mono-energetic Hamilton–Jacobi equation for this case is

$$E = \frac{1}{2m} \left\{ \left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right\} + mgy \quad (19.88)$$

One obtains an initial patch with momenta

$$p_1(\tau) = \alpha \quad p_2(\tau) = \sqrt{2mE - \alpha^2} \quad (19.89)$$

Carrying out the calculation (Exercise 19.4) leads to

$$W(\tau, \beta, E) = \alpha\tau + \frac{\alpha^2\beta}{m} - \frac{1}{3mg} \left\{ \left(\sqrt{2mE - \alpha^2} - mg\beta \right)^3 - \left(\sqrt{2mE - \alpha^2} \right)^3 \right\} \quad (19.90)$$

where

$$\tau(x, y) = x - \frac{\alpha\beta}{m} \quad \beta(x, y) = \frac{\sqrt{2mE - \alpha^2}}{mg} \left(1 - \sqrt{1 - \frac{2m^2gy}{2mE - \alpha^2}} \right) \quad (19.91)$$

Figure 19.4 shows a contour plot of the resulting $W(x, y, E)$, obtained when eqn (19.91) is substituted into eqn (19.90), together with the projection onto the x - y plane of the characteristic curves (solutions of the Hamilton characteristic equations). Note that the characteristic curves are perpendicular to the contours of constant W , as predicted by eqn (19.79).

19.12 The Optical Analogy

The work of Hamilton (Hamiltonian, Hamilton equations, Hamilton's Principle, the Hamilton–Jacobi equation) to which analytical mechanics owes such a debt began as a study of the analogy between mechanics and optics.¹¹⁷ In Hamilton's analogy, the characteristic curves of the mono-energetic Hamilton–Jacobi equation play the role of optical rays.

The ray approximation in optics can be obtained by substituting trial functions, like $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \exp i(\mathcal{W}(\mathbf{r}) - \omega t)$ for the electric field, into the Maxwell equations.¹¹⁸ Assuming a monochromatic field with angular frequency ω , and taking a short wavelength limit, one obtains the relation

$$n = \|\nabla\mathcal{W}\| = \sqrt{\left(\frac{\partial\mathcal{W}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{W}}{\partial y}\right)^2 + \left(\frac{\partial\mathcal{W}}{\partial z}\right)^2} \quad (19.92)$$

where $n = n(\mathbf{r})$ is the index of refraction of the medium. The function \mathcal{W} is called the *eikonal*. The optical rays are then defined to be perpendicular to iso-surfaces of \mathcal{W} and hence to obey

$$\nabla\mathcal{W} = n \frac{d\mathbf{r}}{ds} = n\hat{\mathbf{t}} \quad (19.93)$$

where s is the arc length along the ray and $\hat{\mathbf{t}}$ is the unit tangent vector discussed in Section A.12.

The optical ray model is monochromatic with only a trivial time dependence. Its analog in analytical mechanics is the mono-energetic Hamilton–Jacobi theory. The mechanical analog of the eikonal function \mathcal{W} is Hamilton's characteristic function W .

In three dimensions x, y, z the mono-energetic Hamilton–Jacobi equation, eqn (19.86), becomes

$$E = \frac{1}{2m} \left\{ \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 \right\} + U(x, y, z) \quad (19.94)$$

Writing this as

$$\sqrt{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2} = \sqrt{2m(E - U)} \quad (19.95)$$

and comparing to eqn (19.92) shows that the expression $\sqrt{2m(E - U)}$ is the mechanical analog of the index of refraction n .

Thus surfaces of constant W are the analogs of surfaces of constant eikonal in optics, and the characteristic curves (solutions of Hamilton equations) are the analog of

¹¹⁷Hamilton was an Irish patriot who initially would publish only in little-read Irish journals. His ideas on optics were later developed by Jacobi into a general theory of first-order partial differential equations. Although Jacobi gave profuse credit to Hamilton, there was a sentiment among his contemporaries that Hamilton's actual ideas on optics were neglected. See Hankins (1980).

¹¹⁸See Chapter 3 of Born and Wolf (1980).

optical rays. Although interesting in its own right, this analogy is only partial. On the optical side, one is not using the full Maxwell equations but only a monochromatic, short-wavelength approximation to them. And on the mechanical side, the monoenergetic action function in eqn (19.86) is only a subclass of possible Hamilton–Jacobi action functions. (For example, S_1 and the general integral S in eqn (19.78) are not of this form.) A more interesting analogy for our purposes here is the one between analytical mechanics and another wave theory: quantum mechanics. This analogy is pursued in Section 19.14.

19.13 The Relativistic Hamilton–Jacobi Equation

The manifestly covariant extended Hamiltonian for a particle in an external electromagnetic field was written in eqn (16.56) as

$$\mathcal{K}_a = \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) \cdot \left(\underline{\mathbf{p}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) + m^2 c^2 \quad (19.96)$$

The covariant components of the canonical momentum fourvector $\underline{\mathbf{p}}$ are derived in Section 16.10. They are related to the partial derivatives of the Hamilton–Jacobi action function as

$$p_0 = \frac{1}{c} \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{t}} = \frac{1}{c} \frac{\partial S}{\partial t} \quad \text{and} \quad \underline{\mathbf{p}} = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{\mathbf{r}}} = \frac{\partial S}{\partial \mathbf{r}} \quad (19.97)$$

which can be summarized as the fourvector equation

$$\underline{\mathbf{p}} = \frac{\partial S}{\partial \mathbf{r}} = \partial S \quad (19.98)$$

where the fourvector gradient ∂S is defined in Section 15.12. The invariant Hamilton–Jacobi equation is obtained by substituting this relation into eqn (19.96), and applying the dependency relation $\mathcal{K}_a = 0$, giving

$$\left(\frac{\partial S}{\partial \mathbf{r}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) \cdot \left(\frac{\partial S}{\partial \mathbf{r}} - \frac{q^{(\text{ch})}}{c} \mathbf{A} \right) + m^2 c^2 = 0 \quad (19.99)$$

Since the Hamilton–Jacobi action is defined by a manifestly covariant differential equation, a solution can be found such that S itself is a Lorentz scalar field of the sort discussed in Section 15.21.

19.14 Schroedinger and Hamilton–Jacobi Equations

Soon after Schroedinger’s pioneering papers, it was noticed that the Schroedinger equation of nonrelativistic quantum mechanics could be decomposed into two real-valued, coupled, partial differential equations, one of which bears a close resemblance

to the Hamilton–Jacobi equation.¹¹⁹ Several decades later, this idea was revived by David Bohm as the basis of a re-interpretation of quantum mechanics.¹²⁰

Define the real field functions $\mathcal{R} > 0$ and S by writing the normalized Schroedinger wave function in polar form as

$$\Psi(\mathbf{r}, t) = \mathcal{R}(\mathbf{r}, t) \exp(iS(\mathbf{r}, t)/\hbar) \quad (19.100)$$

With that substitution, the Schroedinger equation of eqn (4.65) reduces to the two real-valued differential equations

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right\} + U + U_{qm} = 0 \quad (19.101)$$

$$\frac{\partial (\mathcal{R}^2)}{\partial t} + \nabla \cdot \left(\frac{\mathcal{R}^2 \nabla S}{m} \right) = 0 \quad (19.102)$$

The first of these equations is identical to the Hamilton–Jacobi equation except for the addition of a term that Bohm calls the *quantum potential* U_{qm} . Its definition

$$U_{qm}(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \mathcal{R}(\mathbf{r}, t)}{\mathcal{R}(\mathbf{r}, t)} \quad (19.103)$$

has the effect of raising the effective potential in regions where \mathcal{R}^2 reaches maximum values. Since the probability distribution function in quantum mechanics is $\mathcal{P} = \Psi^* \Psi = \mathcal{R}^2$, the second equation, eqn (19.102), can be written as

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot \mathcal{J} = 0 \quad (19.104)$$

which has the form of a conservation equation for probability, with $\mathcal{J} = (\nabla S/m) \mathcal{P}$ playing the role of a probability current density.¹²¹

The Bohm interpretation of quantum mechanics considers the characteristic curves of the Hamilton–Jacobi equation, eqn (19.101), (including the added quantum potential U_{qm}) to be real phase-space trajectories of the quantum particle. Soon after Bohm’s papers appeared, there was a telling exchange of letters between Bohm and Otto Halpern.¹²² Halpern criticized Bohm’s approach because the Schroedinger wave function and eqn (19.100) provide only a general field function $S(\mathbf{r}, t)$, and fail to provide a set of constants of integration. It would therefore be impossible to apply the techniques of Sections 19.7 or 19.9 to extract a trajectory from it. Bohm conceded

¹¹⁹See Section 2.3 of Jammer (1974). In his original papers, Schroedinger quoted Hamilton–Jacobi theory in a heuristic derivation of the Schroedinger equation.

¹²⁰Bohm, D. (1952) *Phys. Rev.* **85**, p. 166 and p. 180.

¹²¹Those unfamiliar with this form for conservation equations might review the analogous equation in electrodynamics. A useful introduction is Chapter 12 of Wangsnest (1986).

¹²²Halpern, O. (1952) *Phys. Rev.* **87**, p. 389 and Bohm, D. (1952) *Phys. Rev.* **87**, p. 389.

the point, but replied that it was nonetheless true that the momenta $\mathbf{p} = \partial S / \partial \mathbf{r}$ define trajectories along which the Hamilton characteristic equations are satisfied (of course when the U_{qm} is included). Translated into the language of the present chapter, Halpern's criticism was that Bohm's method does not provide a *complete* integral, and Bohm's reply was that it does provide a *general* integral whose associated characteristic curves are valid phase-space trajectories.

19.15 The Quantum Cauchy Problem

The analogy between the Schroedinger and Hamilton–Jacobi equations is so close that one could in principle, if not as a practical method, solve the Schroedinger equation in simple cases using only the methods of classical analytical mechanics and Bohm's quantum potential. We present only an outline of a possible method here.

To exploit the analogy in this way, first use eqn (19.104) to write the rate of change of \mathcal{P} along any characteristic curve. Assuming a Hamilton–Jacobi equation of the form eqn (19.101), the rate of change of $\mathcal{P}(\mathbf{r}, t)$ is

$$\frac{d\mathcal{P}}{d\beta} = \frac{\partial \mathcal{P}}{\partial t} \dot{t} + \dot{\mathbf{r}} \cdot \nabla \mathcal{P} = \frac{\partial \mathcal{P}}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla \mathcal{P} = -\frac{\mathcal{P}}{m} \nabla^2 S \quad (19.105)$$

where $\mathbf{p} = \nabla S$ and eqn (19.104) were used to get the last equality, and where we have taken $\beta = t$. Now consider an iterative procedure in which one first solves the Cauchy problem entirely classically, with no quantum potential, given some initial strip assignment such as that in eqn (19.67). This zeroth-order calculation gives a general integral $\mathcal{S}^{(0)}(\mathbf{r}, t)$ like that in eqn (19.78). Then assign some initial probability distribution along the initial strip, writing $\mathcal{P} = \mathcal{P}(\tau)$ for some arbitrarily chosen probability distribution function \mathcal{P} . Expressing $\nabla^2 \mathcal{S}^{(0)}$ in terms of α, β using equations like eqn (19.75) and putting that result into eqn (19.105), one can then integrate that equation to obtain $\mathcal{P}^{(0)}(\tau, \beta)$. Inverse functions like eqn (19.77) can be used to convert this probability distribution function to $\mathcal{P}^{(0)}(\mathbf{r}, t)$. Then $\mathcal{R}^{(0)} = \sqrt{\mathcal{P}^{(0)}}$ and eqn (19.103) allow a quantum potential $U_{qm}^{(0)}$ to be calculated.

The next iteration repeats the solution of the Cauchy problem, but now with the quantum potential $U_{qm}^{(0)}$ included. The result is a general integral $\mathcal{S}^{(1)}(\mathbf{r}, t)$ from which first-order values $\mathcal{P}^{(1)}(\mathbf{r}, t)$ and $U_{qm}^{(1)}$ can be obtained. The process is then repeated. If the iteration process converges adequately, the result will be a self-consistent solution of the quantum Cauchy problem consisting of a general integral $\mathcal{S}(\mathbf{r}, t)$, probability assignment $\mathcal{P}(\mathbf{r}, t)$ and quantum potential U_{qm} which solve eqns (19.101, 19.102), and hence the quantum Schroedinger equation. This solution will, by construction, have a general integral that includes the chosen initial strip, and that has the chosen initial probability distribution along it.

The literature on the Bohm model typically works in the opposite logical direction, however. One first solves the Schroedinger equation by the usual methods of quantum mechanics. Then one uses eqn (19.100) to extract $\mathcal{S}(\mathbf{r}, t)$ and $\mathcal{P}(\mathbf{r}, t) = \sqrt{\mathcal{R}}$ that satisfy eqns (19.101, 19.102) by construction. The characteristic curves defined by solving the extended Hamilton equations (with the quantum potential U_{qm} included in the Hamiltonian) are then taken to be possible trajectories of the quantum particle.

19.16 The Bohm Hidden Variable Model

The Bohm interpretation seeks to reduce quantum mechanics to an underlying stratum of classical quantities called *hidden variables*. The hidden variables are those describing the particle trajectories determined by the Hamilton–Jacobi theory with the added quantum potential. There are in general many possible trajectories, with the particle in one or the other of them. The quantum potential is taken to be a real, physical potential that has the effect of pushing the particle trajectories into paths consistent with the quantum mechanical result.

The Bohm interpretation is persuasive because of the close analogy between the Schroedinger equation and Hamilton–Jacobi theory, as detailed in Sections 19.14 and 19.15. It has a rich literature, which the reader may consult. Besides reviews by Bohm himself (Bohm, 1980; Bohm and Hiley, 1993), one may also consult a recent review of the subject in Holland (1993).

The Bohm hidden variable model may be criticized on several grounds, however. First, measurement of a variable in quantum mechanics results in what is called the collapse of the wave function. The fields \mathcal{P} and \mathcal{S} change discontinuously in such a collapse, and therefore so does the quantum potential. Thus a measurement at one physical point could lead to an instantaneous change in the quantum potential everywhere, a violation of the relativistic principle that nothing real can propagate at speeds faster than the speed of light.

Another serious problem has to do with the reality of the particle trajectories that are at the core of the Bohm interpretation. If these trajectories are real, with the particle occupying one or the other of them, then we might expect that the probability of occupation of a given trajectory would be constant as the particle moves along it. But as eqn (19.105) has shown, the probability of occupation of a trajectory is not constant as β increases.¹²³ A given trajectory may begin with a high probability of occupation and evolve into a low-probability region, or vice versa. The idea that the different particle trajectories are assigned classical probabilities¹²⁴ of occupation and carry that assigned probability with them is therefore untenable. This leads to some doubt about the physical reality of the trajectories.

A third problem has to do with the interpretation of the quantum potential. Quantum mechanics is a wave theory. The application of Fourier analysis to this wave theory predicts that any wave that is localized in space must consist of the superposition

¹²³In later work (D. Bohm and J.V. Vigier (1954) *Phys. Rev.* **96**, p. 208), Bohm introduced the idea that the probability distribution undergoes a stochastic process that converges to the quantum value \mathcal{P} . The fact remains, however, that the quantum value to which it converges is not conserved along individual particle trajectories.

Another possibility would be to abandon the idea of individual trajectories and to assign probabilities to bundles of trajectories forming flux tubes, much like the treatment of steady currents in electrodynamics. But even then, the flux of probability out of a tube would equal the flux into it only in the static case with $\partial\mathcal{P}/\partial t = 0$.

¹²⁴Given a set of mutually exclusive possibilities (here occupation of particle trajectories), classical Bayesian probability theory assigns a probability measure (a number between zero and one) to each possibility, to represent our knowledge or lack of knowledge about the system. These are sometimes called “ignorance probabilities”.

of many momentum values. The result is that a localized wave will spontaneously broaden, an effect known in quantum theory as the *spread of the wave function*. This phenomenon is reproduced in the Bohm theory by the quantum potential, which has the effect of raising the effective potential in regions where the probability distribution has peaks. It is an unproved assertion that this “potential” actually represents dynamical “forces” moving the particle trajectories this way and that. It is possible that it is simply a kinematical term that replicates the spread of the wave function, an effect actually coming from the wave nature of matter rather than from real forces.

Despite these objections, the Bohm hidden variable model deserves attention because it is at least asking the right questions. How can a wave theory (quantum mechanics) be reconciled with a theory of discrete particle trajectories (Hamiltonian analytical mechanics)?

19.17 Feynman Path-Integral Technique

Richard Feynman developed a technique for solving the Schroedinger equation by means of action integrals.¹²⁵ To understand his work, we must begin with a description of quantum mechanical propagators. The Schroedinger equation is equivalent to an integral equation linking the wave function at time t to the wave function at some initial time $t^{(1)}$. This integral equation can be written

$$\Psi(x, t) = \int_{-\infty}^{\infty} U(x, t; x^{(1)}, t^{(1)}) \Psi(x^{(1)}, t^{(1)}) dx^{(1)} \quad (19.106)$$

where, for simplicity, we restrict ourselves to one-dimensional problems and nonrelativistic quantum mechanics. The kernel of this integral equation, $U(x, t; x^{(1)}, t^{(1)})$, is called the *propagator*.

Feynman showed that the propagator can be written in terms of the same action function I as was used in stating Hamilton’s principle in eqn (6.4),

$$U(x, t; x^{(1)}, t^{(1)}) = \sum_{\text{all paths } P} B \exp\left(i I_P(x, t; x^{(1)}, t^{(1)})/\hbar\right) \quad (19.107)$$

where

$$I_P(x, t; x^{(1)}, t^{(1)}) = \int_{x^{(1)}, t^{(1)}}^{x, t} L(x, \frac{dx}{dt}, t) dt \quad (19.108)$$

and where B is a smoothly varying function that can be determined. The integral defining each I_P is to be taken along some specific varied path P connecting the end points. In Hamilton’s principle, and in the calculus of variations in general, these varied paths are important only for comparison with the unvaried path. But in the Feynman method all varied paths are significant and all contribute to the sum over “all paths.”

¹²⁵See Feynman and Hibbs (1965). A brief, accessible introduction is Chapter 8 of Shankar (1994).

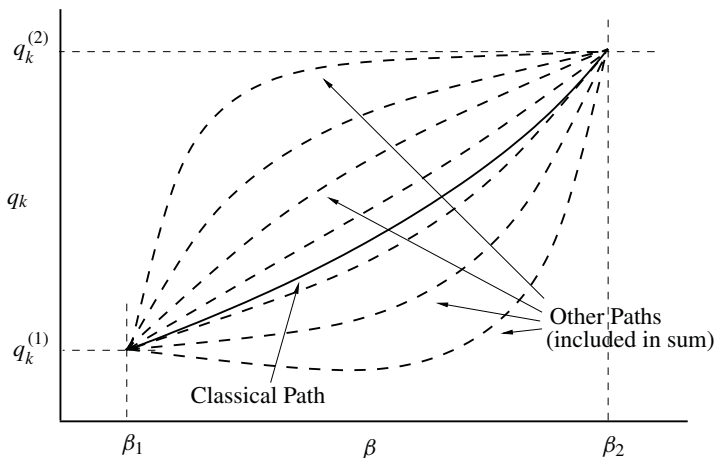


FIG. 19.5. Illustration of Feynman's sum over "all paths". In simple cases, most of the contribution to the sum comes from paths close to the classical path (shown as solid line).

If the potential appearing in the Schrodinger equation is not too complicated, the main contributions to the sum over all paths come from those near to the classical path. In this case, the sum in eqn (19.107) reduces to

$$U(x, t; x^{(1)}, t^{(1)}) = A \exp\left(i S_1(x, t, x^{(1)}, t^{(1)})/\hbar\right) \quad (19.109)$$

where S_1 is the integral of the Lagrangian L along the unique classical path between the given end points. This integral is identical to the Hamilton–Jacobi action function S_1 defined earlier in eqn (19.1). The A is again a slowly varying quantity that can be determined.

It follows that the S_1 action function of Hamilton–Jacobi theory can be used to write the quantum mechanical propagator, up to a function A that must be determined. For example, the propagator for a free particle in one dimension is obtained from eqn (19.18) as

$$U(x, t; x^{(1)}, t^{(1)}) = A \exp\left(\frac{i}{\hbar} \frac{m(x - x^{(1)})^2}{2(t - t^{(1)})}\right) \quad (19.110)$$

The Feynman method illustrates once again the close connection between quantum mechanics and Hamilton–Jacobi theory. The method is generally considered to be a calculational device, analogous to the Huygens–Fresnel construction in optics, and not an alternate interpretation of quantum mechanics. Suitably generalized, it has proved an extremely valuable tool in advanced quantum theory.

19.18 Quantum and Classical Mechanics

From the ubiquitous use of the Hamiltonian in quantum theory, to Schrodinger's original invocation of Hamilton–Jacobi theory in his derivation of the equation that

bears his name, to the Bohm model and Feynman method, there is obviously a close connection between analytical mechanics and quantum mechanics. Some feel that this connection is too close, that quantum mechanics is all too familiar. Rather than being strange, quantum mechanics is not strange enough.

The sense of surprise that quantum mechanics so closely resembles classical analytical mechanics comes from the one essential and irreducible difference between the two theories: quantum mechanics is a wave theory and classical particle mechanics is not. The Schrodinger equation is a linear wave equation with a superposition principle. This wave nature of matter in quantum theory is completely alien to classical particle mechanics.

For example, in quantum mechanics if Ψ_1 and Ψ_2 are solutions of the Schrodinger equation, then $\Psi = \alpha\Psi_1 + \beta\Psi_2$ is also an equally valid solution. But classical mechanics has no such concept as the superposition of two phase-space trajectories. If we decompose Ψ according to eqn (19.100), we find that the quantum mechanical probability distribution function is

$$\mathcal{P} = \alpha^2\mathcal{P}_1 + \beta^2\mathcal{P}_2 + \alpha\beta\sqrt{\mathcal{P}_1\mathcal{P}_2} \cos\left(\frac{\mathcal{S}_1 - \mathcal{S}_2}{\hbar}\right) \neq \alpha^2\mathcal{P}_1 + \beta^2\mathcal{P}_2 \quad (19.111)$$

where we assume for simplicity that the α, β are real numbers chosen so that Ψ is normalized. The cosine term is due to wave interference between the two quantum wave functions. Attempts to reduce quantum mechanics to a system of trajectories with an assigned probability distribution over them are blocked by such interference effects. The probabilities do not add in a classical manner. In the two-slit interference experiment, for example, the probability distribution \mathcal{P} with both slits open is not the renormalized sum of the two probability distributions, each with one slit open and the other closed. In spite of the closeness of the classical-quantum analogies, wave interference phenomena remain inaccessible to classical analytical mechanics.

This book is written in the hope that a reader may one day emerge from the thicket of classical-quantum analogies with ideas for a more comprehensive theory of matter—one that explains why an electron propagates as a wave while still requiring for its description classical variables like spin, momentum, and mass, variables that seem more appropriate for rocks and planets than for waves.

19.19 Exercises

Exercise 19.1 A projectile of mass m moves in three dimensions under the influence of a uniform gravitational force $m\mathbf{g} = -mg\hat{\mathbf{e}}_3$ acting downwards in the negative- z direction. The traditional Lagrangian is thus $L = (m/2) \|\mathbf{d}\mathbf{r}/dt\|^2 - mgz$.

(a) Show that the classical path between end points and times $\mathbf{r}^{(1)}, t^{(1)}$ and $\mathbf{r}^{(2)}, t^{(2)}$ is given by

$$\mathbf{r}(t) = \mathbf{r}^{(1)} + \mathbf{v}^{(1)} \left(t - t^{(1)} \right) - \frac{1}{2} \mathbf{g} \left(t - t^{(1)} \right)^2 \quad (19.112)$$

where

$$\mathbf{v}^{(1)} = \frac{\mathbf{r}^{(2)} - \mathbf{r}^{(1)}}{t^{(2)} - t^{(1)}} + \frac{\mathbf{g}}{2} \left(t^{(2)} - t^{(1)} \right) \quad (19.113)$$

(b) Use eqn (19.2) to derive $S_1(\mathbf{r}^{(2)}, t^{(2)}, \mathbf{r}^{(1)}, t^{(1)})$.

(c) Show that the partial derivatives in Section 19.2 using this S_1 have the values that you would expect from elementary mechanics.

Exercise 19.2

(a) For the same system as Exercise 19.1, perform the Legendre transformation as described in Section 19.3 to derive the S_2 action function.

(b) Show that the partial derivatives in Section 19.3 using this S_2 have the values that you would expect from elementary mechanics.

Exercise 19.3 (a) Verify that the general solution in eqn (19.78) satisfies the Hamilton–Jacobi equation eqn (19.66).

(b) Verify that this general solution contains the initial curve eqn (19.67).

Exercise 19.4

(a) Following the pattern in Section 19.10, carry out the details of the solution of the Cauchy problem for the mono-energetic integral in Section 19.11.

(b) Verify that the solution in eqns (19.90, 19.91) satisfies the mono-energetic Hamilton–Jacobi equation eqn (19.88) and contains the initial curve eqn (19.87).

Exercise 19.5

(a) Use eqn (19.1) and Lemma 11.4.1 to demonstrate that, since the free-particle momenta are constant along any particular trajectory, the S_1 action function for a single free particle may be written as

$$S_1(q^{(2)}, q^{(1)}) = \sum_{k=0}^3 p_k (q_k^{(2)} - q_k^{(1)}) \quad (19.114)$$

(b) Writing the constant momenta p_k in terms of the end values $q_k^{(2)}$ and $q_k^{(1)}$, and using the relativistic coordinates with $q_0 = ct$ described at the end of Section 16.9, show that $S_1(q^{(2)}, q^{(1)})$ for a single free particle may be written as the Lorentz scalar expression

$$S_1(\mathbf{r}^{(2)}, \mathbf{r}^{(1)}) = -mc \sqrt{-(\mathbf{r}^{(2)} - \mathbf{r}^{(1)}) \cdot (\mathbf{r}^{(2)} - \mathbf{r}^{(1)})} \quad (19.115)$$

(c) Dropping the superscript (2) on the upper limit, as was done in Section 19.5, demonstrate that eqn (19.115) is a solution of the covariant Hamilton–Jacobi equation eqn (19.99) when $q^{(\text{ch})} = 0$.

Exercise 19.6

(a) Show that the gauge transformation defined in Exercise 2.7 and Exercise 11.8 may be written as the Lorentz covariant expressions

$$\mathbf{A}' = \mathbf{A} + \partial\chi \quad \text{and hence} \quad \underline{\mathbf{p}}' = \underline{\mathbf{p}} + \frac{q^{(\text{ch})}}{c} \partial\chi \quad (19.116)$$

where $\partial\chi = \partial\chi/\partial\mathbf{r}$ is the fourvector gradient as defined in Section 15.12 and $\underline{\mathbf{p}}$ is the canonical momentum fourvector defined in Section 16.10.

(b) Show that if S is a solution of eqn (19.99), then

$$S' = S + \frac{q^{(\text{ch})}\chi}{c} \quad (19.117)$$

is a solution of the same equation with \mathbf{A} replaced by the gauge transformed potential \mathbf{A}' .

Exercise 19.7 A harmonic oscillator in one dimension has the extended Hamiltonian

$$\mathcal{K}(q, p) = p_0 + \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1 \quad (19.118)$$

(a) Write the Hamilton–Jacobi differential equation for the action function S .

(b) Use the method of separation of variables to find a complete solution to the Hamilton–Jacobi equation involving one non-additive constant a_1 .

(c) Identify your a_1 with the momentum P_1 as described in Section 19.9 and derive the generating function $F_2(q, P)$ that appears in Exercise 18.8.

Exercise 19.8 Substitute eqn (19.100) into the nonrelativistic Schrodinger equation in eqn (4.65)

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{-\hbar^2}{2m} \nabla^2 \psi + U\psi \quad (19.119)$$

and verify the results in eqn (19.101, 19.102, 19.103).

Exercise 19.9 Consider a one-parameter family of planes parameterized by α as in

$$0 = F(x, y, z, \alpha) = ax \cos \alpha + ay \sin \alpha + bz \quad (19.120)$$

where $a^2 + b^2 = 1$ and constants a, b are positive and nonzero.

(a) Show that these planes all pass through the origin of coordinates.

(b) Using the methods of Section D.37, find the curve of intersection of this family of planes for a given value of α . Show that for $z > 0$ it is a line having spherical polar coordinates $\theta = \arctan(b/a)$, $\phi = \pi + \alpha$, and r varying.

(c) Find the envelope of this family of planes. For $z > 0$ show that it is a right circular cone of half angle θ .

(d) Find the curve of intersection and the envelope for the region with $z < 0$. Show that the full envelope consists of two cones, one inverted and one upright, with their common vertex at the origin of coordinates.

Exercise 19.10 In Section 19.8 we used separation of variables to find a complete integral of the Hamilton–Jacobi equation for a simple projectile.

(a) Use eqn (19.54) and the methods of Section 19.9 to write an F_2 generating function for this system.

(b) Use this F_2 generating function to verify eqn (19.56), and also to find the evolution of the other variables x, y, p_0, p_x, p_y, p_z as functions of $q_0 = t$ and the constant coordinates Q_j and P_j .

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Part III

Mathematical Appendices

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APPENDIX A

VECTOR FUNDAMENTALS

The reader is assumed to be familiar with three vectors in Cartesian, spherical-polar, and cylindrical-polar coordinates. We summarize here some basic facts that will be used throughout the text. Even those who feel secure with vector algebra should read this material, since some ideas familiar to them may be presented here in a novel way.

A.1 Properties of Vectors

The interval between two points in three-dimensional space can be thought of as a directed line running from the first to the second point. This displacement can be represented graphically by an arrow whose direction is the same as the displacement and whose length is proportional to the represented distance. The properties of vectors in general are abstracted from these displacement vectors. For example, two vectors are added by placing the tail of the second addend at the head of the first. The sum is then the vector from the tail of the first to the head of the second.

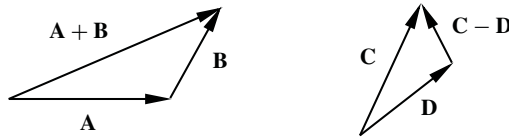


FIG. A.1. Addition and subtraction of displacement vectors.

Associated with every vector \mathbf{V} is its norm or magnitude V , a non-negative number which is zero only when the vector is the null vector. A vector can be multiplied by a number α . The result is a vector having magnitude $|\alpha|V$, and the same direction as \mathbf{V} , unless α is negative in which case the product has the opposite direction. Each vector has an associated unit vector $\hat{\mathbf{V}} = \mathbf{V}/V = (1/V)\mathbf{V}$, a vector of unit norm. The unit vector has no units, these being carried by the magnitude V . The formula $\mathbf{V} = V\hat{\mathbf{V}}$ can be thought of as factoring \mathbf{V} into a magnitude V times a pure direction $\hat{\mathbf{V}}$.

Vector equations must balance, that is they must be of the form $\mathbf{V} = \mathbf{W}$ where both sides are vectors. Equations like $\mathbf{V} = 3$ are meaningless. The only exception is that the null vector, which strictly should be written $\mathbf{0}$, is universally written as the scalar 0. Thus $\mathbf{V} = 0$ is permitted.

A.2 Dot Product

The arrow representing a vector can be moved about in diagrams without changing the vector represented, so long as the direction and length are kept fixed. Two such

vectors define a plane, as can be seen graphically by so moving them until their arrows are tail-to-tail.

The dot product of two vectors is defined as

$$\mathbf{V} \cdot \mathbf{W} = VW \cos \theta_{VW} \quad (\text{A.1})$$

where θ_{VW} is the smaller angle between the two vectors drawn in the plane that they define. It is symmetric: $\mathbf{W} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{W}$. The magnitude or norm of a vector can be expressed in terms of the dot product of \mathbf{V} with itself,

$$\|\mathbf{V}\| = V = \sqrt{\mathbf{V} \cdot \mathbf{V}}. \quad (\text{A.2})$$

Two vectors are orthogonal if and only if their dot product vanishes.

If $\hat{\mathbf{n}}$ is any unit vector, the perpendicular projection of vector \mathbf{V} onto the direction of $\hat{\mathbf{n}}$ is $V_n = V \cos \theta_{Vn} = \mathbf{V} \cdot \hat{\mathbf{n}}$. Using this projection, \mathbf{V} can be decomposed into the sum of two vectors, \mathbf{V}_{\parallel} parallel to $\hat{\mathbf{n}}$, and \mathbf{V}_{\perp} perpendicular to it,

$$\mathbf{V} = \mathbf{V}_{\parallel} + \mathbf{V}_{\perp} \quad (\text{A.3})$$

where

$$\mathbf{V}_{\parallel} = V_n \hat{\mathbf{n}} = (\mathbf{V} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} \quad \text{and} \quad \mathbf{V}_{\perp} = \mathbf{V} - (\mathbf{V} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}. \quad (\text{A.4})$$

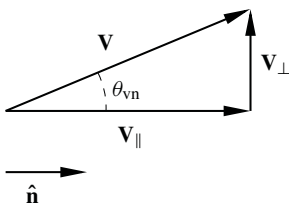


FIG. A.2. Projections parallel and perpendicular to $\hat{\mathbf{n}}$.

A.3 Cross Product

The cross product of \mathbf{V} and \mathbf{W} , denoted $\mathbf{C} = \mathbf{V} \times \mathbf{W}$, is another vector \mathbf{C} whose direction is perpendicular to the plane defined by \mathbf{V} and \mathbf{W} , with the sense of the perpendicular being the direction of the thumb of one's right hand when the fingers are curled from \mathbf{V} to \mathbf{W} . The magnitude of \mathbf{C} is $C = VW \sin \theta_{VW}$, where θ_{VW} is the smaller angle between the two vectors. This definition implies that the cross product of two parallel or anti-parallel vectors is zero, and that $\mathbf{W} \times \mathbf{V} = -\mathbf{V} \times \mathbf{W}$.

A.4 Linearity

Simple geometrical proofs show that the definitions of scalar multiplication, dot product, and cross product all imply the property of linearity (here α, β are scalars),

$$\alpha (\mathbf{W} + \mathbf{X}) = \alpha \mathbf{W} + \alpha \mathbf{X} \quad (\text{A.5})$$

$$\mathbf{V} * (\alpha \mathbf{W} + \beta \mathbf{X}) = \alpha \mathbf{V} * \mathbf{W} + \beta \mathbf{V} * \mathbf{X} \quad (\text{A.6})$$

where the $*$ symbol stands for either \cdot or \times .

An important consequence of this linearity is that the product rule for differentiation applies to vector products. If vectors \mathbf{V} and \mathbf{W} are functions of some variable β , then

$$\frac{d}{d\beta} (\mathbf{V} * \mathbf{W}) = \frac{d\mathbf{V}}{d\beta} * \mathbf{W} + \mathbf{V} * \frac{d\mathbf{W}}{d\beta} \tag{A.7}$$

where the $*$ symbol again stands for either \cdot or \times . A similar product rule applies to products of scalars and vectors.

A.5 Cartesian Basis

We denote the mutually orthogonal, unit basis vectors of a Cartesian coordinate system as $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. (This triad of vectors is often denoted $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ in other texts.) The three vectors are “normalized”, by which we mean that they are unit vectors, and are often referred to as an “orthonormal” basis. The orthonormality of the basis unit vectors implies

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \tag{A.8}$$

where the *Kroeneker delta function* δ_{ij} defined by eqn (A.8) is 1 when $i=j$ and 0 otherwise.

The component $V_i = \mathbf{V} \cdot \hat{\mathbf{e}}_i = V \cos \theta_i$ (where θ_i is the angle between \mathbf{V} and $\hat{\mathbf{e}}_i$) is the projection of \mathbf{V} onto the direction of $\hat{\mathbf{e}}_i$, and $\cos \theta_i$ is often called the *i th direction cosine* of \mathbf{V} . Any vector \mathbf{V} can be expanded in the $\hat{\mathbf{e}}_i$ basis as

$$\mathbf{V} = \sum_{i=1}^3 V_i \hat{\mathbf{e}}_i, \quad \text{where} \quad V_i = \mathbf{V} \cdot \hat{\mathbf{e}}_i \tag{A.9}$$

are its components. The two equations in eqn (A.9) are often combined as

$$\mathbf{V} = \sum_{i=1}^3 (\mathbf{V} \cdot \hat{\mathbf{e}}_i) \hat{\mathbf{e}}_i \quad \text{or} \quad \mathbf{V} = \sum_{i=1}^3 \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \cdot \mathbf{V}) \tag{A.10}$$

where the equivalence of the two forms is due to the symmetry of dot products and to the fact that scalars and vectors can be written in any order.

We sometimes define a vector by listing its components, as in

$$\mathbf{V} : \{V_1, V_2, V_3\}_o \tag{A.11}$$

which states that vector \mathbf{V} has components V_1, V_2, V_3 relative to some orthonormal basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ labeled as basis o . The same vector \mathbf{V} can have quite different components V'_1, V'_2, V'_3 relative to a different orthonormal basis $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$ labeled as o' ,

$$\mathbf{V} : \{V'_1, V'_2, V'_3\}_{o'} \tag{A.12}$$

so we use the colon rather than an equal sign in these expressions to avoid the mistake of equating the vector to its components. Components depend on the coordinate system; invariant objects like vectors do not. Equations like $\mathbf{V} = \{V_1, V_2, V_3\}$ are

therefore incorrect. Although it may seem a small matter, this distinction, like the above-mentioned prohibition of $\mathbf{V} = 3$, can avoid a good deal of grief.

The Cartesian basis vectors are assumed to be a right-handed system, by which we mean that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$ with the above definition of cross products. It is convenient to define what is called the *Levi-Civita function* ε_{ijk} by

$$\varepsilon_{ijk} = (\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) \quad (\text{A.13})$$

The second equality may be verified directly in all cases, assuming a right-handed system, with the results that $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$ and $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$, and that ε_{ijk} vanishes if any pair of the i, j, k is equal. Note that exchange of any pair of indices of ε_{ijk} changes its sign. Since cyclic permutation of three indices is equivalent to two exchanges, this implies that $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$ for any choice of indices i, j, k . Expanding the cross product of two basis vectors using eqn (A.10) gives

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) = \sum_{k=1}^3 ((\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k) \hat{\mathbf{e}}_k = \sum_{k=1}^3 \varepsilon_{ijk} \hat{\mathbf{e}}_k. \quad (\text{A.14})$$

It follows from linearity eqn (A.6), using eqn (A.14) and the interchangeability of finite sums, that

$$\mathbf{C} = \mathbf{V} \times \mathbf{W} = \sum_{i=1}^3 \sum_{j=1}^3 V_i W_j \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{k=1}^3 \left(\sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ijk} V_i W_j \right) \hat{\mathbf{e}}_k \quad (\text{A.15})$$

or, in other words,

$$\mathbf{C} = \sum_{k=1}^3 C_k \hat{\mathbf{e}}_k \quad \text{where} \quad C_k = \sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ijk} V_i W_j \quad (\text{A.16})$$

This same linearity, applied now to dot products, gives

$$\mathbf{V} \cdot \mathbf{W} = \sum_{i=1}^3 \sum_{j=1}^3 V_i W_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 V_i W_j \delta_{ij} = \sum_{i=1}^3 V_i W_i. \quad (\text{A.17})$$

from which we also derive an equation for the magnitude in terms of components,

$$\|\mathbf{V}\| = V = \sqrt{\mathbf{V} \cdot \mathbf{V}} = \sqrt{\sum_{i=1}^3 V_i^2} \quad (\text{A.18})$$

A.6 The Position Vector

Position in space will be indicated by vector \mathbf{r} from the origin of coordinates to the denoted point. This *position vector* is also often referred to as the *radius vector*. The

Cartesian coordinates x_1, x_2, x_3 (which we occasionally also refer to as x, y, z) of a point in space are the dot products of this vector with the Cartesian unit vectors $\hat{\mathbf{e}}_i$,

$$\mathbf{r} = \sum_{i=1}^3 x_i \hat{\mathbf{e}}_i \quad \text{where} \quad x_i = \mathbf{r} \cdot \hat{\mathbf{e}}_i \tag{A.19}$$

It might make more sense to use r_i rather than x_i in the preceding equation, but we bow here to the general custom.

A.7 Fields

Scalar and vector fields are functions of the position vector. For example, the temperature in a room is a scalar quantity that varies from place to place (and also in time, of course, although we ignore that here). We denote such a scalar field by writing $T = T(\mathbf{r})$. The wind velocity is an example of a vector field. At every point it has a particular intensity and direction. We write it as $\mathbf{v} = \mathbf{v}(\mathbf{r})$. A constant vector can be thought of as a particular case of a vector field, a “uniform” vector field that happens not to vary as \mathbf{r} varies.

A.8 Polar Coordinates

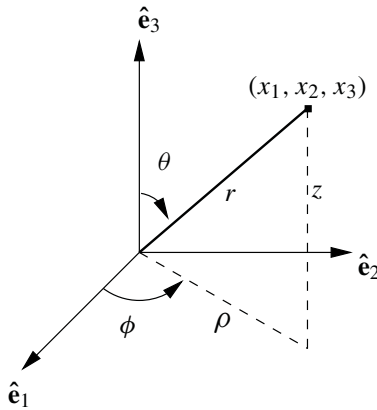


FIG. A.3. Spherical polar and cylindrical polar coordinates.

Cylindrical polar coordinates ρ, ϕ, z and spherical polar coordinates r, θ, ϕ are defined in terms of Cartesian coordinates x_1, x_2, x_3 by

$$x_1 = r \sin \theta \cos \phi = \rho \cos \phi \tag{A.20}$$

$$x_2 = r \sin \theta \sin \phi = \rho \sin \phi \tag{A.21}$$

$$x_3 = r \cos \theta = z \tag{A.22}$$

or, inversely, by

$$\rho = \sqrt{x_1^2 + x_2^2} \quad (\text{A.23})$$

$$\phi = \arctan\left(\frac{x_2}{x_1}\right) \quad (\text{A.24})$$

$$z = x_3 \quad (\text{A.25})$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (\text{A.26})$$

$$\theta = \arccos\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \quad (\text{A.27})$$

where $0 \leq \theta \leq \pi$ and the quadrant of ϕ is determined by the equations

$$x_1 = \rho \cos \phi \quad \text{and} \quad x_2 = \rho \sin \phi \quad (\text{A.28})$$

where ρ is a positive quantity. The position of a point can then be specified by giving the three numbers x_1, x_2, x_3 (Cartesian coordinates), the three numbers ρ, ϕ, z (cylindrical polar coordinates), or the three numbers r, θ, ϕ (spherical polar coordinates).

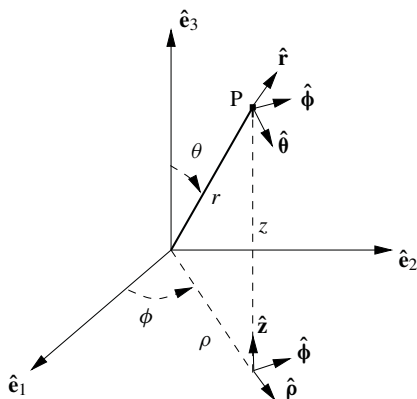


FIG. A.4. Illustration of unit vectors. Point P has Cartesian components x_1, x_2, x_3 .

The basis unit vectors of polar coordinates point in the directions that the denoted point moves when the corresponding polar coordinates are incremented individually,

$$\hat{\rho} = \hat{e}_1 \cos \phi + \hat{e}_2 \sin \phi \quad (\text{A.29})$$

$$\hat{\phi} = -\hat{e}_1 \sin \phi + \hat{e}_2 \cos \phi \quad (\text{A.30})$$

$$\hat{z} = \hat{e}_3 \quad (\text{A.31})$$

$$\hat{r} = \hat{\rho} \sin \theta + \hat{z} \cos \theta \quad (\text{A.32})$$

$$\hat{\theta} = \hat{\rho} \cos \theta - \hat{z} \sin \theta \quad (\text{A.33})$$

Inverse relations can be derived from these, as

$$\hat{\mathbf{e}}_1 = \hat{\boldsymbol{\rho}} \cos \phi - \hat{\boldsymbol{\phi}} \sin \phi \quad (\text{A.34})$$

$$\hat{\mathbf{e}}_2 = \hat{\boldsymbol{\rho}} \sin \phi + \hat{\boldsymbol{\phi}} \cos \phi \quad (\text{A.35})$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{z}} \quad (\text{A.36})$$

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{r}} \sin \theta + \hat{\boldsymbol{\theta}} \cos \theta \quad (\text{A.37})$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta \quad (\text{A.38})$$

The orthonormal basis vectors of cylindrical polar coordinates are $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$. The orthonormal basis vectors of spherical polar coordinates are $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$. The position vector in the three systems, Cartesian, cylindrical polar, and spherical polar, is

$$\mathbf{r} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 \quad (\text{A.39})$$

$$\mathbf{r} = \rho \hat{\boldsymbol{\rho}} + z \hat{\mathbf{z}} \quad (\text{A.40})$$

$$\mathbf{r} = r \hat{\mathbf{r}} \quad (\text{A.41})$$

The differential of position $d\mathbf{r}$ is written in the three systems by

$$d\mathbf{r} = dx_1 \hat{\mathbf{e}}_1 + dx_2 \hat{\mathbf{e}}_2 + dx_3 \hat{\mathbf{e}}_3 \quad (\text{A.42})$$

$$d\mathbf{r} = d\rho \hat{\boldsymbol{\rho}} + \rho d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}} \quad (\text{A.43})$$

$$d\mathbf{r} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}} \quad (\text{A.44})$$

Dividing each term of these three equations by dt will also give the velocity vector $\mathbf{v} = d\mathbf{r}/dt$,

$$\mathbf{v} = \dot{x}_1 \hat{\mathbf{e}}_1 + \dot{x}_2 \hat{\mathbf{e}}_2 + \dot{x}_3 \hat{\mathbf{e}}_3 \quad (\text{A.45})$$

$$\mathbf{v} = \dot{\rho} \hat{\boldsymbol{\rho}} + \rho \dot{\phi} \hat{\boldsymbol{\phi}} + \dot{z} \hat{\mathbf{z}} \quad (\text{A.46})$$

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}} + r \sin \theta \dot{\phi} \hat{\boldsymbol{\phi}} \quad (\text{A.47})$$

where the dots denote total derivatives of the coordinates with respect to time.

The orthonormal basis vectors of Cartesian coordinates are constants; they do not vary with \mathbf{r} . The orthonormal basis vectors of spherical polar and cylindrical polar coordinates are nonuniform vector fields. We know them only when we know the position \mathbf{r} being referred to. The differential changes in the polar coordinate basis-vector fields when the coordinates are incremented are

$$d\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} d\theta + \hat{\boldsymbol{\phi}} \sin \theta d\phi \quad (\text{A.48})$$

$$d\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} d\theta + \hat{\boldsymbol{\phi}} \cos \theta d\phi \quad (\text{A.49})$$

$$d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}} d\phi = -(\hat{\mathbf{r}} \sin \theta + \hat{\boldsymbol{\theta}} \cos \theta) d\phi \quad (\text{A.50})$$

$$d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}} d\phi \quad (\text{A.51})$$

from which one can read partial derivatives such as $\partial \hat{\boldsymbol{\theta}} / \partial \phi = \hat{\boldsymbol{\phi}} \cos \theta$, etc., by letting one coordinate at a time be varied on the right-hand sides of the equations.

A.9 The Algebra of Sums

Sums in this book will use explicit summation signs. The Einstein summation convention (automatic summation over repeated indices) will not be used.

The reader is assumed familiar with the rules of manipulation of multiple sums. These rules closely mirror similar rules for multiple integrals. For example, the factorization in

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ik} b_{ij} c_k = c_k \sum_{i=1}^3 a_{ik} \sum_{j=1}^3 b_{ij} \quad (\text{A.52})$$

resembles that in

$$\int_0^a dx \int_0^a dy f(x, z) g(x, y) h(z) = h(z) \int_0^a dx f(x, z) \int_0^a dy g(x, y). \quad (\text{A.53})$$

Sums that differ only by the universal change of dummy indices are equal, as in

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ik} b_{ij} c_k = \sum_{l=1}^3 \sum_{m=1}^3 a_{lk} b_{lm} c_k. \quad (\text{A.54})$$

The order of finite sums can be exchanged so long as the range of indices summed over is kept the same, as in

$$\sum_{i=1}^3 \sum_{j=1}^i a_{ik} b_{ij} c_k = \sum_{j=1}^3 \sum_{i=j}^3 a_{ik} b_{ij} c_k. \quad (\text{A.55})$$

Sums involving the Kroeneker delta are used frequently, the result usually being the collapse of a sum to a single value as in

$$\sum_{j=1}^3 \delta_{ij} W_j = W_i. \quad (\text{A.56})$$

A.10 Miscellaneous Vector Formulae

It follows from eqn (A.13) and the linearity of dot and cross products that the triple scalar product of three vectors is

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 A_i B_j C_k \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 A_i B_j C_k \varepsilon_{ijk} \quad (\text{A.57})$$

with the same result when dot and cross are exchanged,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \quad (\text{A.58})$$

Both of these expressions are equal to the determinant of a matrix that has the components of the vectors as its columns. To see this, apply the definition of determinant in eqn (B.37) with $A_{i1} = A_i$, $A_{j2} = B_j$, and $A_{k3} = C_k$ and note that ε_{ijk} is zero

for permutations in which any two of its indices are equal. Hence

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}. \quad (\text{A.59})$$

Determinant properties 1 and 4 from Section B.11, then show that the triple scalar product is zero if any two vectors in it are parallel, and that it changes sign under any exchange of vectors.

The triple vector product of three unit vectors $\hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k)$ can be seen (by examination of all twenty-seven possible cases) to be zero unless either $i = j$ or $i = k$, and in those cases to be

$$\hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \hat{\mathbf{e}}_j \delta_{ik} - \hat{\mathbf{e}}_k \delta_{ij}. \quad (\text{A.60})$$

The linearity of dot and cross products then gives what is sometimes called the ‘‘bac-cab’’ expansion,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 A_i B_j C_k \hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}). \quad (\text{A.61})$$

The product $(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot (\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_l)$ can be seen by inspection to be zero except when the pair ij matches the pair kl . In those cases, it is seen (again by examining all the possibilities) that

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot (\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_l) = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}. \quad (\text{A.62})$$

The linearity of cross and dot products then gives

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C}) (\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D}) (\mathbf{B} \cdot \mathbf{C}). \quad (\text{A.63})$$

Applying eqn (A.14) to eqn (A.62) allows a useful identity to be derived,

$$\sum_{m=1}^3 \varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \quad (\text{A.64})$$

from which it also follows that

$$\sum_{l=1}^3 \sum_{m=1}^3 \varepsilon_{ilm} \varepsilon_{klm} = 2\delta_{ik}. \quad (\text{A.65})$$

It should be mentioned that all of the formulas in this section and Section A.5 have exactly the same form in Cartesian, cylindrical polar, and spherical polar coordinates. If one replaces $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in formulas by the orthonormal triad $\hat{\mathbf{f}}_1 = \hat{\mathbf{r}}, \hat{\mathbf{f}}_2 = \hat{\boldsymbol{\theta}}, \hat{\mathbf{f}}_3 = \hat{\boldsymbol{\phi}}$ of spherical polar coordinates or the orthonormal triad $\hat{\mathbf{g}}_1 = \hat{\boldsymbol{\rho}}, \hat{\mathbf{g}}_2 = \hat{\boldsymbol{\phi}}, \hat{\mathbf{g}}_3 = \hat{\mathbf{z}}$ of cylindrical polar coordinates, the formulas for expansions of vectors, dot and cross products, etc., retain exactly the same forms.

A.11 Gradient Vector Operator

The gradient vector operator is defined as

$$\nabla = \sum_{i=1}^3 \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \mathbf{r}} \quad (\text{A.66})$$

The second equality gives a useful alternate notation which lists explicitly the quantity with respect to which the differentiation is done.

If $f = f(x, y, z, t)$ is a time-varying scalar field function of position $\mathbf{r} = \mathbf{r}(t)$ which is itself a function of time, then the chain rule of partial differentiation yields

$$\frac{df}{dt} = \mathbf{v} \cdot \nabla f + \frac{\partial f(x, y, z, t)}{\partial t} = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\partial f(x, y, z, t)}{\partial t} \quad (\text{A.67})$$

where we have written out the functional dependence of f to emphasize the meaning of the time partial derivative. The \mathbf{v} here is the velocity vector defined by

$$\mathbf{v} = \sum_{i=1}^3 \dot{x}_i \hat{\mathbf{e}}_i \quad \text{where} \quad \dot{x}_i = \frac{dx_i}{dt} \quad (\text{A.68})$$

If \mathbf{C} is any vector which is not a function of \mathbf{r} (e.g. it might be a constant vector), then a useful identity is

$$\mathbf{C} = \nabla (\mathbf{C} \cdot \mathbf{r}) = \frac{\partial}{\partial \mathbf{r}} (\mathbf{C} \cdot \mathbf{r}) \quad (\text{A.69})$$

A gradient with respect to velocity components can also be defined

$$\nabla_v = \sum_{i=1}^3 \hat{\mathbf{e}}_i \frac{\partial}{\partial \dot{x}_i} = \frac{\partial}{\partial \mathbf{v}} \quad (\text{A.70})$$

Then, if \mathbf{C} is any vector which is not a function of \mathbf{v} , an identity similar to eqn (A.69) holds,

$$\mathbf{C} = \nabla_v (\mathbf{C} \cdot \mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} (\mathbf{C} \cdot \mathbf{v}) \quad (\text{A.71})$$

It follows from eqn (A.71) that

$$\frac{\partial}{\partial \mathbf{v}} v^2 = \frac{\partial}{\partial \mathbf{v}} (\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \quad (\text{A.72})$$

since the product rule for differentiation considers one of the \mathbf{v} factors to be a constant like \mathbf{C} in each differentiation.

The notations $\partial/\partial \mathbf{r}$ for ∇ , and $\partial/\partial \mathbf{v}$ for ∇_v , are very useful since they allow one to list explicitly the variables being differentiated with respect to. However, they must be used carefully. The expression $\partial f/\partial \mathbf{r}$ for ∇f is correct when f is a scalar function. But $\partial \mathbf{A}/\partial \mathbf{r}$ with a vector \mathbf{A} would be ambiguous and should be avoided. Instead, one should write a divergence $\nabla \cdot \mathbf{A}$ as $(\partial/\partial \mathbf{r}) \cdot \mathbf{A}$ and a curl $\nabla \times \mathbf{A}$ as $(\partial/\partial \mathbf{r}) \times \mathbf{A}$.

A.12 The Serret–Frenet Formulae

In mechanics, we pay special attention to motion along a line, the path of a point mass for example. A line may be defined parametrically by letting each component x_i depend on some parameter, such as the time t (or some other parameter that varies continuously and monotonically as the point moves along the line). Denoting the monotonic parameter by β , we have

$$x_1 = x_1(\beta) \quad x_2 = x_2(\beta) \quad x_3 = x_3(\beta) \quad (\text{A.73})$$

which together define $\mathbf{r} = \mathbf{r}(\beta)$. The differential of \mathbf{r} is

$$d\mathbf{r} = \frac{d\mathbf{r}}{d\beta} d\beta. \quad (\text{A.74})$$

The arc-length measured from some point on the curve in the sense of increasing β is usually denoted by the letter s . Since

$$ds = \left\| \frac{d\mathbf{r}}{d\beta} \right\| d\beta, \quad (\text{A.75})$$

we may define a unit vector $\hat{\mathbf{t}}$, called the unit-tangent vector, by

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/d\beta}{ds/d\beta} = \frac{d\mathbf{r}/d\beta}{\|d\mathbf{r}/d\beta\|}. \quad (\text{A.76})$$

This unit tangent vector is a function of position along the line, always tangent to it at the current point $\mathbf{r}(\beta)$, and always pointing in the direction of increasing β and s .

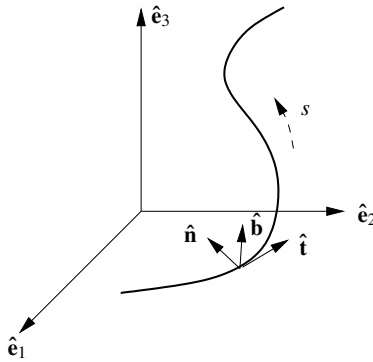


FIG. A.5. Serret–Frenet vectors along a curve.

The derivative of $\hat{\mathbf{t}}$ may be used to define a non-negative magnitude ρ and a unit vector $\hat{\mathbf{n}}$. Define $\rho = \left\| d\hat{\mathbf{t}}/ds \right\|$ and $\hat{\mathbf{n}} = (1/\rho)(d\hat{\mathbf{t}}/ds)$ so that

$$\frac{d\hat{\mathbf{t}}}{ds} = \rho \hat{\mathbf{n}} \quad (\text{A.77})$$

Vector $\hat{\mathbf{n}}$ is called the unit normal vector. It varies with \mathbf{r} , but remains always perpendicular to $\hat{\mathbf{t}}$, as can be seen by differentiating the expression $\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1$ with respect

to s . Magnitude ρ is called the curvature, and its inverse $1/\rho$ is called the radius of curvature. A third unit vector $\hat{\mathbf{b}}$ can now be defined by $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, so that $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$ form a right-handed, mutually orthogonal triad of unit vectors. Vector $\hat{\mathbf{b}}$ is called the unit bi-normal vector.

The derivatives of $\hat{\mathbf{b}}$ and $\hat{\mathbf{n}}$ can be calculated in turn. Taking the derivatives of the expressions $\hat{\mathbf{b}} \cdot \hat{\mathbf{b}} = 1$ and $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$ with respect to s shows that $d\hat{\mathbf{b}}/ds$ is perpendicular to both $\hat{\mathbf{b}}$ and $\hat{\mathbf{t}}$. Its remaining component along $\hat{\mathbf{n}}$ is denoted by $-\kappa$,

$$\frac{d\hat{\mathbf{b}}}{ds} = -\kappa \hat{\mathbf{n}} \quad (\text{A.78})$$

where κ is a positive or negative quantity called the torsion. Differentiating the identity $\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}$ with respect to s and using eqns (A.77, A.78) then gives

$$\frac{d\hat{\mathbf{n}}}{ds} = -\rho \hat{\mathbf{t}} + \kappa \hat{\mathbf{b}} \quad (\text{A.79})$$

As β increases, a nonzero ρ always makes the triad of unit vectors rotate in a right-hand sense about $\hat{\mathbf{b}}$. If κ is positive(negative), the triad also rotates in a right(left)-hand sense about $\hat{\mathbf{t}}$, as for example in a right(left)-handed helix. A curve is planar if and only if its torsion κ vanishes. Bi-normal vector $\hat{\mathbf{b}}$ is then constant and perpendicular to the plane of the curve. If we denote the three unit vectors by $\hat{\mathbf{g}}_k$ where $\hat{\mathbf{g}}_1 = \hat{\mathbf{t}}$, $\hat{\mathbf{g}}_2 = \hat{\mathbf{n}}$, and $\hat{\mathbf{g}}_3 = \hat{\mathbf{b}}$, eqns (A.77, A.78, A.79) may be summarized as

$$\frac{d\hat{\mathbf{g}}_k}{ds} = \boldsymbol{\omega} \times \hat{\mathbf{g}}_k \quad \text{where} \quad \boldsymbol{\omega} = \kappa \hat{\mathbf{t}} + \rho \hat{\mathbf{b}} \quad (\text{A.80})$$

Equations (A.76) through (A.79) are called the Serret–Frenet formulae.¹²⁶ One interesting use of them is to expand a general vector field $\mathbf{V}(\mathbf{r})$ evaluated at a given point $\mathbf{r}(\beta)$ along the curve in terms of the orthogonal basis at that point, as in

$$\mathbf{V}(\mathbf{r}(\beta)) = V_t \hat{\mathbf{t}} + V_n \hat{\mathbf{n}} + V_b \hat{\mathbf{b}} \quad (\text{A.81})$$

where

$$V_t = \hat{\mathbf{t}} \cdot \mathbf{V}(\mathbf{r}(\beta)) \quad V_n = \hat{\mathbf{n}} \cdot \mathbf{V}(\mathbf{r}(\beta)) \quad V_b = \hat{\mathbf{b}} \cdot \mathbf{V}(\mathbf{r}(\beta)) \quad (\text{A.82})$$

in direct analogy to eqn (A.9). Then the rate of change in \mathbf{V} as we move distance ds along the curve can be written as

$$\begin{aligned} \frac{d\mathbf{V}}{ds} &= \frac{dV_t}{ds} \hat{\mathbf{t}} + V_t \frac{d\hat{\mathbf{t}}}{ds} + \frac{dV_n}{ds} \hat{\mathbf{n}} + V_n \frac{d\hat{\mathbf{n}}}{ds} + \frac{dV_b}{ds} \hat{\mathbf{b}} + V_b \frac{d\hat{\mathbf{b}}}{ds} \\ &= \left(\frac{dV_t}{ds} \hat{\mathbf{t}} + \frac{dV_n}{ds} \hat{\mathbf{n}} + \frac{dV_b}{ds} \hat{\mathbf{b}} \right) + \boldsymbol{\omega} \times \mathbf{V} \end{aligned} \quad (\text{A.83})$$

where the Serret–Frenet formulae in the form of eqn (A.80) were used. If the derivatives of components dV_t/ds , dV_n/ds , and dV_b/ds all vanish at all points along the

¹²⁶For more detail, see Chapter I of Struik (1961).

curve, that is if $\mathbf{V}(\mathbf{r}) = c_1 \hat{\mathbf{t}} + c_2 \hat{\mathbf{n}} + c_3 \hat{\mathbf{b}}$ along the curve, where the three c_i are constants, then the value of the vector field $\mathbf{V}(\mathbf{r})$ along the curve is entirely determined by its value at some initial point. We can say that such a vector field is in some sense rigidly attached to the curve and is carried along with it. Such a vector field would not be uniform, of course, not even for points along the curve, because the unit vectors themselves are nonuniform vector fields.

A point of inflection is a point on the curve with $\rho = 0$ but $d\rho/ds \neq 0$. At points of inflection the sense of $\hat{\mathbf{n}}$ and hence $\hat{\mathbf{b}}$ may change discontinuously. At such points, a vector such as $\mathbf{V}(\mathbf{r}) = c_1 \hat{\mathbf{t}} + c_2 \hat{\mathbf{n}} + c_3 \hat{\mathbf{b}}$ would be discontinuous.

In the special case in which parameter β is the time t , eqn (A.75) becomes $ds = \|\mathbf{v}\| dt$ where \mathbf{v} is the velocity vector $\mathbf{v} = d\mathbf{r}/dt$, and $\|\mathbf{v}\| = v$ is its magnitude, the speed of motion along the line. Then eqn (A.76) becomes

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{(d\mathbf{r}/dt)}{(ds/dt)} = \frac{\mathbf{v}}{v} \quad (\text{A.84})$$

And eqn (A.77) becomes

$$\rho \hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \frac{1}{v} \frac{d}{dt} \left(\frac{\mathbf{v}}{v} \right) = \frac{1}{v^2} \mathbf{a} - \frac{1}{v^3} \mathbf{v} \frac{dv}{dt} = \frac{1}{v^2} \left(\mathbf{a} - \hat{\mathbf{t}} \frac{dv}{dt} \right) \quad (\text{A.85})$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the acceleration vector. Solving for \mathbf{a} then gives the acceleration

$$\mathbf{a} = \frac{dv}{dt} \hat{\mathbf{t}} + \frac{v^2}{(1/\rho)} \hat{\mathbf{n}} \quad (\text{A.86})$$

broken down into its tangential and normal components along the curve. If we imagine a particle of mass m moving along a curve under the influence of a force field $\mathbf{f}(\mathbf{r}, t)$, Newton's second law gives

$$\mathbf{f}(\mathbf{r}(t), t) = m\mathbf{a} = m \frac{dv}{dt} \hat{\mathbf{t}} + \frac{mv^2}{(1/\rho)} \hat{\mathbf{n}} \quad (\text{A.87})$$

We note that \mathbf{f} has no component in the $\hat{\mathbf{b}}$ direction. This is *not* because the curve of particle motion lacks torsion. Particles often move in helices, for example. It is rather because the natural motion $\mathbf{r}(t)$ of a mass under a force field $\mathbf{f}(\mathbf{r}, t)$ will always adjust itself to keep $\mathbf{f}(\mathbf{r}(t), t)$ confined to its $\hat{\mathbf{t}}, \hat{\mathbf{n}}$ plane. This is in turn a consequence of the remarkable simplicity Newton's second law, which contains only the acceleration vector \mathbf{a} .

APPENDIX B

MATRICES AND DETERMINANTS

We summarize here some of the standard properties of matrices and determinants that are used throughout the text. Proofs not given here can be found in most surveys of linear algebra, for example Birkhoff and MacLane (1977), Mirsky (1961), and Strang (1998).

B.1 Definition of Matrices

An $M \times N$ matrix is a rectangular array of numbers placed in M rows and N columns. These numbers are called *matrix elements* and are indexed by subscripts ij where $i = 1, \dots, M$ is the row index and $j = 1, \dots, N$ the column index. Thus a 3×4 matrix \mathbf{A} would be written as

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix} \quad (\text{B.1})$$

If all of the matrix elements A_{ij} are real numbers, then \mathbf{A} is a *real* matrix. If one or more of the matrix elements are complex numbers, the matrix is a *complex* matrix.

B.2 Transposed Matrix

The *transpose* of an $M \times N$ matrix \mathbf{A} is an $N \times M$ matrix denoted as \mathbf{A}^T or $\tilde{\mathbf{A}}$. It is defined by listing its matrix elements for all values $i = 1, \dots, M$ and $j = 1, \dots, N$,

$$A_{ij}^T = A_{ji} \quad (\text{B.2})$$

where A_{ij}^T denotes the ij th matrix element of the matrix \mathbf{A}^T . For example, if

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{then} \quad \mathbf{B}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad (\text{B.3})$$

By definition, the transpose of the transpose is just the original matrix,

$$\left(\mathbf{A}^T\right)^T = \mathbf{A} \quad (\text{B.4})$$

B.3 Column Matrices and Column Vectors

A special case of importance is the $M \times 1$ matrix called a *column matrix*. Since these matrices are often used to represent the components of a vector, they are also called *column vectors*. They are denoted by enclosing their label in square brackets and suppressing the second subscript of their matrix elements,

$$[V] = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_M \end{pmatrix} \quad (\text{B.5})$$

The transpose of an $M \times 1$ column matrix is a $1 \times M$ row matrix or row vector,

$$[V]^T = (V_1 \ V_2 \ \cdots \ V_M) \quad (\text{B.6})$$

The column vector all of whose matrix elements are zeroes is called the null vector and is denoted by $[0]$, although sometimes it is also denoted by just the scalar 0,

$$[0] = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{B.7})$$

B.4 Square, Symmetric, and Hermitian Matrices

If $M = N$, the matrix is called an N -rowed *square matrix*. The transpose of a square matrix is also a square matrix, with the same number of rows. The transpose of a square matrix can be thought of as a reflection of the matrix elements about the diagonal from upper left to lower right. Thus, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad (\text{B.8})$$

If a square matrix \mathbf{S} is equal to its transpose, then it is said to be *symmetric*. Such symmetric matrices have

$$\mathbf{S}^T = \mathbf{S} \quad \text{and hence} \quad S_{ij}^T = S_{ji} = S_{ij} \quad (\text{B.9})$$

for all ij values. For example,

$$\mathbf{S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad (\text{B.10})$$

is a real, symmetric matrix.

If the square matrix \mathbf{C} is equal to the negative of its transpose, then it is called *anti-symmetric* or *skew-symmetric*,

$$\mathbf{C} = -\mathbf{C}^T \quad \text{and hence} \quad C_{ij} = -C_{ji} \quad (\text{B.11})$$

The diagonal elements of anti-symmetric matrices are zeroes. For example,

$$\mathbf{C} = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -6 \\ -3 & 6 & 0 \end{pmatrix} \quad (\text{B.12})$$

is a real, anti-symmetric matrix with $N = 3$.

The complex conjugate of a complex matrix \mathbf{H} is the matrix all of whose matrix elements are complex conjugated. Thus \mathbf{H}^* has matrix elements H_{ij}^* where $*$ denotes complex conjugation.

The *Hermitian conjugate* of a complex matrix \mathbf{H} is denoted \mathbf{H}^\dagger and is defined by

$$\mathbf{H}^\dagger = (\mathbf{H}^T)^* = (\mathbf{H}^*)^T \quad \text{and hence} \quad H_{ij}^\dagger = H_{ji}^* \quad (\text{B.13})$$

where H_{ij}^\dagger denotes the ij th matrix element of the matrix \mathbf{H}^\dagger .

For complex matrices, the concept of symmetric matrices is generalized to that of Hermitian matrices. A square matrix is Hermitian if it is equal to its Hermitian conjugate,

$$\mathbf{H}^\dagger = \mathbf{H} \quad \text{and hence} \quad H_{ij}^\dagger = H_{ji}^* = H_{ij} \quad (\text{B.14})$$

for all ij values. A matrix that is both Hermitian and real is therefore symmetric, since complex conjugation has no effect on real numbers.

Anti-Hermitian matrices can also be defined, by analogy to the anti-symmetric matrices above. A matrix \mathbf{G} is anti-Hermitian if

$$\mathbf{G}^\dagger = -\mathbf{G} \quad (\text{B.15})$$

B.5 Algebra of Matrices: Addition

Two matrices are equal if and only if they have the same number of rows and columns and all of their matrix elements are equal. Thus if \mathbf{A} and \mathbf{B} both are $M \times N$ matrices then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad A_{ij} = B_{ij} \quad (\text{B.16})$$

for all possible values $i = 1, \dots, M$ and $j = 1, \dots, N$.

A matrix may be pre- or post-multiplied by a number α . If the matrix \mathbf{A} has matrix elements A_{ij} then the matrix $\alpha \mathbf{A}$ has matrix elements αA_{ij} . Note that *each* matrix element is multiplied by α . Thus, using the matrix of eqn (B.1) as an example,

$$\mathbf{A}\alpha = \alpha\mathbf{A} = \begin{pmatrix} \alpha A_{11} & \alpha A_{12} & \alpha A_{13} & \alpha A_{14} \\ \alpha A_{21} & \alpha A_{22} & \alpha A_{23} & \alpha A_{24} \\ \alpha A_{31} & \alpha A_{32} & \alpha A_{33} & \alpha A_{34} \end{pmatrix} \quad (\text{B.17})$$

Matrices of the same number of rows and columns may be added. The result will be a matrix also of the same type. Each matrix element of the sum is just the sum of

the corresponding matrix elements of the addends. Thus, if \mathbf{A} and \mathbf{B} both are $M \times N$ matrices then

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{if and only if} \quad C_{ij} = A_{ij} + B_{ij} \quad (\text{B.18})$$

for all possible values $i = 1, \dots, M$ and $j = 1, \dots, N$.

For example, with $M = N = 2$,

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad (\text{B.19})$$

give

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix} \quad (\text{B.20})$$

A null matrix $\mathbf{0}$ is defined as the matrix all of whose elements are zeroes. It has the property that

$$\mathbf{0} + \mathbf{A} = \mathbf{A} = \mathbf{A} + \mathbf{0} \quad (\text{B.21})$$

for all matrices \mathbf{A} . The null matrix $\mathbf{0}$ is often denoted by just the number 0.

Matrix addition is associative,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \quad (\text{B.22})$$

B.6 Algebra of Matrices: Multiplication

Matrices may also be multiplied by each other. If \mathbf{A} is an $M \times N$ matrix and \mathbf{B} is an $N \times P$ matrix, then $\mathbf{C} = \mathbf{A} \mathbf{B}$ is an $M \times P$ matrix with matrix elements for all $i = 1, \dots, M$ and $j = 1, \dots, P$ given by

$$C_{ij} = \sum_{k=1}^N A_{ik} B_{kj} \quad (\text{B.23})$$

Notice that the second index of \mathbf{A} and the first index of \mathbf{B} are both set equal to k and then summed over $k = 1, \dots, N$. This is possible only if the number of columns in \mathbf{A} is the same as the number of rows in \mathbf{B} .

A useful graphical device is to write the two matrices down in the order in which they are to be multiplied. Then run a left-hand finger over the i^{th} row of \mathbf{A} and simultaneously run a right-hand finger over the j^{th} column of \mathbf{B} , imagining the matrix elements touched to be multiplied and added. The result will be C_{ij} .

It follows from eqns (B.2, B.23) that the transpose of a matrix product is a product of the transposes in reverse order,

$$(\mathbf{A} \mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T \quad (\text{B.24})$$

Matrices \mathbf{A} and \mathbf{B} can often be multiplied in either order, $\mathbf{A} \mathbf{B}$ or $\mathbf{B} \mathbf{A}$, but the resulting matrix products will in general be different. We say that matrix multiplication is in general not *commutative*. If two particular matrices happen to give the same

product when they are multiplied in opposite orders, then we say that they *commute*. Only square matrices with the same number of rows can commute. For such matrices, the *commutator* of the two matrices is defined as

$$[A, B]_c = A B - B A \quad (\text{B.25})$$

The two matrices commute if and only if their commutator is the null matrix. (The subscript c is to distinguish commutators from Poisson brackets.)

Matrix multiplication is associative. Thus

$$A (B C) = (A B) C \quad (\text{B.26})$$

Besides noncommutativity, another important difference between the algebra of matrices and the algebra of numbers concerns null products, sometimes called divisors of zero. For numbers, $\alpha\beta=0$ implies that either $\alpha = 0$ or $\beta = 0$, or both. However, for matrices $A B = 0$ is possible even when both A and B are non-null.

B.7 Diagonal and Unit Matrices

The matrix elements with the same row and column indices A_{ii} are called *diagonal* elements. A square matrix whose only nonzero elements are the diagonal ones is called a *diagonal matrix*. Thus

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \quad (\text{B.27})$$

is a three-rowed diagonal matrix.

A diagonal matrix whose diagonal elements are all ones is called the *unit matrix*. For example, the unit matrix with $N = 3$ is

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{B.28})$$

As can be seen from eqn (B.23) and the fact that $U_{ij} = \delta_{ij}$, the Kroeneker delta, pre- or post-multiplication of any matrix A by U leaves it unchanged,

$$U A = A = A U \quad (\text{B.29})$$

A diagonal matrix all of whose diagonal elements are equal is called a *scalar matrix*. Thus, for some number α ,

$$S = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad (\text{B.30})$$

is a scalar matrix with $N = 3$. Pre- or post-multiplication of any square matrix A by a scalar matrix of the same size has the same effect as multiplying A by the diagonal number. Thus, if the diagonal elements of scalar matrix S are all equal to the number α ,

$$S A = \alpha A = A S \quad (\text{B.31})$$

B.8 Trace of a Square Matrix

The trace of an N -rowed square matrix \mathbf{A} is denoted $\text{Tr } \mathbf{A}$ and is defined as the sum of the diagonal elements,

$$\text{Tr } \mathbf{A} = \sum_{i=1}^N A_{ii} \quad (\text{B.32})$$

The trace of a product of matrices is unchanged by a cyclic permutation of them. Thus, for example,

$$\text{Tr}(\mathbf{A} \mathbf{B} \mathbf{C}) = \text{Tr}(\mathbf{C} \mathbf{A} \mathbf{B}) = \text{Tr}(\mathbf{B} \mathbf{C} \mathbf{A}) \quad (\text{B.33})$$

B.9 Differentiation of Matrices

Suppose that each matrix element of an $M \times N$ matrix is a function of some parameter β . Thus $A_{ij} = A_{ij}(\beta)$ and $\mathbf{A} = \mathbf{A}(\beta)$. Then $d\mathbf{A}(\beta)/d\beta$ is defined as that matrix whose matrix elements are $dA_{ij}(\beta)/d\beta$. If

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \quad \text{then} \quad \frac{d\mathbf{A}}{d\beta} = \begin{pmatrix} \frac{dA_{11}}{d\beta} & \frac{dA_{12}}{d\beta} & \frac{dA_{13}}{d\beta} \\ \frac{dA_{21}}{d\beta} & \frac{dA_{22}}{d\beta} & \frac{dA_{23}}{d\beta} \end{pmatrix} \quad (\text{B.34})$$

Note that each matrix element is individually differentiated.

The product rule holds for matrix products. If

$$\mathbf{C} = \mathbf{A} \mathbf{B} \quad \text{then} \quad \frac{d\mathbf{C}}{d\beta} = \frac{d\mathbf{A}}{d\beta} \mathbf{B} + \mathbf{A} \frac{d\mathbf{B}}{d\beta} \quad (\text{B.35})$$

as can be seen by applying the ordinary rules of differentiation to eqn (B.23). Note that the order of the factors in the matrix product must be preserved as the product rule is applied.

B.10 Determinants of Square Matrices

The *determinant* of an N -rowed square matrix \mathbf{A} is a single number, calculated from the elements of the matrix and denoted $\det \mathbf{A}$ or $|\mathbf{A}|$.

The rule for calculating the determinant uses the idea of a *permutation* (k_1, k_2, \dots, k_N) of the integers $(1, 2, \dots, N)$. For example, for $N = 3$, two such permutations might be $(3, 1, 2)$ and $(1, 3, 2)$. Each such permutation is either even or odd. If an even(odd) number of exchanges is required to go from $(1, 2, \dots, N)$ to the final arrangement (k_1, k_2, \dots, k_N) , then the permutation is even(odd). In the above example, $(3, 1, 2)$ is even and $(1, 3, 2)$ is odd. The function $\varepsilon(k_1, k_2, \dots, k_N)$ is defined to be $+1$ for even permutations and -1 for odd ones.

The determinant is defined as a sum of products of elements A_{ij} . Each product in this sum has one matrix element chosen from each row, with the choices of column being given by the permutation (k_1, k_2, \dots, k_N) . The sum is over all $N!$ possible permutations,

$$|\mathbf{A}| = \sum_{(k_1, k_2, \dots, k_N)} \varepsilon(k_1, k_2, \dots, k_N) A_{1k_1} A_{2k_2} \cdots A_{Nk_N} \quad (\text{B.36})$$

Alternately, the same determinant can be written using one element from each column and permutations of the rows, as

$$|\mathbf{A}| = \sum_{(k_1, k_2, \dots, k_N)} \varepsilon(k_1, k_2, \dots, k_N) A_{k_1 1} A_{k_2 2} \cdots A_{k_N N} \quad (\text{B.37})$$

If \mathbf{A} has $N = 2$,

$$|\mathbf{A}| = A_{11}A_{22} - A_{12}A_{21} \quad (\text{B.38})$$

If $N = 3$,

$$\begin{aligned} |\mathbf{A}| = & A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ & - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} \end{aligned} \quad (\text{B.39})$$

which can be remembered by a simple mnemonic device: Starting at the upper left, diagonals down to the right from each element of the first row give the positive products. Then, starting again at the lower left, diagonals up and to the right from each element of the last row give the negative products. However, this mnemonic fails for $N = 4$ or greater, and one of the expansion theorems listed below becomes the method of choice for evaluating the determinant.

B.11 Properties of Determinants

We list here some of the important properties of the determinants of square matrices. The proofs can be found in any standard text on linear algebra.

1. If any two rows of the matrix are exchanged, or any two columns of the matrix are exchanged, the new determinant is -1 times the old one.
2. If any row(column) of the matrix is all zeroes, then the determinant is zero.
3. If any row(column) of the matrix is multiplied by any constant α and then added to any other row(column), the value of the determinant is unchanged.
4. If any two rows(columns) are identical, or can be made identical by multiplying all of the elements in one of them by the same number, then the determinant is zero.
5. Transposing a matrix does not change its determinant. For any square matrix, $|\mathbf{A}^T| = |\mathbf{A}|$.
6. The determinant of a diagonal matrix is equal to the product of its diagonal elements. That is, if $A_{ij} = \lambda_i \delta_{ij}$, then $|\mathbf{A}| = \lambda_1 \lambda_2 \cdots \lambda_N$.

7. The determinant of the unit matrix is the number one, $|\mathbf{U}| = 1$.
8. The determinant of the null matrix is the number zero, $|\mathbf{0}| = 0$.
9. If an N -rowed square matrix \mathbf{A} is multiplied by a number α , then its determinant is multiplied by α^N , that is $|\alpha \mathbf{A}| = \alpha^N |\mathbf{A}|$.
10. The determinant of a product is the product of the determinants. If $\mathbf{C} = \mathbf{A} \mathbf{B}$ then $|\mathbf{C}| = |\mathbf{A}| |\mathbf{B}|$, which can be generalized to the product of any number of square matrices.
11. A matrix whose determinant is zero is called a *singular matrix*. If $\mathbf{A} \mathbf{B} = \mathbf{0}$ then $|\mathbf{A}| |\mathbf{B}| = 0$. Hence $\mathbf{A} \mathbf{B} = \mathbf{0}$ implies that at least one of \mathbf{A} or \mathbf{B} must be singular.

B.12 Cofactors

Let \mathbf{A} be any N -rowed square matrix. Suppose that we delete all elements in its i th row and all elements in its j th column and collect the remaining matrix elements into a new matrix, preserving their relative orders. The result will be an $(N - 1)$ -rowed square matrix, which we denote by $\underline{\mathbf{A}}^{(ij)}$. The N -rowed square matrix \mathbf{a} is then defined by letting its ij th matrix element be the determinant of $\underline{\mathbf{A}}^{(ij)}$ multiplied by a factor of plus or minus one,

$$a_{ij} = (-1)^{i+j} \left| \underline{\mathbf{A}}^{(ij)} \right| \quad (\text{B.40})$$

The quantity a_{ij} is called the *cofactor* of the matrix element A_{ij} and the matrix \mathbf{a} is called the *matrix of cofactors*. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad (\text{B.41})$$

then

$$\underline{\mathbf{A}}^{(23)} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \quad \text{and} \quad a_{23} = (-1)^5 (-6) = 6 \quad (\text{B.42})$$

with the other elements of \mathbf{a} defined similarly.

The matrix of cofactors has the property that

$$\sum_{k=1}^N A_{ik} a_{jk} = \delta_{ij} |\mathbf{A}| = \sum_{k=1}^N A_{ki} a_{kj} \quad (\text{B.43})$$

where δ_{ij} is the Kroeneker delta function.

B.13 Expansion of a Determinant by Cofactors

If we set $i = j$ and use the first equality in eqn (B.43), then we obtain an expansion for $|\mathbf{A}|$ by cofactors along its i th row. Each element A_{ik} of the i th row is multiplied

by the determinant $|\underline{\mathbf{A}}^{(ik)}|$ and the factor $(-1)^{i+k}$, and then summed over k ,

$$|\mathbf{A}| = \sum_{k=1}^N A_{ik} a_{ik} = \sum_{k=1}^N (-1)^{i+k} A_{ik} |\underline{\mathbf{A}}^{(ik)}| \quad (\text{B.44})$$

Notice that the $(-1)^{i+k}$ factors can be remembered easily since they have the pattern of a checkerboard.

Similarly, the determinant can also be calculated by expansion along its i^{th} column, using the second equality in eqn (B.43),

$$|\mathbf{A}| = \sum_{k=1}^N A_{ki} a_{ki} = \sum_{k=1}^N (-1)^{i+k} A_{ki} |\underline{\mathbf{A}}^{(ki)}| \quad (\text{B.45})$$

These expansion theorems allow the determinant of any N -rowed matrix to be reduced to a sum whose terms involve only determinants of size $N-1$. This expansion becomes particularly useful when one precedes it by a judicious use of Property 3 of Section B.11 to make several of the elements along a particular row(column) become zeroes.

Modern computer algebra systems automate the numerical, and even the symbolic, evaluation of large determinants. But these expansion theorems remain important. For example, the proof of Theorem D.23.1 in Appendix D makes use of them.

B.14 Inverses of Nonsingular Matrices

The *inverse* of an N -rowed square matrix \mathbf{A} , if it exists, is denoted \mathbf{A}^{-1} and is defined by the property

$$\mathbf{A}^{-1} \mathbf{A} = \mathbf{U} = \mathbf{A} \mathbf{A}^{-1} \quad (\text{B.46})$$

where \mathbf{U} is the N -rowed unit matrix.

Theorem B.14.1: Matrix Inverses

A square matrix \mathbf{A} has an inverse if and only if $|\mathbf{A}| \neq 0$, that is if and only if \mathbf{A} is nonsingular.

Proof: First we assume that the inverse exists and prove that $|\mathbf{A}| \neq 0$. Taking the determinant of both sides of the left equality of eqn (B.46) gives

$$|\mathbf{A}^{-1}| |\mathbf{A}| = |\mathbf{U}| = 1 \quad (\text{B.47})$$

Thus both $|\mathbf{A}|$ and $|\mathbf{A}^{-1}|$ must be nonzero.

Conversely, assume that $|\mathbf{A}| \neq 0$ and define \mathbf{A}^{-1} by giving its matrix elements

$$A_{ij}^{-1} = \frac{1}{|\mathbf{A}|} a_{ji} \quad (\text{B.48})$$

where a_{ij} are the cofactors defined in Section B.12. It then follows from the second

equality in eqn (B.43) that, for all ij ,

$$\left(\mathbf{A}^{-1}\mathbf{A}\right)_{ij} = \sum_{k=1}^N A_{ik}^{-1} A_{kj} = \frac{1}{|\mathbf{A}|} \sum_{k=1}^N a_{ki} A_{kj} = \frac{1}{|\mathbf{A}|} \delta_{ji} |\mathbf{A}| = \delta_{ij} = U_{ij} \quad (\text{B.49})$$

and hence that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{U}$ as was to be proved. Similarly, the first equality in eqn (B.43) proves that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{U}$ for the product in reverse order. \square

This theorem for the existence of matrix inverses is of great importance in Lagrangian mechanics and in the general calculus of many variables.

B.15 Partitioned Matrices

A matrix may be divided into *partitions*, each of which is itself a matrix. This is best seen by an example, which can be generalized to more complicated cases. Suppose that we have a 5×5 matrix \mathbf{E} that is composed of four partitions: A 3×3 matrix \mathbf{A} , a 3×2 matrix \mathbf{B} , a 2×3 matrix \mathbf{C} , and a 2×2 matrix \mathbf{D} ,

$$\mathbf{E} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} \\ A_{21} & A_{22} & A_{23} & B_{21} & B_{22} \\ A_{31} & A_{32} & A_{33} & B_{31} & B_{32} \\ C_{11} & C_{12} & C_{13} & D_{11} & D_{12} \\ C_{21} & C_{22} & C_{23} & D_{21} & D_{22} \end{pmatrix} \quad (\text{B.50})$$

It follows from the basic definitions in Section B.10 that the determinants of partitioned matrices containing null partitions can sometimes be written in terms of the determinants of their non-null parts. For example, if the matrix \mathbf{B} is a null matrix (all zeroes) then $|\mathbf{E}| = |\mathbf{A}||\mathbf{D}|$, which does not depend on the elements C_{ij} . Also, if matrix \mathbf{C} is a null matrix (all zeroes) then $|\mathbf{E}| = |\mathbf{A}||\mathbf{D}|$, which does not depend on the elements B_{ij} .

If both \mathbf{B} and \mathbf{C} are null matrices, then the matrix is what is called a *block diagonal* matrix. For example,

$$\mathbf{F} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} \\ 0 & 0 & 0 & D_{21} & D_{22} \end{pmatrix} \quad (\text{B.51})$$

is a block diagonal matrix with two blocks of different size. Then $|\mathbf{F}| = |\mathbf{A}||\mathbf{D}|$, a result that can be generalized to block matrices with any number of diagonal blocks.

One extremely important example of the partitioning of matrices is to consider each of the N columns of an $M \times N$ matrix to be an $M \times 1$ column vector. Thus the

3×4 matrix in eqn (B.1) can be written as

$$\mathbf{A} = \left(\left(\begin{array}{c} A_{11} \\ A_{21} \\ A_{31} \end{array} \right) \left(\begin{array}{c} A_{12} \\ A_{22} \\ A_{32} \end{array} \right) \left(\begin{array}{c} A_{13} \\ A_{23} \\ A_{33} \end{array} \right) \left(\begin{array}{c} A_{14} \\ A_{24} \\ A_{34} \end{array} \right) \right) \quad (\text{B.52})$$

where we have put brackets around the columns to emphasize the partitioning.

An $M \times N$ matrix can also be partitioned in a similar way into row vectors. Each of the M rows can be considered to be a $1 \times N$ row vector.

The multiplication of two matrices, each of which is partitioned into square blocks of the same dimension, can be done by the standard rule eqn (B.23) with numbers replaced by blocks. For example, if the blocks \mathbf{A}_k , \mathbf{B}_k , \mathbf{C}_k , and \mathbf{D}_k are all $N \times N$ square matrices, then

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{pmatrix} \begin{pmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 \mathbf{A}_2 + \mathbf{B}_1 \mathbf{C}_2 & \mathbf{A}_1 \mathbf{B}_2 + \mathbf{B}_1 \mathbf{D}_2 \\ \mathbf{C}_1 \mathbf{A}_2 + \mathbf{D}_1 \mathbf{C}_2 & \mathbf{C}_1 \mathbf{B}_2 + \mathbf{D}_1 \mathbf{D}_2 \end{pmatrix} \quad (\text{B.53})$$

B.16 Cramer's Rule

Suppose that we wish to solve a system of N linear equations in N unknowns, of the form

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N &= b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N &= b_2 \\ &\vdots \\ &\vdots \\ A_{N1}x_1 + A_{N2}x_2 + \cdots + A_{NN}x_N &= b_N \end{aligned} \quad (\text{B.54})$$

where A_{ij} and b_i are given numbers, and the x_j are the unknowns to be solved for. Defining an $N \times N$ square matrix by

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \quad (\text{B.55})$$

and column vectors by

$$[x] = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad \text{and} \quad [b] = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \quad (\text{B.56})$$

eqn (B.54) can be written in matrix form as $\mathbf{A}[x] = [b]$. If \mathbf{A} is nonsingular, the solution is obtained by applying \mathbf{A}^{-1} to both sides of this equation to get

$$[x] = \mathbf{A}^{-1}[b] \quad (\text{B.57})$$

Cramer's rule is a method of getting that same solution without the labor of calculating the inverse of \mathbf{A} directly. To apply it, define matrix $\mathbf{A}^{(j)}$ to be matrix \mathbf{A}

partitioned into its columns in the style of eqn (B.52), but with the j^{th} column replaced by the column vector $[b]$. Then, for all $j = 1, \dots, N$, the matrix elements of $[x]$, in other words the solution, are given by

$$x_j = \frac{|\mathbf{A}^{(j)}|}{|\mathbf{A}|} \quad (\text{B.58})$$

Thus, for example,

$$x_1 = \frac{\begin{vmatrix} b_1 & A_{12} & \cdots & A_{1N} \\ b_2 & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ b_N & A_{N2} & \cdots & A_{NN} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{vmatrix}} \quad (\text{B.59})$$

B.17 Minors and Rank

Starting from an $M \times N$ matrix \mathbf{A} , suppose that we delete any $M - R$ rows (which need not be contiguous). We then delete any $N - R$ columns (which also need not be contiguous). Collecting all elements that remain, and re-assembling them without changing their relative orders, gives an $R \times R$ square matrix \mathbf{m} , where of course $R \leq \min\{M, N\}$. The determinant of this matrix $|\mathbf{m}|$ is called an R -rowed *minor* of \mathbf{A} .

The *rank* of matrix \mathbf{A} is denoted $R(\mathbf{A})$. It is the largest value of R for which there exists a *nonzero* R -rowed minor of \mathbf{A} . A nonzero minor with $R = R(\mathbf{A})$ is called a *critical minor* of \mathbf{A} . By the definition of $R(\mathbf{A})$, all minors with $R > R(\mathbf{A})$ will be zero.

As an example, starting from the 3×4 matrix in eqn (B.1), we may delete rows 3 and 4, and columns 1 and 3. The corresponding two rowed minor of \mathbf{A} is then

$$|\mathbf{m}| = \begin{vmatrix} A_{12} & A_{14} \\ A_{22} & A_{24} \end{vmatrix} \quad (\text{B.60})$$

If it happens that $(A_{12}A_{24} - A_{14}A_{22}) \neq 0$, then $|\mathbf{m}|$ is a two rowed nonzero minor of \mathbf{A} , and that matrix must have rank $R(\mathbf{A})$ of at least two. To see if its rank is possibly more than two, we would have to investigate its three rowed minors to see if at least one of them is nonzero. If so, the rank of the matrix would be three, the maximum possible value for a 3×4 matrix.

If an N -rowed square matrix is nonsingular, then its determinant itself constitutes a critical minor with $R = N$. Thus it has rank $R(\mathbf{A}) = N$. On the other hand, if a square non-null matrix \mathbf{A} is singular, it must have $0 < R(\mathbf{A}) < N$.

B.18 Linear Independence

Consider the set of $M \times 1$ column vectors $[V^{(1)}], [V^{(2)}], \dots, [V^{(N)}]$. If there exist N numbers $\alpha_1, \alpha_2, \dots, \alpha_N$, not all zero, such that

$$\alpha_1[V^{(1)}] + \alpha_2[V^{(2)}] + \dots + \alpha_N[V^{(N)}] = [0] \quad (\text{B.61})$$

where $[0]$ is the null column vector containing M zeroes, then this set of column vectors is *linearly dependent (LD)*. Otherwise, we say that the set of column vectors is *linearly independent (LI)*.

The following properties relate to linear independence and the rank of matrices.

1. If eqn (B.61) implies that $\alpha_1 = \alpha_2 = \dots = \alpha_N = 0$, then the set of vectors is LI. Otherwise the set is LD.
2. The maximum number of $M \times 1$ column vectors that can appear in any LI set is M . We say that general $M \times 1$ column vectors form a *vector space of dimension M* . Any particular set of M LI column vectors is said to *span* this space and form a *basis* for it. Any $M \times 1$ column vector $[V]$ can be expanded in this basis as $[V] = \alpha_1[V^{(1)}] + \dots + \alpha_M[V^{(M)}]$.
3. A test for the linear independence of the column vectors in eqn (B.61) is to assemble them into an $M \times N$ matrix, like, for example, eqn (B.52). Call this matrix \mathbf{A} . Then the column vectors are linearly independent if and only if $R(\mathbf{A}) = N$.
4. Conversely, suppose we have any $M \times N$ matrix \mathbf{A} of rank $R(\mathbf{A})$. If we partition it into N column vectors, we can always find $R(\mathbf{A})$ of them, and not more than $R(\mathbf{A})$ of them, that are linearly independent.
5. A square matrix is nonsingular if and only if all of its columns(rows) are linearly independent.
6. Statements similar to the above can also be made for sets of $1 \times N$ row vectors $(V^{(1)}), (V^{(2)}), \dots, (V^{(M)})$.

B.19 Homogeneous Linear Equations

Equation (B.54) with $b_1 = b_2 = \dots = b_N = 0$ is called a set of N *homogeneous* equations for the N unknowns x_1, x_2, \dots, x_N . These homogeneous equations may be written in matrix form as

$$\mathbf{A}[x] = [0] \quad (\text{B.62})$$

It follows at once from eqn (B.57) that eqn (B.62) and $|\mathbf{A}| \neq 0$ imply $[x] = \mathbf{A}^{-1}[0] = [0]$. If the square matrix \mathbf{A} is nonsingular, then the only solution to eqn (B.62) is the trivial one $[x] = [0]$ in which $x_i = 0$ for $i = 1, \dots, N$. The following theorem generalizes eqn (B.62) to the case of M equations in N unknowns.

Theorem B.19.1: Homogeneous Linear Equations

If \mathbf{A} is an $M \times N$ matrix, then the homogeneous linear equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N &= 0 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N &= 0 \\ &\vdots \\ A_{M1}x_1 + A_{M2}x_2 + \cdots + A_{MN}x_N &= 0 \end{aligned} \tag{B.63}$$

will have k linearly independent solutions, $[x] = [c^{(i)}]$ for $i = 1, \dots, k$, if and only if matrix \mathbf{A} has rank $R = N - k$. The general solution, when it exists, can then be written in the form

$$[x] = \lambda_1[c^{(1)}] + \cdots + \lambda_k[c^{(k)}] \tag{B.64}$$

where the λ_i may take any values.

The proof of this theorem can be found in Chapter 5 of Mirsky (1961). The case in which $R = N - 1$, so that $k = 1$, is noteworthy because its general solution $[x] = \lambda_1[c^{(1)}]$ has the property that the ratios $x_j/x_l = c_j^{(1)}/c_l^{(1)}$ of the unknowns to some nonzero x_l are uniquely determined functions of the matrix elements A_{ij} .

The following corollary applies to square matrices.

Corollary B.19.2: Homogeneous Equations with Square Matrices

For square matrices with $M = N$ eqn (B.63) has a nontrivial solution with $[x] \neq [0]$ if and only if \mathbf{A} is singular with $|\mathbf{A}| = 0$. This result can also be stated: A matrix \mathbf{A} is nonsingular with $|\mathbf{A}| \neq 0$ if and only if $\mathbf{A}[x] = 0$ implies the trivial solution $[x] = 0$.

B.20 Inner Products of Column Vectors

An inner product of two real, M -rowed column vectors may be defined by analogy with the dot product of vectors in Cartesian three space. If $[x] = (x_1, \dots, x_M)^T$ and $[y] = (y_1, \dots, y_M)^T$ are two such column vectors, then we define

$$[x] \cdot [y] = [x]^T[y] = \sum_{i=1}^M x_i y_i \tag{B.65}$$

It follows that

$$[x] \cdot [y] = [y] \cdot [x] \quad \text{and that} \quad [x] \neq [0] \quad \text{implies} \quad [x] \cdot [x] > 0 \tag{B.66}$$

If $[x] \cdot [y] = 0$, then we say that $[x]$ and $[y]$ are *orthogonal*.

If each member of a set of vectors $[V^{(1)}], [V^{(2)}], \dots, [V^{(N)}]$ is orthogonal to all the others, then this set is LI. If there are M mutually orthogonal vectors in the set, then it spans the space of $M \times 1$ column vectors and any column vector $[V]$ can be expanded

as

$$[V] = \alpha_1[V^{(1)}] + \alpha_2[V^{(2)}] + \dots + \alpha_M[V^{(M)}] \quad (\text{B.67})$$

where, for $i = 1, \dots, M$,

$$\alpha_i = \frac{[V^{(i)}] \cdot [V]}{\sqrt{[V^{(i)}] \cdot [V^{(i)}]}} \quad (\text{B.68})$$

Conversely, if we are given any LI set of column vectors $[V^{(1)}], [V^{(2)}], \dots, [V^{(N)}]$, we can always construct a mutually orthogonal set by the *Schmidt orthogonalization procedure*. Starting from an arbitrary member of the set, which for simplicity we may take to be the first one $[V^{(1)}]$, define

$$\begin{aligned} [W^{(1)}] &= [V^{(1)}] \\ [W^{(2)}] &= [V^{(2)}] - \frac{[V^{(2)}] \cdot [W^{(1)}]}{[W^{(1)}] \cdot [W^{(1)}]} [W^{(1)}] \\ [W^{(3)}] &= [V^{(3)}] - \frac{[V^{(3)}] \cdot [W^{(1)}]}{[W^{(1)}] \cdot [W^{(1)}]} [W^{(1)}] - \frac{[V^{(3)}] \cdot [W^{(2)}]}{[W^{(2)}] \cdot [W^{(2)}]} [W^{(2)}] \end{aligned} \quad (\text{B.69})$$

and so on, following the same pattern until $[W^{(N)}]$ is reached. Then the set $[W^{(1)}], [W^{(2)}], \dots, [W^{(N)}]$ will be mutually orthogonal by construction.

Any non-null column vector can be *normalized*. If $[y]$ is the original vector, the corresponding normalized vector is defined by

$$[x] = \frac{1}{\sqrt{[y] \cdot [y]}} [y] \quad (\text{B.70})$$

which has the property $[x] \cdot [x] = 1$. If the vectors of a set $[V^{(1)}], [V^{(2)}], \dots, [V^{(N)}]$ of mutually orthogonal vectors are all normalized, it is called an *orthonormal set* and obeys the orthonormality condition

$$[V^{(k)}] \cdot [V^{(l)}] = \delta_{kl} \quad (\text{B.71})$$

for all $k, l = 1, \dots, N$.

If $[V^{(1)}], [V^{(2)}], \dots, [V^{(M)}]$ are an orthonormal set, then $[V^{(k)}] \cdot [V^{(k)}] = 1$ and so eqn (B.67) can be written as

$$[V] = \sum_{k=1}^M [V^{(k)}] \left\{ [V^{(k)}] \cdot [V] \right\} \quad (\text{B.72})$$

where $[V^{(k)}]$ is post-multiplied by the scalar $\alpha_k = [V^{(k)}] \cdot [V]$ to enhance the clarity of the expression. The α_k is called the *component* of $[V]$ in the *orthonormal basis* $[V^{(1)}], [V^{(2)}], \dots, [V^{(M)}]$. Such an orthonormal basis is often referred to as a *complete orthonormal set*.

B.21 Complex Inner Products

For complex column vectors, the inner product must be generalized. It becomes

$$[x]^* \cdot [y] = [x]^\dagger [y] = \sum_{i=1}^M x_i^* y_i \tag{B.73}$$

Then $[y]^* \cdot [x] = \{[x]^* \cdot [y]\}^*$ and, as for real vectors, $[x]^* \cdot [x] > 0$ for any non-null column vector $[x]$.

All results of the previous Section B.20 apply also to complex column vectors when all inner products there are replaced by the generalized ones, adding an $*$ to the left-hand vectors of the products. Thus, for example, in an orthonormal basis obeying

$$[V^{(k)}]^* \cdot [V^{(l)}] = \delta_{kl} \tag{B.74}$$

for all $k, l = 1, \dots, N$, eqn (B.72) becomes

$$[V] = \sum_{k=1}^M [V^{(k)}] \{ [V^{(k)}]^* \cdot [V] \} \tag{B.75}$$

and the component becomes $\alpha_k = [V^{(k)}]^* \cdot [V]$.

B.22 Orthogonal and Unitary Matrices

A real, square matrix M is *orthogonal* if it is nonsingular and its transpose is its inverse. Then

$$M^{-1} = M^T \quad \text{and so} \quad M^T M = U = M M^T \tag{B.76}$$

As may be seen by writing out the second of eqn (B.76) as a sum, if an N -rowed real, square matrix is partitioned into its N columns(rows), the resulting column(row) vectors will be an orthonormal set obeying eqn (B.71) if and only if the matrix is orthogonal.

Theorem B.22.1: Proof of Orthogonality

To prove a matrix orthogonal, it is sufficient to show either that $M^T M = U$ or that $M M^T = U$.

Proof: It follows from the first stated condition that

$$1 = |U| = |M^T M| = |M^T| |M| = |M|^2 \tag{B.77}$$

and hence that $|M| = \pm 1$. Thus M is nonsingular and has an inverse. Applying its inverse to $M^T M = U$ then proves that

$$M^T = M^T M M^{-1} = U M^{-1} = M^{-1} \tag{B.78}$$

which is the condition for M to be orthogonal. A similar argument demonstrates the sufficiency of the second stated condition. □

The generalization to complex matrices involves the Hermitian conjugate in place of the transpose. A matrix D is *unitary* if it is nonsingular and its Hermitian conjugate is its inverse. Then

$$D^{-1} = D^\dagger \quad \text{and so} \quad D^\dagger D = U = D D^\dagger \quad (\text{B.79})$$

As may be seen by writing out the second of eqn (B.79) as a sum, the columns (rows) of a square, complex matrix are an orthonormal set and obey eqn (B.74) if and only if the matrix is unitary. Notice that the generalized definition of inner product must be used for the complex vectors coming from complex, unitary matrices.

The above theorem applies also to unitary matrices, with a similar proof.

Theorem B.22.2: Proof of Unitarity

To prove a matrix unitary, it is sufficient to show either that $D^\dagger D = U$ or that $D D^\dagger = U$.

Proof: For unitary matrices, eqn (B.77) becomes

$$1 = |U| = |D^\dagger D| = |D^\dagger| |D| = |D|^* |D| \quad (\text{B.80})$$

which shows that, for some real number θ , $|D| = \exp(i\theta) \neq 0$. The rest of the proof is the same as for orthogonal matrices. \square

B.23 Eigenvalues and Eigenvectors of Matrices

An N -rowed square matrix A , is said to have an *eigenvalue* λ_k and corresponding *eigenvector* $[x^{(k)}]$ if

$$A[x^{(k)}] = \lambda_k[x^{(k)}] \quad (\text{B.81})$$

where the λ_k are simply numbers multiplying the vectors on the right side. Equation (B.81) may be rewritten as

$$(A - \lambda_k U)[x^{(k)}] = [0] \quad (\text{B.82})$$

where U is the unit matrix, and $[0]$ is the null column vector (all zeroes). As noted in Section B.19, this equation will have a nontrivial solution $[x^{(k)}] \neq [0]$ if and only if the matrix multiplying $[x^{(k)}]$ is singular. Thus the condition for this singularity,

$$|A - \lambda U| = 0 \quad (\text{B.83})$$

is an N th order polynomial equation in λ . It is called the *characteristic equation*. Its N roots,

$$\lambda_1, \lambda_2, \dots, \lambda_N \quad (\text{B.84})$$

are the eigenvalues, the values of λ that will give nontrivial eigenvector solutions.

Equation (B.83) in expanded form is

$$\begin{vmatrix} (A_{11} - \lambda) & A_{12} & \cdots & A_{1N} \\ A_{21} & (A_{22} - \lambda) & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & (A_{NN} - \lambda) \end{vmatrix} = 0 \tag{B.85}$$

Each of its roots λ_k is substituted into eqn (B.82) to obtain the homogeneous equation for the components $x_i^{(k)}$ of the k^{th} eigenvector. In expanded form, this equation is

$$\begin{aligned} (A_{11} - \lambda_k)x_1^{(k)} + A_{12}x_2^{(k)} + \cdots + A_{1N}x_N^{(k)} &= 0 \\ A_{21}x_1^{(k)} + (A_{22} - \lambda_k)x_2^{(k)} + \cdots + A_{2N}x_N^{(k)} &= 0 \\ &\vdots \\ A_{N1}x_1^{(k)} + A_{N2}x_2^{(k)} + \cdots + (A_{NN} - \lambda_k)x_N^{(k)} &= 0 \end{aligned} \tag{B.86}$$

If the eigenvalue λ_k is unique, then eqn (B.86) can be solved for a unique set of ratios $x_i^{(k)}/x_1^{(k)}$ of the i th component to some nonzero one, here taken to be the first one although in practice some other one may be used. The component $x_1^{(k)}$ can then be determined from the normalization condition $[x^{(k)}]^* \cdot [x^{(k)}] = 1$. But, even after normalization, the eigenvectors are not completely determined. Equation (B.81) is linear in the eigenvectors; if $[x^{(k)}]$ is a normalized eigenvector corresponding to λ_k , so is $e^{i\theta}[x^{(k)}]$ where θ is any real number.

If some eigenvalue λ_k is not unique, that is if that root of the characteristic equation has multiplicity $\kappa > 1$, then that eigenvalue is said to be *degenerate*. As will be shown below, there is a large class of matrices called normal matrices for which eqn (B.86) for a κ -fold root will have κ linearly independent solutions, each one a set of ratios of the sort just described. Either by a lucky guess in the original determination, or by the use of the Schmidt orthogonalization procedure of Section B.20, these κ linearly independent solutions can be made to produce κ mutually orthogonal ones (of course, using the complex inner product of Section B.21 when the eigenvectors are complex).

B.24 Eigenvectors of Real Symmetric Matrix

The eigenvalue problem for real, symmetric matrices is a special case of great importance in mechanics. For example, it is used to find principal axes of rigid bodies. In the present section, we will assume that the matrix \mathbf{A} is an N -rowed, real, symmetric matrix, and that the techniques of Section B.23 are being used to find its eigenvalues and eigenvectors. We begin with two lemmas, and then state the main theorem.

Lemma B.24.1: Real Eigenvalues

All eigenvalues of a real, symmetric matrix are real numbers.

Proof: Multiply eqn (B.81) from the left by $[x^{(k)}]^\dagger$ to obtain

$$[x^{(k)}]^\dagger \mathbf{A} [x^{(k)}] = \lambda_k [x^{(k)}]^\dagger [x^{(k)}] \quad (\text{B.87})$$

Now take the Hermitian conjugate of eqn (B.81) and multiply it from the right by $[x^{(k)}]$ which gives

$$[x^{(k)}]^\dagger \mathbf{A}^\dagger [x^{(k)}] = \lambda_k^* [x^{(k)}]^\dagger [x^{(k)}] \quad (\text{B.88})$$

But \mathbf{A} is both real and symmetric, and hence $\mathbf{A}^\dagger = \mathbf{A}$. Then, subtracting eqn (B.88) from eqn (B.87) gives

$$0 = (\lambda_k - \lambda_k^*) [x^{(k)}]^\dagger [x^{(k)}] \quad (\text{B.89})$$

Since, for a nontrivial solution,

$$[x^{(k)}]^\dagger [x^{(k)}] = \sum_{i=1}^N x_i^{(k)*} x_i^{(k)} > 0 \quad (\text{B.90})$$

it follows that $(\lambda_k - \lambda_k^*) = 0$ and so λ_k is real. \square

Since the eigenvalues are all real, and since the matrix elements of \mathbf{A} are real by assumption, we assume from now on that the eigenvectors have been chosen to be real column vectors. Also, we will assume that the eigenvectors have been normalized so that each of them obeys $[x^{(k)}] \cdot [x^{(k)}] = 1$.

Lemma B.24.2: Orthogonal Eigenvectors

For a real, symmetric matrix, the eigenvectors corresponding to two distinct eigenvalues $\lambda_k \neq \lambda_l$ will be orthogonal.

Proof: Multiply eqn (B.81) from the left by $[x^{(l)}]^\text{T}$ to obtain

$$[x^{(l)}]^\text{T} \mathbf{A} [x^{(k)}] = \lambda_k [x^{(l)}]^\text{T} [x^{(k)}] \quad (\text{B.91})$$

Now take the transpose of eqn (B.81), but with k replaced by l . Multiply this from the right by $[x^{(k)}]$ to obtain

$$[x^{(l)}]^\text{T} \mathbf{A}^\text{T} [x^{(k)}] = \lambda_l [x^{(l)}]^\text{T} [x^{(k)}] \quad (\text{B.92})$$

Subtract eqn (B.92) from eqn (B.91), and use $\mathbf{A}^\text{T} = \mathbf{A}$ to obtain

$$0 = (\lambda_k - \lambda_l) [x^{(l)}]^\text{T} [x^{(k)}] = (\lambda_k - \lambda_l) [x^{(l)}] \cdot [x^{(k)}] \quad (\text{B.93})$$

where the definition of inner product from Section B.20 has been used. Since, by assumption, $(\lambda_k - \lambda_l) \neq 0$ it follows that $[x^{(l)}] \cdot [x^{(k)}] = 0$ and so the two eigenvectors are orthogonal. \square

The main theorem may now be stated.

Theorem B.24.3: Complete Orthonormal Set of Eigenvectors

A real, N -rowed, symmetric matrix \mathbf{A} has real eigenvalues λ_k and N real, normalized, mutually orthogonal eigenvectors $[x^{(k)}]$,

$$\mathbf{A}[x^{(k)}] = \lambda_k[x^{(k)}] \quad \text{where} \quad [x^{(k)}] \cdot [x^{(l)}] = \delta_{kl} \tag{B.94}$$

for all $k, l = 1, \dots, N$.

Proof: The reality of the eigenvalues and eigenvectors has already been proved in B.24.1. If all N of the eigenvalues are distinct, then the existence of N mutually orthogonal eigenvectors has also been established, by B.24.2. But some eigenvalue roots of eqn (B.83) may be degenerate, so a more general proof is required.

The proof is by induction, and exploits the symmetry of \mathbf{A} . The theorem is trivially true for matrices with $N = 1$. We assume the theorem true for $N - 1$ and prove it true for N . Thus it will be proved true for any N .

Let $[x^{(1)}]$ be some real, normalized eigenvector of \mathbf{A} with eigenvalue λ_1 so that

$$\mathbf{A}[x^{(1)}] = \lambda_1[x^{(1)}] \tag{B.95}$$

Let $[y^{(2)}], \dots, [y^{(N)}]$ be $N - 1$ normalized and mutually orthogonal vectors, all of which are also orthogonal to $[x^{(1)}]$. Then $[x^{(1)}]$ is also orthogonal to each of the vectors $\mathbf{A}[y^{(2)}], \dots, \mathbf{A}[y^{(N)}]$. To see this, take the transpose of eqn (B.95) and multiply it from the right by $[y^{(l)}]$ to obtain

$$[x^{(1)}]^T \mathbf{A}^T [y^{(l)}] = \lambda_1 [x^{(1)}]^T [y^{(l)}] = \lambda_1 [x^{(1)}] \cdot [y^{(l)}] = 0 \tag{B.96}$$

where $l = 2, \dots, N$. But by assumption $\mathbf{A}^T = \mathbf{A}$ and hence eqn (B.96) may be written

$$[x^{(1)}] \cdot (\mathbf{A}[y^{(l)}]) = 0 \tag{B.97}$$

Since both $[y^{(l)}]$ and $\mathbf{A}[y^{(l)}]$ are orthogonal to $[x^{(1)}]$ we can consider the eigenvalue problem separately in the $N - 1$ dimensional space spanned by $[y^{(2)}], \dots, [y^{(N)}]$. Define the $(N - 1)$ -rowed, square, symmetric matrix \mathbf{B} by

$$B_{lm} = [y^{(l)}]^T \mathbf{A}[y^{(m)}] \tag{B.98}$$

for $l, m = 2, \dots, N$. By the induction hypothesis, the theorem is true for $(N - 1)$ -rowed matrices. So $N - 1$ normalized and mutually orthogonal $(N - 1) \times 1$ eigenvectors $[z^{(2)}], [z^{(3)}], \dots, [z^{(N)}]$ can be found that obey

$$\mathbf{B}[z^{(k)}] = \lambda_k [z^{(k)}] \quad \text{and} \quad [z^{(k)}] \cdot [z^{(k')}] = \delta_{kk'} \tag{B.99}$$

for $k, k' = 2, \dots, N$.

Now the original eigenvector $[x^{(1)}]$ and the vectors

$$[x^{(k)}] = \sum_{l=2}^N z_l^{(k)} [y^{(l)}] \tag{B.100}$$

for $k = 2, \dots, N$, are N normalized and mutually orthogonal eigenvectors of \mathbf{A} , as was to be proved. □

B.25 Eigenvectors of Complex Hermitian Matrix

Proofs very similar to those in Section B.24 show that any N -rowed, square, complex, Hermitian matrix H has real eigenvalues and N mutually orthogonal eigenvectors. The difference is that the eigenvectors are now complex, so the complex inner product of eqn (B.73) must be used. In the present section, we assume that the eigenvalues and eigenvectors of an N -rowed, complex, Hermitian matrix H are being found by the techniques of Section B.23.

We state, without further proof, some results for Hermitian matrices corresponding to those of Section B.24 for real, symmetric ones:

1. A Hermitian matrix has real eigenvalues.
2. If two eigenvalues are unequal $\lambda_k \neq \lambda_l$, then the corresponding eigenvector solutions are orthogonal, $[x^{(k)}]^* \cdot [x^{(l)}] = 0$.
3. A complex, N -rowed, Hermitian matrix H has real eigenvalues λ_k and N normalized, mutually orthogonal eigenvectors $[x^{(k)}]$ such that, for all $k, l = 1, \dots, N$,

$$H[x^{(k)}] = \lambda_k[x^{(k)}] \quad \text{where} \quad [x^{(k)}]^* \cdot [x^{(l)}] = \delta_{kl} \quad (\text{B.101})$$

Another theorem of importance, stated without proof, is:

Theorem B.25.1: Common Eigenvectors

Two complex (or real) N -rowed Hermitian matrices H_1 and H_2 have a common set of N mutually orthogonal eigenvectors $[x^{(k)}]$, such that

$$H_1[x^{(k)}] = \lambda_k[x^{(k)}] \quad \text{and} \quad H_2[x^{(k)}] = \gamma_k[x^{(k)}] \quad (\text{B.102})$$

if and only if they commute,

$$[H_1, H_2]_c = 0 \quad (\text{B.103})$$

Note that the real eigenvalues λ_k, γ_k are in general different, even though the eigenvectors are the same.

B.26 Normal Matrices

Since N -rowed real symmetric and complex Hermitian matrices both have N mutually orthogonal eigenvectors, one might wonder if other species such as orthogonal or unitary matrices do also. The answer is yes. The general rule is that an N -rowed matrix has N mutually orthogonal eigenvectors if and only if it commutes¹²⁷ with its Hermitian conjugate. Such matrices are called *normal* matrices. By definition, a matrix is normal if and only if the matrix and its Hermitian conjugate commute,

$$[A, A^\dagger]_c = A A^\dagger - A^\dagger A = 0 \quad (\text{B.104})$$

¹²⁷The commutator of two matrices is defined as $[A, B]_c = AB - BA$. See the analogous definition for operators in Section 7.1. If the commutator vanishes, the two matrices are said to commute.

Theorem B.26.1: Eigenvectors of Normal Matrices

The N -rowed, square matrix \mathbf{A} has N mutually orthogonal eigenvectors $[x^{(k)}]$ with

$$\mathbf{A}[x^{(k)}] = \lambda_k[x^{(k)}] \quad \text{where} \quad [x^{(k)}]^* \cdot [x^{(l)}] = \delta_{kl} \quad (\text{B.105})$$

for all $k, l = 1, \dots, N$, if and only if it is normal.

Proof: To prove that eqn (B.104) implies the existence of N mutually orthogonal eigenvectors, note that any square matrix \mathbf{A} can be written as a sum of two Hermitian matrices as

$$\mathbf{A} = \mathbf{A}_R + i \mathbf{A}_I \quad \text{where} \quad \mathbf{A}_R = \frac{1}{2} (\mathbf{A} + \mathbf{A}^\dagger) \quad \text{and} \quad \mathbf{A}_I = \frac{-i}{2} (\mathbf{A} - \mathbf{A}^\dagger) \quad (\text{B.106})$$

By Theorem B.25.1, the two Hermitian matrices will have a common set of N mutually orthogonal eigenvectors if and only if

$$0 = [\mathbf{A}_R, \mathbf{A}_I]_c = \frac{1}{2i} [\mathbf{A}, \mathbf{A}^\dagger]_c \quad (\text{B.107})$$

where the second equality follows from the definitions in eqn (B.106). A vector $[x^{(k)}]$ that is a common eigenvector of $\mathbf{A}_R, \mathbf{A}_I$ with eigenvalues λ_k, γ_k , respectively, will be an eigenvector of \mathbf{A} with eigenvalue $(\lambda_k + i\gamma_k)$, which completes the proof. The proof of the converse, that the existence of a set of N orthonormal eigenvectors satisfying eqn (B.105) implies eqn (B.104), follows immediately from eqn (B.119) of Section B.27 and the fact that any two diagonal matrices commute. \square

The eigenvalue and eigenvectors of normal matrices are found using the same techniques as described in Section B.23 and used earlier for real symmetric and complex Hermitian matrices. The eigenvalues of normal matrices are derived from the same determinant condition as in eqn (B.85). The eigenvalues of normal matrices may in general be complex rather than real, but the eigenvector solutions are still obtained from eqn (B.86). And the following lemma remains true.

Lemma B.26.2: Orthogonality of Eigenvectors of Normal Matrix

If a normal matrix \mathbf{A} has two unequal eigenvalues $\lambda_k \neq \lambda_l$, then the corresponding eigenvectors will be orthogonal, $[x^{(k)}]^* \cdot [x^{(l)}] = 0$. Two complex eigenvalues are considered unequal if either their real or their imaginary parts differ.

Since complex conjugation has no effect on real matrices, the condition for a real matrix to be normal reduces to

$$[\mathbf{A}, \mathbf{A}^T]_c = \mathbf{A} \mathbf{A}^T - \mathbf{A}^T \mathbf{A} = 0 \quad (\text{B.108})$$

Then examination of the conditions in eqns (B.104, B.108) shows that real symmetric, real anti-symmetric, real orthogonal, complex Hermitian, complex anti-Hermitian, and complex unitary matrices are all normal.

B.27 Properties of Normal Matrices

Normal matrices have a complete orthonormal set of eigenvectors. A number of important results follow from this fact. We present some of these results here, assuming that the eigenvectors and eigenvalues may be complex. But the same formulas may also be used for real, symmetric matrices with real eigenvectors and eigenvalues by ignoring the complex conjugation signs “*”, replacing all Hermitian conjugate signs “†” by transpose signs “T”, and replacing the word “unitary” by the word “orthogonal.” Thus the results of the present section apply in particular to real symmetric, real anti-symmetric, orthogonal, complex Hermitian, complex skew-Hermitian, and complex unitary matrices.

Given a normal, N -rowed matrix \mathbf{A} and its N orthonormal eigenvectors $[x^{(k)}]$, let us define a matrix \mathbf{D} whose columns are the eigenvectors. The matrix elements of \mathbf{D} are therefore

$$D_{ik} = x_i^{(k)} \quad (\text{B.109})$$

for $i, k = 1, \dots, N$, and so

$$\mathbf{D} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(N)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(N)} \\ \vdots & \vdots & \vdots & \vdots \\ x_N^{(1)} & x_N^{(2)} & \cdots & x_N^{(N)} \end{pmatrix} \quad (\text{B.110})$$

As was discussed in Section B.22, since its columns are an orthonormal set of vectors this \mathbf{D} will be a unitary matrix. But it is useful here to prove its unitarity directly. From the orthonormality of the $[x^{(k)}]$ in eqn (B.105),

$$\begin{aligned} (\mathbf{D}^\dagger \mathbf{D})_{kl} &= \sum_{i=1}^N D_{ki}^\dagger D_{il} = \sum_{i=1}^N D_{ik}^* D_{il} \\ &= \sum_{i=1}^N x_i^{(k)*} x_i^{(l)} = [x^{(k)}]^\dagger [x^{(l)}] = [x^{(k)}]^* \cdot [x^{(l)}] = \delta_{kl} \end{aligned} \quad (\text{B.111})$$

Since the unit matrix has $U_{kl} = \delta_{kl}$, this proves that all matrix elements of $\mathbf{D}^\dagger \mathbf{D}$ are identical to those of \mathbf{U} and hence that

$$\mathbf{D}^\dagger \mathbf{D} = \mathbf{U} \quad (\text{B.112})$$

As proved in B.22.2, this is sufficient to prove \mathbf{D} unitary.

Since \mathbf{D} is unitary, it follows also that $\mathbf{U} = \mathbf{D} \mathbf{D}^\dagger$. In terms of the components, this can be written

$$U_{ij} = \delta_{ij} = (\mathbf{D} \mathbf{D}^\dagger)_{ij} = \sum_{k=1}^N x_i^{(k)} x_j^{(k)*} = \left(\sum_{k=1}^N [x^{(k)}] [x^{(k)}]^\dagger \right)_{ij} \quad (\text{B.113})$$

It follows that

$$\mathbf{U} = \sum_{k=1}^N [x^{(k)}][x^{(k)}]^\dagger \tag{B.114}$$

which expands the unit matrix in terms of eigenvectors. This expansion is called a *resolution of unity*. Applying this expression to an arbitrary vector $[V]$ gives

$$[V] = \mathbf{U}[V] = \sum_{k=1}^N [x^{(k)}][x^{(k)}]^\dagger [V] = \sum_{k=1}^N [x^{(k)}] \left([x^{(k)}]^* \cdot [V] \right) \tag{B.115}$$

which should be compared to eqn (B.75).

Now consider the matrix \mathbf{E} defined by the matrix product

$$\mathbf{E} = \mathbf{D}^\dagger \mathbf{A} \mathbf{D} \tag{B.116}$$

Expanding this product and using eqn (B.105) gives

$$\begin{aligned} E_{kl} &= \sum_{i=1}^N \sum_{j=1}^N D_{ki}^\dagger A_{ij} D_{jl} = \sum_{i=1}^N \sum_{j=1}^N x_i^{(k)*} A_{ij} x_j^{(l)} \\ &= [x^{(k)}]^\dagger \mathbf{A} [x^{(l)}] = [x^{(k)}]^* \cdot \lambda_l [x^{(l)}] = \delta_{kl} \lambda_l \end{aligned} \tag{B.117}$$

Thus \mathbf{E} is a diagonal matrix with the eigenvalues of \mathbf{A} as its diagonal elements. We say that \mathbf{D} *reduces* \mathbf{A} to the diagonal matrix \mathbf{E}

$$\mathbf{E} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} \tag{B.118}$$

Equation (B.116) can be inverted, using the unitarity of \mathbf{D} to write

$$\mathbf{A} = \mathbf{D} \mathbf{E} \mathbf{D}^\dagger \tag{B.119}$$

which can be written out as an expansion of \mathbf{A} in terms of its eigenvalues and eigenvectors

$$A_{ij} = \sum_{k=1}^N \sum_{l=1}^N D_{ik} E_{kl} D_{lj}^\dagger = \sum_{k=1}^N \sum_{l=1}^N x_i^{(k)} \delta_{kl} \lambda_l x_j^{(l)*} = \sum_{k=1}^N x_i^{(k)} \lambda_k x_j^{(k)*} \tag{B.120}$$

In matrix notation, this equation becomes

$$\mathbf{A} = \sum_{k=1}^N [x^{(k)}] \lambda_k [x^{(k)}]^\dagger \tag{B.121}$$

which is referred to as an *eigen-dyadic expansion* of matrix \mathbf{A} in terms of its eigenvectors and eigenvalues.

Equation (B.119) also allows the trace and determinant of a normal matrix to be found from its eigenvalues.

Theorem B.27.1: Trace and Determinant of Normal Matrices

If matrix \mathbf{A} is an N -rowed, normal matrix (either real or complex), with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ then its trace and determinant are

$$\text{Tr } \mathbf{A} = \lambda_1 + \lambda_2 + \dots + \lambda_N \quad \text{and} \quad |\mathbf{A}| = \lambda_1 \lambda_2 \dots \lambda_N \quad (\text{B.122})$$

Proof: From eqn (B.119), and the invariance of the trace under cyclic permutations discussed in Section B.8, it follows that

$$\text{Tr } \mathbf{A} = \text{Tr}(\mathbf{D} \mathbf{E} \mathbf{D}^\dagger) = \text{Tr}(\mathbf{E} \mathbf{D}^\dagger \mathbf{D}) = \text{Tr } \mathbf{E} = \lambda_1 + \lambda_2 + \dots + \lambda_N \quad (\text{B.123})$$

From eqn (B.119), and Properties 5 and 11 of Section B.11, it follows that

$$|\mathbf{A}| = |\mathbf{D} \mathbf{E} \mathbf{D}^\dagger| = |\mathbf{D}| |\mathbf{E}| |\mathbf{D}^\dagger| = |\mathbf{E}| = \lambda_1 \lambda_2 \dots \lambda_N \quad (\text{B.124})$$

□

Equation (B.119) also allows the characteristic equation of normal matrix \mathbf{A} to be expressed in terms of the eigenvalues corresponding to its N mutually orthogonal eigenvectors.

Theorem B.27.2: Form of Characteristic Equation

If matrix \mathbf{A} is an N -rowed, normal matrix (either real or complex), with eigenvectors $[x^{(1)}], [x^{(2)}], \dots, [x^{(N)}]$ and corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ then the characteristic equation eqn (B.83) is

$$0 = |\mathbf{A} - \lambda \mathbf{U}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_N - \lambda) \quad (\text{B.125})$$

with one factor for each eigenvector, regardless of the uniqueness or degeneracy of that eigenvalue.

Proof: Using eqn (B.119) and the unitarity of \mathbf{D} ,

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{U}| &= |\mathbf{D} \mathbf{E} \mathbf{D}^\dagger - \lambda \mathbf{D} \mathbf{U} \mathbf{D}^\dagger| = |\mathbf{D} (\mathbf{E} - \lambda \mathbf{U}) \mathbf{D}^\dagger| \\ &= |\mathbf{D}| |\mathbf{E} - \lambda \mathbf{U}| |\mathbf{D}^\dagger| = |\mathbf{E} - \lambda \mathbf{U}| \end{aligned} \quad (\text{B.126})$$

But both \mathbf{E} and \mathbf{U} are diagonal matrices, so

$$|\mathbf{E} - \lambda \mathbf{U}| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_N - \lambda) \quad (\text{B.127})$$

which establishes eqn (B.125). □

Theorem B.27.2 establishes that, for normal matrices, the multiplicity of a particular eigenvalue in the characteristic equation is equal to the number of orthogonal eigenvectors having that eigenvalue. Thus eqn (B.86) for a degenerate eigenvalue λ_k with multiplicity κ will always yield κ mutually orthogonal eigenvector solutions, as was asserted at the end of Section B.23.

B.28 Functions of Normal Matrices

Matrix functions of normal matrices may be defined by using the dyadic expansion in eqn (B.121) of the previous Section B.27.

Let \mathbf{A} be an N -rowed normal matrix with eigenvalues λ_k and mutually orthogonal eigenvectors $[x^{(k)}]$, for $k = 1, \dots, N$. Let $f = f(z)$ be a complex function of the complex variable z that is well defined when z is equal to any one of the λ_k . Then the matrix function $\mathbf{F} = f(\mathbf{A})$ may be defined as

$$\mathbf{F} = f(\mathbf{A}) = \sum_{k=1}^N [x^{(k)}] f(\lambda_k) [x^{(k)}]^\dagger \tag{B.128}$$

This definition has several useful consequences:

1. The matrix \mathbf{F} is also a normal matrix.
2. The eigenvectors $[x^{(k)}]$ of \mathbf{A} are also eigenvectors of \mathbf{F} . The corresponding eigenvalues of matrix \mathbf{F} are $f(\lambda_k)$. That is,

$$\mathbf{F} [x^{(k)}] = \gamma_k [x^{(k)}] \quad \text{where} \quad \gamma_k = f(\lambda_k) \tag{B.129}$$

3. As may be seen by repeated use of eqn (B.121) using the orthonormality of the eigenvectors, if $f(z) = z^n$ for any positive integer n , then $\mathbf{F} = \mathbf{A}^n$ where $\mathbf{A}^n = \underbrace{\mathbf{A} \mathbf{A} \cdots \mathbf{A}}_{n \text{ times}}$.

4. It follows from consequence 3 and the resolution of unity in eqn (B.114) that, when $f(z)$ has a power-series expansion $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and all λ_k lie in the circle of convergence of the power series, then the power series

$$\mathbf{F} = a_0 \mathbf{U} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \cdots \tag{B.130}$$

converges to the same \mathbf{F} as that defined in eqn (B.128).

5. A characteristic equation like eqn (B.83) is used to find the eigenvalues λ_k of any N -rowed normal matrix \mathbf{A} . As proved by the last theorem in Section B.27, this characteristic equation may be written as

$$0 = |\mathbf{A} - \lambda \mathbf{U}| = (\lambda_1 - \lambda) (\lambda_2 - \lambda) \cdots (\lambda_N - \lambda) \tag{B.131}$$

where the $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues corresponding to its N eigenvectors. As may be seen by repeated use of eqns (B.114, B.121), using the orthonormality of the eigenvectors, the normal matrix itself satisfies its own characteristic equation. With \mathbf{A} substituted for the unknown λ , and $\lambda_k \mathbf{U}$ for the numbers λ_k ,

$$0 = (\lambda_1 \mathbf{U} - \mathbf{A}) (\lambda_2 \mathbf{U} - \mathbf{A}) \cdots (\lambda_N \mathbf{U} - \mathbf{A}) \tag{B.132}$$

6. It follows from consequence 5 that the N th power \mathbf{A}^N of any N -rowed square, normal matrix \mathbf{A} may be written as a polynomial $p(\mathbf{A})$ of degree $(N - 1)$ containing only powers of \mathbf{A} less than N , so that $\mathbf{A}^N = p(\mathbf{A})$. Thus, for any N -rowed square, normal matrix, the power series eqn (B.130) can be reduced to an expression containing only powers of \mathbf{A} up to and including $\mathbf{A}^{(N-1)}$.

APPENDIX C

EIGENVALUE PROBLEM WITH GENERAL METRIC

The theory of small vibrations in Chapter 10 requires a generalization of the matrix eigenvalue methods of Appendix B. The generalized eigenvalue equation is of the form

$$\mathbf{A} [z^{(k)}] = \theta_k \mathbf{g} [z^{(k)}] \quad (\text{C.1})$$

where column vector $[z^{(k)}]$ is an eigenvector of matrix \mathbf{A} with eigenvalue θ_k . The only difference between this equation and the standard eigenvalue expression in eqn (B.81) is the presence of a positive-definite matrix \mathbf{g} on the right side of eqn (C.1). The matrix \mathbf{g} also serves as a metric, allowing a generalization of the inner products of two column vectors.

C.1 Positive-Definite Matrices

The real, symmetric $N \times N$ matrix \mathbf{g} is called a *positive definite matrix* if, for any real column vector $[x]$,

$$[x] \neq [0] \quad \text{implies that} \quad [x]^T \mathbf{g} [x] > 0 \quad (\text{C.2})$$

Lemma C.1.1: Eigenvalues of a Positive-Definite Matrix

A real symmetric matrix is positive definite if and only if all of its eigenvalues are nonzero, positive numbers. Such a matrix is nonsingular.

Proof: Suppose that the matrix \mathbf{g} is positive definite and has eigenvalues γ_k and eigenvectors $[y^{(k)}]$. If we choose the arbitrary column vector in eqn (C.2) to be the k th normalized eigenvector of \mathbf{g} , then

$$0 < [y^{(k)}]^T \mathbf{g} [y^{(k)}] = \gamma_k [y^{(k)}]^T [y^{(k)}] = \gamma_k \quad (\text{C.3})$$

which shows that γ_k cannot be zero or negative.

Conversely, assume all $\gamma_k > 0$ and let $[x]$ be an arbitrary non-null column vector. Since a real symmetric matrix \mathbf{g} is a normal matrix, it follows from eqn (B.121) and the orthonormality and hence completeness of its eigenvectors $[y^{(k)}]$ that

$$[x]^T \mathbf{g} [x] = [x]^T \left(\sum_{k=1}^N [y^{(k)}] \gamma_k [y^{(k)}]^T \right) [x] = \sum_{k=1}^N \gamma_k \left(\sum_{i=1}^N y_i^{(k)} x_i \right)^2 > 0 \quad (\text{C.4})$$

as was to be proved.

If a real symmetric matrix is positive definite, it follows from the just-proved positive definiteness of its eigenvalues and from Theorem B.27.1 that $|\mathbf{g}| = \gamma_1 \cdots \gamma_N > 0$. Therefore the matrix is nonsingular. \square

Note that the condition $|\mathbf{g}| > 0$ is a necessary but not a sufficient condition for \mathbf{g} to be positive definite. For example, a 4×4 diagonal matrix with diagonal elements $(1, -1, 1, -1)$ has a positive determinant but is not a positive definite matrix.

Since the γ_k are all positive, we can define the following real, symmetric matrices,

$$\mathbf{g} = \sum_{k=1}^N [y^{(k)}] \gamma_k [y^{(k)}]^T \tag{C.5}$$

$$\mathbf{g}^{1/2} = \sum_{k=1}^N [y^{(k)}] \sqrt{\gamma_k} [y^{(k)}]^T \tag{C.6}$$

$$\mathbf{g}^{-1/2} = \sum_{k=1}^N [y^{(k)}] \frac{1}{\sqrt{\gamma_k}} [y^{(k)}]^T \tag{C.7}$$

$$\mathbf{g}^{-1} = \sum_{k=1}^N [y^{(k)}] \frac{1}{\gamma_k} [y^{(k)}]^T \tag{C.8}$$

The first of these, eqn (C.5), is just an application of the dyadic eqn (B.121) to the matrix \mathbf{g} . The others are defined by analogy. By construction, these matrices have the following properties,

$$\mathbf{g}^{1/2} \mathbf{g}^{1/2} = \mathbf{g} \qquad \mathbf{g}^{-1/2} \mathbf{g}^{-1/2} = \mathbf{g}^{-1} \qquad \mathbf{g}^{-1/2} \mathbf{g}^{1/2} = \mathbf{U} = \mathbf{g}^{1/2} \mathbf{g}^{-1/2} \tag{C.9}$$

C.2 Generalization of the Real Inner Product

If \mathbf{g} is an M -rowed, real, symmetric, positive-definite matrix, a generalized inner product of two $M \times 1$ column vectors may be defined by

$$[x] \bullet [y] = [x]^T \mathbf{g} [y] = \sum_{i=1}^M \sum_{j=1}^M x_i g_{ij} y_j \tag{C.10}$$

This inner product has properties similar to that of the ordinary real inner product in Section B.20, that

$$[x] \bullet [y] = [y] \bullet [x] \quad \text{and that} \quad [x] \neq [0] \quad \text{implies} \quad [x] \bullet [x] > 0 \tag{C.11}$$

This generalized inner product also has other properties similar to those in Section B.20. If a set of $M \times 1$ vectors $[V^{(1)}], [V^{(2)}], \dots, [V^{(M)}]$ is orthonormal in the generalized sense,

$$[V^{(k)}] \bullet [V^{(l)}] = \delta_{kl} \tag{C.12}$$

for all $k, l = 1, \dots, M$, then that set is LI and forms a basis for the space of $M \times 1$

vectors. Any vector $[V]$ can be expanded as

$$[V] = \beta_1[V^{(1)}] + \beta_2[V^{(2)}] + \cdots + \beta_M[V^{(M)}] \quad (\text{C.13})$$

where the components are, for $k = 1, \dots, M$, given by

$$\beta_k = [V^{(k)}] \bullet [V] \quad (\text{C.14})$$

If a set of vectors $[V^{(1)}], [V^{(2)}], \dots, [V^{(N)}]$ is initially LI but not orthogonal, a mutually orthogonal set $[W^{(1)}], [W^{(2)}], \dots, [W^{(N)}]$ can be found by a generalization of the Schmidt orthogonalization procedure outlined in Section B.20,

$$\begin{aligned} [W^{(1)}] &= [V^{(1)}] \\ [W^{(2)}] &= [V^{(2)}] - \frac{[V^{(2)}] \bullet [W^{(1)}]}{[W^{(1)}] \bullet [W^{(1)}]} [W^{(1)}] \end{aligned} \quad (\text{C.15})$$

and so on, following the pattern of eqn (B.69), but with the ordinary inner product “.” replaced by the generalized one “•” throughout. The vectors can then be normalized, again using the generalized inner product, so that they become a generalized orthonormal set obeying $[W^{(i)}] \bullet [W^{(j)}] = \delta_{ij}$ for all i, j values.

C.3 The Generalized Eigenvalue Problem

In the Lagrangian theory of small vibrations, we are asked to solve a generalized eigenvalue problem, to find eigenvectors $[z^{(k)}]$ and eigenvalues θ_k that are solutions of

$$\mathbf{A}[z^{(k)}] = \theta_k \mathbf{g}[z^{(k)}] \quad (\text{C.16})$$

where \mathbf{A} is a real, symmetric matrix, and \mathbf{g} is a real, symmetric, positive-definite matrix. This equation can be rewritten as

$$(\mathbf{A} - \theta_k \mathbf{g})[z^{(k)}] = [0] \quad (\text{C.17})$$

and the eigenvalues found from

$$|\mathbf{A} - \theta \mathbf{g}| = 0 \quad (\text{C.18})$$

These equations differ from the standard eigenvalue equations in B.23 only by the replacement of the unit matrix \mathbf{U} by a positive definite matrix \mathbf{g} .

Before stating and proving the main theorem, we present a preliminary result.

Lemma C.3.1: Transformed Eigenvector Problem

Equation (C.16) is true if and only if

$$\mathbf{B}[x^{(k)}] = \theta_k [x^{(k)}] \quad (\text{C.19})$$

where

$$[x^{(k)}] = \mathbf{g}^{1/2}[z^{(k)}] \quad [z^{(k)}] = \mathbf{g}^{-1/2}[x^{(k)}] \quad \mathbf{B} = \mathbf{g}^{-1/2} \mathbf{A} \mathbf{g}^{-1/2} \quad (\text{C.20})$$

and the definitions in Section C.2 have been used for $\mathbf{g}^{1/2}$ and $\mathbf{g}^{-1/2}$.

Proof: Substituting the second of eqn (C.20) into eqn (C.19) gives

$$\mathbf{A} \mathbf{g}^{-1/2} [x^{(k)}] = \theta_k \mathbf{g} \mathbf{g}^{-1/2} [x^{(k)}] \tag{C.21}$$

Then pre-multiplying both sides by $g^{-1/2}$ and using eqn (C.9) gives eqn (C.16).

Conversely, substituting the first and last of eqn (C.20) into eqn (C.16) gives

$$\mathbf{g}^{-1/2} \mathbf{A} \mathbf{g}^{-1/2} \mathbf{g}^{1/2} [z^{(k)}] = \theta_k \mathbf{g}^{1/2} [z^{(k)}] \tag{C.22}$$

Pre-multiplying both sides by $g^{\frac{1}{2}}$ and using eqn (C.9) then gives eqn (C.19). Thus the two equations are equivalent, as was to be proved. \square

We now state the main theorem.

Theorem C.3.2: Generalized Eigenvector Theorem

If \mathbf{A} is an N -rowed, real, symmetric matrix and \mathbf{g} is a real, symmetric, positive definite matrix of the same size, then the eigenvalue equation

$$\mathbf{A} [z^{(k)}] = \theta_k \mathbf{g} [z^{(k)}] \tag{C.23}$$

has N real eigenvalues $\theta_1, \theta_2, \dots, \theta_N$, and N real eigenvectors $[z^{(1)}], [z^{(2)}], \dots, [z^{(N)}]$ that are normalized and mutually orthogonal according to the generalized inner product of Section C.2,

$$[z^{(k)}] \bullet [z^{(l)}] = \delta_{kl} \tag{C.24}$$

Proof: Since the matrix \mathbf{B} defined in the last of eqn (C.20) is real and symmetric, we know from Theorem B.24.3 that it has N real eigenvalues and N real eigenvectors $[x^{(k)}]$ that obey the ordinary definition of orthonormality $[x^{(k)}] \cdot [x^{(l)}] = \delta_{kl}$. And the above Lemma C.3.1 proves that for each of these $[x^{(k)}]$, the vector $[z^{(k)}]$ defined in the second of eqn (C.20) is a generalized eigenvector of matrix \mathbf{A} obeying eqn (C.23). Thus there are N generalized eigenvectors. It only remains to investigate their generalized orthogonality.

Substituting the first of eqn (C.20) into $[x^{(k)}] \cdot [x^{(l)}] = \delta_{kl}$ gives

$$\delta_{kl} = [x^{(k)}] \cdot [x^{(l)}] = [x^{(k)}]^T [x^{(l)}] = [z^{(k)}]^T \mathbf{g}^{1/2} \mathbf{g}^{1/2} [z^{(k)}] = [z^{(k)}]^T \mathbf{g} [z^{(k)}] = [z^{(k)}] \bullet [z^{(l)}] \tag{C.25}$$

which establishes eqn (C.24). Thus the theorem is proved. \square

C.4 Finding Eigenvectors in the Generalized Problem

We now know that a real, symmetric matrix \mathbf{A} has N generalized eigenvectors. To find them, the procedure is similar to the ordinary eigenvector solution. Written out, eqn (C.18) is

$$\begin{vmatrix} (A_{11} - \theta g_{11}) & (A_{12} - \theta g_{12}) & \cdots & (A_{1N} - \theta g_{1N}) \\ (A_{21} - \theta g_{21}) & (A_{22} - \theta g_{22}) & \cdots & (A_{2N} - \theta g_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ (A_{N1} - \theta g_{N1}) & (A_{N2} - \theta g_{N2}) & \cdots & (A_{NN} - \theta g_{NN}) \end{vmatrix} = 0 \tag{C.26}$$

which can be solved for $\theta_1, \theta_2, \dots, \theta_N$. We know from Section C.3 that these eigenvalues will all be real.

The eigenvector(s) corresponding to a particular eigenvalue are found from eqn (C.17) which may be written as

$$\begin{aligned}
 (A_{11} - \theta_k g_{11}) z_1^{(k)} + (A_{12} - \theta_k g_{12}) z_2^{(k)} + \cdots + (A_{1N} - \theta_k g_{1N}) z_N^{(k)} &= 0 \\
 (A_{21} - \theta_k g_{21}) z_1^{(k)} + (A_{22} - \theta_k g_{22}) z_2^{(k)} + \cdots + (A_{2N} - \theta_k g_{2N}) z_N^{(k)} &= 0 \\
 &\vdots \\
 (A_{N1} - \theta_k g_{N1}) z_1^{(k)} + (A_{N2} - \theta_k g_{N2}) z_2^{(k)} + \cdots + (A_{NN} - \theta_k g_{NN}) z_N^{(k)} &= 0 \quad (C.27)
 \end{aligned}$$

Just as for the ordinary eigenvector solution, if the eigenvalue is unique, then these equations can be solved for a unique set of ratios $z_i^{(k)}/z_1^{(k)}$. The value of $z_1^{(k)}$ can then be obtained from the normalization condition,

$$[z^{(k)}] \bullet [z^{(k)}] = 1 \quad (C.28)$$

If the eigenvalue is a multiple root of degeneracy κ then there will be κ LI roots of eqn (C.27). These can be made orthogonal in the generalized sense by using the generalized Schmidt orthogonalization procedure outlined in eqn (C.15). The resulting set of eigenvector solutions will then obey the orthonormality condition eqn (C.24).

C.5 Uses of the Generalized Eigenvectors

The main use of the generalized eigenvalue problem is simultaneously to reduce the matrix **A** to a diagonal matrix, and the matrix **g** to the unit matrix. Let us define a matrix **C** whose k th column is the k th eigenvector from the generalized eigenvalue problem of Section C.3,

$$C_{ik} = z_i^{(k)} \quad (C.29)$$

so that

$$\mathbf{C} = \begin{pmatrix} z_1^{(1)} & z_1^{(2)} & \cdots & z_1^{(N)} \\ z_2^{(1)} & z_2^{(2)} & \cdots & z_2^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{(1)} & z_N^{(2)} & \cdots & z_N^{(N)} \end{pmatrix} \quad (C.30)$$

Theorem C.5.1: Reduction to Diagonal Form

Let **U** be the unit matrix, and define **F** to be a diagonal matrix whose diagonal elements are the eigenvalues of the generalized eigenvalue problem of Section C.3,

$$\mathbf{F} = \begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_N \end{pmatrix} \quad (C.31)$$

With **C** the matrix defined in eqn (C.29), it follows that

$$\mathbf{C}^T \mathbf{g} \mathbf{C} = \mathbf{U} \quad \text{and} \quad \mathbf{C}^T \mathbf{A} \mathbf{C} = \mathbf{F} \quad (C.32)$$

Proof: To prove the first of eqn (C.32), use eqn (C.29) to write eqn (C.24) as

$$\delta_{kl} = [z^{(k)}] \bullet [z^{(l)}] = \sum_{i=1}^N \sum_{j=1}^N z_i^{(k)} g_{ij} z_j^{(l)} = \sum_{i=1}^N \sum_{j=1}^N C_{ik} g_{ij} C_{jl} = \left(\mathbf{C}^T \mathbf{g} \mathbf{C} \right)_{kl} \quad (\text{C.33})$$

Thus $\mathbf{C}^T \mathbf{g} \mathbf{C}$ has the same matrix elements as the unit matrix $U_{kl} = \delta_{kl}$ and so the two are equal, as was to be proved.

To prove the second of eqn (C.32), replace k by l in eqn (C.23) and then multiply both sides of it from the left by $[z^{(k)}]^T$ to obtain

$$[z^{(k)}]^T \mathbf{A} [z^{(l)}] = \theta_l [z^{(k)}]^T \mathbf{g} [z^{(l)}] = \theta_l [z^{(k)}] \bullet [z^{(l)}] = \theta_l \delta_{kl} = F_{kl} \quad (\text{C.34})$$

Thus

$$F_{kl} = \sum_{i=1}^N \sum_{j=1}^N z_i^{(k)} A_{ij} z_j^{(l)} = \sum_{i=1}^N \sum_{j=1}^N C_{ik} A_{ij} C_{jl} = \left(\mathbf{C}^T \mathbf{A} \mathbf{C} \right)_{kl} \quad (\text{C.35})$$

holds for every value of kl and so the two matrices are equal, as was to be proved. \square

APPENDIX D

THE CALCULUS OF MANY VARIABLES

We summarize here some standard results from the calculus of many variables. These theorems will be invoked frequently throughout the text. In particular, the reader should review this material before beginning the chapter on Lagrangian methods. More background on this topic can be found in many standard calculus texts; a particularly accessible source is the two volumes of Courant (1936a,b). The mathematical appendices of Desloges (1982) are also very useful.

D.1 Basic Properties of Functions

A function y of the set of variables x_1, x_2, \dots, x_N will be written in either of the two equivalent forms,

$$y = y(x_1, x_2, \dots, x_N) \quad \text{OR} \quad y = y(x) \quad (\text{D.1})$$

In the second form, the single unsubscripted variable x denotes the whole set, $x = x_1, x_2, \dots, x_N$. Note that the same letter y is used to denote both quantity and function. Whereas a calculus book might write $y = f(x)$, we write $y = y(x)$. This notation has the advantage of clarity and alphabetic economy. It will be clear from context when letters like y denote the function and when they denote the value of the function. Usually, a letter on the left of an equal sign denotes the value, the same letter before the (x) denotes the function that produces that value.

D.2 Regions of Definition of Functions

Functions are assumed to be single valued for all x_1, x_2, \dots, x_N in a region R . A simple example of such a region is an *open rectangle* R_\perp defined by $a_k < x_k < b_k$ for $k = 1, \dots, N$, where the $a_k < b_k$ are some constants.

The region R is *connected* if any two points in R can be joined by a curve lying entirely in R . The region is *simply connected* if it is connected, and if every closed curve lying entirely in R can be continuously shrunk to a point without leaving R . For example, the interior of a ball is simply connected, the interior of a doughnut is not.

Region R is called *open* if, for every point in R , there is some (possibly quite small) open rectangle lying entirely in R and containing the point. Such an open rectangle will be called an *open neighborhood* of the point x and will be denoted N_x . See Chapter I of Spivak (1965) for more definitions.

D.3 Continuity of Functions

The definition of continuity is similar to that for functions of one variable.¹²⁸ Define an arbitrary increment h_k for each variable x_k for $k = 1, \dots, N$. Then function y is continuous at point x if, for every possible choice of the h_k ,

$$\lim_{\theta \rightarrow 0} y(x_1 + \theta h_1, x_2 + \theta h_2, \dots, x_N + \theta h_N) = y(x_1, x_2, \dots, x_N) \quad (\text{D.2})$$

For example, there is no assigned value of $y(0, 0)$ for which $y = y(x_1, x_2) = (x_1^2 - x_2^2) / (x_1^2 + x_2^2)$ would be continuous at the point $x_1 = 0, x_2 = 0$, since the left side of eqn (D.2) would have the limit $+1$ when $h_1 = 1, h_2 = 0$ but the limit -1 when $h_1 = 0, h_2 = 1$.

D.4 Compound Functions

Suppose we have a function

$$y = y(x_1, x_2, \dots, x_N) \quad (\text{D.3})$$

and each of the x_1, x_2, \dots, x_N is a function of another set of variables r_1, r_2, \dots, r_M as in

$$x_k = x_k(r_1, r_2, \dots, r_M) \quad (\text{D.4})$$

Then y is a *compound function* of the variables r . This compound function $y = y(r_1, r_2, \dots, r_M)$ is defined by direct substitution of the x_k of eqn (D.4) into eqn (D.3). This substitution will be denoted by

$$y = y(r_1, r_2, \dots, r_M) = y(x_1(r_1, r_2, \dots, r_M), x_2(r_1, r_2, \dots, r_M), \dots, x_N(r_1, r_2, \dots, r_M)) \quad (\text{D.5})$$

or, in a somewhat shorter notation,

$$y = y(r) = y(x_1(r), x_2(r), \dots, x_N(r)) \quad (\text{D.6})$$

or, even more simply,

$$y = y(r) = y(x(r)) \quad (\text{D.7})$$

D.5 The Same Function in Different Coordinates

A scalar field T is often represented by functions of different kinds of coordinates. For example,

$$T = T(\mathbf{r}) = T_C(x_1, x_2, x_3) = T_C(\rho, \phi, z) = T_S(r, \theta, \phi) \quad (\text{D.8})$$

where the subscripts refer to Cartesian, cylindrical polar, and spherical polar coordinates, respectively. Of course, eqn (D.8) contains the implicit assumption that the various coordinates in it must be linked by the relations in Section A.8, and so represent the same point \mathbf{r} . The function denoted $T_C(x_1, x_2, x_3)$ is plainly not the same

¹²⁸Definitions, and proofs of theorems stated here without proof, may be found in Chapter II of Courant (1936b).

function of its arguments x_1, x_2, x_3 as $T_s(r, \theta, \phi)$ is of its arguments r, θ, ϕ . For example, suppose $T_C(x_1, x_2, x_3) = ax_2/x_1$ which leads to $T_s(r, \theta, \phi) = a \tan \phi$. But they both represent the same underlying scalar field function $T = T(\mathbf{r})$. The almost universal custom in physics texts is to omit the subscripts “C”, “c”, and “s” in eqn (D.8) and write simply

$$T = T(\mathbf{r}) = T(x_1, x_2, x_3) = T(\rho, \phi, z) = T(r, \theta, \phi) \quad (\text{D.9})$$

The argument list of a function is taken as an adequate label for it.

The various functions in eqn (D.9) are thought of as the *same function*. The value T is really a function of the underlying field point \mathbf{r} and is only being *represented* in Cartesian, cylindrical polar, or spherical polar form. This physics custom is especially evident in Lagrangian and Hamiltonian mechanics where one sees expressions such as $L(s, \dot{s}, t) = L(q, \dot{q}, t)$ for functions expressing the same underlying physical quantity in terms of different variable sets.

We will follow the physics custom in this book, and in fact have already done so by using the same letter y to denote both $y(x_1, x_2, \dots, x_N)$ and the compound function $y(r_1, r_2, \dots, r_M)$ in Section D.4 above. Note that the different functional forms in eqn (D.9) can be defined as compound functions. For example, $T(r, \theta, \phi)$ may be derived from $T(x_1, x_2, x_3)$ by

$$T(r, \theta, \phi) = T(x_1(r, \theta, \phi), x_2(r, \theta, \phi), x_3(r, \theta, \phi)) \quad (\text{D.10})$$

D.6 Partial Derivatives

Lagrangian mechanics makes extensive use of partial derivatives. Given a function $y = y(x_1, x_2, \dots, x_N)$, a partial derivative with respect to variable x_k is defined as

$$\begin{aligned} & \frac{\partial y(x_1, x_2, \dots, x_N)}{\partial x_k} \\ &= \lim_{h_k \rightarrow 0} \left\{ \frac{y(x_1, x_2, \dots, x_{k-1}, x_k + h_k, x_{k+1}, \dots, x_N) - y(x_1, x_2, \dots, x_N)}{h_k} \right\} \quad (\text{D.11}) \end{aligned}$$

which says to take an ordinary derivative with respect to x_k as if variables $x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_N$ were constants. It does not say that they *are* constants, only that they are to be treated as such in calculating the derivative.

Partial derivatives thus depend on the *list* of variables as well as on the variable being differentiated with respect to, since only that list tells what to hold constant as y is differentiated. The often-seen notation $\partial y / \partial x_k$ is inherently ambiguous, unless one happens, as here, to know from context what list of variables is intended. In this text we will give the list of variables in all partial derivatives whenever there is any cause for doubt as to what that list might be, often using the shorthand form like $\partial y(x) / \partial x_k$ in which x stands for the whole list x_1, x_2, \dots, x_N .

One should note that the list of variables $x = x_1, x_2, \dots, x_N$ in a partial derivative is just to indicate those variables to be treated as constants when a derivative is taken. It does not indicate that y necessarily depends on each member of the list in every

case. Thus, we might have $y = 2x_1 + x_3$ as the function, even though the list is x_1, x_2, \dots, x_5 . In this case $\partial y(x)/\partial x_2 = 0$, $\partial y(x)/\partial x_4 = 0$, and $\partial y(x)/\partial x_5 = 0$ for all x values.

Conversely, as discussed in Corollary D.10.2 in Section D.10 below, if $\partial y(x)/\partial x_n = 0$ for all x values in region R , then y does not depend on x_n , and we may choose to expunge x_n from the list of variables and write $y = y(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$. For example, the list of variables for the above function might be shortened to x_1, x_2, x_3 , with x_4 and x_5 dropped but the x_2 retained.

Partial derivatives are themselves functions of the *same* variable list x_1, x_2, \dots, x_N as was the function y being differentiated. Thus second and higher derivatives can be taken by repeated application of rules like eqn (D.11). If they exist, these higher derivatives are denoted by expressions like

$$\frac{\partial}{\partial x_j} \left(\frac{\partial y(x)}{\partial x_k} \right) = \frac{\partial^2 y(x)}{\partial x_j \partial x_k} \quad \frac{\partial}{\partial x_i} \left(\frac{\partial^2 y(x)}{\partial x_j \partial x_k} \right) = \frac{\partial^3 y(x)}{\partial x_i \partial x_j \partial x_k} \quad (\text{D.12})$$

D.7 Continuously Differentiable Functions

If all first partial derivatives of a function $y = y(x_1, x_2, \dots, x_N)$ exist and are continuous functions of x , then the function y itself is continuous. Such functions are called *continuously differentiable*. If all partial derivatives up to and including the n th order exist and are continuous functions of x , the function is called *continuously differentiable to n th order*. Unless specifically stated otherwise, we will assume that all functions used in the present text are continuously differentiable to any order.

D.8 Order of Differentiation

If the second partial derivatives exist and are continuous functions for all x in R (i.e. if $y(x)$ is continuously differentiable to second order), then the order of the second-order partial derivatives is unimportant since, for all i, j ,

$$\frac{\partial^2 y(x)}{\partial x_i \partial x_j} = \frac{\partial^2 y(x)}{\partial x_j \partial x_i} \quad (\text{D.13})$$

Generalizing, if y is continuously differentiable to n th order, then all partial derivatives of that order or less are also independent of the order in which they are taken.

D.9 Chain Rule

Let y be a compound function of r as defined in eqn (D.5). Assume that all partial derivatives of the form $\partial y(x)/\partial x_k$ and $\partial x_k(r)/\partial r_j$ exist. Then the partial derivatives of y with respect to the r variables exist and may be written using what is called the

chain rule of partial differentiation,

$$\frac{\partial y(r_1, r_2, \dots, r_M)}{\partial r_j} = \sum_{k=1}^N \frac{\partial y(x_1, x_2, \dots, x_N)}{\partial x_k} \frac{\partial x_k(r_1, r_2, \dots, r_M)}{\partial r_j} \quad (\text{D.14})$$

or, in shorter but equivalent notation,

$$\frac{\partial y(r)}{\partial r_j} = \sum_{k=1}^N \frac{\partial y(x)}{\partial x_k} \frac{\partial x_k(r)}{\partial r_j} \quad (\text{D.15})$$

D.10 Mean Values

Theorem D.10.1: The Mean Value Theorem

Suppose that a function $y = y(x_1, x_2, \dots, x_N)$ is continuously differentiable in a region R and has partial derivatives $\partial y(x)/\partial x_k = g_k(x)$. Let h_k be increments added to x_k and assume that point $(x_1 + \eta h_1, x_2 + \eta h_2, \dots, x_N + \eta h_N)$ lies in region R for all η in the range $0 \leq \eta \leq 1$. Then

$$\begin{aligned} y(x_1 + h_1, x_2 + h_2, \dots, x_N + h_N) - y(x_1, x_2, \dots, x_N) \\ = \sum_{k=1}^N h_k g_k(x_1 + \theta h_1, x_2 + \theta h_2, \dots, x_N + \theta h_N) \end{aligned} \quad (\text{D.16})$$

for some θ in the range $0 < \theta < 1$.

Corollary D.10.2: Constancy of Functions

If $\partial y(x)/\partial x_k = 0$ for all $k = 1, \dots, N$ and for all x in R , then function $y(x)$ is constant in that region.

Corollary D.10.3: Non-dependence on Variables

If a function $y = y(x_1, x_2, \dots, x_N)$ has $\partial y(x)/\partial x_n = 0$ for some n and for all x in R , then function $y(x)$ does not depend on the variable x_n and so can be written as $y = y(x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$ in that region.

D.11 Orders of Smallness

We often want to compare two functions as a variable approaches some limit L . A useful notation is found in Chapter I of Titchmarsh (1939). It is

$$f(x) = o(\phi(x)) \text{ as } x \rightarrow L \quad \text{which means} \quad \lim_{x \rightarrow L} \left(\frac{f(x)}{\phi(x)} \right) = 0 \quad (\text{D.17})$$

In words, "Function f is of *smaller* order than ϕ as x approaches L ."

The following notation is also used.

$$f(x) = g(x) + o(\phi(x)) \text{ as } x \rightarrow L \quad \text{which means} \quad \lim_{x \rightarrow L} \left(\frac{f(x) - g(x)}{\phi(x)} \right) = 0 \quad (\text{D.18})$$

In words, "The difference between f and g is of *smaller* order than ϕ as x approaches L ."

For example, with $L = 0$,

$$1 - \cos x = o(x) \quad \text{and} \quad \sin x = x + o(x^2) \quad \text{as } x \rightarrow 0 \quad (\text{D.19})$$

and, with $L = \infty$,

$$\ln x = o(x) \quad \text{and} \quad x^n = o(e^x) \quad \text{as } x \rightarrow \infty \quad (\text{D.20})$$

D.12 Differentials

If $y = y(x)$ is a function of one variable, the change in y as the independent variable is incremented from x to $x + dx$ is $\Delta y = y(x + dx) - y(x)$. The increment dx in this expression is *not* assumed to be small. It may take any value. Assuming that the function is differentiable and has a finite derivative at x , the *differential* dy at point x is defined as the linear approximation to Δy based on the tangent line to the curve at point x ,

$$dy = \frac{dy(x)}{dx} dx \quad (\text{D.21})$$

Note to the Reader: This definition of the differential, and that in Section D.13, avoid any use of phrases like “infinitesimally small” or “infinitesimal”. The differential dy should be thought of as a function of two variables, the point x and the increment dx , both of which can take any value.

The approximation of Δy by dy may be good or bad, of course. But we are guaranteed that the difference between the two vanishes for small enough dx . More precisely, it follows from the definition of the derivative that $\lim_{dx \rightarrow 0} \{(\Delta y - dy)/dx\} = 0$, or equivalently, that $\Delta y = dy + o(dx)$ as $dx \rightarrow 0$. In other words, the difference between Δy and dy is of smaller order than dx , as dx approaches zero.

D.13 Differential of a Function of Several Variables

The definition of differential can be extended to functions of more than one variable. Our definition of this differential follows that of pages 66–69 of Courant (1936*b*). Note that he calls it the *total differential*.

If $y = y(x_1, x_2, \dots, x_N)$ and each variable is independently incremented from x_k to $x_k + dx_k$ for all $k = 1, \dots, N$, then the change in y is

$$\Delta y = y(x_1 + dx_1, x_2 + dx_2, \dots, x_N + dx_N) - y(x_1, x_2, \dots, x_N) \quad (\text{D.22})$$

and the differential dy is defined as the linear approximation to Δy given by

$$dy = \sum_{k=1}^N \frac{\partial y(x)}{\partial x_k} dx_k \quad (\text{D.23})$$

As in the one variable case, the increments dx_k may be large or small. If the set x_1, x_2, \dots, x_N is a set of independent variables, then the dx_k are independent and may take any values.

In the many-variable case, just as for the functions of one-variable discussed above, the approximation of Δy by dy may be good or bad. Define $h = \max_k \{|dx_k|\}$. If $y(x)$ is a continuously differentiable function of x , then the mean value theorem, Theorem D.10.1, implies that

$$\lim_{h \rightarrow 0} \left(\frac{\Delta y - dy}{h} \right) = 0 \quad \text{or equivalently} \quad \Delta y = dy + o(h) \text{ as } h \rightarrow 0 \quad (\text{D.24})$$

where Δy and dy are defined in eqns (D.22, D.23).

The differential in eqn (D.23) is well defined for any values of dx_k . Nonetheless, since eqn (D.24) says that, as $h = \max_k \{|dx_k|\}$ goes to zero, the difference between Δy and dy is of smaller order than h , it is also legitimate to think of the differential as the small change in the value of y as the independent variables are incremented by small amounts. The differential is often used heuristically in this way.

D.14 Differentials and the Chain Rule

The differential of a compound function may be constructed by direct substitution. As in Section D.4, suppose we have a function $y = y(x_1, x_2, \dots, x_N)$ whose differential is

$$dy = \sum_{k=1}^N \frac{\partial y(x)}{\partial x_k} dx_k \quad (\text{D.25})$$

and suppose that each x_k is in turn a function of r so that, for $k = 1, \dots, N$,

$$x_k = x_k(r_1, r_2, \dots, r_M) \quad \text{and} \quad dx_k = \sum_{j=1}^M \frac{\partial x_k(r)}{\partial r_j} dr_j \quad (\text{D.26})$$

Substituting eqn (D.26) into eqn (D.25), gives

$$dy = \sum_{k=1}^N \frac{\partial y(x)}{\partial x_k} \sum_{j=1}^M \frac{\partial x_k(r)}{\partial r_j} dr_j = \sum_{j=1}^M \frac{\partial y(r)}{\partial r_j} dr_j \quad (\text{D.27})$$

where the chain rule eqn (D.15) was used to obtain the last equality. But the last expression in eqn (D.27) is precisely the differential of the compound function $y = y(r_1, r_2, \dots, r_M)$. This example illustrates that differentials provide a clear and correct notation for manipulating the chain rule of partial differentiation.

D.15 Differentials of Second and Higher Orders

The first order differential defined in eqn (D.23) is itself a function of the variables $x_1, \dots, x_N, dx_1, \dots, dx_N$. Thus, the second-order differential may be defined as the differential of the first-order differential, using the same increments. Writing eqn

(D.23) using an operator formalism as

$$dy = \sum_{k=1}^N \frac{\partial y(x)}{\partial x_k} dx_k = \left(dx_1 \frac{\partial}{\partial x_1} + \cdots + dx_N \frac{\partial}{\partial x_N} \right) y(x) \quad (\text{D.28})$$

we may define

$$\begin{aligned} d^2 y &= \left(dx_1 \frac{\partial}{\partial x_1} + \cdots + dx_N \frac{\partial}{\partial x_N} \right)^2 y(x) \\ &= \left(dx_1 \frac{\partial}{\partial x_1} + \cdots + dx_N \frac{\partial}{\partial x_N} \right) \left(dx_1 \frac{\partial}{\partial x_1} + \cdots + dx_N \frac{\partial}{\partial x_N} \right) y(x) \\ &= \sum_{i=1}^N \sum_{j=1}^N dx_i dx_j \frac{\partial^2 y(x)}{\partial x_i \partial x_j} \end{aligned} \quad (\text{D.29})$$

assuming that the second-order partial differentials exist. Generalizing to n th order gives

$$\begin{aligned} d^n y &= \left(dx_1 \frac{\partial}{\partial x_1} + \cdots + dx_N \frac{\partial}{\partial x_N} \right)^n y(x) \\ &= \sum_{k_1=1}^N \cdots \sum_{k_n=1}^N dx_{k_1} \cdots dx_{k_n} \frac{\partial^n y(x)}{\partial x_{k_1} \cdots \partial x_{k_n}} \end{aligned} \quad (\text{D.30})$$

where we assume that the partial differentials to n th order exist.

D.16 Taylor Series

The Taylor theorem may be thought of as a generalization of the mean value theorem of Section D.10.

Theorem D.16.1: Taylor Series

If function $y(x_1, \dots, x_N)$ is continuously differentiable to $(n + 1)$ st order, then

$$y(x_1 + dx_1, \dots, x_N + dx_N) = y(x_1, \dots, x_N) + dy + \frac{1}{2!} d^2 y + \cdots + \frac{1}{n!} d^n y + R_n \quad (\text{D.31})$$

where all of the differentials dy , $d^2 y$, etc., are to be evaluated using the same increments dx_k and where the remainder R_n is

$$R_n = \frac{1}{(n + 1)!} \sum_{k_1=1}^N \cdots \sum_{k_{n+1}=1}^N dx_{k_1} \cdots dx_{k_{n+1}} g_{k_1 \dots k_{n+1}}(x_1 + \theta dx_1, \dots, x_N + \theta dx_N) \quad (\text{D.32})$$

where

$$g_{k_1 \dots k_{n+1}}(x) = \frac{\partial^{n+1} y(x)}{\partial x_{k_1} \cdots \partial x_{k_{n+1}}} \quad (\text{D.33})$$

and θ is some number in the range $0 < \theta < 1$.

D.17 Higher-Order Differential as a Difference

Suppose that the conditions of the Taylor theorem of Section D.16 apply, and define the difference Δy as

$$\Delta y = y(x_1 + dx_1, \dots, x_N + dx_N) - y(x_1, \dots, x_N) \quad (\text{D.34})$$

Then inspection of eqns (D.31, D.32) shows that

$$\Delta y = dy + \frac{1}{2!}d^2y + \dots + \frac{1}{n!}d^ny + o(h^n) \quad (\text{D.35})$$

where $h = \max_k \{|dx_k|\}$ for $1 \leq k \leq N$.

D.18 Differential Expressions

Given N functions $a_k(x)$ of the set of independent variables x_1, x_2, \dots, x_N , we may form *differential expressions* like

$$\sum_{k=1}^N a_k(x) dx_k \quad (\text{D.36})$$

which may or may not be the actual differential of some function. These expressions may be manipulated by the usual rules of algebra (think of the dx_k simply as finite increments).

We adopt the usual convention that an equality involving differential expressions includes the implicit assumption that it holds for all possible dx_k values. Since the increments dx_k of independent variables x_1, x_2, \dots, x_N can take any values, it follows that the dx_k may be set nonzero one at a time, which leads to the following Lemmas.

Lemma D.18.1: Zero Differential Expressions

The differential expression is zero

$$\sum_{k=1}^N a_k(x) dx_k = 0 \quad (\text{D.37})$$

if and only if $a_k(x) = 0$ for all $k = 1, \dots, N$.

Lemma D.18.2: Equal Differential Expressions

Two differential expressions are equal,

$$\sum_{k=1}^N a_k(x) dx_k = \sum_{k=1}^N b_k(x) dx_k \quad (\text{D.38})$$

if and only if $a_k(x) = b_k(x)$ for all $k = 1, \dots, N$.

Lemma D.18.3: Differential Expression and Differential

It follows from eqn (D.38) together with the definition of the differential in eqn (D.23) that, if we are given a function $f = f(x_1, x_2, \dots, x_N)$ of the independent variables x_1, x_2, \dots, x_N and a differential expression $\sum_{k=1}^N a_k(x) dx_k$, the equality

$$df = \sum_{k=1}^N a_k(x) dx_k \quad (\text{D.39})$$

holds if and only if $\partial f(x)/\partial x_k = a_k(x)$ for all $k = 1, \dots, N$.

Note to the Reader: Not all differential expressions can be set equal to the differential of a function, as was assumed in the previous Lemma. Those that can are called perfect differentials and are discussed in Section D.20.

Lemma D.18.4: Zero Differential

It follows from Lemma D.18.1 that a function $f = f(x_1, x_2, \dots, x_N)$ of the independent variables x_1, x_2, \dots, x_N will have $df = 0$ at a point x_1, x_2, \dots, x_N if and only if $\partial f(x)/\partial x_k = 0$ for all $k = 1, \dots, N$ at that point.

In some cases of interest, the variables x will be functions of another set of variables $r = r_1, r_2, \dots, r_M$ as in the discussion of compound functions in Section D.4. In that case, using the chain rule from Section D.14, the differential expression becomes

$$\sum_{k=1}^N a_k(x) dx_k = \sum_{k=1}^N a_k(x) \sum_{j=1}^M \frac{\partial x_k(r)}{\partial r_j} dr_j = \sum_{j=1}^M A_j(r) dr_j \quad (\text{D.40})$$

where

$$dx_k = \sum_{j=1}^M \frac{\partial x_k(r)}{\partial r_j} dr_j \quad \text{and} \quad A_j(r) = \sum_{k=1}^N a_k(x) \frac{\partial x_k(r)}{\partial r_j} \quad (\text{D.41})$$

In some applications, it is important to know if the Lemmas above still apply to the differential expression $\sum_{k=1}^N a_k(x) dx_k$ when one assumes *only* that the r variables are independent.

Theorem D.18.5: Compound Differential Expressions

Let the variables r be assumed to be independent so that the increments dr_j can be set equal to zero one at a time in eqn (D.40). Then Lemmas D.18.1 through D.18.4 will continue to hold for the differential expression $\sum_{k=1}^N a_k(x) dx_k$ if and only if $M = N$ and the determinant condition

$$\left| \frac{\partial x(r)}{\partial r} \right| \neq 0 \quad (\text{D.42})$$

is satisfied.

Proof: Theorem D.24.1 below shows that the determinant condition in eqn (D.42) is the necessary and sufficient condition for the transformation $x \rightarrow r$ to be invertible.

Thus, one can use the inverse matrix from Section D.25 to write

$$dr_j = \sum_{k=1}^N \left(\frac{\partial x_k(r)}{\partial r} \right)_{jk}^{-1} dx_k \quad (\text{D.43})$$

from which a choice of the independent increments dr_j can be found that will make the dx_k have any value desired. Thus the dx_k are also arbitrary and independent, and can be set nonzero one at a time, which is the condition needed. \square

D.19 Line Integral of a Differential Expression

A curve can be defined in region R by making each x_k be a function of some monotonically varying parameter β , so that $x_k = x_k(\beta)$ for $k = 1, \dots, N$. The integral

$$I_{01} = \int_{\beta_0}^{\beta_1} \sum_{k=1}^N a_k(x(\beta)) \frac{dx_k(\beta)}{d\beta} d\beta \quad (\text{D.44})$$

is called a *line integral* of the differential expression $\sum_{k=1}^N a_k(x) dx_k$ along a portion of that curve.

The line integral in eqn (D.44) is often denoted more simply, as just the integral of a differential expression with integration along a particular curve being understood. Thus, one often sees I_{01} denoted as

$$I_{01} = \int_0^1 \sum_{k=1}^N a_k(x) dx_k \quad \text{or even} \quad I_{01} = \int_0^1 \vec{a} \cdot d\vec{x} \quad (\text{D.45})$$

where the latter form treats the differential expression as a dot product of two vectors in an N -dimensional Cartesian space with the x_k as its coordinates.

D.20 Perfect Differentials

Differential expressions $\sum_{k=1}^N a_k(x) dx_k$ for which a function f exists satisfying $df = \sum_{k=1}^N a_k(x) dx_k$ are called *perfect differentials*. Line integrals and perfect differentials are treated in Chapter V of Courant (1936b) and Appendix 11 in Volume I of Desloges (1982).

The function f is sometimes called a *potential function* since the $a_k(x)$ can be derived from it by partial differentiation in a way analogous to the derivation of the electric field components from the electric potential. A condition for $\sum_{k=1}^N a_k(x) dx_k$ to be a perfect differential is given by the following theorem.

Theorem D.20.1: Condition for a Perfect Differential

Assume the variables x_1, x_2, \dots, x_N to lie in an open rectangle R_{\perp} . Given a set of continuously differentiable functions $a_k(x)$ for $k = 1, \dots, N$, there exists a potential function

$f = f(x_1, x_2, \dots, x_N)$ such that the following two equivalent conditions are satisfied,

$$\frac{\partial f(x)}{\partial x_k} = a_k(x) \quad \text{for all } k = 1, \dots, N \quad \text{or} \quad df = \sum_{k=1}^N a_k(x) dx_k \quad (D.46)$$

if and only if, for all x in R_\perp and all pairs of indices $i, j = 1, \dots, N$,

$$\frac{\partial a_i(x)}{\partial x_j} = \frac{\partial a_j(x)}{\partial x_i} \quad (D.47)$$

Proof: First we prove that eqn (D.46) implies eqn (D.47). For if an f exists satisfying eqn (D.46), then, using eqn (D.13) gives

$$\frac{\partial a_i(x)}{\partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial a_j(x)}{\partial x_i} \quad (D.48)$$

To prove that eqn (D.47) implies eqn (D.46) we construct a suitable f explicitly. Starting from some arbitrary point $x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)}$, we perform a line integral along a series of straight line segments, first along x_1 , then along x_2 , etc., until the final point 1 at x_1, x_2, \dots, x_N is reached. Such a path will lie entirely in R_\perp , and the integral I_{10} will be the sum of the integrals along its segments. Along the j th segment, all x_i for $i \neq j$ are held constant but x_j varies as $x_j = \beta$, giving $dx_k(\beta)/d\beta = \delta_{jk}$. Inserting this result into eqn (D.44), and setting $f(x_1, x_2, \dots, x_N)$ equal to the integral I_{10} gives

$$f(x_1, x_2, \dots, x_N) = \sum_{j=1}^N \int_{x_j^{(0)}}^{x_j} a_j(x_1, x_2, \dots, x_{j-1}, \beta, x_{j+1}^{(0)}, \dots, x_N^{(0)}) d\beta \quad (D.49)$$

Since any point x can be reached by this integration, the function f is defined for all x in R_\perp .

The partial derivatives of this f may be written as

$$\begin{aligned} \frac{\partial f(x)}{\partial x_i} &= a_i(x_1, x_2, \dots, x_i, x_{i+1}^{(0)}, \dots, x_N^{(0)}) \\ &+ \sum_{j=i+1}^N \int_{x_j^{(0)}}^{x_j} \frac{\partial a_j(x_1, x_2, \dots, x_{j-1}, \beta, x_{j+1}^{(0)}, \dots, x_N^{(0)})}{\partial x_i} d\beta \end{aligned} \quad (D.50)$$

With x_j temporarily replaced by β , the assumption stated in eqn (D.47) implies that

$$\frac{\partial a_j(x_1, x_2, \dots, x_{j-1}, \beta, x_{j+1}^{(0)}, \dots, x_N^{(0)})}{\partial x_i} = \frac{\partial a_i(x_1, x_2, \dots, x_{j-1}, \beta, x_{j+1}^{(0)}, \dots, x_N^{(0)})}{\partial \beta} \quad (D.51)$$

Thus

$$\begin{aligned}
 \frac{\partial f(x)}{\partial x_i} &= a_i(x_1, x_2, \dots, x_i, x_{i+1}^{(0)}, \dots, x_N^{(0)}) \\
 &\quad + \sum_{j=i+1}^N \int_{x_j^{(0)}}^{x_j} \frac{\partial a_i(x_1, x_2, \dots, x_{j-1}, \beta, x_{j+1}^{(0)}, \dots, x_N^{(0)})}{\partial \beta} d\beta \\
 \frac{\partial f(x)}{\partial x_i} &= a_i(x_1, x_2, \dots, x_i, x_{i+1}^{(0)}, \dots, x_N^{(0)}) \\
 &\quad + \sum_{j=i+1}^N a_i(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}^{(0)}, \dots, x_N^{(0)}) \\
 &\quad - \sum_{j=i+1}^N a_i(x_1, x_2, \dots, x_{j-1}, x_j^{(0)}, x_{j+1}^{(0)}, \dots, x_N^{(0)}) \\
 \frac{\partial f(x)}{\partial x_i} &= a_i(x_1, x_2, \dots, x_N) \tag{D.52}
 \end{aligned}$$

which shows that the f constructed in eqn (D.49) does have the required property eqn (D.46). \square

The line integral eqn (D.49) used in this proof is of great practical use. It will be used, for example, to compute generating functions for canonical transformations.

The theorem proved in this section (but not, of course, the proof of it given here) remains true even when open rectangle R_{\perp} is replaced by a more general region R which is only assumed to be open and simply connected. See Chapter V of Courant (1936b) for details.

D.21 Perfect Differential and Path Independence

An alternate, and equivalent, condition for a differential expression to be a perfect differential is that its line integral between two end points be independent of the particular choice of the path between them.

Theorem D.21.1: Path Independence

Assume a given set of functions $a_k(x)$ for $k = 1, \dots, N$, continuously differentiable in an open and simply connected region R . The differential expression $\sum_{k=1}^N a_k(x) dx_k$ is a perfect differential with a potential function f satisfying the equivalent conditions eqn (D.46),

$$\frac{\partial f(x)}{\partial x_k} = a_k(x) \quad \text{for all } k = 1, \dots, N \quad \text{or} \quad df = \sum_{k=1}^N a_k(x) dx_k \tag{D.53}$$

if and only if the line integral between any two points in R

$$I_{01} = \int_{\beta_0}^{\beta_1} \sum_{k=1}^N a_k(x(\beta)) \frac{dx_k(\beta)}{d\beta} d\beta \tag{D.54}$$

depends only on x at the endpoints of the integration $x_1(\beta_0), x_2(\beta_0), \dots, x_N(\beta_0)$

and $x_1(\beta_1), x_2(\beta_1), \dots, x_N(\beta_1)$ and is therefore independent of the path $x_k(\beta)$ taken between those endpoints.

Proof: First assume eqn (D.53) and prove the path independence of eqn (D.54). If a potential function f exists, then eqn (D.54) becomes

$$I_{01} = \int_{\beta_0}^{\beta_1} \sum_{k=1}^N \frac{\partial f(x)}{\partial x_k} \frac{dx_k}{d\beta} d\beta = \int_{\beta_0}^{\beta_1} \frac{df(x)}{d\beta} d\beta = f(x(\beta_2)) - f(x(\beta_1)) \quad (D.55)$$

which depends only on the value of f at the end points and hence is independent of the path, as was to be proved. For proof of the converse, see Chapter V of Courant (1936b). \square

It follows from Theorem D.21.1 that integration along any path between the points $x_1^{(0)}, x_2^{(0)}, \dots, x_N^{(0)}$ and x_1, x_2, \dots, x_N would have given the same integral as the particular path used in the proof of Theorem D.20.1.

The path independence of line integrals between any two points is equivalent to the vanishing of line integrals around closed paths. The following Corollary is given without proof.

Corollary D.21.2: Closed Paths

The line integral I_{01} between any two points is independent of the path if and only if the line integral around any closed path is zero.

D.22 Jacobians

Suppose that we are given $y_k = y_k(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) = y_k(x, z)$ for $k = 1, \dots, N$, where P may have any non-negative value, including the value zero (which would indicate that the extra z variables are absent). Then we may form an $N \times N$ matrix of partial derivatives denoted $(\partial y(x, z)/\partial x)$ and defined by

$$\left(\frac{\partial y(x, z)}{\partial x} \right)_{ij} = \frac{\partial y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)}{\partial x_j} \quad (D.56)$$

This matrix is called the *Jacobian matrix*, and its determinant is called the *Jacobian determinant*, or simply the *Jacobian*. It is variously denoted. We give here the two forms we will use, and the determinant itself,

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} = \left| \frac{\partial y(x, z)}{\partial x} \right| = \begin{vmatrix} \frac{\partial y_1(x, z)}{\partial x_1} & \frac{\partial y_1(x, z)}{\partial x_2} & \dots & \frac{\partial y_1(x, z)}{\partial x_N} \\ \frac{\partial y_2(x, z)}{\partial x_1} & \frac{\partial y_2(x, z)}{\partial x_2} & \dots & \frac{\partial y_2(x, z)}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_N(x, z)}{\partial x_1} & \frac{\partial y_N(x, z)}{\partial x_2} & \dots & \frac{\partial y_N(x, z)}{\partial x_N} \end{vmatrix} \quad (D.57)$$

The first form, which is the traditional one, does not specify the list of variables to be held constant when the partial derivatives are taken. It is implicit in the notation that

all of the variables listed in the denominator are on that list, but there may be others, as here. Usually the intended list is obvious, but in cases of doubt we will modify the traditional notation and use expressions like

$$\frac{\partial (y_1(x, z), y_2(x, z), \dots, y_N(x, z))}{\partial (x_1, x_2, \dots, x_N)} \quad (\text{D.58})$$

in which the list is explicitly stated. Jacobians are treated in Volume I, Appendix 3 of Desloges (1982) and in Chapter III of Courant (1936b).

Lemma D.22.1: Jacobian of a Compound Function

If the variables x_i are themselves functions of another set of variables r , as well as of the same extra variables z as above, $x_i = x_i(r_1, r_2, \dots, r_N, z_1, z_2, \dots, z_P)$ for $i = 1, \dots, N$, then the compound functions $y_k(r_1, r_2, \dots, r_N, z_1, z_2, \dots, z_P)$ may be defined by

$$y_k = y_k(r_1, r_2, \dots, r_N, z_1, z_2, \dots, z_P) = y_k(x_1(r, z), x_2(r, z), \dots, x_N(r, z), z_1, z_2, \dots, z_P) \quad (\text{D.59})$$

Then the Jacobians obey the relation

$$\left| \frac{\partial y(r, z)}{\partial r} \right| = \left| \frac{\partial y(x, z)}{\partial x} \right| \left| \frac{\partial x(r, z)}{\partial r} \right| \quad (\text{D.60})$$

or, in the traditional notation,

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (r_1, r_2, \dots, r_N)} = \frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} \frac{\partial (x_1, x_2, \dots, x_N)}{\partial (r_1, r_2, \dots, r_N)} \quad (\text{D.61})$$

Proof: The chain rule of partial differentiation gives

$$\frac{\partial y_k(r, z)}{\partial r_i} = \sum_{j=1}^N \frac{\partial y_k(x, z)}{\partial x_j} \frac{\partial x_j(r, z)}{\partial r_i} \quad (\text{D.62})$$

which may be written as a matrix equation

$$\left(\frac{\partial y(r, z)}{\partial r} \right) = \left(\frac{\partial y(x, z)}{\partial x} \right) \left(\frac{\partial x(r, z)}{\partial r} \right) \quad (\text{D.63})$$

Equating the determinant of both sides of eqn (D.63) gives eqn (D.60), as was to be proved. \square

Lemma D.22.2: Jacobian of an Augmented Variable Set

If $y_k = y_k(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) = y_k(x, z)$ for $k = 1, \dots, N$ as before, the following identity holds

$$\frac{\partial (y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P)}{\partial (x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)} = \frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} \quad (\text{D.64})$$

Proof: The left expression in eqn (D.64) is the determinant of an $(N + P) \times (N + P)$ matrix whose last P rows consist of N zeroes followed by an element of the $P \times P$ identity matrix. Its determinant is thus the determinant of the $N \times N$ upper left-hand block, which is the right expression in eqn (D.64). \square

Lemma D.22.3: Jacobian of an Inverse Function

If a set of functions $y_k = y_k(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)$ can be solved for x , giving $x_i = x_i(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P)$ then

$$\left| \frac{\partial y(x, z)}{\partial x} \right| \left| \frac{\partial x(y, z)}{\partial y} \right| = 1 \tag{D.65}$$

or, in traditional form,

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} \frac{\partial (x_1, x_2, \dots, x_N)}{\partial (y_1, y_2, \dots, y_N)} = 1 \tag{D.66}$$

Proof: As is proved in Theorem D.24.1 below, the necessary and sufficient condition for $y_k = y_k(x, z)$ to be solved for $x_i = x_i(y, z)$ is the determinant condition

$$\left| \frac{\partial y(x, z)}{\partial x} \right| \neq 0 \tag{D.67}$$

Then, as discussed in Section D.25, the following matrix equation holds

$$\left(\frac{\partial y(x, z)}{\partial x} \right) \left(\frac{\partial x(y, z)}{\partial y} \right) = \mathbf{U} \tag{D.68}$$

where \mathbf{U} is the $N \times N$ unit matrix. Taking the determinant of both sides gives eqn (D.65). □

Lemma D.22.4: Change of Variable in an Integral

Given a set of continuously differentiable functions $y_k = y_k(x_1, x_2, \dots, x_N)$ for $k = 1, \dots, N$, whose Jacobian does not vanish in the range of integration,

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} = \left| \frac{\partial y(x)}{\partial x} \right| \neq 0 \tag{D.69}$$

the multiple integral

$$I = \int dy_N \cdots \int dy_2 \int dy_1 f(y_1, y_2, \dots, y_N) \tag{D.70}$$

may be transformed into an integral over the variables x

$$I = \int dx_N \cdots \int dx_2 \int dx_1 f(x_1, x_2, \dots, x_N) \frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} \tag{D.71}$$

where the compound function $f(x_1, x_2, \dots, x_N)$ is

$$f(x_1, x_2, \dots, x_N) = f(y_1(x), y_2(x), \dots, y_N(x)) \tag{D.72}$$

and the limits of integration in eqn (D.71) are chosen so that x ranges over the inverse image of the range of y .

In practice, these limits of integration in eqn (D.71) are usually chosen so that the two integrals would have the same value, including the same sign, for the case in which f is replaced by the number 1.

D.23 Global Inverse Function Theorem

We often need to invert a set of functions like $y_i = y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)$ where $i = 1, \dots, N$, that is, to solve them for the variables $x_j = x_j(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P)$. The following global and local inverse function theorems are of great importance.

The inverses proved by the present theorem are called *global*, because the same, unique inverse functions apply to the whole of an open rectangle. This open rectangle may be indefinitely large. For example, in the transformation from plane polar coordinates ρ, ϕ to plane Cartesian coordinates x, y , the open rectangle might be $0 < \rho < \infty$ and $-\pi < \phi < \pi$.

Theorem D.23.1: The Global Inverse Function Theorem

Assume that all points $x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P$ lie in an open rectangle R_\perp , and that

$$y_i = y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) \quad (\text{D.73})$$

for $i = 1, \dots, N$, are a set of continuously differentiable functions of the stated variables. If for all x, z in R_\perp , the Jacobian determinant

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} = \left| \frac{\partial y(x, z)}{\partial x} \right| \quad (\text{D.74})$$

is nonzero and has a persistent, nested set of critical minors,¹²⁹ then functions $y_i = y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)$ can be solved for the inverse functions $x_j = x_j(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P)$ for $j = 1, \dots, N$ so that

$$\begin{aligned} x_1 &= x_1(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P) \\ &\vdots \\ x_N &= x_N(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P) \end{aligned} \quad (\text{D.75})$$

These inverse functions will be unique and continuously differentiable in the range covered by variables y, z as variables x, z range over R_\perp .

Proof: This proof is adapted from Volume II, Appendix 18 of Desloges (1982). The proof is by induction. First prove the theorem for $N = 1$. Then prove that, if the theorem is true for $N = K - 1$, it must be true for $N = K$. It follows that the theorem must be true for any integer N .

For $N = 1$, $(\partial y(x, z)/\partial x) = \partial y_1(x_1, z)/\partial x_1$. By assumption, this partial derivative is nonzero in a range of values $a_1 < x_1 < b_1$. Considering the z variables as fixed parameters, the inverse function theorem of ordinary one-variable calculus applies. Since $\partial y_1(x_1, z)/\partial x_1$ is continuous and nonzero, it must have the same sign throughout the range. Thus $y_1(x_1, z)$ is monotonic and has a unique inverse $x_1 = x_1(y_1, z)$. Since

¹²⁹“Persistent” here means that the nonzero Jacobian determinant in eqn (D.74) must have the same $N - 1$ rowed critical (i.e. nonzero) minor throughout R_\perp . “Nested” means that this persistent $N - 1$ rowed critical minor must, in turn, have the same $N - 2$ rowed critical minor throughout R_\perp , etc.

$\partial x_1(x_1, z)/\partial y_1 = (\partial y_1(x_1, z)/\partial x_1)^{-1}$ it follows that $x_1 = x_1(y_1, z)$ is also a continuously differentiable function.

For $N = K$, the Jacobian determinant is

$$\frac{\partial (y_1, y_2, \dots, y_K)}{\partial (x_1, x_2, \dots, x_K)} = \begin{vmatrix} \frac{\partial y_1(x, z)}{\partial x_1} & \frac{\partial y_1(x, z)}{\partial x_2} & \dots & \frac{\partial y_1(x, z)}{\partial x_K} \\ \frac{\partial y_2(x, z)}{\partial x_1} & \frac{\partial y_2(x, z)}{\partial x_2} & \dots & \frac{\partial y_2(x, z)}{\partial x_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_K(x, z)}{\partial x_1} & \frac{\partial y_K(x, z)}{\partial x_2} & \dots & \frac{\partial y_K(x, z)}{\partial x_K} \end{vmatrix} \tag{D.76}$$

Since by assumption this determinant is nonzero, it must have a critical $(K - 1)$ rowed minor. By assumption, the same minor is nonzero for all x, z in R_1 . For simplicity (and without loss of generality since the functions and variables can be relabeled in any way) we assume that this is the minor with the first row and first column removed. Then

$$\frac{\partial (y_2, \dots, y_N)}{\partial (x_2, \dots, x_N)} = \begin{vmatrix} \frac{\partial y_2(x, z)}{\partial x_2} & \dots & \frac{\partial y_2(x, z)}{\partial x_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_K(x, z)}{\partial x_2} & \dots & \frac{\partial y_K(x, z)}{\partial x_K} \end{vmatrix} \neq 0 \tag{D.77}$$

By the induction assumption, the theorem is true for $N = K - 1$. So eqn (D.77), and its assumed persistent nested critical minors, imply that inverse functions exist of the form

$$\begin{aligned} x_2 &= x_2(x_1, y_2, \dots, y_K, z) \\ &\vdots \\ x_K &= x_K(x_1, y_2, \dots, y_K, z) \end{aligned} \tag{D.78}$$

where the set z_1, z_2, \dots, z_P is now being represented by the single unsubscripted letter z . Substitute eqn (D.78) into y_1 to obtain the compound function $y_1 = y_1(x_1, y_2, \dots, y_K, z)$ defined by

$$y_1 = y_1(x_1, y_2, \dots, y_K, z) = y_1(x_1, x_2(x_1, y_2, \dots, y_K, z), \dots, x_K(x_1, y_2, \dots, y_K, z), z) \tag{D.79}$$

We now show that this function can be solved for x_1 . Using the chain rule and eqn

(D.79), the partial derivative $\partial y_1(x_1, y_2, \dots, y_K, z)/\partial x_1$ can be expanded as

$$\begin{aligned} \frac{\partial y_1(x_1, y_2, \dots, y_K, z)}{\partial x_1} &= \frac{\partial y_1(x, z)}{\partial x_1} + \sum_{i=2}^K \frac{\partial y_1(x, z)}{\partial x_i} \frac{\partial x_i(x_1, y_2, \dots, y_K, z)}{\partial x_1} \\ &= \frac{\partial y_1(x, z)}{\partial x_1} + \sum_{i=2}^K \frac{\partial y_1(x, z)}{\partial x_i} \frac{\partial(x_i, y_2, \dots, y_K, z_1, z_2, \dots, z_P)}{\partial(x_1, y_2, \dots, y_K, z_1, z_2, \dots, z_P)} \end{aligned} \quad (\text{D.80})$$

where eqn (D.64) has been used. Now multiply each term in eqn (D.80) by the Jacobian determinant

$$\frac{\partial(x_1, y_2, \dots, y_K, z_1, z_2, \dots, z_P)}{\partial(x_1, x_2, \dots, x_K, z_1, \dots, z_P)} = \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} \quad (\text{D.81})$$

and use eqn (D.61) to obtain

$$\begin{aligned} \frac{\partial y_1(x_1, y_2, \dots, y_K, z)}{\partial x_1} \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} \\ = \frac{\partial y_1(x, z)}{\partial x_1} \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} + \sum_{i=2}^K \frac{\partial y_1(x, z)}{\partial x_i} \frac{\partial(x_i, y_2, \dots, y_K, z_1, z_2, \dots, z_P)}{\partial(x_1, x_2, \dots, x_K, z_1, z_2, \dots, z_P)} \end{aligned} \quad (\text{D.82})$$

But eqn (D.64) and the usual rules for sign change when rows or columns of a determinant are exchanged, show the last Jacobian determinant in eqn (D.82) to be

$$\frac{\partial(x_i, y_2, \dots, y_K, z_1, z_2, \dots, z_P)}{\partial(x_1, x_2, \dots, x_K, z_1, z_2, \dots, z_P)} = (-1)^{(i+1)} \frac{\partial(y_2, \dots, y_K)}{\partial(x_1, x_2, x_{i-1}, x_{i+1}, \dots, x_K)} \quad (\text{D.83})$$

Thus

$$\begin{aligned} \frac{\partial y_1(x_1, y_2, \dots, y_K, z)}{\partial x_1} \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} \\ = \frac{\partial y_1(x, z)}{\partial x_1} \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} + \sum_{i=2}^K (-1)^{(i+1)} \frac{\partial y_1(x, z)}{\partial x_i} \frac{\partial(y_2, \dots, y_K)}{\partial(x_1, x_2, x_{i-1}, x_{i+1}, \dots, x_K)} \end{aligned} \quad (\text{D.84})$$

The right side of eqn (D.84) is just the Jacobian determinant $\partial(y_1, y_2, \dots, y_K)/\partial(x_1, x_2, \dots, x_K)$ evaluated using expansion by cofactors along its first row. Thus eqn (D.82) becomes

$$\frac{\partial y_1(x_1, y_2, \dots, y_K, z)}{\partial x_1} \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} = \frac{\partial(y_1, y_2, \dots, y_K)}{\partial(x_1, x_2, \dots, x_K)} \quad (\text{D.85})$$

Since both

$$\frac{\partial(y_1, y_2, \dots, y_K)}{\partial(x_1, x_2, \dots, x_K)} \neq 0 \quad \text{and} \quad \frac{\partial(y_2, \dots, y_K)}{\partial(x_2, \dots, x_K)} \neq 0 \quad (\text{D.86})$$

by assumption, this proves that the partial derivative

$$\frac{\partial y_1(x_1, y_2, \dots, y_K, z)}{\partial x_1} \neq 0 \tag{D.87}$$

Since R_\perp is an open rectangle, eqn (D.87) will hold for any fixed values of $y_2, \dots, y_K, z_1, z_2, \dots, z_P$ and for all $a_1 < x_1 < b_1$ where a_1 and b_1 are the least and greatest value of x_1 in the open rectangle. Thus, by the same reasoning as was used for the case $N = 1$ above, $y_1 = y_1(x_1, y_2, \dots, y_K, z)$ may be inverted to yield the unique, continuously differentiable inverse function $x_1 = x_1(y_1, y_2, \dots, y_K, z)$. Substituting that equation into eqn (D.78) gives the desired result: $x_i = x_i(y_1, y_2, \dots, y_K, z)$ for all $i = 1, \dots, K$.

Since the truth of the theorem for $N = K - 1$ proves its truth for $N = K$, the theorem must be true for any N . □

D.24 Local Inverse Function Theorem

In many cases, global inverses provided by Theorem D.23.1 are not needed, and indeed may not be available. In these cases, we can still define *local* inverse functions. These local inverses will be proved to exist only in some open neighborhood $N_{x,z}$ surrounding point x, z .

Local inverses are important because they may exist at points of a wider class of regions than the R_\perp assumed in Theorem D.23.1. And, of course, if a global inverse does exist in a region, then local inverses will exist at each point of that region also.

Theorem D.24.1: The Local Inverse Function Theorem

Assume that point $x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P$ lies in an open region R , and that

$$y_i = y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) \quad \text{for } i = 1, \dots, N \tag{D.88}$$

are continuously differentiable functions of the stated variables. If the Jacobian determinant

$$\frac{\partial (y_1, y_2, \dots, y_N)}{\partial (x_1, x_2, \dots, x_N)} = \left| \frac{\partial y(x, z)}{\partial x} \right| \tag{D.89}$$

is nonzero at the point x, z , then there is some open neighborhood $N_{x,z}$ of this point in which functions $y_i = y_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P)$ can be solved for the inverse functions

$$x_j = x_j(y_1, y_2, \dots, y_N, z_1, z_2, \dots, z_P) \tag{D.90}$$

for $j = 1, \dots, N$. These inverse function will be unique and continuously differentiable in the range covered by variables y, z as variables x, z vary over $N_{x,z}$.

Proof: Since R is open, every point x, z is in some open neighborhood $N_{x,z}$ which is contained entirely in R . Since the function y is assumed to be continuously differentiable, the Jacobian eqn (D.74), and all of its minors, are continuous functions of x, z . Thus, since the Jacobian (and hence a set of nested minors) are nonzero at x, z by assumption, we may shrink the open neighborhood $N_{x,z}$ until these determinants

are nonzero over the whole of $N_{x,z}$. Since, by definition, the open neighborhood is a (possibly small) open rectangle, the conditions of Theorem D.23.1 are satisfied, and the local inverse functions exist, as was to be proved. A direct proof of this theorem, not based on the global inverse function theorem, is given on page 152 of Courant (1936b). \square

D.25 Derivatives of the Inverse Functions

If inverse functions, from either the global or local inverse function theorems, exist at some point x, z , then the partial derivatives of the inverse functions x_j can be expressed in terms of the partial derivatives of the original functions y_i at that point.

Substituting eqn (D.90) into eqn (D.88) gives the compound function

$$y_i(y, z) = y_i(x_1(y, z), \dots, x_N(y, z), z) \quad (\text{D.91})$$

The chain rule then gives

$$\delta_{ij} = \frac{\partial y_i(y, z)}{\partial y_j} = \sum_{k=1}^N \frac{\partial y_i(x, z)}{\partial x_k} \frac{\partial x_k(y, z)}{\partial y_j} \quad (\text{D.92})$$

which in matrix form is

$$\mathbf{U} = \left(\frac{\partial y(x, z)}{\partial x} \right) \left(\frac{\partial x(y, z)}{\partial y} \right) \quad (\text{D.93})$$

where \mathbf{U} denotes the $N \times N$ identity matrix. It follows that both product matrices are nonsingular and that

$$\left(\frac{\partial x(y, z)}{\partial y} \right) = \left(\frac{\partial y(x, z)}{\partial x} \right)^{-1} \quad (\text{D.94})$$

and hence that, for any i, j

$$\frac{\partial x_j(y, z)}{\partial y_i} = \left(\frac{\partial y(x, z)}{\partial x} \right)^{-1}_{ji} \quad (\text{D.95})$$

which expresses the partials of x_j with respect to the y_i as functions of the partials of the original functions $y_i(x, z)$.

Similarly, applying the chain rule to the differentiation of eqn (D.91) with respect to z_n gives

$$0 = \frac{\partial y_i(y, z)}{\partial z_n} = \frac{\partial y_i(x, z)}{\partial z_n} + \sum_{k=1}^N \frac{\partial y_i(x, z)}{\partial x_k} \frac{\partial x_k(y, z)}{\partial z_n} \quad (\text{D.96})$$

which leads to the matrix equation

$$\left(\frac{\partial x(y, z)}{\partial z} \right) = - \left(\frac{\partial y(x, z)}{\partial x} \right)^{-1} \left(\frac{\partial y(x, z)}{\partial z} \right) \quad (\text{D.97})$$

and hence

$$\frac{\partial x_j(y, z)}{\partial z_n} = - \sum_{i=1}^N \left(\frac{\partial y(x, z)}{\partial x} \right)^{-1}_{ji} \frac{\partial y_i(x, z)}{\partial z_n} \quad (\text{D.98})$$

which expresses the partials of x_j with respect to the z_n in terms of the partials of the

original functions $y_i(x, z)$.

D.26 Implicit Function Theorem

It often happens that a set of functions $y_j = y_j(x_1, x_2, \dots, x_P)$, for $j = 1, \dots, N$, is not given directly, but rather in *implicit* form. One defines other functions $f_i(x, y)$, for $i = 1, \dots, N$, and requires all x, y values to be those that will make these f_i identically zero,

$$\begin{aligned} 0 &= f_1(x_1, x_2, \dots, x_P, y_1, y_2, \dots, y_N) \\ &\vdots \qquad \qquad \qquad \vdots \\ 0 &= f_N(x_1, x_2, \dots, x_P, y_1, y_2, \dots, y_N) \end{aligned} \tag{D.99}$$

The following theorem gives the conditions under which such identities actually specify the implicit functions y_j .

Theorem D.26.1: Implicit Function Theorem

Assume that $f_i(x_1, x_2, \dots, x_P, y_1, y_2, \dots, y_N)$ for $i = 1, \dots, N$, are continuously differentiable functions. If the Jacobian determinant

$$\frac{\partial (f_1, f_2, \dots, f_N)}{\partial (y_1, y_2, \dots, y_N)} = \left| \frac{\partial f(x, y)}{\partial y} \right| \tag{D.100}$$

is nonzero at point x, y , then there is an open neighborhood N_{xy} of the point x, y in which the identities

$$0 = f_i(x_1, x_2, \dots, x_P, y_1, y_2, \dots, y_N) \tag{D.101}$$

for $i = 1, \dots, N$, can be solved for the implicit functions $y_j = y_j(x_1, x_2, \dots, x_P)$ for $j = 1, \dots, N$. These functions will be unique and continuously differentiable in the open neighborhood.

Proof: Apply the local inverse function theorem, Theorem D.24.1, to solve $f_i = f_i(x_1, x_2, \dots, x_P, y_1, y_2, \dots, y_N)$ for $y_j = y_j(f_1, \dots, f_N, x_1, \dots, x_P)$. Then apply the identity to set $f_i = 0$ for each i , and so obtain

$$y_j(x_1, x_2, \dots, x_P) = y_j(0, 0, \dots, 0, x_1, x_2, \dots, x_P) \tag{D.102}$$

which are the desired functions. □

D.27 Derivatives of Implicit Functions

Applying eqn (D.98) of Section D.25 with the replacements $y \rightarrow f, x \rightarrow y, z \rightarrow x$, and f then set equal to zero, gives the partial derivatives of the implicit functions in terms of the partial derivatives of the f_i . Thus, for all $j = 1, \dots, N$ and $n = 1, \dots, P$,

$$\frac{\partial y_j(x)}{\partial x_n} = - \sum_{i=1}^N \left(\frac{\partial f(x, y)}{\partial y} \right)^{-1}_{ji} \frac{\partial f_i(x, y)}{\partial x_n} \tag{D.103}$$

D.28 Functional Independence

Consider the M continuously differentiable functions of N variables $f_k(x_1, x_2, \dots, x_N)$ for $k = 1, \dots, M$. These functions are *functionally dependent* and have a *dependency relation* at point x if there is a continuously differentiable function F with at least one nonzero partial derivative $\partial F(f)/\partial f_i \neq 0$ for which

$$0 = F(f_1, f_2, \dots, f_M) \quad (\text{D.104})$$

holds identically in an open neighborhood N_x of point x .

If the functions have no dependency relation at point x , they are said to be *functionally independent* at that point. We now give a condition for a set of functions f_k to be functionally independent.

Theorem D.28.1: Condition for Functional Independence

Consider M continuously differentiable functions of N variables, $f_k(x_1, x_2, \dots, x_N)$ for $k = 1, \dots, M$. Let the $M \times N$ matrix $(\partial f(x)/\partial x)$ be defined by its matrix elements

$$\left(\frac{\partial f(x)}{\partial x} \right)_{ki} = \frac{\partial f_k(x_1, x_2, \dots, x_N)}{\partial x_i} \quad (\text{D.105})$$

If, for some x , the rank r of this matrix is $r = M$ (which is possible only if $M \leq N$ since r cannot be greater than N), then the functions f_k are functionally independent at x .

Proof: We show that the existence of a dependency relation implies that $r < M$, and therefore that $r = M$ implies functional independence. Differentiating an assumed dependency relation eqn (D.104) with respect to x_i using the chain rule gives

$$0 = \sum_{k=1}^M \frac{\partial F(f)}{\partial f_k} \frac{\partial f_k(x)}{\partial x_i} \quad (\text{D.106})$$

Defining an $M \times 1$ column vector $[\partial F(f)/\partial f]$ by

$$\left[\frac{\partial F(f)}{\partial f} \right] = \left(\frac{\partial F(f)}{\partial f_1} \quad \frac{\partial F(f)}{\partial f_2} \quad \dots \quad \frac{\partial F(f)}{\partial f_M} \right)^T \quad (\text{D.107})$$

we may rewrite eqn (D.106) as the matrix equation

$$0 = \left(\frac{\partial f(x)}{\partial x} \right)^T \left[\frac{\partial F(f)}{\partial f} \right] \quad (\text{D.108})$$

By assumption the column vector $[\partial F(f)/\partial f]$ will have at least one nonzero element. Thus Corollary B.19.2 requires that the matrix in eqn (D.108), and hence its transpose $(\partial f(x)/\partial x)$, must have rank less than M . Since the existence of a dependency relation implies $r < M$, it follows that $r = M$ at point x implies the non-existence of a dependency relation, as was to be proved. \square

D.29 Dependency Relations

When functions are functionally dependent, there may be one or more dependency relations among them. It is often important to know how many dependency relations there are, as specified in the following theorem.

Theorem D.29.1: Dependency Relations

Consider M continuously differentiable functions,

$$f_k(x_1, x_2, \dots, x_N) \tag{D.109}$$

of N variables $x = x_1, x_2, \dots, x_N$. Consider again the $M \times N$ matrix $(\partial f(x)/\partial x)$ defined by its matrix elements in eqn (D.105). If at the point x the rank r of this matrix is $r < M$, then there are $M - r$ functionally independent dependency relations among the f_k which hold in an open neighborhood N_x of point x ,

$$\begin{aligned} 0 &= F^{(1)}(f_1, \dots, f_M) \\ &\vdots \\ 0 &= F^{(M-r)}(f_1, \dots, f_M) \end{aligned} \tag{D.110}$$

For a proof of this theorem, see Volume I, Appendix 14 of Desloges (1982).

D.30 Legendre Transformations

We often have a function like $f(x_1, \dots, x_M, y_1, \dots, y_N) = f(x, y)$ that is important mainly because its partial derivatives yield desired functions $u_i = u_i(x, y)$ and $w_j = w_j(x, y)$, as in

$$\frac{\partial f(x, y)}{\partial x_i} = u_i(x, y) \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y_j} = w_j(x, y) \tag{D.111}$$

for $i = 1, \dots, M$ and $j = 1, \dots, N$.

It is sometimes useful to have a different, but related, function

$$g(x_1, \dots, x_M, w_1, \dots, w_N) = g(x, w) \tag{D.112}$$

whose partial derivatives are

$$\frac{\partial g(x, w)}{\partial x_i} = -u_i(x, w) \quad \text{and} \quad \frac{\partial g(x, w)}{\partial w_j} = y_j(x, w) \tag{D.113}$$

again for $i = 1, \dots, M$ and $j = 1, \dots, N$. Notice that the roles of y_j and w_j are interchanged in the last partial derivatives in eqns (D.111, D.113), while the partials with respect to x_i yield the same function u_i , except for the minus sign and its expression in the new x, w variable set. This transformation from f to g is called a *Legendre transformation*.

The Legendre transformation is effected by defining g to be

$$g(x, y) = \sum_{j=1}^N y_j \frac{\partial f(x, y)}{\partial y_j} - f(x, y) = \sum_{j=1}^N y_j w_j(x, y) - f(x, y) \quad (\text{D.114})$$

But the g in eqn (D.114) is not yet expressed in the correct set of variables x, w . In order to complete the Legendre transformation, we must prove that the functions $w_j = w_j(x, y)$ defined in the second of eqn (D.111) can be inverted to give $y_j = y_j(x, w)$. By Section D.24, the condition for this inversion to be possible is that

$$\left| \frac{\partial w(x, y)}{\partial y} \right| \neq 0 \quad (\text{D.115})$$

where the matrix $(\partial w(x, y)/\partial y)$ is defined by

$$\left(\frac{\partial w(x, y)}{\partial y} \right)_{jk} = \frac{\partial w_j(x, y)}{\partial y_k} = \frac{\partial^2 f(x, y)}{\partial y_k \partial y_j} \quad (\text{D.116})$$

In general, eqn (D.115) must be proved in each individual case to which the Legendre transformation is to be applied. Assume now that this has been done.

The inverse function $y_j = y_j(x, w)$ then allows us to write g as a compound function of the variable set x, w as

$$g(x, w) = g(x, y(x, w)) = \sum_{j=1}^N y_j(x, w) w_j - f(x, y(x, w)) \quad (\text{D.117})$$

Now consider the differential of function g . Equation (D.114) gives

$$\begin{aligned} dg &= \sum_{j=1}^N (y_j dw_j + w_j dy_j) - df \\ &= \sum_{j=1}^N \left(y_j dw_j + w_j dy_j - \frac{\partial f(x, y)}{\partial y_j} dy_j \right) - \sum_{i=1}^M \frac{\partial f(x, y)}{\partial x_i} dx_i \\ &= \sum_{j=1}^N y_j dw_j - \sum_{i=1}^M u_i dx_i \end{aligned} \quad (\text{D.118})$$

where the second of eqn (D.111) has been used to cancel the dy_j terms, and the first of eqn (D.111) has been used to get the u_i . We know from eqn (D.117) that $g(x, w)$ exists and is well defined, and hence has the differential

$$dg = \sum_{j=1}^N \frac{\partial g(x, w)}{\partial w_j} dw_j + \sum_{i=1}^M \frac{\partial g(x, w)}{\partial x_i} dx_i \quad (\text{D.119})$$

Equations (D.118, D.119) are two expressions for the same differential dg and hence equal to each other. Since $|\partial w(x, y)/\partial y| \neq 0$ by assumption, Theorem D.18.5

shows that the differentials dw, dx may be considered as arbitrary and independent. Thus the equality of eqns (D.118, D.119) implies the equality of each of the coefficients of the differentials dw_j and dx_i in the two equations. Thus eqn (D.113) holds, as desired.

D.31 Homogeneous Functions

Let $f(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) = f(x, z)$ be a continuously differentiable function of the stated variables, defined over a region R . Function $f(x, z)$ is *homogeneous of degree k in the set of variables x_1, x_2, \dots, x_N* if and only if, for some k and any positive number $\lambda > 0$,

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_N, z_1, z_2, \dots, z_P) = \lambda^k f(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) \quad (\text{D.120})$$

An alternate, and equivalent, definition of homogeneous functions is given in the following theorem, whose proof can be found in Chapter II of Courant (1936b).

Theorem D.31.1: Euler Condition

Function $f(x, z)$ is homogeneous of degree k in the set of variables x_1, x_2, \dots, x_N as defined in eqn (D.120) if and only if

$$\sum_{i=1}^N x_i \frac{\partial f(x, z)}{\partial x_i} = k f(x, z) \quad (\text{D.121})$$

D.32 Derivatives of Homogeneous Functions

In Lagrangian mechanics, it is important also to consider the homogeneity of the partial derivatives of homogeneous functions.

Theorem D.32.1: Derivatives of Homogeneous Functions

Function f is homogeneous of degree $k + 1$ in the set of variables x_1, x_2, \dots, x_N , that is

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_N, z_1, z_2, \dots, z_P) = \lambda^{(k+1)} f(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) \quad (\text{D.122})$$

if and only if all partial derivatives $\partial f(x, z)/\partial x_i = g_i(x, z)$ for $i = 1, \dots, N$ are homogeneous of degree k , that is

$$g_i(\lambda x_1, \lambda x_2, \dots, \lambda x_N, z_1, z_2, \dots, z_P) = \lambda^k g_i(x_1, x_2, \dots, x_N, z_1, z_2, \dots, z_P) \quad (\text{D.123})$$

Proof: First, we assume eqn (D.122) and prove eqn (D.123). Taking partial derivatives, and using eqn (D.122) gives

$$\begin{aligned} g_i(\lambda x, z) &= \frac{\partial f(\lambda x, z)}{\partial (\lambda x_i)} = \frac{1}{\lambda} \frac{\partial}{\partial x_i} f(\lambda x, z) \\ &= \frac{1}{\lambda} \frac{\partial}{\partial x_i} \left(\lambda^{(k+1)} f(x, z) \right) = \lambda^k \frac{\partial f(x, z)}{\partial x_i} = \lambda^k g_i(x, z) \end{aligned} \quad (\text{D.124})$$

which establishes eqn (D.123).

Conversely, assume eqn (D.123) and prove eqn (D.122). Equation (D.123) can be written

$$\frac{\partial f(\lambda x, z)}{\partial (\lambda x_i)} = \lambda^k \frac{\partial f(x, z)}{\partial x_i} \quad (\text{D.125})$$

Hence

$$\frac{1}{\lambda} \frac{\partial}{\partial x_i} f(\lambda x, z) = \frac{\partial}{\partial x_i} \lambda^k f(x, z) \quad (\text{D.126})$$

and so

$$\frac{\partial}{\partial x_i} \left(f(\lambda x, z) - \lambda^{(k+1)} f(x, z) \right) = 0 \quad (\text{D.127})$$

It follows from Corollary D.10.2 that $f(\lambda x, z) - \lambda^{(k+1)} f(x, z) = C$. Setting $\lambda = 1$ proves that the constant $C = 0$, which is equivalent to eqn (D.122). \square

D.33 Stationary Points

For functions of one variable $f = f(x)$, maxima, minima, and points of inflection (collectively called *stationary points*) are those points at which $df(x)/dx = 0$.

For functions of many variables, stationary points may be defined similarly as points at which, for all $k = 1, \dots, N$,

$$\frac{\partial f(x_1, x_2, \dots, x_N)}{\partial x_k} = 0 \quad (\text{D.128})$$

Since the variables x and hence their differentials dx_k for $k = 1, \dots, N$ are assumed independent, Lemma D.18.4 implies that this definition may be stated more simply, and equivalently, as

$$df = 0 \quad (\text{D.129})$$

Since we know that this differential approximates the difference Δf when the differentials dx_j are small, it is also valid to say that the extremum is a point such that f is constant to first order for small excursions from it in any direction.

In one sense, the many variable case is much more complex than the single variable one. A function of many variables may have maxima in some variables, minima in others, etc. We avoid this complexity here by assuming that it will be clear from context in most practical examples whether the stationary point is a maximum, minimum, or some more complicated mixture of the two.

D.34 Lagrange Multipliers

Often, we want to find the stationary points of a function $f(x)$ given certain constraints on the allowed values of x_1, x_2, \dots, x_N and their differentials. These constraints are expressed by defining a functionally independent set of functions G_a and setting them equal to zero, $0 = G_a(x_1, x_2, \dots, x_N)$ for $a = 1, \dots, C$.

For example, with

$$f(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (\text{D.130})$$

defined to be the distance from the origin in a three-dimensional Cartesian space, eqn (D.128) gives a stationary point at $(0, 0, 0)$, obviously a minimum. But suppose that

we want the stationary points of f subject to the constraint that x lie on a plane at distance Λ from the origin, which can be expressed by

$$0 = G_1(x) = \alpha x_1 + \beta x_2 + \gamma x_3 - \Lambda \quad (\text{D.131})$$

where constants α, β, γ obey $\alpha^2 + \beta^2 + \gamma^2 = 1$. The Lagrange multiplier theorem gives an elegant method for solving such problems.

Theorem D.34.1: Lagrange Multiplier Theorem

Let the values of the independent variables x_1, x_2, \dots, x_N in some open region R be constrained by equations of the form

$$0 = G_a(x_1, x_2, \dots, x_N) \quad (\text{D.132})$$

for $a = 1, \dots, C$, where the G_a are continuously differentiable functions. Assume that the $C \times N$ Jacobian matrix \mathbf{g} defined by

$$g_{ak} = \left(\frac{\partial G_a(x)}{\partial x} \right)_{ak} = \frac{\partial G_a(x_1, x_2, \dots, x_N)}{\partial x_k} \quad (\text{D.133})$$

has rank C so that the functions G_a are functionally independent and represent C independent constraints.

A continuously differentiable function $f(x_1, x_2, \dots, x_N)$ has a stationary point at x , subject to these constraints, if and only if there exist Lagrange multipliers $\lambda_a = \lambda_a(x)$ such that, at the stationary point,

$$\frac{\partial f(x)}{\partial x_k} = \sum_{a=1}^C \lambda_a g_{ak} \quad (\text{D.134})$$

for all $k = 1, \dots, N$.

Proof: The necessary and sufficient condition for a stationary point is that $df = 0$ subject to the constraints on the possible dx_k values given by the vanishing of the differentials of eqn (D.132)

$$0 = dG_a = \sum_{k=1}^N g_{ak} dx_k \quad (\text{D.135})$$

for all $a = 1, \dots, C$. Since the matrix \mathbf{g} defined in eqn (D.133) has rank C , it must have an C -rowed critical minor. The variables x can be relabeled in any way, so we lose no generality by assuming that this critical minor is the determinant containing the C rows and the last C columns. Call the corresponding matrix $\mathbf{g}^{(b)}$ so that $g_{aj}^{(b)} = g_{a(N-C+j)}$ for $a, j = 1, \dots, C$. By construction, the determinant of this matrix is a critical minor, and so $|\mathbf{g}^{(b)}| \neq 0$. In the following, we will also use the shorthand notations $x^{(f)}$ for what will be called the *free* variables x_1, \dots, x_{N-C} and $x^{(b)}$ for the *bound* variables x_{N-C+1}, \dots, x_N . And x will continue to denote the full set of variables x_1, x_2, \dots, x_N .

Using these notations, eqn (D.135) may be written with separate sums over the free and bound variables

$$0 = dG_a = \sum_{i=1}^{N-C} g_{ai} dx_i^{(f)} + \sum_{j=1}^C g_{aj}^{(b)} dx_{(N-C+j)}^{(b)} \quad (\text{D.136})$$

Since $\mathbf{g}^{(b)}$ is nonsingular, it has an inverse $\mathbf{g}^{(b)-1}$, which may be used with eqn (D.136) to write the bound differentials $dx^{(b)}$ in terms of the free ones.

$$dx_{(N-C+j)}^{(b)} = - \sum_{a=1}^C \sum_{i=1}^{N-C} g_{ja}^{(b)-1} g_{ai} dx_i^{(f)} \quad (\text{D.137})$$

The differential df can also be written with separate sums over the free and bound variables

$$df = \sum_{k=1}^N \frac{\partial f(x)}{\partial x_k} dx_k = \sum_{i=1}^{N-C} \frac{\partial f(x)}{\partial x_i} dx_i^{(f)} + \sum_{j=1}^C \frac{\partial f(x)}{\partial x_{(N-C+j)}} dx_{(N-C+j)}^{(b)} \quad (\text{D.138})$$

Substituting eqn (D.137) into this expression gives

$$\begin{aligned} df &= \sum_{i=1}^{N-C} \frac{\partial f(x)}{\partial x_i} dx_i^{(f)} - \sum_{j=1}^C \frac{\partial f(x)}{\partial x_{(N-C+j)}} \sum_{a=1}^C \sum_{i=1}^{N-C} g_{ja}^{(b)-1} g_{ai} dx_i^{(f)} \\ &= \sum_{i=1}^{N-C} \left(\frac{\partial f(x)}{\partial x_i} - \sum_{a=1}^C \lambda_a g_{ai} \right) dx_i^{(f)} \end{aligned} \quad (\text{D.139})$$

where the λ_a in the last expression are defined, for $a = 1, \dots, C$, as

$$\lambda_a = \sum_{j=1}^C \frac{\partial f(x)}{\partial x_{(N-C+j)}} g_{ja}^{(b)-1} \quad (\text{D.140})$$

The solution, eqn (D.137), for the bound differentials $dx^{(b)}$ reduces eqn (D.135) to an identity. Hence no constraint is placed on the free differentials $dx^{(f)}$. Setting these free differentials nonzero one at a time in eqn (D.139), the condition $df = 0$ for x to be a stationary point implies and is implied by

$$\frac{\partial f(x)}{\partial x_i} - \sum_{a=1}^C \lambda_a g_{ai} = 0 \quad (\text{D.141})$$

for all $i = 1, \dots, (N - C)$, which establishes the theorem for those values of the index.

For indices in the range $(N - C + 1), \dots, N$, with the same choice of λ_a as defined in eqn (D.140), the eqn (D.134) is satisfied identically at all points, including

the stationary one. To demonstrate this, let $j = 1, \dots, C$ and use eqn (D.140) to write

$$\begin{aligned} \frac{\partial f(x)}{\partial x_{(N-C+j)}} - \sum_{a=1}^C \lambda_a g_{a(N-C+j)} &= \frac{\partial f(x)}{\partial x_{(N-C+j)}} - \sum_{a=1}^C \sum_{l=1}^C \frac{\partial f(x)}{\partial x_{(N-C+l)}} g_{la}^{(b)-1} g_{a(N-C+j)} \\ &= \frac{\partial f(x)}{\partial x_{(N-C+j)}} - \sum_{l=1}^C \frac{\partial f(x)}{\partial x_{(N-C+l)}} \delta_{lj} = \frac{\partial f(x)}{\partial x_{(N-C+j)}} - \frac{\partial f(x)}{\partial x_{(N-C+j)}} = 0 \end{aligned} \tag{D.142}$$

Thus eqn (D.134) holds for all index values if and only if x is a stationary point, as was to be proved. \square

Note that the N equations of eqn (D.134) together with the C equations of eqn (D.132), are $N + C$ equations in the $N + C$ unknowns $x_1, x_2, \dots, x_N, \lambda_1, \dots, \lambda_C$, and so can be solved to find the stationary points.

D.35 Geometry of the Lagrange Multiplier Theorem

When applied to the simple example in eqns (D.130, D.131) of Section D.34, the three equations of the Lagrange multiplier condition eqn (D.134) can be written as a single vector equation, the equality of two gradient vectors,

$$\nabla f = \lambda_1 \nabla G_1 \tag{D.143}$$

Since, as described in Section D.37, vector $\nabla G_1 = \alpha \hat{e}_1 + \beta \hat{e}_2 + \gamma \hat{e}_3$ is perpendicular to the surface of constraint $G_1 = 0$, eqn (D.143) says that ∇f must be in the same or opposite direction, and so also perpendicular to that constraint surface.

If we denote by $d\mathbf{r} = dx_1 \hat{e}_1 + dx_2 \hat{e}_2 + dx_3 \hat{e}_3$ the differential displacement vector whose Cartesian components are the differentials dx_i , the chain rule gives

$$df = d\mathbf{r} \cdot \nabla f \quad \text{and} \quad dG_1 = d\mathbf{r} \cdot \nabla G_1 \tag{D.144}$$

The constraint $G_1 = 0$, and hence $dG_1 = 0$, constrains the vector $d\mathbf{r}$ to have a zero dot product with ∇G_1 , and so to be perpendicular to the perpendicular to the surface of constraint. Thus $d\mathbf{r}$ must lie *in* the surface of constraint.

If there were no constraint, the condition for f to have a stationary point would simply be that

$$0 = df = d\mathbf{r} \cdot \nabla f \tag{D.145}$$

Since without a constraint the displacement $d\mathbf{r}$ can take any direction, eqn (D.145) implies that $\nabla f = 0$, which is equivalent to the three equations of eqn (D.128).

However, with the constraint, eqn (D.145) does not imply that $\nabla f = 0$, but only that ∇f must have no components in possible $d\mathbf{r}$ directions. Since $d\mathbf{r}$ can lie anywhere

in the surface of constraint, this means that ∇f must be perpendicular to that surface, in other words that ∇f must be parallel or anti-parallel to ∇G_1 , as eqn (D.143) states.

If another constraint were added, then eqn (D.143) would become

$$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2 \quad (\text{D.146})$$

Adding the second constraint further restricts the possible directions of $d\mathbf{r}$ and hence *increases* the possible directions that ∇f can have while still maintaining the condition $d\mathbf{r} \cdot \nabla f = 0$.

D.36 Coupled Differential Equations

A basic theorem of one-variable calculus is that a first-order differential equation $dx/d\beta = f(\beta, x)$, with the initial condition that $x = b$ when the independent variable is $\beta = \beta_0$, has a unique solution $x = x(\beta, b)$. That result is generalized to a set of N functions of β by the following theorem.

Theorem D.36.1: Coupled Differential Equations

Consider a set of N unknown functions x_i for $i = 1, \dots, N$ obeying the coupled, first-order differential equations

$$\begin{aligned} \frac{dx_1}{d\beta} &= f_1(\beta, x_1, \dots, x_N) \\ &\vdots \\ \frac{dx_N}{d\beta} &= f_N(\beta, x_1, \dots, x_N) \end{aligned} \quad (\text{D.147})$$

and the initial conditions that $x_i = b_i$ when the independent variable is $\beta = \beta_0$ (where b_i for $i = 1, \dots, N$ are arbitrarily chosen constants). Assume that the functions f_i are continuously differentiable. These equations have a unique solution depending on β and the set of constants b_1, \dots, b_N . The solution is the set of equations

$$x_i = x_i(\beta, b_1, \dots, b_N) \quad (\text{D.148})$$

for $i = 1, \dots, N$. The same solution can also be written in implicit form by solving eqn (D.148) for the b and writing

$$b_i = \phi_i(\beta, x_1, \dots, x_N) \quad \text{for} \quad i = 1, \dots, N \quad (\text{D.149})$$

where the ϕ_i are functionally independent, and are called integrals of the set of differential equations.

The proof of this standard theorem can be found, for example, in Chapter 6 of Ford (1955). Note that the initial value of the independent variable β_0 is *not* included in the list of constants upon which the x_i depend. It is simply the value of β at which the integration constants b_i are specified.

The special case in which the functions f_i in eqn (D.147) do not depend explicitly on β is of great importance in analytical mechanics.

Theorem D.36.2: Sets of Equations Without the Independent Variable

Consider the set of differential equations which are the same as eqn (D.147) but with the independent variable β not appearing explicitly in the functions f_i

$$\frac{dx_i}{d\beta} = f_i(x_1, \dots, x_N) \quad \text{for } i = 1, \dots, N \tag{D.150}$$

for $i = 1, \dots, N$. Assume that $N \geq 2$ and that there is some index l for which $f_l \neq 0$ in some region of interest. For simplicity, but without loss of generality, assume the variables relabeled so that $l = 1$. Given the initial conditions that, for $j = 2, \dots, N$, $x_j = b_j$ when $x_1 = b_1$, these equations have a unique solution

$$x_j = x_j(x_1, b_2, \dots, b_N) \tag{D.151}$$

for $j = 2, \dots, N$, in which the role of independent variable has been assumed by x_1 and the number of arbitrary integration constants has been reduced by one.

This solution also can be written in implicit form by solving eqn (D.151) for the b and writing

$$b_j = \phi_j(x_1, \dots, x_N) \tag{D.152}$$

for $j = 2, \dots, N$. The ϕ_j are the functionally independent integrals of eqn (D.150). There are only $N - 1$ integrals ϕ_j , and they do not depend explicitly on β .

Proof: Assuming $f_1 \neq 0$, divide the last $N - 1$ of eqn (D.150) by the first one, giving

$$\frac{dx_j}{dx_1} = \frac{dx_j/d\beta}{dx_1/d\beta} = \frac{f_j(x_1, \dots, x_N)}{f_1(x_1, \dots, x_N)} = A_j(x_1, \dots, x_N) \tag{D.153}$$

for $j = 2, \dots, N$. The functions A_j defined in eqn (D.153) are continuously differentiable, and hence Theorem D.36.1 can be applied with $f \rightarrow A$, the replacement $\beta \rightarrow x_1$ for the independent variable, and the range of unknown functions now running from x_2 to x_N . The theorem follows immediately. \square

Note that the initial value $x_1 = b_1$ for the new independent variable is *not* included among the arbitrary integration constants b_2, \dots, b_N . As in the original theorem with β_0 , it is simply the value of the new independent variable x_1 at which the arbitrary integration constants $x_j = b_j$, for $j = 2, \dots, N$, are specified.

The variable β has disappeared from the solutions, eqns (D.151, D.152). Instead of having all the x_i as functions of some β , we have solutions with all of the x_j for $j \neq 1$ given as functions of one of them, the x_1 .

But sometimes, even under the conditions of Theorem D.36.2, it is convenient to have a solution with all of the unknowns written as functions of β . Such a *parametric* expression can always be found, as is shown in the following Corollary.

Corollary D.36.3: Recovery of the Parameter

Assume that the conditions in Theorem D.36.2 hold. Using solutions eqn (D.151), a set of functions

$$x_i = x_i((\beta - \beta_0), b_1, \dots, b_N) \tag{D.154}$$

for $i = 1, \dots, N$, can always be found that are solutions of the original equations, eqn (D.150). They depend only on the difference $(\beta - \beta_0)$ as indicated. Thus no generality is

lost by simply taking $\beta_0 = 0$, as is frequently done. The solutions in eqn (D.154) can be constructed to obey the initial conditions that $\beta = \beta_0$ implies $x_i = b_i$ for all $i = 1, \dots, N$.

Proof: The first of eqn (D.150) is

$$\frac{dx_1}{d\beta} = f_1(x_1, x_2, \dots, x_D) = f_1(x_1, x_2(x_1, b_2, \dots, b_N), \dots, x_N(x_1, b_2, \dots, b_N)) \quad (\text{D.155})$$

where eqn (D.151) has been used to get the last equality. Thus we have a differential equation with one unknown function x_1

$$d\beta = \frac{dx_1}{f_1(x_1, x_2(x_1, b_2, \dots, b_N), \dots, x_N(x_1, b_2, \dots, b_N))} \quad (\text{D.156})$$

Integrating, with x_1 assigned the value b_1 at $\beta = \beta_0$,

$$\begin{aligned} \beta - \beta_0 &= \int_{b_1}^{x_1} \frac{dx_1}{f_1(x_1, x_2(x_1, b_2, \dots, b_N), \dots, x_N(x_1, b_2, \dots, b_N))} \\ &= F(x_1, b_2, \dots, b_N) - F(b_1, b_2, \dots, b_N) \end{aligned} \quad (\text{D.157})$$

Denoting by F^{-1} the inverse to function F for fixed values of the constants b then gives

$$x_1 = F^{-1}(F(b_1, b_2, \dots, b_N) + \beta - \beta_0) = x_1((\beta - \beta_0), b_1, \dots, b_N) \quad (\text{D.158})$$

which has, by construction, the value $x_1 = b_1$ when $\beta = \beta_0$. Substituting this x_1 into the solutions eqn (D.151) then gives

$$x_j = x_j(x_1((\beta - \beta_0), b_1, \dots, b_N), b_2, \dots, b_N) = x_j((\beta - \beta_0), b_1, \dots, b_N) \quad (\text{D.159})$$

for $j = 2, \dots, N$. Since the solutions eqn (D.151) have the property that $x_1 = b_1$ implies $x_j = b_j$, the x_j defined in eqn (D.159) must take those same values b_j when $\beta = \beta_0$. Equations (D.158, D.159) are the desired eqn (D.154) and the corollary is proved. \square

A note of caution: Even though eqn (D.154) does write the solution to eqn (D.150) in a form that appears to have N arbitrary integration constants b_1, \dots, b_N , the b_1 is not actually an integration constant. It is the initial value of the independent variable x_1 in solution eqn (D.151).

D.37 Surfaces and Envelopes

A two-dimensional surface in three dimensions may be written either as

$$z = \phi(x, y) \quad \text{or} \quad F(x, y, z) = 0 \quad (\text{D.160})$$

Any surface in the first form can be written in the second form as $F(x, y, z) = \phi(x, y) - z = 0$. The second form is preferable since it often leads to simpler expressions. For example, writing the unit sphere as $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ avoids

the need to use different signs of a square root for the upper and lower hemispheres as the first form would require.

The differential change in F resulting from differential displacement $d\mathbf{r}$ is given by the chain rule as $dF = d\mathbf{r} \cdot \nabla F$, which is maximum when $d\mathbf{r} \parallel \nabla F$ and zero when $d\mathbf{r} \perp \nabla F$. Since a small displacement along the surface keeps $F=0$ and so must have $dF = 0$, it follows that vector ∇F is perpendicular to the surface at point x, y, z . The tangent plane touching the surface at \mathbf{r} thus consists of the points \mathbf{r}' where

$$(\mathbf{r}' - \mathbf{r}) \cdot \nabla F = 0 \quad (\text{D.161})$$

and ∇F is normal to the tangent plane.

Consider now what is called a *one-parameter family* of surfaces, defined by

$$F(x, y, z, a) = 0 \quad (\text{D.162})$$

where each different value of parameter a in general gives a different surface. Often (but not always) the two surfaces with parameter values $a + da$ and $a - da$ in the limit $da \rightarrow 0$ will intersect in a curved line called a *curve of intersection*. This is also called a *characteristic curve* by some authors, but we use the term “curve of intersection” from Courant and Hilbert (1962) to avoid confusion with other curves in the theory of partial differential equations that are also called characteristic curves.

The curve of intersection lies in both surfaces and hence obeys both $F(x, y, z, a + da) = 0$ and $F(x, y, z, a - da) = 0$. But these two equations are satisfied if and only if both

$$F(x, y, z, a+da) + F(x, y, z, a-da) = 0 \quad \text{and} \quad F(x, y, z, a+da) - F(x, y, z, a-da) = 0 \quad (\text{D.163})$$

Taking the limit $da \rightarrow 0$, the curve of intersection can therefore be defined as the solution of the two equations

$$F(x, y, z, a) = 0 \quad \text{and} \quad \frac{\partial F(x, y, z, a)}{\partial a} = 0 \quad (\text{D.164})$$

It is the intersection of the surfaces $F = 0$ and $G = 0$ where

$$G(x, y, z, a) = \frac{\partial F(x, y, z, a)}{\partial a} \quad (\text{D.165})$$

As an example, consider the family of spheres $F = x^2 + y^2 + z^2 - a^2 = 0$. Since the surfaces in this family are concentric spheres, the surfaces with $a + da$ and $a - da$ never intersect and so there is no curve of intersection.

A better example for our purposes is the family of unit spheres with center at point a on the z -axis, $F = x^2 + y^2 + (z - a)^2 - 1 = 0$. Then $\partial F / \partial a = 2(z - a) = 0$ shows that the curve of intersection for a given a is the intersection of the plane $z=a$ with the sphere $x^2 + y^2 + (z - a)^2 - 1 = 0$. It is a unit circle lying in a plane parallel to the x - y plane and at height $z=a$. This circle is also the equator of the sphere for that a value.

A suitable one-parameter family of surfaces has a curve of intersection for every value of a . The *envelope* of the one-parameter family is a surface defined as the set of all points of all possible curves of intersection. It may be thought of as the surface swept out by the curve of intersection as a is varied. It is found by solving the second of eqn (D.164) for a as a function of x, y, z and substituting this $a(x, y, z)$ into the first of eqn (D.164), thus eliminating a between the two equations. For a given value of a , the surface $F = 0$ and the envelope are in contact along the curve of intersection defined by that a value.

In the above example, the envelope is got by using $\partial F/\partial a = 2(z - a) = 0$ to get $a(x, y, z) = z$ and then substituting that a into $F = x^2 + y^2 + (z - a)^2 - 1 = 0$ to obtain $x^2 + y^2 - 1 = 0$. The envelope is thus a right circular cylinder of unit radius with its symmetry line along the z -axis. This envelope is the surface swept out by all of the curves of intersections (circles of unit radius at height a) as a is varied. For any value of a , the sphere is in contact with the envelope along the equator of the sphere. The sphere thus contacts the envelope along the curve of intersection for that a value.

APPENDIX E

GEOMETRY OF PHASE SPACE

In Section 18.9, we asserted without proof that a binary number α can always be found that will make the matrix $(\partial Y/\partial p)$ defined in eqn (18.64) nonsingular. Proof of that fact requires an excursion into the geometry of phase space.

In the present reference chapter we define general abstract linear vector spaces and state some of their important properties. We then apply the general theory to define a vector space in phase space, and prove some theorems of importance for the theory of canonical transformations.

E.1 Abstract Vector Space

A *linear vector space* is a set of objects called *vectors* that obey the following axioms, abstracted from the properties of the three-dimensional Cartesian displacement vectors defined in Appendix A. In the following, we will denote vectors in an abstract vector space by the same bold, serif typeface as has been used for the special case of Cartesian threevectors.

E.1.1 Axioms

1. Closure under addition and scalar multiplication: If \mathbf{v} and \mathbf{w} are any vectors, then addition is defined by some rule such that $\mathbf{v} + \mathbf{w}$ is also a vector. If α is any scalar (assumed here to be real numbers) and \mathbf{v} any vector, then scalar multiplication is defined by some rule such that $\alpha\mathbf{v}$ and $\mathbf{v}\alpha$ are also vectors.
2. Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ and $\alpha\mathbf{v} = \mathbf{v}\alpha$.
3. Associativity of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. Associativity of scalar multiplication: $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$.
5. Existence of additive identity: There is a vector $\mathbf{0}$ such that every vector \mathbf{v} satisfies the equation $\mathbf{v} + \mathbf{0} = \mathbf{v}$. Vector $\mathbf{0}$ is also referred to as the *null vector*.
6. Existence of additive inverse: For every vector \mathbf{v} , there is a vector denoted $-\mathbf{v}$ that is a solution of the equation $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
7. Linearity in scalars: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.
8. Linearity in vectors: $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$.
9. Multiplication by one: $1\mathbf{v} = \mathbf{v}$.

E.1.2 Derived properties

The axioms¹³⁰ can be used to prove the following additional properties of linear vector spaces:

¹³⁰See Chapter 2 of Mirsky (1961) or Chapter 7 of Birkhoff and MacLane (1977). Different texts differ as to which are the axioms and which are the derived properties. However, there is general agreement that linear vector spaces do satisfy the whole list of axioms and derived properties given here.

1. The additive identity is unique: If $\mathbf{v} + \mathbf{u} = \mathbf{v}$ for all vectors \mathbf{v} , then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse is unique: If $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. Law of cancellation: $\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w}$ implies $\mathbf{u} = \mathbf{w}$.
4. Unique solution: If \mathbf{u} satisfies the equation $\mathbf{v} + \mathbf{u} = \mathbf{w}$ then $\mathbf{u} = \mathbf{w} + (-\mathbf{v})$.
5. Multiplication of a vector by scalar zero: $0\mathbf{v} = \mathbf{0}$.
6. Multiplication of additive identity by a scalar: $\alpha\mathbf{0} = \mathbf{0}$.
7. Multiplication by a negative number: $(-1)\mathbf{v} = -\mathbf{v}$.
8. Inverse of the inverse: $-(-\mathbf{v}) = \mathbf{v}$.
9. Simplified notations: Since the above axioms and additional properties imply that $\alpha\mathbf{v} + \beta(-\mathbf{w}) = \alpha\mathbf{v} + (-\beta)\mathbf{w}$, we need not be so careful about placement of the minus sign. Both of these equal expressions are commonly denoted simply as $\alpha\mathbf{v} - \beta\mathbf{w}$. Thus the solution in derived property 4 is written $\mathbf{u} = \mathbf{w} - \mathbf{v}$. Also, properties 5 and 6 allow one to drop the distinction between the scalar 0 and the additive identity $\mathbf{0}$. The equation in axiom 5 becomes $\mathbf{v} + 0 = \mathbf{v}$. Leading zeros are usually dropped; for example, $0 - \mathbf{v}$ is written $-\mathbf{v}$.

E.1.3 Linear independence and dimension

The set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ is *linearly dependent* (LD) if and only if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_r$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_r\mathbf{v}_r = \mathbf{0}. \quad (\text{E.1})$$

In the opposite case, the set is *linearly independent* (LI) if and only if eqn (E.1) implies that $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$.

The vector space has dimension N if and only if it contains an LI set of N vectors, but every set in it with more than N vectors is LD. The dimension of a linear vector space is often indicated by labeling the space as V_N .

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N$ be any LI set of N vectors in a vector space V_N of dimension N . Such an LI set is called a *basis* for the vector space, and is said to *span* it. Any vector \mathbf{x} in V_N can be expanded as

$$\mathbf{x} = \sum_{i=1}^N x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_N \mathbf{e}_N. \quad (\text{E.2})$$

The numbers x_i are called *components* of vector \mathbf{x} relative to this basis. For a given basis, the components x_i of a vector \mathbf{x} are uniquely determined.

A vector space may have many different bases. But all of these bases will have the same number of LI vectors in them. And the vector space can be specified uniquely by listing any one of its bases.

Vector equations are equivalent to equations between components (in a given basis). The equation $\mathbf{u} = \alpha\mathbf{v} + \beta\mathbf{w}$ is true if and only if $u_i = \alpha v_i + \beta w_i$ is true for all $i = 1, \dots, N$. This latter relation is often written as an equation of column matrices.

If one defines

$$[v] = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \quad (\text{E.3})$$

with a similar definition for other vectors, the equality of components may be written as $[u] = \alpha[v] + \beta[w]$.

Any LI set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ for $m \leq N$ can be *extended to a basis* in V_N . If $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N$ is some known basis for V_N then vectors $\mathbf{e}_{k_{m+1}}, \mathbf{e}_{k_{m+2}}, \dots, \mathbf{e}_{k_N}$ can be selected from it so that $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{e}_{k_{m+1}}, \mathbf{e}_{k_{m+2}}, \dots, \mathbf{e}_{k_N}$ is an LI set of N vectors and so forms a basis in V_N .

E.2 Subspaces

A *subspace* of V_N is a linear vector space all of whose member vectors are also members of V_N . Denoting a subspace as \mathbb{R} , it follows from the closure axiom for vector spaces that if \mathbf{x} and \mathbf{y} are members of \mathbb{R} , then $(\alpha\mathbf{x} + \beta\mathbf{y})$ must also be a member of \mathbb{R} . Thus subspaces are different from subsets in set theory. If one constructs a subspace by listing a set of vectors, then the subspace must also contain all linear sums of those vectors and not just the vectors in the original list.

Subspaces may be specified by listing an LI set of vectors that spans the space. For example, if V_N has a basis $\mathbf{e}_1, \dots, \mathbf{e}_N$ then the set of all vectors $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_4 + \gamma\mathbf{e}_5$, where α, β, γ may take any values, forms a subspace.

Subspaces may also be specified by stating some criterion for the inclusion of vectors in the subspace. For example, the set of threevectors with components $(0, y, z)$ forms a subspace of V_3 , but the set with components $(1, y, z)$ does not since the sum of two such vectors does not have a 1 as its first component.

The null vector is a member of every subspace. If \mathbb{R} contains only the null vector, then we write $\mathbb{R} = 0$.

E.2.1 Linear Sum and Intersection

If \mathbb{R} and \mathbb{S} are subspaces of V_N , then we define their linear sum $(\mathbb{R} + \mathbb{S})$ as the set of all vectors $\mathbf{x} = \mathbf{r} + \mathbf{s}$ where $\mathbf{r} \in \mathbb{R}$ (which is read as “ \mathbf{r} is a member of \mathbb{R} ”) and $\mathbf{s} \in \mathbb{S}$. The linear sum $(\mathbb{R} + \mathbb{S})$ of two subspaces is itself a subspace of V_N .

If \mathbb{R} and \mathbb{S} are subspaces of V_N , then we define their intersection $(\mathbb{R} \cap \mathbb{S})$ as the set of all vectors \mathbf{x} that are members of both subspaces, $\mathbf{x} \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{S}$. The intersection $(\mathbb{R} \cap \mathbb{S})$ of two subspaces is itself a subspace of V_N .¹³¹

Subspaces are vector spaces. Just as was done for V_N , the dimension $N_{\mathbb{R}}$ of \mathbb{R} is defined as the number of vectors in the largest LI set it contains, and this set then

¹³¹It might appear that a distributive rule would hold for linear sums and intersections, but it does not. In general $\mathbb{R} \cap (\mathbb{S} + \mathbb{T}) \neq \mathbb{R} \cap \mathbb{S} + \mathbb{R} \cap \mathbb{T}$. For example, consider vectors in the Cartesian x - y plane V_2 . Let \mathbb{R} be the one-dimensional subspace with Cartesian components (r, r) , \mathbb{S} with $(s, 0)$, and \mathbb{T} with $(0, t)$. Then $\mathbb{R} \cap \mathbb{S} = 0$ and $\mathbb{R} \cap \mathbb{T} = 0$, so $(\mathbb{R} \cap \mathbb{S} + \mathbb{R} \cap \mathbb{T}) = 0$. But $\mathbb{R} \cap (\mathbb{S} + \mathbb{T}) = \mathbb{R}$ since $(\mathbb{S} + \mathbb{T})$ spans the whole of V_2 .

forms a basis for \mathbb{R} . The notation $\dim(\mathbb{R}) = N_R$ will also be used, meaning that the dimension of \mathbb{R} is N_R . This notation can be used to write the dimension rule,

$$\dim(\mathbb{R} + \mathbb{S}) = \dim(\mathbb{R}) + \dim(\mathbb{S}) - \dim(\mathbb{R} \cap \mathbb{S}) \quad (\text{E.4})$$

If $(\mathbb{R} \cap \mathbb{S}) = 0$ then we say that \mathbb{R} and \mathbb{S} are *disjoint*. Two subspaces are disjoint when their only common vector is the null vector.

In the special case in which $(\mathbb{R} + \mathbb{S}) = V_N$ and $(\mathbb{R} \cap \mathbb{S}) = 0$, we say that \mathbb{R} and \mathbb{S} are *complementary subspaces* or *complements* relative to V_N . This complementarity relation will be denoted $\mathbb{R} \overset{V_N}{\longleftrightarrow} \mathbb{S}$. Complementary subspaces are disjoint and their linear sum is the whole of the space V_N . Since the empty subspace (consisting only of the null vector) has zero dimension by definition, the dimension rule gives $N = N_R + N_S$ for complements.

If \mathbb{R} and \mathbb{S} are complements, and if \mathbb{R} has a basis $\mathbf{r}_1, \dots, \mathbf{r}_{N_R}$ and \mathbb{S} has a basis $\mathbf{s}_1, \dots, \mathbf{s}_{N_S}$, then the set $\mathbf{r}_1, \dots, \mathbf{r}_{N_R}, \mathbf{s}_1, \dots, \mathbf{s}_{N_S}$ is an LI set and forms a basis for V_N . It follows that every vector $\mathbf{x} \in V_N$ can then be written as $\mathbf{x} = \mathbf{r} + \mathbf{s}$ where \mathbf{r} and \mathbf{s} are unique vectors in \mathbb{R} and \mathbb{S} , respectively.

Conversely, suppose that V_N has some basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N$ and we segment that basis into two disjoint LI sets of vectors, say $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{e}_{n+1}, \dots, \mathbf{e}_N$. Then if we define \mathbb{R} to be the subspace spanned by $\mathbf{e}_1, \dots, \mathbf{e}_n$ and define \mathbb{S} to be the subspace spanned by $\mathbf{e}_{n+1}, \dots, \mathbf{e}_N$, it follows that \mathbb{R} and \mathbb{S} are complements relative to V_N .

Every subspace \mathbb{R} has some subspace \mathbb{S} such that \mathbb{R} and \mathbb{S} are complements. But this \mathbb{S} is not unique. So it is incorrect to speak of “the” complement of a given subspace.

The notation $\mathbb{R} \supset \mathbb{S}$ means that every $\mathbf{x} \in \mathbb{S}$ is also a member of \mathbb{R} . We say that \mathbb{R} contains \mathbb{S} . Hence $(\mathbb{R} + \mathbb{S}) \supset \mathbb{S}$, $(\mathbb{R} + \mathbb{S}) \supset \mathbb{R}$, $\mathbb{R} \supset (\mathbb{R} \cap \mathbb{S})$, $\mathbb{S} \supset (\mathbb{R} \cap \mathbb{S})$.

E.3 Linear Operators

A linear operator \mathcal{A} in vector space V_N operates on vectors $\mathbf{x} \in V_N$ and produces other vectors $\mathbf{y} \in V_N$. We write this operation as $\mathbf{y} = \mathcal{A}\mathbf{x}$. The assumed linearity property is

$$\mathcal{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathcal{A}\mathbf{x} + \beta\mathcal{A}\mathbf{y} \quad (\text{E.5})$$

It follows that, for a given basis $\mathbf{e}_1, \dots, \mathbf{e}_N$, there is a unique N -rowed square matrix \mathbf{A} associated with each operator \mathcal{A} . Since any $\mathbf{x} \in V_N$ may be expanded in the basis, as $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_N\mathbf{e}_N$ it follows from linearity that

$$\mathbf{y} = \mathcal{A}\mathbf{x} = \mathcal{A}(x_1\mathbf{e}_1 + \dots + x_N\mathbf{e}_N) = x_1\mathcal{A}\mathbf{e}_1 + \dots + x_N\mathcal{A}\mathbf{e}_N \quad (\text{E.6})$$

But $\mathcal{A}\mathbf{e}_k$ is also a vector in V_N and hence may also be expanded in the basis, as

$$\mathcal{A}\mathbf{e}_k = \sum_{i=1}^N \mathbf{e}_i A_{ik} \quad (\text{E.7})$$

where the *matrix elements* A_{ik} are uniquely determined. Then

$$\sum_{i=1}^N y_i \mathbf{e}_i = \mathbf{y} = \sum_{k=1}^N x_k \sum_{i=1}^N \mathbf{e}_i A_{ik} = \sum_{i=1}^N \left(\sum_{k=1}^N A_{ik} x_k \right) \mathbf{e}_i \tag{E.8}$$

together with the uniqueness of the expansion of \mathbf{y} in the basis implies that

$$y_i = \sum_{k=1}^N A_{ik} x_k \quad \text{or, in matrix form,} \quad [\mathbf{y}] = \mathbf{A}[\mathbf{x}] \tag{E.9}$$

E.3.1 Nonsingular Operators

Since canonical transformations are nonsingular, we will be interested here in nonsingular operators. By definition, an operator \mathcal{A} is *nonsingular* if it satisfies the condition that $\mathcal{A}\mathbf{x} = 0$ only when $\mathbf{x} = 0$.

An operator \mathcal{A} is nonsingular if and only if it possesses an inverse. For nonsingular operators, the equation $\mathbf{y} = \mathcal{A}\mathbf{x}$ can be solved uniquely for \mathbf{x} . This inverse relation is denoted as $\mathbf{x} = \mathcal{A}^{-1}\mathbf{y}$ where the operator \mathcal{A}^{-1} is called the inverse of \mathcal{A} . The inverse \mathcal{A}^{-1} is also nonsingular.

In matrix form, the definition of a nonsingular operator translates to the requirement that $\mathbf{A}[\mathbf{x}] = 0$ only when $[\mathbf{x}] = 0$. In Corollary B.19.2, such nonsingular matrices were shown to have a nonzero determinant and hence to possess an inverse matrix \mathbf{A}^{-1} . If \mathbf{A} is the matrix of \mathcal{A} in some basis, then \mathbf{A}^{-1} is the matrix of the inverse operator \mathcal{A}^{-1} .

E.3.2 Nonsingular Operators and Subspaces

If \mathbb{R} is a subspace of V_N and \mathcal{A} is a nonsingular linear operator, then $\mathcal{A}\mathbb{R}$ is also subspace. It consists of all vectors $\mathbf{y} = \mathcal{A}\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}$. The following properties hold for nonsingular operators acting on subspaces

$$\mathcal{A}(\mathbb{R} + \mathbb{S}) = ((\mathcal{A}\mathbb{R}) + (\mathcal{A}\mathbb{S})) \quad \mathcal{A}(\mathbb{R} \cap \mathbb{S}) = ((\mathcal{A}\mathbb{R}) \cap (\mathcal{A}\mathbb{S})) \tag{E.10}$$

If $\mathbf{r}_1, \dots, \mathbf{r}_{N_R}$ are a basis in \mathbb{R} then $\mathcal{A}\mathbf{r}_1, \dots, \mathcal{A}\mathbf{r}_{N_R}$ are an LI set and form a basis in $\mathcal{A}\mathbb{R}$. Thus, nonsingular operators also have the property

$$\dim(\mathcal{A}\mathbb{R}) = \dim(\mathbb{R}) \tag{E.11}$$

It follows from eqns (E.10, E.11) that if \mathbb{R} and \mathbb{S} are complements and \mathcal{A} is nonsingular, then $\mathcal{A}\mathbb{R}$ and $\mathcal{A}\mathbb{S}$ are also complements

$$\mathbb{R} \xleftrightarrow{V_N} \mathbb{S} \quad \text{if and only if} \quad (\mathcal{A}\mathbb{R}) \xleftrightarrow{V_N} (\mathcal{A}\mathbb{S}) \tag{E.12}$$

E.4 Vectors in Phase Space

A set of values for the variables $q_0, \dots, q_D, p_0, \dots, p_D$ of extended phase space defines a point in what is called a *differentiable manifold*. We want to establish a linear vector space such that the *differentials* of these variables $dq_0, \dots, dq_D, dp_0, \dots, dp_D$ are components of vectors in an abstract linear vector space of dimension $2D + 2$.¹³² Introducing basis vectors $\mathbf{q}_0, \dots, \mathbf{q}_D, \mathbf{p}_0, \dots, \mathbf{p}_D$ a typical vector may be written as

$$d\boldsymbol{\gamma} = dq_0\mathbf{q}_0 + \dots + dq_D\mathbf{q}_D + dp_0\mathbf{p}_0 + \dots + dp_D\mathbf{p}_D \quad (\text{E.13})$$

We may also use the symplectic notation introduced in Section 17.3 to write eqn (E.13) as

$$d\boldsymbol{\gamma} = d\gamma_0\boldsymbol{\gamma}_0 + \dots + d\gamma_{(2D+1)}\boldsymbol{\gamma}_{(2D+1)} \quad (\text{E.14})$$

where $\mathbf{q}_0, \dots, \mathbf{q}_D = \boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_D$ and $\mathbf{p}_0, \dots, \mathbf{p}_D = \boldsymbol{\gamma}_{(D+1)}, \dots, \boldsymbol{\gamma}_{(2D+1)}$.

So far in this chapter, we have made no mention of inner products of vectors. The natural definition for the inner product in phase space is what will be called the *symplectic inner product*, or *symplectic metric*. It can be defined by defining the products of the basis vectors to be, for $i, j = 0, \dots, (2D + 1)$,

$$\boldsymbol{\gamma}_i \circ \boldsymbol{\gamma}_j = s_{ij} \quad (\text{E.15})$$

where s_{ij} are the matrix elements of the symplectic matrix \mathbf{s} defined in Section 17.4. We use the small circle between the vectors rather than the usual dot to emphasize that this is not the standard Cartesian form of the inner product.

Inner products or metrics introduced into linear vector spaces are required to obey certain properties.

1. Linearity: $\mathbf{x} \cdot (\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha\mathbf{x} \cdot \mathbf{y} + \beta\mathbf{x} \cdot \mathbf{z}$.
2. Non-degeneracy: $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x} \in V_N$ if and only if $\mathbf{y} = 0$.
3. Symmetry: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
4. Positive definiteness: If $\mathbf{x} \neq 0$ then $\mathbf{x} \cdot \mathbf{x} > 0$.

The inner product defined in eqn (E.15) is assumed to have the first property. It has the second property since the matrix \mathbf{s} is nonsingular. But it does not satisfy properties 3 and 4. Equation (E.15) therefore does not define a proper metric but what is referred to in mathematics texts as a *structure function*. However, we will continue to refer to it as a metric, following the somewhat looser usage in physics. (The inner product of two timelike or lightlike fourvectors in Minkowski space, for example, violates property 4 and yet g_{ij} is generally called a metric.)

¹³²In modern differential geometry, a curve through a given point of the manifold is defined by writing $\gamma_i = \gamma_i(\beta)$ for each coordinate $i = 0, \dots, (2D + 1)$. Then the derivatives $d\gamma_i(\beta)/d\beta$, taken at the given point, are used as the components of what is called a *tangent vector*. The differentials that we use for vector components are related to these tangent vectors by $d\gamma_i = (d\gamma_i(\beta)/d\beta)d\beta$. In either case, the geometrical idea is that the $d\boldsymbol{\gamma}$ is to represent an arbitrary displacement starting from some point of the manifold. Since we have used differentials (in the sense of the word defined in Section D.12) throughout this text, we will continue to use them here. Those more familiar with the differential-geometric notation may mentally replace the dq_k , etc., by components of the corresponding tangent vectors.

Using the assumed linearity of the inner product, we can write the inner product of any two vectors as

$$d\boldsymbol{\gamma} \circ d\boldsymbol{\phi} = \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} d\gamma_i s_{ij} d\phi_j \quad (\text{E.16})$$

from which it follows that properties 3 and 4 above are replaced by:

3'. Anti-symmetry: $d\boldsymbol{\gamma} \circ d\boldsymbol{\phi} = -d\boldsymbol{\phi} \circ d\boldsymbol{\gamma}$

4'. Nullity: $d\boldsymbol{\gamma} \circ d\boldsymbol{\gamma} = 0$ for every vector $d\boldsymbol{\gamma}$.

E.5 Canonical Transformations in Phase Space

In Section 17.3 a canonical transformation $q, p \rightarrow Q, P$ was written in symplectic coordinates as $\boldsymbol{\gamma} \rightarrow \boldsymbol{\Gamma}$. Then a $(2D+2) \times (2D+2)$ Jacobian matrix with components $J_{ij} = \partial\Gamma_i(\boldsymbol{\gamma})/\partial\gamma_j$ was defined. It follows from the chain rule that the differentials of the symplectic coordinates transform with this Jacobian matrix \mathbf{J} just as the derivatives were shown to do in eqn (17.24)

$$d\Gamma_i = \sum_{j=0}^{2D+1} J_{ij} d\gamma_j \quad (\text{E.17})$$

In discussing canonical transformations in phase space it is simplest to adopt the *active* definition of canonical transformations discussed in Section 18.14. Thus the transformation of differentials in eqn (E.17) is assumed to transform the vector $d\boldsymbol{\gamma}$ in eqn (E.14) into a new vector

$$d\boldsymbol{\Gamma} = \mathcal{J}d\boldsymbol{\gamma} = \sum_{i=0}^{2D+1} d\Gamma_i \boldsymbol{\gamma}_i \quad (\text{E.18})$$

in the same basis. As is always the case for active transformations, the vector $d\boldsymbol{\gamma}$ is transformed into a new vector $d\boldsymbol{\Gamma}$ while the basis vectors $\boldsymbol{\gamma}_i$ are not transformed.

The symplectic inner product defined in Section E.4 has the following important property.

Theorem E.5.1: Invariance of Symplectic Inner Product

The symplectic inner product is invariant under canonical transformations. If \mathcal{J} is the operator of any canonical transformation and $d\boldsymbol{\gamma}$ and $d\boldsymbol{\phi}$ are any two phase-space vectors, then

$$(\mathcal{J}d\boldsymbol{\gamma}) \circ (\mathcal{J}d\boldsymbol{\phi}) = d\boldsymbol{\gamma} \circ d\boldsymbol{\phi} \quad (\text{E.19})$$

Proof: Writing eqn (E.19) in component form, we want to prove that

$$\sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} \left(\sum_{k=0}^{2D+1} J_{ik} d\gamma_k \right) s_{ij} \left(\sum_{l=0}^{2D+1} J_{jl} d\phi_l \right) = \sum_{k=0}^{2D+1} \sum_{l=0}^{2D+1} d\gamma_k \left(\mathbf{J}^T \mathbf{s} \mathbf{J} \right)_{kl} d\phi_l \quad (\text{E.20})$$

is equal to

$$\sum_{k=0}^{2D+1} \sum_{l=0}^{2D+1} d\gamma_k s_{kl} d\phi_l \tag{E.21}$$

The Lagrange bracket condition, eqn (17.54), states that any canonical transformation has $J^T s J = s$, which proves the present theorem. □

E.6 Orthogonal Subspaces

If two phase-space vectors have a zero inner product $d\boldsymbol{\gamma} \circ d\boldsymbol{\phi} = 0$ then we will say that they are orthogonal. But we must note that “orthogonality” in a vector space with a symplectic metric will not have the same geometrical meaning as in the space of threectors. For example, as seen in property 4' of Section E.4, every vector is orthogonal to itself! We will adopt the notation $d\boldsymbol{\gamma} \perp d\boldsymbol{\phi}$ to indicate that the vectors are orthogonal and obey $d\boldsymbol{\gamma} \circ d\boldsymbol{\phi} = 0$. Thus $d\boldsymbol{\gamma} \perp d\boldsymbol{\gamma}$ for every vector $d\boldsymbol{\gamma}$.

This idea of orthogonality can be extended to subspaces. If we have two subspaces \mathbb{R} and \mathbb{S} then we will write $\mathbb{R} \perp \mathbb{S}$ if $d\mathbf{r} \circ d\mathbf{s} = 0$ for every $d\mathbf{r} \in \mathbb{R}$ and $d\mathbf{s} \in \mathbb{S}$. Also a subspace can be self-orthogonal, with $\mathbb{R} \perp \mathbb{R}$, which means that $d\mathbf{x} \circ d\mathbf{y} = 0$ for every $d\mathbf{x}$ and $d\mathbf{y}$ in \mathbb{R} .

It follows from Theorem E.5.1 that mutual- and self-orthogonality of subspaces is invariant under canonical transformations. If, as we did in Section E.3, we denote by $\mathcal{J}\mathbb{R}$ the set of all vectors $d\mathbf{x} = \mathcal{J}d\mathbf{r}$ where $d\mathbf{r} \in \mathbb{R}$, then it follows that for any canonical transformation \mathcal{J} and any subspaces \mathbb{R} and \mathbb{S}

$$\mathbb{R} \perp \mathbb{S} \quad \text{if and only if} \quad (\mathcal{J}\mathbb{R}) \perp (\mathcal{J}\mathbb{S}) \tag{E.22}$$

Thus, in the special case in which $\mathbb{R} = \mathbb{S}$, we also have that

$$\mathbb{R} \perp \mathbb{R} \quad \text{if and only if} \quad (\mathcal{J}\mathbb{R}) \perp (\mathcal{J}\mathbb{R}) \tag{E.23}$$

Lemma 17.7.1 proved that the matrix of any canonical transformation is nonsingular. Hence eqn (E.11) applies to canonical transformations, with the result that

$$\dim(\mathcal{J}\mathbb{R}) = \dim(\mathbb{R}) \tag{E.24}$$

for any subspace \mathbb{R} .

E.7 A Special Canonical Transformation

In Section 18.6 we defined a transformation (linear, with constant coefficients) from the variables Q, P to the new variables X, Y . Taking the differentials of the defining eqn (18.40) it can be written as

$$dX_k = \bar{\alpha}_k dQ_k - \alpha_k dP_k \quad \text{and} \quad dY_k = \alpha_k dQ_k + \bar{\alpha}_k dP_k \tag{E.25}$$

If we define symplectic coordinates $d\Gamma_0, \dots, d\Gamma_{(2D+1)} = dQ_0, \dots, dQ_D, dP_0, \dots, dP_D$ as was done in Section 17.3 and also define $d\Lambda_0, \dots, d\Lambda_{(2D+1)} = dX_0, \dots, dX_D,$

dY_0, \dots, dY_D using a similar pattern, then eqn (E.25) may be written as

$$d\Lambda_i = \sum_{j=0}^{2D+1} A_{ij}d\Gamma_j \quad \text{or} \quad d\Lambda = \mathcal{A}d\Gamma \tag{E.26}$$

where the $(2D + 2) \times (2D + 2)$ matrix \mathbf{A} is composed of diagonal $(D + 1) \times (D + 1)$ blocks \mathbf{a} and $\bar{\mathbf{a}}$. These blocks are defined in terms of the α_k binary digits in eqn (18.38) by $a_{ij} = \delta_{ij}\alpha_i$ and $\bar{a}_{ij} = \delta_{ij}\bar{\alpha}_j$. The matrix \mathbf{A} is

$$\mathbf{A} = \begin{pmatrix} \bar{\mathbf{a}} & -\mathbf{a} \\ \mathbf{a} & \bar{\mathbf{a}} \end{pmatrix} \tag{E.27}$$

As the reader may verify, the matrix \mathbf{A} is both canonical (obeying the definition eqn (17.37), for example) and orthogonal. Thus the operator \mathcal{A} is nonsingular, and defines an active canonical transformation. The inverse matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} \bar{\mathbf{a}} & \mathbf{a} \\ -\mathbf{a} & \bar{\mathbf{a}} \end{pmatrix} \tag{E.28}$$

E.8 Special Self-Orthogonal Subspaces

Before proving the main theorems of this chapter, we require some preliminary definitions and lemmas.

1. Define a subspace \mathbb{Q} to be all vectors of the form $d\mathbf{y}^{(q)} = \sum_{i=0}^D dq_i \mathbf{q}_i$. As can be confirmed using eqn (E.16), $\mathbb{Q} \perp \mathbb{Q}$. Since \mathbb{Q} is spanned by $(D + 1)$ basis vectors, $\dim(\mathbb{Q}) = D + 1$.
2. Define a subspace \mathbb{P} to be all vectors of the form $d\mathbf{y}^{(p)} = \sum_{i=0}^D dp_i \mathbf{p}_i$. As can be confirmed using eqn (E.16), $\mathbb{P} \perp \mathbb{P}$. Since \mathbb{P} is spanned by $(D + 1)$ basis vectors, $\dim(\mathbb{P}) = D + 1$.
3. Given any canonical transformation \mathcal{J} , define a subspaces $\mathbb{M} = \mathcal{J}\mathbb{Q}$ and $\mathbb{N} = \mathcal{J}\mathbb{P}$. It follows from the results in Section E.6 that $\mathbb{M} \perp \mathbb{M}$, $\mathbb{N} \perp \mathbb{N}$, and $\dim(\mathbb{M}) = D + 1 = \dim(\mathbb{N})$.
4. Given any canonical transformation \mathcal{J} , and the special canonical transformation \mathcal{A} for any choice of the α_k , define subspaces $\mathbb{W} = \mathcal{A}\mathbb{M} = \mathcal{A}\mathcal{J}\mathbb{Q}$ and $\mathbb{Z} = \mathcal{A}\mathbb{N} = \mathcal{A}\mathcal{J}\mathbb{P}$. It follows from the results in Section E.6 that $\mathbb{W} \perp \mathbb{W}$, $\mathbb{Z} \perp \mathbb{Z}$, and $\dim(\mathbb{W}) = D + 1 = \dim(\mathbb{Z})$.
5. Given the special canonical transformation \mathcal{A} for any choice of the α_k , define the subspaces $\mathbb{E} = \mathcal{A}^{-1}\mathbb{Q}$ and $\mathbb{F} = \mathcal{A}^{-1}\mathbb{P}$. It follows from the results in Section E.6 that $\mathbb{E} \perp \mathbb{E}$, $\mathbb{F} \perp \mathbb{F}$, and $\dim(\mathbb{E}) = D + 1 = \dim(\mathbb{F})$.
6. By construction $\mathbb{Q} \cap \mathbb{P} = 0$ and $\mathbb{Q} + \mathbb{P} = V_{(2D+2)}$ where $V_{(2D+2)}$ denotes the whole of the vector space, of dimension $(2D + 2)$. Thus \mathbb{Q} and \mathbb{P} are complements relative to $V_{(2D+2)}$, denoted $\mathbb{Q} \overset{V_{(2D+2)}}{\iff} \mathbb{P}$. It then follows from eqn (E.12) that the following pairs are also complements: $\mathbb{M} \overset{V_{(2D+2)}}{\iff} \mathbb{N}$, $\mathbb{W} \overset{V_{(2D+2)}}{\iff} \mathbb{Z}$, $\mathbb{E} \overset{V_{(2D+2)}}{\iff} \mathbb{F}$.

7. Define an operator \mathcal{S} whose associated matrix is \mathbf{s} , the symplectic matrix. Then, as can be verified by writing out the component equation, $\mathbb{P} = \mathcal{S}\mathbb{Q}$. From the definitions 5 ($\mathbb{E} = \mathcal{A}^{-1}\mathbb{Q}$ and $\mathbb{F} = \mathcal{A}^{-1}\mathbb{P}$) it follows that $\mathbb{F} = \mathcal{A}^{-1}\mathcal{S}\mathcal{A}\mathbb{E} = \mathcal{A}^T\mathcal{S}\mathcal{A}\mathbb{E} = \mathcal{S}\mathbb{E}$ where we used the orthogonality of \mathbf{A} to write $\mathbf{A}^{-1} = \mathbf{A}^T$ and the Lagrange bracket condition eqn (17.54) to write $\mathbf{A}^T\mathbf{S}\mathbf{A} = \mathbf{S}$ for the canonical transformation \mathbf{A} .

The scheme of subspace relations can be summarized as

$$\begin{array}{ccccccc} \mathbb{E} & \xleftarrow{\mathcal{A}^{-1}} & \mathbb{Q} & \xrightarrow{\mathcal{J}} & \mathbb{M} & \xrightarrow{\mathcal{A}} & \mathbb{W} \\ \mathcal{S} \downarrow & & \mathcal{S} \downarrow & & & & \\ \mathbb{F} & \xleftarrow{\mathcal{A}^{-1}} & \mathbb{P} & \xrightarrow{\mathcal{J}} & \mathbb{N} & \xrightarrow{\mathcal{A}} & \mathbb{Z} \end{array} \tag{E.29}$$

where subspaces in the upper row and the corresponding subspaces in the lower row are all complements.

The following lemma will also be required.

Lemma E.8.1: Maximum Dimension of Self-Orthogonal Subspace

If \mathbb{R} is any self-orthogonal subspace, with $\mathbb{R} \perp \mathbb{R}$, then $\mathbb{R} \cap (\mathcal{S}\mathbb{R}) = 0$ and $N_{\mathbb{R}} = \dim(\mathbb{R}) \leq (D + 1)$.

Proof: The operator \mathcal{S} defined in property 7 above is nonsingular and is a canonical transformation. Its nonsingularity is proved in Section 17.4. As is proved in eqn (17.29), \mathcal{S} is also orthogonal, with $\mathcal{S}^{-1} = \mathcal{S}^T$. Thus $\mathcal{S}\mathcal{S}\mathcal{S}^T = \mathcal{S}$, which shows that \mathcal{S} satisfies the Poisson bracket condition eqn (17.37) and is a canonical transformation.

Let $d\mathbf{x}$ be any vector in $\mathbb{R} \cap (\mathcal{S}\mathbb{R})$, so that $d\mathbf{x}$ is in both \mathbb{R} and $\mathcal{S}\mathbb{R}$. Since $d\mathbf{x} \in \mathbb{R}$, it follows that $\mathcal{S}d\mathbf{x}$ is in $\mathcal{S}\mathbb{R}$. But $d\mathbf{x}$ is also in $\mathcal{S}\mathbb{R}$. Hence both $d\mathbf{x}$ and $\mathcal{S}d\mathbf{x}$ are in $\mathcal{S}\mathbb{R}$.

It follows from eqn (E.23) that $\mathcal{S}\mathbb{R} \perp \mathcal{S}\mathbb{R}$. Hence $d\mathbf{x} \circ \mathcal{S}d\mathbf{x} = 0$ must hold. Writing this equation out in terms of components and using $(\mathbf{s})^2 = -\mathbf{U}$ gives

$$0 = \sum_{i=0}^{2D+1} \sum_{j=0}^{2D+1} dx_i s_{ij} \left(\sum_{k=0}^{2D+1} s_{jk} dx_k \right) = - \sum_{i=0}^{2D+1} (dx_i)^2 \tag{E.30}$$

which implies $dx_i = 0$ for every i value. Hence $d\mathbf{x}$ is the null vector, which implies that $\mathbb{R} \cap (\mathcal{S}\mathbb{R}) = 0$.

Since \mathcal{S} is a nonsingular operator, eqn (E.11) implies that $\dim(\mathcal{S}\mathbb{R}) = \dim(\mathbb{R}) = N_{\mathbb{R}}$. It then follows from the dimension rule eqn (E.4) that

$$\dim(\mathbb{R} + (\mathcal{S}\mathbb{R})) = \dim(\mathbb{R}) + \dim(\mathcal{S}\mathbb{R}) = 2N_{\mathbb{R}} \tag{E.31}$$

But every subspace in $V_{(2D+2)}$ must have a dimension less than or equal to $(2D + 2)$. Thus $2N_{\mathbb{R}} \leq (2D + 2)$ and so $N_{\mathbb{R}} \leq (D + 1)$ as was to be proved. \square

E.9 Arnold's Theorem

The following theorem allows us to prove that every canonical transformation has a mixed generating function.

Theorem E.9.1: Arnold's Theorem

If \mathbb{N} is any self-orthogonal subspace of dimension $(D + 1)$, there is some choice of the binary digits α_k used to define \mathcal{A} in Section E.7 such that \mathbb{N} and the self-orthogonal subspace \mathbb{E} defined in property 5 of Section E.8 are disjoint,

$$\mathbb{N} \cap \mathbb{E} = 0 \tag{E.32}$$

Proof: The proof first gives a definite procedure for choosing the α_k values. We then prove that the resulting \mathbb{E} has the property eqn (E.32).¹³³

Starting with \mathbb{N} and the self-orthogonal subspace \mathbb{P} defined in property 2 of Section E.8, consider the intersection $\mathbb{N} \cap \mathbb{P}$. This intersection is a self-orthogonal subspace since $(\mathbb{N} \cap \mathbb{P}) \subset \mathbb{P}$ and \mathbb{P} is self-orthogonal. Denote the dimension of $\mathbb{N} \cap \mathbb{P}$ by n . It thus has a basis consisting of an LI set of vectors $\mathbf{x}_0, \dots, \mathbf{x}_{(n-1)}$. Since $(\mathbb{N} \cap \mathbb{P}) \subset \mathbb{P}$, this basis can be extended to a basis for \mathbb{P} by adding vectors $\mathbf{p}_{k_n}, \dots, \mathbf{p}_{k_D}$ selected from the full \mathbb{P} basis $\mathbf{p}_0, \dots, \mathbf{p}_D$ as needed. Then

$$\mathbf{x}_0, \dots, \mathbf{x}_{(n-1)}, \mathbf{p}_{k_n}, \dots, \mathbf{p}_{k_D} \tag{E.33}$$

are $(D + 1)$ linearly independent vectors that form a basis for \mathbb{P} .

Choose the binary digits α_k so that $1 = \alpha_{k_0} = \dots = \alpha_{k_{(n-1)}}$ and $0 = \alpha_{k_n} = \dots = \alpha_{k_D}$ where $k_0, \dots, k_{(n-1)}$ are all those indices *not* selected for the \mathbf{p}_{k_i} in eqn (E.33). (The k_0, \dots, k_D are therefore an arrangement of the integers $0, 1, \dots, D$.) With this definition of the α_k , the definitions in Section E.7 may be used to verify that the $(D + 1)$ -dimensional subspace $\mathbb{E} = \mathcal{A}^{-1}\mathbb{Q}$ will be spanned by the basis

$$\mathbf{q}_{k_0}, \dots, \mathbf{q}_{k_{(n-1)}}, \mathbf{p}_{k_n}, \dots, \mathbf{p}_{k_D} \tag{E.34}$$

Using this definition of \mathbb{E} , we now proceed to the proof of eqn (E.32).

Inspection of eqn (E.34) shows that the intersection $\mathbb{E} \cap \mathbb{P}$ is a self-orthogonal subspace of dimension $(D + 1 - n)$ spanned by $\hat{\mathbf{p}}_{k_n}, \dots, \hat{\mathbf{p}}_{k_D}$. Also, we have already seen that $\mathbb{N} \cap \mathbb{P}$ is a self-orthogonal subspace of dimension n spanned by $\mathbf{x}_0, \dots, \mathbf{x}_{(n-1)}$. Thus, eqn (E.33) shows that \mathbb{P} has a basis consisting of a basis of $\mathbb{N} \cap \mathbb{P}$ concatenated with a basis of $\mathbb{E} \cap \mathbb{P}$. It follows from the discussion in Section E.2.1 that $\mathbb{N} \cap \mathbb{P}$ and $\mathbb{E} \cap \mathbb{P}$ are complements relative to \mathbb{P} and that

$$\mathbb{P} = (\mathbb{N} \cap \mathbb{P}) + (\mathbb{E} \cap \mathbb{P}) \quad \text{and} \quad (\mathbb{N} \cap \mathbb{P}) \cap (\mathbb{E} \cap \mathbb{P}) = 0 \tag{E.35}$$

Since $(\mathbb{N} \cap \mathbb{P}) \subset \mathbb{P}$ and \mathbb{P} is self-orthogonal, it follows that $(\mathbb{N} \cap \mathbb{P}) \perp \mathbb{P}$. Similarly, $(\mathbb{E} \cap \mathbb{P}) \perp \mathbb{P}$. Also, since $\mathbb{N} \cap \mathbb{E}$ and $\mathbb{N} \cap \mathbb{P}$ are both contained in the self-orthogonal

¹³³This theorem and its proof are derived from Chapters 8 and 9 of Arnold (1978).

subspace \mathbb{N} , it follows that $(\mathbb{N} \cap \mathbb{E}) \perp (\mathbb{N} \cap \mathbb{P})$. Similarly, $(\mathbb{N} \cap \mathbb{E}) \perp (\mathbb{E} \cap \mathbb{P})$ since \mathbb{E} is self-orthogonal. Therefore

$$(\mathbb{N} \cap \mathbb{E}) \perp \{(\mathbb{N} \cap \mathbb{P}) + (\mathbb{E} \cap \mathbb{P})\} = \mathbb{P} \tag{E.36}$$

where eqn (E.35) was used to get the last equality.

Now \mathbb{P} is a self-orthogonal subspace of dimension $(D + 1)$. If $\mathbb{N} \cap \mathbb{E} \neq 0$, and if it is not true that $(\mathbb{N} \cap \mathbb{E}) \subset \mathbb{P}$, then there will be a vector not in \mathbb{P} that is symplectically orthogonal to all vectors in \mathbb{P} . This would constitute a self-orthogonal subspace of dimension $(D + 2)$ which is proved impossible by Lemma E.8.1. Thus, whether $\mathbb{N} \cap \mathbb{E} = 0$ or not, it is true that

$$(\mathbb{N} \cap \mathbb{E}) \subset \mathbb{P} \tag{E.37}$$

Thus, using eqn (E.35),

$$(\mathbb{N} \cap \mathbb{E}) = (\mathbb{N} \cap \mathbb{E}) \cap \mathbb{P} = (\mathbb{N} \cap \mathbb{P}) \cap (\mathbb{E} \cap \mathbb{P}) = 0 \tag{E.38}$$

as was to be proved. □

E.10 Existence of a Mixed Generating Function

In Section 18.9 we demonstrated that every canonical transformation can be generated by a mixed generating function $F(q, Y)$. A crucial point in that demonstration was the assertion, without proof, that the binary number α defined in Section 18.6 can always be chosen so that the matrix $(\partial Y / \partial p)$ defined in eqn (18.64) is nonsingular. Using Arnold’s theorem, we can now provide the proof of this assertion.

Theorem E.10.1: Existence of Mixed Generating Function

Assuming a general canonical transformation $q, p \rightarrow Q, P$, the variables X, Y were defined in eqn (18.63) as

$$X_k = X_k(q, p) = \bar{\alpha}_k Q_k(q, p) - \alpha_k P_k(q, p) \quad Y_k = Y_k(q, p) = \alpha_k Q_k(q, p) + \bar{\alpha}_k P_k(q, p) \tag{E.39}$$

The digits α_k of the binary number $\alpha = \alpha_D \cdots \alpha_1 \alpha_0$ used in this definition can always be chosen so that $(D + 1)$ -dimensional square matrix $(\partial Y / \partial p)$ defined by its matrix elements

$$\left(\frac{\partial Y}{\partial p} \right)_{ij} = \frac{\partial Y_i(q, p)}{\partial p_j} \tag{E.40}$$

will have a nonzero determinant and hence be nonsingular.

Proof: As in Theorem E.9.1, we continue to use the subspace definitions from Section E.8. From Arnold’s theorem, we know that there is some choice of α such that $\mathbb{N} \cap \mathbb{E} = 0$. Multiplying by the nonsingular operator \mathcal{A} defined in Section E.7 and using eqn (E.10) gives

$$0 = \mathcal{A}(\mathbb{N} \cap \mathbb{E}) = (\mathcal{A}\mathbb{N}) \cap (\mathcal{A}\mathbb{E}) = \mathbb{Z} \cap \mathbb{Q} \tag{E.41}$$

By its definition in property 4 of Section E.8, $\mathbb{Z} = \mathcal{A}\mathcal{J}\mathbb{P}$, where \mathbb{P} is the subspace defined in property 2, consisting of vectors of the form $d\mathbf{y}^{(p)} = \sum_{i=0}^D dp_i \mathbf{p}_i$. Using the definition of symplectic coordinates from eqn (E.26), vectors in \mathbb{Z} are

$$d\Lambda^{(p)} = \sum_{i=0}^{2D+1} d\Lambda_i^{(p)} \boldsymbol{\gamma}_i = \sum_{i=0}^D \left(dX_i^{(p)} \mathbf{q}_i + dY_i^{(p)} \mathbf{p}_i \right) \tag{E.42}$$

where the components come from the matrix equation

$$[d\Lambda^{(p)}] = \begin{pmatrix} [dX^{(p)}] \\ [dY^{(p)}] \end{pmatrix} = \mathbf{A} \mathbf{J} \begin{pmatrix} [0] \\ [dp] \end{pmatrix} \tag{E.43}$$

Now suppose that at least one of the dp_i is nonzero, so that $[dp] \neq 0$. If $[dY^{(p)}] = 0$ in that case, it would follow that a vector in \mathbb{Z} would be of the form $\sum_{i=0}^D dX_i^{(p)} \mathbf{q}_i$. But such a vector is entirely expressed in the $\mathbf{q}_0, \dots, \mathbf{q}_D$ basis and therefore is a vector in \mathbb{Q} . This is impossible, since, by eqn (E.41), there are no vectors in both \mathbb{Z} and \mathbb{Q} . Therefore it is impossible to have $[dp] \neq 0$ and $[dY^{(p)}] = 0$.

Examination of eqn (E.39) shows that we can also use the chain rule and eqn (E.40) to write $[dY^{(p)}]$ as

$$[dY^{(p)}] = \left(\frac{\partial Y}{\partial p} \right) [dp] \tag{E.44}$$

We have proved above that $[dY^{(p)}] = 0$ implies $[dp] = 0$. By Corollary B.19.2, this is the necessary and sufficient condition for $(\partial Y / \partial p)$ in eqn (E.44) to be nonsingular and have a nonzero determinant. Thus $|\partial Y / \partial p| \neq 0$ as was to be proved. \square

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