Instability in Models Connected with Fluid Flows II

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Instability in Models Connected with Fluid Flows II

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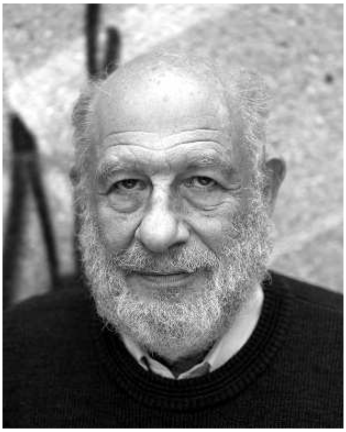

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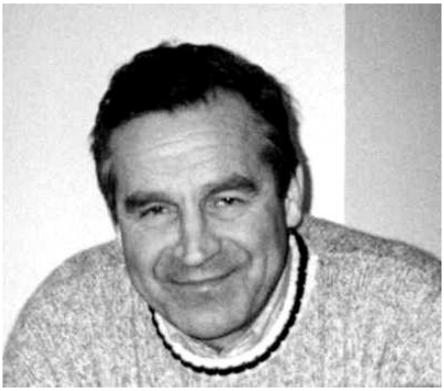

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Justifying Asymptotics for $3D$ Water–Waves

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The main steps of a general method to fully justify asymptotic models for $3D$ water-waves are sketched. The key step is to prove a large time existence result for the nondimensionalized water-waves equations written in terms of the water elevation and the velocity potential at the surface. The theorem also furnishes a bound on a special energy introduced to have uniform control on the solution (with respect to the nondimensionalization parameters). We then describe a systematic way to provide asymptotic expansions on the Dirichlet–Neumann operator involved in the water-waves equations, and deduce asymptotic models in different physical regimes. Full justification of $2DH$ Boussinesq systems, $2DH$ shallow water equations, and the Kadomtsev–Petviashvili approximation are presented as an illustration. Bibliography: 32 titles.

1. Introduction

1.1. General setting.

The motion of a perfect incompressible and irrotational fluid under the influence of gravity is described by the free surface Euler (or water-waves) equations. These equations have a very rich structure, and many famous equations of mathematical physics can be obtained as asymptotic limits: the Korteweg-de Vries (KdV) and Kadomtsev–Petviashvili (KP) equations, the Boussinesq systems, the shallow water equations, deep water models, etc. Each of these asymptotic limits corresponds to a very specific physical regime which determines its range of validity as a tool in oceanography.

While the derivations of these models goes back to the XIXth century, their mathematical justification is a much more recent concern (by mathematical justification, we mean a rigorous proof that the solution of the water-waves equations is well approximated by the solution of the asymptotic model corresponding to the physical regime under consideration). So far, the only asymptotic models fully justified are the KdV equations and $1DH$-Boussinesq systems (see [**8, 24, 4**]) and some variants in presence of surface tension [**25**], bottom topography [**13**], or higher order terms [**27**]. Note also that Kano and Nishida [**15**] gave a justification of the $1DH$-shallow water equations under some restrictions (analytic and small data). For the $2DH$-case and other regimes mentioned above, there is no full rigorous justification; one of the reasons for this is the complexity of the water-waves equations for which local well-posedness and error estimates are nontrivial. Following the pioneer works in $1DH$ of Ovsjannikov [**23**] and Nalimov [**22**] (see also Yosihara [**30, 31**]), Craig [**8**] and Kano and Nishida [**15, 16**] managed to give the first justification of the KdV and $1DH$ Boussinesq and shallow water approximations, but the comprehension of the well-posedness theory for the water-waves equations hindered the perspective of justifying the other asymptotic regimes until the breakthroughs of Wu ([**28**] and [**29**] respectively for the $1DH$ and $2DH$ case, in infinite depth, and without restrictive assumptions). Since when, the literature on free surface Euler equations has been very active: the case of finite depth was proved in [**17**], and in the related case of the study of the free surface of a liquid in vacuum with zero gravity, Lindblad [**19, 20**] and more recently Coutand and Shkoller [**7**] and Shatah and Zeng [**26**] managed to remove the irrotationality condition and/or took into account surface tension effects.

Though quite numerous now, the results on the well-posedness of the water-waves equations cannot be directly applied to justify rigorously asymptotic models because the estimates they give on the existence time are far too rough and only provide existence over an interval of time asymptotically shrinking to zero (relatively to the pertinent time scale). This difficulty is well illustrated by the early works of Kano and Nishida [**15**] and Kano [**14**] where the KdV and KP approximations are justified (for analytic and small data) for times $t = O(1)$ while the relevant time scale for the asymptotics is $t = O(1/\varepsilon)$ (with the notation used in the present paper). In [**8**], this confusion is not made, and the proof relies on a large time (i.e., $O(1/\varepsilon)$) existence theorem for the water-waves equations in the particular "long-waves" scaling. It was recently shown in [**4**] that the $2DH$ Boussinesq systems are justified with sharp error estimates if solutions to the water-waves equations in the long-waves regime exist over the time scale $t = O(1/\varepsilon)$ and are bounded in regular enough Sobolev spaces. Similarly, it is proved in [**18**] that the rigorous justification of the KP approximation follows from such a large time existence theorem and from (unexpected enough) bounds on the solution.

Regardless of the physical regime investigated, the key steps in the process of justification of asymptotic equations is thus the following:

(1) Formally derive the asymptotic equations and identify the relevant time scale of their dynamics.

(2) Prove an existence result for the water-waves equations for this time scale (this is what we call here "large time" existence) and bounds on the solution.

(3) Perform error estimates to control the error between the exact solution of the water-waves equations and the solution furnished by the asymptotic model.

The first step of this procedure can be done at the formal level, while the third one can be done assuming the second one (as in [**4, 6, 18**]). Therefore, it turns out that the proof of a large time existence theorem is the key step of the process. Before explaining the approach developed here, let us review quickly existing results for the main physical regimes. In order to do this, denote by a the typical amplitude of the waves, by h the mean depth, and by λ the wavelength of the waves.

- Shallow-water equations (i.e., $h^2/\lambda^2 \ll 1$). In $1DH$, steps 1 to 3 of the above procedure are done in [**15**], with some restrictions in Step 2 (analytic and small data). In $2DH$, Steps 2 and 3 remain open.
- Long-wave regime (i.e., $h^2/\lambda^2 \sim a/h \ll 1$). The justification process is complete in $1DH$ [**8, 24, 4, 13**]; in $2DH$, Steps 1 and 3 are done in [**4**] (flat bottom) and [**6**] (uneven bottom), but Step 2 is open.
- KP or weakly transverse regime (this regime is the same as the $2DH$ long-wave regime, but with a wavelength in the transverse direction

much larger than in the longitudinal direction). As said above, [**18**] shows that only Step 2 remains to be done.

- Serre approximation (i.e., $h/\lambda \sim a/h \ll 1$). These equations are commonly used in oceanography (see, for instance, [**12**, Chs. 5 and 7]), but no mathematical justification exists.
- Deep water models. In deep water ($h^2/\lambda^2 \gg 1$), the asymptotic expansions are commonly made in terms of the slope of the waves ($a/\lambda \ll 1$). For instance, Matsuno [**21**] proposed (without justification) a model with full dispersion valid for deep water in $1D$.

Instead of developing an existence theory for each physical scaling, we develop here a global method which allows us one to justify all the asymptotics mentioned above at once. In order to do that, we nondimensionalize the water-waves equations, and keep track of the four physical quantities which characterize the dynamics of the water-waves: amplitude, depth, wavelength in the longitudinal direction and wavelength in the transverse direction (for the sake of simplicity, we only consider in this note flat bottoms; in the case of uneven bottoms, a fifth parameter must be introduced, the amplitude of the bottom variations).

Our main theorem gives an estimate of the existence time of the solution of the water-waves equations in terms of these four parameters. It is worth remarking that this estimate is *uniform* with respect to these parameters (though they may grow to infinity or decay to zero, depending on the physical regime investigated). In order to prove this theorem we introduce an energy which involves the aforementioned parameters and use it to construct our solution by an iterative scheme. This energy provides moreover bounds on the solution which appear to be exactly those needed for the error estimates of Step 3.

Having proved such a large time existence result, we derive the asymptotic models for the regimes mentioned above in a systematic way, and use the bounds on the solution provided by the energy to proceed with Step 3.

1.2. Presentation of the results.

Parameterizing the free surface by $z = \zeta(t, X)$ (with $X = (x, y) \in \mathbb{R}^2$) and the bottom by $z = -h$ (with $h > 0$ constant – uneven bottoms are considered in [**1**]), one can use the incompressibility and irrotationality conditions to write the water-waves equations under Bernouilli's formulation, in terms of a velocity potential Φ (i.e., the velocity field is given by $\mathbf{v} = \nabla_{X,z}\Phi$):

$$\begin{aligned}
&\partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi = 0, && -h \leqslant z \leqslant \zeta,\\
&\partial_n\Phi = 0, && z = -h,\\
&\partial_t\zeta + \nabla\zeta\cdot\nabla\Phi = \partial_z\Phi, && z = \zeta,\\
&\partial_t\Phi + \frac{1}{2}\big(|\nabla\Phi|^2 + (\partial_z\Phi)^2\big) + \zeta = 0, && z = \zeta,
\end{aligned} \tag{1.1}$$

where $\nabla = (\partial_x, \partial_y)^T$ and $\partial_n\Phi$ is the normal derivative.

The qualitative study of the water-waves equations is made easier by the introduction of dimensionless variables and unknowns. This requires the introduction of various orders of magnitude linked to the physical regime under consideration. As said in the introduction, a is the order of amplitude of the waves, h is the mean depth, λ is the wavelength of the waves in the x direction, and λ/γ is the wavelength of the waves in the y direction.

We also introduce the following dimensionless parameters

$$\varepsilon = \frac{a}{h}, \quad \mu = \frac{\lambda^2}{h^2}, \quad \nu = \frac{1}{1+\sqrt{\mu}}; \tag{1.2}$$

the parameter ε is often called the *nonlinearity* parameter, while μ is the *shallowness* parameter. The parameter ν is the *transition* parameter which takes into account the fact that different nondimensionalizations are used in shallow and deep water.

Zakharov [**32**] remarked that the system (1.1) could be written in the Hamiltonian form in terms of the free surface elevation ζ and the trace of the velocity potential at the surface $\psi = \Phi_{|_{z=\zeta}}$. Craig, Sulem, and Sulem [**11**] and Craig, Schanz, and Sulem [**10**] used the fact that (1.1) could be reduced to a system of two evolution equations on ζ and ψ to prove the consistency of the Schrödinger and Davey–Stewartson approximation; this formulation has commonly been used since when. In Section 2, we derive the following dimensionless form of this formulation which involves the parameters introduced in (1.2):

$$\begin{aligned}
&\partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = 0,\\
&\partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0,
\end{aligned} \tag{1.3}$$

where $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$ and $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = (\partial_z\Phi - \mu\varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\Phi)_{|_{z=\varepsilon\zeta}}$, with Φ solving the boundary value problem

$$\begin{aligned}&\partial_z^2\Phi + \mu\partial_x^2\Phi + \gamma\mu\partial_y^2\Phi = 0, \qquad -1 < z < \varepsilon\zeta \\ &\Phi_{|_{z=\varepsilon\zeta}} = \psi, \qquad \partial_z\Phi_{|_{z=-1}} = 0.\end{aligned} \tag{1.4}$$

Section 3 is devoted to the asymptotic expansion of the Dirichlet–Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of the parameters ε, γ, and μ. We show how to get explicit expansions in the cases mentioned above.

Section 4 is devoted to the study of the well-posedness of the water-waves equations for large times. With the above notation, we show that solutions to (1.3) exist and are unique over times $t = O(\frac{1}{\varepsilon/\nu})$. We also prove that the energy

$$|\zeta|_{H^s} + \Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s} \quad (\text{with } |D^\gamma| = \sqrt{D_x^2 + \gamma^2 D_y^2})$$

remains bounded over this time scale. It turns our that this existence time and this bound on the solution are exactly those needed to justify rigorously all the models described above. This is sketched in Section 5 for three asymptotic models: the shallow water equations, Boussinesq system, and Kadomtsev–Petviashvili (KP) approximation.

1.3. Notation.

When we want to insist on the dependence of some constant C on various parameters $p_1, p_2, \dots$, we write $C = C(p_1, p_2, \dots)$, and always assume that the dependence on the parameters is *nondecreasing*.

For all tempered distribution $u \in \mathfrak{S}'(\mathbb{R}^d)$ we denote by $\widehat{u}$ its Fourier transform.

Fourier multipliers: For all rapidly decaying $u \in \mathfrak{S}(\mathbb{R}^d)$ and all $f \in C(\mathbb{R}^d)$ with tempered growth, $f(D)$ is the distribution defined by

$$\forall \xi \in \mathbb{R}^d, \qquad \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi) \tag{1.5}$$

(this definition can be extended to wider spaces of functions).

We write $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $\Lambda = \langle D\rangle$, and $\xi^\gamma = (\xi, \gamma\xi_2)$.

For all $1 \leqslant p \leqslant \infty$, $|\cdot|_p$ denotes the classical norm of $L^p(\mathbb{R}^d)$ while $\|\cdot\|_p$ stands for the canonical norm of $L^p(\mathcal{S})$, with $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$.

For all $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is the classical Sobolev space defined as

$$H^s(\mathbb{R}^d) = \{u \in \mathfrak{S}'(\mathbb{R}^d), |u|_{H_s} := |\Lambda^s u|_2 < \infty\}.$$

For all $\gamma > 0$ we write $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$ so that ∇^γ coincides with the usual gradient when $\gamma = 1$. We also use the Fourier multiplier $|D^\gamma|$ defined as $|D^\gamma| = \sqrt{D_x^2 + \gamma^2 D_y^2}$, as well as the anisotropic divergence operator $\mathrm{div}_\gamma = (\nabla^\gamma)^T$. We write $X = (x, y)$ and $\nabla_{X,z} = (\partial_x, \partial_y, \partial_z)^T$.

The notation $a \lesssim b$ means that $a \leqslant Cb$ for some nonnegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).

We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s > \underline{s}} \tag{1.6}$$

to say that $A_s = B_s$ if $s \leqslant \underline{s}$ and $A_s = B_s + C_s$ if $s > \underline{s}$.

When the notation $\partial_n u_{|\partial\Omega}$ is used for boundary conditions of an elliptic equation of the form $\nabla_{X,z} \cdot P\nabla_{X,z} u = h$ in some open set Ω, it stands for the *conormal derivative* associated to this operator, namely,

$$\partial_n u_{|\partial\Omega} = \mathbf{n} \cdot P\nabla_{X,z} u_{|\partial\Omega}, \tag{1.7}$$

$\mathbf{n}$ standing for the *outward* unit normal vector to $\partial\Omega$.

2. Nondimensionalization(s) of Equations

Depending on the value of μ, two distinct nondimensionalizations are commonly used in oceanography (see, for instance, [**12**]). Namely, with dimensionless quantities denoted with a prime:

Shallow-water, i.e., $\mu \ll 1$, one writes

$$\begin{aligned} &x = \lambda x', \quad y = \tfrac{\lambda}{\gamma} y', && z = hz', \quad t = \tfrac{\lambda}{\sqrt{gh}} t', \\ &\zeta = a\zeta', \quad \Phi = \tfrac{a}{h}\lambda\sqrt{gh}\Phi'. \end{aligned}$$

Deep-water, i.e., $\mu \gg 1$, one writes

$$\begin{aligned} &x = \lambda x', \quad y = \tfrac{\lambda}{\gamma} y', && z = \lambda z', \quad t = \tfrac{\lambda}{\sqrt{g\lambda}} t', \\ &\zeta = a\zeta', \quad \Phi = a\sqrt{g\lambda}\Phi'. \end{aligned}$$

Remarking that when $\mu \sim 1$, i.e, $\lambda \sim h$, both nondimensionalizations are equivalent, we introduce the following general nondimensionalization, which is valid for all $\mu > 0$:

$$x = \lambda x', \quad y = \frac{\lambda}{\gamma} y', \qquad z = h\nu z', \quad t = \frac{\lambda}{\sqrt{gh\nu}} t',$$
$$\zeta = a\zeta', \quad \Phi = \frac{a}{h}\lambda\sqrt{\frac{gh}{\nu}}\Phi', \quad b = Bb',$$

where $\nu = \dfrac{1}{1+\sqrt{\nu}}$ is a smooth function of μ such that $\nu \sim 1$ when $\mu \ll 1$ and $\nu \sim \mu^{-1/2} (= \lambda/h)$ when $\mu \gg 1$ (in [**21**], the parameter $\varkappa$ plays a similar role).

The equations of motion (1.1) then become (after dropping the primes for the sake of clarity):

$$\begin{cases} \nu^2\mu\partial_x^2\Phi + \nu^2\gamma^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, & -\frac{1}{\nu} \leqslant z \leqslant \frac{\varepsilon}{\nu}\zeta, \\ -\nu^2\mu\nabla^\gamma(\frac{\beta}{\nu}b)\cdot\nabla^\gamma\Phi + \partial_z\Phi = 0, & z = -\frac{1}{\nu}, \\ \partial_t\zeta - \frac{1}{\mu\nu^2}\big(-\nu^2\mu\nabla^\gamma(\frac{\varepsilon}{\nu}\zeta)\cdot\nabla^\gamma\Phi + \partial_z\Phi\big) = 0, & z = \frac{\varepsilon}{\nu}\zeta, \\ \partial_t\Phi + \frac{1}{2}\big(\frac{\varepsilon}{\nu}|\nabla^\gamma\Phi|^2 + \frac{\varepsilon}{\mu\nu^3}(\partial_z\Phi)^2\big) + \zeta = 0, & z = \frac{\varepsilon}{\nu}\zeta \end{cases} \tag{2.1}$$

with $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$.

In order to reduce this set of equations into a system of two evolution equations, define the Dirichlet–Neumann operator $\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\cdot$ as

$$\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi = \sqrt{1+|\nabla(\frac{\varepsilon}{\nu}\zeta)|^2}\,\partial_n\Phi_{|_{z=\frac{\varepsilon}{\nu}\zeta}}$$

with Φ solving the boundary value problem

$$\begin{cases} \nu^2\mu\partial_x^2\Phi + \nu^2\gamma^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, & -\frac{1}{\nu} \leqslant z \leqslant \frac{\varepsilon}{\nu}\zeta, \\ \Phi_{|_{z=\frac{\varepsilon}{\nu}\zeta}} = \psi, \qquad \partial_n\Phi_{|_{z=\frac{1}{\nu}(-1+\beta b)}} = 0 \end{cases}$$

(as always in this paper, $\partial_n\Phi$ stands for the outward conormal derivative associated to the elliptic equation). As remarked in [**32, 11, 10**], the equations (2.1) are equivalent to a set of two equations on the free surface parametrization ζ and the trace of the velocity potential at the surface $\psi = \Phi_{|_{z=\varepsilon/\nu\zeta}}$

involving the Dirichlet–Neumann operator $\mathcal{G}^{\nu}_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]$. Namely,

$$\begin{aligned}
&\partial_t\zeta - \frac{1}{\mu\nu^2}\mathcal{G}^{\nu}_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi = 0,\\
&\partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu^3}\frac{(\frac{1}{\mu}\mathcal{G}^{\nu}_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi + \nu\nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0.
\end{aligned} \tag{2.2}$$

In order to derive the system (1.3), let $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\cdot$ be the Dirichlet–Neumann operator $\mathcal{G}^{\nu}_{\mu,\gamma}[\varepsilon\zeta]\cdot$ corresponding to the case $\nu = 1$. One will easily check that

$$\forall\nu > 0, \qquad \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\cdot = \frac{1}{\nu}\mathcal{G}^{\nu}_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\cdot,$$

so that plugging this relation into (2.2) yields

$$\begin{aligned}
&\partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = 0,\\
&\partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0.
\end{aligned}$$

3. Asymptotic Expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$

Throughout this section, we assume that the water height is always positive, i.e.,

$$\exists h_0 > 0, \qquad \inf_{\mathbb{R}^d}(1+\varepsilon\zeta) \geqslant h_0. \tag{3.1}$$

3.1. The case of small amplitude waves ($\varepsilon \ll 1$).

Expansions of the Dirichlet–Neumann operator for small amplitude waves has been developed in [**11, 10**]. This method is very efficient to compute the formal expansion, but instead of adapting it in the present case to give uniform estimates on the truncation error, we rather propose a very simple method based on the following explicit formula for the derivative of the mapping $\zeta \mapsto \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\underline{\psi}$, which is a particular case of Theorem 3.20 of [**17**].

Theorem 3.1. *Let $t_0 > 1$, $s \geqslant t_0$, and $\underline{\zeta} \in H^{s+3/2}(\mathbb{R}^2)$ be such that* (3.1) *is satisfied for some $h_0 > 0$. For all $\underline{\psi} \in H^{s+3/2}(\mathbb{R}^2)$ the mapping*

$$\zeta \mapsto \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\underline{\psi} \in H^{s+1/2}(\mathbb{R}^2)$$

is well defined and differentiable in a neighborhood of $\underline{\zeta}$ in $H^{s+3/2}(\mathbb{R}^2)$, and

$$\forall h \in H^{s+3/2}(\mathbb{R}^2), \qquad d_{\underline{\zeta}}\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\underline{\psi}\cdot h = -\varepsilon\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](h\underline{Z}) - \varepsilon\mu\nabla^\gamma\cdot(h\underline{\mathbf{v}})$$

with

$$\underline{Z} = \frac{1}{1+\varepsilon^2\mu|\nabla^\gamma\underline{\zeta}|^2}(\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\underline{\psi} + \varepsilon\mu\nabla^\gamma\underline{\zeta}\cdot\nabla^\gamma\underline{\psi}),$$
$$\underline{\mathbf{v}} = \nabla^\gamma\underline{\psi} - \varepsilon\underline{Z}\nabla^\gamma\underline{\zeta}.$$

We can now state the following proposition, which gives an expansion of the Dirichlet–Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of ε, and uniform with respect to $\gamma \in (0,1]$ and $\mu > 0$.

Proposition 3.1. *Let* $s \geqslant t_0 > 1$, $\psi \in H^{s+4}(\mathbb{R}^2)$, *and* $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ *be such that* (3.1) *is satisfied for some* $h_0 > 0$. *Then*

$$\big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi - \big[\mathcal{G}_{\mu,\gamma}[0] - \varepsilon\mathcal{G}_{\mu,\gamma}[0]\big(\zeta(\mathcal{G}_{\mu,\gamma}[0]\psi)\big) - \varepsilon\mu\nabla^\gamma\cdot(\zeta\nabla^\gamma\psi)\big]\big|_{H^s}$$
$$\leqslant \varepsilon^2\mu^{3/2}C\Big(\frac{1}{h_0}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+9/2}}, \Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^{s+7/2}}\Big).$$

Proof. An order two expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ gives

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = \mathcal{G}_{\mu,\gamma}[0]\psi + d_0\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\psi\cdot\zeta + \int_0^1 (1-z)d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta)dz.$$

Using Theorem 3.1, one computes

$$d_0\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\psi\cdot\zeta = -\varepsilon\mathcal{G}_{\mu,\gamma}[0]\big(\zeta(\mathcal{G}_{\mu,\gamma}[0]\psi)\big) - \varepsilon\mu\nabla^\gamma\cdot(\zeta\nabla^\gamma\psi),$$

and, with some more work, one also gets an explicit expression for the second derivative $d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta)$. It appears that one can write

$$d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta) = \varepsilon^2\mu^{3/2}F(z,\varepsilon,\mu,\gamma,\zeta,\psi)$$

and

$$|F(z,\varepsilon,\mu,\gamma,\zeta,\psi)|_{H^s} \leqslant C\Big(\varepsilon\sqrt{\mu}, \frac{1}{h_0}, |\zeta|_{H^{s+9/2}}, \Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^{s+7/2}}\Big),$$

uniformly with respect to all the parameters; an important step in the above estimate is the following estimate on the operator norm of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]$:

$$\forall s \geqslant t_0 > 1,\ \Big|\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi\Big|_{H^{s-1/2}} \leqslant C\Big(\frac{1}{h_0}, |\zeta|_{H^{s+1}}\Big)\Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s}.$$

$\square$

We can now give asymptotic expansions of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in the different regimes mentioned in the introduction. The first one is the long-waves regime (see also [**4**] for a different proof based on a BKW expansion of the velocity potential).

Corollary 3.1 (long-waves regime)**.** *Let* $\varepsilon_0 > 0$*,* $s \geqslant t_0 > 1$*,* $\psi \in H^{s+6}(\mathbb{R}^2)$*, and* $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ *be such that* (3.1) *is satisfied for some* $h_0 > 0$*. If* $\gamma = 1$*, then for all* $0 < \varepsilon = \mu < \varepsilon_0$

$$\begin{aligned}\Big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi \quad &- \quad \Big[-\varepsilon\Delta\psi - \varepsilon^2\Big(\frac{1}{3}\Delta^2\psi + \nabla\cdot(\zeta\nabla\psi)\Big)\Big]\Big|_{H^s} \\ &\leqslant \quad \varepsilon^3 C\Big(\frac{1}{h_0}, \varepsilon_0, |\zeta|_{H^{s+9/2}}, \big|\nabla\psi\big|_{H^{s+5}}\Big).\end{aligned}$$

Proof. Under the long-waves regime, one can compute explicitly $\mathcal{G}_{\mu,\gamma}[0]\psi = \mathcal{G}_{\varepsilon,1}[0]\psi$ (see Proposition 4.1 below for the computation):

$$\mathcal{G}_{\varepsilon,1}[0]\psi = \sqrt{\varepsilon}|D|\tanh(\sqrt{\varepsilon}|D|)\psi,$$

and a second order Taylor expansion of the function $\varepsilon \mapsto \sqrt{\varepsilon}z\tanh(\sqrt{\varepsilon}z)$ at the origin gives therefore

$$\Big|\ \mathcal{G}_{\varepsilon,1}[0]\psi - \Big[-\varepsilon\Delta\Psi - \varepsilon^2\frac{1}{3}\Delta^2\Psi\Big]\Big|_{H^s} \lesssim \varepsilon^3|\nabla\psi|_{H^{s+5}}.$$

Since $\mu = \varepsilon$, $\gamma = 1$ and $\nu \sim 1$ in the present scaling, one also deduces that

$$\Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s} \leqslant \big|\nabla\psi|_{H^s}$$

uniformly with respect to ε, and the corollary follows therefore from Proposition 3.1. $\square$

In the case of the KP regime (or weakly transverse long-waves), which is the same as the long-wave regime described above, but with $\gamma = \sqrt{\varepsilon}$, one has (see also [**18**]):

Corollary 3.2 (KP regime)**.** *Let* $\varepsilon_0 > 0$*,* $s \geqslant t_0 > 1$*,* $\psi \in H^{s+6}(\mathbb{R}^2)$*, and* $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ *be such that* (3.1) *is satisfied for some* $h_0 > 0$*.*

Then for all $0 < \varepsilon = \mu = \gamma^2 < \varepsilon_0$

$$\begin{aligned}\Big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi \quad &- \quad \Big[-\varepsilon\partial_x^2\psi - \varepsilon^2\Big(\frac{1}{3}\partial_x^4\psi + \partial_x(\zeta\partial_x\psi) + \partial_y^2\psi\Big) - \varepsilon^3\partial_y(\zeta\partial_y\psi)\Big]\Big|_{H^s} \\ &\leqslant \quad \varepsilon^3 C\Big(\frac{1}{h_0}, \varepsilon_0, |\zeta|_{H^{s+9/2}}, \big|\nabla\psi\big|_{H^{s+5}}\Big).\end{aligned}$$

Proof. In this regime, $\mu = \varepsilon = \gamma^2$ so that $\mathcal{G}_{\mu,\gamma}[0]\psi = \mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi$, which can be explicitly computed (see Proposition 3.1):

$$\mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi = \sqrt{\varepsilon}|D^{\sqrt{\varepsilon}}| \tanh\big(\sqrt{\varepsilon}|D^{\sqrt{\varepsilon}}|\big)\psi,$$

where we recall that $|D^{\sqrt{\varepsilon}}| = \sqrt{D_x^2 + \varepsilon D_y^2}$. An order 3 Taylor expansion at the origin of this expression gives

$$\Big|\mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi - \Big[-\varepsilon(\partial_x^2 + \varepsilon\partial_y^2)\Psi - \varepsilon^2\frac{1}{3}(\partial_x^2 + \varepsilon\partial_y^2)^2\Psi\Big]\Big|_{H^s} \lesssim \varepsilon^3\big||D^{\sqrt{\varepsilon}}|\psi|_{H^{s+5}}. \tag{3.2}$$

Remarking that under the present scaling,

$$\Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s} \leqslant \big||D^{\sqrt{\varepsilon}}|\psi|_{H^s} \leqslant |\partial_x\psi|_{H^s} + |\sqrt{\varepsilon}\partial_y\psi|_{H^s},$$

the corollary follows from Proposition 3.1 and (3.2). □

Remark 3.1. (i) The method used above to give an expansion of the Dirichlet–Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ is general and can be used for other scalings, and in particular for the Serre approximation mentioned in the introduction, and for which $\gamma = 1$, $\mu = \varepsilon^2 \ll 1$.

(ii) The two corollaries given above concern shallow–water models ($\mu \ll 1$), but Proposition 3.1 is also valid in deep water. In this case, $\nu \sim \mu^{-1/2}$ and the quantity one has to expand in the first equation of (1.3) is therefore $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$. Remarking also that $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[0]$ is uniformly bounded (as an operator of order 1), Proposition 3.1 furnishes an expansion of $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of $\varepsilon\sqrt{\mu}$. Going back to the definition of ε and μ, one can check that $\varepsilon\sqrt{\mu} = a/\lambda$. This is the *slope* of the wave, used in oceanography as small parameter in deep water.

3.2. The case of large amplitude waves ($\varepsilon = 1$).

The shallow–water regime (for instance) assumes that $\mu \ll 1$, but deals with waves of large amplitude for which $\varepsilon = 1$. In this kind of situation, we cannot use Proposition 3.1 to obtain an expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = \mathcal{G}_{\mu,1}[\zeta]\psi$. However, one can quite easily construct by a standard BKW procedure an (explicit) approximation Φ_{app} of the velocity potential Φ (which solves (1.4)). We then write

$$\begin{aligned}\mathcal{G}_{\mu,1}[\zeta]\psi &= \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi|_{z=0}\\ &= \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_{app}|_{z=0} + \sqrt{1+|\nabla\zeta|^2}\partial_n\big(\Phi - \Phi_{app}\big)|_{z=0}.\end{aligned}$$

The first component of the last equality gives the asymptotic expansion on $\mathcal{G}_{\mu,1}[\zeta]\psi$ since the formula giving Φ_{app} is explicit. The second term of the equality is the truncation error and can be controlled by elliptic estimates (see [**6**] for such estimates within a quite general framework). With this method, one obtains the following assertion.

Proposition 3.2 (shallow water regime). *Let $\mu_0 > 0$, $\varepsilon = \gamma = 1$, $s \geqslant t_0 > 1$, $\psi \in H^{s+4}(\mathbb{R}^2)$, and $\zeta \in H^{s+1}(\mathbb{R}^2)$ be such that* (3.1) *is satisfied for some $h_0 > 0$. Then for all $0 < \mu < \mu_0$*

$$\left|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi-\mu\big(-(1+\zeta)\Delta\psi-\nabla\psi\cdot\nabla\zeta\big)\right|_{H^s} \leqslant \mu^2 C\Big(\frac{1}{h_0},\mu_0,|\zeta|_{H^{s+1}},\left|\nabla\psi\right|_{H^{s+3}}\Big).$$

4. A Large Time Existence Result for the Water–Waves Equations

This section is devoted to the proof of the main theorem of this article. We state it below and refer to [**1**] for full details on the proof. We just hint here at the main steps of the proof. Section 4.1 explains the structure of the linearized equations (we do it in Section 4.1.1 with elementary tools when the reference state is zero, and explain how to treat the general case in Section 4.2). The main steps of the proof of the theorem are given in Section 4.3.

Before stating the theorem, let us introduce the energy $\mathcal{E}^s$:

$$\forall s \geqslant 0, \qquad \mathcal{E}^s\big((\zeta,\psi)\big) := |\zeta|_{H^s} + \left|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\right|_{H^s}. \tag{4.1}$$

Theorem 4.1. *Let $s \geqslant 0$ and $U^0 = (\zeta^0,\psi^0)$ be such that $\mathcal{E}^{\underline{s}}(U^0) < \infty$ for some $\underline{s} = \underline{s}(s)$ large enough. If moreover $\inf_{\mathbb{R}^d}(1+\varepsilon\zeta^0) = h_0 > 0$, then there exists $T = T(\mathcal{E}^{\underline{s}}(U^0), \frac{1}{h_0}, \varepsilon\sqrt{\mu}) > 0$ and a unique solution $U = (\zeta,\psi)$ to* (1.3) *with $(\zeta,\psi-\psi^0) \in C([0,\frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^d))$; moreover,*

$$\sup_{0\leqslant t\leqslant \frac{T}{\varepsilon/\nu}} \mathcal{E}^s(U(t)) \leqslant C\Big(T,\mathcal{E}^{\underline{s}}(U^0),\frac{1}{h_0},\varepsilon\sqrt{\mu}\Big).$$

Remark 4.1. (i) The “large time” evoked in the title of this section is thus $O(\frac{1}{\varepsilon/\nu})$. In the shallow–water regime, $\varepsilon/\nu = 1$ so that the existence time furnished by the theorem is $O(1)$. It is however “large” in the sense that it is uniform with respect to $\mu \ll 1$ (and, in particular, it does not shrink to zero as $\mu \to 0$).

(ii) The scale $O(\frac{1}{\varepsilon/\nu})$ appears to be the pertinent scale of the dynamics of the asymptotics in all the regimes mentioned in the introduction.

(iii) The theorem requires that $\varepsilon\sqrt{\mu}$ remains bounded in order to have a useful control of the energy. As remarked previously, $\varepsilon\sqrt{\mu} = a/\lambda$ is the slope of the waves and it is not restrictive at all to assume that it remains bounded (in all the regimes considered here, $\varepsilon\sqrt{\mu} \ll 1$).

4.1. The linearized equations.

Let us rewrite the water-waves equations (1.3) in condensed form as

$$\partial_t U + \mathcal{F}_{\varepsilon,\mu,\gamma}[U] = 0$$

with $U = (\zeta, \psi)^T$ and $\mathcal{F}_{\varepsilon,\mu,\gamma}[U]$ given by

$$\mathcal{F}_{\varepsilon,\mu,\gamma}[U] = \Big(-\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi, \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)}\Big)^T.$$

By definition, the linearized operator $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ around some reference state $(\underline{\zeta}, \underline{\psi})^T$ is given by

$$\mathcal{L}_{(\underline{\zeta},\underline{\psi})} = \partial_t + d_{\underline{U}}\mathcal{F}_{\varepsilon,\mu,\gamma}.$$

The goal of this section is to give energy estimates on the initial value problem

$$\mathcal{L}_{(\underline{\zeta},\underline{\psi})}U = \frac{\varepsilon}{\nu}G, \quad U_{|_{t=0}} = U^0. \tag{4.2}$$

4.1.1. The linearized equations around the rest state. We assume that $\underline{U}$ is the rest state: $\underline{U} = (0,0)^T$. In this particular case, $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ can be directly computed:

$$\mathcal{L}_{(0,0)} = \partial_t + \begin{pmatrix} 0 & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[0]\cdot \\ 1 & 0 \end{pmatrix};$$

moreover, one has an explicit expression for $\mathcal{G}_{\mu,\gamma}[0]\cdot$.

Proposition 4.1. *The operator $\mathcal{G}_{\mu,\gamma}[0]\cdot$ is given by the Fourier multiplier*

$$\mathcal{G}_{\mu,\gamma}[0]\cdot = \sqrt{\mu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot.$$

Proof. By the definition of the operator $\mathcal{G}_{\mu,\gamma}[0]\cdot$, one has $\mathcal{G}_{\mu,\gamma}[0]\psi = \partial_z\Phi_{|_{z=0}}$, where Φ solves the Laplace equation

$$\partial_z^2\Phi^2 + \mu\partial_x^2\Phi + \gamma^2\mu\partial_y^2\Phi = 0,$$
$$\Phi_{|_{z=0}} = \psi, \qquad \partial_z\Phi_{|_{z=-1}} = 0.$$

One can take the Fourier transform in the horizontal variables, which yields the second order ODE on $\widehat{\Phi}$:

$$\partial_z^2\widehat{\Phi} - \mu|\xi^\gamma|^2\widehat{\Phi} = 0,$$

which can be explicitly solved thanks to the boundary conditions. Taking the inverse Fourier transform of the solution then yields

$$\Phi(\cdot, z) = \frac{\cosh(\sqrt{\mu}(z+1)|D^\gamma|)}{\cosh(\sqrt{\mu}|D^\gamma|)}\psi,$$

so that one gets by direct computation $\partial_z\Phi_{|z=0} = \sqrt{\mu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\psi$, which proves the proposition. □

From the proposition it follows that $\mathcal{L}_{(0,0)}$ takes the explicit form

$$\mathcal{L}_{(0,0)} = \partial_t + \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot \\ 1 & 0 \end{pmatrix}.$$

Quite obviously, this operator is non strictly hyperbolic (0 is a double eigenvalue of the principal symbol, and there is a Jordan block), and any symmetrizer, if it exists will be non-homogeneous (thus inducing a shift of derivatives in the energy – see, for instance, [**9**] for a discussion on this point). Here, a symmetrizer is obviously given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot \end{pmatrix},$$

which motivates the following choice of the energy

$$\begin{aligned} \mathfrak{E}^s(U)^2 &= (\Lambda^s U, S\Lambda^s U) \\ &= |\zeta|_{H^s}^2 + \Big(\Lambda^s\psi, \frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\Lambda^s\Psi\Big). \end{aligned}$$

It worth noticing that $\mathfrak{E}^s(U) \sim \mathcal{E}^s(U)$ (and the equivalence is *uniform* with respect to the parameters μ and γ). This shows in particular that this energy controls the truncation error in Proposition 3.1.

Remark 4.2. It is true that one has the following equivalence:

$$\mathfrak{E}^s(U) \sim |U|_{H^s\times H^{s+1/2}}.$$

However, this equivalence is completely useless because the equivalence is not uniform with respect to γ and μ. The fact that one cannot use such an equivalence complicates considerably the proof and compels us to use more structural properties of the water-waves equations than in [**17**] for instance.

By very standard techniques, one then obtains the following assertion.

Proposition 4.2. *Assume that* $\underline{U} = (0,0)$ *and* $s \geqslant 0$. *Assume also that* $G \in C([0, \frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^2))$ *and* $U^0 \in H^s \times H^{s+1/2}(\mathbb{R}^2)$. *Then there exists a unique solution* $U \in C([0, \frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^2))$ *to* (4.2) *and*

$$\forall t \in [0, \frac{\nu}{\varepsilon}T], \qquad \mathfrak{E}^s(U(t)) \leqslant \mathfrak{E}^s(U^0) + T \sup_{t\in[0,\nu T/\varepsilon]} \mathfrak{E}^s(G(t)).$$

Remark 4.3. One could of course have solved (4.2) explicitly and deduce the estimates, but our purpose here is to introduce the methods used in the study of (4.2) when $\underline{U}$ is not necessarily zero.

4.2. The linearized equations in the general case.

In the general case, i.e., when $\underline{U}$ is not zero, the computation of $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ is not straightforward. However, assuming that $\underline{U}$ is such that the assumptions of Theorem 3.1 are satisfied, one obtains, as in [**17**], the explicit expression

$$\mathcal{L}_{(\underline{\zeta},\underline{\psi})} = \partial_t + \begin{pmatrix} \frac{\varepsilon}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](\underline{Z}\cdot) + \frac{\varepsilon}{\nu}\nabla^\gamma\cdot(\cdot\underline{\mathbf{v}}) & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \\ \frac{\varepsilon^2}{\mu\nu}\underline{Z}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](\underline{Z}\cdot) + (1+\frac{\varepsilon^2}{\nu}\underline{Z}\nabla^\gamma\cdot\underline{\mathbf{v}}) & \frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\nabla^\gamma\cdot - \frac{\varepsilon}{\mu\nu}\underline{Z}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \end{pmatrix},$$

where $\underline{\mathbf{v}}$ and $\underline{Z}$ are the same as in the statement of Theorem 3.1.

The study of the principal symbol of this operator shows that, as for the linearization around zero, $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ is not strictly hyperbolic (the double eigenvalue is now $\frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\xi^\gamma$). As was shown in [**17**, Proposition 4.2], a simple change of basis can be used to put the principal symbol of $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ under a canonical trigonal form. This result is generalized to the present case. More precisely, with

$$\underline{\mathfrak{a}} = 1 + \frac{\varepsilon}{\nu}\underline{\mathfrak{b}}, \quad \text{ and } \quad \underline{\mathfrak{b}} = \varepsilon\underline{\mathbf{v}}\cdot\nabla^\gamma\underline{Z} + \nu\partial_t\underline{Z}, \tag{4.3}$$

and defining the operator $\mathcal{M}_{(\underline{\zeta},\underline{\psi})} = \partial_t + M_{(\underline{\zeta},\underline{\psi})}$ with

$$M_{(\underline{\zeta},\underline{\psi})} = \begin{pmatrix} \frac{\varepsilon}{\nu}\nabla^\gamma\cdot(\cdot\underline{\mathbf{v}}) & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \\ \underline{\mathfrak{a}} & \frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\nabla^\gamma\cdot \end{pmatrix}, \tag{4.4}$$

one reduces the study of (4.2) to the study of the initial value problem

$$\mathcal{M}_{(\underline{\zeta},\underline{\psi})}V = \frac{\varepsilon}{\nu}H, \quad V_{|t=0} = V^0, \tag{4.5}$$

as shown in the following proposition (whose proof relies on simple computations and is omitted).

Proposition 4.3. *The following assertions are equivalent*:

(1) *The pair* $U = (\zeta, \psi)^T$ *solves* (4.2).
(2) *The pair* $V = (\zeta, \psi - \varepsilon\underline{Z}\zeta)^T$ *solves* (4.5) *with* $H = (G_1, G_2 - \varepsilon\underline{Z}G_1)^T$ *and* $V^0 = (\zeta^0, \psi^0 - \varepsilon\underline{Z}_{|t=0}\zeta^0)^T$.

In view of this proposition, it is a key step to understand (4.5), and the rest of this subsection shows the way to prove energy estimates for this initial value problem.

The first remark that a symmetrizer for $\mathcal{M}_{(\underline{\zeta},\underline{\psi})}$ is given by

$$S = \begin{pmatrix} \underline{\mathfrak{a}} & 0 \\ 0 & \frac{\varepsilon^2}{\nu^2} + \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \end{pmatrix}, \tag{4.6}$$

so that (provided that $\underline{\mathfrak{a}}$ is nonnegative), a natural energy for the ivp (4.5) is given by

$$\begin{aligned} E^s(V)^2 &= (\Lambda^s V, S\Lambda^s V) \\ &= |\sqrt{\underline{\mathfrak{a}}}\Lambda^s V_1|_2^2 + \frac{\varepsilon^2}{\nu^2}|V_2|_{H^s}^2 + \Big(\Lambda^s V_2, \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\Lambda^s V_2\Big). \end{aligned} \tag{4.7}$$

Remark 4.4. The term $\frac{\varepsilon^2}{\nu^2}|V_2|_{H^s}^2$ in (4.7) is due to the $\frac{\varepsilon^2}{\nu^2}$ in the second coefficient of the diagonal of (4.6). Removing this term would not affect the energy estimate given below; however, thanks to it, the energy controls the low frequencies of V_2, which is very important in the iterative scheme used to solve to full water-waves equations.

The energy (4.7) is the right one to obtain energy estimates on (4.5), but the reference state $\underline{U}$ must be admissible in the following sense.

Definition 4.1. Let $t_0 > 1$ and $T > 0$. We say that $\underline{U} = (\underline{\zeta}, \underline{\psi})$ is *admissible on* $[0, \frac{T}{\varepsilon/\nu}]$ if

- $(\underline{\zeta}, \nabla\underline{\psi}) \in C^2([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^d)^{1+2})$,
- the surface parametrization $\underline{\zeta}$ satisfies (3.1) for some $h_0 > 0$, uniformly on $[0, \frac{T}{\varepsilon/\nu}]$,
- there exists $c_0 > 0$ such that $\underline{\mathfrak{a}} \geqslant c_0$ uniformly on $[0, \frac{T}{\varepsilon/\nu}]$.

We can now give the energy estimate associated to (4.5); it can be seen as a generalization of Proposition 4.2. We refer to [**1**] for the proof (in the statement of the proposition, $E^s_T(H)$ stands for $E^s_T(H) = \sup_{0\leqslant \nu T/\varepsilon} E^s(H(t)))$.

Proposition 4.4. *Let* $t_0 > 1$, $T > 0$. *Assume that* $\underline{U} = (\underline{\zeta}, \underline{\psi})$ *is admissible on* $[0, \frac{T}{\varepsilon/\nu}]$ *for some* $h_0 > 0$ *and* $c_0 > 0$. *Then, for all* $H \in C([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^2)^2)$, *there exists a unique solution* $V \in C([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^2)^2)$ *to* (4.5) *and for all* $s \geqslant 0$ *and* $0 \leqslant t \leqslant \frac{T}{\varepsilon/\nu}$

$$E^s(V(t)) \leqslant \underline{C} \times [E^s(V^0) + T E^s_T(H) + \langle (E^{t_0+1}(V^0) + T E^{t_0+1}_T(H)) \underline{D}_s \rangle_{s > t_0+1}],$$

where $\underline{D}_s = \mathcal{E}^{s+7/2}_T(\underline{U}) + \mathcal{E}^{s+2}_T(\partial_t \underline{U})$ *and*

$$\underline{C} = C\Big(T, \frac{1}{h_0}, \frac{1}{c_0}, \frac{\varepsilon}{\nu}, \mathcal{E}^{t_0+9/2}_T(\underline{U}), \mathcal{E}^{t_0+2}_T(\partial_t \underline{U}), \mathcal{E}^{t_0+3/2}_T(\partial_t^2 \underline{U})\Big).$$

4.3. Main steps of the proof of Theorem 4.1.

The energy estimate given in Proposition 4.4 concerns the initial value problem (4.5). Using Proposition 4.3 once again, but with the other side of the equivalence, we deduce an energy estimate for the initial value problem (4.2). This energy estimate does not allow us a standard Picard iterative scheme because it exhibits losses of derivatives (in Proposition 4.4 for instance, one needs an energy of order $s + 7/2$ on $\underline{U}$ to control an energy of order s on V). However, this energy is *tame* in the sense that the s-dependent terms on the right hand side are all linear. This allows us, as in [**17**], to use a Nash–Moser type iterative scheme. The order $O(\frac{1}{\varepsilon/\nu})$ of the existence time furnished by the Nash–Moser fixed point theorem follows from the fact that the energy estimate of Proposition 4.4 depends only on T for times $\frac{T}{\varepsilon/\nu}$ (note that we use here a special Nash–Moser theorem with parameters for evolution equations developed in [**2**]).

The last points to comment on are the two conditions required in the definition of an admissible reference state. Quite obviously, the condition on the water depth will remain true for T small enough (but uniformly with respect to the parameters) if it is initially true. The second condition, on the sign of $\underline{\mathfrak{a}}$, is not that clear. In fact, it follows from the works of Wu [**28, 29**], generalized in [**17**] for the case of finite depth, that one has necessarily $\underline{\mathfrak{a}} > 0$ for *exact solutions* of the water-waves equations. Choosing the first term of the iterative scheme in such a way that it solves the water-waves equations at $t = 0$, there exists c_0 such that $\underline{\mathfrak{a}}(t = 0) > 2c_0$; it is then possibly to maintain the condition $\underline{\mathfrak{a}}(t) > c_0$ on $[0, \frac{T}{\varepsilon/\nu}]$ (taking a smaller T if necessary).

5. Asymptotics for 3D Water–Waves

As an illustration of the methods developed in this note, we sketch here how to give a full justification of asymptotic models for 3D water-waves in three different regimes: shallow water, long waves, and KP regime.

5.1. Shallow-water equations.

We recall that the so-called "shallow–water" regime corresponds to the condition $\mu \ll 1$ (so that $\nu \sim 1$) and that $\varepsilon = \gamma = 1$. It follows therefore from Theorem 4.1 that there exists $T > 0$ independent of μ such that solutions to (1.3) exist on $[0, T]$. Moreover, the energy bound provided by the theorem ensures that ζ and $V := \nabla\psi$ are uniformly bounded on $[0, T]$ in Sobolev spaces. Plugging the expansion furnished by Proposition 3.2 into (1.3) and taking the gradient of the second equations in order to obtain a system of equations on ζ and $V = \nabla\psi$, one gets

$$\begin{aligned} &\partial_t V + \nabla\zeta + \tfrac{1}{2}\nabla|V|^2 = \mu R_1^\mu, \\ &\partial_t \zeta + \nabla\cdot(1+\zeta V) = \mu R_2^\mu \end{aligned} \tag{5.1}$$

with R_1^μ and R_2^μ uniformly bounded in Sobolev spaces on the time interval $[0, T]$. An energy estimate on (5.1) thus shows that the error made by using exact solutions of the shallow water equations (namely, (5.1) with zero on the right-hand side) instead of (1.3) is $O(\mu)$ on $[0, T]$. In other words, the 2DH shallow water model is fully justified.

5.2. Long-waves regime.

The long-wave regime is characterized by the scaling $\gamma = 1$, $\mu = \varepsilon \ll 1$ so that $\nu \sim 1$. From Theorem 4.1 it follows that there exists $T > 0$ independent of ε and a unique solution $U = (\zeta, \psi)$ to (1.3) on the time interval $[0, T/\varepsilon]$ such that the energy

$$|\zeta(t)|_{H^s} + \left| \frac{|D|}{(1+\sqrt{\varepsilon}|D|)^{1/2}}\psi(t) \right|_{H^s}$$

remains bounded on $[0, T/\varepsilon]$. Defining $V = \nabla\psi$, this implies that ζ and V remain bounded on $[0, T/\varepsilon]$ in Sobolev spaces. This is exactly the condition that was needed in [**4**] to fully justify the 2DH Boussinesq systems. For the sake of completeness, we recall briefly the strategy of [**4**].

Plugging the asymptotic expansion of the Dirichlet–Neumann operator given by Corollary 3.1 into (1.3) and taking the gradient of the equation on ψ, one gets

$$\begin{aligned} &\partial_t V + \nabla\zeta + \varepsilon\tfrac{1}{2}\nabla|V|^2 = \varepsilon^2 R_1^\varepsilon, \\ &\partial_t\zeta + \nabla\cdot V + \varepsilon\big(\tfrac{1}{3}\Delta\nabla\cdot V + \nabla\cdot(\zeta V)\big) = \varepsilon^2 R_2^\varepsilon, \end{aligned} \tag{5.2}$$

where, as a consequence of Corollary 3.1 and Theorem 4.1, $R_j^\varepsilon = R_j^\varepsilon(\zeta,\psi)$ ($j=1,2$) are uniformly bounded on the time interval $[0,T/\varepsilon]$ in Sobolev spaces. The Boussinesq system (5.2), however, is not well-posed, and one cannot directly conclude as in the shallow–water regime. Using linear manipulations (set forth in a systematic way in [**5**]) and a nonlinear change of variables introduced in [**4**], one can construct an infinity of Boussinesq systems, formally equivalent to (5.2), some of which being well-posed. Making the energy estimates on such a well-posed system, one can show that the approximations furnished by the Boussinesq systems have a precision of order $O(\varepsilon^2 t)$ on $[0,T/\varepsilon]$.

5.3. The Kadomtsev–Petviashvili approximation.

We recall that the KP regime is the same as the long-waves regime, but with $\gamma = \sqrt{\varepsilon}$. Theorem 4.1 then furnishes a solution of (1.3) on a time interval $[0,T/\varepsilon]$; moreover, the energy bound shows that ζ, $\partial_x\psi$ and $\sqrt{\varepsilon}\partial_y\psi$ are bounded on $[0,T/\varepsilon]$ in Sobolev spaces. This was exactly the assumption made in [**18**] to justify the Kadomtsev–Petviashvili equations which states that the water elevation ζ is approximated on $[0,T/\varepsilon]$ by

$$\zeta(t,x) \sim \zeta_+(\varepsilon t, x-t, \sqrt{\varepsilon}y) + \zeta_-(\varepsilon t, x+t, \sqrt{\varepsilon}y),$$

where $\zeta_\pm(\tau,\widetilde{x})$ solves

$$\partial_\tau\zeta_\pm \pm \frac{1}{2}\partial_{\widetilde{x}}^{-1}\partial_y^2\zeta_\pm \pm \frac{1}{6}\partial_{\widetilde{x}}^3\zeta_\pm + \frac{3}{2}\zeta_\pm\partial_{\widetilde{x}}\zeta_\pm = 0.$$

The strategy of [**18**] to justify the KP approximation from the large time existence theorem and the bounds on ζ, $\partial_x\psi$, and $\sqrt{\varepsilon}\partial_y\psi$ consists in justifying first a class of *weakly transverse* Boussinesq systems along the lines described in Section 5.2. The KP approximation is then justified from these systems with nonlinear optics methods, as in [**3**].

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Generalized Solutions of the Cauchy Problem for a Transport Equation with Discontinuous Coefficients

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The well-posedness theory of the Cauchy problem for linear transport equations with only bounded measurable coefficients is presented. In the case of one spatial variable, the existence and uniqueness of generalized and renormalized solutions are established, the notion of generalized characteristics is introduced. This theory is also applied to prove the existence and uniqueness of strong entropy solutions of the Cauchy problem for systems of Keyfitz–Kranzer type. Bibliography: 20 titles.

1. Introduction

Transport equations are linear first order partial differential equations of the form

$$Au_t + (B, \nabla_x u) = f, u = u(t,x), \quad (t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^n. \tag{1.1}$$

Hereinafter, $\mathbb{R}_+ = (0, +\infty)$,$(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$, $A = A(t,x)$, $B = B(t,x) = (B_1(t,x), \dots, B_n(t,x)) \in \mathbb{R}^n$, and $f = f(t,x,u)$ is the source function. We study the Cauchy problem for Equation (1.1) with initial condition

$$u(0,x) = u_0(x). \tag{1.2}$$

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If A, B, f, and u_0 are smooth, there exists a unique solution of the problem (1.1), (1.2), and this solution can be constructed by the method of characteristics (see, for example, [**8**]). More exactly, suppose that $A, B_i \in C^1(\Pi)$, $i = 1, \dots, n$, $f(t, x, u) \in C^1(\Pi \times \mathbb{R})$, $|A| > \delta > 0$. Dividing by A, we reduce Equation (1.1) to the form

$$u_t + (C, \nabla_x u) = g, \quad C = B/A, \quad g = f/A \tag{1.3}$$

with the smooth vector of coefficients $C = (C_1, \dots, C_n)$ and source function $g = g(t, x, u)$. For the sake of simplicity we suppose that C and g are bounded. The characteristics of Equation (1.3) are integral curves $x = x(t) \in \mathbb{R}^n$ of the system of ordinary differential equations

$$\dot{x} = C(t, x), \tag{1.4}$$

which are well defined for all $t \geqslant 0$ since the vector C is bounded. A smooth function $u(t, x)$ satisfies Equation (1.3) if and only if for any characteristic $x(t)$

$$\begin{aligned} \frac{d}{dt} u(t, x(t)) &= [u_t + (\dot{x}, \nabla_x u)](t, x(t)) = [u_t + (C, \nabla_x u)](t, x(t)) \\ &= g(t, x(t), u(t, x(t))). \end{aligned}$$

Thus, along any characteristic $x = x(t)$, the solution $u = u(t, x)$ must satisfy the ordinary differential equation $\dot{u} = g = g(t, x, u)$. This, together with (1.4), leads to the system

$$\begin{aligned} \dot{x} &= C(t, x), \\ \dot{u} &= g(t, x, u). \end{aligned} \tag{1.5}$$

Denote by $x(t; t_0, x_0)$ the characteristic passing through the point $(t_0, x_0) \in \Pi$, i.e., the solution of (1.4) satisfying the Cauchy condition $x(t_0) = x_0$. Let $y(t_0, x_0) = x(0; t_0, x_0)$ be the source point of this characteristic. By the known properties of ordinary differential equations, $y(t, x) \in C^1(\Pi)$. Denote by $x(t; y_0)$, $u(t; y_0, u_0)$ the solution of the Cauchy problem for the system (1.5) with initial data $x(0) = y_0$, $u(0) = u_0$. It is obvious that $x(t; y_0) = x(t; t_0, x_0)$ for $y_0 = y(t_0, x_0)$. If $u(t, x)$ is a solution of the problem (1.3), (1.2) with $u_0(x) \in C^1(\mathbb{R})$, then $u(t, x(t; y_0)) = u(t; y_0, u_0(y_0))$. From this relation it follows that the solution of the problem (1.3), (1.2) is uniquely determined by the equality $u(t, x) = u(t; y(t, x), u_0(y(t, x)))$. In the particular case $g = g(t, x)$, we have

$$u(t, x) = u_0(y(t, x)) + \int_0^t g(s, x(s; t, x)) ds, \tag{1.6}$$

and for homogeneous equation (with $g \equiv 0$) the equality (1.6) takes the simple form $u(t,x) = u_0(y(t,x))$ showing that the solution is constant along the characteristics.

In the classical case of smooth input data, the method of characteristics allows us to reduce the problem (1.1), (1.2) to the Cauchy problem for some system of ordinary differential equations. But this method cannot be applied to equations with nonsmooth (and even discontinuous) coefficients which actually arises in various applications. We give an interesting example in the following section.

2. Hyperbolic Systems of Keyfitz–Kranzer Type

Consider the system of Keyfitz–Kranzer type

$$u_t + (\varphi(|u|)u)_x = 0, \tag{2.1}$$
$$u = u(t,x) \in \mathbb{R}^n, \quad |u| = (u_1^2 + \cdots + u_n^2)^{1/2}.$$

Such systems arise in numerous applications and are widely investigated (see, for example, [**3, 9, 15**]). The system (2.1) is a nonstrictly hyperbolic system of conservation laws. We consider the Cauchy problem for (2.1) with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}, \mathbb{R}^n). \tag{2.2}$$

Assume that $\varphi(r) \in C(\mathbb{R}_+)$ satisfies the condition

$$r\varphi(r) \to 0 \quad \text{as } r \to 0+. \tag{2.3}$$

From this relation it follows that the function

$$f(r) = \begin{cases} \varphi(|r|)r, & r \neq 0, \\ 0, & r = 0 \end{cases}$$

is continuous. Suppose for a moment that $\varphi(r) \in C^1([0,+\infty))$, $u = u(t,x) \in \mathbb{R}^n$ is a smooth solution of (2.1) and $r = |u| > 0$. Then $(\varphi(r)u)_x = \varphi(r)u_x + \varphi'(r)(u/r, u_x)u$. This implies that

$$\big(u/r, (\varphi(r)u)_x\big) = (\varphi(r) + r\varphi'(r))(u/r, u_x) = f'(r)(u/r, u_x)$$

and

$$r_t + (f(r))_x = (u/r, u_t) + f'(r)(u/r, u_x) = (u/r, u_t + (\varphi(r)u)_x) = 0.$$

This relation justifies the notion of a strong generalized entropy solution of the problem (2.1), (2.2) introduced in [**15**].

Definition 1. A bounded measurable vector-function $u = u(t, x)$ is called a *strong generalized entropy solution* of the problem (2.1), (2.2) if $u_t + (\varphi(r)u)_x = 0$ in the sense of distributions on Π (in $\mathcal{D}'(\Pi, \mathbb{R}^n)$), while the function $r = |u(t, x)|$ is a generalized entropy solution (in the sense of Kruzhkov [**10**]) of the scalar problem

$$r_t + (f(r))_x = 0, \quad r(0, x) = r_0 = |u_0(x)| \tag{2.4}$$

and

$$\operatorname*{ess\,lim}_{t\to 0+} u(t, \cdot) = u_0 \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n), \tag{2.5}$$

i.e., there exists a set $E \subset (0, +\infty)$ of full Lebesgue measure such that for $t \in E$ $u(t, \cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, and $u(t, \cdot) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ as $t \to 0$, $t \in E$.

The notion of a strong generalized entropy solution coincides with that of a *renormalized solution* introduced in [**3**]. As was shown in [**15**], any strong generalized entropy solution of the problem (2.1), (2.2) is also a generalized entropy solution of this problem, i.e., it satisfies the Kruzhkov–Lax entropy relations for all convex entropies of (2.1) (these entropies are completely described in [**15**]). As was shown in [**15**], the strong generalized entropy solutions generate the natural correctness class for the problem (2.1), (2.2).

If $u = u(t, x)$ is a strong generalized entropy solution of the problem (2.1), (2.2) then, by Definition 1, the function $r = |u(t, x)|$ is a unique generalized entropy solution of the scalar problem (2.4). Note that the flux function $f(r)$ is here only continuous, and the unconditional uniqueness of a generalized entropy solution is valid only in the case of one spatial variable (see [**11, 12, 16**] for details). We see that if u is a generalized solution of the problem (2.1), then the functions $v_i = u_i/r$, $i = 1, \ldots, n$ (for $r = 0$ the values of these functions are not essential) must satisfy in $\mathcal{D}'(\Pi)$ the linear transport equation

$$(Av)_t + (Bv)_x = 0 \tag{2.6}$$

with, in general, discontinuous coefficients $A = A(t, x) = r(t, x)$, $B = B(t, x) = f(r(t, x))$ and initial condition

$$v(0, x) = v_{0i} = u_{0i}/r_0. \tag{2.7}$$

Based on the theory of generalized solutions of the problem (1.1), (1.2) developed in [**13, 14, 15**], we can prove the existence and uniqueness of a strong generalized entropy solution of the problem (2.1), (2.2) (see [**15**] and Theorem 4 below).

3. Homogeneous Transport Equations. Generalized Solutions of the Cauchy Problem

3.1. Preliminaries.

Let $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$ be a half-space, and let a vector $(A, B) \in L^\infty(\Pi, \mathbb{R}^{n+1})$ be such that

$$\forall \varepsilon > 0 \quad |B| \leqslant N(\varepsilon) \cdot (A + \varepsilon) \text{ a.e. on } \Pi,\ \varepsilon N(\varepsilon) \to 0 \text{ as } \varepsilon \to 0+, \tag{3.1}$$

$$A_t + \operatorname{div}_x B = 0 \text{ in } \mathcal{D}'(\Pi). \tag{3.2}$$

In the condition (3.1), $|B|$ is the Euclidean norm of a vector $B = (B_1, \dots, B_n)$.

As is known (see, for example, [**8**]), for a piecewise smooth field $v = (A, B)$ of coefficients the condition $\operatorname{div} v = A_t + \operatorname{div}_x B = 0$ in $\mathcal{D}'(\Pi)$ is satisfied if and only if $\operatorname{div} v = 0$ in the classical sense in domains where v is smooth and, on any discontinuity surface S, the Rankine–Hugoniot condition

$$([v], \nu) = [A]\nu_t + ([B], \nu_x) = 0 \tag{3.3}$$

is satisfied. In (3.3), $[v] = v_+ - v_-$ denotes the jump of the vector v on S, $v_\pm$ are one-sided limits of v on the surface S, and $\nu = (\nu_t, \nu_x) \in \mathbb{R}^{n+1}$ is the normal vector on S. In the one-dimensional case $n = 1$, the condition (3.3) on the discontinuity curve $x = x(t)$ can be written in the form $\dot{x} = [B]/[A]$.

We show that the condition (3.1) can be written in the form

$$A(t,x) \geqslant 0 \text{ a.e. on } \Pi, \quad |B(t,x)| \leqslant \Phi(A(t,x)) \text{ a.e. on } \Pi \tag{3.4}$$
$$\text{for some } \ \Phi(r) \in C([0, +\infty)),\ \Phi(0) = 0.$$

Proposition 1. *The conditions* (3.1) *and* (3.4) *are equivalent.*

Proof. Suppose that the condition (3.1) is satisfied. If $A(t,x) < 0$ on a set of positive measure, then there is a positive constant ε such that $A(t,x) < -\varepsilon$ on a set of positive measure. But this contradicts the condition $N(\varepsilon)(A+\varepsilon) \geqslant |B(t,x)| \geqslant 0$ a.e. on Π. Thus, $A(t,x) \geqslant 0$ a.e. on Π. Let $\mathbb{Q}_+$ be the set of positive rational numbers, and let $\Phi(r) = \inf\limits_{\varepsilon \in \mathbb{Q}_+} N(\varepsilon)(r + \varepsilon)$, $r \geqslant 0$. The function $\Phi(r)$ is nonnegative and concave since it is the infimum of a family of affine functions $r \to N(\varepsilon)(r + \varepsilon)$, $\varepsilon \in \mathbb{Q}_+$. Therefore, it is continuous for $r > 0$. We show that $\Phi(r)$ is also continuous at $r = 0$. It is clear that $\Phi(0) = \inf\limits_{\varepsilon \in \mathbb{Q}_+} N(\varepsilon)\varepsilon = 0$ in view of (3.1). By (3.1), for any positive

δ there is $\varepsilon_0 \in \mathbb{Q}_+$ such that $N(\varepsilon_0)\varepsilon_0 < \delta/2$. Then for all $r \in [0, \varepsilon_0)$ we have

$$\Phi(r) < N(\varepsilon_0)(r + \varepsilon_0) < 2N(\varepsilon_0)\varepsilon_0 < \delta.$$

Hence $\lim\limits_{r\to 0} \Phi(r) = 0 = \Phi(0)$. We conclude that $\Phi(r) \in C([0, +\infty))$, $\Phi(0) = 0$. Further, since $\mathbb{Q}_+$ is countable, for all $\varepsilon \in \mathbb{Q}_+$ we can choose the same set of full Lebesgue measure, where the inequality (3.1) holds. On this set, we have

$$|B(t, x)| \leqslant \inf_{\varepsilon\in\mathbb{Q}_+} N(\varepsilon)(A(t, x) + \varepsilon) = \Phi(A(t, x)),$$

i.e., the condition (3.4) is satisfied.

Conversely, assume that (3.4) is satisfied. We set

$$M = \|A\|_\infty, \quad C = \max_{r\in[0,M]} \Phi(r), \quad N(\varepsilon) = \sup_{r\in[0,M]} \frac{\Phi(r)}{r + \varepsilon}, \ \varepsilon > 0.$$

It is obvious that for a.e. $(t, x) \in \Pi$ we have $A(t, x) \in [0, M]$ and

$$|B(t, x)| \leqslant \Phi(A(t, x)) \leqslant \frac{\Phi(A(t, x))}{A(t, x) + \varepsilon}\big(A(t, x) + \varepsilon\big) \leqslant N(\varepsilon)\big(A(t, x) + \varepsilon\big).$$

To complete the proof, it remains to verify that $\varepsilon N(\varepsilon) \to 0$ as $\varepsilon \to 0+$. By assumption, $\lim\limits_{r\to 0} \Phi(r) = 0$ and for any $\delta > 0$ there is $r_0 > 0$ such that $\Phi(r) < \delta$ for all $r \in [0, r_0]$. Therefore, $\dfrac{\Phi(r)}{r + \varepsilon} < \dfrac{\delta}{\varepsilon}$ for $r \in [0, r_0]$ and $\dfrac{\Phi(r)}{r + \varepsilon} \leqslant \dfrac{C}{r_0 + \varepsilon}$ for $r \in [r_0, M]$. . Hence

$$N(\varepsilon) \leqslant \max\left(\frac{\delta}{\varepsilon}, \frac{C}{r_0 + \varepsilon}\right) \leqslant \frac{\delta}{\varepsilon} + \frac{C}{r_0 + \varepsilon},$$

which implies that $\limsup\limits_{\varepsilon\to 0} \varepsilon N(\varepsilon) \leqslant \delta$. Since $\delta > 0$ is arbitrary we conclude that $\lim\limits_{\varepsilon\to 0} \varepsilon N(\varepsilon) = 0$. □

Remark 1. For Equation (2.6) the conditions (3.1) and (3.2) are satisfied. Indeed, $A = r$ is a generalized entropy solution of the problem (2.4) with nonnegative initial data. By the maximum principle [**11, 12, 16**], this solution is also nonnegative: $A \geqslant 0$. Since $B = f(A)$ and $f(0) = 0$, the condition (3.1) in the form (3.4) is satisfied. Finally, $A_t + B_x = r_t + (f(r))_x = 0$ by the assumption that r is a generalized entropy solution of the scalar problem (2.4).

We need the following simple result.

Lemma 1. *Suppose that* $A = A(t,x) \in L^\infty(\Pi)$, $B = B(t,x) \in L^\infty(\Pi, \mathbb{R}^n)$, *and* $A_t + \operatorname{div}_x B = 0$ *in* $\mathcal{D}'(\Pi)$. *Then* $A(t,\cdot)$ *is weakly-$*$ continuous in* $L^\infty(\mathbb{R}^n)$ *with respect to* t *up to the point* $t = 0$ (*after a correction of* A *on a set of zero measure, if necessary*).

Proof. Since $A_t + \operatorname{div}_x B = 0$ in $\mathcal{D}'(\Pi)$, for every $h(x) \in C_0^\infty(\mathbb{R}^n)$

$$\frac{\partial}{\partial t} \int A(t,x)h(x)dx = \int (B(t,x), \nabla h(x))dx \text{ in } \mathcal{D}'(\mathbb{R}).$$

We see that the function

$$I_h(t) = \int A(t,x)h(x)dx$$

has the bounded derivative in the sense of distributions. Therefore, this function, well-defined on the set of full measure

$$E = \{t > 0 | (t,x) \text{ is a Lebesgue point of } A(t,x) \text{ for a.e. } x \in \mathbb{R}^n\}, \tag{3.5}$$

is Lipschitz continuous. This implies that there exists a unique extension of $I_h(t)$ as a continuous function on the whole ray $[0, +\infty)$. It is clear that $|I_h(t)| \leqslant \|A\|_\infty \cdot \|h\|_1$ and for every $t \geqslant 0$ the correspondence $h \to I_h(t)$ is extended as a continuous linear functional on $L^1(\mathbb{R}^n)$. Since the dual space $(L^1(\mathbb{R}^n))^*$ is $L^\infty(\mathbb{R}^n)$, for all $t \in [0, +\infty)$ there exists a function $F(t,\cdot) \in L^\infty(\mathbb{R}^n)$ such that $I_h(t) = \displaystyle\int h(x)F(t,x)dx$. It is clear that $F(t,x) = A(t,x)$ a.e. on $\mathbb{R}^n$ for all $t \in E$ and the mapping $t \to F(t,\cdot) \in L^\infty(\mathbb{R}^n)$ is weakly-$*$ continuous, which easily follows from the boundedness of $\|F(t,\cdot)\|_\infty$ and continuity of the functions $I_h(t) = \langle F(t,\cdot), h \rangle$ for each $h = h(x)$ in the dense subspace $C_0^\infty(\mathbb{R}^n)$. of $L^1(\mathbb{R}^n)$

If we correct $A(t,x)$ on a set of zero measure $t \notin E$ by setting $A(t,x) = F(t,x)$ for such t, we obtain the desired property of continuity. □

In particular, from Lemma 1 it follows that $A(0,x)$ can be defined in such a way that for some set $E \subset \mathbb{R}_+$ of full Lebesgue measure

$$A(t,x) \to A(0,x) \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^n) \text{ as } t \to 0,\ t \in E. \tag{3.6}$$

3.2. Definition of a generalized solution.

We consider the Cauchy problem for the transport equation $Au_t + (B, \nabla_x u) = 0$ written in the divergence form (by the condition (3.2))

$$(Au)_t + \operatorname{div}_x Bu = 0, \tag{3.7}$$

with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^n). \tag{3.8}$$

Definition 2. A function $u = u(t,x) \in L^\infty(\Pi)$ is called a *generalized solution* of the Cauchy problem (3.7), (3.8) if for any test function $h = h(t,x)$ in the space $C_0^\infty(\bar\Pi)$ of infinitely differentiable functions with compact support in $\bar\Pi = [0,+\infty) \times \mathbb{R}^n$ we have

$$\int\limits_{\Pi} [Auh_t + (Bu, \nabla_x h)]dtdx + \int\limits_{\mathbb{R}^n} A(0,x)u_0(x)h(0,x)dx = 0. \tag{3.9}$$

Below we give one useful equivalent definition of a generalized solution.

Proposition 2. *A function $u = u(t,x) \in L^\infty(\Pi)$ is a generalized solution of the problem* (3.7), (3.8) *if and only if* $(Au)_t + \mathrm{div}_x Bu = 0$ *in* $\mathcal{D}'(\Pi)$ *and*

$$\operatorname*{ess\,lim}_{t\to 0+} A(t,x)u(t,x) = A(0,x)u_0(x) \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^n). \tag{3.10}$$

Proof. Let $u = u(t,x)$ be a generalized solution of the problem (3.7), (3.8). Taking $h \in C_0^\infty(\Pi)$ in (3.9), we find $(Au)_t + \mathrm{div}_x Bu = 0$ in $\mathcal{D}'(\Pi)$. Let $g(x) \in C_0^\infty(\mathbb{R}^n)$. Choose $\gamma(s) \in C_0^\infty(\mathbb{R})$ such that $\gamma(s) \geqslant 0$, $\mathrm{supp}\,\gamma \subset [0,1]$, $\int \gamma(s)ds = 1$. For $\nu \in \mathbb{N}$ we set $\delta_\nu(s) = \nu\gamma(\nu s)$. It is clear that $\delta_\nu \in C_0^\infty(\mathbb{R})$, $\delta_\nu \geqslant 0$, $\mathrm{supp}\,\delta_\nu \subset [0, 1/\nu]$, $\int \delta_\nu(s)ds = 1$. Therefore, the sequence δ_ν converges to the Dirac δ-function in the space of distributions $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$. Let $\chi_\nu(t) = \int\limits_{-\infty}^{t} \delta_\nu(s)ds$. Then $\chi_\nu(t)$ converges pointwise to the Heaviside function as $\nu \to \infty$. Then $h_\nu(t,x) = g(x)\chi_\nu(t_0 - t) \in C_0^\infty(\bar\Pi)$ for $t_0 > 0$ and, by Definition 2,

$$\int\limits_{\Pi} [Au(h_\nu)_t + (Bu, \nabla_x h_\nu)]dtdx + \int\limits_{\mathbb{R}^n} A(0,x)u_0(x)g(x)dx = 0$$

for sufficiently large ν. We transform this equality as follows:

$$\int\limits_{\Pi} A(t,x)u(t,x)g(x)dx\delta_\nu(t_0 - t)dt - \int\limits_{\mathbb{R}^n} A(0,x)u_0(x)g(x)dx$$

$$= \int_{\Pi} (B(t,x)u(t,x), \nabla g(x))\chi_\nu(t_0 - t)dtdx. \tag{3.11}$$

We can find a set of full measure $E \subset \mathbb{R}_+$ consisting of common Lebesgue points of the functions $t \to \int_{\mathbb{R}^n} A(t,x)u(t,x)g(x)dx$ for all $g(x) \in C_0^\infty(\mathbb{R}^n)$. For example, the set E can be defined similarly to (3.5):

$$E = \{t > 0 | (t,x) \ \text{ is a Lebesgue point of } \ A(t,x)u(t,x) \ \text{ for a.e. } \ x \in \mathbb{R}^n\}.$$

Let $t_0 \in E$. Passing to the limit as $\nu \to \infty$ in (3.11), we find

$$\int_{\mathbb{R}^n} A(t_0,x)u(t_0,x)g(x)dx - \int_{\mathbb{R}^n} A(0,x)u_0(x)g(x)dx$$
$$= \int_{(0,t_0]\times\mathbb{R}^n} (B(t,x)u(t,x), \nabla g(x))dtdx \to 0, \quad t_0 \to 0.$$

Thus, for all $g(x) \in C_0^\infty(\mathbb{R})$

$$\int_{\mathbb{R}^n} A(t,x)u(t,x)g(x)dx \to \int_{\mathbb{R}^n} A(0,x)u_0(x)g(x)dx$$

as $t \to 0$, $t \in E$. Since $A(t,\cdot)u(t,\cdot)$ are bounded uniformly in $t \in E$ and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, we conclude that $A(t,\cdot)u(t,\cdot) \to A(0,\cdot)u_0$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.

Conversely, assume that $u(t,x)$ satisfies (3.7) in $\mathcal{D}'(\Pi)$ and $A(t,x)u(t,x) \to A(0,x)u_0(x)$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, belonging to some set $E_1 \subset \mathbb{R}_+$ of full measure. Assume that $h = h(t,x) \in C_0^\infty(\bar{\Pi})$. Then $f_\nu(t,x) = h(t,x)\chi_\nu(t-t_0)$ belongs to $C_0^\infty(\Pi)$ for $t_0 > 0$ and, since $(Au)_t + (Bu)_x = 0$ in $\mathcal{D}'(\Pi)$, we have the equality

$$\int_{\Pi} [Au(f_\nu)_t + (Bu, \nabla_x f_\nu)]dtdx = 0$$

which can be transformed as follows:

$$\int_{\Pi} A(t,x)u(t,x)h(t,x)dx\delta_\nu(t-t_0)dt$$
$$+ \int_{\Pi} [Auh_t + (Bu, \nabla_x h)]\chi_\nu(t-t_0)dtdx = 0.$$

This implies

$$\int\limits_{[t_0,\infty)\times\mathbb{R}^n} [Auh_t + (Bu, \nabla_x h)]dtdx + \int\limits_{\mathbb{R}^n} A(t_0,x)u(t_0,x)h(t_0,x)dx = 0$$

as $\nu \to \infty$ if $t_0 \in E$. Passing to the limit as $t_0 \to 0$, $t_0 \in E \cap E_1$, and taking into account the weak-$*$ convergence $A(t,x)u(t,x) \to A(0,x)u_0(x)$, we derive (3.9). Thus, $u(t,x)$ is a generalized solution of (3.7), (3.8). $\square$

Remark 2. If $u = u(t,x) \in L^\infty(\Pi)$ is a generalized solution of (3.7), then it always has a weak trace, i.e., there exists a function $u_0 \in L^\infty(\mathbb{R}^n)$ such that

$$\operatorname*{ess\,lim}_{t\to 0+} A(t,x)u(t,x) = A(0,x)u_0(x) \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^n).$$

Indeed, applying Lemma 1 to the divergence free field (Au, Bu), we conclude that there exists a weak trace v of $Au(t,\cdot)$ at $t = 0$. Let $M = \|u\|_\infty$. Passing to the weak limit as $t \to 0$ in the inequality $-MA \leqslant Au \leqslant MA$, we find that $-MA(0,x) \leqslant v(x) \leqslant MA(0,x)$ a.e. on $\mathbb{R}^n$. Then $v(x) = A(0,x)u_0(x)$ for some function $u_0(x) \in L^\infty(\mathbb{R}^n)$ such that $\|u_0\|_\infty \leqslant M$. Thus, any bounded generalized solution of (3.7) is a generalized solution of some Cauchy problem (3.7), (3.8).

In the case of smooth coefficients, any generalized solution must be constant along the characteristics, as in the classical case. More exactly, suppose that $\Omega \subset \mathbb{R}\times\mathbb{R}^n$ is an open domain, $A \in C^1(\Omega)$, $A > 0$, $B \in C^1(\Omega, \mathbb{R}^n)$, $A_t + \operatorname{div}_x B = 0$ in Ω. Consider Equation (3.7) in Ω. We call a function $u = u(t,x) \in L^1_{\text{loc}}(\Omega)$ a *generalized solution* of this equation if $(Au)_t + \operatorname{div}_x Bu = 0$ in $\mathcal{D}'(\Omega)$ (no initial conditions are prescribed).

Proposition 3. *A function $u = u(t,x) \in L^1_{\text{loc}}(\Omega)$ is a generalized solution of (3.7) if and only if $u(t,x)$ is constant along the characteristics, upon a correction on a set of null Lebesgue measure.*

Proof. The statement has local character, and we can assume that the domain $\Omega = \{(t,x) | t = t(\tau;y), x = x(\tau;y), |\tau| < h, |y - x_0| < r\}$, where $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^n$ is a fixed point and $t = t(\tau;y)$, $x = x(\tau;y)$, is the characteristic passing through the point (t_0,y) at $\tau = 0$, i.e., it is the integral curve of the solution to the Cauchy problem for the characteristic system

$$\dot{t} = A(t,x), \quad \dot{x} = B(t,x); \quad t(0) = t_0,\ x(0) = y,$$

which is written in the parametric form. We assume that $h, r > 0$ are chosen so small that any solution of this system is determined in the interval $(-h, h)$.

Introduce the cylinder $\tilde{\Omega} = (-h, h) \times V_r(x_0)$ with $V_r(x_0) = \{y \in \mathbb{R}^n \| y - x_0| < r\}$ and the mapping $F : \tilde{\Omega} \to \Omega$, $F(\tau, y) = (t(\tau; y), x(\tau; y))$. It is clear that F is a diffeomorphism from $\tilde{\Omega}$ onto Ω and, since $A_t + \mathrm{div}_x B = 0$, this mapping conserves the Lebesgue measure. For $u(t, x) \in L^1_{\mathrm{loc}}(\Omega)$ we set $\tilde{u} = (u \circ F)(\tau, y) \in L^1_{\mathrm{loc}}(\tilde{\Omega})$. If $\tilde{h} = \tilde{h}(\tau, y) \in C^1_0(\tilde{\Omega})$ is a test function, then $\tilde{h} = (h \circ F)(\tau, y)$, where $h = h(t, x) = (\tilde{h} \circ F^{-1})(t, x) \in C^1_0(\Omega)$. Since

$$\tilde{h}_\tau(\tau, y) = [h_t t_\tau + (\nabla_x h, x_\tau)](\tau, y) = [Ah_t + (B, \nabla_x h)](\tau, y),$$

we have

$$\begin{aligned} \int_{\tilde{\Omega}} \tilde{u}(\tau, y) \tilde{h}_\tau(\tau, y) d\tau dy &= \int_{\tilde{\Omega}} \tilde{u}(\tau, y)[Ah_t + (B, \nabla_x h)](\tau, y) d\tau dy \\ &= \int_{\Omega} u(t, x)[Ah_t + (B, \nabla_x h)](t, x) dt dx. \end{aligned} \tag{3.12}$$

In the last equality, we make the change of variables $(\tau, y) \to (t, x) = F(\tau, y)$ and take into account that the mapping F conserves the Lebesgue measure. In (3.12), $\tilde{h} \in C^1_0(\tilde{\Omega})$ and, respectively, $h \in C^1_0(\Omega)$ can be arbitrary. Hence $(Au)_t + \mathrm{div}_x Bu = 0$ in $\mathcal{D}'(\Omega)$ if and only if $\tilde{u}_\tau = 0$ in $\mathcal{D}'(\tilde{\Omega})$. The last identity means that, after a correction on a set of null Lebesgue measure, $\tilde{u}$ is independent of τ, i.e., $u(t, x)$ is constant along each characteristic $t = t(\tau; y)$, $x = x(\tau; y)$. $\square$

In the case of smooth coefficients, from Proposition 3 it follows that a generalized solution $u = u(t, x)$ satisfies the *renormalization property*, i.e., the function $p \circ u$ is also a generalized solution for any $p\,(z) \in C(\mathbb{R})$.

4. Existence, Uniqueness, and Renormalization Property

4.1. Existence and nonuniqueness in multi-dimensional case.

We prove the existence of a generalized solution in the case of an arbitrary dimension n by using the regularization of coefficients.

Theorem 1. *There exists a generalized solution $u(t,x)$ of the problem* (3.7), (3.8).

Proof. Let $\gamma(x) \in C_0^\infty(\mathbb{R})$ be such that $\gamma(x) \geqslant 0$, $\operatorname{supp}\gamma \subset [-1,0]$,

$$\int \gamma(x)dx = 1.$$

For $\nu \in \mathbb{N}$, $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n$ we set

$$\delta_\nu(\tau,\xi) = \nu^{n+1}\gamma(\nu\tau)\cdot\prod_{i=1}^n \gamma(\nu\xi_i).$$

It is clear that $\delta_\nu \in C_0^\infty(\mathbb{R}^{n+1})$, $\delta_\nu \geqslant 0$, $\operatorname{supp}\delta_\nu \subset [-1/\nu, 0]^{n+1}$,

$$\int \delta_\nu(\tau,\xi)d\tau d\xi = 1$$

(therefore, δ_ν converges to the Dirac δ-function in $\mathcal{D}'(\mathbb{R}^{n+1})$ as $\nu\to\infty$). Introduce the following approximation sequences for the coefficients $A(t,x)$, $B(t,x)$:

$$A_\nu(t,x) = (A*\delta_\nu)(t,x) + \frac{1}{\nu} = \int A(t-\tau, x-\xi)\delta_\nu(\tau,\xi)d\tau d\xi + \frac{1}{\nu};$$

$$B_\nu(t,x) = (B*\delta_\nu)(t,x) = \int B(t-\tau, x-\xi)\delta_\nu(\tau,\xi)d\tau d\xi \in \mathbb{R}^n.$$

Note that the convolutions are well defined for $(t,x)\in\Pi$ because $t-\tau \geqslant 0$ for $(\tau,\xi)\in\operatorname{supp}\delta_\nu$. By the known properties of averaged functions, $A_\nu, B_\nu \in C^\infty(\Pi)$, $A_\nu \geqslant 1/\nu$ (since $A \geqslant 0$ a.e. on Π) and $A_\nu \to A$, $B_\nu \to B$ in $L^1_{\mathrm{loc}}(\Pi)$ as $\nu\to\infty$. From (3.1), (3.2) it follows that

$$\begin{aligned} |B_\nu(t,x)| &\leqslant \int |B(t-\tau,x-\xi)|\delta_\nu(\tau,\xi)d\tau d\xi \\ &\leqslant N(1/\nu)\int (A(t-\tau,x-\xi)+1/\nu)\delta_\nu(\tau,\xi)d\tau d\xi \\ &= N(1/\nu)A_\nu(t,x), \end{aligned} \tag{4.1}$$

$$(A_\nu)_t + \operatorname{div}_x B_\nu = (A_t + \operatorname{div}_x B)*\delta_\nu = 0. \tag{4.2}$$

Let $C_\nu(t,x) = B_\nu(t,x)/A_\nu(t,x) \in C^\infty(\Pi,\mathbb{R}^n)$ (recall that $A_\nu \geqslant 1/\nu$). By (4.1), we have $|C_\nu| \leqslant N(1/\nu)$. Choose a sequence $u_{0\nu}(x) \in C^1(\mathbb{R}^n)$ such that $u_{0\nu} \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\nu\to\infty$ and $\|u_{0\nu}\|_\infty \leqslant M = \|u_0\|_\infty$. We consider the classical solution $u_\nu(t,x)\in C^1(\Pi)$ of the Cauchy problem

$$u_t + (C_\nu, \nabla_x u) = 0, \quad u(0,x) = u_{0\nu}(x). \tag{4.3}$$

The solution $u_\nu(t,x)$ of the problem (4.3) is uniquely determined by the condition that it remains constant along the characteristics (see Section 1): $u_\nu(t,x) = u_{0\nu}(y_\nu(t,x))$, where $y_\nu(t,x)$ is the source point of the characteristic of (4.3) passing through the point (t,x). It is obvious that the solutions u_ν satisfy the following "maximum principle:"

$$\forall (t,x) \in \Pi \qquad |u_\nu(t,x)| \leqslant \|u_{0\nu}\|_\infty \leqslant M. \tag{4.4}$$

Multiplying the equality $(u_\nu)_t + (C_\nu, \nabla_x u_\nu) = 0$ by A_ν, we find that

$$A_\nu (u_\nu)_t + (B_\nu, \nabla_x u_\nu) = 0$$

pointwise. Taking into account (4.2), we can write this equality in the divergence form:

$$(A_\nu u_\nu)_t + \operatorname{div}_x (B_\nu u_\nu) = 0 \text{ pointwise and in } \mathcal{D}'(\Pi). \tag{4.5}$$

From (4.5) we obtain the following identity for all $h = h(t,x) \in C_0^\infty(\bar{\Pi})$:

$$\int\limits_{\Pi} [A_\nu u_\nu h_t + (B_\nu u_\nu, \nabla_x h)] dt dx + \int\limits_{\mathbb{R}^n} A_\nu(0,x) u_{0\nu}(x) h(0,x) dx = 0. \tag{4.6}$$

By (4.4), the sequence u_ν is bounded in $L^\infty(\Pi)$. Therefore, extracting a subsequence, we can assume that $u_\nu \to u$ weakly-$*$ in $L^\infty(\Pi)$ as $\nu \to \infty$, where $u = u(t,x) \in L^\infty(\Pi)$, $\|u\|_\infty \leqslant M$. Passing to the limit in (4.6) as $\nu \to \infty$ and taking into account the relations $A_\nu \to A$ in $L^1_{\text{loc}}(\Pi)$, $B_\nu \to B$ in $L^1_{\text{loc}}(\Pi, \mathbb{R}^n)$; $u_{0\nu} \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^n)$; $A_\nu(0,x) \to A(0,x)$ weakly-$*$ in $L^\infty(\mathbb{R})$ (the latter easily follows from (3.6)), we derive that the limit function $u(t,x)$ satisfies (3.9), i.e., it is a generalized solution of the problem (3.7), (3.8). $\square$

Unfortunately, the uniqueness of a generalized solution and the important renormalization property (see Theorem 3 below) do not hold in general for $n > 1$ (see recent examples in [**5**], [**6**]). Below we give a construction based on a modification of the example from [**6**].

Example 1. Let $n = 2$. We first introduce the field of coefficients $v = (A, B_1, B_2)$ in the layer $2 \leqslant t \leqslant 4$. Assume that it is 4-periodic with respect to the spatial variables, i.e., $v(t, x_1 + 4, x_2) = v(t, x_1, x_2 + 4) = v(t, x_1, x_2)$. To define the field v in the domain $x_1 \in [0,4)$, $x_2 \in [0,4)$, we set

$$v = \begin{cases} (5-t, 0, x_2), & x_1 \in [0,2), (5-t)x_2 \in [0,4), \\ (5-t, 0, x_2 - 4), & x_1 \in [2,4), (5-t)(4-x_2) \in (0,4], \\ 0 & \text{otherwise} \end{cases}$$

in the strip $3 \leqslant t \leqslant 4$ and

$$v = \begin{cases} (t-1, -x_1, 0), & x_2 \in [0,2), (t-1)x_1 \in [0,4), \\ (t-1, 4-x_1, 0), & x_2 \in [2,4), (t-1)(4-x_1) \in (0,4], \\ 0, & \text{otherwise} \end{cases}$$

in the strip $2 \leqslant t \leqslant 3$.

It is directly verified that the field v is piecewise smooth, is divergence free in the domains of smoothness, and is tangent to the surfaces of discontinuity. This implies that the Rankine–Hugoniot relations hold on these surfaces. Hence $\operatorname{div} v = 0$ in $\mathcal{D}'((2,4) \times \mathbb{R}^2)$. Note also that either $A = B_1 = B_2 = 0$ or $A = A(t) \geqslant 1$, $|B| \leqslant 4$, $B = (B_1, B_2)$. Thus, $|B| \leqslant 4A$ and the condition (3.1) is satisfied with $N(\varepsilon) \equiv 4$.

The characteristics corresponding to the field v in the domains of its smoothness are easily computed. In the region $t \in [3,4]$, $x_1 \in [0,4)$, $x_2 \in [0,4)$, they are the hyperbolas defined by $(5-t)x_2 = \text{const}$, $x_1 = \text{const} \in [0,2)$, and $(5-t)(4-x_2) = \text{const}$, $x_1 = \text{const} \in [2,4)$. The flow along characteristics transforms the rectangles V_1 and V_2 in Fig. 1 into the squares S_1 and S_2 sqeezing the rectangles in the x_2 direction. Similarly, for $t \in [2,3]$ the characteristics starting for $t = 3$ at points of the squares S_1 and S_2 are the hyperbolas $(t-1)x_1 = \text{const}$, $x_2 = \text{const} \in [0,2)$, and $(t-1)(4-x_1) = \text{const}$, $x_2 = \text{const} \in [2,4)$ respectively. The flow along these characteristics expand the squares S_1, S_2 in the x_1-direction producing two horizontal rectangles H_1 and H_2 as in Fig. 1.

Now we define the field v for all $t > 0$ by setting $v(t,x) = v(2^k t, 2^k x)$ if

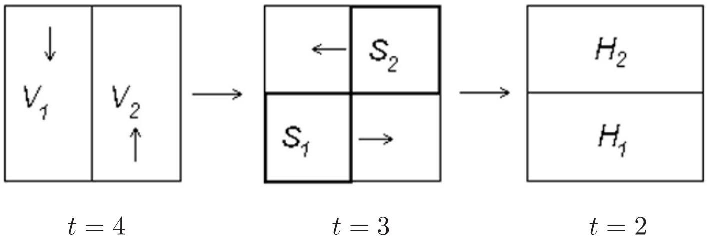


Figure 1. The flow along the characteristics.

$t \in [2 \cdot 2^{-k}, 4 \cdot 2^{-k}]$, $k = 0, 1, 2, \ldots$, and $v(t,x) = (1,0,0)$ if $t \geqslant 4$. Since $A(2,\cdot) = A(4, cdot) = 1$, the coefficient $A(t,\cdot)$ is weakly-$*$ continuous in

$L^\infty(\mathbb{R}^2)$ with respect to $t > 0$, which implies that the field v remains divergence free in the whole half-space $t > 0$. We see also that the condition $|B| \leqslant 4A$ is satisfied. Since the map $t \to A(t, \cdot)$ is weakly-$*$ continuous and $A(2^{-k}, \cdot) = 1$, $k \in \mathbb{N}$, we have $A(0, \cdot) \equiv 1$.

Let $\varphi(s)$ be a 4-periodic function such that $\varphi(s) = -1$ for $s \in [0, 2)$,

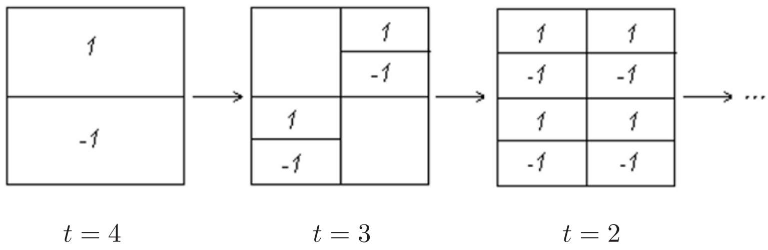


FIGURE 2. The generalized solution $u(t, x)$.

and $\varphi(s) = 1$ for $s \in [2, 4)$. Using the method of characteristics and the properties of the corresponding flow, we can construct a generalized solution of the equation $Au_t + B_1 u_{x_1} + B_2 u_{x_2} = 0$ satisfying the Cauchy condition $u(4, x) = \varphi(x_2)$. This generalized solution is weakly-$*$ continuous in $L^\infty(\mathbb{R}^2)$ with respect to t up to the boundary $t = 0$, and $Au(4 \cdot 2^{-k}, x) = \varphi(2^k x_2)$, $k \in \mathbb{N}$ (see Fig. 2). Thus, the weak trace of $Au(t, \cdot)$ coincides with the weak limit of the sequence $\varphi(2^k x_2)$, which is equal to zero. We conclude that u is a nontrivial generalized solution of our equation with zero initial data. Therefore, the generalized solution of the Cauchy problem under consideration is not unique.

4.2. One-dimensional case.

The above example shows that additional regularity assumptions are necessary to guarantee the well-posedness of the Cauchy problem for the multidimensional transport equation. For example, in the famous paper by DiPerna and Lions [**7**], the uniqueness of a generalized solution and renormalization property were established under the condition that the coefficients belong to the Sobolev space. Similar results were obtained in [**1, 2**] for coefficients in the BV-space. Some other regularity assumptions were considered in [**4, 6, 17, 18**]. It turns out that in the one-dimensional situation $n = 1$, no regularity assumptions are required. Apparently, this fact was first observed in [**19**] (see also [**20**]), where the existence of a generalized solution satisfying the renormalization property was established for the transport equations arising in the study of plane electromagnetic waves. The theory of generalized solutions in the one-dimensional case was also developed in [**13**]–[**15**] as a technical tool for the study of some nonstrictly hyperbolic systems of conservation laws. Below we present some results (including many new results) of this theory.

We restrict ourselves to the one-dimensional case $n = 1$ where the problem (3.7), (3.8) has the simple form

$$(Au)_t + (Bu)_x = 0, (t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}, \tag{4.7}$$

$$u(0,x) = u_0(x), \tag{4.8}$$

where $A, B \in L^\infty(\Pi)$ satisfy the conditions (3.1), (3.2), i.e.,

$$\forall \varepsilon > 0 \ |B| \leqslant N(\varepsilon) \cdot (A + \varepsilon) \text{ a.e. on } \Pi \ , \ \varepsilon N(\varepsilon) \to 0 \text{ as } \varepsilon \to 0+, \tag{4.9}$$

$$A_t + B_x = 0 \text{ in } \mathcal{D}'(\Pi). \tag{4.10}$$

4.2.1. Preliminaries. We need some technical assertions.

Lemma 2. *Suppose that*

$$\alpha_t + \beta_x \leqslant 0 \ \textit{in } \mathcal{D}'(\Pi), \ \alpha = \alpha(t,x), \ \beta = \beta(t,x) \in L^\infty(\Pi),$$
$$\operatorname*{ess\,lim}_{t \to 0+} \alpha(t,x) = \alpha(0,x) \ \textit{weakly-}* \textit{ in } L^\infty(\mathbb{R})$$

and for all $\varepsilon > 0$

$$|\beta(t,x)| \leqslant C(\varepsilon)(\alpha(t,x) + \varepsilon) \quad \textit{a.e. in } \Pi,$$

where $C(\varepsilon) \geqslant 1$. *Let* $c > 0$. *Then for a.e.* $t > 0$

$$\int \alpha(t,x) e^{-c|x|} dx \leqslant e^{ct} \cdot \inf_{\varepsilon > 0} \left(\int \alpha(0,x) e^{-c|x|/C(\varepsilon)} dx + 2\varepsilon C(\varepsilon)/c \right).$$

Proof. Let $\varepsilon > 0$. Multiplying the inequality $(\alpha + \varepsilon)_t + \beta_x \leqslant 0$ by the nonnegative function $g_\varepsilon(x) = e^{-c|x|/C(\varepsilon)}$ and integrating over $x \in \mathbb{R}$, we find

$$\frac{d}{dt} \int (\alpha(t,x) + \varepsilon) g_\varepsilon(x) dx \leqslant \int \beta(t,x) g'_\varepsilon(x) dx \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

Since $|\beta(t,x)| \leqslant C(\varepsilon)(\alpha(t,x) + \varepsilon)$ a.e. on Π and $|g'_\varepsilon(x)| = c g_\varepsilon(x)/C(\varepsilon)$, we have

$$\int \beta(t,x) g'_\varepsilon(x) dx \leqslant c \int (\alpha(t,x) + \varepsilon) g_\varepsilon(x) dx \quad \text{for a.e. } t > 0$$

and

$$H'_\varepsilon(t) \leqslant cH_\varepsilon(t) \quad \text{in } \mathcal{D}'(\mathbb{R}_+),$$

where

$$H_\varepsilon(t) = \int (\alpha(t,x) + \varepsilon) g_\varepsilon(x) dx.$$

By the Gronwall lemma, for a.e. $t > 0$ we have

$$H_\varepsilon(t) \leqslant e^{ct} \cdot H_\varepsilon(0),$$

where

$$H_\varepsilon(0) = \operatorname*{ess\,lim}_{t\to 0+} H_\varepsilon(t) = \int (\alpha(0,x) + \varepsilon) g_\varepsilon(x) dx.$$

Further, by the condition $C(\varepsilon) \geqslant 1$, we have $g_\varepsilon(x) \geqslant e^{-c|x|}$ and, consequently, for a.e. $t > 0$

$$\begin{aligned} \int \alpha(t,x) e^{-c|x|} dx \leqslant H_\varepsilon(t) &\leqslant e^{ct} \cdot H_\varepsilon(0) \\ &= e^{ct}\Big(\int \alpha(0,x) g_\varepsilon(x) dx + \varepsilon \int g_\varepsilon(x) dx \Big). \end{aligned}$$

Since

$$\int g_\varepsilon(x) dx = \int e^{-c|x|/C(\varepsilon)} dx = 2C(\varepsilon)/c, \quad \text{for a.e. } t > 0,$$

we have

$$\int \alpha(t,x) e^{-c|x|} dx \leqslant e^{ct}\Big(\int \alpha(0,x) e^{-c|x|/C(\varepsilon)} dx + 2\varepsilon C(\varepsilon)/c \Big).$$

To complete the proof, it remains to observe that $\varepsilon > 0$ is arbitrary. $\square$

Lemma 3. *Suppose that $A = A(t,x), B = B(t,x) \in L^p_{\text{loc}}(\Pi)$, $1 \leqslant p \leqslant +\infty$, $A_t + B_x = 0$ in $\mathcal{D}'(\Pi)$, $1 \leqslant q \leqslant +\infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $u = u(t,x)$ belongs to the Sobolev space $W^1_{q,loc}$. Then $(Au)_t + (Bu)_x = Au_t + Bu_x$ in $\mathcal{D}'(\Pi)$.*

Proof. Let $\gamma(x) \in C_0^\infty(\mathbb{R})$ be such that $\gamma(x) \geqslant 0$, $\operatorname{supp}\gamma \subset [-1,0]$, and $\int \gamma(x) dx = 1$. As in the proof of Theorem 1, for $\nu \in \mathbb{N}$, $(\tau,\xi) \in \mathbb{R}^2$ we set $\delta_\nu(\tau,\xi) = \nu^2 \gamma(\nu\tau) \cdot \gamma(\nu\xi)$. Then

$$\delta_\nu \in C_0^\infty(\mathbb{R}^2),\ \delta_\nu \geqslant 0,\ \operatorname{supp}\delta_\nu \subset [-1/\nu, 0] \times [-1/\nu, 0],\ \int \delta_\nu(\tau,\xi) d\tau d\xi = 1$$

(thus, the sequence δ_ν converges to the Dirac δ-function in $\mathcal{D}'(\mathbb{R}^2)$). Introduce the sequence of averaged functions

$$u_\nu(t,x) = (u * \delta_\nu)(t,x) = \int u(t-\tau, x-\xi)\delta_\nu(\tau,\xi)d\tau d\xi.$$

By the known properties of averaged functions, $u_\nu \in C^\infty(\Pi)$, $u_\nu \to u$ in $W^1_{q,loc}$ as $\nu \to \infty$ if $q < \infty$. For $q = \infty$ the sequence u_ν is bounded in $W^1_{\infty,loc}$, and $u_\nu \to u$ in $W^1_{1,loc}$ as $\nu \to \infty$. In any case,

$$Au_\nu \to Au, Bu_\nu \to Bu, A(u_\nu)_t \to Au_t,\ B(u_\nu)_x \to Bu_x \text{ in } L^1_{\text{loc}}(\Pi) \quad (4.11)$$

as $\nu \to \infty$ (in the case $q = \infty$, we can use the Lebesgue theorem on dominated convergence).

Now, we choose a test function $h \in C_0^\infty(\Pi)$. Then $hu_\nu \in C_0^\infty(\Pi)$ for all $\nu \in \mathbb{N}$ and applying the equality $A_t + B_x = 0$ to these test functions, we get, after simple transformations

$$\int_\Pi [Au_\nu h_t + Bu_\nu h_x]dtdx + \int_\Pi [A(u_\nu)_t + B(u_\nu)_x]hdtdx = 0.$$

Passing to the limit in this relation as $\nu \to \infty$ and taking into account (4.11), we derive that

$$\int_\Pi [Auh_t + Buh_x]dtdx = -\int_\Pi [Au_t + Bu_x]hdtdx.$$

Since a test function $h \in C_0^\infty(\Pi)$ is arbitrary, we conclude that $(Au)_t + (Bu)_x = Au_t + Bu_x$ in $\mathcal{D}'(\Pi)$. $\square$

4.2.2. Uniqueness of a generalized solution. Now, we are ready to prove the uniqueness of a generalized solution of the problem (4.7), (4.8).

Theorem 2 (uniqueness). *If $A(0,x)u_0(x) = 0$ almost everywhere on $\mathbb{R}$, then $A(t,x)u(t,x) = 0$ almost everywhere on Π.*

Proof. Let $A(0,x)u_0(x) = 0$ a.e. on $\mathbb{R}$. Since $(Au)_t + (Bu)_x = 0$ in $\mathcal{D}'(\Pi)$, there exists a Lipschitz function $Q(t,x) \in W^1_\infty(\Pi)$ (a potential) which is uniquely determined by the conditions

$$Q_t = -Bu, Q_x = Au \text{ in } \mathcal{D}'(\Pi), Q(0,0) = 0. \quad (4.12)$$

Let $r(z) \in C^1(\mathbb{R})$, $0 < r(z) \leqslant 1$, for $z \neq 0$, $r(0) = 0$; $|r'(z)| \leqslant 1$ (for example, we can take $r(z) = z^2/(1+z^2)$. It is clear that $r(Q) \in W^1_\infty(\Pi)$ and

$$r(Q)_t = r'(Q)Q_t = -r'(Q)Bu, \quad r(Q)_x = r'(Q)Q_x = r'(Q)Au.$$

Then $A(r(Q))_t + B(r(Q))_x = r'(Q) \cdot (-ABu + ABu) = 0$ and, by Lemma 3 (with arbitrary p and q),

$$(Ar(Q))_t + (Br(Q))_x = 0 \text{ in } \mathcal{D}'(\Pi). \tag{4.13}$$

We need to prove that $Q \equiv 0$. We first show that $Q(0,x) \equiv 0$. For this purpose, we choose functions $g(x) \in C_0^\infty(\mathbb{R})$, $f(t) \in C^\infty([0,+\infty))$, $f(0) = 1$, $f(t) = 0$, $t \geqslant 1$, and set $h(t,x) = f(\nu t)g(x)$, $\nu \in \mathbb{N}$ in (3.9).

Since $A(0,x)u_0(x) = 0$ a.e. on $\mathbb{R}$, we obtain the equality

$$\nu \int\limits_{\Pi} A(t,x)u(t,x)g(x)f'(\nu t)dtdx = -\int\limits_{\Pi} B(t,x)u(t,x)g'(x)f(\nu t)dtdx.$$

By the equality $Au = Q_x$ in $\mathcal{D}'(\Pi)$, we can transform the first integral by integrating by parts. Then

$$\nu \int\limits_{\Pi} Q(t,x)g'(x)f'(\nu t)dtdx = \int\limits_{\Pi} B(t,x)u(t,x)g'(x)f(\nu t)dtdx.$$

Taking into account the continuity of Q and properties of f, we see that the left-hand side of the above equality converges to $-\int\limits_{\mathbb{R}} Q(0,x)g'(x)dx$ as $\nu \to \infty$ while the modulus of the right integral is bounded by const/ν and therefore converges to zero. Thus, $\int\limits_{\mathbb{R}} Q(0,x)g'(x)dx = 0$ and, since $g(x) \in C_0^\infty(\mathbb{R})$ is arbitrary, $\dfrac{d}{dx}Q(0,x) = 0$ in $\mathcal{D}'(\mathbb{R})$. Consequently, $Q(0,x) \equiv Q(0,0) = 0$.

To prove the identity $Q \equiv 0$, we note that

$$|Br(Q)| \leqslant N(\varepsilon)(A+\varepsilon)r(Q) \leqslant N(\varepsilon)(Ar(Q)+\varepsilon) \ \forall \varepsilon > 0$$

in view of (4.9) and the estimate $r(Q) \leqslant 1$. We can assume that $N(\varepsilon) \geqslant 1$. Thus, $\alpha(t,x) = A(t,x)r(Q(t,x))$ and $\beta(t,x) = B(t,x)r(Q(t,x))$ satisfy the assumptions of Lemma 2 (we also should take into account (4.13)) with $\alpha(0,x) = A(0,x)r(Q(0,x)) \equiv 0$ and $C(\varepsilon) = N(\varepsilon)$. By Lemma 2 (with $c = 1$) and (4.9), for a.e. $t > 0$

$$\int A(t,x)r(Q(t,x))e^{-|x|}dx \leqslant 2e^t \inf_{\varepsilon>0} \varepsilon N(\varepsilon) = 0.$$

This means that $A(t,x)r(Q(t,x)) = 0$ a.e. on Π, which implies $Aur(Q) = Bur(Q) = 0$ a.e. on Π (we take into account that $B = 0$ a.e. on the set, where $A = 0$). Let $R(z)$ be a primitive for $r(z)$ so that $R'(z) = r(z)$. It is obvious that $R(z)$ strictly increases. Then, in the sense of distributions,

$$R(Q)_t = r(Q)Q_t = -Bur(Q) = 0, R(Q)_x = r(Q)Q_x = Aur(Q) = 0,$$

which implies $R(Q) = \text{const}$. Using the strict monotonicity of $R(z)$, we conclude that $Q \equiv Q(0,0) = 0$. This implies $Au = Q_x = 0$ a.e. on Π. $\square$

Let $\rho(x) = e^{-|x|}$. We introduce finite measures $\mu = A(0,x)\rho(x)dx$ and $m = A(t,x)\rho(x)e^{-t}dtdx$ on $\mathbb{R}$ and Π respectively. By Definition 2, in the setting of the problem (4.7), (4.8), the behavior of $u_0(x)$ and $u(t,x)$ is essential only on the sets where $A(0,x) \neq 0$ and $A(t,x) \neq 0$ respectively. We also should take into account the implication $A(t,x) = 0 \Rightarrow B(t,x) = 0$ for a.e. (t,x), which directly follows from Proposition 1. Thus, we can suppose that $u_0 \in L^\infty(\mathbb{R}, d\mu)$, $u \in L^\infty(\Pi, dm)$. In such a setting, Theorems 1 and 2 can be formulated as the existence and uniqueness of a generalized solution in $L^\infty(\Pi, dm)$.

Remark 3. In the proof of Theorem 1, we also obtain the maximum principle: $\|u\|_{L^\infty(\Pi,dm)} \leqslant \|u_0\|_{L^\infty(\mathbb{R},d\mu)}$. This implies that, if $a \leqslant u_0(x) \leqslant b$ μ-a.e. on $\mathbb{R}$, $a, b \in \mathbb{R}$, then $a \leqslant u(t,x) \leqslant b$ m-a.e. on Π. Indeed, it suffices to apply the maximum principle to the generalized solution $u - (a+b)/2$. Suppose that $u = u(t,x)$ and $v = v(t,x)$ are two generalized solutions of the problem (4.7), (4.8) with initial data $u_0(x)$, $v_0(x)$, and $u_0 \leqslant v_0$ μ-a.e. on $\mathbb{R}$. Then $v - u$ is a generalized solution of the problem (4.7), (4.8) with initial function $v_0 - u_0 \geqslant 0$. Therefore, $u \leqslant v$ m-a.e. on Π. We see that generalized solutions satisfy the *comparison principle*, i.e., they depend monotonically on their initial data.

4.2.3. The renormalization property.

Lemma 4. *Let D be the set of absolutely continuous functions $u(x)$ on $\mathbb{R}$ such that $u'(x) = A(0,x)v(x)$ with $v(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then D is dense in $L^2(\mathbb{R}, d\mu)$.*

Proof. It suffice to prove that there are no nontrivial linear continuous functionals $l = l(u)$, $u \in L^2(\mathbb{R}, d\mu)$, such that $l(u) = 0$ on D. By the Riesz theorem, the functional l can be represented as follows:

$$l(u) = l_h(u) = \int u(x)h(x)d\mu(x), \tag{4.14}$$

where $h(x) \in L^2(\mathbb{R}, d\mu)$.

Let

$$H(x) = \int_{-\infty}^{x} h(y)d\mu(y) = \int_{-\infty}^{x} h(y)A(0,y)\rho(y)dy.$$

Note that $h(y) \in L^2(\mathbb{R}, d\mu) \subset L^1(\mathbb{R}, d\mu)$ because the measure μ is finite. Thus, the function $H(x)$ is well defined. The equality (4.14) can be written in the form $l(u) = \int u(x)dH(x)$. If $u = u(x) \in D$, then $u'(x) = A(0,x)v(x)$, where $v(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, and integrating by parts, we find

$$0 = l(u) = \int u(x)dH(x) = -\int H(x)A(0,x)v(x)dx. \tag{4.15}$$

The absence of the term $\big[u(x)H(x)\big]_{-\infty}^{+\infty}$ in (4.15) follows from the relations $\lim\limits_{x\to-\infty} H(x) = 0$ and $\lim\limits_{x\to+\infty} H(x) = \int dH(x) = l(1) = 0$ (note that the constants belong to D) and the boundedness of $u(x)$ in view of the integrability of its derivative.

Since a function $v(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is arbitrary in (4.15), we see that $H(x)A(0,x) = 0$ a.e. on $\mathbb{R}$. This implies that $A(0,x) = 0$ for a.e. x from the open (in view of the continuity of H) set $E = \{x \in \mathbb{R} | H(x) \neq 0\}$. Therefore, $H'(x) = h(x)A(0,x)\rho(x) = 0$ a.e. in E, which implies that H is constant on connected components of E. Since the set of these components is at most countable and $H \equiv 0$ on $\mathbb{R} \setminus E$, the continuous function H takes at most countable set of values on connected domain $\mathbb{R}$, and we conclude that $H \equiv$ const. Thus, $dH(x) = h(x)d\mu(x) = 0$ and $l = 0$ by (4.14). $\square$

Theorem 3. *Let $u(t,x)$ be a generalized solution of the problem* (4.7), (4.8). *Then the following assertions hold.*

(i) *For each function $p\,(z) \in C(\mathbb{R})$ the function $(p \circ u)(t,x)$ is a generalized solution of the problem* (4.7), (4.8) *with initial data $p\,(u_0(x))$* (*the renormalization property*).

(ii) *If $u_0^k(x)$, $k \in \mathbb{N}$ is a bounded in $L^\infty(\mathbb{R}, d\mu)$ sequence such that $A(0,x)(u_0^k(x) - u_0(x)) \to 0$ in $L^1_{\text{loc}}(\mathbb{R})$ as $k \to \infty$ and $u^k(t,x)$ is the corresponding sequence of generalized solutions of the problem* (4.7), (4.8) *with initial functions $u_0^k(x)$, then $A(t,x)(u^k(t,x) - u(t,x)) \to 0$ in $L^1_{\text{loc}}(\Pi)$ as $k \to \infty$.*

Proof. Since $u_0(x) \in L^\infty(\mathbb{R}, d\mu) \subset L^2(\mathbb{R}, d\mu)$, from Lemma 4 if follows that for all $k \in \mathbb{N}$ there exists an absolutely continuous function $u_0^k(x) \in D$, $(u_0^k)'(x) = A(0,x)v_0^k(x)$, $v_0^k(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that

$\|u_0 - u_0^k\|_2 \leqslant 1/k$ (hereinafter, $\|\cdot\|_2$ denotes the norm in the space $L^2(\mathbb{R}, d\mu)$). We also assume that $\|u_0^k\|_\infty \leqslant M = \|u_0\|_\infty$, which can be always achieved by making the change u_0^k with $p\,(u_0^k)$, where $p\,(z) = \max(-M, \min(M, z))$ is a cut-off function. It is easy to see that the above properties of u_0^k remain true under such a replacement. By Theorem 1, there exists a generalized solution $v^k(t,x)$ of the problem (4.7), (4.8) with initial data v_0^k. From the equality $(Av^k)_t + (Bv^k)_x = 0$ in $\mathcal{D}'(\Pi)$ it follows that there exists the Lipschitz potential $u^k(t,x)$ determined by the relations

$$(u^k)_t = -Bv^k, (u^k)_x = Av^k \text{ in } \mathcal{D}'(\Pi), \quad u^k(0,0) = u_0^k(0). \tag{4.16}$$

Using the Lipschitz continuity of the function u^k and Lemma 3, we see that for any $p\,(z) \in C^1(\mathbb{R})$

$$(Ap\,(u^k))_t + (Bp\,(u^k))_x = -ABp'(u^k)v^k + ABp'(u^k)v^k = 0 \text{ in } \mathcal{D}'(\Pi).$$

By the Lipschitz continuity of u^k and relations

$$(u^k)_x(0,x) = A(0,x)v^k(0,x) = (u_0^k)'(x), \quad u^k(0,0) = u_0^k(0)$$

(see (4.16)), we have $u^k(t,\cdot) \to u_0^k$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \to 0$. This, together with (3.6), implies that for bounded $p\,(z) \in C^1(\mathbb{R})$

$$\operatorname*{ess\,lim}_{t\to 0+} A(t,x)p\,(u^k(t,x)) = A(0,x)p\,(u_0^k(x)) \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}).$$

By Proposition 2, we conclude that $p\,(u^k)$ is a generalized solution of the problem (4.7), (4.8) with initial data $p\,(u_0^k(x))$. Taking $p\,(z) = \arctan z$ and applying the maximum principle (see Remark 3), we find

$$|\arctan u^k| \leqslant \|\arctan u_0^k\|_\infty \leqslant \arctan M \quad m\text{-a.e. on} \quad \Pi,$$

which implies $u^k \in L^\infty(\Pi, dm)$ and $\|u^k\|_\infty \leqslant M$. Therefore, in the above arguments, the functions $p\,(z) \in C^1(\mathbb{R})$ can be arbitrary, and we conclude that

$$\begin{gathered}\forall p\,(z) \in C^1(\mathbb{R}) \text{ the function } p\,(u^k(t,x)) \text{ is a generalized solution}\\ \text{of the problem (4.7), (4.8) with initial data } p\,(u_0^k(x)).\end{gathered} \tag{4.17}$$

In particular, $u^k(t,x)$ is a generalized solution of the problem (4.7), (4.8) with initial data u_0^k.

Now, we take $k, r \in \mathbb{N}$ and multiply $A(u^k - u^r)_t + B(u^k - u^r)_x = 0$ by $2(u^k - u^r)$. Taking into account that $u^k - u^r \in W^1_\infty(\Pi)$ and

$$A(u^k - u^r)_t^2 + B(u^k - u^r)_x^2 = 2(u^k - u^r)[A(u^k - u^r)_t + B(u^k - u^r)_x] = 0$$

a.e. on Π, we derive the relation

$$(A(u^k - u^r)^2)_t + (B(u^k - u^r)^2)_x = A(u^k - u^r)_t^2 + B(u^k - u^r)_x^2 = 0 \tag{4.18}$$

in $\mathcal{D}'(\Pi)$ (according to Lemma 3). Further, by (4.9), for $k, r \in \mathbb{N}$ and all $\varepsilon > 0$

$$|B|(u^k - u^r)^2 \leqslant N(\varepsilon)(A+\varepsilon)(u^k - u^r)^2 \leqslant N(\varepsilon)(A(u^k - u^r)^2 + \varepsilon \cdot (u^k - u^r)^2)$$
$$\leqslant N(\varepsilon)(A(u^k - u^r)^2 + 4M^2\varepsilon) \leqslant C(\varepsilon)(A(u^k - u^r)^2 + \varepsilon)$$

a.e. in Π, where $C(\varepsilon) = \max(4M^2, 1)N(\varepsilon) + 1$ (we added 1 to satisfy the condition $C(\varepsilon) \geqslant 1$).

Using (4.18), we conclude that

$$\alpha(t,x) = A \cdot (u^k - u^r)^2, \quad \beta(t,x) = B \cdot (u^k - u^r)^2$$

satisfy the assumptions of Lemma 2 with $\alpha(0,x) = A(0,x) \cdot \big(u_0^k(x) - u_0^r(x)\big)^2$. By Lemma 2 with parameter $c = 1$, we have

$$\int A(t,x) \cdot \big(u^k(t,x) - u^r(t,x)\big)^2 \rho(x) dx \leqslant e^t \omega_{k,r} \tag{4.19}$$

for almost all $t > 0$, where

$$\omega_{k,r} = \inf_{\varepsilon > 0} \bigg(\int A(0,x) \cdot \big(u_0^k(x) - u_0^r(x)\big)^2 e^{-|x|/C(\varepsilon)} dx + 2\varepsilon C(\varepsilon) \bigg). \tag{4.20}$$

The sequence u_0^k converges in $L^2(\mathbb{R}, d\mu)$ to u_0 as $k \to \infty$. Since $\|u_0^k\|_\infty \leqslant M$, we also have $u_0^k \to u_0$ in $L^2(\mathbb{R}, d\mu_\varepsilon)$, $d\mu_\varepsilon = A(0,x)e^{-|x|/C(\varepsilon)}dx$ for all $\varepsilon > 0$. Therefore,

$$F_{k,r} = \int A(0,x) \cdot \big(u_0^k(x) - u_0^r(x)\big)^2 e^{-|x|/C(\varepsilon)} dx \to 0 \text{ as } k, r \to \infty.$$

Formula (4.20) implies that

$$\limsup_{k,r \to \infty} \omega_{k,r} \leqslant \lim_{k,r \to \infty} F_{k,r} + 2\varepsilon C(\varepsilon) = 2\varepsilon C(\varepsilon)$$

for all $\varepsilon > 0$. By (4.9), $\varepsilon C(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Hence $\lim\limits_{k,r \to \infty} \omega_{k,r} = 0$.

Formula (4.19) implies that the sequence $u^k(t,x)$ is fundamental in the spaces $L^2([0,T] \times \mathbb{R}, dm)$, $T > 0$. By the Cauchy criterion, this sequence converges in these spaces to some function $\bar{u} = \bar{u}(t,x)$. It is clear that $\bar{u}(t,x)$ is independent of T if $T > t$. We show that $\bar{u} = u$ in $L^\infty(\Pi, dm)$. By the maximum principle (see Remark 3), the sequence $u_k(t,x)$ is bounded in $L^\infty(\Pi, dm)$. Hence $\bar{u}(t,x) \in L^\infty(\Pi, dm)$. We can assume ("correcting" $\bar{u}$ on a set of m-measure zero, if necessary) that $\bar{u}(t,x) \in L^\infty(\Pi)$. Passing to the limit as $k \to \infty$ in (3.9) with $u = u^k$, $u_0 = u_0^k$, we conclude that $\bar{u}(t,x)$ is a generalized solution of the problem (4.7), (4.8) with initial data $u_0(x)$. Since a generalized solution is unique, $\bar{u} = u$ in $L^\infty(\Pi, dm)$, as was to be shown.

Further, for any $p\,(z) \in C^1(\mathbb{R})$ we have $A \cdot (p\,(u^k) - p\,(u)) \to 0$, $B \cdot (p\,(u^k) - p\,(u)) \to 0$ in $L^1_{\mathrm{loc}}(\Pi)$ as $k \to \infty$ (the second limit relation follows from the first one since u^k is bounded and $A = 0 \Rightarrow B = 0$ a.e. on Π). This enables us to prove assertion (i) for $p\,(z) \in C^1(\mathbb{R})$ by passing to the limit in the integral identity corresponding to (4.17). Finally, every continuous function $p\,(z) \in C(\mathbb{R})$ can be approximated by a sequence $p_k(z) \in C^1(\mathbb{R})$, $k \in \mathbb{N}$, so that $p_k(z) \to p\,(z)$ uniformly on compact sets of $\mathbb{R}$. As was established above, for all $k \in \mathbb{N}$ functions $p_k(u(t,x))$ are generalized solution s of (4.7), (4.8) with initial data $p_k(u_0(x))$. Since $p_k(u(t,x)) \to p\,(u(t,x))$ in $L^1_{\mathrm{loc}}(\Pi)$, $p_k(u_0(x)) \to p\,(u_0(x))$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$, we conclude, passing to the limit in the integral identity corresponding to the generalized solution $p_k(u(t,x))$ as $k \to \infty$, that $p\,(u(t,x))$ is a generalized solution of the problem (4.7), (4.8) with initial data $p\,(u_0(x))$.

To prove assertion (ii), we note that the function $(u^k - u)^2$ is a generalized solution of the problem (4.7), (4.8) with initial data $(u_0^k - u_0)^2$, which follows from assertion (i) and the obvious fact that generalized solutions form a linear space. Repeating the arguments of the proof of (4.19), (4.20), we obtain the bound

$$\int A(t,x) \cdot \big(u^k(t,x) - u(t,x)\big)^2 \rho(x)dx \leqslant e^t \omega_k, \tag{4.21}$$

where

$$\omega_k = \inf_{\varepsilon>0}\bigg(\int A(0,x) \cdot \big(u_0^k(x) - u_0(x)\big)^2 e^{-|x|/C(\varepsilon)}dx + 2\varepsilon C(\varepsilon)\bigg),$$

$C(\varepsilon) = \mathrm{const} \cdot N(\varepsilon) + 1$. By assumption, $A(0,x)(u_0^k(x) - u_0(x)) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$. Hence

$$\lim_{k\to\infty} \int A(0,x) \cdot \big(u_0^k(x) - u_0(x)\big)^2 e^{-|x|/C(\varepsilon)}dx = 0.$$

This implies that $\lim\limits_{k\to\infty} \omega_k = 0$ (see the above arguments concerning $\omega_{k,r}$). From (4.21) we obtain the desired relation $A(t,x)(u^k(t,x) - u(t,x)) \to 0$ in $L^1_{\mathrm{loc}}(\Pi)$. □

Corollary 1. *Let $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ be generalized solutions of* (4.7), (4.8) *with initial data $u_1^0 = u_1^0(x)$, $u_2^0 = u_2^0(x)$ respectively. Then their product u_1u_2 is also a generalized solution of the same problem with initial function $u_1^0u_2^0$.*

PROOF. By assertion (i) of Theorem 3, the function $u_1u_2 = [(u_1 + u_2)^2 - (u_1)^2 - (u_2)^2]/2$ is a generalized solution of the problem (4.7), (4.8) with initial data $u_1^0u_2^0 = [(u_1^0 + u_2^0)^2 - (u_1^0)^2 - (u_2^0)^2]/2$. □

By Corollary 1, the generalized solutions form a subalgebra of the algebra $L^\infty(\Pi, dm)$.

Corollary 2. *If $u(t,x)$ is a generalized solution of* (4.7), (4.8), *then the initial condition* (4.8) *is satisfied in the following strong sense*:

$$\operatorname*{ess\,lim}_{t\to 0} A(t,\cdot)|u(t,\cdot) - u_0| = 0 \ \textit{in}\ L^1_{\mathrm{loc}}(\mathbb{R}).$$

In particular, if $A(t,x)$ has the strong trace at $t = 0$, i.e., $\operatorname*{ess\,lim}_{t\to 0} A(t,x) = A(0,x)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$, then $Au(t,\cdot)$ has a strong trace $A(0,\cdot)u_0$ at $t = 0$:

$$\operatorname*{ess\,lim}_{t\to 0} A(t,\cdot)u(t,\cdot) = A(0,\cdot)u_0 \ \textit{in}\ L^1_{\mathrm{loc}}(\mathbb{R}).$$

PROOF. By assertion (i) of Theorem 3, u^2 is a generalized solution of the problem (4.7), (4.8) with initial function u_0^2. From Proposition 2 it follows that

$$\operatorname*{ess\,lim}_{t\to 0} A(t,\cdot)u(t,\cdot) = A(0,\cdot)u_0, \ \operatorname*{ess\,lim}_{t\to 0} A(t,\cdot)u^2(t,\cdot) = A(0,\cdot)u_0^2$$

weakly-$*$ in $L^\infty(\mathbb{R})$. Let $\rho(x) = e^{-|x|}$. Then, by (3.6) and the above relations, we have

$$\begin{aligned}\int\limits_{\mathbb{R}} A(t,x)[u(t,x) - u_0(x)]^2\rho(x)dx = &\int\limits_{\mathbb{R}}[A(t,x)u^2(t,x) - A(0,x)u_0^2(x)]\rho(x)dx \\ &- 2\int\limits_{\mathbb{R}}[A(t,x)u(t,x) - A(0,x)u_0(x)]u_0(x)\rho(x)dx \\ &+ \int\limits_{\mathbb{R}}(A(t,x) - A(0,x))u_0^2(x)\rho(x)dx \to 0\end{aligned}$$

essentially as $t \to 0$. Hence, by the Cauchy–Schwartz inequality,

$$\begin{aligned}&\int\limits_{\mathbb{R}} A(t,x)|u(t,x) - u_0(x)|\rho(x)dx \\ &\leqslant \mathrm{const}\cdot\Bigg(\int\limits_{\mathbb{R}} A(t,x)[u(t,x) - u_0(x)]^2\rho(x)dx\Bigg)^{1/2} \to 0,\end{aligned}$$

as required. If, in addition, $A(t,x)$ has the strong trace at $t=0$, then

$$\int_{\mathbb{R}} |A(t,x)u(t,x) - A(0,x)u_0(x)|\rho(x)dx$$
$$\leqslant \int_{\mathbb{R}} A(t,x)|u(t,x)-u_0(x)|\rho(x)dx + \int_{\mathbb{R}} |A(t,x)-A(0,x)||u_0(x)|\rho(x)dx \to 0$$

essentially as $t \to 0$. □

Corollary 3. *Suppose that* $u_0 = u_0(x) \in L^\infty(\mathbb{R})$, $A(0,x)u_0(x) \in L^1(\mathbb{R})$, *and* $u(t,x)$ *is a generalized solution of* (4.7), (4.8) *with initial function* u_0. *Then for a.e.* $t>0$

$$A(t,\cdot)u(t,\cdot) \in L^1(\mathbb{R}), \quad \|A(t,\cdot)u(t,\cdot)\|_1 = \|A(0,x)u_0(x)\|_1.$$

Proof. By Theorem 3, $|u(t,x)|$ is a generalized solution of the problem (4.7), (4.8) with initial data $|u_0(x)|$. We see that $\alpha = A(t,x)|u(t,x)|$ and $\beta = B(t,x)|u(t,x)|$ satisfy the assumptions of Lemma 2 with $\alpha_0 = A(0,x)|u_0(x)|$, $C(\varepsilon) = \|u\|_\infty \cdot N(\varepsilon) + 1$. By this lemma, for a.e. $t>0$

$$\int A(t,x)|u(t,x)|e^{-c|x|}dx$$
$$\leqslant e^{ct} \cdot \inf_{\varepsilon>0}\left(\int A(0,x)|u_0(x)|e^{-c|x|/C(\varepsilon)}dx + 2\varepsilon C(\varepsilon)/c\right)$$
$$\leqslant e^{ct}\int A(0,x)|u_0(x)|dx + 2e^{ct}\inf_{\varepsilon>0}\varepsilon C(\varepsilon)/c = e^{ct}\int A(0,x)|u_0(x)|dx.$$

It is clear that the set of full measure $E \subset \mathbb{R}_+$ such that for $t \in E$ the above relation holds can be chosen the same for rational values of c. Then, passing to the limit in this relation as $c \to 0$, $c \in \mathbb{Q}$, and using the Levy theorem on monotone convergence, we find that for a.e. $t>0$

$$\int A(t,x)|u(t,x)|dx \leqslant \int A(0,x)|u_0(x)|dx. \tag{4.22}$$

By Lemma 1, without loss of generality, we can assume that the mappings $t \to A(t,\cdot)$, $t \to (Au)(t,\cdot)$, $t \geqslant 0$, are weakly-$*$ continuous in $L^\infty(\mathbb{R})$. It is obvious that the estimate (4.22) becomes valid for all $t>0$. Let $\tau>0$, $\tilde{A}(t,x) = A(|\tau-t|,x)$, $\tilde{B}(t,x) = \operatorname{sign}(t-\tau)B(|\tau-t|,x)$. It is easy to verify that the coefficients $\tilde{A}$ and $\tilde{B}$ satisfy the conditions (4.9), (4.10) and $v = v(t,x) = u(|\tau-t|,x)$ is a generalized solution of the Cauchy problem

$$(\tilde{A}v)_t + (\tilde{B}v)_x = 0, \quad v(0,x) = v_0(x) = u(\tau,x).$$

Applying the estimate (4.22) to this solution with $t = \tau$, we find

$$\int A(0,x)|u_0(x)|dx = \int \tilde{A}(\tau,x)|v(\tau,x)|dx$$

$$\leqslant \int \tilde{A}(0,x)|v_0(x)|dx = \int A(\tau,x)|u(\tau,x)|dx$$

and, since $\tau > 0$ is arbitrary, we derive the inequality

$$\int A(0,x)|u_0(x)|dx \leqslant \int A(t,x)|u(t,x)|d,x$$

inverse to (4.22). □

Assertion (i) of Theorem 3, known as *the renormalization property*, was introduced for generalized solutions of multi-dimensional transport equations by DiPerna and Lions [**7**]. This property can be extended in the following way.

Proposition 4. *The renormalization property is satisfied by any bounded Borel function $p(z)$.*

Proof. Let $C > 0$. Denote by F_C the space of Borel functions $p(z)$ on $\mathbb{R}$ satisfying the renormalization property and such that $|p(z)| \leqslant C$. Let us show that F_C is a Borel family, i.e., F_C contains pointwise limits of sequences in F_C. Suppose that $p_n(z) \in F_C$, $n \in \mathbb{N}$, and $p_n(z) \to p(z)$ pointwise as $n \to \infty$. It is clear that $p(z)$ is a Borel function and $|p(z)| \leqslant C$. If $u(t,x)$ is a generalized solution of the problem (4.7), (4.8) with initial data $u_0(x)$, then, in view of the condition $p_n(z) \in F_C$, for any $n \in \mathbb{N}$ $p_n(u(t,x))$ is a generalized solution of the problem (4.7), (4.8) with initial data $p(u_0(x))$. Therefore, for all $h = h(t,x) \in C_0^\infty(\bar{\Pi})$

$$\int\limits_{\Pi} p_n(u)[Ah_t + Bh_x]dtdx + \int\limits_{\mathbb{R}} A(0,x)p_n(u_0(x))h(0,x)dx = 0.$$

Since $p_n(u(t,x)) \to p(u(t,x))$ and $p_n(u_0(x)) \to p(u_0(x))$ pointwise as $n \to \infty$ and these sequences are bounded, we can pass to the limit as $n \to \infty$ in the last relation (using the Lebesgue theorem on dominated convergence) and derive that

$$\int\limits_{\Pi} p(u)[Ah_t + Bh_x]dtdx + \int\limits_{\mathbb{R}} A(0,x)p(u_0(x))h(0,x)dx = 0.$$

This means that $p(u(t,x))$ is a generalized solution of the problem (4.7), (4.8) with initial data $p(u_0(x))$. Since u is an arbitrary generalized solution, the renormalization property is satisfied and $p(z) \in F_C$. Since F_C is a Borel family, F_C contains all Borel functions $p(z)$, $|p(z)| \leqslant C$. To complete the proof, it remains to note that $C > 0$ is arbitrary. $\square$

By Proposition 4, we can establish the strict maximum principle.

Proposition 5. *Assume that* $u_0(x) \in L^\infty(\mathbb{R})$*,* $u_0(x) < M$ *μ-a.e. on* $\mathbb{R}$*, and* $u(t,x)$ *is a generalized solution of* (4.7), (4.8) *with initial data* $u_0(x)$*. Then* $u(t,x) < M$ *m-a.e. on* Π*.*

PROOF. Introduce the Heaviside function

$$\theta(s) = \begin{cases} 1, & s \geqslant 0, \\ 0, & s < 0. \end{cases}$$

By Proposition 4, $\theta(u(t,x) - M)$ is a generalized solution of the problem (4.7), (4.8) with initial function $\theta(u_0(x) - M) = 0$ μ-a.e. on $\mathbb{R}$. By the uniqueness of a generalized solution, $\theta(u(t,x) - M) = 0$ m-a.e. on Π. $\square$

Remark 4. In the multi-dimensional case $n > 1$, it may happen that the renormalization property fails even if any generalized solution of the problem (3.7), (3.8) is unique. Indeed, suppose that $n = 2$, $x = (x_1, x_2)$, (A, B_1, B_2) is the divergence free field constructed in Example 1, $\tilde{A} = A(4-t, x)$, $\tilde{B}_1 = -B_1(4-t,x)$, $\tilde{B}_2 = -B_2(4-t,x)$, $0 < t < 4$, $\tilde{A} = 1$, $\tilde{B}_1 = \tilde{B}_2 = 0$, $t \geqslant 4$. Since $A(0,x) \equiv 1$, the field $(\tilde{A}, \tilde{B}_1, \tilde{B}_2)$ is divergence free. Moreover, $|\tilde{B}| \leqslant 4A$, where $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)$. Consider the Cauchy problem

$$(\tilde{A}u)_t + (\tilde{B}_1 u)_{x_1} + (\tilde{B}_2 u)_{x_2} = 0, \quad u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^2). \tag{4.23}$$

In the layer $0 < t < 4$, the field of coefficients is piecewise smooth and tangent to discontinuity surfaces. Using Proposition 3, we see that a generalized solution $u = u(t,x)$ of (4.23) is uniquely determined for $0 < t < 4$ under the condition that it remains constant along the characteristics. By Lemma 1, there exists the weak trace $\tilde{A}u(4,x) = A(0,x)v(x) = v(x)$ at the plane $t = 4$. Hence $u(t,x) = v(x)$ for $t > 4$ because the equation takes the form $u_t = 0$ in this domain. We conclude that a generalized solution of (4.23) is unique for any initial data u_0. Now choose $u_0(x) = \varphi(x_2)$, where the function $\varphi(s)$ is defined in Example 1. Let $\tilde{u}(t,x)$ be the corresponding generalized solution. Then $\tilde{u}(t,x) = u(4-t,x)$ in the layer $0 < t < 4$, where $u(t,x)$ is the generalized solution constructed in Example 1. As was shown in Example 1, $\tilde{A}\tilde{u}(4,x) = Au(0,x) = 0$. In particular, $\tilde{u}(t,x) = 0$ for $t > 4$. Since $(u_0(x))^2 \equiv 1$, we see that $\tilde{u}^2 = 1$ for $0 < t < 4$, whereas $\tilde{u}^2 = 0$ for $t > 4$. Therefore, $\tilde{u}^2$ has a jump at the plane $t = 4$ and it is not a generalized solution. Thus, the renormalization property fails.

4.2.4. Application to the Keyfitz–Kranzer system. Now we can use the above results to prove the existence and uniqueness of a strong generalized entropy solution of the problem (2.1), (2.2). Suppose that $r = r(t, x)$ is a unique generalized entropy solution of the scalar problem (2.4).

Theorem 4. *Suppose that $A = r$, $B = f(r)$, and $v_i = v_i(t, x)$ be generalized solutions of the problem* (2.6), (2.7) *(such solutions exist by Theorem* 1*) with initial data $v_{0i} = u_{0i}/r_0$, $i = 1, \dots, n$ (for $r_0 = 0$ we can take $v_{0i} = 0$). Then the vector-function $u = rv$, $v = (v_1, \dots, v_n)$ is a unique strong generalized entropy solution of the problem* (2.1), (2.2).

PROOF. Since $\|r\|_\infty = M < \infty$, we have $u = rv \in L^\infty(\Pi, \mathbb{R}^n)$. By our assumptions, the vector $v(t, x)$ satisfies Equation (2.6). Therefore,

$$u_t + (\varphi(r)u)_x = (rv)_t + (f(r)v)_x = 0 \text{ in } \mathcal{D}'(\Pi). \tag{4.24}$$

Recall that $r = r(t, x)$ is a generalized entropy solution of (2.4). Taking into account the renormalization property in Theorem 3, we see that the function $|v|^2 = v_1^2 + \dots + v_n^2$ is a generalized solution of the transport equation (2.6) with initial function $|v_0|^2$. Since $r_0|v_0|^2 \equiv r_0 \cdot 1$, we conclude that $v \equiv 1$ is another generalized solution of the same problem. By uniqueness (see Theorem 2), $r|v|^2 = r$ a.e. on Π. We see that $|u|^2 = r^2|v|^2 = r^2$ and $|u| = r$ a.e. on Π. We conclude that $|u|$ is a generalized entropy solution of (2.4). Moreover, since $r = |u|$ in (4.24), we see that u satisfies (2.1) in the sense of distributions. By Corollary 2, the initial condition (2.5) is also satisfied. Thus, $u = u(t, x)$ is a strong generalized entropy solution of (2.1), (2.2).

To prove the uniqueness, suppose that $u_1 = u_1(t, x)$, $u_2 = u_2(t, x)$ are two solutions of the problem (2.1), (2.2). Then $|u_1| = |u_2| = r$ is a unique solution of the scalar problem (2.4). But then $v_1 = u_1/r$, $v_2 = u_2/r$ are two (vector-valued) solutions of the problem (2.6), (2.7) with the same initial data. By uniqueness, $u_1 = rv_1 = rv_2 = u_2$ a.e. on Π. □

5. Generalized Characteristics

Generally speaking, a generalized solution of (4.7), (4.8) is not continuous even if $u_0(x) \in C^\infty(\mathbb{R})$. We confirm this by the following example.

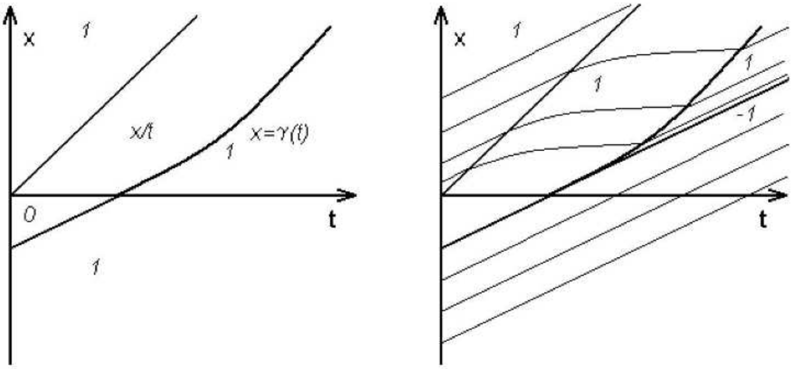
Example 2. Let us take in (4.7) the field of coefficients (A, B) such that $B = A^2/2$ and

$$A(t,x) = \begin{cases} \max(x/t, 0), & \gamma(t) < x < t, \\ 1 & \text{otherwise}, \end{cases}$$

where (see Fig. 3)

$$\gamma(t) = \begin{cases} (t-2)/2, & 0 \leqslant t \leqslant 2, \\ t - \sqrt{2t}, & t > 2. \end{cases}$$

It is clear that the coefficients A and B satisfy the assumption (3.6),



the coefficient A — the map of characteristics and the solution $u(t,x)$

FIGURE 3

where $A(0,x)$ is the characteristic function of the set, where $|2x+1| \geqslant 1$. They also satisfy (4.9) since $B = A^2/2$ (see Proposition 1). To verify the divergence free condition (4.10), we note that this condition is satisfied in the classical sense inside the domains of smoothness of A and B, whereas, on the single discontinuity line $x = \gamma(t)$, we have the Rankine–Hugoniot condition $\dot\gamma(t) = [B]/[A] = 1/2, t \leqslant 2$, $\dot\gamma(t) = (t - \sqrt{2t})' = [B]/[A] = (x+t)/2t$, $t > 2$.

We consider the problem (4.7), (4.8) with initial function $u_0(x) \in C^\infty(\mathbb{R})$ such that $u_0(x) = 1$ for $x \geqslant 0$ and $u_0(x) = -1$ for $x \leqslant -1$. We show that this problem has no continuous generalized solutions. Assuming the contrary, we can find a continuous generalized solution $u(t,x)$. By Proposition 3, this solution must be constant along the characteristics in the regions, where the coefficients are smooth and $A > 0$. On the set, where

$A = 1$, the characteristics are the lines $2x - t = \text{const}$ (see Fig. 3). This and the continuity of u imply that $u(t,x) = 1$ for $x \geqslant t$ and $u(t,x) = -1$ for $2x + 2 \leqslant t$. In the domain D, where $\max(\gamma(t), 0) < x < t$, $A(t,x) = x/t$ and the characteristics are computed as solutions of the differential equation $\dot{x} = B/A = x/2t$. Solving this equation, we obtain $x(t) = C\sqrt{t}$, where $C > 0$ is an arbitrary constant. By the condition $(t, x(t)) \in D$, the characteristic $x(t) = C\sqrt{t}$ is defined for $t \in (C^2, (C + \sqrt{2})^2)$. By the continuity of u, we have $u(t, C\sqrt{t}) = u(C^2, C^2) = 1$ for all $t \in [C^2, (C + \sqrt{2})^2]$. Taking $t = (C + \sqrt{2})^2$, we have $u(t, \gamma(t)) = 1$ for all $t \geqslant 2$. In the domain, where $(t-2)/2 < x < \gamma(t)$, the characteristics are the lines $2x - t = \text{const} > -2$. Therefore, for some function $v(s)$, $s > -2$, we have $u(t,x) = v(2x - t)$. Taking $x = \gamma(t)$, we derive that $v(t - 2\sqrt{2t}) = u(t, \gamma(t)) = 1$ for all $t > 2$, which implies $v \equiv 1$. Hence $u(t,x) = 1$ for $2x + 2 > t > 2$. As was shown above, $u(t,x) = -1$ for $2x + 2 \leqslant t$. We see that the line $2x + 2 = t > 2$ is a discontinuity line, which contradicts our assumption. Observe that the Rankine–Hugoniot condition $\dot{x} = 1/2 = [Bu]/[Au]$ is really satisfied on the line $2x + 2 = t > 2$. Hence the constructed above solution, which can be written in the form $u(t,x) = \operatorname{sign}(2x + 2 - t)$ (we put $u = 1$ on the set, where $A = 0$), is the unique generalized solution of our problem, and it is not continuous.

In the above example $A(t,x) = 0$ on a nontrivial open set. Now we consider the case where

$$\forall t \geqslant 0 \quad \text{the set } \{x \in \mathbb{R} | A(t,x) > 0\} \text{ has positive measure on any nondegenerate interval.} \tag{5.1}$$

We assume that $A(t, \cdot)$ is weakly-$*$ continuous in $L^\infty(\mathbb{R})$, which can be always attained by correcting A on a set of zero measure (see Lemma 1). Let $\hat{\mathbb{R}} = [-\infty, +\infty]$ be the two-point compactification of $\mathbb{R}$, and let $C(\hat{\mathbb{R}})$ be the space of continuous functions on $\hat{\mathbb{R}}$. It is clear that $C(\hat{\mathbb{R}})$ is identified with the space of continuous functions on $\mathbb{R}$ with finite limits at $\pm\infty$. We wish to show that, under the condition (5.1), for initial data $u_0 \in C(\hat{\mathbb{R}})$ the generalized solution $u(t,x)$ of the problem (4.7), (4.8) remains continuous for $t > 0$. For this purpose, we need the following analog of Lemma 4.

Lemma 5. *Let D be the subset of the space of absolutely continuous functions, introduced in Lemma 4. Then D is dense in $C(\hat{\mathbb{R}})$.*

Proof. Since the functions $u(x) \in D$ have finite total variations, they have finite limits at $\pm\infty$ and therefore belong to $C(\hat{\mathbb{R}})$. If $u_1, u_2 \in D$, then $(u_i)'(x) = A(0,x)v_i(x)$ in $\mathcal{D}'(\mathbb{R})$, where $v_i \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $i = 1, 2$. It follows that $(u_1u_2)'(x) = A(0,x)(v_1(x)u_2(x) + v_2(x)u_1(x))$ and, since $v_1u_2 + v_2u_1 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, we conclude that $u_1u_2 \in D$. The established property means that D is a subalgebra of $C(\hat{\mathbb{R}})$. It is clear that $1 \in D$. Let $y < x$. We can choose a function $v \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $v(x) > 0$ on the interval (y, x). Then the function $u(x) \in D$ such that $u'(x) = A(0,x)v(x)$ separates the points x and y, as it follows from the equality

$$u(x) - u(y) = \int_y^x A(0,s)v(s)ds$$

and condition (5.1). By the Stone–Weierstrass theorem, the set D is dense in $C(\hat{\mathbb{R}})$. $\square$

Theorem 5. *Let $u_0 \in C(\hat{\mathbb{R}})$. Then the generalized solution $u = u(t,x)$ of the problem* (4.7), (4.8) *is continuous on $\bar{\Pi}$. Moreover, $u(t,\cdot) \in C(\hat{\mathbb{R}})$ for all $t \geqslant 0$, and $T_tu_0 = u(t,\cdot)$ are isomorphisms of the algebra $C(\hat{\mathbb{R}})$.*

Proof. Let $u_0 \in C(\hat{\mathbb{R}})$. By Lemma 5, we can find a sequence $u_{0k} \in D$, $k \in \mathbb{N}$, uniformly convergent to u_0. By the definition of D, we have $(u_{0k})'(x) = A(0,x)v_{0k}(x)$, $v_{0k}(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Let $u_k = u_k(t,x)$ be a generalized solution of the problem (4.7), (4.8) with initial function u_{0k}. As was shown in the proof of Theorem 3, the solutions u_k are Lipschitz continuous on $\bar{\Pi}$. Moreover, $(u_k)'(t,\cdot) = (Av_k)(t,\cdot)$, where $v_k = v_k(t,x)$ is a generalized solution of the problem (4.7), (4.8) with initial function v_{0k}. By Corollary 3, we have $(Av_k)(t,\cdot) \in L^1(\mathbb{R})$ and $\|(Av_k)(t,\cdot)\|_1 \leqslant \text{const}$ for almost all $t > 0$. This implies that $u_k(t,x)$ has a bounded total variation with respect to x for all $t \geqslant 0$ (because of the continuity of $u_k(t,x)$). Therefore, $u_k(t,\cdot) \in C(\hat{\mathbb{R}})$.

It is clear that $u_k - u$ is a generalized solution of the problem (4.7), (4.8) with initial data $u_{0k} - u_0$. By the maximum principle, $|u_k(t,x) - u(t,x)| \leqslant \|u_{0k} - u_0\|_\infty \to 0$ as $k \to \infty$. Thus, $u_k \to u$ as $k \to \infty$ uniformly on $\bar{\Pi}$. Therefore, the limit function $u(t,x)$ is continuous on $\bar{\Pi}$. Moreover, since $u_k(t,\cdot) \in C(\hat{\mathbb{R}})$, the same is true for the limit function: $u(t,\cdot) \in C(\hat{\mathbb{R}})$. By the maximum principle, $\|u(t,\cdot)\|_\infty \leqslant \|u_0\|_\infty$. Thus, the mappings T_t, defined by the equality $T_tu_0 = u(t,\cdot)$, are bounded linear operators on $C(\hat{\mathbb{R}})$. By Corollary 1, they satisfy the condition $T_t(uv) = T_tuT_tv$, i.e., the operators T_t are homomorphisms of the algebra $C(\hat{\mathbb{R}})$. To prove that T_t are isomorphisms, we fix $t_0 > 0$ and consider the backward Cauchy problem for Equation (4.7) in the half-plane $t < t_0$ with the initial condition $u|_{t=t_0} = v(x) \in C(\hat{\mathbb{R}})$. It is obvious that this problem can be reduced to the canonical form by the change of the time variable $t_0 - t \to t$. Thus, the analogs of our main results remain true. In particular, there exists a unique continuous generalized solution $u(t,x)$ of the backward problem. We set $Sv = u(0,x) \in C(\hat{\mathbb{R}})$. By the uniqueness of a generalized solution, $ST_{t_0}u = u$, $T_{t_0}Sv = v$ for all $u, v \in C(\hat{\mathbb{R}})$, i.e., T_{t_0} is invertible and $S = (T_{t_0})^{-1}$. Since t_0 is arbitrary, the operators T_t are isomorphisms. $\square$

As is known, any isomorphism T of the algebra $C(\hat{\mathbb{R}})$ is generated by a unique homeomorphism F of the compact set $\hat{\mathbb{R}}$ such that $(Tu)(x) = u(F(x))$. Since every homeomorphism of $\hat{\mathbb{R}}$ corresponds to a unique homeomorphism of $\mathbb{R}$, we can consider F as a mapping $F : \mathbb{R} \to \mathbb{R}$ of an "usual" line into itself. The above equality can be written in the equivalent form $(Tu)(G(x)) = u(x)$, where $G = F^{-1}$ is the inverse homeomorphism. In particular, from Theorem 5 it follows that there exists a one-parameter family $G(t,\cdot)$ of homeomorphisms of $\mathbb{R}$ such that $(T_t u_0)(G(t,x)) = u_0(x)$. Hence for any $u_0 \in C(\hat{\mathbb{R}})$ the corresponding generalized solution satisfies the equality $u(t, G(t,x)) = u_0(x)$. We see that generalized solutions are constant along the curves $x = x(t;x_0) = G(t,x_0)$, $x_0 \in \mathbb{R}$. Therefore, these curves can be considered as the characteristics of our equations. In other words, they are trajectories of the ordinary differential equation $\dot{x} = B(t,x)/A(t,x)$. It is obvious that $x(0;x_0) = x_0$. Therefore, the characteristic $x(t;x_0)$ starts at the point x_0 and the function $x(t) = x(t;x_0)$ is a generalized solution of the Cauchy problem

$$\dot{x} = B(t,x)/A(t,x), \quad x(0) = x_0.$$

By construction, this solution exists and is unique. We do not discuss here, in what sense the characteristics $x(t)$ satisfy the above equation, but indicate several important properties of characteristics.

Theorem 6. *The functions $x(t;y)$ are continuous with respect to both variables and depend on source point as an increasing function*: $x(t;y_1) > x(t;y_2)$ *for* $y_1 > y_2$.

Proof. Suppose that $t_0 > 0$, $y_0 \in \mathbb{R}$, $x_0 = x(t_0;y_0)$ and $v(x) \in C(\hat{\mathbb{R}})$ is a function such that $v(x_0) = 0$ and $v'(x) = A(t_0,x)w(x)$, where $w(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $w(x) > 0$. As follows from this condition and (5.1), the function $v(x)$ is strictly increasing. By Theorem 1, there exists a unique generalized solution $w(t,x) \in L^\infty(\Pi)$ of (4.7) such that $w(t_0,x) = w(x)$.

Indeed, to prove this fact, we have to solve the Cauchy problems in the domains $t < t_0$ and $t > t_0$ with initial data $w(x)$ at the time $t = t_0$. By the maximum principle, $w \geqslant 0$ a.e. in Π. Consider the Lipschitz function $u(t, x)$ uniquely determined by the conditions $u_x = Aw$, $u_t = -Bw$, $u(t_0, x_0) = 0$. Then u is a continuous generalized solution of (4.7) and

$$u(t_0, x) = \int_{x_0}^{x} A(t_0, x)w(x)dx = v(x).$$

Since $v(x) \in C(\hat{\mathbb{R}})$, from Theorem 5 it follows that $u(t, \cdot) \in C(\hat{\mathbb{R}})$ for all $t \geqslant 0$. Note that $u_x = Aw \geqslant 0$. Therefore, u increases with respect to x. We fix $\varepsilon > 0$ and set $c = \min(v(x_0 + \varepsilon), -v(x_0 - \varepsilon)) > 0$ (recall that $v(x)$ strictly increases and $v(x_0 - \varepsilon) < v(x_0) = 0 < v(x_0 + \varepsilon)$). Then $u(t_0, x_0 + \varepsilon) = v(x_0 + \varepsilon) \geqslant c$ and $u(t_0, x_0 - \varepsilon) = v(x_0 - \varepsilon) \leqslant -c$. By the continuity of $u(t, x)$, we can find a value $\delta_1 > 0$ such that $u(t, x_0 + \varepsilon) > c/2$, $u(t, x_0 - \varepsilon) < -c/2$ for all $t \in (t_0 - \delta_1, t_0 + \delta_1)$. Since $u(t, x)$ increases with respect to x, we see that

$$|u(t, x)| > c/2, \quad |x - x_0| \geqslant \varepsilon,\ |t - t_0| < \delta_1. \tag{5.2}$$

Let $x(t; y_0)$ be a characteristic passing through the point (t_0, x_0) so that $x(t_0; y_0) = x_0$. Since the functions $y \to x(t_0, y) = G(t_0, y)$ and $v(x)$ are continuous and $v(x_0) = 0$, there exists $\delta_2 > 0$ such that $|v(x(t_0; y))| < c/2$ for $|y - y_0| < \delta_2$. Since the solution $u(t, x)$ remains constant along the characteristic $x(t; y)$, we conclude that $|u(t, x(t; y))| = |v(x(t_0; y))| < c/2$ for all $|y - y_0| < \delta_2$, $|t - t_0| < \delta_1$. By (5.2), $|x(t; y) - x(t_0; y_0)| = |x(t; y) - x_0| < \varepsilon$ in the neighborhood $|y - y_0| < \delta_2$, $|t - t_0| < \delta_1$ of the point (t_0, y_0). This completes the proof of the continuity of characteristics.

To prove the last assertion, suppose that $y_1 > y_2$ and set $h(t) = x(t; y_1) - x(t; y_2)$. As follows from the continuity of $x(t; y)$, $h(t)$ is continuous for $t \geqslant 0$ and $h(t) \neq 0$ because the mapping $y \to x(t; y) = G(t, y)$ is invertible. This, together with the obvious relation $h(0) = y_1 - y_2 > 0$, implies that $h(t) > 0$ for all $t \geqslant 0$, i.e., $x(t; y_1) > x(t; y_2)$. $\square$

Using the above theorem, we derive the following property.

Theorem 7. *Let $a, b \in \mathbb{R}$, $a < b$, and $u(t, x)$ be a generalized solution of* (4.7), (4.8) *with initial data $u_0(x) \in L^\infty(\mathbb{R})$. Then for almost all $t > 0$*

$$\int_{x(t;a)}^{x(t;b)} A(t, x)u(t, x)dx = \int_{a}^{b} A(0, x)u_0(x)dx. \tag{5.3}$$

PROOF. We choose a function $v_0(x) \in C(\mathbb{R})$ such that $v_0(x) > 0$ in the interval $x \in (a,b)$ and $v_0(x) = 0$ for $x \notin (a,b)$. Let $v(t,x)$ be a generalized solution of the problem (4.7), (4.8) with initial function $v_0(x)$. Since $v(x) \in C(\hat{\mathbb{R}})$, from Theorem 5 it follows that $v(t,x) \in C(\bar{\Pi})$ and $v(t,x)$ is constant along characteristics. As follows from Theorem 6, the characteristics $x(t;y)$ started at points $y \in (a,b)$ fill for $t > 0$ the whole domain $\Omega(a,b) = \{(t,x) | t > 0, x(t;a) < x < x(t;b)\}$. This implies that $v(t,x) > 0$ in $\Omega(a,b)$ and $v(t,x) = 0$ outside $\Omega(a,b)$. Choosing a sequence of initial functions $v_{0n} \in C(\mathbb{R})$ with the above properties converging to the indicator function w_0 of the interval (a,b) as $n \to \infty$, we see that the corresponding sequence $v_n(t,x)$ of generalized solutions converges pointwise to the indicator function w of the domain $\Omega(a,b)$. By Theorem 3, w is a generalized solution of the problem (4.7), (4.8) with initial function w_0. By Corollary 1, the product uw is a generalized solution of the problem (4.7), (4.8) with initial function $u_0 w_0$. Suppose that $T > 0$, $m = \min\limits_{t\in[0,T]} x(t;a)$, $M = \max\limits_{t\in[0,T]} x(t;b)$, $h = h(x) \in C_0^\infty(\mathbb{R})$ are such that $h(x) = 1$ on $[m,M]$. Multiplying $(Auw)_t + (Buw)_x = 0$ by a test function h, we find

$$\frac{\partial}{\partial t} \int\limits_{x(t;a)}^{x(t;b)} A(t,x)u(t,x)h(x)dx = \frac{\partial}{\partial t} \int A(t,x)u(t,x)w(t,x)h(x)dx$$

$$= -\int B(t,x)u(t,x)w(t,x)h'(x)dx = 0$$

in $\mathcal{D}'((0,T))$ because $h'(x) = 0$ on $[m,M]$ and $w = 0$ for $x \notin [m,M]$. This implies the desired identity (5.3) (we also take into account that $T > 0$ is arbitrary). $\square$

From Theorem 7 we obtain an important property of a finite domain of dependence of a generalized solution on its initial data.

Corollary 4. *Suppose that $u(t,x)$ is a generalized solution of the problem (4.7), (4.8) and $A(0,x)u_0(x) = 0$ a.e. on (a,b). Then $A(t,x)u(t,x) = 0$ a.e. on $\Omega(a,b)$.*

It suffices to note that, by Theorem 3, $|u(t,x)|$ is a generalized solution of the problem (4.7), (4.8) with initial data $|u_0(x)|$ and to apply (5.3) to this generalized solution.

6. Unbounded Solutions. Notion of a Renormalized Solution

Note that the identity (3.9) makes sense even for unbounded functions $u(t,x)$, $u_0(x)$ such that $A(0,x)u_0(x) \in L^1_{\mathrm{loc}}(\mathbb{R})$, $Au, Bu \in L^1_{\mathrm{loc}}(\bar{\Pi})$. This enables us to extend Definition 2 of a generalized solution of the problem (4.7), (4.8) to this general case. These solutions are referred to as L^1_{loc}-*generalized solutions.* Many properties of generalized solutions remain valid in the unbounded case. In particular, the following assertion holds.

Theorem 8. *Let* $u = u(t,x)$ *be an* L^1_{loc}*-generalized solution of the problem* (4.7), (4.8) *with initial function* $u_0(x)$*. Then the following assertions hold.*

(i) *The generalized solution* u *is unique*: *if* $u_0(x)A(0,x) = 0$ *a.e. on* $\mathbb{R}$*, then* $u(t,x)A(t,x) = 0$ *a.e. on* Π.

(ii) *If* $p(z) \in C(\mathbb{R})$, $|p(z)| \leqslant \mathrm{const}(|z|+1)$*, then* $p(u(t,x))$ *is an* L^1_{loc}*-generalized solution of the problem* (4.7), (4.8) *with initial function* $p(u_0(x))$ (*the renormalization property*)*.*

Proof. The uniqueness is proved in the same way as in the corresponding part of the proof of Theorem 2. Namely, assume that $Au, Bu \in L^1_{\mathrm{loc}}(\bar{\Pi})$ and $A(0,x)u_0(x) = 0$ a.e. on $\mathbb{R}$. From the relation $(Au)_t + (Bu)_x = 0$ in $\mathcal{D}'(\Pi)$ it follows that there exists a potential $Q(t,x)$ in the Sobolev space $W^1_{1,loc}(\Pi)$ which is uniquely determined by the conditions

$$Q_t = -Bu, Q_x = Au \text{ in } \mathcal{D}'(\Pi), \qquad \int_{-1}^{1} Q(0,x)dx = 0.$$

Here, the function $Q(0,x) \in L^1_{\mathrm{loc}}(\mathbb{R})$ is well defined by the trace property for functions in the Sobolev space. By the initial condition (3.10), for the solution $u(t,x)$ we have

$$\frac{d}{dx}Q(0,x) = A(0,x)u_0(x) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R})$$

(to prove this assertion, it suffices to repeat arguments of the proof of Theorem 2). This and the condition $\int_{-1}^{1} Q(0,x)dx = 0$ imply $Q(0,x) = 0$. Let $r(z) \in C^1(\mathbb{R})$, $0 < r(z) \leqslant 1$ for $z \neq 0$, $r(0) = 0$, $|r'(z)| \leqslant 1$ for all $z \in \mathbb{R}$. Then, using the assertion of Lemma 3 with $p = \infty$, we derive the following equality similar to (4.13):

$$(Ar(Q))_t + (Br(Q))_x = A(r(Q))_t + B(r(Q))_x = r'(Q) \cdot (-ABu + ABu) = 0$$

in $\mathcal{D}'(\Pi)$ and $r(Q)(0, x) = r(0) = 0$. We see that $r(Q)$ is a bounded generalized solution of (4.7), (4.8) with zero initial data. By Theorem 2, $Ar(Q) = 0$ a.e. on Π. From Proposition 1 it follows that $Br(Q) = 0$ a.e. on Π. Let $R(z)$ be a primitive of $r(z)$, i.e., $R'(z) = r(z)$. It is clear that $R(z)$ strictly increases and

$$R(Q)_x = Ar(Q)u = 0, R(Q)_t = Br(Q)u = 0 \text{ in } \mathcal{D}'(\Pi).$$

By these relations, $R(Q) = C = \text{const}$ a.e. on Π. Taking into account that $R(z)$ strictly increases, we conclude that $Q = R^{-1}(C)$ a.e. on Π, which implies the desired result $u(t, x)A(t, x) = Q_x = 0$ a.e. on Π.

To prove the second assertion, we put

$$v_k^0(x) = \begin{cases} 1, & |u_0(x)| \leqslant k, \\ 0, & |u_0(x)| > k, \end{cases} \qquad k \in \mathbb{N}.$$

Let $v_k(t, x) \in L^\infty(\Pi)$ be a generalized solution of the problem (4.7), (4.8) with initial data $v_k^0(x)$. By the maximum principle, we can assume that $|v_k(t, x)| \leqslant 1$. (Applying assertion (i) of Theorem 3 with $p\,(z) = z(1 - z)$, we see that $A(t, x)v_k(t, x)(1 - v_k(t, x)) = 0$ a.e. on Π and can suppose that $v_k(t, x)$ is the characteristic function of some set.) Since $v_k^0(x) \to 1$ in $L^1_{\text{loc}}(\mathbb{R})$ as $k \to \infty$ and $\tilde{u}(t, x) \equiv 1$ is a generalized solution of the problem (4.7), (4.8) with initial data $\tilde{u}_0(x) \equiv 1$, assertion (ii) of Theorem 3 implies

$$A(t, x)(v_k(t, x) - 1) \to 0 \text{ in } L^1_{\text{loc}}(\Pi) \tag{6.1}$$

as $k \to \infty$. If $v = v(t, x)$ is a bounded generalized solution of the problem (4.7), (4.8), then uv is also a generalized solution of this problem with initial data u_0v_0 (cf. Corollary 1). Indeed, if $v(t, x)$ is Lipschitz continuous, this follows from Lemma 3 applied for the vector of coefficients (Au, Bu):

$$(Auv)_t + (Buv)_x = (Av_t + Bv_x)u = 0 \text{ in } \mathcal{D}'(\Pi).$$

In the general case, we can use the approximation of $v(t, x)$ in $L^1_{\text{loc}}(\Pi)$ by a sequence of Lipschitz continuous generalized solutions which is bounded in $L^\infty(\Pi)$. The sequence is constructed in a similar way as in the proof of Theorem 3.

Thus, the functions $u_k = v_ku$ are generalized solutions of the problem (4.7), (4.8) with bounded initial data $u_k^0 = v_k^0u_0$, $|u_k^0| \leqslant k$, $k \in \mathbb{N}$. By uniqueness, u_k coincides with the unique bounded generalized solution. Hence $u_k \in L^\infty(\Pi, dm)$, and $\|u_k\|_\infty \leqslant k$ by the maximum principle.

Let $p(z) \in C(\mathbb{R})$, $|p(z)| \leqslant \mathrm{const}(|z|+1)$. By assertion (i) of Theorem 3, $p(u_k)$ is a generalized solution of (4.7), (4.8) with initial function $p(u_k^0)$ for any $k \in \mathbb{N}$. Further, by the relation (6.1), $Au_k \to Au$ in $L^1_{\mathrm{loc}}(\bar{\Pi})$ as $k \to \infty$. Passing to a subsequence (if necessary), we can assume that $Au_k \to Au$ as $k \to \infty$ almost everywhere on Π. Then

$$Ap(u_k) \to Ap(u) \quad Bp(u_k) \to Bp(u) \text{ as } k \to \infty \quad \text{a.e. on } \Pi;$$
$$|Ap(u_k)| \leqslant cA \cdot (|u_k|+1) \leqslant cA \cdot (|u|+1) \in L^1_{\mathrm{loc}}(\bar{\Pi}),$$
$$|Bp(u_k)| \leqslant c|B| \cdot (|u|+1) \in L^1_{\mathrm{loc}}(\bar{\Pi}), \quad c = \mathrm{const} > 0.$$

It is also clear that

$$A(0,\cdot)p(u_k^0) \to A(0,\cdot)p(u_0) \text{as } k \to \infty \quad \text{a.e. on } \mathbb{R},$$
$$|A(0,\cdot)p(u_k^0)| \leqslant cA(0,\cdot)(|u_k^0|+1) \leqslant cA(0,\cdot)(|u_0|+1) \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

By the Lebesgue theorem on dominated convergence, we can pass to the limit as $k \to \infty$ in the identity (3.9) corresponding to the generalized solution $p(u_k)$. Thus, $p(u)$ is an L^1_{loc}-generalized solution of the problem (4.7), (4.8) with initial data $p(u_0)$. $\square$

In the case where $u_0(x)$ is not bounded, it may happen that the problem (4.7), (4.8) has no generalized solutions (even if $A(0,x)u_0(x) \in L^\infty(\mathbb{R})$). Let us confirm this by the following example.

Example 3. In the half-plane, we consider the field of coefficients

$$A(t,x) = \begin{cases} (1-t)^2/x^2, & t<1, x<t-1, \\ 1, & t<1, x \geqslant t-1, \\ \theta(x+2-2t), & t \geqslant 1, \end{cases} \quad B(t,x) = 2\sqrt{A(t,x)},$$

where

$$\theta(r) = \begin{cases} 1, & s \geqslant 0, \\ 0, & s < 0 \end{cases}$$

is the Heaviside function. One can directly verify that the coefficients A and B satisfy the conditions (4.9), (4.10). To prove (4.10), we should take into account that on the single discontinuity line $x = 2t-2$ the Rankine–Hugoniot condition $\dot{x} = [B]/[A] = [2-0]/[1-0]$ is satisfied. Consider the Cauchy problem for the equation

$$(Au)_t + (Bu)_x = 0, \quad u = u(t,x), \quad (t,x) \in \Pi, \tag{6.2}$$

with initial data

$$u(0,x) = u_0(x) = \begin{cases} x^2, & x < -1, \\ 1, & x \geqslant -1. \end{cases} \tag{6.3}$$

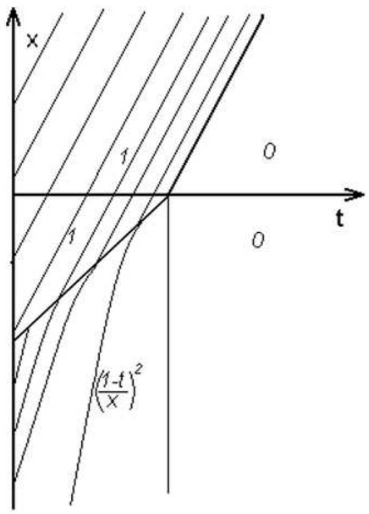


FIGURE 4. The coefficient A and the mapping of characteristics.

We see that $A(0,x)u_0(x) \equiv 1 \in L^\infty(\mathbb{R})$. Let us show that the problem under consideration has no generalized solutions. Assuming the contrary, we can find an $L^1_{\rm loc}$-generalized solution $u(t,x)$ of (6.2), (6.3). In the domain $D = \{(t,x)|\ 0 \leqslant t < 1, x < t-1\}$, where A, B are smooth and $A > 0$, the solution $u(t,x)$ is uniquely determined by the requirement that it is constant along the characteristics (by Proposition 3). These characteristics are solutions of the ordinary differential equation $\dot{x} = B/A = -2x/(1-t)$. A direct computation shows that $x(t;t_0,x_0) = y(1-t)^2$, where $y = y(t_0,x_0) = x_0/(1-t_0)^2$, $(t_0,x_0) \in D$, $0 \leqslant t < 1+(1-t_0)^2/x_0$ (note that $1+(1-t_0)^2/x_0 > t_0$). The mapping of characteristics is indicated in Fig. 4. By the initial condition (6.3), we have $u(t,x) = (y(t,x))^2 = x^2/(1-t)^4$ for $(t,x) \in D$. Then $Au = (1-t)^{-2}$, $Bu = 2|x|(1-t)^{-3} \geqslant 2(1-t)^{-2}$. Since these functions are not integrable in neighborhoods of points $(1,x)$, $x < 0$, we conclude that

$Au, Bu \notin L^1_{\mathrm{loc}}(\Pi)$, which contradicts our assumption. Thus, the problem (6.2), (6.3) has no L^1_{loc}-generalized solutions.

We introduce the notion of a renormalized solution.

The above example shows that for studying the case of unbounded solutions we need to extend the class of solutions. As in [**7**], we introduce the class of renormalized solutions. Suppose that $u_0(x)$ is measurable.

Definition 3. A measurable function $u(t,x)$ on Π is called a *renormalized solution* of the problem (4.7), (4.8) if for any bounded continuous function $p\,(z)$ the function $p\,(u(t,x))$ is a generalized solution of the problem (4.7), (4.8) with initial function $p\,(u_0(x))$.

By Theorem 8(ii), any L^1_{loc}-generalized solution is renormalized.

Theorem 9. *There exists a unique renormalized solution of the problem* (4.7), (4.8). *Moreover, for every Borel function $p\,(z)$ the function $p\,(u(t,x))$ is a renormalized solution of the problem* (4.7), (4.8) *with initial data $p\,(u_0(x))$.*

Proof. Consider a strictly increasing continuous function $h(z)$ such that $\lim\limits_{z\to-\infty} h(z) = 0$ and $\lim\limits_{z\to+\infty} h(z) = 1$ (for example, we can take $h(z) = (\pi + 2\operatorname{arc\,tan} z)/2\pi$). By Theorem 1, there exists a bounded generalized solution $v(t,x)$ with initial data $v_0 = h(u_0(x)) \in L^\infty(\mathbb{R})$. It is clear that $0 < v_0 < 1$ and, applying the strict maximum principle (see Proposition 5) to the generalized solution $\pm v$, we see that $0 < v(t,x) < 1$ m-a.e. on Π. "Correcting" $v(t,x)$ on a set of zero m-measure, we can assume that the last relation holds for all $(t,x) \in \Pi$. Thus, we can define $u(t,x) = h^{-1}(v(t,x))$, where $h^{-1} : (0,1) \to \mathbb{R}$ is the inverse function to h. It is obvious that $u(t,x)$ is measurable. Now, we choose a bounded Borel function $p\,(z)$ and set $q(v) = p\,(h^{-1}(v))$ if $v \in (0,1)$ and $q(v) = 0$ if $v \notin (0,1)$. It is easy to see that $q(v)$ is a bounded Borel function and $p\,(u(t,x)) = q(v(t,x))$, $p\,(u_0(x)) = q(v_0(x))$. From Proposition 4 it follows that $p\,(u(t,x))$ is a generalized solution of the problem (4.7), (4.8) with initial function $p\,(u_0(x))$. In particular, the above property is satisfied for all bounded continuous functions $p\,(z)$ which means that $u(t,x)$ is a renormalized solution. Moreover, we also proved the last assertion of the theorem. Indeed, for all bounded continuous functions $f(z)$ functions $(f \circ p)(v)$ are bounded and Borel and, as was shown above, $f(p\,(u(t,x)))$ is a generalized solution of the problem (4.7), (4.8) with corresponding initial data. Hence $p\,(u(t,x))$ is a renormalized solution of the problem (4.7), (4.8) with initial function $p\,(u_0)$.

To prove uniqueness, we assume that $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ are two renormalized solutions of (4.7), (4.8). Then for any bounded function $p(z) \in C(\mathbb{R})$ we have $p(u_1) = p(u_2)$ m-a.e. on Π by the uniqueness of bounded generalized solutions. Hence $u_1 = u_2$ m-a.e. on Π. $\square$

7. Nonhomogeneous Transport Equations and Renormalization Property

In this section, we study the Cauchy problem for the linear nonhomogeneous transport equation

$$(Au)_t + (Bu)_x = f, \tag{7.1}$$

with initial condition

$$u(0,x) = u_0(x). \tag{7.2}$$

We suppose that $A(0,x)u_0(x) \in L^1_{\text{loc}}(\mathbb{R})$, $f = f(t,x) \in L^1_{\text{loc}}(\bar{\Pi})$. The definition of a generalized solution of the problem (7.1), (7.2) is similar to Definition 2.

Definition 4. A measurable function $u = u(t,x)$ such that $Au, Bu \in L^1_{\text{loc}}(\bar{\Pi})$ is called a *generalized solution* of the Cauchy problem (7.1), (7.2) if for any test function $h = h(t,x) \in C_0^\infty(\bar{\Pi})$

$$\int\limits_{\Pi} [Auh_t + Buh_x + fh]dtdx + \int\limits_{\mathbb{R}} A(0,x)u_0(x)h(0,x)dx = 0. \tag{7.3}$$

It is clear that a generalized solution of the problem (7.1), (7.2) is unique since the difference of two solutions u_1 and u_2 is a generalized solution of the homogeneous problem (4.7), (4.8) with zero initial data. By Theorem 8, $A(u_1 - u_2) = 0$ a.e. on Π, i.e., $u_1 = u_2$ m-a.e. on Π. To prove the existence of a generalized solution and analogs of the renormalization property, we need auxiliary results.

Proposition 6. *Let $u = u(t,x)$ be a generalized solution of the problem* (7.1), (7.2), *and let $v = v(t,x) \in L^\infty(\Pi)$ be a generalized solution of the homogeneous problem* (4.7), (4.8) *with initial function $v_0 = v_0(x)$. Then $w = uv$ is a generalized solution of the problem* (7.1), (7.2) *with initial data $w_0 = u_0v_0$ and source function fv.*

Proof. As was shown in the proof of Theorem 3, there exists a bounded in $L^\infty(\Pi)$ sequence $v_k = v_k(t,x)$ of Lipschitz continuous generalized solutions of the problem (4.7), (4.8) such that $A(v_k)_t + B(v_k)_x = 0$ a.e. on Π, and $v_k \to v$, $v_k(0,x) \to v_0$ as $k \to \infty$ in the spaces $L^1_{\text{loc}}(\Pi, dm)$ and $L^1_{\text{loc}}(\mathbb{R}, d\mu)$ respectively. Let $h = h(t,x) \in C_0^\infty(\bar\Pi)$. Then hv_k is a function with compact support in $\bar\Pi$ and $hv_k \in W^1_\infty(\Pi)$. It is clear that one can choose test functions of such a kind in Definition 4. By (7.3), we have

$$\int_\Pi [Auv_kh_t + Buv_kh_x + fv_kh]dtdx$$
$$+ \int_\Pi [A(v_k)_t + B(v_k)_x]uhdtdx + \int_{\mathbb{R}} A(0,x)u_0(x)v_k(0,x)h(0,x)dx = 0.$$

Since $A(v_k)_t + B(v_k)_x = 0$ a.e. on Π¡ we have

$$\int_\Pi [Auv_kh_t + Buv_kh_x + fv_kh]dtdx + \int_{\mathbb{R}} A(0,x)u_0(x)v_k(0,x)h(0,x)dx = 0.$$

Passing to the limit in this equality as $k \to \infty$ (using, for example, the Lebesgue theorem on dominated convergence), we conclude that for all $h \in C_0^\infty(\bar\Pi)$

$$\int_\Pi [Auvh_t + Buvh_x + fvh]dtdx + \int_{\mathbb{R}} A(0,x)u_0(x)v_0(x)h(0,x)dx = 0,$$

i.e., uv is a generalized solution of the problem (7.1), (7.2) with initial data u_0v_0 and source function fv. □

Corollary 5. *For the existence of a generalized solution of* (7.1), (7.2) *it is necessary that* $f(t,x) = 0$ *a.e. on the set, where* $A = 0$.

Proof. Let $E = A^{-1}(0)$ be the set, where $A = 0$ and $v = v(t,x)$ be the indicator function of E. Then $Av = Bv = 0$ a.e. on Π, which implies that v is a generalized solution of the problem (7.1), (7.2) with zero initial data $A(0,x)v(0,x)$. Suppose that $u = u(t,x)$ is a generalized solution of the problem (7.1), (7.2). By Proposition 6, uv is a generalized solution of the problem (7.1), (7.2) with zero initial data and source function fv. In particular, $(Avu)_t + (Bvu)_x = fv$ in $\mathcal{D}'(\Pi)$. But $Avu = Bvu = 0$ a.e. on Π, and we conclude that $fv = 0$ a.e. on Π. □

We see that the problem (7.1), (7.2) can be well-posed only if

$$f(t,x) = A(t,x)g(t,x) \tag{7.4}$$

with some measurable function $g(t,x)$. Hereinafter, we assume that the condition (7.4) is satisfied.

7.1. The Duhamel principle and properties of generalized solutions.

As was shown in Example 3, for unbounded initial data u_0 a generalized solution of the problem (7.1), (7.2) does not necessarily exist (even if $f \equiv 0$). But for bounded u_0 and g a generalized solution exists and can be constructed by approximation arguments in a similar way as in the proof of Theorem 1. Another method for proving the existence is the Duhamel principle. Let us describe the construction. Without loss of generality, we can assume that $g = 0$ on the set, where $A(t,x) = 0$. Then $g \in L^\infty(\Pi)$. Denote by E the set of $t > 0$ such that for a.e. $x \in \mathbb{R}$ (t,x) is a Lebesgue point of g. It is clear that E is a set of full Lebesgue measure. We also can assume that the mapping $t \to A(t,\cdot)$ is weakly-$*$ continuous in $L^\infty(\mathbb{R})$ (see Lemma 1). Let $\tau \in E$, $g(t,x)$ be the function from (7.4), and let $v(t,x;\tau)$ be a generalized solution of the Cauchy problem for the homogeneous equation (4.7) in the half-plane $t > \tau$ with initial function $g(\tau,x)$ (for $t = \tau$). By Theorem 1, such a generalized solution exists, and for fixed τ the function $v(t,x;\tau) \in L^\infty(\Pi'_\tau)$, where $\Pi'_\tau = (\tau,+\infty) \times \mathbb{R}$. We first show that $v(t,x;\tau)$ can be chosen to be measurable as a function of all variables (t,x,τ). Denote by D the set of $(t,x,\tau) \in \Pi \times \mathbb{R}_+$ such that $t > \tau$.

Lemma 6. *There exists a function $\tilde v(t,x;\tau) \in L^\infty(D)$ such that for almost all τ $v(t,x;\tau) = \tilde v(t,x;\tau)$ almost everywhere on Π'_τ.*

Proof. We consider the two-dimensional transport equation

$$\tilde A w_t + \tilde B_1 w_x + \tilde B_2 w_\tau = 0, \quad t > 0,\ (x,\tau) \in \mathbb{R}^2, \tag{7.5}$$

with coefficients $\tilde A = A(t+|\tau|,x)$, $\tilde B_1 = B(t+|\tau|,x)$, $\tilde B_2 \equiv 0$. One can directly verify that the coefficients $\tilde A$, $\tilde B = (\tilde B_1, \tilde B_2)$ satisfy the conditions (3.1) and (3.2). By Theorem 1, there exists a generalized solution $w(t,x,\tau) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ of the Cauchy problem for Equation (7.5) with initial data

$$w(0,x,\tau) = g(|\tau|,x). \tag{7.6}$$

Since $\tilde B_2 \equiv 0$ then the function $w(t,x,\tau)$ is a generalized solution of the one-dimensional Cauchy problem

$$A(t+\tau,x)w_t + B(t+\tau,x)w_x = 0, \quad w(0,x) = g(\tau,x)$$

for all positive τ from some subset $E_1 \subset E$ of full measure. By the uniqueness of this generalized solution, $w(t,x,\tau) = v(t+\tau,x;\tau)$ a.e. on Π (we agree that both functions vanish on the set, where $A(t+\tau,x)=0$). To complete the proof, we can set $\tilde{v}(t,x;\tau) = w(t-\tau,x,\tau)$. □

Now we can prove the following result.

Theorem 10. *Suppose that $u_0(x) \in L^\infty(\mathbb{R},d\mu)$, $g(t,x) \in L^\infty(\Pi,dm)$, where $g(t,x)$ is the function from* (7.4)*. Then there exists a generalized solution $u = u(t,x) \in L^\infty_{\mathrm{loc}}(\Pi,dm)$ of the problem* (7.1), (7.2)*. Moreover, for any $p(z) \in C^1(\mathbb{R})$ the function $p(u)$ is a generalized solution of the problem* (7.1), (7.2) *with initial function $p(u_0)$ and source function $p'(u)f$ (the renormalization property).*

Proof. Since $v(t,x;\tau) \in L^\infty(D)$, for a.e. $(t,x) \in \Pi$ the function $v(t,x;\tau)$ is bounded and measurable with respect to $\tau \in (0,t)$. Therefore, we can define the function

$$I(t,x) = \int_0^t v(t,x;\tau)d\tau.$$

It is clear that $I(t,x) \in L^\infty$ in any layer $(0,T)\times\mathbb{R}$. According to the Duhamel principle, we show that $I(t,x)$ is a generalized solution of the problem (7.1) with zero initial data. Let $h = h(t,x) \in C_0^\infty(\bar{\Pi})$ be a test function. Using the fact that for a.e. $\tau > 0$ the function $v(t,x;\tau)$ is a generalized solution of the homogeneous equation (4.7) in the half-plane $t > \tau$ with initial function $g(\tau,x)$, we derive that

$$\begin{aligned}
&\int_\Pi [AIh_t + BIh_x]dtdx \\
&= \int_D [A(t,x)v(t,x;\tau)h_t(t,x) + B(t,x)v(t,x;\tau)h_x(t,x)]dtdxd\tau
\end{aligned}$$

$$= \int_0^{+\infty}\left(\int_{t>\tau}[A(t,x)v(t,x;\tau)h_t(t,x) + B(t,x)v(t,x;\tau)h_x(t,x)]dtdx\right)d\tau$$

$$= -\int_0^{+\infty}\int_{\mathbb{R}} A(\tau,x)g(\tau,x)h(\tau,x)d\tau dx = -\int_\Pi f(t,x)h(t,x)dtdx.$$

Thus, for all $h \in C_0^\infty(\bar{\Pi})$

$$\int\limits_{\Pi} [AIh_t + BIh_x + fh]dtdx = 0,$$

and, according to Definition 4, $I(t,x)$ is a generalized solution of the problem (7.1), (7.2) with zero initial data. Let $\tilde{u}(t,x) \in L^\infty(\Pi, dm)$ be a generalized solution of the homogeneous problem (4.7), (4.8). The solution exists by Theorem 1. Then the sum

$$u(t,x) = \tilde{u}(t,x) + I(t,x) = \tilde{u}(t,x) + \int\limits_0^t v(t,x;\tau)d\tau \tag{7.7}$$

is a locally bounded generalized solution of the problem (7.1), (7.2). The existence is proved.

To prove the last assertion of the theorem, we suppose that $u(t,x)$ is a generalized solution of the problem (7.1), (7.2). Denote by $w(t,x;\tau)$ the generalized solution of the Cauchy problem for the homogeneous equation (4.7) considered in Π'_τ with initial condition $w(\tau,x) = u(\tau,x)$. Then the difference $u(t,x) - w(t,x;\tau)$ is a generalized solution of the problem (7.1), (7.2) in the half-plane $t > \tau$ with zero initial data. By formula (7.7) with the initial "time" 0 replaced with τ, we have

$$u(t,x) - w(t,x;\tau) = \int\limits_\tau^t v(t,x;s)ds.$$

Therefore,

$$w(t,x;\tau) = u(t,x) - \int\limits_\tau^t v(t,x;s)ds. \tag{7.8}$$

In view of Theorem 3 and Corollary 1, for any $p\,(z) \in C^1(\mathbb{R})$ the function $p'(w(t,x;\tau))v(t,x;\tau)$ is a generalized solution of (4.7) in Π'_τ with initial data $p'(u(\tau,x))g(\tau,x)$. By (7.7), the function

$$q(t,x) = p\,(w(t,x;0)) + \int\limits_0^t p'(w(t,x;\tau))v(t,x;\tau)d\tau \tag{7.9}$$

is a generalized solution of the problem (7.1), (7.2) with source $p'(u(t,x))f(t,x)$ and initial data $p\,(u_0(x))$. From (7.8) it follows that

$$p'(w(t,x;\tau))v(t,x;\tau) = p'\left(u(t,x) - \int\limits_{\tau}^{t} v(t,x;s)ds \right) v(t,x;\tau)$$

$$= \frac{\partial}{\partial \tau} p\left(u(t,x) - \int\limits_{\tau}^{t} v(t,x;s)ds \right)$$

in the sense of distributions on D. This implies that for a.e. $(t,x) \in \Pi$

$$\int\limits_{0}^{t} p'(w(t,x;\tau))v(t,x;\tau)d\tau$$

$$= p\left(u(t,x)\right) - p\left(u(t,x) - \int\limits_{0}^{t} v(t,x;s)ds \right) = p\left(u(t,x)\right) - p\left(w(t,x;0)\right).$$

From this and (7.9) we obtain the equality $p\left(u(t,x)\right) = q(t,x)$. We conclude that $p\left(u(t,x)\right)$ is a generalized solution of the problem (7.1), (7.2) with the source $p'(u(t,x))f(t,x)$ and initial data $p\left(u_0(x)\right)$. □

Remark 5. It is clear that Theorem 10 remains valid for source functions Ag, where g is bounded in any layer $\Pi_T = (0,T) \times \mathbb{R}$, $T > 0$. By (7.7), the corresponding generalized solution is also bounded in any layer Π_T.

Remark 6. For a generalized solution of (7.1) the same trace properties (see Remark 2 and Corollary 2) are satisfied as in the homogeneous case. For instance, let us show that any bounded generalized solution $u = u(t,x)$ of (7.1) in a layer Π_T with $f = Ag$, $g \in L^\infty(\Pi)$ is a generalized solution of the Cauchy problem (7.1), (7.2) with some initial function u_0. For this purpose, we take the generalized solution $v = v(t,x)$ of the problem (7.1), (7.2) with zero initial data. Then $v \in L^\infty(\Pi_T, dm)$ (see Remark 5) and the difference $u - v$ is a generalized solution of homogeneous equation (4.7). By Remark 2, it admits a weak trace $u_0 \in L^\infty(\mathbb{R}, d\mu)$. Hence $u = u - v + v$ is a generalized solution of the problem (7.1), (7.2) with initial function u_0.

Remark 7. Suppose that the coefficient A satisfies the condition (5.1) and $u_0, g(t,\cdot) \in C(\hat{\mathbb{R}})$. Then the functions $\tilde{u}(t,x)$ and $v(t,x;\tau)$ from (7.7) must be constant along generalized characteristics (see Section 5): $\tilde{u}(t,x) = u_0(y(t,x))$, where $y(t,x) = x(0;t,x)$ is a source point of the characteristic passing through the point (t,x), and similarly $v(t,x;\tau) = g(\tau; x(\tau;t,x))$.

Hence (7.7) reduces to the form (1.6):

$$u(t,x) = u_0(y(t,x)) + \int_0^t g(\tau; x(\tau; t,x))d\tau.$$

Corollary 6. *Let* u_1, u_2 *be a generalized solution of the problem* (7.1), (7.2) *with initial data* $u_{01} \in L^1_{\mathrm{loc}}(\mathbb{R}, d\mu)$, $u_{02} \in L^\infty(\mathbb{R}, d\mu)$ *and source functions* $f_1 = Ag_1, f_2 = Ag_2$ *such that* $f_1 \in L^1_{\mathrm{loc}}(\bar{\Pi})$, $g_2 \in L^\infty(\Pi, dm)$. *Then the product* u_1u_2 *is a generalized solution of the problem* (7.1), (7.2) *with initial function* $u_{01}u_{02}$ *and source function* $u_1f_2 + u_2f_1$.

Proof. In the case $u_{01}, u_{02} \in L^\infty(\mathbb{R}, d\mu)$, $g_1, g_2 \in L^\infty(\Pi, dm)$, the conclusion of Corollary 6 easily follows from the equality $u_1u_2 = [(u_1 + u_2)^2 - (u_1)^2 - (u_2)^2]/2$ and the renormalization property with $p\,(z) = z^2$. In the general situation, from (7.7) it follows that

$$u_1u_2 = u_1(t,x)\tilde{u}(t,x) + \int_0^t u_1(t,x)v(t,x;\tau)d\tau,$$

where $\tilde{u}(t,x) \in L^\infty(\Pi, dm)$ is a generalized solution of the homogeneous problem (4.7), (4.8) with initial data $u_{02}(x)$, and $v(t,x;\tau)$ is a generalized solution of (4.7) in the half-plane $t > \tau$ with initial function $g_2(\tau,x)$ at $t = \tau$. By Proposition 6, $u_1(t,x)\tilde{u}(t,x)$ is a generalized solution of the problem (7.1), (7.2) with initial data $u_{01}u_{02}$ and source function $F_1 = f_1\tilde{u}$. By Proposition 6, $w(t,x;\tau) = u_1(t,x)v(t,x;\tau)$ is a generalized solution of (7.1), (7.2) in the domain $t > \tau$ with initial data $u_1(\tau,x)g_2(\tau,x)$ and source function f_1v. Let

$$J = J(t,x) = \int_0^t u_1(t,x)v(t,x;\tau)d\tau,$$

$h = h(t,x) \in C_0^\infty(\bar{\Pi})$. Then, using the identity (7.3) for the generalized solution $w(t,x;\tau)$, which is written as

$$\int\limits_{t>\tau} [A(t,x)w(t,x;\tau)h_t(t,x) + B(t,x)w(t,x;\tau)h_x(t,x)$$
$$+ f_1(t,x)v(t,x;\tau)]dtdx + \int\limits_{\mathbb{R}} A(\tau,x)u_1(\tau,x)g_2(\tau,x)h(\tau,x)dx = 0,$$

for any test function $h \in C_0^\infty(\bar{\Pi})$, we find

$$\begin{aligned}
&\int_{\Pi}[AJh_t + BJh_x]dtdx \\
&= \int_0^{+\infty}\left(\int_{t>\tau}[A(t,x)w(t,x;\tau)h_t(t,x) + B(t,x)w(t,x;\tau)h_x(t,x)]dtdx\right)d\tau \\
&= -\int_0^{+\infty}\left(\int_{t>\tau} f_1(t,x)v(t,x;\tau)h(t,x)dtdx\right)d\tau \\
&\quad - \int_0^{+\infty}\int_{\mathbb{R}} A(\tau,x)u_1(\tau,x)g_2(\tau,x)h(\tau,x)d\tau dx \\
&= -\int_{\Pi}\left(\int_0^t v(t,x;\tau)d\tau\right)f_1(t,x)h(t,x)dtdx - \int_{\Pi} u_1(t,x)f_2(t,x)h(t,x)dtdx.
\end{aligned}$$

Thus, for all $h \in C_0^\infty(\bar{\Pi})$

$$\int_{\Pi}[AJh_t + BJh_x + F_2h]dtdx = 0,$$

where we denote

$$F_2 = f_1(t,x)\int_0^t v(t,x;\tau)d\tau + u_1(t,x)f_2(t,x)$$

and, according to Definition 4, $J(t,x)$ is a generalized solution of the problem (7.1), (7.2) with zero initial data and source F_2. We conclude that $u_1u_2 = u_1\tilde{u} + J$ is a generalized solution of the problem (7.1), (7.2) with initial data $u_{01}u_{02}$ and source function $F_1 + F_2 = f_1u_2 + u_1f_2$. $\square$

Now, we establish one useful growth estimate for a generalized solution of the problem (7.1), (7.2).

Proposition 7. *Suppose that $u_0 \in L^\infty(\mathbb{R}, d\mu)$, $g \in L^\infty(\Pi_T, dm)$ for any $T > 0$, and $u = u(t,x)$ is a generalized solution of the problem* (7.1), (7.2) *with initial data u_0 and source function $f = Ag$. Then for a.e. $t > 0$*

$$\|u(t,\cdot)\|_\infty \leqslant \|u_0\|_\infty + \int_0^t \|g(\tau,\cdot)\|_\infty d\tau. \tag{7.10}$$

To simplify the formulation, we agree that the functions u_0 and u, g vanish on the sets, where $A(0,\cdot)=0$ and $A=0$ respectively.

PROOF. Consider the set $E=\{t>0|(t,x)$ is a Lebesgue point of $A(t,x)$, $A(t,x)u(t,x)$ for a.e. $x\in\mathbb{R}\}$ similar to (3.5). It is clear that $E\subset\mathbb{R}_+$ is a set of full measure. Suppose that $t_0\in E$ and a function $\bar v(x)\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R},A(t_0,x)dx)$ is such that $\|A(t_0,x)\bar v\|_1\leqslant 1$.

Take a unique generalized solution $v=v(t,x)$ of the backward Cauchy problem $(Av)_t+(Bv)_x=0$, $v(t_0,x)=\bar v(x)$ in the strip Π_{t_0}. Taking into account Remark 2, we see that $v(t,x)$ satisfies the initial condition $v(0,x)=v_0(x)$ for some $v_0\in L^\infty(\mathbb{R})$. By Lemma 1, we can assume that the mapping $t\to A(t,\cdot)v(t,\cdot)$ is weakly-$*$ continuous on $[0,t_0]$. By Corollary 3, for all $t\in[0,t_0]$ we have $\|A(t,\cdot)v(t,\cdot)\|_1\leqslant\|A(t_0,\cdot)\bar v\|_1\leqslant 1$. By Proposition 6, the product uv is a generalized solution of the problem (7.1), (7.2) with initial data u_0v_0 and source function fv. Moreover, uv takes the Cauchy data $u(t_0,x)\bar v$ at $t=t_0$. Indeed, from the condition $t_0\in E$ and weak continuity of $A(t,\cdot)u(t,\cdot)$ it follows that $A(t,\cdot)u(t,\cdot)\to A(t_0,\cdot)u(t_0,\cdot)$ as $t\to t_0$ weakly-$*$ in $L^\infty(\mathbb{R})$ while (see Corollary 2) $A(t,x)(v(t,\cdot)-\bar v)\to 0$ in $L^1_{\rm loc}(\mathbb{R})$. Choose a function $\rho(x)\in C_0^\infty(\mathbb{R})$ such that $\rho(x)=1$ for $|x|\leqslant 1$. Multiplying $(Auv)_t+(Buv)_x=fv=Agv$ by $\rho(\alpha x)$, $\alpha>0$, and integrating over $x\in\mathbb{R}$, we find

$$\begin{aligned}\frac{\partial}{\partial t}\int(Auv)(t,x)\rho(\alpha x)dx&=\alpha\int(Buv)(t,x)\rho'(\alpha x)dx\\&+\int(Agv)(t,x)\rho(\alpha x)dx \text{ in } \mathcal{D}'(\mathbb{R}).\end{aligned}\tag{7.11}$$

Now we pass to the limit in this equality as $\alpha\to 0$. Taking into account that $\|Av(t,\cdot)\|_1\leqslant 1$, $\rho(\alpha x)\to 1$ as $\alpha\to 0$, and for a.e. $t\in(0,t_0)$

$$\begin{aligned}\alpha\left|\int(Buv)(t,x)\rho'(\alpha x)dx\right|&\leqslant\alpha N(\alpha)\int((A+\alpha)|uv|)(t,x)\rho'(\alpha x)dx\\&\leqslant\alpha N(\alpha)(\|u\|_\infty\cdot\|Av\|_1\cdot\|\rho'\|_\infty+\|uv\|_\infty\cdot\|\rho'\|_1)\to 0 \text{ as } \alpha\to 0\end{aligned}$$

(here, the condition (4.9) and the simple equality

$$\alpha\int|\rho'(\alpha x)|dx=\int|\rho'(y)|dy$$

are used), from (7.11) we obtain the relation

$$\frac{\partial}{\partial t}\int(Auv)(t,x)dx=\int(Agv)(t,x)dx\leqslant\|g(t,\cdot)\|_\infty\cdot\|Av(t,\cdot)\|_1\leqslant\|g(t,\cdot)\|_\infty$$

in $\mathcal{D}'(\mathbb{R})$. This, together with the weak continuity of $Auv(t,\cdot)$ at the points $t=0,t_0$ and the inequality $\|A(0,\cdot)v_0\|_1 \leqslant 1$, implies the inequality

$$\int A(t_0,x)u(t_0,x)\bar{v}(x)dx \leqslant \int A(0,x)u_0(x)v_0(x) + \int\limits_0^{t_0} \|g(t,\cdot)\|_\infty dt$$

$$\leqslant \|u_0\|_\infty + \int\limits_0^{t_0} \|g(t,\cdot)\|_\infty dt.$$

Since

$$\sup\left\{ \int A(t_0,x)u(t_0,x)\bar{v}(x)dx \middle| \bar{v}(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}, A(t_0,x)dx),\right.$$
$$\left. \|A(t_0,x)\bar{v}\|_1 \leqslant 1 \right\} = \|u(t_0,\cdot)\|_{L^\infty(\mathbb{R},A(t_0,x)dx)},$$

the proof is complete. □

7.2. Approximate solutions and renormalized solutions.

Suppose that $u_0(x)$ is an arbitrary measurable function and $f = Ag \in L^1_{\rm loc}(\bar{\Pi})$. We construct a "solution" of the problem (7.1), (7.2) using approximation of the input data by sequences of bounded functions. Suppose that $u_{0n} = u_{0n}(x)$ and $g_n = g_n(t,x)$ are sequences of bounded functions such that $u_{0n} \to u_0$ in measure $\mu = e^{-|x|}dx$ on $\mathbb{R}$, $Ag_n \to f = Ag$ in $L^1_{\rm loc}(\bar{\Pi})$ as $n \to \infty$. By Theorem 10, there exists a unique generalized solution $u_n = u_n(t,x)$ of the problem (7.1), (7.2) with initial data u_{0n} and source functions Ag_n. The following assertion justifies the limit procedure.

Theorem 11. *The sequence u_n converges to a measurable function $u=u(t,x)$ in measure $m = Ae^{-t-|x|}dtdx$. This limit function is independent of the choice of sequences u_{0n}, g_n and for all $p(z) \in C^1(\mathbb{R})$ such that $p(z)$ and $p'(z)$ are bounded the function $p(u(t,x))$ is a generalized solution of the problem* (7.1), (7.2) *with initial data $p(u_0)$ and source function $p'(u)f$ (the renormalization property).*

Proof. Take $p(z) = \dfrac{2}{\pi}\arctan z$. It is clear that for all $u_1,u_2 \in \mathbb{R}$

$$p(u_1) - p(u_2)| = \frac{2}{\pi}\left|\int\limits_{u_1}^{u_2} \frac{du}{1+u^2}\right| \leqslant \frac{2}{\pi} \int\limits_{-|u_1-u_2|/2}^{|u_1-u_2|/2} \frac{du}{1+u^2}$$

$$= 2p(|u_1 - u_2|/2). \tag{7.12}$$

The function $q(z) = 2p(|z|/2)$ is Lipschitz continuous and the function $q'(z) = p'(|z|/2)\,\mathrm{sign}(z)$ is piecewise continuous and bounded with only one jump point $z = 0$. Let us show that $q(u_n - u_m) \to 0$ in $L^1(\Pi, dm)$ as $n, m \to \infty$. Assuming the contrary, we can choose $\varepsilon > 0$ and sequences $n_k, m_k \to \infty$ such that for $v_k = u_{n_k} - u_{m_k}$

$$\int q(v_k)dm(t,x) > \varepsilon. \tag{7.13}$$

The function v_k is a generalized solution of the problem (7.1), (7.2) with initial data $v_{0k} = u_{0n_k} - u_{0m_k}$ and source function $F_k = Ag_{n_k} - Ag_{m_k}$. By Theorem 10, $q(v_k)$ is a generalized solution of this problem with initial data $q(v_{0k})$ and source term $q'(v_k)F_k$. More exactly, $q(z) \notin C^1(\mathbb{R})$ and we cannot directly use the renormalization property. To verify that this property nevertheless holds, we should approximate $q(z)$ by a sequence of smooth functions $q_r(z)$, $r \in \mathbb{N}$, such that $q_r(z) \to q(z)$ uniformly on $\mathbb{R}$ while $q_r'(z) \to q'(z) = p'(|z|/2)\,\mathrm{sign}(z)$ pointwise, and then pass to the limit as $r \to \infty$ in the integral identities (7.3) corresponding to the generalized solution $q_r(v_k)$.

Extracting a subsequence, if necessary, we can assume that the sequence $q(v_k)$ converges to some function $w \in L^\infty(\Pi)$ weakly-$*$ in $L^\infty(\Pi)$ as $k \to \infty$. Passing to the limit as $k \to \infty$ in the identity

$$\int\limits_\Pi [Aq(v_k)h_t + Bq(v_k)h_x + q'(v_k)F_k h]dtdx$$
$$+ \int\limits_{\mathbb{R}} A(0,x)q(v_{0k})h(0,x)dx = 0, \quad h \in C_0^\infty(\bar{\Pi})$$

and taking into account the limit relations $q(v_{0k}) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ (which easily follows from the condition that $v_{0k} \to 0$ in measure μ and the boundedness and continuity of q), $F_k \to 0$ in $L^1_{\mathrm{loc}}(\bar{\Pi})$, we derive that for all $h = h(t,x) \in C_0^\infty(\bar{\Pi})$

$$\int\limits_\Pi [Awh_t + Bwh_x]dtdx = 0,$$

i.e., $w = w(t,x)$ is a generalized solution of the problem (7.1), (7.2) with zero input data. By uniqueness, we have $Aw = 0$. However, this contradicts (7.13), which implies

$$\int w(t,x)dm(t,x) > \varepsilon \quad \text{as } k \to \infty.$$

Thus, $q(u_n - u_m) \to 0$ in $L^1(\Pi, dm)$ as $n, m \to \infty$. From the estimate (7.12) it follows that the sequence $p\,(u_n)$ is fundamental in $L^1(\Pi, dm)$. Therefore, it converges to a function $\bar{p} = \bar{p}\,(t,x) \in L^\infty(\Pi)$ in this space. Let $\bar{u}_{0n}$, $\bar{g}_n$ be another pair of sequences possessing the limit properties described before the theorem, and let $\bar{u}_n = \bar{u}_n(t,x)$ be the corresponding sequence of generalized solutions to the problem (7.1), (7.2) with input data $\bar{u}_n$, $A\bar{g}_n$. Then we can generate new sequences $\tilde{u}_{0n}$ and $\tilde{g}_n$ by setting $\tilde{u}_{02k} = u_k$, $\tilde{u}_{02k-1} = \bar{u}_k$, $\tilde{g}_{2k} = g_k$, and $\tilde{g}_{2k-1} = \bar{g}_k$ for $k \in \mathbb{N}$. It is clear that this sequence satisfies the required approximation properties. Therefore, for the corresponding sequence $\tilde{u}_n = \tilde{u}_n(t,x)$ of generalized solutions of the problem (7.1), (7.2) with input data $\tilde{u}_n$, $A\tilde{g}_n$ the sequence $p\,(\tilde{u}_n(t,x))$ converges in $L^1(\Pi, dm)$. Since $\tilde{u}_{2k} = u_k$ and $\tilde{u}_{2k-1} = \bar{u}_k$, the limit functions for $p\,(\bar{u}_n(t,x))$ must coincide with $\bar{p}\,(t,x)$ m-a.e. on Π and, consequently, they are independent of the choice of approximation sequences u_{0n} and g_n.

It is obvious that $|\bar{p}| \leqslant 1$ and, extracting a subsequence, we can assume that $p\,(u_n) \to \bar{p}$ m-a.e. on Π. Since $p\,(z) \in C(\hat{\mathbb{R}})$ strictly increases, $p\,(\pm\infty) = \pm 1$, we see that $u_n(t,x) \to u(t,x)$ a.e. on Π, where $u(t,x) = p^{-1}(\bar{p}\,(t,x))$, $p^{-1}(v) = \tan(\pi v/2)$ and we agree that $\tan(\pm\pi/2) = \pm\infty$. We show that $u(t,x)$ is finite a.e. on Π. Indeed, by Theorem 10, for any $\alpha > 0$ the function $p\,(\alpha u_n)$ is a generalized solution of the problem (7.1), (7.2) with initial data $p\,(\alpha u_{0n}(x))$ and source function $\alpha p'(\alpha u_n)f$. Passing to the limit as $n \to \infty$ and then as $\alpha \to 0$, we conclude that a function that is equal to ± 1 on the sets, where $u = \pm\infty$, is a generalized solution of the problem (7.1), (7.2) with zero initial and source functions. Therefore, this function must coincide with zero, i.e., $u(t,x)$ is finite a.e. on Π and $\bar{p} = p\,(u(t,x))$. From the condition $p\,(u_n) \to p\,(u)$ in $L^1(\Pi, dm)$ it follows that $u_n \to u$ in measure m. Since $\bar{p} = p\,(u)$ is independent of the choice of the approximation sequence, the same is true for the function u.

Finally, if $p\,(z) \in C^1(\mathbb{R})$ is such that $p\,(z)$ and $p'(z)$ are bounded, then $p\,(u_n) \to p\,(u)$ in $L^1(\Pi, dm)$ and $p\,(u_n)$ are generalized solutions of the problem (7.1), (7.2) with initial data $p\,(u_{0n})$ and source functions $p'(u_n)f_n$. Taking into account that $p\,(u_{0n}) \to p\,(u_0)$ in $L^1_{\text{loc}}(\mathbb{R})$, $p'(u_n)f_n \to p'(u)f$ in

$L^1_{\text{loc}}(\bar{\Pi})$ as $n \to \infty$, we can pass to the limit in the identities

$$\int_{\Pi} [Ap\,(u_n)h_t + Bp\,(u_n)h_x + p'(u_n)f_n h]dtdx + \int_{\mathbb{R}} A(0,x)p\,(u_{0n})h(0,x)dx = 0.$$

We find that for all $h \in C_0^\infty(\bar{\Pi})$

$$\int_{\Pi} [Ap\,(u)h_t + Bp\,(u)h_x + p'(u)fh]dtdx + \int_{\mathbb{R}} A(0,x)p\,(u_0)h(0,x)dx = 0,$$

i.e., $p\,(u(t,x))$ is a generalized solution of the problem (7.1), (7.2) with initial data $p\,(u_0)$ and source function $p'(u)f$. □

The solutions constructed in Theorem 11 are referred to as *approximate solutions* of the problem (7.1), (7.2).

Based on the renormalization property, one can naturally expand the notion of a renormalized solution to the nonhomogeneous case.

Definition 5. A measurable function $u(t,x)$ is called a *renormalized solution* of the problem (7.1), (7.2) if for any function $p\,(z) \in C^1(\mathbb{R})$ such that $|p\,(z)| + |p'(z)|$ is bounded $p\,(u(t,x))$ is a generalized solution of the problem (7.1), (7.2) with source $p'(u(t,x))f(t,x)$ and initial data $p\,(u_0(x))$.

As follows from Theorem 11, any approximate solution is a renormalized solution. In particular, a renormalized solution always exists. To prove the uniqueness, we need the following assertion.

Lemma 7. *Let u be a renormalized solution of the problem* (7.1), (7.2) *with measurable initial data u_0 and source $f_1 = Ag_1 \in L^1_{\text{loc}}(\bar{\Pi})$. Let $v = v(t,x)$ be a unique generalized solution of the problem* (7.1), (7.2) *with input data v_0, $f_2 = Ag_2$, where v_0 and g_2 are bounded. Then $u - v$ is a renormalized solution of the problem* (7.1), (7.2) *with corresponding initial data $u_0 - v_0$ and source function $f_1 - f_2$.*

Proof. We first assume that u_0 and u are bounded. In this case, the renormalization property holds for all $p\,(z) \in C^1(\mathbb{R})$ (since the values of $p\,(z)$ and $p'(z)$ are not essential for large $|z|$). In particular, all the powers u^k, $k = 0,1,2,\ldots$, are generalized solutions of the problem (7.1), (7.2) with the corresponding input data. By Theorem 10, the same is true for v: all the powers v^l, $l = 0,1,2,\ldots$, are generalized solutions of the problem (7.1), (7.2). By Corollary 6, $u^k v^l$ is a generalized solution of the problem (7.1), (7.2) with initial data $(u_0)^k(v_0)^l$ and source $ku^{k-1}v^l f_1 + lu^k v^{l-1} f_2$. In turn,

$$(u-v)^m = \sum_{l=0}^{m} C_m^l (-1)^l u^{m-l} v^l$$

($C_m^l = \dfrac{m!}{l!(m-l)!}$ are binomial coefficients) is a generalized solution of the problem (7.1), (7.2) with initial data $(u_0 - v_0)^m$ and source function

$$\begin{aligned} & f_1 \sum_{l=0}^{m-1} C_m^l (-1)^l (m-l) u^{m-l-1} v^l + f_2 \sum_{l=1}^{m} C_m^l (-1)^l l u^{m-l} v^{l-1} \\ & = m f_1 \sum_{l=0}^{m-1} C_{m-1}^l (-1)^l u^{m-1-l} v^l - m f_2 \sum_{k=l-1=0}^{m-1} C_{m-1}^k (-1)^k u^{m-1-k} v^k \\ & = m(u-v)^{m-1}(f_1 - f_2). \end{aligned}$$

Thus, $u - v$ satisfies the renormalization properties for $p(z) = z^m$, $m = 0, 1, 2, \ldots$. Then this property is satisfied for any polynomial $p(z)$. Since polynomials are dense in $C^1(\mathbb{R})$, for any $p(z) \in C^1(\mathbb{R})$ we can choose a sequence of polynomials $p_n(z)$, $n \in \mathbb{N}$, such that $p_n(z) \to p(z)$ and $p_n'(z) \to p'(z)$ as $n \to \infty$ uniformly on any compact subset of $\mathbb{R}$. Passing to the limit as $n \to \infty$ in the integral identities (7.3) corresponding to the generalized solution $p_n(u-v)$ and taking into account that $u-v$ is locally bounded, we conclude that $p(u-v)$ is a generalized solution of the problem (7.1), (7.2) with the corresponding input data. Since $p(z) \in C^1(\mathbb{R})$ is arbitrary, $u - v$ is a renormalized solution.

For an arbitrary renormalized solution $u(t,x)$ we consider a sequence $q_n(z) \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $|q_n'(z)| \leqslant 1$ and $q_n(z) \to z$, $q_n'(z) \to 1$ pointwise as $n \to \infty$. Then for $u = u(t,x)$ the functions $q_n(u)$ are bounded renormalized solutions and, as was proved, the difference $q_n(u) - v$ is also a renormalized solution. Therefore, for any $p(z) \in C^1(\mathbb{R})$ such that $|p(z)| + |p'(z)| \leqslant \text{const}$ the function $p(q_n(u) - v)$ is a generalized solution of the problem (7.1), (7.2) with initial data $p(q_n(u_0) - v_0)$ and source function $p'(q_n(u)-v)(q_n'(u)f_1 - f_2)$. It is clear that $p(q_n(u)-v) \to p(u-v)$, $p'(q_n(u)-v)(q_n'(u)f_1 - f_2) \to p'(u-v)(f_1-f_2)$ in $L^1_{\text{loc}}(\overline{\Pi})$ and $p(q_n(u_0)-v_0) \to p(u_0 - v_0)$ in $L^1_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. Taking into account these limit relations, we can pass to the limit as $n \to \infty$ in the integral identities (7.3) corresponding to the generalized solution $p(q_n(u)-v)$ and derive that $p(u-v)$ is a generalized solution of the problem (7.1), (7.2) with initial data $p(u_0 - v_0)$ and source function $p'(u-v)(f_1 - f_2)$. Thus, $u - v$ is a renormalized solution. $\square$

Theorem 12. *A renormalized solution of the problem* (7.1), (7.2) *coincides with the unique approximate solution of this problem.*

Proof. Let $u = u(t,x)$ and $\bar{u} = \bar{u}(t,x)$ be a renormalized solution and an approximate solution of the problem (7.1), (7.2) respectively. We show that the difference $u - \bar{u}$ is a renormalized solution. For this purpose, we choose sequences $u_{0n} \in L^\infty(\mathbb{R})$, $g_n \in L^\infty(\Pi)$, $n \in \mathbb{N}$, approximating the input data u_0, Ag in the above sense. Let $u_n = u_n(t,x)$ be a unique generalized solution of the problem (7.1), (7.2) with input data u_{0n}, Ag_n. By Theorem 11, $u_n \to \bar{u}$ as $n \to \infty$ in measure m. By Lemma 7, $u - u_n$ is a renormalized solution which means that for any function $p(z) \in C^1(\mathbb{R})$ such that $|p(z)| + |p'(z)| \leqslant$ const the composition $p(u - u_n)$ is a generalized solution of the problem (7.1), (7.2) with initial data $p(u_0 - u_{0n})$ and source term $p'(u - u_n)(f - f_n)$. Passing to the limit as $n \to \infty$, we see that $p(u - \bar{u})$ is a generalized solution of the problem (7.1), (7.2) with constant initial function $p(0)$ and zero source term. Thus, $u - \bar{u}$ is a renormalized solution of the problem (7.1), (7.2) with zero input data. Applying the renormalization property with the function $p(z) = z^2/(1 + z^2)$, we derive that $Ap(u - \bar{u}) = 0$ is a unique generalized solution of the problem (7.1), (7.2) with zero input data. This implies that $Au = A\bar{u}$. □

Corollary 7. *Let $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ are renormalized solutions of the problem* (7.1), (7.2) *with initial data u_{01}, u_{02} and source functions $f_1 = Ag_1$, $f_2 = Ag_2$. Then the following assertions hold.*

(i) *For any $\alpha, \beta \in \mathbb{R}$ the function $\alpha u_1 + \beta u_2$ is a renormalized solution of the problem* (7.1), (7.2) *with input data $\alpha u_{01} + \beta u_{02}$ and $\alpha f_1 + \beta f_2$. Thus, renormalized solutions generate a linear space.*

(i) *If $u_{01} \leqslant u_{02}$, $f_1 \leqslant f_2$, then $u_1 \leqslant u_2$ m-a.e. on* Π (*comparison principle*).

Proof. It is clear that (i) and (ii) hold for the generalized solution for bounded input data. In particular, the comparison principle follows from Remark 3 and formula (7.7). By Theorem 12, any renormalized solution can be constructed as a limit in measure of such a generalized solution. This implies (i) and (ii) in the general case. □

7.3. Renormalized solutions and generalized solutions.

If $u = u(t,x)$ is a renormalized solution of the problem (7.1), (7.2) and $Au, Bu \in L^1_{\mathrm{loc}}(\bar{\Pi})$, $A(0,x)u_0(x) \in L^1_{\mathrm{loc}}(\mathbb{R})$, then u is also a generalized solution of the same problem. To prove this assertion, we can use the renormalization property with $p = p_n(z) = nz/(n + z^2)$ and pass to the limit as $n \to \infty$ in the corresponding integral identity. Now, we prove the converse assertion.

Theorem 13. *Any generalized solution of the problem* (7.1), (7.2) *is a renormalized solution.*

PROOF. Suppose that $u = u(t, x)$ is a generalized solution of the problem (7.1), (7.2), $\bar{u} = \bar{u}(t, x)$ is a unique renormalized solution of the same problem, and $v = v(t, x)$ is a renormalized solution of the problem (7.1), (7.2) with input data $|u_0|$, $|f|$. We need to prove that $u = \bar{u}$ m-a.e. on Π. Introduce the cut-off functions $u_{0n} = \max(-n, \min(u_0(x), n))$ and $g_n = \max(-n, \min(g(t, x), n))$, $n \in \mathbb{N}$, and consider the corresponding sequences of generalized solutions $u_n = u_n(t, x)$, $v_n = v_n(t, x)$ of the problem (7.1), (7.2) with input data $u_{0n}, f_n = Ag_n$ and $|u_{0n}|, |f_n|$ respectively. Since the input data are bounded, u_n, v_n are also bounded in any layer Π_T, $T > 0$.

By Theorems 11 and 12, the sequences u_n and v_n converge to the renormalized solution $\bar{u}, v$ in measure m as $n \to \infty$. Since $-|u_{0n}| \leqslant u_{0n} \leqslant |u_{0n}|$ and $-|f_n| \leqslant f_n \leqslant |f_n|$, by the comparison principle, we have $-v_n \leqslant \bar{u}_n \leqslant v_n$, i.e., $|u_n| \leqslant v_n$ m-a.e. on Π. By Corollary 6, the functions

$$w_{1n} = \frac{u_n(t, x)}{1 + v_n(t, x)}, \quad w_{2n} = \frac{u(t, x)}{1 + v_n(t, x)}$$

are generalized solutions of (7.1), (7.2) with initial data

$$\frac{u_{0n}}{1 + |u_{0n}|}, \quad \frac{u_0}{1 + |u_{0n}|}$$

and source functions

$$\frac{f_n}{1 + v_n} - \frac{|f_n|}{(1 + v_n)^2} u_n = \frac{f_n}{1 + v_n} - \frac{|f_n|}{1 + v_n} w_{1n},$$
$$\frac{f}{1 + v_n} - \frac{|f_n|}{(1 + v_n)^2} u = \frac{f}{1 + v_n} - \frac{|f_n|}{1 + v_n} w_{2n}$$

respectively. Note that the difference $w_n = w_{1n} - w_{2n}$ is a generalized solution of the problem (7.1), (7.2) with initial data

$$w_{0n}(x) = \frac{u_{0n} - u_0}{1 + |u_{0n}|}$$

and source function $q_n - p_n w_n$, where

$$q_n = q_n(t,x) = \frac{f_n(t,x) - f(t,x)}{1 + v_n(t,x)}, \quad p_n = p_n(t,x) = \frac{|f_n(t,x)|}{1 + v_n(t,x)}.$$

Let $T > 0$ and $\rho_n = \rho_n(t,x)$ be a generalized solution of the backward Cauchy problem

$$(A\rho)_t + (B\rho)_x = p_n, \quad \rho(T,x) = 0.$$

It is obvious that $\rho_n(T-t,-x)$ is a generalized solution of the equation

$$(A(T-t,-x)\rho)_t + (B(T-t,-x)\rho)_x = -p_n(T-t,-x) \leqslant 0$$

with zero initial data. By the comparison principle, $\rho_n \leqslant 0$. Let $\rho_{0n} = \rho_n(0,x)$ in the sense of the trace property in Remark 6. Using Corollary 6 again, we see that $w_n e^{\rho_n}$ is a generalized solution of the problem (7.1), (7.2) with initial data $w_{0n}e^{\rho_{0n}}$ and source function $(q_n - p_n w_n)e^{\rho_n} + w_n e^{\rho_n} p_n = q_n e^{\rho_n}$. By Theorems 11 and 12, $\rho_n \to \rho$ as $n \to \infty$ in measure m, where $\rho = \rho(t,x)$ is a renormalized solution of the problem

$$(A\rho)_t + (B\rho)_x = p = \frac{|f(t,x)|}{1 + v(t,x)}, \quad \rho(T,x) = 0.$$

Since $\rho_n \leqslant 0$, we see that $e^{\rho_n} \leqslant 1$. Therefore, $e^{\rho_n} \to e^{\rho}$ in $L^1_{\mathrm{loc}}(\bar{\Pi}_T, dm)$ as $n \to \infty$. Here, $\bar{\Pi}_T = [0,T) \times \mathbb{R}$. Taking into account the above property and estimates

$$\frac{1}{1+v_n} \leqslant 1, \quad \frac{u_n}{1+v_n} \leqslant 1,$$

we find

$$q_n e^{\rho_n} = \frac{f_n - f}{1 + v_n} e^{\rho_n} \to 0 \text{ as } n \to \infty \text{ in } L^1_{\mathrm{loc}}(\bar{\Pi}_T),$$

$$w_n e^{\rho_n} = \frac{u_n - u}{1+v_n} e^{\rho_n} = \left(\frac{u_n}{1+v_n} - \frac{u}{1+v_n} \right) e^{\rho_n} \to \frac{\bar{u} - u}{1+v} e^{\rho}$$
$$\text{as } n \to \infty \text{ in } L^1_{\mathrm{loc}}(\bar{\Pi}_T, dm),$$

$$w_{0n} e^{\rho_{0n}} = \frac{u_{0n} - u_0}{1 + |u_{0n}|} e^{\rho_{0n}} \to 0 \text{ as } n \to \infty \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}, d\mu).$$

These relations allow us to pass to the limit in the identity (7.3) corresponding to the generalized solution $w_n e^{\rho_n}$ and derive that $\dfrac{\bar{u} - u}{1+v} e^{\rho}$ is a generalized solution of the problem (7.1), (7.2) with zero input data. By uniqueness, this generalized solution must be trivial, i.e., $u = \bar{u}$ m-a.e. on Π_T. Since $T > 0$ is arbitrary, this completes the proof. $\square$

7.4. Transport equations with linear source term.

We consider the Cauchy problem for the general linear equation

$$(Au)_t + (Bu)_x = f_1 u + f_2, \tag{7.14}$$

with initial condition (7.2). A generalized solution of this problem is naturally defined as a generalized solution of the problem (7.1), (7.2) with source function $f = f_1 u + f_2$. In accordance with the condition (7.4), we assume that $f_1 = Ag_1$ and $f_2 = Ag_2$. For the sake of simplicity, we suppose that $u_0 \in L^\infty(\mathbb{R}, d\mu)$ and $g_1, g_2 \in L^\infty(\Pi, dm)$. The main result concerning the problem (7.14), (7.2) is formulated as follows.

Theorem 14. *There exists a unique generalized solution of the problem* (7.14), (7.2).

PROOF. Let $u_1 = u_1(t,x)$ be a generalized solution of the problem

$$(Au)_t + (Bu)_x = f_1, \quad u(0,x) = 0. \tag{7.15}$$

By Theorem 10 and Remark 5, this generalized solution exists and is bounded in any layer Π_T. Taking into account Remark 5 again, we see that there exists a generalized solution $u_2 = u_2(t,x)$ of the problem

$$(Au)_t + (Bu)_x = f_2 e^{-u_1}, \quad u(0,x) = u_0(x).$$

By Theorem 10 and Corollary 6, the locally bounded function $u = u_2 e^{u_1}$ is a generalized solution of the problem (7.1), (7.2) with initial data $u_0(x)$ and source function $u_2 e^{u_1} f_1 + f_2 e^{-u_1} e^{u_1} = f_1 u + f_2$, i.e., $u = u(t,x)$ is a generalized solution of the problem (7.14), (7.2).

To prove the uniqueness of this generalized solution, we consider two generalized solutions $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ and set $\tilde{u} = u_1(t,x) - u_2(t,x)$. Then $\tilde{u}$ is a generalized solution of the homogeneous problem $(Au)_t + (Bu)_x = f_1 u$, $u(0,x) = 0$. Let u be a generalized solution of the problem (7.15). Then u is locally bounded, and, by Theorem 10 and Corollary 6, the product $\tilde{u}e^{-u}$ is a generalized solution of the problem (7.1), (7.2) with zero initial data and the source $e^{-u} f_1 \tilde{u} - \tilde{u} e^{-u} f_1 = 0$. By uniqueness, we conclude that this solution is trivial, i.e., $Au_1 = Au_2$ a.e. on Π. $\square$

We give one application of the above results to the system of Keyfitz–Kranzer type with a linear tangential source term

$$u_t + (\varphi(|u|)u)_x = S(t,x)u, \quad u = u(t,x) \in \mathbb{R}^n, \tag{7.16}$$

where $S(t,x)$ is a skew-symmetric $n \times n$-matrix with bounded measurable components. In particular, the Euclidean norm

$$\|S(t,x)\| \leqslant C = \text{const} \ \text{ a.e. on } \Pi. \tag{7.17}$$

We consider the Cauchy problem for the system (7.16) with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}, \mathbb{R}^n). \tag{7.18}$$

The notion of a strong generalized entropy solution of (7.16), (7.18) is defined similarly to the homogeneous case. Namely, the strong generalized entropy solution of (7.16), (7.18) is a vector $u = u(t,x) \in L^\infty_{\text{loc}}(\bar{\Pi}, \mathbb{R}^n)$ satisfying (7.16) in the sense of distributions, and the conditions (2.4), (2.5).

Theorem 15. *There exists a unique strong generalized entropy solution of the problem* (7.16), (7.18).

Proof. Let $r = r(t,x)$ be a unique generalized entropy solution of the scalar problem (2.4). Then the coefficients $A = r$, $B = f(r) = r\varphi(|r|)$ satisfy all our assumptions (see Remark 1). We consider the Cauchy problem for the linear system of transport equations

$$(Av)_t + (Bv)_x = ASv \tag{7.19}$$

with initial data

$$v(0,x) = v_0(x) = \begin{cases} u_0(x)/r_0(x), & r_0(x) > 0, \\ 0, & r_0(x) = 0. \end{cases} \tag{7.20}$$

Let us show that this problem admits a unique generalized solution $v = v(t,x) \in X$, where X is a Banach space of vector-valued functions v such that $e^{-2Ct}v \in L^\infty(\Pi)$ with the norm $\|v\|_X = \|e^{-2Ct}v\|_\infty$, where C is the constant from the estimate (7.17). It is obvious that a generalized solution of the problem (7.19), (7.20) is a fixed point of the mapping $F : X \to X$, defined by the requirement that $v = F(w)$ is a generalized solution of the Cauchy problem for the system

$$(Av)_t + (Bv)_x = ASw \tag{7.21}$$

with initial data (7.20). Since this problem is decoupled into n independent scalar problems like (7.1), (7.2), there exists a unique generalized solution $v = v(t,x)$ of this problem and it is bounded in any layer Π_T, $T > 0$ (see Theorem 10 and Remark 5). As we show later, $v \in X$ and the mapping F is well defined.

We prove that F is a contraction. Suppose that $w_1, w_2 \in X$ and $v_1 = F(w_1)$, $v_2 = F(w_2)$. Then $v = v_1 - v_2$ is a generalized solution of (7.21) with zero initial data and $w = w_1 - w_2$. By the renormalization property, from Theorem 10 it follows that $|v|^2 = \sum_{i=1}^{n} v_i^2$ is a generalized solution of the scalar problem (7.1), (7.2) with zero initial data and source function $f = 2A \sum_{i,j=1}^{n} S_{ij} w_j v_i = 2A(Sw, v)$. By (7.10) and the bound $|(Sw, v)| \leqslant \|S\| \cdot |v| \cdot |w| \leqslant C|v| \cdot |w|$, for almost all $t > 0$ we have

$$\begin{aligned}\||v|^2(t,\cdot)\|_\infty &\leqslant 2C \int_0^t \|v(\tau,\cdot)\|_\infty \cdot \|w(\tau,\cdot)\|_\infty d\tau \\ &\leqslant 2C \int_0^t e^{4C\tau} \|e^{-2C\tau} v(\tau,\cdot)\|_\infty \cdot \|e^{-2C\tau} w(\tau,\cdot)\|_\infty d\tau.\end{aligned}$$

By the above inequality, for $M(t) = \|e^{-2Ct} v\|_{L^\infty(\Pi_t)}$ we derive

$$(M(t))^2 \leqslant 2Ce^{-4Ct} \|w\|_X M(t) \int_0^t e^{4C\tau} d\tau \leqslant \frac{1}{2} M(t) \|w\|_X,$$

which implies that $v = v_1 - v_2 \in X$ and

$$\|v_1 - v_2\|_X \leqslant \frac{1}{2} \|w_1 - w_2\|_X. \tag{7.22}$$

If we apply this estimate to $w_1 = w \in X$, $w_2 = 0$ and take into account that $v_2 = F(0) \in L^\infty(\Pi) \subset X$ (as a generalized solution of the homogeneous problem) we find $v_1 = F(w) = (v_1 - v_2) + v_2 \in X$. By (7.22), F is a contraction of X. By the Banach theorem, there exists a unique fixed point $v \in X$ of F, i.e., the problem (7.19), (7.20) has a unique generalized solution $v = v(t, x)$. Using again the renormalization property from Theorem 10, we find that $|v|^2$ is a generalized solution of (7.1), (7.2) with initial data $|v_0|^2$ and source function $2A(Sv, v) = 0$. By the identity $A(0, x)|v_0(x)|^2 = A(0, x)$ and Theorem 2, $A(t, x)|v(t, x)|^2 = A(t, x) = r(t, x)$ a.e. on Π, which implies $r^2|v|^2 = r^2$, i.e., $|u| = r \in L^\infty(\Pi)$ for $u = rv$. Since $Av = u$, $Bv = \varphi(r)rv = \varphi(|u|)u$, and $ASv = Srv = Su$, we see that $u_t + (\varphi(|u|)u)_x = Su$ in $\mathcal{D}'(\Pi, \mathbb{R}^n)$ and the vector $u = u(t, x)$ satisfies (7.16) in the sense of distributions. By construction, $r = |u|$ satisfies the condition (2.4). Finally, the initial condition (2.5) holds, which easily follows from Corollary 2 (obviously, the assertion of this corollary remains true for generalized solutions of nonhomogeneous problems, see Remark 6). Thus, $u(t, x)$ is a strong generalized entropy solution of (7.16), (7.18). To prove the uniqueness of this solution, we only need to repeat arguments from the corresponding part of the proof of Theorem 4. □

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Irreducible Chapman–Enskog Projections and Navier–Stokes Approximations

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Problems of Navier–Stokes approximations of kinetic equations are studied in terms of irreducible Chapman–Enskog projections. Properties of the Chapman–Enskog projection for the Cauchy problem for moment approximations of kinetic equations and, in particular, the Boltzmann equation and the Boltzmann–Peierls equation are described. The existence conditions for Chapman–Enskog projections are formulated in terms of the solvability of the Riccati matrix equations for which necessary and sufficient existence conditions are obtained. Bibliography: 25 titles.

1. Introduction

In this paper, we consider mathematical aspects arising in the study of the system of conservation laws with relaxation [**4**]

$$\begin{gathered}\partial_t u + \partial_x f(u, v) = 0, \\ \partial_t v + \partial_x g(u, v) + b(u)v = 0,\end{gathered} \tag{1.1}$$

Instability in Models Connected with Fluid Flows. II. Edited by Claude Bardos and Andrei Fursikov / International Mathematical Series, Vol. 7, Springer, 2008

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{d_c}$, $v \in \mathbb{R}^{d^\perp}$, $d = d_c + d^\perp$, b is the relaxation $d^\perp \times d^\perp$-matrix, $f(u,v)$ and $g(u,v)$ are flow $d_c \times d$-matrix and $d^\perp \times d$-matrix respectively, and d_c is the number of conservative variables.

The initial goal was to treat the so-called ultraviolet catastrophe (see, for example, [**2**], [**9**]), which turned out to be a banal mathematical fact caused by "bad" uniform polynomial approximations of internal layer functions (kink type solutions). However, the analysis of the instability of post-Navier–Stokes approximations for moment approximations of kinetic equations highlights rigid structural singularities of moment approximations (in general, hyperbolic systems with relaxation [**4, 18**]). This effect is connected with "basic" dynamics of the processes under consideration at large times and the role of conservative and nonequillibrium variables.

The leading part of the system (1.1) is nonstrictly hyperbolic in the sense of the following definition.

Definition 1.1. We say that (1.1) is a *nonstrictly hyperbolic* system if the characteristic matrix

$$\tau E + \xi \cdot \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix} \tag{1.2}$$

has only real roots $\tau = \tau_j(\xi, u, v)$, $j = 1, \dots, n$, where E is the identity matrix.

The nonstrict hyperbolicity condition is satisfied if the system (1.1) is simmetrizable (see [**8**]).

Let $U = (u(t), v(t))$ be a homogeneous solution to (1.1) depending only on t. Then some dependent variables are conservative, i.e., $u = u_0 = \text{const}$.

Definition 1.2. We say that (1.1) is a *system with relaxation* if the origin is an asymptotically stable equilibrium point of the system of ordinary differential equations

$$\partial_t v(t) + b(t, u_0) v(t) = 0$$

relative to the nonequillibrium variables $v(t)$.

For an example of a system with relaxation we can take moment systems of kinetic equations, which is the simplest hyperbolic regularization of the Euler isentropic gas dynamics model [**19**]. In particular, moment approximations of the *Boltzmann kinetic equation* describing nonequillibrium processes in hydrodynamics, the *Fokker–Planck equation* describing dynamics of Brownian particles, the *Boltzmann–Peierls equation* describing heat transfer processes in crystals can be regarded as systems with relaxation.

As in problems with viscosity, the system (1.1) regularizes discontinuous solutions to the locally equilibrium limit

$$\partial_t u + \partial_x f(u, 0) = 0.$$

But, unlike the viscous regularization, sufficiently small discontinuities are removed in this case.

The difficulties arising in the study of nonequillibrium processes are mostly caused by the behavior of nonequillibrium variables v at large times. Comparing exact solutions to the Cauchy problem for kinetic equations with their moment approximations, we see that for a reasonably small residue a large number of nonequilibrium variables is required. However, a number of boundary conditions, that could be reasonable from the physical point of view, is not sufficient for worthy formulations of boundary-value problems. Therefore, it is necessary to understand how the initial and boundary conditions should be interpreted.

The Chapman–Enskog conjecture [**3**] reads: *for well-posed models in continuum mechanics the influence of higher order moments is inessential.* We do not list different versions of the notion of "well-posedness" from the physical point of view, but discuss the expression "the influence of higher order moments is inessential." Following Chapman and Enskog, this means the following.

1. *Projection.* Nonequilibrium variables are expressed in terms of conservative variables, i.e., there exist an operator correspondence

$$v = \Pi_{\mathrm{ChEns}}(u) \tag{1.3}$$

such that the system of projections

$$\partial_t w + \partial_x f(w, \Pi_{\mathrm{ChEns}}(w)) = 0, \quad w\big|_{t=0} = w^0 \tag{1.4}$$

into the phase space of conservative variables remains in the class of hyperbolic systems with relaxation.

2. *Separation of dynamics.* There is an operator connecting the initial data of the original system with those of the system of projections

$$w^0 = \tau(u^0, v^0) \tag{1.5}$$

in such a way that special solutions $U_{\mathrm{ChEns}} = (w, \Pi_{\mathrm{ChEns}}(w))$ to the Cauchy problem for (1.1) determined by solutions to the Cauchy problem (1.4), (1.5) form an attracting invariant manifold (the definition can be found, for example, in [**1, 6**].

In other words, the dynamics of the process must be divided into the dynamics in the phase space of conservative variables and the (inessential) dynamics in the phase space of all variables (u, v) corresponding to the attracting invariant set of special solutions.

3. *Irreducible projections.* Projections that are not representable as the composition of projections must correspond to the basic (characteristic) dynamics of the simulated process. Moreover, if relations for relaxation times are different (i.e., the so-called time relaxation ranges are different, see [**3, 18**]), then the corresponding attracting invariant manifolds are also different.

As we will show below, the phase space of irreducible projections can contain not only conservative variables, but also the so-called consolidated variables, i.e., variables of the phase space of an irreducible Chapman–Enskog Projection.

To understand the nature of the Chapman–Enskog projections, we consider, following [**3**] (see also [**4**]), a regular asymptotics of the solution to the system (1.1) with rigid relaxation and small parameter $\varepsilon > 0$. For this purpose, we consider the system

$$\partial_t u + \partial_x f(u, v) = 0,$$
$$\partial_t u + \partial_x g(u, v) + \frac{1}{\varepsilon}\, b(u)v = 0,$$

where $\varepsilon = 1/Kn$, Kn is the Knudsen number. For the sake of simplicity, we assume that the matrix $b(u)$ is invertible for all $u \in R^{d_c}$. Then

$$u = u_0 + \varepsilon\, u_1 + \dots, \quad v = 0 + \varepsilon\, v_1 + \dots,$$

where $v_1 = -b^{-1}(u_0)\partial_x g(u_0, 0)$. Formally, we obtain the following equation for u_0:

$$\partial_t u_0 + \partial_x f(u_0, -\varepsilon\, b^{-1}(u_0)\partial_x g(u_0, 0)) = 0.$$

The system

$$\begin{gathered} \partial_t u + \partial_x f(u, -\varepsilon\, b^{-1}(u)\partial_x g(u, 0)) = 0, \\ v = -b^{-1}(u)\partial_x g(u, 0) \end{gathered} \tag{1.6}$$

is called the *Navier–Stokes approximation* of (1.1). It is easy to see that the stability condition for the linearizations of (1.1) on constants can be expressed as follows: *the system* (1.6) *is parabolic and the linearizations of* (1.6) *are stable on constants.* However, as was observed in [**2**], the linearizations of the so-called post-Navier–Stokes approximations on constants (ε^2,

$\varepsilon^3, \dots$) are unstable. We emphasize that this happens in spite of the stability of the linearizations of the original system (1.1) on constants. This phenomenon is referred to as the *ultraviolet catastrophe*.

What is a reason of this phenomenon? Whether the conjecture of the existence of a projection to the phase space of conservative variables fails or the Navier–Stokes approximations are not sufficiently well justified?

2. Linear Analysis

We describe properties of the Chapman–Enskog projection (1.4) by considering a simple example.

2.1. Boltzmann–Peierls kinetic equation.

We begin with the simplest phonon gas model [**16**]. As is known, under the assumption that interatomic potentials in crystals are harmonic, vibrations of atoms around the equilibrium state can be represented as eigenvibrations or eigenmodes. Then N atoms determine $3N$ eigenmodes with frequencies ω_s, $s = 1, \dots, 3N$,. The energy of each mode is given by the relation $e_s = (n_s + 1/2)\hbar\omega_s$, $n_s = 0, 1, 2, \dots$, where $\hbar$ is the Planck constant and n_s is the number of s-modes of the energy quantum $\hbar\omega_s$. Based on the classical specific heat theory for dielectric solids, Peierls [**16**] suggested to use the phonon nature for describing heat transfer processes in crystals. Within the framework of this approach, we say that there are n_s phonons with energy $\hbar\omega_s$. The phonons behave themselves as particles subject to the Bose statistics. Using a representation of eigenmodes, Peierls showed that waves with neighboring wave vectors in the volume $[\overline{k}, \overline{k} + \Delta\overline{k}]$ can be represented as wave packets localized in the space volume $[x, x + \Delta\, x]$ such that $|\Delta\overline{k}\,|\delta\, x| = 2\pi$. Such a packet contains a number of phonons with energy $\hbar\omega(k)$. $\hbar\omega(k)$. The function $\omega(k)$, called a *dispersion relation*, is, in general, nonisotropic and depends on the crystal structure and interatomic interaction. Even in the case of the simplest lattices, it is not easy to compute $\omega(k)$. Therefore, the isotropic dispersion relation $\omega = c\,|k|$ from the Debye phonon model [**5**]) is often used for calculations; here

$$\frac{3}{c^3} = \sum_{\alpha=1}^{3} \frac{1}{c_\alpha^3}, \quad |k|^2 = k_1^2 + k_2^2 + k_3^2,$$

and α denotes three wave modes: one longitudinal and two transverse waves propagating with velocities c_α. As is known, wave packets propagate with group velocity $\partial\omega/\partial k$. The same is true for the corresponding phonons. Based on these facts, Peierls conjectured that nonequillibrium variables of crystals can be described by analogy with the kinetic theory of gases. The state space for phonons is determined by the moment of a phonon $\hbar\overline{k}$, its position x, t in space and time, and phase density $f(x,t,\overline{k})$, which determines the number of phonons in a neighborhood of x and $\overline{k}$ at the moment of time t. The time-wise dynamics of gas density is determined by the *Boltzmann–Peierls kinetic equation*

$$\partial_t f + \partial_{k_j}\omega\,\partial_{x_j} f = S(f).$$

The collision operator $S(f)$ takes account of collisions between phonons, between phonons and lattice defects, and between phonons and crystal edges. We recall that there are two different mechanisms of phonon interaction that contribute to the collision operator: the N- and R-processes with which two basic dynamics of heat transfer processes in crystals are associated. Both processes conserve energy, while the normal process also conserves the moment. The corresponding contributions to S are denoted by $S_N(f)$ and $S_R(f)$. In this notation, we can write

$$S(f) = S_N(f) + S_R(f),$$

$$\int \hbar\omega\, S_N(f) d\overline{k} = \int \hbar\,\omega\, k_j S_N(f) d\overline{k} = 0,$$

$$\int \hbar\,\omega\, S_R(f) d\overline{k} = 0,$$

where it may occur

$$\int \hbar\,\omega\, k_j S_R(f) d\overline{k} \neq 0.$$

Consequently, e and p_j are conservative variables in the N-process, whereas only e, generally speaking, is a conservative variable in the R-process.

The distribution of phonon energy and its flux is described by the relations

$$e(x,t) = \int \hbar\,\omega(k) f(x,t,\overline{k})\ d\overline{k},$$

$$Q_j(x,t) = \int \hbar\,\omega(k)\partial_{k_j}\omega(k) f(x,t,\overline{k})\ d\overline{k} = c^2 p_j.$$

HIgher order moments are introduced in a similar way as in the kinetic theory of gases.

2.2. Moment approximations of the Boltzmann–Peierls kinetic equation.

Numerical investigations of the phonon model and their comparison with experiment [**16, 5**] show that hyperbolic systems with relaxation written in terms of systems of moments of the Boltzmann–Peierls kinetic equation serve as a perfect tool for describing heat transfer in crystals. In particular, the number of nonequilibrium variables (the number of equations in the moment approximation) required to obtain a satisfactory description of experimental data was given in [**5**]: at least *forty equations* are required in the one-dimensional case.

However, as was mentioned above, the incorporation of higher order moments with no direct physical interpretation gives rise to the problem of choosing the initial and boundary data. The Chapman–Enskog approach allows us to remain within the framework of initial and boundary data only for basic variables because the crucial point of this method is to find an operator dependence of nonequilibrium variables on consolidated variables of an irreducible projection

We will show that for moment approximations of phonon gas model there are only two irreducible projections: to the phase space of the conservative variable e and to the phase space of the variables $(e.p)$. Thus, for the Cauchy problem for the 3-moment system of phonon gas (one–dimensional case) we have (see [**5, 15**])

$$
\begin{aligned}
&\partial_t\, e + \partial_x p = 0,\\
&\partial_t p + \alpha_1 \partial_x e + \partial_x N + \frac{1}{\tau_R} p = 0, \\
&\partial_t N + \alpha_2 \partial_x p + \frac{1}{\tau_{NR}} = 0, \qquad \frac{1}{\tau_{NR}} > \frac{1}{\tau_R}
\end{aligned}
\tag{2.1}
$$

with only one conservative variable e for a diffusion process.

To pass from the wave process to a diffusion process, we need to study the existence conditions for a diffusion Chapman–Enskog projection $p = q(\partial_x)e$, $N = \mu(\partial_x)e$, $1/\tau_{NR} = 1/\tau_R + 1/\tau_N$ to the phase space of conservative variable e. Here, $\tau_R > 0$ is the relaxation time of the R-normal process and $\tau_N > 0$ is the relaxation time of the N-normal process, $\alpha_1 = c^2/3$, $\alpha_2 = 4c^2/(15)$, c is the Debye sound velocity [**15**], μ and q are pseudodifferential operators of at most zero order.

2.3. Chapman–Enskog projection.

To confirm the appearance of the ultraviolet catastrophe, we consider regular asymptotics for the Chapman–Enskog projection of the system (2.1) with rigid relaxation at large times:

$$\begin{aligned} &\partial_t e + \partial_x p = 0, \\ &\partial_t p + \alpha_1 \partial_x e + \partial_x N + \frac{1}{\varepsilon}\frac{1}{\tau_R} p = 0, \\ &\partial_t N + \alpha_2 \partial_x p + \frac{1}{\varepsilon}\frac{1}{\tau_{NR}} N = 0, \end{aligned} \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter. We look for a Chapman–Enskog projection in the form

$$\begin{aligned} &p = q(\varepsilon\partial_x) = \varepsilon\, q_1(\partial_x)e + \varepsilon^2 q_2(\partial_x)e + \ldots, \\ &N = N(\varepsilon\partial_x)\varepsilon\, \mu_1(\partial_x)e + \varepsilon^2 \mu_2(\partial_x)e + \ldots\ . \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) and equating terms at the same powers of ε, we obtain

$$\begin{aligned} q_1 &= -\tau_R \alpha_1 \partial_x, \\ q_3 &= -\tau_{NR}\partial_x \mu_2 + \partial_x q_1^2 = -(\tau_{NR}^2 \tau_R \alpha_2 \alpha_1 + \alpha_1^2 \tau_R^2)\partial_x^2, \end{aligned}$$

which imply the so-called *Navier–Stokes approximation*

$$\partial_t e = \varepsilon \tau_R \alpha_1 \partial_x^2 e$$

and *post-Navier–Stokes approximation*

$$\partial_t e = \varepsilon \tau_R \alpha_1 \partial_x^2 e + \varepsilon^3 (\tau_{NR}^2 \tau_R \alpha_2 \alpha_1 + \alpha_1^2 \tau_R^2)\partial_x^3 e$$

of the Boltzmann–Peierls kinetic equation. The first approximation is stable and the second approximation is unstable, whereas the system of moments (2.1) is stable. The dispersion equation of the system (2.2)

$$\begin{aligned} \mathcal{D}_3 &= \tau\Big(\tau - \frac{i}{\tau_{NR}}\Big)\Big(\tau - \frac{i}{\tau_R}\Big) - \xi^2\Big((\alpha_1+\alpha_2)\tau - \alpha_1 \frac{i}{\tau_{NR}}\Big) \\ &= P_0 - i\gamma_1 P_1 - \gamma_2 P_2 = 0, \end{aligned} \tag{2.4}$$

where

$$P_0 = \omega\Big(\omega^2 - (\alpha_1 + \alpha_2)\xi^2\Big),$$
$$P_1 = \omega^2 - \alpha_1 \frac{\tau_R}{\tau_{NR} + \tau_R}\xi^2, \quad P_2 = \omega,$$
$$\gamma_1 = \Big(\frac{1}{\tau_{NR}} + \frac{1}{\tau_R}\Big), \quad \gamma_2 = \frac{1}{\tau_{NR}\tau_R},$$

satisfies the following conditions for the stability of hyperbolic pencils [**22, 23**]:

(1) P_0, P_1, and P_2 are hyperbolic,

(2) the roots of neighboring polynomials of the pencil are strictly separated.

Such a situation is often observed in quantum mechanics and statistical physics. It is clear that we should take sufficiently many terms in (2.3). However, the main question is: How many?

The study of the regular expansion

$$e = e_0 + \varepsilon\, e_1 + \dots, \quad p = \varepsilon\, p_1 + \dots, \quad N = \varepsilon\, p_1 + \dots,$$

shows (see [**15**]), that the first terms in (2.3) can be collected into the following groups:

$$p_\varepsilon = \partial_x q_N(-\varepsilon^2\partial_x^2)e_\varepsilon, \quad N = \partial_x^2 \mu_{N-1}(-\varepsilon^2\partial_x^2)e_\varepsilon, \tag{2.5}$$

where q_N and μ_N are the Taylor series of the symbols $q(-\partial_x^2)$ and $\mu(-\partial_x^2)$ are pseudodifferential operators of order -2 and 0 respectively. Thus, "quantization" can be observed: the order of moment $\Leftrightarrow$ the order of Chapman–Enskog projection. We show that (2.5) is realized.

We formally construct a Chapman–Enskog projection in the linear case. Consider the linear system of n equations

$$\partial_t u + A\partial_x u + Bu = 0, \tag{2.6}$$

where A and B are matrices with constant entries such that $b_{ij} = 0$, $i = 1, \dots, m_c$, $j = 1, \dots, n$; $i = 1, \dots, n$, $j = 1, \dots, m_c$, and m_c, $1 \leqslant m_c < n$, is the number of conservative variables. We look for a projection to m equations ($m \geqslant m_c$) in the form

$$u = Pu_c. \tag{2.7}$$

The variables of projection $u_c = (u_1, \dots, u_m, 0, \dots, 0)^T$ including the conservative variables or coinciding with conservative variables are said to be *consolidated*. The matrix corresponding to the operator P has the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \tag{2.8}$$

where $P_{11} = E_m$ is the identity matrix of order m, $P_{22} = 0_{n-m}$ is the zero quadratic matrix of order $n-m$. Let the matrix of the operator $\Lambda = Ai\xi + B$ have the form

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \tag{2.9}$$

where Λ_{ij} have the same size as the matrices P_{ij}. Since P is a projection, $P^2 = P$ and

$$P\partial_t u_c + AP\partial_x u_c + BPu_c = 0, \tag{2.10}$$

$$P\partial_t u_c + PAP\partial_x u_c + PBPu_c = 0. \tag{2.11}$$

Subtracting (2.11) from (2.10), we find

$$(E - P)(A\partial_x + B)Pu_c = 0. \tag{2.12}$$

We denote by Π the Fourier image of P. Then (2.12) can be written in terms of Fourier images as follows:

$$(E - \Pi)\Lambda\Pi v_c = 0, \tag{2.13}$$

i.e., $\Lambda\Pi v_c \in \operatorname{Ker}(E-\Pi)$. Representing $v \in \operatorname{Ker}(E-\Pi)$ as $v^T = (v_m^T, v_{n-m}^T)$, $v_k \in \mathbb{R}^k$, we obtain the equality $v_{n-m} = \Pi_{21}v_m$. Hence we arrive at the system

$$\Pi_{21}(\Lambda_{11} + \Lambda_{12}\Pi_{21}) = \Lambda_{21} + \Lambda_{22}\Pi_{21}, \tag{2.14}$$

which completely determines the projection P.

A situation is interesting from the physical point of view if it is possible to pass from the system (2.14) with solutions depending on complex parameter ξ to a system of the same form, but with real-valued solutions depending on $|\xi|^2$. In such a case, the system under consideration remains in the class of hyperbolic first order systems with relaxation. Below we discuss an algorithm allowing such a passage.

Definition 2.1. A function

$$d(x) = \begin{cases} 1, & x \neq 0, ix \in \mathbb{R}, \\ 0, & x \in \mathbb{R}, \end{cases}$$

where $i^2 = -1$, is called a *recognizing function.*

Let $\Pi_{21} = ((i\xi)^{m_{kl}} p_{kl})_{k=1,\dots,n-m,\ l=1,\dots,m}$, and let $\Lambda_{st} = (a^{st}_{kl})$. Then for $m_{kl} \in \mathbb{Z}_2$ we obtain the following *quantization system* over $\mathbb{Z}_2$:

$$m_{kl} + d(a^{12}_{lj}) + m_{jt} = d(a^{21}_{kt}),$$
$$k=1,\dots,n-m,\ j=1,\dots,n-m,\ l=1,\dots,m,\ t=1,\dots,m$$

$$d(a^{22}_{ks}) + m_{st} = d(a^{21}_{kt}), \tag{2.15}$$
$$k=1,\dots,n-m,\ s=1,\dots,n-m,\ t=1,\dots,m,$$

$$m_{kq} + d(a^{11}_{qt}) = d(a^{21}_{kt}),$$
$$k=1,\dots,n-m,\ q=1,\dots,m,\ t=1,\dots,m.$$

The quantization system (2.15) is solvable only if a recognizing function is defined for all entries λ of the matrix A. Thus, the necessary condition for the solvability of the quantization system (2.15) can be formulated as follows.

Condition 2.1. $a_{ij}b_{ij} = 0$ for all i, j, where A and B are the matrices of the system (2.6).

Proposition 2.1. *If the system* (2.6) *admits projections to* m^1 *equations and to* m^2 *equations,* $k < m^1$, $k < m^2$, *then* $m^1_{jk} = m^2_{jk}$ *for all* j, *i.e., the solution to the quantization system* (2.15) *yields the same values for* m_{jk}.

Proposition 2.2. *Let the system* (2.6) *admit projections to one equation and to* m *equations,* $1 < k \leqslant m$. *If* $u_k = (i\xi)^{\varkappa_k} p_k u_1$ *in the case of the projection to one equation, then* $m_{jk} = m_{j1} + \varkappa_k$ *in the case of the projection to* m *equations.*

In the case of the projection to one equation, under some additional restrictions, we can formulate the necessary and sufficient solvability conditions for the system (2.15).

Proposition 2.3. *In the case of the projection to one equation, which corresponds to the system* (2.14) *with* $A_{11} = a_{11} \neq 0$, $a^{21}_j \neq 0$ *for all* j, *the quantization system* (2.15) *is solvable if and only if the following two conditions hold*:

$$\forall k : a^{12}_k \neq 0 : d(a^{12}_k) + d(a^{21}_k) + d(a_{11}) = 0 (\mathrm{mod}\ 2), \tag{2.16}$$

$$\forall j,k : a^{22}_{jk} \neq 0 : d(a^{22}_{jk}) + d(a^{21}_j) + d(a^{12}_k) + d(a_{11}) = 0 (\mathrm{mod}\ 2). \tag{2.17}$$

Proof. In this case, the quantization system (2.15) contains the equality $m_{j1} + d(a_{11}) = d(a^{21}_j)$, which yields an expression for m_{j1}. $\square$

Lemma 2.1. *In the case of the projection to one equation, the quantization system* (2.15) *with* $a_{11} \neq 0$, $a_j^{21} \neq 0$ *for all* j *is solvable if and only if* $a_{kk}^{22} d(a_{kk}^{22}) = 0$ *for all* k.

We note that the system (2.15) is overdetermined and for the solvability of (2.15) it is necessary to impose certain conditions on A and B.

Now, we reduce the matrix equation to the block form. Equation (2.14) determines invariant subspaces that are smooth with respect to $|\xi|^2$. We find the corresponding canonical form of the system (2.6).

Lemma 2.2. *Let an invertible matrix* S *be divided into blocks* S_{ij}, $i, j = 1, 2$, *where* S_{11} *and* S_{22} *are quadratic matrices. Suppose that* $FS = SF = E$ *and the matrix* F *is divided into blocks of the same size. If* $S_{11} = E$, *then the matrix* F_{22} *is invertible.*

Proof. Assume the contrary. Since $FS = E$, we have

$$F_{21} + F_{22} S_{21} = 0. \tag{2.18}$$

Since F_{22} is not invertible, there is a row $h \neq 0$ such that $hF_{22} = 0$. Taking into account (2.18), we get $hF_{21} = 0$. But, in this case, the last rows of the matrix F are linearly dependent: there is a row v such that $v \neq 0$ and $vF = 0$. Consequently, the matrix F is not invertible. However, F is the inverse of S, and we arrive at a contradiction. □

Theorem 2.1. *If the matrix* Λ *is divided into blocks* Λ_{ij}, $i, j, = 1, 2$, *then the quadratic matrix equation*

$$P_{21}\Lambda_{12}P_{21} - \Lambda_{22}P_{21} + P_{21}\Lambda_{11} - \Lambda_{21} = 0 \tag{2.19}$$

is solvable if and only if there exists a matrix S *satisfying the following three conditions*:

(1) S *is invertible,*

(2) $S_{11} = E$,

(3) $(S^{-1}\Lambda\, S)_{21} = 0$.

Proof. Assume that there exists a matrix S satisfying conditions (1)–(3) and introduce the notation $F = S^{-1}$. Then

$$F_{21} + F_{22}S_{21} = 0,$$
$$F_{21}(\Lambda_{11} + \Lambda_{12}S_{21}) + F_{22}(\Lambda_{21} + \Lambda_{22}S_{21}) = 0.$$

Expressing F_{21} from the first equation and substituting the expression obtained into the second equation, we find

$$F_{22}(-S_{21}(\Lambda_{11}+\Lambda_{12}S_{21}))+F_{22}(\Lambda_{21}+\Lambda_{22}S_{21})=0.$$

Since the matrix S satisfies the assumptions of Lemma 2.2, we have

$$(-S_{21}(\Lambda_{11}+\Lambda_{12}S_{21}))+(\Lambda_{21}+\Lambda_{22}S_{21})=0,$$

i.e., S_{21} satisfies (2.19).

Assume that (2.19) is solvable and set $S_{11}=E$, $S_{12}=0$, $S_{21}=P_{21}$, $S_{22}=E$. It is easy to check that there exists the inverse matrix $S^{-1}=2E-S$.

The constructed matrix S satisfies conditions (1) and (2). Computing $(S^{-1}\Lambda\, S)_{21}$, we find

$$\begin{aligned}(S^{-1}\Lambda\, S)_{21} &= F_{21}(\Lambda_{11}+\Lambda_{12}S_{21})+F_{22}(\Lambda_{21}+\Lambda_{22}S_{21})\\ &= (-P_{21})(\Lambda_{11}+\Lambda_{12}P_{21})+(\Lambda_{21}+\Lambda_{22}P_{21})=0\end{aligned}$$

since P_{21} is a solution to (2.19). Thus, condition (3) is also satisfied. □

Thus, the existence of a Chapman–Enskog projection is equivalent to the reduction of the system to the block form, which allows us to separate dynamics.

2.4. Examples of moment approximations for Boltzmann–Peierls equation.

2.4.1. Projections of three equations to one equation. Making the change of variables $(t,x) \to (t,x)/\tau_{NR}$, we reduce (2.1) to the case of a single parameter $q=\tau_{NR}/\tau_R<1$:

$$\begin{gathered}\partial_t e+\partial_x p=0,\\ \partial_t p+\alpha_1\partial_x e+\partial_x N+q\,p=0,\\ \partial_t N+\alpha_2\partial_x p+N=0.\end{gathered} \tag{2.20}$$

By definition,

$$\Lambda=\begin{pmatrix} 0 & i\xi & 0\\ \alpha_1 i\xi & q & i\xi\\ 0 & \alpha_2 i\xi & 1\end{pmatrix}. \tag{2.21}$$

Then

$$\Lambda_{11}=0,\quad \Lambda_{12}=\begin{pmatrix} i\xi & 0\end{pmatrix},\quad \Lambda_{21}=\begin{pmatrix}\alpha_1 i\xi\\ 0\end{pmatrix},\quad \Lambda_{22}=\begin{pmatrix} q & i\xi\\ \alpha_2 i\xi & 1\end{pmatrix}.$$

In this case, the quantization system has a unique solution

$$m_{11} = 1, \quad m_{21} = 0.$$

The system (2.14) takes the form

$$\begin{aligned} &\xi^2 p_1^2 + q\, p_1 + p_2 + \alpha_1 = 0, \\ &\xi^2 p_1 p_2 - \alpha_2 \xi^2 p_1 + p_2 = 0. \end{aligned}$$

We set $Q = \xi^2 p_1$ and $Y = \xi^2 p_2$. Then

$$(Q+q)Q = -Y - \alpha_1 \xi^2, \quad (Q+1)Y = \alpha_2 Q.$$

Consequently, the equation

$$(Q+1)(Q+q)Q + \xi^2((\alpha_2+\alpha_1)Q + \alpha_1) = 0 \tag{2.22}$$

has a real-valued monotone decreasing solution, smooth with respect to ξ^2, such that $Q(0) = 0$ and $Q(|\xi|^2) \to -\dfrac{\alpha_1}{\alpha_1+\alpha_2}$ as $|\xi| \to \infty$ if and only if

$$q > \alpha_1/(\alpha_1+\alpha_2). \tag{2.23}$$

Using the transformation $(e, p, N)^\top = S(\widehat{e}, \widehat{p}, \widehat{N})^\top$, we reduce the system (2.1) to the block form

$$\begin{aligned} &\partial_t \widehat{e} - Q\widehat{e} + \partial_x \widehat{p} = 0, \\ &\partial_t \widehat{p} - (Q+q)\widehat{p} + \partial_x \widehat{N} = 0, \\ &\partial_t \widehat{N} + (\alpha_2 - p_2)\partial_x \widehat{p} + \widehat{N} = 0. \end{aligned} \tag{2.24}$$

The projection equation takes the form

$$\partial_t\ w - Q\ w = 0 \tag{2.25}$$

and

$$\mathcal{M}_{\text{ChEns}} = \{(1,0,0)w\},$$

where w is a solution to the Cauchy problem for (2.25). A solution to the Cauchy problem for (2.24) can be represented as $\widehat{U} = \widehat{U}_1 + \widehat{U}_2$, where $\widehat{U}_1 = (1,0,0)^\top w_1 \in \mathcal{M}_{\text{ChEns}}$, w_1 is a solution to the Cauchy problem for (2.25) with initial condition $w_1\big|_{t=0} = \widehat{e}^0$, $U^0 = (e^0, p^0, N^0)^\top = S\ (\widehat{e}^0, \widehat{p}^0, \widehat{N}^0)^\top$, $\widehat{U}_2$ is a solution to the Cauchy problem for (2.24) with initial condition $\widehat{U}_2\big|_{t=0} = (0, \widehat{p}^0, \widehat{N}^0)^\top$.

Now, we construct a corrector $U_{\text{cor}} \in \mathcal{M}_{\text{ChEns}}$ such that

$$\|(U - U_1 - U_{\text{cor}})(t,\cdot)\|_{L^2(\mathbb{R})} = o(\|(U_1 + U_{\text{cor}})(t,\cdot)\|_{L^2(\mathbb{R})}). \tag{2.26}$$

Then $U_{\mathrm{ChEns}} = U_1 + U_{\mathrm{cor}} \in \mathcal{M}_{\mathrm{ChEns}}$ is the projection of U to the invariant manifold $\mathcal{M}_{\mathrm{ChEns}}$ determined by (2.25) with initial condition

$$w|_{t=0} = \tau(e^0, p^0, N^0) = \widehat{e}^0 + \widehat{e}^0_{\mathrm{cor}}.$$

We note that the solution $Q(|\xi|^2)$ to the equation for the generating function is connected with a purely imaginary root $\omega(|\xi|^2) = -iQ(|\xi|^2)$ of the dispersion equation

$$\begin{aligned} D_3(\omega,\xi^2) &= i\omega(i\omega+1)(i\omega+q) + \xi^2((\alpha_2+\alpha_1)i\omega+\alpha_1) \\ &= (\omega+iQ)[(i\omega+q+Q)(i\omega+1)+(\alpha_2-p_2)\xi^2] = 0, \end{aligned} \tag{2.27}$$

where the second factor coincides with the dispersion polynomial of the system consisting of the last two equations in (2.24) whose roots ω_1^b and ω_2^b are of boundary-layer type and are conjugate, i.e., $i\omega_1 = -i\overline{\omega_2}$. Let R_j be the corresponding eigenvectors. Then a solution to the Cauchy problem for the system

$$\begin{aligned} &\partial_t\widehat{p} + \partial_x\widehat{N} + (q+Q(-\Delta))\widehat{p} = 0, \\ &\partial_t\widehat{N} + (\alpha_2-p_2)\partial_x\widehat{p} + \widehat{N} = 0, \\ &\widehat{p}\big|_{t=0} = \widehat{p}^0, \quad \widehat{N}|_{t=0} = \widehat{N}^0 \end{aligned}$$

can be written in terms of Fourier images as follows:

$$(\widetilde{\widehat{p}},\ \widetilde{\widehat{N}})^\top = C_1(\xi)R_1(\xi)\ e^{i\omega_1\ t} + C_2(\xi)R_2(\xi)\ e^{i\omega_1\ t}.$$

The first component of the solution $\widehat{U_2}$ is written as

$$\begin{aligned} \widetilde{\widehat{e}}(t,\xi) = &\int_0^t i\xi\ (C_1(\xi)R_{11}(\xi)e^{i\omega_1\tau} + C_2(\xi)R_{2,1}(\xi)e^{i\omega_1\tau})e^{Q(|\xi|^2)(t-\tau)}\,d\tau \\ &= e^{Q(|\xi|^2)t}i\xi\Big[C_1(\xi)R_{11}(\xi)\int_0^t e^{i(\omega_1+iQ(|\xi|^2))\ \tau} \\ &\quad + C_2(\xi)R_{2,1}(\xi)\int_0^t e^{i(\omega_1+iQ(|\xi|^2))\ \tau}\ d\tau\Big] \\ &= e^{Q(|\xi|^2)t}i\xi[C_1(\xi)R_{11}(\xi)\int_0^\infty e^{i(\omega_1+iQ(|\xi|^2))\ tau}d\tau \end{aligned}$$

$$+ C_2(\xi)R_{2,1}(\xi) \int_0^\infty e^{i(\omega_1 + iQ(|\xi|^2))\tau}\, d\tau]$$

$$- e^{Q(|\xi|^2)t} i\xi \Big[C_1(\xi)R_{11}(\xi) \int_t^\infty e^{i(\omega_1 + iQ(|\xi|^2))\tau} d\tau$$

$$+ C_2(\xi)R_{2,1}(\xi) \int_t^\infty e^{i(\omega_1 + iQ(|\xi|^2))\ \tau}\ d\tau \Big].$$

Hence

$$\widetilde{\widehat{e^0_{\text{cor}}}} = i\xi\ [C_1(\xi)R_{11}(\xi) \int_0^\infty e^{i(\omega_1 + iQ(|\xi|^2))\ \tau}\, d\tau$$

$$+ C_2(\xi)R_{2,1}(\xi) \int_0^\infty e^{i(\omega_1 + iQ(|\xi|^2))\ \tau}\ d\tau].$$

Furthermore, (2.26) holds provided the following integrals are finite:

$$\int_0^\infty e^{i(\omega_1 + iQ(|\xi|^2))\ \tau}\ d\tau, \quad \int_0^\infty e^{i(\omega_1 + iQ(|\xi|^2))\ \tau}\ d\tau,$$

which is true in our case since Im $\omega_j + Q > 0$ for all $\xi \neq 0$.

2.4.2. Projection of (e, p) to the phase space. We consider the projection of (e, p) to the phase space. We have

$$\Lambda_{11} = \begin{pmatrix} 0 & i\xi \\ i\xi\alpha_1 & q \end{pmatrix},\ \Lambda_{12} = \begin{pmatrix} 0 \\ i\xi \end{pmatrix},$$

$$\Pi_{21} = (p_1, p_2),\ \Lambda_{21} = (0, i\xi\alpha_2),\ \Lambda_{22} = (1)$$

We obtain the following system for $X = \xi^2 p_1(|\xi|^2)$ and $Q = \xi^2 p_2(|\xi|^2)$:

$$X(Q+1) + \alpha_1\xi^2 Q = 0,$$
$$Q(Q+1-q) + \alpha_2\xi^2 - X = 0.$$

Then

$$Q(Q+1-q)(Q+1) + \xi^2((\alpha_1+\alpha_2)Q + \alpha_2) = 0.$$

A smooth real-valued monotone decreasing solution to this equation such that $Q(0) = 0$ and $Q(|\xi|^2) \to -\dfrac{\alpha_2}{\alpha_1+\alpha_2}$ as $|\xi| \to \infty$ exists if and only if

$$1 - q > \frac{\alpha_2}{\alpha_1+\alpha_2} \Rightarrow q < \frac{\alpha_1}{\alpha_1+\alpha_2}. \tag{2.28}$$

Hence we obtain the inequality opposite to (2.23). The solution Q is connected with a boundary-layer purely imaginary root $\omega(|\xi|^2) = i(1+Q(|\xi|^2))$ of the dispersion equation (2.27). In this case, making the change of variables $(\widehat{e}, \widehat{p}, \widehat{N})^\top = S^{-1}\,(e, p, N)^\top$, we can write the system (2.1) as follows:

$$\partial_t\widehat{e} + \partial_x\widehat{p} - Q(-\Delta)\widehat{e} = 0,$$
$$\partial_t\widehat{p} + (\alpha_1 + p_1)\partial_x\widehat{e} + (q - Q(-\Delta))\widehat{p} + \partial_x\widehat{N} = 0,$$
$$\partial_t\widehat{N} + (1 + Q(-\Delta))\widehat{N} = 0.$$

We have $\mathcal{M}_{\text{ChEns}} = \{(\widehat{e}, \widehat{p}, 0, 0)^\top$, where $(\widehat{e}, \widehat{p})^\top$ is a solution to the Cauchy problem for the system

$$\begin{aligned}\partial_t\widehat{e} + \partial_x\widehat{p} - Q(-\Delta)\widehat{e} = 0,\\ \partial_t\widehat{p} + (\alpha_1 + p_1)\partial_x\widehat{e} + (q - Q(-\Delta))\widehat{p} = 0.\end{aligned} \tag{2.29}$$

As above, the solution $\widehat{U}$ to the Cauchy problem for (2.4.2) can be represented as the sum $\widehat{U}_1 + \widehat{U}_2$, where $\widehat{U}_1 \in \mathcal{M}_{\text{ChEns}}$ is the solution to the Cauchy problem for (2.4.2) with initial condition $\widehat{U}_1|_{t=0} = (\widehat{e}^0, \widehat{p}^0, 0)^\top$ and $\widehat{U}_2 = \widehat{U}_{\text{cor}} + \widehat{U}_3$, $\widehat{U}_{\text{cor}} \in \mathcal{M}_{\text{ChEns}}$,

$$\|(\widehat{U} - \widehat{U}_1 - \widehat{U}_{\text{cor}})(t,\cdot)\|_{L^2(R)} = o(\|(\widehat{U}_1 + \widehat{U}_{\text{cor}})(t,\cdot)\|_{L^2(R)}), \quad t \to \infty.$$

Consequently, the projection of the solution to the Cauchy problem for (2.4.2) to the invariant manifold $\mathcal{M}_{\text{ChEns}}$ is the vector-valued function

$$\widehat{U}_{\text{ChEns}} = \widehat{U}_1 + \widehat{U}_{\text{cor}} \in \mathcal{M}_{\text{ChEns}}$$

determined as the solution to the Cauchy problem for (2.4.2) with initial condition $\widehat{U}_{\text{ChEns}}|_{t=0} = \tau(\widehat{e^0, p^0, N^0}) = (\widehat{e}^0 + \widehat{p}^0_{\text{cor}},\ \widehat{p}^0 + \widehat{p}^0_{\text{cor}},\ 0)^\top$.

2.5. Diffusion and boundary-layer type Chapman–Enskog projections.

By the above arguments, we obtain the following assertion.

Proposition 2.4. *For*

$$q > \frac{\alpha_1}{\alpha_1+\alpha_2} \tag{2.30}$$

there exists a smooth bounded branch $Q(|\xi|^2)$ *of a root of the equation*

$$Q(Q+q)(Q+1)+\xi^2((\alpha_1+\alpha_2)Q+\alpha_1)=0 \tag{2.31}$$

stabilizing at infinity as $|\xi|\to\infty$ *and satisfying the conditions* $Q(0)=0$ *and* $Q'(0)\neq 0$.

From Proposition 2.4 we obtain the following assertion.

Theorem 2.2. *For*

$$q \ > \ \frac{\alpha_1}{\alpha_1+\alpha_2} \tag{2.32}$$

there exists a diffusion type Chapman–Enskog projection

$$p=\partial_x p_1(-\Delta_x)e, \quad N=p_2(-\Delta_x)e \tag{2.33}$$

of the system (2.1) *to the phase space of conservative variable* e. *The Cauchy problem for the projection equations* (*the quotient equation of the Chapman–Enskog projection*) *has the form*

$$\partial_t w(t,x)-Q(-\partial_x^2)w(t,x)=0, \quad w\big|_{t=0}=w^0(x) \tag{2.34}$$

and is stable in view of the properties of the generating function Q.

Thus, using the Chapman–Enskog projection, we remain within the framework of hyperbolic systems with relaxation.

The graph of the generating function $Q(|\xi|^2)=|\xi|^2p_1(|\xi|^2)$ is presented in Fig. 1. The function $Q(|\xi|^2)$ stabilizes to the constant $Q(\infty)=-\alpha_1/(\alpha_1+\alpha_2)$ as $|\xi|\to\infty$ and has a clearly expressed interior layer with abrupt drop of the profile. Such functions are called *kinks*.

Let us write (2.34) in the form

$$\partial_t w(t,x)=\partial_x^2 p_1(-\partial_x^2)w(t,x) \tag{2.35}$$

Remark 2.1. 1. Functions Q and p_1 are not well approximated by the first terms of the Taylor expansion at the origin in the uniform norm. Therefore, there are no satisfactory approximations of solutions even for regular asymptotics. This situation is similar to the well-known Gibbs phenomenon concerning the nonuniform convergence of partial sums of the Fourier series in a neighborhood of a discontinuity point.

2. The coefficient $p_1(-\partial_x^2)$ in (2.35) plays the role of the operator diffusion coefficient. The behavior of the solution to the Cauchy problem for (2.35) at large times is the same as in the case of the Cauchy problem for a parabolic equation. The question on separating dynamics has positive

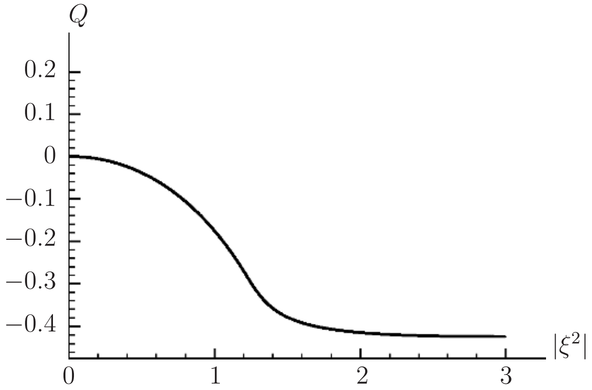

FIGURE 1

answer in the L^2-norm over the cut-sections $t = \text{const}$, which justifies the Navier–Stokes approximation at large times.

3. Making the change $Q(|\xi|^2) = i\omega(|\xi|^2)$, we transform (2.22) to the dispersion polynomial (2.27), and the existence of a generating function Q is equivalent to the existence of a branch, smooth in $|\xi|^2$, of a purely imaginary root $\omega(|\xi|^2)$ of the dispersion equation of diffusion type (2.22), i.e., $\omega(0) = 0$ and $\omega'(0) \neq 0$.

For the inequality opposite to (2.32) the projection has the form

$$N = p_1(-\partial_x^2)e + \partial_x p_2(-\partial_x^2)p. \tag{2.36}$$

Proposition 2.5. *In the parameter range* (2.28), *there exists a real-valued smooth bounded* $Q(|\xi|^2)$ *branch of a root of the equation*

$$Q(Q + 1 - q)(Q + 1) + \xi^2((\alpha_1 + \alpha_2)Q + \alpha_2) = 0 \tag{2.37}$$

stabilizing that stabilizes at infinity, $Q(|\xi|^2) \to -\alpha_2/(\alpha_1 + \alpha_2)$ *as* $|\xi| \to \infty$, *and satisfying the conditions* $Q(0) = 0$ *and* $Q'(0) \neq 0$.

Theorem 2.3. *For*

$$q < \frac{\alpha_1}{\alpha_1 + \alpha_2} \tag{2.38}$$

there exists a Chapman–Enskog projection of the second sound velocity of the system (2.1) *to the phase space of* (e, p). *The system of projection equations*

$$\begin{aligned} &\partial_t e + \partial_x p = 0, \\ &\partial_t p + (\alpha_1 + p_1)\partial_x e + \Big(\frac{1}{\tau_R} - Q\Big)p = 0 \end{aligned} \tag{2.39}$$

is a hyperbolic pseudodifferential first order system with relaxation.

Thus, in this case, we also remain in the class of hyperbolic systems with relaxation.

Indeed, by properties of the generating function Y, the system (2.39) is a hyperbolic pseudodifferential first order system with relaxation. The corresponding dispersion equation

$$D_{\text{ChEns}} = \omega(\omega - i(q - Y)) - (\alpha_1 + \sigma)\xi^2 = 0 \tag{2.40}$$

determines a stable hyperbolic pencil provided that

$$\alpha_1 + p_1(|\xi|^2) > 0, \quad q - Q(|\xi|^2) > 0 \quad \forall |\xi| \geqslant 0. \tag{2.41}$$

The second condition in (2.41) holds because the generating function Q is nonnegative. Since Q is monotone, p_1 monotonnically increases to the limit values $p_1^\infty = \alpha_2$ and $p_1(0) = 0$. Thus, p_1 is a positive kink-like function, and the first inequality in (2.41) is obviously satisfied.

Making the change $U = SV$, $V = (V_1, V_2, V_3)^\top$, we reduce the system (2.1) to the block form

$$\begin{aligned} &\partial_t V_1 + \partial_x V_2 = 0, \\ &\partial_t V_2 + (\alpha_1 + p_1)\partial_x V_2 + (q - Q)V_2 + \partial_x V_3 = 0, \\ &\partial_t V_3 + (1 - Q)V_3 = 0. \end{aligned} \tag{2.42}$$

A solution to the Cauchy problem for (2.42) has the form

$$V = W_1 + W_2,$$

where W_1 belongs to the invariant manifold $\mathcal{M}_{\text{ChEns}}$ of solutions to the Cauchy problem for (2.42) with initial condition $W_1^0 = (V_1^0, V_2^0, 0)^\top$, $V^0 = S^{-1}\, U^0$, and W_2 is the solution to the Cauchy problem for (2.42) with initial condition $W_2^0 = (0, 0, V_3^0)^\top$.

From the Duhamel principle, in terms of Fourier images, we can write

$$(\widetilde{W_2})_3 = e^{-(1-Q)t}\; V_3^0,$$

$$\begin{pmatrix}(\widetilde{W_2})_1\\(\widetilde{W_2})_2\end{pmatrix}=\int\limits_0^t(e^{i\omega_1(t-\tau)}C_1(\tau)R_1+e^{i\omega_2(t-\tau)}C_2(\tau)R_2)d\tau$$

$$=e^{i\omega_1 t}R_1\int\limits_0^\infty e^{-i\omega_1\tau}C_1(\tau)\,d\tau+e^{i\omega_1 t}R_2\int\limits_0^\infty e^{-i\omega_2\tau}C_2(\tau)\,d\tau$$

$$=-\int\limits_t^\infty(e^{i\omega_1(t-\tau)}C_1(\tau)R_1+e^{i\omega_2(t-\tau)}C_2(\tau)R_2)d\tau,$$

where R_j are eigenvectors of the roots ω_j of the dispersion equation (2.40) of the system

$$C_1(\tau)R_1+C_2(\tau)R_2=(0,-i\xi\,e^{-(1-Q)\tau}\;V_3^0)^\top.$$

Consequently, the projection of the solution V to the manifold $\mathcal{M}_{\mathrm{ChEns}}$ is represented as the sum

$$V_{\mathrm{ChEns}}=V_1+V_{\mathrm{cor}},$$

where V_{cor} is the solution to the Cauchy problem for (2.42) with

$$(\widetilde{V^0_{\mathrm{cor}}})_j=R_{j1}\int\limits_0^\infty e^{-i\omega_1\tau}C_1(\tau)\,d\tau+R_{j2}\int\limits_0^\infty e^{-i\omega_2\tau}C_2(\tau)\,d\tau,\;j=1,2,$$

$$(\widetilde{V^0_{\mathrm{cor}}})_3=0.$$

We say that the roots $\omega_1,\ \omega_2,\ \omega_3=i(1-Q)$ of the dispersion equation (2.27) satisfy the gap condition if $\mathrm{Im}\,\omega_3>\ \mathrm{Im}\,\omega_j$, $j=1,2$, for all $|\xi|\geqslant 0$. For the polynomial (2.27) this condition is satisfied. This means the separation of dynamics in this case.

The origin of the term "second sound velocity" could be explained as follows. One of the roots of the dispersion equation $\tau(\tau^2-(\alpha_1+\alpha_2))=0$ of the leading part of the system (2.1) determines the characteristic velocity $\sqrt{\alpha_1+\alpha_2}<c$ which is less than the sound velocity in the Debye phonon model. We note that the characteristic velocity of the quotient system satisfies the relation

$$\sqrt{\alpha_1+\sigma(|\xi|^2)}<\sqrt{\alpha_1+\alpha_2}\ \forall|\xi|<\infty,$$

and approaches to the second sound velocity only at high frequencies.

2.6. Chapman–Enskog projection and Schrödinger Approximation

Following [**25**], we consider the following extension of the Maxwell system (see [**12**]):

$$
\begin{aligned}
i\alpha\frac{\partial U}{\partial t} &= \operatorname{rot} U - \beta\nabla\ \varrho - \gamma_0 U,\\
i\alpha_1\frac{\partial \rho}{\partial t} &= \operatorname{div} U - \gamma_1\rho,
\end{aligned}
\tag{2.43}
$$

where α, α_1, β, γ_0, γ_1 are constants. The system (2.43) possesses a number of remarkable properties.

We first discuss how the extended hyperbolic Maxwell system (2.43) is connected with basic equations in quantum mechanics. Setting $\alpha_1 = 0$, $\beta = 0$, and $\gamma_0 = 0$ in (2.43), we obtain the Maxwell system

$$
\begin{aligned}
&i\alpha\frac{\partial U}{\partial t} = \operatorname{rot}\ U - f,\\
&\operatorname{div}\ U = \gamma_1\rho.
\end{aligned}
$$

Indeed, introduce the notation $\alpha = \sqrt{\varepsilon\mu}/c$,

$$U = U_1 + iU_2,$$

where $U_1 = \sqrt{\varepsilon}E$ and $U_2 = \sqrt{\mu}H$,

$$f = f_1 + if_2,$$

where $f_1 = -\dfrac{4\pi}{c}\ \sqrt{\varepsilon}\ j_m$ and $f_2 = \dfrac{4\pi}{c}\ \sqrt{\mu}\ j_e$,

$$\rho = \rho_1 + i\rho_2,$$

where $\rho_1 = \dfrac{4\pi}{\sqrt{\varepsilon}}\ \rho_e$ and $\rho_2 = \dfrac{4\pi}{\sqrt{\mu}}\ \rho_m$. Then we obtain a "symmetrized" Maxwell system which differs from the classical one by the presence of the "magnetic" charge ρ_m introduced by Dirac and the "magnetic" flow j_m introduced by Schwinger.

If $\gamma_0 = \alpha_1 = 0$, we have the system

$$
\begin{aligned}
&i\alpha\frac{\partial U}{\partial t} = \operatorname{rot}\ U - \beta\operatorname{grad}\ \rho - f,\\
&\operatorname{div}\ U = \gamma_1\rho.
\end{aligned}
\tag{2.44}
$$

Applying div to the first equation in (2.44), we obtain the nonhomogeneous Schrödinger equation

$$i\alpha\partial_t\ \operatorname{div} U + \frac{\beta}{\gamma_1}\ \Delta\ \operatorname{div} U + \operatorname{div}\ f = 0.$$

Setting $m = \dfrac{1}{2}\dfrac{\sqrt{\varepsilon\mu}}{\beta}\gamma_1$ and div $f = 0$, we obtain the so-called free Schrödinger equation

$$i\hbar\,\partial_t \operatorname{div} U + \frac{\hbar^2}{2m}\,\Delta \operatorname{div} U = 0.$$

Thus, div U coincides in limit with the wave Schrödinger function.

The system (2.44) has the same number of equations as the Dirac system, but these systems are not equivalent. We show that the former system is closely connected with the Schrödinger equation, namely, the latter appears as the limit of such systems. Depending on the ratio γ_0/γ_1, we introduce approximations, called Schrödinger approximations, which are similar to Navier–Stokes approximations. We use the method of regular asymptotic expansions. Consider two cases.

Case 1. $\gamma_0 = 1/\varepsilon$, $\varepsilon \ll 1$, $\gamma_1 = O(1)$. For the longitudinal wave in the first approximation, from the first equation in (2.43) we find

$$\operatorname{div} U = -\varepsilon\beta\Delta\varrho. \tag{2.45}$$

From the second equation in (2.43) it follows that

$$\alpha_1\partial_t\varrho + \varepsilon\beta\Delta\varrho + \gamma_1\varrho = 0. \tag{2.46}$$

As we can see, the system (2.45), (2.46), called a Schrödinger approximation, is similar to a Navier–Stokes approximation [**15, 21**].

Case 2. $\gamma_1 = 1/\varepsilon$, $\varepsilon \ll 1$, $\gamma_0 = O(1)$. From the second equation in (2.43) we find

$$\varepsilon \operatorname{div} U = \gamma_1\varrho. \tag{2.47}$$

From the first equation for div U in the first approximation we obtain

$$\alpha\gamma_1\partial_t \operatorname{div} U + \varepsilon\beta\Delta \operatorname{div} U - \gamma_0\gamma_1 \operatorname{div} U = 0. \tag{2.48}$$

In this case, the system (2.47), (2.48) is the Schrödinger approximation.

In the case $\alpha_1\gamma_0 - \alpha\gamma_1 \neq 0$, two projections are possible: to one equation and to three equations. Making the change $(\varrho, U)^\top = S(r, V)^\top$, we can consider both cases. Note that the blocks of FAS satisfy the relations

$$\begin{aligned}
(FAS)_{11} &= A_{11} + A_{12}S_{21},\\
(FAS)_{12} &= A_{12},\\
(FAS)_{21} &= 0,\\
(FAS)_{22} &= -S_{21}A_{12} + A_{22}.
\end{aligned}$$

In the case of the projection to one equation, we have

$$A = \begin{pmatrix} -\dfrac{i\gamma_1}{\alpha_1} & -\dfrac{\xi_1}{\alpha_1} & -\dfrac{\xi_2}{\alpha_1} & -\dfrac{\xi_3}{\alpha_1} \\ -\dfrac{\beta\xi_1}{\alpha} & -\dfrac{i\gamma_0}{\alpha} & \dfrac{\xi_3}{\alpha} & -\dfrac{\xi_2}{\alpha} \\ -\dfrac{\beta\xi_2}{\alpha} & -\dfrac{\xi_3}{\alpha} & -\dfrac{i\gamma_0}{\alpha} & \dfrac{\xi_1}{\alpha} \\ -\dfrac{\beta\xi_3}{\alpha} & \dfrac{\xi_2}{\alpha} & -\dfrac{\xi_1}{\alpha} & -\dfrac{i\gamma_0}{\alpha} \end{pmatrix}. \tag{2.49}$$

Therefore, setting

$$S_{21} = (i\xi_1 R_1,\ i\xi_2 R_2,\ i\xi_3 R_3)^T,$$
$$Q = -i\alpha_1 A_{12} S_{21} = \sum_{j=1}^{3} \xi_j^2 R_j$$

and using the above formulas, we find

$$(FAS)_{11} = \frac{iQ - i\gamma_1}{\alpha_1},$$

$$(FAS)_{22} = \begin{pmatrix} -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\xi_1^2 R_1}{\alpha_1} & \dfrac{\xi_3}{\alpha} + \dfrac{i\xi_1\xi_2 R_1}{\alpha_1} & -\dfrac{\xi_2}{\alpha} + \dfrac{i\xi_1\xi_3 R_1}{\alpha_1} \\ -\dfrac{\xi_3}{\alpha} + \dfrac{i\xi_1\xi_2 R_2}{\alpha_1} & -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\xi_2^2 R_2}{\alpha_1} & \dfrac{\xi_1}{\alpha} + \dfrac{i\xi_2\xi_3 R_2}{\alpha_1} \\ \dfrac{\xi_2}{\alpha} + \dfrac{i\xi_1\xi_3 R_3}{\alpha_1} & -\dfrac{\xi_1}{\alpha} + \dfrac{i\xi_2\xi_3 R_3}{\alpha_1} & -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\xi_3^2 R_3}{\alpha_1} \end{pmatrix}$$

From (2.19) it follows that

$$\alpha Q^2 - (\alpha\gamma_1 - \alpha_1\gamma_0)Q - \alpha_1\beta|\xi|^2 = 0. \tag{2.50}$$

Thus, if we have the projection to one equation, the system (2.43) can be written as

$$i\alpha_1 \partial_t\, r = \operatorname{div} V + (Q(-\Delta) - \gamma_1)\, r, \tag{2.51}$$

$$i\alpha\ \partial_t V = \operatorname{rot} V - \gamma_0 V + \frac{\alpha}{\alpha_1} B_1 V, \tag{2.52}$$

where the Fourier image of B_1 has the form

$$\begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix} \xi\xi^T. \tag{2.53}$$

In the new variables, the invariant manifold $\mathcal{M}_{\text{ChEns}}$ determined by the Chapman–Enskog projection is written as $\mathcal{M}_{\text{ChEns}} = (1,0,0,0)^\top\, r(x,t)$, where $r(x,t)$ is the solution to the Cauchy problem for the equation

$$i\alpha_1\partial_t\, r = (Q(-\Delta) - \gamma_1)\, r \tag{2.54}$$

with initial condition $r_1\big|_{t=0} = r^0$, $(\varrho^0, U^0) = S(r^0, V^0)$.

The second invariant manifold $\mathcal{M}$ is determined by the Cauchy problem for the system (2.51), (2.52) with initial condition $V|_{t=0} = V^0$, $r|_{t=0} = 0$. The direct sum of these manifolds coincides with the phase space of variables (V, r).

We recall the dispersion equation of the system (2.43):

$$D_4 = ((\alpha\omega - \gamma_0)^2 - |\xi|^2)\;((\alpha\omega - \gamma_0)(\alpha_1\omega - \gamma_1) - \beta|\xi|^2) = 0. \tag{2.55}$$

In the case of real coefficients α, α_1, β, γ_0, γ_1, the second factor in (2.55) does not have multiple roots, i.e.,

$$(\alpha_1\gamma_0 + \alpha\gamma_1)^2 - 4\alpha\alpha_1(\gamma_1\gamma_0 - \beta|\xi|^2) = (\alpha_1\gamma_0 - \alpha\gamma_1)^2 + 4\alpha\alpha_1\beta|\xi|^2 > 0$$

if $\alpha_1\gamma_0 - \alpha\gamma_1 \neq 0$. The factors in (2.55) have a common root for $|\xi|$ such that

$$\Big(\omega - \frac{\gamma_0}{\alpha}\Big)^2 + \Big(\omega - \frac{\gamma_0}{\alpha}\Big)\Big(\frac{\gamma_0}{\alpha} - \frac{\gamma_1}{\alpha_1}\Big) - \frac{\beta}{\alpha\alpha_1}|\xi|^2$$
$$= \pm\,|\xi|\Big(\frac{\gamma_0}{\alpha} - \frac{\gamma_1}{\alpha_1}\Big) + \Big(\frac{1}{\alpha} - \frac{\beta}{\alpha\alpha_1}\Big)|\xi|^2 = 0,$$

i.e.,

$$\pm\Big(\frac{\gamma_0}{\alpha} - \frac{\gamma_1}{\alpha_1}\Big) + \Big(\frac{1}{\alpha} - \frac{\beta}{\alpha\alpha_1}\Big)|\xi| = 0.$$

Consequently, for $\beta = \alpha_1$, $\gamma_0\alpha_1 - \gamma_1\alpha \neq 0$ the factors in (2.55) have no common roots.

The dispersion equation of the system (2.51), (2.52) has the form

$$D_3 = ((\alpha\omega - \gamma_0)^2 - |\xi|^2)\;\Big(\omega - \frac{\alpha_1\gamma_0 - \alpha\, Q}{\alpha\alpha_1}\Big).$$

We represent the solution to the system (2.51), (2.52) in the form

$$(r, V) = (r_1, 0) + (r_2, V_2),$$

where r_1 is the solution to the Cauchy problem for (2.54) with initial condition $r_1|_{t=0} = r^0$ and (r_2, V_2) is the solution to the Cauchy problem for (2.51), (2.52) with initial condition $r_2|_{t=0} = 0$, $V|_{t=0} = V^0$. By the Duhamel principle,

$$r_2(x,t) = \int\limits_0^t e^{i\frac{\gamma_1}{\alpha_1}\ (t-\tau)}\ e^{-i\frac{Q(-\Delta)}{\alpha_1}\ (t-\tau)}\,(\operatorname{div}\,V)|_{t=\tau}\ d\tau.$$

In the case of the projection to three equations, we set

$$S_{21} = (i\xi_1 P_1, i\xi_2 P_2, i\xi_3 P_3),$$

$$Q' = -\frac{i\alpha}{\beta} S_{21} A_{12} = \sum_{j=1}^{3} \xi_j^2 P_j.$$

Further, we replace columns of the matrix A with its rows in the way corresponding to transfer of ϱ to the last position. Then

$$(FAS)_{11} = \begin{pmatrix} -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\beta\xi_1^2 P_1}{\alpha} & \dfrac{\xi_3}{\alpha} + \dfrac{i\beta\xi_1\xi_2 P_2}{\alpha} & -\dfrac{\xi_2}{\alpha} + \dfrac{i\beta\xi_1\xi_3 P_3}{\alpha} \\[2ex] -\dfrac{\xi_3}{\alpha} + \dfrac{i\beta\xi_1\xi_2 P_1}{\alpha} & -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\beta\xi_2^2 P_2}{\alpha} & \dfrac{\xi_1}{\alpha} + \dfrac{i\beta\xi_2\xi_3 P_3}{\alpha} \\[2ex] \dfrac{\xi_2}{\alpha} + \dfrac{i\beta\xi_1\xi_3 P_1}{\alpha} & -\dfrac{\xi_1}{\alpha} + \dfrac{i\beta\xi_2\xi_3 P_2}{\alpha} & -\dfrac{i\gamma_0}{\alpha} + \dfrac{i\beta\xi_3^2 P_3}{\alpha} \end{pmatrix},$$

$$(FAS)_{22} = -\frac{i\beta}{\alpha} Q' - \frac{i\gamma_1}{\alpha_1},$$

and the equation for Q' takes the form

$$\alpha|\xi|^2 - (\alpha\gamma_1 - \alpha_1\gamma_0)Q' - \alpha_1\beta Q'^2 = 0,$$

which implies

$$Q' = \frac{|\xi|^2}{Q}.$$

In accordance with the above calculations, we transform the system (2.43) to the form

$$i\alpha\,\partial_t V = \operatorname{rot} V + \beta\ \nabla r - \gamma_0 V + \beta\ B_2 V, \tag{2.56}$$

$$i\alpha_1 \partial_t\, r = \beta\ Q'(-\Delta)\, r - \gamma_1\, r; \tag{2.57}$$

the Fourier image of B_2 is written as

$$\xi\xi^T \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix}. \tag{2.58}$$

Since $(U, \varrho)^\top = S(V, r)^\top$, we have

$$\mathcal{M}_{\mathrm{ChEns}} = \{(1,0,0,0)^\top\ V_1 + (0,1,0,0)^\top\ V_2 + (0,0,1,0)^\top\ V_3\},$$

where $V = (V_1, V_2, V_3)^\top$ is the solution to the Cauchy problem

$$i\alpha\,\partial_t V = \mathbf{rot} V - \gamma_0 V + \beta\ B_2 V, \tag{2.59}$$
$$V|_{t=0} = V^0,$$

$(U^0, \varrho^0) = S(V^0, r^0)$. A solution to the Cauchy problem for the system (2.56), (2.57) with initial data $(V^0, r^0)^\top$ can be represented in the form

$$(V, r)^\top = (V^{(1)}, 0)^\top + (V_1^{(2)}, V_2^{(2)}, V_3^{(2)}, r_2),$$

where $V^{(1)}$ is the solution to the Cauchy problem for (2.59) and $(V^{(2)}, r_2)$ is the solution to the Cauchy problem for the system 2.56, 2.57 with initial condition $V^{(2)}|_{t=0} = 0$, $r_2|_{t=0} = r^0$. Thus, we obtain $\mathcal{M}$.

The direct sum of $\mathcal{M}$ and $\mathcal{M}_{\mathrm{ChEns}}$ is the entire phase space of variables (V, r). By the Duhamel principle,

$$V^{(2)}(t, x) = \int_0^t W(t, x; \tau)\ d\tau,$$

where $W(t, x; \tau)$ is the solution to the Cauchy problem for the system (2.59) with $t > \tau$ with initial condition $W(t, x; \tau)|_{t=\tau} = \nabla r|_{t=\tau}$.

Thus, if $\alpha\gamma_1 - \alpha_1\gamma_0 \neq 0$ is an invariant manifold of solutions to the Cauchy problem for (2.43) which are generated at large times by the solution to the Schrödinger equation (with respect to density ϱ or div U of potential solutions).

For the critical value $q_{\mathrm{cr}} = \gamma_0/\gamma_1 = \alpha/\alpha_1$ we have $\gamma_1\alpha - \alpha_1\gamma_0 = 0$ and a dispersion equation of the form

$$D_4 = ((\alpha\omega - \gamma_0)^2 - |\xi|^2)\left((\alpha\omega - \gamma_0)^2 - \frac{\alpha\beta}{\alpha_1}|\xi|^2\right) = 0. \tag{2.60}$$

Thus, we find only wave type roots

$$\omega = \frac{\gamma_0}{\alpha} \pm \frac{1}{\alpha}|\xi|, \quad \alpha\omega = \frac{\gamma_0}{\alpha} \pm \sqrt{\frac{\beta}{\alpha\alpha_1}}\ |\xi|,$$

and the passage at large times to the Schrödinger approximation is impossible. Bifurcation of roots at the critical value $q_{\rm cr}$ is expressed by passing a boundary-layer root and a diffusion type root of the second factor in (2.60) to two wave type roots. In this case, the generating functions have the form

$$Q(|\xi|^2) = -\sqrt{\frac{\alpha_1\beta}{\alpha}}\,|\xi|, \quad Q'(|\xi|^2) = \sqrt{\frac{\alpha}{\alpha_1\beta}}\,\frac{1}{|\xi|},$$

which leads, as in the phonon gas model, to singularities of the coefficients of the projection Π_{21} at $|\xi| = 0$.

Remark 2.2. The question concerning separating dynamics for the system (2.43) is more delicate and is not discussed here.

3. Existence of Chapman–Enskog Projections. Necessary and Sufficient Conditions

This section is devoted to the solvability conditions for the Riccati matrix equation

$$P_{21}\Lambda_{12}P_{21} - \Lambda_{22}P_{21} + P_{21}\Lambda_{11} - \Lambda_{21} = 0 \tag{3.1}$$

arising in the study of the Chapman–Enskog projection (see [**15, 17, 24**]) for the Cauchy problem and the mixed problem for moment approximations of kinetic equations. Here, P_{21} is a complex-valued $(n-m)\times m$-matrix, $\Lambda_{11} \in M_{m,m}(\mathbb{C})$, $\Lambda_{21} \in M_{n-m,m}(\mathbb{C})$, $\Lambda_{12} \in M_{m,n-m}(\mathbb{C})$, $\Lambda_{22} \in M_{n-m,n-m}(\mathbb{C})$, $n > m$, $M_{l,s}(\mathbb{C})$ is a complex-valued $l \times s$-matrix.

The solvability of the Riccati equation (3.1) is a rather complicated question (see, for example, [**7**]). Even in the case of simple problems considered below, it is required to generalize some well-known results.

Let a matrix P be a solution to the projection problem formulated in terms of Fourier images, i.e., let P consist of four blocks P_{ij} such that $P_{11} = E$, $P_{12} = 0$, $P_{22} = 0$, and P_{21} satisfies (3.1). Our study is based on the following assertion.

Lemma 3.1. *A matrix P is a solution to the projection problem if and only if*

$$(E - P)\Lambda P = 0, \tag{3.2}$$

where E is the identity matrix.

Proof. We compute explicitly the product $(E-P)\Lambda P$. Since the first m rows of the matrix $(E-P)$ are zero, the first m rows of the product $(E-P)\Lambda P$ are also zero. Since the last $n-m$ columns of the matrix P are zero, the last $n-m$ columns of the product $(E-P)\Lambda P$ are also zero. For $((E-P)\Lambda P)_{21}$ we have

$$\begin{aligned}
&((E-P)\Lambda P)_{21}\\
&= (E-P)_{21}(\Lambda_{11}P_{11}+\Lambda_{12}P_{21}) + (E-P)_{22}(\Lambda_{21}P_{11}+\Lambda_{22}P_{21})\\
&= -P_{21}\Lambda_{11} - P_{21}\Lambda_{12}P_{21} + \Lambda_{21} + \Lambda_{22}P_{21} = 0,
\end{aligned}$$

where the last equality holds in view of (3.1). Thus, $(E-P)\Lambda P = 0$. Consequently, the relation (3.2) is a necessary and sufficient condition for the solvability of the Riccati matrix equation (3.1). □

3.1. Solvability of matrix equations.

It is very surprised that for such a classical object as a matrix equation there is no complete theory yet! This fact was also remarked in [**7**]. To a single reference on this topic indicated in [**7**], we can add only two references: [**11**] and [**20**].

Here, we obtain necessary and sufficient conditions for the solvability of a general matrix equation of the form (3.1). Thereby we generalize the corresponding results of the above-mentioned papers. Since our goal is to study Chapman–Enskog projections, we are interested in the case of a singular matrix Λ_{12}, i.e., $\det \Lambda_{12} = 0$. This case was not considered in [**7**] and [**11**].

To give a geometric interpretation, we introduce, following [**7**], two algebraic manifolds, denoted by $\mathcal{E}$ and $\mathcal{S}$. The manifold $\mathcal{E}$ is the set of all second order equations (3.1) over matrices of second order. It consists of pairs (B,Q) and coincides with the 8-dimensional affine space $\mathbb{C}^8$. The manifold $\mathcal{S}$ consists of triples (X,B,Q), where X is a solution. It is nonlinear and belongs to $\mathbb{C}^{12}$. Our goal is to describe, under the natural projection $\mathcal{S} \to \mathcal{E}$, the layers over different points or, equivalently, to find a number of solutions to a scalar quadratic equation. In a general position, there are exactly six preimages of the projection $\mathcal{S} \to \mathcal{E}$. However, there are points without preimages and, at the same time, there are points with infinitely many preimages. For an example we will consider the case, where a preimage is a cone. It is natural to try to work out an algorithm for determining a number of solutions to a given equation. For a scalar equation such an algorithm is simple: there exists a unique solution provided that the discriminant vanishes and there exist two solution otherwise.

Example 3.1 (see [**7**])**.** We consider the quadratic matrix equation

$$X^2 + BX + Q = 0, \quad B, Q \in M_{2,2}(\mathbb{C}). \tag{3.3}$$

To illustrate the unusual character of this classical object, we consider two special cases of (3.3).

1. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 0$ and $a + d = 0$ is a solution to the equation

$$X^2 = 0.$$

There are infinitely many such matrices, and they form a two-dimensional cone in $\mathbb{C}^4$.

2. There are no solutions to the equation

$$X^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

since a matrix has only the zero eigenvalue if the squared matrix possesses this property, i.e., X is nilpotent and the squared nilpotent matrix of second order vanishes.

Obviously, both cases are extraordinary. So, the question arises: What should be understood as a general position? In the scalar case, the system (3.3) consists of four second order equations. By the Bezout theorem, there are $2^4 = 16$ solutions in the general position.

Conjecture 3.1 (see [**7**])**.** The matrix equation (3.3) can have a finite number of solutions, from 0 to 6, or infinitely many solutions forming a two-dimensional cone or a two-dimensional hyperboloid.

3.1.1. Necessary conditions.

Proposition 3.1. *Let a matrix P_{21} be a solution to the Riccati equation* (3.1)*, and let $X = \Lambda\, P$, where $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ is a quadratic matrix of order n, P_{11} is the identity matrix of order m, and P_{12}, P_{22} are zero matrices. Then X is a solution to the quadratic matrix equation*

$$X^2 - \Lambda\, X = 0. \tag{3.4}$$

Multiplying (3.2) from the left by Λ and making the change of variables $X = \Lambda P$, we obtain (3.4). Thus, the question about the solvability of (3.1) is reduced to the case of the system consisting of two equations in (3.4) and the equation $X = \Lambda P$.

Lemma 3.2. *Let* $\det(\Lambda) \neq 0$*, and let* $h_1, \dots, h_n$ *be a Jordan basis for a matrix* X*, a solution to Equation* (3.4)*. Then there exists* $K \geqslant 0$ *such that* $h_1, \dots, h_K$ *are a part of a Jordan basis for the matrix* Λ *with preserving the adjunction order* (*i.e. if* h_j *is such that* $X h_j = \lambda h_j + h_{j-1}$*, then* $\Lambda h_j = \lambda h_j + h_{j-1}$) *and* $h_{K+1}, \dots, h_n$ *are the eigenvectors corresponding to the zero eigenvalue.*

PROOF. Let v be an eigenvector of the unknown matrix X corresponding to an eigenvalue λ. Multiplying (3.4) from the right by v, we find $\lambda^2 v - \lambda\Lambda v = 0$, which implies $\det(\lambda^2 E - \lambda\Lambda) = 0$ since $v \neq 0$. But, in this case, only an eigenvalue of the matrix Λ or zero can be an eigenvalue of the matrix X. Moreover, if $\lambda \neq 0$, from the equation for v it follows that v is an eignevector of the matrix Λ corresponding to the same eigenvalue. If $\lambda = 0$, then v is arbitrary.

We show that the matrix X cannot have Jordan cells of order more than 1 corresponding to the zero eigenvalue. Let v_1 be the adjoined eigenvector corresponding to the zero eigenvalue. Then

$$X v_1 = \lambda v_1 + v = v.$$

Multiplying (3.4) from the right by v_1, we find

$$X v - \Lambda v = -\Lambda v = 0,$$

i.e., $v \in \mathrm{Ker}(\Lambda)$. However, $\det(\Lambda) \neq 0$ by assumption, and we obtain a contradiction.

Let v_1 be the first adjoined eigenvector corresponding to a nonzero eigenvalue λ of the matrix X, i.e., $X v_1 = \lambda v_1 + v$, $X v = \lambda v$. Then $\lambda^2 v_1 + 2\lambda v - \lambda\Lambda v_1 - \Lambda v = 0$, which implies $\Lambda v_1 = \lambda v_1 + v$, i.e., v_1 is also the first adjoined eigenvector of the matrix Λ. Similarly, if v_k is the kth adjoined eigenvector corresponding to an eigenvalue λ of the matrix X, then the same true for v_k relative to the matrix Λ. $\square$

Definition 3.1. Let A and B be quadratic matrices of the same size, and let λ be their common eigenvalule. We say that the matrices A and B are *consistent* with respect to λ up to k if there exists a bijection Φ from the set of Jordan cells of the matrix A corresponding to λ into the set of Jordan cells of the matrix B corresponding to the same λ such that

1) with an h order cell of the matrix A corresponds to a cell of the matrix B of order at least $h-k$,

2) if $v_1, \ldots, v_h$ is the Jordan chain of length h corresponding to the cell $K_h(A)$ of the matrix A and $w_1, \ldots, w_L$ is the Jordan chain corresponding to the cell of order L of the matrix B which is the image of $K_h(A)$ under the mapping Φ, then $w_1 = v_1, \ldots, w_{h-k} = v_{h-k}$.

Lemma 3.3. *Let X be a solution to* (3.4). *If $\lambda = 0$ is a common eigenvalue of the matrices X and Λ, then the matrices X and Λ are consistent up to* 1 *with respect to λ.*

Proof. We subsequently consider several variants. Let v_0 be the eigenvector of the matrix X corresponding to the Jordan cell of order 1. Then the equation for v_0 implies that v_0 is arbitrary. Let v_1 is the first adjoined eigenvector of the matrix X corresponding to the eigenvector v_0 and the zero eigenvalue. Multiplying (3.4) from the right by v_1, we find

$$X^2 v_1 - \Lambda\, X v_1 = 0, \quad X v_0 - \Lambda\, v_0 = 0,$$

which means that v_0 is an eigenvector of the matrix Λ corresponding to the zero eigenvalue. Further, if $v_0, \ldots, v_{k+2}$ are a part of the Jordan chain of the matrix X such that $X v_j = v_{j-1}$ (i.e., v_j correspond to the zero eigenvalue), then, multiplying (3.4) from the right by v_{k+2}, we find $X v_k - \Lambda\, v_{k+1} = 0$, which means that $v_0, \ldots, v_{k+1}$ are a part of the Jordan chain of the matrix Λ; moreover, $\Lambda\, v_0 = 0$ and $\Lambda\, v_j = v_{j-1}$ for all $j > 0$. $\square$

As a consequence, we obtain the following assertion.

Lemma 3.4. *Let* $\det(\Lambda) \neq 0$. *If P_{21} is a solution to* (3.1), *then the corresponding matrix X satisfies the following conditions*:

(1) *X has exactly $n-m$ zero eigenvalues with corresponding eigenvectors e_j, $j > m$, where e_j, $j > m$, are basis vectors with* 1 *at the jth position,*

(2) *every nonzero eigenvalue of X is an eigenvalue of Λ,*

(3) *if h_j is an eigenvector of X corresponding to an eigenvalue λ, then it is an eigenvector of Λ and $\Lambda\, h_j = \lambda\, h_j$,*

(4) *if h_j is an adjoined eigenvector of X, then it is also an adjoined eigenvector of Λ, and the adjunction order is preserved.*

Lemma 3.5. *If the matrix equation* (3.1) *is solvable, then there exist two solutions X_1 and X_2 to the corresponding matrix quadratic equation* (3.4) *such that*

$$\begin{aligned}
&X_1 e_j = 0 \quad \forall j > m, \\
&e_j^T X_2 = e_j^T \Lambda \quad \forall j = 1, \ldots, m, \\
&\Lambda X_2 = X_1 \Lambda.
\end{aligned} \tag{3.5}$$

PROOF. If P_{21} is a solution to (3.1), then the corresponding matrix P generates two solutions $X_1 = \Lambda P$ and $X_2 = P\Lambda$. Indeed if P_{12} is a solution to (3.1), then P satisfies (3.2). Multiplying (3.2) from the left by Λ, we see that $X_1 = \Lambda P$ is a solution to (3.4). Multiplying (3.2) from the right by Λ, we see that $X_2 = P\Lambda$ is a solution to (3.4). The first and second equations in (3.5) follow from the structure of the matrix P. Further, $X_1\Lambda = \Lambda P \Lambda = \Lambda X_2$. □

As a consequence, we obtain the following assertion.

Proposition 3.2. *Let* $\det(\Lambda) \neq 0$*. If a solution to* (3.1) *exists, then it can be represented in the form* $P_{21} = (\Lambda^{-1}X)_{21}$*, where the matrix* X *satisfies conditions* (1)–(4) *of Lemma* 3.4*. Moreover, since there are finitely many matrices satisfying these conditions, the solution* (*if it exists*) *to* (3.1) *can be found by enumeration of finitely many variants.*

By Proposition 3.1, if (3.1) has a solution, then the corresponding matrix $X = \Lambda P$ is a solution to (3.4). Furthermore, since the last $n - m$ columns of the matrix P are zero, the basis vectors e_j, $j > m$, satisfy the condition $Xe_j = 0$. Since for any column $(P)_j$ of the matrix P and $j \leqslant m$ we have $X(P)_j \neq 0$ (since $\det(\Lambda) \neq 0$) and the vectors $(P)_1, \ldots, (P)_m, e_{m+1}, \ldots, e_n$ form a basis, we conclude that $\operatorname{codim}(\operatorname{Ker}(X)) = m$. By Lemma 3.5, we can describe all matrices X corresponding to the solution P_{21} to the matrix equation (3.1).

3.1.2. Sufficient conditions. Using Lemmas 3.4 and 3.5, we can construct all solutions to the Riccati matrix equation (3.1) with an arbitrary matrix Λ. We describe the corresponding algorithm.

STEP I. 1. Based on the coefficients of the matrix equation (3.1), we construct an equation of the form (3.4).

2. We solve (3.4) by using the algorithm described in Lemma 3.4 and obtain a finite number of classes of solutions X. It is convenient to consider only the classes such that $Xe_j = 0$, $j = m+1, \ldots, n$, because the remaining classes do not correspond to any solution to the original problem in view of Lemma 3.5.

Step II. 1. For each of the obtained classes we solve the linear matrix equation $X = \Lambda P$ with respect to P and obtain a finite (possibly, empty) set of classes of solutions P. If this set is empty, the matrix equation (3.1) has no solutions.

2. Among the obtained classes of solutions P we select the classes satisfying the conditions

$$e_j^T P = e_j^T \quad \forall j = 1, \dots, m, \quad P e_j = 0 \quad \forall j = m+1, \dots, n.$$

If none of the classes P satisfies these conditions, then the matrix equation (3.1) has no solutions.

3. With each of the selected classes of solutions P we associate a class of submatrices P_{21} (we deal with classes of solutions since the Jordan basis for the matrix X can contain arbitrary vectors). The union of all such parameter classes is the set of all solutions to the matrix equation (3.1).

Lemma 3.6. *Let* $\det(\Lambda) \neq 0$*, and let* X *be a solution to the matrix equation* (3.4)*. Suppose that* $K \geqslant 0$*, the matrix* X *has Jordan basis* $h_1, \dots, h_n$*, vectors* $h_1, \dots, h_K$ *are a part of a Jordan basis for the matrix* Λ *with preserving the adjunction order, and* $h_{K+1}, \dots, h_n$ *are regarded as the eigenvectors corresponding to the zero eigenvalue with the Jordan cell of order* 1*. Then* $h_{m+1} = e_{m+1}, \dots, h_n = e_n$ *implies* $X e_j = 0$*,* $j = m+1, \dots, n$*.*

Proof. Let M be the matrix with columns h_j, and let $J(X)$ be the Jordan form of the matrix X. Then $X = MJ(X)M^{-1}$. Substituting this expression for X into (3.4), we can write the left-hand side in the form $MJ^2(X)M^{-1} - \Lambda\, MJ(X)M^{-1}$. Let $f(v) = (MJ^2(X)M^{-1} - \Lambda\, MJ(X)M^{-1})v$. Note that $f(v)$ is a linear function. Furthermore, $f(h_j) = 0$ for all $j = 1, \dots, n$. Thus, $MJ^2(X)M^{-1} - \Lambda\, MJ(X)M^{-1} = 0$, i.e., $X = MJ(X)M^{-1}$ is a solution to the matrix equation (3.4). □

Theorem 3.1. *Let* $\det \Lambda \neq 0$*. Then the matrix equation* (3.1) *is solvable if there exist two solutions* X_1*,* X_2 *to the corresponding quadratic matrix equation* (3.4) *such that*

$$\begin{gathered} X_1 e_j = 0 \quad \forall j > m, \\ e_j^T X_2 = e_j^T \Lambda \quad \forall j = 1, \dots, m, \\ \Lambda X_2 = X_1 \Lambda. \end{gathered} \tag{3.6}$$

Proof. Assume that there exist two solutions X_1 and X_2 to the matrix equation (3.4) satisfying (3.6). We set $P = \Lambda^{-1} X_1$. From the first equation in (3.6) it follows that

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}.$$

From the second and third equations in (3.6) it follows that

$$e_j^T P = e_j^T \Lambda^{-1} X_1 = e_j^T X_2 \Lambda^{-1} = e_j^T, \quad j = 1, \ldots, m,$$

i.e., $P_{11} = E$. Therefore, $P_{21} = (\Lambda^{-1} X_1)_{21}$ is a solution to the matrix equation (3.1). □

3.1.3. Two examples from [7]. EXAMPLE 1. We consider the following special case of the Riccati matrix equation (3.1):

$$P_{21}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

Then the matrix Λ has the form

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the above algorithm and Lemma 3.4, we can show that all the eigenvalues of the matrix X are zero. Moreover, since the Jordan form of the matrix Λ consists of two Jordan cells of order 2, we can find the following classes of solutions X to the corresponding quadratic matrix equation (3.4).

1. The Jordan form of X consists of four cells of order 1. Then $X = 0$, and the Jordan basis is arbitrary.

2. The Jordan form of X consists of two cells of order 1 and one cell of order 2. Then either the eigenvector e_1 or the eigenvector e_2 corresponds to the cell of order 2.

3. The Jordan form of X consists of two cells of order 2. Then e_1 and e_2 are eigenvectors in the Jordan basis.

4. The Jordan form of X consists of one cell of order 3 and one cell of order 1. Then either the eigenvector e_1 and the first adjoined eigenvector e_3 or the eigenvector e_2 and the first adjoined eigenvector e_4 correspond to the Jordan cell of order 3.

5. The Jordan form of X cannot consist of one cell of order 4 in view of Lemma 3.4,(3).

According In view of Lemma 3.5, for our purpose it suffices to consider only those classes for which $Xe_3 = 0$ and $Xe_4 = 0$, i.e., e_3 and e_4 are eigenvectors of the matrix X. Since e_3 and e_4 are not eigenvectors of the matrix Λ, they correspond to the Jordan cells of order 1 by Lemma 3.4, i.e., the matrix X has at least two cells of order 1. Thus, it suffices to consider only cases 1 and 2. In case 2, we find two different classes of solutions:

$$X_1 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \neq 0$ is an arbitrary parameter. In case 1, we have one class of a single solution $X_0 = 0$.

Using the above algorithm, for every class we find a set of matrices P such that $X_j = \Lambda P_j$. We have

$$P_0 = \begin{pmatrix} p_{11}^0 & p_{12}^0 & p_{13}^0 & p_{14}^0 \\ p_{21}^0 & p_{22}^0 & p_{23}^0 & p_{24}^0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{for class } X_0 \\ (p_{1j}^0 \text{ arbitrary}) \end{array}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + P_0 \quad \text{for class } X_1,$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix} + P_0 \quad \text{for class } X_2.$$

Further, from the conditions $e_1^T P = e_1^T$, $e_2^T P = e_2^T$, $Pe_3 = 0$, and $Pe_4 = 0$ we find the coefficients p_{ij}^0. Hence only the classes

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

correspond to a solution to (3.7). This means that all the solutions to (3.7) are quadratic matrices of one of the following forms:

$$P_{21} = 0,$$

$$P_{21} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix},$$

$$P_{21} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

moreover, in the last two cases, $\alpha \neq 0$ is arbitrary, i.e., Equation (3.7) has infinitely many solutions.

Example 2. We consider the following special case of the Riccati matrix equation (3.1):

$$P_{21}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

Then the matrix Λ has the form

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As in Example 1, all the eigenvalues of the matrix Λ are zero, but, in this case, the Jordan form of the matrix Λ consists of one cell of order 4: $e_1 \to 0$,

$e_3 \to e_1$, $e_2 \to e_3$, $e_4 \to e_2$. As in Example 1, all the eigenvalues of the matrix solution X to the corresponding equation (3.4) are zero. Furthermore, it suffices to look for only those matrices X for which $Xe_3 = 0$ and $Xe_4 = 0$. However, since e_3 and e_4 are not eigenvectors of the matrix Λ, they correspond to Jordan cells of order 1 in view of Lemma 3.4. Thus, with the solution to the matrix equation (3.8) only two classes can be associated.

1. $X_0 = 0$.

2. The Jordan form of X consists of one cell of order 2 with which the eigenvector e_1 is associated and two cells of order 1 with which the eigenvalues e_3 and e_4 are associated. Furthermore, the matrix X has the form

$$X_1 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \neq 0$ is an arbitrary parameter.

According to the above algorithm, for the obtained classes we must solve the matrix equations $X_j = \Lambda P_j$. We find

$$P_0 = \begin{pmatrix} p^0_{11} & p^0_{12} & p^0_{13} & p^0_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{for class } X_0 \\ (p^0_{1j} \text{ arbitrary}) \end{array}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + P_0 \quad \text{for class } X_1.$$

The next step of algorithm is the verification of the conditions $e_1^T P = e_1^T$, $e_2^T P = e_2^T$, $Pe_3 = 0$, $Pe_4 = 0$. However, for both matrices P_0 and P_1 we have $e_2^T P_j = 0$. Thus, none of the above-selected classes satisfies these conditions. Consequently, the matrix equation (3.8) has no solutions. The further results on the solvability conditions for the matrix equation (3.1) were obtained by my former student Palin [**14**].

3.2. Solvability of quantization system.

To discuss results concerning the quantization system (2.15), we need to introduce some definitions and notation.

Definition 3.2. Two columns $(A)_k$ and $(A)_l$ of the matrix A are *connected* if there exists s such that $a_{sk}a_{sl} \neq 0$. Similarly, two rows $[A]_k$ and $[A]_l$ are *connected* if there exists s such that $a_{ks}a_{ls} \neq 0$.

Definition 3.3. The *sth distance* between connected columns $(A)_k$ an $(A)_l$ is defined by the formula

$$\mathrm{dist}_s((A)_k, (A)_l) = d(a_{sk}) + d(a_{sl}) (\mathrm{mod}\ 2), \quad a_{sk}a_{sl} \neq 0.$$

The *sth distance* between rows is defined in a similar way.

Definition 3.4. If $\mathrm{dist}_{s_1}((A)_k, (A)_l) = \mathrm{dist}_{s_2}((A)_k, (A)_l)$ for all s_1, s_2, $a_{s_j k}a_{s_j l} \neq 0$, $j = 1, 2$, then $\varrho((A)_k, (A)_l) = \mathrm{dist}_{s_1}((A)_k, (A)_l)$ is the *distance between connected columns* $(A)_k$ and $(A)_l$. The *distance between connected rows* is defined in a similar way.

Definition 3.5. 1. A *path* is a sequence $a^{(j)}$ of columns (rows) such that $a^{(j)}$ and $a^{(j+1)}$ are connected for all j. The *length of path* is defined by the formula

$$l(a^{(1)}, \dots, a^{(n)}) = \sum_{k=1}^{n-1} \varrho(a^{(k)}, a^{(k+1)}) (\mathrm{mod}\ 2).$$

2. If two any paths S_1 and S_2 connecting columns (rows) a and b have the same length, then $\varrho(a, b) = l(S_1)$ is the *distance between a and b*.

Definition 3.6. A set of columns (rows) M is a *connection component* if for any $a, b \in M$ there exists a path S connecting a and b and for any S there exists a path connecting $a \in M$ and $b \in M$.

Definition 3.7. 1. The *size* of a connection component is the number, diminished by 1, of elements of the maximal path without cycles which starts in this component.

2. The *horisontal* connection $\#(A)$ is the sum of sizes of all connected components of a matrix A. The *vertical* connection $\#[A]$ is defined in a similar way.

3. The columns (rows) labeled by k and l are *connected in the system of matrices* A and B if they are connected in at least one of these matrices.

Definition 3.8. Let columns labeled by k and l are connected in A and B, and let $\varrho((A)_k, (A)_l) = \varrho((B)_k, (B)_l)$. Then the distance between the matrices A and B is *consistent.*

For the consistency distance we can introduce the distance between rows (columns) of a system of matrices.

Definition 3.9. 1. A *connection matrix component* μ for the problem (2.14) is a set of pairs (i, j) such that for all $(i, j), (k, l) \in \mu$ the rows i and k are connected in the system of matrices A_{11} and A_{12}, the columns j and l are connected in the system of matrices A_{12} and A_{22}.

2. The problem (2.14) is *nondegenerate* if $(\det(A_{11}))^2 + (\det(A_{22}))^2 > 0$ for every $\xi > 0$, and *esentially nondegenerate* if for any connection matrix component μ there are i and j such that

(1) $(i, j) \in \mu$,

(2) $a^{21}_{ji} \neq 0$,

(3) there exists s such that $(i, s) \in \mu, (a^{22}_{js})^2 + (a^{11}_{si})^2 > 0$.

3.2.1. Necessary conditions. We prove two assertions concerning the necessary existence conditions for the quantization system (2.15).

Theorem 3.2. *Let the quantization system* (2.15) *for the nondegenerate problem* (2.14) *has a solution. Then it is possible to introduce the distance between the columns of the system of matrices* A_{12}, A_{22} *and between rows of the system of matrices* A_{11}, A_{12}. *Moreover, if* m_{ij} *is a solution to the quantization system* (2.15), *then the following equalities hold*:

$$m_{ik} = m_{jk} + \varrho((A_{22})_i, (A_{22})_j), \tag{3.9}$$

$$m_{ij} = m_{ik} + \varrho([A_{11}]_j, [A_{11}]_k), \tag{3.10}$$

$$m_{ik} = m_{jk} + \varrho((A_{12})_i, (A_{12})_j), \tag{3.11}$$

$$m_{ij} = m_{ik} + \varrho([A_{12}]_j, [A_{12}]_k). \tag{3.12}$$

Proof. We begin by proving that the it is possible to introduce the consistent distance. We consider only the case of columns since the proof for rows is the same.

Assume the contrary, i.e., it is impossible to introduce the distance between columns of the system of matrices A_{12}, A_{22}. Then three cases can happen.

CASE 1. It is impossible to introduce the distance between columns of the matrix A_{12}. Then there are i, j, k, and l such that

$$d(a_{ij}^{12}) + d(a_{il}^{12}) = 0,$$

$$d(a_{kj}^{12}) + d(a_{kl}^{12}) = 1.$$

But (2.15) implies $m_{ri} + d(a_{ij}^{12}) + m_{js} = m_{ri} + d(a_{il}^{12}) + m_{ls}$. Hence $m_{js} = m_{ls}$. Similarly, $m_{rk} + d(a_{kj}^{12}) + m_{js} = m_{rk} + d(a_{kl}^{12}) + m_{ls}$, which implies $m_{js} + 1 = m_{ls}$. Thus, $m_{ls} = m_{js} = m_{js} + 1$, which is impossible.

CASE 2. It is impossible to introduce the distance between columns of the matrix A_{22}. Then there are i, j, k, and l such that

$$d(a_{ij}^{22}) + d(a_{il}^{22}) = 0,$$

$$d(a_{kj}^{22}) + d(a_{kl}^{22}) = 1.$$

But from the second equation in (2.15) we find $d(a_{ij}^{22}) + m_{js} = d(a_{il}^{22}) + m_{ls}$, which implies $m_{js} = m_{ls}$. Similarly, $d(a_{kj}^{22}) + m_{js} = d(a_{kl}^{22}) + m_{ls}$, which implies $m_{js} + 1 = m_{ls}$. Thus, $m_{ls} = m_{js} = m_{js} + 1$, which is impossible.

CASE 3. The distance is not consistent for the matrices A_{12} and A_{22}. Then there are i, j, k, and l such that

$$d(a_{ij}^{12}) + d(a_{il}^{12}) = 0,$$

$$d(a_{kj}^{22}) + d(a_{kl}^{22}) = 1.$$

From (2.15) we find $m_{ri} + d(a_{ij}^{12}) + m_{js} = m_{ri} + d(a_{il}^{12}) + m_{ls}$, which implies $m_{js} = m_{ls}$. On the other hand, from the second equation in (2.15) it follows that $d(a_{kj}^{22}) + m_{js} = d(a_{kl}^{22}) + m_{ls}$, which implies $m_{js} + 1 = m_{ls}$. Thus, $m_{ls} = m_{js} = m_{js} + 1$, which is impossible.

Thus, in each of the above cases, the quantization system (2.15) for the problem (2.14) has no solutions. However, by the assumptions of the theorem, there exists a solution to the quantization system. Consequently, none of the above cases is possible, and the distance can be defined between columns of the system of matrices A_{12}, A_{22}.

We prove that (3.9)–(3.12) hold. Assume that the columns in the matrix A_{22} labeled by i and j are connected. Then there is s such that $a_{si}^{22}a_{sj}^{22} \neq 0$, and from (2.15) we find $d(a_{si}^{22}) + m_{ik} = d(a_{sj}^{22}) + m_{jk}$, which implies $m_{ik} = d(a_{si}^{22}) + d(a_{sj}^{22}) + m_{jk} = \varrho((A_{22})_i, (A_{22})_j) + m_{jk}$. Hence (3.9) holds. Let $(A_{22})_i, (A_{22})_j$ be two columns in the same connection component (i.e., the distance between them can be defined). Then there is a path $a^{(1)}, \dots, a^{(n)}$ connecting these columns, i.e., $a^{(1)} = (A_{22})_i$ and

$a^{(n)} = (A_{22})_j$. For any two neighboring columns of this path the equality (3.9) is valid. Further, summarizing all such equalities and using the fact that the distance is additive in $\mathbb{Z}_2$, we conclude that (3.9) holds.

The remaining equalities are proved in a similar way. □

Theorem 3.3. *Let the quantization system* (2.15) *for the nondegenerate problem* (2.14) *has a solution. Then for any connection matrix component* μ *and all* $(i,j),(k,l) \in \mu$

$$m_{ji} + m_{lk} = \varrho([A_{11}, A_{12}]_i, [A_{11}, A_{12}]_k) + \varrho((A_{12}, A_{22})_j, (A_{12}, A_{22})_l). \tag{3.13}$$

Proof. Note that the assumptions of Theorem 3.2 hold and, consequently, the equalities (3.9)–(3.12) are valid. Since the distance for the corresponding pairs of matrices is consistent, we have

$$\varrho((A_{22})_i, (A_{22})_j) = \varrho((A_{12})_i, (A_{12})_j) = \varrho((A_{12}, A_{22})_i, (A_{12}, A_{22})_j),$$
$$\varrho([A_{12}]_i, [A_{12}]_j) = \varrho([A_{11}]_i, [A_{11}]_j) = \varrho([A_{11}, A_{12}]_i, [A_{11}, A_{12}]_j).$$

Using (3.9) and (3.10), we find

$$m_{ji} = m_{li} + \varrho((A_{22})_j, (A_{22})_l),$$
$$m_{lk} = m_{li} + \varrho([A_{11}]_i, [A_{11}]_k).$$

Summarizing both equalities and taking into account the consistency of distance, we obtain the required assertion. □

3.2.2. Sufficient conditions. We prove two assertions concerning the sufficient solvability conditions for the quantization system.

Theorem 3.4. *Suppose that the problem* (2.14) *is nondegenerate, there exist* i *and* j *such that* $a_{ij}^{21} \neq 0$, *and* $\#(A_{12}, A_{22}) = n - m - 1$, $\#[A_{11}, A_{12}] = m - 1$. *Let the assumptions of Theorem* 3.3 *be satisfied. Then the quantization system* (2.15) *of the problem* (2.14) *has a unique solution.*

Proof. From the conditions on the horizontal and vertical connection of the corresponding systems of columns and rows it follows that there is the distance between any two columns (rows) of the system of matrices A_{12}, A_{22} (A_{11}, A_{12}). Since the problem (2.14) is nondegenerate, we have either $\det(A_{11}) \neq 0$ or $\det(A_{22}) \neq 0$. For the sake of definiteness, we assume that $\det(A_{22}) \neq 0$. Then there exists s such that $a_{is}^{22} \neq 0$. Then from the second equation in (2.15) we find $m_{sj} = d(a_{is}^{22}) + d(a_{ij}^{21})$.

We construct a solution to the quantization system (2.15) by the following rule:

$$m_{rt} = d(a^{22}_{is})+d(a^{21}_{ij})+\varrho([A_{11},A_{12}]_j,[A_{11},A_{12}]_t)+\varrho((A_{12},A_{22})_r,(A_{12},A_{22})_s).$$

It is easy to check that $\{m_{rt}\}$ is a solution to the quantization system (2.15). Thus, the existence of solutions is proved.

The uniqueness of a solution follows from Subsection 2.3 and the fact that m_{sj} must be the same for all solutions.

In the case $\det(A_{11}) \neq 0$, the arguments are similar. □

Theorem 3.5. *Let the problem* (2.14) *be essentially nondegenerate, and let for any connection μ-matrix component there exist $(i,j) \in \mu$ such that $a^{21}_{ji} \neq 0$. Suppose that the assumptions of Theorem* 3.3 *are satisfied. Then the quantization system* (2.15) *for the problem* (2.14) *has a unique solution.*

Proof. We note that any column does not belong to two connection components simultaneously. Let μ be an arbitrary connection matrix component. By assumption, there exists $(i,j) \in \mu$ such that $a^{21}_{ji} \neq 0$. Since the problem (2.14) is essentially nondegenerate and a nonzero entry of the matrix A_{21} labeled by μ is unique, there is s such that either $a^{22}_{js} \neq 0$ or $a^{11}_{si} \neq 0$. For the sake of definiteness, we assume that $a^{22}_{js} \neq 0$. We set $m_{si} = d(a^{22}_{js}) + d(a^{21}_{ji})$. Further, for all $(t,r) \in \mu$ we set

$$m_{rt} = d(a^{22}_{js})+d(a^{21}_{ji})+\varrho([A_{11},A_{12}]_i,[A_{11},A_{12}]_t)+\varrho((A_{12},A_{22})_r,(A_{12},A_{22})_s).$$

Performing the same procedure for all connection matrix components μ, we obtain a collection $\{m_{rt}\}$ solving the quantization system. We note that for every connection matrix component m_{si} is uniquely determined and, consequently, the solution is unique on this component. Hence a solution to the quantization system is unique. □

3.2.3. The essentially nondegenerate case.

Theorem 3.6. *Let the problem* (2.14) *be essentially nondegenerate, and let the following conditions hold*:

(1) *the distance is defined between columns of the system of matrices A_{12}, A_{22} and between rows of the system of matrices A_{11}, A_{12},*

(2) *for any connection μ-matrix component there exists $(i,j) \in \mu$ such that $a^{21}_{ji} \neq 0$,*

(3) *for any connection μ-matrix components and all* $(i,j),(k,l)\in\mu$, $a^{21}_{ji}a^{21}_{lk}\neq 0$ *we have*

$$\begin{aligned}&d(a^{21}_{ji})+d(a^{21}_{lk})\\&=\varrho([A_{11},A_{12}]_i,[A_{11},A_{12}]_k)+\varrho((A_{12},A_{22})_j,(A_{12},A_{22})_l).\end{aligned}$$

Then the quantization system (2.15) *for the problem* (2.14) *has a unique solution.*

Conversely, if the essentially nondegenerate problem (2.14) *has a unique solution, then conditions* (1)–(3) *are satisfied.*

Proof. The existence and uniqueness of a solution is established in the same way as above. If the quantization system of an essentially nondegenerate problem has a unique solution, the condition (1) is satisfied.

We show that condition (2) is satisfied. Assume the contrary. Let μ be a connection matrix component such that $a^{21}_{ji}=0$ for all $(i,j)\in\mu$. By the strong nondegeneracy condition, there is s such that $(i,s)\in\mu$, $a^{22}_{js}\neq 0$ or $a^{11}_{si}\neq 0$. For the sake of definiteness, let $a^{22}_{js}\neq 0$. Let $\{m_{rt}\}$ be a solution to the quantization system (2.15). We set $m'_{si}=m_{si}+1$. Then we set

$$m_{rt}=d(a^{22}_{js})+d(a^{21}_{ji})+\varrho([A_{11},A_{12}]_i,[A_{11},A_{12}]_t)+\varrho((A_{12},A_{22})_r,(A_{12},A_{22})_s)$$

for any $(t,r)\in\mu$. We consider the collection $\{m'_{rt}\}$ obtained from $\{m_{rt}\}$ by replacing m_{rt} with m'_{rt} for all $(t,r)\in\mu$. It is easy to check that this collection also yields a solution to (2.15) and does not coincide with $\{m_{rt}\}$, i.e., the solution $\{m_{rt}\}$ is not unique. WE arrive at a contradiction.

We show that condition (3) is satisfied. We note that there exist (s,t) such that $a^{12}_{st}\neq 0$. Let i, j, k, l be such as in condition (3). Then the first equation in (2.15) implies

$$\begin{aligned}m_{js}+d(a^{12}_{st})+m_{ti}&=d(a^{21}_{ji}),\\ m_{ls}+d(a^{12}_{st})+m_{tk}&=d(a^{21}_{lk}).\end{aligned}$$

Adding the last two equalities, we find

$$m_{js}+m_{ls}+m_{ti}+m_{tk}=d(a^{21}_{ji})+d(a^{21}_{lk}).$$

We note that

$$\begin{aligned}m_{js}+m_{ls}&=\varrho([A_{11},A_{12}]_s,[A_{11},A_{12}]_s)+\varrho((A_{12},A_{22})_j,(A_{12},A_{22})_l)\\&=\varrho((A_{12},A_{22})_j,(A_{12},A_{22})_l).\end{aligned}$$

Similarly,

$$m_{ti}+m_{tk}=\varrho([A_{11},A_{12}]_i,[A_{11},A_{12}]_k).$$

Adding the last two equalities, we arrive at condition (3). □

3.2.4. Examples of solvable quantization systems.

Theorem 3.7. *The quantization system corresponding to the projection to one equation for moment approximations of the Boltzmann–Peierls equation has a unique solution.*

Proof. For the moment system of the Boltzmann–Peierls equation the matrix A_{22} is three-diagonal, i.e., $a^{22}_{kk}a^{22}_{k,k+1}a^{22}_{k,k-1} \neq 0$ for all k. Consequently, in the case of n equations, $\#(A_{12}, A_{22}) = n-2$. Note that $\#[A_{11}, A_{12}] = 0$ since $m = 1$. Moreover, $\det(A_{22}) \neq 0$ since the matrix A_{22} is three-diagonal. Thus, the corresponding problem (2.14) is nondegenrate. Furthermore, $A_{21} = (\alpha_1 i\xi, 0, \dots, 0)^T$, i.e., there exists a pair (i,j) such that $a^{21}_{ij} \neq 0$. Thus, the assumptions of Theorem 3.6 hold and the corresponding quantization system has a unique solution. □

Theorem 3.8. *The quantization system corresponding to the projection to two equations for moment approximations of the Boltzmann–Peierls equation has a unique solution.*

Proof. In this case, the matrix A_{22} is three-diagonal, which means $\det(A_{22}) \neq 0$ and $\#(A_{12}, A_{22}) = n-3$. Further, since

$$A_{11} = \begin{pmatrix} 0 & i\xi \\ \alpha_1 i\xi & \frac{1}{\tau_R} \end{pmatrix}, \tag{3.14}$$

the size of a single connection component for rows of the system of matrices A_{11}, A_{12} is equal to 1. It remains to note that $a^{21}_{ij} \neq 0$ implies $i = 1, j = 2$. Thus, the assumptions of Theorem 3.6 hold and the corresponding quantization system has a unique solution. □

Theorem 3.6 provided us with necessary and sufficient conditions for the existence of a unique solution to the quantization system for an essentially nondegenerate problem. The following natural question arises: What happens if the problem (2.14) is not essentially nondegenerate? We consider three examples of such a situation: (1) a solution exists, but is not unique; (2) there are no solutions, and (3) there exists a unique solution.

We consider the problem

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ i\xi & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} i\xi \\ 0 \end{pmatrix} = 0. \tag{3.15}$$

In case (1), $M = (i\xi \quad 0)$. Then the quantization system is reduced to a single equation $m_2 = m_1 + 1$ which has two different solutions. We note that conditions (1)–(3) of Theorem 3.6 hold.

In case (2), $M = (0 \quad i\xi)$. Then the quantization system has the form $m_1+m_2 = 0$, $m_1+1 = 1$, $m_2 = 1$ and, obviously, has no solutions. However, conditions (1)–(3) of Theorem 3.6 hold.

In case (3), $M = (i\xi \quad 1)$. Then the quantization system has the form $m_1+m_2 = 1$, $m_1+1 = 0$, $m_2 = 0$ and its single solution is $m_1 = 1$, $m_2 = 0$.

4. Other Examples of Construction of Chapman–Enskog Projections

4.1. Multi-dimensional case. Hierarchy of moment systems.

A multi-dimensional linearization of the M order moment system of the Boltzmann–Peierls kinetic equation [**5**] can be written as follows:

$$
\begin{aligned}
&\partial_t \widetilde{e} + \partial_{x_k} p_k = 0,\\
&\partial_t p_j + \frac{1}{3} c^2 \partial_{x_j} \widetilde{e} + \partial_{x_k} N_{\langle jk\rangle} + \frac{1}{\tau_R} p_j = 0,\\
&\partial_t N_{\langle ij\rangle} + \frac{2}{5} c^2 \partial_{x_{j\rangle}} p_{\langle i} + \frac{1}{\tau_{NR}} N_{\langle ij\rangle} = 0,\\
&\cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots\\
&\partial_t N_{\langle i_1\ldots\, i_n\rangle} + \frac{n}{2n+1} c^2 \partial_{x_{i_n\rangle}} N_{\langle\langle i_1\ldots\, i_{n-1}\rangle} + \frac{1}{\tau_{NR}} N_{\langle i_1\ldots\, i_n\rangle} = 0,\\
&\qquad\qquad i_1 \leqslant i_2 \leqslant \ldots i_n, \quad i_k = 1,2,3, \quad 1 < n \leqslant M.
\end{aligned}
\tag{4.1}
$$

Here, we used the notation $\partial_{x_{j\rangle}} p_{\langle <i}$, $\partial_{x_{i_n\rangle}} N_{\langle\langle i_1\ldots\, i_{n-1}\rangle}$ for symmetric traceless tensors [**3**]; δ_{ij} is the Kronecker symbol, c is the Debye sound velocity [**5**], and e is the distribution of phonon energy. In (4.1) and below we adopt the rule of summation relative repeated indices. For example, $\partial_{x_{j\rangle}} p_{\langle i} = \partial_{x_j} p_i + \partial_{x_i} p_j - 2\, \mathrm{div}_x\, \overline{p}\, \delta_{ij}/n$.

As was proved in [**15**], this system has only three irreducible Chapman–Enskog projections: to the phase space of conservative variable e (diffusion type projection), to the phase space of moments of order less than 1 (boundary-layer type projection), and to the phase space of variables e, $\overline{p} = (p_1, \ldots, p_n)^\top \in \mathbb{R}^n$ (projection of the second sound velocity).

Theorem 4.1. *Let* $q = \tau_{NR}/\tau_R \in (0,1)$. *Consider a multi-dimensional* $(d = 2,3)$ *moment system of order* $2k$ *of the Boltzmann–Peierls equation. Then there exist the critical values of parameter* q,

$$q^b_{2k+1} < q^d_{2k+1} \tag{4.2}$$

such that the following assertions hold.

1. *For* $q > q^d_{2k+1}$ *the pencil of the dispersion equation is diffusively connected, i.e., there exists a diffusion type Chapman–Enskog projection*

$$p_j = \partial_{x_j} q_j(\nabla_x)\widetilde{e}, \quad j = 1,2,3,$$
$$N_{i_1,\ldots,i_k} = \partial_{x_{i_1}} \ldots \partial_{x_{i_k}} q_{i_1,\ldots,i_k}(\nabla_x)\widetilde{e}, \quad i_1,\ldots,i_k = 1,2,3, \quad 2 \leqslant k \leqslant M,$$

with smooth symbols $q_{i_1,\ldots,i_k}(\xi)$, $\xi \in \mathbb{R}^d$, *of order* $-(M-k)$, $k = 1,\ldots,M$.

2. *For* $q < q^b_{2k+1}$ *the pencil of the dispersion equation is boundary-layer connected, i.e., there exists a bounded curve of stable purely imaginary roots of the boundary-layer type dispersion equation stabilizing at infinity.*

Due to this theorem, we can find a boundary-layer type Chapman–Enskog projection.

The solvability of the system of algebraic equations for the symbols $q_1,\ldots,q_d,\ldots,q_{i_1},\ldots,q_{i_k}$, $i_k \in \{1,\ldots,d\}$, $d = 1,2$, is connected with the solvability of the equation for the generating function

$$Q(|\xi|^2) = \xi_1^2 q_1(|\xi|^2) + \ldots + \xi_d^2 q_d(|\xi|^2),$$

where $\tau = iQ$ is a purely imaginary root.

The question whether (4.2) may be equality remains still open. For the system of moments of order $2k+1 = 5,7,9$ the inequality (4.2) becomes equality. Thus, for $k = 5$ we have the critical value

$$q_5 = \frac{\alpha_2(\alpha_3+\alpha_4)}{\alpha_3\alpha_1 + \alpha_4(\alpha_1+\alpha_2)}.$$

For the system of moments of odd order the proof is similar, but more complicated from the technical point of view. There is $q^c_{2(k+1)} < q^b_{2k+1}$ such that for $0 < q < q^c_{2(k+1)}$ there exists the Chapman–Enskog projection of the second sound velocity

$$\begin{aligned} N_{i_1,\ldots,i_k} &= \partial_{x_{i_1}} \ldots \partial_{x_{i_k}} q_{i_1,\ldots,i_k}(\nabla_x)\widetilde{e} \\ &\quad + \partial_{x_{i_1}} \ldots \partial_{x_{i_k}} \sum_{j=1}^{3} \mu_{j;i_1,\ldots,i_k}(\nabla_x)\partial_{x_j} p_j, \\ &i_1,\ldots,i_k = 1,2,3, \quad 2 \leqslant k \leqslant M, \end{aligned} \tag{4.3}$$

with smooth symbols $q_{i_1,\dots,i_k}(\xi)$, $\mu_{j;i_1,\dots,i_k}(\xi)$, $\xi \in \mathbb{R}^d$, of order $-(M-k)$, $k = 1,\dots,M$ and $-1-(M-k)$, $k = 1,\dots,M$ respectively.

4.2. Chapman–Enskog projection for mixed problem.

We consider the one-dimensional third order moment system of the Boltzmann–Peierls kinetic equation in the quarter space $\mathbb{R}^2_{++} = \{(x,t), x > 0, t > 0\}$:

$$\begin{aligned}
&\partial_t\ e + \partial_x p = 0,\\
&\partial_t p + \alpha_1 \partial_x e + \partial_x N + q\,p = 0,\\
&\partial_t N + \alpha_2 \partial_x p + \partial_x N_1 + N = 0,\\
&\partial_t N_1 + \alpha_3 \partial_x N + N_1 = 0,
\end{aligned} \tag{4.4}$$

where

$$\alpha_j = \frac{j^2}{(4j^2-1)} : \quad \alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{4}{15},\ \alpha_3 = \frac{9}{35}.$$

Denoting $\mathcal{U} = (e, p, N, N_1)^T$, we can write the system (4.4) in the form

$$E \cdot \partial_t \mathcal{U} + A_x \cdot \partial_x \mathcal{U} + B \cdot \mathcal{U} = 0,$$

where E is the identity matrix,

$$A_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & 1 & 0 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & \alpha_3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We reduce the system (4.4) to the normal form relative to ∂_x. For this purpose, we multiply the equations from the left by A_x^{-1}, where

$$A_x^{-1} = \begin{pmatrix} 0 & \dfrac{1}{\alpha_1} & 0 & -\dfrac{1}{\alpha_1\alpha_3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\alpha_3} \\ -\alpha_2 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\widehat{B} = A_x^{-1} \cdot B = \begin{pmatrix} 0 & \dfrac{q}{\alpha_1} & 0 & -\dfrac{1}{\alpha_1\alpha_3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\alpha_3} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

i.e., we obtain the system

$$E \cdot \partial_x \mathcal{U} + A_x^{-1} \cdot \partial_t \mathcal{U} + \widehat{B} \cdot \mathcal{U}$$

or

$$\begin{aligned} &\partial_x e + \frac{1}{\alpha_1}(\partial_t + q)p - \frac{1}{\alpha_1\alpha_3}(\partial_t + 1)N_1 = 0, \\ &\partial_x p + \partial_t e = 0, \\ &\partial_x N + \frac{1}{\alpha_3}(\partial_t + 1)N_1 = 0, \\ &\partial_x N_1 - \alpha_2 \partial_t e + (\partial_t + 1)N = 0. \end{aligned} \tag{4.5}$$

We make the Fourier transform with respect to t and set

$$\Lambda_b = A_x^{-1} i\tau + \widehat{B} = \begin{pmatrix} 0 & \dfrac{i\tau + q}{\alpha_1} & 0 & -\dfrac{i\tau + 1}{\alpha_1\alpha_3} \\ i\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{i\tau + 1}{\alpha_3} \\ -\alpha_2 i\tau & 0 & i\tau + 1 & 0 \end{pmatrix}$$

The dispersion equation has the form

$$\begin{aligned} \lambda^4 &- \left(\frac{(i\tau)(i\tau + q)}{\alpha_1} + \frac{(i\tau + 1)^2}{\alpha_3} + \alpha_2 \frac{(i\tau)(i\tau + 1)}{\alpha_1\alpha_3}\right)\lambda^2 \\ &+ \frac{(i\tau)(i\tau + q)(i\tau + 1)^2}{\alpha_1\alpha_3} = 0, \end{aligned}$$

$$
\begin{aligned}
\lambda^2 = {} & \frac{1}{2}\Big[\frac{(i\tau)(i\tau+q)}{\alpha_1} + \frac{(i\tau+1)^2}{\alpha_3} + \alpha_2\frac{(i\tau)(i\tau+1)}{\alpha_1\alpha_3}\Big] \\
& \pm \frac{1}{2}\Big\{\Big[\frac{(i\tau)(i\tau+q)}{\alpha_1} + \frac{(i\tau+1)^2}{\alpha_3} + \alpha_2\frac{(i\tau)(i\tau+1)}{\alpha_1\alpha_3}\Big]^2 \\
& - 4\frac{(i\tau)(i\tau+q)(i\tau+1)^2}{\alpha_1\alpha_3}\Big\}^{1/2}.
\end{aligned}
$$

Denote by $\lambda^b_\pm$ two roots corresponding to "+" and by $\lambda_\pm$ two roots corresponding to "$-$." The eigenvectors take the form

$$
R = \begin{pmatrix} 1 \\ i\tau/\lambda \\ -\alpha_1 + \dfrac{i\tau(i\tau+q))}{\lambda^2} \\ -\lambda\dfrac{\alpha_1\alpha_3}{i\tau+1} + \dfrac{\alpha_3 i\tau(i\tau+q)}{\lambda(i\tau+1)} \end{pmatrix}.
$$

Since

$$
\begin{aligned}
\lambda^2 = {} & \frac{1}{2}\Big\{\Big[\frac{1}{\alpha_3} + \Big(\frac{q}{\alpha_1} + \frac{2}{\alpha_3} + \frac{\alpha_2}{\alpha_1\alpha_3}\Big)i\tau\Big] \\
& \pm \Big[\frac{1}{\alpha_3} + \Big(-\frac{q}{\alpha_1} + \frac{2}{\alpha_3} + \frac{\alpha_2}{\alpha_1\alpha_3}\Big)i\tau\Big]\Big\} + O(\tau^2)
\end{aligned}
$$

as $\tau \to 0$, we have

$$
\lambda^b_\pm = \pm\frac{1}{\sqrt{\alpha_3}} + O(i\tau), \quad \lambda_\pm = \pm\sqrt{\frac{q}{\alpha_1}}\sqrt{i\tau} + O(i\tau).
$$

The eigenvectors corresponding to the eigenvalues $\lambda^b_\pm$ as $\tau \to 0$ take the form

$$
R^b_\pm = \begin{pmatrix} -1/\alpha_1 \\ 0 \\ 1 \\ \pm\sqrt{\alpha_3} \end{pmatrix}
$$

The eigenvectors $R_\pm$ corresponding to the eigenvalues $\lambda_\pm$ go to the eigenvector R_1 and the adjoined eigenvector R'_1 corresponding to the two-multiple eigenvalue $\lambda = 0$, as $\tau \to 0$:

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad R_1' = \begin{pmatrix} 0 \\ \dfrac{\alpha_1}{q} \\ 0 \\ 0 \end{pmatrix}$$

Indeed,

$$R_\pm \to \begin{pmatrix} 1 \\ 0 \\ -\alpha_1 + \dfrac{q}{q/\alpha_1} 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{R_+ - R_-}{\sqrt{i\tau}} \to \begin{pmatrix} 0 \\ 2\sqrt{\alpha_1/q} \\ 0 \\ 2(-\sqrt{\frac{q}{\alpha_1}}\alpha_1\alpha_3 + \frac{\alpha_3 q}{\sqrt{\frac{q}{\alpha_1}}}) \end{pmatrix} = \begin{pmatrix} 1 \\ 2\sqrt{\alpha_1/q} \\ 0 \\ 0 \end{pmatrix}.$$

Proposition 4.1. *In a neighborhood of $\tau = 0$, there is only one projection to the phase space of consolidated variables (e, p).*

By Theorem 3.5, for $\tau \neq 0$ in a neighborhood of $\tau = 0$ with each pair of eigenvectors

$$(R_+, R_-),\ (R_+^b, R_+),\ (R_+^b, R_-),\ (R_-^b, R_-),\ (R_-^b, R_+),\ (R_+^b, R_-^b)$$

we can associate the projection to the phase space of consolidated variables (e, p). But only one projection, corresponding to the pair (R_+, R_-), is continuous as $\tau \to 0$ when the pair of eigenvalues (R_+, R_-) goes to the pair (R_1, R_1'), where R_1 is an eigenvector and R_1' is an adjoined eigenvector corresponding to the eigenvalue $\lambda = 0$.

4.2.1. Projection. It is equivalent to look for the projection

$$N = p_{11} \cdot e + p_{12} \cdot p,$$

$$N_1 = p_{21} \cdot e + p_{22} \cdot p$$

or a solution to the matrix equation

$$(E - P)\Lambda_b P = 0,$$

where

$$P = \begin{pmatrix} E & 0 \\ P_{21} & 0 \end{pmatrix}, \quad P_{21} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

which is equivalent to the equation

$$P_{21}\Lambda_{12}P_{21} - \Lambda_{22}P_{21} + P_{21}\Lambda_{11} - \Lambda_{21} = 0,$$

where Λ_{ij}, $i,j = 1,2$, are the corresponding 2×2–blocks of the matrix Λ_b. Making the change of variables $\mathcal{U} = S\widehat{U}$, where

$$S = \begin{pmatrix} E & 0 \\ P_{21} & E \end{pmatrix}$$

and multiplying the system

$$S\partial_x \widehat{U} + \Lambda_b S\widehat{U} = 0$$

from the left by S^{-1}, where

$$S^{-1} = 2E - S = \begin{pmatrix} E & 0 \\ -P_{21} & E \end{pmatrix},$$

we find

$$\partial_x \widehat{U} + S^{-1}\Lambda_b S\widehat{U},$$

where

$$S^{-1}\Lambda_b S = \begin{pmatrix} \Lambda_{11} + \Lambda_{12}P_{21} & \Lambda_{12} \\ -P_{21}\Lambda_{11} + \Lambda_{21} - P_{21}\Lambda_{12}P_{21} + \Lambda_{22}P_{21} & -P_{21}\Lambda_{12} + \Lambda_{22} \end{pmatrix}.$$

Since the lower left block vanishes, we can reduce the system to the block form

$$\partial_x \widehat{U} + M_b \widehat{U} = 0,$$

where

$M^b = S^{-1}\Lambda_b S$

$$= \begin{pmatrix} -\dfrac{i\tau+1}{\alpha_1\alpha_3}p_{21} & \dfrac{i\tau+q}{\alpha_1} - \dfrac{i\tau+1}{\alpha_1\alpha_3}p_{22} & 0 & -\dfrac{i\tau+1}{\alpha_1\alpha_3} \\ i\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{i\tau+1}{\alpha_3}\Big(\dfrac{1}{\alpha_1}p_{11}+1\Big) \\ 0 & 0 & i\tau+1 & \dfrac{i\tau+1}{\alpha_1\alpha_3}p_{21} \end{pmatrix}$$

or

$$
\begin{aligned}
&\partial_x \widehat{e} - \frac{1}{\alpha_1\alpha_3}(\partial_t + 1)p_{21}\widehat{e} + \Big(\frac{i\tau + q}{\alpha_1} - \frac{i\tau+1}{\alpha_1\alpha_3}p_{22}\Big)\widehat{p} \\
&\qquad\qquad\qquad\qquad - \frac{1}{\alpha_1\alpha_3}(\partial_t + 1)\widehat{N_1} = 0, \\
&\partial_x \widehat{p} + (\partial_t)\widehat{e} = 0, \\
&\partial_x \widehat{N} + \frac{1}{\alpha_3}(\partial_t + 1)\Big(\frac{1}{\alpha_1}p_{11} + 1\Big)\widehat{N_1} = 0, \\
&\partial_x \widehat{N_1} + (\partial_t + 1)\widehat{N} + \frac{1}{\alpha_1\alpha_3}(\partial_t + 1)p_{21}\widehat{N_1} = 0.
\end{aligned} \tag{4.6}
$$

The dispersion equation has the form

$$
\begin{aligned}
\det(\mu E - M_b) = &\Big[\mu\Big(\mu + \frac{i\tau+1}{\alpha_1\alpha_3}p_{21}\Big) - i\tau\Big(\frac{i\tau+\gamma+q}{\alpha_1} - \frac{i\tau+1}{\alpha_1\alpha_3}p_{22}\Big)\Big] \\
&\times\Big[\mu\Big(\mu - \frac{i\tau+1}{\alpha_1\alpha_3}p_{21}\Big) - \frac{(i\tau+1)^2}{\alpha_3}\Big(\frac{1}{\alpha_1}p_{11} + 1\Big)\Big] = 0.
\end{aligned} \tag{4.7}
$$

If the matrix P_{21} is a solution to the quadratic matrix equation

$$P_{21}\Lambda_{12}P_{21} - \Lambda_{22}P_{21} + P_{21}\Lambda_{11} - \Lambda_{21} = 0,$$

then the matrix $X = \Lambda\, P$, where $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, P is a square matrix of order n, P_{11} is the identity matrix of order m, and P_{12}, P_{22} are zero matrices, is a solution to the quadratic matrix equation

$$X^2 - \Lambda\, X = 0.$$

Hence

$$
X = \Lambda_b P = \begin{pmatrix}
-\dfrac{i\tau+1}{\alpha_1\alpha_3}p_{21} & \dfrac{i\tau+q}{\alpha_1} - \dfrac{i\tau+1}{\alpha_1\alpha_3}p_{22} & 0 & 0 \\
i\tau & 0 & 0 & 0 \\
\dfrac{i\tau+1}{\alpha_3}p_{21} & \dfrac{i\tau+1}{\alpha_3}p_{22} & 0 & 0 \\
-\alpha_2 i\tau + (i\tau+1)p_{11} & (i\tau+1)p_{12} & 0 & 0
\end{pmatrix}.
$$

We have two zero eigenvalues of the matrix X: $\lambda^x_{3,4} = 0$. For the corresponding eigenvectors e_3 and e_4 we have $Xe_3 = 0$ and $Xe_4 = 0$. The upper left block of the matrix X coincides with the upper left block of the matrix

M_b, i.e., $\mu_{1,2} = \lambda^x_{1,2}$. Since a unique correct projection is the projection to the pair (R_+, R_-),, we need to choose a solution P_{21} to the equation

$$P_{21}\Lambda_{12}P_{21} - \Lambda_{22}P_{21} + P_{21}\Lambda_{11} - \Lambda_{21} = 0$$

such that $\mu_{1,2} = \lambda^x_{1,2} = \lambda_\pm$, i.e.,

$$\lambda_\pm^2 + \frac{i\tau+1}{\alpha_1\alpha_3}p_{21}\lambda_\pm - i\tau(\frac{i\tau+q}{\alpha_1} - \frac{i\tau+1}{\alpha_1\alpha_3}p_{22}) = 0$$

Since $\lambda_+ = -\lambda_-$, we have $p_{21} = 0$. Thus, for p_{ij} we have the system

$$\begin{aligned}
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{11}p_{21} + \frac{1}{\alpha_3}(i\tau+1)p_{21} = i\tau p_{12},\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{11}p_{22} + \frac{1}{\alpha_3}(i\tau+1)p_{22} = \frac{1}{\alpha_1}(i\tau+q)p_{11},\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{21}^2 - i\tau\, p_{22} - i\tau\alpha_2 + (i\tau+1)p_{11} = 0,\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{22}p_{21} - \frac{1}{\alpha_1}(i\tau+q)p_{21} + (i\tau+1)p_{12} = 0
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{\alpha_3}(i\tau+1)\Big(\frac{1}{\alpha_1}p_{11}+1\Big)p_{21} = i\tau\, p_{12},\\
&\frac{1}{\alpha_3}(i\tau+1)\Big(\frac{1}{\alpha_1}p_{11}+1\Big)p_{22} = \frac{1}{\alpha_1}(i\tau+q)p_{11},\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{21}^2 - i\tau\, p_{22} - i\tau\alpha_2 + (i\tau+1)p_{11} = 0,\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{22}p_{21} - \frac{1}{\alpha_1}(i\tau+q)p_{21} + (i\tau+1)p_{12} = 0.
\end{aligned}$$

We set $Z = \frac{1}{\alpha_1}p_{11} + 1$. Then the above system can be written in the form

$$\begin{aligned}
&\frac{1}{\alpha_3}(i\tau+1)Zp_{21} = i\tau\, p_{12},\\
&\frac{1}{\alpha_3}(i\tau+1)Zp_{22} = \frac{1}{\alpha_1}(i\tau+q)\alpha_1(Z-1),\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{21}^2 - i\tau\, p_{22} - i\tau\alpha_2 + (i\tau+1)\alpha_1(Z-1) = 0,\\
&\frac{1}{\alpha_1\alpha_3}(i\tau+1)p_{22}p_{21} - \frac{1}{\alpha_1}(i\tau+q)p_{21} + (i\tau+1)p_{12} = 0.
\end{aligned}$$

Since $p_{21}=0$, we have $p_{12}=0$. Now, two equations remain:

$$\frac{1}{\alpha_3}(i\tau+1)Zp_{22}=\frac{1}{\alpha_1}(i\tau+q)\alpha_1(Z-1),$$

$$p_{22}=-\alpha_2+\frac{i\tau+1}{i\tau}\alpha_1(Z-1)=0,$$

We obtain the quadratic equation with respect to Z:

$$\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}Z^2-\Big(\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}+\frac{\alpha_2}{\alpha_3}(i\tau+1)+i\tau+q\Big)Z+i\tau+q=0,$$

which implies

$$Z=\frac{1}{2}\frac{\alpha_3 i\tau}{\alpha_1(i\tau+1)^2}\Bigg[-\Big(\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}+\frac{\alpha_2}{\alpha_3}(i\tau+1)+i\tau+q\Big)$$

$$\pm\sqrt{\Big(\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}+\frac{\alpha_2}{\alpha_3}(i\tau+1)+i\tau+q\Big)^2-4\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}(i\tau+q)}\Bigg].$$

From the characteristic equation for the matrix M_b we find

$$\mu_{3,4}=\pm\frac{i\tau+1}{\sqrt{\alpha_3}}\sqrt{Z}.$$

Choosing the sign "+" in the expression for Z, we obtain the case of a *correct projection*

$$\mu_{3,4}=\lambda^b_{\pm},\quad \mu_{1,2}=\lambda_{\pm}.$$

Under the choice "−," we have

$$\mu_{3,4}=\lambda_{\pm},\quad \mu_{1,2}=\lambda^b_{\pm}.$$

In the case $p_{21}\neq 0$, from the equations relative to p_{ij} we can additionally obtain four solutions corresponding to four remaining partitions of the eigenvalues $\lambda_{\pm},\lambda^b_{\pm}$ into pairs.

Let us prove that $\Delta=\mathrm{Re}\,(\lambda^b_{+}-\lambda_{+})>0$ for all $\tau\geqslant 0$ (the gap condition). We have $\Delta\to\dfrac{1}{\sqrt{\alpha_3}}$ as $\tau\to 0$.

Introduce the notation

$$Z_1=\frac{1}{2}\sqrt{\frac{i\tau+1}{\alpha_3}\left(\left[\sqrt{\frac{\alpha_3 i\tau(i\tau+q)}{\alpha_1(i\tau+1)}}+\sqrt{(i\tau+1)}\right]^2+i\tau\frac{\alpha_2}{\alpha_1}\right)},$$

$$Z_2 = \frac{1}{2}\sqrt{\frac{i\tau+1}{\alpha_3}\left(\left[\sqrt{\frac{\alpha_3 i\tau(i\tau+q)}{\alpha_1(i\tau+1)}} - \sqrt{(i\tau+1)}\right]^2 + i\tau\frac{\alpha_2}{\alpha_1}\right)}.$$

A direct calculation shows that

$$(Z_1+Z_2)^2 = (\lambda^b_\pm)^2, \quad (Z_1-Z_2)^2 = (\lambda_\pm)^2.$$

Consequently,

$$\begin{gathered} Z_1+Z_2 = \lambda^b_+, \quad -Z_1-Z_2 = \lambda^b_-, \\ Z_1-Z_2 = \lambda_+, \quad -Z_1+Z_2 = \lambda_- \end{gathered}$$

Hence $\Delta = 2Z_2$. Let us show that Re $\Delta > 0$ as $\tau \to \infty$. We have

$$\Delta = \sqrt{\frac{i\tau+1}{\alpha_3}\left(\left[\sqrt{\frac{\alpha_3 i\tau(i\tau+q)}{\alpha_1(i\tau+1)}} - \sqrt{(i\tau+1)}\right]^2 + i\tau\frac{\alpha_2}{\alpha_1}\right)}$$

$$= \frac{i\tau+1}{\alpha_3}\sqrt{\frac{\alpha_3}{i\tau+1}\left[\frac{\alpha_3 i\tau(i\tau+q)}{\alpha_1(i\tau+1)} + i\tau+1-2\sqrt{i\tau(i\tau+q)\frac{\alpha_3}{\alpha_1}} + \frac{\alpha_2}{\alpha_1}i\tau\right]}$$

$$= \frac{i\tau+1}{\alpha_3}\sqrt{\alpha_3\left[\frac{\alpha_3(1+\frac{q}{i\tau})}{\alpha_1(1+\frac{1}{i\tau})^2} + 1 - 2\sqrt{\frac{1+\frac{q}{i\tau}}{(1+\frac{1}{i\tau})^2}\frac{\alpha_3}{\alpha_1}} + \frac{\alpha_2}{\alpha_1}\frac{1}{1+\frac{1}{i\tau}}\right]}$$

$$\sim \frac{i\tau+1}{\alpha_3}\sqrt{\alpha_3\left[\frac{\alpha_3}{\alpha_1}\left(1+\frac{q-2}{i\tau}\right)+1-2\sqrt{\frac{\alpha_3}{\alpha_1}}\left(1+\frac{q-2}{2i\tau}\right)+\frac{\alpha_2}{\alpha_1}\left(1-\frac{1}{i\tau}\right)+O\left(\frac{1}{\tau^2}\right)\right]}$$

$$= \frac{i\tau+1}{\alpha_3}\sqrt{\left[\alpha_3\left(\frac{\alpha_3}{\alpha_1}+1-2\sqrt{\frac{\alpha_3}{\alpha_1}}+\frac{\alpha_2}{\alpha_1}\right)+\alpha_3\frac{(q-2)\left(\frac{\alpha_3}{\alpha_1}-2\sqrt{\frac{\alpha_3}{\alpha_1}}\right)-\frac{\alpha_2}{\alpha_1}}{i\tau}+O\left(\frac{1}{\tau^2}\right)\right]}$$

$$\sim Ci\tau + \sqrt{\frac{\frac{\alpha_3}{\alpha_1} + 1 - 2\sqrt{\frac{\alpha_3}{\alpha_1}} + \frac{\alpha_2}{\alpha_1}}{\alpha_3}} + \frac{(q-2)\left(\frac{\alpha_3}{\alpha_1} - 2\sqrt{\frac{\alpha_3}{\alpha_1}}\right) - \frac{\alpha_2}{\alpha_1}}{2\alpha_3^2\left(\frac{\alpha_3}{\alpha_1} + 1 - 2\sqrt{\frac{\alpha_3}{\alpha_1}} + \frac{\alpha_2}{\alpha_1}\right)} + O\Big(\frac{1}{\tau}\Big).$$

Since $\alpha_1 = \dfrac{1}{3}$, $\alpha_2 = \dfrac{4}{15}$, $\alpha_3 = \dfrac{9}{35}$, $0 < q < 1$, a direct calculation shows that $\mathrm{Re}\,\Delta > 0$ as $\tau \to \infty$.

It remains to prove that $\mathrm{Re}\,\Delta \neq 0$ for all τ. We have

$$\Delta = \sqrt{\frac{i\tau + 1}{\alpha_3}\left(\left[\sqrt{\frac{\alpha_3 i\tau(i\tau + q)}{\alpha_1(i\tau + 1)}} - \sqrt{(i\tau + 1)}\right]^2 + i\tau\frac{\alpha_2}{\alpha_1}\right)}.$$

But $\mathrm{Re}\,\sqrt{z} = 0$ only if $\mathrm{Im}\,(z) = 0$ and $\mathrm{Re}\,(z) < 0$. Denote by z the expression under the root sign. Let us prove that $Im(z) \neq 0$ for all τ. We have

$$\alpha_3 z = \frac{\alpha_3}{\alpha_1} i\tau(i\tau + q) + 1 - \tau^2 + 2i\tau - \frac{\alpha_2}{\alpha_1}\tau^2 + \frac{\alpha_2}{\alpha_1} i\tau$$

$$- 2\sqrt{\frac{\alpha_3}{\alpha_1}}\sqrt{-\tau^2 + i\tau q} - 2i\tau\sqrt{\frac{\alpha_3}{\alpha_1}}\sqrt{-\tau^2 + i\tau q}$$

Assume that

$$\mathrm{Im}\,(\alpha_3 z) = 0 = \tau\Big(\frac{\alpha_3}{\alpha_1} q + 2 + \frac{\alpha_2}{\alpha_1}\Big) - 2\sqrt{\frac{\alpha_3}{\alpha_1}} B - 2\sqrt{\frac{\alpha_3}{\alpha_1}}\tau A,$$

where $A + iB = \sqrt{-\tau^2 + i\tau q}$. To find A and B, we can argue as follows. If $\sqrt{x + iy} = A + iB$, then $x + iy = A^2 - B^2 + 2iAB$. Hence

$$A^2 - B^2 = x, \quad 2AB = y,$$

$$A = \frac{y}{2B},$$

$$4B^4 + 4B^2 x - y^2 = 0,$$

$$B^2 = \frac{\sqrt{x^2 + y^2} - x}{2}.$$

In our case, $x = -\tau^2$, $y = \tau q$. Consequently,

$$B^4 - \tau^2 B^2 - \frac{1}{4}\tau^2 q^2$$

From the equation $\mathrm{Im}\,(\alpha_3 z) = 0$ we have

$$B \cdot C\tau = B^2 + \frac{1}{2}\tau^2 q,$$

where

$$C = \frac{\frac{\alpha_3}{\alpha_1}q + 2 + \frac{\alpha_2}{\alpha_1}}{2\sqrt{\frac{\alpha_3}{\alpha_1}}}$$

and $A = \dfrac{y}{2B} = \dfrac{\tau q}{2B}$. Raising to square, we have

$$B^4 + B^2(\tau^2 q - \tau^2 C^2) + \frac{1}{4}\tau^4 q^2 = 0.$$

Comparing two equations for B, we conclude that

$$\tau^2 q - \tau^2 C^2 = -\tau^2, \quad \frac{1}{4}\tau^4 q^2 = -\frac{1}{4}\tau^2 q^2$$

So, only the case $\tau = 0$ is possible.

4.2.2. On boundary conditions. Thus, making the change of variables $\mathcal{U} = S\widehat{U}$, where

$$S = \begin{pmatrix} E & 0 \\ P_{21} & E \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p_{11} & p_{12} & 1 & 0 \\ p_{21} & p_{22} & 0 & 1 \end{pmatrix}$$

$$p_{12} = p_{21} = 0, \quad p_{11} = \alpha_1(Z-1), \quad p_{22} = \frac{i\tau + 1}{i\tau}\alpha_1(Z-1) - \alpha_2$$

and

$$Z = \frac{1}{2}\frac{\alpha_3 i\tau}{\alpha_1(i\tau+1)^2}\Big[- \left(\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau} + \frac{\alpha_2}{\alpha_3}(i\tau+1) + i\tau + q\right)$$
$$+ \sqrt{\left(\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau} + \frac{\alpha_2}{\alpha_3}(i\tau+1) + i\tau + q\right)^2 - 4\frac{\alpha_1(i\tau+1)^2}{\alpha_3 i\tau}(i\tau+q)}\Big],$$

we can reduce the system under consideration to the block form

$$\partial_x \widehat{U} + M_b \widehat{U} = 0,$$

where

$$M^b = S^{-1}\Lambda_b S = \begin{pmatrix} 0 & \dfrac{i\tau+q}{\alpha_1} - \dfrac{i\tau+1}{\alpha_1\alpha_3}p_{22} & 0 & -\dfrac{i\tau+1}{\alpha_1\alpha_3} \\ i\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{i\tau+1}{\alpha_3}\big(\dfrac{1}{\alpha_1}p_{11}+1\big) \\ 0 & 0 & i\tau+1 & 0 \end{pmatrix}$$

Then we have the eigenvalues

$$\mu_1 = \lambda_+, \quad \mu_2 = \lambda_-, \quad \mu_3 = \lambda_+^b, \quad \mu_4 = \lambda_-^b$$

and the eigenvectors

$$\widehat{R_{1,2}} = \begin{pmatrix} 1 \\ \dfrac{i\tau}{\mu_{1,2}} \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{R_{3,4}} = \begin{pmatrix} 1 \\ i\tau/\mu_{3,4} \\ \left[\frac{i\tau}{\mu_{3,4}}\left(\frac{i\tau+q}{\alpha_1} - \frac{(i\tau+1)p_{22}}{\alpha_1\alpha_3}\right) - \mu_{3,4}\right]\frac{\alpha_1\alpha_3\mu_{3,4}}{(i\tau+1)^2} \\ \left[\frac{i\tau}{\mu_{3,4}}\left(\frac{i\tau+q}{\alpha_1} - \frac{(i\tau+1)p_{22}}{\alpha_1\alpha_3}\right) - \mu_{3,4}\right]\frac{\alpha_1\alpha_3}{i\tau+1} \end{pmatrix}.$$

We represent bounded solutions in the form

$$\widehat{U} = C_1\widehat{R_1}e^{-\mu_1 x} + C_2\widehat{R_3}e^{-\mu_3 x}$$

The original boundary conditions are given by the formula

$$B \cdot \mathcal{U} = \varphi,$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

Making the change of variables $\mathcal{U} = S\widehat{U}$, we find

$$\widehat{B}\widehat{U} = \varphi$$

where

$$\widehat{B} = B \cdot S = \begin{pmatrix} b_{11}+b_{13}p_{11}+b_{14}p_{21} & b_{12}+b_{13}p_{12}+b_{14}p_{22} & b_{13} & b_{14} \\ b_{21}+b_{23}p_{11}+b_{24}p_{21} & b_{22}+b_{23}p_{12}+b_{24}p_{22} & b_{23} & b_{24} \end{pmatrix}$$

Assume that $\widehat{B}$ is a block matrix:

$$\widehat{b_{21}} = b_{21} + b_{23}p_{11} + b_{24}p_{21} = 0,$$

$$\widehat{b_{22}} = b_{22} + b_{23}p_{12} + b_{24}p_{22} = 0.$$

Then

$$\widehat{B}\widehat{U_0} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where

$$\widehat{B} = \begin{pmatrix} \widehat{b_{11}} & \widehat{b_{12}} & \widehat{b_{13}} & \widehat{b_{14}} \\ 0 & 0 & \widehat{b_{23}} & \widehat{b_{24}} \end{pmatrix},$$

$$\widehat{U_0} = (\widehat{u_1}, \widehat{u_2})^\top\big|_{x=0} = \begin{pmatrix} C_1\widehat{r_{11}} + C_2\widehat{r_{13}} \\ C_2\widehat{r_{23}} \end{pmatrix},$$

where $\widehat{R_1} = (\widehat{r_{11}}, 0)^\top$ and $\widehat{R_3} = (\widehat{r_{13}}, \widehat{r_{23}})^\top$. Hence we obtain the equation

$$C_2\widehat{B_{22}}\widehat{r_{23}} = \varphi_2.$$

We write out the Lopatinskii condition $\widehat{B_{22}}\widehat{r_{23}} \neq 0$, i.e.,

$$\frac{\mu_3}{i\tau + 1}b_{23} + b_{24} \neq 0.$$

Hence

$$C_2 = \varphi_2/(\widehat{B_{22}}\widehat{r_{22}}),$$

$$C_1\widehat{B_{11}}\widehat{r_{11}} + C_2(\widehat{B_{11}}\widehat{r_{13}} + \widehat{B_{12}}\widehat{r_{23}}) = \varphi_1,$$

i.e.,

$$C_1\widehat{B_{11}}\widehat{r_{11}} = -C_2(\widehat{B_{11}}\widehat{r_{13}} + \widehat{B_{12}}\widehat{r_{23}}) + \varphi_1.$$

We write out the second Lopatinskii condition:

$$\widehat{B_{11}}\widehat{r_{11}} = b_{11} + b_{13}p_{11} + b_{14}p_{21} + (b_{12} + b_{13}p_{12} + b_{14}p_{22})\frac{i\tau}{\mu_1} \neq 0$$

and give an example of matrix B satisfying the above conditions:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -p_{11} & 0 & 1 & 0 \end{pmatrix}$$

By the gap condition $\operatorname{Re}(\mu_3 - \mu_1) > 0$, we have

$$\widehat{U} = C_1\widehat{R_1}e^{-\mu_1 x} + C_2e^{-\mu_1 x}(\widehat{R_3}e^{(\mu_1-\mu_3)x}),$$

where $(\widehat{R_3}e^{(\mu_1-\mu_3)x}) = o(1)$. Finally, we have

$$\widehat{U} = C_1\widehat{R_1}e^{-\mu_1 x} + C_2e^{-\mu_1 x}(\widehat{R_3}e^{(\mu_1-\mu_3)x})$$

$$= \Big(\frac{\varphi_1}{\widehat{B_{11}}\widehat{r_{11}}} - \frac{\varphi_2(\widehat{B_{11}}\widehat{r_{13}} + \widehat{B_{12}}\widehat{r_{23}})}{\widehat{B_{11}}\widehat{r_{11}}\widehat{B_{22}}\widehat{r_{23}}}\Big)\widehat{R_1}e^{-\mu_1 x} + \frac{\varphi_2}{\widehat{B_{22}}\widehat{r_{23}}}e^{-\mu_1 x}(\widehat{R_3}e^{(\mu_1-\mu_3)x}),$$

where

$$\frac{\varphi_2}{\widehat{B_{22}}\widehat{r_{23}}} e^{-\mu_1 x}(\widehat{R_3} e^{(\mu_1-\mu_3)x}) = o(U_1).$$

4.3. Nonlinear analysis.

In general, the construction of a nonlinear Chapman–Enskog projection is analogous to the construction of an attracting invariant manifold of a dissipative hyperbolic equation (see [**1**], [**10**]). In this paper, we do not discuss this analogy in detail. Instead, we demonstrate difficulties arising in the justification of Navier–Stokes approximations by considering the simplest example.

We consider the hyperbolic regularization of the one-dimensional isentropic Euler system

$$\begin{aligned}
&\partial_t \varrho + \partial_x(\varrho u) = 0,\\
&\partial_t(\varrho u) + \partial_x(\varrho u^2 + p(\varrho) + \alpha \varrho u = 0, \quad \alpha > 0,\\
&\varrho|_{t=0} = \varrho^0, \quad u|_{t=0} = u^0,\\
&\varrho^0 > 0, \quad \varrho^0, u^0) \to \ (\varrho^{\pm}, u^{\pm}\ t \to \pm\infty,
\end{aligned} \tag{4.8}$$

where ϱ is a conservative variable and u is a nonequillibrium variable.

The condition $p'(s) > 0$ for all $s \geqslant 0$ is the hyperbolicity condition, and $\alpha > 0$ is the relaxation (stability) condition. The system degenerates in a vacuum $\varrho = 0$. We establish the existence of a global smooth solution to the Cauchy problem (4.8) such that the condition of the uniform nondegeneracy of the initial distribution of density $\varrho^0 > 0$ for all $x \in R$ guarantees the nondegeneracy of density for any $t > 0$.

Our goal is to justify the Navier–Stokes approximation, i.e., in accordance with the Chapman–Enskog conjecture, we must show that the Navier–Stokes approximation determines asymptotically, at large times, the principal part of the solution to the Cauchy problem (4.8).

For simplicity, we pass to the Lagrange coordinates in (4.8):

$$\begin{aligned}
&\partial_t v - \partial_x u = 0,\\
&\partial_t u + \partial_x p(v) + u = 0,\\
&p'(v) < 0
\end{aligned} \tag{4.9}$$

with friction u in the moment equation. Here, v is a conservative variable and u is a nonequillibrium variable. This is a model of a compressible fluid in a porous medium. Both systems, (4.8) and (4.9), degenerate in a vacuum $\varrho = v = 0$ and are equivalent outside the vacuum. We establish the existence of a c^1-solution such that $v(x,t) > 0$ for all $t > 0$ provided that $v^0(x) > 0$ for all $x \in R$. (see [**21**]).

We consider the smooth initial data $(u_0(x),\, v_0(x)) \ \rightarrow \ (u^{\pm},\, v^{\pm})$ as $x \rightarrow \ \pm\infty$. As is known, the dissipation prevents the formation of shock waves provided that the initial data are not "too sharp" (of mean force in the sense of Lax). We are interested in the "diffusion phenomenon" caused by wave decay.

The Navier–Stokes approximation $u = \Pi_{NS}(w)$, $v = w$ is found from the system

$$u = \partial_x p(w), \tag{4.10}$$

$$\partial_t w = -\partial_x^2 p(w), \ \ w|_{t=0} = w^0, \tag{4.11}$$

Note that the closure of the first equation in (4.9) is known as the Darcy law. Thereby we have justify the Darcy law.

We look for an approximation of the solution to the Cauchy problem (4.8) in the form

$$v_{as}(x,t) = \overline{v}(x,t) + v_{\rm cor}(x,t), \ \ u_{as}(x,t) = \overline{u}(x,t) + u_{\rm cor}(x,t)$$

We find $\overline{v}(x,t), \overline{u}(x,t)$ by using the properties of the Navier–Stokes approximation which is invariant under the transformation $(x,t) \ \rightarrow \ (cx, c^2 t)$, $c > 0$. There exists an automodel solution $\overline{v}(x,t) = \varphi(x + x^*/\sqrt{t})$, $x^* \in R$, of the nonlinear diffusion equation (4.11). Here, $\varphi(\xi)$ is a unique monotone bounded solution to the ordinary differential equation

$$\varphi''(\xi) + \frac{-p''(\varphi)\,\varphi' + \frac{1}{2}\xi}{p'(\varphi)}\ \varphi' = 0 \tag{4.12}$$

with boundary condition

$$\varphi(\xi) \rightarrow \ v^{\pm}, \ \xi \rightarrow \ \pm\infty \tag{4.13}$$

with the exponential convergence rate, and

$$\begin{aligned} &|\varphi''(\xi)| + |\varphi'(\xi)| + |\varphi(\xi) - \varrho^+, \xi > 0| + |\varphi(\xi) - \varrho^-,\, \xi < 0| \\ &\leqslant \ C\ |\varrho^+ - \varrho^-|\ e^{-\gamma\alpha\,\xi^2}, \end{aligned}$$

$$\gamma = \frac{1}{4}\min(1/|p'(\varrho^+)|,\, 1/|p'(\varrho^+)|) > 0, \quad \int\limits_{-\infty}^{\infty} \varphi'(\xi)d\xi = v^+ - v^-.$$

According to (4.10), we set

$$\overline{u}(x,t) = \frac{\partial_x p(\overline{\varrho}(x,t))}{\overline{\varrho}(x,t)}$$

Using the properties of automodel solutions, we obtain the exact decay estimates for v:

$$\begin{aligned}
&\|\partial_x \overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\ t^{-1/2},\\
&\|\partial_t \overline{v}(\cdot,t)\| + \|\partial_x^2 \overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\ t^{-3/2},\\
&\|\partial_x \partial_t \overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\ t^{-5/2}.
\end{aligned} \tag{4.14}$$

We will show that the shift x_* is uniquely determined from the consistency condition

$$\int\limits_{-\infty}^{\infty} \Big(v^0(x) - \varphi(x+x_*) - v_{\mathrm{cor}}(x,0)\Big)\,dx = 0$$

of the initial data v^0 and the smoothing wave $\overline{v}(x,0)$.

Our next goal is to prove the estimate

$$\|w(\cdot,t)\|_{H^1} + \|z(\cdot,t)\|_{H^1} \leqslant\ C\,\delta\,(1+t)^{-1/2}$$

for the residuals

$$\begin{aligned}
w(x,t) &= v(x,t) - \overline{v}(x,t) - v_{\mathrm{cor}},\\
z(x,t) &= u(x,t) - \overline{u}(x,t) - u_{\mathrm{cor}}
\end{aligned} \tag{4.15}$$

if

$$|\varrho^+ - \varrho^-| + |u^+ - u^-| + \|\varrho^0 - \overline{\varrho}(x,0)\|_{H^2} + \|u^0 - \overline{u}(x,0) - u_{\mathrm{cor}}(x,0)\|_{H^2} \leqslant\ \delta$$

with sufficiently small δ. The estimate for δ is exact. For large jumps the solution blows up in finite time.

4.4. Estimate for residual.

Now, we introduce the correctors u_{cor} and v_{cor}. We study the system in variations on the initial approximation $u = \overline{u}$, $v = \overline{v}$:

$$\begin{aligned}
&\partial_t v_{\mathrm{cor}} - \partial_x u_{\mathrm{cor}} = 0,\\
&\partial_t u_{\mathrm{cor}} + \alpha\, u_{\mathrm{cor}} = -\partial_x[p(\overline{v} + v_{\mathrm{cor}}) - p(\overline{v})] - \partial_t \overline{u}.
\end{aligned} \tag{4.16}$$

We recall that for the leading part of approximations of v the following exact decay estimate holds:

$$\begin{aligned}
&\|\partial_x\overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\; t^{-1/2},\\
&\|\partial_t\overline{v}(\cdot,t)\| + \|\partial_x^2\overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\; t^{-3/2},\\
&\|\partial_x\partial_t\overline{v}(\cdot,t)\| \leqslant\ C\,|v^+ - v^-|\; t^{-5/2}.
\end{aligned}\tag{4.17}$$

These estimates yield the correct decay rate of the solution at large times, which allows us to extract lower order terms of the system (4.16).

Based on the estimates (4.17), we make several remarks.

1. The last estimate in (4.17) implies that $\partial_t\overline{u}$ is a lower order term.

2. Since the limit values agree at $x = \pm\infty$, we see that v_{cor} is a soliton type function (i.e., a smooth function such that $v_{\mathrm{cor}} \to 0$ as $x \to \pm\infty$). The same is true for $\partial_x[p(\overline{v} + v_{\mathrm{cor}}) - p(\overline{v})]$, i.e., it belongs to $H^3(R)$ relative to the spatial variables. To show that it is also a lower order term, it suffices to estimate its decay as $t \to \infty$.

3. Because of the Navier–Stokes approximation, $\overline{u}(x,t)$ is a soliton type function. Consequently, we have a residual with initial condition which should be removed by the corrector u_{cor}.

We construct a corrector as follows. Consider a soliton type function

$$m_0(x) \in\ C_0^\infty,\quad \int\limits_{-\infty}^{\infty} m_0(x)dx = 1$$

and define the corrector with respect to the variable v by the formula

$$v_{\mathrm{cor}} = -\frac{u_+ - u_-}{\alpha}\ m_0(x)\ e^{-\alpha\, t}.$$

Then the corrector for u has the form

$$u_{\mathrm{cor}}(x,t) = \Big(u^- + \int\limits_{-\infty}^{x}\ (u^+ - u^-)m_0(s)ds\Big)\ e^{-\alpha\, t}.$$

Then the right-hand side of (4.16) contains lower order terms. For

$$\begin{aligned}
w(x,t) &= v(x,t) - \overline{v}(x,t) - v_{\mathrm{cor}},\\
z(x,t) &= u(x,t) - \overline{u}(x,t) - u_{\mathrm{cor}},
\end{aligned}$$

we obtain the system

$$\begin{aligned}&\partial_t w - \partial_x z = 0,\\&\partial_t z + \partial_x[p(w + \overline{v} + e^{-\alpha t} v_{\rm cor}) - p(\overline{v})] + \partial_t \overline{u} + \alpha z = 0.\end{aligned} \tag{4.18}$$

We look for a solution to the system (4.18) such that $w, z \in C^1(R \times (0,\infty)) \cap C^0((0,\infty); H^2(R))$. Now, we can uniquely determine the shift x_0 by the formula

$$\int_{-\infty}^{\infty} (v_0(x) - \overline{v}(x + x_0, 0))dx = -\int_{-\infty}^{\infty} v_{\rm cor}(x, 0)\ dx = \frac{u_+ - u_-}{\alpha}. \tag{4.19}$$

As we will show below, this condition is connected with the existence of the potential $w(x,t) = \partial_x y(x,t)$, $z(x,t) = \partial_t y(x,t)$.

Now, we can make the following conclusions.

1. The corrector $u_{\rm cor}$ is an exponentially decaying function; moreover, the velocity u goes exponentially rapidly to the equilibrium state at infinity (the jump $[u_{\rm cor}]_-^+$ of the limit values on $x = \pm\infty$ exponentially tends to zero). It is a general fact for nonequillibrium variables.

2. Exponential decay of correctors of all nonequillibrium variables is not a general fact (the soliton part decays slower). As we have shown above, for the moment approximations of the kinetic equations of phonon gas some higher order moments, called consolidated, give an essential contribution to the corrector. Note that consolidated variables are select few nonequilibrium variables admitting the physical interpretation (can be determined by experiments).

This means that we have to correct Navier–Stokes approximation. In this case, we have a more complicated asymptotics in time separation of dynamics into an inessential dynamics of the basic part of nonequillibrium variables and dynamics of conservative variables and heat flux in the phase space, where basic dynamics are separated.

3. For the Navier–Stokes approximation the consistency condition (4.19) determines the initial data $w^0 \asymp \varphi(x + x_0)$ for the Cauchy problem. Thus, we have found the Navier–Stokes approximation Π_{NS} of the operator (1.5)

$$w^0 = \Pi(u^0, v^0)$$

connecting the initial data in the Chapman–Enskog projection.

4.5. Solvability of the system for residuals.

We prove the existence of a solution in C^1 to the system for residuals

$$\begin{aligned}&\partial_t w - \partial_x z = 0,\\ &\partial_t z + \partial_x[p(w+\overline{v}+e^{-\alpha\, t}v_{\rm cor}) - p(\overline{v})] + \partial_t\overline{u} + \alpha\, z = 0.\end{aligned} \tag{4.20}$$

It is easy to see that tit is a potential. We set

$$y(x,t) = \int\limits_{-\infty}^{x} w(\eta,t)d\eta.$$

Then

$$\partial_x y = w,\ \partial_t y = \int\limits_{-\infty}^{x} \partial_t w(\eta,t)d\eta = \int\limits_{-\infty}^{x} \partial_x z(\eta,t)d\eta = z$$

Thus, we obtain the Cauchy problem for the potential

$$\begin{aligned}&\partial_t^2 y + \partial_x[p(\partial_x y + \overline{v} + e^{-\alpha\, t}v_{\rm cor}) - p(\overline{v})] + \alpha\partial_t y - \frac{1}{\alpha}\,\partial_x\partial_t(p(\overline{v})) = 0,\\ &y(x,0) = \int\limits_{-\infty}^{x}(v^0(s) - \overline{v}(s,0) - v_{\rm cor}(s))ds,\\ &\partial_t y(x,0) = u^0(x) - \overline{u}(x,0) - u_{\rm cor}(x).\end{aligned} \tag{4.21}$$

The consistency condition (4.19) is equivalent to the condition

$$y(x,0) = \int\limits_{-\infty}^{x}[v^0(s) - \overline{v}(s,0) - v_{\rm cor}(s,0)]ds \to\ 0 \text{ as } x \to\ \pm\infty$$

and follows from the conservation laws. By the definition of $u_{\rm cor}$, we have $\partial_t y(x,0) \to\ 0$ as $x \to\ \pm\infty$.

Lemma 4.1. *Let* $|u^+ - u^-| + |v^+ - v^-|$ *be so small that*

$$|u^+ - u^-|v^+ - v^-| + \|y(x,0)\|_{H^3} + \|\partial_t y(x,0)\|_{H^2} \leqslant\ \delta \tag{4.22}$$

for sufficiently small δ*. Then there exists a solution* $y \in\ C^0((0,\infty);C^2(R))$*,* $\partial_t y \in\ C^0((0,\infty);C^1(R))$*, to the Cauchy problem* (4.21) *satisfying the estimate* (4.22) *for any* $t > 0$

The proof of the existence of a smooth solution to the Cauchy problem (4.21) is standard. We assume that $|u^+ - u^-| + |v^+ - v^-|$ is so small that (4.22) holds for sufficiently small δ. Then $y(x,t)$ satisfies the estimate (4.22) for any $t > 0$.

As a consequence, we obtain the following result.

Proposition 4.2. *Under the assumptions of Lemma* 4.1*, there exists a solution to the Cauchy problem* (4.21) *such that* $v \to \overline{v}$ *and* $u \to \overline{u}$ *in the following sense*:

$$\begin{aligned}
&\|v(x,t) - \overline{v}(x,t) - e^{-\alpha t} v_{\rm cor}(x,t)\|_{L_2} \\
&\qquad + \|v(x,t) - \overline{v}(x,t) - e^{-\alpha t} v_{\rm cor}(x,t)\|_{L_\infty} = O(t^{-1/2}), \\
&\|u(x,t) - \overline{u}(x,t) - e^{-\alpha t} u_{\rm cor}(x,t)\|_{L_2} \\
&\qquad + \|u(x,t) - \overline{u}(x,t) - e^{-\alpha t} u_{\rm cor}(x,t)\|_{L_\infty} = O(t^{-1/2}).
\end{aligned}$$

By the definition of y,

$$\begin{aligned}
&\|(u - \overline{u} - e^{-\alpha t} u_{\rm cor})(\cdot, t)\|_{H^1(R)} \\
&\qquad + \|(v - \overline{v} - e^{-\alpha t} v_{\rm cor})(\cdot, t)\|_{H^1(R)} \leqslant C\delta(1+T)^{-1/2}.
\end{aligned}$$

Using the Sobolev embedding theorem, we find the point-wise decay

$$\begin{aligned}
&\sup_x(|(u - \overline{u} - e^{-\alpha t} u_{\rm cor}|(x,t) \\
&\qquad + |v - \overline{v} - e^{-\alpha t} v_{\rm cor}|(x,t)) \leqslant C'\delta(1+T)^{-1/2}.
\end{aligned}$$

We note that the above constructions are rather universal in applications to the one-dimensional conservation laws with relaxation. Thus, in the case of the one-dimensional 13-moment system of the Boltzmann kinetic equation, there exists an automodel solution to the Navier–Stokes approximation, which allows us to justify the separation of dynamics and show that a special solution, determined by the Chapman–Enskog projection, asymptotically yields the leading part of the solution to the Cauchy problem for the 13-moment system at large times.

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Exponential Mixing for Randomly Forced Partial Differential Equations: Method of Coupling

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We describe a coupling method that enables one to study ergodic properties of random dynamical systems associated with stochastic partial differential equations. A general criterion for the uniqueness of a stationary measure and an exponential mixing property are established. The method is illustrated by an example of a complex Ginzburg–Landau equation. Bibliography: 30 titles.

0. Introduction

The method of coupling was introduced by Doeblin [**4**] for studying ergodic properties of Markov chains. We briefly describe the Doeblin approach in the simplest situation.

Let X be a compact metric space, and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X parametrized by the initial point $u \in X$. We denote by $P_k(u,\Gamma)$ the transition function associated with the Markov family, i.e.,

$$P_k(u,\Gamma) = \mathbb{P}_u\{u_k \in \Gamma\} \quad \text{for } k \geqslant 0,\, \Gamma \in \mathcal{B}_X,$$

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where $\mathcal{B}_X$ is the Borel σ-algebra on X. Recall that a probability measure μ on the space $(X, \mathcal{B}_X)$ is said to be *stationary for* $(u_k, \mathbb{P}_u)$ if

$$\mu(\Gamma) = \int_X P_1(u, \Gamma)\mu(du) \quad \text{for any } \Gamma \in \mathcal{B}_X. \tag{0.1}$$

Suppose that there is a constant $\gamma < 1$ such that

$$\|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \leqslant \gamma \tag{0.2}$$

for any $u, u' \in X$, where $\|\cdot\|_{\text{var}}$ denotes the total variation distance. In this case, one can use the following argument to prove that the family $(u_k, \mathbb{P}_u)$ has a unique stationary measure.[1)]

Let $(\mathcal{R}(u, u', \cdot), \mathcal{R}'(u, u', \cdot))$ be a pair of random variables depending on $u, u' \in X$ such that the laws of $\mathcal{R}$ and $\mathcal{R}'$ coincide with $P_1(u, \cdot)$ and $P_1(u', \cdot)$ respectively and

$$\mathbb{P}\big\{\mathcal{R}(u, u') \neq \mathcal{R}'(u, u')\big\} = \|P_1(u, \cdot) - P_1(u', \cdot)\|_{\text{var}} \quad \text{for all } u, u' \in X. \tag{0.3}$$

It can be shown that such random variables exist (see [**19**]). Denote by Ω the direct product of countably many independent copies of the probability space on which $\mathcal{R}$ and $\mathcal{R}'$ are defined and consider a family of Markov chains $\{U_k\}$ in $\boldsymbol{X} = X \times X$ given by the rule

$$U_0(\omega) = U, \qquad U_k(\omega) = (\mathcal{R}(U_{k-1}, \omega_k), \mathcal{R}'(U_{k-1}, \omega_k)) \quad \text{for } k \geqslant 1, \tag{0.4}$$

where $\omega = (\omega_j, j \geqslant 1) \in \Omega$ denotes the random parameter and $U \in \boldsymbol{X}$ is an initial point. Writing $U = (u, u')$ and $U_k = (u_k, u_k')$, from (0.2) and (0.3) we derive that

$$\mathbb{P}_U\{u_{k+1} \neq u'_{k+1} \,|\, \mathcal{F}_k\} \leqslant \gamma \quad \text{for any } U \in \boldsymbol{X},\ k \geqslant 0, \tag{0.5}$$

where $\mathcal{F}_k$ denotes the σ-algebra generated by $U_1, \dots, U_k$ and the subscript U indicates that we consider the trajectory starting from U. Iterating the inequality (0.5), we obtain the estimate

$$\mathbb{P}_U\{u_k \neq u_k\} \leqslant \gamma^k \quad \text{for any } U \in \boldsymbol{X},\ k \geqslant 0, \tag{0.6}$$

which implies

$$\|P_k(u, \cdot) - P_k(u', \cdot)\|_{\text{var}} \leqslant \gamma^k. \tag{0.7}$$

Combining (0.7) with (0.1) and the Kolmogorov–Chapman relation, we can easily show that there is at most one stationary measure. Moreover,

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1) It would be easier to observe that the right-hand side of (0.1) defines a contraction in the space of probability measures on X (endowed with the total variation distance) and therefore has a unique fixed point. However, we use a longer coupling argument whose development is applied in the paper.

from (0.7) it follows that the sequence $\{P_k(u,\cdot)\}$ converges to a limiting measure μ, which is stationary for $(u_k, \mathbb{P}_u)$.

The Doeblin argument can be used to prove the uniqueness of a stationary measure for stochastic differential equations with nondegenerate diffusion on a compact manifold. At the same time, an application of the above scheme to stochastic differential equations in $\mathbb{R}^n$ encounters an obstacle related to the fact that the phase space of the problem is not compact, and inequality (0.2) cannot be satisfied uniformly in u and u', unless some restrictive conditions are imposed on the drift. However, one can overcome this difficulty with the help of the following modification of the Doeblin approach.

Let X be a separable Banach space with a norm $\|\cdot\|$, and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X. Retaining the above notation, suppose that we can find a closed subset $B \subset X$ for which the two properties below are satisfied:

(i) The inequality (0.2) holds for any $u, u' \in B$ and a constant $\gamma < 1$.

(ii) The first hitting time τ_B of the set B is almost surely finite for any initial point $u \in X$, and there is $\delta > 0$ such that

$$\mathbb{E}_u \exp\bigl(\delta\tau_B\bigr) < \infty \quad \text{for all } u \in X. \tag{0.8}$$

Let $(\mathcal{R}, \mathcal{R}')$ be the family of random variables in $\boldsymbol{X}$ defined above, and let $\{U_k\}$ be the family of Markov chains given by (0.4). Denote by ρ_n the nth instant when the trajectory U_k enters the set $\boldsymbol{B} := B \times B$. Then, using (0.2), (0.3), and the strong Markov property, it can be shown that (cf. (0.5))

$$\mathbb{P}\{u_{\rho_n+1} \neq u'_{\rho_n+1} \,|\, \mathcal{F}_{\rho_n}\} \leqslant \gamma \quad \text{for any } U \in \boldsymbol{X},\ n \geqslant 1, \tag{0.9}$$

where $\mathcal{F}_{\rho_n}$ denotes the σ-algebra associated with the Markov time ρ_n. Iteration of (0.9) results in (cf. (0.6))

$$\mathbb{P}_U\{u_{\rho_n+1} \neq u'_{\rho_n+1}\} \leqslant \gamma^n \quad \text{for any } U \in \boldsymbol{X},\ n \geqslant 1.$$

Combining this with (0.8), one can prove the inequality (0.7) with a larger constant $\gamma < 1$. Thus, the Doeblin method applies also in the case of an unbounded phase space, provided that the inequality (0.2) is satisfied on a subset that can be reached exponentially fast from any initial point. However, it should be noted that the inequality (0.2) is rather restrictive for Markov chains in an infinite-dimensional space. For instance, in the case of stochastic partial differential equations, it is satisfied only if the diffusion is "very rough." The goal of this paper is to establish a general criterion for the uniqueness of a stationary measure and exponential mixing and to show how to apply it to a complex Ginzburg–Landau equation. Without going into details, let us describe our scheme in the case of discrete time.

As before, we consider a Markov family $(u_k, \mathbb{P}_u)$ in a separable Banach space X and denote by $P_k(u, \Gamma)$ its transition function. Suppose that we can construct a family of Markov chains $(U_k, \mathbb{P}_U)$, $U_k = (u_k, u'_k)$, in the product space $\boldsymbol{X}$ such that the laws of u_k and u'_k under $\mathbb{P}_U$, $U = (u, u')$, coincide with $P_k(u, \cdot)$ and $P_k(u', \cdot)$ respectively, and the following two properties hold (cf. properties (i) and (ii) above):

(i') Let $\sigma = \min\{k \geqslant 1 : \|u_k - u'_k\| > \gamma^k\}$, where $\gamma < 1$ is a positive constant and the minimum over the empty set is $+\infty$. Then there is a subset $\boldsymbol{B} \subset \boldsymbol{X}$ and positive constants C and $\alpha < 1$ such that

$$\mathbb{P}_U\{\sigma = +\infty\} \geqslant \tfrac{1}{2}, \quad \mathbb{P}_U\{\sigma = k\} \leqslant C\alpha^k \quad \text{for } U = (u, u') \in \boldsymbol{B}.$$

(ii') Let $\tau_{\boldsymbol{B}} = \min\{k \geqslant 0 : U_k \in \boldsymbol{B}\}$. Then there is $\delta > 0$ such that

$$\mathbb{E}_U \exp\bigl(\delta \tau_{\boldsymbol{B}}\bigr) < \infty \quad \text{for any } U \in \boldsymbol{X}.$$

In this case, the difference $P_k(u, \cdot) - P_k(u', \cdot)$, regarded as a signed measure in X, goes to zero in the dual Lipschitz norm $\|\cdot\|_{\mathcal{L}}^*$ exponentially fast. (See Notation for the definition of $\|\cdot\|_{\mathcal{L}}^*$.) Indeed, from (i') it follows that, each time the process is in $\boldsymbol{B}$, with probability $\geqslant \frac{1}{2}$ we have $\sigma = +\infty$, which means that the difference $\Delta_k = \|u_k - u'_k\|$ goes to zero exponentially fast. Let us consider a sequence of stopping times ρ_k defined by the following rule. Denote by ρ_0 the first hitting time of $\boldsymbol{B}$ (i.e., $\rho_0 = \tau_{\boldsymbol{B}}$). With probability $\geqslant \frac{1}{2}$ we have $\sigma = +\infty$ for the chain starting from U_{ρ_0}, and in this case we set $\rho_k = +\infty$ for $k \geqslant 2$. Otherwise, we denote by ρ the first instant after σ when U_{ρ_0+k} hits $\boldsymbol{B}$ and define ρ_1 by the formula $\rho_1 = \rho_0 + \rho$. In general, if ρ_k is already defined, then $\rho_{k+1} = \rho_k + \rho$, where ρ is the first instant after σ when the chain starting from U_{ρ_k} hits $\boldsymbol{B}$. As in the case of ρ_0, with probability $\geqslant \frac{1}{2}$ we have $\rho_l = +\infty$ for $l \geqslant k+1$.

The above construction implies that, if $\rho_k < +\infty$ and $\rho_{k+1} = +\infty$, then $\Delta_{\rho_k+m} \leqslant \gamma^m$ for all $m \geqslant 0$. Using the strong Markov property and assertions (i') and (ii'), it can be shown that $\mathbb{P}_U\{\rho_k < +\infty\} \leqslant 2^{-k}$. What has been said implies that with probability $\geqslant 1 - 2^{-k-1}$ we have

$$\|u_k - u'_k\| \leqslant \gamma^{k-\rho_k} \quad \text{for all } k \geqslant \rho_k. \tag{0.10}$$

Moreover, further analysis enables one to show that

$$\mathbb{P}_U\bigl\{k/2 \leqslant \rho_k < \infty\bigr\} \leqslant C\beta^k, \tag{0.11}$$

where C and $\beta < 1$ are positive constants. Combining (0.10) and (0.11), we see that

$$\mathbb{P}_U\big\{\|u_k - u_k'\| > \gamma^{k/2}\big\} \leqslant 2^{-k-1} + C\beta^k \quad \text{for } k \geqslant 1.$$

Thus, the difference $\|u_k - u_k'\|$ converges to zero in probability exponentially fast. This property implies the uniqueness of a stationary measure.

Let us mention that the problem of ergodicity for randomly forced equations of mathematical physics was in the focus of attention of many researchers during the last ten–fifteen years, and first results in this direction were obtained in [**29, 8, 13, 6, 2**]. We refer the reader to the review papers [**7, 17, 3, 27**] and the book [**18**] for a detailed account of the results obtained so far. The coupling technique described above is a modified version of the one used in [**14, 15, 25**]. Related approaches were also developed in [**20, 23, 10, 24**].

The paper is organized as follows. In Section 1, we give a description of random dynamical systems studied in this work and introduce the concept of an extension for random dynamical systems. A general criterion (in terms of extension) for the uniqueness of a stationary measure and exponential mixing is presented in Section 2. In Section 3, we give some simple sufficient conditions under which one of the hypotheses of our criterion is satisfied. The fourth section is devoted to the application of these results to complex Ginzburg–Landau equation with random perturbation. We also formulate an open question. Finally, in Appendix, we present two auxiliary results used in the main text.

Notation.

Let X be a separable Banach space endowed with its Borel σ-algebra $\mathcal{B}_X$. Denote by B_R the ball in X of radius R centered at origin, by $\mathcal{P}(X)$ the set of probability measures on $(X, \mathcal{B}_X)$, by $C(X)$ the space of continuous functions $f : X \to \mathbb{R}$, and by $\mathcal{L}(X)$ the space of functions $f \in C(X)$ such that

$$\|f\|_{\mathcal{L}} := \sup_{u \in X} |f(u)| + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|} < \infty,$$

where $\|\cdot\|$ stands for the norm in X. The space $\mathcal{P}(X)$ is endowed with either the total variation distance,

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} := \sup_{\Gamma \in \mathcal{B}_X} |\mu_1(\Gamma) - \mu_2(\Gamma)|,$$

or the dual Lipschitz distance,

$$\|\mu_1 - \mu_2\|_{\mathcal{L}}^* := \sup_{\|f\|_{\mathcal{L}} \leqslant 1} \bigl|(f, \mu_1) - (f, \mu_2)\bigr|,$$

where (f, μ) denotes the integral of the function f with respect to the measure μ. The space $\mathcal{P}(X)$ is complete with respect to both metrics $\|\cdot\|_{\mathrm{var}}$ and $\|\cdot\|_{\mathcal{L}}^*$ (see [**5**]).

Let $D \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary ∂D, and let $T > 0$ be a constant. We use the following functional spaces.

$L^2 = L^2(D, \mathbb{C})$ is the space of complex-valued square-integrable functions on D.

$H^1 = H^1(D, \mathbb{C})$ is the Sobolev space of order 1.

$H_0^1 = H_0^1(D, \mathbb{C})$ is the space of functions $u \in H^1$ vanishing on ∂D.

$C^k(0, T; X)$ is the space of continuous functions $u : [0, T] \to X$ that are k times continuously differentiable. In the case $k = 0$, we write $C(0, T; X)$.

$L^2(0, T; X)$ is the space of Bochner-measurable square-integrable functions on the interval $[0, T]$ with range in X.

If a and b are real numbers, then $a \vee b$ $(a \wedge b)$ stands for their maximum (minimum). For a random variable ξ we denote by $\mathcal{D}(\xi)$ its distribution. If A is a subset in a given space, then I_A stands for its indicator function and A^c denotes its complement. We denote by $\mathbb{R}_+$ the half-line $[0, \infty)$.

1. Description of the Class of Problems

1.1. A class of random dynamical systems.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathcal{F}_t$, $t \geqslant 0$, and a semigroup of measure-preserving transformations $\theta_t : \Omega \to \Omega$ such that $\theta_t^{-1}\mathcal{F}_s \subset \mathcal{F}_{t+s}$. We always assume that $\mathcal{F}_t$ is augmented with respect to $(\mathcal{F}, \mathbb{P})$, i.e., the σ-algebra $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

We consider a random dynamical system whose trajectories form a Markov process. More precisely, let X be a separable Banach space with a norm $\|\cdot\|$, let $\mathcal{B}_X$ be the Borel σ-algebra on X, and let $S_t(u, \omega)$, $t \geqslant 0$, $\omega \in \Omega$, $u \in X$, be a continuous random dynamical system over θ_t (see Definitions 1.1.1 and 1.1.2 in [**1**]). We *always* assume that the following two properties hold.

- For almost all $\omega \in \Omega$ the trajectories $S_t(u, \omega)$, $u \in X$, are continuous in $t \geqslant 0$.
- For any $u \in X$ the random process $S_t(u, \omega)$, $t \geqslant 0$, is Markov with respect to the filtration $\mathcal{F}_t$, i.e., for any $\Gamma \in \mathcal{B}_X$ and $t, s \geqslant 0$ we have

$$\mathbb{P}\left(S_{t+s}(u, \cdot) \in \Gamma \mid \mathcal{F}_t\right) = P_s(S_t(u, \omega), \Gamma), \tag{1.1}$$

where the equality holds for almost all $\omega \in \Omega$, and $P_s(u, \Gamma)$ is the transition function defined by the formula

$$P_t(u, \Gamma) = \mathbb{P}\{S_t(u, \cdot) \in \Gamma\}, \quad u \in X, \quad \Gamma \in \mathcal{B}_X. \tag{1.2}$$

In what follows, random dynamical systems satisfying the above properties (in particular, the continuity condition with respect to time) will be said to be *Markov*. With every Markov random dynamical system we associate a family of Markov processes parametrized by the initial point $u \in X$. To fix notation, let us briefly recall the corresponding construction.

We set

$$\Omega' = X \times \Omega, \quad \mathcal{F}' = \mathcal{B}_X \otimes \mathcal{F}, \quad \mathcal{F}'_t = \mathcal{B}_X \otimes \mathcal{F}_t, \quad \mathbb{P}_u = \delta_u \otimes \mathbb{P},$$

where $\delta_u \in \mathcal{P}(X)$ is the Dirac measure concentrated at $u \in X$ and $\otimes$ denotes the direct product of measures and σ-algebras. For $\omega' = (u, \omega) \in \Omega'$ we set

$$S'_t(\omega') = S_t(u, \omega), \quad \theta'_t\omega' = (S_t(u, \omega), \theta_t\omega).$$

We thus obtain a Feller [2] family $(S'_t, \mathbb{P}'_u)$ of homogeneous Markov processes in the phase space X with the transition function (1.2) and the corresponding Markov semigroups

$$\mathfrak{P}_t f(u) = \int_X P_t(u, dv) f(v), \quad \mathfrak{P}^*_t \mu(\Gamma) = \int_X P_t(u, \Gamma)\mu(du), \tag{1.3}$$

where $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$. In what follows, we drop the prime from the notation and write $\omega, \Omega, S_t, \mathcal{F}, \mathcal{F}_t, \theta_t$ instead of $\omega', \Omega', S'_t, \mathcal{F}', \mathcal{F}'_t, \theta'_t$.

In this paper, we consider the Markov random dynamical system associated with the randomly forced complex Ginzburg–Landau equation

$$\dot{u} - (\nu + i)\Delta u + i|u|^{2p}u = h(x) + \dot{\zeta}(t, x), \quad x \in D, \tag{1.4}$$

$$u\big|_{\partial D} = 0, \tag{1.5}$$

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[2] The Feller property of the transition function follows from the continuity of $S_t(u, \omega)$ with respect to u and the Lebesgue theorem on dominated convergence.

where $u = u(t,x)$ is a complex-valued unknown function, $D \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂D, $h \in L^2(D,\mathbb{C})$ stands for a deterministic function, and $\zeta(t,x)$ is a complex-valued colored Wiener process. We show that the problem in question has a unique stationary measure and possesses a property of exponential mixing. We refer the reader to Section 4.2 for an exact formulation of the result.

1.2. Extension of random dynamical systems.

Let X be a separable Banach space, and let $S_t(u,\omega)$ be a Markov random dynamical system in X over a semigroup θ_t. We define the product space $\boldsymbol{X} = X \times X$ endowed with the usual norm and denote by $\mathcal{B}_{\boldsymbol{X}}$ its Borel σ-algebra. Write $\boldsymbol{u} = (u,u')$ and denote by

$$\Pi_X : \boldsymbol{u} \mapsto u, \quad \Pi'_X : \boldsymbol{u} \mapsto u'$$

the natural projections to the components of $\boldsymbol{u}$. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ be a complete probability space endowed with a filtration $\widehat{\mathcal{F}}_t$, $t \geqslant 0$, which is assumed to be augmented with respect to $(\widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, and let $\hat{\theta}_t : \widehat{\Omega} \to \widehat{\Omega}$ be a semigroup of measure-preserving transformations such that $\theta_t^{-1}\widehat{\mathcal{F}}_s \subset \widehat{\mathcal{F}}_{t+s}$. Consider a Markov random dynamical system $\boldsymbol{S}_t(\boldsymbol{u},\hat{\omega})$ in $\boldsymbol{X}$ over $\hat{\theta}_t$.

Definition 1.1. A Markov random dynamical system $\boldsymbol{S}_t$ in $\boldsymbol{X}$ defined on the half-line $t \geqslant 0$ is called an *extension of* S_t if for any $\boldsymbol{u} = (u,u') \in \boldsymbol{X}$ the distributions of the random processes $\Pi_X \boldsymbol{S}_t(\boldsymbol{u},\hat{\omega})$ and $\Pi'_X \boldsymbol{S}_t(\boldsymbol{u},\hat{\omega})$ regarded as random variables in $C(\mathbb{R}_+, X)$ coincide with those of $S_t(u,\omega)$ and $S_t(u',\omega)$ respectively.

In what follows, if S_t is a random dynamical system and $\boldsymbol{S}_t$ is its extension, then we denote the corresponding stochastic bases by the same symbol $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$. Moreover, abusing the notation, we write $\boldsymbol{S}_t(\boldsymbol{u},\omega) = (S_t(\boldsymbol{u},\omega), S'_t(\boldsymbol{u},\omega))$. Finally, we denote by $(\boldsymbol{S}_t, \mathbb{P}_{\boldsymbol{u}})$ the family of Markov processes associated with $\boldsymbol{S}_t$ and parametrized by the initial point $\boldsymbol{u} \in \boldsymbol{X}$.

We note that, if $\boldsymbol{S}_t$ is an extension of S_t, then for any $f \in C(X)$ and $\boldsymbol{u} = (u,u') \in \boldsymbol{X}$ we have

$$\mathbb{E}_{\boldsymbol{u}} f(\Pi_X \boldsymbol{S}_t) = \mathfrak{P}_t f(u), \quad \mathbb{E}_{\boldsymbol{u}} f(\Pi'_X \boldsymbol{S}_t) = \mathfrak{P}_t f(u'). \tag{1.6}$$

This observation, which is a simple consequence of the definition of extension, will be important in the next section (see the proof of Theorem 2.3).

We also need an auxiliary concept of *extension on a finite time interval*. More precisely, let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u},\omega) = (\mathcal{R}_t(\boldsymbol{u},\omega), \mathcal{R}'_t(\boldsymbol{u},\omega))$ be a continuous

Markov random dynamical system defined for $t \in [0, T]$, where $T > 0$ is a constant independent of $(\boldsymbol{u}, \omega)$. (In other words, the properties entering the definition of a Markov random dynamical system hold on the interval $[0, T]$; see [**1**, Definitions 1.1.1 and 1.1.2].)

Definition 1.2. The random dynamical system $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}'_t)$ in $\boldsymbol{X}$ is called an *extension of* S_t *on* $[0, T]$ if for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ the distributions of the random processes $\mathcal{R}_t(\boldsymbol{u}, \cdot)$ and $\mathcal{R}'_t(\boldsymbol{u}, \cdot)$ regarded as random variables in $C(0, T; X)$ coincide with those of $S_t(u, \cdot)$ and $S_t(u', \cdot)$ respectively.

Given an extension $\boldsymbol{\mathcal{R}}_t$ of S_t on an interval $[0, T]$, we can iterate it to construct an extension defined on the half-line $t \geqslant 0$. To this end, we denote by $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k, \mathcal{F}_t^k, \theta_t^k)$, $k \geqslant 1$, a countable family of independent copies of the stochastic bases on which $\boldsymbol{\mathcal{R}}_t$ is defined. We consider a new stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$ defined by the following rules.

- The space Ω is the product of Ω^k, $k \geqslant 1$, and its points are denoted by $\omega = (\omega_1, \omega_2, \dots)$.
- The σ-algebra $\mathcal{F}$ is the direct product of $\mathcal{F}^k$, $k \geqslant 1$, completed with respect to the product measure $\mathbb{P} = \mathbb{P}^1 \otimes \mathbb{P}^2 \otimes \cdots$.
- If $t = (k-1)T + s$, where $k \geqslant 1$ is an integer and $0 \leqslant s < T$, then $\mathcal{F}_t$ is the augmentation (with respect to $(\mathcal{F}, \mathbb{P})$) of the σ-algebra generated by the sets of the form

$$\Gamma = \big\{\omega = (\omega_1, \omega_2, \dots) : \omega_m \in \Gamma_m \text{ for } m = 1, \dots, k\big\},$$

where $\Gamma_m \in \mathcal{F}_T^m$ for $m = 1, \dots, k-1$ and $\Gamma_k \in \mathcal{F}_s^k$. Furthermore, the shift operator θ_t is given by the formula

$$\theta_t \omega = \theta_t(\omega_1, \omega_2, \dots) = (\theta_s^k \omega_k, \theta_s^{k+1} \omega_{k+1}, \dots).$$

An extension $\boldsymbol{S}_t$ on $t \geqslant 0$ is now defined by induction. Namely, for $0 \leqslant t \leqslant T$ we set

$$\boldsymbol{S}_t(\boldsymbol{u}, \omega) = \boldsymbol{\mathcal{R}}_t(\boldsymbol{u}, \omega_1). \tag{1.7}$$

If $\boldsymbol{S}_t$ is already defined for $0 \leqslant t \leqslant kT$, where $k \geqslant 1$ is an integer, then for $0 \leqslant s \leqslant T$ we set

$$\boldsymbol{S}_{kT+s}(\boldsymbol{u}, \omega) = \boldsymbol{\mathcal{R}}_s(\boldsymbol{S}_{kT}(\boldsymbol{u}, \omega), \omega_{k+1}). \tag{1.8}$$

It is a matter of direct verification to show that $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ is a continuous Markov random dynamical system in $\boldsymbol{X}$ over θ_t and that it is an extension of S_t.

2. Coupling Hypothesis

2.1. Markov random dynamical system satisfying a coupling condition.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, \theta_t)$ be a stochastic basis satisfying the conditions formulated in Section 1, let $S_t(u,\omega)$ be a Markov random dynamical system in a separable Banach space X, and let $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ be the corresponding Markov semigroups (see (1.3)). Recall that $\mu \in \mathcal{P}(X)$ is called a *stationary measure* for $S_t(u,\omega)$ if $\mathfrak{P}_t^*\mu = \mu$ for all $t \geqslant 0$.

Definition 2.1. We say that S_t is *exponentially mixing* if it has a unique stationary measure $\mu \in \mathcal{P}(X)$ and there is a constant $\gamma > 0$ and an increasing function $V\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $u \in X$ we have

$$\|P_t(u,\cdot) - \mu\|_{\mathcal{L}}^* \leqslant V(\|u\|)e^{-\gamma t}, \quad t \geqslant 0. \tag{2.1}$$

Let $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ be an extension of $S_t(u,\omega)$ (see Section 1.2). We fix positive constants C, β and a closed subset $\boldsymbol{B} \subset \boldsymbol{X}$ and introduce the stopping times

$$\tau_{\boldsymbol{B}} = \tau_{\boldsymbol{B}}(\boldsymbol{u},\omega) = \inf\bigl\{t \geqslant 0 : \boldsymbol{S}_t(\boldsymbol{u},\omega) \in \boldsymbol{B}\bigr\}, \tag{2.2}$$

$$\sigma = \sigma(\boldsymbol{u},\omega) = \inf\bigl\{t \geqslant 0 : \|S_t(\boldsymbol{u},\omega) - S_t'(\boldsymbol{u},\omega)\| \geqslant C\,e^{-\beta t}\bigr\}, \tag{2.3}$$

where $\boldsymbol{u} = (u,u')$ and the infimum over the empty set is $+\infty$. In other words, $\tau_{\boldsymbol{B}}$ is the first hitting time of the closed set $\boldsymbol{B}$ for the trajectory $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ and σ is the first instance when the curves $S_t(\boldsymbol{u},\omega)$ and $S_t'(\boldsymbol{u},\omega)$ "stop converging" to each other exponentially fast. In particular, if $\sigma(\boldsymbol{u},\omega) = \infty$, then

$$\|S_t(\boldsymbol{u},\omega) - S_t'(\boldsymbol{u},\omega)\| \leqslant C\,e^{-\beta t} \quad \text{for } t \geqslant 0. \tag{2.4}$$

Definition 2.2. We say that the random dynamical system $S_t(u,\omega)$ satisfies the *coupling hypothesis* if it has an extension $\boldsymbol{S}_t(\boldsymbol{u},\omega)$ possessing the following properties.

(i) There is a constant $\delta > 0$, a closed set $\boldsymbol{B} \subset \boldsymbol{X}$, and an increasing function $g(r) \geqslant 1$ of the variable $r \geqslant 0$ such that

$$\mathbb{E}_{\boldsymbol{u}} \exp\bigl(\delta\tau_{\boldsymbol{B}}\bigr) \leqslant G(\boldsymbol{u}) \quad \text{for all } \boldsymbol{u} = (u,u') \in \boldsymbol{X}, \tag{2.5}$$

where $G(\boldsymbol{u}) = g(\|u\|) + g(\|u'\|)$.

(ii) There are positive constants δ_1, δ_2, c, K, and $q > 1$ such that

$$\mathbb{P}_{\boldsymbol{u}}\bigl\{\sigma = \infty\bigr\} \geqslant \delta_1, \tag{2.6}$$

$$\mathbb{E}_{\boldsymbol{u}}\bigl\{I_{\{\sigma<\infty\}} \exp\bigl(\delta_2\sigma\bigr)\bigr\} \leqslant c, \tag{2.7}$$

$$\mathbb{E}_{\boldsymbol{u}}\bigl\{I_{\{\sigma<\infty\}} G(\boldsymbol{S}_\sigma)^q\bigr\} \leqslant K \tag{2.8}$$

for any $\boldsymbol{u} \in \boldsymbol{B}$.

Any extension of S_t satisfying properties (i) and (ii) will be called a *mixing extension*.

Before formulating the main result of this section, we wish to make some comments on the above definition. We take an arbitrary initial point $\boldsymbol{u} \in \boldsymbol{B}$. Then, in view of (2.6), with probability $\geqslant \delta_1$ we have $\sigma = \infty$, and therefore, with the same probability, the trajectories $S_t(\boldsymbol{u}, \omega)$ and $S_t'(\boldsymbol{u}, \omega)$ converge to each other exponentially fast (see (2.4)). On the other hand, if they do not, the inequality (2.7) says that the first instant $\sigma(\boldsymbol{u}, \omega)$ when the trajectories "stop converging" to each other is not very large. Moreover, by (2.8), we have some control over $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ at the instant $t = \sigma(\boldsymbol{u}, \omega)$. If the initial point $\boldsymbol{u} \in \boldsymbol{X}$ does not belong to $\boldsymbol{B}$, we cannot claim that the above properties hold. However, we know that with probability 1 any trajectory hits the set $\boldsymbol{B}$, and by (2.5), the first hitting time $\tau_{\boldsymbol{B}}$ has a finite exponential moment.

These observations make it plausible that for any initial point $\boldsymbol{u} \in \boldsymbol{X}$ the trajectories $S_t(\boldsymbol{u}, \omega)$ and $S_t'(\boldsymbol{u}, \omega)$ converge to each other exponentially fast. In fact, we have the following result, whose proof is given in the next subsection.

Theorem 2.3. *Let $S_t(u, \omega)$ be a continuous Markov random dynamical system satisfying the coupling hypothesis, and let $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ be a mixing extension for S_t. Then there is a random time $\ell = \ell(\boldsymbol{u}, \omega)$ such that*

$$\|S_t(\boldsymbol{u}, \omega) - S_t'(\boldsymbol{u}, \omega)\| \leqslant C_1 e^{-\beta(t-\ell(\boldsymbol{u},\omega))} \quad \text{for } t \geqslant \ell(\boldsymbol{u}, \omega), \tag{2.9}$$

$$\mathbb{E}_{\boldsymbol{u}} e^{\alpha\ell} \leqslant C_1\bigl(g(\|u\|) + g(\|u'\|)\bigr), \tag{2.10}$$

where $\boldsymbol{u} \in \boldsymbol{X}$ is an arbitrary initial point, $g(r)$ is the function in Definition 2.2, and C_1, α, and β are positive constants independent of $\boldsymbol{u}$ and t. If, in addition, there is an increasing function $\tilde{g}(r) \geqslant 1$, $r \geqslant 0$, such that

$$\mathbb{E}_u\, g\bigl(\|S_t\|\bigr) \leqslant \tilde{g}(\|u\|) \quad \text{for } u \in X,\ t \geqslant 0, \tag{2.11}$$

then $S_t(u, \omega)$ is exponentially mixing, and the inequality (2.1) *holds with*

$$V(r) = 3C_1\bigl(g(r) + \tilde{g}(0)\bigr). \tag{2.12}$$

2.2. Proof of Theorem 2.3.

We first note that the inequalities (2.9), (2.10), and (2.11) imply that $S_t(u,\omega)$ is exponentially mixing. Indeed, to prove this, let us show that for any $u, u' \in X$ we have

$$\big\|P_t(u,\cdot) - P_t(u',\cdot)\big\|_{\mathcal{L}}^* \leqslant 3C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\gamma t}, \quad t \geqslant 0. \tag{2.13}$$

To this end, we fix an arbitrary functional $f \in \mathcal{L}(X)$ with $\|f\|_{\mathcal{L}} \leqslant 1$ and note that, in view of (1.6),

$$\begin{aligned}\big|\big(f, P_t(u,\cdot) - P_t(u',\cdot)\big)\big| &= \big|\mathbb{E}_{\boldsymbol{u}}\big(f(S_t) - f(S_t')\big)\big| \leqslant \mathbb{E}_{\boldsymbol{u}}\big|f(S_t) - f(S_t')\big| \\ &\leqslant 2\mathbb{P}_{\boldsymbol{u}}\big\{\ell > \tfrac{t}{2}\big\} + \mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\ell \leqslant \frac{t}{2}\}}\big|f(S_t) - f(S_t')\big|\big\}.\end{aligned} \tag{2.14}$$

In view of (2.10) and the Chebyshev inequality, we have

$$\mathbb{P}_{\boldsymbol{u}}\big\{\ell > \tfrac{t}{2}\big\} \leqslant C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\frac{\alpha t}{2}}. \tag{2.15}$$

Furthermore, from the condition $\|f\|_{\mathcal{L}} \leqslant 1$ and the inequality (2.9) it follows that the second term on the right-hand side of (2.14) does not exceed

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\ell \leqslant \frac{t}{2}\}}\|S_t - S_t'\|\big\} \leqslant C_1 e^{-\frac{\beta t}{2}}. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14), we obtain

$$\big|\big(f, P_t(u,\cdot) - P_t(u',\cdot)\big)\big| \leqslant 2C_1\big(g(\|u\|) + g(\|u'\|)\big)\, e^{-\frac{\alpha t}{2}} + C_1 e^{-\frac{\beta t}{2}},$$

which implies the required inequality (2.13) with $\gamma = \frac{1}{2}(\alpha \wedge \beta)$.

We now use (2.13) to show that S_t is exponentially mixing. Let us fix arbitrary points $u, u' \in X$ and a functional $f \in \mathcal{L}(X)$ such that $\|f\|_{\mathcal{L}} \leqslant 1$. By the Kolmogorov–Chapman relation and the inequality (2.13), for $t \leqslant s$ we have

$$\begin{aligned}\big|\big(f, P_t(u,\cdot) - P_s(u',\cdot)\big)\big| &= \bigg|\int_X P_{s-t}(u',dz)\int_X \big(P_t(u,dv) - P_t(z,dv)\big) f(v)\bigg| \\ &\leqslant 3C_1 e^{-\gamma t}\int_X P_{s-t}(u',dz)\big[g(\|u\|) + g(\|z\|)\big] \\ &= 3C_1 e^{-\gamma t}\big[g(\|u\|) + \mathbb{E}_{u'}\, g(\|S_{s-t}\|)\big].\end{aligned}$$

Taking into account (2.11), we conclude that

$$\big\|P_t(u,\cdot) - P_s(u',\cdot)\big\|_{\mathcal{L}}^* \leqslant 3C_1\big(g(\|u\|) + \tilde{g}(\|u'\|)\big)\, e^{-\gamma t}. \tag{2.17}$$

By the Prokhorov theorem (see [**5**, Corollary 11.5.5]), $\mathcal{P}(X)$ is a complete metric space with respect to the norm $\|\cdot\|_{\mathcal{L}}^*$. Hence $P_t(u,\cdot)$ converges as $t \to +\infty$ to a measure $\mu \in \mathcal{P}(X)$, which does not depend on u and is stationary. Setting $u' = 0$ in (2.17) and passing to the limit as $s \to +\infty$, we obtain the inequality (2.1) with V given by (2.12).

Thus, we need to establish the inequalities (2.9) and (2.10). Their proof is divided into four steps.

Step 1. We introduce the stopping time

$$\rho = \sigma + \tau_{\boldsymbol{B}} \circ \theta_\sigma = \sigma(\boldsymbol{u}, \omega) + \tau_{\boldsymbol{B}}\big(\boldsymbol{S}_{\sigma(\boldsymbol{u},\omega)}(\boldsymbol{u}, \omega), \theta_{\sigma(\boldsymbol{u},\omega)}\omega)\big). \tag{2.18}$$

In other words, we wait until the first instant σ when the trajectories S_t and S_t' "stop converging" to each other and denote by ρ the first hitting time of $\boldsymbol{B}$ after σ. Let δ, δ_1, and δ_2 be the constants in (2.5), (2.6), and (2.7). We claim that for any $\boldsymbol{u} \in \boldsymbol{B}$

$$\mathbb{P}_{\boldsymbol{u}}\{\rho = \infty\} \geqslant \delta_1, \tag{2.19}$$

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\alpha\rho}\big\} \leqslant a, \tag{2.20}$$

where $\alpha \leqslant \delta_2 \wedge \delta$ and $a < 1$ are positive constants independent of $\boldsymbol{u}$. Indeed, the definition of $\rho(\boldsymbol{u}, \omega)$ (see (2.18)) implies that $\{\rho = \infty\} = \{\sigma = \infty\}$, and therefore (2.19) is an immediate consequence of (2.6).

To prove (2.20), we first show that

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\delta_3\rho}\big\} \leqslant M \quad \text{for any } \boldsymbol{u} \in \boldsymbol{B}, \tag{2.21}$$

where $\delta_3 = \frac{(q-1)(\delta_2 \wedge \delta)}{q}$ and $M > 0$ is a constant independent of $\boldsymbol{u}$. Indeed, using the relation (2.18), the strong Markov property, and the inequality (2.5), we derive

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\delta_3\rho}\big\} = \mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\sigma<\infty\}} e^{\delta_3\sigma}\big(\mathbb{E}_{\boldsymbol{S}_\sigma}\, e^{\delta_3\tau_{\boldsymbol{B}}}\big)\big\} \leqslant \mathbb{E}\big\{I_{\{\sigma<\infty\}} e^{\delta_3\sigma} G(\boldsymbol{S}_\sigma)\big\}.$$

Combining this with (2.7) and (2.8), we conclude that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\delta_3\rho}\big\} &\leqslant \big(\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\sigma<\infty\}} e^{\delta_2\sigma}\big\}\big)^{\frac{q-1}{q}} \big(\mathbb{E}_{\boldsymbol{u}}\,\big\{I_{\{\sigma<\infty\}} G(\boldsymbol{S}_\sigma)^q\big\}\big)^{\frac{1}{q}} \\ &\leqslant (c^{q-1}K)^{\frac{1}{q}} =: M.\end{aligned}$$

To derive (2.20), let us set $\alpha = \varepsilon\delta_3$ and note that, in view of (2.19) and (2.21), we have

$$\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\alpha\rho}\big\} \leqslant \big(\mathbb{P}_{\boldsymbol{u}}\{\rho < \infty\}\big)^{1-\varepsilon} \big(\mathbb{E}_{\boldsymbol{u}}\big\{I_{\{\rho<\infty\}} e^{\delta_3\rho}\big\}\big)^{\varepsilon} \leqslant (1-\delta_1)^{1-\varepsilon} M^{\varepsilon}.$$

The right-hand side of this inequality is less than 1 if $\varepsilon > 0$ is sufficiently small.

Step 2. We now consider the iterations of ρ. Namely, we define a sequence of stopping times $\rho_k = \rho_k(\boldsymbol{u},\omega)$ by the formulas

$$\rho_0 = \tau_{\boldsymbol{B}}, \quad \rho_k = \rho_{k-1} + \rho \circ \theta_{\rho_{k-1}}, \quad k \geqslant 1.$$

We claim that

$$\mathbb{E}_{\boldsymbol{u}} \bigl\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \bigr\} \leqslant a^k G(\boldsymbol{u}) \quad \text{for any } \boldsymbol{u} \in \boldsymbol{X}. \tag{2.22}$$

Indeed, since $\boldsymbol{S}_{\rho_k(\boldsymbol{u},\omega)}(\boldsymbol{u},\omega) \in \boldsymbol{B}$, the inequality (2.20) and the strong Markov property imply that

$$\begin{aligned} \mathbb{E}_{\boldsymbol{u}} \bigl\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \bigr\} &\leqslant \mathbb{E}_{\boldsymbol{u}} \Bigl\{ I_{\{\rho_{k-1} < \infty\}} e^{\alpha \rho_{k-1}} \sup_{\boldsymbol{v} \in \boldsymbol{B}} \mathbb{E}_{\boldsymbol{v}} \bigl(I_{\{\rho < \infty\}} e^{\alpha \rho} \bigr) \Bigr\} \\ &\leqslant a \, \mathbb{E}_{\boldsymbol{u}} \bigl\{ I_{\{\rho_{k-1} < \infty\}} e^{\alpha \rho_{k-1}} \bigr\} \leqslant a^k \mathbb{E}_{\boldsymbol{u}} e^{\alpha \tau_{\boldsymbol{B}}}. \end{aligned}$$

The required inequality (2.22) follows now from (2.5) and the fact that $\alpha \leqslant \delta$.

Step 3. We now note that, if $\rho_k(\boldsymbol{u},\omega) < \infty$ and $\rho_{k+1}(\boldsymbol{u},\omega) = \infty$ for an integer $k \geqslant 0$, then

$$\|S_t(\boldsymbol{u},\omega) - S'_t(\boldsymbol{u},\omega)\| \leqslant C \, e^{-\beta(t - \rho_k(\boldsymbol{u},\omega))} \quad \text{for } t \geqslant \rho_k(\boldsymbol{u},\omega). \tag{2.23}$$

For any $\boldsymbol{u} \in \boldsymbol{X}$ let us set

$$\bar{k} = \bar{k}(\boldsymbol{u},\omega) = \sup\{k \geqslant 0 : \rho_k(\boldsymbol{u},\omega) < \infty\}.$$

We wish to show that

$$\bar{k} < \infty \quad \text{for } \mathbb{P}_{\boldsymbol{u}}\text{-almost every } \omega. \tag{2.24}$$

To this end, note that, in view of (2.19) and the strong Markov property,

$$\mathbb{P}_{\boldsymbol{u}}\{\rho_k < \infty\} \leqslant (1-\delta_1)\mathbb{P}_{\boldsymbol{u}}\{\rho_{k-1} < \infty\} \leqslant (1-\delta_1)^k \mathbb{P}_{\boldsymbol{u}}\{\rho_0 < \infty\} \leqslant (1-\delta_1)^k.$$

Hence the Borel–Cantelli lemma implies (2.24).

Step 4. Let us set

$$\ell = \ell(\boldsymbol{u},\omega) = \begin{cases} \rho_{\bar{k}(\boldsymbol{u},\omega)}(\boldsymbol{u},\omega) & \text{if } \bar{k}(\boldsymbol{u},\omega) < \infty, \\ +\infty & \text{if } \bar{k}(\boldsymbol{u},\omega) = \infty. \end{cases}$$

The inequality (2.9) follows immediately from (2.23), the definition of ρ_k, and the fact that $\rho_{\ell+1} = \infty$. To prove (2.10), we write

$$\mathbb{E}_{\boldsymbol{u}} e^{\alpha \ell} = \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{u}} \bigl\{ I_{\{\bar{k} = k\}} e^{\alpha \rho_k} \bigr\} \leqslant \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{u}} \bigl\{ I_{\{\rho_k < \infty\}} e^{\alpha \rho_k} \bigr\} \leqslant (1-a)^{-1} G(\boldsymbol{u}),$$

where we used the inequality (2.22) and the fact that $\ell(\boldsymbol{u},\omega) < \infty$ for $\mathbb{P}_{\boldsymbol{u}}$-a.a. ω. This completes the proof of Theorem 2.3.

Remark 2.4. Analyzing the proof given above, it is not difficult to see that Theorem 2.3 remains valid if $\sigma(\boldsymbol{u},\omega)$ is replaced with any other stopping time $\tilde{\sigma} \leqslant \sigma$. In other word, if the inequalities (2.6)–(2.9) hold with σ replaced by $\tilde{\sigma}$, then the conclusion of Theorem 2.3 is true. To see this, it suffices to repeat the arguments above, replacing everywhere σ by $\tilde{\sigma}$.

3. Dissipative Random Dynamical Systems and Their Extensions

In this section, we give sufficient conditions for the existence of an extension satisfying the inequality (2.5). These results will be used in the next section to prove exponential mixing for the complex Ginzburg–Landau equation.

3.1. Lyapunov function.

Let $S_t(u,\omega)$ be a Markov random dynamical system in a separable Banach space X, and let $F(u) \geqslant 1$ be a continuous functional on X tending to $+\infty$ as $\|u\| \to \infty$. Suppose that S_t satisfies the following condition.

(H_1) *Lyapunov function.* There are positive constants t_*, R_*, C_*, and $a < 1$ such that

$$\mathbb{E}_u F\big(S_{t_*}\big) \leqslant a\,F(u) \qquad \text{for } \|u\| \geqslant R_*, \tag{3.1}$$

$$\mathbb{E}_u F\big(S_t\big) \leqslant C_* \qquad \text{for } \|u\| \leqslant R_*,\ t \geqslant 0. \tag{3.2}$$

In what follows, we call F a *Lyapunov function* for S_t. An important property of a Markov random dynamical system possessing a Lyapunov function is that the first hitting time of sufficiently large balls in the phase space is almost surely finite for any initial condition and has a finite exponential moment. Namely, we have the following result.

Proposition 3.1. *Let $S_t(u,\omega)$ be a Markov random dynamical system satisfying Hypothesis* (H_1)*, and let $\tau_R(u,\omega)$ be the first hitting time of the ball $B_R = \{u \in X : \|u\| \leqslant R\}$, where $R \geqslant R_*$. Then*

$$\mathbb{P}_u\{\tau_R < \infty\} = 1 \quad \textit{for all } u \in X. \tag{3.3}$$

Moreover, there are positive constants δ and C independent of R and u such that

$$\mathbb{E}_u \exp(\delta\tau_R) \leqslant 1 + CK_R^{-1}F(u), \tag{3.4}$$

where

$$K_R = \inf_{\|v\| \geqslant R} F(v). \tag{3.5}$$

Proposition 3.1 can be established by a standard argument (see [**22**]). However, for the sake of completeness, we give its proof.

Proof of Proposition 3.1. *Step 1.* The result is trivial for $\|u\| \leqslant R$ since, in this case, $\tau_R(u,\omega) = 0$ for $\mathbb{P}_u$-almost every ω. We fix arbitrary $u \in X$ with $\|u\| > R$ and consider an auxiliary stopping time defined by the formula

$$\bar\tau = \bar\tau(u,\omega) = \min\bigl\{t = mt_* : \|S_t\| \leqslant R,\ m \geqslant 0 \text{ is an integer}\bigr\}.$$

For any integer $k \geqslant 0$ and $v \in X$ we set

$$p_k(v) = \mathbb{E}_v\bigl\{I_{\{\bar\tau > kt_*\}} F(S_{kt_*})\bigr\}. \tag{3.6}$$

We claim that

$$p_k(u) \leqslant a^k F(u) \quad \text{for all } k \geqslant 0. \tag{3.7}$$

Indeed, the Markov property (1.1) and the inequality (3.1) imply that

$$\begin{aligned} p_{k+1}(u) &\leqslant \mathbb{E}_u\bigl\{I_{\{\bar\tau > kt_*\}} \mathbb{E}_u\bigl(F(S_{(k+1)t_*}) \,|\, \mathcal{F}_{kt_*}\bigr)\bigr\} \\ &= \mathbb{E}_u\bigl\{I_{\{\bar\tau > kt_*\}} \mathbb{E}_{S_{kt_*}} F(S_{t_*})\bigr\} \\ &\leqslant a\,\mathbb{E}_u\bigl\{I_{\{\bar\tau > kt_*\}} F(S_{kt_*})\bigr\} = a p_k(u), \end{aligned} \tag{3.8}$$

where we used the nonnegativity of F and the fact that $\|S_{kt_*}\| > R \geqslant R_*$ on the set $\{\bar\tau > kt_*\}$. Iterating (3.8) and noting that $\mathbb{E}_u\{I_{\{\bar\tau>0\}} F(S_0)\} \leqslant F(u)$, we arrive at (3.7).

Step 2. From (3.6) and (3.7) it follows that

$$\mathbb{P}_u\{\bar\tau > kt_*\} \leqslant K_R^{-1} \mathbb{E}_u\bigl\{I_{\{\bar\tau > kt_*\}} F(S_{kt_*})\bigr\} \leqslant a^k K_R^{-1} F(u). \tag{3.9}$$

Combining this with the Borel–Cantelli lemma, we see that

$$\mathbb{P}_u\{\bar\tau < \infty\} = 1 \quad \text{for any } u \in X. \tag{3.10}$$

Furthermore, if $\delta > 0$ is so small that $b := e^{\delta t_*} a < 1$, then, by (3.9), we have

$$\begin{aligned} \mathbb{E}_u e^{\delta\bar\tau} &\leqslant 1 + \sum_{k=1}^{\infty} \mathbb{E}_u\bigl\{I_{\{\bar\tau = kt_*\}} e^{\delta\bar\tau}\bigr\} \leqslant 1 + \sum_{k=1}^{\infty} e^{\delta kt_*} \mathbb{P}_u\{\bar\tau > (k-1)t_*\} \\ &\leqslant 1 + K_R^{-1} F(u) \sum_{k=1}^{\infty} e^{\delta kt_*} a^{k-1} = 1 + C K_R^{-1} F(u), \end{aligned} \tag{3.11}$$

where $C = e^{\delta t_*}(1-b)^{-1}$. It remains to note that $\bar{\tau} \geqslant \tau_R$, and hence (3.10) and (3.11) imply (3.3) and (3.4). $\square$

A result similar to Proposition 3.1 is true for any extension of S_t. More precisely, let $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ be an extension of a Markov random dynamical system satisfying Hypothesis (H_1), and let [3)]

$$\tau_R = \min\{t \geqslant 0 : \|S_t(\boldsymbol{u}, \omega)\| \vee \|S'_t(\boldsymbol{u}, \omega)\| \leqslant R\}. \tag{3.12}$$

Let $R^* > 0$ be the smallest constant such that $K_{R^*} \geqslant \frac{2C_*}{1-a}$, where a and C_* are the constants in Hypothesis (H_1) and K_R is defined by (3.5). The assertion below can be established by repeating the arguments in the proof of Proposition 3.1.

Proposition 3.2. *Let $S_t(u, \omega)$ be a Markov random dynamical system satisfying Hypothesis* (H_1)*, and let $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ be its extension. Then there are positive constants δ and C such that for any $\boldsymbol{u} \in \boldsymbol{X}$ and $R \geqslant R^*$ we have*

$$\mathbb{P}_{\boldsymbol{u}}\{\tau_R < \infty\} = 1, \tag{3.13}$$

$$\mathbb{E}_{\boldsymbol{u}} \exp(\delta \tau_R) \leqslant 1 + C K_R^{-1}\big(F(u) + F(u')\big). \tag{3.14}$$

3.2. Dissipation.

Let $S_t(u, \omega)$ be a continuous Markov random dynamical system in a separable Banach space X, and let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u}, \omega)$ be its extension on an interval $[0, T]$. Suppose that $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}'_t)$ satisfies the following condition.

(H_2) *Dissipation.* For any $R > 0$ there is a constant $q \in (0,1)$ and an increasing function $\varepsilon(d) > 0$ defined for $d > 0$ such that for any $\boldsymbol{u} = (u, u') \in \boldsymbol{X}$ with $\|u\| \vee \|u'\| \leqslant R$ and any $d > 0$ we have

$$\mathbb{P}_{\boldsymbol{u}}\big\{\|\mathcal{R}_T(\boldsymbol{u}, \cdot)\| \vee \|\mathcal{R}'_T(\boldsymbol{u}, \cdot)\| \leqslant \{q(\|u'\| \vee \|u'\|)\} \vee d\big\} \geqslant \varepsilon(d). \tag{3.15}$$

In other words, the dissipation condition (H_2) means that for any $d > 0$, with positive probability, any ball in X of radius $R \geqslant d/q$ centered at zero is pushed into a ball of radius qR by the maps $\mathcal{R}_T$ and $\mathcal{R}'_T$. Therefore, it is reasonable to expect that, if $\boldsymbol{S}_t$ is the extension of S_t constructed by iteration of $\boldsymbol{\mathcal{R}}_t$ (see (1.7) and (1.8)), then for any initial point $\boldsymbol{u} \in \boldsymbol{X}$ the trajectory $\boldsymbol{S}_t(\boldsymbol{u}, \omega)$ will hit, in a finite time, any ball of given radius centered at zero. We have in fact the following result which shows that the existence

[3)] The stopping time (3.12) is different from the one defined in Proposition 3.1 for the original random dynamical system. However, we retained the same notation since they play similar roles for $\boldsymbol{S}_t$ and S_t.

of a Lyapunov function combined with the dissipation property (H_2) implies that the first hitting time of any ball centered at zero has a finite exponential moment (cf. (2.5)).

Proposition 3.3. *Let $S_t(u,\omega)$ be a Markov random dynamical system possessing a Lyapunov function $F(u)$ in the sense of* (H_1), *and let $\boldsymbol{\mathcal{R}}_t(\boldsymbol{u},\omega)$ be its extension defined on an interval $[0,T]$ and satisfying Hypothesis* (H_2). *Then for any $d>0$ there are positive constants C and ν such that for the extension $\boldsymbol{S}_t$ constructed by iteration of $\boldsymbol{\mathcal{R}}_t$ we have*

$$\mathbb{E}_{\boldsymbol{u}} \exp(\nu\tau_d) \leqslant C\big(F(u)+F(u')\big), \quad \boldsymbol{u}=(u,u')\in \boldsymbol{X}, \tag{3.16}$$

Proof. We first describe the main idea, which is well known; for instance, see Sections 3.7 and 4.2 in [**11**] or Section 13 in [**30**]. By Proposition 3.2, the first hitting time of the set

$$\boldsymbol{B}_R=\{\boldsymbol{u}\in \boldsymbol{X} : \|u\|\vee\|u'\|\leqslant R\} \tag{3.17}$$

has a finite exponential moment for $R\geqslant R^*$, and by the dissipation property (H_2), each time the process $\boldsymbol{S}_t$ is in $\boldsymbol{B}_R$, with positive probability it hits $\boldsymbol{B}_d$ in finite (deterministic) time. Combining these two observations with the Markov property, we can prove the required result. An accurate proof is divided into four steps.

Step 1. Let R^* and q be the constants in Proposition 3.2 and Hypothesis (H_2). We fix arbitrary $d>0$ and set $l_d=\min\{l\geqslant 0 : q^l R^*\leqslant d\}$. It follows from the inequality (3.15) and the Markov property that for any $\boldsymbol{u}\in \boldsymbol{B}_{R^*}$ we have

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{l_dT}\in \boldsymbol{B}_d\big\}\geqslant p_d:=\varepsilon(d)^{l_d}>0. \tag{3.18}$$

Step 2. We set $\tau=\tau_{R^*}$ and define two sequences of stopping times by the formulas

$$\rho_1'=\tau, \quad \rho_1=\tau+l_dT, \quad \rho_m'=\rho_{m-1}+\tau\circ\theta_{\rho_{m-1}}, \quad \rho_m=\rho_m'+l_dT, \quad m\geqslant 2.$$

Consider the events $\Gamma_m=\big\{\boldsymbol{S}_{\rho_n}\notin \boldsymbol{B}_d \text{ for } n=1\dots,m\big\}$. Let us show that for any $\boldsymbol{u}\in\boldsymbol{X}$ the sequence $P_m(\boldsymbol{u})=\mathbb{P}_{\boldsymbol{u}}(\Gamma_m)$ satisfies the inequality

$$P_m(\boldsymbol{u})=(1-p_d)^m, \quad m\geqslant 1. \tag{3.19}$$

Indeed, by the strong Markov property, for any $m\geqslant 1$ we have[4)]

$$\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{\rho_m}\notin \boldsymbol{B}_d \,\big|\, \mathcal{F}_{\rho_m'}\big\}=\mathbb{P}_{\boldsymbol{S}(\rho_m')}\big\{\boldsymbol{S}_{l_dT}\notin \boldsymbol{B}_d\big\}\leqslant 1-p_d, \tag{3.20}$$

[4)] We write $\boldsymbol{S}(\rho_m')$ instead of $\boldsymbol{S}_{\rho_m'}$ to avoid a double subscript.

where we used the inequality (3.18) and the fact that $\boldsymbol{S}_{\rho'_m} \in \boldsymbol{B}_{R^*}$. Therefore, using again the strong Markov property, we derive

$$P_m(\boldsymbol{u}) = \mathbb{E}_{\boldsymbol{u}}\Big(I_{\Gamma_{m-1}}\mathbb{P}_{\boldsymbol{u}}\big\{\boldsymbol{S}_{\rho_m} \notin \boldsymbol{B}_d \,\big|\, \mathcal{F}_{\rho'_m}\big\}\Big) \leqslant (1-p_d)\,P_{m-1}(\boldsymbol{u}).$$

Iterating this inequality and using (3.20) with $m=1$, we obtain (3.19).

Step 3. We now show that for any $d>0$ there is a constant $K \geqslant 1$ such that

$$\mathbb{E}_{\boldsymbol{u}} e^{\delta\rho_m} \leqslant K^m\big(F(u)+F(u')\big), \quad m \geqslant 1, \tag{3.21}$$

where $\delta>0$ is the constant in (3.14). Indeed, applying the strong Markov property and the inequalities (3.14) and (3.2) (with $t = l_d T$), we derive

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{u}} e^{\delta\rho'_m} &= \mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta\rho_{m-1}}\mathbb{E}_{\boldsymbol{S}(\rho_{m-1})}(e^{\delta\tau})\big\}\\
&\leqslant C_1\mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta\rho_{m-1}}\big(F(S_{\rho_{m-1}})+F(S'_{\rho_{m-1}})\big)\big\}\\
&\leqslant C_1 e^{\delta l_d T}\mathbb{E}_{\boldsymbol{u}}\big\{e^{\delta\rho'_{m-1}}\mathbb{E}_{\boldsymbol{S}(\rho'_{m-1})}\big(F(S_{l_dT})+F(S'_{l_dT})\big)\big\}\\
&\leqslant C_2 e^{\delta l_d T}\mathbb{E}_{\boldsymbol{u}} e^{\delta\rho'_{m-1}},
\end{aligned}$$

where we used the fact that $\boldsymbol{S}_{\rho_{m-1}} \in \boldsymbol{B}_{R^*}$. Iterating this inequality and using again (3.14), we obtain (3.21).

Step 4. We can now prove the inequality (3.16) with sufficiently small $\nu>0$. To this end, we define the random integer

$$\hat{n} = \min\{n \geqslant 1 : \boldsymbol{S}_{\rho_n} \in \boldsymbol{B}_d\}$$

and note that $\tau_d \leqslant \rho_{\hat{n}}$. Moreover, from (3.19) and the Borel–Cantelli lemma it follows that $\mathbb{P}_{\boldsymbol{u}}\{\hat{n}<\infty\}=1$ for any $\boldsymbol{u} \in \boldsymbol{X}$. Hence for any $\nu>0$

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{u}} e^{\nu\tau_d} \leqslant \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_{\hat{n}}} &= \sum_{n=1}^{\infty}\mathbb{E}_{\boldsymbol{u}}\big(I_{\{\hat{n}=n\}}e^{\nu\rho_n}\big) \leqslant \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_1} + \sum_{n=2}^{\infty}\mathbb{E}_{\boldsymbol{u}}\big(I_{\Gamma_{n-1}}e^{\nu\rho_n}\big)\\
&\leqslant \mathbb{E}_{\boldsymbol{u}} e^{\nu\rho_1} + \sum_{m=1}^{\infty} P_m(\boldsymbol{u})^{\frac12}\big(\mathbb{E}_{\boldsymbol{u}} e^{2\nu\rho_{m+1}}\big)^{\frac12}\\
&\leqslant K\Big(1+\sum_{m=1}^{\infty}(1-p_d)^{\frac{m}{2}}K^{\frac{\nu m}{\delta}}\Big)\big(F(u)+F(u')\big).
\end{aligned} \tag{3.22}$$

Comparing this inequality with (3.19) and (3.21), we see that for a sufficiently small $\nu>0$ the right-hand side of (3.22) can be estimated by $C(F(u)+F(u'))$. This completes the proof of Proposition 3.3. □

4. Complex Ginzburg–Landau Equation

4.1. Cauchy problem and a priori estimates.

Let $D \subset \mathbb{R}^n$ ($n = 3$ or 4) be a bounded domain with smooth boundary ∂D, and let $L^2 = L^2(D, \mathbb{C})$ be the space of square-integrable complex-valued functions on D. We regard L^2 as a real Hilbert space and endow it with the scalar product

$$(u, v) = \operatorname{Re} \int_D u(x)\bar{v}(x)\,dx$$

and the corresponding norm $\|\cdot\|$. Let $\{e_j\}$ be a complete set of L^2-normalized eigenfunctions of the Dirichlet Laplacian, and let $\{\alpha_j\}$ be the corresponding set of eigenvalues indexed in an increasing order.

We consider the problem

$$\dot{u} - (\nu + i)\Delta u + i|u|^{2p}u = h(x) + \eta(t, x), \tag{4.1}$$

$$u\big|_{\partial D} = 0, \tag{4.2}$$

$$u(0, x) = u_0(x), \tag{4.3}$$

where $\nu > 0$ and $p \geqslant 0$ are some constants, $h \in L^2$ is a deterministic function, and η is an H^1-valued random force. More precisely, we assume that

$$\eta(t, x) = \frac{\partial}{\partial t}\zeta(t, x), \quad \zeta(t, x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{4.4}$$

where $\beta_j(t) = \beta_{j1}(t) + i\beta_{j2}(t)$ are complex-valued independent Brownian motions and $b_j \geqslant 0$ are some constant satisfying the condition

$$B_1 := \sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty.$$

In what follows, we always assume that $0 \leqslant p \leqslant \frac{2}{n}$. For any function $u(t, x)$ let us set

$$\mathcal{E}_u(t) = \|u(t)\|^2 + \nu \int_0^t \|u(s)\|_1^2 ds. \tag{4.5}$$

The theorem below establishes the well-posedness of the problem (4.1)–(4.3) in appropriate functional spaces. We refer the reader to [**12, 21, 16, 28**] for proofs of similar (and more general) results.

Theorem 4.1. *Suppose that the above-mentioned conditions are fulfilled. Let u_0 be an L^2-valued random variable that is independent of ζ and satisfies the condition $\mathbb{E}\,\|u_0\|^2 < \infty$. Then the following statements hold.*

(i) *There is a random process $u(t) = u(t,x)$, $t \geqslant 0$, whose almost every trajectory belongs to the space $\mathcal{X} := C(\mathbb{R}_+; L^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H_0^1)$ and satisfies Equations* (4.1) *and* (4.3) *in the sense that*

$$u(t) = u_0 + \int_0^t \big((\nu + i)\Delta u(s) - i|u(s)|^{2p}u(s)\big)\,ds + th + \zeta(t), \quad t \geqslant 0.$$

Moreover, the random process $u(t,x)$ is adapted to the filtration $\mathcal{F}_t$ generated by u_0 and ζ.

(ii) *The process $u(t)$ constructed in* (i) *is unique in the sense that if $\tilde{u}(t)$ is another random process satisfying* (i)*, then with probability* 1 *we have $u(t) = \tilde{u}(t)$ for all $t \geqslant 0$.*

(iii) *We have the a priori estimates*

$$\mathbb{E}\,\|u(t)\|^2 + \nu \int_0^t \mathbb{E}\,\|u(s)\|_1^2 ds \leqslant \mathbb{E}\,\|u_0\|^2 + Ct \quad \textit{for } t \geqslant 0, \tag{4.6}$$

$$\mathbb{P}\Big\{\sup_{t\geqslant 0}\big(\mathcal{E}_u(t) - Lt\big) \geqslant \|u_0\|^2 + \rho\Big\} \leqslant e^{-\varkappa\rho} \quad \textit{for } \rho > 0, \tag{4.7}$$

where C, L, and $\varkappa$ are positive constants independent of u_0.

4.2. Formulation of the result and an open question.

We denote by $S_t(u_0,\omega)$ the solution of (4.1)–(4.3) constructed in Theorem 4.1. Using a standard argument (see, for example, [**12, 21**]), it is not difficult to show that $S_t(u_0,\omega)$ can be regarded as a Markov random dynamical system in L^2, and we denote by $(u_t, \mathbb{P}_u)$ the corresponding Markov family (cf. Section 1.1). The transition function and Markov operators associated with $(u_t, \mathbb{P}_u)$ will be denoted by $P_t(u,\Gamma)$, $\mathfrak{P}_t$, and $\mathfrak{P}_t^*$. The following theorem is the main result of this section.

Theorem 4.2. *Suppose that the assumptions of Theorem* 4.1 *are satisfied and*

$$b_j \neq 0 \quad \textit{for all } j \geqslant 1. \tag{4.8}$$

Then for any $\nu > 0$ the Markov random dynamical system associated with (4.1), (4.2) *has a unique stationary measure $\mu \in \mathcal{P}(L^2)$. Moreover, there* *are positive constants C and γ such that*

$$|\mathfrak{P}_t f(u) - (f,\mu)| \leqslant C\,\|f\|_{\mathcal{L}}(1+\|u\|^2)\,e^{-\gamma t} \quad \textit{for any } t \geqslant 0,\ u \in L^2, \quad (4.9)$$

where $f \in \mathcal{L}(L^2)$ is an arbitrary functional.

To prove this theorem, we construct an extension $\boldsymbol{S}_t$ for S_t that satisfies the coupling hypothesis in the sense of Definition 2.2, and the application of Theorem 2.3 will imply the required result. Moreover, using the regularizing property for the complex Ginzburg–Landau equation and the associated Markov semigroup (see [**28**, Proposition 4]), it is not difficult to show that the stationary measure μ is concentrated on the space H^1, and the exponential convergence to μ holds also for continuous functionals on H_0^1. At the same time, the following question remains open.

Open Question. The complex Ginzburg–Landau equation is well posed in the space H_0^1 for $n = 3$ or $n = 4$ and $p \leqslant \frac{2}{n-2}$. Prove the uniqueness of a stationary measure and exponential mixing property for these values of p.

The rest of this section is organized as follows. In Section 4.3, we construct an extension for S_t. Section 4.4 is devoted to verification of Hypotheses (H_1) and (H_2) (see Section 3). In Section 4.5, we prove the inequalities (2.6) and (2.7). The proof of Theorem 4.2 is completed in Section 4.6.

4.3. Construction of an extension.

We wish to construct an extension for S_t that satisfies the coupling hypothesis described in Definition 2.2. As was explained in Section 1.2, if we have an extension $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}_t')$ on a time interval $[0,T]$, then its iteration results in an extension defined on the half-line $\mathbb{R}_+$. Our construction of $\boldsymbol{\mathcal{R}}_t$ will depend on $T \geqslant 1$ and an integer $N \geqslant 1$. Both parameters will be fixed later.

Step 1. Let H_N be the $2N$-dimensional subspace in L^2 spanned by the vectors e_j, ie_j, $1 \leqslant j \leqslant N$, and let $H_N^\perp$ be its orthogonal complement in L^2. Denote by P_N and Q_N the orthogonal projections in L^2 onto the subspaces H_N and $H_N^\perp$ respectively.

We set $v = \mathsf{P}_N u$, $w = \mathsf{Q}_N u$ and rewrite Equation (1.4) in the form

$$\dot v - (\nu + i)\Delta v + F_N(v+w) = \mathsf{P}_N h + \dot\varphi(t), \qquad (4.10)$$

$$\dot w - (\nu + i)\Delta w + G_N(v+w) = \mathsf{Q}_N h + \dot\psi(t), \qquad (4.11)$$

where $\varphi = \mathsf{P}_N\zeta$, $\psi = \mathsf{Q}_N\zeta$, $F_N(u) = i\mathsf{P}_N(|u|^{2p}u)$, $G_N(u) = i\mathsf{Q}_N(|u|^{2p}u)$. Equations (4.10) and (4.11) are supplemented with the initial conditions

$$v(0) = v_0, \tag{4.12}$$

$$w(0) = w_0, \tag{4.13}$$

where $v_0 \in H_N$ and $w_0 \in H_N^\perp$. Using standard arguments, it is not difficult to check that for any functions $w_0 \in H_N^\perp$, $v \in C(0,T;H_N)$, $\psi \in C(0,T;H_N^\perp \cap H_0^1)$ the problem (4.11), (4.13) has a unique solution $w \in \mathcal{X}_N(T) := C(0,T;H_N^\perp) \cap L^2(0,T;H_N^\perp \cap H_0^1)$. We denote by

$$\mathcal{W} : H_N^\perp \times C(0,T;H_N) \times C(0,T;H_N^\perp \cap H_0^1) \to \mathcal{X}_N(T), \quad (w_0, v, \psi) \mapsto w,$$

the resolving operator for the problem (4.11), (4.13) and by $\mathcal{W}_t$ its restriction to the time t. The operators $\mathcal{W}$ and $\mathcal{W}_t$ are uniformly Lipschitz with respect to (w_0, v, ψ) on bounded subsets, and it is easy to see that $\mathcal{W}_t(w_0, v, \psi)$ depends only on the restriction of v and ψ to the interval $[0,t]$.

Step 2. We now fix an arbitrary function $\chi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi(t) = 1$ for $t \leqslant 0$, $\chi(t) = 0$ for $t \geqslant 1$. Let us take any initial points $u_0, u_0' \in L^2$ and set $f_N(u_0, u_0') = \mathsf{P}_N(u_0' - u_0)$. Denote by $\lambda_T(u_0, u_0')$ and $\lambda_T'(u_0, u_0')$ the laws of the processes

$$\left\{\begin{pmatrix} \mathsf{P}_N u(t) \\ \mathsf{Q}_N\zeta(t) \end{pmatrix}, t \in [0,T]\right\}, \left\{\begin{pmatrix} \mathsf{P}_N u'(t) - f_N(u_0, u_0')\chi(t) \\ \mathsf{Q}_N\zeta(t) \end{pmatrix}, t \in [0,T]\right\} \tag{4.14}$$

respectively, where $u(t) = S_t(u_0, \omega)$ and $u(t) = S_t(u_0', \omega)$. Thus, $\lambda_T(u_0, u_0')$ and $\lambda_T'(u_0, u_0')$ are probability measures on the separable Banach space $C(0,T;L^2)$. Let $(U(u_0, u_0'), U'(u_0, u_0'))$ be a maximal coupling for $(\lambda_T(u_0, u_0'), \lambda_T'(u_0, u_0'))$.[5] By Proposition 5.2, such a pair of random variables exists and is a measurable function of its arguments. Let

$$\mathcal{R}_t(u_0, u_0') = \mathsf{P}_N U_t + \mathcal{W}_t(\mathsf{Q}_N u_0, \mathsf{P}_N U, \mathsf{Q}_N U), \tag{4.15}$$

$$\begin{aligned} \mathcal{R}_t'(u_0, u_0') = \mathsf{P}_N U_t &+ f_N(u_0, u_0')\chi(t) \\ &+ \mathcal{W}_t(\mathsf{Q}_N u_0', \mathsf{P}_N U' + f_N(u_0, u_0')\chi, \mathsf{Q}_N U'), \end{aligned} \tag{4.16}$$

where U_t stands for the restriction of $U(u_0, u_0')$ to the time t and U_t' is defined in a similar way. We claim that $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}_t')$ is an extension of S_t on the interval $[0,T]$.

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[5] See Section 5.2 for the definition of maximal coupling.

Indeed, we need to show that the laws of the processes $\{\mathcal{R}_t(u_0,u_0')\}$ and $\{\mathcal{R}_t'(u_0,u_0')\}$ coincide with those of $\{S_t(u_0,\omega)\}$ and $\{S_t(u_0',\omega)\}$ respectively. To this end, let us set $\mathcal{X}(T)=C(0,T;L^2)\cap L^2(0,T;H_0^1)$ and introduce an operator $\varUpsilon: H_N^\perp\times C(0,T;H_N)\times C(0,T;H_N^\perp\cap H_0^1)\to\mathcal{X}(T)$ by the relation

$$\varUpsilon(w_0,v,\psi)=v+\mathcal{W}(w_0,v,\psi). \tag{4.17}$$

The definition of $\mathcal{W}$ implies that

$$\{S_t(u_0,\omega),t\in[0,T]\}=\varUpsilon\big(\mathsf{Q}_Nu_0,\mathsf{P}_NS.(u_0,\omega),\mathsf{Q}_N\zeta(\cdot)\big). \tag{4.18}$$

Thus, the law of $\{S_t,t\in[0,T]\}$ coincides with the image of the law of the first process in (4.14) under the mapping $\varUpsilon(\mathsf{Q}_Nu_0,\cdot,\cdot)$. Furthermore, from (4.15) it follows that the distribution $\mathcal{D}(\mathcal{R}.(u_0,u_0'))$ is the image of $\lambda_T(u_0,u_0')$ under $\varUpsilon(\mathsf{Q}_Nu_0,\cdot,\cdot)$. By construction, the law of the first process in (4.14) coincides with $\lambda_T(u_0,u_0')$, and we conclude that $\mathcal{D}(\mathcal{R}.(u_0,u_0'))=\mathcal{D}(S.(u_0,\cdot))$. A similar argument proves that $\mathcal{D}(\mathcal{R}'.(u_0,u_0'))=\mathcal{D}(S.(u_0',\cdot))$.

Our next goal is to check that Hypotheses (H_1) and (H_2) are satisfied for S_t and $\boldsymbol{\mathcal{R}}_t$. In view of Propositions 3.2 and 3.3, this will imply that property (i) of Definition 2.2 is true for the extension $\boldsymbol{S}_t$.

4.4. Lyapunov function and dissipation.

We show that S_t satisfies Hypothesis (H_1) with $F(u)=\|u\|^2$ and any $t_*>0$. Indeed, from (4.6) and the Gronwall inequality it follows that

$$\mathbb{E}_uF(S_t)\leqslant e^{-\nu t}F(u)+C\nu^{-1},\quad t\geqslant 0.$$

In particular, fixing any constant $a\in(e^{-\nu t_*},1)$, we see that (3.1) and (3.2) hold with

$$R_*=\left(\frac{C}{\nu(a-e^{-\nu t_*})}\right)^{1/2},\quad C_*=R_*^2+C\nu^{-1}.$$

We now show that the extension $\boldsymbol{\mathcal{R}}_t$ satisfies Hypothesis (H_2) for sufficiently large N and T. Note that, in view of (4.8), the distribution of $\{\zeta(t),0\leqslant t\leqslant T\}$ is a nondegenerate Gaussian measure on $C(0,T;H_0^1)$. Combining this with the obvious property of approximate controllability of the complex Ginzburg–Landau equation (1.4) with a control force $\zeta\in C^1(0,T;H_0^1)$, for any $R>0$, $q\in(0,1)$, and $d>0$ we can find $\alpha(R,q,d)>0$ such that (see, for example, [**8, 26**])

$$\mathbb{P}_u\{\|S_T(u,\cdot)\|\leqslant(q\|u\|)\vee d\}\geqslant\alpha(R,q,d)\quad\text{for any } u\in L^2,\ \|u\|\leqslant R. \tag{4.19}$$

Moreover, using the existence of a Lyapunov function for S_t, the constant $\alpha(R,q,d)$ can be made independent of $T \geqslant 1$. Since $\boldsymbol{\mathcal{R}}_t$ is an extension for S_t, we conclude from (4.19) that

$$\begin{aligned} &\mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}_T(u,u')\| \leqslant (q\|u\|) \vee d\} \geqslant \alpha(R,q,d), \\ &\mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}'_T(u,u')\| \leqslant (q\|u'\|) \vee d\} \geqslant \alpha(R,q,d) \end{aligned} \tag{4.20}$$

for any $(u,u') \in L^2 \times L^2$ with $\|u\| \vee \|u'\| \leqslant R$. The inequalities (4.20) would imply (3.15) with $\varepsilon(d) = \alpha(R,q,d)^2$ and any $T \geqslant 1$, if the processes $\mathcal{R}_t$ and $\mathcal{R}'_t$ were independent. However, this is not the case, and we have to proceed differently.

Step 1. To prove (3.15), it suffices to show that for any $\delta > 0$ there is $c_\delta > 0$ such that

$$P_\delta := \mathbb{P}_{\boldsymbol{u}}\{\|\mathcal{R}_T(u,u')\| \vee \|\mathcal{R}'_T(u,u')\| \leqslant q_1(\|u\| \vee \|u'\|) + \delta\} \geqslant c_\delta \tag{4.21}$$

for $u, u' \in B_R$, where $q_1 \in (0,1)$ is a constant and B_R denotes the ball in L^2 of radius R centered at origin. Indeed, suppose that (4.21) is already proved and fix any $d > 0$. Setting $\delta = \frac{1-q_1}{1+q_1} d$ and $q = \frac{1+q_1}{2}$, we derive

$$q_1\|v\| + \delta = (q\|v\|) \vee d \quad \text{for any } v \in L^2.$$

It follows that the probability on the left-hand side of (3.15) is bounded below by P_δ. Since δ depends only on d and q_1, this proves (3.15).

Step 2. We now prove (4.21). In view of the existence of a Lyapunov function for S_t, we can assume that $u, u' \in B_{R_*}$ for some $R_* > 0$. Introduce the events

$$\begin{aligned} G_\delta &= \{\|\mathcal{R}_T(u,u')\| \leqslant q_1(\|u\| \vee \|u'\|) + \delta\}, \\ G'_\delta &= \{\|\mathcal{R}'_T(u,u')\| \leqslant q_1(\|u\| \vee \|u'\|) + \delta\}, \\ E_\rho &= \{\mathcal{E}_{\mathcal{R}}(t) + \mathcal{E}_{\mathcal{R}'}(t) \leqslant 2(R_*^2 + Lt) + \rho \text{ for all } t \geqslant 0\}, \end{aligned}$$

where $\mathcal{E}_u$ is defined by (4.5). We need to estimate from below the expression $\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta)$. It follows from (4.19) that

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta) \geqslant \varkappa_\delta, \quad \mathbb{P}_{\boldsymbol{u}}(G'_\delta) \geqslant \varkappa_\delta \quad \text{for any } u, u' \in B_{R_*}, \tag{4.22}$$

where $\varkappa_\delta > 0$ is a constant independent of u, u', and T. Moreover, the inequality (4.7) implies that

$$\mathbb{P}_{\boldsymbol{u}}(E_\rho) \geqslant 1 - \beta_\rho \quad \text{for any } u, u' \in B_{R_*}, \tag{4.23}$$

where $\beta_\rho \to 0$ as $\rho \to \infty$. Now recall that (see (4.15) and (4.16))

$$\mathcal{R}_t(u,u') = \Upsilon_t(\mathsf{Q}_N u, U), \quad \mathcal{R}'_t(u,u') = \Upsilon_t(\mathsf{Q}_N u, U' + \tilde{f}_N(u,u')\chi), \tag{4.24}$$

where (U, U') is a maximal coupling for the pair $(\lambda_T(u,u'), \lambda'_T(u,u'))$, the operator Υ is defined in (4.17), Υ_t stands for its restriction to the time t, and $\tilde f_N(u,u') = \binom{f_N(u,u')}{0}$. Without loss of generality, we can assume that

$$\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}^c) \leqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c), \tag{4.25}$$

where $\mathcal{N} = \{U(u,u') \neq U'(u,u')\}$ and $\mathcal{N}^c$ denotes the complement of $\mathcal{N}$. The case in which the opposite inequality is satisfied can be treated by a similar argument.

Suppose that we already shown that

$$G_{\delta/2}E_\rho\mathcal{N}^c \subset G_\delta G'_\delta \quad \text{for any } \rho > 0 \text{ and } T \geqslant T_\rho, \tag{4.26}$$

where $T_\rho \geqslant 1$ depends only on ρ. In this case, we can write

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta) &= \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta \mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta \mathcal{N}) \\
&\geqslant \mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta E_\rho \mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta \,|\, \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta \,|\, \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(\mathcal{N}) \\
&\geqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2} E_\rho \mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta \mathcal{N}),
\end{aligned}$$

where we used the inclusion (4.26) and the independence of U and U' conditioned on $\mathcal{N}$. Combining this inequality with (4.23), we derive

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta G'_\delta) \geqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c) + \mathbb{P}_{\boldsymbol{u}}(G_\delta \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta \mathcal{N}) - \beta_\rho. \tag{4.27}$$

We claim that if $\rho > 0$ is so large that $\beta_\rho \leqslant \frac{1}{8}\varkappa^2_{\delta/2}$, then (4.21) holds with $c_\delta = \frac{1}{8}\varkappa^2_{\delta/2}$. Indeed, if $\mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N}^c) \geqslant \frac{1}{4}\varkappa^2_{\delta/2}$, then (4.21) follows immediately from (4.27). In the opposite case, the inequalities (4.22) and (4.25) imply that

$$\varkappa^2_{\delta/2} \leqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}) \leqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}) + \frac{3}{4}\varkappa^2_{\delta/2}.$$

Hence

$$\mathbb{P}_{\boldsymbol{u}}(G_\delta \mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_\delta \mathcal{N}) \geqslant \mathbb{P}_{\boldsymbol{u}}(G_{\delta/2}\mathcal{N})\mathbb{P}_{\boldsymbol{u}}(G'_{\delta/2}\mathcal{N}) \geqslant \frac{1}{4}\varkappa^2_{\delta/2}.$$

Combining this with (4.27), we obtain (4.21) with $c_\delta = \frac{1}{8}\varkappa^2_{\delta/2}$.

Step 3. It remains to prove (4.26). The construction implies that if $\omega \in \mathcal{N}^c$, then the processes $\mathcal{R}_t(u,u')$ and $\mathcal{R}'_t(u,u')$ belong to the space $\mathcal{X}(T)$ and satisfy Equation (1.4) with some right-hand sides $\zeta, \zeta' \in C(0,T;H^1_0)$. Moreover, we have the relations (cf. (5.1), (5.2))

$$\mathsf{P}_N\mathcal{R}_t(u,u') = \mathsf{P}_N\mathcal{R}'_t(u,u') - f_N(u,u')\chi(t), \tag{4.28}$$

$$\mathsf{Q}_N\zeta(t) = \mathsf{Q}_N\zeta'(t) \tag{4.29}$$

for $0 \leqslant t \leqslant T$. Furthermore, if $\omega \in G_{\delta/2}E_\rho$, then

$$\int_0^t \big(\|\mathcal{R}_s(u,u')\|^2 + \|\mathcal{R}'_s(u,u')\|^2\big)\,ds \leqslant 2(R^2+Lt)+\rho,\ 0 \leqslant t \leqslant T, \quad (4.30)$$

$$\|\mathcal{R}_T(u,u')\| \leqslant \delta/2 + q_1\big(\|u\| \vee \|u'\|\big). \quad (4.31)$$

Applying Proposition 5.3 and using (4.28) and (4.30), we see that

$$\begin{aligned}\|\mathcal{R}_t(u,u') - \mathcal{R}'_t(u,u')\| &= \|\mathsf{Q}_N(\mathcal{R}_t(u,u') - \mathcal{R}'_t(u,u'))\| \\ &\leqslant C_1 \exp\big\{-\nu\alpha_{N+1}(t-1) + C_1 t + 2R_*^2 + \rho\big\}\|u-u'\|,\end{aligned}$$

where $C_1 > 0$ is a constant independent of u, u', and N. It follows that if N is sufficiently large, then for any $\rho > 0$ we can choose $T_\rho \geqslant 1$ such that

$$\|\mathcal{R}_T(u,u') - \mathcal{R}'_T(u,u')\| \leqslant \tfrac{\delta}{2} \quad \text{for } u,u' \in B_{R_*},\ T \geqslant T_\rho. \quad (4.32)$$

Combining this with (4.31), we obtain the inequality

$$\|\mathcal{R}_T(u,u')\| \vee \|\mathcal{R}'_T(u,u')\| \leqslant q_1\big(\|u\| \vee \|u'\|\big) + \delta,$$

which shows that $G_{\delta/2}E_\rho\mathcal{N}^c \subset G_\delta G'_\delta$. This completes the verification of Hypothesis (H_2).

4.5. Squeezing: verification of (2.6) and (2.7).

We recall that the extension $\boldsymbol{S}_t = (S_t, S'_t)$ is obtained by the iteration of $\boldsymbol{\mathcal{R}}_t = (\mathcal{R}_t, \mathcal{R}'_t)$ and that the random processes $S_t(\boldsymbol{u},\omega)$ and $S'_t(\boldsymbol{u},\omega)$ satisfy Equation (1.4) with some right-hand sides $\zeta = \zeta(t,u,u')$ and $\zeta = \zeta(t,u,u')$ respectively. Introduce the Markov times

$$\sigma_1(\boldsymbol{u},\omega) = \inf\{t \geqslant 0 : \mathsf{P}_N S_t \neq \mathsf{P}_N S'_t - f_N(u,u')\chi(t) \text{ or } \mathsf{Q}_N\zeta(t) \neq \mathsf{Q}_N\zeta'(t)\},$$

$$\sigma_2(\boldsymbol{u},\omega) = \inf\{t \geqslant 0 : \mathcal{E}_{S_\cdot}(t) + \mathcal{E}_{S'_\cdot}(t) \geqslant \|\boldsymbol{u}\|^2 + 2(L+M)t + 2\rho\},$$

where M and ρ are positive parameters which will be chosen later. We set

$$\tilde\sigma(\boldsymbol{u},\omega) = \sigma_1(\boldsymbol{u},\omega) \wedge \sigma_2(\boldsymbol{u},\omega).$$

The Foiaş–Prodi estimate (5.3) implies that if $N \gg 1$ and $u, u' \in B_1$, then (cf. the derivation of (4.32))

$$\|S_t(\boldsymbol{u},\omega) - S'_t(\boldsymbol{u},\omega)\| \leqslant C\,e^{-t} \quad \text{for } 0 \leqslant t \leqslant \tilde\sigma(\boldsymbol{u},\omega), \quad (4.33)$$

where $C > 0$ does not depend on u and u'. It follows that $\tilde\sigma \leqslant \sigma$, where σ is defined by the relation (2.3) with $\beta = 1$. We show that if $N \gg 1$, $\rho \gg 1$, and $\boldsymbol{B} = B_d \times B_d$ with $d \ll 1$, then $\tilde\sigma$ satisfies (2.6) and (2.7).

Step 1. We set $Q_k = \{\tilde\sigma(\boldsymbol{u},\omega) \in I_k\}$, $I_k = [(k-1)T, kT]$. Suppose that we already shown that

$$\mathbb{P}_{\boldsymbol{u}}(Q_k) \leqslant 2e^{-2k} \quad \text{for any } k \geqslant 1,\ \boldsymbol{u} \in \boldsymbol{B}. \tag{4.34}$$

In this case, we can write

$$\mathbb{P}_{\boldsymbol{u}}\{\tilde\sigma = \infty\} = 1 - \sum_{k=1}^{\infty} \mathbb{P}_{\boldsymbol{u}}(Q_k) \geqslant 1 - 2\sum_{k=1}^{\infty} e^{-2k} =: \delta_1 > 0,$$

$$\mathbb{E}_{\boldsymbol{u}}\left(I_{\{\tilde\sigma<\infty\}} e^{\delta_2\tilde\sigma}\right) \leqslant \sum_{k=1}^{\infty} \mathbb{P}_{\boldsymbol{u}}(Q_k) e^{\delta_2 Tk} \leqslant 2\sum_{k=1}^{\infty} e^{-(2-\delta_2 T)k} \leqslant K,$$

where $\delta_2 < T^{-1}$. Thus, it suffices to prove (4.34).

Step 2. To prove (4.34), we need the following result. Recall that the measures $\lambda_T(u,u')$ and $\lambda_T'(u,u')$ are defined in Section 4.3.

Proposition 4.3. *There is an integer $N_0 \geqslant 1$ such that if $N \geqslant N_0$, then*

$$\left\|\lambda_T(u,u') - \lambda_T'(u,u')\right\|_{\mathrm{var}} \leqslant Ce^{-cR^2} + C_N d e^{CR^2} \tag{4.35}$$

for any $u, u' \in B_R$ such that $\|u-u'\| \leqslant d$. Here, C_N, C, and c are positive constants independent of R and d.[6)]

The proof of this result is based on a well-known argument using the Girsanov theorem (see [**6, 15**]). The case of the complex Ginzburg–Landau equation is technically more complicated; however, the main ideas remain the same, and therefore we omit the proof. We refer the reader to [**28**, Proposition 3] for a weaker version of (4.35).

The inequality (4.34) is proved by induction on k. Denote by A_k the set of $\omega \in \Omega$ for which $\mathsf{P}_N S_t = \mathsf{P}_N S_t' - f_N(u,u')\chi(t)$, $\mathsf{Q}_N\zeta(t) = \mathsf{Q}_N\zeta'(t)$ for $t \in I_k$. For $k=1$ we have

$$Q_1 = \{\sigma_2 \in [0,T]\} \cup A_1^c. \tag{4.36}$$

From (4.7) it follows that

$$\mathbb{P}_{\boldsymbol{u}}\{\sigma_2 \in [0,T]\} \leqslant 2e^{-\varkappa\rho} \leqslant e^{-2} \quad \text{for } \rho \geqslant 4/\varkappa. \tag{4.37}$$

Furthermore, Proposition 4.3 and the definition of maximal coupling imply that

$$\mathbb{P}_{\boldsymbol{u}}(A_1^c) \leqslant Ce^{-cR^2} + C_N d e^{CR^2}. \tag{4.38}$$

The right-hand side of this inequality is smaller than e^{-2} if

$$R \geqslant c^{-1}(\ln C + 4), \quad d \leqslant (2C_N)^{-1} e^{-CR^2}. \tag{4.39}$$

6) However, they may depend on T.

Combining (4.36)–(4.38), we arrive at (4.34) for $k = 1$.

We now assume that $k = l + 1 \geqslant 2$ and the inequality (4.34) is established for $1 \leqslant k \leqslant l$. Denote by $\bar{A}_l$ the intersection of $A_1, \dots, A_l$. We have

$$Q_{l+1} \subset \{\sigma_2 \in I_{l+1}\} \cup D_{l+1}, \tag{4.40}$$

where $D_{l+1} = \bar{A}_l \cap A^c_{l+1} \cap \{\sigma_2 \geqslant (l+1)T\}$. Let us estimate the probabilities of the events on the right-hand side of (4.40). The inequality (4.7) implies

$$\mathbb{P}_{\boldsymbol{u}}\{\sigma_2 \in I_{l+1}\} \leqslant 2e^{-\varkappa(\rho+Ml)} \leqslant e^{-2(l+1)}, \tag{4.41}$$

provided that

$$M \geqslant 2/\varkappa, \quad \rho \geqslant 4/\varkappa. \tag{4.42}$$

Furthermore, using the inequality (4.34) for $0 \leqslant k \leqslant l$, we derive

$$\mathbb{P}_{\boldsymbol{u}}\big(\bar{A}_l \cap \{\sigma_2 \geqslant lT\}\big) \geqslant \mathbb{P}_{\boldsymbol{u}}\{\tilde{\sigma} \geqslant lT\} \geqslant 1 - 2\sum_{k=1}^{l} e^{-2k} \geqslant 1/2 \tag{4.43}$$

for $\boldsymbol{u} \in \boldsymbol{B}$. The Foiaş–Prodi inequality (5.3) implies that for any $P > 0$ and sufficiently large N we have (cf. the derivation of (4.32))

$$\begin{aligned} &\|S_{lT}\| \vee \|S'_{lT}\| \leqslant C_1(\rho + MTl)^{1/2}, \\ &\|S_{lT} - S'_{lT}\| \leqslant C_2 d\, e^{C_2\rho - PTl} \end{aligned}$$

on the set $\bar{A}_l \cap \{\sigma_2 \geqslant lT\}$, where C_1 and C_2 are positive constants independent of N, d, and l. Applying now the Markov property and using the inequalities (4.35) and (4.43), we obtain

$$\begin{aligned} \mathbb{P}_{\boldsymbol{u}}(D_{l+1}) &\leqslant \mathbb{P}_{\boldsymbol{u}}\big(A^c_{l+1} \,|\, \bar{A}_l \cap \{\sigma_2 \geqslant lT\}\big)\, \mathbb{P}_{\boldsymbol{u}}\big(\bar{A}_l \cap \{\sigma_2 \geqslant lT\}\big) \\ &\leqslant Ce^{-c\,C_1^2(\rho+MTl)} + C_N C_2 d \exp\big\{\rho(CC_1^2 + C_2) + (CC_1^2 M - P)Tl\big\}. \end{aligned} \tag{4.44}$$

The right-hand side of this inequality is smaller than $e^{-2(l+1)}$ if

$$\begin{aligned} &M \geqslant (2c\,C_1^2 T)^{-1}, \qquad \rho \geqslant \frac{\ln C + 2}{c\,C_1^2}, \\ &P \geqslant CC_1^2 M + 2, \qquad d \leqslant (C_N C_2)^{-1} e^{-\rho(CC_1^2 + C_2) - 1}. \end{aligned} \tag{4.45}$$

Note that the conditions imposed on the parameters M, ρ, P, and d by the inequalities (4.39), (4.42), and (4.45) are compatible. Combining (4.40), (4.41), and (4.44), we arrive at (4.34) for $k = l + 1$. This completes the proof of (4.34).

4.6. Completion of the proof of Theorem 4.2.

We have thus shown that the random dynamical system associated with the complex Ginzburg–Landau equation (4.2) possesses an extension $\boldsymbol{S}_t = (S_t, S_t')$ that satisfies (2.5)–(2.7) with $\sigma = \tilde\sigma$, $\boldsymbol{B} = B_d \times B_d$, $g(r) = r^2$, where $d > 0$ is sufficiently small. If we show that

$$\mathbb{E}_{\boldsymbol{u}}\bigl\{I_{\{\tilde\sigma<\infty\}}\|\boldsymbol{S}_{\tilde\sigma}\|^{2q}\bigr\} \leqslant K \quad \text{for any } \boldsymbol{u} \in \boldsymbol{B}, \tag{4.46}$$

where K and q are positive constants independent of $\boldsymbol{u}$, then the application of Theorem 2.3 and Remark 2.4 will prove that the problem (4.2), (4.3) possesses a unique stationary measure $\mu \in \mathcal{P}(L^2)$ and the inequality (4.9) holds.

To prove (4.46), note that if $\tilde\sigma < \infty$, then

$$\|S_{\tilde\sigma}\|^2 + \|S_{\tilde\sigma}'\|^2 \leqslant 2(d^2 + L\tilde\sigma) + \rho \quad \text{for } u, u' \in B_d.$$

It follows that

$$\|\boldsymbol{S}_{\tilde\sigma}\|^{2q} \leqslant C_q(\tilde\sigma^2 + 1) \quad \text{for any } q > 1,$$

where $C_q > 0$ depends only on L, d, and ρ. Multiplying this inequality by $I_{\{\tilde\sigma<\infty\}}$, taking the mean value, and using (2.7), we arrive at (4.46). The proof of Theorem 4.2 is complete.

5. Appendix

5.1. Maximal coupling of measures.

Let X be a Polish space, and let μ, μ' be two probability Borel measures on X. Recall that a pair (ξ, ξ') of X-valued random variables defined on the same probability space is called a *coupling for* (μ, μ') if $\mathcal{D}(\xi) = \mu$ and $\mathcal{D}(\xi') = \mu'$.

Definition 5.1. A coupling (ξ, ξ') for (μ, μ') is said to be *maximal* if

$$\mathbb{P}\{\xi \neq \xi'\} = \|\mu - \mu'\|_{\mathrm{var}}$$

and the random variables ξ and ξ' conditioned on the event $\mathcal{N} = \{\xi \neq \xi'\}$ are independent, i.e., $\mathbb{P}\{\xi \in \Gamma, \xi' \in \Gamma' \,|\, \mathcal{N}\} = \mathbb{P}\{\xi \in \Gamma \,|\, \mathcal{N}\}\,\mathbb{P}\{\xi' \in \Gamma' \,|\, \mathcal{N}\}$ for any $\Gamma, \Gamma' \in \mathcal{B}_X$.

In Section 4.3, we have used the following result on the existence of maximal coupling for measures depending on a parameter. Let Y be a Polish space endowed with its Borel σ-algebra $\mathcal{B}_Y$, and let $\{\mu_y\}_{y\in Y}$ be a family of measures on X. We say that μ_y *measurable depends on* $y \in Y$ if the function $y \mapsto \mu_y(\Gamma)$ is $(\mathcal{B}_Y, \mathcal{B}_{\mathbb{R}})$-measurable for any $\Gamma \in \mathcal{B}_X$.

Proposition 5.2. *Let* $\{\mu_y\}, \{\mu'_y\} \subset \mathcal{P}(X)$ *be two families that measurably depend on* $y \in Y$. *Then there is a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and two measurable functions* $\xi : Y \times \Omega \to X$ *and* $\xi' : Y \times \Omega \to X$ *such that* $(\xi(y,\cdot), \xi'(y,\cdot))$ *is a maximal coupling for* (μ_y, μ'_y) *for any* $y \in Y$.

In the case $X = \mathbb{R}^n$, the proof can be found in [**14**]. In the general case, it suffices to use the fact that any Polish space is measurably isomorphic to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

5.2. Foiaş–Prodi estimate.

In this section, we present an estimate for the difference between two solutions of the problem (1.4), (1.5) in which $\zeta : \mathbb{R}_+ \to H^1$ is a deterministic continuous function. Recall that $\{e_j\} \subset H$ is the complete set of eigenfunctions for the Dirichlet Laplacian in the domain D, H_N is the $2N$-dimensional subspace in L^2 generated by $\{e_j, ie_j, 1 \leqslant j \leqslant N\}$, and $H_N^\perp$ is the orthogonal complement of H_N in L^2. Denote by $\mathsf{P}_N : L^2 \to H_N$ and $\mathsf{Q}_N : L^2 \to H_N^\perp$ the corresponding orthogonal projections.

The following result provides a Foiaş–Prodi type estimate for the difference between two solutions whose projections to H_N coincide (cf. [**9**]). The proof can be found in [**28**, Section 4].[7)]

Proposition 5.3. *Let* $n = 3$ *or* 4, *let* $p \leqslant \frac{2}{n}$, *and let*

$$u_1, u_2 \in \mathcal{X}(T) = C(0,T; L^2) \cap L^2(0,T; H_0^1)$$

be two solutions of the problem (1.4), (1.5) *that correspond to deterministic functions* $\zeta_1, \zeta_2 \in C(0,T; H_0^1)$ *and* $h \in L^2(D, \mathbb{C})$. *Suppose that*

$$\mathsf{P}_N u_1(t) = \mathsf{P}_N u_2(t) \quad \text{for } t_0 \leqslant t \leqslant T, \tag{5.1}$$

$$\mathsf{Q}_N \zeta_1(t) = \mathsf{Q}_N \zeta_2(t) \quad \text{for } 0 \leqslant t \leqslant T, \tag{5.2}$$

where $t_0 \in [0,T]$ *and* $N \geqslant 1$ *is an integer. Then there is a constant* $C > 0$ *independent of* u_1, u_2, t_0, *and* N *such that*

7) The estimate established in [**28**] is slightly different. However, a similar argument enables one to prove (5.3).

$$\|\mathsf{Q}_N(u_1(t)-u_2(t))\|^2 \leqslant \exp\{-\nu\alpha_{N+1}t+q(t)\}\biggl(\|\mathsf{Q}_N(u_1(0)-u_2(0))\|^2 + Ce^{\nu\alpha_{N+1}t_0+q(t_0)}\int_0^t (\|u_1(s)\|_1+\|u_2(s)\|_1)^{(4p-2)\vee 0}\|\mathsf{P}_N(u_1(s)-u_2(s))\|_1^2 ds\biggr) \tag{5.3}$$

for $0 \leqslant t \leqslant T$, *where*

$$q(t) = C\int_0^t \bigl(\|u_1(s)\|_1^2+\|u_2(s)\|_1^2+1\bigr)\,ds.$$

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On Problem of Stability of Equilibrium Figures of Uniformly Rotating Viscous Incompressible Liquid

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Recent results on the stability of a rotating capillary viscous incompressible liquid bounded by a free surface are presented. It is established that the regime of a rigid rotation is stable if the second variation of the energy functional is positive definite and is instable if the second variation can take negative values. The proof is based on the study of the spectrum of the corresponding linear problem. Extensions of these results to the multi-dimansional case are discussed. Bibliography: 26 titles.

1. Introduction

We study the stability of some special solutions of the free boundary problem for the Navier–Stokes equations governing the evolution of an isolated mass of a viscous incompressible liquid subjected to the capillary and self-gravitation forces. The problem consists in the determination of a bounded domain $\Omega_t \in R^3$, the velocity vector field $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$, and the

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pressure function $p(x,t)$, $x \in \Omega_t$, satisfying the relations

$$\begin{gathered}
\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} + \nabla p = 0, \\
\nabla \cdot \boldsymbol{v} = 0, \qquad x \in \Omega_t, \quad t > 0, \\
T(\boldsymbol{v}, p)\boldsymbol{n} = (\sigma H(x,t) + \varkappa U(x,t))\boldsymbol{n}, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \quad x \in \Gamma_t \equiv \partial \Omega_t, \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0.
\end{gathered} \tag{1.1}$$

Here, ν and σ are positive constant coefficients of viscosity and the surface tension respectively, $\varkappa \geqslant 0$ is a gravitational constant, $T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor,

$$S(\boldsymbol{v}) = \Big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\Big)_{j,k=1,2,3}$$

is the doubled rate-of-strain tensor, H is the doubled mean curvature of Γ_t negative for convex domains, V_n is the velocity of evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$, and

$$U(x,t) = \int\limits_{\Omega_t} \frac{dz}{|x - z|}$$

is the Newtonian potential depending on the unknown domain Ω_t. The density of a liquid is assumed to be equal to one. The domain Ω_0 is given.

The solvability of this problem in a finite time interval is proved in [**14**]. In the present paper, we are concerned with the stability of the solution corresponding to the rigid rotation of a liquid about the x_3-axis with constant angular velocity ω. In this case, the velocity and pressure are given by the formula

$$\begin{aligned}
\boldsymbol{V}(x) &= \omega(\boldsymbol{e}_3 \times \boldsymbol{x}) = \omega(-x_2, x_1, 0), \\
P(x) &= \frac{\omega^2}{2}|x'|^2 + p_0
\end{aligned} \tag{1.2}$$

where $x' = (x_1, x_2, 0)$, $p_0 = \text{const}$, and $\boldsymbol{e}_3$ is a unit vector directed along the x_3-axis. The domain $\mathcal{F}$ occupied by the rotating liquid, called an *equilibrium figure*, is determined by the equation

$$\sigma \mathcal{H} + \frac{\omega^2}{2}|x'|^2 + \varkappa \mathcal{U} + p_0 = 0, \quad x \in \mathcal{G} = \partial \mathcal{F}. \tag{1.3}$$

where $\mathcal{H}$ is the doubled mean curvature of $\mathcal{G}$ and

$$\mathcal{U}(x) = \int\limits_{\mathcal{F}} \frac{dz}{|x - z|}.$$

Let $\boldsymbol{N}(x)$ be the exterior normal to $\mathcal{G}$. If $\mathcal{F}$ is axially symmetric with respect to the x_3-axis, then

$$\boldsymbol{V}(x) \cdot \boldsymbol{N}(x) = \omega(\boldsymbol{e}_3 \times \boldsymbol{x}) \cdot \boldsymbol{N}(x) = 0, \quad x \in \mathcal{G}, \tag{1.4}$$

which means that $\Omega_t = \mathcal{F}$ is independent of t and the functions (1.2) given in $\mathcal{F}$ represent a stationary solution of the problem (1.1). If $\mathcal{F}$ is nonsymmetric, then equation (1.3) determines a one parameter family of equilibrium figures $\mathcal{F}_\theta$ obtained by rotation of one of them, $\mathcal{F}_0$, around the x_3-axis of the angle θ. In this case functions (1.2) given in $\mathcal{F}_{\omega t + \varphi_0}$ define a periodic solution of (1.1). The function $V_n = \boldsymbol{V} \cdot \boldsymbol{N}|_{\mathcal{G}}$ is different from identical zero.

The problem of the stability of equilibrium figures of rotating liquid has drawn attention of many generations of great mathematicians, beginning with I.Newton. The review of results obtained in the past and of some recent contributions can be found in [**1, 7**]. A generally accepted criterion of the stability of $\mathcal{F}$ is the positivity of the second variation $\delta^2 R$ of the energy functional

$$R = \sigma|\Gamma| + \frac{\beta^2}{2\int\limits_{\Omega} (x_1^2 + x_2^2)dx} - \frac{\varkappa}{2} \int\limits_{\Omega}\int\limits_{\Omega} \frac{dx}{|x - y|} - p_0|\Omega|, \quad \Gamma = \partial\Omega \tag{1.5}$$

with respect to the normal deformation of $\mathcal{G}$. By Ω we mean a domain in R^3 close to $\mathcal{F}$ with the same volume $|\Omega|$ and the same position of the barycenter as $\mathcal{F}$, $\Gamma = \partial\Omega$, $|\Gamma| = \mathrm{meas}\Gamma$, and

$$\beta = \omega \int\limits_{\mathcal{F}} (x_1^2 + x_2^2)dx$$

is the magnitude of the total angular momentum of the rotating liquid. We show that if $\delta^2 R > 0$, then the problem (1.1) with $\boldsymbol{v}_0$ close to $\boldsymbol{V}$ and Ω_0 close to $\mathcal{F}$ is solvable in the infinite time interval $t > 0$ and the solution tends to the rigid rotation as $t \to \infty$. If $\delta^2 R$ can take negative values, then for some initial data $(\boldsymbol{v}_0, \Omega_0)$ arbitrarily close to $(\boldsymbol{V}, \mathcal{F})$ the solution of the problem (1.1) leaves sooner or later a certain neighborhood of $(\boldsymbol{V}, \mathcal{F})$. The proof of these assertions, the exact form of which will be given below, rests upon the analysis of a linearized problem.

The fact that the rigid rotation (1.2) can be the limit for the solution of the problem (1.1) as $t \to \infty$ was discovered in [**12, 13**] in the case of a slow motion. As was shown in [**10**], the convergence of the solution of the problem (1.1) to this limit is exponential. In [**11**], the condition of the smallness of $\boldsymbol{V}$ was replaced with the condition of positivity of the second variation of the functional

$$G = \sigma|\Gamma| - \frac{\omega^2}{2}\int\limits_{\Omega}(x_1^2 + x_2^2)dx - \frac{\varkappa}{2}\int\limits_{\Omega}\int\limits_{\Omega}\frac{dx}{|x-y|} - p_0|\Omega|, \quad \Gamma = \partial\Omega,$$

under the assumption that $\mathcal{F}$ and Ω_t are star-shaped domains (the functional G is also considered very often in the theory of equilibrium figures). In [**21**], the functional R was invoked, which is more natural for the free motion of the liquid, and the class of domains $\mathcal{F}$ was significantly enlarged. This has required some modifications of the proofs. Concerning $\mathcal{F}$, the axial symmetry was always assumed. The case of nonsymmetric $\mathcal{F}$ was considered in [**20**]. In all these papers, the stability of the rigid rotation under the assumption $\delta^2 R > 0$ was established.

The instability of the solution (1.2) when the second variation of R can take negative values was proved in [**24, 22**]. The proof rests upon the analysis of the spectrum of the linearized problem that turns out to contain a finite number of eigenvalues with positive real parts. The proof of this fact given in [**23**] is based on the idea of Kopachevskii presented in [**4**, Ch. 9]: at first, the existence of such eigenvalues is established for a large viscosity coefficient ν, and then it is shown that they cannot leave the right complex half-plane when this coefficient is changed continuously. Another proof given in [**25, 26**] for $\sigma = 0$ rests upon the application of the Pontryagin–M. Krein–Langer–Azizov theorem on the invariant spaces of dissipative operators in the Pontryagin space with indefinite metrics. Unfortunately, the proof in [**25**] contains an error; it was corrected in [**26**].

In what follows, a short exposition of these results is given and, in Section 6, their possible extension to the case of higher spatial dimension n is discussed. Some arguments are made more transparent. The proof of stability presented in Sections 4 and 5 is different from that given in [**21, 20**]; it is based on the study of the spectrum of the linearized problem.

The results of [**26**] are presented for the case $\sigma > 0$. The case $n > 3$, although it has no physical meaning, is interesting from the mathematical point of view, in particular, due to the fact that the set of rigid motions is much more rich than in the three-dimensional case. We restrict ourselves to the extension of Theorem 4.1 on the stability of rigid motion to the case of arbitrary n. The case, where this motion is unstable, is studied in the forthcoming paper of the author and Padula.

We work in the Hölder spaces of functions because it is easier to estimate the nonlinear terms in the Hölder norms. In this way, we also avoid an application of embedding theorems depending on n. But the same type of analysis could be carried out in the Sobolev spaces, as it is done, for instance, in [**13**]. Important estimates of solutions of related linear problems in the Hölder norms are obtained in [**15, 16**]. The proof of these estimates is omitted.

2. Auxiliary Relations and Transformation of Problem (1.1)

We start with the proof of some useful relations for the equilibrium figure $\mathcal{F}$ defined by Equation (1.3). It is always assumed to be a bounded domain in R^3 with connected smooth boundary. Following Lyapunov [**6**], we show that the vector of the total angular momentum of rotating liquid,

$$\boldsymbol{\beta} = \int_{\mathcal{F}} \boldsymbol{x} \times \boldsymbol{V}(x) dx, \tag{2.1}$$

is directed along the x_3-axis. Multiplying (1.3) by $N_j x_3 - N_3 x_j$, $j = 1, 2,$, integrating over $\mathcal{G}$, and taking into account the equations

$$\int_{\mathcal{G}} \mathcal{U}(x)(N_j x_3 - N_3 x_j) dS = \int_{\mathcal{F}} \int_{\mathcal{F}} \Big(x_3 \frac{z_j - x_j}{|x-z|^3} - x_j \frac{z_3 - x_3}{|x-z|^3} \Big) dx dz$$

$$= \int_{\mathcal{F}} \int_{\mathcal{F}} \Big(z_3 \frac{z_j - x_j}{|x-z|^3} - z_j \frac{z_3 - x_3}{|x-z|^3} \Big) dx dz = 0$$

and

$$\int_{\mathcal{G}} \mathcal{H}(x)(N_j x_3 - N_3 x_j)dS = \int_{\mathcal{G}} (x_3 \Delta_{\mathcal{G}} x_j - x_j \Delta_{\mathcal{G}} x_3)dS = 0,$$

where $\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on $\mathcal{G}$, we find

$$\frac{\omega^2}{2} \int_{\mathcal{F}} x_3 \frac{\partial}{\partial x_j} |x'|^2 dx = \omega^2 \int_{\mathcal{F}} x_3 x_j dx = 0, \quad j = 1, 2.$$

Hence

$$\int_{\mathcal{F}} \boldsymbol{x} \times \boldsymbol{V} dx = \beta \boldsymbol{e}_3,$$

where

$$\beta = \omega \int_{\mathcal{F}} |x'|^2 dx.$$

Similarly, multiplying (1.3) by N_j and integrating, we obtain the equation

$$\omega^2 \int_{\mathcal{F}} x_j dx = 0,$$

which shows that the barycenter of $\mathcal{F}$ is located at the axis of rotation. Without loss of generality, we can assume that this equation is satisfied also for $j = 3$, i.e., that the barycenter coincides with the origin of the coordinate system x_1, x_2, x_3. Finally, the multiplication of (1.3) by $\boldsymbol{x} \cdot \boldsymbol{N}$ and integration leads to the expression for p_0:

$$p_0 = \frac{2\sigma|\mathcal{G}|}{3|\mathcal{F}|} - \frac{5}{6|\mathcal{F}|}\Big(\omega^2 \int_{\mathcal{F}} |x'|^2 dx + \varkappa \int_{\mathcal{F}} \mathcal{U} dx\Big).$$

For $\sigma = 0$ it was obtained in [**6**].

In fact, p_0 is the Lagrange multiplier corresponding to the prescription of the volume of $\mathcal{F}$; other multipliers corresponding to the prescription of the position of the barycenter vanish (see [**17**]).

If $\mathcal{F}$ is axially symmetric, then

$$\int_{\mathcal{F}} x_1 x_2 dx = 0. \tag{2.2}$$

In the case of the absence of symmetry, this equation can be satisfied by rotation of the equilibrium figure about the x_3-axis of a certain appropriate angle θ. Indeed, if $x = \mathcal{Z}(\theta)y$, where

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

then θ is determined from

$$\int\limits_{\mathcal{F}_\theta} x_1 x_2 dx = \cos\theta \sin\theta \int\limits_{\mathcal{F}_0} (y_1^2 - y_2^2) dy + (\cos^2\theta - \sin^2\theta) \int\limits_{\mathcal{F}_0} y_1 y_2 dy = 0. \tag{2.4}$$

Now, we go back to the problem (1.1) and recall that the solution of this problem satisfies the following "conservation laws" which are easily verified:

$$|\Omega_t| = |\Omega_0|,$$

$$\int\limits_{\Omega_t} \boldsymbol{v}(x,t)dx = \int\limits_{\Omega_0} \boldsymbol{v}_0(x)dx,$$

$$\int\limits_{\Omega_t} (\boldsymbol{x} \times \boldsymbol{v}(x,t))dx = \int\limits_{\Omega_0} (\boldsymbol{x} \times \boldsymbol{v}_0(x))dx.$$

Since we study the problem of stability of a given equilibrium figure $\mathcal{F}$, we should assume that

$$|\Omega_0| = |\mathcal{F}|,$$

$$\int\limits_{\Omega_0} \boldsymbol{v}_0(x)dx = \int\limits_{\mathcal{F}} \boldsymbol{V}(x)dx = 0,$$

$$\int\limits_{\Omega_0} (\boldsymbol{x} \times \boldsymbol{v}_0(x))dx = \int\limits_{\mathcal{F}} (\boldsymbol{x} \times \boldsymbol{V}(x))dx = \beta \boldsymbol{e}_3.$$

As a consequence, we have

$$|\Omega_t| = |\mathcal{F}|, \tag{2.5}$$

$$\int_{\Omega_t} x_i dx = 0, \quad i = 1, 2, 3, \tag{2.6}$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t) dx = 0, \tag{2.7}$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_i(x) dx = \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx = \delta_{i3}\beta, \quad i = 1, 2, 3, \tag{2.8}$$

where $\boldsymbol{\eta}_i(x) = \boldsymbol{e}_i \times \boldsymbol{x}$, and $\boldsymbol{e}_i$ is a unit vector directed along the x_i-axis.

From now on until Section 5 we assume that $\mathcal{F}$ is axially symmetric.

It is convenient to work with the problem for the perturbations of the velocity and pressure,

$$\boldsymbol{v}_r(x,t) = \boldsymbol{v}(x,t) - \boldsymbol{V}(x), \quad p_r(x,t) = p(x,t) - P(x),$$

written in the coordinate system rotating about the x_3-axis with the angular velocity ω. We introduce new coordinates y_i and new unknown functions $(\boldsymbol{w},\, q)$ by the formulas

$$\begin{gathered} x = \mathcal{Z}(\omega t) y, \\ \boldsymbol{w}(y,t) = \mathcal{Z}^{-1}(\omega t) \boldsymbol{v}_r(\mathcal{Z}(\omega t) y, t), \\ q(y,t) = p_r(\mathcal{Z}(\omega t) y, t) \end{gathered}$$

and transform the problem (1.1) into

$$\begin{gathered} \boldsymbol{w}_t + (\boldsymbol{w} \cdot \nabla) \boldsymbol{w} + 2\omega(e_3 \times \boldsymbol{w}) - \nu \nabla^2 \boldsymbol{w} + \nabla q = 0, \\ \nabla \cdot \boldsymbol{w} = 0, \quad y \in \Omega'_t, \quad t > 0, \end{gathered} \tag{2.9}$$

$$T(\boldsymbol{w}, p)\boldsymbol{n}' = (\sigma H(y) + \frac{\omega^2}{2}|y'|^2 + \varkappa U'(y,t) + p_0)\boldsymbol{n}', \tag{2.10}$$

$$\begin{gathered} V'_n = \boldsymbol{w} \cdot \boldsymbol{n}', \quad y \in \Gamma'_t, \\ \boldsymbol{w}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y \in \Omega_0, \end{gathered} \tag{2.11}$$

where $\Omega'_t = \mathcal{Z}^{-1}(\omega t)\Omega_t$, $\Gamma'_t = \partial\Omega'_t$, $\boldsymbol{n}'$ is the exterior normal to Γ'_t, and

$$U' = \int\limits_{\Omega'_t} |y - z|^{-1} dz.$$

The conditions (2.5)–(2.8) take the form

$$\begin{gathered} |\Omega'_t| = |\mathcal{F}|, \\ \int\limits_{\Omega'_t} y'_i dy = 0, \quad i = 1, 2, 3, \\ \int\limits_{\Omega'_t} \boldsymbol{w}(y, t) dy = 0, \end{gathered} \tag{2.12}$$

$$\begin{aligned} \int\limits_{\Omega'_t} \boldsymbol{w}(y, t) \cdot \boldsymbol{\eta}_i(y) dy + \omega \int\limits_{\Omega'_t} \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i(y) dy = \omega \int\limits_{\mathcal{F}} \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_i(z) dz& \\ = \beta\delta_{i3}, \quad i = 1, 2, 3.& \end{aligned} \tag{2.13}$$

To the solution (1.2) of the problem (1.1) there corresponds the zero solution of the problem (2.9)–(2.13) whose stability we should analyze.

In what follows, we consider only this last problem, without addressing to the problem (1.1) any more. Therefore, we return to the previous notation: we omit primes and denote by x a point of $\Omega_t = \Omega'_t$. Further, we write (2.9)–(2.13) as the nonlinear problem in a fixed domain $\mathcal{F}$ for $\boldsymbol{w}$, q and one more unknown function ρ given on $\mathcal{G}$. We assume that Ω_0 (and Ω_t, $t > 0$) is sufficiently close to $\mathcal{F}$ and, as a consequence, Γ_t can be prescribed by the equation

$$x = y + \boldsymbol{N}(y)\rho(y, t), \quad y \in \mathcal{G} \tag{2.14}$$

with a small function $\rho(y, t)$ given on $\mathcal{G}$. We extend $\boldsymbol{N}$ and ρ from $\mathcal{G}$ into $\mathcal{F}$ with the preservation of class and so that the extended functions $\boldsymbol{N}^*$ and ρ^* satisfy the conditions

$$\begin{aligned} &\frac{\partial}{\partial N} \boldsymbol{N}^*(x, t)|_{\mathcal{G}} = 0, \\ &\frac{\partial}{\partial N} \rho^*(x, t)|_{\mathcal{G}} = 0, \\ &|\rho^*(\cdot, t)|_{C^1(\mathcal{F})} \leqslant \delta \ll 1. \end{aligned} \tag{2.15}$$

We map $\mathcal{F}$ onto Ω_t by the transformation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) \equiv e_\rho(y), \quad y \in \mathcal{F}. \tag{2.16}$$

If δ is small enough, then the transformation (2.16) is invertible. Let $\mathcal{L} = \left(\dfrac{\partial e_\rho}{\partial y}\right)$ be the Jacobi matrix of this transformation with entries

$$l_{ij} = \delta_{ij} + \frac{\partial}{\partial y_j} N_i(y)\rho(y,t) \tag{2.17}$$

and the determinant L. By l^{ij} and $\widehat{L}_{ij}$, $i.j = 1,2,3,$, we denote the entries of the inverse matrix $\mathcal{L}^{-1}$ and the cofactor matrix $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ respectively. It is clear that the mapping e_ρ transforms the operator ∇_x into $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$, where $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$ and T means transposition; moreover, we have

$$\sum_{j=1}^{3} \frac{\partial}{\partial y_j} \widehat{L}_{ji} = 0$$

and, as a consequence,

$$0 = L\mathcal{L}^{-T}\nabla_y \cdot \widetilde{\boldsymbol{w}}(y,t) = \widehat{\mathcal{L}}^T \nabla_y \cdot \widetilde{\boldsymbol{w}} = \nabla_y \cdot \widehat{\mathcal{L}}\widetilde{\boldsymbol{w}},$$

where $\widetilde{\boldsymbol{w}}(y,t) = \boldsymbol{w}(e_\rho(y),t)$. Since

$$\begin{aligned}\frac{\partial \widetilde{\boldsymbol{w}}(y,t)}{\partial t} &= \frac{\partial \boldsymbol{w}(x,t)}{\partial t} + \sum_{k=1}^{3} \frac{\partial \boldsymbol{w}(x,t)}{\partial x_k} N_k^* \frac{\partial \rho^*(y,t)}{\partial t} \\ &= \frac{\partial \boldsymbol{w}(x,t)}{\partial t} + \frac{\partial \rho^*}{\partial t}(\mathcal{L}^{-1}N^* \cdot \nabla_y)\widetilde{\boldsymbol{w}}(y,t),\end{aligned}$$

we can write the basic system of Equations (2.9) in the form

$$\begin{aligned}&\frac{\partial}{\partial t}\widetilde{\boldsymbol{w}} - \frac{\partial \rho^*}{\partial t}(\mathcal{L}^{-1}N^* \cdot \nabla)\widetilde{\boldsymbol{w}} + (\mathcal{L}^{-1}\widetilde{\boldsymbol{w}} \cdot \nabla)\widetilde{\boldsymbol{w}} \\ &\qquad + 2\omega(\boldsymbol{e}_3 \times \widetilde{\boldsymbol{w}}) - \nu\widetilde{\nabla} \cdot \widetilde{\nabla}\widetilde{\boldsymbol{w}} + \widetilde{\nabla}\widetilde{q} = 0, \quad \nabla_y \cdot \widehat{\mathcal{L}}\widetilde{\boldsymbol{w}} = 0,\end{aligned} \tag{2.18}$$

where $\widetilde{q}(y,t) = q(e_\rho(y),t)$.

The dynamic boundary condition (2.10) can be written in the following equivalent form (in the case $\boldsymbol{n}(e_\rho) \cdot \boldsymbol{N}(y) > 0$, i.e., for small ρ):

$$\Pi_0 \Pi S(\boldsymbol{w}(x,t))\boldsymbol{n} = 0,$$

$$-q(x,t)+\nu\boldsymbol{n}\cdot S(\boldsymbol{w})\boldsymbol{n}=\sigma H(x)+\frac{\omega^2}{2}|x'|^2+\varkappa U(x,t)+p_0. \tag{2.19}$$

Here,

$$\Pi\boldsymbol{h}=\boldsymbol{h}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{h}),\quad \Pi_0\boldsymbol{h}=\boldsymbol{h}-\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{h}).$$

We also note that $\boldsymbol{n}(e_\rho(y))$ is related to $\boldsymbol{N}(y)$ by

$$\boldsymbol{n}(e_\rho)=\frac{\widehat{\mathcal{L}}^T\boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T\boldsymbol{N}(y)|} \tag{2.20}$$

and that $S(\boldsymbol{w})=\nabla_x\boldsymbol{w}+(\nabla_x\boldsymbol{w})^T$ is transformed by the mapping (2.16) into

$$\widetilde{S}(\boldsymbol{w}(e_\rho(y),t))=\widetilde{\nabla}\boldsymbol{w}(e_\rho(y),t)+(\widetilde{\nabla}\boldsymbol{w}(e_\rho(y),t))^T.$$

Hence we can make use of (1.3) and write (2.19) as

$$\begin{aligned}&\Pi_0\Pi\widetilde{S}(\widetilde{\boldsymbol{w}}(y,t))\widehat{\mathcal{L}}^T\boldsymbol{N}=0,\quad y\in\mathcal{G},\\ &-\widetilde{q}(y,t)+\nu\boldsymbol{n}\cdot\widetilde{S}(\widetilde{\boldsymbol{w}})\boldsymbol{n}=\Big(\sigma(H(x)-\mathcal{H}(y))\\ &+\frac{\omega^2}{2}(|x'|^2-|y'|^2)+\varkappa(U(x,t)-\mathcal{U}(y))\Big)\Big|_{x=e_\rho(y)},\end{aligned} \tag{2.21}$$

where $\boldsymbol{n}$ is the vector field (2.20).

By (2.20), the kinematic boundary condition $V_n=\boldsymbol{w}\cdot\boldsymbol{n}$ is equivalent to the following equality:

$$\rho_t(y,t)=\frac{\widetilde{\boldsymbol{w}}(y,t)\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}(y)}{\boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}(y)},\quad y\in\mathcal{G}. \tag{2.22}$$

From the calculations carried out in [**17**] it follows that

$$\sum_{i=1}^3\widehat{L}_{ij}N_i=N_j(y)\Lambda(y,\rho)-\frac{\partial\rho}{\partial y_j}(1-\rho\mathcal{H}(y))+\rho\sum_{m=1}^n\frac{\partial\rho}{\partial y_m}\frac{\partial N_m}{\partial y_j}, \tag{2.23}$$

where

$$\Lambda(y,\rho)=\boldsymbol{N}\cdot\widehat{\mathcal{L}}\boldsymbol{N}=1-\rho\mathcal{H}(y))+\rho^2\mathcal{K}(y) \tag{2.24}$$

and $\mathcal{K}$ is the Gaussian curvature of $\mathcal{G}$. Hence (2.22) is equivalent to the following equality:

$$\begin{aligned}\rho_t(y,t) &= \widetilde{\boldsymbol{w}}\cdot\boldsymbol{N} - \Lambda^{-1}(y,\rho)\\ &\quad\times\sum_{j=1}^{3}\widetilde{w}_j\Big(\frac{\partial\rho}{\partial y_j}(1-\rho\mathcal{H}(y)) - \rho\sum_{m=1}^{n}\frac{\partial\rho}{\partial y_m}\frac{\partial N_m}{\partial y_j}\Big).\end{aligned} \tag{2.25}$$

Now, we pass to the conditions (2.12) and (2.13). In terms of ρ, (2.12) can be written as

$$\int_{\mathcal{G}}\varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}}\psi_i(y,\rho)dS = 0, \quad i=1,2,3, \tag{2.26}$$

(see [**17**]), where

$$\begin{aligned}\varphi(y,\rho) &= \rho - \frac{\rho^2}{2}\mathcal{H}(y) + \frac{\rho^3}{3}\mathcal{K}(y),\\ \psi_i(y,\rho) &= \varphi(y,\rho)y_i + N_i(y)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(y) + \frac{\rho^4}{4}\mathcal{K}(y)\Big).\end{aligned} \tag{2.27}$$

Finally, the orthogonality conditions (2.13) take the form

$$\begin{aligned}&\int_{\mathcal{F}}\widetilde{\boldsymbol{w}}(y,t)L dy = 0,\\ &\int_{\mathcal{F}}L\widetilde{\boldsymbol{w}}(y,t)\cdot\boldsymbol{\eta}_i(e_\rho(y))dy = -\omega\int_{\mathcal{F}}L\boldsymbol{\eta}_3(e_\rho(y),t)\cdot\boldsymbol{\eta}_i(e_\rho(y))dy\\ &\qquad\qquad + \omega\int_{\mathcal{F}}\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy, \quad i=1,2,3.\end{aligned} \tag{2.28}$$

Hence the problem (2.9)–(2.13) is transformed into (2.18), (2.21), (2.25), (2.26), and (2.28).

We can now write the corresponding linearized problem. For this we should omit all the nonlinear terms with respect to $(\widetilde{\boldsymbol{w}},\widetilde{q},\rho)$ in (2.18), (2.21), (2.25) and replace the differences $H(e_\rho(y)) - \mathcal{H}(y)$ etc. with their first variations with respect to ρ. We use the well known formula

$$\delta(H(e_\rho(y)) - \mathcal{H}(y)) = \frac{d}{ds}H(e_{s\rho}(y))\Big|_{s=0} = \Delta_{\mathcal{G}}\rho(y,t) + b_1(y)\rho(y,t), \tag{2.29}$$

where $b_1(y) = \mathcal{H}^2(y) - 2\mathcal{K}(y) = c^2(y)$ is the sum of squares of principal curvatures of $\mathcal{G}$, as well as

$$\frac{1}{2}\delta(|x'|^2 - |y'|^2)\Big|_{x=e_\rho(y)} = \boldsymbol{N}(y)\cdot\boldsymbol{y}'\rho(y,t), \tag{2.30}$$

$$\delta(U(e_\rho(y),t) - \mathcal{U}(y)) = \rho(y,t)\frac{\partial\mathcal{U}(y)}{\partial N} + \int\limits_{\mathcal{G}}\frac{\rho(z,t)dS_z}{|z-y|}, \tag{2.31}$$

$$\delta\Big(\int\limits_{\Omega_t} f(x)dx - \int\limits_{\mathcal{F}} f(y)dy\Big) = \int\limits_{\mathcal{G}} f(y)\rho(y,t)dS, \tag{2.32}$$

(see [**17, 18**]). As a result, we obtain the following linear problem for $\boldsymbol{v}(x,t)$, $p(x,t)$, $x\in\mathcal{F}$, and $\rho(x,t)$, $x\in\mathcal{G}$:

$$\begin{gathered}
\boldsymbol{v}_t + 2\omega(e_3\times\boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \quad x\in\mathcal{F},\\
\Pi_0 S(\boldsymbol{v})\boldsymbol{N} = 0,\\
\boldsymbol{N}\cdot T(\boldsymbol{v},p)\boldsymbol{N} + B_0\rho = 0,\\
\rho_t = \boldsymbol{v}\cdot\boldsymbol{N}, \quad \rho(x,0) = \rho_0(x), \quad x\in\mathcal{G},\\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\mathcal{F}.
\end{gathered} \tag{2.33}$$

By B_0 we mean an integro-differential operator on $\mathcal{G}$ defined by

$$B_0\rho = -\sigma\Delta_{\mathcal{G}}\rho(x,t) - b(x)\rho(x,t) - \varkappa\int\limits_{\mathcal{G}}\frac{\rho(z,t)dS}{|x-z|} \tag{2.34}$$

with

$$b(x) = \sigma(\mathcal{H}^2(x) - 2\mathcal{K}(x)) + \frac{\omega^2}{2}\frac{\partial}{\partial N}|x'|^2 + \varkappa\frac{\partial\mathcal{U}(x)}{\partial N}. \tag{2.35}$$

Equations (2.33) should be supplemented with the orthogonality conditions

$$\int\limits_{\mathcal{G}}\rho(y,t)dS = 0, \quad \int\limits_{\mathcal{G}}\rho(y,t)y_i dS = 0, \quad i=1,2,3, \tag{2.36}$$

$$\int\limits_{\mathcal{F}}\boldsymbol{v}(x,t)dx = 0,$$

$$\int\limits_{\mathcal{F}}\boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int\limits_{\mathcal{G}}\rho(x,t)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS = 0, \quad i=1,2,3, \tag{2.37}$$

obtained by the linearization of (2.26), (2.28). It can be easily verified that if these conditions hold at the initial moment $t = 0$ for $(\boldsymbol{v}_0, \rho_0)$, then they are satisfied for all $t > 0$.

3. Linear Problem

We start with some important definitions and auxiliary relations. If the surface $\Gamma = \partial\Omega$ in (1.5) is defined by Equation (2.4): $x = y + \boldsymbol{N}(y)\rho(y)$, then the functional R can be considered as the functional depending on ρ and its first and second variations are defined by

$$\delta R = \frac{d}{ds} R[s\rho]\Big|_{s=0}, \quad \delta^2 R = \frac{d^2}{ds^2} R[s\rho]\Big|_{s=0}.$$

Let us introduce the operators

$$B\rho = B_0\rho + \frac{\omega^2 |\boldsymbol{\eta}_3(x)|^2}{\|\boldsymbol{\eta}_3\|^2_{L_2(\mathcal{F})}} \int\limits_{\mathcal{G}} \rho |\boldsymbol{\eta}_3(y)|^2 dS \tag{3.1}$$

and

$$\widehat{B}\rho = B\rho - \frac{1}{|\mathcal{G}|} \int\limits_{\mathcal{G}} B\rho dS. \tag{3.2}$$

It is not hard to verify that

$$\delta^2 R = \int\limits_{\mathcal{G}} \rho(y) B\rho(y) dS$$

(see [**17, 18, 21**]). Thus, if

$$\int\limits_{\mathcal{G}} \rho dS = 0,$$

then

$$\delta^2 R = \int\limits_{\mathcal{G}} \rho(y) B\rho(y) dS = \int\limits_{\mathcal{G}} \rho(y) \widehat{B}\rho(y) dS. \tag{3.3}$$

Moreover, due to (1.3), $\delta R = 0$, which means that (1.3) is the Euler equation for R.

For an arbitrary vector field $\boldsymbol{\eta}(x) = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}$ of rigid motion with $\boldsymbol{a}, \boldsymbol{b} = \text{const}$ we have

$$B_0(\boldsymbol{\eta} \cdot \boldsymbol{N}) = B(\boldsymbol{\eta} \cdot \boldsymbol{N}) = -\omega^2 \boldsymbol{x}' \cdot \boldsymbol{\eta}(x), \quad x \in \mathcal{G}. \tag{3.4}$$

Let

$$\widetilde{\mathcal{S}} = \int_{\mathcal{F}} (x_1^2 - x_3^2) dS = \int_{\mathcal{F}} (x_2^2 - x_3^2) dS = \int_{\mathcal{F}} ((x_1 \cos\varphi + x_2 \sin\varphi)^2 - x_3^2) dx$$

for all $\varphi \in [0, 2\pi)$. Since

$$\begin{aligned} \int_{\mathcal{G}} B(\boldsymbol{\eta}_1 \cdot \boldsymbol{N}) \boldsymbol{\eta}_1 \cdot \boldsymbol{N} dS &= \omega^2 \int_{\mathcal{G}} x_3 x_2 \boldsymbol{\eta}_1 \cdot \boldsymbol{N} dS = \omega^2 \widetilde{\mathcal{S}}, \\ \int_{\mathcal{G}} B(\boldsymbol{\eta}_2 \cdot \boldsymbol{N}) \boldsymbol{\eta}_2 \cdot \boldsymbol{N} dS &= -\omega^2 \int_{\mathcal{G}} x_3 x_1 \boldsymbol{\eta}_2 \cdot \boldsymbol{N} dS = \omega^2 \widetilde{\mathcal{S}}, \end{aligned} \tag{3.5}$$

the condition

$$\widetilde{\mathcal{S}} > 0 \tag{3.6}$$

is necessary for the positivity of $\delta^2 R$, and we assume that it is satisfied.

Arbitrary $\rho \in L_2(\mathcal{G})$ can be represented in the form

$$\rho(x) = \rho_0(x) + \rho_1(x) \tag{3.7}$$

where

$$\rho_0(x) = \widetilde{\mathcal{S}}^{-1} \Big(\boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) \int_{\mathcal{G}} \rho(y) y_3 y_2 dS - \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) \int_{\mathcal{G}} \rho(y) y_3 y_1 dS \Big)$$

and ρ_1 satisfies the orthogonality conditions

$$\int_{\mathcal{G}} \rho_1(y) y_3 y_1 dS = \int_{\mathcal{G}} \rho_1(y) y_3 y_2 dS = 0. \tag{3.8}$$

The proof of these statements can be found in [**20, 23**]. In particular, (3.7) follows from the elementary formulas already used in (3.5), namely,

$$\int_{\mathcal{G}} \boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) x_2 x_3 dS = \widetilde{\mathcal{S}}, \quad \int_{\mathcal{G}} \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) x_1 x_3 dS = -\widetilde{\mathcal{S}},$$

$$\int\limits_{\mathcal{G}} \boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) x_1 x_3 dS = \int\limits_{\mathcal{G}} \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) x_2 x_3 dS = 0.$$

Equation (3.7) defines a nonorthogonal projection Q onto the subspace of $L_2(\mathcal{G})$ satisfying conditions (3.8): $\rho_1 = Q\rho$.

Let us consider the nonhomogeneous linear problem

$$\begin{gathered} \boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\ \nabla \cdot \boldsymbol{v} = f = \nabla \cdot \boldsymbol{F}, \quad x \in \mathcal{F}, \\ \Pi_0 S(\boldsymbol{v})\boldsymbol{N} = \boldsymbol{b}(x,t), \quad \boldsymbol{N} \cdot T(\boldsymbol{v},p)\boldsymbol{N} + B_0\rho = d(x,t), \\ \rho_t - \boldsymbol{v} \cdot \boldsymbol{N} = g(x,t), \quad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F}. \end{gathered} \tag{3.9}$$

The following theorem on the solvability of this problem in anisotropic Hölder spaces is of a fundamental importance.

Theorem 3.1. *Let the data of the problem* (2.1)–(2.5) *satisfy the following assumptions*: $\boldsymbol{f}(\cdot,t) \in C^{\alpha}(\mathcal{F})$, $\alpha \in (0,1)$, $f(\cdot,t) \in C^{1+\alpha}(\mathcal{F})$, $\boldsymbol{F}_t(\cdot,t) \in C^{\alpha}(\mathcal{F})$ *for all* $t \in [0,T]$, $\boldsymbol{b} \in C^{1+\alpha,(1+\alpha)/2}(G_T)$, $G_T = \mathcal{G} \times [0,T]$, $d(\cdot,t) \in C^{1+\alpha}(\mathcal{G})$, $g(\cdot,t) \in C^{2+\alpha}(\mathcal{G})$ *for all* $t \in [0,T]$, $\boldsymbol{v}_0 \in C^{2+\alpha}(\mathcal{F})$, $\rho_0 \in C^{3+\alpha}(\mathcal{G})$. *Let the compatibility conditions*

$$\begin{gathered} \nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad \boldsymbol{b}(x,t) \cdot \boldsymbol{N}(x) = 0, \\ \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N}(x) = \boldsymbol{b}(x,0), \quad x \in \mathcal{G} \end{gathered} \tag{3.10}$$

be satisfied. Then the problem (3.9) *has a unique solution* $\boldsymbol{v}(\cdot,t) \in C^{2+\alpha}(\mathcal{F})$ *with* $\boldsymbol{v}_t(\cdot,t) \in C^{\alpha}(\mathcal{F})$, $p(\cdot,t) \in C^{1+\alpha}(\mathcal{F})$, $\rho(\cdot,t) \in C^{3+\alpha}(\mathcal{G})$ *for all* $t \in [0,T]$, *and the inequality*

$$Y_t(\boldsymbol{v},p,\rho) \leqslant c(t) M_t \tag{3.11}$$

holds for arbitrary $t \in [0,T]$, $c(t)$ *being a nondecreasing function of* t,

$$\begin{aligned} Y_t(\boldsymbol{v},p,\rho) = {} & \sup_{\tau \leqslant t} |\boldsymbol{v}_t(\cdot,\tau)|_{C^{\alpha}(\mathcal{F})} + \sup_{\tau \leqslant t} |\boldsymbol{v}(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{F})} \\ & + \sup_{\tau \leqslant t} |p(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{F})} + \sup_{\tau \leqslant t} |\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} \end{aligned} \tag{3.12}$$

and

$$M_t = \sup_{\tau \leqslant t} |\boldsymbol{f}(\cdot,\tau)|_{C^{\alpha}(\mathcal{F})} + |\boldsymbol{b}|_{C^{1+\alpha,(1+\alpha)/2}(G_t)} + \sup_{\tau \leqslant t} |f(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{F})}$$

$$+\sup_{\tau\leqslant t}|\boldsymbol{F}_t(\cdot,\tau)|_{C^\alpha(\mathcal{F})}+\sup_{\tau\leqslant t}|d(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{G})}$$
$$+\sup_{\tau\leqslant t}|g(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{G})}+|\boldsymbol{v}_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})}. \tag{3.13}$$

The proof of this theorem (see [**24**]) relies on the analysis of the initial-boundary value problem for the Stokes equation with the nonstandard boundary conditions

$$\Pi_0 S(\boldsymbol{v})\boldsymbol{N}=\boldsymbol{b}(x,t),$$
$$\boldsymbol{N}\cdot T(\boldsymbol{v})\boldsymbol{N}-\sigma\boldsymbol{N}\cdot\Delta_{\mathcal{G}}\int_0^t\boldsymbol{v}(x,\tau)d\tau=d(x,t) \tag{3.14}$$

carried out in [**15, 16**]. Note that the orthogonality conditions (2.36), (2.37) are not required here.

If $\boldsymbol{f}=0,\ f=0,\ b=0,\ d=0,\ g=0$, then (3.11) implies

$$|\boldsymbol{v}(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})}+|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}$$
$$\leqslant c(t)(|\boldsymbol{v}_0|_{C^{2+\alpha}(\mathcal{F})}+|\rho_0|_{C^{3+\alpha}(\mathcal{G})})\quad\forall t\leqslant T. \tag{3.15}$$

It turns out that in the case of positivity of δ^2R the solution of the same problem in the subspace (2.36), (2.37) satisfies (3.15) with $c(t)=e^{-bt}$, $b=\text{const}>0$. To prove this, we should consider the spectral problem

$$\lambda\boldsymbol{v}+2\omega(e_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=0,\quad\nabla\cdot\boldsymbol{v}=0,\quad x\in\mathcal{F},$$
$$\Pi_0S(\boldsymbol{v})\boldsymbol{N}=0,$$
$$\boldsymbol{N}\cdot T(\boldsymbol{v},p)\boldsymbol{N}+B_0\rho=0,$$
$$\lambda\rho=\boldsymbol{v}\cdot\boldsymbol{N},\quad x\in\mathcal{G}, \tag{3.16}$$
$$\int_{\mathcal{G}}\rho(y)dS=0,\qquad\int_{\mathcal{G}}\rho(y)y_idS=0,\quad i=1,2,3, \tag{3.17}$$
$$\int_{\mathcal{F}}\boldsymbol{v}(x)dx=0,$$
$$\int_{\mathcal{F}}\boldsymbol{v}(x)\cdot\boldsymbol{\eta}_i(x)dx+\omega\int_{\mathcal{G}}\rho(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS=0,\quad i=1,2,3. \tag{3.17$'$}$$

We assume that the spectral parameter λ is a complex number and $\boldsymbol{v},p,\rho$ are complex-valued functions. Following [**3, 8**], we can write Equations (3.16) in an abstract form

$$\lambda\varphi=A\varphi, \tag{3.18}$$

where $\varphi = (\boldsymbol{v}, \rho)^T$. We take the orthogonal in $L_2(\mathcal{F})$ projection P_J of the equation

$$\lambda \boldsymbol{v} + 2\omega(e_3 \times \boldsymbol{v}) - \nu\nabla^2 \boldsymbol{v} + \nabla p = 0$$

onto the space J of divergence-free vector fields defined in $\mathcal{F}$ and obtain

$$\lambda \boldsymbol{v} + 2\omega P_J(e_3 \times \boldsymbol{v}) - \nu\nabla^2 \boldsymbol{v} + \nabla s = 0$$

where s is a harmonic function in $\mathcal{F}$ satisfying the same boundary condition as p:

$$s(x,t) = \nu \boldsymbol{N}(x) \cdot S(\boldsymbol{v})\boldsymbol{N} + B_0\rho(x,t), \quad x \in \mathcal{G}.$$

We set $A = (A_{ij})_{i,j=1,2}$ so that

$$\begin{gathered} A\varphi = (A_{11}\boldsymbol{v} + A_{12}\rho, \quad A_{21}\boldsymbol{v})^T, \\ A_{11}\boldsymbol{v} = -2\omega P_J(e_3 \times \boldsymbol{v}) + \nu\nabla^2\boldsymbol{v} - \nabla s_1, \\ A_{12}\rho = -\nabla s_2, \\ A_{21}\boldsymbol{v} = \boldsymbol{v} \cdot N|_{\mathcal{G}}, \quad A_{22}\rho = 0. \end{gathered}$$

By s_i we mean harmonic functions in $\mathcal{F}$ satisfying the conditions

$$s_1(x,t) = \nu \boldsymbol{N}(x) \cdot S(\boldsymbol{v})\boldsymbol{N}(x), \quad s_2 = B_0\rho(x,t), \quad x \in \mathcal{G},$$

so that $s_1 + s_2 = s$. Thus, the pressure is excluded. For the domain of A we can choose a subspace of the Sobolev space $W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$ characterized by the conditions

$$\nabla \cdot \boldsymbol{v}(x) = 0, \quad \Pi_0 S(\boldsymbol{v})\boldsymbol{N}(x)|_{\mathcal{G}} = 0 \tag{3.18$'$}$$

and the orthogonality conditions (3.17), (3.17$'$) . If $(\boldsymbol{v}, p, \rho)$ satisfy (3.16)–(3.17$'$), then $(\boldsymbol{v}, \rho)$ satisfy (3.18) and, conversely, to every solution $(\boldsymbol{v}, \rho)$ of (3.18) corresponds the solution $(\boldsymbol{v}, p, \rho)$ of (3.16)–(3.17$'$) with $p = s + s'$,

$$\nabla^2 s'(x,t) = -2\omega\nabla \cdot (\boldsymbol{e}_3 \times \boldsymbol{v}), \quad x \in \mathcal{F}, \quad s'|_{x\in\mathcal{G}} = 0.$$

The evolution problem (3.9), (3.17), (3.17$'$) (with $\boldsymbol{f} = 0$, $f = 0$, $b = 0$, $d = 0$, $g = 0$) is equivalent to

$$\frac{d\varphi}{dt} - A\varphi = 0, \quad \varphi|_{t=0} = \varphi_0 \equiv (\boldsymbol{v}_0, \rho_0)^T. \tag{3.19}$$

Operator A possesses the following properties.

1. The range $R(A)$ of A is the set of functions $F=(\boldsymbol{f},g)^T\subset L_2(\mathcal{F})\times W_2^{3/2}(\mathcal{G})$ satisfying the conditions

$$\nabla\cdot\boldsymbol{f}(x)=0,\quad x\in\mathcal{F},$$

$$\int\limits_{\mathcal{G}} g(y)dS=0,\quad \int\limits_{\mathcal{G}} g(y)y_i dS=0,\quad i=1,2,3,$$

$$\int\limits_{\mathcal{F}}\boldsymbol{f}(x)dx=0,$$

$$\int\limits_{\mathcal{F}}\boldsymbol{f}(x)\cdot\boldsymbol{\eta}_i(x)dx+\omega\int\limits_{\mathcal{G}} g(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS=0,\quad i=1,2,3. \tag{3.20}$$

2. If $\mathrm{Re}\lambda\geqslant a\gg 1$, then the equation $\lambda\varphi-A\varphi=F$ is uniquely solvable for arbitrary $F\in R(A)$ and the solution satisfies the inequality

$$\begin{aligned}|\lambda|\|\boldsymbol{v}\|_{L_2(\mathcal{F})}+\|\boldsymbol{v}\|_{W_2^2(\mathcal{F})}+|\lambda|\|\rho\|_{W_2^{3/2}(\mathcal{G})}+\|\rho\|_{W_2^{5/2}(\mathcal{G})}&\\ \leqslant c(a)\Big(\|\boldsymbol{f}\|_{L_2(\mathcal{F})}+\|g\|_{W_2^{3/2}(\mathcal{G})}\Big);&\end{aligned} \tag{3.21}$$

moreover, if $F\in R(A)\cap(W_2^l(\mathcal{F})\times W_2^{l+3/2}(\mathcal{G}))$ for all $l>0$, then $\varphi\in D(A)\cap(W_2^{l+2}(\mathcal{F})\times W_2^{l+5/2}(\mathcal{G}))$ and

$$\begin{aligned}&|\lambda|^{l/2+1}\|\boldsymbol{v}\|_{L_2(\mathcal{F})}+\|\boldsymbol{v}\|_{W_2^{l+2}(\mathcal{F})}+|\lambda|^{l/2+1}\|\rho\|_{W_2^{3/2}(\mathcal{G})}+\|\rho\|_{W_2^{l+5/2}(\mathcal{G})}\\ &\leqslant c(a)(|\lambda|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F}}+\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})}+|\lambda|^{l/2}\|g\|_{W_2^{3/2}(\mathcal{G})}+\|g\|_{W_2^{l+3/2}(\mathcal{G})}).\end{aligned} \tag{3.22}$$

The operator $(\lambda I-A)^{-1}$, $\mathrm{Re}\lambda\geqslant a$, is compact.

3. The spectrum of A consists of a countable number of eigenvalues with the only accumulation point at infinity. Purely imaginary non-zero λ belong to a resolvent set of A. $\lambda=0$ can be an eigenvalue; the corresponding eigenfunctions are of the form $\varphi=(d_3(\rho)\boldsymbol{\eta}_3,\rho)^T$ with $\rho\in\mathrm{Ker}\widehat{B}$, i.e., $\widehat{B}\rho=0$, (d_3 is defined below in (3.24)) and there are no associated eigenfunctions.

4. If the form (3.3) is positive definite on the set of functions ρ satisfying (2.36), then all the eigenvalues of A have the negative real part.

5. If the form (3.3) can take negative values for some ρ satisfying (2.36), then A has a finite number of eigenvalues with the positive real part.

The proofs of these statements can be found in [**23, 19**]. We restrict ourselves to the presentation of their main ideas. We concentrate on the properties of the spectrum of A. The properties mentioned in Assertions 3 and 4 are verified in an elementary way. Let $\boldsymbol{v}, p, \rho$ be a solution of the problem (3.16)–(3.17′). By (3.17′),

$$\boldsymbol{v}(x) = \boldsymbol{v}^{\perp}(x) + \sum_{j=1}^{3} d_j(\rho)\boldsymbol{\eta}_i(x),$$

where

$$d_j(\rho) = -\frac{\omega}{\|\boldsymbol{\eta}_i\|^2_{L_2(\mathcal{F})}} \int_{\mathcal{G}} \rho(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dS \tag{3.24}$$

and $\boldsymbol{v}^{\perp}$ is a solenoidal vector field orthogonal to arbitrary $\boldsymbol{\eta}(x) = \boldsymbol{a}\times\boldsymbol{x}+\boldsymbol{b}$. Multiplying the first equation in (3.16) by $\boldsymbol{v}$, integrating over $\mathcal{F}$, and taking into account the boundary conditions, we obtain

$$\lambda\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}+2\omega\int_{\mathcal{F}}(v_1\bar{v}_2-v_2\bar{v}_1)dx+\bar{\lambda}\int_{\mathcal{G}}\bar{\rho}B_0\rho dS+\frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathcal{F})} = 0, \tag{3.25}$$

which implies

$$\mathrm{Re}\lambda\Big(\|\boldsymbol{v}^{\perp}\|^2_{L_2(\mathcal{F})} + \sum_{j=1}^{3}|d_j|^2\|\boldsymbol{\eta}_j\|^2_{L_2(\mathcal{F})} + \int_{\mathcal{G}}\bar{\rho}B_0\rho dS\Big) + \frac{\nu}{2}\|S(\boldsymbol{v}^{\perp})\|^2_{L_2(\mathcal{F})} = 0, \tag{3.26}$$

If $\mathrm{Re}\lambda > 0$ and $\delta^2 R > 0$, then (3.26) yields $\rho = 0$, $\boldsymbol{v} = 0$. If $\mathrm{Re}\lambda = 0$ and $\lambda \neq 0$, from (3.26) it follows that $S(\boldsymbol{v}^{\perp}) = 0$. By the Korn inequality, $\boldsymbol{v}^{\perp} = 0$, and (3.16) reduces to

$$\lambda\sum_{j=1}^{3} d_j\boldsymbol{\eta}_j + 2\omega(\boldsymbol{e}_3\times\sum_{j=1}^{3} d_j\boldsymbol{\eta}_j) + \nabla p = 0, \quad x\in\mathcal{F}, \tag{3.27}$$

$$\lambda\rho = \sum_{j=1}^{3} d_j\boldsymbol{\eta}_j(x)\cdot\boldsymbol{N}(x), \quad x\in\mathcal{G}. \tag{3.28}$$

The last equation implies

$$d_3(\rho) = -\frac{\omega}{\lambda\|\boldsymbol{\eta}_3\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}}\sum_{j=1}^{3} d_j\boldsymbol{\eta}_j(x)\cdot\boldsymbol{N}(x)|\boldsymbol{\eta}_3|^2 dS = 0, \tag{3.29}$$

$$\lambda d_1 = -\frac{\omega\widetilde{\mathcal{S}}}{\|\boldsymbol{\eta}_1\|^2_{L_2(\mathcal{F})}}d_2, \quad \lambda d_2 = \frac{\omega\widetilde{\mathcal{S}}}{\|\boldsymbol{\eta}_2\|^2_{L_2(\mathcal{F})}}d_1.$$

Applying the operation rot to (3.27), we obtain

$$\lambda d_1 = \omega d_2, \quad \lambda d_2 = -\omega d_1. \tag{3.30}$$

From (3.29) and (3.30) we conclude that $d_1 = d_2 = 0$. Hence $\rho = 0$, $\nabla p = 0$, $p|_{\mathcal{G}} = 0$, $p = 0$, which is required to prove.

If $\lambda = 0$, then the same argument yields

$$\boldsymbol{v}^{\perp} = 0, \quad \boldsymbol{v} = \sum_{j=1}^{3} d_j \boldsymbol{\eta}_j.$$

The condition (3.28), i.e.,

$$\sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x) \cdot \boldsymbol{N}(x)|_{\mathcal{G}} = 0$$

implies $d_1 = d_2 = 0$. Hence $\boldsymbol{v} = d_3\boldsymbol{\eta}_3$. Equations (3.16) yield $\nabla p = 2d_3\omega\boldsymbol{x}'$, $p = \omega d_3|x'|^2 + C$, and

$$-p + B_0\rho = B_0\rho - d_3\omega|x'|^2 - C = B\rho - C = 0, \quad x \in \mathcal{G},$$

i.e., $\widehat{B}\rho = 0$. Let $\varphi_1 = (\boldsymbol{v}_1, \rho_1)^T$ be an associated eigenfunction, i.e., let φ_1 satisfy $A\varphi_1 = \varphi_0$, where $\varphi_0 = (d_3(\rho_0)\boldsymbol{\eta}_3(x), \rho_0)^T$ is an eigenfunction (consequently, $\rho_0 \in \mathrm{Ker}\widehat{B}$). This means that $(\boldsymbol{v}_1, p_1, \rho_1)$ satisfy the relations

$$2\omega(e_3 \times \boldsymbol{v}_1) - \nu\nabla^2\boldsymbol{v}_1 + \nabla p_1 = d_3(\rho_0)\boldsymbol{\eta}_3(x), \quad \nabla \cdot \boldsymbol{v}_1 = 0, \quad x \in \mathcal{F},$$

$$\Pi_0 S(\boldsymbol{v}_1)\boldsymbol{N} = 0,$$

$$\boldsymbol{N} \cdot T(\boldsymbol{v}_1, p_1)\boldsymbol{N} + B_0\rho_1 = 0,$$

$$\boldsymbol{v}_1 \cdot \boldsymbol{N} = \rho_0, \quad x \in \mathcal{G}$$

and the orthogonality conditions (3.17), (3.17′), so

$$\boldsymbol{v}_1 = \boldsymbol{v}_1^{\perp} + \sum_{j=1}^{3} d_j(\rho_1)\boldsymbol{\eta}_j.$$

The functions $\boldsymbol{u}_1 = \boldsymbol{v}_1 - d_3(\rho_1)\boldsymbol{\eta}_3$, $s_1 = p_1 - d_3\omega|x'|^2 - C_1$, ρ_1 satisfy

$$\begin{gathered} 2\omega(e_3 \times \boldsymbol{u}_1) - \nu\nabla^2\boldsymbol{u}_1 + \nabla s_1 = d_3(\rho_0)\boldsymbol{\eta}_3(x), \\ \nabla \cdot \boldsymbol{u}_1 = 0, \quad x \in \mathcal{F}, \\ \Pi_0 S(\boldsymbol{u}_1)\boldsymbol{N} = 0, \quad \boldsymbol{N} \cdot T(\boldsymbol{u}_1, s_1)\boldsymbol{N} + \widehat{B}\rho_1 = 0, \\ \boldsymbol{u}_1 \cdot \boldsymbol{N} = \rho_0, \quad x \in \mathcal{G} \end{gathered} \tag{3.31}$$

(if the constant C_1 is chosen in an appropriate way) and

$$\int_{\mathcal{G}} \rho_1(y)dS = 0, \quad \int_{\mathcal{G}} \rho_1(y)y_i dS = 0, \quad i = 1,2,3,$$

$$\int_{\mathcal{F}} \boldsymbol{u}_1(x)dx = 0, \quad \int_{\mathcal{F}} \boldsymbol{u}_1(x)\cdot\boldsymbol{\eta}_3(x)dx = 0,$$

$$\int_{\mathcal{F}} \boldsymbol{u}_1(x)\cdot\boldsymbol{\eta}_\alpha(x)dx + \omega\int_{\mathcal{G}} \rho_1(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_\alpha(x)dS = 0, \quad \alpha = 1,2.$$

We have again $\boldsymbol{v}_1^\perp = 0$ and

$$\boldsymbol{u}_1\cdot\boldsymbol{N}|_{\mathcal{G}} = \sum_{\alpha=1}^{2} d_\alpha(\rho_1)\boldsymbol{\eta}_\alpha\cdot\boldsymbol{N}|_{\mathcal{G}} = \rho_0(x).$$

Since $\rho_0 \in \mathrm{Ker}\widehat{B}$ is orthogonal to x_1x_3 and to x_2x_3, we can use (3.5) and prove that $d_\alpha(\rho_1) = 0$. Hence $\rho_0 = 0$, φ_1 satisfies $A\varphi_1 = 0$ and belongs to the subspace of eigenfunctions φ_0. It follows that the dimension of the root space of A at the point $\lambda = 0$ is equal to dim $\mathrm{Ker}\widehat{B}$.

Let us turn to Statement 5. We assume that the form (3.3) can take negative values for some ρ satisfying (3.17). We should conclude that the spectral problem (3.16)–(3.17′) (or (3.18)) has nontrivial solutions for some λ with $\mathrm{Re}\lambda > 0$. It is convenient to pass from (3.16)–(3.17′) to the problem

$$\begin{gathered}
\lambda\boldsymbol{u} + 2\omega(e_3\times\boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = -\lambda d_3(\rho)\boldsymbol{\eta}_3(x) = -d_3(\boldsymbol{u}\cdot\boldsymbol{N})\boldsymbol{\eta}_3,\\
\nabla\cdot\boldsymbol{u} = 0, \quad x\in\mathcal{F},\\
\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = 0,\\
\boldsymbol{N}\cdot T(\boldsymbol{u},q)\boldsymbol{N} + \widehat{B}\rho = 0,\\
\lambda\rho = \boldsymbol{u}\cdot\boldsymbol{N}, \quad x\in\mathcal{G},
\end{gathered} \tag{3.32}$$

$$\int_{\mathcal{G}} \rho(y)dS = 0, \quad \int_{\mathcal{G}} \rho(y)y_i dS = 0, \quad i = 1,2,3, \tag{3.33}$$

$$\int_{\mathcal{F}} \boldsymbol{u}(x)dx = 0, \quad \int_{\mathcal{F}} \boldsymbol{u}(x)\cdot\boldsymbol{\eta}_3(x)dx = 0, \tag{3.34}$$

$$\int_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_i(x)dx + \omega \int_{\mathcal{G}} \rho(x)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x)dS = 0, \quad i = 1, 2, \tag{3.35}$$

where $\boldsymbol{u} = \boldsymbol{v} - d_3(\rho)\boldsymbol{\eta}_3(x)$ and $q = p - \omega d_3(\rho)|x'|^2 + \text{const}$ (a similar passage was already done; see (3.31)).

We introduce the following spaces:

H: the subspace of functions $\rho \in L_2(\mathcal{G})$ satisfying (3.33);

$\mathrm{Ker}\widehat{B}$: the set of functions $\rho \in H \cap W_2^2(\mathcal{G})$ satisfying $\widehat{B}\rho = 0$;

$H_0 = H \ominus \mathrm{Ker}\widehat{B}$;

$\widetilde{J}$: the subspace of J (i.e., of the space of solenoidal vector fields $\boldsymbol{u} \in L_2(\mathcal{F})$) whose elements satisfy (3.34);

H_1: the subspace of functions $\rho \in H_0$ satisfying the additional orthogonality conditions

$$\int_{\mathcal{G}} \rho \boldsymbol{\eta}_j \cdot \boldsymbol{N} dS = 0, \quad j = 1, 2,$$

$X = \widetilde{J} \times H_0$;

Y: subspace of elements $\psi = (\boldsymbol{u}, \rho)^T \in X$ satisfying the conditions

$$\int_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_j dx + \omega \int_{\mathcal{G}} \rho(x)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{N}(x)dS = 0, \quad j = 1, 2,$$

$Z = \widetilde{J} \times H_1$.

We denote by $\widetilde{P}$, P_0, and P_1 the orthogonal projections onto $\widetilde{J}$, H_0, and H_1 respectively. We set

$$(\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathcal{F}} = \int_{\mathcal{F}} \boldsymbol{u}_1(x) \cdot \boldsymbol{u}_2(x)dx, \quad (\rho_1, \rho_2)_{\mathcal{G}} = \int_{\mathcal{G}} \rho_1(x)\bar{\rho}_2(x)dS$$

and

$$(\psi_1, \psi_2)_X = (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathcal{F}} + (\rho_1, \rho_2)_{\mathcal{G}}$$

where $\psi_j = (\boldsymbol{u}_j, \rho_j)^T$, $j = 1, 2$. We recall that all the functions are complex-valued and the scalar products are Hermitian.

Since $2\omega P_J(\boldsymbol{e}_3 \times \boldsymbol{u}) = 2\omega \widetilde{P}(\boldsymbol{e}_3 \times \boldsymbol{u}) - d_3(\boldsymbol{u} \cdot \boldsymbol{N})\boldsymbol{\eta}_3$, we can write the problem (3.32)–(3.35) in the form

$$\begin{gathered}
\lambda \boldsymbol{u} + 2\omega \widetilde{P}(e_3 \times \boldsymbol{u}) - \nu \nabla^2 \boldsymbol{u} + \nabla r = 0, \quad \nabla \cdot \boldsymbol{u} = 0, \quad x \in \mathcal{F}, \\
\nabla^2 r = 0, \quad x \in \mathcal{F}, \\
r|_{\mathcal{G}} = \nu \boldsymbol{N} \cdot S(\boldsymbol{u})\boldsymbol{N} + \widehat{B}\rho, \\
\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = 0, \\
\lambda \rho = \boldsymbol{u} \cdot \boldsymbol{N}, \quad x \in \mathcal{G},
\end{gathered} \tag{3.36}$$

plus the orthogonality conditions (3.33)–(3.35). Moreover, we modify the last equation in (3.36) and consider the auxiliary spectral problem

$$\begin{gathered}
\lambda \boldsymbol{u} + 2\omega \widetilde{P}(e_3 \times \boldsymbol{u}) - \nu \nabla^2 \boldsymbol{u} + \nabla r = 0, \quad \nabla \cdot \boldsymbol{u} = 0, \quad x \in \mathcal{F}, \\
\nabla^2 r = 0, \quad x \in \mathcal{F}, \\
r|_{\mathcal{G}} = \nu \boldsymbol{N} \cdot S(\boldsymbol{u})\boldsymbol{N} + \widehat{B}\rho, \\
\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = 0, \\
\lambda \rho = P_0 \boldsymbol{u} \cdot \boldsymbol{N}, \quad x \in \mathcal{G},
\end{gathered} \tag{3.37}$$

complemented with (3.33)–(3.35). If $(\boldsymbol{u}, q, \rho)$ is a solution of the last problem with a nonzero λ, then $(\boldsymbol{u}, q, \rho + \lambda^{-1}(I - P_0)\boldsymbol{u} \cdot \boldsymbol{N})$ is a solution of the problem (3.33)–(3.36) with the same λ.

The problem (3.33)–(3.35), (3.37) can be also written in an abstract form similar to (3.18), namely,

$$\lambda \psi = A' \psi,$$

where $\psi = (\boldsymbol{u}, \rho)$, $A' = (A'_{ij})_{i,j=1,2}$, so that

$$\begin{gathered}
A'\psi = \Big(A'_{11}\boldsymbol{u} + A'_{12}\rho, \quad A'_{21}\boldsymbol{u} \Big)^T, \\
A'_{11}\boldsymbol{u} = -2\omega \widetilde{P}(e_3 \times \boldsymbol{u}) + \nu \nabla^2 \boldsymbol{u} - \nabla s_1, \quad A'_{12}\rho = -\nabla s_2, \\
A'_{21}\boldsymbol{u} = P_0 \boldsymbol{u} \cdot N|_{\mathcal{G}}, \quad A'_{22}\rho = 0
\end{gathered}$$

and s_1, s_2 are harmonic functions in $\mathcal{F}$ satisfying the boundary conditions

$$s_1(x,t) = \nu \boldsymbol{N}(x) \cdot S(\boldsymbol{u})\boldsymbol{N}(x), \quad s_2 = \widehat{B}\rho(x,t), \quad x \in \mathcal{G}.$$

For the domain $D(A')$ of A' we take the set of elements $\psi = (\boldsymbol{u}, \rho)^T \in (W_2^2(\mathcal{F}) \cap \widetilde{J}) \times (W_2^{5/2}(\mathcal{G}) \cap H_0)$ satisfying the conditions

$$\Pi_0 S(\boldsymbol{u})\boldsymbol{N}(x)|_{\mathcal{G}} = 0$$

and the orthogonality conditions (3.35). If $\psi \in D(A')$, then $A'\psi \in Y$.

We use the decomposition (3.7) of arbitrary $\rho \in H$. If $\rho \in H_0$, i.e., $\rho = P_0\rho$, then $\rho = P_0\rho_0 + P_0Q\rho$. As was shown [**26**], $P_0Q\rho$ is uniquely representable as $P_0Q\rho = P_0Qr$ with $r = P_1Q\rho \in H_1$. Hence

$$\rho = P_0\rho_0 + P_0Qr \quad \forall \rho \in H_0, \quad r = P_1Q\rho.$$

Moreover, by (3.4), we have

$$(\widehat{B}\rho, \rho)_{\mathcal{G}} = (\widehat{B}P_0\rho_0, P_0\rho_0)_{\mathcal{G}} + (\widehat{B}P_0Qr, P_0Qr)_{\mathcal{G}} = (\widehat{B}\rho_0, \rho_0)_{\mathcal{G}} + (\widehat{B}Qr, Qr)_{\mathcal{G}}$$

$$= \frac{\omega^2}{\widetilde{\mathcal{S}}} \sum_{j=1}^{2} \Big| \int_{\mathcal{G}} \rho(x) x_j x_3 dS \Big|^2 + (\widehat{B}Qr, Qr)_{\mathcal{G}}.$$

If (3.6) holds, then the first term on the right-hand side is nonnegative, which shows that

$$(\widehat{B}\rho, \rho)_{\mathcal{G}} \geqslant (\widehat{B}Qr, Qr)_{\mathcal{G}}. \tag{3.37$'$}$$

In particular, if the form (3.3) is negative for some $\rho \in H_0$, then the same is true for $(\widehat{B}Qr, Qr)_{\mathcal{G}}$, which means that the elliptic operator

$$B_1 = P_1 Q^* \widehat{B} Q P_1$$

has a finite number of negative eigenvalues $\lambda_k^{(-)}$ and a countable number of positive eigenvalues $\lambda_k^{(+)}$. It can be shown that $\mathrm{Ker} B_1 = \varnothing$ (see [**26**]). For arbitrary $r \in H_1 \cap W_2^2(\mathcal{G})$, taking into account the multiplicity of $\lambda_k^{(\pm)}$, we have

$$B_1 r = \sum_{k=1}^{m} \lambda_k^{(-)} (r, \varphi_k)_{\mathcal{G}} \varphi_k + \sum_{k=m+1}^{m} \lambda_k^{(+)} (r, \varphi_k)_{\mathcal{G}} \varphi_k$$

where $\varphi_k(x)$ are eigenfunctions of B_1. The spaces

$$H_- = \mathrm{Span}(\varphi_1, \dots, \varphi_m), \quad H_+ = \mathrm{Span}(\varphi_{m+1}, \dots)$$

are orthogonal to each other and $H_- \oplus H_+ = H_0$. We introduce the orthogonal projections onto these spaces P_-, P_+ and the operators $|B_1|$ and $|B_1|^{1/2}$ defined by the standard formulas

$$|B_1|\rho = \sum_{k=1}^{m} |\lambda_k^{(-)}| (\rho, \varphi_k) \varphi_k + \sum_{k=m+1}^{m} \lambda_k^{(+)} (\rho, \varphi_k) \varphi_k,$$

$$|B_1|^{1/2}\rho = \sum_{k=1}^{m} |\lambda_k^{(-)}|^{1/2} (\rho, \varphi_k) \varphi_k + \sum_{k=m+1}^{m} \lambda_k^{(+)1/2} (\rho, \varphi_k) \varphi_k.$$

It is easy to verify that

$$B_1 = |B_1|^{1/2} S |B_1|^{1/2}, \tag{3.38}$$

where $S = P_+ - P_-$. It is clear that $(Sr, r)_{\mathcal{G}} < 0$ for nonzero $r \in H_-$ and $(Sr, r)_{\mathcal{G}} > 0$ for nonzero $r \in H_+$.

Now, we pass to the spaces X, Y, Z. By (3.35), an arbitrary element $\psi = (\boldsymbol{u}, \rho)^T \in Y$ can be represented in the form

$$(\boldsymbol{u}, \rho)^T = (\boldsymbol{u}, P_0 Q\rho + P_0\rho_0)^T = (\boldsymbol{u}, P_0 Q r + P_0 \Sigma \boldsymbol{u})^T = \mathcal{L}\varphi,$$

where $r = P_1 Q\rho \in H_1$, $\varphi = (\boldsymbol{u}, r)^T \in Z$,

$$\Sigma \boldsymbol{u} = (\omega \widetilde{\mathcal{S}})^{-1} \Big(\boldsymbol{\eta}_1 \cdot \boldsymbol{N} \int\limits_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_2(x) dx - \boldsymbol{\eta}_2 \cdot \boldsymbol{N} \int\limits_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_1(x) dx \Big),$$

and

$$\mathcal{L} = \begin{pmatrix} I & 0 \\ P_0\Sigma & P_0 Q \end{pmatrix}.$$

The element $\varphi \in Z$ is expressed in terms of ψ by $\varphi = \mathcal{M}\psi$, where

$$\mathcal{M} = \begin{pmatrix} I & 0 \\ 0 & P_1 Q \end{pmatrix}.$$

We have $\mathcal{L}\mathcal{M}\psi = \psi$, $\mathcal{M}\mathcal{L}\varphi = \varphi$ and

$$\|\psi\|_X^2 \equiv \|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + \|\rho\|^2_{L_2(\mathcal{G})} \leqslant c_1 \|\varphi\|_X^2 \leqslant c_2 \|\psi\|_X^2.$$

Let

$$\mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & \widehat{B} \end{pmatrix}, \quad \mathcal{B}_1 = \mathcal{L}^* \mathcal{B} \mathcal{L}.$$

A direct computation shows that

$$\mathcal{B}_1 = \begin{pmatrix} I + \Sigma^* P_0 \widehat{B} P_0 \Sigma & \Sigma^* \widehat{B} P_0 Q \\ Q^* P_0 \widehat{B} \Sigma & Q^* P_0 \widehat{B} P_0 Q \end{pmatrix} = \begin{pmatrix} I + \Sigma^* \widehat{B} \Sigma & \Sigma^* \widehat{B} Q \\ Q^* \widehat{B} \Sigma & Q^* \widehat{B} Q \end{pmatrix}$$

$$= \operatorname{diag}(I + \Sigma^* \widehat{B} \Sigma, B_1)$$

because $Q^* \widehat{B} \Sigma = 0$ and $\Sigma^* \widehat{B} Q = 0$. By (3.38),

$$\mathcal{B}_1 = \mathcal{D} \mathcal{J} \mathcal{D},$$

where $\mathcal{D} = \mathrm{diag}(I, |B_1|^{1/2})$ and $\mathcal{J} = \mathrm{diag}(I + \Sigma^* \widehat{B}\Sigma, S)$. This operator is bounded, selfadjoint, invertible and

$$(\mathcal{J}\varphi, \varphi)_X = \|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} + q(\boldsymbol{u}) + (Sr, r)_{\mathcal{G}}, \tag{3.38$'$}$$

where $\varphi = (\boldsymbol{u}, r)^T$ is an arbitrary element of Z and

$$q(\boldsymbol{u}) = (\widehat{B}\Sigma\boldsymbol{u}, \Sigma\boldsymbol{u})_{\mathcal{G}} = \frac{1}{\widetilde{\mathcal{S}}} \sum_{j=1}^{2} \Big| \int\limits_{\mathcal{F}} \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_j dx \Big|^2 \geqslant 0.$$

Hence $(\mathcal{J}\varphi, \varphi)_X < 0$ for φ in a finite- dimensional space $Z_- = 0 \times H_-$ and $(\mathcal{J}\varphi, \varphi)_X > 0$ for $\varphi \in Z_+ = \widetilde{J} \times H_+$. The spaces Z_-, Z_+ are orthogonal and $Z_- \oplus Z_+ = Z$. The Hilbert space with the indefinite scalar product $(\mathcal{J}\varphi, \varphi)$ possessing the above properties is referred to as the Pontryagin space [**4**].

It is easy to check (see [**26**]) that for arbitrary $\psi \in D(A')$

$$\mathrm{Re}(\mathcal{B}A'\psi, \psi)_X \leqslant 0.$$

Let $A_1 = \mathcal{M}A'\mathcal{L}$, $\psi = \mathcal{L}\varphi$. We have

$$\begin{aligned}(\mathcal{B}A'\psi, \psi)_X &= (\mathcal{B}\mathcal{L}\mathcal{M}A'\mathcal{L}\varphi, \mathcal{L}\varphi)_X = (\mathcal{B}_1 A_1 \varphi, \varphi)_X \\ &= (\mathcal{D}\mathcal{J}\mathcal{D}A_1\mathcal{D}^{-1}\mathcal{D}\varphi, \varphi)_X = (\mathcal{J}\mathcal{D}A_1\mathcal{D}^{-1}\chi, \chi)_X,\end{aligned}$$

where $\chi = \mathcal{D}\varphi \in Z$. It follows that the operator $-i\mathcal{D}A_1\mathcal{D}^{-1} \equiv A_2$ satisfies the inequality

$$\mathrm{Im}(\mathcal{J}A_2\chi, \chi)_X \geqslant 0$$

for arbitrary $\chi \in Z$, i.e., it is $\mathcal{J}$-dissipative in the Pontryagin space Z. Since the spectrum of A (and of A_2) is discrete, it is maximal dissipative, and we can apply the M. Krein–Langer–Azizov theorem [**5, 2**]. According to this theorem, A_2 has m-dimensional invariant space $L \subset Z$ and all the eigenvalues of $A_2\Big|_L$ have nonpositive imaginary part. As a consequence, A' has a finite-dimensional invariant subspace $L' \subset Y$ and the eigenvalues of $A'\Big|_{L'}$ have nonnegative real part. In exactly the same way as it was done above for the operator A, it is possible to show that they cannot be purely imaginary, and this proves Assertion 5. We have thus shown that the "instability index" of the operator A is equal to m.

Remark. If (3.6) does not hold and $\widetilde{\mathcal{S}} < 0$, then Assertion 5 takes the following form: If $(\widehat{B}Qr, Qr)_{\mathcal{G}}$ can be negative for some $r \in H_1$, then the operator A has a finite number of eigenvalues with positive real part. The above arguments are still valid, although the inequality (3.37$'$) cannot be used any more and the form $q(\boldsymbol{u})$ in (3.38$'$) is not positive. It vanishes for

$\boldsymbol{u}$ in the space $J^\perp \subset \widetilde{J}$ orthogonal to $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ and is negative for nonzero $\boldsymbol{u} \in \widetilde{J} \ominus J^\perp$. Hence the definition of $Z_\pm$ should be modified as follows:

$$Z_+ = J^\perp \times H_+, \quad Z_- = (\widetilde{J} \ominus J^\perp) \times H_-.$$

Let us go back to the evolution linear problem (2.33), (3.36), (2.37). Assume that $\delta^2 R > 0$. Then the spectrum of A is located in the left complex half-plane, and from semigroup theory it follows that the solution of the problem (3.19) satisfies the inequality

$$\|\boldsymbol{v}(\cdot,t)\|_{L_2(\mathcal{F})} + \|\rho(\cdot,t)\|_{W_2^{3/2}(\mathcal{F})} \leqslant ce^{-b_1 t}(\|\boldsymbol{v}_0\|_{L_2(\mathcal{F})} + \|\rho_0\|_{W_2^{3/2}(\mathcal{F})}) \quad (3.39)$$

with a certain positive b_1. In [**10**]–[**18**], [**20, 21**], inequalities of this type are obtained by construction of a special "generalized energy." This method is presented below in Section 6. With the help of a local estimate for the problem (3.19) it is possible to deduce from (3.39) the inequality (3.15) with $c(t) = e^{-bt}$, as it was done in [**17**]–[**21**].

Theorem 3.2. *If $\delta^2 R > 0$ for arbitrary ρ satisfying* (3.33)*, then the problem* (2.33), (2.36), (2.37) *with $\boldsymbol{v}_0$ satisfying the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0 = 0, \quad \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N}|_{\mathcal{G}} = 0$$

is solvable in the infinite time interval $t > 0$, and

$$\begin{aligned} y_t(\boldsymbol{v}, q, \rho) \equiv{}& |\boldsymbol{v}_t(\cdot,t)|_{C^\alpha(\mathcal{F})} + |\boldsymbol{v}(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \\ &+ |p(\cdot,t)|_{C^{1+\alpha}(\mathcal{F})} + |\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} \\ &\leqslant ce^{-bt}(|\boldsymbol{v}_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}) \end{aligned} \quad (3.40)$$

with the constants independent of t.

Proof. We obtain estimate (3.40) that is uniform with respect to t. The solvability of the problem (2.33), (2.26), (2.37) in the infinite time interval follows from this estimate and Theorem 3.1. Let $t_0 \geqslant 1$, $\lambda \in (0, 1/2)$ and let $\zeta_\lambda(t)$ be a smooth cut-off function of time such that $\zeta_\lambda(t) = 1$ for $t > t_0 - 1 + \lambda$, $\zeta_\lambda(t) = 0$ for $t < t_0 - 1 + \lambda/2$, $\left|\dfrac{\partial^k \zeta_\lambda(t)}{\partial t^k}\right| \leqslant c\lambda^{-k}$. The functions $\boldsymbol{u} = \boldsymbol{v}\zeta_\lambda$, $r = \rho\zeta_\lambda$, $q = p\zeta_\lambda$ satisfy the relations

$$\begin{gathered} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{v}\zeta'_{\lambda t}, \quad \nabla \cdot \boldsymbol{u} = 0, \\ \Pi_0 S(\boldsymbol{u})\boldsymbol{N} = 0, \\ \boldsymbol{N} \cdot T(\boldsymbol{u}, q)\boldsymbol{N} + B_0 r = 0, \\ r_t - \boldsymbol{u} \cdot \boldsymbol{N} = \rho\zeta'_{\lambda t}, \quad r(x, t_0 - 1) = 0, \quad x \in \mathcal{G}, \end{gathered}$$

$$\boldsymbol{u}(x,t_0-1)=0,\quad x\in\mathcal{F}.$$

Let

$$Y_{t_1,t_2}(\boldsymbol{v},p,\rho)=\sup_{\tau\in(t_1,t_2)}|\boldsymbol{v}_t(\cdot,\tau)|_{C^\alpha(\mathcal{F})}+\sup_{\tau\in(t_1,t_2)}|\boldsymbol{v}(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{F})}$$

$$+\sup_{\tau\in(t_1,t_2)}|p(\cdot,\tau)|_{C^{1+\alpha}(\mathcal{F})}+\sup_{\tau\in(t_1,t_2)}|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}.$$

By (3.11),

$$\begin{aligned}&Y_{t_0-1+\lambda,t_0}(\boldsymbol{v},p,\rho)\leqslant Y_{t_0-1+\lambda/2,t_0}(\boldsymbol{u},q,r)\\&\leqslant c(\sup_{\tau\in(t_0-1,t_0)}|\boldsymbol{v}\zeta'_{\lambda t}|_{C^\alpha(\mathcal{F})}+\sup_{\tau\in(t_0-1,t_0)}|\rho\zeta'_{\lambda t}|_{C^{2+\alpha}(\mathcal{F})})\\&\leqslant c(\lambda^{-1}\sup_{\tau\in(t_0-1+\lambda/2,t_0)}|\boldsymbol{v}|_{C^\alpha(\mathcal{F})}+\lambda^{-1}\sup_{\tau\in(t_0-1,t_0)}|\rho|_{C^{2+\alpha}(\mathcal{G})})\end{aligned}$$

with the constant independent of t_0.

Using the interpolation inequalities, we can estimate the norms on the right-hand side by higher order norms

$$|\boldsymbol{v}(\cdot,\tau)|_{C^{2+\alpha}(\mathcal{F})},\quad |\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}$$

and by the L_2-norms of the same functions. We find

$$Y_{t_0-1+\lambda,t_0}(\boldsymbol{v},p,\rho)\leqslant\varepsilon Y_{t_0-1+\lambda/2,t_0}(\boldsymbol{v},p,\rho)$$

$$+c\lambda^{-M}(\sup_{\tau\in(t_0-1,t_0)}\|\boldsymbol{v}\|_{L_2(\mathcal{F})}+\sup_{\tau\in(t_0-1,t_0)}\|\rho\|_{L_2(\mathcal{G})})$$

with some $M>0$. This implies

$$y(\lambda)\leqslant 2^M\varepsilon y(\lambda/2)+K,$$

where $y(\lambda)=\lambda^M Y_{t_0-1+\lambda,t_0}(\boldsymbol{v},p,\rho)$ and

$$K=c(\sup_{\tau\in(t_0-1,t_0)}\|\boldsymbol{v}\|_{L_2(\mathcal{F})}+\sup_{\tau\in(t_0-1,t_0)}\|\rho\|_{L_2(\mathcal{G})}).$$

Choosing $\varepsilon\leqslant 2^{-M-1}$, we obtain

$$y(\lambda)\leqslant\frac{1}{2}y(\lambda/2)+K\leqslant\ldots\leqslant 2K.$$

Setting $\lambda=1/2$ and using (3.31), we find

$$|\boldsymbol{v}_t(\cdot,t_0)|_{C^\alpha(\mathcal{F})}+|\boldsymbol{v}(\cdot,t_0)|_{C^{2+\alpha}(\mathcal{F})}+|p(\cdot,t_0)|_{C^{1+\alpha}(\mathcal{F})}+|\rho(\cdot,t_0)|_{C^{3+\alpha}(\mathcal{G})}$$

$$\leqslant cK\leqslant ce^{-bt_0}(\|\boldsymbol{v}_0\|_{L_2(\mathcal{F})}+\|\rho_0\|_{L_2(\mathcal{G})}).$$

This inequality, together with (3.15) in the case $t_0 < 1$, yields the desired estimate (3.40). $\square$

In the case considered in statement 5 above, the problem (2.33), (2.36). (2.37) has a finite-dimensional subspace of solutions exponentially growing as $t \to \infty$. It is clear that they are smooth functions of x and t.

4. Stability and Instability of Zero Solution of Problem (2.9)–(2.13)

Using (2.18), (2.21), (2.25), we can write the problem (2.9)–(2.13) in the form (3.9), namely,

$$\begin{gathered}\boldsymbol{w}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{v} + \nabla q = \boldsymbol{f},\\ \nabla\cdot\boldsymbol{w} = f = \nabla\cdot\boldsymbol{F}, \quad y \in \mathcal{F},\\ \Pi_0 S(\boldsymbol{w})\boldsymbol{N} = \boldsymbol{b},\\ \boldsymbol{N}\cdot T(\boldsymbol{w}, q)\boldsymbol{N} + B_0\rho = d,\\ \rho_t - \boldsymbol{w}\cdot\boldsymbol{N} = g, \quad \rho(y,0) = \rho_0(y), \quad y \in \mathcal{G},\\ \boldsymbol{w}(y,0) = \boldsymbol{w}_0(e_\rho(y)), \quad y \in \mathcal{F},\end{gathered} \tag{4.1}$$

where $\boldsymbol{f}$, f, $\boldsymbol{F}$, $\boldsymbol{b}$, d, g are nonlinear functions of $\boldsymbol{w}$, p, ρ:

$$\begin{gathered}\boldsymbol{f} = \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{w} - (\mathcal{L}^{-1}\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + \nu(\widetilde{\nabla}^2\boldsymbol{w} - \nabla^2\boldsymbol{w}) + (\nabla - \widetilde{\nabla})q,\\ f = (I - \widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{w},\\ \boldsymbol{F} = (I - \widehat{\mathcal{L}})\boldsymbol{w},\\ \boldsymbol{b} = \Pi_0(\Pi_0 S(\boldsymbol{w})\boldsymbol{N} - \Pi\widetilde{S}(\boldsymbol{w})\widehat{\mathcal{L}}^T\boldsymbol{N}),\end{gathered} \tag{4.2}$$

$$\begin{aligned} d =& \nu(\boldsymbol{N}\cdot S(\boldsymbol{w})\boldsymbol{N} - |\widehat{\mathcal{L}}^T\boldsymbol{N}|^{-2}\widehat{\mathcal{L}}^T\boldsymbol{N}\cdot\widetilde{S}(\boldsymbol{w})\widehat{\mathcal{L}}^T\boldsymbol{N}) + \Big(\sigma(H(x) - \mathcal{H}(y))\\ &+ \frac{\omega^2}{2}(|x'|^2 - |y'|^2) + \varkappa(U(x,t) - \mathcal{U}(y))\Big)\Big|_{x=e_\rho(y)} + B_0\rho\\ &= \nu(\boldsymbol{N}\cdot S(\boldsymbol{w})\boldsymbol{N} - |\widehat{\mathcal{L}}^T\boldsymbol{N}|^{-2}\widehat{\mathcal{L}}^T\boldsymbol{N}\cdot\widetilde{S}(\boldsymbol{w})\widehat{\mathcal{L}}^T\boldsymbol{N})\\ &+ (\sigma(H(x) - \mathcal{H}(y) - \sigma\delta((H(x) - \mathcal{H}(y)))\\ &+ \frac{\omega^2}{2}(|x'|^2 - |y'|^2 - \delta(|x'|^2 - |y'|^2))\\ &+ \varkappa(U(x,t) - \mathcal{U}(y) - \delta(U(x,t) - \mathcal{U}(y))))\Big|_{x=e_\rho(y)}, \end{aligned}$$

$$g = -\Lambda^{-1}(y,\rho)\sum_{j=1}^{3} w_j\Big(\frac{\partial \rho}{\partial y_j}(1-\rho\mathcal{H}(y)) - \rho\sum_{m=1}^{n}\frac{\partial \rho}{\partial y_m}\frac{\partial N_m}{\partial y_j}\Big)$$

(we omitted the tilde over $\boldsymbol{w}$ and q). The orthogonality conditions (2.26), (2.28) can be written as

$$\int_{\mathcal{G}} \rho(y,t)dS = l(t), \quad \int_{\mathcal{G}} \rho(y,t)y_i dS = l_i(t), \quad i=1,2,3, \tag{4.3}$$

$$\int_{\mathcal{F}} \boldsymbol{w}(y,t)dy = \boldsymbol{m}(t),$$

$$\int_{\mathcal{F}} \boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_i(y)dy + \omega\int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dS = M_i(t), \quad i=1,2,3, \tag{4.4}$$

where

$$l(t) = \int_{\mathcal{G}} (\rho(y,t) - \varphi(y,\rho))dS,$$

$$l_i(t) = \int_{\mathcal{G}} (\rho(y,t)y_i - \psi_i(y,\rho))dS,$$

$$m_i(t) = \int_{\mathcal{F}} \boldsymbol{w}(y,t)(1-L)dx, \tag{4.5}$$

$$M_i(t) = \int_{\mathcal{F}} (\boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_i(y) - L\boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_i(e_\rho(y)))dy + \omega\Big(\int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy$$

$$-\int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y))\cdot\boldsymbol{\eta}_i(e_\rho(y))dy + \int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_i(y)\cdot\boldsymbol{\eta}_3(y)dS\Big)$$

$$= \int_{\mathcal{F}} (\boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_i(y) - L\boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_i(e_\rho(y)))dy$$

$$- \omega\Big(\int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y)) \cdot \boldsymbol{\eta}_i(e_\rho(y))dy - \int_{\mathcal{F}} \boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dy$$

$$+ \omega\delta\Big(\int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y)) \cdot \boldsymbol{\eta}_i(e_\rho(y))dy - \int_{\mathcal{F}} \boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dy\Big)\Big).$$

The initial data $(\boldsymbol{w}_0, \rho_0)$ (we write $\boldsymbol{w}_0(y)$ instead of $\boldsymbol{w}_0(e_\rho(y))$) satisfy these conditions at $t = 0$; moreover, the compatibility conditions also hold:

$$\boldsymbol{b}(y,t) \cdot \boldsymbol{N}(y) = 0, \quad \nabla \cdot \boldsymbol{w}_0 = f_0, \quad \Pi_0 S(\boldsymbol{w}_0)\boldsymbol{N} = \boldsymbol{b}_0(y) \tag{4.6}$$

with $f_0 = f|_{t=0}$ and $\boldsymbol{b}_0 = \boldsymbol{b}|_{t=0}$.

We need to estimate the nonlinear terms (4.2). We assume that the extension ρ^* of ρ into $\mathcal{F}$ is made by a linear operation such that (2.15) holds,

$$\begin{aligned} &|\rho^*(\cdot,t)|_{C^{3+\alpha}(\mathcal{F})} \leqslant c|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}, \\ &|\rho^*_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \leqslant c|\rho_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{G})} \end{aligned} \tag{4.7}$$

and $\mathcal{N}^*$ is smooth enough.

Proposition 4.1. *If* (4.7) *holds and*

$$\sup_{t<T} |\boldsymbol{w}(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} + \sup_{t<T} |\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} \leqslant \varepsilon_1, \tag{4.8}$$

where ε_1 *is a certain sufficiently small positive number, then the functions* (4.2) *satisfy the inequalities*

$$\begin{aligned} &\sup_{\tau\leqslant t} |\boldsymbol{f}(\boldsymbol{w},q,\rho)(\cdot,\tau)|_{C^{\alpha}(\mathcal{F})} + \sup_{\tau\leqslant t} |f(\boldsymbol{w},\rho)|_{C^{1+\alpha}(\mathcal{F})} \\ &+ \sup_{\tau\leqslant t} |\boldsymbol{F}_t(\boldsymbol{w},\rho)|_{C^{\alpha}(\mathcal{F})} + |\boldsymbol{b}(\boldsymbol{w},\rho)|_{C^{1+\alpha,(1+\alpha)/2}(G_t)} \\ &+ \sup_{\tau\leqslant t} |d(\boldsymbol{w},\rho)|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau\leqslant t} |g(\boldsymbol{w},\rho)|_{C^{2+\alpha}(\mathcal{G})} \leqslant cY_t^2(\boldsymbol{w},q,\rho), \end{aligned} \tag{4.9}$$

where $G_t = \mathcal{G} \times [0,t]$ *and* $t \in [0,T]$ *is arbitrary. Moreover, if* $(\boldsymbol{w}_1, \rho_1)$ *and* $(\boldsymbol{w}_2, \rho_2)$ *satisfy* (4.8), *then*

$$\begin{aligned} &\sup_{\tau\leqslant t} |\boldsymbol{f}(\boldsymbol{w}_1,q_1,\rho_1) - \boldsymbol{f}(\boldsymbol{w}_2,q_2,\rho_2)|_{C^{\alpha}(\mathcal{F})} + \sup_{\tau\leqslant t} |f(\boldsymbol{w}_1,\rho_1) - f(\boldsymbol{w}_2,\rho_2)|_{C^{1+\alpha}(\mathcal{F})} \\ &+ \sup_{\tau\leqslant t} |\boldsymbol{F}_t(\boldsymbol{w}_1,\rho_1) - \boldsymbol{F}_t(\boldsymbol{w}_2,\rho_2)|_{C^{\alpha}(\mathcal{F})} + |\boldsymbol{b}(\boldsymbol{w}_1,\rho_1) - \boldsymbol{b}(\boldsymbol{w}_2,\rho_2)|_{C^{1+\alpha,(1+\alpha)/2}(G_t)} \\ &+ \sup_{\tau\leqslant t} |d(\boldsymbol{w}_1,\rho_1) - d(\boldsymbol{w}_2,\rho_2)|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau\leqslant t} |g(\boldsymbol{w}_1,\rho_1) - g(\boldsymbol{w}_2,\rho_2)|_{C^{2+\alpha}(\mathcal{G})} \\ &\leqslant cY_t(\boldsymbol{w}_1 - \boldsymbol{w}_2, q_1 - q_2, \rho_1 - \rho_2)(Y_t(\boldsymbol{w}_1,q_1,\rho_1) + Y_t(\boldsymbol{w}_2,q_2,\rho_2)). \end{aligned} \tag{4.10}$$

The proof of (4.9), (4.10) is transparent but lengthy and is omitted (see [**18, 21, 16**]). Proposition 4.1 and Theorem 3.1 allow one to prove the local solvability of the problem (4.1). We will use the following proposition.

Proposition 4.2. *For arbitrary $T > 0$ there exists a number $\varepsilon_2(T) > 0$ such that, in the case*

$$|\boldsymbol{w}_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})} \leqslant \varepsilon_2 \tag{4.11}$$

the problem (4.1) *with initial data $\boldsymbol{w}_0 \in C^{2+\alpha}(\mathcal{F})$, $\rho_0 \in C^{3+\alpha}(\mathcal{G})$ satisfying the compatibility conditions*

$$\nabla \cdot \boldsymbol{w}_0 = f(\boldsymbol{w}_0, \rho_0) \equiv f_0(y), \quad \Pi_0 S(\boldsymbol{w})\boldsymbol{N} = \boldsymbol{b}(\boldsymbol{w}_0, \rho_0) \equiv \boldsymbol{b}_0(y), \tag{4.12}$$

is uniquely solvable in the interval of time $t \in [0, T]$ and the solution satisfies the inequality

$$Y_t(\boldsymbol{w}, q, \rho) \leqslant c(|\boldsymbol{w}_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}), \tag{4.13}$$

where Y_t is defined in (3.12).

The conditions (4.14) are equivalent to the compatibility conditions

$$\nabla \cdot \boldsymbol{w}_0 = 0, \quad S(\boldsymbol{w}_0)\boldsymbol{n}_0 - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot S(\boldsymbol{w}_0)\boldsymbol{n}_0 = 0$$

in the problem (2.9)-(2.11) ($\boldsymbol{n}_0$ is the normal to Γ_0). Along with (4.1), we consider the problem

$$\begin{gathered}
\boldsymbol{w}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{v} + \nabla q = \boldsymbol{f}(\boldsymbol{w}' + \boldsymbol{w}, q' + q, \rho' + \rho) \\
\nabla \cdot \boldsymbol{w} = f(\boldsymbol{w}' + \boldsymbol{w}, \rho' + \rho) = \nabla \cdot \boldsymbol{F}(\boldsymbol{w}' + \boldsymbol{w}, \rho' + \rho), \quad y \in \mathcal{F}, \\
\Pi_0 S(\boldsymbol{w})\boldsymbol{N} = \boldsymbol{b}(\boldsymbol{w}' + \boldsymbol{w}, \rho' + \rho), \\
\boldsymbol{N} \cdot T(\boldsymbol{w}, q)\boldsymbol{N} + B_0\rho = d(\boldsymbol{w}' + \boldsymbol{w}, \rho' + \rho), \\
\rho_t - \boldsymbol{w} \cdot \boldsymbol{N} = g(\boldsymbol{w}' + \boldsymbol{w}, \rho' + \rho), \quad \rho(y, 0) = \rho_0(y), \quad y \in \mathcal{G}, \\
\boldsymbol{w}(y, 0) = \boldsymbol{w}_0(y), \quad y \in \mathcal{F},
\end{gathered} \tag{4.14}$$

Proposition 4.3. *For arbitrary $T > 0$ there exist numbers $\varepsilon_2(T) > 0$ and $\varepsilon_3(T) > 0$ such that, in the case*

$$|\boldsymbol{w}_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})} \leqslant \varepsilon_2,$$

$$Y_t(\boldsymbol{w}', q', \rho') \leqslant \varepsilon_3$$

the problem (4.14) *with initial data* $\boldsymbol{w}_0 \in C^{2+\alpha}(\mathcal{F})$, $\rho_0 \in C^{3+\alpha}(\mathcal{G})$ *satisfying the compatibility conditions*

$$\begin{aligned} \nabla \cdot \boldsymbol{w}_0 &= f(\boldsymbol{w}'|_{t=0} + \boldsymbol{w}_0, \rho'|_{t=0} + \rho_0) \\ \Pi_0 S(\boldsymbol{w})\boldsymbol{N} &= \boldsymbol{b}(\boldsymbol{w}'|_{t=0} + \boldsymbol{w}_0, \rho'|_{t=0} + \rho_0) \end{aligned} \tag{4.15}$$

is uniquely solvable in the interval of time $t \in [0, T]$ *and the solution satisfies the inequality*

$$Y_t(\boldsymbol{w}, q, \rho) \leqslant c(|w_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})} + Y_t^2(\boldsymbol{w}', q', \rho')) \tag{4.16}$$

The proof can be carried out by successive approximations on the basis of estimates (3.11), (4.9), (4.10). Note that the orthogonality conditions are not required in the last two propositions.

The following proposition is an analog of Proposition 2.2 in [**24**].

Proposition 4.4. *For arbitrary number* l *vectors* $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{M}$ *a function* $f_0 \in C^{1+\alpha}(\mathcal{F})$ *and a tangential vector field* $\boldsymbol{b}_0 \in C^{1+\alpha}(\mathcal{G})$, *i.e., such that* $\boldsymbol{b} \cdot \boldsymbol{N} = 0$, *there exist* $r \in C^{3+\alpha}(\mathcal{G})$ *and* $\boldsymbol{u} \in C^{2+\alpha}(\mathcal{F})$ *satisfying the conditions*

$$\begin{gathered} \int_{\mathcal{G}} r(y)dS = l, \\ \int_{\mathcal{G}} r(y)y_i dS = l_i, \quad i = 1, 2, 3, \qquad (4.17) \\ \int_{\mathcal{F}} \boldsymbol{u}(y)dy = \boldsymbol{m}, \end{gathered}$$

$$\begin{gathered} \int_{\mathcal{F}} \boldsymbol{u}(y) \cdot \boldsymbol{\eta}_i(y)dy + \omega \int_{\mathcal{G}} r(y)\boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dS = M_i, \quad i = 1, 2, 3, \\ \nabla \cdot \boldsymbol{u}(x) = f_0(x), \quad x \in \mathcal{F} \qquad (4.18) \\ \Pi_0 S(\boldsymbol{u})\boldsymbol{N}(x) = \boldsymbol{b}_0(x), \quad x \in \mathcal{G} \end{gathered}$$

and the inequality

$$|r|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{u}|_{C^{2+\alpha}(\mathcal{F})} \leqslant c(|l| + |\boldsymbol{l}| + |\boldsymbol{m}| + |\boldsymbol{M}| + |f_0|_{C^{1+\alpha}(\mathcal{F})} + |\boldsymbol{b}_0|_{C^{1+\alpha}(\mathcal{G})}). \quad (4.19)$$

PROOF. We put

$$r(y) = \frac{l\boldsymbol{N}(y)\cdot\boldsymbol{y}}{3|\mathcal{F}|} + \frac{1}{|\mathcal{F}|}\boldsymbol{l}\cdot\boldsymbol{N}(y), \quad y \in \mathcal{G}.$$

This function satisfies (4.17). Next, we construct the vector field $\boldsymbol{u}_1(y)$ satisfying the equations

$$\nabla\cdot\boldsymbol{u}_1(y) = f_0(y), \quad y \in \mathcal{G}, \quad \boldsymbol{u}_1(y)\cdot\boldsymbol{N}(y) = f_1(y), \quad y \in \mathcal{G},$$

with

$$f_1(y) = \frac{\boldsymbol{N}(y)\cdot\boldsymbol{y}}{3|\mathcal{F}|}\int\limits_{\mathcal{F}} f_0(z)dz + \frac{1}{|\mathcal{F}|}\Big(\int\limits_{\mathcal{F}} f_0(z)\boldsymbol{z}dz + \boldsymbol{m}\Big)\cdot\boldsymbol{N}(y)$$

and the inequality

$$|\boldsymbol{u}_1|_{C^{2+\alpha}(\mathcal{F})} \leqslant c(|f_0|_{C^{1+\alpha}(\mathcal{F})} + |\boldsymbol{f}_1|_{C^{2+\alpha}(\mathcal{G})}) \leqslant c(|f_0|_{C^{1+\alpha}(\mathcal{F})} + |\boldsymbol{m}|).$$

It is clear that

$$\int\limits_{\mathcal{G}} f_1(y)dS = \int\limits_{\mathcal{F}} f_0(y)dy, \quad \int\limits_{\mathcal{G}} f_1(y)y_i dS = \int\limits_{\mathcal{F}} f_0(y)y_i dy + m_i.$$

On the other hand,

$$\int\limits_{\mathcal{G}} f_1(y)y_i dS = \int\limits_{\mathcal{F}} \nabla\cdot\boldsymbol{u}(y)y_i dy + \int\limits_{\mathcal{F}} u_{1i}(y)dy, \quad i = 1,2,3.$$

Hence

$$\int\limits_{\mathcal{F}} \boldsymbol{u}_1(y)dy = \boldsymbol{m}.$$

Now, we argue as in Proposition 2.2 in [**24**] and construct $\boldsymbol{u}_2 \in C^{2+\alpha}(\mathcal{F})$ such that

$$\Pi_0 S(\boldsymbol{u}_2)\boldsymbol{N}(y) = \boldsymbol{b}_0(y) - \Pi_0 S(\boldsymbol{u}_1)\boldsymbol{N}(y) \equiv \boldsymbol{b}'(y).$$

We take it in the form $\boldsymbol{u}_2(y) = \mathrm{rot}\boldsymbol{\Phi}(y,t)$ with $\boldsymbol{\Phi} \in C^{3+\alpha}(\mathcal{F})$ satisfying the conditions

$$\boldsymbol{\Phi}(y) = \frac{\partial\boldsymbol{\Phi}}{\partial N} = 0, \quad \frac{\partial^2\boldsymbol{\Phi}}{\partial N^2} = \boldsymbol{b}'(y)\times\boldsymbol{N}(y), \quad y \in \mathcal{G},$$

and the estimate

$$|\boldsymbol{\Phi}|_{C^{3+\alpha}(\mathcal{F})} \leqslant c|\boldsymbol{b}'|_{C^{1+\alpha}(\mathcal{G})}.$$

It is clear that $\boldsymbol{u}_2(y) = 0$ on $\mathcal{G}$ and

$$\int\limits_{\mathcal{F}} \boldsymbol{u}_2(y)dy = 0;$$

moreover,

$$\frac{\partial \boldsymbol{u}_2(y)}{\partial N} = \boldsymbol{N}(y) \times \frac{\partial^2 \boldsymbol{\Phi}(y,t)}{\partial N^2}, \quad y \in \mathcal{G},$$

which implies $\boldsymbol{N} \cdot \dfrac{\partial \boldsymbol{u}_2}{\partial N}\Big|_{\mathcal{G}} = 0$ and

$$\Pi_0 S(\boldsymbol{u}_2)\boldsymbol{N} = \frac{\partial \boldsymbol{u}_2}{\partial N} = \boldsymbol{N} \times [\boldsymbol{b}' \times \boldsymbol{N}] = \boldsymbol{b}', \quad y \in \mathcal{G}.$$

Finally, we set

$$\boldsymbol{u}_3(y) = \sum_{i=1}^{3} \widehat{M}_i \mathrm{rot}\boldsymbol{e}_i A(y)$$

where $A \in C_0^\infty(\mathcal{F})$,

$$\int\limits_{\mathcal{F}} A(y)dy = 1/2$$

and

$$\widehat{M}_i = M_i - \int\limits_{\mathcal{F}} (\boldsymbol{u}_1(y) + \boldsymbol{u}_2(y)) \cdot \boldsymbol{\eta}_i(y)dy - \omega \int\limits_{\mathcal{G}} r(y)\boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dS.$$

Since $\mathrm{rot}\boldsymbol{\eta}_i = 2\boldsymbol{e}_i$, we have

$$\int\limits_{\mathcal{F}} \boldsymbol{u}_3(y) \cdot \boldsymbol{\eta}_i(y)dy = \sum_{j=1}^{3} \widehat{M}_j \boldsymbol{e}_j \cdot \boldsymbol{e}_i = \widehat{M}_i.$$

We also have the estimate

$$|\boldsymbol{u}_3|_{C^{2+\alpha}(\mathcal{F})} \leqslant c \sum_{j=1}^{3} |\widehat{M}_j|.$$

The above-defined function $r(y)$ and $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$ satisfy all the necessary requirements. □

Now, we pass to the analysis of the stability of the zero solution of the problem (4.1)–(4.4).

Theorem 4.1. *If the form*

$$\int\limits_{\mathcal{G}} \rho \widehat{B} \rho dS$$

is positive definite for all ρ *satisfying* (3.17)*, then the problem* (4.1)–(4.4) *with the initial data satisfying the smallness condition*

$$M_0 \equiv |\rho_0|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}_0|_{C^{2+\alpha}(\mathcal{F})} \leqslant \varepsilon \ll, 1 \tag{4.19'}$$

as well as the orthogonality and compatibility conditions (4.3)–(4.6) *for* $t = 0$ *has a unique solution* $\boldsymbol{w}(\cdot, t) \in C^{2+\alpha}(\mathcal{F})$ *with* $\boldsymbol{w}_t(\cdot, t) \in C^{\alpha}(\mathcal{F})$*,* $q(\cdot, t) \in C^{1+\alpha}(\mathcal{F})$*,* $\rho(\cdot, t) \in C^{3+\alpha}(\mathcal{G})$ *for all* $t \geqslant 0$*, that satisfies the inequality*

$$y_t(\boldsymbol{v}, p, \rho) \leqslant e^{-bt} M_0, \quad b = \text{const} > 0 \tag{4.20}$$

($y_t(\boldsymbol{v}, p, \rho)$ is defined in (3.40)).

Proof. We look for $(\boldsymbol{w}, q, \rho)$ in the form

$$\boldsymbol{w} = \boldsymbol{w}' + \boldsymbol{w}'', \quad q = q' + q'', \quad \rho = \rho' + \rho'', \tag{4.21}$$

where $(\boldsymbol{w}', q', \rho')$ is a solution of the linear problem

$$\begin{gathered}
\boldsymbol{w}'_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}') - \nu\nabla^2 \boldsymbol{w}' + \nabla q' = 0, \\
\nabla \cdot \boldsymbol{w}' = 0, \quad y \in \mathcal{F}, \\
\Pi_0 S(\boldsymbol{w}')\boldsymbol{N} = 0, \\
\boldsymbol{N} \cdot T(\boldsymbol{w}')\boldsymbol{N} + B_0 \rho' = 0, \\
\rho'_t - \boldsymbol{w}' \cdot \boldsymbol{N} = 0, \quad \rho'(y, 0) = \rho'_0(y), \quad y \in \mathcal{G}, \\
\boldsymbol{w}'(y, 0) = \boldsymbol{w}'_0(y), \quad y \in \mathcal{F},
\end{gathered} \tag{4.22}$$

and $(\boldsymbol{w}'', p'', \rho'')$ is a solution of the nonlinear problem

$$\begin{gathered}
\boldsymbol{w}''_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}'') - \nu\nabla^2 \boldsymbol{w}'' + \nabla q'' = \boldsymbol{f}(\boldsymbol{w}' + \boldsymbol{w}'', q' + q'', \rho' + \rho''), \\
\nabla \cdot \boldsymbol{w}'' = f(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho'') = \nabla \cdot \boldsymbol{F}(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''), \ y \in \mathcal{F}, \\
\Pi_0 S(\boldsymbol{w}'')\boldsymbol{N} = \boldsymbol{b}(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''), \\
\boldsymbol{N} \cdot T(\boldsymbol{w}'')\boldsymbol{N} + B_0 \rho'' = d(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho'')), \\
\rho''_t - \boldsymbol{w}'' \cdot \boldsymbol{N} = g(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''), \quad \rho''(y, 0) = \rho''_0(y), \ y \in \mathcal{G}, \\
\boldsymbol{w}''(y, 0) = \boldsymbol{w}''_0(y), \quad y \in \mathcal{F}.
\end{gathered} \tag{4.23}$$

We define $(\rho''(y), \boldsymbol{w}''_0(y))$ as in Proposition 4.4 with $l = l(0)$, $\boldsymbol{l} = \boldsymbol{l}(0)$, $\boldsymbol{m} = \boldsymbol{m}(0)$, $\boldsymbol{M} = \boldsymbol{M}(0)$, $f_0(x) = f_0(x, 0)$, $\boldsymbol{b}_0(x) = \boldsymbol{b}(x, 0)$, where $l(0)$, $\boldsymbol{l}(0)$, $\boldsymbol{m}(0)$, $\boldsymbol{M}(0)$ are the same functions as in (4.3), (4.4) (with t=0) and $f_0(x, t)$, $\boldsymbol{b}(x, t)$ are the functions in the compatibility conditions (4.6). From

(4.19) and the definition of $l(0)$, $\boldsymbol{l}(0)$, $\boldsymbol{m}(0)$, $\boldsymbol{M}(0)$, $f_0(x)$, $\boldsymbol{b}_0(x)$ it follows that

$$|\rho_0''|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}_0''|_{C^{2+\alpha}(\mathcal{F})} \leqslant c_1(|\rho_0|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}_0|_{C^{2+\alpha}(\mathcal{F})})^2, \tag{4.24}$$

The differences

$$\rho_0'(y) = \rho_0(y) - \rho_0''(y),$$
$$\boldsymbol{w}_0'(y) = \boldsymbol{w}_0(y) - \boldsymbol{w}_0''(y)$$

satisfy the homogeneous compatibility and orthogonality conditions

$$\nabla \cdot \boldsymbol{w}_0'(y) = 0, \quad y \in \mathcal{F}, \quad \Pi_0 S(\boldsymbol{w}_0')\boldsymbol{N}\Big|_{\mathcal{G}} = 0,$$

$$\int\limits_{\mathcal{G}} \rho_0'(y)dS = 0, \quad \int\limits_{\mathcal{G}} \rho_0'(y)y_i dS = 0, \quad i = 1,2,3,$$

$$\int\limits_{\mathcal{F}} \boldsymbol{w}_0'(y)dy = 0, \tag{4.25}$$

$$\int\limits_{\mathcal{F}} \boldsymbol{w}_0'(y) \cdot \boldsymbol{\eta}_i(y)dy + \omega \int\limits_{\mathcal{G}} \rho_0'(y)\boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y)dS = 0, \quad i = 1,2,3.$$

Hence the problem (4.22) is solvable in an infinite time interval $t > 0$ and the solution satisfies the inequality

$$y_t(\boldsymbol{w}', q', \rho') \leqslant ce^{-bt}(|\rho_0'|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}_0'|_{C^{2+\alpha}(\mathcal{F})}) \leqslant c_2 e^{-bt} M_0,$$
$$Y_T(\boldsymbol{w}', q', \rho') \leqslant c_2 M_0. \tag{4.26}$$

We fix T in such a way that

$$c_2 e^{-bT} \leqslant \frac{1}{3}$$

and require

$$c_2 M_0 \leqslant \varepsilon_3(T), \quad c_1 M_0^2 \leqslant \varepsilon_2.$$

Then

$$|\rho_0''|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}_0''|_{C^{2+\alpha}(\mathcal{F})} \leqslant \varepsilon_2, \quad Y_T(\boldsymbol{w}', q', \rho') \leqslant \varepsilon_3,$$

and we can use Proposition 4.3. By this proposition, the problem (4.23) is solvable in the time interval $(0, T)$ and, by (4.16), (4.24), and (4.26),

$$Y_T(\boldsymbol{w}'', q'', \rho'') \leqslant c_3 M_0^2,$$

which implies

$$y_T(\boldsymbol{w}, q, \rho) \leqslant \frac{1}{3} M_0 + c_3 M_0^2.$$

Now, if we subject ε to one more restriction

$$c_3\varepsilon \leqslant \frac{1}{3},$$

then

$$y_T(\boldsymbol{w}, q, \rho) \leqslant \frac{2}{3} M_0.$$

Since $(\boldsymbol{w}(x,t), \rho(x,t))$ satisfy the conditions (4.3), (4.4) with $t = T$, the equations

$$\nabla \cdot \boldsymbol{w}(x,T) = f(x,T), \quad \Pi_0 S(\boldsymbol{w}(x,T))\boldsymbol{N}(x) = \boldsymbol{b}(x,T,) \tag{4.27}$$

and the inequality

$$|\rho(\cdot,T)|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}(\cdot,T)|_{C^{2+\alpha}(\mathcal{F})} \leqslant \frac{2}{3}\varepsilon,$$

we can repeat the above procedure in the interval $t \in [T, 2T]$ etc. and show at the end that

$$y_{kT}(\boldsymbol{w}, q, \rho) \leqslant \left(\frac{2}{3}\right)^k M_0, \quad k = 1, \dots,$$

which implies (4.20). □

Let us consider the case of the instability of the zero solution of the problem (4.1)–(4.4).

Theorem 4.2. *Let* (3.6) *be satisfied, and let the form*

$$\int_{\mathcal{G}} \rho \widehat{B} \rho dS$$

take negative values for some ρ satisfying (3.17)*. Then there exist initial data $(\boldsymbol{w}_0, \rho_0)$ with arbitrarily small norm* (4.11) *such that the corresponding solution of* (4.1)–(4.4) *leaves sooner or later a certain neighborhood of zero, i.e., for some $t > 0$, $\varepsilon > 0$*

$$|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \geqslant \varepsilon.$$

PROOF. We again represent the solution in the form (4.21) and consider the problem (4.22), (4.25) as the Cauchy problem (3.19) with the initial condition

$$\varphi|_{t=0} = \varphi' = (\boldsymbol{w}_0', \rho_0')^T.$$

We introduce the space $\mathcal{H} = C^{2+\alpha}(\mathcal{F}) \times C^{3+\alpha}(\mathcal{G})$ of elements $\varphi = (\boldsymbol{w}, \rho)$ with the norm

$$|\varphi|_{\mathcal{H}} = |\boldsymbol{w}|_{C^{2+\alpha}(\mathcal{F})} + |\rho|_{C^{3+\alpha}(\mathcal{G})}$$

and the subspace $\mathcal{H}_0 \subset \mathcal{H}$ whose elements satisfy the compatibility and orthogonality conditions (4.25). Equation

$$\frac{d\varphi}{dt} = A\varphi$$

has a finite-dimensional set of solutions $(\boldsymbol{w}', \rho') \in \mathcal{H}_0$ growing exponentially as $t \to \infty$. For arbitrary fixed $T > 0$ the spectrum of the operator $V = e^{TA}$, $\sigma(V)$, consists of two parts, $\sigma_1(V)$ and $\sigma_2(V)$, where $\sigma_1(V)$ is a finite set of eigenvalues $\mu \in C$ with $|\mu| > 1$, whereas the eigenvalues $\mu \in \sigma_2(V)$ satisfy $|\mu| \leqslant 1$. By the Riesz formula, V can be represented in the form $V = V_1 + V_2$ where

$$V_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} \mu(\mu I - V)^{-1} d\mu, \quad k = 1, 2,$$

and γ_k are nonintersecting contours enclosing $\sigma_k(V)$. Replacing V_1 with

$$V_1^n = \frac{1}{2\pi i} \int\limits_{\gamma_1} \mu^n (\mu I - V)^{-1} d\mu,$$

if necessary (i.e., choosing T large enough), it is possible to satisfy the inequalities

$$|V_1\psi|_{\mathcal{H}} \geqslant b_1 |\psi|_{\mathcal{H}}, \quad b_1 > 1 \quad \forall \psi \in V_1,$$
$$\|V_2\|_{\mathcal{H}\to\mathcal{H}} \leqslant b_2 < b_1.$$

With the above-defined decomposition of V we can associate the decomposition of the space $\mathcal{H}_0$ into the direct sum

$$\mathcal{H}_0 = \mathcal{H}_1 + \mathcal{H}_2.$$

The operators

$$P_k = \frac{1}{2\pi i} \int\limits_{\gamma_k} (\mu I - V)^{-1} d\mu, \quad k = 1, 2,$$

are the projections onto $\mathcal{H}_k$, and the following relations hold:

$$P_1 P_2 = P_2 P_1 = 0, \quad P_k^2 = P_k,$$

$$V_k = V_k P_k = P_k V_k, \quad P_1 V_2 = P_2 V_1 = 0.$$

We assume that the solution of the problem (4.1)–(4.4) is defined for all $t > 0$ and satisfies the inequality

$$|\varphi|_{\mathcal{H}} \leqslant c\varepsilon \leqslant \min(\varepsilon_2(T), \varepsilon_3(T)) \quad \forall t \geqslant 0 \tag{4.28}$$

with small $\varepsilon > 0$ subjected to the restrictions given below, and we show that this is not possible if the initial data $\varphi_0 \neq 0$ satisfy the additional constraint

$$|P_1 R\varphi_0|_{\mathcal{H}} \geqslant 2|P_2 Q\varphi_0|_{\mathcal{H}}, \tag{4.29}$$

where $R\varphi_0 = (\boldsymbol{w}_0', \rho_0')^T$. From (2.24) and (4.29) it follows that

$$\begin{aligned} |\varphi_0 - R\varphi_0|_{\mathcal{H}} &\leqslant c|\varphi_0|^2_{\mathcal{H}}, \\ |\varphi_0|_{\mathcal{H}} &\leqslant |P_1 R\varphi_0|_{\mathcal{H}} + |P_2 R\varphi|_{\mathcal{H}} + |\varphi_0 - R\varphi_0|_{\mathcal{H}} \\ &\leqslant \frac{3}{2}|P_1 R\varphi_0|_{\mathcal{H}} + c_1|\varphi_0|^2_{\mathcal{H}} \leqslant \frac{3}{2}|P_1 R\varphi_0|_{\mathcal{H}} + c_1\varepsilon|\varphi_0|_{\mathcal{H}}; \end{aligned} \tag{4.30}$$

and, if

$$c_1\varepsilon < 1/2, \tag{4.31}$$

then

$$|\varphi_0|_{\mathcal{H}} \leqslant 3|P_1 R\varphi_0|_{\mathcal{H}}. \tag{4.32}$$

Let W and W' be operators making correspond to φ_0 the solutions of the problems (4.1)–(4.4) and (4.23) respectively at the moment $t = T$:

$$(\boldsymbol{w}, \rho)^T|_{t=T} = W\varphi_0, \quad (\boldsymbol{w}'', \rho'')^T|_{t=T} = W'\varphi_0.$$

It is clear that

$$W\varphi_0 = VR\varphi_0 + W'\varphi_0.$$

We show that if ε is sufficiently small, then $\varphi_1 \equiv W\varphi_0$ also satisfies (4.29). Since

$$\begin{aligned} P_1 RW\varphi_0 &= P_1(R - I)W\varphi_0 + P_1 V_1 R\varphi_0 + P_1 RW'\varphi_0 \\ &= V_1 P_1 R\varphi_0 + P_1(R - I)W\varphi_0 + P_1 RW'\varphi_0 \end{aligned}$$

and

$$P_2 RW\varphi_0 = V_2 P_2 R\varphi_0 + P_2(R - I)W\varphi_0 + P_2 RW'\varphi_0,$$

we have, by (4.29) and (4.32),

$$\begin{aligned} &|P_1 RW\varphi_0|_{\mathcal{H}} - 2|P_2 RW\varphi_0|_{\mathcal{H}} \\ &\quad \geqslant b_1|P_1 R\varphi_0|_{\mathcal{H}} - 2b_2|P_2 R\varphi_0|_{\mathcal{H}} - |P_1 RW'\varphi_0|_{\mathcal{H}} - 2|P_2 RW'\varphi_0|_{\mathcal{H}} \\ &\quad\quad - |P_1(R - I)W\varphi_0|_{\mathcal{H}} - 2|P_2(R - I)W\varphi_0|_{\mathcal{H}} \end{aligned}$$

$$\geqslant (b_1 - b_2)|P_1 R\varphi_0|_{\mathcal{H}} - c_2|\varphi_0|^2_{\mathcal{H}} \geqslant (b_1 - b_2 - 3c_2\varepsilon)|P_1 R\varphi_0|_{\mathcal{H}} \geqslant 0$$

if

$$b_1 - b_2 - 3c_2\varepsilon > 0. \tag{4.33}$$

Finally, we estimate $P_1 RW\varphi_0$ from below, again with the help of (4.29), (4.32):

$$|P_1 RW\varphi_0|_{\mathcal{H}} \geqslant |V_1 P_1 R\varphi_0|_{\mathcal{H}} - |P_1 RW'\varphi_0|_{\mathcal{H}} - |P_1(R - I)W\varphi_0|_{\mathcal{H}}$$
$$\geqslant b_1|P_1 R\varphi_0|_{\mathcal{H}} - c_3|\varphi_0|^2_{\mathcal{H}} \geqslant (b_1 - 3c_3\varepsilon)|P_1 R\varphi_0|_{\mathcal{H}}.$$

We assume that

$$b_1' = b_1 - 3c_3\varepsilon > 1. \tag{4.34}$$

Then

$$|P_1 RW\varphi_0|_{\mathcal{H}} \geqslant b_1'|P_1 R\varphi_0|_{\mathcal{H}}.$$

Since $\varphi_1 = (\boldsymbol{w}(x,T), \rho(x,T))$ satisfies the conditions (4.3), (4.4), (4.27), (4.29) for $t = T$, we can repeat our argument for the time interval $(T, 2T)$ and then for $t \in (kT, (k+1)T)$, $k > 0$. Then we arrive at the following conclusion: If (4.28) holds with ε satisfying (4.31), (4.33), (4.34), then $\varphi_k = \varphi|_{t=kT}$, $k \geqslant 1$, satisfy (4.29) and

$$|P_1 R\varphi_k|_{\mathcal{H}} \geqslant b_1'|P_1 R\varphi_{k-1}|_{\mathcal{H}},$$

which implies

$$|P_1 R\varphi_k|_{\mathcal{H}} \geqslant b_1'^k|P_1 R\varphi_0|_{\mathcal{H}}.$$

For large k this contradicts (4.28), which proves the theorem. □

5. Case of Nonsymmetric $\mathcal{F}$

If $\mathcal{F}$ does not possess the property of axial symmetry with respect to the x_3-axis, then Equation (1.3) defines a one-parameter family of equilibrium figures, $\mathcal{F}_\theta$, obtained by rotation of the angle θ of one of them, $\mathcal{F}_0$, about the x_3-axis. It is clear that $\theta \in R$ and $\mathcal{F}_{\theta+2\pi} = \mathcal{F}_\theta$. The condition $\delta^2 R > 0$ cannot hold for all ρ satisfying (2.36) because $B(\boldsymbol{\eta}_3 \cdot \boldsymbol{N}) = 0$ and $\boldsymbol{\eta}_3 \cdot \boldsymbol{N} \neq 0$ for nonsymmetric $\mathcal{F}$. In this case, (2.36) should be supplemented with one more condition

$$\int\limits_{\mathcal{G}} \rho(y)\boldsymbol{\eta}_3 \cdot \boldsymbol{N}(y)dS = 0, \tag{5.1}$$

and the main result consists in the following: The regime of rigid rotation, where $(\boldsymbol{v}, p)$ are given by (1.2) and $\Omega_t = \mathcal{G}_{\omega t+\varphi}$, is stable if the form (3.3) is positive definite for all ρ satisfying (2.36), (5.1) and unstable if the form (3.3) can take negative values for some ρ satisfying the above conditions.

If the surface Γ_t is close to a certain $\mathcal{G}_\theta$, then it is also close to $\mathcal{G}_{\theta'}$ with $\theta' - \theta$ small, and the problem arises how to choose an optimal representation of Γ_t in the form (2.14), i.e.,

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho_\theta(y,t), \quad y \in \mathcal{G}_\theta\}. \tag{5.2}$$

We make such a choice minimizing the integral

$$\int\limits_{\mathcal{G}} \rho_\theta^2(z)dS \tag{5.3}$$

among all the possible representations (5.2). The following proposition is proved in [**22**].

Proposition 5.1. *Let Γ_t be represented by the equation*

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t), \quad y \in \mathcal{G}_0\}, \tag{5.4}$$

with ρ satisfying

$$|\rho(\cdot,t)|_{C^1(\mathcal{G}_0)} \leqslant \delta \ll 1. \tag{5.5}$$

Then there exists a function $\theta(t)$ such that Γ_t can be given by (5.2) *with $\theta = \theta(t)$ and the integral* (5.3) *takes a minimal value. The function $\theta(t)$ is defined for all t for which Γ_t stays in a certain δ_1-neighborhood $\mathcal{U}$ of $\mathcal{G}_0$ and*

$$|\theta_t(t)| \leqslant c \int\limits_{\mathcal{G}_\theta(t)} \Big|\frac{\partial \rho_\theta(y,t)}{\partial t}\Big| dS. \tag{5.6}$$

We give the main ideas of the proof. Let us consider a fixed closed surface Γ given by the equation

$$x = y + \boldsymbol{N}(y)\rho(y)$$

and the rotated surface $\Gamma(\lambda) = \mathcal{Z}(\lambda)\Gamma$,where $\mathcal{Z}$ is the matrix (2.3). If $\Gamma(\lambda)$ is contained in $\mathcal{U}$ (which is the case for small λ), then it is prescribed by a similar equation

$$\Gamma(\lambda) = \{z + \boldsymbol{N}(z)\rho(z,\lambda) \equiv X(z,\lambda), \quad z \in \mathcal{G}_0 \equiv \mathcal{G}\} \tag{5.7}$$

where $X(z,\lambda)=\mathcal{Z}(\lambda)x$ and $x=y+\boldsymbol{N}(y)\rho(y)\in\Gamma$. We look for the value λ_0 of λ such that the function

$$f(\lambda)=\int\limits_{\mathcal{G}}\rho(z,\lambda)\rho_\lambda(z,\lambda)dS$$

vanishes for $\lambda=\lambda_0$: $f(\lambda_0)=0$. For the derivative ρ_λ the formula

$$\frac{\partial\rho(z,\lambda)}{\partial\lambda}=\boldsymbol{\eta}_3(z)\cdot\boldsymbol{N}(z)+\boldsymbol{h}(z,\rho(z,\lambda))\cdot\nabla_{\mathcal{G}}\rho(z,\lambda) \tag{5.8}$$

with

$$\boldsymbol{h}(z,\rho)=-\frac{\boldsymbol{\eta}_3(X(z,\lambda))-\boldsymbol{N}(z)(\boldsymbol{N}(z)\cdot\boldsymbol{\eta}_3(X(z,\lambda)))}{\Lambda(z,\rho)}(1-\rho\mathcal{H}(z))$$
$$+\rho(z,\lambda)\frac{((\boldsymbol{\eta}_3(X(z,\lambda))\cdot\nabla_{\mathcal{G}})\boldsymbol{N}(z)}{\Lambda(z,\rho)}$$

was obtained in [**20, 22**]. This allows us to conclude that for small ρ the derivative

$$f_\lambda(\lambda)=\int\limits_{\mathcal{G}}(\rho_\lambda^2(z,\lambda)+\rho(z,\lambda)\rho_{\lambda\lambda}(z,\lambda))dS$$

is bounded from below:

$$f_\lambda(\lambda)\geqslant\frac{1}{2}\int\limits_{\mathcal{G}}(\boldsymbol{\eta}_3\cdot\boldsymbol{N}(z))^2dS>0.$$

Since

$$|f(0)|=\left|\int\limits_{\mathcal{G}}\rho\rho_\lambda|_{\lambda=0}dS\right|\leqslant c\|\rho\|_{L_1(\mathcal{G})},$$

the equation $f(\lambda)=0$ has a unique solution $\lambda=\lambda_0$ in a certain interval $|\lambda|\leqslant\lambda_1$, if ρ is small enough. Hence the functional

$$I(\lambda)=\int\limits_{\mathcal{G}}\rho^2(z,\lambda)dS$$

attains the unique minimum in this interval.

Let us consider one-parameter family of surfaces Γ_t given by (5.4) with small ρ,and the family $\Gamma_t(\lambda) = \mathcal{Z}(\lambda)\Gamma_t$ given by (5.7) with $\rho = \rho(z,t,\lambda)$. There exists the value $\lambda_0 \equiv \lambda(t)$ such that

$$f(t,\lambda) = \int\limits_{\mathcal{G}} \rho(z,t,\lambda)\rho_\lambda(z,t,\lambda)dS = 0 \tag{5.9}$$

for $\lambda = \lambda(t)$. Differentiating (5.9) we obtain $f_t + \lambda_t f_\lambda = 0$, i.e.,

$$\lambda_t(t) = -\frac{f_t(\lambda,t)}{f_\lambda(\lambda,t)}\Big|_{\lambda=\lambda(t)} = -\frac{\int\limits_{\mathcal{G}} (\rho_t\rho_\lambda + \rho\rho_{\lambda t})dS}{\int\limits_{\mathcal{G}} (\rho_\lambda^2 + \rho\rho_{\lambda\lambda})dS}\Bigg|_{\lambda=\lambda(t)} \tag{5.10}$$

where the derivatives ρ_t and $\rho_{\lambda t}$ are computed with λ fixed. The angle $\theta(t)$ mentioned in the statement of the proposition is related to λ by $\theta(t) = -\lambda(t)$, so (5.6) follows from (5.10), (5.8).

Now, we consider the free boundary problem (2.9)–(2.13) with the initial data satisfying the assumptions of Theorem 4.1 and, in addition, the condition (5.9), i.e.,

$$\int\limits_{\mathcal{G}_0} \rho_0(y)(\boldsymbol{\eta}_3 \cdot \boldsymbol{N}(y) + \boldsymbol{h}(y,\rho_0(y)) \cdot \nabla_{\mathcal{G}}\rho_0(y)dS = 0. \tag{5.11}$$

This problem has a unique solution defined on some finite time interval $(0,T)$. Let $\lambda(t)$ be the function constructed in Proposition 5.1 for the family Γ_t. We make the change of variables

$$z = \mathcal{Z}(\lambda(t))x,$$

which maps Ω_t onto $\widetilde{\Omega}_t = \mathcal{Z}(\lambda(t))\Omega_t$ and Γ_t onto $\widetilde{\Gamma}_t = \partial\widetilde{\Omega}_t$, and we introduce the functions

$$\widetilde{\boldsymbol{w}}(z,t) = \mathcal{Z}(\lambda(t))\boldsymbol{w}(\mathcal{Z}^{-1}(\lambda(t))z,t),$$
$$\widetilde{q}(z,t) = q(\mathcal{Z}^{-1}(\lambda(t))z,t).$$

An elementary calculation shows that $\widetilde{\boldsymbol{w}}$ and $\widetilde{q}$ satisfy the relations

$$
\begin{gathered}
\widetilde{\boldsymbol{w}}_t + (\widetilde{\boldsymbol{w}} \cdot \nabla)\widetilde{\boldsymbol{w}} + 2\omega(\boldsymbol{e}_3 \times \widetilde{\boldsymbol{w}}) - \lambda_t(t)(\boldsymbol{e}_3 \times \widetilde{\boldsymbol{w}}) \\
+ \lambda_t(t)(\boldsymbol{\eta}_3(z) \cdot \nabla)\widetilde{\boldsymbol{w}} - \nu\nabla^2\widetilde{\boldsymbol{w}} + \nabla\widetilde{q} = 0, \\
\nabla \cdot \widetilde{\boldsymbol{w}} = 0, \quad z \in \widetilde{\Omega}_t, \quad t > 0, \\
T(\widetilde{\boldsymbol{w}}, \widetilde{q})\widetilde{\boldsymbol{n}} = (\sigma\widetilde{H}(z) + \frac{\omega^2}{2}|z'|^2 + \varkappa\widetilde{U}(z,t) + p_0)\widetilde{\boldsymbol{n}}, \qquad (5.12) \\
V_n = \widetilde{\boldsymbol{w}} \cdot \widetilde{\boldsymbol{n}} + \lambda_t(t)\boldsymbol{\eta}_3(z) \cdot \widetilde{\boldsymbol{n}}(z), \quad z \in \widetilde{\Gamma}_t \\
\widetilde{\boldsymbol{w}}(z,0) = \boldsymbol{v}_0(z), \quad z \in \Omega_0.
\end{gathered}
$$

Here, $\widetilde{\boldsymbol{n}}$ is the exterior normal to $\widetilde{\Gamma}_t$, $\widetilde{H}(z)$ is the doubled mean curvature of $\widetilde{\Gamma}_t$ and

$$\widetilde{U}(y,t) = \int\limits_{\widetilde{\Omega}_t} |y-z|^{-1}dy.$$

The surface $\widetilde{\Gamma}_t$ is given by

$$z = y + \boldsymbol{N}(y)\widetilde{\rho}(y,t), \quad y \in \mathcal{G} \equiv \mathcal{G}_0 \tag{5.13}$$

where $\widetilde{\rho}(y,t) = \rho(y,t,\lambda(t))$. The orthogonality and compatibility conditions remain unchanged. By the definition of $\lambda(t)$, we have

$$\int\limits_{\mathcal{G}_0} \widetilde{\rho}(z,t)(\boldsymbol{\eta}_3 \cdot \boldsymbol{N}(z) + \boldsymbol{h}(z,\widetilde{\rho}(z,t)) \cdot \nabla_{\mathcal{G}}\widetilde{\rho}(z,t)dS = 0. \tag{5.14}$$

The time derivative of $\lambda(t)$ is expressed by formula (5.10), where $\rho_t = \rho_t(z,t,\lambda(t))$ and $\rho_{t\lambda}$ are calculated with λ fixed. In particular, due to (2.22), we have

$$\widetilde{\rho}_t(z,t) = \frac{\widetilde{\boldsymbol{w}}(z,t) \cdot \widehat{\mathcal{L}}^T\boldsymbol{N}(z)}{\boldsymbol{N}(z) \cdot \widehat{\mathcal{L}}^T\boldsymbol{N}(z)}, \quad z \in \mathcal{G}. \tag{5.15}$$

The corresponding linearized problem has the form

$$
\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times v) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\
\nabla \cdot \boldsymbol{v} = 0, \quad x \in \mathcal{F}, \\
T(\boldsymbol{v},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = 0, \qquad (5.16) \\
\rho_t = \boldsymbol{v} \cdot \boldsymbol{N} - h_0(x)\int\limits_{\mathcal{G}} h_0(z)\boldsymbol{v}(z) \cdot \boldsymbol{N}(z)dS, \quad x \in \mathcal{G} \\
\rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}, \quad \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F},
\end{gathered}
$$

$$\int_{\mathcal{G}} \rho dS = 0, \quad \int_{\mathcal{G}} \rho x_i dS = 0, \quad i = 1, 2, 3,$$

$$\int_{\mathcal{F}} \boldsymbol{v} dS = 0, \quad \int_{\mathcal{F}} \boldsymbol{v} \cdot \boldsymbol{\eta}_i dS + \omega \int_{\mathcal{G}} \rho \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dS = 0, \quad i = 1, 2, 3, \tag{5.17}$$

$$\int_{\mathcal{G}} \rho(x,t) h_0(x) dS = 0, \tag{5.18}$$

where

$$h_0(x) = \frac{\boldsymbol{\eta}_3(x) \cdot \boldsymbol{N}(x)}{\|\boldsymbol{\eta}_3 \cdot \boldsymbol{N}\|_{L_2(\mathcal{G})}}.$$

It coincides with (2.33)–(2.37) for symmetric $\mathcal{F}$. Equation (5.18) is obtained by the linearization of (5.14). If (5.18) is satisfied at $t = 0$ for ρ_0, then it holds for all $t > 0$.

Theorem 5.1. *Assume that the form*

$$\int_{\mathcal{G}} \rho \widehat{B} \rho dS$$

is positive definite for all ρ *satisfying the conditions* (2.36), (5.1). *Then the problem* (5.12) *with the initial data satisfying the smallness condition* (4.19′), *the compatibility and orthogonality conditions* (4.3)–(4.6), *and the condition* (5.11) *has a unique solution* $\widetilde{\boldsymbol{w}}(\cdot,t) \in C^{2+\alpha}(\mathcal{F})$ *with* $\widetilde{\boldsymbol{w}}_t(\cdot,t) \in C^{\alpha}(\mathcal{F})$, $\widetilde{q}(\cdot,t) \in C^{1+\alpha}(\mathcal{F})$, $\widetilde{\rho}(\cdot,t) \in C^{3+\alpha}(\mathcal{G})$ *for all* $t \geqslant 0$ *that satisfies the inequality*

$$y_t(\widetilde{\boldsymbol{w}}, \widetilde{q}, \widetilde{\rho}) \leqslant e^{-bt} M_0, \quad b = \text{const} > 0. \tag{5.19}$$

The surface Γ_t *is given by Equation* (5.13). *The function* $\theta(t) = -\lambda(t)$ *is defined for all* $t > 0$ *and satisfies the inequality* (5.6) *with* $\rho = \widetilde{\rho}$.

This theorem is proved in the same way as Theorem 4.1. We make the transformation (2.16) (with ρ replaced by $\widetilde{\rho}$) and write the problem (5.12) in the form similar to (4.1), namely,

$$\boldsymbol{w}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu \nabla^2 \boldsymbol{w} + \nabla q = \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{w} = f = \nabla \cdot \boldsymbol{F}, \quad y \in \mathcal{F},$$
$$\Pi_0 S(\boldsymbol{w}) \boldsymbol{N} = \boldsymbol{b},$$

$$
\begin{aligned}
&\boldsymbol{N}\cdot T(\boldsymbol{w},q)\boldsymbol{N}+B_0\rho=d,\\
&\rho_t=\boldsymbol{w}\cdot\boldsymbol{N}-h_0(x)\int\limits_{\mathcal{G}}h_0(z)\boldsymbol{w}\cdot\boldsymbol{N}dS+g,\\
&\rho(y,0)=\rho_0(y),\quad y\in\mathcal{G},\\
&\boldsymbol{w}(y,0)=\boldsymbol{w}_0(e_\rho(y)),\quad y\in\mathcal{F}
\end{aligned}
\tag{5.20}
$$

(as above, we denote by $\boldsymbol{w},q,\rho$ the transformed functions). The functions $\boldsymbol{f}$, f, $\boldsymbol{F}$, $\boldsymbol{b}$, d, and g have the form

$$
\begin{aligned}
\boldsymbol{f}&=\rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{w}-(\mathcal{L}^{-1}\boldsymbol{w}\cdot\nabla)\boldsymbol{w}+\nu(\widetilde{\nabla}^2\boldsymbol{w}-\nabla^2\boldsymbol{w})\\
&+(\nabla-\widetilde{\nabla})q+\lambda_t(t)(\boldsymbol{e}_3\times\boldsymbol{w})-\lambda_t(t)(\boldsymbol{\eta}_3(e_\rho(z))\cdot\widetilde{\nabla})\boldsymbol{w},
\end{aligned}
$$

$$
\begin{aligned}
&f=(I-\widehat{\mathcal{L}}^T)\nabla\cdot\boldsymbol{w},\quad \boldsymbol{F}=(I-\widehat{\mathcal{L}})\boldsymbol{w},\\
&\boldsymbol{b}=\Pi_0(\Pi_0S(\boldsymbol{w})\boldsymbol{N}-\Pi\widetilde{S}(\boldsymbol{w})\widehat{\mathcal{L}}^T\boldsymbol{N}),
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
d&=\nu\Big(\boldsymbol{N}\cdot S(\boldsymbol{w})\boldsymbol{N}-|\widehat{\mathcal{L}}^T\boldsymbol{N}|^{-2}\widehat{\mathcal{L}}^T\boldsymbol{N}\cdot\widetilde{S}(\boldsymbol{w})\widehat{\mathcal{L}}^T\boldsymbol{N})\\
&+\Big(\sigma(H(x)-\mathcal{H}(y)-\sigma\delta((H(x)-\mathcal{H}(y)))+\frac{\omega^2}{2}(|x'|^2-|y'|^2\\
&-\delta(|x'|^2-|y'|^2)\Big)+\varkappa(U(x,t)-\mathcal{U}(y)-\delta(U(x,t)-\mathcal{U}(y))))\Big|_{x=e_\rho(y)},
\end{aligned}
$$

$$
\begin{aligned}
g=&-\Lambda^{-1}(y,\rho)\sum_{j=1}^{3}w_j\Big(\frac{\partial\rho}{\partial y_j}(1-\rho\mathcal{H}(y))+\rho\sum_{m=1}^{n}\frac{\partial\rho}{\partial y_m}\frac{\partial N_m}{\partial y_j}\Big)\\
&+\lambda_t(t)\boldsymbol{\eta}_3(e_\rho(z))\cdot\widetilde{\boldsymbol{n}}+h_0(z)\int\limits_{\mathcal{G}}\boldsymbol{w}\cdot\boldsymbol{N}h_0dS.
\end{aligned}
$$

The orthogonality conditions have the same form (4.3)–(4.5), but, in addition, we have

$$
\int\limits_{\mathcal{G}}\rho h_0dS=\int\limits_{\mathcal{G}}\rho_0(h_0-\rho_{0\lambda})dS\equiv l_0(t)
$$

where $\rho_{0\lambda}$ is expressed by (5.8). The functions (5.21) satisfy the inequalities (4.9) and (4.10).

Propositions 4.2 and 4.3 remain true for the nonlinear problem (5.20), (5.21). Proposition 4.4 should be slightly modified because, in addition to (4.17) and (4.18), we should guarantee

$$\int_{\mathcal{G}} r h_0 dS = l_0(0) \equiv l_0.$$

The function r should be taken in the form

$$r(y) = \frac{l\boldsymbol{N}(y)\cdot\boldsymbol{y}}{3|\mathcal{F}|} + \frac{1}{|\mathcal{F}|}\boldsymbol{l}\cdot\boldsymbol{N}(y) + l' h_0(y), \quad y \in \mathcal{G}$$

with

$$l' = l_0(0) - \int_{\mathcal{G}} h_0(y)\Big(\frac{l\boldsymbol{N}(y)\cdot\boldsymbol{y}}{3|\mathcal{F}|} + \frac{1}{|\mathcal{F}|}\boldsymbol{l}\cdot\boldsymbol{N}(y)\Big)dS.$$

The rest of the proof of this proposition is unchanged.

We look for the solution of the problem (5.20), (5.21) in the form (4.21) where $(\boldsymbol{w}', q', \rho')$ is a solution to a linear problem

$$\begin{gathered}
\boldsymbol{w}'_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}') - \nu\nabla^2\boldsymbol{w}' + \nabla q' = 0, \\
\nabla\cdot\boldsymbol{w}' = 0, \quad y \in \mathcal{F}, \\
\Pi_0 S(\boldsymbol{w}')\boldsymbol{N} = 0, \\
\boldsymbol{N}\cdot T(\boldsymbol{w}')\boldsymbol{N} + B_0\rho' = 0, \\
\\
\rho'_t = \boldsymbol{w}'\cdot\boldsymbol{N} - h_0\int_{\mathcal{G}} \boldsymbol{w}'\cdot\boldsymbol{N} h_0 dS, \\
\rho'(y,0) = \rho'_0(y), \quad y \in \mathcal{G}, \\
\boldsymbol{w}'(y,0) = \boldsymbol{w}'_0(y), \quad y \in \mathcal{F},
\end{gathered} \tag{5.22}$$

and $(\boldsymbol{w}'', p'', \rho'')$ is a solution of the nonlinear problem

$$\begin{gathered}
\boldsymbol{w}''_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}'') - \nu\nabla^2\boldsymbol{w}'' + \nabla q'' = \boldsymbol{f}(\boldsymbol{w}' + \boldsymbol{w}'', q' + q'', \rho' + \rho''), \\
\nabla\cdot\boldsymbol{w}'' = f(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho'') = \nabla\cdot\boldsymbol{F}(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''), \quad y \in \mathcal{F}, \\
\Pi_0 S(\boldsymbol{w}'')\boldsymbol{N} = \boldsymbol{b}(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''), \\
\boldsymbol{N}\cdot T(\boldsymbol{w}'')\boldsymbol{N} + B_0\rho'' = d(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho'')),
\end{gathered} \tag{5.23}$$

$$\rho_t'' - \boldsymbol{w}'' \cdot \boldsymbol{N} + \int\limits_{\mathcal{G}} \boldsymbol{w}'' \cdot \boldsymbol{N} h_0 dS = g(\boldsymbol{w}' + \boldsymbol{w}'', \rho' + \rho''),$$

$$\rho''(y, 0) = \rho_0''(y), \quad y \in \mathcal{G},$$
$$\boldsymbol{w}''(y, 0) = \boldsymbol{w}_0''(y), \quad y \in \mathcal{F}.$$

We define $(\rho''(y), \boldsymbol{w}_0''(y))$ as in a modified Proposition 4.4 with $l = l(0)$, $\boldsymbol{l} = \boldsymbol{l}(0)$, $\boldsymbol{m} = \boldsymbol{m}(0)$, $\boldsymbol{M} = \boldsymbol{M}(0)$, $f_0(x) = f_0(x, 0)$, $\boldsymbol{b}_0(x) = \boldsymbol{b}(x, 0)$. $l_0 = l_0(0)$. It is clear that the differences

$$\rho_0'(y) = \rho_0(y) - \rho_0''(y),$$
$$\boldsymbol{w}_0'(y) = \boldsymbol{w}_0(y) - \boldsymbol{w}_0''(y)$$

satisfy the homogeneous compatibility and orthogonality conditions

$$\nabla \cdot \boldsymbol{w}_0'(y) = 0, \quad y \in \mathcal{F}, \quad \Pi_0 S(\boldsymbol{w}_0')\boldsymbol{N}\Big|_{\mathcal{G}} = 0,$$

$$\int\limits_{\mathcal{G}} \rho_0'(y) dS = 0, \quad \int\limits_{\mathcal{G}} \rho_0'(y) y_i dS = 0, \quad i = 1, 2, 3,$$

$$\int\limits_{\mathcal{G}} \rho' h_0 dS = 0, \quad \int\limits_{\mathcal{F}} \boldsymbol{w}_0'(y) dy = 0,$$

$$\int\limits_{\mathcal{F}} \boldsymbol{w}_0'(y) \cdot \boldsymbol{\eta}_i(y) dy + \omega \int\limits_{\mathcal{G}} \rho_0'(y) \boldsymbol{\eta}_3(y) \cdot \boldsymbol{\eta}_i(y) dS = 0, \quad i = 1, 2, 3.$$

As in Section 3, it can be verified that, under the above orthogonality conditions, the problem (5.22) is solvable in the infinite time interval $t > 0$ and the solution satisfies (4.26). This allows us to obtain the estimate (5.19) exactly in the same way as it was done in Section 4. From (5.19) it follows that the solution $(\boldsymbol{v}, p, \Omega_t)$ of the problem (1.1) tends to the periodic solution $(\boldsymbol{V}, P, \mathcal{G}_{t\omega+\varphi})$, where $\varphi = \lim_{t\to\infty} \theta(t)$. The existence of this limit is a consequence of (5.6) and (5.19).

Let us formulate an analog of Theorem 4.2.

Theorem 5.2. *Assume that $\mathcal{F}$ satisfies the condition*

$$\min_{|\theta| \leqslant \pi} \int\limits_{\mathcal{F}} ((x_1 \cos\theta + x_2 \sin\theta)^2 - x_3^2) dx > 0, \tag{5.24}$$

and let the form

$$\int_{\mathcal{G}} \rho \widehat{B} \rho dS$$

take negative values for some ρ satisfying (3.17), (5.1). *Then there exist initial data $(\boldsymbol{w}_0, \rho_0)$ with arbitrarily small norm M_0 such that the corresponding solution of* (5.20), (5.21) *leaves sooner or later a certain neighborhood of zero, i.e., for some $t > 0$, $\varepsilon > 0$*

$$|\widetilde{\rho}(\cdot, t)|_{C^{3+\alpha}(\mathcal{G})} + |\boldsymbol{w}(\cdot, t)|_{C^{2+\alpha}(\mathcal{F})} \geqslant \varepsilon. \tag{5.25}$$

The condition (5.24) replaces (3.6) because, if $\mathcal{F}$ is nonsymmetric and satisfies (2.4), then, instead of (3.5), we have

$$\int_{\mathcal{G}} B(\boldsymbol{\eta}_1 \cdot \boldsymbol{N})\boldsymbol{\eta}_1 \cdot \boldsymbol{N} dS = \omega^2 \widetilde{\mathcal{S}}_2,$$

$$\int_{\mathcal{G}} B(\boldsymbol{\eta}_2 \cdot \boldsymbol{N})\boldsymbol{\eta}_2 \cdot \boldsymbol{N} dS = \omega^2 \widetilde{\mathcal{S}}_1,$$

where

$$\widetilde{\mathcal{S}}_j = \int_{\mathcal{F}} (x_j^2 - x_3^2) dx, \quad j = 1, 2,$$

and (3.7) takes the form

$$\rho(x) = \frac{\boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x)}{\widetilde{\mathcal{S}}_2} \int_{\mathcal{G}} \rho(x) x_2 x_3 dS - \frac{\boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x)}{\widetilde{\mathcal{S}}_1} \int_{\mathcal{G}} \rho(x) x_1 x_3 dS + \rho_1(x)$$

with $\rho_1 = Q\rho$ satisfying (3.8). Assertion 5 in Section 3 remains true, as well as the remark after the proof of this assertion. Theorem 5.2 is proved by the same arguments as Theorem 4.2, i.e., by representing the solution of the problem (5.20), (5.21) in the form (4.21).

The inequality (5.25) means that either $\boldsymbol{w}$ does not tend to zero as $t \to \infty$ or Γ_t stays away from $\mathcal{G}_{\theta(t)}$. Since the integral (5.3) attains the minimal value for $\theta = \theta(t)$, this means that Ω_t stays away from $\cup_\theta \mathcal{G}_\theta$, i.e., the periodic solution $(\boldsymbol{V}, P, \mathcal{G}_{t\omega+\varphi})$ of the problem (1.1) is unstable.

6. Multi-Dimensional Case

In this section, we discuss the extension of the above results to the case of arbitrary space dimension $n > 3$. We consider the evolution free boundary problem (1.1), where we set $\varkappa = 0$ for simplicity:

$$\begin{gathered}\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0,\\ \nabla\cdot\boldsymbol{v} = 0, \quad x\in\Omega_t, \quad t>0,\\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\Omega_0,\\ T(\boldsymbol{v},p)\boldsymbol{n} = \sigma H(x,t)\boldsymbol{n},\\ V_n = \boldsymbol{v}\cdot n, \quad x\in\Gamma_t\equiv\partial\Omega_t.\end{gathered} \tag{6.1}$$

It is necessary to find a bounded domain $\Omega_t \in R^n$, as well as $\boldsymbol{v}(x,t) = (v_1,\dots,v_n)$ and $p(x,t)$ given in Ω_t and satisfying (6.1). By $H(x,t)$ we mean the $n-1$ times mean curvature of $\mathcal{G}$. All other functions in (6.1) are defined in the same way as in the 3-dimensional case in Section 1.

We observe that the solution of the problem (6.1) is subjected to the same "conservation laws" as in the 3-dimensional case, namely,

$$\begin{gathered}|\Omega_t| = |\Omega_0|,\\ \int\limits_{\Omega_t}\boldsymbol{v}(x,t)dx = \int\limits_{\Omega_0}\boldsymbol{v}_0(x)dx,\end{gathered} \tag{6.2}$$

$$\int\limits_{\Omega_t}\boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx = \int\limits_{\Omega_0}\boldsymbol{v}_0(x)\cdot\boldsymbol{\eta}_{ij}(x)dx \equiv m_{ij}, \quad i\neq j, \tag{6.3}$$

where $\boldsymbol{\eta}_{ij}(x) = \boldsymbol{e}_i x_j - \boldsymbol{e}_j x_i$ and $\boldsymbol{e}_j$ is a unit vector in the direction of the x_j-axis. Indeed, (6.2) is easily obtained by the integration of the first equation in (6.1) over Ω_t, which leads to

$$0 = \frac{d}{dt}\int\limits_{\Omega_t}\boldsymbol{v}(x,t)dx - \sigma\int\limits_{\Gamma_t}H(x,t)\boldsymbol{n}dS = \frac{d}{dt}\int\limits_{\Omega_t}\boldsymbol{v}(x,t)dx,$$

because $H(x,t)\boldsymbol{n} = \Delta_{\Gamma_t}\boldsymbol{x}$ and the surface integral vanishes. Equations (6.3) are obtained in a similar way.

We would like to study the stability of solutions corresponding to a rigid motion of the liquid. We say that a motion is *rigid* if the vector field of velocity $\boldsymbol{V}$ given as a function of the Eulerian coordinates x satisfies the relations

$$\frac{\partial V_i(x)}{\partial x_j} + \frac{\partial V_j(x)}{\partial x_i} = 0, \quad i, j = 1, \ldots, n.$$

It is easily seen that this is the case if and only if

$$\boldsymbol{V}(x) = \mathcal{C}\boldsymbol{x} + \boldsymbol{h}, \tag{6.4}$$

where $\mathcal{C}$ is an antisymmetric matrix. We assume that $\boldsymbol{h} = 0$ and the entries C_{ij} of the matrix $\mathcal{C}$ are numbers. The functions

$$\boldsymbol{V}(x) = \mathcal{C}\boldsymbol{x}, \quad P(x) = \frac{1}{2}|\mathcal{C}\boldsymbol{x}|^2 + p_0, \quad p_0 = \text{const}, \tag{6.5}$$

satisfy the Navier–Stokes equations. Substituting $\boldsymbol{V}$ and P into the boundary conditions, we obtain the equation for the equilibrium figure $\mathcal{F}$:

$$\sigma\mathcal{H} + \frac{1}{2}|\mathcal{C}\boldsymbol{x}|^2 + p_0 = 0, \quad x \in \mathcal{G} \equiv \partial\mathcal{F}. \tag{6.6}$$

Without loss of generality, we can assume that the matrix $\mathcal{C}$ has the canonical form

$$\mathcal{C} = \operatorname{diag}\Big(C_1, \ldots, C_l, O\Big), \tag{6.7}$$

where $l \leqslant n/2$, O is an $n - 2l \times n - 2l$ matrix whose entries are zeros and C_k are 2×2 antisymmetric matrices of the form

$$C_k = \begin{pmatrix} 0 & -\omega_k \\ \omega_k & 0 \end{pmatrix}. \tag{6.8}$$

In particular, if $n = 3$, then $l = 1$ and $\boldsymbol{V}$ is the velocity of the liquid rotating as a rigid body about the x_3-axis with the angular velocity ω_1. In the n-dimensional case, there are l "angular velocities" ω_k.

Passing to the Lagrangean coordinates, it is easy to calculate the trajectories of particles whose velocity regarded a function of the Eulerian coordinates is $\boldsymbol{V}(x)$. If $\boldsymbol{x}(0) = \boldsymbol{\xi}$, then

$$x_k(t) = \xi_k \cos\omega_k t - \xi_{k+1}\sin\omega_k t,$$

$$x_{k+1}(t) = \xi_k \sin\omega_k t + \xi_{k+1}\cos\omega_k t, \quad k = 1, \ldots, l,$$

$$x_m = \xi_m, \quad m = l + 1, \ldots, n,$$

i.e., the projection of the trajectory onto the x_k, x_{k+1}-plane is a circle centered at the origin, along which the motion proceeds with a constant velocity proportional to ω_k. This complicated motion is, in general, nonperiodic.

We say that a figure $\mathcal{F}$ is *symmetric* if it is invariant under the transformation

$$x = \mathcal{Z}y,$$

where $\mathcal{Z} = \operatorname{diag}(Z_1, \dots, Z_l, I_{n-2l})$, I_{n-2l} is the unit $n-2l \times n-2l$ matrix and

$$Z_k = \begin{pmatrix} \cos\varphi_k & -\sin\varphi_k \\ \sin\varphi_k & \cos\varphi_k \end{pmatrix}.$$

It is easy to see that the velocity of liquid particles located at the boundary $\mathcal{G}$ of a symmetric $\mathcal{F}$ is tangential to $\mathcal{G}$, i.e.,

$$\mathcal{C}\boldsymbol{x} \cdot \boldsymbol{N}(x)|_{\mathcal{G}} = 0,$$

This means that the functions (6.4) and (6.5) given in a symmetric $\mathcal{F}$ represent a stationary solution of (6.1). We consider here only symmetric $\mathcal{F}$.

By symmetry,

$$\int_{\mathcal{F}} x_j dx = 0, \quad j = 1, \dots, 2l,$$
$$\int_{\mathcal{F}} x_j x_q dx = 0, \quad j = 1, \dots, 2l, \quad q = 1, \dots, n, \ q \neq j \tag{6.9}$$

(some of these relations can be also deduced from Equation (6.6) in the same way as above in the three-dimensional case). Without loss of generality, we can assume that

$$\int_{\mathcal{F}} x_j dx = 0, \quad j = 1, \dots, n. \tag{6.10}$$

If the matrix $\mathcal{C}$ has the canonical form (6.7) and the figure is symmetric, then the corresponding matrix of momenta

$$m_{ij} = \int_{\mathcal{F}} \mathcal{C}\boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dx$$

also has the canonical form. Indeed, since

$$\mathcal{C}\boldsymbol{x} = -\sum_{q=1}^{l} \omega_q \boldsymbol{\eta}_q(x),$$

where $\boldsymbol{\eta}_q(x) = \boldsymbol{\eta}_{2q-1,2q}(x)$, it is easy to verify, using (6.9), that m_{ij} can be different from zero if and only if $i = 2k-1, j = 2k$, $k \leqslant l$, in which case

$$m_{2k-1,2k} = -\omega_k \|\boldsymbol{\eta}_k\|^2_{L_2(\mathcal{F})}.$$

We do not consider the problem of existence and uniqueness of equilibrium figures, as well as their geometry, but we can prove the existence of a symmertic equilibrium figure of a given volume in the case of a slow motion (i.e., with small prescribed momenta $m_{2k-1,2k}$). For $n = 3$ this result was obtained in [**13**].

Let us return to the evolution problem (6.1). As above, we assume that $\mathcal{F}$ is given and

$$|\Omega_t| = |\Omega_0| = |\mathcal{F}|, \quad \int\limits_{\Omega_t} x_j dx = 0, \quad j = 1, \dots, n, \tag{6.11}$$

$$\int\limits_{\Omega_t} \boldsymbol{v}(x,t)dx = \int\limits_{\Omega_0} \boldsymbol{v}_0(x)dx = 0,$$

$$\int\limits_{\Omega_t} \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_{ij}(x)dx = \int\limits_{\Omega_0} \boldsymbol{v}_0(x) \cdot \boldsymbol{\eta}_{ij}(x)dx = \int\limits_{\mathcal{F}} \mathcal{C}\boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x)dx. \tag{6.12}$$

We work with the evolution problem for the perturbations

$$\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{V}, \quad p_r = p - P$$

written in the coordinate system rigidly connected with the liquid whose velocity is given by (6.5). We make the change of variables

$$x = \mathcal{Z}(t)y$$

and the corresponding transformation of the unknown functions

$$\boldsymbol{w}(y,t) = \mathcal{Z}^{-1}(t)\boldsymbol{v}_r(\mathcal{Z}(t)y,t), \quad q(y,t) = p_r(\mathcal{Z}(t)y,t),$$

where $\mathcal{Z}(t) = \mathrm{diag}(Z_1(t), \dots, Z_l(t), I_{n-2l})$, I_{n-2l} is the unit $n-2l \times n-2l$ matrix, and

$$Z_k(t) = \begin{pmatrix} \cos\omega_k t & -\sin\omega_k t \\ \sin\omega_k t & \cos\omega_k t \end{pmatrix}.$$

This leads to a problem similar to (2.9)–(2.11), namely,

$$\begin{gathered}
\boldsymbol{w}_t + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + 2\mathcal{C}\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = 0, \\
\nabla \cdot \boldsymbol{w} = 0, \quad y \in \Omega_t, \quad t > 0, \\
T(\boldsymbol{w},q)\boldsymbol{n} = \Big(\sigma H + \frac{1}{2}|\mathcal{C}\boldsymbol{y}|^2 + p_0\Big)\boldsymbol{n}, \\
V_n = \boldsymbol{w} \cdot \boldsymbol{n}, \quad y \in \Gamma_t,
\end{gathered} \tag{6.13}$$

$$\boldsymbol{w}(y,0)=\boldsymbol{v}_0(y), \quad y\in\Omega_0,$$

in a transformed domain denoted again by Ω_t. The conditions (6.11), (6.12) take the form

$$|\Omega_t|=|\mathcal{F}|, \quad \int\limits_{\Omega_t} x_j dx=0, \quad j=1,\dots,n, \tag{6.14}$$

$$\int\limits_{\Omega_t} \boldsymbol{w}(x,t)dx=0,$$

$$\int\limits_{\Omega_t} \boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx+\int\limits_{\Omega_t} \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx=\int\limits_{\mathcal{F}} \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx. \tag{6.15}$$

Finally, we assume that Γ_t is close to $\mathcal{G}$ and is given by Equation (2.14). We map Ω_t onto $\mathcal{F}$ by the mapping (2.16) and arrive at the system

$$\begin{gathered}
\frac{\partial}{\partial t}\boldsymbol{w}-\frac{\partial\rho^*}{\partial t}(\mathcal{L}^{-1}N^*\cdot\nabla)\boldsymbol{w}+(\mathcal{L}^{-1}\boldsymbol{w}\cdot\nabla)\boldsymbol{w} \\
+2\mathcal{C}\boldsymbol{w}-\nu\widetilde{\nabla}\cdot\widetilde{\nabla}\boldsymbol{w}+\widetilde{\nabla}q=0, \\
\nabla_y\cdot\widehat{\mathcal{L}}\boldsymbol{w}=0, \quad y\in\mathcal{F}, \\
\Pi_0\Pi\widetilde{S}(\boldsymbol{w}(y,t))\widehat{\mathcal{L}}^T\boldsymbol{N}=0, \\
-q(y,t)+\nu\boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{w})\boldsymbol{n}=(\sigma(H(x)-\mathcal{H}(y))+\frac{1}{2}(|\mathcal{C}\boldsymbol{x}|^2-|\mathcal{C}\boldsymbol{y}|^2))\big|_{x=e_\rho(y)}, \\
\rho_t(y,t)=\frac{\boldsymbol{w}(y,t)\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}(y)}{\boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}(y)}, \quad y\in\mathcal{G}, \\
\boldsymbol{w}(y,0)=\boldsymbol{w}_0(e_{\rho_0}(y)), \quad y\in\mathcal{F}, \\
\rho(y,0)=\rho_0(y) \quad y\in\mathcal{G},
\end{gathered} \tag{6.16}$$

where $\mathcal{L}$, $\mathcal{L}^{-1}$, $\widehat{\mathcal{L}}$, $\widetilde{\nabla}$, $\widetilde{S}(\boldsymbol{w})$, ρ^*, $\boldsymbol{N}^*$ are defined as above in the 3-dimensional case. In terms of ρ the restrictions (6.12) can be written in the form (2.26),

where

$$\varphi(y,\rho) = \int_0^1 \rho\Lambda(y,s\rho)ds,$$

$$\psi(y,\rho) = \int_0^1 (y_i + sN_i(y)\rho(y))\rho\Lambda(y,s\rho)ds,$$

$$\Lambda(y,\rho) = \boldsymbol{N}(y)\cdot\widehat{\mathcal{L}}\boldsymbol{N}(y)$$

(see [**17**, formula (2.9)]). Formulas (2.24) and (2.27) do not hold any more, but it can be verified by analyzing the matrix $\widehat{\mathcal{L}}$ that Λ, φ, and ψ are independent of the first order derivatives of ρ. The conditions (6.15) are transformed into

$$\int_{\mathcal{F}} \boldsymbol{w}(y,t)L dy = 0,$$

$$\begin{aligned}\int_{\mathcal{F}} L\boldsymbol{w}(y,t)\cdot\boldsymbol{\eta}_{ij}(e_\rho(y))dy = &-\int_{\mathcal{F}} L\mathcal{C}e_\rho(y)\cdot\boldsymbol{\eta}_{ij}(e_\rho(y))dy \\ &+\int_{\mathcal{F}} \mathcal{C}\boldsymbol{y}\cdot\boldsymbol{\eta}_{ij}(y)dy, \quad i=1,2,\ldots,n.\end{aligned} \tag{6.17}$$

The corresponding linear problem reads

$$\begin{gathered}\boldsymbol{v}_t + 2\mathcal{C}\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \quad x\in\mathcal{F}, \\ \Pi_0 S(\boldsymbol{v})\boldsymbol{N} = 0, \\ \boldsymbol{N}\cdot T(\boldsymbol{v},p)\boldsymbol{N} + B_0\rho = 0, \\ \rho_t = \boldsymbol{v}\cdot\boldsymbol{N}, \quad \rho(x,0) = \rho_0(x), \quad x\in\mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\mathcal{F},\end{gathered} \tag{6.18}$$

where

$$\begin{gathered}-B_0\rho = \sigma\delta(H(x) - \mathcal{H}(y)) + \frac{1}{2}\delta(|\mathcal{C}\boldsymbol{x}|^2 - |\mathcal{C}\boldsymbol{y}|^2) = \sigma\Delta_{\mathcal{G}}\rho + b(y)\rho, \\ b(y) = \sigma c^2(y) + \mathcal{C}\boldsymbol{y}\cdot\mathcal{C}\boldsymbol{N}(y)\end{gathered}$$

and $c^2(y)$ is the sum of the squares of principal curvatures of $\mathcal{G}$ at y.

The linearized orthogonality conditions have the form

$$\int_{\mathcal{G}} \rho(y,t)dS = 0, \quad \int_{\mathcal{G}} \rho(y,t)y_i dS = 0, \quad i = 1,2,\dots,n, \tag{6.19}$$

$$\int_{\mathcal{F}} \boldsymbol{v}(x,t)dx = 0,$$

$$\int_{\mathcal{F}} \boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx + \int_{\mathcal{G}} \rho(x,t)\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dS = 0, \quad i = 1,2,\dots,n. \tag{6.20}$$

Hence

$$\boldsymbol{v}(x,t) = \boldsymbol{v}^{\perp}(x,t) + \sum_{k<m} d_{km}(\rho)\boldsymbol{\eta}_{km}(x), \tag{6.21}$$

where $\boldsymbol{v}^{\perp}$ is orthogonal to all vector fields $\boldsymbol{\eta}$ of the rigid motion. We separate out terms with $\boldsymbol{\eta}_{km} = \boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_l$, where $\boldsymbol{\eta}_q = \boldsymbol{\eta}_{2q-1,2q}$ and write (6.21) in the form

$$\boldsymbol{v}(x,t) = \boldsymbol{v}^{\perp}(x,t) + \boldsymbol{D}'(\rho) + \boldsymbol{D}''(\rho),$$

where

$$\boldsymbol{D}'(\rho) = \sum_{q=1}^{l} d_q(\rho)\boldsymbol{\eta}_q(x)$$

and

$$\boldsymbol{D}'' = \sum_{k<m}{}'' d_{km}(\rho)\boldsymbol{\eta}_{km}$$

is the sum of all other terms in (6.21). By (6.9), $\boldsymbol{\eta}_q$ are orthogonal in $L_2(\mathcal{F})$ to all other $\boldsymbol{\eta}_{km}$. Hence $\boldsymbol{D}'$ and $\boldsymbol{D}''$ are orthogonal to each other and to $\boldsymbol{v}^{\perp}$. It is easy to see that

$$\begin{aligned} d_q(\rho) &= \|\boldsymbol{\eta}_q\|^{-2}_{L_2(\mathcal{F})} \int_{\mathcal{F}} \boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_q(x)dx \\ &= -\|\boldsymbol{\eta}_q\|^{-2}_{L_2(\mathcal{F})} \int_{\mathcal{G}} \rho(x,t)\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_q(x)dS \\ &= \|\boldsymbol{\eta}_q\|^{-2}_{L_2(\mathcal{F})}\omega_q \int_{\mathcal{G}} \rho(x,t)|\boldsymbol{\eta}_q(x)|^2 dS. \end{aligned} \tag{6.22}$$

Other coefficients $d_{km}(\rho)$ can be found from the linear system

$$\sum_{k<m}{}'' d_{km}(\rho) \int\limits_{\mathcal{F}} \boldsymbol{\eta}_{km}(x) \cdot \boldsymbol{\eta}_{ij}(x) dx = \int\limits_{\mathcal{F}} \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_{ij}(x) dx$$

$$= - \int\limits_{\mathcal{G}} \rho(x) \mathcal{C} \boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dS.$$

Since $\boldsymbol{\eta}_{km}$ are linearly independent, the matrix with entries

$$A_{km,ij} = \int\limits_{\mathcal{F}} \boldsymbol{\eta}_{km}(x) \cdot \boldsymbol{\eta}_{ij}(x) dx$$

is nonsingular; moreover, it is positive definite. We have

$$d_{km}(\rho) = -\sum_{i<j}{}'' A^{km,ij} \int\limits_{\mathcal{G}} \rho(x) \mathcal{C} \boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dS$$

where $A^{km,ij}$ are the entries of the inverse matrix. The form

$$Q(\rho) = \sum_{k<m,i<j}{}'' A^{km,ij} \int\limits_{\mathcal{G}} \rho(x) \mathcal{C} \boldsymbol{x} \cdot \boldsymbol{\eta}_{km}(x) dS \int\limits_{\mathcal{G}} \rho(x) \mathcal{C} \boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dS \quad (6.22')$$

is nonnegative.

The operators B and $\widehat{B}$ are defined by the formulas

$$B\rho = B_0\rho + \sum_{q=1}^{l} d_q(\rho) \omega_q |\boldsymbol{\eta}_q(x)|^2$$

$$= B_0\rho + \sum_{q=1}^{l} \omega_q^2 |\boldsymbol{\eta}_q(x)|^2 \|\boldsymbol{\eta}_q\|^{-2}_{L_2(\mathcal{F})} \int\limits_{\mathcal{G}} \rho(y) |\boldsymbol{\eta}_q(y)|^2 dS$$

and

$$\widehat{B}\rho = B\rho - \frac{1}{|\mathcal{G}|} \int\limits_{\mathcal{G}} B\rho dS.$$

For the problem (6.18) and the corresponding nonhomogeneous problem Theorem 3.1 and the estimates (3.11), (3.15) hold. Our next objective is to establish the exponential decay of the solution of the problem (6.18)–(6.20) in the case, where the form

$$\int_{\mathcal{G}} \rho \widehat{B} \rho dS \tag{6.23}$$

is positive. We achieve this by constructing a special function, the so-called "generalized energy" $E(t)$ playing the role of the Lyapunov function. The basic idea is due to Padula [**9**]. At first, we prove the following n-dimensional analog of Lemma 4.1 in [**21**].

Proposition 6.1. *Let $\rho(x,t)$ be a function in $W_2^{1/2}(\mathcal{G})$ for all $t \in (0,T)$ possessing the derivative $\rho_2 \in L_2(\mathcal{G})$ and satisfying the condition*

$$\int_{\mathcal{G}} \rho(x,t) dS = 0.$$

Then there exists a solenoidal vector field $\boldsymbol{W}(x,t)$, $x \in \mathcal{F}$, orthogonal to all rigid rotations $\boldsymbol{\eta}_{km}$:

$$\int_{\mathcal{F}} \boldsymbol{W}(x,t) \cdot \boldsymbol{\eta}_{km}(x) dx = 0, \tag{6.24}$$

and satisfying the boundary condition

$$\boldsymbol{W}(x,t) \cdot \boldsymbol{N}(x) = \rho(x,t), \quad x \in \mathcal{G} \tag{6.25}$$

and the inequalities

$$\|\boldsymbol{W}(\cdot,t)\|_{W_2^1(\mathcal{F})} \leqslant c\|\rho(\cdot,t)\|_{W_2^{1/2}(\mathcal{G})}, \tag{6.26}$$

$$\|\boldsymbol{W}(\cdot,t)\|_{L_2(\mathcal{F})} \leqslant c\|\rho(\cdot,t)\|_{L_2(\mathcal{G})}, \tag{6.27}$$

$$\|\boldsymbol{W}_t(\cdot,t)\|_{L_2(\mathcal{F})} \leqslant c\|\rho_t(\cdot,t)\|_{L_2(\mathcal{G})}. \tag{6.28}$$

PROOF. We find $\boldsymbol{W}$ in the form

$$\boldsymbol{W}(x,t) = \boldsymbol{W}_0(x,t) + \boldsymbol{W}_1(x,t)$$

where $\boldsymbol{W}_0$ is a solenoidal vector field satisfying (6.25)–(6.28) and $\boldsymbol{W}_1$ is a correcting term responsible for the condition (6.24). The construction of $\boldsymbol{W}_0$ is a quite standard problem. The vector field $\boldsymbol{W}_1$ can be taken in the form

$$\boldsymbol{W}_1(x,t) = \sum_{i<j} \Big(\boldsymbol{e}_i \frac{\partial}{\partial x_j} - \boldsymbol{e}_j \frac{\partial}{\partial x_i}\Big) c_{ij}(t) A(x),$$

where

$$c_{ij}(t) = \int_{\mathcal{F}} \boldsymbol{W}_0(x,t)\cdot \boldsymbol{\eta}_{ij}(x)dx$$

and $A(x)$ is a smooth function with compact support in $\mathcal{F}$ such that

$$\int_{\mathcal{F}} A(x)dx = 1/2.$$

We have

$$\int_{\mathcal{F}} \boldsymbol{W}_1(x,t)\cdot \boldsymbol{\eta}_{km}(x)dx = -\sum_{i<j} c_{ij}(t) \int_{\mathcal{F}} A(x)\Big(\boldsymbol{e}_i \frac{\partial}{\partial x_j} - \boldsymbol{e}_j \frac{\partial}{\partial x_i}\Big)\cdot \boldsymbol{\eta}_{km} dx$$
$$= -c_{km}(t)$$

which implies (6.24). It is clear that $\boldsymbol{W}_1$ satisfies (6.26)–(6.28). □

The next proposition concerns the structure of the vector fields $\mathcal{C}\boldsymbol{\eta}_{km}(x)$.

Proposition 6.2. *For arbitrary $k, m \leqslant n$, $k < m$ the vector field $\mathcal{C}\boldsymbol{\eta}_{km}(x)$ can be represented in the form*

$$2\mathcal{C}\boldsymbol{\eta}_{km}(x) = -\nabla(\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x)) + \boldsymbol{R}_{km}(x), \tag{6.29}$$

where $\boldsymbol{R}_{km}$ is a linear combination of $\boldsymbol{\eta}_{ij}$.

Proof. It is clear that, in the case $k, m > 2l$, both expressions in (6.29) containing $\mathcal{C}$ vanish and $\boldsymbol{R}_{km}$ should be taken equal to zero. For $k = 2q-1$, $m = 2q$ the equality (6.29) with $\boldsymbol{R}_{km} = 0$ is easily verified. In other cases, the following possibilities can take place: (1) both k and m are even, (2) both k and m are odd, (3) k is even and m is odd, (4) k is odd and m even, (5) $m > 2l$ and k is even or odd. We obtain (6.29) by a direct calculation. We consider cases (1)–(4) assuming that $n = 4$. Hence

$$\mathcal{C} = \operatorname{diag}(\mathcal{C}_1, \mathcal{C}_2), \quad \mathcal{C}_\alpha = \begin{pmatrix} 0 & -\omega_\alpha \\ \omega_\alpha & 0 \end{pmatrix}, \quad \alpha = 1,2,$$

and we verify (6.29) for $\boldsymbol{\eta}_{km} = \boldsymbol{\eta}_{13}, \boldsymbol{\eta}_{14}, \boldsymbol{\eta}_{23}, \boldsymbol{\eta}_{24}$. We have

$$2\mathcal{C}\boldsymbol{\eta}_{13} = 2(\omega_1\boldsymbol{e}_2 x_3 - \omega_2\boldsymbol{e}_4 x_1) = \omega_1\boldsymbol{\eta}_{23} + \omega_2\boldsymbol{\eta}_{14} + \nabla(\omega_1 x_2 x_3 - \omega_2 x_4 x_1),$$

and

$$\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{13} = (-\omega_1\boldsymbol{\eta}_{12} - \omega_2\boldsymbol{\eta}_{34})\cdot\boldsymbol{\eta}_{13} = -\omega_1 x_3 x_2 + \omega_2 x_1 x_4,$$
$$2\mathcal{C}\boldsymbol{\eta}_{14} = 2(\omega_1\boldsymbol{e}_2 x_4 + \omega_2\boldsymbol{e}_3 x_1) = \omega_1\boldsymbol{\eta}_{24} - \omega_2\boldsymbol{\eta}_{13} + \nabla(\omega_1 x_2 x_4 + \omega_2 x_3 x_1),$$
$$\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{14} = -\omega_1 x_4 x_2 - \omega_2 x_1 x_3,$$
$$2\mathcal{C}\boldsymbol{\eta}_{23} = 2(-\omega_1\boldsymbol{e}_1 x_3 - \omega_2\boldsymbol{e}_4 x_2) = -\omega_1\boldsymbol{\eta}_{13} + \omega_2\boldsymbol{\eta}_{24} - \nabla(\omega_1 x_1 x_3 + \omega_2 x_2 x_4),$$
$$\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{23} = \omega_1 x_1 x_3 + \omega_2 x_2 x_4,$$
$$2\mathcal{C}\boldsymbol{\eta}_{24} = 2(-\omega_1\boldsymbol{e}_1 x_4 + \omega_2\boldsymbol{e}_3 x_2) = -\omega_1\boldsymbol{\eta}_{14} - \omega_2\boldsymbol{\eta}_{23} - \nabla(\omega_1 x_1 x_4 - \omega_2 x_3 x_2),$$
$$\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{24} = \omega_1 x_4 x_1 - \omega_2 x_2 x_3.$$

The last case $k \leqslant 2l$, $n > 2l$ occurs when $n = 3$, $\mathcal{C} = \mathrm{diag}(\mathcal{C}_1, 0)$, $\boldsymbol{\eta}_{km} = \boldsymbol{\eta}_{13}, \boldsymbol{\eta}_{23}$. We have

$$2\mathcal{C}\boldsymbol{\eta}_{13} = 2\omega_1\boldsymbol{e}_2 x_3 = \omega_1\boldsymbol{\eta}_{23} + \nabla\omega_1 x_3 x_2, \quad \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{13} = -\omega_1 x_3 x_2,$$
$$2\mathcal{C}\boldsymbol{\eta}_{23} = -2\omega_1\boldsymbol{e}_1 x_3 = -\omega_1\boldsymbol{\eta}_{13} - \nabla\omega_1 x_3 x_1, \quad \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{23} = \omega_1 x_3 x_1.$$

We see that (6.29) holds in all these cases. It is clear that the same arguments are true in the general case. □

Now, we consider the evolution problem (6.18)–(6.20).

Proposition 6.3. *If the form* (6.23) *is positive definite, i.e.,*

$$\int\limits_{\mathcal{G}} \rho\widehat{B}\rho dS \geqslant c\|\rho\|^2_{W^1_2(\mathcal{G})}, \tag{6.30}$$

then the solution of the problem (6.18)–(6.20) *satisfies the inequality*

$$\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F})} + \|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})} \leqslant ce^{-bt}\Big(\|\boldsymbol{v}_0\|^2_{L_2(\mathcal{F})} + \|\rho_0\|^2_{W^1_2(\mathcal{G})}\Big), \tag{6.31}$$

where $b = \mathrm{const} > 0$.

Proof. Multiplying the first equation in (6.18) by $\boldsymbol{v}$ and integrating over $\mathcal{F}$, we obtain the energy relation

$$\frac{d}{dt}\Big(\frac{1}{2}\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F})} + \int\limits_{\mathcal{G}} \rho(x,t)B_0\rho(x,t)dS\Big) + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathcal{F})} = 0. \tag{6.32}$$

Further, we write the same equation in the form

$$\boldsymbol{v}_t^\perp + 2\mathcal{C}\boldsymbol{v}^\perp + 2\mathcal{C}\boldsymbol{D}' + 2\mathcal{C}\boldsymbol{D}'' - \nu\nabla^2\boldsymbol{v}^\perp + \nabla p = -\boldsymbol{D}'_t - \boldsymbol{D}''_t \tag{6.33}$$

and we observe that

$$2\mathcal{C}\boldsymbol{D}' = \nabla p'(x,t), \quad 2\mathcal{C}\boldsymbol{D}'' = \nabla p''(x,t) + \boldsymbol{R}(x,t),$$

where

$$p'(x,t) = \sum_{q=1}^{l} d_q(\rho)\omega_q|\boldsymbol{\eta}_q(x)|^2, \quad p''(x,t) = -{\sum_{k,m}}'' d_{km}(\rho)\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x)$$

and $\boldsymbol{R}$ is a linear combination of $\boldsymbol{\eta}_{ij}$. Multiplying (6.33) by $\boldsymbol{W}$, integrating over $\mathcal{F}$, and taking into account (6.24), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{F}} \boldsymbol{v}^\perp\cdot\boldsymbol{W}dx - \int_{\mathcal{F}} \boldsymbol{v}^\perp\cdot\boldsymbol{W}_t dx + 2\int_{\mathcal{F}} \mathcal{C}\boldsymbol{v}^\perp\cdot\boldsymbol{W}dx\\ &+\frac{\nu}{2}\int_{\mathcal{F}} S(\boldsymbol{v}^\perp):S(\boldsymbol{W})dx + \int_{\mathcal{G}}(B_0\rho + p' + p'')\rho dS = 0.\end{aligned} \tag{6.34}$$

The last surface integral is equal to

$$\int_{\mathcal{G}} \rho\widehat{B}\rho dS + Q(\rho) \tag{6.35}$$

where $Q(\rho)$ is defined in (6.22′).

Now, we add (6.32) and (6.34) multiplied by a small positive number γ. This gives

$$\frac{dE(t)}{dt} + E_1(t) = 0 \tag{6.36}$$

where

$$\begin{aligned}E(t) &= \frac{1}{2}\Big(\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})} + \int_{\mathcal{G}} \rho B_0\rho dx\Big) + \gamma\int_{\mathcal{F}} \boldsymbol{v}^\perp\cdot\boldsymbol{W}dx\\ &= \frac{1}{2}\Big(\|\boldsymbol{v}^\perp\|^2_{L_2(\mathcal{F})} + \int_{\mathcal{G}} \rho\widehat{B}\rho dx + \|\boldsymbol{D}''\|^2_{L_2(\mathcal{F})} + 2\gamma\int_{\mathcal{F}} \boldsymbol{v}^\perp\cdot\boldsymbol{W}dx\Big),\end{aligned}$$

$$E_1(t) = \frac{\nu}{2}\|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})} - \gamma \int\limits_{\mathcal{F}} \boldsymbol{v}^\perp \cdot \boldsymbol{W}_t dx + 2\gamma \int\limits_{\mathcal{F}} \mathcal{C}\boldsymbol{v}^\perp \cdot \boldsymbol{W} dx$$

$$+ \frac{\gamma\nu}{2} \int\limits_{\mathcal{F}} S(\boldsymbol{v}^\perp) : S(\boldsymbol{W}) dx + \gamma\Big(\int\limits_{\mathcal{G}} \rho \widehat{B} \rho dS + Q(\rho)\Big).$$

If γ is small enough, then, by the Korn inequality and (6.26)–(6.28), (6.30), we have

$$E_1(t) \geqslant bE(t)$$

and $E(t)$ is estimated from below and from above by

$$\text{const } (\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F})} + \|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})}).$$

Hence (6.36) implies

$$\frac{dE(t)}{dt} + bE(t) \leqslant 0, \quad E(t) \leqslant e^{-bt}E(0),$$

which completes the proof of the proposition. □

As in the case $n = 3$, from (6.31) we can deduce the inequality (3.15) with $c(t) = e^{-bt}$ and prove the exponential stability of the zero solution of the problem (6.13)–(6.15) by the arguments of Section 4. Thus, Theorem 4.1 is extended to the n-dimensional symmetric case.

The estimate (6.31) was obtained under apparently weaker than (6.30) assumption of the positivity of the sum (6.35), where $Q(\rho) \geqslant 0$. A similar phenomenon has occurred in the three-dimensional case, but it was proved (see [**20**]) that both assumptions are equivalent.

The above theory applies to the case $n = 2$ (considered in [**12**]). In this case, there exists only one vector field of a rigid rotation, $\boldsymbol{\eta}_{12}$,

$$\mathcal{C} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \boldsymbol{V}(x) = \omega\boldsymbol{\eta}_{12}(x),$$

the equilibrium figure is a disc $|x| \leqslant R_0$,

$$B_0\rho = \frac{\sigma}{R_0^2}\Big(\frac{d^2\rho}{d\varphi^2} + \rho\Big) + \omega^2 R_0\rho, \quad B\rho = B_0\rho + \frac{2\omega^2}{\pi} \int\limits_{|x|=R_0} \rho dS,$$

where $\varphi \in [0, 2\pi)$ is a polar angle.

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Weak Spatially Nondecaying Solutions of 3D Navier–Stokes Equations in Cylindrical Domains

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The weighted energy theory for the Navier–Stokes equations in 3D cylindrical domains is developed. Based on this theory, the existence of a weak solution belonging to the uniformly local phase space (without any spatial decaying assumptions), its dissipativity and existence of the so-called trajectory attractor are verified. In particular, this phase space contains the 3D Poiseuille flows. Bibliography: 37 titles.

1. Introduction

It is well known that the Navier–Stokes system

$$\begin{aligned} &\partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \\ &\operatorname{div} u = 0, \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{aligned} \tag{1.1}$$

in a *bounded* 2D domain $\Omega \subset\subset \mathbb{R}^2$ is well posed and generates a dissipative semigroup $S(t)$ in the appropriate phase space (of square integrable divergence-free vector fields). It is also known that, in the case of bounded 3D domains, we have only the global existence of weak solutions (without uniqueness) and local in time existence of strong solutions (with uniqueness), see [**6, 27, 28**] and references therein. These results are strongly based on the so-called energy estimate. To obtain this energy estimate, one multiplies Equation (1.1) by u, integrate over Ω, and uses the fact that the

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nonlinear term disappears:

$$((u,\nabla_x)u,u) := \int\limits_{x\in\Omega} (u(x),\nabla_x)u(x).u(x)\,dx \equiv 0 \tag{1.2}$$

for every divergence-free vector field with Dirichlet boundary conditions.

The situation becomes much more difficult when the domain Ω is unbounded. Moreover, although there exists a highly developed theory of dissipative PDEs in unbounded domains (mainly based on the so-called weighted energy estimates, see [**7**]–[**12**], [**20, 21, 32, 33, 34, 35**] and references therein), during the long time, it was not clear how to apply it to the Navier–Stokes problem in unbounded domains because of several principal obstacles.

Indeed, in contrast to bounded domains, in unbounded ones, the space of square integrable (divergence-free) vector fields is not a convenient phase space since the assumption $u \in L^2(\Omega)$ imposes too restrictive *decay* conditions on $u(x)$ as $x \to \infty$. So, under this choice of the phase space, many classical hydrodynamical objects, like Poiseuille flows, Couette–Taylor flows, Kolmogorov flows, etc. are automatically out of consideration. Thus, following the general theory, it is reasonable to replace the assumption $u \in L^2(\Omega)$ by a more relevant condition: $u \in L^2_b(\Omega)$, where the uniformly local Sobolev spaces $W^{l,p}_b(\Omega)$ are defined via the following standard expression:

$$W^{l,p}_b(\Omega) := \{u \in D'(\Omega),\ \|u\|_{W^{l,p}_b(\Omega)} := \sup_{x_0\in\Omega} \|u\|_{W^{l,p}(\Omega\cap B^1_{x_0})} < \infty\}.$$

Here, $B^1_{x_0}$ denotes the ball of radius one of $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ and $W^{l,p}$ means the classical Sobolev space. But then the main difficulty arises: how to obtain a priori estimates for the solution $u(t)$ in uniformly local spaces?

Indeed, since $u(t)$ is not square integrable any more, we cannot simply multiply (1.1) by u and use the identity (1.2) (the integrals do not have sense). So, following the general strategy, we need to multiply it by φu, where $\varphi = \varphi(x)$ is an appropriate *weight* function. But in this case, the nonlinear term does not vanish and produces an additional cubic term like $\varphi' u^3$. We note that this cubic term *is not of fixed sign* and the remaining terms in the energy equality are at most quadratic with respect to u, so it was not clear how to control this cubic term in order to produce reasonable a priori estimate.

Another obstacle is related with the fact that φu is not *divergence-free.* Hence the pressure p does not disappear in the weighted energy equality and one should be able to control the term $(\varphi' p, u)$. Of course, this problem is closely related with finding a reasonable extension of the Helmholtz projector (to divergence-free vector fields) to uniformly local spaces.

The above-mentioned difficulties stimulated the development of alternative methods for studying the Navier–Stokes equations in unbounded domains. In particular, in the 2D case, the following so-called vorticity equation is very helpful:

$$\partial_t \omega - \Delta_x \omega + (u, \nabla_x)\omega = \partial_{x_2} g_1 - \partial_{x_1} g_2, \tag{1.3}$$

where $\omega := \partial_{x_2} u_1 - \partial_{x_1} u_2$. If Ω does not contain boundary, for example, $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is a circle (like in the Kolmogorov problem), the maximum principle applied to (1.3) allows us to obtain a global a priori estimate for the vorticity ω which, together with the accurate analysis of the explicit formulas for the Helmholtz projectors, allows us to obtain global in time a priori estimates for the solution $u(t)$ and thereby to prove the global solvability of the Navier–Stokes equation in uniformly local phase spaces (see [**2, 14**]). Unfortunately, the a priori estimate for vorticity obtained from the maximum principle grows linearly in time, so all the further estimates also grow in time (to the best of our knowledge, in the case $\Omega = \mathbb{R}^2$, it yields double exponential ($\sim e^{Ce^{Ct}}$) growth rate and polynomial ($\sim t^3$) growth rate for $\Omega = \mathbb{S}^1 \times \mathbb{R}$). The other essential drawback is that this method seems to be inapplicable to the problems with boundary (for example, for a cylindrical domain Ω) and does not work in the 3D case.

Another attractive possibility to avoid direct weighted energy estimates is to use the bifurcation analysis. Indeed, in the situation where the basic steady state of the Navier–Stokes problem is slightly above the instability threshold, the solutions remaining close to that steady state can be described in terms of the so-called *modulation* equations which are essentially simpler than the initial Navier–Stokes problem (usually it is Ginzburg–Landau or Swift–Hohenberg equations), see [**1, 15, 16, 17, 19**] and references therein. Since the well-posedness and dissipativity of these modulation equations is well understood, the standard perturbation methods allow us sometimes to obtain global in time estimates for solutions of the initial Navier–Stokes problem starting from the small neighborhood of the basic steady state. In particular, the global existence and dissipativity of such solutions for the 3D Couette–Taylor flow is obtained in [**23**] and "almost global solvability" (on the exponentially long with respect to perturbation parameter time interval) for the case of Poiseuille flow can be found in [**24**].

It is worth to emphasize that, in the case where the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, possesses the Friedrichs inequality

$$\|u\|^2_{L^2(\Omega)} \leqslant \lambda_1 \|\nabla_x u\|^2_{L^2(\Omega)}, \ u \in W^{1,2}_0(\Omega), \tag{1.4}$$

with *positive* λ_1 and under the restrictive assumption that *u is square integrable*, all the above-mentioned obstacles disappear and the Navier–Stokes problem (1.1) possesses a standard (unweighted) energy theory similar to the case of bounded domains, see [**5, 28**]. We also mention the survey paper [**3**] on the existence of spatially decaying solutions of the Navier–Stokes problem in various domains (not necessarily satisfying (0.4)), see also [**13, 30**].

Recently, the above-mentioned obstacles for applying the general weighted energy theory to Navier–Stokes equations in unbounded domains were overcome [**37**] in the case of 2D cylindrical domains. This result allowed us to verify the global existence, uniqueness and dissipativity of the 2D Navier–Stokes equations in the classes of spatially nondecaying solutions. Moreover, this result embeds the 2D Navier–Stokes problem in a strip into the general scheme of investigating dissipative PDEs in unbounded domains, including the study of dimension and Kolmogorov's entropy of attractors, topological entropies, spatial and temporal chaos, etc. (see [**36**]).

The main goal of this paper is to extend (up to uniqueness and further regularity) this result to the case of 3D cylindrical domains. Although the general strategy of the paper is similar to [**37**], there are several essential differences and complications in comparison to the 2D case. Namely, in the 3D case, we do not have a scalar stream function and, consequently, we cannot reduce the study of the Helmholtz projector and Stokes operator in weighted spaces to simple model problems for the Laplace and bi-Laplace equations and should use the theory of general elliptic problems.

Further, because of the lack of uniqueness for the 3D Navier–Stokes equations, we cannot directly apply the methods of [**37**], but should first consider the regularizing Leray approximations to the Navier–Stokes equations, prove the existence of spatially nondecaying solutions for these equations, and, after that, obtain the required solution by passing to the limit. Finally, again because of the lack of uniqueness, we cannot construct a usual *global* attractor for the problem considered and should use the so-called *trajectory* approach (see [**8, 25, 31, 9**] and references therein).

The paper is organized as follows. In Sections 2 and 3, we recall some basic facts on the theory of weighted spaces and the regularity of elliptic boundary value problems in these spaces which will be systematically used throughout the paper.

Section 4 is devoted to the study of the Helmholtz projector Π and stationary Stokes equations in weighted and uniformly local Sobolev spaces. The results of this section are somehow close to [**4, 5**] (and are, factually, inspired by these papers).

In Section 5, we study the following auxiliary linear nondivergence-free problem:

$$\begin{aligned} &-\partial_t v = \Delta_x v + \nabla_x q, \quad \Pi v\big|_{t=T} = 0,\\ &\operatorname{div} v = \varphi' u, \quad v\big|_{\partial\Omega} = 0, \end{aligned} \tag{1.5}$$

where $\varphi(x)$ is an appropriate weight function and $u(t)$ is a solution of the Navier–Stokes problem. This auxiliary problem is necessary in order to overcome the obstacle related with the appearance of the term containing pressure in the weighted energy equality. Roughly speaking, we will multiply Equation (1.1) by $\varphi u(t) - v(t)$, where v solves (1.5). Then, since $\operatorname{div}(\varphi u - v) = 0$, the pressure term disappears (and the derivative of our weights is small, so the corrector v is also small and does not produce any essential difficulties in its estimating, see Sections 5 and 6 for details).

It is not clear how to overcome this obstacle in a more simple way. Indeed, the "most natural" multiplication by $\Pi(\varphi u)$ does not work since $\Pi(\varphi u)$ has nonzero trace at the boundary, which leads to additional uncontrollable boundary terms under the integration by parts in $(\Delta_x u, \Pi(\varphi u))$. Another possibility is to construct a new "projector" Q to divergence-free vector fields which preserves the boundary conditions and multiply the equation by $Q(\varphi u)$. However, this leads to essential difficulties with the term $(\partial_t u, Q(\varphi u))$ which should be a complete time derivative from something. We also note that the multiplication of the equation by the combination of $\varphi\partial_t u$ and $\varphi\Pi\Delta_x u$ (as in [**4**] and [**5**]) is useless for us since it works *only* if the unweighted L^2-norm of $\Delta_x u$ is a priori known.

In Section 6, we verify the basic (uniform with respect to $\alpha \to 0$) a priori estimate and prove the global existence of solutions of the Leray–Navier–Stokes problem

$$\begin{aligned} &\partial_t u + (\Pi w, \nabla_x) u + c\partial_{x_1} u = \Delta_x u - \nabla_x p + g,\\ &w - \alpha\Delta_x w = u,\\ &\operatorname{div} u = 0, \ \mathbb{S}u_1 = c,\\ &u\big|_{\partial\Omega} = w\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{aligned} \tag{1.6}$$

where Π is a Helmholtz projector to the divergence-free vector fields, $\alpha > 0$ is a small parameter, $\mathbb{S}$ is the averaging operator with respect to the cross section $x' := (x_2, x_3)$:

$$\mathbb{S}v := \frac{1}{|\omega|} \int\limits_{x' \in \Omega} v(x')\, dx',$$

and c is a given constant.

The additional projector Π is necessary since, in contrast to the spatially periodic case, w is no more divergence-free and we will not have zero integral analogous to (1.2) without this projector. The term $c\partial_{x_1} u$ appears in order to have the classical Navier–Stokes problem as $\alpha = 0$ (since, due to our choice of projector Π, the mean flux of Πw is equal to zero and, consequently, $\Pi u = u - (c, 0, 0)$).

To obtain the required estimate, we use, following [**37**], the special weights

$$\theta_{\varepsilon, x_0}(x) := (1 + \varepsilon^2 |x - x_0|^2)^{1/2} \tag{1.7}$$

with very small ε which factually depends on the solution u. Then a careful analysis of the obtained weighted energy inequality allows us to obtain the globally in time bounded a priori estimate of the L^2_b-norm of $u(t)$. Based on this a priori estimate, we then establish the existence of such a solution. In fact, we first consider the case of zero flux $c = 0$ (see Theorem 6.5) and after that reduce the general case to this particular case using the trick with the auxiliary "energy stable" equilibrium (see Theorem 6.6).

The uniqueness of such solutions is verified in Section 7 (see Theorem 7.1). Moreover, we also verify the *dissipative* estimate, uniform with respect to $\alpha \to 0$, for these solutions and the existence of global attractors $\mathcal{A}_\alpha$ for the approximating problems (1.6).

Finally, in Section 8, we establish the existence of a dissipative weak solution for the classical Navier–Stokes problem by passing to the limit $\alpha \to 0$. Moreover, an appropriate *trajectory* attractor $\mathcal{A}_{\mathrm{tr}}$ for the Navier-stokes problem is also constructed here. Using the proper scaling, we obtain the following estimate for the size of attractor in the L^2_b-norm in terms of the kinematic viscosity ν:

$$\|\mathcal{A}_{\mathrm{tr}}\|_{L^\infty(\mathbb{R}_+, L^2_b(\Omega))} \leqslant C\nu^{-3}(|c|^3\nu + \|g\|^2_{L^2_b(\Omega)} + \nu^4), \tag{1.8}$$

where the constant C is independent of ν, c, and g. We recall that, in bounded domains (in square integrable case), the best known estimate is the following:

$$\|\mathcal{A}_{\mathrm{tr}}\|_{L^\infty(\mathbb{R}_+, L^2(\Omega))} \leqslant C\nu^{-1}\|g\|_{L^2(\Omega)}. \tag{1.9}$$

We see that, although the estimate (1.8) is "worse" than (1.9), but it remains polynomial as $\nu \to 0$ (with a reasonable degree 3). Thus, our method is not "extremely rough" and can be used for obtaining reasonable *quantitative* bounds for the solutions.

2. Function Spaces

In this section, we briefly recall the definitions and basic properties of weight functions and weighted function spaces which will be systematically used throughout the paper (see also [**11, 33**] for details). We start with the class of admissible weight functions.

Definition 2.1. A function $\varphi \in C_{loc}(\mathbb{R}^n)$ is a weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\varphi(x+y) \leqslant C_\varphi \varphi(x) e^{\mu|y|}, \quad \varphi(x) > 0 \quad \forall x, y \in \mathbb{R}^n. \tag{2.1}$$

The following proposition collects the evident properties of such weights.

Proposition 2.2. 1. *Let φ be a weight function with exponential growth rate μ. Then, for every $\varepsilon > \mu$, φ is a weight function of exponential growth rate ε (with the same constant C_φ).*

2. *Let φ and ψ be weight functions of exponential growth rate μ. Then the functions $\Psi_1 = \varphi(x)\psi(x)$ and $\Psi_2 = \varphi(x)/\psi(x)$ are weight functions of exponential growth rate 2μ with the constant $C_{\Psi_i} \leqslant C_\varphi C_\psi$.*

3. *Let φ be a weight function of exponential growth rate μ, and let $\psi \in C_{loc}(\mathbb{R}^n)$ satisfy*

$$C_1\varphi(x) \leqslant \psi(x) \leqslant C_2\varphi(x), \ x \in \mathbb{R}^n. \tag{2.2}$$

Then ψ is also a weight function of exponential growth rate μ and $C_\psi \leqslant C_1^{-1}C_2C_\varphi$.

4. *Let $\varepsilon > 0$, and let $\varphi(x)$ be a weight function of exponential growth rate μ. Then the function $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ is of exponential growth rate $\varepsilon\mu$ and $C_{\varphi_\varepsilon} = C_\varphi$.*

All the assertions of Proposition 2.2 are simple consequences of the estimate (2.1).

A natural example of such weights is the following:

$$\varphi_{\mu,x_0}(x) := e^{-\mu|x-x_0|}, \ x_0 \in \mathbb{R}^n, \ \mu \in \mathbb{R}. \tag{2.3}$$

It is obvious that they are of exponential growth rate $|\mu|$ and the constant $C_{\varphi_{\mu,x_0}} = 1$ (independent of $x_0 \in \mathbb{R}^n$). However, these weights are non-smooth at $x = x_0$. To overcome this drawback, it is natural to use the following equivalent weights:

$$\varphi_{\mu,x_0}(x) := e^{-\mu\sqrt{1+|x-x_0|^2}},\ x_0 \in \mathbb{R}^n. \tag{2.4}$$

Since $|x| \leqslant \sqrt{x^2+1} \leqslant |x|+1$, these weights satisfy

$$e^{-|\mu|}\varphi_{\mu,x_0}(x) \leqslant \varphi_{\mu,x_0}(x) \leqslant e^{|\mu|}\varphi_{\mu,x_0}(x),\ x \in \mathbb{R}^n \tag{2.5}$$

and, consequently, φ_{μ,x_0} are also weight functions of exponential growth rate μ (with $C_{\varphi_{\mu,x_0}} = e^{2|\mu|}$). Moreover, in contrast to (2.3). these weights are smooth and satisfy for $\mu \leqslant 1$ the additional obvious inequality

$$|D_x^k\varphi_{\mu,x_0}(x)| \leqslant C_k|\mu|\varphi_{\mu,x_0}(x), \quad x \in \mathbb{R}^n, \tag{2.6}$$

where $k \in \mathbb{N}$, D_x^k denotes the collection of all x-derivatives of order k and the constant C_k is independent of x and μ. This inequality is crucial for obtaining the regularity estimates in weighted spaces (see [**11, 12, 32, 33, 34, 35**] and Section 3 below).

Another important class of weight functions is the so-called polynomial ones:

$$\theta_{x_0}^m(x) := (1+|x-x_0|^2)^{-m/2},\ m \in \mathbb{R}. \tag{2.7}$$

It is not difficult to verify that these weights are of exponential growth rate μ for every $\mu > 0$ with the constant $C_{\theta_{m,x_0}}$ depending on μ and m, but independent of $x_0 \in \Omega$.

We now introduce a class of weighted Sobolev spaces in a regular unbounded domain Ω associated with weights introduced above. Since we need below only the case where $\Omega := \mathbb{R} \times \omega$ is a cylinder with regular boundary, we do not formulate precise assumptions on the boundary $\partial\Omega$ (which can be found, for example, in [**11**] or [**12**]) in order to avoid the technicalities.

Definition 2.3. Let Ω be a regular domain, and let φ be a weight function of exponential growth rate. Then for every $1 \leqslant p \leqslant \infty$ we set

$$L^p_\varphi(\Omega) := \left\{u \in L^p_{loc}(\Omega),\ \|u\|^p_{L^p_\varphi} := \int_\Omega \varphi(x)^p|u(x)|^p\,dx < \infty\right\} \tag{2.8}$$

and

$$L^p_{b,\varphi}(\Omega) := \{u \in L^p_{loc}(\Omega),\ \|u\|_{L^p_{b,\varphi}} := \sup_{x_0\in\Omega}(\varphi(x_0)\|u\|_{L^p(\Omega\cap B^1_{x_0})}) < \infty\}. \tag{2.9}$$

Hereinafter, $B^r_{x_0}$ denotes an r-ball of $\mathbb{R}^n$ centered at x_0 and we write L^p_b instead of $L^p_{b,1}$.

Moreover, for every $l \in \mathbb{N}$, we define the weighted Sobolev spaces $W^{l,p}_\varphi(\Omega)$ and $W^{l,p}_{b,\varphi}(\Omega)$ as spaces of distributions whose derivatives up to order l belong to $L^p_\varphi(\Omega)$ and $L^p_{b,\varphi}(\Omega)$ respectively.

Furthermore, the weighted Sobolev spaces $W^{l,p}_\varphi(\partial\Omega)$ and $W^{l,p}_{b,\varphi}(\partial\Omega)$ on the boundary $\partial\Omega$ can be defined in a similar way; only the integral over Ω (respectively, supremum in (2.9)) in (2.8) should be replaced with the integral (respectively, supremum) over the boundary $\partial\Omega$ (see [**11, 12**]).

Remark 2.4. In the sequel, we also use functions $u(t)$ with the values in the weighted Sobolev spaces defined above. In slight abuse the notations, we denote by $L^p_b(\mathbb{R}, W^{l,p}_b)$ the space generated by the norm

$$\|u\|_{L^p_b(\mathbb{R},W^{l,p}_b)} := \sup_{x_0\in\Omega}\sup_{T\in\mathbb{R}} \|u\|_{L^p([T,T+1],W^{l,p}(\Omega\cap B^1_{x_0}))}. \tag{2.10}$$

The following proposition presents some useful facts on the spaces introduced above.

Proposition 2.5. *Let Ω be a regular domain, and let φ be a weight of exponential growth rate μ. Then the following assertions hold.*

1) *For any $r > 0$ and $u \in L^p_\varphi(\Omega)$, $1 \leqslant p < \infty$,*

$$C_r^{-1}\|u\|_{L^p_\varphi(\Omega)} \leqslant \Big(\int\limits_{x_0\in\Omega} \varphi^p(x_0)\|u\|^p_{L^p(\Omega\cap B^r_{x_0})}\,dx_0\Big)^{1/p} \leqslant C_r\|u\|_{L^p_\varphi(\Omega)}, \tag{2.11}$$

where the constant C_r depends on r, μ, and the constant C_φ from (2.1)*, but is independent of φ and of the choice of φ.*

2) *For any $\alpha > \mu$, $q \in [1,\infty]$, and $u \in L^1_\varphi(\Omega)$*

$$\Big(\int\limits_{x_0\in\Omega} \varphi(x_0)^q\Big(\int\limits_{x\in\Omega} e^{-\alpha|x-x_0|}|u(x)|\,dx\Big)^q dx_0\Big)^{1/q} \leqslant C_\alpha\|u\|_{L^1_\varphi(\Omega)}, \tag{2.12}$$

where C_α depends on α, μ, and C_φ, but is independent of u and the choice of φ and q.

3) *For any $\alpha > \mu$ and $u \in L^p_{b,\varphi}(\Omega)$*

$$\begin{aligned} C_\alpha^{-1}\|u\|^p_{L^p_{b,\varphi}(\Omega)} &\leqslant \sup_{x_0\in\Omega}\{\varphi(x_0)^p \int\limits_{x\in\Omega} e^{-\alpha p|x-x_0|}|u(x)|^p\,dx\} \\ &\leqslant C_\alpha\|u\|^p_{L^p_{b,\varphi}(\Omega)}, \end{aligned} \tag{2.13}$$

where the constant C_α depends on α, μ, and C_φ, but is independent of u and the choice of φ.

The proof of the estimates is given in [**11**] (see also [**12, 30**]).

Remark 2.6. As we will see below, the estimate (2.11) allows us to reduce the proof of embedding and interpolation theorems for weighted Sobolev spaces to the classical unweighted case in a bounded domain. Estimates (2.12) and (2.13) allow us, in turns, to obtain the elliptic regularity in weighted spaces with *arbitrary* weights of exponential growth rate if analogous result for the *special* weights $e^{-\alpha|x-x_0|}$ (or, which is the same, for the equivalent smooth weights (2.4)) is known (see Section 3). Moreover, these estimates allow us to control the dependence of the constants in embedding, interpolation, and regularity theorems on the choice of weights, which is crucial in our study of the nondecaying solutions of the NS equations.

We introduce the weighted Sobolev spaces with fractional derivatives. We first recall that, in the unweighted case, the space $W^{l+s,p}(\Omega)$ for $s \in (0,1)$ and $l \in \mathbb{Z}_+$ is usually defined via

$$\|u\|^p_{W^{l+s,p}(\Omega)} := \|u\|^p_{W^{l,p}(\Omega)} + \int\limits_{x\in\Omega}\int\limits_{y\in\Omega} \frac{|D^l_x u(x) - D^l_x u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \qquad (2.14)$$

and, for negative l the space $W^{l,p}(\Omega)$ is defined as the conjugate space of $W_0^{-l,q}(\Omega)$, $1/p + 1/q = 1$ (see [**18, 29**]). Then the estimate (2.11) justifies the following definition.

Definition 2.7. Let Ω be a regular domain, and let φ be a weight function of exponential growth rate. For any $1 < p \leqslant \infty$ and $l \in \mathbb{R}$ we define the space $W^{l,p}_\varphi(\Omega)$ as a subspace of distributions with finite norm

$$\|u\|^p_{W^{l,p}_\varphi(\Omega)} := \int\limits_{x_0\in\Omega} \varphi(x_0)^p \|u\|^p_{W^{l,p}(\Omega\cap B^r_{x_0})}\,dx_0, \qquad (2.15)$$

where r is a positive number (it is not difficult to verify that this space is independent of r). Similarly, the norm in $W^{l,p}_{b,\varphi}$ is defined via

$$\|u\|^p_{W^{l,p}_{b,\varphi}(\Omega)} := \sup_{x_0\in\Omega}\{\varphi(x_0)^p \|u\|^p_{W^{l,p}(\Omega\cap B^r_{x_0})}\}. \qquad (2.16)$$

For the sake of simplicity, we fix below $r = 1$ in the definitions (2.15) and (2.16) of weighted norms.

According to (2.11), we see that for $l \in \mathbb{Z}_+$ the spaces thus defined coincide with the spaces from Definition 2.1. Moreover, it is not difficult to verify, using the explicit formula (2.14) that, in the unweighted case $\varphi = 1$, the norm (2.15) is equivalent to (2.14).

The following proposition describes the weighted negative Sobolev spaces in terms of conjugate spaces.

Proposition 2.8. *Let Ω be a regular domain, and let φ be a weight function of exponential growth rate μ. Then for any $l > 0$ and $1 < p, q < \infty$, $1/p + 1/q = 1$,*

$$W_\varphi^{-l,p}(\Omega) = [W_{0,\varphi^{-1}}^{l,q}(\Omega)]^*, \tag{2.17}$$

where $W_{0,\varphi}^{l,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the $W_\varphi^{l,q}$-norm and $*$ means the conjugate space (with respect to the standard inner product in $L^2(\Omega)$). Moreover,

$$C_1 \|u\|_{W_\varphi^{-l,p}(\Omega)} \leqslant \|u\|_{[W_{0,\varphi^{-1}}^{l,q}(\Omega)]^*} \leqslant C_2 \|u\|_{W_\varphi^{-l,p}(\Omega)} \tag{2.18}$$

where the constants C_1 and C_2 depend on μ, l, p, and C_φ, but are independent of the choice of u and C_φ.

Proof. In order to avoid the technicalities, we give below the proof of (2.18) only in the case of a cylindrical domain $\Omega := \mathbb{R} \times \omega$, where ω is a smooth bounded domain of $\mathbb{R}^{n-1}$ (only this case will be used in the sequel), although a slightly modified proof works for a general regular domain. In this particular case, we can restrict ourselves to consider only one-dimensional weights $\varphi \in C_{loc}(\mathbb{R})$. Since ω is bounded, (2.1) implies that

$$C_1 \varphi(s, \xi_0) \leqslant \varphi(s, \xi) \leqslant C_2 \varphi(s, \xi_0), \;\; s \in \mathbb{R}, \;\; \xi \in \omega, \tag{2.19}$$

where $\xi_0 \in \omega$ is some fixed point and, consequently, the weight $\varphi(s, \xi)$ is equivalent to $\varphi_{\xi_0}(s) := \varphi(s, \xi_0)$. Moreover, it is more convenient to use, instead of balls $B_{x_0}^r$, the finite cylinders $\Omega_s := (s, s+1) \times \omega$, i.e., to define the norm in $W_\varphi^{l,p}(\Omega)$ via

$$\|u\|_{W_\varphi^{l,p}(\Omega)}^p = \int\limits_{s \in \mathbb{R}} \varphi(s)^p \|u\|_{W^{l,p}(\Omega_s)}^p \, ds \tag{2.20}$$

(since the norms (2.15) are equivalent for different r and ω is bounded, (2.15) and (2.20) are also equivalent).

We first verify the right inequality of (2.18). To this end, we introduce a partition of unity $\{\psi_y\}_{y \in \mathbb{R}} \in C_0^\infty(\mathbb{R})$ such that

$$
\begin{aligned}
&1) \qquad \operatorname{supp}\psi_y \subset (y, y+1),\\
&2) \qquad \int_{y\in\mathbb{R}} \psi_y(s)\,dy \equiv 1,\\
&3) \qquad |D_s^k\psi_y(s)| \leqslant C_k,
\end{aligned}
\tag{2.21}
$$

where the constant C_k is independent of $s \in \mathbb{R}$ (such a partition of unity exists and can be chosen in a smooth way with respect to $y \in \mathbb{R}$).

Let $u \in [W^{l,q}_{0,\varphi^{-1}}(\Omega)]^*$ be a functional over $W^{l,q}_{0,\varphi^{-1}}(\Omega)$, and let v be an arbitrary test function from this space. Using (2.21) and the Hölder inequality, we find

$$
\begin{aligned}
|\langle u, v\rangle| &\leqslant \int_{y\in\mathbb{R}} |\langle u, \psi_y v\rangle|\,dy \leqslant \int_{y\in\mathbb{R}} \|u\|_{W^{-l,p}(\Omega_y)}\|\psi_y v\|_{W^{l,q}(\Omega_y)}\,dy\\
&\leqslant C\int_{y\in\mathbb{R}} \varphi(y)\|u\|_{W^{-l,p}(\Omega_y)}\cdot\varphi(y)^{-1}\|v\|_{W^{l,q}(\Omega_y)}\,dy\\
&\leqslant C\|u\|_{W^{-l,p}_{\varphi}(\Omega)}\|v\|_{W^{l,q}_{\varphi^{-1}}(\Omega)},
\end{aligned}
\tag{2.22}
$$

which, together with the definition of the norm in the conjugate space gives the right inequality in (2.18).

We verify the left inequality. Let $u \in W^{-l,p}_{\varphi}(\Omega)$. We fix a family of functions $v_y \in W^{l,q}_0(\Omega_y)$ such that

$$
\langle u, v_y\rangle = \|u\|_{W^{-l,p}(\Omega_y)}\|v_y\|_{W^{l,q}(\Omega_y)} \tag{2.23}
$$

and normalize these functions as follows:

$$
\|v_y\|_{W^{l,q}(\Omega_y)} = \varphi(y)^p\|u\|^{p-1}_{W^{-l,p}(\Omega_y)}. \tag{2.24}
$$

Since the spaces $W^{l,q}(\Omega_y)$ are uniformly convex, these families are uniquely defined and, moreover, continuous with respect to $y \in \mathbb{R}$.

We define the function $v(x)$ as follows:

$$
v(x) := \int_{y\in\mathbb{R}} v_y(x)\,dy. \tag{2.25}
$$

We claim that $v \in W^{l,q}_{0,\varphi^{-1}}(\Omega)$. Since $v_y \in W^{l,q}_0(\Omega_y)$, it can be naturally continued by zero to a function $v_y \in W^{l,q}_0(\Omega)$ with $\operatorname{supp} v_y \subset \Omega_y$. Thus, the integral (2.25) is well posed and defines a function $v \in W^{l,q}_{loc}(\overline{\Omega})$ vanishing at the boundary $\partial\Omega$. So, we only need to estimate the $W^{l,q}_{\varphi^{-1}}(\Omega)$-norm of this function.

Since $\|v_y\|_{W^{l,q}(\Omega_s)} = 0$ if $|s-y| \geqslant 1$, we have

$$\begin{aligned}\|v\|_{W^{l,q}(\Omega_s)} &\leqslant \int\limits_{|s-y|\leqslant 1} \|v_y\|_{W^{l,q}(\Omega_y)}\,dy = \int\limits_{|s-y|\leqslant 1} \varphi(y)^p \|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy \\ &\leqslant C\varphi(s)^p \int\limits_{|s-y|\leqslant 1} \|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy \\ &\leqslant C_1\varphi(s)^p \int\limits_{y\in\mathbb{R}} e^{-\alpha|s-y|}\|u\|^{p-1}_{W^{-l,p}(\Omega_y)}\,dy \end{aligned} \tag{2.26}$$

where the constant $\alpha > 2p\mu/q$ can be arbitrary (we implicitly used (2.1) in order to estimate $\varphi(y)$ via $\varphi(s)$). Taking the qth power of both sides of the relation, applying the Hölder inequality, and using that $q(p-1) = p$, we arrive at the inequality

$$\varphi(s)^{-q}\|v\|^q_{W^{l,q}(\Omega_s)} \leqslant C\varphi(s)^p \int\limits_{y\in\mathbb{R}} e^{\alpha q|s-y|/2}\|u\|^p_{W^{-l,p}(\Omega_y)}\,dy.$$

Integrating over $s \in \mathbb{R}$ and using (2.12), we finally infer

$$\|v\|^q_{W^{l,q}_{\varphi^{-1}}(\Omega)} \leqslant C_2\|u\|^p_{W^{-l,p}_{\varphi}(\Omega)}. \tag{2.27}$$

We are now ready to complete the proof of the proposition. By (2.23)–(2.25), we have

$$\langle u, v\rangle = \int\limits_{y\in\Omega} \|u\|_{W^{-l,p}(\Omega_y)}\|v_y\|_{W^{l,q}(\Omega_y)}\,dy = \|u\|^p_{W^{-l,p}_{\varphi}(\Omega)}$$

and, consequently, due to (2.27),

$$\|u\|_{[W^{l,q}_{0,\varphi^{-1}}(\Omega)]^*} \geqslant \frac{\langle u, v\rangle}{\|v\|_{W^{l,q}_{\varphi^{-1}}(\Omega)}} \geqslant C\|u\|^{p(1-1/q)}_{W^{-l,p}_{\varphi}(\Omega)}. \tag{2.28}$$

Since $p(1-1/q) = 1$, (2.28) implies the left inequality in (2.18). □

Remark 2.9. Proposition 2.8 shows, in particular, that in the case $\varphi = 1$, the spaces $W^{l,p}(\Omega)$ introduced in Definition 2.7, coincide with the standard Sobolev spaces for any $l \in \mathbb{R}$. Moreover, arguing Similarly, to the proof of Proposition 2.8, one can verify the interpolation representation of the weighted spaces $W^{l+\alpha,p}_{\varphi}(\Omega)$ with fractional derivatives ($l \in \mathbb{Z}$, $\alpha \in (0,1)$)

$$W^{l+\alpha,p}_{\varphi}(\Omega) = \big(W^{l,p}_{\varphi}(\Omega), W^{l+1,p}_{\varphi}(\Omega)\big)_{\alpha,p} \tag{2.29}$$

in a complete analogy with the unweighted case (see, for example, [**29**]).

For the sake of convenience, we will show how to obtain weighted analogs of the interpolation and embedding inequalities.

Proposition 2.10. *Let Ω be a regular domain, and let φ_1 and φ_2 be two weight functions of exponential growth rate μ, $0 \leqslant l_1, l_2 < \infty$, $1 < p_1, p_2 < \infty$. Let $\theta \in [0,1]$ be arbitrary and*

$$l := \theta l_1 + (1-\theta) l_2, \quad \frac{1}{p} := \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \varphi := \varphi_1^\theta \cdot \varphi_2^{1-\theta}.$$

Then $W^{l,p}_{\varphi}(\Omega) \subset W^{l_1,p_1}_{\varphi_1}(\Omega) \cap W^{l_2,p_2}_{\varphi_2}(\Omega)$ and

$$\|u\|_{W^{l,p}_{\varphi}} \leqslant C \|u\|^{\theta}_{W^{l_1,p_1}_{\varphi_1}} \cdot \|u\|^{1-\theta}_{W^{l_2,p_2}_{\varphi_2}},$$

where the constant C depends on l_i, p_i, μ, C_{φ_i} and on some regularity constant of the domain Ω, but is independent of the choice of weights φ_i and of the form of domain Ω. Moreover, similar estimate holds for the spaces $W^{l,p}_{b,\varphi}$.

Proof. As in the proof of Proposition 2.8, we restrict ourselves to the case of a cylindrical domain $\Omega := \mathbb{R} \times \omega$, one-dimensional weights, and the equivalent norms (2.20). Moreover, we consider only the spaces $W^{l,p}_{\varphi}$ ($W^{l,p}_{b,\varphi}$ can be considered in a similar way).

According to the standard unweighted interpolation inequality for domains Ω_s, we have

$$\|u\|^p_{W^{l,p}(\Omega_s)} \leqslant c_1 \|u\|^{p\theta}_{W^{l_1,p_1}(\Omega_s)} \|u\|^{p(1-\theta)}_{W^{l_2,p_2}(\Omega_s)}$$

where the constant C_1 is independent of s, see [**29**]. Multiplying this inequality by the weight $\varphi^p(s)$, integrating over $s \in \mathbb{R}$ and using (2.11), we get

$$\|u\|^p_{W^{l,p}_{\varphi}(\Omega)} \leqslant C_2 \int_{s\in\mathbb{R}} \left(\varphi_1 \|u\|_{W^{l_1,p_1}(\Omega_s)}\right)^{p\theta} \left(\varphi_2 \|u\|_{W^{l_2,p_2}(\Omega_s)}\right)^{p(1-\theta)} ds.$$

Applying the Hölder inequality with exponents $\dfrac{p_1}{p\theta}$ and $\dfrac{p_2}{p(1-\theta)}$ to the right-hand side of this inequality and using the estimate (2.11) once more, we deduce the required weighted interpolation inequality and finish the proof of the proposition. $\square$

The following proposition gives a weighted analog of embedding and trace inequalities.

Proposition 2.11. *Let Ω be a regular domain, and let φ be a weight function of exponential growth rate μ. Then the following assertions hold.*

1) *For any $1 < p_1 \leqslant p_2 < \infty$ and $0 \leqslant l_2 \leqslant l_1$ satisfying*

$$\frac{1}{p_2} - \frac{l_2}{n} \geqslant \frac{1}{p_1} - \frac{l_1}{n} \tag{2.30}$$

there is a continuous embedding $W^{l_1,p_1}_\varphi(\Omega) \subset W^{l_2,p_2}(\Omega)$ and the norm of the embedding operator depends on l_i, p_i, μ, and C_φ, but is independent of the form of weight function φ. If the inequality (2.30) is strict, we can take also $p_2 = \infty$.

2) *For every $m \in \mathbb{Z}_+$, $1 < p < \infty$, and $l > m + 1/p$ the trace operator $\Pi^m_{\partial\Omega}$*

$$\Pi^m_\Omega u := (u\big|_{\partial\Omega}, \partial_n u\big|_{\partial\Omega}, \cdots, \partial^m_n u\big|_{\partial\Omega}) \tag{2.31}$$

(where $\partial_n u$ denotes the normal derivative of u at $\partial\Omega$) maps $W^{l,p}_\varphi(\Omega)$ to $\overset{m}{\underset{k=0}{\otimes}} W^{l-k-1/p,p}_\varphi(\partial\Omega)$ and there exists the associated extension operator $[\Pi^m_{\partial\Omega}]^{-1}$ (the right inverse of $\Pi^m_{\partial\Omega}$) with the norm depending on l, m, p, μ and C_φ, but independent of the choice of weight φ.

The above results also hold for the family of spaces $W^{l,p}_{b,\varphi}(\Omega)$.

Proof. As above, we restrict ourselves to the case of a cylindrical domain $\Omega := \mathbb{R} \times \omega$, one-dimensional weights, and the equivalent norms (2.20). Moreover, we consider only the spaces $W^{l,p}_\varphi$ (the spaces $W^{l,p}_{b,\varphi}$ can be considered in a similar way).

Let $u \in W^{l_1,p_1}_\varphi(\Omega)$. According to the classical Sobolev embedding theorem (see [**29**]), we have

$$\|u\|_{W^{l_2,p_2}(\Omega_s)} \leqslant C\|u\|_{W^{l_1,p_1}(\Omega_s)}, \tag{2.32}$$

where the constant C is independent of s. Taking the power p_2 of both sides of the inequality, we transform it to the following form (for the sake of simplicity, we consider only the case $p_2 < \infty$):

$$\begin{aligned}\|u\|^{p_2}_{W^{l_2,p_2}(\Omega_s)} &\leqslant C^{p_2}\|u\|^{p_2}_{W^{l_1,p_1}(\Omega_s)} \\ &\leqslant C_1\Big(\int\limits_{s\in\mathbb{R}} e^{-\alpha p_1|s-y|}\|u\|^{p_1}_{W^{l_1,p_1}(\Omega_y)}\,dy\Big)^{p_2/p_1},\end{aligned}$$

where $\alpha > \mu$ is arbitrary and C_1 is independent of u. Multiplying by $\varphi(s)^{p_2}$, integrating by $s \in \mathbb{R}$, and using (2.12), we infer

$$\|u\|^{p_2}_{W^{l_2,p_2}_\varphi(\Omega)} \leqslant C_2\|u\|^{p_2}_{W^{l_1,p_1}_\varphi(\Omega)},$$

which proves the first assertion of the proposition.

Now, we verify the second assertion. The existence and boundedness of the trace operator $\Pi^m_{\partial\Omega}$ can be verified by using an analogous property for domains Ω_s as above. Thus, we only need to construct an extension operator $[\Pi^m_{\partial\Omega}]^{-1}$. Let $U := \{u_k\}_{k=0}^m \in \bigotimes_{k=0}^m W^{l-k-1/p,p}_\varphi(\partial\Omega)$ be arbitrary. Using the partition of unity (2.21), we construct the family $U_s := \psi_s U = \{\psi_s u_k\}_{k=0}^m$. Since all these functions vanish at the origin of Ω_s, there exists an extension operator $[\Pi^m_{\partial\Omega_s}]^{-1}$ for bounded domain Ω_s which maps U_s to $W^{l,p}(\Omega_s)$ and whose norm is independent of U and s (see [**29**]). The required extension operator $[\Pi^m_{\partial\Omega}]^{-1}$ can be constructed as follows:

$$[\Pi^m_{\partial\Omega}]^{-1}U := \int_{s\in\mathbb{R}} [\Pi^m_{\partial\Omega_s}]^{-1}U_s\, ds. \tag{2.33}$$

The fact that this operator is well defined and the uniform (with respect to φ) estimate for its norm regarded as a map from $\bigotimes_{k=0}^m W^{l-k-1/p,p}_\varphi(\partial\Omega)$ to $W^{l,p}_\varphi(\Omega)$ can be proved in the same way as in the estimate (2.27) for the function (2.25) in the proof of Proposition 2.8. □

Our following task is formulate trace theorems for classes of less smooth functions which are closely related with the theory of NS equations. For this purpose, we need the following definition.

Definition 2.12. Let Ω be a regular domain of $\mathbb{R}^n$, and let φ be a weight function of exponential growth rate μ, $1 < p < \infty$. Define the space $E^p_\varphi(\Omega)$ of vector-valued functions $u := (u^1, \cdots, u^n) \in [D'(\Omega)]^n$ by the norm

$$\|u\|^p_{E^p_\varphi(\Omega)} := \|u\|^p_{[L^p_\varphi(\Omega)]^n} + \|\operatorname{div} u\|^p_{L^p_\varphi(\Omega)}. \tag{2.34}$$

The spaces $E^p_{b,\varphi}(\Omega)$ are defined in a similar way. Moreover, for every sufficiently smooth vector-valued function $u := (u^1, \cdots, u^n)$, we denote by $l_n u := (\vec{u}, \vec{n})\big|_{\partial\Omega}$ the normal component of this function at the boundary.

Proposition 2.13. *Let Ω be a regular domain, and let φ be a weight function of exponential growth rate μ. Then the operator $l_n : E^p_\varphi(\Omega) \to W^{-1/p,p}_\varphi(\partial\Omega)$ is well defined and*

$$\|l_n u\|_{W^{-1/p,p}_\varphi(\partial\Omega)} \leqslant C\|u\|_{E^p_\varphi(\Omega)}, \tag{2.35}$$

where the constant C depends on μ and C_φ, but is independent of the choice of weight function φ. A similar result holds for $E^p_{b,\varphi}(\Omega)$.

Proof. As above, we verify the estimate (2.35) only for cylindrical domains. Let u and v_s be smooth functions in Ω_s. By the Green formula,

$$(l_n u, v)_{\partial\Omega_s} := (\operatorname{div} u, v)_{\Omega_s} + (u, \nabla_x v)_{\Omega_s}. \tag{2.36}$$

As usual, we see that the right-hand side of (2.36) is well defined for all $u \in E^p(\Omega_s)$ and $v \in W^{1,q}(\Omega_s)$ where $1/p + 1/q = 1$. Moreover, due to the classical trace theorems, there exists an extension operator $[\Pi_s]^{-1}$: $W^{1-1/q,q}(\partial\Omega_s) \to W^{1,q}(\Omega_s)$ whose norm is independent of s. Thus, (2.36) shows that the functional $l_n u$ is well defined and satisfies

$$\|l_n u\|_{W^{-1/p,p}(\partial\Omega_s)} = \|l_n u\|_{[W^{1-1/q,q}(\partial\Omega_s)]^*} \leqslant C\|u\|_{E^p(\Omega_s)}. \tag{2.37}$$

Multiplying this relation by $\varphi(s)^p$ and integrating over $s \in \mathbb{R}$, we deduce (2.35) and complete the proof of the proposition.

Here, we implicitly used that

$$\|l_n u\|_{W^{-1/p,p}((s,s+1)\times\partial\omega)} \leqslant \|l_n u\|_{W^{-1/p,p}(\partial\Omega_s)}.$$

The estimate for $E^p_{b,\varphi}(\Omega)$ can be obtained in a similar way by using the supremum instead of the integral over $s \in \mathbb{R}$. □

As was already mentioned, the estimates of Proposition 2.5 allow us to reduce the proofs of elliptic regularity in arbitrary weighted spaces to the particular case of special weights (2.4). The following evident proposition will be useful in order to reduce the case of special weights to the classical unweighted case $\varphi = 1$.

Proposition 2.14. *Let Ω be a regular domain, and let $\mathbb{T}_{\mu,x_0}$ be the multiplication by weight $\varphi_{\mu,x_0}(x)$ (i.e., $(\mathbb{T}_{\mu,x_0}u)(x) := \varphi_{\mu,x_0}(x)u(x)$). Then for any $l \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$ this operator realizes an isomorphism between the spaces $W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)$ and $W^{l,p}(\Omega)$. Moreover,*

$$C^{-1}\|u\|_{W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)} \leqslant \|\mathbb{T}_{\mu,x_0}u\|_{W^{l,p}(\Omega)} \leqslant C\|u\|_{W^{l,p}_{\varphi_{\mu,x_0}}(\Omega)} \tag{2.38}$$

where the constant C depends on l, p, and μ, but is independent of u and $x_0 \in \mathbb{R}^n$.

This estimate is an immediate consequence of the inequalities (2.6) and Definition 2.7 of the corresponding weighted spaces.

We conclude by formulating some useful results on the weighted and local topologies on bounded sets of $W^{l,p}_b(\Omega)$.

Proposition 2.15. *Suppose that Ω is a bounded domain $l \in \mathbb{R}$, $p \in [1,\infty]$, and $\mathbb{B}$ is a bounded subset of $W_b^{l,p}(\Omega)$. Then, for every weight function φ of exponential growth rate μ satisfying*

$$\|\varphi\|_{L^p(\mathbb{R}^n)} < \infty \tag{2.39}$$

the set $\mathbb{B}$ belongs to $W_\varphi^{l,p}(\Omega)$ and the topology generated on $\mathbb{B}$ by this embedding is independent of the weight φ and coincides with the local topology on $\mathbb{B}$ generated by embedding to $W_{loc}^{l,p}(\overline{\Omega})$.

PROOF. By (2.39), we have

$$\|u\|^p_{W_\varphi^{l,p}(\Omega)} = \int\limits_{x_0\in\Omega} \varphi^p(x_0)\|u\|^p_{W^{l,p}(\Omega\cap B^1_{x_0})}\,dx_0 \leqslant \|\varphi\|^p_{L^p(\mathbb{R}^n)}\|u\|^p_{W_b^{l,p}(\Omega)},$$

which shows that $W_b^{l,p}(\Omega) \subset W_\varphi^{l,p}(\Omega)$. Let $u_n \to u$ in $W_{loc}^{l,p}(\overline{\Omega})$. This means that for any $x_0 \in \Omega$ and $R \in \mathbb{R}_+$

$$\lim_{n\to\infty} \|u_n - u\|_{W^{l,p}(\Omega\cap B^R_{x_0})} = 0. \tag{2.40}$$

Assume that $u_n, u \in \mathbb{B}$ and φ is an integrable (in the sense of (2.39)) weight. Since the set $\mathbb{B}$ is assumed to be bounded in $W_b^{l,p}(\Omega)$,

$$\lim_{R\to\infty} \|u_n\|_{W_\varphi^{l,p}(\Omega\setminus B^R_0)} = 0 \tag{2.41}$$

uniformly with respect to $n \in \mathbb{N}$. Formulas (2.40) and (2.41) imply that $u_n \to u$ in $W_\varphi^{l,p}(\Omega)$. The embedding $W_\varphi^{l,p}(\Omega) \subset W_{loc}^{l,p}(\overline{\Omega})$ is obvious, and Proposition 2.15 is proved. □

3. Elliptic Regularity in Weighted Spaces

In this section, we recall some standard elliptic regularity results in weighted Sobolev spaces which are necessary to deals with the Navier–Stokes equations in unbounded domains. For the sake of simplicity, we restrict ourselves to the case of a 3D cylinder $\Omega := \mathbb{R} \times \omega$, where ω is a bounded smooth domain in $\mathbb{R}^2$ ($x := (x_1, x_2, x_3) \in \Omega$, $x_1 \in \mathbb{R}$, $x' := (x_2, x_3) \in \omega$). although some of the results of this section remain true for general regular domains (see [**11, 12, 32, 33, 34, 35**] for details).

We start with a weighted regularity estimate for the Laplacian with Dirichlet boundary conditions.

Proposition 3.1. *Consider the Dirichlet problem in a cylinder* Ω

$$\Delta_x u = h, \ u\big|_{\partial\Omega} = 0. \tag{3.1}$$

For every $1 < p < \infty$ *and* $l = -1, 0, 1$ *there exists positive* $\mu_0 = \mu_0(p)$ *such that for any weight function* φ *with sufficiently small exponential growth rate* μ ($\mu \leqslant \mu_0$) *and* $h \in W^{l,p}_{\varphi}(\Omega)$ *Equation* (2.1) *possesses a unique solution* $u \in W^{l+2,p}_{\varphi}(\Omega)$ *and the following estimate holds*:

$$\|u\|_{W^{l+2,p}_{\varphi}(\Omega)} \leqslant C\|h\|_{W^{l,p}_{\varphi}(\Omega)}, \tag{3.2}$$

where the constant C *depends on* C_φ, *but is independent of the choice of weight* φ. *Moreover, an analogous estimate holds for* $W^{l,p}_{b,\varphi}(\Omega)$.

Proof. We restrict ourselves to the a priori estimate (3.2) (the existence and uniqueness of a solution can be then verified in a standard way; see, for example, [**11, 12**]).

As was already mentioned, by the estimates (2.12) and (2.13), it suffices to verify the estimate (3.2) only for the special class of weights $\varphi_{\mu_0,x_0}(x)$ introduced in (2.4). If we have the estimate (3.2) for such weights with the constant C independent of x_0, then

$$\begin{aligned}\|u\|^p_{W^{l+2,p}(\Omega_s)} &\leqslant C_{\mu_0}\|u\|^p_{W^{l+2,p}_{\varphi_{\mu_0,s}}(\Omega)} \leqslant C_1\|h\|^p_{W^{l,p}_{\varphi_{\mu_0,s}}(\Omega)} \\ &\leqslant C_2 \int\limits_{y\in\mathbb{R}} e^{-p\mu_0|s-y|}\|h\|^p_{W^{l,p}(\Omega_y)}\,dy,\end{aligned} \tag{3.3}$$

where the constant C_2 is also independent of $s \in \mathbb{R}$. Multiplying the estimate (3.3) by $\varphi(s)^p$ (where φ is a weight function with exponential growth rate $\mu < \mu_0$), integrating over $s \in \mathbb{R}$, and using the estimate (2.12), we infer the required estimate (2.2). Similarly, the estimate (3.2) for the spaces $W^{l,p}_{b,\varphi}$ can be obtained by multiplying (3.3) by $\varphi(s)^p$, taking the supremum over $s \in \mathbb{R}$, and using the estimate (2.13).

Thus, it remains to verify (3.2) for the special weights $\varphi_{\mu_0,s}$ with a sufficiently small positive μ_0 and any $s \in \mathbb{R}$. In turns, by Proposition 2.14 and the estimates (2.6), the case of special weights $\varphi_{\mu_0,s}$ can be reduced to the unweighted case $\varphi \equiv 1$. The function $u \in W^{l+2,p}_{\varphi_{\mu_0}}(\Omega)$ solves (3.1) if and only if the function $v := \varphi_{\mu_0,s}u \in W^{l+2,p}(\Omega)$ solves the following perturbed version of the problem (3.1):

$$\begin{aligned}\Delta_x v &= \varphi_{\mu_0,s}h + 2\varphi'_{\mu_0,s}\partial_{x_1}(\varphi_{-\mu_0,s}v) + \varphi''_{\mu_0,s}\varphi_{-\mu_0,s}v \\ &:= \mathbb{T}_{\mu_0,s}h + h_{\mu_0}(v), \quad v\big|_{\partial\Omega} = 0.\end{aligned} \tag{3.4}$$

We recall that, due to (2.6),

$$\|h_{\mu_0}(v)\|_{W^{l,p}(\Omega)} \leqslant C\mu_0\|v\|_{W^{l+2,p}(\Omega)}, \tag{3.5}$$

where the constant C is independent of s and μ_0. Thus, if the estimate (3.2) for $\varphi \equiv 1$ is known, then applying it to Equation (3.4) and using (3.5), we infer

$$\|\mathbb{T}_{\mu_0,s}u\|_{W^{l+2,p}(\Omega)} \leqslant C(\|\mathbb{T}_{\mu_0,s}h\|_{W^{l,p}(\Omega)} + \mu_0\|v\|_{W^{l+2,p}(\Omega)})$$

with the constant C independent of μ_0 and s. Fixing μ_0 to be small enough such that $C\mu_0 < 1/2$, from the last estimate we deduce that

$$\|v\|_{W^{l+2,p}(\Omega)} \leqslant 2C\|\mathbb{T}_{-\mu_0,s}h\|_{W^{l,p}(\Omega)} \tag{3.6}$$

which, together with Proposition 2.14, implies the estimate (3.2) for special weights $\varphi_{\mu_0,s}$.

Thus, we reduced the proof of the regularity estimate (3.2) in weighted spaces to the unweighted case $\varphi \equiv 1$. It remains to note that (3.2) with $\varphi \equiv 1$ is a classical L^p-regularity estimate for solutions of the Laplace operator (see, for example, [**18, 29**]). Proposition 3.1 is proved. $\square$

Remark 3.2. The regularity estimate (3.2) holds not only for $l = -1, 0, 1$, but we need it in the sequel only for these values of l. We also note that the estimate (3.2) holds for the unweighted space since the spectrum of the Laplacian in a cylinder with Dirichlet boundary conditions is strictly negative.

The following proposition providing some uniform estimate for the singular perturbed Laplace equation will be useful for approximating the 3D Navier–Stokes problem.

Proposition 3.3. *Suppose that $\alpha > 0$ is small, φ is a weight function with exponential growth rate μ, and u solves the problem*

$$u - \alpha\Delta_x u = h, \quad u\big|_{\partial\Omega} = 0 \tag{3.7}$$

for some $h \in L^p_\varphi(\Omega)$. Then the following estimate holds:

$$\alpha\|u\|_{W^{2,p}_\varphi(\Omega)} + \|u\|_{L^p_\varphi(\Omega)} \leqslant C\|h\|_{L^p_\varphi(\Omega)}, \tag{3.8}$$

where the constant C depends only on μ and C_φ, but is independent of α and the form of weight φ. A similar result holds for $W^{l,p}_{b,\varphi}$.

Proof. Indeed, after the scaling $\bar{x} := \alpha^{-1/2}x$, Equation (3.7) reads

$$\bar{u} - \Delta_{\bar{x}}\bar{u} = \bar{h}, \ \bar{u}\big|_{\partial\bar{\Omega}} = 0, \ \bar{\Omega} := \alpha^{-1/2}\Omega$$

and the weight φ should be replaced by $\bar\varphi(\bar s) := \varphi(\alpha^{1/2}\bar x)$. It is clear that the regularity constant of the domain $\bar\Omega$ is at least not worse than for Ω (if α is small enough) and the weight $\bar\varphi$ will be of exponential growth rate $\alpha^{1/2}\mu \leqslant \mu$ with $C_{\bar\varphi} = C_\varphi$. By this reason, the estimate (3.2) of Proposition 3.1 hold for the scaled equation uniformly with respect to α, i.e.,

$$\|\bar u\|_{W^{2,p}_{\bar\varphi}(\bar\Omega)} \leqslant C\|\bar h\|_{L^p_{\bar\varphi}(\Omega)}$$

(see. for example, [**12, 30**] for details). Returning back to the variable x, we obtain the desired estimate (3.8) and complete the proof of the proposition. □

Remark 3.4. An analog of Proposition 3.3 for more regular external forces $h \in W^{l,p}_\varphi(\Omega)$, $l > 0$, is not true since boundary layer terms may appear. In the simplest 1D case, we have

$$y(x) - \alpha^2 y''(x) = 1,\ x \in [0,1],\ y(0) = y(1) = 0,$$

the external force belongs to C^∞, and the associated solution

$$y(x) = 1 - \frac{\sinh(\alpha^{-1}x)}{\sinh(\alpha^{-1})} - \frac{\sinh\alpha^{-1}(1-x)}{\sinh(\alpha^{-1})}$$

is a typical boundary layer solution which does not uniformly bounded in any C^β, $\beta > 0$, as $\alpha \to 0$.

Consider the Neumann type boundary value problems for the Laplacian in a cylinder Ω. The main difficulty here is the fact that, in contrast to the Dirichlet problems considered above, the Neumann problem for the Laplacian has an essential spectrum at $\lambda = 0$, which makes the situation much more delicate. We however start with the regularized Neumann type problem, where the spectrum remains strictly negative.

Proposition 3.5. *Let Ω be a cylinder. Consider the following boundary value problem in Ω :*

$$\Delta_x u - u = 0,\ \partial_n u\big|_{\partial\Omega} = h_0, \tag{3.9}$$

Then for every $1 < p < \infty$ and $l = 0, 1, 2$ there exists $\mu_0 = \mu_0(p)$ such that for every weight function of sufficiently small exponential growth rate μ ($\mu \leqslant \mu_0$) and every $h_0 \in W^{l-1/p,p}_\varphi(\partial\Omega)$ the problem (3.9) *has a unique solution $u \in W^{l+1,p}_\varphi(\Omega)$ and the following estimate holds*:

$$\|u\|_{W^{l+1,p}_\varphi(\Omega)} \leqslant C\|h_0\|_{W^{l-1/p,p}_\varphi(\partial\Omega)}, \tag{3.10}$$

where the constant C depends on C_φ, but is independent of the choice of weight function φ. A similar result holds for $W^{l,p}_{b,\varphi}$.

Proof. In the case $l = 1, 2$, the estimate (3.10) can be verified exactly as in Propositions 3.1 and 3.3 (by reducing to the homogeneous and unweighted case). In the case $l = 0$, the situation is slightly more delicate since we do not formulate the extension theorem f or the space $W_\varphi^{-1/p,p}(\partial\Omega)$ in Proposition 2.11 and, consequently, we need to work with a nonhomogeneous boundary value problem. Nevertheless, the reduction to the unweighted case based on introducing the function $v := \varphi_{\mu_0,s}u$ works in this case as well. Indeed, this function satisfies

$$\Delta_x v - v = h_{\mu_0}(v), \quad \partial_n v\big|_{\partial\Omega} := \mathbb{T}_{-\mu_0,s}h_0 \tag{3.11}$$

and

$$\|h_{\mu_0}(v)\|_{L^p(\Omega)} \leqslant C\mu_0\|v\|_{W^{1,p}(\Omega)} \tag{3.12}$$

Thus, we can represent the solution v of (3.11) as the sum $v = v_1 + v_2$, where v_1 solves the homogeneous problem

$$\Delta_x v_1 - v_1 = h_{\mu_0}(v), \ \partial_n v_1\big|_{\partial\Omega} = 0 \tag{3.13}$$

and v_2 solves an analog of (3.9) with h_0 replaced by $\mathbb{T}_{-\mu_0,s}h_0$. We see that the right-hand side of (3.11) belongs to $L^p(\Omega)$ and, consequently, due to the classical L^p-regularity, we have

$$\|v_1\|_{W^{2,p}(\Omega)} \leqslant C\|h_{\mu_0}(v)\|_{L^p(\Omega)} \leqslant C_1\mu_0\|v\|_{W^{1,p}(\Omega)}. \tag{3.14}$$

Assuming that the estimate (3.10) for the unweighted case $\varphi = 1$ and $l = 0$ is known and using (3.14), we infer

$$\begin{aligned}\|v\|_{W^{1,p}(\Omega)} &\leqslant \|v_1\|_{W^{1,p}(\Omega)} + \|v_2\|_{W^{1,p}(\Omega)} \\ &\leqslant C\|\mathbb{T}_{\mu_0,s}h_0\|_{W^{-1/p.p}(\partial\Omega)} + C\mu_0\|v\|_{W^{1,p}(\Omega)},\end{aligned}$$

which implies the estimate

$$\|v\|_{W^{1,p}(\Omega)} \leqslant 2C\|\mathbb{T}_{\mu_0,s}h_0\|_{W^{-1/p,p}(\partial\Omega)} \tag{3.15}$$

if μ_0 is small. Thus, the case of a general weight naturally reduces to the case of $\varphi \equiv 1$ for $l = 0$. It remains to recall that for $\varphi \equiv 1$ the estimate (3.10) is a classical L^p-regularity result for the Laplacian (see [**29**]). Proposition 3.5 is proved. □

To treat the Neumann problem without the regularizing term $-u$, we introduce the following averaging operator with respect to the variable x' ($(x_1, x') \in \mathbb{R} \times \omega := \Omega$):

$$(\mathbb{S}u)(x_1) := \frac{1}{|\omega|}\int_{s\in\omega} u(x_1, s)\, ds. \tag{3.16}$$

The following proposition gives the solvability of the Neumann problem for some natural closed subspace of the space of external forces h.

Proposition 3.6. *Let Ω be a cylinder. Consider the boundary value problem in Ω*

$$\Delta_x u = h, \quad \partial_n u\big|_{\partial\Omega} = 0. \tag{3.17}$$

Then for every $1 < p < \infty$ and $l = 0, 1, 2$ there exists $\mu_0 = \mu_0(p)$ such that for every weight function of a sufficiently small exponential growth rate μ ($\mu \leqslant \mu_0$) and every $h \in W^{l,p}_{\varphi}(\Omega)$ satisfying

$$\mathbb{S}h \equiv 0,$$

the problem (3.17) *has a unique solution $u \in W^{l+2,p}_{\varphi}(\Omega)$, $\mathbb{S}u \equiv 0$ and the following estimate holds*:

$$\|u\|_{W^{l+2,p}_{\varphi}(\Omega)} \leqslant C\|h\|_{W^{l,p}_{\varphi}(\Omega)}, \tag{3.18}$$

where the constant C depends on C_φ, but is independent of the choice of weight function φ. A similar result holds for $W^{l,p}_{b,\varphi}$.

Proof. Note that the operator $\mathbb{S}$ commutes with the multiplication operator $\mathbb{T}_{\mu_0}$ and with the x_1-derivatives ∂_{x_1}. Arguing as above, we can reduce the proof of (3.18) to the unweighted case $\varphi \equiv 1$. Therefore, we will prove (3.18) only in the case $\varphi \equiv 1$.

We begin with the case $p = 2$. We can multiply Equation (3.17) by u and, integrating by parts, find

$$\|\nabla_x u\|^2_{L^2(\Omega)} \leqslant \|h\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}. \tag{3.19}$$

Since we additionally assumed that $\mathbb{S}u \equiv 0$, we have the Friedrichs inequality

$$\|u\|_{W^{1,2}(\Omega)} \leqslant C\|\nabla_x u\|_{L^2(\Omega)} \tag{3.20}$$

which, together with (3.19), implies

$$\|u\|_{W^{1,2}(\Omega)} \leqslant C\|h\|_{L^2(\Omega)}. \tag{3.21}$$

To prove the estimate (3.18) for $p = 2$ and $\varphi \equiv 1$, we use the following standard interior regularity estimate:

$$\|u\|^2_{W^{l+2,2}(\Omega_s)} \leqslant C(\|u\|^2_{W^{1,2}(\Omega_{s-1}\cup\Omega_s\cup\Omega_{s+1})} + \|h\|^2_{W^{l,2}(\Omega_s)})$$
$$\leqslant C_1 \int_{y\in\Omega} e^{-\alpha|s-y|}(\|u\|^2_{W^{1,2}(\Omega_y)} + \|h\|^2_{W^{l,2}(\Omega_y)})\,dy. \tag{3.22}$$

Integrating this estimate over $s \in \mathbb{R}$ and using (2.12) and (3.21), we infer the unweighted estimate (3.18) for $p = 2$. Thus, due to the trick with the multiplication operator $\mathbb{T}_{\mu_0,s}$, the estimate (3.18) is verified for $p = 2$ and all weights with sufficiently small exponential growth rate. Moreover, we also have an analog of the estimate (3.18) with $p = 2$ for $W^{l,p}_{b,\varphi}(\Omega)$.

Consider the case $p \neq 2$. We first consider the case $p > 2$ and prove the estimate (3.18) for the spaces $W^{l,p}_b(\Omega)$. Since $W^{l,p}_b(\Omega) \subset W^{l,2}_b(\Omega)$, we already have the estimate

$$\|u\|_{W^{1,2}_b(\Omega)} \leqslant C\|h\|_{L^2_b(\Omega)} \leqslant C_1\|h\|_{L^p_b(\Omega)}. \tag{3.23}$$

Using the interior regularity estimate

$$\begin{aligned}\|u\|_{W^{l+2,p}(\Omega_s)} &\leqslant C(\|u\|_{W^{1,2}(\Omega_{s-1}\cup\Omega_s\cup\Omega_{s+1})} + \|h\|_{W^{l,p}(\Omega_s)})\\ &\leqslant C_1 \sup_{y\in\mathbb{R}}\{e^{-\alpha|s-y|}(\|u\|_{W^{1,2}(\Omega_y)} + \|h\|_{W^{l,p}(\Omega_y)})\},\end{aligned}$$

taking the supremum over $s \in \mathbb{R}$ of both sides of this inequality, and using (2.3) and (3.23), we finally get

$$\|u\|_{W^{l+2,p}_b(\Omega)} \leqslant C\|h\|_{W^{l,p}_b(\Omega)}. \tag{3.24}$$

Let $1 < p < 2$. We split the solution u of (3.17) as $u = u_1 + u_2$, where u_1 solves the problem

$$\Delta_x u_1 - u_1 = h, \quad \partial_n u_1\big|_{\partial\Omega} = 0 \tag{3.25}$$

and u_2 solves the problem

$$\Delta_x u_2 = -u_1, \quad \partial_n u_2\big|_{\partial\Omega} = 0. \tag{3.26}$$

By the L^p-regularity (see Proposition 3.5), for Equation (3.25) we have

$$\|u_1\|_{W^{l+2,p}_b(\Omega)} \leqslant C\|h\|_{W^{l,p}_b(\Omega)}. \tag{3.27}$$

Moreover, applying the operator $\mathbb{S}$ to both sides of Equation (3.25) and using $\mathbb{S}h \equiv 0$, we find

$$(\mathbb{S}u_1)'' - \mathbb{S}u_1 \equiv 0 \text{ and, consequently, } \mathbb{S}u_1 \equiv 0. \tag{3.28}$$

Furthermore, by the embedding theorem (see Proposition 2.11),

$$\|u_1\|_{W^{l,2}(\Omega)} \leqslant C\|u_1\|_{W^{l+2,p}_b(\Omega)} \tag{3.29}$$

for every $1 < p < 2$. Thus, we can apply the estimate (3.23) to Equation (3.26) which, together with (3.27), yields (3.24) for $1 < p < 2$.

Thus, the estimate (3.24) is verified for all $1 < p < \infty$. Due to the above described trick with the multiplication operator $\mathbb{T}_{\mu_0,s}$, we can deduce the estimate (3.18) for the spaces $W^{l+2,p}_{b,\varphi}(\Omega)$ for all weight functions of sufficiently small exponential growth rate.

It remains to obtain the estimate for the spaces $W^{l,p}_{\varphi}(\Omega)$. Note that (3.18) for the spaces $W^{l,p}_{b,\varphi_{\mu_0,s}}(\Omega)$ implies, in particular, that

$$\begin{aligned}\|u\|^p_{W^{l+2,p}(\Omega_s)} &\leqslant C\sup_{y\in\mathbb{R}}\{e^{-\mu_0 p|s-y|}\|h\|^p_{W^{l,p}(\Omega_y)}\}\\ &\leqslant C_1\int_{y\in\Omega} e^{-\mu_0 p|s-y|}\|h\|^p_{W^{l,p}(\Omega)}\,dy.\end{aligned}\tag{3.30}$$

Multiplying (3.30) by $\varphi(s)^p$, integrating over $s\in\mathbb{R}$, and using (2.12), we finally obtain the estimate (3.18) and complete the proof of Proposition 3.6. □

Remark 3.7. As we see from the proof of Proposition 3.6, the weighted regularity estimates can be deduced not only from the unweighted estimates in $W^{l,p}(\Omega)$, but also from their analogs in the spaces $W^{l,p}_b(\Omega)$. The last scale of spaces is sometimes (for example, in the proof of Proposition 3.6) more convenient since, in contrast to spaces $L^p(\Omega)$, the spaces $L^p_b(\Omega)$ have usual (for bounded domains) embedding properties ($L^{p_1}_b(\Omega)\subset L^{p_2}_b(\Omega)$ for $p_1\geqslant p_2$).

Note that the assumption $\mathbb{S}h\equiv 0$ in Proposition 3.6 is essential for the weighted estimate (3.18). In the general case $\mathbb{S}h\neq 0$, for $\mathbb{S}u=(\mathbb{S}u)(x_1)$ we have the following equation:

$$(\mathbb{S}u)(x_1)''=(\mathbb{S}h)(x_1),\quad x_1\in\mathbb{R},\tag{3.31}$$

whose solution $\mathbb{S}u$ does not possess any weighted regularity estimates for general h. Fortunately, for problems arising in the weighted regularity theory for the Helmholtz operator, the function $\mathbb{S}h$ has a special structure, which allows us to take one primitive of it remaining in weighted Sobolev classes. To be more precise, the following proposition holds.

Proposition 3.8. *Let Ω be a cylinder. Consider the Neumann boundary value problem in Ω*

$$\Delta_x u=0,\ \partial_n u\big|_{\partial\Omega}=l_n g,\tag{3.32}$$

where $g\in[L^p(\Omega)]^2$ is a divergence-free vector field

$$\operatorname{div} g\equiv 0.\tag{3.33}$$

Then for every $1<p<\infty$ and $l=0,1,2$ there exists $\mu_0=\mu_0(p)$ such that, for every weight function of a sufficiently small exponential growth rate μ

($\mu \leqslant \mu_0$) and every $g \in W^{l,p}_{\varphi}(\Omega)$ satisfying (3.33)*, the problem* (3.32) *has a unique solution* (*up to adding a constant*) *satisfying $\nabla_x u \in W^{l,p}_{\varphi}(\Omega)$, and*

$$(\mathbb{S}u)(x_1)' = (\mathbb{S}g_1)(x_1), \ x_1 \in \mathbb{R} \tag{3.34}$$

and the following estimate holds:

$$\|\nabla_x u\|_{W^{l,p}_{\varphi}(\Omega)} \leqslant C\|g\|_{W^{l,p}_{\varphi}(\Omega)}, \tag{3.35}$$

where the constant C depends on C_{φ}, but is independent of the choice of weight function φ. A similar result holds for $W^{l,p}_{b,\varphi}$.

Proof. For the sake of simplicity, we deduce only the a priori estimate (3.35). The existence and uniqueness of a solution can be established in a standard way (see also [**4**]).

We first define an auxiliary function v as a solution of the problem

$$\Delta_x v - v = 0, \ \partial_n v\big|_{\partial\Omega} = l_n g. \tag{3.36}$$

By Propositions 3.5 and 2.13,

$$\|v\|_{W^{l+1,p}_{\varphi}(\Omega)} \leqslant C\|l_n g\|_{W^{l-1/p,p}_{\varphi}(\partial\Omega)} \leqslant C_2\|g\|_{W^{l,p}_{\varphi}(\Omega)}. \tag{3.37}$$

Applying the x'-averaging operator $\mathbb{S}$ to Equation (3.36), we find

$$(\mathbb{S}v)(x_1)'' - (\mathbb{S}v)(x_1) = -\frac{1}{|\omega|}\int\limits_{s\in\partial\omega} (\vec{n}, g(x_1, s))\, ds, \ x_1 \in \mathbb{R}. \tag{3.38}$$

Furthermore, since the vector field g is divergence free, we have

$$\frac{1}{|\omega|}\int\limits_{s\in\partial\omega} (\vec{n}, g(x_1, s))\, ds = (\mathbb{S}[\partial_{x_2} g_2 + \partial_{x_3} g_3])(x_1) = -(\mathbb{S}g_1)(x_1)'$$

and, consequently,

$$(\mathbb{S}v)(x_1)'' - (\mathbb{S}v)(x_1) = (\mathbb{S}g_1)(x_1)'. \tag{3.39}$$

Consider the remainder $w := u - v$ which satisfies the problem

$$\Delta_x w = -v, \ \partial_n w\big|_{\partial\Omega} = 0. \tag{3.40}$$

By Proposition 3.6, the function $\bar{w} := w - \mathbb{S}w$ satisfies the following estimate:

$$\|\bar{w}\|_{W^{l+1,p}_{\varphi}(\Omega)} \leqslant C\|\bar{v}\|_{W^{l,p}_{\varphi}(\Omega)} \leqslant C_1\|g\|_{W^{l,p}_{\varphi}(\Omega)}. \tag{3.41}$$

It remains to consider the equation for $\mathbb{S}w$, i.e.,

$$(\mathbb{S}w)(x_1)'' = -(\mathbb{S}v)(x_1)$$

which, together with (3.39), yields

$$(\mathbb{S}u)(x_1)'' = (\mathbb{S}g_1)(x_1)'. \tag{3.42}$$

This relation shows that we can take one primitive and satisfy the condition (3.34). It remains to note that the function $(\mathbb{S}u)(x_1)$ is independent of x' and, consequently,

$$\nabla_x u = \nabla_x \bar{u} + ((\mathbb{S}u)', 0, 0). \tag{3.43}$$

Thus, the estimates (3.37), (3.41), together with the obvious fact that

$$\|\mathbb{S}g\|_{W^{l,p}_{\varphi}(\mathbb{R})} \leqslant C\|g\|_{W^{l,p}_{\varphi}(\Omega)}, \tag{3.44}$$

imply (3.35), which completes the proof of Proposition 3.8. $\square$

4. The Helmholtz Projector and Stationary Stokes Problem

In this section, we discuss a weighted analog of the classical Helmholtz decomposition of the space $[L^2(\Omega)]^2$ to divergence-free and gradient vector fields, which is necessary for excluding the pressure from Navier–Stokes equations. We need to define the corresponding spaces of divergence-free vector fields.

Definition 4.1. Let Ω be a cylinder. For any $l \geqslant 0$, $1 < p < \infty$ and weight function φ of exponential growth rate we define the space of divergence-free vector fields

$$\mathcal{H}^{l,p}_{\varphi}(\Omega) := \{v \in [W^{l,p}_{\varphi}(\Omega)]^3, \;\; \operatorname{div} v \equiv 0, \; l_n v\big|_{\partial\Omega} = 0, \; \mathbb{S}v_1 \equiv 0\} \tag{4.1}$$

which is considered as a closed subspace of $W^{l,p}_{\varphi}(\Omega)$ and is endowed by the norm induced by this embedding. Here, the normal component $l_n v$ of the trace on the boundary is well defined due to Proposition 2.13 and the x'-averaging operator $\mathbb{S}$ is defined by (3.16). The spaces $\mathcal{H}^{l,p}_{b,\varphi}(\Omega)$ can be defined in a similar way. For the sake of simplicity, we write $\mathcal{H}^{p}_{\varphi}(\Omega)$ and $\mathcal{H}^{p}_{b,\varphi}(\Omega)$ instead of $\mathcal{H}^{0,p}_{\varphi}(\Omega)$ and $\mathcal{H}^{0,p}_{b,\varphi}(\Omega)$ respectively.

We define the space $\mathcal{V}^{p}_{\varphi}(\Omega)$ as follows:

$$\mathcal{V}^{p}_{\varphi}(\Omega) := \{v \in \mathcal{H}^{1,p}_{\varphi}(\Omega), \; v\big|_{\partial\Omega} = 0\}.$$

The space $\mathcal{V}^{p}_{b,\varphi}(\Omega)$ is defined in a similar way.

The following natural proposition clarifies the additional conditions $l_n v\big|_{\partial\Omega} = 0$ and $\mathbb{S}v_1 \equiv 0$ in formula (4.1).

Proposition 4.2. *Let Ω be a cylinder, and let φ be a weight function of exponential growth rate μ, $1 < p < \infty$. Then the space $\mathcal{H}^p(\Omega)$ coincides with the closure of all divergence-free vector fields $v \in [\mathcal{D}(\Omega)]^3$ in the topology of $[L^p_\varphi(\Omega)]^3$:*

$$\mathcal{H}^p_\varphi(\Omega) = \big[v \in [\mathcal{D}(\Omega)]^3,\ \operatorname{div} v = 0\big]_{[L^p_\varphi(\Omega)]^2}, \tag{4.2}$$

where $[\cdot]_V$ denotes the closure in the topology of the space V.

Proof. Let v be a divergence-free vector field in $[D(\Omega)]^3$. It is obvious that $l_n v\big|_{\partial\Omega} = 0$. Integrating the relation $\partial_{x_1} v_1 = -\partial_{x_2} v_2 - \partial_{x_3} v_3$, we infer that $\mathbb{S} v_1 \equiv const = 0$ (since v_1 has finite support). Since all these properties are preserved under taking the closure (see Proposition 2.13), the right-hand side of (4.2) is a subset of the left one.

It remains to approximate every function in $u \in \mathcal{H}^p_\varphi(\Omega)$ by divergence-free vector fields in $[\mathcal{D}(\Omega)]^3$. Since the assertion is well known for bounded domains (see, for example, [**27**]), it suffices to approximate u by functions $u_n \in \mathcal{H}^p_\varphi(\Omega)$ with *bounded* support. For this purpose, we introduce a family of cut-off functions θ_n such that $\theta_n(s) \in [0,1]$, $\theta_n(s) = 1$ for $s \in [-n,n]$, $\theta_n(s) = 0$ for $s \notin [-n-1, n+1]$, and $\varphi_n'(s)$ is uniformly bounded with respect to s and n.

Consider the vector-field $\tilde u^n(x) := \theta_n(x_1) u(x)$. It is obvious that $\tilde u^n \to u$ in $[L^p_\varphi(\Omega)]^3$, the support of $\tilde u^n$ is bounded, is contained in the subdomain $\Omega_{[-n-1,n+1]} := [-n-1, n+1] \times \omega$, and has zero trace of the normal component on the boundary and zero mean flux. The only problem is that vector field is not divergence-free:

$$\begin{aligned}\operatorname{div} \tilde u^n = \varphi_n' u_1 := h^n(x) &= h^n(x)\chi_{\Omega_{-n-1}}(x) + h^n(x)\chi_{\Omega_n}(x)\\ &:= h^n_+(x) + h^n_-(x)\end{aligned}$$

(we implicitly used the fact that $\operatorname{supp}\theta_n' \subset [-n-1,-n] \cup [n, n+1]$). Moreover, since u_1 has the zero mean, we conclude that

$$\int\limits_{\Omega_{-n-1}} h^n_-(x)\,dx = \int\limits_{\Omega_n} h^n_+(x)\,dx = 0.$$

Thus, there exist vector fields $u^n_- \in [W^{1,2}_0(\Omega_{-n-1})]^3$ and $u^n_+ \in [W^{1,2}_0(\Omega)]^3$ such that

$$\begin{aligned}&\operatorname{div} u^n_\pm = h^n_\pm,\ \ \|u^n_-\|_{W^{1,p}(\Omega_{-n-1})} \leqslant C\|u_1\|_{L^p(\Omega_{-n-1})},\\ &\|u^n_+\|_{W^{1,p}(\Omega_n)} \leqslant C\|u_1\|_{L^p(\Omega_n)},\end{aligned} \tag{4.3}$$

where the constant C is independent of n (see [**27**]). Extending the vector fields $u^n_\pm$ by zero outside $\Omega_{-n-1}\cup\Omega_n$, we obtain the vector fields $u^n_\pm$ defined in the entire cylinder Ω and satisfying (4.3) (we used the zero boundary conditions). Finally, the estimates (4.3) show that $u^n_\pm \in L^p_\varphi(\Omega)$ (and even $W^{1,p}_\varphi(\Omega)$) tend to zero as $n\to\infty$ in these spaces. Setting

$$u^n := \tilde u^n - u^n_+ - u^n_-, \tag{4.4}$$

we obtain the desired converging sequence of divergence-free vector fields with finite support and thereby complete the proof. □

Remark 4.3. Arguing as above, one can verify the description for $\mathcal{V}^p_\varphi(\Omega)$:

$$\mathcal{V}^p_\varphi(\Omega) = \big[v\in[\mathcal{D}(\Omega)]^3,\ \operatorname{div} v = 0\big]_{[W^{1,p}_\varphi(\Omega)]^2} \tag{4.5}$$

As usual, we define an operator $\Pi : [L^2(\Omega)]^3 \to \mathcal{H}^2(\Omega)$ as the orthoprojector to the divergence-free vector fields. As is known (see, for example, [**27**] or [**28**]), every vector field $u\in[L^2(\Omega)]^3$ can be uniquely split into the sum of a divergence-free vector field $v\in\mathcal{H}^2(\Omega)$ and a potential field $\nabla_x p\in[L^2(\Omega)]^3$ for an appropriate $p\in H^1_{loc}(\overline{\Omega})$:

$$u = v + \nabla_x p, \quad \operatorname{div} v = 0, \quad v := \Pi u. \tag{4.6}$$

The following theorem shows that an analogous splitting holds in weighted spaces.

Theorem 4.4. *Let Ω be a cylinder, and let Π be the orthoprojector defined above. Then, for every $1<p<\infty$ and $l=0,1,2$, there exists a sufficiently small positive μ_0 such that for every weight function with exponential growth rate $\mu\leqslant\mu_0$ this projector can be uniquely extended by continuity to a bounded operator from $[W^{l,p}_\varphi(\Omega)]^3$ to $\mathcal{H}^{l,p}_\varphi(\Omega)$ and the following estimate holds*:

$$\|\Pi u\|_{\mathcal{H}^{l,p}_\varphi(\Omega)} \leqslant C\|u\|_{[W^{l,p}_\varphi(\Omega)]^3}, \tag{4.7}$$

where the constant C depends only on p, l, and C_φ, but is independent of the choice of weight φ. Thus, for every $u\in[W^{l,p}_\varphi(\Omega)]^3$ there is a unique decomposition in the form of (4.6) *with $v\in\mathcal{H}^{l,p}_\varphi(\Omega)$ and $p\in W^{l+1,p}_{loc}(\overline{\Omega})$. In this formula, $v=\Pi u$. A similar result holds for $W^{l,p}_{b,\varphi}$.*

Proof. Let $u\in[W^{l,p}_\varphi(\Omega)]^3$. We construct the pressure p in the decomposition (4.6). Taking formally the divergence of both sides of (4.6), we get

$$\Delta_x p = \operatorname{div} u. \tag{4.8}$$

Using $l_n v\big|_{\partial\Omega} = 0$, we infer the boundary condition for p:

$$\partial_n p\big|_{\partial\Omega} = l_n u\big|_{\partial\Omega}. \tag{4.9}$$

We however note that the right-hand side of (4.9) is ill-posed for arbitrary $u \in [L^p(\Omega)]^3$. To overcome this difficulty, we (following [**4**]) introduce an auxiliary function p_1 which solves the problem

$$\Delta_x p_1 = \operatorname{div} u, \quad p_1\big|_{\partial\Omega} = 0 \tag{4.10}$$

and then the remainder $\bar p := p - p_1$ solves the problem

$$\Delta_x \bar p = 0,\ \partial_n \bar p\big|_{\partial\Omega} = l_n(u - \nabla_x p)\big|_{\partial\Omega}. \tag{4.11}$$

Note that $\operatorname{div}(u - \nabla_x p_1) = 0$ and, consequently, by Proposition 2.13, the trace $l_n(u - \nabla_x p_1)$ on the boundary is well defined and we can apply Proposition 3.8 which provides the unique solvability (up to a constant) of (4.11) and the estimate (3.35) for the gradient of $\bar p$. It remains to note that the condition (3.34) now reads

$$\partial_{x_1}\mathbb{S}\bar p = \mathbb{S}u_1 - \partial_{x_1}\mathbb{S}p_1 \text{ and, thus } \mathbb{S}\partial_{x_1} p = \mathbb{S}u_1,$$

which shows that p is well defined ($\mathbb{S}v_1 = \mathbb{S}u_1 - \mathbb{S}\partial_{x_1} p = 0$, $\operatorname{div} v = 0$ and $l_n v = 0$). From the estimates (3.35) for $\nabla_x \bar p$ and (3.2) for $\nabla_x p_1$, we immediately obtain an analog of the estimate (4.7) for $\nabla_x p$. Since $\Pi u := v = u - \nabla_x p$, we have the estimate (4.7) for Π. $\square$

Corollary 4.5. *Let the assumptions of Theorem* 4.4 *hold, and let* $v \in \mathcal{H}^p_\varphi(\Omega)$. *Then for every potential vector field* $w = \nabla_x p$ *such that* $w \in [L^q_{\varphi^{-1}}(\Omega)]^3$

$$(v, w)_{[L^2(\Omega)]^3} = 0. \tag{4.12}$$

By Proposition 4.2, the function v can be approximated (in the metric of $L^p_\varphi(\Omega)$) by a sequence of smooth divergence-free vector fields with compact support. Since for such vector fields (4.12) is obvious, we can pass to the limit and obtain (4.12) for all $v \in \mathcal{H}^p_\varphi(\Omega)$.

The following proposition gives the estimate for the weighted norms of the commutator of Π and the multiplication operator $\mathbb{T}_{\mu,x_0}$ introduced in Proposition 2.14.

Proposition 4.6. *Let* Ω *be a cylinder,* $1 < p < \infty$, $l = 0, 1, 2$ *and* $\mathbb{T}_{\mu,x_0}$ *is a multiplication by the special weight* $\varphi_{\mu,x_0}(x_1)$. *Then there exists* $\mu_0 = \mu_0(p) > 0$ *such that, for every weight function of exponential growth rate* $\varepsilon \leqslant \mu_0$, *every* $\mu \leqslant \mu_0$ *and every* $x_0 \in \mathbb{R}$, *we have*

$$\|(\mathbb{T}_{\mu,x_0} \circ \Pi - \Pi \circ \mathbb{T}_{\mu,x_0})u\|_{W^{l+1,p}_{\varphi(\varphi_{\mu,x_0})^{-1}}(\Omega)} \leqslant C\mu \|u\|_{W^{l,p}_\varphi(\Omega)} \tag{4.13}$$

where the constant C depends on C_φ, but is independent of μ, u, x_0 and on the choice of the weight φ. Moreover, a similar result holds for $W^{l,p}_{b,\varphi}(\Omega)$.

PROOF. Let $u \in W^{l,p}_\varphi(\Omega)$ be arbitrary, and let

$$u = v + \nabla_x p, \ \varphi u = v_\varphi + \nabla_x p_\varphi \tag{4.14}$$

be decompositions (4.6) for functions u and φu respectively (which exist due to Theorem 4.4) and $\varphi := \varphi_{\mu,x_0}$. Then

$$\Delta_x p_\varphi = \operatorname{div}(\varphi u), \ \Delta_x p = \operatorname{div} u, \ \partial_n p = l_n u, \ \partial_n p_\varphi = \varphi l_n u.$$

Let $P_\varphi := p_\varphi - \varphi p$. Then this function solves the problem

$$\Delta_x P_\varphi = \varphi' u_1 - 2\varphi' \partial_{x_1} p - \varphi'' p, \quad \partial_n P_\varphi = 0. \tag{4.15}$$

However, from Proposition 3.6 we obtain the weighted estimate of $\bar{P}_\varphi := P_\varphi - \mathbb{S}P_\varphi$ only. Moreover, from Theorem 4.4 we are able to extract the weighted estimate only for $\bar{p} := p - \mathbb{S}p$. To overcome this difficulty, we recall that for proving (4.13) we only need that

$$\|v_\varphi - \varphi v\|_{W^{l+1,p}_{\varphi\varphi^{-1}}(\Omega)} = \|\nabla_x p_\varphi - \varphi \nabla_x p\|_{W^{l+1,p}_{\varphi\varphi^{-1}}(\Omega)} \leqslant C\mu \|u\|_{W^{l,p}_\varphi(\Omega)} \tag{4.16}$$

We claim that for estimating this quantity it suffices to have the proper estimates for $\bar{P}_\varphi$ and $\bar{p}$ only. Indeed,

$$\nabla_x p_\varphi - \varphi \nabla_x p = \begin{pmatrix} \partial_{x_1} \bar{P}_\varphi + \varphi' \bar{p} - \partial_{x_1} \mathbb{S}P_\varphi - \varphi' \mathbb{S}p \\ \nabla_{x'} \bar{P}_\varphi p \end{pmatrix}.$$

Furthermore, since the mean flux of v and v_φ are equal to zero, from (4.14) we conclude that

$$\partial_{x_1} \mathbb{S}p_\varphi = \varphi \mathbb{S}u_1, \quad \partial_{x_1} \mathbb{S}p = \mathbb{S}u_1 \tag{4.17}$$

and, consequently,

$$\begin{aligned} \partial_{x_1} \mathbb{S}P_\varphi + \varphi' \mathbb{S}p &= \partial_{x_1}(\mathbb{S}p_\varphi - \varphi \mathbb{S}p) + \varphi' \mathbb{S}p = \partial_{x_1} \mathbb{S}p_\varphi - \varphi \partial_{x_1} \mathbb{S}p \\ &= \varphi \mathbb{S}u_1 - \varphi \mathbb{S}u_1 = 0. \end{aligned}$$

Thus, to complete the proof, we need only estimates for $\bar{P}_\varphi$ and $\bar{p}$.

As in the proof of Theorem 4.4, we have the estimate

$$\|\bar{p}\|_{W^{l+1,p}_\varphi(\Omega)} \leqslant C\|u\|_{W^{l,p}_\varphi(\Omega)} \tag{4.18}$$

which, together with the estimate (2.6) for φ', give the required estimate for $\varphi' \bar{p}$. So, we only need to estimate $\nabla_x \bar{P}_\varphi$. For this purpose, applying the operator $(\mathrm{Id} - \mathbb{S})$ to Equation (4.15), we get

$$\Delta_x \bar{P}_\varphi = H_u(x) := \varphi' \bar{u}_1 - 2\varphi' \partial_{x_1} \bar{p} - \varphi'' \bar{p}, \ \partial_n \bar{P}_\varphi = 0. \tag{4.19}$$

Using the estimate (4.18), together with the inequality (2.6) for weights φ, we conclude that

$$\|H_u\|_{W^{l,p}_{\varphi^{-1}\varphi}(\Omega)} \leqslant C\mu\|u\|_{W^{l,p}_{\varphi}(\Omega)},$$

where C is independent of μ. Applying the result of Proposition 3.6 to Equation (4.19) ($\mathbb{S}\bar{P}_\varphi = 0!$), we finally find

$$\|\bar{P}_\varphi\|_{W^{l+2,p}_{\varphi^{-1}\varphi}(\Omega)} \leqslant C\mu\|u\|_{W^{l,p}_{\varphi}(\Omega)}$$

which completes the proof of Proposition 4.6 for the spaces $W^{l,p}_\varphi$. The case of spaces $W^{l,p}_{b,\varphi}$ is treated in a similar way. □

Now, we start to study the Stokes operator $A := \Pi\Delta_x$ in weighted spaces. For this purpose, we need to define the spaces of distributions associated with divergence-free vector fields. the corresponding function spaces.

Definition 4.7. Let Ω be a cylinder, and let $\mathcal{D}_{\rm div}(\Omega)$ be the space of all smooth divergence-free vector fields in Ω with compact support. As usual, we denote by $\mathcal{D}'_{\rm div}(\Omega)$ the space of all linear continuous functionals on $\mathcal{D}_{\rm div}(\Omega)$. We denote also by $\mathcal{H}^{-1,p}(\Omega_s) \subset \mathcal{D}'_{\rm div}(\Omega_s)$ the conjugate space to $\mathcal{V}^q(\Omega_s)$ equipped with the standard norm.

Finally, for every weight function φ of exponential growth rate μ we define the spaces $\mathcal{H}^{-1}_\varphi(\Omega)$ and $\mathcal{H}^{-1}_{b,\varphi}(\Omega)$ as subspaces of $\mathcal{D}'_{\rm div}(\Omega)$ with the following finite norms

$$\|u\|^p_{\mathcal{H}^{-1,p}_\varphi(\Omega)} := \int_{s\in\Omega} \varphi(s)^p\|u\|^p_{\mathcal{H}^{-1,p}(\Omega_s)}\,ds < \infty,$$

$$\|u\|_{\mathcal{H}^{-1,p}_{b,\varphi}(\Omega)} := \sup_{s\in\mathbb{R}}\{\varphi(s)\|u\|_{\mathcal{H}^{-1,p}(\Omega_s)}\} < \infty.$$

Arguing as in Proposition 2.8, one can show that

$$\mathcal{H}^{-1,p}_\varphi(\Omega) = [\mathcal{V}^q_{\varphi^{-1}}(\Omega)]^*. \tag{4.20}$$

We however note that the spaces $\mathcal{H}^{-1,p}_\varphi(\Omega)$ *are not* subspaces of usual distributions $\mathcal{D}'(\Omega)$ and, in fact, larger than the corresponding spaces $[W^{-1,p}_\varphi(\Omega)]^2$ of distributions. Nevertheless, there is a natural map from $[W^{-1,p}_\varphi(\Omega)]^2$ to $\mathcal{H}^{-1,p}_\varphi(\Omega)$ (which is usually considered as an extension of the projector Π to the negative Sobolev spaces and is also denoted by Π)

$$\langle \Pi u, v\rangle_{\rm div} := \langle u, v\rangle\,, \quad \operatorname{div} v = 0, \tag{4.21}$$

where, on the left-hand side, we have the pairing in $\mathcal{D}'_{\mathrm{div}}(\Omega) \times \mathcal{D}_{\mathrm{div}}(\Omega)$ and, on the right-hand side, there is the standard pairing in the sense of distributions.

Thus, the Stokes operator $A = \Pi\Delta_x$ can be naturally extended to the operator from $\mathcal{V}^p_\varphi(\Omega)$ to $\mathcal{H}^{-1,p}_\varphi(\Omega)$ (and, analogously, in the spaces $\mathcal{V}^p_{b,\varphi}(\Omega)$).

We will study the linear Stokes equation in a cylinder Ω:

$$\Delta_x u + \nabla_x p = g, \ u\big|_{\partial\Omega} = 0, \quad \operatorname{div} u = 0 \tag{4.22}$$

or, in the equivalent operator form,

$$Au = \Pi g.$$

We first recall one standard regularity result for the unweighted case.

Proposition 4.8. *For any $g \in W^{l,p}(\Omega)$, where $l \geqslant -1$ is integer and $1 < p < \infty$, there exists a unique solution (u, p) (up to adding a constant to p; in the sense of distributions) such that $u \in \mathcal{V}^p(\Omega) \cap \mathcal{H}^{l+2,p}(\Omega)$, $\bar{p} := p - \mathbb{S}p \in W^{l+1,p}(\Omega)$ such that*

$$\|u\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p\|_{W^{l,p}(\Omega)} + \|\bar{p}\|_{W^{l+1,p}(\Omega)} \leqslant C\|g\|_{W^{l,p}(\Omega)}, \tag{4.23}$$

where the constant C is independent on u.

Indeed, at least for bounded domains, this assertion is well known (see, for example, [**27**, Proposition I.2.3]) even for the nonhomogeneous Stokes problem. In the Hilbert case $p = 2$, $l = -1$, the result follows immediately from the energy estimate. For other l and p it can be easily verified by reducing the problem to the case of bounded domains via the standard localization technique (see also [**3**] and references therein).

Note that, in contrast to the case of bounded domains, we are now able to control the L^p-norms of pressure function $\bar{p} := p - \mathbb{S}p$ and $\partial_{x_1}\mathbb{S}p$ and the mean pressure $\mathbb{S}p$ may even grow as $|x_1| \to \infty$.

Our goal is to obtain a weighted analog of Proposition 4.8. For this purpose, it is convenient to consider a more general nonhomogeneous analog of this equation:

$$\begin{aligned} &\Delta_x u + \nabla_x p = g, \quad \mathbb{S}u_1 \equiv 0, \quad u\big|_{\partial\Omega} = 0, \\ &\operatorname{div} u = h \end{aligned} \tag{4.24}$$

for a function h such that

$$\mathbb{S}h \equiv 0. \tag{4.25}$$

Theorem 4.9. *For any integer $l \geqslant -1$ and $1 < p < \infty$ there exists $\mu_0 = \mu_0(p,\omega) > 0$ such that for any weight function φ of exponential growth rate $\mu \leqslant \mu_0$, $g \in W^{l,p}_\varphi(\Omega)$, and $h \in W^{l+1,p}_\varphi(\Omega)$ satisfying the zero flux condition* (4.25) *the problem* (4.24) *possesses a unique solution (u,p) such that $u \in W^{l+2,p}_\varphi(\Omega)$, $\bar p \in W^{l+1,p}_\varphi(\Omega)$ and the following estimate holds*:

$$\|u\|_{W^{l+2,p}_\varphi(\Omega)} + \|\nabla_x p\|_{W^{l,p}_\varphi(\Omega)} + \|\bar p\|_{W^{l+1,p}_\varphi(\Omega)} \leqslant C(\|g\|_{W^{l,p}_\varphi(\Omega)} + \|h\|_{W^{l+1,p}_\varphi(\Omega)}), \tag{4.26}$$

where the constant C depends on C_φ, but is independent of the form of φ and g and h. A similar result holds for $W^{l,p}_{b,\varphi}$.

Proof. We first consider the unweighted nonhomogeneous case $\varphi = 1$ and $h \neq 0$. In the case $l = -1$, it can be easily reduced to the homogeneous case $h = 0$ by substructing a function $\tilde u \in W^{1,p}_0(\Omega)$ satisfying $\operatorname{div} \tilde u = h$. To construct such a function, it suffices to solve the problem

$$\operatorname{div} u_s = h, \ u\big|_{\partial\Omega_s} = 0, \ x \in \Omega_s.$$

Since $\mathbb{S}h \equiv 0$, the mean value of h in Ω_s is also equal to zero and, consequently, this problem has a solution u_s such that

$$\|u_s\|_{W^{1,p}(\Omega_s)} \leqslant C\|h\|_{L^p(\Omega_s)},$$

where C is independent of s. The required function $\tilde u$ can be defined as follows:

$$\tilde u(x) := \tilde u_n(x), \ x \in \Omega_n.$$

Thus, the assertion of the theorem is verified in the case $l = -1$ and $\varphi = 1$. Using this estimate and the localization technique, one can verify the estimate for every integer $l \geqslant -1$.

Consider the weighted case $\varphi \neq 0$. As usual, it suffice to verify the estimate (4.26) for special exponential weights $\varphi_{\mu,x_0}(x_1)$ (a general result follows then from the representations (2.11) and (2.13)). We restrict ourselves to the a priori estimate (4.26) (the existence of a solution follows then in a standard way, for example, by approximating g and h by functions with finite support and passing to the limit).

Let (u,p) be a desired solution of the problem (4.24), and let $\varphi := \varphi_{\mu,x_0}(x_1)$ for some $x_0 \in \mathbb{R}$. We set

$$u_\varphi := \varphi u, \ p_\varphi := \varphi p - \int_0^{x_1} \varphi'(y)(\mathbb{S}p)(y)\,dy.$$

In the new variables, the problem (4.24) takes the form

$$\begin{aligned} &\Delta_x u_\varphi + \nabla_x p_\varphi = g_\varphi := \varphi g + 2\varphi' \partial_{x_1} u_1 + \varphi'' u_1 + \varphi' \bar{p} \vec{e}_1, \\ &\operatorname{div} u_\varphi = h_\varphi := \varphi h + \varphi' u_1, \end{aligned} \tag{4.27}$$

where $\vec{e}_1 := (1,0,0)$. We see that $\mathbb{S} h_\varphi \equiv 0$ (since $\mathbb{S} u_1 \equiv 0$). Moreover, using the inequalities (2.6) and the obvious fact that $\bar{p}_\varphi = \varphi \bar{p}$, we find

$$\begin{aligned} \|g_\varphi\|_{W^{l,p}(\Omega)} + \|h_\varphi\|_{W^{l+1,p}(\Omega)} &\leqslant C(\|g\|_{W^{l,p}_\varphi(\Omega)} + \|h\|_{W^{l+1,p}_\varphi(\Omega)}) \\ &+ C\mu(\|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\bar{p}_\varphi\|_{W^{l+1,p}(\Omega)}), \end{aligned} \tag{4.28}$$

where the constant C is independent of μ (we implicitly used Proposition 2.14). Applying the unweighted estimate (4.26), which is already proved, to Equation (4.27), we find

$$\begin{aligned} \|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p_\varphi\|_{W^{l,p}(\Omega)} + \|\bar{p}_\varphi\|_{W^{l+1,p}(\Omega)} \\ \leqslant C(\|g_\varphi\|_{W^{l,p}(\Omega)} + \|h_\varphi\|_{W^{l+1,p}(\Omega)}). \end{aligned}$$

Combining this estimate with (4.28) and fixing $\mu = \mu_0$ small enough, we arrive at the inequality

$$\begin{aligned} \|u_\varphi\|_{W^{l+2,p}(\Omega)} + \|\nabla_x p_\varphi\|_{W^{l,p}_\varphi(\Omega)} + \|\bar{p}_\varphi\|_{W^{l+1,p}(\Omega)} \\ \leqslant C(\|g\|_{W^{l,p}_\varphi(\Omega)} + \|h\|_{W^{l+1,p}_\varphi(\Omega)}) \end{aligned}$$

which, together with Proposition 2.14, yields (4.26) and completes the proof of the theorem. □

We conclude the section by formulating several useful corollaries of the theorem.

Corollary 4.10. *Let Ω be a cylinder, and let $A := \Pi\Delta_x$. Then for every $1 < p < \infty$ and $l = 0, -1$ there exists positive $\mu_0 = \mu_0(p)$ such that for every weight function of a sufficiently small exponential growth rate ($\mu \leqslant \mu_0$) operator A realizes an isomorphism between the spaces $\mathcal{V}^p_\varphi(\Omega) \cap \mathcal{H}^{l+2,p}_\varphi(\Omega)$ and $\mathcal{H}^{l,p}_\varphi(\Omega)$ and the following estimate holds*:

$$C^{-1}\|u\|_{\mathcal{H}^{l+2,p}_\varphi(\Omega)} \leqslant \|\Pi\Delta_x u\|_{\mathcal{H}^{l,p}_\varphi(\Omega)} \leqslant C\|u\|_{\mathcal{H}^{l+2,p}_\varphi(\Omega)}, \tag{4.29}$$

where the constant C depends on C_φ, but is independent of the choice of the weight function φ. A similar result holds for $\mathcal{H}^{l,p}_{b,\varphi}(\Omega)$.

Proof. The right estimate in (4.29) follows from Theorem 4.4 (in the case $l = 0$) and the definition of Π for Sobolev spaces with negative exponent ($l = -1$). The left one immediately follows from Theorem 4.9 with $h \equiv 0$ by applying the projector Π to both sides of (4.22) (in the case $l = -1$, we use the obvious fact that for every $g \in \mathcal{H}_\varphi^{-1,p}(\Omega)$ there exists an extension $\tilde{g} \in W_\varphi^{-1,2}(\Omega)$ such that $\Pi\tilde{g} = g$). $\square$

Corollary 4.11. *Let the assumptions of Corollary* 4.10 *hold, and let* $p = 2$. *Then, for every weight function with a sufficiently small growth rate* μ

$$C^{-1}(\varphi\Delta_x u, \varphi\Delta_x u) \leqslant (\varphi\Pi\Delta_x u, \varphi\Pi\Delta_x u) \leqslant C(\varphi\Delta_x u, \varphi\Delta_x u), \tag{4.30}$$

where $(\cdot,\cdot)$ *denotes the standard inner product in* $[L^2(\Omega)]^3$ *and the constant* C *is independent of the choice of weight* φ *and* $u \in \mathcal{V}_\varphi^2(\Omega) \cap \mathcal{H}_\varphi^{2,2}(\Omega)$.

Indeed, the estimate (4.30) is an immediate consequence of (4.29) with $p = 2$ and the following elliptic regularity estimate for the Laplacian in Ω with Dirichlet boundary conditions (see Proposition 3.1):

$$C^{-1}\|u\|_{W_\varphi^{2,2}(\Omega)} \leqslant \|\Delta_x u\|_{L_\varphi^2(\Omega)} \leqslant C\|u\|_{W_\varphi^{2,2}(\Omega)}. \tag{4.31}$$

5. Auxiliary Nonstationary Stokes Problem

In this section, we study the following nonstationary linear Stokes problem in a cylindrical domain Ω:

$$\begin{aligned} &\partial_t w = \Delta_x w - \nabla_x q, \\ &\operatorname{div} w = h(t),\ \mathbb{S}w_1 \equiv 0, \\ &w\big|_{\partial\Omega} = 0,\ \Pi w\big|_{t=0} = 0, \end{aligned} \tag{5.1}$$

where $h(t) = h(t,x)$ is a given function such that

$$\mathbb{S}h(t)(x_1) \equiv 0, \quad t \in [0,T],\ x_1 \in \mathbb{R}. \tag{5.2}$$

This auxiliary problem is essentially used in the following section for obtaining the weighted energy estimates for weak solutions of the nonlinear Navier–Stokes system.

The following theorem yields a priori estimates and the solvability result for the problem (5.1).

Theorem 5.1. *There exists a positive* μ_0 *such that for any weight function* φ *of sufficiently small exponential growth rate* μ ($\mu \leqslant \mu_0$) *and*

$$h \in L^2([0,T], W_\varphi^{1,2}(\Omega)) \cap C([0,T], L_\varphi^2(\Omega)) \tag{5.3}$$

for which (5.2) *is satisfied the problem* (5.1) *possesses a unique solution* w *belonging to the class*

$$w \in L^2([0,T], W^{2,2}_\varphi(\Omega)) \cap C([0,T], W^{1,2}_\varphi(\Omega)),$$
$$\partial_t \Pi w \in L^2([0,T], L^2_\varphi(\Omega)),\ q \in \mathcal{D}'([0,T]\times\Omega) \quad (5.4)$$

and satisfying the estimates

$$\int_0^T e^{-\alpha|t-s|}(\|\partial_t \Pi w(s)\|^2_{L^2_\varphi(\Omega)} + \|w(s)\|^2_{W^{2,2}_\varphi(\Omega)})\,ds$$
$$\leqslant C\int_0^T e^{-\alpha|t-s|}\|h(s)\|^2_{W^{1,2}_\varphi(\Omega)}\,ds, \quad (5.5)$$
$$\|w(t)\|^2_{W^{1,2}_\varphi(\Omega)} \leqslant C\Big(\|h(t)\|^2_{L^2_\varphi(\Omega)} + \int_0^T e^{-\alpha|t-s|}\|h(s)\|^2_{W^{1,2}_\varphi(\Omega)}\,ds\Big),$$

where α *is a sufficiently small positive constant depending only on* μ_0 *and the constant* C *depends on* C_φ*, but is independent of the choice of weight* φ*.*

PROOF. To solve (5.1), we will reduce it to the divergence-free case. For this purpose, for every $t \in [0,T]$ we introduce $v(t) = K_{h(t)}$ as a solution of the stationary Stokes problem

$$\Delta_x v - \nabla_x r = 0, \quad \operatorname{div} v = h(t), \quad v\big|_{\partial\Omega} = 0. \quad (5.6)$$

By Theorem 4.9, there exists positive μ_0 such that, for every weight function of sufficiently small exponential growth rate μ ($\mu \leqslant \mu_0$) and every $h \in W^{l,2}_\varphi(\Omega)$, $l = 0, 1$ satisfying (5.2), Equation (5.6) possesses a unique solution $v \in W^{l+1,2}_\varphi(\Omega)$, $\mathbb{S}v_1 \equiv 0$, so the operator $K_{h(t)}$ is well defined. Moreover, the following estimate holds:

$$\|v\|_{W^{l+1,2}_\varphi(\Omega)} \leqslant C\|h\|_{W^{l,2}_\varphi(\Omega)}, \quad (5.7)$$

where the constant C depends on C_φ, but is independent of the choice of weight φ.

Introduce a new dependent variable $\bar{w}(t) := w(t) - v(t)$. This function satisfies the equation

$$\partial_t(\bar{w} + v) = \Delta_x \bar{w} - \nabla_x \bar{q},\ \operatorname{div}\bar{w} = 0,\ \bar{w}\big|_{\partial\Omega} = 0,\ \bar{w}\big|_{t=0} = -\Pi v\big|_{t=0}. \quad (5.8)$$

Applying the projector Π to both sides of (5.8), we infer

$$\partial_t(\bar{w} + \Pi v) = \Pi\Delta_x \bar{w},\ \operatorname{div}\bar{w} = 0,\ \bar{w}\big|_{\partial\Omega} = 0,\ \bar{w}\big|_{t=0} = -\Pi v\big|_{t=0}. \quad (5.9)$$

To obtain a priori estimate for solutions of (5.9), we multiply it by the expression

$$\varphi_{2\mu,x_0}(x_1)(\partial^2_{x_2}+\partial^2_{x_3})(\bar w+\Pi v)+\partial_{x_1}[\varphi_{2\mu,x_0}(x_1)\partial_{x_1}(\bar w+\Pi v)],$$

where $x_0\in\mathbb{R}$ is arbitrary, $\mu>0$ is small enough, and the weight φ is defined by (2.4). Then

$$\begin{aligned}&1/2\partial_t(\varphi_{2\mu,x_0},|\nabla_x(\bar w+\Pi v)|^2)+(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\bar w)\\&=(\varphi'_{2\mu,x_0}\Pi\Delta_x\bar w,\partial_{x_1}\bar w)-(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\Pi v)\\&-(\varphi'_{2\mu,x_0}\Pi\Delta_x\bar w,\partial_{x_1}\Pi v).\end{aligned}\tag{5.10}$$

We estimate the second term on the left-hand side of (5.10) using the estimates (4.13), (4.7), and (4.30) in the following way:

$$\begin{aligned}&(\varphi_{2\mu,x_0}\Pi\Delta_x\bar w,\Delta_x\bar w)\\&=(\varphi_{2\mu,x_0},|\Pi\Delta_x\bar w|^2)-(\Pi\Delta_x\bar w,(\varphi_{2\mu,x_0}\circ\Pi-\Pi\circ\varphi_{2\mu,x_0})\Delta_x\bar w)\\&\geqslant C(\varphi_{2\mu,x_0},|\Delta_x\bar w|^2)-C_1(\varphi_{-2\mu,x_0},|(\varphi_{2\mu,x_0}\circ\Pi-\Pi\circ\varphi_{2\mu,x_0})\Delta_x\bar w|^2)\\&\geqslant(C_2-C_3\mu)\|\Delta_x\bar w\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)},\end{aligned}\tag{5.11}$$

where the constants C_i are independent of μ and x_0. Fixing μ small enough, estimating the right-hand side of (5.10) by the Hölder inequality, and using (4.7) and (4.29), we find

$$\begin{aligned}&\partial_t(\|\nabla_x(\bar w+\Pi v)\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)})+\alpha(\|\Delta_x\bar w\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)}\\&+\|\nabla_x(\bar w+\Pi v)\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)})\leqslant C\|v\|^2_{W^{2,2}_{\varphi_{\mu.x_0}}(\Omega)},\end{aligned}\tag{5.12}$$

where the positive constants α and C are independent of $x_0\in\mathbb{R}$. Here, we implicitly used the inequality

$$\|\nabla_x(\bar w+\Pi v)\|_{L^2_{\varphi_{\mu,x_0}}(\Omega)}\leqslant C(\|\nabla_x\bar w\|_{L^2_{\varphi_{\mu,x_0}}(\Omega)}+\|v\|_{W^{2,2}_{\varphi_{\mu,x_0}}(\Omega)}).$$

Applying the Gronwall inequality to (5.12) and using the estimate (5.7) with $l=1$ (for every fixed t), we get

$$\begin{aligned}&\|\nabla_x(\bar w(t)+\Pi v(t))\|^2_{L^2_{\varphi_{\mu,x_0}}(\Omega)}+\int\limits_0^t e^{-\alpha(t-s)}\|\bar w(s)\|^2_{W^{2,2}_{\varphi_{\mu,x_0}}(\Omega)}\,ds\\&\qquad\leqslant C\int\limits_0^t e^{-\alpha(t-s)}\|h(s)\|^2_{W^{1,2}_{\varphi_{\mu,x_0}}(\Omega)}\,ds\end{aligned}\tag{5.13}$$

(we used the equality $\bar{w}(0) + \Pi v(0) = \Pi u(0) = 0$). Since the constant C in (5.13) is independent of $x_0 \in \mathbb{R}$, multiplying (5.13) by $\varphi^2(x_0)$, integrating over $x_0 \in \mathbb{R}$ and using (2.12), we obtain (exactly as in Section 3) an analog of the estimate (5.13) not only for the special weights φ_{μ,x_0}, but also for an arbitrary weight φ of exponential growth rate $\varepsilon < \mu$.

To deduce the a priori estimate (5.5) from (5.13), it remains to recall that $w = \bar{w} + v$ and (due to (5.7) with $l = 0$)

$$\|w(t)\|_{W^{1,2}_{\varphi}(\Omega)} \leqslant C(\|\nabla_x(\bar{w}(t) + \Pi v(t))\|_{L^2_{\varphi}(\Omega)} + \|h(t)\|_{L^2_{\varphi}(\Omega)}).$$

This estimate, together with (5.13), yields the required estimate for the $W^{1,2}_{\varphi}$-norm of $w(t)$; the estimate for the $W^{2,2}_{\varphi}$-norm of w is also an immediate consequence of (5.13) and (5.7) with $l = 1$. Finally, the required estimate for $\partial_t \Pi w = \partial_t(\bar{w} + \Pi v)$ can be obtained from Equation (5.9). Thus, the a priori estimate (5.5) is proved.

We also note that,

$$(\mathrm{Id} - \Pi)\partial_t w(t) = \partial_t (Id - \Pi) K_{h(t)} = (Id - \Pi) K_{\partial_t h(t)}, \tag{5.14}$$

and we see that, in contrast to the divergence free component of $\partial_t w$ its potential component *does not* belong to $L^2_{\varphi}(\Omega)$ for general external forces h, but if, in addition, we have $\partial_t h \in L^2_{\varphi}(\Omega)$, then (5.14) shows that $\partial_t u$ belongs to $L^2_{\varphi}(\Omega)$ and Equation (5.1) can be naturally understood as an equality in $L^2([0,T], L^2_{\varphi}(\Omega))$.

The above observation gives a natural way for constructing the required solution $w(t)$ of (5.1) based on the obtained a priori estimate.

Indeed, let us approximate the external force $h \in C([0,T], L^2_{\varphi}(\Omega)) \cap L^2([0,T], W^{1,2}_{\varphi}(\Omega))$ by a sequence of smooth (with respect to t and x) functions h^n with compact support in x_1 and satisfying (5.2). Having such h^n, we construct the associated functions $v^n \in C^1([0,T], W^{2,2}(\Omega))$ by Theorem 4.9. Then the associated equation (5.8) for $\bar{w}^n$ will be the standard nonstationary Stokes equation with external forces $\partial_t v(t)$ belonging to the unweighted space $C([0,T], W^{2,2}(\Omega))$.

As is well known, for such external forces the nonstationary Stokes equation possesses a unique solution

$$\bar{w}^n \in W^{1,2}([0,T], L^2(\Omega)) \cap L^2([0,T], W^{2,2}(\Omega))$$

(see, for example, [**4**] or [**5**]). Thus, the approximating sequence of solutions w^n is constructed. Note that, since $w^n(t)$ belongs to $L^2(\Omega)$ and is divergence-free,

$$\mathbb{S}\bar{w}^n_1 \equiv 0 \text{ and, consequently, } \mathbb{S}w^n_1 \equiv 0. \tag{5.15}$$

Moreover, since h^n have compact support in x_1, then the a priori estimate (5.5) holds for w^n uniformly with respect to $n \to \infty$. Passing now to the limit $n \to \infty$ and using (5.15) we construct the required solution $w(t)$. Theorem 5.1 is proved. $\square$

Remark 5.2. Condition $\mathbb{S}w_1 \equiv 0$ is essential for the uniqueness part of Theorem 5.1. As we will see below, for every function $c(t) \in C_b(R)$, Equation (5.1) possesses a solution w satisfying $\mathbb{S}w_1(t) \equiv c(t)$.

We conclude this section by preparing some technical tools for obtaining the energy estimates for the nonlinear Navier–Stokes equation in a cylindrical domain. For this purpose, we introduce function spaces.

Definition 5.3. Let Ω be a cylinder, and let $\mathbb{W}_b([0,T]\times\Omega)$ consist of vector fields $u \in L^2_b([0,T],\mathcal{V}^2_b(\Omega))$ (see Remark 2.4) such that the t-derivative $\partial_t u$ belongs to $\mathcal{D}'_{\mathrm{div}}(\Omega)$ a.e. and satisfies the condition

$$\partial_t u \in L^2_b([0,T],\mathcal{H}_b^{-1,2}(\Omega)). \tag{5.16}$$

Consider also an arbitrary weight function θ of a sufficiently small exponential growth rate μ and a smooth nonnegative function φ satisfying the following assumptions:

$$|\varphi'(s)| + \varphi(s) \leqslant C\theta(s),\ \ s \in \mathbb{R},\quad \int_{s\in\mathbb{R}} \theta^2(s)\,ds < \infty. \tag{5.17}$$

To obtain the weighted energy estimates for the solution $u \in \mathbb{W}_b([0,T]\times\Omega)$ of the Navier–Stokes equation in $L^2_\varphi(\Omega)$ (which contains $L^2_b(\Omega)$ due to the integrability assumption on φ), it would be natural to multiply it by the function $\varphi^2 u$ and integrate over Ω. However, unfortunately, this function is no more divergence-free and, consequently, this way does not allow us to exclude the pressure. Instead of that, we multiply, following [**37**]), by the function $\varphi^2 u - v$, where $v(t) := (\mathbb{P}_\varphi u)(t)$ is an appropriate corrector which makes this multiplier divergence-free. To this end, the function $v(t)$ should satisfy

$$\operatorname{div} v(t) \equiv h_u(t) := 2\varphi\varphi' u_1(t) \tag{5.18}$$

(here we have used that $\operatorname{div} u = 0$). Due to the integrability assumption on φ, the function $h \in L^2([0,T],W^{1,2}_{\theta^{-1}}(\Omega))$; moreover, since $\mathbb{S}u_1 \equiv 0$, we have $\mathbb{S}h \equiv 0$ and (5.2) is satisfied.

Furthermore, it is convenient for us to fix the corrector $v(t) := (\mathbb{P}_\varphi u)(t)$ as a solution of the following auxiliary nonstationary Stokes problem in Ω:

$$\begin{aligned} &-\partial_t v = \Delta_x v - \nabla_x q, \quad \operatorname{div} v(t) = h_u(t),\\ &v\big|_{\partial\Omega} = 0, \quad \Pi v\big|_{t=T} = 0. \end{aligned} \tag{5.19}$$

This equation can be reduced to (5.1) by the time change $t \to T - t$. Thus, Theorem 5.1 and the estimate (5.5) hold for this equation as well. The following theorem justifies our choice of the corrector $\mathbb{P}_\varphi$ and gives the main technical tool for the weighted energy estimates of the Navier–Stokes equations.

Theorem 5.4. *Let Ω be a cylinder, and let φ be a smooth nonnegative function, satisfying* (5.17) *for some square integrable weight θ of sufficiently small exponential growth rate μ. Then*

$$\mathbb{W}_b([0,T] \times \Omega) \subset C([0,T], L^2_\theta(\Omega)). \tag{5.20}$$

Let also $\mathbb{P}_\varphi$ be defined as the solving operator for the problem (5.19). *Then*

$$\begin{aligned} &\frac{d}{dt}\left[1/2(\varphi^2 u(t), u(t)) - (u(t), (\mathbb{P}_\varphi u)(t))\right] + (\nabla_x u(t), \nabla_x(\varphi^2 u(t)))\\ &= (\partial_t u(t) - \Pi\Delta_x u(t), \varphi^2 u - (\mathbb{P}_\varphi u)(t)), \end{aligned} \tag{5.21}$$

which means that the function $1/2(\varphi^2 u, u) - (u, \mathbb{P}_\varphi u)$ is absolutely continuous as a scalar function on $[0,T]$ and (5.21) *holds almost everywhere.*

Proof. We give below only the formal derivation of (5.21) which can be justified in a standard way (see [**37**]; the detailed proof of embedding (5.20) also can be found there).

Since $\partial_t \Pi v + \Pi\Delta_x v \equiv 0$ and $\operatorname{div} u = \operatorname{div}(\varphi^2 u - v) = 0$, we can integrate by parts and find

$$\begin{aligned} &(\partial_t u - \Pi\Delta_x u, \varphi^2 u - v) = (\partial_t u - \Delta_x u, \varphi^2 u - v)\\ &= \partial_t[1/2(\varphi^2 u, u) - (u, v)] + (\nabla_x u, \nabla_x(\varphi^2 u)) + (u, \partial_t v + \Delta_x v)\\ &= \partial_t[1/2(\varphi^2 u, u) - (u, v)] + (\nabla_x u, \nabla_x(\varphi^2 u)). \end{aligned} \tag{5.22}$$

Theorem 5.4 is proved. $\square$

6. Leray Approximations to Navier–Stokes Equations

The goal of this section is to verify the existence of spatially nondecaying solutions for the following Leray–Navier–Stokes equation in a cylindrical unbounded domain Ω:

$$\begin{aligned}&\partial_t u + (\Pi w, \nabla_x)u + \mathbb{S}u_1\partial_{x_1}u = \Delta_x u - \nabla_x p + g,\\ &w - \alpha\Delta_x w = u, \quad w\big|_{\partial\Omega} = u\big|_{\partial\Omega} = 0, \quad \operatorname{div} u = 0,\\ &u\big|_{t=0} = u_0,\end{aligned} \tag{6.1}$$

where $\alpha > 0$ is a small regularizing parameter which is now fixed. To obtain the unique solvability, we endow the problem by the additional mean flux assumption

$$\mathbb{S}u_1(t) \equiv c, \tag{6.2}$$

where c is a given constant which plays the role of a "boundary" condition at $x_1 = \pm\infty$.

The additional term $\mathbb{S}u_1\partial_{x_1}u = c\partial_{x_1}u$ is related with the fact that, in the case $\alpha = 0$, on one hand, we should have the classical Navier–Stokes problem and, on the other hand, $w(t) = u(t)$ and $\Pi w(t) = u(t) - (c, 0, 0)$.

For the sake of simplicity, we start with the case of zero flux

$$\mathbb{S}u_1(t) \equiv 0. \tag{6.3}$$

The case of general flux c will be considered at the end of this section. We assume that

$$g \in L^2_b(\mathbb{R}_+, L^2_b(\Omega)), \quad u_0 \in \mathcal{H}^2_b(\Omega), \tag{6.4}$$

and the solution u satisfies the condition

$$u \in \mathbb{W}_b([0,T]\times\Omega) \tag{6.5}$$

(see Definition 5.3) and Equation (6.1) in the sense of distributions $\mathcal{D}'_{\mathrm{div}}(\Omega)$ over the divergence-free vector fields.

Remark 6.1. Due to Theorem 5.4,

$$u \in L^\infty([0,T], \mathcal{H}^2_b(\Omega)) \cap C([0,T], \mathcal{H}^2_\varphi(\Omega))$$

for every square integrable weight function of exponential growth rate. Hence the initial condition $u\big|_{t=0} = u_0$ is well defined. Moreover, since $u \in L^\infty([0,T], L^2_b(\Omega)) \cap L^2_b([0,T], \mathcal{V}^2_b(\Omega))$, we conclude that, due to Proposition 3.3, Theorem 4.4, and the embedding $W^{2,2} \subset L^\infty$,

$$\Pi w \in L^\infty([0,T]\times\Omega). \tag{6.6}$$

Then the inertial term $(\Pi w, \nabla_x u)$ satisfies the condition

$$(\Pi w, \nabla_x)u \in L^2_b([0,T]\times\Omega) \subset L^2_b([0,T], \mathcal{H}_b^{-1,2}(\Omega)). \tag{6.7}$$

Thus,

$$\Pi[(\Pi w, \nabla_x)u] \in L^2_b([0,T], \mathcal{H}_b^{-1,2}(\Omega)), \tag{6.8}$$

where Π is the projector on the divergence-free vector fields introduced in Section 4. Applying this projector to Equation (6.1), we obtain

$$\partial_t u = \Pi\Delta_x u - \Pi[(\Pi w, \nabla_x)u] + \Pi g, \tag{6.9}$$

which shows that the derivative $\partial_t u$ satisfies the condition

$$\partial_t u \in L^2_b([0,T], \mathcal{H}_b^{-1,2}(\Omega)) \tag{6.10}$$

(see Corollary 4.10 for $\Pi\Delta_x u$). This shows that the definition of a solution u in the form (6.5) is not contradictory and Equation (6.1) can be understood as the equality (6.9) in the space (6.10). We also note that the zero flux assumption (6.3) is now incorporated into the definition of the space $\mathbb{W}_b([0,T]\times\Omega)$.

Introduce a special family of polynomial weight functions $\theta_\varepsilon(s) = \theta_{\varepsilon,x_0}(s)$ by the formula

$$\theta_{\varepsilon,x_0}(s) := \left(1+\varepsilon^2|s-x_0|^2\right)^{-1/2}, \quad \varepsilon>0,\ s,x_0\in\mathbb{R}. \tag{6.11}$$

It is obvious that these functions are weight functions of exponential growth rate μ for every $\mu > 0$ with a constant C_{θ_ε} that depends on μ, but is independent of $x_0 \in \Omega$ and $\varepsilon \in [0,1]$. This means that all the weighted estimates formulated in the previous sections hold for the weights (6.11) with constants independent of $\varepsilon \to 0$, which is crucial for our method. Moreover, these weights satisfy the following improved version of (5.17):

$$|\varphi'_{\varepsilon,x_0}(s)| \leqslant \varepsilon[\varphi_{\varepsilon,x_0}(s)]^2, \quad \|\varphi_\varepsilon\|_{L^2(\mathbb{R})} < \infty. \tag{6.12}$$

Thus, Theorem 5.4 holds for these weights.

The following proposition gives a basic *uniform* with respect to α a priori estimate for the solutions of (6.1).

Proposition 6.2. *Let the above assumptions hold, and let* $u \in \mathbb{W}_b([0,T]\times\Omega)$ *be a solution of the Leray–Navier–Stokes problem* (6.1)*. Then the following estimate holds*:

$$\sup_{s\in[0,T]}\{e^{-\beta|t-s|}\|u(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\}$$
$$+ (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds$$
$$\leqslant C_3 e^{-\beta t}\|u(0)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C_3\int_0^T e^{-\beta|t-s|}\|g(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\,ds, \tag{6.13}$$

where the positive constants β *and* C_i*,* $i = 1, 2, 3$*, are independent of small* $\alpha > 0$*,* u*,* u_0*,* g*,* $\varepsilon \to 0$*,* T*, and* x_0 *(we recall that we write* θ_ε *instead of* θ_{ε,x_0} *for brevity).*

Proof. Let u be a solution of (6.9) in the above class. By Theorem 5.4, we have

$$\frac{d}{dt}[1/2(\theta_\varepsilon^2 u(t), u(t)) - (u(t), v(t))] + (\nabla_x u(t), \nabla_x(\theta_\varepsilon^2 u(t)))$$
$$= -(\theta_\varepsilon^2 u(t) - v(t), (\Pi w(t), \nabla_x)u(t) - g(t))), \tag{6.14}$$

where $v := \mathbb{P}_{\theta_\varepsilon} u$ solves the auxiliary problem (5.19). Using (6.12) and the inequality $\|u\|_{L^2_{\theta_\varepsilon}(\Omega)} \leqslant C\|\nabla_x u\|_{L^2_{\theta_\varepsilon}(\Omega)}$, we transform (6.14) as follows:

$$\frac{d}{dt}R_u(t) + \beta R_u(t) + 1/2\|u(t)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$
$$\leqslant |(\theta_\varepsilon^2 u(t), (\Pi w(t), \nabla_x u(t))| + |(v(t), (\Pi w(t), \nabla_x)u(t))|$$
$$+ C\|g(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\|v\|^2_{L^2_{[\theta_\varepsilon]^{-1}}(\Omega)} := H_u(t), \tag{6.15}$$

where $R_u(t) := 1/2\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} - (u(t), v(t))$. Applying the Gronwall inequality to (6.15), we infer

$$R_u(t) + \int_0^t e^{-\beta(t-s)}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds$$
$$\leqslant Ce^{-\beta t}R_u(0) + C\int_0^t e^{-\alpha(t-s)}H_u(s)\,ds. \tag{6.16}$$

Now, we need to estimate the auxiliary function $v(t)$. Note that, due to (6.11), the function $h_u(t) := 2\theta_\varepsilon\theta'_\varepsilon u(t)$ satisfies the estimate

$$\|h_u(t)\|_{W^{l,2}_{[\theta_\varepsilon]-2}(\Omega)} \leqslant C\varepsilon\|u(t)\|_{W^{l,2}_{\theta_\varepsilon}(\Omega)}, \tag{6.17}$$

where the constant C is independent of $\varepsilon \to 0$. Applying Theorem 5.1 to the auxiliary equation (5.19), we deduce the estimate

$$\begin{gathered}\|v(t)\|^2_{W^{1,2}_{[\theta_\varepsilon]-2}(\Omega)} \leqslant C\varepsilon^2\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\varepsilon^2\int\limits_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds,\\ \int\limits_0^T e^{-\beta|t-s|}\|v(s)\|^2_{W^{2,2}_{[\theta_\varepsilon]-2}(\Omega)}\,ds \leqslant C\varepsilon^2\int\limits_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds,\end{gathered} \tag{6.18}$$

where $\beta > 0$ is small enough and the constant C are independent of α and $\varepsilon \to 0$. Inserting these estimates into (6.16), after simple transformations, we get

$$\begin{gathered}\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + \int\limits_0^t e^{-\beta(t-s)}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds \leqslant Ce^{-\beta t}\|u_0\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\\ + C\varepsilon^2\int\limits_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds + \int\limits_0^T e^{-\beta|t-s|}H_u(s)\,ds.\end{gathered} \tag{6.19}$$

In turns, this estimate, implies in a standard way that for sufficiently small $\varepsilon > 0$

$$\begin{gathered}\sup_{s\in[0,T]}\{e^{-\beta|t-s|}\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\} + C'\int\limits_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds\\ \leqslant Ce^{-\beta t}\|u_0\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + C\int\limits_0^T e^{-\beta|t-s|}\|g(s)\|^2_{L^2_{\theta_\varepsilon}(\Omega)}\,ds\end{gathered}$$

$$
\begin{aligned}
&+ C \int_0^T e^{-\beta|t-s|} |(\theta_\varepsilon^2 u(s), (u(s), \nabla_x) u(s))| \, ds \\
&+ C \int_0^T e^{-\beta|t-s|} |(v(s), (u(s), \nabla_x) u(s))| \, ds \\
&:= I_{u_0} + I_g + I_1 + I_2. \qquad (6.20)
\end{aligned}
$$

To estimate the first term on the left-hand side of (6.20), it suffices to multiply (6.19) by $e^{-\gamma|t_1-t|}$, where $\gamma < \beta$, take the supremum over $t \in [0, T]$, and use Proposition 2.5. Similarly, to estimate the second term, we need to integrate over $t \in [0, T]$ instead of taking the supremum (rigorously speaking, we obtain (6.20) for some new exponent γ which is less than β (say, $\gamma = \beta/2$), but, in order to simplify the notation, we denote this new exponent by β as well).

Thus, to complete the proof of Proposition 6.2, we only need to estimate the integrals I_1 and I_2 on the right-hand side of (6.20) uniformly with respect to $\alpha \to 0$. For this purpose, we use the uniform (with respect to α) estimate

$$
\|\Pi w(t)\|_{L^2_{\theta_\varepsilon}(\Omega)} \leqslant C \|u(t)\|_{L^2_{\theta_\varepsilon}(\Omega)} \qquad (6.21)
$$

which is an immediate consequence of Proposition 3.3 and Theorem 4.4.

For I_1, integrating by parts in $(\theta_\varepsilon^2 u, (\Pi w, \nabla_x) u)$ and using the fact that $\operatorname{div} u = 0$, (6.21), and the inequality (6.12), we have

$$
\begin{aligned}
|(\theta_\varepsilon^2 u, (\Pi w, \nabla_x) u)| &= |(\theta_\varepsilon \theta_\varepsilon' (\Pi w)_1, |u|^2)| \leqslant C\varepsilon([\theta_\varepsilon]^3 |\Pi w|, |u|^2) \\
&\leqslant C_1 \varepsilon \|\Pi w\|_{L^2_{\theta_\varepsilon}(\Omega)} \|u\|^2_{L^4_{\theta_\varepsilon}(\Omega)} \leqslant C_2 \varepsilon \|u\|_{L^2_{\theta_\varepsilon}(\Omega)} \|u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}, \qquad (6.22)
\end{aligned}
$$

where the constant C_2 is independent of ε and α (we implicitly used the embedding $W^{1,2}_{\theta_\varepsilon}(\Omega) \subset L^4_{\theta_\varepsilon}(\Omega)$, where the embedding constant is independent of ε, see Proposition 2.11).

Inserting this estimate into the expression for I_1, we obtain

$$
I_1 \leqslant C_3 \varepsilon \|u\|_{L^\infty([0,T], L^2_{\theta_\varepsilon}(\Omega))} \int_0^T e^{-\beta|t-s|} \|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \, ds. \qquad (6.23)
$$

To estimate the integral I_2, we use the following embedding estimate of Proposition 2.11:

$$
\|v\|_{L^\infty_{[\theta_\varepsilon]^{-2}}(\Omega)} \leqslant C \|v\|_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)},
$$

where again the constant C is independent of ε. Thus, we can estimate I_2 as follows:

$$I_2 \leqslant C \int_0^T e^{-\beta|t-s|} \|\Pi w(s)\|_{L^2_{\theta_\varepsilon}(\Omega)} \|\nabla_x u(s)\|_{L^2_{\theta_\varepsilon}(\Omega)} \|v(s)\|_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)}\, ds$$

$$\leqslant C\|\Pi w(s)\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} \int_0^T e^{-\beta|t-s|} (\varepsilon\|u(t)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$

$$+ \varepsilon^{-1}\|v(s)\|^2_{W^{2,2}_{[\theta_\varepsilon]^{-2}}(\Omega)})\, ds. \tag{6.24}$$

Using (6.18) and (6.21), we finally obtain

$$I_2 \leqslant C_3\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} \int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\, ds. \tag{6.25}$$

Inserting the estimates (6.23) and (6.25) into the right-hand side of (6.20), we obtain (6.13) and complete the proof of Proposition 6.2. □

To deduce the existence of a solution $u \in \mathbb{W}_b([0,T]\times\Omega)$ of the problem (6.1) from the a priori estimate (6.13), we need the following simple proposition.

Proposition 6.3. *Let $w \in L^2_b(\Omega)$, and let $\theta_\varepsilon = \theta_{\varepsilon,x_0}$ be the weight function defined by* (6.11). *Then the following estimate holds*:

$$\|w\|_{L^2_{\theta_\varepsilon}(\Omega)} \leqslant C\varepsilon^{-1/2}\|w\|_{L^2_b(\Omega)}, \tag{6.26}$$

where the constant C is independent of $\varepsilon \to 0$ and $x_0 \in \mathbb{R}$.

PROOF. According to (2.11), we have

$$\|w\|^2_{L^2_{\theta_\varepsilon}(\Omega)} \leqslant C \int_{s\in\mathbb{R}} \theta_\varepsilon(s)^2\|w\|^2_{L^2(\Omega_s)}\, ds$$

$$\leqslant C\|w\|^2_{L^2_b(\Omega)} \int_{s\in\mathbb{R}} (1+\varepsilon^2|s-x_0|^2)^{-1}\, ds$$

$$= C\|w\|^2_{L^2_b(\Omega)}\varepsilon^{-1} \int_{s\in\mathbb{R}} (1+|s|^2)^{-1}\, ds = C_1\varepsilon^{-1}\|w\|^2_{L^2_b(\Omega)},$$

and Proposition 6.3 is proved. □

Proposition 6.3 allows us to simplify the basic a priori estimate (6.13) as follows.

Corollary 6.4. *Let the assumptions of Proposition* 6.2 *hold, and let* $u \in \mathbb{W}_b([0,T]\times\Omega)$ *be a solution of* (6.1)*. Then the following estimate holds*:

$$\|u\|^2_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} + (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\|u\|^2_{L^2_b([0,T],W^{1,2}_{\theta_\varepsilon}(\Omega))}$$
$$\leqslant C_3\varepsilon^{-1}(\|u(0)\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T],L^2_b(\Omega))}), \tag{6.27}$$

where the positive constants α *and* C_i*,* $i = 1, 2, 3$*, are independent of* u*,* u_0*,* g*,* $\varepsilon \to 0$*,* T*, and* x_0 (*we recall that we write* θ_ε *instead of* θ_{ε,x_0} *for brevity*).

To deduce (6.27) from (6.13), it suffices to use (6.26), take the supremum over $t \in [0,T]$, and use (2.13).

We are now ready to prove the existence of a bounded solution of the Leray–Navier–Stokes problem (6.1).

Theorem 6.5. *Let the above assumptions hold. Then the problem* (6.1) *possesses at least one solution* $u \in \mathbb{W}_b([0,T]\times\Omega)$ *which satisfies the following estimate*:

$$\|u\|_{L^\infty([0,T],L^2_b(\Omega))\cap L^2_b([0,T],W^{1,2}_b(\Omega))}$$
$$\leqslant C(1 + \|u_0\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T]\times\Omega)}), \tag{6.28}$$

where the constant C *is independent of small* $\alpha > 0$ T*,* g*, and* u_0.

Proof. The idea based on the following observation. Let

$$K_{u_0,g} := (1 + \|u_0\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T]\times\Omega)})^{1/2}. \tag{6.29}$$

Then the a priori estimate (6.27) gives the following conditional result. Let the solution u a priori satisfy the estimate

$$\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} \leqslant \frac{C_1}{2C_2\varepsilon}. \tag{6.30}$$

Then

$$\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega)} + C_1/2\|u\|_{L^2_b([0,T],W^{1,2}_b(\Omega))} \leqslant C_3^{1/2}\varepsilon^{-1/2}K_{u_0,g}. \tag{6.31}$$

We fix $\varepsilon \ll 1$ such that

$$C_3^{1/2}\varepsilon^{-1/2}K_{g,u_0} < \frac{C_1}{2C_2\varepsilon} \tag{6.32}$$

or, which is the same,

$$\varepsilon \sim [K_{u_0,g}]^{-2}. \tag{6.33}$$

In this case, from the estimates (6.30) and (6.31) we can deduce estimate of the form (6.28) using the standard continuation by parameter arguments. Let u^s, $s \in [0,1]$, be a continuous curve of solutions of (6.1) such that

$$K_{u_0^s, g^s} \leqslant K_{u_0^1, g^1} \tag{6.34}$$

and the estimate (6.31) is satisfied for $s = 0$. Then it is satisfied for $s = 1$ as well since, due to (6.32), we cannot achieve the bound (6.30) before crossing the bound (6.31) and, consequently, the continuity arguments show that (6.31) holds for every $s \in [0,1]$.

Let us proceed in a more rigorous way. For this purpose, we first prove the estimate (6.28) in the square integrable case:

$$u_0 \in \mathcal{H}^2(\Omega), \quad g \in L^2([0,T], L^2(\Omega)). \tag{6.35}$$

In this case, the Leray–Navier–Stokes problem has a unique square integrable solution u,

$$u \in C([0,T], L^2(\Omega)) \cap L^2([0,T], W^{1,2}(\Omega)), \tag{6.36}$$

which can be obtained in the same way as in the case of bounded domains. Moreover, this solution depends continuously (in the metric of (6.36)) on the initial data u_0 and external forces g, which can be verified in a standard way since the α-regularization makes the inertial term subordinated to the linear part of the equation (see, for example, [**4, 5, 28**]).

Thus, the solutions u^s, $s \in [0,1]$, associated with the initial data $u_0^s := su_0$, $g^s := sg$ generate a continuous curve in the space (6.36) and, evidently, (6.31) is satisfied for $u^0 \equiv 0$. Therefore, by the above continuity arguments, we have the estimate (6.31) for $s = 1$. Taking into account (6.33), we can write it as follows:

$$\|u\|_{L^\infty([0,T], L^2_{\theta_\varepsilon, x_0}(\Omega)) \cap L^2_b([0,T], W^{1,2}_{\theta_\varepsilon, x_0}(\Omega))} \leqslant C[K_{u_0,g}]^2, \tag{6.37}$$

where the constant C is independent of $x_0 \in \mathbb{R}$. Using the obvious estimate

$$\|v\|_{W^{l,2}_b(\Omega)} \leqslant C \sup_{x_0 \in \mathbb{R}} \|v\|_{W^{l,2}_{\theta_\varepsilon, x_0}(\Omega)}, \quad l = 0, 1,$$

where C is independent of $\varepsilon \ll 1$, we deduce the required estimate (6.28).

Thus, the assertion of the theorem is verified in the square integrable case (6.35). Consider the general case of u_0 and g satisfying only the assumption (6.4). For this purpose, we approximate u_0 and g by a sequence of square integrable functions u_0^n and g^n satisfying (6.35). Moreover, we assume that

$$\|u_0^n\|_{\mathcal{H}^2_b(\Omega)} + \|g^n\|_{L^2_b([0,T]\times\Omega)} \leqslant C, \tag{6.38}$$

where C is independent of n and

$$u_0^n \to u_0 \text{ in } L^2_{loc}(\overline{\Omega}), \quad g^n \to g \text{ in } L^2_{loc}([0,T]\times\overline{\Omega}). \tag{6.39}$$

Then, due to already proved part of the estimate (6.28), the associated solution u^n of the Navier–Stokes equation (belonging to the class (6.36)) satisfies the estimate

$$\|u^n\|_{L^\infty([0,T],L^2_b(\Omega))} + \|u^n\|_{L^2_b([0,T],W^{1,2}_b(\Omega))} \leqslant C_1, \tag{6.40}$$

where C_1 is independent of n. Moreover, from Equation (6.9), we infer

$$\|\partial_t u^n\|_{L^2_b([0,T],\mathcal{H}^{-1,2}_b(\Omega))} \leqslant C. \tag{6.41}$$

Passing to a subsequence if necessary, we can assume without loss of generality that the sequence u^n converges weakly to some $u \in \mathbb{W}_b([0,T]\times\Omega)$ in the local topology, i.e., for every square integrable weight φ satisfying (5.17)

$$u^n \to u \text{ weakly in } \mathbb{W}_\varphi([0,T]\times\Omega). \tag{6.42}$$

Moreover, due to the embedding $\mathbb{W}_\varphi([0,T]\times\Omega) \subset C([0,T],L^2_\varphi(\Omega))$ (see Theorem 5.4), the limit function u satisfies the initial condition $u(0)=u_0$.

Thus, we only need to verify that the constructed function u satisfy Equation (6.1) (or, which is the same, Equation (6.9)) in the sense of distributions, i.e., we need to verify that for every $U \in C_0^\infty((0,T)\times\Omega)$ with $\operatorname{div} U = 0$

$$-\langle u, \partial_t U\rangle = \langle u, \Delta_x U\rangle - \langle (\Pi w, \nabla_x)u, U\rangle + \langle g, U\rangle \tag{6.43}$$

(the passage to the limit in the linear equation $w^n - \alpha\Delta_x w^n = u^n$ is obvious). Since u^n solves the Leray–Navier–Stokes equations, we have

$$-\langle u^n, \partial_t U\rangle = \langle u^n, \Delta_x U\rangle - \langle (\Pi w^n, \nabla_x)u^n, U\rangle + \langle g^n, U\rangle\,. \tag{6.44}$$

The passage to the limit $n\to\infty$ in all the linear terms of (6.44) is evident, and we only need to pass to the limit in the inertial term $(\Pi w^n, \nabla_x)u^n$. It suffices to show that

$$u^n \to u \ \text{ strongly in the space } \ L^2_{loc}([0,T]\times\overline{\Omega}) \tag{6.45}$$

since $\nabla_x u^n \to \nabla_x u$ weakly in $L^2_{loc}([0,T]\times\overline{\Omega})$. By Proposition 3.1, Theorem 4.4, and the convergence (6.45), we have the analogous strong convergence of Πw^N to Πw. This implies the weak convergence

$$(\Pi w^n, \nabla_x)u^n \to (\Pi w, \nabla_x)u \text{ in } L^1_{loc}([0,T]\times\overline{\Omega}).$$

To prove (6.45), we note that, by (6.42), for every integrable weight φ

$$\partial_t u^n \to \partial_t u \text{ weakly in } L^2([0,T],\mathcal{H}^{-1,2}_{\varphi^2}(\Omega)). \tag{6.46}$$

Furthermore, by (6.42), we have

$$u^n \to u \text{ weakly in } L^2([0,T], \mathcal{V}^2_\varphi(\Omega)). \tag{6.47}$$

Since we have the standard embeddings

$$\mathcal{V}^2_\varphi(\Omega) \subset\subset \mathcal{H}^2_{\varphi^2}(\Omega) \subset \mathcal{H}^{-1,2}_{\varphi^2}(\Omega)$$

and the first embedding is compact, by the compactness theorem (see, for example, [**27**]), we have the strong convergence

$$u^n \to u \text{ in } L^2([0,T], \mathcal{H}^2_{\varphi^2}(\Omega)).$$

Thus, the convergence (6.45) holds and, consequently, Theorem 6.5 is proved. $\square$

We now return to the general case of nonzero flux $c \neq 0$ in (6.2). Then the Leray–Navier–Stokes equation (6.1) with $g = 0$ possesses the classical Poiseuille solution

$$\vec{v}_c(x) := c\left(\frac{|\omega|}{\|v_c\|_{L^1(\omega)}} v_c(x'), 0, 0\right), \tag{6.48}$$

where the scalar function $v_c = v_c(x')$ solves the Laplace equation in ω:

$$\Delta_{x'} v_c = 1, \quad v_c\big|_{\partial\omega} = 0. \tag{6.49}$$

Indeed, the associated vector field $\vec{w}_c$ has the form $\vec{w}_c(x) = (w_c(x'), 0, 0)$, where $w_c(x')$ solves the problem

$$w_c - \alpha\Delta_{x'} w_c = v_c, \quad w_c\big|_{\partial\omega} = 0. \tag{6.50}$$

In particular, the vector field $\vec{w}_c$ is divergence-free and, consequently,

$$\Pi\vec{w}_c = \vec{w}_c - (\mathbb{S}w_c, 0, 0) = (\bar{w}_c, 0, 0). \tag{6.51}$$

Using (6.51), one can immediately verify that $\vec{v}_c$ solves the Leray–Navier–Stokes problem (6.1), (6.2) with $g \equiv 0$.

Therefore, the difference $u - \vec{v}_c$ has zero flux and, consequently, it is natural to define a weak solution of (6.1) as a function $u \in \vec{v}_c + \mathbb{W}_b([0,T]\times\Omega)$ which satisfies (6.1) in the sense of distributions over the divergence-free vector fields. Moreover, the assumption on u_0 should be also naturally replaced with $u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega)$. The following assertion is an analog of Theorem 6.5 in the case of nonzero flux.

Theorem 6.6. *Let the above assumptions hold. Then for every $c \in \mathbb{R}$, $u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega)$, and $g \in L^2_b([0,T] \times \Omega)$ the Navier–Stokes problem* (6.1), (6.2) *possesses at least one weak solution* $u \in v_c + \mathbb{W}_b([0,T]\times\Omega)$ *which satisfies the following estimate*:

$$\begin{aligned}&\|u\|_{L^\infty([0,T],L^2_b(\Omega))\cap L^2_b([0,T],W^{1,2}_b(\Omega))}\\&\leqslant C(1+|c|^3+\|u_0\|^2_{L^2_b(\Omega)}+\|g\|^2_{L^2_b([0,T]\times\Omega)}),\end{aligned}\tag{6.52}$$

where the constant C *is independent of* α, T, u_0, g, *and* c.

Proof. We want to reduce the general case to the particular case of zero flux considered above. The most natural way to do so is to make the change of variables $\bar u := u - \vec v_c$, where $\vec v_c$ is the Poiseuille flow. However, this scheme does not work since the Poiseuille flow can be unstable. Instead, we construct a special solution of the stationary Navier–Stokes problem (6.1), (6.2) of the form $V_c(x) := (V_c(x'),0,0)$, $V_c\big|_{\partial\omega} = 0$ (with the appropriate nonzero external force g_c) and introduce a new unknown $\bar u := u - V_c$. Then this function belongs to $\mathbb{W}_b([0,T]\times\Omega)$ and solves the problem

$$\begin{cases}\partial_t\bar u + (\Pi\bar w,\nabla_x)\bar u = \Delta_x\bar u + L_{V_c}\bar u - \nabla_x p + g - g_c,\\ \bar w - \alpha\Delta_x\bar w = \bar u,\quad \bar w\big|_{\partial\omega} = 0,\\ \operatorname{div}\bar u = 0,\quad \bar u\big|_{\partial\Omega} = 0,\quad \mathbb{S}\bar u_1 \equiv 0,\\ \bar u\big|_{t=0} = \bar u_0 := u_0 - V_c,\end{cases}\tag{6.53}$$

which differs from (6.1) by the presence of the linear operator L_{V_c},

$$L_{V_c}z := (\Pi W_c,\nabla_x)z + (\Pi w,\nabla_x)V_c - c\partial_{x_1}z,\; w - \alpha\Delta_x w = z.\tag{6.54}$$

The following assertion specifies the choice of the special function V_c.

Lemma 6.7. *Let* $c\in\mathbb{R}$ *be arbitrary. Then there exists a vector field* $V_c(x) = (V_c(x'),0,0)$, $V_c\big|_{\partial\omega} = 0$ *such that*

$$(L_{V_c}z,z)\leqslant 1/2\|z\|^2_{W^{1,2}(\Omega)},\ \forall w\in W^{1,2}_0(\Omega)\tag{6.55}$$

and

$$\|V_c\|_{C(\omega)}\leqslant \varkappa|c|,\ \|\nabla_{x'}V'_c\|_{L^2(\omega)}\leqslant\varkappa(|c|^{3/2}+|c|),\tag{6.56}$$

where the constant $\varkappa$ *is independent of* c *and* α, *and* $g_c = -\Delta_{x'}V_c$.

Proof. Suppose that $\delta > 0$ is small and $\omega_\delta := \{x'\in\omega,\ \operatorname{dist}(x',\partial\omega) < \delta\}$. As is known, ω_δ is a smooth subdomain of ω if δ is small enough.

We seek for the required function $V_c(x')\in W^{1,2}_0(\omega)$ in the form

$$V_c(z) = \begin{cases}\lambda, & z\in\omega\backslash\omega_\delta,\\ \lambda\delta^{-1}\operatorname{dist}(z,\partial\omega), & z\in\omega_\delta,\end{cases}\tag{6.57}$$

where $\delta \ll 1$ is a small positive constant and λ is some parameter. To satisfy the flux condition, we need

$$|\omega|c = \int\limits_{\omega} V_c(z)\,dz = \lambda(|\omega| - |\omega_\delta| + \delta^2/2|\partial\omega|). \tag{6.58}$$

We fix below $\delta \sim |c|^{-1}$. Then formula (6.58) shows that $\lambda = c + o(c^{-1})$.

So, we need to fix δ such that (6.55) is satisfied. Let $w \in [W_0^{1,2}(\Omega)]^2$. A direct calculation gives

$$\begin{aligned}(L_{V_c} z, z) &= ((\Pi w)_2 \partial_{x_2} V_c + (\Pi w)_3 \partial_{x_3} V_c, z_1) \\ &= \lambda\delta^{-1}(\Pi w.\vec{n}, z_1)_{L^2(\mathbb{R}\times\omega_\delta)},\end{aligned} \tag{6.59}$$

where $\vec{n}(x') := \nabla_{x'} \operatorname{dist}(x', \partial\omega)$. Since $\vec{n}\big|_{\partial\omega}$ coincides with a normal vector to $\partial\omega$, the function $Z := \Pi w.\vec{n}$ has zero trace at $\partial\omega$: $Z\big|_{\partial\omega} = 0$ (we implicitly used that $l_n u = 0$ for every $u \in \mathcal{H}^2(\Omega)$). Thus, (6.59) can be written in the form

$$|(L_{V_c} z, z)| \leqslant \lambda\delta^{-1} \int\limits_{\mathbb{R}} \int\limits_{\omega_\delta} |Z(x_1, x')|^2 + |z(x_1, x')|^2\,dx'\,dx_1. \tag{6.60}$$

Using the standard estimate

$$\int\limits_{\omega_\delta} |u(x')|^2\,dx' \leqslant C\delta^2 \int\limits_{\omega_\delta} |\nabla_{x'} u(x)|^2\,dx'$$

which holds for every $u \in W^{1,2}(\omega_\delta)$ such that $u\big|_{\partial\omega} = 0$, we transform (6.60) as follows:

$$|(L_{V_c} w, w)| \leqslant C\lambda\delta(\|Z\|^2_{W^{1,2}(\Omega)} + \|z\|^2_{W_0^{1,2}(\Omega)}), \tag{6.61}$$

where C is independent of λ and δ.

Furthermore, according to Theorem 4.4, we have

$$\|Z\|_{W^{1,2}(\Omega)} \leqslant C\|w\|_{W^{1,2}(\Omega)}.$$

We now recall that w solves the Laplace equation

$$w - \alpha\Delta_x w = z,\ \ z\big|_{\partial\Omega} = 0. \tag{6.62}$$

Therefore, multiplying (6.62) by $\Delta_x w$ and using that $u\big|_{\partial\Omega} = 0$, we infer

$$\|w\|_{W^{1,2}(\Omega)} \leqslant C\|z\|_{W^{1,2}(\Omega)}.$$

where the constant C is independent of α. Finally,

$$|(L_{V_c} z, z)| \leqslant C'\lambda\delta\|z\|^2_{W_0^{1,2}(\Omega)}$$

with C' independent of α, λ and δ. So, it remains to fix δ such that $C'\lambda\delta \leqslant 1/2$. Since λ should be close to c, this gives the following estimate for the desired δ:

$$\delta \sim C|c|^{-1} \tag{6.63}$$

with some constant C independent of c and λ. It is not difficult to verify that the function V_c thus defined satisfies also the inequalities (6.56). Lemma 6.7 is proved. □

We are now ready to complete the proof of Theorem 6.6. The arguments repeat with minor modifications the proof of Theorem 6.5 in the case of zero flux. The only difference is that we now have the additional linear term $L_{V_c}\bar{u}$ in Equation (6.53), which is not essential because of the estimate (6.55).

Indeed, proving an analog of the basic a priori estimate (6.13), we have the additional terms

$$(L_{V_c}\bar{u}, \theta_\varepsilon^2\bar{u}) - (L_{V_c}\bar{u}, v) - (\nabla_{x'}V_c, \nabla_{x'}(\theta_\varepsilon^2\bar{u} - v)) \tag{6.64}$$

on the right-hand side of (6.15). To estimate the first term in (6.64), we use the commutation relation

$$|(\theta_\varepsilon L_{V_c}\bar{u} - L_{V_c}(\theta_\varepsilon\bar{u}), \theta_\varepsilon\bar{u})| \leqslant \varkappa|c|\varepsilon\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \tag{6.65}$$

for some $\varkappa$ independent of α, ε and c. We begin with the most complicated second term in the expression (6.54) for L_{V_c}. Let w_θ solve the problem

$$w_\theta - \alpha\Delta_x w_\theta = \theta_\varepsilon\bar{u}, \quad w_\theta\big|_{\partial\Omega} = 0.$$

Then the difference $\theta_\varepsilon w - w_\theta$ solves the equation

$$(\theta_\varepsilon w - w_\theta) - \alpha\Delta_x(\theta_\varepsilon - w_\varepsilon) = H_\varepsilon := -2\alpha\theta_\varepsilon'\partial_{x_1}w - \alpha\theta_\varepsilon'' w. \tag{6.66}$$

Using the estimate (6.12) for the derivatives of θ_ε and Proposition 3.3 for estimating w, we infer

$$\|H_\varepsilon\|_{W^{1,2}(\Omega)} \leqslant \varkappa\varepsilon\|\bar{u}\|^2_{L^2(\Omega)},$$

where $\varkappa$ is independent of α and ε. Moreover, since $H_u\big|_{\partial\Omega} = 0$, multiplying Equation (6.66) by $\Delta_x(\theta_\varepsilon - w_\theta)$, we can write

$$\|\theta_\varepsilon w - w_\theta\|_{W^{1,2}(\Omega)} \leqslant C\|H_\varepsilon\|_{W^{1,2}(\Omega)} \leqslant \varkappa_1\varepsilon\|\bar{u}\|_{L^2_{\theta_\varepsilon}(\Omega)}.$$

Furthermore, using this estimate together with Theorem 4.4 and an analog of Proposition 4.6 for θ_ε, we find

$$\|\theta_\varepsilon\Pi w - \Pi w_\theta\|_{W^{1,2}(\Omega)} \leqslant \varkappa_2\varepsilon\|\bar{u}\|_{L^2_{\theta_\varepsilon}(\Omega)}. \tag{6.67}$$

Thus, for the commutator of the second term in the expression (6.54), we get

$$|((\theta_\varepsilon \Pi w - \Pi w_\theta, \nabla_x V_c, \theta_\varepsilon \bar{u})|$$
$$\leqslant 2\|V_c\|_{L^\infty}\|\|\theta_\varepsilon \Pi w - \Pi w_\theta\|_{W^{1,2}(\Omega)}\|\bar{u}\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)} \leqslant \varkappa|c|\varepsilon\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$

where we integrated by parts in order to avoid the terms $\nabla_x V_c$ and used the estimate (6.56). Thus, the estimate (6.66) is verified for the second term in (6.54). This estimate is obvious for the third term. Regarding the first term, it suffices to note that $\|\Pi W_c\|_{L^\infty(\omega)} \leqslant \varkappa|c|$ by (6.56) and the maximum principle applied to the equation for W_c. Thus, the estimate (6.66) is proved.

Using the estimate (6.66) and Lemma 6.7, we estimate the first additional term in (6.64) as follows:

$$|(L_{V_c}\bar{u}, \theta_\varepsilon^2 \bar{u})| \leqslant |(L_{V_c}(\theta_\varepsilon \bar{u}), \theta_\varepsilon \bar{u})| + \varkappa|c|\varepsilon\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}$$
$$\leqslant 1/2\|\nabla_x(\theta_\varepsilon \bar{u})\|^2_{L^2(\Omega)} + \varkappa|c|\varepsilon\|u\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}, \tag{6.68}$$

where the constant $\varkappa$ is independent of $\bar{u}$, c, ε, and α.

The second additional term of (6.64) can be then estimated in the following way:

$$|L_{V_c}\bar{u}, v)| \leqslant |((\Pi w, \nabla_x)V_c, v)| + |((\Pi W_c, \nabla_x)\bar{u}, v)| + |c(\partial_{x_1} u, v)|$$
$$\leqslant 2\|V_c\|_{L^\infty}\|\Pi w\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}$$
$$+ \|\Pi W_c\|_{L^\infty}\|\bar{u}\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)} + |c|\|\bar{u}\|_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\|v\|_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}$$
$$\leqslant \varkappa(|c|+1)(\varepsilon\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \varepsilon^{-1}\|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}). \tag{6.69}$$

where the constant $\varkappa$ is independent of ε, c, α, u, and v.

Finally, the third additional term of (6.64) can be estimated with the help of (6.56) and the Hölder inequality:

$$|(\nabla x' V_c, \nabla_{x'}(\theta_\varepsilon^2 \bar{u} - v))|$$
$$\leqslant C_\beta\|\nabla_{x'} V_c\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + \beta(\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)})$$
$$\leqslant \varkappa_\beta(c^3+1)\varepsilon^{-1} + \beta(\|\bar{u}\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)} + \|v\|^2_{W^{1,2}_{[\theta_\varepsilon]^{-1}}(\Omega)}), \tag{6.70}$$

where $\beta > 0$ is arbitrary and the constant $\varkappa_\beta$ depends on δ, but is independent of c, ε, α, u, and v.

The estimates (6.68)–(6.70) show that, under the additional assumption

$$|c|\varepsilon \leqslant \varkappa, \tag{6.71}$$

where $\varkappa > 0$ is a sufficiently small number independent of c, α, and ε (we recall that $v \sim \varepsilon[\theta_\varepsilon]^2\bar{u}$ by (6.18)), we can repeat word-by-word the proof of (6.13) to obtain the following analog of (6.27):

$$\begin{aligned}\|\bar{u}\|^2_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))} + (C_1 - C_2\varepsilon\|\bar{u}\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\|\bar{u}\|^2_{L^2_b([0,T],W^{1,2}_{\theta_\varepsilon}(\Omega))}\\ \leqslant C_3\varepsilon^{-1}(1+|c|^3+\|\bar{u}(0)\|^2_{L^2_b(\Omega)} + \|g\|^2_{L^2_b([0,T],L^2_b(\Omega))}),\end{aligned} \tag{6.72}$$

where the positive constants and C_i, $i = 1, 2, 3$, are independent of u, u_0, g, α $\varepsilon \to 0$, T, c, and x_0.

Furthermore, arguing exactly as in the proof of estimate (6.28), we deduce the a priori estimate (6.52) (see (6.29)–(6.33)). The existence of a solution can be than verified exactly as in the case of zero flux c. Theorem 6.6 is proved. □

Remark 6.8. Arguing in the same way as above, we can establish the existence of a solution of the following more general Leray–Navier–Stokes problem with the *nonautonomous* flux

$$\mathbb{S}u_1(t) \equiv c(t), \tag{6.73}$$

where $c \in C^1([0,T])$ is an arbitrary given function. Moreover, the assumption on the external force g is also can be relaxed till

$$g \in L^2_b([0,T], \mathcal{H}^{-1,2}_b(\Omega)). \tag{6.74}$$

Furthermore, the weighted theory developed in this section allows us to consider not only bounded with respect to $x_1 \to \infty$ solutions, but also slowly growing solutions of the NS equation (growing not faster than $|x_1|^{1/2-\delta}$, where $\delta > 0$ is arbitrary). We however will not use these facts in the sequel and, by this reason, do not give their rigorous proofs here.

7. Leray–Navier–Stokes Equations: Uniqueness and Dissipativity

In this section, we prove the uniqueness of a spatially nondecaying solution of the Leray approximations and verify the dissipativity of this system in

$\mathcal{H}_b^2(\Omega)$. We start with the uniqueness which is now almost obvious (due to the regularization of the inertial term).

Theorem 7.1. *Let the assumptions of Theorem* 6.6 *hold. Then there exists positive* μ *such that for any two solutions* $u_1, u_2 \in \vec{v}_c + \mathbb{W}_b([0,T]\times\Omega)$ *of the problem* (6.1) *and every weight function* φ *of sufficiently small exponential growth rate* $\varepsilon \leqslant \mu$ *the following estimate holds*:

$$\|u_1(t) - u_2(t)\|_{L^2_\varphi(\Omega)} \leqslant Ce^{Kt}\|u_1(0) - u_2(0)\|_{L^2_\varphi(\Omega)}, \tag{7.1}$$

where the constants K *and* C *depend on the* L^2_b*-norms of* $u_1(0)$ *and* $u_2(0)$, g, $\alpha > 0$, *and the constant* C_φ, *but are independent of the choice of* u_1, u_2, *and* φ.

In particular, the energy solution of the Leray–Navier–Stokes is unique. Moreover, a similar estimate holds for $L^2_{b,\varphi}$.

Proof. The arguments are based on the solvability result for the linear Stokes problem

$$\begin{aligned} &\partial_t v - \Delta_x v + \nabla_x q = h(t), \quad v\big|_{\partial\Omega} = 0, \quad \operatorname{div} v = 0,\\ &\mathbb{S}v_1 = 0, \quad v\big|_{t=0} = v_0. \end{aligned} \tag{7.2}$$

Lemma 7.2. *Let* Ω *be a cylindrical domain. Then there exists* $\mu_0 > 0$ *such that for any weight function* φ *of exponential growth rate* $\mu \leqslant \mu_0$, $v_0 \in \mathcal{H}^2_\varphi(\Omega)$, *and* $h \in L^2_{loc}(\mathbb{R}_+, \mathcal{H}^{-1,2}_\varphi(\Omega))$ *Equation* (7.2) *has a unique weighted energy solution* v *and the following estimate holds*:

$$\begin{aligned} &\|v(T)\|^2_{L^2_\varphi(\Omega)} + \int_0^T e^{-\beta(T-s)}\|v(s)\|^2_{W^{1,2}_\varphi(\Omega)}\,ds\\ &\qquad \leqslant Ce^{-\beta T}\|v(0)\|^2_{L^2_\varphi(\Omega)} + C\int_0^T e^{-\beta(T-s)}\|h(s)\|^2_{\mathcal{H}^{-1,2}_\varphi(\Omega)}\,ds, \end{aligned} \tag{7.3}$$

where the positive constants C *and* β *depend on* Ω *and* C_φ, *but are independent of* v *and* h.

Proof. The a priori estimate (7.3) can be verified on the basis of the energy identity of Theorem 5.4 in the same way as in the proof of (6.19). The existence of a solution can be proved in the same way as in Theorem 6.5. Thus, Lemma 7.2 is proved. □

Now, we are ready to complete the proof of the theorem. Let u_1 and u_2 be two solutions of Equation (6.1) in $\vec{v}_c+\mathbb{W}_b(\mathbb{R}\times\Omega)$. Then the difference $v := u_1 - u_2$ belongs to $\mathbb{W}_b(\mathbb{R}\times\Omega)$ and solves Equation (7.2) with

$$h(t) := -(\Pi w_1, \nabla_x)v - (\Pi w, \nabla_x)u_2 - c\partial_{x_1}v, \tag{7.4}$$

where $w_i - \alpha\Delta_x w_i = u_i$, $i = 1, 2$, $w := w_1 - w_2$. Now, we estimate the function h in the $W^{-1,2}_{\varphi_\varepsilon}(\Omega)$-norm with $\varphi_\varepsilon = \varphi_{\varepsilon,x_0}$. Integrating by parts and using the Hölder inequality and embedding $W^{1,2} \subset L^6$, we infer

$$\begin{aligned}|((\Pi w_1, \nabla_x)v, \varphi_\varepsilon^2 W)| \leqslant |\nabla_x \Pi w_1\|_{W^{1,3}_b(\Omega)}\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}\|W\|_{W^{1,2}_{\varphi_\varepsilon}(\Omega)}\\ + \|\Pi w_1\|_{L^\infty(\Omega)}\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}\|Z\|_{W^{1,2}_{\varphi_\varepsilon}(\Omega)},\end{aligned}$$

which holds for every $Z \in W^{1,2}(\Omega)$ and, consequently,

$$\|(\Pi w_1, \nabla_x)v\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \leqslant (\|\Pi w_1\|_{W^{1,3}_b(\Omega)} + \|\Pi w_1\|_{L^\infty(\Omega)})\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}.$$

Moreover, since u_1 is bounded in $L^\infty(\mathbb{R}, L^2_b(\Omega))$ (we recall that $u_1 \in \vec{v}_c + \mathbb{W}(\mathbb{R}_+)$, see Theorem 5.4), w_1 is bounded in $L^\infty(\mathbb{R}_+, W^{2,2}_b(\Omega))$ and, consequently, by Theorem 4.4,

$$\|\Pi w_1\|_{W^{1,3}_b(\Omega)} + \|\Pi w\|_{L^\infty(\Omega)} \leqslant C\|u_1\|_{L^2_b(\Omega)} \leqslant C_1, \tag{7.5}$$

where C_1 depends on α. Thus,

$$\|(\Pi w_1, \nabla_x)v\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \leqslant C_1\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}. \tag{7.6}$$

The second term in (7.4) can be estimated in a similar way; the only difference is that one should use a weighted analog of (7.5) and the unweighted estimate for u_2. This gives us the estimate (7.6) for the second term. Finally, a similar estimate for the third term is immediate, and we have

$$\|h(t)\|_{W^{-1,2}_{\varphi_\varepsilon}(\Omega)} \leqslant C\|v\|_{L^2_{\varphi_\varepsilon}(\Omega)}.$$

Using this estimate together with (7.3), we find

$$\|v(T)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)} \leqslant C\|v(0)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)} + C\int_0^T e^{-\beta(T-s)}\|v(s)\|^2_{L^2_{\varphi_\varepsilon}(\Omega)}\,ds.$$

Applying the Gronwall inequality to this estimate, we complete the proof. $\square$

We recall the uniformly-local analog of the smoothing property for solutions of the Leray–Navier–Stokes equations.

Theorem 7.3. *Let the assumptions of Theorem* 6.6 *be satisfied, and let* $u \in v_c + \mathbb{W}_b([0,T] \times \Omega)$ *be a weak solution of* (6.1) *constructed in this theorem. Then*

$$t^{1/2}u(t) \in L^\infty([0,T], W_b^{1,2}(\Omega)) \cap L_b^2([0,T], W_b^{2,2}(\Omega)) \tag{7.7}$$

and the following estimate holds:

$$t\|u(t)\|^2_{W_b^{1,2}(\Omega)} \leqslant Q(\|u_0\|_{L_b^2(\Omega)} + \|g\|_{L_b^2([0,T]\times\Omega)}), \quad t \in [0,1], \tag{7.8}$$

where the monotone function Q *depends on* α*, but is independent of* u*.*

The proof of this theorem is more or less standard and is based on the multiplication of Equation (6.1) by the expression

$$t\Pi\left(\partial_{x_1}(\varphi_\varepsilon^2 \partial_{x_1} u) + \varphi_\varepsilon^2 \partial_{x_2}^2 u + \varphi_\varepsilon^2 \partial_{x_3}^2 u\right) \tag{7.9}$$

(see the proof of Theorem 5.1). Therefore, we omit a rigorous proof of this result.

Our goal is to verify that the Leray–Navier–Stokes problem (6.1) generates a dissipative dynamical system in the corresponding phase space and obtain a *dissipative* estimate for the solutions of the problem which will be *uniform* with respect to $\alpha \to 0$. This dissipative estimate will be used in the following section in order to prove the dissipativity of the classical Navier–Stokes problem with $\alpha = 0$.

For the sake of simplicity, we restrict ourselves to the autonomous case:

$$g(t) \equiv g \in [L_b^2(\Omega)]^2. \tag{7.10}$$

By Theorems 6.6 and 7.1, the Leray–Navier–Stokes problem (6.1) generates (for all $\alpha > 0$) semigroups $S_\alpha(t) = S_{c,\alpha}(t)$ in the phase spaces

$$\Phi_b := \Phi_b(c) = \vec{v}_c + \mathcal{H}_b^2(\Omega) \tag{7.11}$$

via the standard expression

$$S_\alpha(t)u_0 := u(t), \quad S_\alpha(t_1 + t_2) = S_\alpha(t_1) \circ S_\alpha(t_2), \quad t_1, t_2 \geqslant 0. \tag{7.12}$$

The following theorem, which gives a dissipative estimate for the solutions of the Leray–Navier–Stokes problem, can be considered as the main result of the section.

Theorem 7.4. *Suppose that the assumptions of Theorem* 6.6 *hold and, in addition,* (7.10) *is satisfied. Then there exist positive constants* β *and* K *and a monotone function* Q *such that for every weak energy solution*

$u(t)$ of the Leray–Navier–Stokes problem (6.1)–(6.2) *the following uniform estimate holds*:

$$\|u(t)\|_{L^2_b(\Omega)} \leqslant Q(\|u(0)\|_{L^2_b(\Omega)} + \|g\|_{L^2_b(\Omega)})e^{-\beta t} \\ + K(1+|c|^3 + \|g\|^2_{L^2_b(\Omega)}) \tag{7.13}$$

(*we emphasize that the constant K in* (7.13) *is independent of α, t, $\|u(0)\|_{L^2_b(\Omega)}$, and the flux $c = \mathbb{S}u_1(0)$*).

Proof. To verify (7.13), it suffices to prove that the ball

$$\mathcal{B} := \{u_0 \in [L^2_b(\Omega)]^3,\ \|u_0\|_{L^2_b(\Omega)} \leqslant K(1+|c|^3 + \|g\|^2_{L^2_b(\Omega)})\} \tag{7.14}$$

is an absorbing set for the Leray–Navier–Stokes problem (6.1), i.e., for every bounded subset $B \subset \Phi$ there exists $T = T(\|B\|_\Phi, \|g\|_{L^2_b(\Omega)})$ such that

$$S(t)B \subset \mathcal{B} \quad \forall t \geqslant T. \tag{7.15}$$

For the sake of simplicity, we restrict ourselves to the case of zero flux $c = 0$. The general case can be reduced to this particular case in the same way as in Theorem 6.6.

The proof of the embedding (7.15) requires a little more detailed analysis of the basic a priori estimate (6.13) which can be written in the following more convenient way:

$$\|u(t)\|^2_{L^2_{\theta_\varepsilon}(\Omega)} + (C_1 - C_2\varepsilon\|u\|_{L^\infty([0,T],L^2_{\theta_\varepsilon}(\Omega))})\int_0^T e^{-\beta|t-s|}\|u(s)\|^2_{W^{1,2}_{\theta_\varepsilon}(\Omega)}\,ds$$

$$\leqslant C_3^2(e^{-\beta t}\|u(0)\|_{L^2_{\theta_\varepsilon}(\Omega)} + \|g\|_{L^2_{\theta_\varepsilon}(\Omega)})^2, \tag{7.16}$$

where the positive constants β and C_i, $i = 1, 2, 3$, are independent of α, u, u_0, g, $\varepsilon \to 0$, T, and x_0 (in order to deduce (7.16) from (6.13), it suffices to take $s = t$ on the left-hand side).

Lemma 7.5. *Let the assumptions of Theorem* 6.5 *hold and let the initial data $u(0)$ for the problem* (6.1) *satisfy the following conditions*:

$$C_1 - 2C_2C_3\varepsilon(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}) \geqslant 0, \tag{7.17}$$

where all the constants are the same as in (7.16). *Then the associated energy solutions $u(t)$ of the Leray–Navier–Stokes problem* (*with zero flux $c = 0$*) *satisfies the estimate*

$$\|u(t)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} \leqslant C_3(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}e^{-\beta t} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}) \tag{7.18}$$

for all $t \geqslant 0$.

PROOF. The estimate (7.16) implies (7.18) under the additional assumption that

$$C_1 - C_2\varepsilon\|u\|_{L^\infty(\mathbb{R}_+,L^2_{\theta_{\varepsilon,x_0}}(\Omega))} \geqslant 0. \tag{7.19}$$

On the other hand, (7.18) yields

$$\|u\|_{L^\infty(\mathbb{R}_+,L^2_{\theta_{\varepsilon,x_0}}(\Omega))} \leqslant C_3(\|u(0)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}), \tag{7.20}$$

which *formally* implies (7.19). Thus, using the continuity arguments (similar to the proof of Theorem 6.5), we can verify that (7.18) holds if the initial data satisfies (7.17), and Lemma 7.5 is proved. □

Note that, although (7.18) looks like a dissipative estimate (in the phase space $L^2_{\theta_{\varepsilon,x_0}}(\Omega)$), it is not sufficient to complete the proof of the theorem since the exponent $\varepsilon > 0$ *depends on* $\|u(0)\|_{L^2_b(\Omega)}$ (through the assumption (7.17)); namely,

$$\varepsilon \leqslant \varepsilon_0 := C(\|u(0)\|_{L^2_b(\Omega)} + \|g\|_{L^2_b(\Omega)} + 1)^{-2} \tag{7.21}$$

for some positive C (see the proof of Theorem 6.5).

Thus, we need to be able to increase the exponent ε as $t \to \infty$ which is guaranteed by the following assertion.

Lemma 7.6. *Let the above assumptions hold. Then for every bounded subset $B \subset \Phi$ there exists $T = T(\|B\|, \|g\|)$ such that for every $x_0 \in \mathbb{R}$*

$$C_1 - 2C_2C_3\varepsilon(\|u(T)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}) \geqslant 0 \tag{7.22}$$

with $\varepsilon \geqslant \bar{\varepsilon} := L(1 + \|g\|_{L^2_b(\Omega)})^{-2}$ (where the constant L is independent of α, u_0, and g) if $u(0) \in B$.

PROOF. We proceed by an iteration procedure. Let $T_0 = 0$, and let $\varepsilon = \varepsilon_0$ be given by (7.21). Then the estimate (7.22) is satisfied with $\varepsilon = \varepsilon_0$ and $T = T_0$. Assume that (7.22) is already proved for some $T_k > 0$ and $\varepsilon_k := 2^k\varepsilon_0 < \bar{\varepsilon}$. Then we need to prove the existence of $T_{k+1} > T_k$ such that (7.22) is satisfied with $\varepsilon = \varepsilon_{k+1} := 2\varepsilon_k$ and $T = T_{k+1}$. For this purpose, we note that

$$\theta_{2\varepsilon,x_0}(x) := (1 + 4\varepsilon^2|x - x_0|^2)^{-1/2} \leqslant 2(1 + \varepsilon^2|x - x_0|^2)^{1/2} = 2\theta_{\varepsilon,x_0}(x)$$

and, consequently,

$$\|v\|_{L^2_{\theta_{2\varepsilon,x_0}}(\Omega)} \leqslant 2\|v\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}. \tag{7.23}$$

We fix $T_{k+1} > T_k$ such that

$$\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)} \leqslant 2C_3\|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)} \tag{7.24}$$

for all $u(t)$ such that $u(0) \in B$ (it is possible because of our assumptions on ε_k and the "dissipative" estimate (7.18)). The estimates (7.23) and (7.24), together with (6.26), yield

$$\begin{aligned}
&\varepsilon_{k+1}(\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_{k+1},x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon_{k+1},x_0}}(\Omega)})\\
&\leqslant 4\varepsilon_k(\|u(T_{k+1})\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)} + \|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)})\\
&\leqslant 4(2C_3+1)\varepsilon_k\|g\|_{L^2_{\theta_{\varepsilon_k,x_0}}(\Omega)} \leqslant 4(2C_3+1)C\varepsilon_k^{1/2}\|g\|_{L^2_b(\Omega)}\\
&\leqslant 4(2C_3+1)CL^{1/2}\|g\|_{L^2_b(\Omega)}(1+\|g\|_{L^2_b(\Omega)})^{-1} \leqslant 4C(2C_3+1)L^{1/2}.
\end{aligned}$$

Thus, if the constant L is small enough to satisfy $C_1 - 8C_2CC_3(2C_3 + 1)L^{1/2} \geqslant 0$, then the estimate (7.22) is satisfied with $T = T_{k+1}$ and $\varepsilon = \varepsilon_{k+1} = 2\varepsilon_k$. Thus, the iteration completes the proof of the lemma. $\square$

Now, it is not difficult to complete the proof of the theorem. By Lemma 7.6 and the estimate (7.18), there exists $T = T(\|B\|, \|g\|)$ such that

$$\|u(t)\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} \leqslant 2C_3\|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)}, \quad t \geqslant T, \tag{7.25}$$

where $\varepsilon \geqslant \bar\varepsilon := L(1+\|g\|_{L^2_b(\Omega)})^{-2}$, uniformly with respect to $x_0 \in \mathbb{R}$. Taking the supremum over $x_0 \in \mathbb{R}$ of both sides of the inequality (7.25) and using again (6.26), we arrive at the estimate

$$\|u(t)\|_{L^2_b(\Omega)} \leqslant 2C_3CL^{-1/2}\|g\|_{L^2_b(\Omega)}(1+\|g\|_{L^2_b(\Omega)}), \quad t \geqslant T, \tag{7.26}$$

which shows that the ball (7.14) is an absorbing set if $K \geqslant 2C_3CL^{-1/2}$. Theorem 7.4 is proved. $\square$

Remark 7.7. The intermediate estimate (7.25) gives a slightly more information on the solutions than the final estimate (7.26). Assume that $c = 0$ and g is square integrable $g \in [L^2(\Omega)]^3$. Then, instead of (6.26), we have $\|g\|_{L^2_{\theta_{\varepsilon,x_0}}(\Omega)} \leqslant C\|g\|_{L^2(\Omega)}$, where the constant C is *independent* of ε. Thus, instead of (7.26), we have the following better estimate:

$$\|u(t)\|_{L^2_b(\Omega)} \leqslant 2C_3C\|g\|_{L^2(\Omega)}, \quad t \geqslant T,$$

for the radius of the absorbing set (which grows *linearly* with respect to g, in contrast to the quadratic growth rate in the general case).

Now, we are in a position to prove the existence of a global attractor for semigroups (7.12) associated with the Leray–Navier–Stokes equation. Note that, in contrast to the dissipative systems in bounded domains, in the unbounded case, the global attractor is usually *not compact* in the initial phase space (Φ_b in our case). That is reason why one needs to use the following weaker definition of a global attractor (following [**6, 11, 20**]).

Definition 7.8. A set $\mathcal{A} = \mathcal{A}_\alpha \subset \Phi_b$ is a *locally compact* (*global*) *attractor* for a semigroup $S_\alpha(t) : \Phi_b \to \Phi_b$ if the following assumptions are satisfied.

1) $\mathcal{A}$ is bounded in Φ_b and compact in $\Phi_{loc} := \vec{v}_c + \mathcal{H}^2_{loc}(\overline{\Omega})$, i.e., the restriction $\mathcal{A}\big|_{\Omega_1}$ of the attractor $\mathcal{A}$ to any bounded sub-domain Ω_1 of Ω is compact in $L^2(\Omega_1)$.

2) $\mathcal{A}$ is strictly invariant: $S_\alpha(t)\mathcal{A} = \mathcal{A}$.

3) $\mathcal{A}$ is an attracting set for the semigroup $S_\alpha(t)$, i.e., for any neighborhood $\mathcal{O}(\mathcal{A})$ (in the local topology of the space Φ_{loc}) and bounded (in Φ_b) subset B there exists $T = T(\mathcal{O}, B)$ such that

$$S_\alpha(t)B \subset \mathcal{O}(\mathcal{A}) \quad \forall t \geqslant T. \tag{7.27}$$

Corollary 7.9. *Under the assumptions of Theorem* 7.4*, the semigroup* (7.12) *associated with the Leray–Navier–Stokes problem* (6.1), (6.2) *possesses a locally compact attractor $\mathcal{A}_\alpha = \mathcal{A}^c_\alpha$ which is bounded in $\vec{v}_c + \mathcal{V}^2_b(\Omega)$. Moreover, the following uniform estimate holds*:

$$\|\mathcal{A}_\alpha\|_{L^2_b(\Omega)} \leqslant K(1 + |c|^3 + \|g\|^2_{L^2_b(\Omega)}), \tag{7.28}$$

where the constant K is independent of α, c, and g.

Proof. As usual, to verify the attractor existence, we need to check the standard conditions; namely, the existence of a compact absorbing set and the continuity (see, for example, [**6**]).

By Theorem 7.4, the semigroup (7.11) possesses an absorbing set $\mathcal{B} \subset \Phi$ which is, however, not compact in the space Φ_{loc}. But, by Theorem 7.3, the set $S_\alpha(1)\mathcal{B}$ is bounded in $\vec{v}_c + \mathcal{V}^2_b(\Omega)$ and, consequently, is compact in Φ_{loc}. Thus, a compact absorbing set $\mathcal{B}_1 := S_\alpha(1)\mathcal{B}$ for the semigroup (7.11) is constructed. Moreover, due to Theorem 7.1, the operators $S_\alpha(t) : \mathcal{B}_1 \to \Phi$ are continuous (in the topology of Φ_{loc}) for every fixed $t > 0$. Thus, due to the standard attractor existence theorem, the semigroup $S_\alpha(t)$ possesses a global attractor $\mathcal{A}_\alpha \subset \mathcal{B}_1 \cap \mathcal{B}$. Now, the estimate (7.28) is an immediate consequence of Theorem 7.4. □

8. Classical Navier–Stokes Problem

In this concluding section, we construct (by passing to the limit $\alpha \to 0$) a dissipative weak solution of the classical 3D Navier–Stokes equation in a cylindrical domain Ω:

$$\partial_t u + (u, \nabla_x)u = \Delta_x u - \nabla_x p + g,\ \operatorname{div} u = 0,\ \mathbb{S}u_1 = c,\ u\big|_{t=0} = u_0. \quad (8.1)$$

The following theorem can be considered as the main result of the paper.

Theorem 8.1. *For any $c \in \mathbb{R}$, $g \in L^2_b(\Omega)$, and $u_0 \in \vec{v}_c + \mathcal{H}^2_b(\Omega)$ there exists at least one weak solution $u = u(t)$,*

$$u \in \Psi_b := L^\infty(\mathbb{R}_+, \mathcal{H}^2_b(\Omega)) \cap L^2_b(\mathbb{R}_+, \mathcal{V}^2_b(\Omega)), \quad (8.2)$$

satisfying the Navier–Stokes equation (8.1) *in the sense of distributions over the divergence-free vector fields, which satisfies the dissipative estimate*

$$\begin{aligned} &\|u(t)\|_{L^2_b(\Omega)} + \|u\|_{L^2_b([t,t+1],W^{1,2}_b(\Omega))} \\ &\leqslant Q(\|u_0\|_{L^2_b(\Omega)})e^{-\beta t} + K(1 + |c|^3 + \|g\|^2_{L^2_b(\Omega)}), \end{aligned} \quad (8.3)$$

where the monotone function Q and positive constants β and K are independent of c, g, and u_0.

Proof. Let $u_\alpha(t)$, $\alpha > 0$, $\alpha \to 0$, be the unique solutions of the Leray approximations (6.1) to the Navier–Stokes equations (with fixed c, g, and u_0), constructed above. By Theorems 6.6 and 7.4, these functions satisfy the estimate (8.3) *uniformly* with respect to $\alpha \to 0$. In particular, u_α are uniformly bounded in Ψ_b. Thus, we can extract a subsequence $u_n := u_{\alpha_n}$ converging to some function $u \in \Psi_b$ in the following sense:

$$\begin{aligned} u_n \to u \ \text{ weakly star in }\ & L^\infty_{loc}(\mathbb{R}_+, L^2_{loc}(\Omega)) \\ \text{and weakly in }\ & L^2_{loc}(\mathbb{R}_+, W^{1,2}_{loc}(\Omega)) \end{aligned} \quad (8.4)$$

(see, for example, [**22**]). Moreover, passing to the weak limit in the estimates (8.3) for u_n, we see that the limit function u also satisfies this estimate. Thus, it suffices to show that u solves the limit Navier–Stokes equation. As usual, for this purpose, we need the *strong* convergence $u_n \to u$ in the appropriate space, which, in turns, requires to control $\partial_t u$ in some negative Sobolev space. Let us obtain such a control.

For the sake of simplicity, we consider the case $c = 0$ (the general case can be easily reduced to this case by making the change of variable

$\bar{u} := u - \vec{v}_c$, see the proof of Theorem 6.6). Applying the projector Π to Equation (6.1), we have

$$\partial_t u_n = \Pi\Delta_x u_n - \Pi(\Pi w_n, \nabla_x)u_n + \Pi g, \tag{8.5}$$

where w_n solves the problem

$$w_n - \alpha_n \Delta_x w_n = u_n, \quad w_n\big|_{\partial\Omega} = 0. \tag{8.6}$$

We see that the first term on the right-hand side of (8.5) belongs to $L^2_b(\mathbb{R}_+, \mathcal{H}^{-1,2}_b(\Omega))$ and is uniformly bounded in this space since u_n are bounded in Ψ_b. To estimate the second term, we note that $\Psi_b \subset L^{10/3}_b(\mathbb{R}_+ \times \Omega)$ due to the standard interpolation theorem (see, for example, [**29**]). Consequently, u_n are uniformly bounded in $L^{10/3}_b(\mathbb{R}_+ \times \Omega)$ and, by Proposition 3.3, w_n are uniformly bounded in $L^{10/3}_b$.

In turns, Theorem 4.4 guarantees that Πw_n are uniformly bounded in $L^{10/3}_b(\mathbb{R}_+ \times \Omega)$. Therefore, using the Hölder inequality, we see that $(\Pi w_n, \nabla_x)u_n$ are uniformly bounded in $L^{5/4}_b(\mathbb{R}_+ \times \Omega)$ (since $4/5 = 3/10 + 1/2$). Using Theorem 4.4 again, we infer that the second term in (8.5) belongs to $L^{5/4}_b(\mathbb{R}_+ \times \Omega)$. Finally, since $L^{5/4} \subset W^{-1,2}$, we have

$$\|\partial_t u_n\|_{L^{5/4}_b(\mathbb{R}_+, \mathcal{H}^{-1,2}_b(\Omega))} \leqslant C\|u_n\|_{\Psi_b} \leqslant C_1, \tag{8.7}$$

where the constants C and C_1 are independent of n.

Arguing in the same way as in the proof of Theorem 6.6, we conclude that

$$u_n \to u \ \text{ strongly in } \ L^2_{loc}(\mathbb{R}_+ \times \Omega). \tag{8.8}$$

We claim that (8.8) implies the analogous strong convergence

$$w_n \to u \text{ strongly in } \ L^2_{loc}(\mathbb{R}_+ \times \Omega). \tag{8.9}$$

We split w_n as $w_n = w^0_n + w^1_n$, where $w^0_n - \alpha_n\Delta_x w^0_n = u_n - u$ and $w^1_n - \alpha_n\Delta_x w^1_n = u$. By (8.8) and Proposition 3.3, the functions w^0_n converge strongly to zero as $n \to \infty$ in the space $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. Therefore, it suffices to study w^1_n. Multiplying the equations for w^1_n by $\varphi^2_{\varepsilon,x_0}\Delta_x w^1_n$, using that $u \in L^2_b(W^{1,2}_b(\Omega))$, $u\big|_{\partial\Omega} = 0$, and arguing in a standard way, it is easy to check that

$$\alpha_n\|\Delta_x w^1_n\|^2_{L^2_b(\mathbb{R}_+\times\Omega)} \leqslant C\|u\|_{L^2_b(\mathbb{R}_+, W^{1,2}_b(\Omega))} \leqslant C_1$$

and, consequently, $\alpha_n\Delta_x w^1_n$ tends to zero in $L^2_b(\mathbb{R}_+ \times \Omega)$. Thus, $w^1_n \to u$ strongly even in $L^2_b(\mathbb{R}_+ \times \Omega)$ and the convergence (8.9) is established.

We are now ready to complete the proof of the theorem. According to Theorem 4.4 and the convergence (8.9), $\Pi w_n \to \Pi u = u$ *strongly* in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$. Since $\nabla_x u_n \to \nabla_x u$ weakly in $L^2_{loc}(\mathbb{R}_+ \times \Omega)$,

$$(\Pi w_n, \nabla_x) u_n \to (u, \nabla_x) u \quad \text{weakly in} \quad L^1_{loc}(\mathbb{R}_+ \times \Omega).$$

As usual, the limit passage (in the sense of distributions) in linear terms of the equations for u_n is obvious, and thereby u solves the classical Navier–Stokes problem (8.1). $\square$

Remark 8.2. As usual, it is not difficult to verify that the space

$$\Theta_b := \{u \in \Psi_b,\ \partial_t u \in L^{5/4}(\mathbb{R}_+, \mathcal{H}_b^{-1,2}(\Omega))\}$$

is compactly embedded, for example, in $C([0,T], \mathcal{H}_\varphi^{-1,2}(\Omega))$ for any $T > 0$ and square integrable weight φ. Thus, the above-constructed solution u satisfies the initial condition $u(0) = u_0$.

Our next goal is to construct an attractor for the classical Navier–Stokes equation in a cylindrical domain. However, in contrast to the previous section, the uniqueness of a solution u is still out of reach of the theory (even in the case of bounded domains) and, consequently, the limit semigroup $S_0(t)$ can be defined only as a semigroup of multi-valued maps. To overcome this difficulty, we use the so-called trajectory approach which allows us to restore the uniqueness by changing the phase space of the problem and to construct a global attractor for the so-called trajectory dynamical system related with the problem under consideration (see [**8, 25, 31, 9**] for details).

We start by constructing the trajectory phase space and trajectory semigroup for the Navier–Stokes problem (8.1).

Definition 8.3. Let $K_{\mathrm{tr}} = K_{\mathrm{tr}}(c)$ be a set of all weak solutions $u \in \Psi_b$ of Equation (8.1) (for all initial data $u_0 \in \vec{v}_c + \mathcal{H}_b^{-1,2}(\Omega)$) which satisfy, additionally, the dissipative estimate

$$\|u(t)\|_{L^2_b(\Omega)} + \|u\|_{L^2_b([t,t+1],W^{1,2}_b(\Omega))} \leqslant C_u e^{-\beta t} + K(|c|^3 + 1 + \|g\|^2_{L^2_b(\Omega)}), \tag{8.10}$$

where the positive constants K and β are the same as in Theorem 8.1 and C_u is an arbitrary constant depending on u. By Theorem 8.1, K_{tr} is not empty. Moreover, since our equation is autonomous, and the estimate (8.10) is invariant under translations, the semigroup of temporal translations acts on K_{tr}:

$$T(t) : K_{\mathrm{tr}} \to K_{\mathrm{tr}}, \quad (T(t)u)(s) := u(t+s),\ t, s \geqslant 0. \tag{8.11}$$

The translation semigroup $(T(t), K_{\rm tr})$ acting on the trajectory phase space $K_{\rm tr}$ will be referred to as a trajectory dynamical system associated with Equation (8.1).

Finally, we endow the trajectory phase space $K_{\rm tr}$ with the topology induced by the embedding

$$K_{\rm tr} \subset \Psi_{loc} := [L^\infty_{loc}(\mathbb{R}_+, L^2_{loc}(\Omega))]^{w^*} \cap [L^2_{loc}(\mathbb{R}_+, W^{1,2}_{loc}(\Omega))]^{w}, \qquad (8.12)$$

where the symbols w^* and w mean weak-star and weak topology respectively. We recall that a sequence u_n converges to u in $\Psi_{\rm loc}$ if for any $T > 0$ and $N > 0$ the restrictions of this sequence to the domain $t \in [0, T]$, $x \in \Omega_{-N,N} := [-N, N] \times \omega$ converge weakly in $L^2([0, T], W^{1,2}(\Omega_N))$ and weakly-star in $L^\infty([0, T], L^2(\Omega_N))$.

Remark 8.4. (i) If we assume that the uniqueness theorem holds, then the solution operator $S : u_0 \to K_{\rm tr}$ generates a one-to-one map between the usual phase space $\Phi_b = \vec{v}_c + \mathcal{H}^2_b(\Omega)$ and the trajectory phase space $K_{\rm tr}$. Moreover, the translation semigroup $T(t)$ on $K_{\rm tr}$ is conjugated with the usual semigroup $S_0(t)$ ($S_0(t)u_0 := u(t)$) by this map:

$$T(t) = S \circ S_0(t) \circ S^{-1}, \quad S^{-1}u := u(0). \qquad (8.13)$$

Thus, in the case of uniqueness, the trajectory dynamical system $(T(t), K_{\rm tr})$ is formally equivalent to the classical system $(S_0(t), \Phi_b)$ and, if the uniqueness fails, can be considered as a natural generalization allowing us to avoid the usage of theory of multi-valued maps.

(ii) We need to include some form of *dissipative* estimate into the definition of a solution in $K_{\rm tr}$ since it is not still known, whether or not there exist other "pathological" weak solutions $u \in \Psi_b$ that do not satisfy energy inequalities and are, possibly, nondissipative. Including the estimate (8.10) into the definition, we automatically exclude such solutions. We also emphasize that the estimate (8.10) is slightly weaker than the estimate (8.3) obtained in the proof of Theorem (8.1); namely, we have an arbitrary constant C_u instead of $Q(\|u(0)\|_{L^2_b(\Omega)})$. This is related with the fact that the estimate (8.3) is not translation-invariant (since it was proved only on the time interval $[0, t]$, but not on $[\tau, t + \tau]$, the weak convergence $u_n(\tau)$ to $u(\tau)$ obtained in the proof of Theorem 8.1 is not sufficient to pass to the limit in $Q(\|u_n(\tau)\|_{L^2_b(\Omega)}))$. By this reason, we cannot use the dissipative estimate with $C_u = Q(\|u(0)\|_{L^2_b})$ for defining $K_{\rm tr}$ (otherwise, the translation semigroup may not act on it) and, following [**9**] use the slightly different estimate (8.10) for which this translation invariance is immediate.

(iii) The rather unusual choice of topology on $K_{\rm tr}$ is motivated, on one hand, by the necessity to have some kind of compactness / asymptotic compactness for the attractor theory and, on the other hand, by the fact that no additional regularity and / or compactness is known for the 3D Navier–Stokes equations even in the case of bounded domains. So, we may speak only about *weak* attractors (i.e., attractors in a weak topology of the phase space, where the proper bounded subsets are automatically precompact). In contrast to that, the choice of a *local* topology on the trajectory phase space $K_{\rm tr}$ is unavoidable for the trajectory approach since, even in the case of uniqueness and continuous dependence on the initial data, the solution map $S : \Phi_b \to K_{\rm tr}$ is a homeomorphism only under such a choice of the topology on $K_{\rm tr}$.

Our next task is to define properly the class of bounded sets in the trajectory phase space $K_{\rm tr}$ (which will be attracted by our trajectory attractor as $t \to \infty$). We first note that the most natural way is to use the topology of the Banach space Ψ_b for defining bounded sets. However, this choice is incompatible with the dissipative estimate (8.10). Since we do not have the relation $C_u = Q(\|u(0)\|_{L^2_b(\Omega)})$, the constant C_u may be, in general, not bounded on bounded subsets of Ψ_b. Therefore, under such a choice of bounded sets, we are not able establish the dissipativity (= the existence of a bounded absorbing set), which is crucial for the attractor theory.

This obstacle is overcome by using the abstract class of "bounded" sets (not related with any Banach or metric space); namely, a subset $B \subset K_{\rm tr}$ is "bounded" if the constant C_u in the estimate (8.10) is uniformly bounded on B

$$C_u \leqslant C_B < \infty \quad \forall u \in K_{\rm tr}.$$

On one hand, this class of "bounded" sets satisfies the property

$$\text{if } B \text{ "bounded" and } B_1 \subset B, \text{ then } B_1 \text{ is also "bounded".} \tag{8.14}$$

On the other hand, since for "reasonable" solutions (for example, constructed in Theorem 8.1) we expect that $C_u = Q(\|u(0)\|_{L^2_b(\Omega)})$, this definition is naturally related with bounded subsets of the classical phase space Φ_b.

We are now ready to introduce a concept of a trajectory attractor associated with the Navier–Stokes equation and to formulate the existence theorem. .

Definition 8.5. A set $\mathcal{A}_{\mathrm{tr}} = \mathcal{A}_{\mathrm{tr}}(c) \subset K_{\mathrm{tr}}$ is a trajectory attractor of the Navier–Stokes system (8.1) (= a global attractor of the trajectory dynamical system $(T(s), K_{\mathrm{tr}})$) if

(i) it is "bounded" and compact in K_{tr} (in the topology of Ψ_{loc}),
(ii) it is strictly invariant: $T(t)\mathcal{A} = \mathcal{A}$, $t \geqslant 0$,
(iii) it attracts the images of all "bounded" subsets of K_{tr} in the topology of Ψ_{loc}, i.e., for any "bounded" subset $B \subset K_{\mathrm{tr}}$ and neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the topology of Ψ_{loc} there exists $T = T(B, \mathcal{A})$ such that $T(t)B \subset \mathcal{O}(\mathcal{A})$ if $t \geqslant T$.

Theorem 8.6. *Let the above assumptions hold. Then for any $c \in \mathbb{R}$ and $g \in L^2_b(\Omega)$ the Navier–Stokes problem* (8.1) *possesses a trajectory attractor $\mathcal{A}_{\mathrm{tr}}(c)$ in the sense of Definition* 8.5*. Moreover,*

$$\|\mathcal{A}_{\mathrm{tr}}(c)\|_{\Psi_b} \leqslant K(|c|^3 + 1 + \|g\|^2_{L^2_b(\Omega)}), \tag{8.15}$$

where the constant K is the same as in (8.3)*.*

Proof. According to the attractor existence theorem for abstract classes of "bounded" sets (see [**26**] and also [**9**]), it is required to verify the following assertions.

1. There exists a "bounded" compact metrizable absorbing set $\mathcal{B}$ for the semigroup $T(t)$ acting on K_{tr}.

2. $T(t)$ is continuous on $\mathcal{B}$ for every fixed t.

The second condition is obvious since $T(t)$ is continuous on the whole K_{tr} considered as a translation semigroup. Let us verify the first condition.

According to the estimate (8.10),

$$\mathcal{B}_\varepsilon = \{u \in K_{\mathrm{tr}},\ C_u \leqslant \varepsilon\} \tag{8.16}$$

are "bounded" absorbing sets for every $\varepsilon > 0$. By the estimate (8.3), the sets $\mathcal{B}_\varepsilon$ are bounded in Ψ_b and, consequently, precompact and metrizable in Ψ_{loc} (see [**22**]). So, we only need to check that $\mathcal{B}_\varepsilon$ are closed in K_{tr}. The fact that the limit point u solves again the Navier–Stokes equation can be verified exactly in the same way as in the proof of Theorem 8.1 (using the additional control of $\partial_t u$ provided by the equation and compactness arguments, see (8.7)). Finally, passing to the limit in the estimates (8.10), wee see that the limit point u should satisfy this estimate. Thus, $u \in K_{\mathrm{tr}}$, $\mathcal{B}_\varepsilon$ is closed and Theorem 8.6 is proved. □

Remark 8.7. (i) Using the fact that the attractor $\mathcal{A}_{\mathrm{tr}}$ is bounded in a stronger space Θ_b, see Remark 8.2 and compactness arguments, one can verify that the weak attraction in Ψ_{loc} implies the following strong local attraction: for any "bounded" subset $B \subset K_{\mathrm{tr}}$, $T > 0$, and $N \in \mathbb{R}_+$

$$\lim_{t\to\infty} \operatorname{dist}(T(t)B\big|_{[0,T]\times\Omega_{-N,N}}, \mathcal{A}_{\mathrm{tr}}\big|_{[0,T]\times\Omega_{-N,N}}) = 0,$$

where the distance is understood in the space

$$C([0,T], \vec{v}_c + \mathcal{H}^{-\delta,2}(\Omega_{-N,N})) \cap L^2([0,T], W^{1-\delta,2}(\Omega_{-N,N})),$$

$\delta > 0$ is arbitrary.

(ii) One can define also a "global" attractor $\mathcal{A}^{gl}$ by projecting the trajectory attractor $\mathcal{A}_{\mathrm{tr}}$ to the classical phase space Φ_b:

$$\mathcal{A}^{gl} := \mathcal{A}_{\mathrm{tr}}\big|_{t=0}.$$

Then it is not difficult to show that the global attractors $\mathcal{A}_\alpha$ of Leray approximations tend to the limit attractor $\mathcal{A}^{gl}$ as $\alpha \to 0$ in the sense of upper semicontinuity in $[L^2_{loc}(\Omega)]^w$. Alternatively, lifting global attractors $\mathcal{A}_\alpha$, $\alpha > 0$ to the equivalent trajectory attractors $\mathcal{A}_{\alpha,tr}$ by the solution map, one has the upper semicontinuity of trajectory attractors $\mathcal{A}_{\alpha,tr}$ as $\alpha \to 0$ in the topology of Ψ_{loc}.

To conclude the paper, we restore the physical parameters in the Navier–Stokes system (6.1), i.e., consider the problem

$$\partial_t u + (u, \nabla_x)u = \nu\Delta_x u - \nabla_x p + g, \quad \operatorname{div} u = 0 \tag{8.17}$$

in a cylindrical domain Ω and study the dependence of the size of attractor on ν.

Corollary 8.8. *The trajectory attractor $\mathcal{A}_{\mathrm{tr}} = \mathcal{A}_{\mathrm{tr}}(c, g, \nu)$ of the problem* (8.17) *satisfies the estimate*

$$\|\mathcal{A}_{\mathrm{tr}}\|_{L^\infty(\mathbb{R}_+, L^2_b(\Omega))} \leqslant C\nu^{-3}(|c|^3\nu + \|g\|^2_{L^2_b(\Omega)} + \nu^4), \tag{8.18}$$

where the constant C is independent of c, g, and ν.

By scaling $t' = \nu t$, $u' = \nu^{-1}u$, we can reduce Equation (8.18) to Equations (6.1)–(6.2) with $c' = \nu^{-1}c$ and $g' = \nu^{-2}g$. Since $\mathcal{A}' = \nu^{-1}\mathcal{A}$, (7.28) implies (8.18).

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On Global in Time Properties of the Symmetric Compressible Barotropic Navier–Stokes–Poisson Flows in a Vacuum

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We consider symmetric self-gravitating flows of a viscous compressible barotropic gas/fluid around a hard core with a free outer boundary in a vacuum; the density is degenerating at the free boundary. For large discontinuous initial data and general state function (including increasing and non-strictly increasing ones) we prove a collection of the global-in-time bounds for solutions and study their large-time behavior both in the Lagrangian mass and Eulerian coordinates. The results on the existence, nonexistence and uniqueness of the corresponding static solutions are also included. Bibliography: 29 titles.

1. Introduction

We consider symmetric self-gravitating flows of a layer of a viscous compressible barotropic gas/fluid around a hard core with a free outer boundary in a vacuum. These are described by the compressible barotropic Navier–Stokes-Poisson equations. The cases of the planar, cylindrical and spherical symmetry are studied simultaneously; the spherical case is the most important but the cylindrical one is also of interest in astrophysics. Our aim is to derive global-in-time bounds for solutions and to study their large-time behavior (a stabilization to stationary solutions). The latter problem (together with the stability of the corresponding rest states) is interesting in astrophysics (see, in particular, [**5, 7, 14, 6**]).

The paper is closely connected to [**9**] (see also [**29**]) and develops the corresponding technique. In contrast to the previous study, we treat the case of flows with the density ρ degenerating at the free boundary and in the absence of an outer pressure. This case is clearly more complicated mathematically but much well adopted in astrophysics. In addition, now initial data can be discontinuous (Lebesgue's versus Sobolev's) so that we deal with global weak (rather than strong) solutions. We emphasize that the initial data can be arbitrarily large, and no other smallness conditions on data are exploited.

We take general state function p and general (strictly positive) viscosity coefficient μ depending on ρ. Though the particular power law $p(\rho) = p_1\rho^\gamma$ with $p_1 > 0$ and $\gamma \geqslant 1$ is widespread, the case of general increasing p allows us to investigate much broader applications (including those in astrophysics; see [**13**]). Note that in particular, we cover the power law for $\gamma > 1$ in the cases of the planar and cylindrical symmetries or for $\gamma > 4/3$ in the case of the spherical symmetry. The borderline exponent $4/3$ is well-known in astrophysics; some results on the necessity of the restrictions on γ are also contained in the paper. On the other hand, the cases of non-strictly increasing piecewise smooth and nonmonotone p are invoked to consider phase transition phenomena; in particular, the former one is involved in some astrophysical models (see, for example, [**11, 19**]). In these cases, few results on global behavior of solutions are available in the literature since they are much more delicate mathematically.

The Poisson equation is excluded from the system by allowing for a specific body force depending both on the Eulerian and Lagrangian mass coordinates. Actually, we analyze general body force of such a kind to cover broader possible applications.

The Lagrangian mass treatment of the problem is the basic one in the paper (in contrast to [**9**]). On the other hand, we present global-in-time bounds and stabilization results in the Eulerian coordinates as well.

The content of the paper is as follows. In Section 2, we present the Eulerian free boundary statement of the evolutionary problem and the corresponding free boundary static problem together with their Lagrangian fixed boundary versions. We define weak solutions to the Lagrangian evolutionary statement and state the corresponding global existence result in Proposition 2.1.

Sections 3, 4, and 6 are devoted to the Lagrangian evolutionary problem. In Section 3, we give the energy conservation law and prove the uniform-in-time energy bound, the uniform upper bound for ρ, the uniform-in-time L^{α}-bound for the specific volume $\eta := 1/\rho$ and the stabilization of kinetic and potential energies respectively in Propositions 3.1–3.4. The bound for η essentially complements the study in [**9**].

In Section 4, we first study an auxiliary functional involving the difference between the true and quasistationary pressures and prove an auxiliary energy-type equality for the difference between the true and quasistationary stresses (in Lemma 4.1, Proposition 4.1 and Lemma 4.2). Both are crucial to derive the main results on the stabilization of solutions. In Theorem 4.1, we prove a stronger than $W^{1,1}$-stabilization of the velocity v to zero (together with its strong bounds, for positive time and globally in time). We outline that all the above-listed results concern general nonmonotone state functions.

In Theorem 4.2, the L^{λ}-stabilization of η and ρ to solutions of the corresponding static problem is studied, for both increasing and nondecreasing state function. The corresponding ω-limit sets are analyzed in Proposition 4.2.

The static problem, both in the Lagrangian mass and Eulerian statements, is considered in Section 5. We prove the existence, nonexistence and uniqueness results respectively in Propositions 5.1–5.3. We outline the uniqueness result in the astrophysical context provided that the first adiabatic exponent Γ_1 of the state function is greater than or equal to 1 or 4/3 respectively in the case of the cylindrical or spherical symmetry (see Corollary 5.2). This solves the problem posed by Kuan and Lin [**15**].

In Section 6, some additional bounds for ρ are proved (including its curious Hölder continuity in time and bounds for its difference in space in Propositions 6.1 and 6.2). The proof of Proposition 2.1 is also put here.

Section 7 is devoted to the Eulerian evolutionary statement of the problem: Proposition 7.1 collects the uniform-in-time bounds for solutions (and auxiliary stabilization results) and Theorem 7.1 contains the stabilization results for v (together with its additional bounds) and for ρ and the free radius R; the latter stabilization result differs from the preceding Lagrangian one. Proposition 4.2 treats the corresponding ω-limit sets.

To complete this brief introduction, we only list some related papers in the field of mathematical astrophysics: [**20, 21, 8, 15, 17, 24, 25, 10**]; other related papers on viscous compressible flows are listed in [**9**].

2. The Eulerian and Lagrangian Mass Statements of the Problem

We consider a system of quasilinear differential equations describing symmetric flows of a viscous compressible barotropic fluid consisting in the continuity and the impulse equations

$$\rho_t + \frac{1}{\varkappa}(\varkappa\rho v)_r = 0, \tag{2.1}$$

$$\rho(v_t + vv_r) = \sigma_r + \rho f[m], \quad \sigma := \frac{\mu(\rho)}{\varkappa}(\varkappa v)_r - p(\rho) \tag{2.2}$$

in a domain $Q := \{(r,t);\, r_0 < r < R(t),\ t > 0\}$ and the free boundary equation

$$R'(t) = v\big|_{r=R(t)}, \quad t > 0. \tag{2.3}$$

The system is supplemented with the fixed left-hand and free right-hand boundary conditions

$$v\big|_{r=r_0} = 0, \quad \sigma\big|_{r=R(t)} = 0, \quad t > 0, \tag{2.4}$$

and the initial conditions

$$\rho\big|_{t=0} = \rho^0, \quad v\big|_{t=0} = v^0 \text{ on } \Omega_0 := (r_0, R^0), \quad R\big|_{t=0} = R^0 > r_0. \tag{2.5}$$

The unknown functions $\rho > 0$, v, and R are the density and the velocity of the gas and the radius of the exterior free boundary, $\sigma, p(\rho)$, and $\mu(\rho)$ are the stress, the pressure ($s \to p(s)$ is the corresponding state function), and the viscosity coefficient.

Hereinafter, $\varkappa(r) := r^k$ with $k = 0, 1$ or $k = 2$ respectively for the planar, cylindrical or spherical symmetry, and $\nu(r) := r^{k+1}/(k+1)$, $\nu_0 := r_0^{k+1}/(k+1)$ and $V := R^{k+1}/(k+1)$; $r_0 > 0$ is the radius of the hard core, and V is the gas volume (up to a constant multiplier).

The function f is a body force of the form $f(r,\chi,t) = f_S(r,\chi) + \Delta f(r,\chi,t)$, where f_S is the static contribution and Δf is its perturbation (of any nature) tending to zero as $t \to \infty$ in a weak sense; also $f[m](r,t) := f(r,m(r,t),t)$. The function

$$m(r,t) := \int_{r_0}^{r} \rho(r_1,t)\varkappa(r_1)\,dr_1 \tag{2.6}$$

is the mass of the gas layer around the hard core, with the external radius r; thus f depends both on the Eulerian and Lagrangian mass coordinates. The mass conservation law

$$m(R(t),t) \equiv M := \int_{\Omega_0} (\rho^0\varkappa)(r)\,dr \text{ for } t \geqslant 0$$

holds for solutions to the problem; M is the total mass of the gas.

In the astrophysical context, the function f_S has the specific form

$$f_S(r,\chi) = f_G(r,\chi) := -G\,\frac{M_0 + i_0\chi}{r^k},$$

where $G > 0$, $M_0 \geqslant 0$ (the mass of the hard core) and $i_0 = 0, 1$. For $i_0 = 1$ the most interesting case with self-gravitation is covered, whereas for $i_0 = 0$ the self-gravitation is neglected (but this case was also of interest in the literature). In the latter case, we suppose that $M_0 > 0$.

The corresponding static problem consists in finding a pair $\{\rho_S, R_S\}$ of static density $\rho_S > 0$ and free radius $R_S \in (r_0,\infty)$ such that the following integro-differential equation holds:

$$p(\rho_S)_r = \rho_S f_S[m_S], \quad m_S(r) := \int_{r_0}^{r} (\rho_S\varkappa)(r_1)\,dr_1 \text{ in } \Omega_S := (r_0, R_S), \tag{2.7}$$

under the free boundary condition and the mass constraint

$$p(\rho_S)(R_S) = 0, \quad \int_{\Omega_S} \rho_S\varkappa\,dr = M. \tag{2.8}$$

Here, $f_S[m_S](r) = f_S(r, m_S(r))$ and $m_S(r)$ is the mass of the static gas layer around the hard core, with the external radius r.

Throughout the paper, we assume that p and μ satisfy the conditions $p, \mu \in W^{1,\infty}_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$ and

$$\begin{gathered} p(0) = 0, \quad p(s) > 0, \ s > 0, \quad \int\limits_0^1 \frac{p(s)}{s^2}\, ds < \infty, \\ p(+\infty) := \lim_{s \to +\infty} p(s) = +\infty, \end{gathered} \tag{2.9}$$

$$0 < \underline{\mu} \leqslant \mu(s), \quad s \geqslant 0. \tag{2.10}$$

Recall that $\varphi \in L^{\lambda}_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$ means that $\varphi \in L^{\lambda}(0, T)$ for any $T > 0$, and $\varphi \in W^{1,\lambda}_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$ means that $\varphi, \varphi' \in L^{\lambda}_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$, where $\lambda \in [1, \infty]$. We impose a more restrictive condition than the third condition (2.9) on p in the main results (see (3.16) below). For comparison, we introduce the power law $p_\gamma(s) = p_1 s^\gamma$, $p_1 > 0$, $\gamma > 1$ and set $\gamma' := \gamma/(\gamma - 1)$.

One can pass to the Lagrangian mass coordinates by considering $\chi = m(r, t)$ (see formula (2.6)) as a new independent variable together with t. Taking into account the mass conservation law, one can transform the free boundary problem (2.1)–(2.3) to the problem

$$D_t \check{\rho} = -\check{\rho}^2 D(\check{\varkappa} \check{v}), \tag{2.11}$$

$$D_t \check{v} = \check{\varkappa} D \check{\sigma} + \check{f}[\check{r}], \quad \check{\sigma} := \mu(\check{\rho}) \check{\rho} D(\check{\varkappa} \check{v}) - p(\check{\rho}), \tag{2.12}$$

$$\check{\nu} := \frac{1}{k+1}\, \check{r}^{k+1} = \nu_0 + \int\limits_0^\chi \frac{d\xi}{\check{\rho}(\xi, t)} \tag{2.13}$$

in the fixed domain $\check{Q} := J \times \mathbb{R}^+$, $J := (0, M)$. The original unknown functions $\rho(r, t)$, $v(r, t)$, $m(r, t)$ and the new ones $\check{\rho}(\chi, t)$, $\check{v}(\chi, t)$, $\check{r}(\chi, t)$ are related by the equalities

$$\rho(r, t) \equiv \check{\rho}(m(r, t), t), \ v(r, t) \equiv \check{v}(m(r, t), t), \ r \equiv \check{r}(m(r, t), t) \text{ in } \bar{Q},$$

where $\check{\rho} > 0$ and $\check{r} \geqslant r_0$. Moreover, D and D_t denote the partial derivatives with respect to the new variables χ and t as well as $\check{\varkappa} = \check{r}^k$ and $\check{f}[\check{r}](\chi, t) = f(\check{r}(\chi, t), \chi, t)$.

The following boundary and initial conditions supplement the equations:

$$\check{v}\big|_{\chi=0} = 0, \quad \check{\sigma}\big|_{\chi=M} = 0 \text{ on } \mathbb{R}^+, \tag{2.14}$$

$$\check{\rho}\big|_{t=0} = \check{\rho}^0, \quad \check{v}\big|_{t=0} = \check{v}^0 \text{ on } J, \tag{2.15}$$

where $\check{\rho}^0(m^0(r)) = \rho^0(r)$ and $\check{v}^0(m^0(r)) = v^0(r)$ on Ω_0 with

$$m^0(r) := \int_{r_0}^{r} \rho^0 \varkappa \, dr_1.$$

The continuity equation (2.11) can be written in the form

$$D_t \check{\eta} = D(\check{\varkappa} \check{v}), \tag{2.16}$$

where $\check{\eta} := 1/\check{\rho}$ is the specific volume. By applying D_t to (2.13) and using (2.16), together with $\check{v}\big|_{\chi=0} = 0$, we get the standard equation

$$D_t \check{r} = \check{v} \tag{2.17}$$

connecting $\check{r}$ and $\check{v}$. It is clear that (2.13) implies the following formula for the gas volume:

$$V(t) = \nu_0 + \int_J \check{\eta}(\chi, t) \, d\chi, \quad t \geqslant 0. \tag{2.18}$$

It is convenient to introduce the function

$$h(\nu, \chi) := \frac{f_S(r, \chi)}{r^k}\bigg|_{r=[(k+1)\nu]^{1/(k+1)}}$$

and set $\check{h}[\check{\nu}](\chi, t) := h(\check{\nu}(\chi, t), \chi)$. Let $\Delta \check{f}[\check{r}](\chi, t) := \Delta f(\check{r}(\chi, t), \chi, t)$.

For $\check{Q}_T := J \times (0, T)$ let $L^{\lambda,s}(\check{Q}_T)$ be the anisotropic Lebesgue space [**16**] equipped with the norm $\|\varphi\|_{L^{\lambda,s}(\check{Q}_T)} = \| \, \|\varphi\|_{L^\lambda(J)} \, \|_{L^s(0,T)}$, $\lambda, s \in [1, \infty]$. The similar space is used for $\check{Q}$ replacing $\check{Q}_T$ as well. In the proofs, we also adopt the abbreviation $\|\cdot\|_G = \|\cdot\|_{L^2(G)}$. For an unbounded domain G we denote by $C_b(\bar{G})$ the space of continuous and bounded functions on $\bar{G}$ equipped with the norm $\|\varphi\|_{C_b(\bar{G})} := \sup_{\bar{G}} |\varphi|$.

We impose the following conditions on the initial data and h

$$\check{\rho}^0 \in L^\infty(J), \; \operatorname{ess\,inf}_{(0,\chi_1)} \check{\rho}^0 > 0 \text{ for any } \chi_1 \in J, \; \frac{1}{\check{\rho}^0} \in L^1(J), \tag{2.19}$$

$$\check{v}^0 \in L^2(J), \; h \in C_b([\nu_0, \infty) \times \bar{J}), \; h \leqslant 0. \tag{2.20}$$

Let $\Delta f(r, \chi, t)$ be measurable on $(r_0, \infty) \times J \times \mathbb{R}^+$, continuous with respect to $\chi \in \bar{J}$ for almost all $(r, t) \in (r_0, \infty) \times \mathbb{R}^+$, continuous with respect to $r \in [r_0, \infty)$ for almost all $(\chi, t) \in J \times \mathbb{R}^+$ and satisfy the bound

$$|\Delta f| \leqslant \bar{f}, \quad \|\bar{f}\|_{L^1(\mathbb{R}^+)} \leqslant N, \; \bar{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^+).$$

Hereinafter, $N > 1$ is an (arbitrarily large) parameter.

We study weak solutions possessing the following properties in $\check{Q}_T$ for any $T>0$:

$$\check{\rho}\in L^\infty(\check{Q}_T),\ \operatorname{ess\,inf}_{(0,\chi_1)\times(0,T)}\check{\rho}>0 \text{ for any } \chi_1\in J,\ \frac{1}{\check{\rho}}\in L^{1,\infty}(\check{Q}_T), \tag{2.21}$$

$$D_t\check{\rho}\in L^2(\check{Q}_T),\ \check{v}\in C([0,T];L^2(J)),\ \sqrt{\check{\rho}}\,D\check{v}\in L^2(\check{Q}_T) \tag{2.22}$$

and the following internal regularity for any $0<t_0<T$:

$$\frac{\check{\sigma}}{\sqrt{\check{\rho}}}\in L^{2,\infty}(\check{Q}_T\backslash\check{Q}_{t_0}),\quad D_t\check{v},D\check{\sigma}\in L^2(\check{Q}_T\backslash\check{Q}_{t_0}). \tag{2.23}$$

The solutions satisfy the continuity equation (2.11) in $L^2(\check{Q}_T)$, the equality (2.13) in $L^\infty(\check{Q}_T)$, and the impulse equation (2.12) in the weak form

$$\int\limits_{\check{Q}_T}(-\check{v}D_t\varphi+\check{\sigma}D\varphi)d\chi dt=\int\limits_J\check{v}^0\varphi|_{t=0}\,d\chi+\int\limits_{\check{Q}_T}\check{f}[\check{r}]\varphi\,d\chi dt, \tag{2.24}$$

for any $\varphi\in H^1(\check{Q}_T)$ such that $\varphi|_{\chi=0}=0$ and $\varphi|_{t=T}=0$, as well as in $L^2(\check{Q}_T\backslash\check{Q}_{t_0})$. The boundary conditions (2.14) and the initial condition $\check{\rho}|_{t=0}=\check{\rho}^0$ are understood in the sense of traces.

We mention the useful inequality

$$\|D\check{v}\|_{L^{1,2}(\check{Q}_T)}\leqslant\|1/\check{\rho}\|^{1/2}_{L^{1,\infty}(\check{Q}_T)}\|\sqrt{\check{\rho}}\,D\check{v}\|_{L^2(\check{Q}_T)}. \tag{2.25}$$

The properties (2.21) and (2.23) imply the additional internal regularity

$$\sqrt{\check{\rho}}\,D\check{v}\in L^{2,\infty}(\check{Q}_T\backslash\check{Q}_{t_0}). \tag{2.26}$$

Note that formally we do not suppose that necessarily $\check{\rho}^0(M)=0$ and $\check{\rho}|_{\chi=M}=0$ (in a suitable weak sense) though namely this case is the most interesting and, accordingly, the boundary condition $\check{\sigma}|_{\chi=M}=0$ (neglecting the second viscosity coefficient) is the most relevant.

Proposition 2.1. 1. *Let the conditions* (2.19), (2.20) *hold, and let* $h_\nu\in L^1(J;C[\nu_0,\nu])$ *for any* $\nu>\nu_0$. *Then there exists a solution to the initial-boundary value problem* (2.11)–(2.15) *having the properties* (2.21)–(2.23) *and*

$$C(T)^{-1}\check{\rho}^0(\chi)\leqslant\check{\rho}(\chi,t)\leqslant C(T)\check{\rho}^0(\chi)\ \textit{in}\ \check{Q}_T, \tag{2.27}$$

where $C(T)>0$ *for any* $T>0$.

2. *Under the additional conditions* $\sqrt{\check{\rho}^0}\,D\check{v}^0\in L^2(J)$, $\check{v}^0(0)=0$, *and* $\bar{f}\in L^2_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$, *the result holds for* (2.23) (*and* (2.26)) *with* $t_0=0$, *i.e., for* $\check{Q}_T$ *replacing* $\check{Q}_T\backslash\check{Q}_{t_0}$.

3. *If, in addition to the assumptions of Claim* 1, $\check{\rho}^0 \in C(\bar{J})$, *then* $\check{\rho} \in C(\bar{J} \times [0, T])$.

The proof is given at the end of Section 6 below.

The static problem corresponding to (2.11)–(2.15) has the form

$$Dp(\bar{\rho}_S) = \check{h}[\nu_S], \quad \nu_S(\chi) := \nu_0 + \int_0^\chi \frac{d\xi}{\bar{\rho}_S(\xi)} \text{ in } J, \tag{2.28}$$

$$p(\bar{\rho}_S)(M) = 0, \tag{2.29}$$

where the unknown function $\bar{\rho}_S > 0$ is the Lagrangian static density and $\check{h}[\nu_S](\chi) = h(\nu_S(\chi), \chi)$. We consider static solutions $\bar{\rho}_S$ such that

$$\bar{\rho}_S \in L^\infty(J), \;\; p(\bar{\rho}_S) > 0 \text{ on } [0, M), \;\; \frac{1}{\bar{\rho}_S} \in L^1(J), \; p(\bar{\rho}_S) \in C^1(\bar{J}). \tag{2.30}$$

For increasing p it is clear that $\bar{\rho}_S \in C(\bar{J})$, $\bar{\rho}_S > 0$ on $[0, M)$, and $\bar{\rho}_S(M) = 0$.

This problem arises also from the Eulerian static problem (2.7), (2.8) after the change of variable $\chi = m_S(r)$ with the relation $\rho_S(r) \equiv \overline{\rho}_S(m_S(r))$ for $r \in \Omega_S$ (see Section 5 below for details).

3. Global in Time Bounds in the Lagrangian Mass Coordinates

The energy conservation law for the problem (2.11)–(2.15) in the Lagrangian mass coordinates has the form

$$(\mathcal{E} + \mathcal{F})' + \int_J \mu(\check{\rho})\check{\rho}[D(\check{\varkappa}\check{v})]^2 \, d\chi = \int_J \Delta \check{f}[\check{r}]\check{v} \, d\chi, \tag{3.1}$$

involving $(\cdot)' = d/dt$, the kinetic and potential energies

$$\mathcal{E} := \frac{1}{2} \int_J \check{v}^2 \, d\chi, \quad \mathcal{F} := \int_J (P^{(0)}(\check{\rho}) - \check{H}[\check{\nu}]) d\chi$$

with $\check{H}[\check{\nu}](\chi, t) = H(\check{\nu}(\chi, t), \chi)$, and the primitive functions

$$P^{(0)}(s) := \int_0^s \frac{p(s_1)}{s_1^2} \, ds_1, \quad H(\nu, \chi) := \int_{\nu_0}^\nu h(\nu_1, \chi) \, d\nu_1;$$

the derivation of the law is recalled in the next proof.

We introduce the integration operations

$$\check{I}\varphi(\chi) := \int_0^\chi \varphi(\chi_1)\,d\chi_1, \quad \check{I}^*\varphi(\chi) := \int_\chi^M \varphi(\chi_1)\,d\chi_1.$$

Proposition 3.1. *Let*

$$\left\|\frac{1}{2}(\check{v}^0)^2 + P^{(0)}(\check{\rho}^0)\right\|_{L^1(J)} \leqslant N, \ h \leqslant 0.$$

Then the following uniform-in-time energy bound holds:

$$\sup_{t\geqslant 0}\left\|\Big(\frac{1}{2}\check{v}^2 + P^{(0)}(\check{\rho})\Big)(\cdot,t)\right\|_{L^1(J)} + \left\|\sqrt{\check{\rho}}\,D(\check{\varkappa}\check{v})\right\|_{L^2(\check{Q})} \leqslant K. \tag{3.2}$$

Hereinafter, we denote by K, possibly with indices, some nondecreasing functions of N that can depend also on p, μ, f_S, M, etc.

Proof of Proposition 3.1. To simplify the notation, we omit "checks" over the functions $\check{\rho}$, $\check{v}$, $\check{r}$, $\check{\varkappa}$, $\check{\nu}$, $\check{\eta}$, and their initial values in the proofs.

The impulse equation (2.12), together with the boundary conditions (2.14), implies

$$\frac{1}{2}\frac{d}{dt}\int_J v^2\,d\chi + \int_J [\mu(\rho)\rho D(\varkappa v) - p(\rho)]D(\varkappa v)\,d\chi$$
$$= \int_J (\check{h}[\nu]\varkappa v + \Delta\check{f}[r]v)d\chi. \tag{3.3}$$

Using the continuity equation (2.11), we get $-p(\rho)D(\varkappa v) = D_t P^{(0)}(\rho)$. Using the continuity equation in the form (2.16), we get

$$D_t\nu = D_t\check{I}\eta = \varkappa v. \tag{3.4}$$

Hence $\check{h}[\nu]\varkappa v = D_t\check{H}[\nu]$. Thus, we obtain the energy conservation law (3.1).

Taking into account the condition (2.10) and the estimate

$$\left|\int_J \Delta\check{f}[r]v\,d\chi\right| \leqslant \bar{f}M^{1/2}\|v\|_J = \bar{f}M^{1/2}(2\mathcal{E})^{1/2}, \tag{3.5}$$

we derive the bound (3.2) from the energy conservation law. □

Proposition 3.2. *Let*

$$\check{\rho}^0 \leqslant N, \ \|\check{v}^0\|_{L^2(J)} \leqslant N, \tag{3.6}$$

$$-N \leqslant h \leqslant 0 \ \textit{in} \ [\nu_0,\infty)\times\bar{J}. \tag{3.7}$$

Then the following pointwise upper bound for $\check{\rho}$ *holds*:

$$\check{\rho}(\chi, t) \leqslant K_\rho \tag{3.8}$$

for almost all $\chi \in J$ *and any* $t \geqslant 0$.

PROOF. Since $D_t r = v$ and thus $D_t \varkappa = (k/r)\varkappa v$, we have

$$\frac{1}{\varkappa} D_t v = D_t \frac{v}{\varkappa} + \frac{k}{r\varkappa} v^2.$$

Therefore, dividing the impulse equation (2.12) by $\varkappa$, we get

$$D_t \frac{v}{\varkappa} + \frac{k}{r\varkappa} v^2 = D\check{\sigma} + \check{h}[\nu] + \frac{\Delta \check{f}[r]}{\varkappa}. \tag{3.9}$$

The equality $\mu(\rho)\rho D(\varkappa v) = -D_t A(\rho)$ holds with the primitive function

$$A(s) := \int_{s_0}^{s} \frac{\mu(\zeta)}{\zeta}\, d\zeta, \quad s > 0 \tag{3.10}$$

with some fixed $s_0 > 0$ (in this proof, one can simply set $s_0 := 1$). Applying the operator $-\check{I}^*$ to Equation (3.9), we obtain the following important equation

$$D_t A(\rho) + p(\rho) = D_t \check{I}^* \frac{v}{\varkappa} + \check{p}_R[\rho] + k\check{I}^* \frac{v^2}{r\varkappa} - \check{I}^* \frac{\Delta \check{f}[r]}{\varkappa}, \tag{3.11}$$

where $\check{p}_R[\rho] := -\check{I}^* \check{h}[\nu]$ is the so-called *quasistationary pressure.*

By the energy bound and the condition (3.7) on h, we have

$$\left\| \check{I}^* \frac{v}{\varkappa} \right\|_{C(\bar{J})} \leqslant r_0^{-k} M^{1/2} \|v\|_J \leqslant K, \tag{3.12}$$

$$\|\check{p}_R[\rho]\|_{C(\bar{J})} \leqslant M \|h\|_{C_b([\nu_0,\infty)\times \bar{J})} \leqslant MN, \tag{3.13}$$

$$\left\| \check{I}^* \frac{v^2}{r\varkappa} \right\|_{C(\bar{J})} \leqslant r_0^{-(k+1)} \|v\|_J^2 \leqslant K, \tag{3.14}$$

$$\left\| \check{I}^* \frac{\Delta \check{f}[r]}{\varkappa} \right\|_{C(\bar{J})} \leqslant M \left\| \frac{\Delta \check{f}[r]}{\varkappa} \right\|_{L^\infty(J)} \leqslant r_0^{-k} M \bar{f}. \tag{3.15}$$

We can consider (3.11) as an ordinary differential equation with respect to t for almost all fixed $\chi \in J$ (such that $D_t \rho(\chi, \cdot) \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$). Using the properties $p(\infty) = +\infty$, $\underline{\mu} \leqslant \mu$ and the estimates (3.12)–(3.15) and applying Lemma 1.3 in [**26**], we derive the bound

$$A(\rho(\chi, t)) \leqslant \max\{A(\rho^0(\chi)),\, K_1\} + K_2 \leqslant K_3 \text{ on } \bar{\mathbb{R}}^+.$$

The bound (3.8) is proved. $\square$

Proposition 3.3. *Let p satisfy the upper bound*

$$p \leqslant p_\gamma \text{ on } [0,1] \text{ with some } \gamma > \gamma^{(k)}, \tag{3.16}$$

where $\gamma^{(0)} = \gamma^{(1)} = 1$, $\gamma^{(2)} = 4/3$, and let

$$\check{\rho}^0 \leqslant N, \quad \|1/\check{\rho}^0\|_{L^\alpha(J)} \leqslant N \text{ for some } \alpha \in (1,\gamma),\ \|\check{v}^0\|_{L^2(J)} \leqslant N, \tag{3.17}$$

$$-N \leqslant h(\nu,\chi) \leqslant -\frac{\alpha_0(\chi)}{\nu^{2k/(k+1)}} \text{ in } [\nu_0,\infty) \times \bar{J} \tag{3.18}$$

for some $\alpha_0 \in C(\bar{J})$ such that $\alpha_0 \geqslant 0$ and $\alpha_0(M) > 0$. Then the following uniform-in-time bound holds:

$$\sup_{t \geqslant 0} \left\| \frac{1}{\check{\rho}(\cdot,t)} \right\|_{L^{\alpha_1}(J)} \leqslant K_{\alpha_1} \text{ for any } \alpha_1 \in [1,\alpha) \tag{3.19}$$

and, consequently, the following uniform upper bound for V holds:

$$\sup_{t \geqslant 0} V(t) \leqslant K_V. \tag{3.20}$$

In the particular case $\mu(s) \equiv \mathrm{const}$, one can take $\alpha_1 = \alpha$; moreover, the result remains valid for $\alpha = \alpha_1 = 1$ when the second condition (3.17) reduces to $V^0 - \nu_0 \leqslant N$, where $V^0 := (R^0)^{k+1}/(k+1)$ (see R^0 in (2.5)).

Proof. We divide the proof in four steps.

1. We begin by passing from the ordinary differential equation (3.11) to the simpler inequality

$$-D_t A(\rho) + a \leqslant p(\rho), \quad a := D_t \check{I}^* \frac{v}{\varkappa} - \check{I}^* \frac{\Delta \check{f}[r]}{\varkappa} + \check{p}_R[\rho]. \tag{3.21}$$

By the estimates (3.12) and (3.15), we get

$$\int_\tau^t a(\chi,\theta)\, d\theta \geqslant -K_1 + (t-\tau)\lambda(\chi,t) \text{ for any } 0 \leqslant \tau \leqslant t \tag{3.22}$$

provided that

$$\min_{0 \leqslant \theta \leqslant t} \check{p}_R[\rho](\chi,\theta) \geqslant \lambda(\chi,t) > 0. \tag{3.23}$$

Consequently, for any number $c > 0$

$$\int_0^t e^{-c \int_\tau^t a(\chi,\theta)\, d\theta}\, d\tau \leqslant e^{cK_1} \frac{1}{c\lambda(\chi,t)}. \tag{3.24}$$

2. For the sake of clarity, we first consider the particular case $\mu(s) \equiv \mu_0 > 0$. In this case, for any $\gamma > 0$

$$-D_t A(\rho) = \frac{\mu_0}{\gamma} \frac{D_t(\eta^\gamma)}{\eta^\gamma},$$

and the inequality (3.21) is transformed to the following:

$$D_t(\eta^\gamma) + \frac{\gamma}{\mu_0} a\eta^\gamma \leqslant \frac{\gamma}{\mu_0} p(\rho)\eta^\gamma.$$

By the condition (3.16) and the bound $\rho \leqslant K_\rho$, we get

$$p(s) \leqslant K_2 s^\gamma, \quad 0 \leqslant s \leqslant K_\rho. \tag{3.25}$$

Consequently,

$$D_t(\eta^\gamma) + \frac{\gamma}{\mu_0} a\eta^\gamma \leqslant \frac{\gamma}{\mu_0} K_2.$$

For almost all $\chi \in J$ such that $D_t\rho(\chi, \cdot) \in L^1_{\text{loc}}(\bar{\mathbb{R}}^+)$, solving this inequality and applying the estimates (3.22) and (3.24), we have

$$\begin{aligned} \eta^\gamma(\chi, t) &\leqslant (\eta^0(\chi))^\gamma e^{-(\gamma/\mu_0)\int\limits_0^t a(\chi,\theta)\, d\theta} + \frac{\gamma}{\mu_0} K_2 \int\limits_0^t e^{-(\gamma/\mu_0)\int\limits_\tau^t a(\chi,\theta)\, d\theta} d\tau \\ &\leqslant e^{(\gamma/\mu_0)K_1} \Big((\eta^0(\chi))^\gamma + K_2 \frac{1}{\lambda(\chi, t)} \Big) \end{aligned} \tag{3.26}$$

with $\eta^0 := 1/\rho^0$ and, after taking the (α/γ)th power of the result,

$$\eta^\alpha(\chi, t) \leqslant K_3 \Big[(\eta^0(\chi))^\alpha + \frac{1}{\lambda^{\alpha/\gamma}(\chi, t)} \Big]. \tag{3.27}$$

3. Using the condition (3.18), we get

$$\min_{0 \leqslant \theta \leqslant t} \check{p}_R[\rho](\chi, \theta) \geqslant \frac{\underline{\alpha}_0 (M - \chi)}{\mathcal{V}^{2k/(k+1)}(t)} =: \lambda(\chi, t), \tag{3.28}$$

where

$$\mathcal{V}(t) := \max_{0 \leqslant \theta \leqslant t} V(\theta), \quad \underline{\alpha}_0 := \inf_{0 \leqslant \chi < M} \frac{1}{M - \chi} \int\limits_\chi^M \alpha_0(\chi_1)\, d\chi_1$$

and $\underline{\alpha}_0 > 0$ by the assumptions on α_0. Note that for $\delta > 1$

$$\Big\| \frac{1}{\lambda^{1/\delta}} \Big\|_{L^1(J)} = \frac{\delta}{\delta - 1} \frac{M^{1-1/\delta}}{\underline{\alpha}_0^{1/\delta}} \mathcal{V}^{2k/[\delta(k+1)]}.$$

Integrating (3.27) over J and using the second condition (3.17) and the last formula with $\delta := \gamma/\alpha > 1$, we find

$$\|\eta\|^{\alpha}_{L^{\alpha}(J)} \leqslant K_3\Big(N + \Big\|\frac{1}{\lambda^{\alpha/\gamma}}\Big\|_{L^1(J)}\Big) \leqslant K_4(1 + \mathcal{V}^{2k\alpha/[\gamma(k+1)]}).$$

Consequently,

$$\max_{0\leqslant\theta\leqslant t} \|\eta(\cdot,\theta)\|_{L^{\alpha}(J)} \leqslant K_4(1 + \mathcal{V}^{2k/[\gamma(k+1)]}(t))$$

for any $t \geqslant 0$. By (2.18), we have

$$V = \nu_0 + \|\eta\|_{L^1(J)} \leqslant \nu_0 + M^{1-(1/\alpha)}\|\eta\|_{L^{\alpha}(J)}.$$

Therefore,

$$\max_{0\leqslant\theta\leqslant t} \|\eta(\cdot,\theta)\|_{L^{\alpha}(J)} \leqslant K_5(1 + \max_{0\leqslant\theta\leqslant t} \|\eta(\cdot,\theta)\|^{2k/[\gamma(k+1)]}_{L^{\alpha}(J)}) \tag{3.29}$$

for any $t \geqslant 0$. Since $2k/[\gamma(k+1)] < 1$ in view of (3.16), the Young inequality leads to the bound

$$\sup_{t\geqslant 0} \|\eta(\cdot,t)\|_{L^{\alpha}(J)} \leqslant K_6.$$

This technique goes back to [**21**].

4. The case of general μ is more delicate. For given $\varepsilon \in (0,1)$ we choose $s_\varepsilon \in (0,1)$ so small that

$$\Big|\frac{\mu(s)}{\mu(0)} - 1\Big| < \varepsilon, \quad 0 \leqslant s \leqslant s_\varepsilon.$$

We set $s_0 := s_\varepsilon$ in the definition of A (see (3.10)), $z_\varepsilon := e^{-A(\rho)/\mu_\varepsilon}$, and $\mu_\varepsilon := (1-\varepsilon)\mu(0)$. Then

$$\mu_\varepsilon \log(s_\varepsilon/s) \leqslant -A(s) \leqslant (1+\varepsilon)\mu(0)\log(s_\varepsilon/s)$$

for $0 < s \leqslant s_\varepsilon$. Thus,

$$s_\varepsilon\eta \leqslant z_\varepsilon \leqslant (s_\varepsilon\eta)^{(1+\varepsilon)/(1-\varepsilon)} \text{ for } \eta \geqslant s_\varepsilon^{-1}. \tag{3.30}$$

We set $\gamma_\varepsilon := (1-\varepsilon)\gamma/(1+\varepsilon) < \gamma$, $\alpha_\varepsilon := (1-\varepsilon)\alpha/(1+\varepsilon) < \alpha$ and choose ε so small that $\gamma_\varepsilon > \gamma^{(k)}$ and $\alpha_\varepsilon > 1$.

Since

$$-D_tA(\rho) = \mu_\varepsilon D_t \log z_\varepsilon = \frac{\mu_\varepsilon}{\gamma_\varepsilon}\frac{D_t(z_\varepsilon^{\gamma_\varepsilon})}{z_\varepsilon^{\gamma_\varepsilon}},$$

the inequality (3.21) can be transformed to the form

$$D_t(z_\varepsilon^{\gamma_\varepsilon}) + \frac{\gamma_\varepsilon}{\mu_\varepsilon} a z_\varepsilon^{\gamma_\varepsilon} \leqslant \frac{\gamma_\varepsilon}{\mu_\varepsilon} p(\rho) z_\varepsilon^{\gamma_\varepsilon}. \tag{3.31}$$

We consider almost all $\chi \in J$ such that $D_t\rho(\chi,\cdot) \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$. Thus, $\rho(\chi,t)$ is continuous on $\bar{\mathbb{R}}^+$. The set $S_\varepsilon := \{t > 0;\ \eta(\chi,t) > s_\varepsilon^{-1}\}$ is open. Hence it is an at most countable collection of disjoint intervals $\mathcal{J}_l = (t_{1l}, t_{2l})$ (depending on χ). Moreover, $\eta(\chi, t_{1l}) = \eta^0(\chi) \geqslant s_\varepsilon^{-1}$ provided that $t_{1l} = 0$; otherwise, $\eta(\chi, t_{1l}) = s_\varepsilon^{-1}$. By the right-hand estimate (3.30) and condition (3.16), we have $p(\rho(\chi,t))z_\varepsilon^{\gamma_\varepsilon}(\chi,t) \leqslant p(\rho(\chi,t))(s_\varepsilon\eta(\chi,t))^\gamma \leqslant p_1 s_\varepsilon^\gamma$ on $\bar{\mathcal{J}}_l$. As in the case (3.27), from (3.31) we derive

$$z_\varepsilon^{\alpha_\varepsilon}(\chi,t) \leqslant K_{3\varepsilon}\Big[z_\varepsilon^{\alpha_\varepsilon}(\chi,t_{1l}) + \frac{1}{\lambda^{\alpha_\varepsilon/\gamma_\varepsilon}(\chi,t)}\Big] \text{ on } \bar{\mathcal{J}}_l, \tag{3.32}$$

where $K_{3\varepsilon} \geqslant 1$. Using the two-sided estimate (3.30), we find

$$s_\varepsilon^{\alpha_\varepsilon}\eta^{\alpha_\varepsilon}(\chi,t) \leqslant K_{3\varepsilon}\Big[(s_\varepsilon\eta^0(\chi))^\alpha + 1 + \frac{1}{\lambda^{\alpha_\varepsilon/\gamma_\varepsilon}(\chi,t)}\Big] \text{ on } \bar{\mathcal{J}}_l \tag{3.33}$$

since $z_\varepsilon(\chi, t_{1l}) \leqslant s_\varepsilon\eta^0(\chi)$ provided that $t_{1l} = 0$ or $z_\varepsilon(\chi,t_{1l}) = 1$ otherwise. Since $K_{3\varepsilon} \geqslant 1$, the inequality (3.33) is also valid on $\bar{\mathbb{R}}^+\backslash S_\varepsilon$ and, finally, on $\bar{\mathbb{R}}^+$.

The argument of the previous step of the proof leads from the estimate (3.33) on $\bar{\mathbb{R}}^+$ to the bound $\sup_{t\geqslant 0}\|\eta(\cdot,t)\|_{L^{\alpha_\varepsilon}(J)} \leqslant K_{6\varepsilon}$. Since $\varepsilon > 0$ is arbitrarily small, the proof is complete. $\square$

Note that the condition (3.18) holds for $f_S = f_G$, i.e., $h = h_G(\nu,\chi) := -\widetilde{G}(M_0 + i_0\chi)/\nu^{2k/(k+1)}$ with $\widetilde{G} := G/(k+1)^{2k/(k+1)}$.

The bound (3.20) leads to the following useful inequalities:

$$r_0^k\|\check{v}\|_{C(\bar{J})} \leqslant \|D(\check{\varkappa}\check{v})\|_{L^1(J)} \leqslant K_V^{1/2}\|\sqrt{\check{\rho}}\,D(\check{\varkappa}\check{v})\|_{L^2(J)}. \tag{3.34}$$

Remark 3.1. Under the hypotheses of Proposition 3.3,

$$\|\check{r}\|_{L^\infty(\check{Q})} + \|D\check{r}\|_{L^{\alpha_1,\infty}(\check{Q})} + \|D_t\check{r}\|_{L^{2,\infty}(\check{Q})\cap L^{\infty,2}(\check{Q})} + \|\sqrt{\check{\rho}}\,DD_t\check{r}\|_{L^2(\check{Q})} \leqslant K.$$

Consequently, $\check{r} \in C_b(\bar{J}\times\bar{\mathbb{R}}^+)$.

Proposition 3.4. *Let the hypotheses of Proposition* 3.3 *be valid. Then the following bounds for the kinetic and potential energies hold*:

$$\|\mathcal{E}\|_{L^1(\mathbb{R}^+)} \leqslant K, \quad \|\mathcal{E}'\|_{L^1(\mathbb{R}^+)+L^2(\mathbb{R}^+)} \leqslant K, \quad \|(\mathcal{E}+\mathcal{F})'\|_{L^1(\mathbb{R}^+)} \leqslant K \tag{3.35}$$

and, consequently, they stabilize

$$\mathcal{E}(t) \to 0,\ \mathcal{F}(t) \to \mathcal{F}^{(S)} \text{ as } t \to \infty. \tag{3.36}$$

Here, $L^1(\mathbb{R}^+) + L^2(\mathbb{R}^+)$ *is the sum of Banach spaces* (*see, for example,* [**3**]).

PROOF. Using (3.3), we get

$$\mathcal{E}' = -\int_J \mu(\rho)\rho(D(\varkappa v))^2\,d\chi + \int_J (p(\rho) - \check{p}_R[\rho])D(\varkappa v)\,d\chi + \int_J \Delta\check{f}[r]v\,d\chi.$$

Using the bounds $\rho \leqslant K_\rho$, (3.13), (3.5), and the energy bound, we find

$$|\mathcal{E}'| \leqslant \|\mu\|_{C[0,K_\rho]} \|\sqrt{\rho}\, D(\varkappa v)\|_J^2 + (\|p\|_{C[0,K_\rho]} + MN)\|D(\varkappa v)\|_{L^1(J)} + K\bar{f}. \tag{3.37}$$

By (3.34) and the energy bound,

$$\mathcal{E} \leqslant r_0^{-2k} M \|D(\varkappa v)\|_{L^1(J)}^2. \tag{3.38}$$

Then we obtain the first and second bounds (3.35) which imply that $\mathcal{E}(t) \to 0$ as $t \to \infty$ (see [**23**, Lemma 2.1]).

From the energy conservation law, simpler than (3.37) we derive the third bound (3.35). Thus, $(\mathcal{E} + \mathcal{F})(t)$ has a limit as $t \to \infty$ and so $\mathcal{F}(t)$ does. □

4. Stabilization of Velocity and Density in the Lagrangian Mass Coordinates

We define the functional

$$\check{\mathcal{P}}[\check{\rho}] := \int\limits_J \frac{1}{\check{\rho}}\, (p(\check{\rho}) - \check{p}_R[\check{\rho}])^2 d\chi = \int\limits_J \frac{1}{\check{\rho}}\, (\Delta p[\check{\rho}])^2 d\chi,$$

where $\Delta p[\check{\rho}] := p(\check{\rho}) - \check{p}_R[\check{\rho}]$ is the difference between the true and quasistationary pressures.

Lemma 4.1. *Let the hypotheses of Proposition* 3.3 *be valid, and let*

$$\|h_\nu\|_{L^1(J;\, C[\nu_0, K_V])} \leqslant C_1(N). \tag{4.1}$$

Then

$$\check{\mathcal{P}}[\check{\rho}] = \mathcal{R}_0' + \mathcal{R}, \tag{4.2}$$

where

$$\mathcal{R}_0 := \int\limits_J \frac{1}{\check{\rho}}\, (\Delta p[\check{\rho}])\, \check{I}^* \frac{\check{v}}{\check{\varkappa}}\, d\chi, \tag{4.3}$$

$$\mathcal{R} := -\int\limits_J \Big\{(\Delta p[\check{\rho}] - \check{\rho} p'(\check{\rho}))D(\check{\varkappa}\check{v}) - \frac{1}{\check{\rho}}\, D_t \check{p}_R[\check{\rho}]\Big\} \check{I}^* \frac{\check{v}}{\check{\varkappa}}\, d\chi + \int\limits_J \Big\{\mu(\check{\rho})D(\check{\varkappa}\check{v}) + \frac{1}{\check{\rho}} \Big(k\check{I}^* \frac{\check{v}^2}{\check{r}\check{\varkappa}} - \check{I}^* \frac{\Delta \check{f}[\check{r}]}{\check{\varkappa}}\Big)\Big\} \Delta p[\check{\rho}]\, d\chi \tag{4.4}$$

with

$$D_t\check{p}_R[\check{\rho}] = -\check{I}^*(\check{\varkappa}\check{v}h_\nu[\check{\nu}]). \tag{4.5}$$

Moreover, the following formula for the derivative holds:

$$(\check{\mathcal{P}}[\check{\rho}])' = \int\limits_J \Big\{(\Delta p[\check{\rho}] - 2\check{\rho}p'(\check{\rho}))D(\check{\varkappa}\check{v}) - \frac{2}{\check{\rho}}\, D_t\check{p}_R[\check{\rho}]\Big\}\Delta p[\check{\rho}]\, d\chi. \tag{4.6}$$

Proof. We write (3.11) as follows:

$$\Delta p[\rho] = D_t\check{I}^*\frac{v}{\varkappa} + \mu(\rho)\rho D(\varkappa v) + k\check{I}^*\frac{v^2}{r\varkappa} - \check{I}^*\frac{\Delta\check{f}[r]}{\varkappa}.$$

Consequently,

$$\begin{aligned}
\check{\mathcal{P}}[\rho] &= \int\limits_J (\Delta p[\rho])\, \eta\Delta p[\rho]\, d\chi \\
&= \frac{d}{dt}\int\limits_J \Big(\check{I}^*\frac{v}{\varkappa}\Big)\eta\Delta p[\rho]\, d\chi - \int\limits_J \Big(\check{I}^*\frac{v}{\varkappa}\Big) D_t(\eta\Delta p[\rho])\, d\chi \\
&+ \int\limits_J \Big\{\mu(\rho)D(\varkappa v) + \frac{1}{\rho}\Big(k\check{I}^*\frac{v^2}{r\varkappa} - \check{I}^*\frac{\Delta\check{f}[r]}{\varkappa}\Big)\Big\}\Delta p[\rho]\, d\chi.
\end{aligned}$$

Further,

$$\begin{aligned}
D_t(\eta\Delta p[\rho]) &= (D_t\eta)\Delta p[\rho] + \eta(p'(\rho)D_t\rho - D_t\check{p}_R[\rho]) \\
&= (D_t\eta)(\Delta p[\rho] - \rho p'(\rho)) - \eta D_t\check{p}_R[\rho].
\end{aligned} \tag{4.7}$$

These two formulas and the equations $D_t\eta = D(\varkappa v)$ and $D_t\nu = \varkappa v$ imply (4.2)–(4.5).

It is easy to see that

$$\begin{aligned}
(\check{\mathcal{P}}[\check{\rho}])' &= \int\limits_J \{(D_t\eta)(\Delta p[\rho])^2 + 2\eta(\Delta p[\rho])D_t\Delta p[\rho]\}d\chi \\
&= \int\limits_J \{2D_t\,(\eta\Delta p[\rho]) - (D_t\eta)\Delta p[\rho]\}\,\Delta p[\rho]\, d\chi.
\end{aligned}$$

Using formula (4.7) once again, we obtain formula (4.6). □

Proposition 4.1. *Let the hypotheses of Proposition* 3.3 *and the condition* (4.1) *be valid. Then*

$$|\mathcal{R}_0| \leqslant K\mathcal{E}^{1/2}, \quad \|\mathcal{R}\|_{L^1(\mathbb{R}^+)+L^2(\mathbb{R}^+)} \leqslant K, \quad \|(\check{\mathcal{P}}[\check{\rho}])'\|_{L^2(\mathbb{R}^+)} \leqslant K \tag{4.8}$$

and, consequently,

$$\check{\mathcal{P}}[\check{\rho}(\cdot,t)] \to 0 \ \textit{as}\ t \to \infty. \tag{4.9}$$

Proof. According to the bounds $\rho \leqslant K_\rho$ and (3.13), we get

$$\|\Delta p[\rho]\|_{L^\infty(J)} \leqslant K.$$

By the condition (4.1),

$$\|D_t\check{p}_R[\rho]\|_{C(\bar{J})} \leqslant K\|\varkappa v\|_{C(\bar{J})} \leqslant K\|D(\varkappa v)\|_{L^1(J)}. \tag{4.10}$$

Using the estimates (3.12), (3.14), (3.15), and $\|\eta\|_{L^1(J)} \leqslant K_V$, we find

$$|\mathcal{R}_0| \leqslant \|\eta\|_{L^1(J)}\|\Delta p[\rho]\|_{L^\infty(J)}\Big\|\check{I}^*\frac{v}{\varkappa}\Big\|_{C(\bar{J})} \leqslant K_1\mathcal{E}^{1/2}$$

and

$$\begin{aligned}
|\mathcal{R}| &\leqslant \{(\|\Delta p[\rho]\|_{L^\infty(J)} + \|sp'(s)\|_{L^\infty(0,K_\rho)})\|D(\varkappa v)\|_{L^1(J)}\\
&\quad + \|\eta\|_{L^1(J)}\|D_t\check{p}_R[\rho]\|_{C(\bar{J})}\}\Big\|\check{I}^*\frac{v}{\varkappa}\Big\|_{C(\bar{J})}\\
&\quad +\{\|\mu\|_{C[0,K_\rho]}\|D(\varkappa v)\|_{L^1(J)}+\|\eta\|_{L^1(J)}(kr_0^{-(k+1)}2\mathcal{E}+r_0^{-k}M\bar{f})\}\|\Delta p[\rho]\|_{L^\infty(J)}\\
&\leqslant K(\|D(\varkappa v)\|_{L^1(J)}\,\mathcal{E}^{1/2} + \|D(\varkappa v)\|_{L^1(J)} + \mathcal{E} + \bar{f}).
\end{aligned}$$

Since $\mathcal{E} \leqslant K_2$ and $\mathcal{E} \leqslant K_2^{1/2}\mathcal{E}^{1/2} \leqslant K_3\|D(\varkappa v)\|_{L^1(J)}$ (see (3.38)), we derive

$$|\mathcal{R}| \leqslant K(\|D(\varkappa v)\|_{L^1(J)} + \bar{f}).$$

Similarly,

$$\begin{aligned}
|(\check{\mathcal{P}}[\check{\rho}])'| &\leqslant \{(\|\Delta p[\rho]\|_{L^\infty(J)} + 2\|sp'(s)\|_{L^\infty(0,K_\rho)})\|D(\varkappa v)\|_{L^1(J)}\\
&\quad + 2\|\eta\|_{L^1(J)}\|D_t\check{p}_R[\rho]\|_{C(\bar{J})}\}\|\Delta p[\rho]\|_{L^\infty(J)} \leqslant K\|D(\varkappa v)\|_{L^1(J)}.
\end{aligned}$$

Recalling (3.34) and the energy bound, we obtain the estimates (4.8). These estimates and the property $\mathcal{E}(t) \to 0$ as $t \to \infty$ imply (4.9) (see [**23**, Lemma 2.1]). □

We introduce the function $\Delta\check{\sigma} := \check{\sigma} + \check{p}_R[\check{\rho}]$, which is the difference between the stress $\check{\sigma}$ and the quasistationary stress $-\check{p}_R[\check{\rho}]$.

Lemma 4.2. *Let the hypotheses of Proposition* 3.3 *and the condition* (4.1) *be valid. Then the function* $\Delta\check{\sigma}$ *satisfies the following energy-type* *equality*:

$$\frac{1}{2}Y' + \int_J (\check{\varkappa} D\Delta\check{\sigma})^2\, d\chi = -\frac{1}{2}\int_J (\beta(\check{\rho}) - \check{\rho}\beta'(\check{\rho}))(D(\check{\varkappa}\check{v}))(\Delta\check{\sigma})^2\, d\chi$$

$$- \int_J k\frac{\check{\varkappa}}{\check{r}}\,\check{v}^2 D\Delta\check{\sigma}\, d\chi - \int_J \Big\{[(\beta p)(\check{\rho}) - \check{\rho}(\beta p)'(\check{\rho})$$

$$- (\beta(\check{\rho}) - \check{\rho}\beta'(\check{\rho}))\check{p}_R[\check{\rho}]]D(\check{\varkappa}\check{v}) - \frac{\beta(\check{\rho})}{\check{\rho}}\, D_t\check{p}_R[\check{\rho}]\Big\}\Delta\check{\sigma}\, d\chi$$

$$- \int_J \Delta\check{f}[\check{r}]\check{\varkappa} D\Delta\check{\sigma}\, d\chi =: \frac{1}{2}S_1 + S_2 + S_3 + S_4, \tag{4.11}$$

where

$$Y := \int_J \frac{\beta(\check{\rho})}{\check{\rho}}\,(\Delta\check{\sigma})^2\, d\chi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+), \quad \beta := \frac{1}{\mu}.$$

PROOF. 1. We write the impulse equation (2.12) in the form

$$D_t v = \varkappa D\Delta\check{\sigma} + \Delta\check{f}[r]. \tag{4.12}$$

Multiplying by $-\varkappa D\Delta\check{\sigma}$, integrating over J, and using the formula $\varkappa D_t v = D_t(\varkappa v) - k(\varkappa/r)v^2$, we get

$$- \int_J (D_t(\varkappa v))\, D\Delta\check{\sigma}\, d\chi + \int_J (\varkappa D\Delta\check{\sigma})^2\, d\chi$$

$$= - \int_J k\frac{\varkappa}{r}\,\check{v}^2 D\Delta\check{\sigma}\, d\chi - \int_J \Delta\check{f}[r]\varkappa D\Delta\check{\sigma}\, d\chi. \tag{4.13}$$

By the definition of $\check{\sigma}$, we can write

$$D(\varkappa v) = \eta\beta(\rho)(\check{\sigma} + p(\rho)) = \eta\beta(\rho)(\Delta\check{\sigma} + \Delta\check{p}_R[\rho]). \tag{4.14}$$

We have

$$- \int_J (D_t(\varkappa v))\, D\Delta\check{\sigma}\, d\chi = \frac{1}{2}\frac{d}{dt}\int_J \eta\beta(\rho)(\Delta\check{\sigma})^2\, d\chi$$

$$+ \frac{1}{2}\int_J \{D_t(\eta\beta(\rho))\}(\Delta\check{\sigma})^2\, d\chi + \int_J \{D_t(\eta\beta(\rho)\Delta\check{p}_R[\rho])\}\Delta\check{\sigma}\, d\chi. \tag{4.15}$$

This formula is justified at the second step of the proof. Formally, it can be obtained by integrating by parts on its left-hand side and using formula (4.14). We calculate

$$D_t(\eta\beta(\rho)) = (D_t\eta)\beta(\rho) + \eta\beta'(\rho)D_t\rho = (\beta(\rho) - \rho\beta'(\rho))D(\varkappa v).$$

Similarly,

$$\begin{aligned} D_t(\eta\beta(\rho)\Delta\check{p}_R[\rho]) &= D_t(\eta(\beta p)(\rho) - \eta\beta(\rho)\check{p}_R[\rho]) \\ &= [(\beta p)(\rho) - \rho(\beta p)'(\rho)]D(\varkappa v) \\ &\quad - (\beta(\rho) - \rho\beta'(\rho))(D(\varkappa v))\,\check{p}_R[\rho] - \eta\beta(\rho)D_t\check{p}_R[\rho]. \end{aligned}$$

Substituting (4.15) into (4.13) and using the last two formulas, we obtain (4.11).

2. To justify formula (4.15), we apply the same technique as in [**27**, Lemma 1]. For the sake of brevity, we set $a := \eta\beta(\rho)$ and $\psi := \eta\beta(\rho)\Delta\check{p}_R[\rho]$. Introduce the following operators for $0 \leqslant t < t+\tau$:

$$S^{(\tau)}y(t) := \frac{1}{\tau}\int_0^\tau y(t+\theta)\,d\theta,\ \ \partial_t^{(\tau)}y(t) := \frac{y(t+\tau)-y(t)}{\tau},\ \ y^{(\tau)}(t) := y(t+\tau).$$

We take $0 \leqslant t_0 < t_1$ and set $\tilde{Q} := J \times (t_0, t_1)$. Integrating by parts and applying formula (4.14) in the form $D(\varkappa v) = a\Delta\check{\sigma} + \psi$, we get

$$\begin{aligned} &-\int_{\tilde{Q}} (\partial_t^{(\tau)}(\varkappa v))\, D\frac{\Delta\check{\sigma} + \Delta\check{\sigma}^{(\tau)}}{2}\, d\chi dt \\ &= \int_{\tilde{Q}} \Big[\frac{1}{2}\,\partial_t^{(\tau)}(a(\Delta\check{\sigma})^2) + \frac{1}{2}(\partial_t^{(\tau)}a)(\Delta\check{\sigma})\Delta\check{\sigma}^{(\tau)} + (\partial_t^{(\tau)}\psi)\frac{\Delta\check{\sigma} + \Delta\check{\sigma}^{(\tau)}}{2}\Big] d\chi dt \\ &= \frac{1}{2}\Big(S^{(\tau)}\int_J a(\Delta\check{\sigma})^2\Big)\Big|_{t=t_0}^{t=t_1} \\ &\quad + \int_{\tilde{Q}} \Big[\frac{1}{2}(S^{(\tau)}D_t a)(\Delta\check{\sigma})\Delta\check{\sigma}^{(\tau)} + (S^{(\tau)}D_t\psi)\frac{\Delta\check{\sigma} + \Delta\check{\sigma}^{(\tau)}}{2}\Big] d\chi dt. \end{aligned}$$

Using the properties of the involved functions, we can pass to the limit as $\tau \to 0^+$ and obtain the equality

$$
\begin{aligned}
&-\int\limits_{\tilde{Q}} (D_t(\varkappa v))D\Delta\check{\sigma}\,d\chi dt \\
&= \frac{1}{2}\int\limits_J a(\Delta\check{\sigma})^2\,d\chi\Big|_{t_0}^{t_1} + \int\limits_{\tilde{Q}} \Big[\frac{1}{2}(D_t a)(\Delta\check{\sigma})^2 + (D_t\psi)\Delta\check{\sigma}\Big]d\chi dt
\end{aligned}
$$

for almost all $0 < t_0 < t_1$. This equality implies that

$$
\int\limits_J a(\Delta\check{\sigma})^2\,d\chi \in W^{1,1}_{\text{loc}}(\mathbb{R}^+),
$$

$$
\begin{aligned}
-\int\limits_J (D_t(\varkappa v))D\Delta\check{\sigma}\,d\chi = \frac{1}{2}\,\frac{d}{dt}\int\limits_J a(\Delta\check{\sigma})^2\,d\chi \\
+ \int\limits_J \Big[\frac{1}{2}(D_t a)(\Delta\check{\sigma})^2 + (D_t\psi)\Delta\check{\sigma}\Big]d\chi
\end{aligned}
$$

almost everywhere (a.e.) on $\mathbb{R}^+$. $\square$

Let $\zeta \in W^{1,\infty}(\mathbb{R})$ be a cut-off function such that

$$
\zeta(t) = 0,\ t \leqslant 0, \quad \zeta(t) = 1,\ t \geqslant t_0 \text{ for some } t_0 > 0, \quad 0 \leqslant \zeta' \leqslant N. \tag{4.16}
$$

Theorem 4.1. *Let the hypotheses of Proposition* 3.3, *the condition* (4.1), *and* $\|\zeta\bar{f}\|_{L^2(\mathbb{R}^+)} \leqslant N$ *be satisfied. Then the following global bounds and the stabilization property for velocity hold*:

$$
\|\zeta D_t\check{v}\|_{L^2(\check{Q})} \leqslant K, \quad \sup_{t\geqslant 0}\|(\zeta\sqrt{\check{\rho}}\,D\check{v})(\cdot,t)\|_{L^2(J)} \leqslant K, \tag{4.17}
$$

$$
\|(\sqrt{\check{\rho}}\,D\check{v})(\cdot,t)\|_{L^2(J)} \to 0 \text{ as } t \to \infty. \tag{4.18}
$$

If, in addition,

$$
\|\sqrt{\check{\rho}^0}\,D\check{v}^0\|_{L^2(J)} \leqslant N, \quad \check{v}^0(0) = 0, \quad \|\bar{f}\|_{L^2(\mathbb{R}^+)} \leqslant N, \tag{4.19}
$$

then the bounds (4.17) *are valid for* $\zeta \equiv 1$.

Remark 4.1. It is clear that the property (4.18) implies $W^{1,1}(J)$- and $C(\bar{J})$-stabilization of $\check{v}(\cdot,t)$ since, as in the case (3.34), we have

$$
\|\check{v}\|_{C(\bar{J})} \leqslant \|D\check{v}\|_{L^1(J)} \leqslant K_V^{1/2}\|\sqrt{\check{\rho}}\,D\check{v}\|_{L^2(J)}.
$$

Proof of Theorem 4.1. 1. We set $\zeta_\tau(t) := \zeta(t-\tau)$, $\tau > 0$. Multiplying the energy-type equality (4.11) by ζ_τ^2, we get

$$\frac{1}{2}\,(\zeta_\tau^2 Y)' + \zeta_\tau^2 \int\limits_J (\check{\varkappa} D\Delta\check{\sigma})^2\,d\chi = \zeta_\tau\zeta_\tau' Y + \zeta_\tau^2\Big(\frac{1}{2}\,S_1 + S_2 + S_3 + S_4\Big). \quad (4.20)$$

We sequentially estimate the terms on the right-hand side. First, recalling (4.14), we get

$$Y \leqslant 2(\|\mu\|_{C[0,K_\rho]}\|\sqrt{\rho}\,D(\varkappa v)\|_J^2 + \underline{\mu}^{-1}\check{\mathcal{P}}[\rho]), \quad (4.21)$$

where $\check{\mathcal{P}}[\rho] \leqslant K$. Second, for any $\varepsilon > 0$ we derive

$$\begin{aligned} |S_1| &\leqslant \|\sqrt{\mu(s)}(\beta(s) - s\beta'(s))\|_{L^\infty(0,K_\rho)} \\ &\quad\times \|\sqrt{\rho}D(\varkappa v)\|_J\|\sqrt{\eta\beta(\rho)}\Delta\check{\sigma}\|_J\|\Delta\check{\sigma}\|_{L^\infty(J)} \\ &\leqslant \varepsilon\|\Delta\check{\sigma}\|^2_{L^\infty(J)} + \varepsilon^{-1}K\|\sqrt{\rho}\,D(\varkappa v)\|_J^2\,Y. \end{aligned}$$

Third, applying the energy bound and the inequalities (3.34), we find

$$|S_2| \leqslant kr_0^{-1}\|v\|_J\|v\|_{L^\infty(J)}\|\varkappa D\Delta\check{\sigma}\|_J \leqslant \varepsilon\|\varkappa D\Delta\check{\sigma}\|_J^2 + \varepsilon^{-1}K\|\sqrt{\rho}\,D(\varkappa v)\|_J^2.$$

Fourth, using the regularity properties of p and μ and the estimates (3.13) for $\check{p}_R[\rho]$ and (4.10) for $D_t\check{p}_R[\rho]$, we also get

$$\begin{aligned} |S_3| &\leqslant \{[\|(\beta p)(s) - s(\beta p)'(s))\|_{L^\infty(0,K_\rho)} \\ &\quad + \|\beta(s) - s\beta'(s))\|_{L^\infty(0,K_\rho)}\|\check{p}_R[\rho]\|_{L^\infty(J)}]K_V^{1/2}\|\sqrt{\rho}\,D(\varkappa v)\|_J \\ &\quad + \underline{\mu}^{-1}K_V\|D_t\check{p}_R[\rho]\|_{L^\infty(J)}\}\|\Delta\check{\sigma}\|_{L^\infty(J)} \\ &\leqslant \varepsilon\|\Delta\check{\sigma}\|^2_{L^\infty(J)} + \varepsilon^{-1}K\|\sqrt{\rho}\,D(\varkappa v)\|_J^2. \end{aligned}$$

Finally,

$$|S_4| \leqslant \bar{f}M^{1/2}\|\varkappa D\Delta\check{\sigma}\|_J \leqslant \varepsilon\|\varkappa D\Delta\check{\sigma}\|_J^2 + \varepsilon^{-1}M\bar{f}^{\,2}.$$

Using the above estimates and the inequalities

$$\|\Delta\check{\sigma}\|_{L^\infty(J)} \leqslant \|D\Delta\check{\sigma}\|_{L^1(J)} \leqslant r_0^{-k}M^{1/2}\|\varkappa D\Delta\check{\sigma}\|_J, \quad (4.22)$$

$\zeta_\tau \leqslant \zeta \leqslant 1$ and choosing ε small enough, we pass from (4.20) to the inequality

$$\begin{aligned} (\zeta_\tau^2 Y)' + \|\varkappa\zeta_\tau D\Delta\check{\sigma}\|_J^2 &\leqslant K(\|\sqrt{\rho}\,D(\varkappa v)\|_J^2\,\zeta_\tau^2 Y + \|\sqrt{\rho}\,D(\varkappa v)\|_J^2 \\ &\quad + \zeta_\tau' + (\zeta\bar{f})^2). \end{aligned} \quad (4.23)$$

Note that

$$Y \leqslant \underline{\mu}^{-1}K_V\|\Delta\check{\sigma}\|^2_{L^\infty(J)} \leqslant K_1\|\varkappa D\Delta\check{\sigma}\|_J^2$$

(see (4.22)), $(\zeta_\tau^2 Y)|_{t=0} = 0$, and

$$\|\|\sqrt{\rho}\,D(\varkappa v)\|_J^2 + \zeta_\tau' + (\zeta\bar{f})^2\|_{L^1(\mathbb{R}^+)} \leqslant K \quad (4.24)$$

by the energy bound and the assumptions on ζ_τ and $\bar{f}$. Consequently,

$$\sup_{t\geqslant 0}(\zeta_\tau^2 Y)(t) \leqslant K, \quad \|\zeta_\tau \varkappa D\Delta\check{\sigma}\|_{\check{Q}} \leqslant K, \tag{4.25}$$

$$(\zeta_\tau^2 Y)(t) \to 0 \text{ as } t \to \infty \tag{4.26}$$

(see, for example, [**22**, Lemma 2.1]). The bounds (4.25) are uniform with respect to τ. Therefore, passing to the limit as $\tau \to 0^+$, we find

$$\sup_{t\geqslant 0}(\zeta^2 Y)(t) \leqslant K, \quad \|\zeta \varkappa D\Delta\check{\sigma}\|_{\check{Q}} \leqslant K. \tag{4.27}$$

Applying formula (4.14) to $D(\varkappa v)$, we have

$$\|\sqrt{\rho}\, D(\varkappa v)\|_J \leqslant \underline{\mu}^{-1/2} Y^{1/2} + \underline{\mu}^{-1}\check{\mathcal{P}}[\rho].$$

Thus, the relations (4.27) and (4.26) and Proposition 4.1 imply that

$$\sup_{t\geqslant 0}\|(\zeta\sqrt{\rho}\, D(\varkappa v))(\cdot,t)\|_J \leqslant K, \quad \|(\sqrt{\rho}\, D(\varkappa v))(\cdot,t)\|_J \to 0 \text{ as } t \to \infty.$$

Moreover, since (by (2.13))

$$Dv = \frac{1}{\varkappa}\Big[D(\varkappa v) - \frac{D\varkappa}{\varkappa}\varkappa v\Big], \quad D\varkappa = \frac{k}{r}\,\eta,$$

we have (by $\|\eta\|_{L^1(J)} \leqslant K_V$)

$$\|\sqrt{\rho}\, Dv\|_J \leqslant K\|\sqrt{\rho}\, D(\varkappa v)\|_J, \tag{4.28}$$

which leads to the second bound (4.17) and the property (4.18).

In addition, according to Equation (4.12), the first bound (3.35), and the second bound (4.27), we have

$$\|\zeta D_t v\|_{\check{Q}} \leqslant \|\zeta'\|_{L^\infty(\mathbb{R}^+)}\|2\mathcal{E}\|^{1/2}_{L^1(\mathbb{R}^+)} + \|\zeta\varkappa D\Delta\check{\sigma}\|_{\check{Q}} + M^{1/2}\|\zeta\bar{f}\|_{\mathbb{R}^+} \leqslant K,$$

which leads to the first bound (4.17).

2. In the case where (4.19) are valid, a similar (and simpler) argument with $\zeta \equiv 1$ can be applied since $Y \in W^{1,1}_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$ and

$$Y|_{t=0} \leqslant 2(\|\mu\|_{C[0,K_\rho]}\|\sqrt{\rho^0}\, D(\varkappa^0 v^0)\|_J^2 + \underline{\mu}^{-1}\check{\mathcal{P}}[\rho^0]) \leqslant K,$$

similar to the estimate (4.21). □

In addition to (4.17), the bound $\|\zeta\check{\rho}^{-3/2}D_t\check{\rho}\|_{L^{2,\infty}(\check{Q})} \leqslant K$ holds according to the continuity equation (2.11). Moreover, under the condition (4.19), one can take $\zeta \equiv 1$.

By the moment, we did not suppose the monotonicity of p in any sense. However, we need this for studying the stabilization of $\check{\rho}$.

Theorem 4.2. *Let the hypotheses of Proposition* 3.3 (*for some* $\alpha \in (1,\gamma)$) *and the condition* (4.1) *be valid. Let also either*

$$p \text{ be increasing on } \bar{\mathbb{R}}^+ \tag{4.29}$$

or

$$p \text{ be nondecreasing on } \bar{\mathbb{R}}^+, \quad f_S(r,\chi) < 0 \text{ for } r \geqslant r_0,\ 0 < \chi \leqslant M. \tag{4.30}$$

Then for any sequence $t_n \to +\infty$ *there exists a subsequence* θ_n *such that*

$$\check{\eta}(\cdot,\theta_n) \to \eta_* \text{ weakly in } L^\lambda(J) \tag{4.31}$$

for any $1 < \lambda < \alpha$. *Moreover, for any sequence* $\theta_n \to \infty$, $\theta_n \geqslant 0$, *such that* (4.31) *takes place for some* $\lambda \in (1,\alpha)$ *the following assertions hold*:

(1) $\bar{\rho}_S := 1/\eta_*$ *serves as a solution to the Lagrangian static problem* (2.28), (2.29); *in addition,*

$$\bar{\mathcal{F}}[\bar{\rho}_S] := \int_J (P^{(0)}(\bar{\rho}_S) - \check{H}[\nu_S])\, d\chi = \mathcal{F}^{(S)} \tag{4.32}$$

(*see* $\mathcal{F}^{(S)}$ *in* (3.36)) *and, for* $\eta_S := \eta_*$,

$$\|\eta_S\|_{L^\lambda(J)} \leqslant K_\lambda \text{ for any } 1 \leqslant \lambda < \alpha; \tag{4.33}$$

(2) *the following strong limit relations hold*:

$$\check{\eta}(\cdot,\theta_n) \to \eta_S(\cdot) \text{ in } L^\lambda(J) \text{ for any } 1 \leqslant \lambda < \alpha, \tag{4.34}$$

$$\check{\rho}(\cdot,\theta_n) \to \bar{\rho}_S(\cdot) \text{ in } L^\lambda(J) \text{ for any } 1 \leqslant \lambda < \infty. \tag{4.35}$$

Proof. We divide the proof in three steps.

1. The property (4.31) follows from the uniform-in-time bound (3.19).

Let this be valid for some $\lambda = \lambda_0 \in (1,\alpha)$. Then we sequentially get

$$\nu|_{t=\theta_n} \to \nu_* := \nu_0 + \check{I}\eta_*, \quad \check{p}_R[\rho]|_{t=\theta_n} \to p_* := -\check{I}^*\check{h}[\nu_*] \text{ in } C(\bar{J}).$$

By Proposition 4.1,

$$\|p(\rho)|_{t=\theta_n} - p_*\|_J \leqslant (K_\rho \check{\mathcal{P}}[\rho](\theta_n))^{1/2} + \|\check{p}_R[\rho]|_{t=\theta_n} - p_*\|_J \to 0.$$

Therefore, for a subsequence τ_n of θ_n we have

$$p(\rho)|_{t=\tau_n} \to p_* \text{ a.e. in } J. \tag{4.36}$$

It is clear that $0 \leqslant p_* \leqslant p(K_\rho)$.

2. We first consider the case of increasing p. Introducing p^{-1}, the inverse of p, we get the pointwise convergence of ρ:

$$\rho|_{t=\tau_n} \to \bar{\rho}_S := p^{-1}(p_*) \text{ a.e. in } J. \tag{4.37}$$

It is clear that $\bar\rho_S \in C(\bar J)$ and $\bar\rho_S \geqslant 0$. By Proposition 3.3 and the Fatou lemma, we have

$$\int_J \frac{1}{\bar\rho_S^\lambda}\, d\chi \leqslant K_\lambda, \text{ for any } 1 \leqslant \lambda < \alpha. \tag{4.38}$$

Consequently, $\bar\rho_S > 0$ almost everywhere in J and the limit relation (4.37) implies

$$\eta|_{t=\tau_n} \to \eta_S := \frac{1}{\bar\rho_S} \text{ a.e. in } J. \tag{4.39}$$

The bound (4.38) implies (4.33). The bound (3.19) and the well-known Lemma 1.1.3 in [**18**] yield

$$\check\eta(\cdot,\tau_n) \to \eta_S \text{ weakly in } L^\lambda(J) \text{ for any } 1 < \lambda < \alpha. \tag{4.40}$$

Comparing with (4.31) for $\lambda = \lambda_0$, we conclude that $\eta_S = \eta_*$. Therefore, recalling the first step of the proof, we get $p(\bar\rho_S) = p_* = -\check I^*\check h[\nu_S]$ and $\nu_S = \nu_0 + \check I\eta_S$. Since $p(\bar\rho_S)$ is nonincreasing on $\bar J$ and $\bar\rho_S > 0$ almost everywhere in J, $\bar\rho_S$ is also nonincreasing on $\bar J$ and $\bar\rho_S > 0$ on $[0, M)$. Thus, $\bar\rho_S$ is a solution to the static problem (2.28), (2.29).

According to (4.40), we get

$$\int_J \eta\big|_{t=\tau_n}\, d\chi \to \int_J \eta_S\, d\chi. \tag{4.41}$$

It is clear that both sides of this relation are positive. Coupling this and (4.39), by the Scheffé theorem (see, for example, [**4**]), we have

$$\frac{\eta|_{t=\tau_n}}{\|\eta\big|_{t=\tau_n}\|_{L^1(J)}} \to \frac{\eta_S}{\|\eta_S\|_{L^1(J)}} \text{ in } L^1(J).$$

Invoking the relation (4.41) once again, we find that $\eta\big|_{t=\tau_n} \to \eta_S$ in $L^1(J)$. Now the multiplicative inequality

$$\|w\|_{L^\lambda(J)} \leqslant \|w\|_{L^1(J)}^{1-\beta}\|w\|_{L^{\bar\lambda}(J)}^\beta, \quad \frac{1}{\lambda} = 1 - \beta + \frac{\beta}{\bar\lambda} \text{ for any } 1 \leqslant \lambda \leqslant \bar\lambda,$$

implies the relation (4.34) for τ_n replacing θ_n.

By the bounds $\rho \leqslant K_\rho$ and $\nu \leqslant \nu_0 + K_V$ and the Lebesgue dominated convergence theorem, we have $\mathcal F(\tau_n) \to \bar{\mathcal F}[\bar\rho_S] = \mathcal F^{(S)}$ (recall that $\mathcal F(t) \to \mathcal F^{(S)}$ as $t \to \infty$); moreover, the property (4.35) holds for τ_n replacing θ_n.

Furthermore, the limit relations (4.34) and (4.35) hold for the whole θ_n. Indeed, assuming, for example, that $\|\rho\big|_{t=\theta_n^{(1)}} - \bar\rho_S\|_{L^1(J)} \geqslant \delta > 0$

for a subsequence $\theta_n^{(1)}$ of θ_n, nevertheless we are able to extract a subsequence $\tau_n^{(1)}$ of $\theta_n^{(1)}$ such that (4.37) holds for $\tau_n^{(1)}$ replacing τ_n. Consequently, $\rho\big|_{t=\tau_n^{(1)}} \to \bar{\rho}_S$ in $L^1(J)$, which yields a contradiction. The case of increasing p is complete.

3. Let the conditions (4.30) be valid. We go back to step 1 and note that $Dp_* = \check{h}[\nu_*]$. We have $\nu_*(\chi) \geqslant \nu_0 + K_\rho^{-1}\chi$ and $\check{h}[\nu_*] < 0$ on $(0, M]$. Therefore, $Dp_* < 0$ in J and p_* is decreasing on $\bar{J}$.

The set

$$\mathcal{S}_p := \{c > 0\colon \text{ the equation } p(s) = c \text{ has a nonunique solution}\}$$

is at most countable. Hence

$$\operatorname{meas}\{\chi \in \bar{J};\, p_*(\chi) \in \mathcal{S}_p\} = 0. \tag{4.42}$$

The inverse function p^{-1} is defined on $\bar{\mathbb{R}}^+ \backslash \mathcal{S}_p$. We set

$$p^{-1}(c) := \sup_{c_1 < c,\, c_1 \notin \mathcal{S}_p} p^{-1}(c_1)$$

for $c \in \mathcal{S}_p$ (for the sake of definiteness). Let J_0 be a subset of J such that $p(\rho)|_{t=\tau_n} \to p_*$ on J_0, $\operatorname{meas}(J\backslash J_0) = 0$ (see (4.36)). Let $c := p_*(\chi) \notin \mathcal{S}_p$ for some $\chi \in J_0$. If c is not a limiting point of $\mathcal{S}_p$, then p^{-1} is continuous in a neighborhood of c (or in a right-hand one provided that $c = 0$). Thus,

$$(p^{-1} \circ p)(\rho(\chi, \tau_n)) \to p^{-1}(p_*(\chi)). \tag{4.43}$$

Moreover, $\rho(\chi, \tau_n) = (p^{-1} \circ p)(\rho(\chi, \tau_n))$ for sufficiently large $n \geqslant n_0(\chi)$. Hence

$$\rho(\chi, \tau_n)) \to p^{-1}(p_*(\chi)). \tag{4.44}$$

If c is a limiting point of $\mathcal{S}_p$, then p^{-1} is discontinuous in any neighborhood of c but is continuous at c. Thus, the property (4.43) remains valid. Moreover, $(p^{-1} \circ p)(\rho(\chi, \tau_n)) - \rho(\chi, \tau_n) \to 0$. Therefore, the property (4.44) remains valid. Finally, the basic property (4.37) of Step 2 holds (if we take into account (4.42)).

After that the rest of the arguments of Step 2 is applicable, where $\bar{\rho}_S$ is decreasing on $\bar{J}$ (instead of $\bar{\rho}_S \in C(\bar{J})$ and $\bar{\rho}_S$ is nonincreasing). $\square$ me

Remark 4.2. A by-product of Theorem 4.2 is the existence of solutions to the Lagrangian static problem (2.28), (2.29) under the above conditions on p and h.

Note that, firstly, the condition (4.30) on f_S holds for $f_S = f_G$; secondly, the property (4.34) is stronger than (4.35) because of the equality $|\check{\rho} - \bar{\rho}_S| = |\check{\eta} - \eta_S|\check{\rho}\bar{\rho}_S$ and the bounds $\check{\rho} \leqslant K_\rho$ and $\bar{\rho}_S \leqslant K_\rho$.

It is clear that the property (4.31) for $\lambda = 1$ ensures us that

$$V(\theta_n) \to \nu_0 + \int_J \eta_* \, d\chi. \tag{4.45}$$

We define the ω-limit set for the specific volume $\mathcal{O}_{\check{\eta}}$ as a set of $\eta_* \in L^\lambda(J)$ such that property (4.31) holds, for some $1 < \lambda < \alpha$, and the ω-limit set for the density $\mathcal{O}_{\check{\rho}}$ as a set of $\rho_* = 1/\eta_*$ such that $\eta_* \in \mathcal{O}_{\check{\eta}}$. Theorem 4.2 implies that these sets are independent of λ and consist of functions respectively $\eta_* = \eta_S = 1/\bar{\rho}_S$ and $\rho_* = \bar{\rho}_S$, where $\bar{\rho}_S$ is a static solution such that (4.32)–(4.35) hold.

Proposition 4.2. *Let the hypotheses of Theorem* 4.2 *be valid. Then $\mathcal{O}_{\check{\eta}}$ and $\mathcal{O}_{\check{\rho}}$ are compact, attracting and connected sets in $L^\lambda(J)$, respectively for $1 \leqslant \lambda < \alpha$ and $1 \leqslant \lambda < \infty$.*

The attracting property, say, for $\mathcal{O}_{\check{\rho}}$, means that

$$\inf_{\bar{\rho}_S \in \mathcal{O}_{\check{\rho}}} \|\check{\rho}(\cdot, t) - \bar{\rho}_S(\cdot)\|_{L^\lambda(J)} \to 0 \ \text{as } t \to +\infty.$$

Proof. All the properties are proved in a standard manner (see, for example, [**9**, Theorem 4]). Note only that the compactness and attracting property of $\mathcal{O}_{\check{\eta}}$ imply the same properties of $\mathcal{O}_{\check{\rho}}$; also η and ρ belong to $C_b(\bar{\mathbb{R}}^+; L^\lambda(J))$, respectively for $1 \leqslant \lambda < \alpha$ and $1 \leqslant \lambda < \infty$ (for $\lambda = 1$, this clearly follows from the properties of $D_t\eta$ and $D_t\rho$ and then, for other λ, this holds because of the bounds (3.19) and $\rho \leqslant K_\rho$). □

The main result of Theorem 4.2 can be essentially simplified under weakened uniqueness assumptions for the static problem (concerning uniqueness, see Proposition 5.3 and Corollary 5.2 below).

Corollary 4.1. *Let, in addition to the hypotheses of Theorem* 4.2, *the Lagrangian static problem* (2.28), (2.29) *have one of the following properties.*

(i) *There exists no continual family of solutions $\bar{\rho}_S$ such that $V_S := \nu_S(M)$ runs over some segment $[\underline{\nu}, \bar{\nu}]$, $\nu_0 < \underline{\nu} < \bar{\nu}$, and $\bar{\mathcal{F}}_S[\bar{\rho}_S] = a$ for some fixed a, and the overdetermined static problem such that some $\nu_S(M) > \nu_0$ and $\bar{\mathcal{F}}_S[\bar{\rho}_S] = a$ are given, has at most one solution.*

(ii) *There exists an at most countable set of solutions $\bar{\rho}_S$ such that $\bar{\mathcal{F}}_S[\bar{\rho}_S] = a$ for any fixed a.*

Then there exists a static solution $\bar{\rho}_S$ such that

$$\check{\eta}(\cdot, t) \to \eta_S(\cdot) \ \text{in } L^\lambda(J) \text{ for any } 1 \leqslant \lambda < \alpha \text{ as } t \to \infty,$$

$$\check{\rho}(\cdot, t) \to \bar{\rho}_S(\cdot) \ \text{in } L^\lambda(J) \text{ for any } 1 \leqslant \lambda < \infty \text{ as } t \to \infty,$$

and the properties (4.32) and (4.33) hold with $\eta_S = 1/\bar{\rho}_S$.

Proof. By the property (4.45), Theorem 4.2, and the bounds $\nu_0 + MK_\rho^{-1} \leqslant V \leqslant K_V$, we conclude that

$$\mathcal{O}_V := [\underline{\nu}, \bar{\nu}], \quad \underline{\nu} := \liminf_{t\to+\infty} V(t),\ \bar{\nu} := \limsup_{t\to+\infty} V(t),$$

is the ω-limit set for V; moreover, for any $\nu \in [\underline{\nu}, \bar{\nu}]$ there exists a static solution $\bar{\rho}_S$ such that $V_S = \nu$ and $\bar{\mathcal{F}}_S[\bar{\rho}_S] = \mathcal{F}^{(S)}$. Properties (i) imply that the set $\mathcal{O}_{\check{\eta}}$ reduces to a point.

This reduction is induced by property (ii) as well since the set $\mathcal{O}_{\check{\eta}}$ is connected. □

Theorem 4.2 allows us to comment on the role of the conditions (3.16) and (3.18) in Proposition 3.3 (see also Proposition 5.2 below).

Corollary 4.2. *Let the hypotheses of Theorem* 4.2 *be valid, excepting* (3.16)*, with* (3.18) *weakened down to* (3.7)*. If the Lagrangian static problem* (2.28), (2.29) *has no solution, then*

$$\limsup_{t\to\infty} \left\| \frac{1}{\check{\rho}(\cdot,t)} \right\|_{L^{\alpha_1}(J)} = \infty \text{ for any } \alpha_1 \in (1, \alpha).$$

Proof. Note that (3.16) and the right-hand condition (3.18) were explicitly used only in the proof of Proposition 3.3. If, in addition to the hypotheses, we assume that $\sup_{t\geqslant 0} \|1/\rho(\cdot,t)\|_{L^{\alpha_1}(J)} < \infty$ for some $\alpha_1 \in (1, \alpha)$, then the argument of Theorem 4.2 remains valid and, in particular, there exists a static solution, which yields a contradiction. Note that the property $1/\rho(\cdot,t) \in L^\alpha(J)$ for any $t \geqslant 0$, is valid by Lemma 6.1. □

5. Static Problem

We set $\hat{p}(\xi) := p(1/\xi)$ for $\xi > 0$.

Lemma 5.1. *Let p be any continuous increasing function on $[0, s_0]$ for some $s_0 > 0$, and let $p(0) = 0$. Then*

$$\int_0^{p(s_0)} \frac{dq}{p^{-1}(q)} = \frac{p(s_0)}{s_0} + \int_0^{s_0} \frac{p(s)}{s^2}\, ds, \tag{5.1}$$

where both integrals are finite or equal to $+\infty$ simultaneously (recall that p^{-1} is the inverse of p).

PROOF. The inverse function of $1/p^{-1}$ is $\hat{p}$ that is continuous and decreasing on $[1/s_0, \infty)$. Considering the graphs of $1/p^{-1}$ and $\hat{p}$, we get

$$\int_0^{p(s_0)} \frac{dq}{p^{-1}(q)} = \frac{p(s_0)}{s_0} + \int_{1/s_0}^{\infty} \hat{p}(\xi)\, d\xi.$$

Making the change of variable $s = 1/\xi$, we obtain the result. □

In the case $\int\limits_0^{s_0} (p(s)/s^2)\, ds < \infty$, there exists $\lim\limits_{s \to 0^+} p(s)/s = 0$ by (5.1).

Corollary 5.1. *Let p satisfy the hypotheses of Lemma 5.1.*

1. *Suppose that $p_S \in C^1[M - \varepsilon_0, M]$ for some $\varepsilon_0 > 0$, $p_S(M) = 0$, and $Dp_S(M) < 0$. Then for sufficiently small $0 < \varepsilon \leqslant \varepsilon_1$*

$$c^{-1}(M - \chi) \leqslant p_S(\chi) \leqslant c(M - \chi) \quad \text{on } [M - \varepsilon, M]$$

with some $c > 0$ and the integrals

$$\int_{M-\varepsilon}^{M} \frac{d\chi}{p^{-1}(p_S)}, \quad \int_0^{s_0} \frac{p(s)}{s^2}\, ds$$

are finite or equal to $+\infty$ simultaneously.

2. *Suppose that p is continuous on $\bar{\mathbb{R}}^+$ and $h(\nu, M) < 0$ for $\nu \geqslant \nu_0$. If*

$$\int_0^{s_0} (p(s)/s^2)\, ds = \infty$$

(*in contrast to the third condition* (2.9)), *then the Lagrangian static problem* (2.28), (2.29) *has no solutions.*

We begin with the existence result for the static problem independently of Theorem 4.2 (see Remark 4.2).

Proposition 5.1. *Let p be increasing on $\bar{\mathbb{R}}^+$, and let the conditions* (3.16) *on p and* (3.18) *on h be valid. Then the Lagrangian static problem* (2.28), (2.29) *has a solution such that $K^{-1}(M - \chi) \leqslant p(\bar{\rho}_S(\chi)) \leqslant N(M - \chi)$ on $\bar{J}$.*

PROOF. Since p is increasing, we can equivalently reduce the static problem (2.28), (2.29) to the fixed point problem

$$p_S = \mathcal{A} p_S \text{ in } C(\bar{J}) \tag{5.2}$$

for the unknown static pressure $p_S := p(\rho_S)$. Here, the nonlinear operator $\mathcal{A}$ is given by the formula

$$\mathcal{A}\varphi := -\check{I}^*\check{h}[\nu_S[\varphi]], \quad \nu_S[\varphi] := \nu_0 + \check{I}\frac{1}{p^{-1}(\varphi)},$$

and we take into account Claim 1 of Corollary 5.1.

We define the nonempty closed convex set in $C(\bar{J})$

$$\mathcal{K}_\varepsilon := \{\varphi \in C(\bar{J});\, \varepsilon(M-\chi) \leqslant \varphi(\chi) \leqslant N(M-\chi) \text{ on } \bar{J}\},$$

where $\varepsilon \in (0, N)$ is a parameter. By (3.16), we have $p(s) \leqslant K_1 s^\gamma$ for $0 \leqslant s \leqslant p^{-1}(NM)$. Consequently, $p^{-1}(q) \geqslant (q/K_1)^{1/\gamma}$ for $0 \leqslant q \leqslant NM$. Therefore, for $\varphi \in \mathcal{K}_\varepsilon$

$$\nu_S[\varphi](\chi) \leqslant \bar{\nu}_{S,\varepsilon} := \nu_0 + \int\limits_0^M \Big[\frac{K_1}{\varepsilon(M-\chi)}\Big]^{1/\gamma} d\chi = \nu_0 + \frac{K_2}{\varepsilon^{1/\gamma}} \text{ on } \bar{J}$$

(with $K_2 := \gamma' K_1^{1/\gamma} M^{1/\gamma'}$) and $\nu_S[\varphi] \in C(\bar{J})$. This means that for $\varphi \in \mathcal{K}_\varepsilon$ the operator $\mathcal{A}\varphi$ is well defined. By (3.18), we have

$$\frac{\underline{\alpha}_0}{\bar{\nu}_{S,\varepsilon}^{2k/(k+1)}}(M-\chi) \leqslant \mathcal{A}\varphi(\chi) \leqslant N(M-\chi) \text{ on } \bar{J},$$

compare with (3.28). It is easy to check that the operator $\mathcal{A}$: $\mathcal{K}_\varepsilon \to C(\bar{J})$ is continuous and compact (because $\|\mathcal{A}\varphi\|_{C^1(\bar{J})} \leqslant K$ for $\varphi \in \mathcal{K}_\varepsilon$).

Since $\gamma > 2k/(k+1)$, the inequalities

$$\frac{\underline{\alpha}_0}{\bar{\nu}_{S,\varepsilon}^{2k/(k+1)}} \geqslant \frac{\underline{\alpha}_0}{K_2^{2k/(k+1)}}\, \varepsilon^{2k/[\gamma(k+1)]} \geqslant \varepsilon \tag{5.3}$$

hold for sufficiently small $0 < \varepsilon \leqslant K_3^{-1}$. For such ε it is clear that $\mathcal{A}$: $\mathcal{K}_\varepsilon \to \mathcal{K}_\varepsilon$. Finally, by the Schauder fixed point theorem, the problem (5.2) has a solution $p_S \in \mathcal{K}_\varepsilon$. $\square$

Remark 5.1. In the case $k = 2$ and the borderline value $\gamma = \gamma^{(2)} = 4/3$ in the condition (3.16), it is easy to see that the inequalities (5.3) remain valid for

$$c_0 K_1 M^{1/3} \leqslant \underline{\alpha}_0, \quad c_0 = 2^{8/3} \tag{5.4}$$

and $K_1 = K_1(NM)$ is nondecreasing with respect to NM (in particular, $K_1 = p_1$ is independent of NM provided that $p \leqslant p_{\gamma^{(2)}}$). Therefore, if $\inf_{M>0} \underline{\alpha}_0 > 0$, then the inequality (5.4) holds and Proposition 5.1 remains valid at least for sufficiently small $0 < M \leqslant \bar{M}(N)$.

But other situations are also possible. For example, let $\alpha_0(\chi) \geqslant \widetilde{G}(M_0 + i_0\chi)$ on $\bar{J}$. Thus, $\underline{\alpha}_0 \geqslant \widetilde{G}(M_0 + i_0(M/2))$. Let $p \leqslant p_{\gamma^{(2)}}$. Then (5.4) follows from the inequality

$$c_0\widetilde{p}_1 M^{1/3} \leqslant M_0 + i_0\frac{M}{2}, \quad \widetilde{p}_1 := \frac{p_1}{\widetilde{G}}. \tag{5.5}$$

For $M_0 > 0$ and $i_0 = 1$, the inequality holds both for $M/M_0 \leqslant \bar{M}(q)$ and $M/M_0 \geqslant \underline{M}(q)$, for $q := \widetilde{p}_1/M_0^{2/3} > q_0 > 0$, or for any M for $q \leqslant q_0$. On the other hand, for $M_0 > 0$, $i_0 = 0$ and $M_0 = 0$, $i_0 = 1$, the inequality is guaranteed respectively for $M \leqslant [M_0/(c_0\widetilde{p}_1)]^3$ and $M \geqslant (2c_0\widetilde{p}_1)^{3/2}$ only.

Now we turn to the Eulerian static problem (2.7), (2.8). We consider positive solutions such that

$$\rho_S \in L^\infty(\Omega_S), \quad p(\rho_S) \in W^{1,\infty}(\Omega_S), \quad p(\rho_S) > 0 \text{ on } [r_0, R_S). \tag{5.6}$$

Note that the nonlinear integro-differential equations

$$p(\rho_S(r)) = -\int_r^{R_S} \rho_S h[m_S]\varkappa\, dr_1 \text{ on } \bar{\Omega}_S \tag{5.7}$$

and

$$p(\bar{\rho}_S(\chi)) = -\int_\chi^M \check{h}[\nu_S]\, d\chi_1 \text{ on } \bar{J} \tag{5.8}$$

replace equivalently Equation (2.7) for ρ_S, together with the condition $p(\rho_S(R_S)) = 0$, and Equation (2.28) for $\bar{\rho}_S$, together with the condition $p(\bar{\rho}_S(M)) = 0$. The properties $p(\rho_S) \in C(\bar{\Omega}_S)$ and $p(\bar{\rho}_S) \in C(\bar{J})$ are able to replace equivalently the corresponding properties in (5.6) and (2.30). Making the change of variable $\chi = m_S(r)$ and the inverse change defined by the formula $\nu = \nu_S(\chi)$, we can transform Equations (5.7) and (5.8) one into another; the condition $R_S < \infty$ corresponds to $1/\bar{\rho}_S \in L^1(J)$. Thus, we have a one-to-one correspondence between solutions to the Eulerian and Lagrangian static problems. Consequently, the (non)existence or the (non)uniqueness for one of the problems implies the same for another.

Proposition 5.2. *Let $k = 2$ and p be increasing and satisfy the lower bound $p \geqslant p_\gamma$ for some $1 < \gamma < 4/3$, in contrast to the condition* (3.16)*. Let f_S satisfy the condition $-\dfrac{\bar{\alpha}_0}{r^2} \leqslant f_S(r,\chi)$ on $[r_0, \infty) \times \bar{J}$. Then the Eulerian static problem* (2.7)*,* (2.8) *can have positive solutions only if*

$$c(\gamma)p_1 r_0^{3(4/3-\gamma)} M^{\gamma-1} < \bar{\alpha}_0, \tag{5.9}$$

where $c(\gamma) = \gamma'[(4-3\gamma)/(\gamma-1)]^{\gamma-1}$.

PROOF. We write Equation (2.7) in the form

$$(\pi_0(\rho_S))_r = f_S[m_S], \quad \pi_0(s) := \int_0^{p(s)} \frac{d\zeta}{p^{-1}(\zeta)}. \tag{5.10}$$

The function π_0 is well defined continuous increasing on $\bar{\mathbb{R}}^+$ and $\pi_0(0) = 0$ according to Lemma 5.1. Consequently, by (2.8), we have

$$\pi_0(\rho_S(r)) = -\int_r^{R_S} f_S[m_S]\, dr_1.$$

Denoting by π_0^{-1} the inverse of π, we have

$$\int_{\Omega_S} \pi_0^{-1}\Big(-\int_r^{R_S} f_S[m_S]\, dr_1\Big) r^2\, dr = M. \tag{5.11}$$

Since $p \geqslant p_\gamma$, we also have $\pi_0 \geqslant \pi_{0\gamma}$ by formula (5.1), where $\pi_{0\gamma}(s) := \pi_0|_{p=p_\gamma}(s) = \gamma' p_1 s^{\gamma-1}$. Therefore, the equality (5.11) implies that

$$M \leqslant \int_{\Omega_S} \pi_{0\gamma}^{-1}\Big(-\int_r^{R_S} f_S[m_S]\, dr_1\Big) r^2\, dr \leqslant \int_{r_0}^{R_S} \pi_{0\gamma}^{-1}\Big(\bar{\alpha}_0\Big(\frac{1}{r} - \frac{1}{R_S}\Big)\Big) r^2\, dr.$$

The integral on the right-hand side is an increasing function of R_S (in particular, its derivative is positive). Therefore,

$$M < \int_{r_0}^{\infty} \pi_{0\gamma}^{-1}\Big(\frac{\bar{\alpha}_0}{r}\Big) r^2\, dr = \frac{\gamma-1}{4-3\gamma}\Big(\frac{\bar{\alpha}_0}{\gamma' p_1 r_0^{3(4/3-\gamma)}}\Big)^{1/(\gamma-1)}.$$

This inequality is equivalent to (5.9). □

In the case where $\bar{\alpha}_0 = G(M_0 + i_0 M)$, the inequality (5.9) is similar to the inequality (5.5) from Remark 5.1 and, consequently, it holds under similar conditions on M.

In the particular case $p = p_\gamma$, $1 < \gamma < 4/3$, and $f_S(r) = -\bar{\alpha}_0/r^2$, $\bar{\alpha}_0 \geqslant 0$, the problem is (uniquely) solvable if and only if the condition (5.9) is satisfied.

In both Propositions 5.1 and 5.2, the continuity of p on $\bar{\mathbb{R}}^+$ is sufficient instead of $p \in W^{1,\infty}_{\text{loc}}(\bar{\mathbb{R}}^+)$ and the condition (4.1) is not imposed, compare with Remark 4.2.

We define the so-called *first adiabatic exponent* of the state function

$$\Gamma_1(s) := \frac{sp'(s)}{p(s)} = -\left.\frac{\xi\hat{p}'(\xi)}{\hat{p}(\xi)}\right|_{\xi=1/s}, \quad s > 0. \tag{5.12}$$

Proposition 5.3. *Suppose that $p' \in C(\mathbb{R}^+)$, $p' > 0$, $h_\nu \in C([\nu_0, \infty) \times \bar{J})$, and there exists a function $h^{(1)}$ such that $h = h_\nu h^{(1)}$ (clearly $h^{(1)} = h/h_\nu$ on the set where $h_\nu \neq 0$) and*

$$h^{(1)}, h_\nu^{(1)}, h_\chi^{(1)} \in L^1(J; C[\nu_0, \nu]) \text{ for any } \nu > \nu_0, \tag{5.13}$$

$$\frac{1}{\Gamma_1(s)} + h_\nu^{(1)}(\nu, \chi) + sh_\chi^{(1)}(\nu, \chi) \leqslant 0 \text{ for any } s > 0,\ \nu > \nu_0,\ \chi \in J, \tag{5.14}$$

$$h^{(1)}(\nu_0, 0) := \lim_{(\nu,\chi)\to(\nu_0^+, 0+)} h^{(1)}(\nu, \chi) < 0. \tag{5.15}$$

Then the Lagrangian static problem (2.28), (2.29) *has at most one solution.*

Proof. We apply shooting method and, following [**29**, Proposition 10], consider the auxiliary Cauchy problem

$$Dp_S = \check{h}[\nu_S], \quad D\nu_S = \hat{p}^{-1}(p_S) \text{ on } (0, M), \tag{5.16}$$

$$p_S|_{\chi=0} = \lambda, \quad \nu_S|_{\chi=0} = \nu_0, \tag{5.17}$$

for the unknown functions $p_S(\chi, \lambda) > 0$ and $\nu_S(\chi, \lambda)$, with shooting parameter $\lambda > 0$. For any $\lambda > 0$ there exists a unique solution to the problem on $[0, \bar{M}(\lambda))$, where either $0 < \bar{M}(\lambda) < M$, $p_S((\bar{M}(\lambda))^-) = 0$, or $\bar{M}(\lambda) = M$.

It is clear that for any solution $\bar{\rho}_S$ to the problem (2.28), (2.29) the functions $p_S := p(\bar{\rho}_S) = \hat{p}(1/\bar{\rho}_S)$ and ν_S satisfy the problem (5.16), (5.17) for some $\lambda > 0$.

We also introduce the derivatives $\dot{p}_S = \partial p_S/\partial\lambda$ and $\dot{\nu}_S = \partial\nu_S/\partial\lambda$ which are well defined and satisfy the linear Cauchy problem

$$D\dot{p}_S = \check{h}_\nu[\nu_S]\dot{\nu}_S, \quad D\dot{\nu}_S = (\hat{p}^{-1})'(p_S)\dot{p}_S \text{ on } [0, \bar{M}(\lambda)), \tag{5.18}$$

$$\dot{p}_S|_{\chi=0} = 1, \quad \dot{\nu}_S|_{\chi=0} = 0 \tag{5.19}$$

(see, for example, [**12**, Sec. V.3]). By (5.19) and $(\hat{p}^{-1})' < 0$, we have

$$\dot{p}_S(\chi, \lambda) > 0, \quad D\dot{\nu}_S(\chi, \lambda) < 0, \quad \dot{\nu}_S(\chi, \lambda) < 0 \quad \text{on } [0, a(\lambda)),$$

where either $0 < a(\lambda) < \bar{M}(\lambda)$, $\dot{p}_S(a(\lambda)) = 0$, or $a(\lambda) = \bar{M}(\lambda)$.

We define the auxiliary function $\psi := p_S D\dot{\nu}_S - \dot{p}_S D(h^{(1)}[\nu_S])$. By the second equations (5.16) and (5.18) and the condition (5.14), we have

$$\psi = -\dot{p}_S\hat{p}^{-1}(p_S)\left\{\frac{1}{\Gamma_1\left(p^{-1}(p_S)\right)} + h_\nu^{(1)}[\nu_S] + p^{-1}(p_S)\, h_\chi^{(1)}[\nu_S]\right\} \geqslant 0 \tag{5.20}$$

for almost all $0 < \chi < a(\lambda)$ and any $\lambda > 0$. By the first equations (5.16) and (5.18), we get $-(Dp_S)\dot{\nu}_S + (D\dot{p}_S)h^{(1)}[\nu_S] = 0$ and, consequently, for any $0 < \chi_0 < \bar{M}(\lambda)$

$$\int_0^{\chi_0} \psi\, d\chi = \Big(p_S\dot{\nu}_S - \dot{p}_S h^{(1)}[\nu_S]\Big)\Big|_0^{\chi_0}, \tag{5.21}$$

where $h^{(1)}[\nu_S]|_{\chi=0} = h^{(1)}(\nu_0, 0) < 0$ by the condition (5.15).

The relations (5.20) and (5.21) imply $a(\lambda) = \bar{M}(\lambda)$ (since, in the case $a(\lambda) < \bar{M}(\lambda)$, it is not difficult to check that

$$\int_0^{a(\lambda)} \psi(\chi,\lambda)\, d\chi < 0$$

according to (5.21), (5.17), (5.19), and $\dot{\nu}_S(a(\lambda),\lambda) < 0$, which yields a contradiction).

Moreover, assume that the problem (2.28), (2.29) has two solutions corresponding to the values $\lambda = \lambda_0, \lambda_1$ such that $0 < \lambda_0 < \lambda_1$. Then $\bar{M}(\lambda)|_{\lambda=\lambda_0,\lambda_1} = M$ and $\nu_S(M^-,\lambda)|_{\lambda=\lambda_0,\lambda_1} < \infty$. Since both $p_S > 0$ and $\dot{p}_S > 0$ on the set $\{(\chi,t);\, 0 \leqslant \chi < \bar{M}(\lambda), \lambda > 0\}$, we have $\bar{M}(\lambda) \equiv M$ for $\lambda_0 \leqslant \lambda \leqslant \lambda_1$. We have

$$\|\dot{p}_S(\chi,\cdot)\|_{L^1(\lambda_0,\lambda_1)} = p_S(\chi,\lambda_1) - p_S(\chi,\lambda_0) \to 0 \text{ as } \chi \to M^-.$$

Further, $\nu_S(\chi,\lambda)$ increases with respect to $0 \leqslant \chi < \bar{M}(\lambda)$. Hence the limit $\nu_S(M^-,\lambda) \leqslant \infty$ exists for any $\lambda_0 \leqslant \lambda \leqslant \lambda_1$. Since $\dot{\nu}_S(\chi,\lambda)$ decreases in $\lambda_0 \leqslant \lambda \leqslant \lambda_1$ for $0 \leqslant \chi < M$, we have $\nu_S(M^-,\lambda) \leqslant \nu_S(M^-,\lambda_0) < \infty$ for $\lambda_0 \leqslant \lambda \leqslant \lambda_1$. Consequently, for any sequence $\chi_n \to M^-$ as $n \to \infty$ there exists a subsequence (not relabelled) such that $\dot{p}_S(\chi_n,\lambda) \to 0$ and $\nu_S(\chi_n,\lambda) \to \nu_S(M^-,\lambda)$ almost everywhere on (λ_0,λ_1). Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^{\chi_n} \psi(\chi,\lambda)\, d\chi \\
&\leqslant -\lim_{n\to\infty} \dot{p}_S(\chi_n,\lambda)h^{(1)}(\nu_S(\chi_n,\lambda),\chi_n) + h^{(1)}(\nu_0,0) \\
&= h^{(1)}(\nu_0,0) < 0,
\end{aligned}$$

which contradicts the inequality (5.20). □

Remark 5.2. If, in addition, $h \leqslant 0$ and the a priori bound $\bar{\rho}_S \leqslant s_0$ is known, then we can take $0 < s \leqslant s_0$ only in condition (5.14) (since $p_S(\chi,\lambda) \leqslant \lambda$ in this case and we can consider $\lambda \leqslant p(s_0)$ only in the proof).

Corollary 5.2. *Suppose that $p' \in C(\mathbb{R}^+)$, $p' > 0$, $f_S = f_G$, $k = 1, 2$. If*

$$\Gamma_1(s) \geqslant \gamma^{(k)} = \frac{2k}{k+1}, \quad 0 < s \leqslant s_0 := p^{-1}\Big(\frac{GM}{r_0^k}\Big(M_0 + i_0\frac{M}{2}\Big)\Big), \qquad (5.22)$$

then the Lagrangian static problem (2.28), (2.29) *has at most one solution.*

Proof. For $f_S = f_G$ and $k = 1, 2$ it is clear that $h^{(1)}(\nu, \chi) = -\nu/\gamma^{(k)}$, the inequality (5.14) reduces to $\Gamma_1(s) \geqslant \gamma^{(k)}$, and the a priori bound $p(\bar{\rho}_S) \leqslant (GM/r_0^k)(M_0 + i_0(M/2))$ holds. □

The condition (5.22) implies the inequality $p(s) \leqslant p(s_0)(s/s_0)^\gamma$ on $[0, s_0]$ for $\gamma = \gamma^{(k)}$, compare with (3.16).

6. Some Additional Bounds for Density and the Existence of Global Weak Solutions

We define the difference and integration operations with respect to t

$$\Delta^{(\tau)} y(t) := y(t+\tau) - y(t) \text{ for } \tau > 0, \quad I_t y(t) := \int_0^t y(\tau)\, d\tau.$$

Proposition 6.1. *Let the hypotheses of Proposition* 3.2 *be valid. Then the following generalized uniform Hölder condition in time of order* 1/4 *holds*:

$$\begin{aligned} \|\Delta^{(\tau)}\check{\rho}\|_{L^\infty(\check{Q})} &\leqslant K_\rho \|\Delta^{(\tau)} \log \check{\rho}\|_{L^\infty(\check{Q})} \\ &\leqslant K(\tau_0)\tau^{1/4} \textit{ for any } 0 < \tau \leqslant \tau_0 \end{aligned} \qquad (6.1)$$

with any $\tau_0 > 0$.

Proof. By Proposition 3.2, we have

$$|\Delta^{(\tau)}\rho| \leqslant K_\rho |\Delta^{(\tau)} \log \rho| \leqslant \frac{K_\rho}{\underline{\mu}} |\Delta^{(\tau)} A(\rho)|. \qquad (6.2)$$

We go back to Equation (3.11) and set $z := \check{I}^*[(v/\varkappa) - I_t(\Delta\check{f}[r]/\varkappa)]$. Then

$$\begin{aligned} &|\Delta^{(\tau)} A(\rho)| \leqslant |\Delta^{(\tau)}(A(\rho) - z)| + |\Delta^{(\tau)} z| \\ &\leqslant \tau S^{(\tau)} |D_t(A(\rho) - z)| + \sqrt{2}\, \|\tau S^{(\tau)}|D_t z|\|_J^{1/2} \|D\Delta^{(\tau)} z\|_J^{1/2} \\ &\leqslant \tau \|D_t(A(\rho) - z)\|_{L^\infty(\check{Q})} + 2\tau^{1/4} \sup_{t>0} \|D_t z\|_{J\times(t,t+\tau)}^{1/2} \|Dz\|_{L^{2,\infty}(\check{Q})}^{1/2}. \end{aligned}$$

Equation (3.11) can be written in the form

$$D_t(A(\rho)-z) = -p(\rho) + \check{p}_R[\rho] + k\check{I}^*\frac{v^2}{r\varkappa}. \tag{6.3}$$

It is clear that

$$-D_t z = D_t(A(\rho)-z) + \mu(\rho)\rho D(\varkappa v), \quad Dz = -\frac{v}{\varkappa} + I_t\frac{\Delta\check{f}[r]}{\varkappa}. \tag{6.4}$$

Using the estimates (3.13)–(3.15), $\rho \leqslant K_\rho$, and the energy bound, we find

$$\|D_t(A(\rho)-z)\|_{L^\infty(\check{Q})} \leqslant K_1,$$

$$\sup_{t>0}\|D_t z\|_{J\times(t,t+\tau)} \leqslant \sqrt{M\tau}K_1 + K_2,$$

$$\|Dz\|_{L^{2,\infty}(\check{Q})} \leqslant K_3.$$

Consequently, $\|\Delta^{(\tau)}A(\rho)\|_{L^\infty(\check{Q})} \leqslant K_4(\tau_0)\tau^{1/4}$. By the inequality (6.2), we obtain the required assertion. $\square$

Remark 6.1. Let the hypotheses of Theorem 4.1 be valid. Then the following generalized Hölder condition in time of order 1/2 holds:

$$\|\zeta\Delta^{(\tau)}\check{\rho}\|_{L^\infty(\check{Q})} \leqslant K_\rho\|\zeta\Delta^{(\tau)}\log\check{\rho}\|_{L^\infty(\check{Q})} \leqslant K(\tau_0)\sqrt{\tau} \text{ for any } 0<\tau\leqslant\tau_0$$

with any $\tau_0 > 0$. Moreover, under the condition (4.19), one can take $\zeta \equiv 1$.

Indeed, note that the equalities (6.3) and (6.4) imply

$$-D_t z = \Delta\check{\sigma} + k\check{I}^*\frac{v^2}{r\varkappa}.$$

Therefore, by the first bound (4.27) and the estimate (3.14), we have

$$\zeta\|\tau S^{(\tau)}|D_t z|\|_J^{1/2} \leqslant \sqrt{\tau}\,\|\zeta D_t z\|_{L^{2,\infty}(\check{Q})}^{1/2} \leqslant K\sqrt{\tau}$$

(taking into account the properties (4.16) of ζ).

Lemma 6.1. *Let the hypotheses of Proposition* 3.2 *be valid. Then for any* $T>0$ *the following auxiliary pointwise two-sided bound for* $\check{\rho}$ *holds*:

$$K(T)^{-1}\check{\rho}^0(\chi) \leqslant \check{\rho}(\chi,t) \leqslant K(T)\check{\rho}^0(\chi) \tag{6.5}$$

for almost all $\chi \in J$ *and any* $0 \leqslant t \leqslant T$.

Proof. Applying the operator I_t to the ordinary differential equation (3.11) for almost all $\chi \in J$ (such that $D_t\rho(\chi,\cdot) \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$) and using the estimates (3.12)–(3.15) and $0 \leqslant p(\rho) \leqslant \|p\|_{C[0,K_\rho]}$, we get

$$|A(\rho(\chi,t)) - A(\rho^0(\chi))| \leqslant K_1 + K_2 T \text{ on } [0,T].$$

Since

$$\underline{\mu} \log \max\{\rho^0/\rho, \rho/\rho^0\} \leqslant |A(\rho) - A(\rho^0)|,$$

we get the bound (6.5) with $K(T) := e^{(K_1+K_2T)/\underline{\mu}}$. □

Remark 6.2. Let the hypotheses of Proposition 4.1 be valid.

1. If the bound $\check{\rho} \leqslant K_0\check{\rho}^0$ in $\check{Q}$ holds, then $\check{\rho}^0$ satisfies the inequality $K^{-1}(M-\chi) \leqslant (\check{\rho}^0(\chi))^\gamma$ on J.

2. Let $c_1 s^{\gamma_1} \leqslant p(s)$ for $0 < s \leqslant 1$ with some $c_1 > 0$ and $\gamma_1 \geqslant \gamma$. If the bound $K_0^{-1}\check{\rho}^0 \leqslant \check{\rho}$ in $\check{Q}$ holds, then $\check{\rho}^0$ satisfies the inequality $(\check{\rho}^0(\chi))^{\gamma_1} \leqslant K(M-\chi)$ on J.

Proof. 1. Using the estimate (3.25) for p, the bounds $\rho \leqslant K_0\rho^0$ and $K^{-1}(M-\chi) \leqslant \check{p}_R[\rho](\chi,t)$ (which follows from (3.28) and (3.20)), we get

$$(p(\rho) - \check{p}_R[\rho])(\chi,t) \leqslant K(\rho^0(\chi))^\gamma - K^{-1}(M-\chi) \text{ in } \check{Q}.$$

Passing to the limit as $t \to \infty$ in $L^2(J)$ and using Proposition 4.1, we have $0 \leqslant K(\rho^0(\chi))^\gamma - K^{-1}(M-\chi)$.

2. The proof of Claim 2 is similar with using the bound $\check{p}_R[\rho](\chi,t) \leqslant N(M-\chi)$, compare with (3.13). □

We define the difference and mollification operations relative to χ

$$\Delta_\delta\varphi(\chi) := \varphi(\chi+\delta) - \varphi(\chi), \quad S_\delta\varphi(\chi) := \frac{1}{\delta}\int_\chi^{\chi+\delta} \varphi(\chi_1)\,d\chi_1, \quad 0 < \delta < M.$$

Proposition 6.2. *Let the hypotheses of Proposition* 3.2 *be valid. Then for any* $T > 0$ *the following auxiliary pointwise uniform bounds hold*:

$$\begin{aligned} |\Delta_\delta\check{\rho}(\chi,t)| &\leqslant K(T)\{|\Delta_\delta\check{\rho}^0(\chi)| + \delta[(S_\delta|v|)(\chi,t) + (S_\delta|v^0|)(\chi) + 1]\} \\ &\leqslant K_1(T)(|\Delta_\delta\check{\rho}^0(\chi)| + \sqrt{\delta}) \end{aligned} \tag{6.6}$$

in $\check{Q}_{M-\delta,T} := (0, M-\delta) \times (0,T)$ *for any* $0 < \delta < M$.

Proof. Applying the operator S_δ to Equation (3.9), we find

$$D_t S_\delta \frac{v}{\varkappa} = \frac{1}{\delta}\,\Delta_\delta\check{\sigma} + S_\delta g, \quad g := \check{h}[\nu] - \frac{k}{r\varkappa}\,v^2 + \frac{\Delta\check{f}[r]}{\varkappa}.$$

We write the equality $\mu(\rho)\rho D(\varkappa v) = -D_t A(\rho)$ in the form

$$D_t w_\delta = -\frac{1}{\delta}\,\Delta_\delta p(\rho) + S_\delta g, \quad w_\delta := S_\delta\frac{v}{\varkappa} + \frac{1}{\delta}\Delta_\delta A(\rho).$$

Then

$$|w_\delta| \leqslant |w_\delta^0| + I_t\Big(\Big\|\frac{sp'(s)}{\mu(s)}\Big\|_{L^\infty(0,K_\rho)}\frac{1}{\delta}|\Delta_\delta A(\rho)| + S_\delta|g|\Big),$$

where $w_\delta^0 := S_\delta(v^0/\varkappa^0) + \delta^{-1}\Delta_\delta A(\rho^0)$. Taking into account the inequality

$$\frac{1}{\delta}|\Delta_\delta A(\rho)| \leqslant |w_\delta| + r_0^{-k} S_\delta |v| \tag{6.7}$$

and the estimates $\|v\|_{L^{\infty,2}(\check{Q})} \leqslant K$, (3.13) and (3.15), we derive that

$$\|g\|_{L^{\infty,1}(\check{Q}_T)} \leqslant NT + K.$$

Hence $|w_\delta| \leqslant |w_\delta^0| + KI_t|w_\delta| + K_1(T)$ in $\check{Q}_{M-\delta,T}$. By the Gronwall lemma, $|w_\delta| \leqslant K_2(T)(|w_\delta^0|+1)$ in $\check{Q}_{M-\delta,T}$. Using the inequality (6.7) again, we find

$$|\Delta_\delta A(\rho)| \leqslant K_3(T)\{|\Delta_\delta A(\rho^0)| + \delta[S_\delta|v^0| + S_\delta|v| + 1]\}$$

in $\check{Q}_{M-\delta,T}$. Therefore,

$$\underline{\mu}\,|\Delta_\delta \log\rho| \leqslant K_3(T)\{\|\mu\|_{C[0,N]}|\Delta_\delta \log\rho^0| + \delta[S_\delta|v^0| + S_\delta|v| + 1]\}. \tag{6.8}$$

We define the divided difference for the logarithmic function for $s > 0$ and $s_1 > 0$:

$$\log(s; s_1) := \frac{\log s - \log s_1}{s - s_1} \text{ for } s \neq s_1,\ \log(s;s) := \frac{1}{s}.$$

It is clear that

$$\log(s; s_1) = \int_0^1 \frac{da}{(1-a)s + as_1}.$$

We have $\Delta_\delta \log\rho = \log(\rho_{(\delta)};\rho)\Delta_\delta\rho$ and $\Delta_\delta \log\rho^0 = \log(\rho^0_{(\delta)};\rho^0)\Delta_\delta\rho^0$, where, for example, $\rho_{(\delta)}(x,t) := \rho(x+\delta,t)$. By the right-hand bound (6.5), we get $\log(\rho^0_{(\delta)};\rho^0) \leqslant K(T)\log(\rho_{(\delta)};\rho)$ in $\check{Q}_{M-\delta,T}$; also $K_\rho^{-1} \leqslant \log(\rho_{(\delta)};\rho)$. Now estimate (6.8) yield the left-hand bound (6.6) and, by the energy bound, the right-hand one as well. □

Corollary 6.1. *Let the hypotheses of Proposition* 3.2 *be valid and* $\check{\rho}^0 \in C(\bar{J})$. *Then* $\check{\rho} \in C(\bar{J} \times \bar{\mathbb{R}}^+)$.

Proof. We consider any $0 < \delta \leqslant \delta_0 < M$, $\tau > 0$ and $T > 0$. The mollification $S_\delta S^{(\tau)}\rho$ is continuous on $[0, M-\delta] \times \bar{\mathbb{R}}^+$ and tends to ρ in $L^\infty((0, M-\delta_0) \times (0,T))$ as both $\delta \to 0$ and $\tau \to 0$ by Propositions 6.1 and 6.2. Consequently, $\rho \in C([0, M-\delta_0] \times [0,T])$. Allowing for $-\delta_0 \leqslant \delta < 0$, we similarly get $\rho \in C([\delta_0, M] \times [0,T])$. □

Proof of Proposition 2.1. In the case $\operatorname{ess\,inf}_J \rho^0 > 0$, the existence of weak solutions can be established by a well-developed technique (see, in particular, [**28**] and [**1, 2**]). Thus this is omitted, and we concentrate on the case $\operatorname{ess\,inf}_J \rho^0 = 0$. We take $\rho_n^0 := \rho^0 + \varepsilon_n$, with some $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, and consider a corresponding weak solution ρ_n and v_n together with $r_n > 0$ such that $r_n^{k+1}/(k+1) = \nu_0 + I(1/\rho_n)$.

We fix any $T > 0$ and consider the domain $\check{Q}_T$. By Propositions 6.1 and 6.2 together Lemma 6.1 and Remark 3.1 we get (after passing to a subsequence, not relabelled)

$$\begin{aligned} &\rho_n \to \rho,\ r_n \to r \text{ in } L^q(\check{Q}_T) \text{ for any } 1 \leqslant q < \infty \text{ and a.e. in } \check{Q}_T, \\ &D_t\rho_n \to D_t\rho,\ D_t r_n \to D_t r \text{ weakly in } L^2(\check{Q}_T). \end{aligned} \tag{6.9}$$

It is clear that the function ρ satisfies the two-sided bounds (2.27). Therefore, $\|1/\rho\|_{L^{1,\infty}(\check{Q}_T)} \leqslant C(T)\|1/\rho^0\|_{L^1(J)}$. Moreover, $\rho|_{t=0} = \rho^0$. Also $r_0 \leqslant r \leqslant C(T)$ and

$$\frac{r_n^{k+1}}{k+1} = \nu_0 + I\frac{1}{\rho_n} \to \frac{r^{k+1}}{k+1} = \nu_0 + I\frac{1}{\rho} \text{ and a.e. in } \check{Q}_T$$

(taking account of the uniform in n bounds $1/\rho_n \leqslant C(T)/\rho_n^0 \leqslant C(T)/\rho^0$), i.e., Equation (2.13) holds.

By Proposition 3.1 and Lemma 6.1 (see also inequality (4.28)), we have (after passing to a subsequence, not relabelled)

$$\begin{aligned} &v_n \to v \text{ weakly star in } L^{2,\infty}(\check{Q}_T), \\ &Dv_n \to Dv \text{ weakly in } L^2_{\rho^0}(\check{Q}_T) \end{aligned} \tag{6.10}$$

taking account of $\rho^0 \leqslant \rho_n^0$. Here $L^2_{\rho^0}(\check{Q}_T)$ is a Hilbert space (a weighted Lebesgue one) with the norm $\|w\|_{L^2_{\rho^0}(\check{Q}_T)} := \|\sqrt{\rho^0}\, w\|_{L^2(\check{Q}_T)}$. The boundary condition $v_n|_{x=0} = 0$ on $(0,T)$ yields that $v|_{x=0} = 0$ on $(0,T)$ too. Passing to the limit in the equation $D_t r_n = v_n$, we derive equation (2.17).

Property (6.10) implies that

$$\rho_n^0 Dv_n \to \rho^0 Dv \text{ weakly in } L^{1,2}(\check{Q}_T)$$

taking account of $\rho_n^0 - \rho^0 \equiv \varepsilon_n$ and the uniform in n bound $\|Dv_n\|_{L^{1,2}(\check{Q}_T)} \leqslant C(T)$, see inequality (2.25). Furthermore, let $\varkappa_n := r_n^k$ and $\varkappa = r^k$. Using the formula $\rho D(\varkappa v) = \rho\varkappa Dv + (k/r)v$, we get

$$\rho_n D(\varkappa_n v_n) - \rho D(\varkappa v) = \Big(\frac{\rho_n}{\rho_n^0}\,\varkappa_n - \frac{\rho}{\rho^0}\,\varkappa\Big)\rho_n^0 Dv_n + \frac{\rho}{\rho^0}\,\varkappa(\rho_n^0 Dv_n - \rho^0 Dv)$$

$$+\,k\Big(\frac{1}{r_n}-\frac{1}{r}\Big)v_n+\frac{k}{r}(v_n-v)\to 0$$

weakly in $L^{1,s}(\check{Q}_T)$ for any $1\leqslant s<2$ since

$$\frac{\rho_n}{\rho_n^0}\,\varkappa_n\leqslant C(T),\quad \frac{\rho}{\rho^0}\,\varkappa\leqslant C(T),\quad \frac{\rho_n}{\rho_n^0}\,\varkappa_n\to\frac{\rho}{\rho^0}\,\varkappa$$

in $L^q(\check{Q}_T)$ for any $1\leqslant q<\infty$, and $\|\rho_n^0 Dv_n\|_{L^2(\check{Q}_T)}\leqslant C(T)$. Applying properties (6.9) for ρ_n once again, we derive that also

$$\begin{aligned}&\rho_n^2 D(\varkappa_n v_n)\to\rho^2 D(\varkappa v),\\ &\check{\sigma}_n:=\mu(\rho_n)\rho_n D(\varkappa_n v_n)-p(\rho_n)\to\check{\sigma}=\mu(\rho)\rho D(\varkappa v)-p(\rho)\\ &\quad\text{weakly in } L^{1,s}(\check{Q}_T)\text{ for any } 1\leqslant s<2.\end{aligned}\tag{6.11}$$

Passing to the limit in the equation $D_t\rho_n=-\rho_n^2 D(\varkappa_n v_n)$, we derive equation (2.12). Since $\sqrt{\rho}\,Dv, 1/\sqrt{\rho}\in L^2(\check{Q}_T)$, actually $\check{\sigma}/\sqrt{\rho}\in L^2(\check{Q}_T)$ too. Passing to the limit in the identity

$$\int\limits_{\check{Q}_T}(-\check{v}_n D_t\varphi+\check{\sigma}_n D\varphi)d\chi dt=\int\limits_J \check{v}^0\varphi|_{t=0}\,d\chi+\int\limits_{\check{Q}_T}\check{f}[\check{r}_n]\varphi\,d\chi dt,$$

for any $\varphi\in H^1(\check{Q}_T)$ such that $\varphi|_{\chi=0}=0$ and $\varphi|_{t=T}=0$, we derive identity (2.24). This implies that $D_t v\in L^2(0,T;H^*)+L^{\infty,1}(\check{Q}_T)$, where $H:=\{\varphi\in L^2(J);\ \sqrt{\rho^0}\,D\varphi\in L^2(J),\ \varphi(0)=0\}$ is a Hilbert space equipped with the norm $\|\varphi\|_H:=\|\sqrt{\rho^0}\,D\varphi\|_{L^2(J)}$ and H^* is its conjugate one; consequently, $v\in C([0,T];L^2(J))$.

By the argument of Theorem 4.1, the uniform in n bound

$$\Big\|\zeta\frac{\check{\sigma}_n}{\sqrt{\rho_n}}\Big\|_{L^{2,\infty}(\check{Q}_T)}+\|\zeta D\check{\sigma}_n\|_{L^2(\check{Q}_T)}+\|\zeta D_t v_n\|_{L^2(\check{Q}_T)}\leqslant C(T)$$

holds. Consequently (after passing to a subsequence, not relabelled), for any $0<t_0<T$

$$\begin{aligned}&\frac{\check{\sigma}_n}{\sqrt{\rho_n}}\to\frac{\check{\sigma}}{\sqrt{\rho}}\text{ weakly star in } L^{2,\infty}(\check{Q}_T\backslash\check{Q}_{t_0}),\\ &D\check{\sigma}_n\to D\check{\sigma}\text{ and } D_t v_n\to D_t v\text{ weakly in } L^2(\check{Q}_T\backslash\check{Q}_{t_0}),\end{aligned}$$

where properties (6.10) for ρ_n and (6.11) for $\check{\sigma}_n$ have been applied. The boundary condition $\check{\sigma}_n|_{x=M}=0$ on (t_0,T) yields that $\check{\sigma}|_{x=M}=0$ on (t_0,T) too. Using also properties (6.9) for r_n, we pass to the limit in the equation $D_t v_n=\varkappa_n D\check{\sigma}_n+\check{f}[r_n]$ and derive equation (2.12).

Under the additional conditions $\sqrt{\check{\rho}^0}\, D\check{v}^0 \in L^2(J)$, $v^0(0) = 0$ and $\bar{f} \in L^2_{\mathrm{loc}}(\bar{\mathbb{R}}^+)$, one can take $\zeta \equiv 1$ and $t_0 = 0$. All of that together with Corollary 6.1 proves Proposition 2.1. □

7. Properties of Solutions in the Eulerian Coordinates

We consider weak solutions ρ, v and R to the Eulerian problem (2.1)–(2.5) arising from the weak solutions $\check{\rho}$ and $\check{v}$ to the Lagrangian problem (2.11)–(2.15) after the inverse change of variable

$$r = \check{r}(\chi, t) = \Big[(k+1)\Big(\nu_0 + \int\limits_0^\chi \frac{1}{\check{\rho}(\chi_1, t)}\, d\chi_1\Big)\Big]^{1/(k+1)}$$

so that

$$\rho(\check{r}(\chi,t),t) \equiv \check{\rho}(\chi,t), \quad v(\check{r}(\chi,t),t) \equiv \check{v}(\chi,t), \ R(t) \equiv \check{r}(M,t) \text{ in } \check{Q}. \quad (7.1)$$

On the initial data and h we impose the conditions

$$\rho^0 \in L^\infty(\Omega_0), \ \operatorname{ess\,inf}_{(r_0,r_1)} \rho^0 > 0 \text{ for any } r_1 \in \Omega_0,$$
$$\sqrt{\rho^0}\, v^0 \in L^2(\Omega_0), \ h \in C_b([\nu_0, \infty) \times \bar{J}),$$

together with the conditions on Δf described in Section 2.

We first collect the main bounds of Sections 3 and 4 restated in the Eulerian coordinates.

Proposition 7.1. 1. *Let*

$$\Big\| \frac{1}{2}\, \rho^0 (v^0)^2 + \rho^0 P^{(0)}(\rho^0) \Big\|_{L^1(\Omega_0)} \leqslant N, \ h \leqslant 0.$$

Then the following uniform-in-time energy bound holds:

$$\sup_{t \geqslant 0} \Big\| \Big(\frac{1}{2}\, \rho v^2 + \rho P^{(0)}(\rho)\Big)(\cdot, t) \Big\|_{L^1(\Omega_t)} + \Big\| \frac{1}{\sqrt{\varkappa}}\, (\varkappa v)_r \Big\|_{L^2(Q)} \leqslant K.$$

2. *Let* $\rho^0 \leqslant N$, $\|\sqrt{\rho^0}\, v^0\|_{L^2(\Omega_0)} \leqslant N$, $-N \leqslant h \leqslant 0$ *in* $[\nu_0, \infty) \times \bar{J}$. *Then the uniform upper bound* $\rho \leqslant K_\rho$ *in* Q *holds.*

3. *Let* p *and the data satisfy the conditions*

$$p \leqslant p_\gamma \ \text{ on } [0,1] \text{ for some } \gamma > \gamma^{(k)}, \quad (7.2)$$

$$\rho^0 \leqslant N, \ \int\limits_{\Omega_0} \frac{\varkappa}{(\rho^0)^{\alpha - 1}}\, dr \leqslant N, \quad \|\sqrt{\rho^0}\, v^0\|_{L^2(\Omega_0)} \leqslant N, \quad (7.3)$$

$$-N \leqslant h(\nu,\chi) \leqslant -\frac{\alpha_0(\chi)}{\nu^{2k/(k+1)}} \ \textit{in } [\nu_0,\infty)\times\bar{J} \tag{7.4}$$

for some $\alpha\in(1,\gamma)$ *and* $\alpha_0\in C(\bar J)$ *such that* $\alpha_0\geqslant 0$ *and* $\alpha_0(M)>0$ (*recall that* $\gamma^{(0)}=\gamma^{(1)}=1$, $\gamma^{(2)}=4/3$). *Then the following uniform-in-time bound holds*:

$$\sup_{t\geqslant 0}\int\limits_{\Omega_t}\frac{\varkappa(r)}{\rho^{\alpha_1-1}(r,t)}\,dr\leqslant K_{\alpha_1}\ \textit{for any } \alpha_1\in[1,\alpha) \tag{7.5}$$

and, consequently, the uniform upper bound $\sup_{t\geqslant 0}R(t)\leqslant K_R$ *holds.*

In the particular case $\mu(s)\equiv\mathrm{const}$, *one can take* $\alpha_1=\alpha$; *moreover, the result remains valid for* $\alpha=\alpha_1=1$ *when the second condition* (7.3) *automatically holds for* $V^0-\nu_0\leqslant N$.

4. *Let the hypotheses of Claim* 3 *be valid. Then, for the kinetic and potential energies*

$$\mathcal{E}(t)=\frac12\int\limits_{\Omega_t}\rho v^2\varkappa\,dr,\quad \mathcal{F}(t)=\int\limits_{\Omega_t}\rho(P^{(0)}(\rho)-H[m])\varkappa\,dr,$$

where $H[m](r,t):=H(\nu,m(r,t))$, *the following bounds hold*:

$$\|\mathcal{E}\|_{L^1(\mathbb{R}^+)}\leqslant K,\quad \|\mathcal{E}'\|_{L^1(\mathbb{R}^+)+L^2(\mathbb{R}^+)}\leqslant K,\quad \|(\mathcal{E}+\mathcal{F})'\|_{L^1(\mathbb{R}^+)}\leqslant K,$$

and, consequently, they stabilize

$$\mathcal{E}(t)\to 0,\quad \mathcal{F}(t)\to\mathcal{F}^{(S)}\ \textit{as } t\to\infty. \tag{7.6}$$

5. *Let the hypotheses of Claim* 3 *and the condition* $\|h_\nu\|_{L^1(J;\,C[\nu_0,K_V])}\leqslant C_1(N)$ *with* $K_V=K_R^{k+1}/(k+1)$ *be valid. Then*

$$\mathcal{P}[\rho(\cdot,t)]=\int\limits_{\Omega_t}(p(\rho)-p_R[\rho])^2\varkappa\,dr\to 0\ \textit{as } t\to\infty,$$

where $p_R[\rho](r,t)=-\displaystyle\int\limits_r^R(\rho f_S[m])(r_1,t)\,dr_1.$

Note that $\|R'\|_{L^2(\mathbb{R}^+)}\leqslant K$ in Claim 3 by (7.1), (3.34), and the energy bound.

To study the stabilization of density and free radius, we extend ρ by setting $\widetilde{\rho}:=\rho$ on $\bar{Q}$ and $\widetilde{\rho}(r,t):=0$ for $r>R(t)$ and $t\geqslant 0$.

Theorem 7.1. *Let the hypotheses of Claim* 5 *of Proposition* 7.1 *be valid.*

1. *Let the condition* $\|\zeta\bar{f}\|_{L^2(\mathbb{R}^+)} \leqslant N$ *be valid for some cut-off function* $\zeta \in W^{1,\infty}(\mathbb{R})$ *satisfying* (4.16). *Then the following global bounds and the stabilization property for velocity holds*:

$$\|\zeta\sqrt{\rho}\, v_t\|_{L^2(Q)} \leqslant K, \quad \sup_{t\geqslant 0}\|(\zeta v)(\cdot,t)\|_{H^1(\Omega_t)} \leqslant K, \tag{7.7}$$

$$\|v(\cdot,t)\|_{H^1(\Omega_t)} \to 0 \text{ as } t \to \infty.$$

If, in addition, $\|v_r^0\|_{L^2(\Omega_0)} \leqslant N$, $v^0(0) = 0$ *and* $\|\bar{f}\|_{L^2(\mathbb{R}^+)} \leqslant N$, *then the bounds* (7.7) *hold for* $\zeta \equiv 1$.

2. *Let* $\alpha \in (1,\gamma)$ *in* (7.3), *and let one of the conditions* (4.29) *or* (4.30) *be valid. Then for any sequence* $t_n \to +\infty$ *there exists a subsequence* θ_n *such that*

$$\widetilde{\rho}(\cdot,\theta_n) \to \rho_* \text{ weakly star in } L^\infty(r_0, K_R) \text{ and } R(\theta_n) \to R_*. \tag{7.8}$$

Moreover, for any sequence $\theta_n \to \infty$, $\theta_n \geqslant 0$, *such that* (7.8) *is valid, the following assertions hold*:

(1) *the restricted pair* $\{\rho_*\big|_{[r_0,R_*]}, R_*\} =: \{\rho_S, R_S\}$ *serves as a positive solution to the Eulerian static problem* (2.7), (2.8); *in addition,*

$$\mathcal{F}_S\{\rho_S, R_S\} := \int\limits_{\Omega_S} \rho_S(P^{(0)}(\rho_S) - H[m_S])\varkappa\, dr = \mathcal{F}^{(S)} \tag{7.9}$$

(*see* $\mathcal{F}^{(S)}$ *in* (7.6)) *and*

$$\int\limits_{\Omega_S} \frac{\varkappa}{\rho_S^{\lambda-1}}\, dr \leqslant K_\lambda \text{ for any } 1 \leqslant \lambda < \alpha; \tag{7.10}$$

(2) *the following strong limit relation holds*:

$$\widetilde{\rho}(\cdot,\theta_n) \to \rho_S(\cdot) \text{ in } L^\lambda(\Omega_S) \text{ for any } 1 \leqslant \lambda < \infty. \tag{7.11}$$

PROOF. 1. It suffices to give two comments on Claim 1. First, we can pass from $\|v_r\|_{\Omega_t}$ to $\|v\|_{H^1(\Omega_t)}$ using the inequality $\|v\|_{C(\bar{\Omega}_t)} \leqslant K_R^{1/2}\|v_r\|_{\Omega_t}$. Second, to prove the first bound (7.7), we use the formula $D_t\check{v} = v_t + vv_r$ to get

$$\|\zeta\sqrt{\rho\varkappa}\, v_t\|_Q \leqslant \|\zeta\sqrt{\rho\varkappa}\,(v_t + vv_r)\|_Q + K_\rho^{1/2} r_0^{-k/2}\|\varkappa v\|_{L^{\infty,2}(Q)}\|\zeta v_r\|_{L^{2,\infty}(Q)} \leqslant K$$

with the help of the energy bound and the second bound (7.7).

2. In the case of Claim 2, the arguments of the proof of Theorem 3 in [**9**] are applicable (exploiting the bound $R \leqslant K_R$, but not (7.5)). The proof

operates by the functional $\mathcal{P}[\rho]$, but does not address directly to the initial-boundary value problem (2.1)–(2.5), and we do not reproduce this here (in that proof, another extension of ρ was used; however, the present extension can be also applied; this fact was already mentioned in [**9**].) Actually, the argument leads to somewhat weaker results because, first, $\{\rho_S, R_S\}$ is not necessarily a positive (possessing the third property (5.6)) static solution and, second, it is required that $0 \leqslant \chi \leqslant M$ in (4.30).

To remove the first drawback, we invoke the bound (7.5). Let $p(\rho_S) > 0$ on $[r_0, R_1)$, but $p(\rho_S) = \rho_S = 0$ on $[R_1, R_S]$. If $R_1 < R_S$, then $R(\theta_n) \geqslant R_2 := (R_1 + R_S)/2$ for $n \geqslant n_0$ since $R(\theta_n) \to R_S$. Therefore, (7.11) implies

$$\|\rho(\cdot, \theta_n)\|_{L^\lambda(R_1,R_2)} \to 0 \text{ for any } 1 \leqslant \lambda < \infty. \tag{7.12}$$

On the other hand, by the Hölder inequality and (7.5), we have

$$\begin{aligned} 0 &< \frac{1}{k+1}\,(R_2^{k+1} - R_1^{k+1}) = \int_{R_1}^{R_2} \rho^\beta \frac{1}{\rho^\beta}\,\varkappa\,dr \\ &\leqslant \Big(\int_{R_1}^{R_2} \rho^{q'\beta}\varkappa\,dr\Big)^{1/q'} \Big(\int_{\Omega_t} \frac{1}{\rho^{q\beta}}\varkappa\,dr\Big)^{1/q} \leqslant K_{\beta q}\Big(\int_{R_1}^{R_2} \rho^{q'\beta}\varkappa\,dr\Big)^{1/q'} \end{aligned}$$

for any $0 < \beta < \alpha - 1$, $1 < q < (\alpha-1)/\beta$ and $q' = q/(q-1)$. For $\lambda = q'\beta \geqslant 1$ (that is valid for q sufficiently close to 1), this contradicts (7.12).

To remove the second drawback, we consider the functions (from [**9**])

$$m_*(r) = \int_{r_0}^{r} \rho_*\varkappa\,dr_1 \leqslant M, \quad p_*(r) := -\int_{r}^{R_*} \rho_* f_S[m_*]\,dr_1 \text{ on } \bar{\Omega}_*,$$

where $\Omega_* := (r_0, R_*)$. It is clear that

$$p_{*r} = \rho_* f_S[m_*] \text{ on } \Omega_* \tag{7.13}$$

and p_* is nonincreasing on $\bar{\Omega}_*$. Let $B_c := \{r \in \bar{\Omega}_*; p_*(r) = c\} = [\alpha_c, \beta_c]$ for $0 < c \leqslant \max_{\bar{\Omega}_*} p_*$. According to the same proof in [**9**], it suffices to check that $\beta_c = \alpha_c$. If $\beta_c > \alpha_c$, then either $\alpha_c = r_0$ and $\rho_*(r) = 0$ almost everywhere on (α_c, β_c) or $\alpha_c > r_0$ and ρ_* is not equivalent to zero on (r_0, α_c) (by (7.13)). In the second case, $m_* > 0$ and $f_S[m_*] < 0$ on $[\alpha_c, \beta_c]$ as well as $\rho_*(r) = 0$ almost everywhere on (α_c, β_c) again (by (7.13)). As was shown in [**9**], this degeneracy of ρ_* on (α_c, β_c) implies $\operatorname{meas} B_c = 0$, i.e., $\beta_c = \alpha_c$.

Finally, by the property $R(\theta_n) \to R_S$ and the bound (7.5), for any $0 < \varepsilon < R_S - r_0$ and sufficiently large $n \geqslant n_0(\varepsilon)$ we get

$$\int\limits_{r_0}^{R_S-\varepsilon} \frac{\varkappa}{\rho^{\lambda-1}|_{t=\theta_n}}\,dr \leqslant \int\limits_{\Omega_{\theta_n}} \frac{\varkappa}{\rho^{\lambda-1}|_{t=\theta_n}}\,dr \leqslant K_\lambda \text{ for any } 1 \leqslant \lambda < \alpha,$$

where K_λ is independent of ε. By the Fatou lemma,

$$\int\limits_{r_0}^{R_S-\varepsilon} \frac{\varkappa}{\rho_S^{\lambda-1}}\,dr \leqslant K_\lambda \text{ for any } 1 \leqslant \lambda < \alpha,$$

and, consequently, (7.10) holds. □

By the ω-limit set $\mathcal{O}_{\rho,R}$ for density and free radius we mean the set of pairs $\{\rho_*, R_*\} \in L^\infty(r_0, K_R) \times \mathbb{R}$ such that (7.8) holds. We extend ρ_S by setting $\widetilde{\rho}_S = \rho_S$ on Ω_S and $\widetilde{\rho}_S = 0$ on (R_S, K_R).

Proposition 7.2. *Let the hypotheses of Theorem* 7.1 *be valid. Then the following assertions hold.*

(1) $\mathcal{O}_{\rho,R}$ *consists of pairs* $\{\rho_*, R_*\} = \{\widetilde{\rho}_S, R_S\}$, *where* $\{\rho_S, R_S\}$ *is a positive static solution such that the properties* (7.9) *and* (7.10) *and the strong limit relation*

$$\widetilde{\rho}(\cdot, \theta_n) \to \widetilde{\rho}_S(\cdot) \textit{ in } L^\lambda(r_0, K_R) \textit{ for any } 1 \leqslant \lambda < \infty \tag{7.14}$$

hold (for the same sequence θ_n *as in the property* $\{\rho_*, R_*\} \in \mathcal{O}_{\rho,R}$).

(2) $\mathcal{O}_{\rho,R}$ *is a compact attracting connected set in* $L^\lambda(r_0, K_R) \times \mathbb{R}$ *for any* $1 \leqslant \lambda < \infty$. *The attracting property means that*

$$\inf_{\{\widetilde{\rho}_S, R_S\} \in \mathcal{O}_{\rho,R}} \|\widetilde{\rho}(\cdot, t) - \widetilde{\rho}_S(\cdot)\|_{L^\lambda(r_0,K_R)} + |R(t) - R_S| \to 0 \textit{ as } t \to +\infty.$$

PROOF. The arguments almost repeat the proof of Theorem 4 in [**9**] and will be omitted. Note only that $\widetilde{\rho} \in C_b(\bar{\mathbb{R}}^+; L^\lambda(r_0, K_R))$, for any $1 \leqslant \lambda < \infty$. For $\lambda = 1$ it is obtained with the help of the estimate

$$\int\limits_{r_0}^{K_R} |\Delta^{(\tau)} \widetilde{\rho}(r,t)|\,dr \leqslant \int\limits_{r_0}^{\min\{R(t),R(t+\tau)\}} |\Delta^{(\tau)} \rho(r,t)|\,dr + K_\rho |\Delta^{(\tau)} R(t)|;$$

for other λ one can also invoke the bound $\rho \leqslant K_\rho$ once more. □

The main result of Theorem 7.1 can be essentially simplified under weakened uniqueness assumptions for the static problem.

Corollary 7.1. *Let, in addition to the hypotheses of Theorem* 7.1*, the Eulerian static problem* (2.7)*,* (2.8) *have one of the following properties*:

(i) *there exists no continual family of positive solutions* $\{\rho_S, R_S\}$ *such that* R_S *runs over some segment* $[\underline{R}, \bar{R}]$*,* $r_0 < \underline{R} < \bar{R}$*,* $\mathcal{F}_S\{\rho_S, R_S\} = a$ *for some fixed* a*, and the overdetermined fixed domain static problem with given* R_S *and* $\mathcal{F}_S\{\rho_S, R_S\} = a$ *has at most one positive solution*;

(ii) *there exists an at most countable set of positive solutions* $\{\rho_S, R_S\}$ *such that* $\mathcal{F}_S\{\rho_S, R_S\} = a$ *for any fixed* a*.*

Then there exists a positive static solution $\{\rho_S, R_S\}$ *such that* $\widetilde{\rho}(\cdot, t) \to \widetilde{\rho}_S(\cdot)$ *in* $L^\lambda(r_0, K_R)$ *for any* $1 \leqslant \lambda < \infty$*,* $R(t) \to R_S$ *as* $t \to +\infty$*, and the properties* (7.9) *and* (7.10) *hold.*

The proof repeats that of Corollary 4.1 (with R replacing V).

Remark 7.1. The static equation (2.7) can be considered as a system of ordinary differential equations

$$(\pi_0(\rho_S))_r = f_S[m_S], \ m_{Sr} = \rho_S \varkappa \text{ on } (r_0, R_S),$$

see (5.10), with $\pi_0'(s) = p'(s)/s$. Suppose that $c_1 s \leqslant p'(s)$ on $(0, s_0)$, $c_1 = c_1(s_0) > 0$ for any $s_0 > 0$ (in particular, for $p = p_\gamma$ with $1 < \gamma \leqslant 2$) and $f_{S\chi} \in L^1((r_0, K_R); C(\bar{J}))$. Then the corresponding backward Cauchy problem with given $\pi_0(\rho_S)(R_S) = 0$ and $m_S(R_S) = M$ (and given R_S) has at most one solution, and thus the overdetermined problem from (i) does.

We can strengthen Corollary 4.2.

Corollary 7.2. *Let the hypotheses of Theorem* 7.1 *be valid, excluding the condition* (7.2)*, with the conditions* (7.3) *for* $\alpha = 1$ *and the condition* (7.4) *weakened down to* (3.7)*. If the Eulerian static problem* (2.7)*,* (2.8) *has no positive solution, then* $\limsup_{t\to\infty} V(t) = \infty$*.*

Proof. We recall that the condition (7.2), the second condition (7.3) for $\alpha \in (1, \gamma)$, and the right-hand condition (7.4) were explicitly used only in the proof of Claim 3 of Proposition 7.1 (see the proof of Proposition 3.3). Thus, under the assumption $\sup_{t \geqslant 0} V(t) < \infty$, the argument of Theorem 7.1 is partially valid and, in particular, there exists a static solution $\{\rho_S, R_S\}$ possessing the first and second properties (5.6) but, possibly, the third property is not satisfied. Since $p(\rho_S)$ is continuous and nonincreasing on $[r_0, R_S]$, there exists $r_0 < \widetilde{R}_S \leqslant R_S$ such that $p(\rho_S) > 0$ on $[r_0, \widetilde{R}_S)$ and $p(\rho_S)(\widetilde{R}_S) = 0$. The pair $\{\rho_S|_{[r_0, \widetilde{R}_S]}, \widetilde{R}_S\}$ is a positive solution, which yields a contradiction. $\square$

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