Instability in Models Connected with Fluid Flows I

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Instability in Models Connected with Fluid Flows I

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Two volumes of the *International Mathematical Series* present various topics on control theory, free boundary problems, the Navier–Stokes equations, attractors, first order linear and nonlinear equations, partial differential equations of fluid mechanics, etc. with the focus on the key question in the study of mathematical models simulating physical processes:

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Does the model ***adequately*** describe the physical process?

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- **Navier-Stokes equations. Existence and smoothness results**

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  Efim Dinaburg and Yakov Sinai, Vol. I

  — The analyticity of periodic solutions of the 2D Boussinesq system.
  Maxim Arnold, Vol. I

  — Navier–Stokes equations in cylindrical domains. Leray approximations, Leray–Navier–Stokes equations, the Helmholtz projector and stationary Stokes problem, the classical Navier–Stokes problem.
  Sergey Zelik, Vol. II

- **First order linear and nonlinear equations**

  — Nonlinear dynamics of a system of particle-like wavepackets, reduction of wavepacket interaction systems to averaged ones, superposition principle and decoupling of wavepacket interaction systems.
  Anatoli Babin and Alexander Figotin, Vol. I

  — Transport equations with discontinuous coefficients, Keyfitz-Kranzer type hyperbolic systems, generalized solutions of the Cauchy problem, existence, uniqueness, and renormalization property.
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  — Exponential mixing for randomly forced partial differential equations (method of coupling), Markov random dynamical system, dissipative random dynamical systems, the complex Ginzburg–Landau equation.
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  — Asymptotics for $3D$ water–waves, large time existence theorems, the Kadomtsev–Petviashvilii approximation.
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  — Stability of a rotating capillary viscous incompressible liquid bounded by a free surface.
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  — Symmetric compressible barotropic Navier–Stokes-Poisson flows in a vacuum, the existence of global weak solutions.
  Alexander Zlotnik, Vol. II

- **Optimal control**

  — Increased stability in the Cauchy problem for some elliptic equations, energy type estimates in low frequency zone, Carleman estimates.
  Victor Isakov, Vol. I

  — Controllability and accessibility of equations of dynamics of incompressible fluids controlled by low-dimensional (degenerate) forcing, controllability of Navier–Stokes / Euler equations on a two-dimensional sphere and on a generic Riemannian surface.
  Andrey Agrachev and Andrey Sarychev, Vol. I

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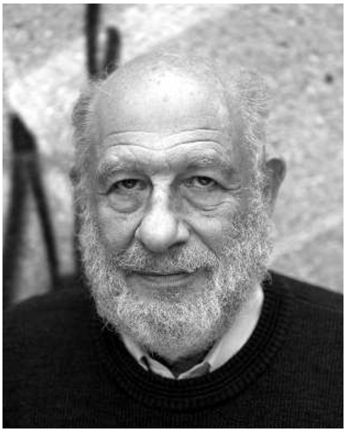

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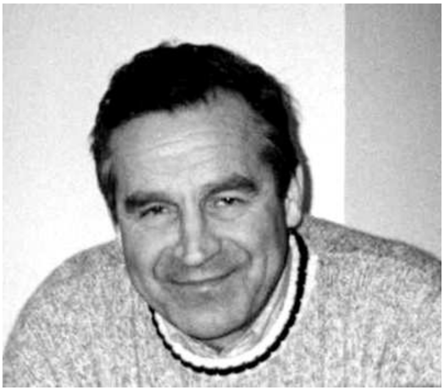

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Preface

1. Overview

These two volumes are devoted to mathematical analysis of equations of continuous media (mostly fluids) describing phenomena for which the basic underlying physics, i.e., their relation with First Principles, is well understood and broadly accepted. One of the most important mathematical issues is how these equations can be used for an accurate description of "matter." At present, this question is especially urgent in virtue of at least three interconnected factors: new engineering problems, advantages of functional analysis, and the emergence of digital computing.

- Modern engineering problems involve physics at different levels of accuracy, corresponding to different equations. The properties of these equations and the relations between them turn out to be important for applications.

  For instance, the Navier–Stokes equations and the Maxwell equations are the most commonly used to compute quantities related to fluids and electromagnetic waves respectively. However, if a medium is rarefied, other (more refined) equations should be used. This is typically the case for the re-entry in the atmosphere of a space vehicle transiting very rapidly from a region where the gas is rarefied to a region of gas with normal density. Then the Boltzmann equation should be used.

  In the same way, the use of the transport kinetic equation is imperative for devices so small that the flux of electrons cannot reach thermal equilibrium. At the other end of the scale spectrum, one confronts issues like climate evolution, and therefore it is necessary to use equations describing the interaction between the ocean and the atmosphere or the stability of very large structures in fluids such as anticyclones and the Jupiter red spot.

- During the evolution of mathematics from the 19th to the 20th century, the emphasis in studying these equations shifted from trying to find an explicit form of solutions to investigating equations by functional analysis methods due to Hilbert, Banach, and others.

- In fact, the systematic use of functional analysis is naturally combined with access to digital computing, also not relying on explicit solutions. Functional analysis is of paramount importance not only for computing error estimates between a real solution and its discrete approximation, but also, most significantly, for constructing a discrete version of the problem that retains the basic properties of the original problem (a necessary condition for convergence). For instance, in fluid mechanics, any discrete approximation should preserve mass, momentum, and energy. As predicted by von Neumann in 1946, digital computation provides information not available through other methods. It is important to note that, combined with mathematical analysis, these computations have led to mathematical discoveries. The most classical examples involve dynamical systems.

  i) The observation of the singular behavior of a discrete version of the Kortweg-de Vries equation made in 1955 by Fermi, Pasta, and Ulam [**4**], which led Lax, in 1968, to the study of the integrability of the Kortweg-de Vries equation by using the so-called Lax pair [**8**].

  ii) The discovery of *strange attractors* by Lorentz [**10**] and Hénon [**6**] on the basis of numerical experiments, which motivated a systematic research on properties of attractors; for fluids, in particular, starting with the contribution of Ladyzhenskaya [**7**] in 1972.

While the range of applications of partial differential equations is extremely large, from quantum theory to biology, the equations of fluid mechanics have a particular status. It turns out that success in the investigation of these equations leads to new results in many other nonlinear problems. Therefore, the equations of fluid mechanics often serve as models in the study of other nonlinear problems arising in applications and as a constant stimulus for new mathematical discoveries.

A striking example is the notion of a weak solution, implicitly presented in the analysis of shocks in conservation laws obeying the *Rankine–Hugoniot condition.* This notion was formalized for the construction of turbulent solutions to the Navier-Stokes equations by Leray [**9**] in 1933 and was ultimately completed with the creation of distribution theory by Sobolev [**16, 17**] in 1935/36 and by Schwartz [**15**] in 1945.

A description of a physical process by PDEs can be adequate only if a certain stability property interpreted depending on the physical problem takes place.

For linear partial differential equations the first formal definition of stability (well-posedness) was given by Hadamard [**5**] in 1904. In 1937, based on the notion of stability in the sense of Hadamard, Petrowsky [**13**] proposed a systematic classification of general systems of PDEs.

The nonlinear structure of equations describing fluid flows dictates different approaches to the introduction of the notion of stability. In addition to the classical stability (well-posedness in the Hadamard sense), there are various definitions of stability reflecting specific mathematical aspects of physical problems. In particular, the following variants will be discussed in these volumes:

— the large time behavior of solutions, which is related to the Lyapunov stability of stationary solutions and attractors

— stability relative to initial data (for example, wave packets)

— stability of averaged models obtained by introducing an infinite-dimensional measure driven by a stochastic process

— stability of free-boundary problems

— stability problems in control theory

2. Classification of Contributions and Comments

The idea was to gather a collection of contributions from experts to cover current approaches to the study of stability of mathematical models simulating processes in fluid flows. We present several directions in this area that are different by methods and problem statements, but all of them are joined by the final goal of research: to clarify whether the mathematical model under consideration possesses the property of stability (instability) in a certain interpretation of this notion.

Below we classify the papers in both volumes according to the selected directions and give our comments on presented results.

2.1. Navier-Stokes equations. General results (existence and smoothness of solutions).

This direction is presented by three papers, where nontrivial situations are considered; in particular, the problem can be stated in an unbounded domain or the solution can be of infinite energy.

[DS] **Efim Dinaburg and Yakov Sinai**, *Existence theorems for the* $3D$ *Navier–Stokes system having as initial conditions sums of plane waves*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 289–300.

In this paper, the existence theorem for the Cauchy problem for the 3D Navier-Stokes equations is proved in the case, where the initial condition is a finite sum of plane waves. The time interval, where the solution exists, depends on the initial condition. We emphasize that the initial condition is not assumed to be of finite energy. The proof is based on the method of power series which is of independent interest. There is also an example, where a solution exists on a time interval independent of the initial condition. We should note that the existence of solutions on an arbitrary time interval was earlier obtained by another method in [**18**] for almost all coefficients of the initial quasiperiodic polynomial with respect to the Lebesgue measure.

[A] **Maxim Arnold**, *Analyticity of periodic solutions of the 2D Boussinesq system*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 37–52.

The paper by Sinai's former student M. D. Arnold is devoted to the proof of the analyticity of periodic solutions to the 2D Boussinesq system, an extension of the Navier-Stokes equations, and uses the method of [**11**].

[Ze] **Sergey Zelik**, *Weak spatially nondecaying solutions of 3D Navier–Stokes equations in cylindrical domains*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 329–376.

Zelik develops an infinite energy theory for the Navier–Stokes equations in unbounded $3D$ cylindrical domains. Based on this theory, he establishes the existence of a weak solution in a uniformly local phase space (without any spatial decay assumptions), the dissipativity of the solution, and the existence of the so-called trajectory attractor. In particular, this phase space contains the 3D Poiseuille flows. Estimates on the size of the attractor in terms of the kinematic viscosity are also obtained.

2.2. First order linear and nonlinear equations.

The difference in statements and approaches presented in the papers of this direction reflects the rich variety of subjects and methods in current investigations of different aspects of stability (instability) in this area.

[BF] **Anatoli Babin and Alexander Figotin**, *Nonlinear dynamics of a system of particle-like wavepackets*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 53–134.

The authors highlight the propagation properties of quasilinear hyperbolic equations by introducing a special class of the so-called *particle-like wave packets*. This notion has a dual nature. On one hand, a particle-like wave packet is a wave with a well-defined principal wave vector. On the other hand, it is a particle in the sense that it can be assigned to a well-defined position in space. As was established in this paper, under this nonlinear evolution, a generic multi-particle wave packet remains a multi-particle wave packet with high accuracy and the constituent single particle-like wave packet not only preserves the principal wave number, but also has a well-defined space position evolving with constant velocity (their group velocity). To prove these results, the authors use properties of the linear (hyperbolic) part of the system under consideration and the particle-like wave packet structure of the initial data. The methods used in [BF] are close to those of [Ch] and [GMN].

[P] **Evgenii Panov**, *Generalized solutions of the Cauchy problem for a transport equation with discontinuous coefficients*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 23–84.

Transport equations with discontinuous coefficients arise in the analysis of various nonlinear systems of conservation and balance laws with linear degeneracy of some components. For example, the system of Keyfitz–Kranzer type, known in magnetohydrodynamics, reduces to a system of such a kind. Furthermore, as is known [**12**], transport equations with discontinuous coefficients appear as the adjoint equations corresponding to hyperbolic systems of conservation laws. Panov presents the well-posedness theory for general nonhomogeneous transport equations which can be applied for establishing the existence and uniqueness of strong entropy solutions to the Cauchy problem for Keyfitz–Kranzer type systems.

[R] **Evgenii Radkevich**, *Irreducible Chapman–Enskog Projections and Navier–Stokes approximations*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 85–154.

In order to derive the viscosity and heat diffusion coefficients from the Boltzmann equation, Chapman and Enskog proposed an approximation of solutions to the Boltzmann equation in terms of macroscopic quantities or *moments* of the solution. This approach works very well for the first-order approximation with respect to the Knudsen number ε. This leads to the compressible Navier–Stokes equation and provides a way to derive the viscosity and heat diffusion coefficients from First Principles. For the next order in ε, the Burnett equation appears, an ill-posed equation in the sense of Hadamard. As was noted in [**2**], a very good model for relaxation to the equilibrium property of the Boltzmann equation is the nonlinear Euler equation with relaxation term of order ε^{-1}. Based on spectral analysis, Radkewich proposed some other derivation. In particular, he proved that, in the case of an odd number of equations, a well-posed approximation of dependent variables of any order can be expressed as an equation of one variable. If the number of equations is even, the approximation can be expressed via two macroscopic variables.

2.3. Finite time instabilities of 3*d* incompressible Euler equations.

The question whether solutions to the 3d incompressible Euler equations with finite energy and smooth initial data may blow up in finite time is still open. However, it is known that a family of smooth initial data may generate growth in the vorticity that, even if not infinite, may be arbitrarily large. Furthermore, even in the 2d case, a family of initial data with nonuniformly bounded vorticity may generate pathological behavior. In [Ch] and [GMN], the reasons leading to such patologies are investigated.

[Ch] **Christophe Cheverry**, *Recent results in large amplitude monophase nonlinear geometric optics*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 267–288.

Using methods of nonlinear geometric optics applied to a family of oscillating initial data, Cheverry shows that the weak limit of the corresponding solutions does not satisfy the Euler equation any more.

[GMN] **Francois Golse, Alex Mahalov, and Basil Nicolaenko**, *Bursting dynamics of the* 3*D Euler equations in cylindrical domains*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 301–338.

To exhibit the stabilizing effect of a fast rotation, the authors consider solutions to the Euler equations in a finite cylinder with initial data that is a bounded perturbation of a large uniform rotation Ω along the axis of the cylinder. Conjugating the solution with the Poincaré–Steklov operator (the rotation in the space of divergence-free functions), they construct a resonant limit system. Special solutions (in particular, periodic and integrable ones) are studied by methods of the classical Hamiltonian mechanics for rigid bodies. Using a shadowing lemma, the authors find that the solutions to the original Euler equation have similar behavior. From the Editors' point of view, the major and remarkable result is the construction of time periodic solutions with large variation of the ratio of the $H^s(t)$ norms between two different times t_1 and t_2 (for any s). Such a *bursting* dynamics, without singularities, corresponds to the so-called *depletion* in the study of the Euler equations.

2.4. Large time asymptotics of solutions.

The analysis of the large time behavior of solutions to the fluid equations covers many applications and is connected with basic physical issues, for instance, the route to turbulence. At the same time, it can be approached through very different aspects. In addition to the contribution presented in this subsection, the papers by Zelik (see Subsection 2.1) and by Zlotnik (see Subsection 2.6 below) are directly related to this topic.

[ChV] **Vladimir Chepyzhov and Mark Vishik**, *Attractors for nonautonomus Navier–Stokes system and other partial differential equations*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 135–266.

As was already mentioned, a description of attractors was a strong stimulus for mathematical research. Beginning with the 80's, the theory of *global attractors* was actively developed by many authors towards different directions, including the estimation of the Hausdorff dimension of attractors by basic scaling numbers (Reynolds, Grasshoff, etc.) of a flow. Attractors for nonautonomous equations were first studied by Chepyzhov and Vishik [**3**] who have made the main contribution to the field.

In the present paper, the authors treat the case of nonautonomous systems. The Hausdoff dimension of the global attractor can be infinite in the nonautonomous case, and, by this reason, the authors use the notion of an ε-entropy introduced by Kolmogorov for estimating the attractor size. Nonautonomous partial differential equations with oscillating external forces are analyzed. In particular, the authors consider the situation, where the amplitude of the oscillation grows infinitely, whereas the attractor remains bounded.

2.5. Statistical approach.

To derive an equation describing an instable movement, it is reasonable to replace unspecified forces by random forces with time-independent increments, instead of omitting unspecified forces altogether. Then one obtains a stochastic equation, i.e., a partial differential equation with white noise on the right-hand side. The presented results of Shirikyan lead to a very interesting setting of the problem that is adequate to described instable physical processes.

[Sh] **Armen Shirikyan**, *Exponential mixing for randomly forced partial differential equations. Method of coupling*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 155–188.

During many years, physicists were firmly convinced that the white noise possesses a smoothing effect on solutions to a partial differential equation. In the case of the complex Ginzburg–Landau equation, this conjecture finds its rigorous justification in the paper by Shirikyan presented in this collection. In fact, Shirikyan proves the ergodicity of stochastic partial differential equations, i.e., the uniqueness of the steady-state statistical solution even in the case, where the same partial differential equation, without white noise on the right-hand side, possesses many individual steady-state solutions belonging to an attractor of complicated structure. The smoothing action of the white noise is precisely to transform the set of individual steady-state solutions into a unique statistical steady-state solution. Using the coupling method, Shirikyan establishes a general criterion for the uniqueness of stationary measures and an exponential mixing property. The latter is understood as a certain kind of the Lyapunov exponential stability of the steady-state statistical solution. The method is then illustrated by the stochastic complex Ginzburg–Landau equation. Note that the results presented in [Sh] are based on an approach developed in a series of papers by Kuksin and Shirikyan (see references in [Sh]).

2.6. Water waves and free boundary problems.

The papers presented in this subsection are devoted to the study of delicate physical situations, where the surface separating a liquid and an external medium is not fixed. There are many different problems of such a kind. Some of them are discussed in our volumes.

[L] **David Lannes**, *Justifying asymptotics for 3D water-waves*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 1–22.

A motion of a perfect incompressible irrotational fluid under the influence of gravity is described by the free surface Euler (or water-wave) equations. These equations have rich structure, and many well-known equations in mathematical physics can be obtained as their asymptotic limits, for example, the Korteweg-de Vries equations, the Kadomtsev–Petviashvilii equations, the Boussinesq systems, the shallow water equations, the deep water models, etc. Lannes studies the validation of such asymptotics. Since the fluid is irrotational, it derives from a potential and therefore leads to the Dirichlet–Neumann operator on the free boundary. An asymptotic analysis of the Dirichlet–Neumann operator yields a linearized version of the problem. To reach the full nonlinear case, the perturbation method employing the Nash–Moser theorem is used.

[S] **Vsevolod Solonnikov**, *On problem of stability of equilibrium figures of uniformly rotating viscous incompressible liquid*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 189–254.

The free boundary problem governing the evolution of an isolated mass of a viscous incompressible fluid, subject to capillary and self-gravitation forces, is considered. The solvability of this problem in a finite time interval was established by the author in his previous publications. In the present paper, Solonnikov studies the stability of the solution corresponding to the rigid rotation of a liquid about the vertical axis with constant angular velocity. The main goal of this investigation is to show that the stability/instability is driven by the second variation of the energy functional, which has been done via analysis of the spectrum of the linearized operator in a neighborhood of the stationary regime. Then the perturbations are estimated in terms of the Hölder norms.

[Zl] **Alexander Zlotnik**, *On global in time properties of the symmetric compressible barotropic Navier–Stokes–Poisson flows in a vacuum*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 329–376.

Unlike the papers [L] and [S] dealing with incompressible fluids (for instance, water) and several spatial dimensions, Zlotnik considers symmetric self-gravitating flows of a viscous compressible barotropic gas/fluid around a hard core with a free outer boundary in a vacuum. The density degenerates at the free boundary. Under spherical symmetry, the problem becomes one-dimensional relative to the spatial variables. Such problems arise in astrophysics. For large discontinuous initial data and general state functions (including increasing and not strictly increasing ones) the global-in-time bounds for solutions are established, which allows one to study of their large-time behavior. Results on the existence, nonexistence, and uniqueness of the corresponding static solutions are also presented.

2.7. Control theory.

Control theory gives the most natural point of view for engineering sciences. Indeed, instead of determining a solution in terms of data, one seeks to find the *most suitable data to produce the desired output.* This approach was first developed for time-dependent ordinary differential equations (see, for example, [**14**]). Due to the use of computers, advantages of functional analysis, and modern technology, this approach is now extended to distributed system. Note that control is closely related to the notion of *observability*, where frequencies of the solution play a crucial role. This fundamental fact was widely used by J.-L. Lions, one of the creators of control theory for PDEs. The main feature of this area is that many control problems arising in applications are ill posed in the sense of Hadamard.

[Sh] **Victor Isakov**, *Increased stability in the Cauchy problem for some elliptic equations*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 339–362.

Variations of the boundary data for elliptic equations generate fluctuations that show up everywhere in the domain. However, according to the regularizing properties of these problems, these fluctuations may be very small and the identification of their source is an ill-posed problem in the sense of Hadamard. It turns out that, in this setting, the most convenient tools for obtaining the best possible estimates are "Carleman estimates." Using these tools, Isakov derives some bounds which can be thought of as the increasing stability of the Cauchy problem for the Helmholtz equation with lower order terms when frequency is growing. These bounds hold under certain pseudoconvexity conditions on the surface for the Cauchy data and on the coefficient of the zero order term in the Helmholtz equation.

[AS] **Andrey Agrachev and Andrey Sarychev**, *Solid controllability in fluid dynamics*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 1–36.

The authors consider the controllability and accessibility properties of the Navier–Stokes and Euler systems controlled by a low-dimensional force on the right hand side. After a survey of recent results, the authors establish new results for these systems on the two-dimensional sphere and generic two-dimensional Riemannian surfaces. They focus on geometric and Lie algebraic ideas, adopting the approach due to Arnold and Khesin [**1**] and making a connection with geometric methods in classical control theory. This paper should be especially interesting for those specialists, familiar with analytical methods, who wish to be introduced to the geometrical approach and to make a step towards more applied points of view.

3. Methods and Tools

To obtain the results presented in the volumes, the authors used well-known methods and their modifications or developed new approaches. Keeping in mind that mathematical methods are often as important as results they produce, we list the main methods and tools used by the contributors and indicate the corresponding references.

- Infinite dimensional geometric approach to fluid dynamics [AS]
- Nash–Moser theorem [L]
- Pseudodifferential calculus and harmonic analysis [L]
- Expansion of nonlinear part in terms of perturbation series [DS], [A]
- Nonlinear optic high frequency approximations [BF], [Ch]
- Poincaré–Sobolev operator [GMN]
- Resonant frequencies [GMN], [BF]
- White noise, stochastic methods, coupling method in particular [Sh]
- Hausdorff dimension, Kolmogorov entropy, attractors [ChV], [Ze].
- Carleman estimates [I]
- Weight Sobolev spaces [Ze]
- Moments of solutions (deterministic and random) [AS], [R], [Sh]
- Free boundary problems [S], [Zl]
- Conservation laws, hyperbolic systems with discontinuous coefficients [P]

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Content of Volume I

Content of Volume II

Solid Controllability in Fluid Dynamics

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We survey results of recent activity towards studying the controllability and accessibility issues for equations of dynamics of incompressible fluids controlled by low-dimensional (degenerate) forcing. New results concerning the controllability of Navier–Stokes / Euler equations on a two-dimensional sphere and on a generic Riemannian surface are presented. Bibliography: 28 titles.

1. Introduction

We survey results of recent activity aimed at studying the controllability and accessibility properties of the Navier–Stokes (NS) equations controlled by

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low-dimensional (degenerate) forcing. This choice of control is the characteristic feature of our statement of the problem. The corresponding equations are as follows:

$$\partial u/\partial t + \nabla_u u + \operatorname{grad} p = \nu \Delta u + F(t,x), \tag{1.1}$$

$$\operatorname{div} u = 0. \tag{1.2}$$

The words "*degenerate* forcing" mean that $F(t,x)$ can be represented as

$$F(t,x) = \sum_{k \in \mathcal{K}^1} v_k(t) F^k(x), \quad \mathcal{K}^1 \text{ is finite.}$$

The word "controlled" means that the functions $v_k(t)$, $t \in [0,T]$, entering the forcing can be chosen freely among measurable essentially bounded functions. In fact, any function space, dense in $L_1[0,T]$, would fit.

The domains considered here include two-dimensional (compact) Riemannian manifolds: a sphere, a torus, a rectangle, a generic Riemannian surface diffeomorphic to a disc. We impose the so-called Lions boundary condition whenever the boundary is nonempty.

Our approach stems from geometric control theory which is based on differential geometry and Lie theory; the geometric control approach proved its effectiveness in studying controlled dynamics in finite dimensions. We report on some ideas of how such methods can be extended to the area of infinite-dimensional dynamics and controlled partial differential equations. Extensions of geometric control theory to the infinite-dimensional case are almost unknown. The classical Lie techniques are not well adapted for the infinite-dimensional case, and several analytic problems are encountered.

In this contribution, we concentrate almost exclusively on geometric and Lie algebraic ideas of the accomplished work. For details on analytic part we refer the interested reader to [**7, 6, 23, 21, 22, 26, 27**].

Applications of geometric theory to the study of the controllability of finite-dimensional systems is a well established subject, although many problems still remain unsolved. Starting point of the activity aimed at controlling the Navier–Stokes equations by degenerate forcing was the study [**13, 4, 6, 25**] of the accessibility and controllability of their finite-dimensional Galerkin approximations on $\mathbb{T}^2$ and $\mathbb{T}^3$ (periodic boundary conditions). One should note that the controllability of finite-dimensional Galerkin approximations of the Navier–Stokes equations on many other domains remains an open question. Answers for generic Riemannian surfaces follow from the results of Section 9.

The study in the infinite-dimensional case started in [**5, 6, 7**], where we dealt with the 2D Navier–Stokes / Euler equations on a 2D torus $\mathbb{T}^2$. In these publications, the notion of the solid controllability in projections and that of the approximate controllability are introduced and sufficient criteria for them are established.

To obtain such criteria, the technique of the so-called Lie extensions in infinite dimensions was suggested. In the context of our problem, this technique can be loosely interpreted as designing the propagation to higher modes of the energy pumped by controlled forcing into the lower modes.

The control functions involved are fast-oscillating, and the analytic part of the study consists of establishing the continuity properties of solutions of the Navier–Stokes equations with respect to the so-called relaxation metric of forcing. Being weaker than the classical metrics, it is adapted for dealing with fast oscillating functions.

An extension of the above-mentioned techniques to the Navier–Stokes equations with the Lions boundary conditions on a rectangle has been accomplished by Rodrigues [**21**]. In the course of this study, both geometric and analytic parts needed to be adjusted: the Lie extensions turn more intricate and the continuity properties need to be reproved. These results are surveyed in Section 8.

A new approach is suggested for establishing the controllability on a Riemannian surface diffeomorphic to a disc (Section 9).

Finally, the study of the Lie algebraic properties of spherical harmonics results in a controllability criterion for the Navier–Stokes / Euler equations on a 2D sphere (Section 10).

The results appearing in Sections 9 and 10 have not been previously published.

An interesting extension of the above described methods to the case of the Navier–Stokes equations on a 3D torus was accomplished by Shirikyan [**26, 27**]. The geometric part of his study essentially coincides with that in [**6**] and [**25**], but many additional analytic difficulties in the 3D case arise. We do not survey these results here, but refer the interested reader to [**26, 27**].

The controllability of the Navier–Stokes and Euler equations was extensively studied, in particular, by means of boundary control. There are various results on the exact local controllability of the 2D and 3D Navier–Stokes equations obtained by Fursikov and Imanuilov, the global exact controllability for the 2D Euler equation obtained by Coron, and the global exact controllability for the 2D Navier–Stokes equations obtained by Coron and Fursikov. We refer the reader to the book [**14**] and surveys [**15, 11**] for the further references.

2. 2D Navier–Stokes / Euler Equations Controlled by Degenerate Forcing. Definitions and Problem Setting

2.1. Navier–Stokes / Euler equations on 2D Riemannian manifold.

The representation of the Navier–Stokes / Euler equations in the form (1.1), (1.2) requires an interpretation whenever one considers the system on a 2D domain M with arbitrary Riemannian metric. There is a general way of representing the Navier–Stokes / Euler equations on any n-dimensional Riemannian manifold (see, for example, [**10**]), but we prefer to remain in two dimensions and advance with some elementary vector analysis in the 2D Riemannian case.

We consider a smooth (or analytic) two-dimensional Riemannian manifold M (with or without boundary) endowed with the Riemannian metric $(\cdot,\cdot)$ and area 2-form σ. All functions, vector fields, and forms will be assumed to be smooth.

Any vector field y on M can be paired with two differential 1-forms

$$y \mapsto y^\flat : \langle y^\flat, \xi\rangle = (y,\xi), \quad y \mapsto y^\sharp : \langle y^\sharp, \xi\rangle = \sigma(y,\xi) \tag{2.1}$$

for each vector field ξ. It is obvious that $\langle y^\sharp, y\rangle = \sigma(y,y) = 0$.

Note that for any 1-form λ

$$\lambda \wedge y^\sharp = \langle \lambda, y\rangle \sigma. \tag{2.2}$$

To prove (2.2), it suffices to compare the values of 2-forms $\lambda \wedge y^\sharp$ and $\langle \lambda, y\rangle\sigma$ on any pair of linearly independent vectors. It is obvious that (2.2) is valid if y (and $y^\sharp$) vanishes. If $y \neq 0$, we take a pair y, z which is linearly independent. Then

$$(\lambda \wedge y^\sharp)(y,z) = \begin{vmatrix} \langle\lambda, y\rangle & \langle y^\sharp, y\rangle \\ \langle \lambda, z\rangle & \langle y^\sharp, z\rangle \end{vmatrix} = \langle \lambda, y\rangle \sigma(y,z).$$

Now, we define the vorticity curl and divergence div of a vector field via the differentials $dy^\flat$ and $dy^\sharp$ which are 2-forms. We put $dy^\flat = (\operatorname{curl} y)\sigma$

and $dy^\sharp = (\operatorname{div} y)\sigma$ or, by abuse of the notation,

$$(\operatorname{curl} y) = dy^\flat/\sigma, \quad (\operatorname{div} y) = dy^\sharp/\sigma. \tag{2.3}$$

The gradient $\operatorname{grad}\varphi$ of a function φ is the vector field paired with $d\varphi$ metrically: $(\operatorname{grad}\varphi)^\flat = d\varphi$.

As in the Euclidean case, the vorticity of the gradient vector field of a function vanishes: $\operatorname{curl}(\operatorname{grad}\varphi) = d(\operatorname{grad}\varphi)^\flat/\sigma = d(d\varphi)/\sigma = 0$.

In the 3D case, curl transforms vector fields into vector fields while, in the 2D case it transforms vector fields into scalar functions (actually, the component of a vector field directed along the additional third dimension). We define the vorticity operator curl on functions. The result of the action of curl on a function φ is a vector field $\operatorname{curl}\varphi$ such that

$$\langle \lambda, \operatorname{curl}\varphi\rangle\sigma = (d\varphi \wedge \lambda)$$

for each 1-form λ. By (2.2) and the nondegeneracy of paring $y \mapsto y^\sharp$, we conclude:

$$(\operatorname{curl}\varphi)^\sharp = -d\varphi. \tag{2.4}$$

As in the Euclidean case, the divergence of the vorticity of a function vanishes:

$$\operatorname{div}(\operatorname{curl}\varphi) = d(\operatorname{curl}\varphi)^\sharp/\sigma = -d(d\varphi)/\sigma = 0. \tag{2.5}$$

Coming back to Equation (1.2), we note that the condition $\operatorname{div} u = 0$ can be written as

$$du^\sharp = 0. \tag{2.6}$$

If M is simply connected, we conclude that $u^\sharp$ must be a differential: $u^\sharp = -d\psi$, where ψ is the so-called *stream function*. By (2.4), $\operatorname{curl}\psi = u$.

For non-simply connected domains we *impose* a condition which guarantees the exactness; in the next subsection we comment on it.

For the symplectic structure on M defined by σ and $(\cdot,\cdot)$ we see that u is the Hamiltonian vector field corresponding to the Hamiltonian $-\psi$: $u = -\overrightarrow{\psi}$.

The nonlinear term $\nabla_u u$ on the right-hand side of (1.1) corresponds to the covariant derivative of the Riemannian (metric torsion-free) connection on M.

Finally, we define the Laplace–Beltrami operator Δ as $\Delta = \operatorname{curl}^2$. In the Hodge theory (see [**10**]), this operator transforms p-forms into p-forms; in our notation Δ transforms vector fields into vector fields and functions into functions.

2.2. Helmholtz form of 2D Navier–Stokes equations.

To obtain the Helmholtz form of the Navier–Stokes equations (1.1), (1.2), we apply the operator curl to both sides of (1.1). As a result, for the vorticity $\operatorname{curl} u = w$ we get the equation

$$\partial w/\partial t + \operatorname{curl}(\nabla_u u) = \nu\Delta w + f(t,x), \tag{2.7}$$

where $f(t,x) = \operatorname{curl} F(t,x)$.

Note that the vorticity of $\operatorname{grad} p$ vanishes and the operator curl commutes with $\Delta = \operatorname{curl}^2$.

To calculate $\operatorname{curl}(\nabla_u u)$ according to formula (2.3), we first compute the 1-form $(\nabla_u u)^\flat$ adapting the argument of [**10**, § IV.1.D].

Let y be a vector field that commutes with u: the Lie–Poisson bracket vanishes, $[u,y]=0$. Then

$$\langle(\nabla_u u)^\flat, y\rangle = (\nabla_u u, y) = L_u(u,y) - (u, \nabla_u y). \tag{2.8}$$

Hereinafter, L_u denotes the Lie derivative. Note that for the covariant derivative of metric connection we have $L_u(u,y) = (\nabla_u u, y) + (u, \nabla_u y)$. Since the connection is torsion-free and $[u,y]=0$, we have $\nabla_u y - \nabla_y u = 0$ and the right-hand side of (2.8) can be represented as

$$L_u\langle u^\flat, y\rangle - (u, \nabla_y u) = L_u\langle u^\flat, y\rangle - \frac{1}{2}\langle d(u,u), y\rangle.$$

Moreover, $L_u\langle u^\flat, y\rangle = \langle L_u u^\flat, y\rangle$ if $L_u y = [u,y] = 0$ and we conclude:

$$\langle(\nabla_u u)^\flat, y\rangle = \langle L_u u^\flat, y\rangle - \frac{1}{2}\langle d(u,u), y\rangle.$$

As far as we can find a vector field y that commutes with u and has any prescribed value at a given point, we conclude:

$$(\nabla_u u)^\flat = L_u u^\flat - \frac{1}{2} d(u,u).$$

Using the definition of curl (2.3), we get

$$\operatorname{curl}(\nabla_u u) = d((\nabla_u u)^\flat)/\sigma = dL_u u^\flat/\sigma = L_u du^\flat/\sigma = L_u(w\sigma)/\sigma = L_u w.$$

Hence $\operatorname{curl}(\nabla_u u) = L_u w$.

If u is a Hamiltonian vector field with Hamiltonian $-\psi$, then $\nabla_u w = -\{\psi, w\}$, where $\{\cdot,\cdot\}$ is the Poisson bracket of functions.

The Helmholtz form of the Navier–Stokes equations (cf. [**10**]) reads

$$\frac{\partial w}{\partial t} - \{\psi, w\} - \nu\Delta w = f(t,x).$$

Note that $w = \operatorname{curl} u = \operatorname{curl}^2 \psi = \Delta\psi$.

The Lions condition written in terms of the vorticity w and the stream function ψ reads

$$\psi\big|_{\partial M} = w\big|_{\partial M} = 0. \tag{2.9}$$

If the boundary ∂M of M is smooth, then the Hamiltonian vector field $u = -\overrightarrow{\psi}$ is tangent to ∂M.

For the vorticity w and boundary conditions (2.9) one can recover in a unique way the velocity field u corresponding to the *exact* 1-form $u^\sharp$. The corresponding formula is $u = \operatorname{curl}\psi$, where ψ is a unique solution of the Dirichlet problem $\Delta\psi = w$ with the boundary condition (2.9). Indeed, such u is divergence-free and the vorticity of u is equal to w by the definition of Δ.

The Navier–Stokes equations can be written as

$$\frac{\partial w}{\partial t} - \{\Delta^{-1}w, w\} - \nu\Delta w = f(t,x). \tag{2.10}$$

This equation looks universal. In fact, its dependence on the domain is encoded in properties of the Laplacian Δ on this domain. It is well explained in [**8, 10**] that the Euler equation for a fluid motion is an infinite-dimensional analog of the Euler equation for rotation of a (multi-dimensional) rigid body, and the Laplacian in (2.10) plays role of the inertia tensor for the rotating rigid body.

2.2.1. Stream function on flat torus. Consider a flat torus $\mathbb{T}^2$ endowed with the standard Riemannian metric and area form σ, both inherited from the covering of $\mathbb{T}^2$ by the Euclidean plane. Let φ_1, φ_2 be the "Euclidean" coordinates on $\mathbb{T}^2$. We proceed in the space of velocities u with vanishing space average: $\int\limits_{\mathbb{T}^2} u d\sigma = 0$ (by flatness, we may think that all velocities belong to the same linear space).

To establish the exactness of the closed 1-form $u^\sharp$ (involved in (2.6)), it suffices to prove that its integral along the generator of a torus vanishes. By the Stokes theorem, the integrals of the closed form $u^\sharp$ along any two homologous paths have the same value.

Taking $u = (u_1, u_2)$, we get $u^\sharp = -u_2 d\varphi_1 + u_1 d\varphi_2$. Integrating $u^\sharp$ along the loop $\Gamma : \varphi_1 = \alpha$, we obtain the value of the integral

$$\int\limits_\Gamma u^\sharp = \int\limits_0^{2\pi} u_1 d\varphi_2 = c(\alpha),$$

which, by the aforesaid, is constant: $c(\alpha) \equiv c$. Integrating it with respect to φ_1, we conclude that

$$2\pi c = \int_0^{2\pi} u_1 d\varphi_2 d\varphi_1 = \int_{\mathbb{T}^2} u_1 d\sigma = 0.$$

Hence

$$\int_\Gamma u^\sharp = c = 0.$$

The same holds for the loops $\Gamma' : \varphi_2 = \text{const}$.

2.3. Controllability. Definitions.

In what follows, we reason in terms of the so-called *modes* which are eigenfunctions $\varphi^k(x)$ of the Laplace–Beltrami operator Δ defined in the space of vorticities w: $\Delta\varphi^k(x) = \lambda_k\varphi^k(x)$.

Representing w and f in (2.10) as series $w(t,x) = \sum_k q_k(t)\varphi^k(x)$ and $f(t,x) = \sum_k v_k(t)\varphi^k(x)$ with respect to the basis of eigenfunctions, we can write the Navier–Stokes equations as an infinite system of ordinary differential equations on the coefficients $q_k(t)$. Assume that

$$\{\varphi^i(x), \varphi^j(x)\} = \sum_k C_k^{ij}\varphi^k(x).$$

Then the system (2.10) can be written in the coordinate form as

$$\dot{q}_k - \sum_{i,j} C_k^{ij}\lambda_i^{-1}q_iq_j - \nu\lambda_k q_k = v_k(t). \tag{2.11}$$

Typically, we will consider a controlled forcing which is applied to few modes $\varphi^k(x)$, $k \in \mathcal{K}^1$, where $\mathcal{K}^1$ is finite. Then, in the system (2.11), the controls enter only the equations indexed by $k \in \mathcal{K}^1$, while $v_k = 0$ for $k \notin \mathcal{K}^1$.

Introduce another finite set $\mathcal{K}^o$ of *observed* modes. We always assume that $\mathcal{K}^o \supset \mathcal{K}^1$. We identify the space of observed modes with $\mathbb{R}^N$ and denote by Π^o the operator of projection of solutions onto the space of observed modes $\text{span}\{\varphi_k \mid k \in \mathcal{K}^o\}$. The coordinates corresponding to the observed modes are reunited in the *observed component* q^o.

A Galerkin $\mathcal{K}^o$-approximation of the 2D Navier–Stokes / Euler equations is the ordinary differential equation for $q^o(t)$ obtained by projecting the

2D Navier–Stokes equations onto the space of observed modes and equating all the components $q_k(t)$, $k \notin \mathcal{K}^o$, to zero. The resulting equation is

$$\frac{\partial q^o}{\partial t} - \Pi^o\{\Delta^{-1}q^o, q^o\} - \nu\Delta q^o = f(t,x). \tag{2.12}$$

If $\mathcal{K}^o \supset \mathcal{K}^1$, i.e., the controlled forcing f only affects a part of observed modes, then $\Pi^o f(t,x) = f(t,x)$.

In the coordinate form, the passage to Galerkin approximations means omitting the equations in (2.11) for the variables q_k with $k \notin \mathcal{K}^o$ and equating these q_k to zero in the remaining equations.

We say that a control $f(t,x)$ *steers* the system (2.10) (or (2.12)) from $\tilde{\varphi}$ to $\hat{\varphi}$ in time T if for the system (2.10) forced by f the solution with the initial condition $\tilde{\varphi}$ at $t=0$ takes the "value" $\hat{\varphi}$ at $t=T$.

The first notion of controllability considered is the controllability of Galerkin approximations.

Definition 2.1 (controllability of Galerkin approximations)**.** A Galerkin $\mathcal{K}^o$-approximation of the 2D Navier–Stokes / Euler equations is *time-T globally controllable* if for any two points $\tilde{q}$ and $\hat{q}$ in $\mathbb{R}^N$ there exists a control that steers in time T this Galerkin approximation from $\tilde{q}$ to $\hat{q}$.

This is a purely finite-dimensional notion. The following notion regards a finite-dimensional component of solutions, but takes into account the complete infinite-dimensional dynamics.

Definition 2.2 (attainable sets of Navier–Stokes equations)**.** An *attainable set* $\mathcal{A}_{\tilde{\varphi}}$ of the Navier–Stokes / Euler equations (2.10) is the set of points in $H^2(M)$ attained from $\tilde{\varphi}$ by means of essentially bounded measurable controls in any positive time. For each $T > 0$ time-T a (*time*-$\leqslant T$) *attainable set* $\mathcal{A}^T_{\tilde{\varphi}}$ ($\mathcal{A}^{\leqslant T}_{\tilde{\varphi}}$) of the Navier–Stokes / Euler equations is the set of points attained from $\tilde{\varphi}$ by means of essentially bounded measurable controls in time T (in time $\leqslant T$). Then the attainable set $\mathcal{A}_{\tilde{\varphi}} = \bigcup\limits_T \mathcal{A}^T_{\tilde{\varphi}}$.

Definition 2.3. The Navier–Stokes / Euler equations are *time-T globally controllable in projection onto $\mathcal{L}$* if for each $\tilde{\varphi}$ the image $\Pi^{\mathcal{L}}\left(\mathcal{A}^T_{\tilde{\varphi}}\right)$ coincides with $\mathcal{L}$.

Definition 2.4. The Navier–Stokes / Euler equations are *time-T L_2-approximately controllable* if $\mathcal{A}^T_{\tilde{\varphi}}$ is L_2-dense in H^2.

Definition 2.5 (accessibility in finite-dimensional projection)**.** Let $\mathcal{L}$ be a finite-dimensional subspace of $H_2(M)$, and let $\Pi^{\mathcal{L}}$ be the L_2-orthogonal projection of $H_2(M)$ onto $\mathcal{L}$. The Navier–Stokes / Euler equations are *time-T accessible in projection on* $\mathcal{L}$ if for any $\tilde{\varphi} \in H_2(M)$ the image $\Pi^{\mathcal{L}}\left(\mathcal{A}^T_{\tilde{\varphi}}\right)$ contains interior points in $\mathcal{L}$.

Definition 2.6. Fix an initial condition $\tilde{\varphi} \in H_2(M)$ for trajectories of the controlled 2D Navier–Stokes / Euler equations. Let $v(\cdot) \in L_\infty([0,T];\mathbb{R}^r)$ be a controlled forcing, and let w_t be the corresponding trajectory of the Navier–Stokes equations.

If the Navier–Stokes / Euler equations are considered on an interval $[0,T]$ $(T < +\infty)$, then $E_T : v(\cdot) \mapsto w_T$ is called an *end-point mapping*, $\Pi^o \circ \mathcal{F}/\mathcal{T}_T$ is called an *end-point component mapping*, and $\Pi^{\mathcal{L}} \circ \mathcal{F}/\mathcal{T}_T$ is called an *$\mathcal{L}$-projected end-point mapping.*

Definition 2.7. Let $\Phi : \mathcal{M}^1 \mapsto \mathcal{M}^2$ be a continuous mapping between two metric spaces, and let $S \subseteq \mathcal{M}^2$ be any subset. We say that Φ *covers S solidly* if $S \subseteq \Phi(\mathcal{M}^1)$ and this inclusion is stable with respect to C^0-small perturbations of Φ, i.e., for some C^0-neighborhood Ω of Φ and each mapping $\Psi \in \Omega$ we have $S \subseteq \Psi(\mathcal{M}^1)$.

Definition 2.8 (solid controllability in finite-dimensional projection)**.** The 2D Navier–Stokes / Euler equations are *time-T solidly globally controllable* in projection on a finite-dimensional subspace $\mathcal{L} \subset H^2(M)$ if for any bounded set S in $\mathcal{L}$ there exists a set of controls B_S such that $(\Pi^{\mathcal{L}} \circ \mathcal{F}/\mathcal{T}_T)(B_S)$ covers S solidly.

2.4. Statement of the problem.

In this paper, we discuss the following questions.

- Under what conditions the 2D Navier–Stokes / Euler equations are globally controllable in the observed component?
- Under what conditions the 2D Navier–Stokes / Euler equations are solidly controllable in a finite-dimensional projection?
- Under what conditions the 2D Navier–Stokes / Euler equations are accessible in a finite-dimensional projection?
- Under what conditions the 2D Navier–Stokes / Euler equations are L_2-approximately controllable?

As we explained above, the geometry of controllability is encoded in the spectral properties of the Laplacian Δ and therefore depends on the geometry of the domain on which the controlled Navier–Stokes equations evolves. Below we provide answers for particular types of domains.

3. Geometric Control. Accessibility and Controllability via Lie Brackets

In this section, we collect some results of geometric control theory regarding the accessibility and controllability of *finite-dimensional* real-analytic control-affine systems of the form

$$\dot{x} = f^0(x) + \sum_{i=1}^{r} f^i(x)v_i(t),\ x(0) = x^0, \quad v_i(t) \in \mathbb{R}, i = 1, \ldots, r. \tag{3.1}$$

The geometric approach is coordinate-free, so that it is adapted for dealing with dynamics on manifolds. However, we assume that the system (3.1) is defined on a finite-dimensional linear space $\mathbb{R}^N$ in order to maintain the parallel with the Navier–Stokes equations which evolve in Hilbert spaces.

We use the standard notation $P_t = e^{tf}$ for the flow corresponding to a vector field f.

3.1. Orbits, Lie rank, and accessibility.

Let $v(\cdot) \in L_\infty([0, T]; \mathbb{R}^r)$ be admissible controls, and let $x(t)$ be the corresponding trajectories of the system $\dot{x} = f^0(x) + \sum_{i=1}^{r} f^i(x)v_i(t)$ with initial point $x(0) = x^0$. We again introduce an end-point mapping $\mathcal{E}_T : v(\cdot) \mapsto x_v(T)$; here $x_v(\cdot)$ is the trajectory of (3.1) corresponding to the control $v(\cdot)$.

For each $T > 0$ the time-T attainable set $\mathcal{A}^T_{x^0}$ from x^0 of the system (3.1) is the image of the set $L_\infty([0, T]; \mathbb{R}^r)$ under the mapping $\mathcal{E}_T$ or, equivalently, the set of points $x(T)$ attained in time T from x^0 by means of admissible controls. The time-$\leqslant T$ attainable set from x^0 is $\mathcal{A}^{\leqslant T}_{x^0} = \bigcup_{t\in[0,T]} \mathcal{A}^t_{x^0}$. The attainable set from x^0 of the system (3.1) is $\mathcal{A}_{x^0} = \bigcup_{T\geqslant 0} \mathcal{A}^T_{x^0}$.

An important notion in geometric control theory is an orbit of a control system.

Definition 3.1 (orbits and zero-time orbits of control systems)**.** The *orbit* of the control system (3.1) passing through x^0 is the set of points obtained from x^0 under the action of (the group of) diffeomorphisms of the form $e^{t_1 f^{u^1}} \circ \cdots \circ e^{t_N f^{u^N}}$, where $t_j \in \mathbb{R}$, $j = 1, \ldots, N$, and $f^{u^j} = f^0 + \sum_{i=1}^{r} f^i(x) u_i^j$ is the right-hand side of (3.1) corresponding to the constant control $u^j = (u_1^j, \ldots, u_r^j) \in \mathbb{R}^r$. The *zero-time orbit* is the subset of the orbit resulting from the action of these diffeomorphisms subject to the condition $\sum_j t_j = 0$.

If we consider the "symmetrization" of the system (3.1),

$$\dot{x} = f^0(x) v_0 + \sum_{i=1}^{r} f^i(x) v_i(t), x(0) = x^0, v_0 \in \mathbb{R}, \ v_i(t) \in \mathbb{R}, i = 1, \ldots, r,$$

then the orbit of (3.1) can be interpreted as the attainable set from x^0 of this symmetrization corresponding to application of piecewise-constant controls.

The famous Nagano theorem relates properties of orbits and Lie algebraic properties of the system. It claims that the orbit and the zero-time orbit of the analytic system (3.1) are immersed manifolds of $\mathbb{R}^N$ and the tangent spaces to these orbits can be calculated via the Lie brackets of vector fields $\{f^0, \ldots, f^m\}$.

Definition 3.2 (Lie rank and zero-time Lie rank)**.** Take the Lie algebra $\mathrm{Lie}\{f^0, \ldots, f^m\}$ generated by $\{f^0, \ldots, f^m\}$ and evaluate vector fields from $\mathrm{Lie}\{f^0, \ldots, f^m\}$ at a point x. The dimension of the resulting linear space $\mathrm{Lie}_x\{f^0, \ldots, f^m\}$ is the *Lie rank* of the system $\{f^0, \ldots, f^m\}$ at x.

Take the Lie ideal generated by $\mathrm{span}\{f^1, \ldots, f^m\}$ in $\mathrm{Lie}\{f^0, \ldots, f^m\}$ and evaluate vector fields from it at x. The dimension of the resulting linear space $\mathrm{Lie}_x^0\{f^0, \ldots, f^m\}$ is the *zero-time Lie rank* of the system $\{f^0, \ldots, f^m\}$ at x.

These Lie ranks either are equal or differ by 1.

The Nagano theorem claims that, in the analytic case, $\mathrm{Lie}_x\{f^0, \ldots, f^m\}$ and $\mathrm{Lie}_x^0\{f^0, \ldots, f^m\}$ are the tangent spaces at each point x of the orbit and zero-time orbit respectively.

The accessibility properties of the analytic control system (3.1) are determined by the Lie ranks of this system. Recall that a system is *accessible* if the attainable set $\mathcal{A}_{x^0}$ has nonempty interior and is *strongly accessible* if for all $T > 0$ the attainable sets $\mathcal{A}_{x^0}^T$ have nonempty interior.

Theorem 3.3 (Jurdjevic–Sussmann (C^ω case) and Krener (C^∞ case))**.** *If the Lie rank of a system of vector fields $\{f^0, \ldots, f^r\}$ at x^0 is equal to n,* *then for all* $T > 0$ *the interior of the attainable set* $\mathcal{A}_{x^0}^{\leqslant T}$ *is nonvoid. Moreover,* $\mathcal{A}_{x^0}^{\leqslant T}$ *possesses the interior which is dense in it. If the zero-time Lie rank at* x^0 *is equal to* n*, then for all* $T > 0$ *the interior of the attainable set* $\mathcal{A}_{x^0}^{T}$ *is nonvoid and is dense in* $\mathcal{A}_{x^0}^{T}$*.*

See [**18, 2**] for the proof.

Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^N$, and let $\Pi^{\mathcal{L}}$ be a projection of $\mathbb{R}^N$ onto $\mathcal{L}$. The control system (3.1) is (*strongly*) *accessible from* x *in projection on* $\mathcal{L}$ if the image $\Pi^{\mathcal{L}}\mathcal{A}_{x^0}$ ($\Pi^{\mathcal{L}}\mathcal{A}_{x^0}^{T}$) contains interior points in $\mathcal{L}$ (for each $T > 0$).

From Theorem 3.3 we easily obtain the following criterion for accessibility in projection.

Theorem 3.4. *If* $\Pi^{\mathcal{L}}$ *maps* $\mathrm{Lie}_x\{f^0, \ldots, f^m\}$ ($\mathrm{Lie}_x^0\{f^0, \ldots, f^m\}$) *onto* $\mathcal{L}$*, then the control system* (3.1) *is accessible* (*strongly accessible*) *at* x *in projection on* $\mathcal{L}$*.*

Proof. Since the proofs of both assertions are similar, we sketch the proof of the first one. Consider the orbit of the system (3.1) passing through x_0. The tangent space to the orbit at each of its points x coincides with $\mathrm{Lie}_x\{f^0, \ldots, f^m\}$.

By Theorem 3.3, the attainable set of the system possesses relative interior with respect to the orbit. Moreover, there are interior points $x_{int} \in \mathcal{A}_{x^0}$ arbitrarily close to x^0 so that $\Pi^{\mathcal{L}}$ maps $\mathrm{Lie}_{x_{int}}\{f^0, \ldots, f^m\}$ onto $\mathcal{L}$. Then sufficiently small neighborhoods of x_{int} in the orbit are contained in $\mathcal{A}_{x^0}$ and are mapped by $\Pi^{\mathcal{L}}$ onto a subset of $\mathcal{L}$ with nonempty interior. $\square$

3.2. Lie extensions and controllability.

Controllability is stronger and much more delicate property than accessibility. For the verification of controllability it does not suffice, in general, to compute the Lie rank which accounts for *all* the Lie brackets. Instead, one should select “good Lie brackets” avoiding “bad Lie brackets” or “obstructions.”

To have a general idea of what good and bad Lie brackets can be like, we consider the following elementary example.

Example. $\dot{x}_1 = v, \dot{x}_2 = x_1^2$. This is the two-dimensional control-affine system (3.1) with $f^0 = x_1^2\partial/\partial x_2$ and $f^1 = \partial/\partial x_1$. The Lie rank of this system is equal to 2 at each point. The system is accessible, but uncontrollable from each point $\hat{x} = (\hat{x}_1, \hat{x}_2)$ given the fact that we cannot achieve points with $x_2 < \hat{x}_2$. One can prove that the attainable set $\mathcal{A}_{\hat{x}}$ coincides with the half-plane $x_2 > \hat{x}_2$ with added point $\hat{x}$. One can see that it is possible to move freely (bidirectionally) along the good vector field f^1, while, along the bad Lie bracket $[f^1[f^1, f^0]] = 2\partial/\partial x_2$, we can move only in one direction.

The good Lie brackets form the *Lie extension* of our control system.

Definition 3.5. A family $\mathcal{F}'$ of real analytic vector fields is

(i) an *extension* of $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$ and the closures of the attainable sets $\mathcal{A}_{\mathcal{F}}(\tilde{x})$ and $\mathcal{A}_{\mathcal{F}'}(\tilde{x})$ coincide,

(ii) a *time-T extension* of $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$ and the closures of the time-T attainable sets $\mathcal{A}^T_{\mathcal{F}}(\tilde{x})$ and $\mathcal{A}^T_{\mathcal{F}'}(\tilde{x})$ coincide,

(iii) a *fixed-time extension* if it is a time-T extension for all $T > 0$.

The vector fields from $\mathcal{F}' \setminus \mathcal{F}$ are called (i) *compatible*, (ii) *compatible in time T*, (iii) *compatible in a fixed time* with $\mathcal{F}$ in cases (i), (ii), and (iii) respectively.

The inclusions $\mathcal{A}_{\mathcal{F}}(\tilde{x}) \subset \mathcal{A}_{\mathcal{F}'}(\tilde{x})$ and $\mathcal{A}^T_{\mathcal{F}}(\tilde{x}) \subset \mathcal{A}^T_{\mathcal{F}'}(\tilde{x})$ are obvious. Less obvious is the following proposition (see [**2**]).

Proposition 3.6. *If an extension $\mathcal{F}'$ of an analytic system $\mathcal{F}$ is globally controllable, then $\mathcal{F}$ is also globally controllable.*

Remark 3.1. Talking about time-T extensions, one can consider also extensions by time-variant vector fields X_t, $t \in [0, T]$. We say that a vector field X_t is *time-T compatible with $\mathcal{F}$* if it drives the system in time T from $\tilde{x}$ to the closure of $\mathcal{A}^T_{\mathcal{F}}(\tilde{x})$.

Our idea is to proceed with a series of extensions of a control system in order to end up with an extended system for which controllability can be verified and then to apply Proposition 3.6.

Obviously, Definition 3.5 is nonconstructive. In what follows, we will use three particular types of extensions.

The first natural type is based on the possibility of taking the topological closure of a set of vector fields, maintaining the closures of attainable sets.

Proposition 3.7. *(see* [**18**, Ch. 3, §2, Theorem 5]*) The topological* (*with respect to the C^∞ convergence on compact sets*) *closure* $\mathrm{cl}(\mathcal{F})$ *of $\mathcal{F}$ is a Lie extension.*

The second type is based on the theory of relaxed (or sliding mode) controls. This theory [**17, 16**] is a far-going development of the pioneering contributions by Young [**28**] and McShane [**20**] in the context of optimal control theory. To introduce the extension, we consider a family of the so-called relaxation seminorms $\|\cdot\|_{s,K}$ of time-variant vector fields X_t, $t \in [0,T]$:

$$\|X.\|^{\mathrm{rx}}_{s,K} = \max_{t\in[0,T]} \left| \int_0^t \|X_\tau\|_{s,K} d\tau \right|, \tag{3.2}$$

where K is a compact set in $\mathbb{R}^N$, $s \geqslant 0$ is an integer, and $\|X_\tau\|_{s,K}$ is the C^k-norm on K. The family of relaxation seminorms defines the relaxation topology (metric) in the set of time-variant vector fields.

Proposition 3.8 (see [**17, 2**])**.** *Let a sequence of time-variant vector fields X_t^j converge to a vector field X_t in the relaxation metric, and let these vector fields have compact support. Then the flows of X_t^j converge to the flow of X_t.*

Based on this result, one can prove the following assertion.

Proposition 3.9. *For the systems $\mathcal{F}$ and*

$$\operatorname{co}\mathcal{F} = \left\{ \sum_{i=1}^m \beta_i f_i, f_i \in \mathcal{F},\ \beta_i \in C^\omega(\mathbb{R}^N), \beta_i \geqslant 0, \sum_{i=1}^m \beta_i \equiv 1,\ i = 1,\dots,m \right\}$$

the closures of their time-T attainable sets coincide. Hence $\operatorname{co}\mathcal{F}$ *is an extension of $\mathcal{F}$.*

The proof of Proposition 3.9 and its modifications can be found in [**2**, Ch. 8], [**18**, Ch. 3],[**17**, Chs. II, III].

The third type of extensions, we will use, relies upon Lie brackets. It appeared in our earlier work on the controllability of the Euler equation for a rigid body in [**3**] and was called there the *reduction* of a *control–affine* system. We present a particular version adapted to our problem. The repeated application of this extension settles the controllability issue for finite-dimensional Galerkin approximations of the Navier–Stokes equations.

Proposition 3.10. *Consider the control-affine analytic system*

$$\dot{x} = f^0(x) + f^1(x)\hat{v}_1 + f^2(x)\hat{v}_2. \tag{3.3}$$

Let

$$[f^1, f^2] = 0, \quad [f^1,[f^1,f^0]] = 0. \tag{3.4}$$

Then the system

$$\dot{x} = f^0(x) + f^1(x)\tilde{v}_1 + f^2(x)\tilde{v}_2 + [f^1, [f^2, f^0]]v_{12},$$

is a fixed-time Lie extension of (3.3).

Sketch of the proof. Take Lipschitz functions $v_1(t)$, $v_2(t)$, $v_1(0) = v_2(0) = 0$, and replace $\hat{v}_1$ and $\hat{v}_2$ in (3.3) with $\varepsilon^{-1}\dot{v}_1(t) + \tilde{v}_1$ and $\varepsilon\dot{v}_2(t) + \tilde{v}_2$ respectively. We obtain the equation

$$\dot{x} = f^0(x) + f^1(x)\tilde{v}_1 + f^2(x)\tilde{v}_2 + \left(\varepsilon^{-1}f^1\dot{v}_1(t) + \varepsilon f^2\dot{v}_2(t)\right). \tag{3.5}$$

Applying the "reduction formula" from [**3**] or, alternatively, the "variation of constants" formula of chronological calculus [**1**], one can represent the flow of (3.5) as the composition of the flow $\tilde{P}_t$ of the equation

$$\dot{y} = e^{\mathrm{ad}\, f^1(y)\varepsilon^{-1}v_1(t) + \mathrm{ad}\, f^2\varepsilon v_2(t)}f^0(y) + f^1(x)\tilde{v}_1 + f^2(x)\tilde{v}_2 \tag{3.6}$$

and the flow

$$P_t = e^{f^1\varepsilon^{-1}v_1(t) + f^2\varepsilon v_2(t)}. \tag{3.7}$$

For the validity of this decomposition the equality $[f^1, f^2] = 0$ is important.

In (3.6), e^{ad_f} is the exponential of the operator ad_f:

$$e^{\mathrm{ad}_f} = \sum_{j=0}^{\infty}(\mathrm{ad}_f)^j/j!.$$

The operator ad_f is determined by the vector field f and acts on vector fields as $\mathrm{ad}_f\, g = [f, g]$, where $[f, g]$ is the Lie bracket of f and g.

By the first relation in (3.4), the operators ad_{f^1} and ad_{f^2} commute and, by the second one, any iterated Lie bracket of the form $(\mathrm{ad}_{f^{i_1}}) \circ \cdots \circ (\mathrm{ad}_{f^{i_m}})f^0$, $i_j = 1, 2$, vanishes whenever it contains ad_{f^1} at least twice.

Taking the expansion of the operator exponential in (3.6) and using these facts, we get [1]

$$\begin{aligned}\dot{y} = {} & f^0(y) + f^1(x)\tilde{v}_1 + f^2(x)\tilde{v}_2 + \varepsilon^{-1}[f^1, f^0](x)v_1(t) \\ & + [f^1, [f^2, f^0]](x)v_1(t)v_2(t) + O(\varepsilon).\end{aligned} \tag{3.8}$$

To obtain the flow of (3.5), we need to compose the flow of (3.8) with the flow (3.7). For any fixed T one can get $P_T = Id$ in (3.7) by choosing $v_1(\cdot), v_2(\cdot)$ such that $v_1(T) = v_2(T) = 0$.

[1] The time-variant vector field abbreviated by $O(\varepsilon)$ in (3.8) is equal to

$$\varepsilon\overline{\varphi}(\varepsilon\,\mathrm{ad}_{v_2(t)f^2})\,\mathrm{ad}_{v_2(t)f^2}\, f^0 + \varepsilon^2\,\mathrm{ad}^2_{v_2(t)f^2}\,\varphi(\varepsilon\,\mathrm{ad}_{v_2(t)f^2})[f^1, f^0],$$

where $\overline{\varphi}(z) = z^{-1}(e^z - 1)$, $\varphi(z) = z^{-2}(e^z - 1 - z)$.

From now on, we deal with a fixed T and the flow of Equation (3.8).

In (3.8), we replace $v_j(t)$ with $v_j(t) = 2^{1/2}\sin(t/\varepsilon^2)\bar{v}_j(t)$, where $\bar{v}_j(t)$, $j = 1, 2$, are functions of bounded variation. The relaxation seminorms of the time-variant vector field $\varepsilon^{-1}[f^1, f^0](x)2^{1/2}\sin(t/\varepsilon^2)\bar{v}_1(t)$ on the right-hand side of (3.8) are $O(\varepsilon)$ as $\varepsilon \to +0$. On the right-hand side of (3.8), we have

$$\begin{aligned}&[f^1, [f^2, f^0]](x)2\sin^2(t/\varepsilon^2)\bar{v}_1(t)\bar{v}_2(t)\\&= [f^1, [f^2, f^0]](x)\bar{v}_1(t)\bar{v}_2(t) - [f^1, [f^2, f^0]](x)\cos(2t/\varepsilon^2)\bar{v}_1(t)\bar{v}_2(t).\end{aligned}$$

The relaxation seminorms of the addend $[f^1, [f^2, f^0]](x)\cos(2t/\varepsilon^2)\bar{v}_1(t)\bar{v}_2(t)$ are $O(\varepsilon^2)$ as $\varepsilon \to +0$.

Hence the right-hand sides of (3.8) with controls

$$v_j(t) = 2^{1/2}\sin(t/\varepsilon^2)\bar{v}_j(t), \quad j = 1, 2,$$

converge in the relaxation metric to the vector field

$$f^0(y) + f^1(x)\tilde{v}_1 + f^2(x)\tilde{v}_2 + [f^1, [f^2, f^0]](x)\bar{v}_1(t)\bar{v}_2(t)$$

as $\varepsilon \to 0$. We can consider the product $\bar{v}_1(t)\bar{v}_2(t)$ as a new control v_{12} and invoke Proposition 3.8. $\square$

4. Computation of Brackets in Finite and Infinite Dimensions. Controlling along "Principal Axes"

In this section, we adjust the statement of Proposition 3.8 for studying the controllability of the systems (2.10) and (2.12).

From the viewpoint of geometric control, the Galerkin approximation (2.12) of the Navier–Stokes / Euler equations is a special case of the control-affine system (3.1). Its state space is finite-dimensional and is generated by a finite number of eigenfunctions of the Laplace–Beltrami operator Δ or *modes*. The dynamics of this control system is determined by the quadratic *drift* vector field

$$f_o^0 = \Pi^o\{\Delta^{-1}q^o, q^o\} + \nu\Delta q^o$$

and by controlled forcing $\sum_{i=1}^{r} f^i(x)u_i$, where f^i are constant (q^o-independent) controlled vector fields.

We start by computing the particular Lie brackets involved in the formulation of Proposition 3.8. For two constant vector fields f^1 and f^2 we have

$$
\begin{aligned}
&[f^i, f^0_o] = \Pi^o(\{f^i, \Delta^{-1}w\} + \{w, \Delta^{-1}f^i\}) + \nu\Delta f^i, \quad i = 1, 2,\\
&[f^1, [f^2, f^0_o]] = \Pi^o(\{f^2, \Delta^{-1}f^1\} + \{f^1, \Delta^{-1}f^2\}),\\
&[[f^1, [f^1, f^0_o]] = 2\Pi^o\{f^1, \Delta^{-1}f^1\}.
\end{aligned}
$$

This computation is finite-dimensional, but the same holds if one considers constant vector fields acting in an infinite-dimensional Hilbert space. Taking the "drift" vector field of (2.10) in infinite dimension,

$$f^0 = \{\Delta^{-1}q, q\} + \nu\Delta q,$$

we obtain the following assertion.

Lemma 4.1. *For two constant vector fields f^1 and f^2*

$$
\begin{aligned}
&[f^i, f^0] = \{f^i, \Delta^{-1}w\} + \{w, \Delta^{-1}f^i\} + \nu\Delta f^i, \quad i = 1, 2,\\
&[f^1, [f^1, f^0]] = 2\{f^1, \Delta^{-1}f^1\},\\
&\mathcal{B}(f^1, f^2) = [f^1, [f^2, f^0]] = \{f^2, \Delta^{-1}f^1\} + \{f^1, \Delta^{-1}f^2\}.
\end{aligned}
\tag{4.1}
$$

Let us clarify what is needed for the assumptions of Proposition 3.10 to hold. As long as f^1 and f^2 are constant and hence commuting, all what we need is the following:

$$[f^1, [f^1, f^0]] = 2\{f^1, \Delta^{-1}f^1\} = 0. \tag{4.2}$$

Regarding the Euler equation for an *ideal* fluid

$$\frac{\partial w}{\partial t} - \{\Delta^{-1}w, w\} = 0,$$

formula (4.2) means that f^1 corresponds to its *steady motion*. In particular, the *eigenfunctions of the Laplace–Beltrami operator* Δ correspond to steady motions and satisfy (4.2). These eigenfunctions will be used as controlled directions.

The eigenfunctions of the Laplacian are analogous to the principal axes of a (multi-dimensional) rigid body.

By Proposition 3.10, for two constant controlled vector fields f^1, f^2, one of which corresponds to a steady motion, we can extend our control system by the new controlled vector field $[f^1, [f^2, f^0]]$ which is again constant.

Our method consists of iterating this procedure. The algebraic/geometric difficulties arising in this way consist of scrutinizing newly obtained controlled directions in order to pick among them the ones which satisfy

(4.2). This will be illustrated in the following sections dealing with particular 2D domains.

Another (analytic) difficulty arises when we pass from the finite-dimensional approximations to the controlled partial differential equation. For the latter the above sketch of the proof of Proposition 3.10 is not valid (for example, one cannot speak about flows). We have to reprove the statement of the proposition in each particular situation. The main idea will be still based on using fast oscillating control and relaxation metric. The analytic difficulties are in proving the continuity of forcing/trajectory mapping with respect to such a metric. We will provide a brief comment later on; the details can be found in [**6, 7, 21, 22**].

5. Controllability and Accessibility of Galerkin Approximations of Navier–Stokes / Euler Equations on $\mathbb{T}^2$

We survey results on the accessibility and controllability of Galerkin approximations of the 2D Navier–Stokes / Euler equations on $\mathbb{T}^2$.

5.1. Accessibility of Galerkin approximations.

The result of the computation (4.1) in the periodic case is easy to visualize when the constant controlled vector fields corresponding to the eigenfunctions of Laplacian Δ on $\mathbb{T}^2$ are written as complex exponentials.

For two different complex eigenfunctions $f^1 = e^{ik\cdot x}$ and $f^2 = e^{i\ell\cdot x}$ of the Laplacian Δ on $\mathbb{T}^2$, $x \in \mathbb{R}^2$, $k, \ell \in \mathbb{Z}^2$, the Poisson bracket in (4.1) is equal to

$$\mathcal{B}\left(e^{ik\cdot x}, e^{i\ell\cdot x}\right) = (k \wedge \ell)(|k|^{-2} - |\ell|^{-2})e^{i(k+\ell)\cdot x}, \tag{5.1}$$

i.e., it again corresponds to an eigenfunction of Δ provided that $|k| \neq |\ell|$, $k \wedge \ell \neq 0$. The conclusion is that for two given pairs of complex exponentials $e^{\pm ik\cdot x}$ and $e^{\pm i\ell\cdot x}$ taken as controlled vector fields one can add to them the controlled vector fields $e^{i(\pm k\pm\ell)\cdot x}$.

Iterating the computation of the Lie–Poisson brackets (4.1) and obtaining new directions, we obtain a (finite or infinite) set of functions which contains $e^{\pm ik\cdot x}, e^{\pm i\ell\cdot x}$ and is invariant under the bilinear operation $\mathcal{B}(\cdot,\cdot)$.

Therefore, in the case of $\mathbb{T}^2$, starting with the controlled vector fields corresponding to the eigenfunctions $e^{ik\cdot x}$, $k \in \mathcal{K}^1 \subset \mathbb{Z}^2$, of the Laplacian, the whole computation of Lie extensions "can be modeled" on the integer lattice $\mathbb{Z}^2$ of "mode indices" k.

Actually, one has to operate with real-valued eigenfunctions of the Laplacian on $\mathbb{T}^2$, i.e., functions of the form $\cos(k \cdot x), \sin(k \cdot x)$. Also, in this case, a computation of the iterated Lie–Poisson brackets (4.1) can be modelled on $\mathbb{Z}^2$ and the addition formulas are similar to those of the complex case.

Proposition 5.1 (bracket generating property)**.** *If*

$$|k| \neq |\ell|, \quad |k \wedge \ell| = 1, \tag{5.2}$$

then the following assertions hold:

(i) *an invariant with respect to $\mathcal{B}$ set of functions, containing $e^{\pm ik\cdot x}$ and $e^{\pm i\ell\cdot x}$, contains all the eigenfunctions $e^{im\cdot x}$, $m \in \mathbb{Z}^2 \setminus 0$,*

(ii) *an invariant with respect to $\mathcal{B}$ set of real functions, containing $\cos(k \cdot x)$, $\sin(k \cdot x)$, $\cos(\ell \cdot x)$, $\sin(\ell \cdot x)$, contains all the eigenfunctions $\cos(m \cdot x)$, $\sin(m \cdot x)$, $m \in \mathbb{Z}^2 \setminus 0$.*

The bracket generating property for a Galerkin approximation of the 2D Navier–Stokes equations with periodic boundary conditions was established by E and Mathingly [**13**]. The following result from [**13**] is an immediate consequence of Proposition 5.1 and Theorem 3.3.

Corollary 5.2 (accessibility by means of four controls)**.** *For any set $\mathcal{M} \subset \mathbb{Z}^2$ there exists a larger set $\mathcal{M}' \supseteq \mathcal{M}$ such that the Galerkin $\mathcal{M}'$-approximation controlled by the forcing*

$$\cos(k \cdot x)v_k(t) + \sin(k \cdot x)w_k(t) + \cos(\ell \cdot x)v_\ell(t) + \sin(\ell \cdot x)w_\ell(t), \tag{5.3}$$

with k and ℓ satisfying (5.2) *is strongly accessible.*

Here, four controls $v_k(t)$, $w_k(t)$, $v_\ell(t)$, $w_\ell(t)$ are used for providing strong accessibility, but actually it can be achieved by a smaller number of controls.

Example (accessibility by means of two controls). Consider the forcing

$$gv(t) + \bar{g}\bar{v}(t), g = \cos(k \cdot x) + \cos(\ell \cdot x),\ \bar{g} = \sin(k \cdot x) - \sin(\ell \cdot x). \tag{5.4}$$

The controlled forcing (5.3) involves four independent controls, each one of which appears in just one of Equations (2.11). The controlled forcing (5.4) involves two controls v and $\bar{v}$, each of which appears in a pair of equations in (2.11).

Assume that $|k| \neq |\ell|, k \wedge \ell \neq 0$. We compute the bilinear form (4.1):

$$\mathcal{B}(g,g) = (-|\ell|^{-2} + |k|^{-2})\{\cos(k \cdot x), \cos(\ell \cdot x)\}$$

Up to a scalar multiplier, $\mathcal{B}(g,g)$ is equal to

$$\begin{aligned}&(-|\ell|^{-2} + |k|^{-2}) \sin(k \cdot x) \sin(\ell \cdot x)\\ &= (k \wedge \ell)(-|\ell|^{-2} + |k|^{-2})(\cos((k-\ell)\cdot x) - \cos((k+\ell)\cdot x)).\end{aligned}$$

Similarly, up to a scalar multiplier, $\mathcal{B}(\bar{g},\bar{g})$ is equal to

$$(k \wedge \ell)(-|\ell|^{-2} + |k|^{-2})(\cos((k-\ell)\cdot x) + \cos((k+\ell)\cdot x)).$$

The span of $\mathcal{B}(g,g)$ and $\mathcal{B}(\bar{g},\bar{g})$ coincides with the span of $g^{01} = \cos((k-\ell)\cdot x)$ and $g^{21} = \cos((k+\ell)\cdot x)$. The direction $\bar{g}^{01} = \sin((k-\ell)\cdot x)$ is obtained from the computation of $\mathcal{B}(g,\bar{g})$. Choosing $k = (1,1)$ and $\ell = (1,0)$, we get $m = k + \ell = (2,1)$ and $n = k - \ell = (0,1)$.

Computing new directions $\mathcal{B}(g^{01},\bar{g})$ and $\mathcal{B}(\bar{g}^{01},g)$, we note that, by the equality $|n| = |\ell|$, they coincide with $\mathcal{B}(g^{01}, \sin(k\cdot x))$ and $\mathcal{B}(\bar{g}^{01}, \cos(k\cdot x))$ respectively, and their span coincides with the span of $\bar{g}^{12} = \sin((k+n)\cdot x))$ and $\bar{g}^{10} = \sin((k-n)\cdot x)) = \sin(\ell \cdot x))$. Similarly, the span of $\mathcal{B}(g^{01},g)$ and $\mathcal{B}(\bar{g}^{01},\bar{g})$ coincides with the span of $g^{12} = \cos((k+n)\cdot x))$ and $g^{10} = \cos((k-n)\cdot x) = \cos(\ell\cdot x)$. Then $g - g^{10} = \cos(k\cdot x)$ and $\bar{g} - \bar{g}^{10} = \sin(k\cdot x)$. These two functions, together with g^{01} and $\bar{g}^{01}$, form a quadruple satisfying the assumptions of Corollary 5.2. Hence our system is accessible by means of 2 controls.

Remark 5.1. It is plausible that the strong accessibility of Galerkin approximations can be achieved by a single control.

5.2. Controllability of Galerkin approximations.

In general, the bracket generating property is not sufficient for controllabilitly. One has to select Lie brackets, which form a Lie extension; meanwhile, in the previous example, $\{g, \Delta^{-1}g\}$ and $\{\bar{g}, \Delta^{-1}\bar{g}\}$ a priori do not correspond to a Lie extension.

Even in the finite-dimensional case, a stronger result of Proposition 3.10 is required for proving the controllability property for Galerkin approximations. This was done in [**4, 25**] in the 2D and 3D cases.

Theorem 5.3. *Let k, ℓ satisfy* (5.2). *For any subset $\mathcal{M} \subset \mathbb{Z}^2$ there exists a larger set $\mathcal{M}'$ such that the Galerkin $\mathcal{M}'$-approximation controlled* *by the forcing*

$$\cos(k\cdot x)v_k(t)+\sin(k\cdot x)w_k(t)+\cos(\ell\cdot x)v_\ell(t)+\sin(\ell\cdot x)w_\ell(t),$$

is globally controllable.

The proof of this controllability result consists of the iterated application of the Lie extension described in Proposition 3.10. At each step, we extend the system by new controlled vector fields corresponding, in accordance with (4.1) and (5.1), to $f^{m\pm\ell}=\cos((m\pm\ell)\cdot x)$ and $\bar{f}^{m\pm\ell}=\sin((m\pm\ell)\cdot x)$. At the end of the iterated procedure, we arrive at a system with an extended set of controls, one for each observed mode. It is evident that this system must be controllable.

An important case, where the controllability of Galerkin approximations is implied by the bracket generating property, regards the 2D Euler equations for an incompressible ideal fluid ($\nu=0$). Indeed, in this case, the drift (zero control) dynamics is *Hamiltonian* and it evolves on a compact energy level. By the Liouville and Poincaré theorems, the Poisson–stable points of this dynamics are dense and one can apply the Lobry–Bonnard theorem [**2, 18**] to establish the following assertion.

Theorem 5.4. *For $\nu=0$ the Galerkin approximation of the $2D$ Euler equations controlled is globally controllable by means of the forcing* (5.4).

In the case of an ideal fluid, the controllability of Galerkin approximations of the 2D Euler system can be achieved by *scalar* control.

6. Steady State Controlled Directions. Abstract Controllability Result for Navier–Stokes Equations

We cannot apply Proposition 3.10 to the infinite-dimensional case directly. However, the main idea of adding new controlled directions is still valid for the Navier–Stokes equations. Now we want to formulate an abstract controllability criterion based on the Lie extensions and computation of Lemma 4.1. This criterion will be employed in the following sections for establishing the controllability of the Navier–Stokes equations on various 2D domains.

Theorem 6.1 (controllability of Navier–Stokes equations via saturation of controls)**.** *Let* $\mathrm{span}\{f^1,\dots,f^r\}=\mathcal{S}=\mathcal{D}^1$ *be a finite-dimensional* *space of controlled directions. Assume that* $f^1, \dots f^r$ *are steady motions of the Euler equation* (4.2)*. For each pair of linear subspaces* $\mathcal{L}^1$ *and* $\mathcal{L}^2$ *we consider the span of the image* $\mathcal{B}(\mathcal{L}^1, \mathcal{L}^2)$ *of the bilinear mapping* (4.1)*. Define successively* $\mathcal{D}^{j+1} = \mathcal{D}^j + \operatorname{span} \mathcal{B}(\mathcal{S}^j, \mathcal{D}^j)$*,* $j = 1, 2, \dots$*, where* $\mathcal{S}^j \subseteq \mathcal{D}^j$ *is the linear subspace spanned by steady motions. If* $\bigcup\limits_j \mathcal{D}^j$ *is dense in the Sobolev space* $H^2(M)$*, then the Navier–Stokes equations are controllable in finite-dimensional projections and are* L_2*-approximately controllable.*

If $\mathcal{D}^1$ consists of steady motions, then $\mathcal{D}^{j+1} \supseteq \mathcal{D}^j + \operatorname{span} \mathcal{B}(\mathcal{D}^1, \mathcal{D}^j)$. Introduce another sequence of spaces

$$\mathcal{D}_1^{j+1} = \mathcal{D}_1^j + \operatorname{span} \mathcal{B}(\mathcal{D}^1, \mathcal{D}_1^j). \tag{6.1}$$

It is evident that $\mathcal{D}_1^j \subseteq \mathcal{D}^j$ and the density of $\bigcup\limits_j \mathcal{D}_1^j$ in $H^2(M)$ guarantees controllability.

Let $D_{f_s} = \{\Delta^{-1} \cdot, f_s\} + \{\Delta^{-1} f_s, \cdot\}$ for $f_s \in \mathcal{D}^1$, $D_{f_s} = \mathcal{B}(f_s, \cdot)$. The iterated computations (6.1) correspond to iterated applications of the operators D_{f_s} to $f^1, \dots f^r$ and taking the linear span.

Corollary 6.2. *Let* $\mathcal{F}$ *be the minimal common invariant linear subspace of the operators* $D_{f_1}, \dots, D_{f_l}$ *which contains* $f_1, \dots, f_k$*. If* $\mathcal{F}$ *is everywhere dense in* $L_2(M)$*, then the system is* L_2*-approximately controllable and is solidly controllable in finite-dimensional projections.*

7. Navier–Stokes and Euler Equations on $\mathbb{T}^2$

In this section, we formulate results regarding the controllability in finite-dimensional projections and the L_2-approximate controllability on $\mathbb{T}^2$. Namely, we describe sets of controlled directions which satisfy a criterion provided by Theorem 6.1.

We take the basis of complex eigenfunctions $\left(e^{ik\cdot x}\right)$ of the Laplacian on $\mathbb{T}^2$ and introduce the Fourier expansion of the vorticity $w(t,x) = \sum\limits_k q_k(t) e^{ik\cdot x}$ and the control $v(t,x) = \sum\limits_{k \in \mathcal{K}^1} v_k(t) e^{ik\cdot x}$. Here, $k \in \mathbb{Z}^2$. If w and f are real-valued, we have $\bar{q}_n = q_{-n}$ and $\bar{v}_n = v_{-n}$. We assume that $v_0 = 0$ and $q_0 = 0$.

Using (5.1), we write the infinite system of ordinary differential equations (2.11) as follows:

$$\dot{q}_k = \sum_{m+n=k,|m|<|n|} (m \wedge n)(|m|^{-2} - |n|^{-2})q_m q_n - |k|^2 q_k + \hat{v}_k(t). \qquad (7.1)$$

The controls $\hat{v}_k$ are nonvanishing only in the equations for the variables q_k indexed by the symmetric set $\mathcal{K}^1 \subset \mathbb{Z}^2 \setminus \{0\}$. For $k \notin \mathcal{K}^1$ the dynamics is as follows:

$$\dot{q}_k = \sum_{m+n=k,|m|<|n|} (m \wedge n)(|m|^{-2} - |n|^{-2})q_m q_n - |k|^2 q_k, k \notin \mathcal{K}^1. \qquad (7.2)$$

There is a symmetric set of those observed modes $\mathcal{K}^o \supset \mathcal{K}^1$, which we want to steer to some preassigned values. In the only interesting case, where $\mathcal{K}^1$ is a proper subset of $\mathcal{K}^o$, the equations indexed by $k \in \mathcal{K}^o \setminus \mathcal{K}^1$ are of the form (7.2). They do not contain controls and have to be *controlled via state variables.*

We give a hint of how this can be done; it is an infinite-dimensional version of Proposition 3.10 for the Navier–Stokes equations on $\mathbb{T}^2$.

Let $r, s \in \mathcal{K}^1, r \wedge s \neq 0, |r| < |s|$, $k = r + s \notin \mathcal{K}^1$. The equations for q_r and q_s contain controls $\hat{v}_r$ and $\hat{v}_s$, while the equation for q_k does not.

Take Lipschitz functions $v_r(t)$, $v_s(t)$, $v_r(0) = v_s(0) = 0$), and substitute $\varepsilon^{-1}\dot{v}_r(t) + \tilde{v}_r$ and $\varepsilon\dot{v}_s(t) + \tilde{v}_s$ for $\hat{v}_r, \hat{v}_s$ into the right-hand sides of Equations (7.1) for the variables q_r, q_s. We obtain

$$\dot{q}_r = \sum_{m+n=r,|m|<|n|} (m \wedge n)(|m|^{-2} - |n|^{-2})q_m q_n - |r|^2 q_r + \varepsilon^{-1}\dot{v}_r(t) + \tilde{v}_r,$$

$$\dot{q}_s = \sum_{m+n=s,|m|<|n|} (m \wedge n)(|m|^{-2} - |n|^{-2})q_m q_n - |s|^2 q_s + \varepsilon\dot{v}_s(t) + \tilde{v}_s.$$

Introduce $q_r^* = q_r - \varepsilon^{-1}v_r(t)$ and $q_s^* = q_s - \varepsilon v_s(t)$. Assuming $v_r(T) = v_s(T) = 0$, we conclude $q_r(T) = q_r^*(T), q_s(T) = q_s^*(T)$.

We write the infinite system of ordinary differential equations (7.1), (7.2) via q_r^* and q_s^* in place of q_r and q_s. The right-hand side of the equation for $q_k = q_{r+s}$ contains the addend

$$(r \wedge s)(|r|^{-2} - |s|^{-2})(q_r^* + \varepsilon^{-1}v_r(t))(q_s^* + \varepsilon v_s(t))$$

and we see that the controls v_r and v_s enter this equation via the product $v_r(t)v_s(t)$. The same v_r, v_s enter this and all other equations linearly.

Substitute $v_j(t)$, $j = r, s$, by $v_j(t) = 2^{1/2}\sin(t/\varepsilon^2)\bar{v}_j(t)$ with $\bar{v}_j(t)$ having bounded variations. Then the right-hand side of the equation for q_k

will gain the product $2\sin^2(t/\varepsilon^2)\bar{v}_r(t)\bar{v}_s(t) = (1-2\cos(2t/\varepsilon^2))\bar{v}_r(t)\bar{v}_s(t)$. If $\varepsilon \to +0$, this product tends to $\bar{v}_r(t)\bar{v}_s(t)$ in the relaxation metric. In all other equations, $\bar{v}_r(t)$ and $\bar{v}_s(t)$ enter linearly and are multiplied by the fast oscillating functions $2^{1/2}\sin(t/\varepsilon^2)$. Hence the corresponding terms tend to 0 in relaxation metric.

Therefore, one can pass (as $\varepsilon \to 0$) to a limit system which now contains the "new" control $\bar{v}_{rs} = \bar{v}_r(t)\bar{v}_s(t)$ in the equation for $q_k = q_{r+s}$. (This control corresponds to the control v_{12} from Proposition 3.10.)

A difficult analytic part is a justification of this passage to the limit. It is accomplished in [**6, 7**] for $\mathbb{T}^2$ and in [**21, 22**] for a rectangular and other kinds of regular 2D domains. We refer the interested reader to these publications.

Note that the new controlled direction corresponds to the complex exponential which is an eigenfunction of the Laplacian on $\mathbb{T}^2$. Hence we can model the Lie extensions and formulate the controllability results in terms of indices $k \in \mathbb{Z}^2$ of controlled modes.

Define iteratively a sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^2$ as follows:

$$\begin{aligned} &j = 2, \dots, \\ &\mathcal{K}^j = \mathcal{K}^{j-1} \bigcup \{m+n | m, n \in \mathcal{K}^{j-1} \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0\}. \end{aligned} \tag{7.3}$$

Definition 7.1. A finite set $\mathcal{K}^1 \subset \mathbb{Z}^2 \setminus \{0\}$ of forcing modes is *saturating* if $\bigcup\limits_{j=1}^{\infty} \mathcal{K}^j = \mathbb{Z}^2 \setminus \{0\}$, where $\mathcal{K}^j$ are defined by (7.3).

Theorem 7.2 (controllability in finite-dimensional projection)**.** *Let $\mathcal{K}^1$ be a saturating set of controlled forcing modes, and let $\mathcal{L}$ be any finite-dimensional subspace of $H^2(\mathbb{T}^2)$. Then for any $T > 0$ the Navier–Stokes / Euler equations on $\mathbb{T}^2$ is time-T solidly controllable in finite-dimensional projections and is time-T L_2-approximately controllable.*

As we see, the saturating property is crucial for controllability. In [**7**], the following characterization of this property was established.

Theorem 7.3. *For a symmetric finite set $\mathcal{K}^1 = \{m^1, \dots, m^s\} \subset \mathbb{Z}^2$ the following properties are equivalent*:

(i) *$\mathcal{K}^1$ is saturating,*

(ii) *the greatest common divisor of the numbers $d_{ij} = m^i \wedge m^j, i, j \in \{1, \dots, s\}$ is equal to 1, and there exist $m^\alpha, m^\beta \in \mathcal{K}^1$ that are not collinear and have different lengths.*

Corollary 7.4. *The set* $\mathcal{K}^1 = \{(1,0), (-1,0), (1,1), (-1,-1)\} \subset \mathbb{Z}^2$ *is saturating. The solid controllability in any finite-dimensional projection and* L_2*-approximately controllability can be achieved by forcing four modes.*

8. Controllability of 2D Navier–Stokes Equations on Rectangular Domain

The study of the controllability in finite-dimensional projections and the L_2-approximate controllability on a rectangular domain has been accomplished by Rodrigues [**21, 22**]. The main idea is similar to that in the periodic case, but computations are more intricate. The reason is twofold: (i) the algebraic properties of the bilinear operation calculated for the eigenfunctions of the Laplacian are more complex and (ii) one needs to care about boundary conditions.

For a velocity field u on a rectangular $\mathcal{R}$ with sides of length a, b, $a \neq b$, we assume that the Lions boundary conditions hold. In terms of the vorticity w, they can be written as (2.9).

The (vorticity) eigenfunctions φ^k of the Laplacian are

$$\varphi^k = \sin\left(\frac{\pi}{a}k_1 x_1\right) \sin\left(\frac{\pi}{b}k_2 x_2\right), \quad (k = (k_1, k_2) \in \mathbb{Z}^2. \tag{8.1}$$

To find an extending controlled direction, one needs to pick two eigenfunctions $f^1 = \varphi^k$, $f^2 = \varphi^\ell$, $k, \ell \in \mathbb{Z}^2$, and to proceed with the computation (4.1). The result is a linear combination of at most four eigenfunctions φ^s.

Then again one can follow Lie extensions on the two-dimensional lattice $\mathbb{Z}^2$ of Fourier exponents $k = (k_1, k_2)$. If the controlled modes are indexed by $k \in \mathcal{K}^1 = \{(k_1, k_2) | 1 \leqslant k_1, k_2 \leqslant 3, k \neq (3,3)\}$, then one can verify that after m Lie extensions the set of extended controlled directions will contain all the modes (k_1, k_2), $k_1, k_2 \leqslant m + 3$, with the exception of $(m+3, m+3)$.

This leads to the following controllability result.

Theorem 8.1 (controllability on rectangular domain)**.** *Let* 8 *controlled directions correspond to the functions* (8.1) *with* $k \in \{(k_1, k_2) | 1 \leqslant k_1, k_2 \leqslant 3, k \neq (3,3)\}$. *Then the Navier–Stokes equations defined on the rectangular domain with the Lions boundary condition are controllable in finite-dimensional projections and are* L_2*-approximately controllable.*

9. Controllability on Generic Riemannian Surface Diffeomorphic to Disc

In this section, we consider the Navier–Stokes equations under the boundary conditions (2.9) on a Riemannian surface M. We manage to prove that for a generic surface (the exact meaning of genericity will be specified below) diffeomorphic to a disc one can choose 3 controlled directions corresponding to the eigenfunctions (modes) of the Laplacian on M, which provides the controllability in finite-dimensional projections.

In what follows, we assume that M has C^∞-smooth boundary and is endowed with a Riemannian metric.

The diffeomorphism $\Phi : M \mapsto D$ induces the C^∞-smooth metric on the disc D and we speak about various Riemannian metrics on the fixed disc D instead of Riemannian surfaces. A *generic* Riemannian surface corresponds to a *generic* smooth Riemannian metric on D, meaning a metric in a residual subset of the topological space of C^∞ metrics. A subset is *residual* if it contains the intersection of countably many open dense subsets of the topological space.

For controlled "directions" we take the modes or eigenfunctions f_s of the Laplace–Beltrami operator Δ corresponding to each metric: $\Delta^{-1} f_s = \lambda_s^{-1} f_s$, $s = 1, \ldots, l$. To apply the abstract controllability criterion (see Corollary 6.2), it suffices to verify that functions of the form $D_{f_{s_1}} \circ \cdots \circ D_{f_{s_m}} f_j$, $m \geqslant 0$; $j \in \{1, \ldots, l\}$, where $D_{f_s} = \{\Delta^{-1} \cdot, f_s\} + \{\Delta^{-1} f_s, \cdot\}$, span a dense subset of $H^2(M)$.

Theorem 9.1. *For a generic Riemannian surface M diffeomorphic to a disc there exist* 3 *eigenfunctions* (*modes*) f_1, f_2, f_3 *of the Laplace–Beltrami operator Δ on M such that the Navier-Stokes / Euler equation on M is controllable in finite-dimensional projections by means of a controlled forcing applied to these modes.*

Sketch of the proof. As was shown [**7**], it suffices to establish the controllability in projection on any finite-dimensional *coordinate* subspace $\mathcal{L}$ spanned by a finite number of eigenfunctions of the Laplace–Beltrami operator. By Corollary 6.2, we need to verify that some determininat $\mathrm{Det}_{\mathcal{L}}$ calculated via the (iterated) Poisson bracket of f_1, f_2, and f_3 does not vanish.

Assume for a moment that for some smooth metric μ_0 on D the determinant $\mathrm{Det}_{\mathcal{L}}$ does not vanish. Consider an *analytic* metric approximating

μ_0 (note that analytic metrics are dense in the space of smooth metrics) for which $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing and denote it by μ_0 again.

Then taking any analytic Riemannian metric μ_1 on D, we construct a linear homotopy μ_t between μ_0 and μ_1:

$$\mu_t|_q(\xi,\xi) = (1-t)\mu_0|_q(\xi,\xi) + t\mu_1|_q(\xi,\xi), \quad 0 \leqslant t \leqslant 1.$$

Recall that the "values" of the Riemannian metrics at each point $q \in M$ are positive definite quadratic forms which form a convex cone.

The t-dependence of the Laplacians $\Delta(t)$ corresponding to the metrics μ_t is analytic.

We want to trace the evolution of a *finite* number of the eigenvalues λ_j^t, $j \in J$ (J is a finite set), and the corresponding eigenfunctions of $\Delta(t)$ with t varying in $[0,1]$. This allows us to study the restriction of $\Delta(t)$ onto a finite-dimensional space (see [**19**, Ch. 7]).

By the classical result of perturbation theory (see [**19**, Chs. 2, 7]), the eigenvalues λ_j^t of an analytic family $t \mapsto A_t$ of linear operators are analytic with respect to t beyond a finite number of exceptional points in $[0,1]$. Any moment t at which the eigenvalues λ_j^t, $j \in J$, are pairwise distinct is nonexceptional. Singularities of the function $t \mapsto \lambda_j^t$ may occur when λ_j^t become multiple. The eigenvectors and corresponding eigenprojections may have poles at exceptional points.

The picture is much more regular for normal operators, in particular, for the Laplacians which are selfadjoint. In this case, the eigenvalues and eigenfunctions are analytic functions of t everywhere on $[0,1]$ ([**19**, Ch.2, Theorem 1.10]). The dependence of the derivatives of eigenfunctions on $t \in [0,1]$ is also analytic. Hence the determinant $\mathrm{Det}_{\mathcal{L}}$ is an analytic function of t. If it does not vanish at the point $t=0$, it may vanish only at finitely many points $t \in [0,1]$.

Take μ_t corresponding to all nonexceptional $t \in [0,1]$ for which $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing. Among nonexceptional t there exist t_s which are arbitrarily close to 1. The metrics μ_{t_s} are arbitrarily close to μ_1 in the C^∞-metrics; for the corresponding $\Delta(t_s)$ the eigenvalues of interest are distinct and $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing. The dependence of the eigenfunctions and their derivatives on the metric μ is continuous in the C^∞-metric in a neighborhood of μ_{t_s}. Hence $\mathrm{Det}_{\mathcal{L}}$ is nonzero for all μ from small C^∞-neighborhoods of these μ_{t_s}. Taking the union of these neighborhoods, we get an open set whose closure contains μ_1. Repeating the homotopy argument for each analytic metric

μ_1 on D, we get an open dense in C^∞ set of metrics for which $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing.

One unsettled problem still remains: To find a metric μ_0 on D for which the determinant $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing.

This problem is by no means minor. To construct such a metric, we use the result mentioned in Remark 10.1 and obtained by Rodrigues [**24**] who established the controllability of Navier–Stokes / Euler equations on the half-sphere $\mathbb{S}^2_+$ with the Navier boundary conditions (in particular, the Lions boundary conditions). The metric on $\mathbb{S}^2_+$ is inherited from the embracing Euclidean space $\mathbb{R}^3$.

The degenerate control is applied to three modes, spherical harmonics which are eigenfunctions of the Laplacian on $\mathbb{S}^2_+$. It is proved that this system is controllable in any finite-dimensional projection.

Mapping $\mathbb{S}^2_+$ onto D analytically, we obtain the corresponding metric μ_0 and Laplacian on D for which the determinant $\mathrm{Det}_{\mathcal{L}}$ is nonvanishing.

Remark 9.1. The construction of the residual set of Riemannian metrics can be transferred (almost) without alterations to the torus $\mathbb{T}^2$ for which we studied the controllability of the Navier–Stokes / Euler equations in Section 7. The conclusion claims that there exists a residual set of smooth Riemannian metrics on $\mathbb{T}^2$ such that the assumptions of Corollary 6.2 are verified and therefore the Navier–Stokes equations is controllable in finite-dimensional projections by forcing four modes on $\mathbb{T}^2$ endowed with any of these metrics.

A pertinent question would be whether the result of Theorem 9.1 holds for a generic subdomain $\mathcal{Q}$ with smooth boundary in $\mathbb{R}^2$ diffeomorphic to a disc and endowed with the Euclidean metric. The corresponding diffeomorphism $\mathcal{Q} \mapsto D$ sends the Euclidean metric on $\mathcal{Q}$ to the metric (9.1) on D which possesses the zero curvature. An approximating analytic metric μ admits the conformal form [**12**, Vol.1, § 11–13])

$$\mu = e^{a(x_1,x_2)}(dx_1^2 + dx_2^2). \tag{9.1}$$

Note that the curvature of (9.1) is equal to

$$K = (1/2)e^{-a(x_1,x_2)}\Big(\frac{\partial^2 a}{\partial x_1^2} + \frac{\partial^2 a}{\partial x_2^2}\Big).$$

Therefore, the plane metrics are distinguished by the condition

$$\frac{\partial^2 a}{\partial x_1^2} + \frac{\partial^2 a}{\partial x_2^2} = 0. \tag{9.2}$$

On the contrary, if D possesses a Riemannian metric μ of the form (9.1) satisfying (9.2), then D can be isometrically and analytically mapped onto a 2D domain $\mathcal{Q}$ with Euclidean metric.

We can define the corresponding homotopy between

$$\mu_0 = e^{a_0(x_1,x_2)}(dx_1^2 + dx_2^2), \quad \mu_1 = e^{a_1(x_1,x_2)}(dx_1^2 + dx_2^2)$$

as follows:

$$\mu_t = e^{(1-t)a_0(x_1,x_2)+ta_1(x_1,x_2)}(dx_1^2 + dx_2^2),$$

and then advance as in the previous proof.

The only problem would be to construct a plane domain $\mathcal{Q}$ with analytic boundary and an Euclidean metric for which the controllability in finite-dimensional projection holds.

A good candidate could be an analytically perturbed (smoothened) rectangular $\mathcal{R}_\varepsilon$, $\varepsilon > 0$. The controllability on the rectangular $\mathcal{R}$ was established in Section 8. We are confident that controllability also holds for $\mathcal{R}_\varepsilon$ with small $\varepsilon > 0$, but there are still some technical problems to be settled in the proof.

10. Navier–Stokes / Euler Equations on Sphere $\mathbb{S}^2$

The controlled vector fields we employ in the case of $\mathbb{S}^2$ correspond to eigenfunctions of the corresponding *spherical Laplacian* or the so-called *spherical harmonics*. We start with a brief description of them.

10.1. Spherical harmonics.

In this subsection, we introduce some notions and results regarding spherical harmonics; our source was mainly the book [**9**, Chs.10, 11] by Arnold.

Consider the sphere $\mathbb{S}^2$ equipped with the Riemannian metric inherited from $\mathbb{R}^3$ and area 2-form σ. The latter defines the symplectic structure on $\mathbb{S}^2$.

The eigenfunctions of the spherical Laplacian are described by the following classical result. Recall that a function g is *homogeneous* of degree s on $\mathbb{R}^n \setminus 0$ if $g(\varkappa x) = \varkappa^s g(x)$ for each $\varkappa > 0$. A function g is *harmonic* in $\mathbb{R}^n \setminus 0$ if $\Delta g = 0$, where Δ is the Euclidean Laplacian. As is known, a harmonic homogeneous function of degree $s > 0$ is extendable by continuity

($g(0) = 0$) to a harmonic function on $\mathbb{R}^n$. This harmonic function is smooth and therefore must be a homogeneous polynomial of integer degree $s > 0$.

Theorem 10.1 ([**9**]). *Constants are eigenfunctions of the spherical Laplacian (of degree 0). If a (smooth) harmonic function defined on $\mathbb{R}^n \setminus 0$ is homogeneous of degree $s > 0$, then its restriction onto the sphere is the eigenfunction of the spherical Laplacian $\tilde{\Delta}$ corresponding to the eigenvalue $-s(s+n-2)$. Vice versa, every eigenfunction of $\tilde{\Delta}$ is a restriction onto $\mathbb{S}^n$ of a homogeneous harmonic polynomial.*

Another famous result is the Maxwell theorem [**9**] which holds in $\mathbb{R}^3$. It states that if $\rho(x) = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, then any spherical harmonic a on $\mathbb{S}^2$ can be represented as the iterated directional derivative of ρ: $a = l_1 \circ \cdots \circ l_n \rho$, where $l_1, \ldots, l_n \in \mathbb{R}^3$ and $\{l_1, \ldots, l_n\}$ is uniquely determined by a.

Our controlled directions will correspond to spherical harmonics on S^2, which are the restrictions to $\mathbb{S}^2$ of homogeneous functions on $\mathbb{R}^3$. In particular, we invoke the so-called zonal spherical harmonics which are the iterated directional derivatives of ρ with respect to a fixed direction l.

Let a, b be smooth (not necessarily homogeneous) functions on $\mathbb{R}^3$. The Poisson bracket of their restrictions to $\mathbb{S}^2$ can be computed as follows:

$$\{a|_{S^2}, b|_{S^2}\}(x) = \langle x, \nabla_x a, \nabla_x b \rangle, \tag{10.1}$$

where $\langle x, \eta, \zeta \rangle$ stands for the "mixed product" in $\mathbb{R}^3$, calculated as the determinant of the 3×3-matrix with columns x, η, and ζ. From now on, we omit the symbol of restriction $|_{\mathbb{S}^2}$ in the notation of the Poisson bracket.

The linear functions (l, x) are, of course, spherical harmonics. We denote by $\vec{l}$ the Hamiltonian field on $\mathbb{S}^2$ associated with Hamiltonian $\langle l, x \rangle$, $x \in \mathbb{S}^2$. Obviously, $\vec{l}$ generates a rotation of the sphere around the l-axis. According to the aforesaid, $\vec{l}a = \langle x, l, \nabla a \rangle$ is the Poisson bracket of the functions $\langle l, x \rangle$ and a restricted to $\mathbb{S}^2$.

The group of rotations acts (by the change of variables) on the space of homogeneous harmonic polynomials of fixed degree n. As is known, this action is irreducible for any n (see [**9**] for a sketch of the proof). In other words, the following result holds.

Proposition 10.2. *For a given homogeneous harmonic polynomial b of a nonzero degree n the space span$\{\vec{l}_1 \circ \cdots \circ \vec{l}_k b : k \geqslant 0\}$ coincides with the space of all homogeneous harmonic polynomials of degree n.*

10.2. Poisson bracket of spherical harmonics and controllability.

Calculating the Lie extensions according to formula (4.1), we obtain the iterated Poisson bracket of spherical harmonic polynomials, which, in general, need not to be harmonic. The following lemma shows that there is a way of finding some harmonic polynomials among the Poisson brackets.

Lemma 10.3. *For each $n > 2$ there exists a harmonic homogeneous polynomial q of degree 2 and a harmonic homogeneous polynomial p of degree $n > 2$ such that their Poisson bracket is again harmonic (and homogeneous of degree $n+1$) polynomial.*

Proof. Consider the so-called quadratic zonal harmonic function $q = \dfrac{\partial^2 \rho}{\partial x_3^2}$. Being restricted to the sphere $\mathbb{S}^2$, this function coincides with the Legendre polynomial $q(x_3) = 3x_3^2 - 1$.

We consider homogeneous harmonic polynomials in variables x_1, x_2. In the polar coordinates, they are represented as $r^m \cos m\varphi$ or, alternatively, $Re(x_1 + ix_2)^m, m = 1, 2, \ldots$. We pick the nth degree polynomial $p(x_1, x_2) = Re(x_1 + ix_2)^n$.

According to (10.1), the Poisson bracket of q, p is equal to

$$\{q, p\} = \langle x, \nabla q, \nabla p\rangle = \begin{vmatrix} x_1 & 0 & p'_{x_1} \\ x_2 & 0 & p'_{x_2} \\ x_3 & 6x_3 & 0 \end{vmatrix} = 6x_3 \begin{vmatrix} x_1 & 0 & p'_{x_1} \\ x_2 & 0 & p'_{x_2} \\ x_3 & 1 & 0 \end{vmatrix}.$$

By (10.1), the determinant, which multiplies $6x_3$ on the right-hand side of the formula, coincides with $\{x_3, p(x_1, x_2)\} = \vec{e}_3 p(x_1, x_2)$, where $e_3 = (0, 0, 1)$ is the standard basis vector of $\mathbb{R}^3$. Hence, by Proposition 10.2, the value of this determinant is a harmonic polynomial of degree n. It is equal to $\tilde{p}(x_1, x_2) = -x_1 p'_{x_2} + x_2 p'_{x_1}$ and therefore does not depend on x_3. Then $\{q, p\} = -6x_3 \tilde{p}(x_1, x_2)$. Since both $-6x_3$ and $\tilde{p}$ are harmonic, we get $\Delta\{q, p\} = 2\nabla(-6x_3) \cdot \nabla\tilde{p} = -12\partial\tilde{p}/\partial x_3 = 0$. □

Theorem 10.4. *Consider the Navier–Stokes / Euler equations on the sphere $\mathbb{S}^2$. Let (constant) controlled vector fields correspond to independent linear spherical harmonics l^1, l^2, l^3, one quadratic harmonic q, and one cubic harmonic c. Then this set of controlled vector fields is saturating and the Navier–Stokes / Euler equations are controllable in finite-dimensional projections.*

Proof. It suffices to verify the assumption of Corollary 6.2. Without lack of generality, we may think that $q = \tilde{q}$ is the second degree zonal harmonic from Lemma 10.3. Indeed, otherwise we may transform q into $\tilde{q}$ by taking the iterated Poisson bracket with the linear harmonics l^1, l^2, l^3.

In fact, taking the iterated Poisson bracket of q and c respectively with l^1, l^2, l^3, we obtain all quadratic and cubic harmonics. Thus, we manage to obtain all the harmonics of degrees $\leqslant 3$.

We proceed by induction on the degree of harmonics. Assume that all harmonics of degree $\leqslant n$ are already obtained by taking the iterated Poisson brackets of $\{l^1, l^2, l^3, q, s\}$. Pick the harmonic polynomial p constructed in Lemma 10.3. The Poisson bracket of p with q is a homogeneous harmonic polynomial $\bar{p}$ of degree $n+1$. Taking the iterated Poisson brackets of $\bar{p}$ with l^1, l^2, l^3, we obtain all polynomials of degree $n+1$. $\square$

Remark 10.1. Following the lines of the previous proof, Rodrigues [**24**] established the controllability of the Navier–Stokes / Euler equations on the half-sphere $\mathbb{S}^2_+$. One can force three modes, spherical harmonics on $\mathbb{S}^2_+$ in order to guarantee the controllability in any finite-dimensional projection. The details will appear elsewhere.

Remark 10.2. Arguing in a similar way as in the previous section, one can conclude that there exists a residual set of Riemannian metrics on $\mathbb{S}^2$ such that the assumptions of Corollary 6.2 are verified and, consequently, the Navier–Stokes equations are controllable in finite-dimensional projections by forcing five modes on $\mathbb{S}^2$ endowed with any of these metrics.

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Analyticity of Periodic Solutions of the $2D$ Boussinesq System

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The Cauchy problem for the $2D$ Boussinesq system with periodic boundary conditions is studied. The global existence and uniqueness of a solution with initial data $(u_{(0)}, \theta_{(0)}) \in \Phi(\alpha)$ is established, where $\Phi(\alpha)$ is the space of functions the kth Fourier coefficients of which decay at infinity as $\dfrac{1}{|k|^\alpha}$, $\alpha > 2$. It is proved that the solution becomes analytic at any positive time. Bibliography: 10 titles.

1. Introduction. The viscous $2D$ Boussinesq system describing dynamics of a homogeneous fluid with temperature transfer has the form

$$\frac{\mathcal{D}}{\partial t} u(x,t) = \nu \Delta u(x,t) - \nabla p(x,t) + \vec{e}_2 \theta(x,t), \tag{BS1}$$

$$\frac{\mathcal{D}}{\partial t} \theta(x,t) = \mu \Delta \theta(x,t), \tag{BS2}$$

$$\operatorname{div} u(x,t) = 0, \tag{BS3}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}_+$ is time, $\vec{e}_2 = (0,1)$, $u(x,t) = (u_1, u_2) : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}^2$ is a 2-dimensional velocity vector, $\theta(x,t) : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{R}$ is temperature, positive integers ν and μ are viscosity and diffusivity coefficients respectively, and the scalar function $p(x,t)$ denotes pressure. On the left-hand side of (BS1), (BS2), we use the notation

$$\frac{\mathcal{D}}{dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{d} u_j(x,t) \frac{\partial}{\partial x_j}.$$

By a *solution* of the Cauchy problem for (BS1)–(BS3) with initial conditions

$$\begin{gathered} u(x,0) = u_{(0)}(x), \quad \theta(x,0) = \theta_{(0)}(x), \\ \operatorname{div} u_{(0)}(x) = 0 \end{gathered} \tag{1}$$

we mean functions $u(x,t)$, $\theta(x,t)$, $p(x,t)$ satisfying (BS1)–(BS3) and (1).

Recent results about the existence and uniqueness of solutions of (BS1)–(BS3) were obtained in [**7, 2**] and were based on the methods developed in [**3, 4, 5**].

The system (BS1)–(BS3) is similar to the $2D$ Navier–Stokes system, and many methods developed for the Navier–Stokes equations can be applied to the Boussinesq system.

In this paper, we prove the analyticity of solutions to (BS1)–(BS3). We use the methods of [**8**], where the global existence and uniqueness of a solution were established for the $2D$ Navier–Stokes system. Note that the arguments in [**8**] are similar to those in [**6**], but are more geometrical.

In [**9, 10**], and [**1**], there was introduced the space $\Phi(\alpha)$ of functions $f(x)$ the Fourier transform of which can be written in the form

$$\mathcal{F}f(k) = \frac{c(k)}{|k|^{\alpha}}, \quad \sup_k |c(k)| = h < \infty$$

The norm in $\Phi(\alpha)$ is defined by the formula $\|f\|_{\alpha} = h = \sup_k |k|^{\alpha} |\mathcal{F}f(k)|$.

In [**8**], the global existence and uniqueness theorems were proved for the Cauchy problem for the $2D$ Navier–Stokes system with periodic boundary conditions and initial conditions in the space $\Phi(\alpha)$. The analyticity of the solution was also established. These results were extended to the continuous case in [**1**]. In [**9, 10**], similar local in time results were obtained in $3D$ statement.

2. Formulation of the results. Consider the system (BS1)–(BS3) for $\theta(x,t)$ and $w(x,t) = \operatorname{curl} u(x,t)$. Since $u(x,t) \in \mathbb{R}^2$, $w(x,t)$ has only one nonzero component $w = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$. The system (BS1)–(BS3) takes the form

$$\begin{aligned}
&\frac{\partial}{\partial t}w(x,t) + \langle u, \nabla\rangle w(x,t) = \nu\Delta w(x,t) + \frac{\partial}{\partial x_1}\theta(x,t),\\
&\frac{\partial}{\partial t}\theta(x,t) + \langle u, \nabla\rangle \theta(x,t) = \mu\Delta\theta(x,t),\\
&\text{div }\, u(x,t) = 0, \quad w(x,t) = \nabla \times u(x,t).
\end{aligned} \tag{2}$$

We expand $w(x,t)$ and $\theta(x,t)$ into the Fourier series:

$$w(x,t) = \sum_{k\in\mathbb{Z}^2} w_k(t)e^{i\langle k,x\rangle}, \quad \theta(x,t) = \sum_{k\in\mathbb{Z}^2} theta_k(t)e^{i\langle k,x\rangle}.$$

Since $w(x,t)$ and $\theta(x,t)$ are real-valued functions and $w_k = i(k_1u_k^{(2)} - k_2u_k^{(1)})$ and $k_1u_k^{(1)} + k_2u_k^{(2)} = 0$, for $w_k(t)$ and $\theta_k(t)$ we get

$$\begin{aligned}
&\frac{d}{dt}w_k(t) = -\nu|k|^2 w_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}w_l(t)w_{k-l}(t) + ik_1\theta_k(t),\\
&\frac{d}{dt}\theta_k(t) = -\mu|k|^2 \theta_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}w_l(t)\theta_{k-l}(t),\\
&w_{-k}(t) = \overline{w}_k(t), \quad \theta_{-k}(t) = \overline{\theta}_k(t),
\end{aligned} \tag{3}$$

where $l^\perp = (l_2, -l_1)$ and w_k, θ_k, $u_k = (u_k^{(1)}, u_k^{(2)})$ denote the kth Fourier coefficient of w, θ, and u respectively. Assume that $\theta_0(t) = w_0(t) = 0$. We consider the system (3) instead of (2).

It is natural to regard the 2D Boussinesq system as the 2D Navier–Stokes system with $\theta(x,t)$ as an external forcing. The only difference is the dependence of $\theta(x,t)$ on $w(x,t)$.

From the results of Mattingly and Sinai [**8**] it follows that for initial data from $\Phi(\alpha)$, $\alpha > 1$, and analytic external force the solution of 2D Navier–Stokes system becomes analytic at any positive moment of time. Thus, it suffices to show that $\theta(x,t)$ becomes analytic, i.e., its Fourier coefficients decay exponentially with $|k|$. Then the results of Mattingly and Sinai can be used.

Introduce the notation

$$\Theta(t) = \|\theta(\cdot,t)\|^2_{L^2} = \sum_{k\in\mathbb{Z}^2}|\theta_k(t)|^2, \quad \mathcal{W}(t) = \|w(\cdot,t)\|^2_{L^2} = \sum_{k\in\mathbb{Z}^2}|w_k(t)|^2.$$

Lemma 1 (termo–convection bound)**.** $\Theta(t)$ is nonincreasing.

Lemma 1 implies the following assertion.

Lemma 2 (enstrophy estimate)**.** For any $\Theta_0 = \|\theta_{(0)}\|^2_{L^2}$ and $\mathcal{W}_0 = \|w_{(0)}\|^2_{L^2}$

$$\mathcal{W}(t) \leqslant \max\Big(\mathcal{W}_0, \frac{\Theta_0}{\nu^2}\Big)$$

Hereinafter, we assume that $\alpha > 2$. Consider the initial conditions

$$w_{(0)}, \theta_{(0)} \in \Phi(\alpha), \quad \|w_{(0)}\|_\alpha = \mathcal{C}, \quad \|\theta_{(0)}\|_\alpha = \mathcal{B}. \tag{4}$$

Since $\alpha > 2$, we have $\|w_{(0)}\|_{L^2} < \infty$ and $\|\theta_{(0)}\|_{L^2} < \infty$. Using Lemmas 1 and 2, we derive a priori estimates for the solution of (3) with initial conditions (4).

Theorem 1. *For any initial data* (4) *there exists a constant* $\mathcal{B}'$ *that depends only on* $\mathcal{B}$, α, Θ_0 *and* $\mathcal{W}_0$ *and is independent of* t *such that for all* $t > 0$ *the solution* $\theta(t)$ *of* (3) *satisfies the estimate* $\|\theta(t)\|_\alpha \leqslant \mathcal{B}'$.

Note that $\mathcal{B}'$ is independent of $\mathcal{C}$. Theorem 1 and [**8**, Theorem 1] imply the following assertion.

Theorem 2. *Under the assumptions of Theorem* 1, *there exists a constant* $\mathcal{C}'$ *depending only on the initial conditions such that* $\|w(t)\|_\alpha \leqslant \mathcal{C}'$ *for all* $t > 0$.

Two theorems below provide the analyticity of the solution.

Theorem 3. *If the initial conditions* (4) *satisfy the estimates*

$$|\theta_k(0)| \leqslant \frac{\mathcal{B}_1}{|k|^\alpha} e^{-\beta|k|}, \quad |w_k(0)| \leqslant \frac{\mathcal{C}_1}{|k|^\alpha} e^{-\beta|k|}$$

for all $k \in \mathbb{Z}^2$, *where* $\mathcal{B}_1$, $\mathcal{C}_1$, *and* β *are constants, then there exist constants* $\mathcal{B}_1'$ *and* $\mathcal{C}_1'$ *such that for all* $t > 0$

$$|\theta_k(t)| \leqslant \frac{\mathcal{B}_1'}{|k|^\alpha} e^{-\beta|k|}, \quad |w_k(t)| \leqslant \frac{\mathcal{C}_1'}{|k|^\alpha} e^{-\beta|k|} \quad \forall\ k \in \mathbb{Z}^2.$$

Theorem 4. *If the initial data satisfy the assumptions of Theorem* 1, *then for any* $t_0 > 0$ *there exist constants* $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{C}}$, $\delta_1 > 0$, *and* $\delta_2 > 0$ *independent of* k *and such that for all* $t > 0$ *and* $k \in \mathbb{Z}^2$

$$|\theta_k(t)| \leqslant \frac{\widetilde{\mathcal{B}}}{|k|^\alpha} e^{-\delta(t)|k|}, \quad |w_k(t)| \leqslant \frac{\widetilde{\mathcal{C}}}{|k|^\alpha} e^{-\delta(t)|k|},$$

where $\delta(t) = \delta_1$ *for* $t < t_0$ *and* $\delta(t) = \delta_2$ *for* $t \geqslant t_0$.

Theorems 3 and 4 assert that if there exists a solution of (3) with initial conditions (4), then it becomes analytic at any positive moment of time. Namely, Theorem 4 implies that the solution becomes analytic and then it remains analytic since Theorem 3 holds. Below we give an independent proof of the local in time existence of a solution to the system (3).

Theorem 5 (local existence)**.** *Suppose that* $w_{(0)}$ *and* $\theta_{(0)}$ *belong to* $\Phi(\alpha)$, $\alpha > 2$, $\|w_{(0)}\|_\alpha$, *and* $\|\theta_{(0)}\|_\alpha \leqslant \mathcal{N}$. *Then there exist constants* $\widetilde{\mathcal{N}}$ *and* $T > 0$ *such that, on* $[0,T]$, *there exists a solution* $(w_k(t), \theta_k(t))$ *of* (3) *such that* $\|w(t)\|_\alpha \leqslant \widetilde{\mathcal{N}}$ *and* $\|\theta(t)\|_\alpha \leqslant \widetilde{\mathcal{N}}$.

3. Proof of the main results. Our arguments are based on the following technical estimate. Let h and g be scalar functions of 2-dimensional variable.

Proposition 1. *Suppose that* $h \in \Phi(\alpha)$, $\|h\|_\alpha = \mathcal{H}$, *and* $g \in L^2$, $\|g\|_{L^2} = \mathcal{G}$. *Then there exists a constant* $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\alpha)$ *such that for any* $k \in \mathbb{Z}^2$

$$\sum_{l\in\mathbb{Z}^2} \left|\frac{\langle k, l^\perp\rangle}{|l|^2}\right| |g_l||h_{k-l}| \leqslant \mathcal{H}\widetilde{\mathcal{H}}\mathcal{G}|k|^{\frac{3}{2}-\alpha}.$$

Note that $\|f\|_{L^2} < \infty$ for any function $f \in \Phi(\alpha)$.

Proof of Lemma 1. Note that $\Theta(t)$ is a positive real-valued function. Thus, it suffices to show that $\dfrac{d}{dt}\Theta(t)$ is nonpositive. By the Plancherel theorem and (3), we can write for $\dfrac{d}{dt}\Theta(t)$:

$$\begin{aligned}\frac{d}{dt}\Theta(t) &= \sum_{k\in\mathbb{Z}^2}\Big(\frac{d}{dt}\theta_k(t)\overline{\theta}_k(t) + \frac{d}{dt}\overline{\theta}_k(t)\theta_k(t)\Big)\\ &= -2\mu\sum_{k\in\mathbb{Z}^2}|k|^2|\theta_k(t)|^2 + 2\sum_{k\in\mathbb{Z}^2}\sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}w_l(t)\theta_{k-l}(t)\overline{\theta}_k(t).\end{aligned}$$

Note that $\theta_{-k}(t) = \overline{\theta}_k(t)$ and $w_{-k}(t) = \overline{w}_k(t)$. Then for every term of the last sum there exists a term

$$\frac{\langle k', l^\perp\rangle}{|l|^2}w_l(t)\theta_{k'-l}(t)\overline{\theta}_{k'}(t), \quad k' = l - k,$$

for which the kernel gives

$$\frac{\langle k', l^\perp\rangle}{|l|^2} = -\frac{\langle k, l^\perp\rangle}{|l|^2}.$$

The factor $w_l(t)\theta_{k'-l}(t)\overline{\theta}_{k'}(t)$ takes the form $w_l(t)\theta_{-k}(t)\theta_{k-l}(t)$. Hence

$$\frac{\langle k', l^{\perp}\rangle}{|l|^2} w_l(t)\theta_{k'-l}(t)\overline{\theta}_{k'}(t) + \frac{\langle k, l^{\perp}\rangle}{|l|^2} w_l(t)\theta_{k-l}(t)\overline{\theta}_{k}(t) = 0.$$

Thus, the last sum vanishes and we have

$$\frac{d}{dt}\Theta(t) = -2\mu \sum_{k\in\mathbb{Z}^2} |k|^2|\theta_k(t)|^2 \leqslant -2\mu\Theta(t).$$

The proof is complete. $\square$

Proof of Lemma 2. By the Plancherel theorem, for $\dfrac{d}{dt}\mathcal{W}(t)$ we have

$$\frac{d}{dt}\mathcal{W}(t) = \sum_{k\in\mathbb{Z}^2} \frac{d}{dt}|w_k(t)|^2 = \sum_{k\in\mathbb{Z}^2} \Big(\frac{dw_k(t)}{dt}\overline{w}_k(t) + \frac{d\overline{w}_k(t)}{dt} w_k(t)\Big).$$

Using (3), we find

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(t) = -2\nu \sum_{k\in\mathbb{Z}^2} |k|^2|w_k(t)|^2 + 2\sum_{k\in\mathbb{Z}^2}\sum_{l\in\mathbb{Z}^2} \frac{\langle k, l^{\perp}\rangle}{|l|^2} w_l(t)\overline{w}_k(t)w_{k-l}(t)\\ + 2i\sum_{k\in\mathbb{Z}^2} k_1\theta_k\overline{w}_k(t).\end{aligned}$$

The second term on the right-hand side vanishes by the same arguments as in the previous proof. To estimate the third term, we use the Cauchy–Schwarz inequality

$$\sum_{k\in\mathbb{Z}^2} k_1\theta_k(t)\overline{w}_k(t) \leqslant \sqrt{\sum_{k\in\mathbb{Z}^2}|\theta_k(t)|^2}\sqrt{\sum_{k\in\mathbb{Z}^2}|k|^2|w_k(t)|^2}.$$

By Lemma 1, the first term on the right-hand side is not greater than $\sqrt{\Theta_0}$. Hence

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(t) &\leqslant 2\sqrt{\sum_{k\in\mathbb{Z}^2}|k|^2|w_k(t)|^2}\Big(\sqrt{\Theta_0} - \nu\sqrt{\sum_{k\in\mathbb{Z}^2}|k|^2|w_k(t)|^2}\Big)\\ &\leqslant 2\sqrt{\sum_{k\in\mathbb{Z}^2}|k|^2|w_k(t)|^2}\Big(\sqrt{\Theta_0} - \nu\sqrt{\mathcal{W}(t)}\Big). \qquad (5)\end{aligned}$$

By (5), $\mathcal{W}(t) > \dfrac{\Theta_0}{\nu^2}$ implies $\dfrac{d}{dt}\mathcal{W}(t) < 0$, which completes the proof. $\square$

Proof of Proposition 1. We divide the summation domain into three parts and derive estimates for each part separately.

1. $|l| \leqslant \dfrac{1}{2}|k|$. In this case, $|k-l| \geqslant \dfrac{1}{2}|k|$. Hence $|f_{k-l}| \leqslant \dfrac{2^\alpha \mathcal{H}}{|k|^\alpha}$. Then

$$\Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| \leqslant \frac{|k|}{|l|}. \qquad \sum_{|l|\leqslant \frac{|k|}{2}} \Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| |g_l||f_{k-l}| \leqslant \mathcal{H}|k|^{1-\alpha} \sum_{|l|\leqslant \frac{|k|}{2}} \frac{|g_l|}{|l|}.$$

By the Cauchy–Schwarz inequality,

$$\sum_{|l|\leqslant \frac{|k|}{2}} \frac{|g_l|}{|l|} \leqslant \sqrt{\sum_{|l|\leqslant \frac{|k|}{2}} |g_l|^2} \sqrt{\sum_{|l|\leqslant \frac{|k|}{2}} \frac{1}{|l|^2}} \leqslant \mathcal{G} B_1 \sqrt{\ln|k|},$$

where B_1 can be found from the inequality

$$\sum_{|l|\leqslant \frac{|k|}{2}} \frac{1}{|l|^2} \leqslant B_1^2 \ln|k|.$$

Collecting the above estimates, we conclude that for $|l| \leqslant \dfrac{1}{2}|k|$

$$\sum_{|l|\leqslant \frac{|k|}{2}} \Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| |g_l||f_{k-l}| \leqslant \mathcal{H}|k|^{1-\alpha} \mathcal{G} B_1 \sqrt{\ln|k|}.$$

2. $\dfrac{1}{2}|k| \leqslant |l| \leqslant 2|k|$. We can estimate $\Big|\dfrac{\langle k, l^\perp\rangle}{|l|^2}\Big|$ by 2 and use again the Cauchy–Schwarz inequality:

$$\begin{aligned}\sum_{\frac{|k|}{2}\leqslant|l|\leqslant 2|k|} \Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| |g_l||f_{k-l}| &\leqslant 2\sqrt{\sum_{\frac{|k|}{2}\leqslant|l|\leqslant 2|k|} |g_l|^2} \sqrt{\sum_{\frac{|k|}{2}\leqslant|l|\leqslant 2|k|} |f_{k-l}|^2} \\ &\leqslant 2\mathcal{G}\mathcal{H} \sqrt{\sum_{|k-l|\leqslant 3|k|} \frac{1}{|k-l|^{2\alpha}}} \leqslant 2\mathcal{G}\mathcal{H} B_2 |k|^{1-\alpha},\end{aligned}$$

where B_2 is the constant in the estimate

$$\sum_{|l|\leqslant 3|k|} \frac{1}{|l|^{2\alpha}} \leqslant B_2^2 |k|^{2-2\alpha}.$$

3. $|l| \geqslant 2|k|$. In this region, $|k-l| \geqslant |l| - |k| \geqslant |l| - \frac{1}{2}|l| = \frac{1}{2}|l|$. Hence

$$\sum_{|l|\geqslant 2|k|} \Big|\frac{\langle k, l^{\perp}\rangle}{|l|^2}\Big| |g_l||f_{k-l}| \leqslant |k| \sqrt{\sum_{|l|\geqslant 2|k|} |g_l|^2} \sqrt{\sum_{|l|\geqslant 2|k|} \frac{2^{2\alpha}\mathcal{H}^2}{|l|^{2+2\alpha}}}$$

$$\leqslant \mathcal{H}\mathcal{G}|k| 2^{\alpha} \sqrt{\sum_{|l|\geqslant 2|k|} \frac{1}{|l|^{2+2\alpha}}} \leqslant 2^{\alpha}\mathcal{H}\mathcal{G}B_3|k|^{1-\alpha},$$

where B_3 is the constant in the estimate

$$\sum_{|l|\geqslant 2|k|} \frac{1}{|l|^{2+2\alpha}} \leqslant B_3^2|k|^{-\alpha}.$$

Collecting the estimates, we find

$$\sum_{l\in\mathbb{Z}^2} \Big|\frac{\langle k, l^{\perp}\rangle}{|l|^2}\Big| |g_l||f_{k-l}| \leqslant \mathcal{G}|k|^{1-\alpha}\Big(B_1\sqrt{\ln|k|} + 2\mathcal{H}B_2 + 2^{\alpha}\mathcal{H}B_3\Big)$$

$$\leqslant \mathcal{H}\mathcal{G}|k|^{\frac{3}{2}-\alpha}\Big(\frac{B_1}{\mathcal{H}}\sqrt{\frac{\ln|k|}{|k|}} + 2B_2 + 2^{\alpha}B_3\Big)$$

Thus, for $\widetilde{\mathcal{H}} = \dfrac{|B_1|}{\mathcal{H}} + 2|B_2| + 2^{\alpha}|B_3|$ the last inequality proves the required assertion. $\square$

Proof of Theorem 1. Denote by $\Omega_{\alpha}^{(A)}$ the set of functions whose $\Phi(\alpha)$-norms are bounded by a constant A: $\Omega_{\alpha}^{(A)} = \{f : \ |f_k| < A/|k|^{\alpha}\}$. By assumption, $\theta(0) \in \Omega_{\alpha}^{(\mathcal{B})}$. We fix a constant K_{crit} depending only on the initial conditions which will be defined later.

By Lemma 1, for $|k| < K_{\mathrm{crit}}$ we have

$$\sup_{|k|\leqslant K_{\mathrm{crit}}} |\theta_k(t)| \leqslant \sqrt{\Theta_0},$$

which implies

$$\sup_{|k|\leqslant K_{\mathrm{crit}}} |\theta_k(t)||k|^{\alpha} \leqslant \sqrt{\Theta_0}K_{\mathrm{crit}}^{\alpha}.$$

We show that for any $k \in \mathbb{Z}^2$, $|k| > K_{\mathrm{crit}}$, the vector field on the boundary $\partial\Omega_{\alpha}^{(\mathcal{B})}$ is directed inward. Then for $\mathcal{B}' = \max(\mathcal{B}, \sqrt{\Theta_0}K_{\mathrm{crit}}^{\alpha})$ the required assertion holds for any $t > 0$.

Suppose that for some $t > 0$ $\theta(t)$ leaves $\Omega_{\alpha}^{(\mathcal{B})}$. Let $t_0 = \inf\{t : \ \theta(t) \notin \Omega_{\alpha}^{(\mathcal{B})}\}$. Then for $t = t_0$ and all $k \in \mathbb{Z}^2$ we have the inequality $|\theta_k(t_0)| \leqslant$

$\mathcal{B}/|k|^\alpha$ which becomes equality for some k. Denote by k_* a point in $\mathbb{Z}^2$ such that the equality takes place at this point and $|k_*| \leqslant |k|$ at any other point $k \in \mathbb{Z}^2$, where equality holds. Then for $\theta_{k_*}(t_0)$ from (3) it follows that

$$\left.\frac{d\theta_{k_*}(t)}{dt}\right|_{t=t_0} = -\mu|k_*|^2\theta_{k_*}(t_0) + \sum_{l\in\mathbb{Z}^2} \frac{\langle k_*, l^\perp\rangle}{|l|^2}\theta_{k_*-l}(t_0)w_l(t_0). \tag{6}$$

An obvious calculations shows that

$$\begin{aligned}
2|\theta_k(t)|\frac{d|\theta_k(t)|}{dt} &= \frac{d|\theta_k(t)|^2}{dt} = \theta_k(t)\frac{d\overline{\theta}_k(t)}{dt} + \overline{\theta}_k(t)\frac{d\theta_k(t)}{dt} \\
&= -2\mu|k|^2\theta_k(t)\overline{\theta}_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}(w_l(t)\theta_{k-l}(t)\overline{\theta}_k(t) + \overline{w}_l(t)\overline{\theta}_{k-l}(t)\theta_k(t)) \\
&\leqslant -2\mu|k|^2|\theta_k(t)|^2 + 2|\theta_k(t)|\sum_{l\in\mathbb{Z}^2}\left|\frac{\langle k, l^\perp\rangle}{|l|^2}\right||w_l(t)||\theta_{k-l}(t)|.
\end{aligned}$$

Thus, for $|\theta_{k_*}(t_0)|$ we can write

$$\frac{d}{dt}|\theta_{k_*}(t_0)| \leqslant -\mu|k_*|^{2-\alpha}\mathcal{B} + \sum_{l\in\mathbb{Z}^2}\left|\frac{\langle k_*, l^\perp\rangle}{|l|^2}\right||\theta_{k_*-l}(t_0)||w_l(t_0)| \tag{7}$$

By Proposition 1, the last term on the right-hand side of (7) is less than or equal to $|k_*|^{\frac{3}{2}-\alpha}\sqrt{\mathcal{W}_0}\mathcal{B}A_1$, where A_1 is a constant that is independent of k_* and t_0 and can depend only on α. Thus,

$$\frac{d}{dt}|\theta_{k_*}(t_0)| \leqslant |k_*|^{\frac{3}{2}-\alpha}\sqrt{\mathcal{W}_0}\mathcal{B}A_1 - \mu|k_*|^{2-\alpha}\mathcal{B} = \mathcal{B}|k_*|^{\frac{3}{2}-\alpha}(\sqrt{\mathcal{W}_0}A_1 - \mu|k_*|^{\frac{1}{2}}).$$

Setting $K_{\mathrm{crit}} = \dfrac{\mathcal{W}_0 A_1^2}{\mu^2}$, from the inequality $|k_*| > K_{\mathrm{crit}}$ we immediately conclude that $\frac{d}{dt}|\theta_{k_*}(t_0)|$ is negative and, consequently, $|\theta_{k_*}(t_0)|$ decreases. Recall the estimate $|\theta_k(t)||k|^\alpha \leqslant \sqrt{\Theta_0}(K_{\mathrm{crit}})^\alpha$ for $|k| \leqslant K_{\mathrm{crit}}$. Setting

$$\mathcal{B}' = \max(\mathcal{B}, \frac{\sqrt{\Theta_0}\mathcal{W}_0^\alpha A_1^{2\alpha}}{\mu^{2\alpha}}),$$

we complete the proof. □

Proof of Theorem 2. We argue in a similar way as in the previous proof. We fix a constant $\widetilde{K}_{\mathrm{crit}}$ which depends only on α and will be defined later. By Lemma 2, for any $|k| \leqslant \widetilde{K}_{\mathrm{crit}}$ we can write

$$|w_k(t)||k|^\alpha \leqslant \widetilde{K}_{\mathrm{crit}}^\alpha|w_k(t)| \leqslant \widetilde{K}_{\mathrm{crit}}^\alpha\sqrt{\mathcal{W}(t)} \leqslant \widetilde{K}_{\mathrm{crit}}^\alpha A_2, \tag{8}$$

where $A_2 = \max\Big(\mathcal{W}_0, \dfrac{\Theta_0}{\nu^2}\Big)$ and thus is a constant which depends only on the initial data.

Now, we show that for some constant $\widetilde{K}_{\text{crit}}$ the inequality $|k| > \widetilde{K}_{\text{crit}}$ implies $|w_k(t)||k|^\alpha \leqslant \mathcal{C}$. By the assumption of the theorem, $w_{(0)} \in \Omega_\alpha^{(\mathcal{C})}$.

Suppose that for some $t > 0$ $w(t)$ leaves the $\Omega_\alpha^{(\mathcal{C})}$. Denote by t_0 the infimum of such t. Then for $t = t_0$ we have the inequality

$$|w_k(t_0)| \leqslant \frac{\mathcal{C}}{|k|^\alpha}$$

which becomes equality for some k. Let k_* be a point where equality takes place. Assume that k_* has the minimal norm among points of such a type.

By (3), for $|w_k(t)|$ we have

$$\begin{aligned}
2|w_k(t)|\frac{d|w_k(t)|}{dt} &= \frac{d|w_k(t)|^2}{dt} = \overline{w}_k(t)\frac{dw_k(t)}{dt} + w_k(t)\frac{d\overline{w}_k(t)}{dt} \\
&= -2\nu|k|^2|w_k(t)|^2 + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}(w_{k-l}(t)w_l(t)\overline{w}_k(t) \\
&\quad + \overline{w}_{k-l}(t)\overline{w}_l(t)w_k(t)) + ik_1(\theta_k(t)\overline{w}_k(t) + \overline{\theta}_k(t)w_k(t)).
\end{aligned}$$

Thus,

$$\frac{d|w_k(t)|}{dt} \leqslant -\nu|k|^2|w_k(t)| + \sum_{l\in\mathbb{Z}^2}\Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big||w_{k-l}(t)||w_l(t)| + |k||\theta_k(t)|. \tag{9}$$

By Proposition 1, the second term on the right-hand side of (9) is not greater than $|k|^{\frac{3}{2}-\alpha}\mathcal{C}\sqrt{\mathcal{W}_0}A_3$, where A_3 is a constant depending only on α. The third term is bounded by $|k|^{1-\alpha}\mathcal{B}'$ because of Theorem 1. Thus, for $|w_{k_*}(t_0)|$ we have

$$\begin{aligned}
\frac{d|w_{k_*}(t_0)|}{dt} &\leqslant -\nu|k_*|^{2-\alpha}\mathcal{C} + |k_*|^{\frac{3}{2}-\alpha}\mathcal{C}\sqrt{\mathcal{W}_0}A_3 + |k_*|^{1-\alpha}\mathcal{B}' \\
&= |k_*|^{2-\alpha}\mathcal{C}\Big(\frac{\sqrt{\mathcal{W}_0}A_3}{\sqrt{|k_*|}} + \frac{\mathcal{B}'}{|k_*|\mathcal{C}} - \nu\Big)
\end{aligned} \tag{10}$$

If we define $\widetilde{K}_{\text{crit}} = \dfrac{\mathcal{W}_0A_3^2}{4\nu^2} - \dfrac{\mathcal{B}'}{\mathcal{C}\nu}$, then for $|k_*| > \widetilde{K}_{\text{crit}}$ the right-hand side of the inequality (10) is negative and, consequently, $|w_{k_*}(t_0)|$ decreases. The theorem is proved for $\mathcal{C}' = \max(\mathcal{C}, A_2(\widetilde{K}_{crit})^\alpha)$. □

Proof of Theorem 3. The required assertion is obtained by applying Theorems 1 and 2 to the functions $\tilde{\theta}_k(t) = e^{\beta|k|}\theta_k(t)$ and $\tilde{w}_k(t) =$

$e^{\beta|k|}w_k(t)$. From (3) we derive for $\tilde{\theta}_k(t)$, $\tilde{w}_k(t)$

$$\frac{d}{dt}\tilde{\theta}_k(t) = -\mu|k|^2\tilde{\theta}_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}\tilde{w}_l(t)\tilde{\theta}_{k-l}(t)e^{\beta(|k|-|l|-|k-l|)},$$

$$\frac{d}{dt}\tilde{w}_k(t) = -\nu|k|^2\tilde{w}_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}\tilde{w}_l(t)\tilde{w}_{k-l}(t)e^{\beta(|k|-|l|-|k-l|)} + ik_1\tilde{\theta}_k(t).$$

Since $|k-l|+|l| \geqslant |k|$ and, consequently, $e^{-\beta(|l|+|k-l|-|k|)} \leqslant 1$, we can replace the corresponding terms with 1 and obtain the estimates

$$\frac{d}{dt}|\tilde{\theta}_k(t)| \leqslant -\mu|k|^2|\tilde{\theta}_k(t)| + \sum_{l\in\mathbb{Z}^2}\left|\frac{\langle k, l^\perp\rangle}{|l|^2}\right||\tilde{w}_l(t)||\tilde{\theta}_{k-l}(t)|,$$

$$\frac{d}{dt}|\tilde{w}_k(t)| \leqslant -\nu|k|^2|\tilde{w}_k(t)| + \sum_{l\in\mathbb{Z}^2}\left|\frac{\langle k, l^\perp\rangle}{|l|^2}\right||\tilde{w}_l(t)||\tilde{w}_{k-l}(t)| + |k||\tilde{\theta}_k(t)|.$$

Then we repeat the proof of Theorems 1 and 2 without any change. □

PROOF OF THEOREM 4. Let $t_0 > 0$. Consider $\hat{\theta}(x,t)$ and $\hat{w}(x,t)$ with the Fourier coefficients $\hat{\theta}_k(t) = e^{\delta_1 t|k|}\theta_k(t)$ and $\hat{w}_k(t) = e^{\delta_1 t|k|}w_k(t)$ respectively. The constant $\delta_1 > 0$ will be defined later.

We have $\|\hat{w}(0)\|_\alpha = \mathcal{C}$ and $\|\hat{\theta}(0)\|_\alpha = \mathcal{B}$. It suffices to show that $\hat{\theta}(t)$ and $\hat{w}(t)$ remain bounded in $\Phi(\alpha)$ with some constants $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$ for all $t \in [0, t_0]$.

Since the inequalities

$$|\hat{w}_k(t_0)| \leqslant \frac{\widetilde{\mathcal{C}}}{|k|^\alpha}, \quad |\hat{\theta}_k(t_0)| \leqslant \frac{\widetilde{\mathcal{B}}}{|k|^\alpha}$$

imply

$$|w_k(t_0)| \leqslant \frac{\widetilde{\mathcal{C}}}{|k|^\alpha}e^{-|k|\delta_1 t_0}, \quad |\hat{\theta}_k(t_0)| \leqslant \frac{\widetilde{\mathcal{B}}}{|k|^\alpha}e^{-|k|\delta_1 t_0},$$

we can use Theorem 3 with the initial data $w(t_0)$ and $\theta(t_0)$. Thus, if at t_0, we have

$$\|\hat{w}(t_0)\|_\alpha \leqslant \widetilde{\mathcal{C}}, \quad \|\hat{\theta}(t_0)\|_\alpha \leqslant \widetilde{\mathcal{B}},$$

then for all $t > t_0$ the decay rate of the Fourier coefficients of the solution remains exponential with k.

By (3), we have

$$\frac{d\hat{\theta}_k(t)}{dt} = \delta_1|k|\hat{\theta}_k(t) - \mu|k|^2\hat{\theta}_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}\hat{w}_l(t)\hat{\theta}_{k-l}(t)e^{\delta_1 t(|k|-|l|-|k-l|)},$$

$$\frac{d\hat{w}_k(t)}{dt} = \delta_1|k|\hat{w}_k(t) - \nu|k|^2\hat{w}_k(t) + \sum_{l\in\mathbb{Z}^2}\frac{\langle k, l^\perp\rangle}{|l|^2}\hat{w}_l(t)\hat{w}_{k-l}(t)e^{\delta_1 t(|k|-|l|-|k-l|)} + ik_1\hat{\theta}_k(t)$$

From the inequality $|k| \leqslant |l|+|k-l|$ it follows that $e^{\delta_1 t(|k|-|l|-|k-l|)} \leqslant 1$. A calculation, similar to that in the proof of Theorems 1 and 2, shows that

$$\begin{aligned}\frac{d|\hat{\theta}_k(t)|}{dt} &\leqslant \delta_1|k||\hat{\theta}_k(t)| - \mu|k|^2|\hat{\theta}_k(t)| + \sum_{l\in\mathbb{Z}^2}\Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big||\hat{w}_l(t)||\hat{\theta}_{k-l}(t)|,\\ \frac{d|\hat{w}_k(t)|}{dt} &\leqslant |k|(|\hat{w}_k(t)|\delta_1 + |\hat{\theta}_k(t)|) - \nu|k|^2|\hat{w}_k(t)| + \sum_{l\in\mathbb{Z}^2}\Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big||\hat{w}_l(t)||\hat{w}_{k-l}(t)|\end{aligned} \tag{11}$$

We fix a constant $\mathcal{K}$ which will be defined later. By Theorems 1 and 2, for any $t>0$ and $k\in\mathbb{Z}^2$ we have $|w_k(t)| \leqslant \dfrac{\mathcal{C}'}{|k|^\alpha}$ and $|\theta_k(t)| \leqslant \dfrac{\mathcal{B}'}{|k|^\alpha}$. Thus, for any $t\in[0,t_0]$ and $|k|\leqslant\mathcal{K}$

$$|\hat{\theta}_k(t)| \leqslant \frac{\mathcal{B}_1}{|k|^\alpha}, \quad |\hat{w}_k(t)| \leqslant \frac{\mathcal{C}_1}{|k|^\alpha} \tag{12}$$

where $\mathcal{B}_1 = \mathcal{B}'e^{\mathcal{K}\delta_1 t_0}$, $\mathcal{C}_1 = \mathcal{C}'e^{\mathcal{K}\delta_1 t_0}$.

As above, $\hat{\theta}(0)\in\Omega_\alpha^{(\mathcal{B})}$ and $\hat{w}(0)\in\Omega_\alpha^{(\mathcal{C})}$. Assume that at some positive $\tau\in[0,t_0]$ the pair $(\hat{\theta}(\tau),\hat{w}(\tau))$ reaches the boundary of $\Omega_\alpha^{(\mathcal{B})}\times\Omega_\alpha^{(\mathcal{C})}$. Then we can estimate the sums on the right-hand side of (11) using Proposition 1:

$$\begin{aligned}\frac{d|\hat{\theta}_k(t)|}{dt}\Big|_{t=\tau} &\leqslant |k|^{1-\alpha}\delta_1\mathcal{B} + |k|^{\frac{3}{2}-\alpha}\sqrt{\mathcal{W}_0}\mathcal{B}A_4 - \mu|k|^2|\hat{\theta}_k(\tau)|,\\ \frac{d|\hat{w}_k(t)|}{dt}\Big|_{t=\tau} &\leqslant |k|^{\frac{3}{2}-\alpha}\sqrt{\mathcal{W}_0}\mathcal{C}A_5 + |k|^{1-\alpha}(\mathcal{B}+\delta_1\mathcal{C}) - \nu|k|^2|\hat{w}_k(\tau)|,\end{aligned} \tag{13}$$

where A_4 and A_5 are constants depending on the initial conditions.

By (13), we can force the right-hand side of (11) to be negative on the boundary of $\Omega_\alpha^{(\mathcal{B})} \times \Omega_\alpha^{(\mathcal{C})}$ for $|k| \geqslant \mathcal{K}$ by choosing $\mathcal{K}$ large enough, and thus for such $|k|$ the vector field on the boundary is directed inward.

Since for all other $|k|$ we have (12), there exist constants $\widetilde{\mathcal{B}} = \max(\mathcal{B}, \mathcal{B}_1)$ and $\widetilde{\mathcal{C}} = \max(\mathcal{C}, \mathcal{C}_1)$ such that for any $t \in [0, t_0]$ the pair $(\hat{\theta}(t), \hat{w}(t))$ belongs to the region $\Omega_\alpha^{(\widetilde{\mathcal{B}})} \times \Omega_\alpha^{(\widetilde{\mathcal{C}})}$. $\square$

Proof of Theorem 5. Consider the classical iteration scheme. We write Equations (3) in the integral form and consider the sequences

$$\theta_k^{(0)}(t) = \theta_k(0), \quad w_k^{(0)}(t) = w_k(0); \tag{14}$$

$$\theta_k^{(n+1)}(t) = e^{-|k|^2 t}\theta_k^{(0)} + \int\limits_0^t e^{-|k|^2(t-s)} \sum_{l\in\mathbb{Z}^2} \frac{\langle k, l^\perp\rangle}{|l|^2} w_l^{(n)}(s)\theta_{k-l}^{(n)}(s)ds,$$

$$\begin{aligned} w_k^{(n+1)}(t) &= e^{-|k|^2 t} w_k^{(0)} \\ &+ \int\limits_0^t e^{-|k|^2(t-s)}\Big(\sum_{l\in\mathbb{Z}^2} \frac{\langle k, l^\perp\rangle}{|l|^2} w_l^{(n)}(s) w_{k-l}^{(n)}(s) + ik_1\theta_k^{(n)}(s)\Big)ds. \end{aligned} \tag{15}$$

It is easy to check that $\theta_k^{(n)}(t) = \overline{\theta}_{-k}^{(n)}(t)$ and $w_k^{(n)}(t) = \overline{w}_{-k}^{(n)}(t)$. Hence Lemmas 1 and 2 can be applied to all the functions $\theta^{(n)}(t)$ and $w^{(n)}(t)$. We proceed by induction. We show that for initial data from $\Phi(\alpha)$ all $\{w_k^{(n)}(t)\}$ and $\{\theta_k^{(n)}(t)\}$ are bounded in $\Phi(\alpha)$ uniformly with respect to n. Then we show that $\{w_k^{(n)}(t)\}$ and $\{\theta_k^{(n)}(t)\}$ are fundamental in the norm $\|\cdot\|_\alpha$ and, consequently, converge in $\Phi(\alpha)$. By (15), the corresponding limits provide a solution of (3).

Uniform bound. Assume that for some $n \in \mathbb{N}$ we have $\|\theta^{(n)}(t)\|_\alpha \leqslant 2\mathcal{N}$ and $\|w^{(n)}(t)\|_\alpha \leqslant 4\mathcal{N}$, where $\mathcal{N}$ is taken from the formulation of the theorem. By Proposition 1,

$$\sum_{l\in\mathbb{Z}^2} \Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| |w_l^{(n)}(s)||\theta_{k-l}^{(n)}(s)| \leqslant 2\mathcal{N} A_6 \sqrt{\mathcal{W}_0} |k|^{\frac{3}{2}-\alpha},$$

$$\sum_{l\in\mathbb{Z}^2} \Big|\frac{\langle k, l^\perp\rangle}{|l|^2}\Big| |w_l^{(n)}(s)||w_{k-l}^{(n)}(s)| \leqslant 4\mathcal{N} A_7 \sqrt{\mathcal{W}_0} |k|^{\frac{3}{2}-\alpha}.$$

Thus, for $\theta_k^{(n+1)}(t)$ and $w_k^{(n+1)}(t)$ from (15), integrating with respect to s, we find

$$|\theta_k^{(n+1)}(t)| \leqslant 2\mathcal{N} A_6 \sqrt{\mathcal{W}_0}|k|^{-\alpha-\frac{1}{2}} + \mathcal{N}|k|^{-\alpha}, \tag{16}$$

$$|w_k^{(n+1)}(t)| \leqslant 4\mathcal{N} A_7 \sqrt{\mathcal{W}_0}|k|^{-\alpha-\frac{1}{2}} + |\theta_k^{(n+1)}| + \mathcal{N}|k|^{-\alpha}. \tag{17}$$

By (16), for $|k| > 4A_6^2\mathcal{W}_0$ we have

$$|\theta_k^{(n+1)}(t)| \leqslant 2\mathcal{N}|k|^{-\alpha}.$$

By (17), for $|k| > 16A_7^2\mathcal{W}_0$ we have

$$|w_k^{(n+1)}(t)| \leqslant 4\mathcal{N}|k|^{-\alpha}.$$

Denote $K = 4\mathcal{W}_0 \max(A_6^2, 4A_7^2)$. Then for $k > K$ the inductive assumption is satisfied.

For $|k| \leqslant K$ Lemmas 1 and 2 imply

$$|\theta_k^{(n+1)}(t)| \leqslant \sqrt{\Theta_0}, \quad |w_k^{(n+1)}(t)| \leqslant \sqrt{\mathcal{W}_0}.$$

Setting $\widetilde{\mathcal{N}} = \max(4\mathcal{N}, \sqrt{\Theta_0}K^\alpha, \sqrt{\mathcal{W}_0}K^\alpha)$, for all k we get

$$\|\theta_k^{(n)}(t)\|_\alpha \leqslant \widetilde{\mathcal{N}} < \quad \|w_k^{(n)}(t)\|_\alpha \leqslant \widetilde{\mathcal{N}}$$

uniformly with respect to n.

Convergence. Consider the functions $g_k^{(n+1)}(t) = \theta_k^{(n+1)}(t) - \theta_k^{(n)}(t)$ and $\tilde{g}_k^{(n+1)}(t) = w_k^{(n+1)}(t) - w_k^{(n)}(t)$. From (15) it follows that

$$g_k^{(n+1)}(t) = \int\limits_0^t e^{-|k|^2(t-s)} \sum_{l\in\mathbb{Z}^3} \frac{\langle l^\perp, k\rangle}{|l|^2} \Big(w_l^{(n)}(s) g_{k-l}^{(n)}(s) + \tilde{g}_l^{(n)}(s)\theta_{k-l}^{(n-1)}(s)\Big)\, ds,$$

$$\begin{aligned}\tilde{g}_k^{(n+1)}(t) = \int\limits_0^t e^{-|k|^2(t-s)} \Bigg[\sum_{l\in\mathbb{Z}^3} \frac{\langle l^\perp, k\rangle}{|l|^2} (w_l^{(n)}(s)\tilde{g}_{k-l}^{(n)}(s) \\ + \tilde{g}_l^{(n)}(s) w_{k-l}^{(n-1)}(s)) + ik_1 g_k^{(n)}(s)\Bigg] ds.\end{aligned} \tag{18}$$

Since, according to the first step of the proof, all $\theta^{(n)}(t)$ and $w^{(n)}(t)$ belong to $\Phi(\alpha)$, we see that $g^{(n)}(t)$ and $\tilde{g}^{(n)}(t)$ also belong to $\Phi(\alpha)$. Using Proposition 1 and integrating over s, we find

$$\|g_k^{(n+1)}(t)\|_\alpha \leqslant \left(A_8\|g^{(n)}(t)\|_\alpha\sqrt{\mathcal{W}_0} + A_9\|\tilde{g}^{(n)}(t)\|_\alpha\sqrt{\Theta_0}\right)|k|^{-\frac{1}{2}}(1-e^{-|k|^2t}), \tag{19}$$

$$\|\tilde{g}_k^{(n+1)}(t)\|_\alpha \leqslant \left(A_{10}\|\tilde{g}^{(n)}(t)\|_\alpha\widetilde{\mathcal{N}}|k|^{-\frac{1}{2}} + |k|^{-1}\|g^{(n)}(t)\|_\alpha\right)(1-e^{-|k|^2t}), \tag{20}$$

where A_8, A_9, and A_{10} are constants independent of k, t, and n. For sufficiently large $|k|t$ we can replace $(1-e^{-|k|^2t})$ with 1 in the estimates (19) and (20).

Since the power of $|k|$ on the right-hand side of these inequalities remains negative, for each given $\varepsilon \in (0,1)$ there exists K_ε such that for $|k| > K_\varepsilon$ we get

$$|g_k^{(n+1)}(t)| \leqslant |k|^{-\alpha}\varepsilon\max(\|g^{(n)}(t)\|_\alpha, \|\tilde{g}^{(n)}(t)\|_\alpha),$$
$$|\tilde{g}_k^{(n+1)}(t)| \leqslant |k|^{-\alpha}\varepsilon\max(\|g^{(n)}(t)\|_\alpha, \|\tilde{g}^{(n)}(t)\|_\alpha).$$

For $|k| \leqslant K_\varepsilon$ there exists $T_\varepsilon = \dfrac{\varepsilon}{(\text{const})K_\varepsilon^2}$ with some positive (const) such that for any $|k| \leqslant K_\varepsilon$ and $t \in [0, T_\varepsilon]$ we can estimate $(1-e^{-|k|^2t})$ by $\dfrac{\varepsilon}{(\text{const})}$. Then (19) and (20) imply that for sufficiently large (const) and $t \in [0, T_\varepsilon]$ we have

$$\begin{aligned} \|g^{(n+1)}(t)\|_\alpha &\leqslant \varepsilon|k|^{-\alpha}\max(\|g^{(n)}(t)\|_\alpha, \|\tilde{g}^{(n)}(t)\|_\alpha) \\ \|\tilde{g}^{(n+1)}(t)\|_\alpha &\leqslant \varepsilon|k|^{-\alpha}\max(\|g^{(n)}(t)\|_\alpha, \|\tilde{g}^{(n)}(t)\|_\alpha) \end{aligned} \tag{21}$$

Note that T_ε depends only on the constants in the inequalities (19), (20) and, consequently, depends only on the initial conditions. By (21),

$$\|g^{(n)}(t)\|_\alpha \leqslant \varepsilon^n\max(\|g^{(0)}(t)\|_\alpha, \|\tilde{g}^{(0)}(t)\|_\alpha),$$
$$\|\tilde{g}^{(n)}(t)\|_\alpha \leqslant \varepsilon^n\max(\|g^{(0)}(t)\|_\alpha, \|\tilde{g}^{(0)}(t)\|_\alpha).$$

Thus, $\{\theta^{(n)}(t)\}$ and $\{w^{(n)}(t)\}$ are Cauchy sequences in $\Phi(\alpha)$. $\square$

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Nonlinear Dynamics of a System of Particle-Like Wavepackets

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This work continues our studies of nonlinear evolution of a system of wavepackets. We study a wave propagation governed by a nonlinear system of hyperbolic PDE's with constant coefficients with the initial data being a multi-wavepacket. By definition, a general wavepacket has a well-defined principal wave vector, and, as we proved in previous works, the nonlinear dynamics preserves systems of wavepackets and their principal wave vectors. Here we study the nonlinear evolution of a special class of wavepackets, namely particle-like wavepackets. A particle-like wavepacket is of a dual nature: on one hand, it is a wave with a well-defined principal wave vector, on the other hand, it is a particle in the sense that it can be assigned a well-defined position in the space. We prove that under the nonlinear evolution a generic multi-particle wavepacket remains to be a multi-particle wavepacket with high accuracy, and every constituting single particle-like wavepacket not only preserves its principal wave number but also it has a well-defined space position evolving with a constant velocity which is its group

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velocity. Remarkably the described properties hold though the involved single particle-like wavepackets undergo nonlinear interactions and multiple collisions in the space. We also prove that if principal wavevectors of multi-particle wavepacket are generic, the result of nonlinear interactions between different wavepackets is small and the approximate linear superposition principle holds uniformly with respect to the initial spatial positions of wavepackets. Bibliography: 41 titles.

1. Introduction

The principal object of our studies here is a general nonlinear evolutionary system which describes wave propagation in homogeneous media governed by hyperbolic PDE's in $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$, is the space dimension, of the form

$$\partial_\tau \mathbf{U} = -\frac{\mathrm{i}}{\varrho}\mathbf{L}(-\mathrm{i}\nabla)\mathbf{U} + \mathbf{F}(\mathbf{U}), \;\; \mathbf{U}(\mathbf{r}, \tau)|_{\tau=0} = \mathbf{h}(\mathbf{r}), \;\; \mathbf{r} \in \mathbb{R}^d, \tag{1.1}$$

where (i) $\mathbf{U} = \mathbf{U}(\mathbf{r}, \tau)$, $\mathbf{r} \in \mathbb{R}^d$, $\mathbf{U} \in \mathbb{C}^{2J}$ is a $2J$-dimensional vector; (ii) $\mathbf{L}(-\mathrm{i}\nabla)$ is a linear selfadjoint differential (pseudodifferential) operator with constant coefficients with the symbol $\mathbf{L}(\mathbf{k})$, which is a Hermitian $2J \times 2J$ matrix; (iii) $\mathbf{F}$ is a general polynomial nonlinearity; (iv) $\varrho > 0$ is a *small parameter*. The properties of the linear part are described in terms of dispersion relations $\omega_n(\mathbf{k})$ (eigenvalues of the matrix $\mathbf{L}(\mathbf{k})$). The form of the equation suggests that the processes described by it involve two time scales. Since the nonlinearity $\mathbf{F}(\mathbf{U})$ is of order one, nonlinear effects occur at times τ of order one, whereas the natural time scale of linear effects, governed by the operator $\mathbf{L}$ with the coefficient $1/\varrho$, is of order ϱ. Consequently, the small parameter ϱ measures the ratio of the slow (nonlinear effects) time scale and the fast (linear effects) time scale. A typical example of an equation of the form (1.1) is the nonlinear Schrödinger equation (NLS) or a system of NLS's. Many more examples including a general nonlinear wave equation and the Maxwell equations in periodic media truncated to a finite number of bands are considered in [**7, 8**].

As in our previous works [**7, 8**], we consider here the nonlinear evolutionary system (1.1) with the initial data $\mathbf{h}(\mathbf{r})$ being the sum of wavepackets. *The special focus of this paper is particle-like localized wavepackets which can be viewed as quasiparticles.* Recall that a general wavepacket is defined as such a function $\mathbf{h}(\mathbf{r})$ that its Fourier transform $\hat{\mathbf{h}}(\mathbf{k})$ is localized in a β-neighborhood of a single wavevector $\mathbf{k}_*$, called *principal wavevector*, where β is a *small parameter*. The simplest example of a wavepacket is a function of the form

$$\hat{\mathbf{h}}(\beta;\mathbf{k}) = \beta^{-d}\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{r}_*}\hat{h}\Big(\frac{\mathbf{k}-\mathbf{k}_*}{\beta}\Big)\mathbf{g}_n(\mathbf{k}_*),\ \ \mathbf{k}\in\mathbb{R}^d, \tag{1.2}$$

where $\mathbf{g}_n(\mathbf{k}_*)$ is an eigenvector of the matrix $\mathbf{L}(\mathbf{k}_*)$ and $\hat{h}(\mathbf{k})$ is a scalar Schwarz function (i.e., it is an infinitely smooth and rapidly decaying one). Note that for $\hat{\mathbf{h}}\ (\beta,\mathbf{k})$ of the form (1.2) we have its inverse Fourier transform

$$\mathbf{h}(\beta;\mathbf{r}) = h(\beta(\mathbf{r}-\mathbf{r}_*))\mathrm{e}^{\mathrm{i}\mathbf{k}_*(\mathbf{r}-\mathbf{r}_*)}\mathbf{g}_n(\mathbf{k}_*),\ \ \mathbf{r}\in\mathbb{R}^d. \tag{1.3}$$

Evidently, $\mathbf{h}(\beta,\mathbf{r})$ described by the above formula is a plane wave $\mathrm{e}^{\mathrm{i}\mathbf{k}_*\mathbf{r}}\mathbf{g}_n(\mathbf{k}_*)$ modulated by a slowly varying amplitude $h(\beta(\mathbf{r}-\mathbf{r}_*))$ obtained from $h(\mathbf{z})$ by a spatial shift along the vector $\mathbf{r}_*$ with a subsequent dilation with a large factor $1/\beta$. Clearly, the resulting amplitude has a typical spatial extension proportional to β^{-1} and the spatial shift produces a noticeable effect if $|\mathbf{r}_*| \gg \beta^{-1}$. The spatial form of the wavepacket (1.3) naturally allows us to interpret $\mathbf{r}_* \in \mathbb{R}^d$ as its position and, consequently, to consider the wavepacket as a particle-like one with the position $\mathbf{r}_* \in \mathbb{R}^d$. But how one can define a position for a general wavepacket? Note that not every wavepacket is a particle-like one. For example, let, as before, the function $h(\mathbf{r})$ be a scalar Schwarz function, and let us consider a slightly more general than (1.3) function

$$\mathbf{h}(\beta;\mathbf{r}) = [h(\beta(\mathbf{r}-\mathbf{r}_{*1})) + h(\beta(\mathbf{r}-\mathbf{r}_{*2}))]\mathrm{e}^{\mathrm{i}\mathbf{k}_*\mathbf{r}}\mathbf{g}_n(\mathbf{k}_*),\ \ \mathbf{r}\in\mathbb{R}^d, \tag{1.4}$$

where $\mathbf{r}_{*1}$ and $\mathbf{r}_{*2}$ are two arbitrary, independent vector variables. The wave $\mathbf{h}(\beta,\mathbf{r})$ defined by (1.4) is a wavepacket with the wave number $\mathbf{k}_*$ for any choice of vectors $\mathbf{r}_{*1}$ and $\mathbf{r}_{*2}$, but it is not a particle-like wavepacket since it does not have a single position $\mathbf{r}_*$, but rather it is a sum of two particle-like wavepackets with two positions $\mathbf{r}_{*1}$ and $\mathbf{r}_{*2}$.

Our way to introduce a general particle-like wavepacket $\mathbf{h}(\beta,\mathbf{k}_*,\mathbf{r}_{*0};\mathbf{r})$ with a position $\mathbf{r}_{*0}$ is by treating it as a single element of a family of wavepackets $\mathbf{h}(\beta,\mathbf{k}_*,\mathbf{r}_*;\mathbf{r})$ with $\mathbf{r}_* \in \mathbb{R}^d$ being another *independent parameter*. In fact, we define the entire family of wavepackets $\mathbf{h}(\beta,\mathbf{k}_*,\mathbf{r}_*;\mathbf{r})$, $\mathbf{r}_* \in \mathbb{R}^d$, subject to certain conditions allowing us to interpret any fixed $\mathbf{r}_* \in \mathbb{R}^d$ as the position of $\mathbf{h}(\beta,\mathbf{k}_*,\mathbf{r}_*;\mathbf{r})$. Since we would like of course a wavepacket to maintain under the nonlinear evolution its particle-like property, it is clear that its definition must be sufficiently flexible to accommodate the wavepacket evolutionary variations. In light of the above discussion, the definition of the particle-like wavepacket with a transparent interpretation of its particle properties turns into the key element of the entire construction. It turns out that there is a precise description of a particle-like wavepacket, which is rather simple and physically transparent and such a description is provided in Definition 2.2 below, see also Remarks 2.4, 2.5. The concept of the position is applicable to very general functions, it does not require a parametrization of the whole family of solutions, which was used, for example, in [**25, 20, 21**].

As in our previous works, we are interested in nonlinear evolution not only a single particle-like wavepacket $\mathbf{h}(\beta, \mathbf{k}_*, \mathbf{r}_*; \mathbf{r})$, but a system $\{\mathbf{h}(\beta, \mathbf{k}_{*l}, \mathbf{r}_{*l}; \mathbf{r})\}$ of particle-like wavepackets which we call *multi-particle wavepacket*. Under certain natural conditions of genericity on $\mathbf{k}_{*l}$, we prove here that under the nonlinear evolution: (i) the multi-particle wavepacket remains to be a multi-particle wavepacket; (ii) the principal wavevectors $\mathbf{k}_{*l}$ remain constant; (ii) the spatial position $\mathbf{r}_{*l}$ of the corresponding wavepacket evolves with the constant velocity which is exactly its group velocity $\frac{1}{\varrho}\nabla\omega_n(\mathbf{k}_{*l})$. The evolution of positions of wavepackets becomes the most simple in the case, where at $\tau = 0$ we have $\mathbf{r}_{*l} = \frac{1}{\varrho}\mathbf{r}_*^0$, i.e., the case, where spatial positions are bounded in the same spatial scale in which their group velocities are bounded. In this case, the evolution of the positions is described by the formula

$$\mathbf{r}_l(\tau) = \frac{1}{\varrho}[\mathbf{r}_*^0 + \tau\nabla\omega_{n_l}(\mathbf{k}_{*l})],\ \tau \geqslant 0. \tag{1.5}$$

The rectilinear motion of positions of particle-like wavepackets is a direct consequence of the spatial homogeneity of the master system (1.1). If the system were not spatially homogeneous, the motion of the positions of particle-like wavepackets would not be uniform, but we do not study that problem in this paper. In the rescaled coordinates $\mathbf{y} = \varrho\mathbf{r}$, the trajectory of every particle is a fixed, uniquely defined straight line defined uniquely if $\varrho/\beta \to 0$ as $\varrho, \beta \to 0$. Notice that under the above-mentioned genericity condition, the uniform and independent motion (1.5) of the positions of all involved particle-like wavepackets $\{\mathbf{h}(\beta, \mathbf{k}_{*l}, \mathbf{r}_{*l}; \mathbf{r})\}$ persists though they can collide in the space. In the latter case, they simply pass through each other without significant nonlinear interactions, and the nonlinear evolution with high accuracy is reduced just to a nonlinear evolution of shapes of the particle-like wavepackets. In the case, where the set of the principal wavevectors $\{\mathbf{k}_{*l}\}$ satisfy certain resonance conditions, some components of the original multi-particle wavepacket can evolve into a more complex structure which can be only partly localized in the space and, for instance, can be needle- or pancake-like. We do not study in detail those more complex structures here.

Now let us discuss in more detail the superposition principle introduced and studied for general multi-wavepackets in [**8**] in the particular case, where initially all $\mathbf{r}_{*l} = 0$. Here we consider multi-particle wavepackets with arbitrary $\mathbf{r}_{*l}$ and develop a new argument based on the analysis of an averaged wavepacket interaction system introduced in [**7**]. Assume that the initial data $\mathbf{h}$ for the evolution equation (1.1) is the sum of a finite number of *wavepackets* (*particle-like wavepackets*) $\mathbf{h}_l$, $l = 1, \ldots, N$, i.e.,

$$\mathbf{h} = \mathbf{h}_1 + \ldots + \mathbf{h}_N, \tag{1.6}$$

where the monochromaticity of every wavepacket $\mathbf{h}_l$ is characterized by another small parameter β. The well-known *superposition principle* is a fundamental property of every linear evolutionary system, stating that the solution $\mathbf{U}$ corresponding to the initial data $\mathbf{h}$ as in (1.6) equals

$$\mathbf{U} = \mathbf{U}_1 + \ldots + \mathbf{U}_N \text{ for } \mathbf{h} = \mathbf{h}_1 + \ldots + \mathbf{h}_N, \tag{1.7}$$

where $\mathbf{U}_l$ is the solution to the same linear problem with the initial data $\mathbf{h}_l$.

Evidently, the standard superposition principle cannot hold exactly as a general principle for a nonlinear system, and, at the first glance, there is no expectation for it to hold even approximately. We show though that, in fact, the superposition principle does hold with high accuracy for general dispersion nonlinear wave systems such as (1.1) provided that the initial data are a sum of generic particle-like wavepackets, and this constitutes one of the subjects of this paper. Namely, the superposition principle for nonlinear wave systems states that the solution $\mathbf{U}$ corresponding to the multi-particle wavepacket initial data $\mathbf{h}$ as in (1.6) satisfies

$$\mathbf{U} = \mathbf{U}_1 + \ldots + \mathbf{U}_N + \mathbf{D} \text{ for } \mathbf{h} = \mathbf{h}_1 + \ldots + \mathbf{h}_N, \text{ where } \mathbf{D} \text{ is small.}$$

A more detailed statement of the superposition principle for nonlinear evolution of wavepackets is as follows. We study the nonlinear evolution equation (1.1) on a finite time interval

$$0 \leqslant \tau \leqslant \tau_*, \text{ where } \tau_* > 0 \text{ is a fixed number} \tag{1.8}$$

which may depend on the L^∞ norm of the initial data $\mathbf{h}$ but, importantly, τ_* *does not depend on* ϱ. *We consider classes of initial data such that wave evolution governed by* (1.1) *is significantly nonlinear on time interval* $[0, \tau_*]$ *and the effect of the nonlinearity* $F(\mathbf{U})$ *does not vanish as* $\varrho \to 0$. We assume that β, ϱ satisfy

$$0 < \beta \leqslant 1, \ 0 < \varrho \leqslant 1, \ \beta^2/\varrho \leqslant C_1 \text{ with some } C_1 > 0. \tag{1.9}$$

The above condition of boundedness on the dispersion parameter β^2/ϱ ensures that the dispersion effects are not dominant and they do not suppress nonlinear effects, see [**7, 8**] for a discussion.

Let us introduce the solution operator $\mathcal{S}(\mathbf{h})(\tau) : \mathbf{h} \to \mathbf{U}(\tau)$ relating the initial data $\mathbf{h}$ of the nonlinear evolution equation (1.1) to its solution $\mathbf{U}(t)$. Suppose that the initial state is a system of particle-like wavepackets or *multi-particle wavepacket*, namely $\mathbf{h} = \sum \mathbf{h}_l$ with $\mathbf{h}_l$, $l = 1, \ldots, N$, being "generic" wavepackets. Then for all times $0 \leqslant \tau \leqslant \tau_*$ the following *superposition principle* holds:

$$\mathcal{S}\Big(\sum_{l=1}^{N} \mathbf{h}_l\Big)(\tau) = \sum_{l=1}^{N} \mathcal{S}(\mathbf{h}_l)(\tau) + \mathbf{D}(\tau), \tag{1.10}$$

$$\|\mathbf{D}(\tau)\|_E = \sup_{0 \leqslant \tau \leqslant \tau_*} \|\mathbf{D}(\tau)\|_{L^\infty} \leqslant C_\delta \frac{\varrho}{\beta^{1+\delta}} \text{ for any small } \delta > 0. \tag{1.11}$$

Obviously, the right-hand side of (1.11) may be small only if $\varrho \leqslant C_1\beta$. There are examples (see [**7**]) in which $\mathbf{D}(\tau)$ is not small for $\varrho = C_1\beta$. In what follows, we refer to a linear combination of particle-like wavepackets as a *multi-particle wavepacket*, and to single particle-like wavepackets which constitutes the multi-particle wavepacket as *component particle wavepackets.*

Very often in theoretical studies of equations of the form (1.1) or ones reducible to it, a functional dependence between ϱ and β is imposed, resulting in a single small parameter. The most common scaling is $\varrho = \beta^2$. The nonlinear evolution of wavepackets for a variety of equations which can be reduced to the form (1.1) was studied in numerous physical and mathematical papers, mostly by asymptotic expansions of solutions with respect to a single small parameter similar to β, see [**10, 12, 16, 18, 22, 24, 26, 32, 34, 36, 37**] and references therein. Often the asymptotic expansions are based on a specific ansatz prescribing a certain form to the solution. In our studies here we do not use asymptotic expansions with respect to a small parameter and do not prescribe a specific form to the solution, but we impose conditions on the initial data requiring it to be a wavepacket or a linear combination of wavepackets. Since we want to establish a general property of a wide class of systems, we apply a general enough dynamical approach. There is a number of general approaches developed for the studies of high-dimensional and infinite-dimensional nonlinear evolutionary systems of hyperbolic type, see [**9, 11, 17, 19, 23, 29, 33, 36, 38, 40, 41**] and references therein. The approach we develop here is based on the introduction of a wavepacket interaction system. We show in [**8**] and here that solutions to this system are in a close relation to solutions of the original system.

The superposition principle implies, in particular, that in the process of nonlinear evolution every single wavepacket propagates almost independently of other wavepackets (even though they may "collide" in physical space for a certain period of time) and the exact solution equals the sum of particular single wavepacket solutions with high precision. In particular, the dynamics of a solution with multi-wavepacket initial data is reduced to dynamics of separate solutions with single wavepacket data. Note that the nonlinear evolution of a single wavepacket solution for many problems is studied in detail, namely it is well approximated by its own nonlinear Schrödinger equation (NLS), see [**16, 22, 26, 27, 36, 37, 38, 7**] and references therein.

Let us give now an elementary physical argument justifying the superposition principle which goes as follows. If there would be no nonlinearity, the system would be linear and, consequently, the superposition principle would hold exactly. Hence any deviation from it is due to the nonlinear interactions between wavepackets, and one has to estimate their impact. Suppose that initially at time $\tau = 0$ the spatial extension s of every involved wavepacket is characterized by the parameter β^{-1} as in (1.3). Assume also (and it is quite an assumption) that the involved wavepackets evolving nonlinearly maintain somehow their wavepacket identities, including the group velocities and the spatial extensions. Then, consequently, the spatial extension of every involved wavepacket is proportional to β^{-1} and its group velocity v_l is proportional to ϱ^{-1}. The difference Δv between any two different group velocities is also proportional to ϱ^{-1}. Then the time when two different wavepackets overlap in the space is proportional to $s/|\Delta v|$ and hence to ϱ/β. Since the nonlinear term is of order one, the magnitude of the impact of the nonlinearity during this time interval should be roughly proportional to ϱ/β, which results in the same order of the magnitude of $\mathbf{D}$ in (1.10)–(1.11). Observe that this estimate is in agreement with our rigorous estimate of the magnitude of $\mathbf{D}$ in (1.11) if we set there $\delta = 0$.

The rigorous proof of the superposition principle presented here is not directly based on the above argument since it already implicitly relies on the principle. Though some components of the physical argument can be found in our rigorous proof. For example, we prove that the involved wavepackets maintain under the nonlinear evolution constant values of their wavevectors with well defined group velocities (the wavepacket preservation). Theorem 6.12 allows us to estimate spatial extensions of particle-like wavepackets under the nonlinear evolution. The proof of the superposition principle for general wavepackets provided in [**8**] is based on general algebraic-functional considerations and on the theory of analytic operator expansions in Banach spaces. Here we develop an alternative approach with a proof based on properties of the wavepacket interaction systems introduced in [**7**].

To provide a flexibility in formulating more specific statements related to the spatial localization of wavepackets, we introduce a few types of wavepackets:

- a single *particle-like wavepacket* w which is characterized by the following properties: (a) its modal decomposition involves only wavevectors from β-vicinity of a single wavevector $\mathbf{k}_*$, where $\beta > 0$ is a small parameter; (b) it is spatially localized in all directions and can be assigned its position $\mathbf{r}_*$;
- a *multi-particle wavepacket* which is a system $\{w_l\}$ of particle-like wavepackets with the corresponding sets of wavevectors $\{\mathbf{k}_{*l}\}$ and positions $\{\mathbf{r}_{*l}\}$;
- a *spatially localized multi-wavepacket* which is a system $\{w_l\}$ with w_l being either a particle-like wavepacket or a general wavepacket.

We would like to note that a more detailed analysis, which is left for another paper, indicates that, under certain resonance conditions, nonlinear interactions of particle-like wavepackets may produce a *spatially localized wavepacket* w characterized by the following properties: (i) its modal decomposition involves only wavevectors from a β-vicinity of a single wavevector $\mathbf{k}_*$, where $\beta > 0$ is a small parameter; (ii) it is only partly spatially localized in some, not necessarily all directions, and, for instance, it can be needle- or pancake-like.

We also would like to point out that the particular form (1.1) of the dependence on the small parameter ϱ is chosen so that appreciable nonlinear effects occur at times of order one. In fact, many important classes of problems involving small parameters can be readily reduced to the framework of (1.1) by a simple rescaling. It can be seen from the following examples. The first example is a system with a *small nonlinearity*

$$\partial_t \mathbf{v} = -\mathrm{i}\mathbf{L}\mathbf{v} + \alpha \mathbf{f}(\mathbf{v}), \ \ \mathbf{v}|_{t=0} = \mathbf{h}, \ 0 < \alpha \ll 1, \tag{1.12}$$

where the initial data is bounded uniformly in α. Such problems are reduced to (1.1) by the time rescaling $\tau = t\alpha$. Note that here $\varrho = \alpha$ and the finite time interval $0 \leqslant \tau \leqslant \tau_*$ corresponds to the *long time interval* $0 \leqslant t \leqslant \tau_*/\alpha$.

The second example is a system with *small initial data* considered on long time intervals. The system itself has no small parameters, but the initial data are small, namely

$$\partial_t \mathbf{v} = -\mathrm{i}\mathbf{L}\mathbf{v} + \mathbf{f}_0(\mathbf{v}),\ \ \mathbf{v}|_{t=0} = \alpha_0 \mathbf{h},\ 0 < \alpha_0 \ll 1,\ \text{where} \tag{1.13}$$
$$\mathbf{f}_0(\mathbf{v}) = \mathbf{f}_0^{(m)}(\mathbf{v}) + \mathbf{f}_0^{(m+1)}(\mathbf{v}) + \dots,$$

where α_0 is a small parameter and $\mathbf{f}^{(m)}(\mathbf{v})$ is a homogeneous polynomial of degree $m \geqslant 2$. After rescaling $\mathbf{v} = \alpha_0 \mathbf{V}$ we obtain the following equation with a small nonlinearity:

$$\partial_t \mathbf{V} = -\mathrm{i}\mathbf{L}\mathbf{V} + \alpha_0^{m-1}\left[\mathbf{f}_0^{(m)}(\mathbf{V}) + \alpha_0 \mathbf{f}^{0(m+1)}(\mathbf{V}) + \dots\right],\ \ \mathbf{V}|_{t=0} = \mathbf{h}, \tag{1.14}$$

which is of the form (1.12) with $\alpha = \alpha_0^{m-1}$. Introducing the slow time variable $\tau = t\alpha_0^{m-1}$, we get from the above an equation of the form (1.1), namely

$$\partial_\tau \mathbf{V} = -\frac{\mathrm{i}}{\alpha_0^{m-1}}\mathbf{L}\mathbf{V} + \left[\mathbf{f}^{(m)}(\mathbf{V}) + \alpha_0 \mathbf{f}^{(m+1)}(\mathbf{V}) + \dots\right],\ \ \mathbf{V}|_{t=0} = \mathbf{h}, \tag{1.15}$$

where the nonlinearity does not vanish as $\alpha_0 \to 0$. In this case, $\varrho = \alpha_0^{m-1}$ and the finite time interval $0 \leqslant \tau \leqslant \tau_*$ corresponds to the long time interval $0 \leqslant t \leqslant \tau_*/\alpha_0^{m-1}$ with small $\alpha_0 \ll 1$.

The third example is related to a *high-frequency carrier wave* in the initial data. To be concrete, we consider the nonlinear Schrödinger equation

$$\begin{gathered} \partial_\tau U - \mathrm{i}\partial_x^2 U + \mathrm{i}\alpha |U|^2 U, \\ U|_{\tau=0} = h_1(M\beta x)e^{\mathrm{i}Mk_{*1}x} + h_2(M\beta x)e^{\mathrm{i}Mk_{*2}x} + c.c., \end{gathered} \tag{1.16}$$

where *c.c.* stands for the complex conjugate of the prior term and $M \gg 1$ is a large parameter. Equation (1.16) can be readily recast into the form (1.1) by the change of variables $y = Mr$ yielding

$$\begin{gathered} \partial_\tau U = -\mathrm{i}\frac{1}{\varrho}\partial_r^2 U + \mathrm{i}\alpha |U|^2 U, \\ U|_{\tau=0} = h_1(\beta r)e^{\mathrm{i}k_{*1}r} + h_2(\beta r)e^{\mathrm{i}k_{*2}r} + c.c., \\ \text{where } \varrho = \frac{1}{M^2} \ll 1. \end{gathered} \tag{1.17}$$

Summarizing the above analysis, we list below important ingredients of our approach.

- The wave nonlinear evolution is analyzed based on the modal decomposition with respect to the linear part of the system. The significance of the modal decomposition to the nonlinear analysis is based on the following properties: (i) the wave modal amplitudes do not evolve under the linear evolution; (ii) the same amplitudes evolve slowly under the nonlinear evolution; (iii) modal decomposition is instrumental to the wavepacket definition including its spatial extension and the group velocity.

- Components of multi-particle wavepacket are characterized by their wavevectors $\mathbf{k}_{*l}$, band numbers n_l, and spatial positions $\mathbf{r}_{*l}$. The nonlinear evolution preserves $\mathbf{k}_{*l}$ and n_l, whereas the spatial positions evolve uniformly with the velocities $\frac{1}{\varrho}\nabla\omega_{n_l}(\mathbf{k}_{*l})$.
- The problem involves two small parameters β and ϱ respectively in the initial data and coefficients of the master equation (1.1). These parameters scale respectively (i) the range of wavevectors involved in its modal composition, with β^{-1} scaling its spatial extension, and (ii) ϱ scaling the ratio of the slow and the fast time scales. We make no assumption on the functional dependence between β and ϱ, which are essentially independent and are subject only to inequalities.
- The nonlinear *evolution is studied for a finite time* τ_* which may depend on, say, the amplitude of the initial excitation, and, importantly, τ_* is long enough to observe appreciable nonlinear phenomena which are not vanishingly small. The superposition principle can be extended to longer time intervals up to blow-up time or even infinity if relevant uniform in β and ϱ estimates of solutions in appropriate norms are available.
- In the chosen slow time scale there are two fast wave processes with typical time scale of order ϱ which can be attributed to the linear operator $\mathbf{L}$: (i) fast time oscillations resulting in time averaging and consequent suppression of many nonlinear interactions; (ii) fast wavepacket propagation with large group velocities resulting in effective weakening of nonlinear interactions which are not time-averaged because of resonances. It is these two processes provide mechanisms leading to the superposition principle.

The rest of the paper is organized as follows. In the following Subsection 2.1, we introduce definitions of wavepackets, multi-wavepackets, and particle wavepackets. In Subsection 2.1, we also formulate and briefly discuss some important results of [**7**] which are used in this paper, and, in Subsection 2.2, we formulate new results. In Section 3, we formulate conditions imposed on the linear and the nonlinear parts of the evolution equation (1.1) and also introduce relevant concepts describing resonance interactions inside wavepackets. In Section 4, we introduce an integral form of the basic evolution equation and study basic properties of involved operators. In Section 5, we introduce a wavepacket interaction system describing the dynamics of wavepackets. In Section 6, we first define an averaged wavepacket interaction system which plays a fundamental role in the analysis of the dynamics of multi-wavepackets and then prove that solutions to this system approximate solutions to the original equation with high accuracy. We also discuss there properties of averaged nonlinearities, in particular, for universally and conditionally universal invariant wavepackets, and prove the fundamental theorems on preservation of multi-particle wavepackets, namely Theorems 6.13 and 2.10. In Section 7, we prove the superposition principle using an approximate decoupling of the averaged wavepacket interaction system. In the last subsection of Section 7, we prove some generalizations to the cases involving nongeneric resonance interactions such as the second harmonic and third harmonic generations.

2. Statement of Results

This section consists of two subsections. In the first one, we introduce basic concepts and terminology and formulate relevant results from [**7**] which are used latter on, and in the second one, we formulate new results of this paper.

2.1. Wavepackets and their basic properties.

Since both linear operator $\mathbf{L}(-\mathrm{i}\nabla)$ and nonlinearity $\mathbf{F}(\mathbf{U})$ are translation invariant, it is natural and convenient to recast the evolution equation (1.1) by applying to it the Fourier transform with respect to the space variables $\mathbf{r}$, namely

$$\partial_\tau \hat{\mathbf{U}}(\mathbf{k}) = -\frac{\mathrm{i}}{\varrho}\mathbf{L}(\mathbf{k})\hat{\mathbf{U}}(\mathbf{k}) + \hat{F}(\hat{\mathbf{U}})(\mathbf{k}), \quad \hat{\mathbf{U}}(\mathbf{k})\Big|_{\tau=0} = \hat{\mathbf{h}}(\mathbf{k}), \tag{2.1}$$

where $\hat{\mathbf{U}}(\mathbf{k})$ is the Fourier transform of $\mathbf{U}(\mathbf{r})$, i.e.,

$$\begin{aligned}\hat{\mathbf{U}}(\mathbf{k}) = \int_{\mathbb{R}^d} \mathbf{U}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{r}\cdot\mathbf{k}}\,\mathrm{d}\mathbf{r}, \ \mathbf{U}(\mathbf{r}) \\ = (2\pi)^{-d}\int_{\mathbb{R}^d} \hat{\mathbf{U}}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}}\,\mathrm{d}\mathbf{r}, \ \text{where } \mathbf{r},\mathbf{k}\in\mathbb{R}^d,\end{aligned} \tag{2.2}$$

and $\hat{F}$ is the Fourier form of the nonlinear operator $\mathbf{F}(\mathbf{U})$ involving convolutions, see (3.9) for details. Equation (2.1) is written in terms of the Fourier modes, and we call it the modal form of the original equation (1.1). The most of our studies are conducted first for the modal form (2.1) of the evolution equation and carried over then to the original equation (1.1).

The nonlinear evolution equations (1.1), (2.1) are commonly interpreted as describing wave propagation in a nonlinear medium. We assume that the linear part $\mathbf{L}(\mathbf{k})$ is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(\mathbf{k})$ and eigenvectors $\mathbf{g}_{n,\zeta}(\mathbf{k})$ satisfying

$$\begin{gathered} \mathbf{L}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}) = \omega_{n,\zeta}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}),\ \zeta = \pm, \\ \omega_{n,+}(\mathbf{k}) \geqslant 0,\ \omega_{n,-}(\mathbf{k}) \leqslant 0,\ n = 1, \ldots, J, \end{gathered} \tag{2.3}$$

where $\omega_{n,\zeta}(\mathbf{k})$ are real-valued, continuous for all nonsingular $\mathbf{k}$ functions and vectors $\mathbf{g}_{n,\zeta}(\mathbf{k}) \in \mathbb{C}^{2J}$ have unit length in the standard Euclidean norm. The functions $\omega_{n,\zeta}(\mathbf{k})$, $n = 1, \ldots, J$, are called *dispersion relations* between the frequency ω and the *wavevector* $\mathbf{k}$ with n being the *band number*. We assume that the eigenvalues are naturally ordered by

$$\omega_{J,+}(\mathbf{k}) \geqslant \ldots \geqslant \omega_{1,+}(\mathbf{k}) \geqslant 0 \geqslant \omega_{1,-}(\mathbf{k}) \geqslant \ldots \geqslant \omega_{J,-}(\mathbf{k}) \tag{2.4}$$

and for *almost every* $\mathbf{k}$ (with respect to the standard Lebesgue measure) the eigenvalues are distinct and, consequently, the above inequalities become strict. Importantly, we also assume the following *diagonal symmetry* condition:

$$\omega_{n,-\zeta}(-\mathbf{k}) = -\omega_{n,\zeta}(\mathbf{k}),\ \zeta = \pm,\ n = 1, \ldots, J, \tag{2.5}$$

which is naturally presented in many physical problems (see also Remark 3.3 below) and is a fundamental condition imposed on the matrix $\mathbf{L}(\mathbf{k})$. Very often we use the abbreviation

$$\omega_{n,+}(\mathbf{k}) = \omega_n(\mathbf{k}). \tag{2.6}$$

In particular, we obtain from (2.5)

$$\omega_{n,-}(\mathbf{k}) = -\omega_n(-\mathbf{k}),\ \omega_{n,\zeta}(\mathbf{k}) = \zeta\omega_n(\zeta\mathbf{k}),\ \zeta = \pm. \tag{2.7}$$

In addition to that, in many examples we also have

$$\mathbf{g}_{n,\zeta}(\mathbf{k}) = \mathbf{g}^*_{n,-\zeta}(-\mathbf{k}),\ \text{where } z^* \text{ is complex conjugate to } z. \tag{2.8}$$

We also use rather often the orthogonal projection $\Pi_{n,\zeta}(\mathbf{k})$ in $\mathbb{C}^{2J}$ onto the complex line defined by the eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$, namely

$$\Pi_{n,\zeta}(\mathbf{k})\hat{\mathbf{u}}(\mathbf{k}) = \tilde{u}_{n,\zeta}(\mathbf{k})\mathbf{g}_{n,\zeta}(\mathbf{k}) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}),\ n = 1, \ldots, J,\ \zeta = \pm. \tag{2.9}$$

As it is indicated by the title of this paper, we study the nonlinear problem (1.1) for initial data $\hat{\mathbf{h}}$ in the form of a properly defined particle-like *wavepackets* or, more generally, a sum of such wavepackets to which we refer as *multi-particle wavepacket.* The simplest example of a wavepacket $\mathbf{w}$ is provided by the following formula:

$$\mathbf{w}(\beta;\mathbf{r}) = \Phi_{+}(\beta(\mathbf{r}-\mathbf{r}_{*}))\mathrm{e}^{\mathrm{i}\mathbf{k}_{*}(\mathbf{r}-\mathbf{r}_{*})}\mathbf{g}_{n,+}(\mathbf{k}_{*}),\ \mathbf{r}\in\mathbb{R}^{d}, \tag{2.10}$$

where $\mathbf{k}_{*} \in \mathbb{R}^{d}$ is a *wavepacket principal wavevector*, n is a *band number*, and $\beta > 0$ is a small parameter. We refer to the pair $(n, \mathbf{k}_{*})$ in (2.10) as a *wavepacket nk-pair* and $\mathbf{r}_{*}$ as a *wavepacket position.* Observe that the space extension of the wavepacket $\mathbf{w}(\beta;\mathbf{r})$ is proportional to β^{-1} and it is large for small β. Notice also that, as $\beta \to 0$, the wavepacket $\mathbf{w}(\beta;\mathbf{r})$ as in (2.10) tends, up to a constant factor, to the elementary eigenmode $\mathrm{e}^{\mathrm{i}\mathbf{k}_{*}\cdot\mathbf{r}}\mathbf{g}_{n,\zeta}(\mathbf{k}_{*})$ of the operator $\mathbf{L}(-\mathrm{i}\nabla)$ with the corresponding eigenvalue $\omega_{n,\zeta}(\mathbf{k}_{*})$. We refer to wavepackets of the simple form (2.10) as *simple wavepackets* to underline the very special way the parameter β enters its representation. The function $\Phi_{\zeta}(\mathbf{r})$, which we call *wavepacket envelope*, describes its shape, and it can be any scalar complex-valued regular enough function, for example, a function from Schwarz space. Importantly, as $\beta \to 0$, the L^{∞} *norm of a wavepacket* (2.10) *remains constant. Hence nonlinear effects in* (1.1) *remain strong.*

Evolution of wavepackets in problems which can be reduced to the form (1.1) was studied for a variety of equations in numerous physical and mathematical papers, mostly by asymptotic expansions with respect to a single small parameter similar to β, see [**10, 12, 16, 18, 22, 24, 26, 32, 34, 36, 37**] and references therein. We are interested in general properties of evolutionary systems of the form (1.1) with wavepacket initial data which hold for a wide class of nonlinearities and all values of the space dimensions d and the number $2J$ of the system components. Our approach is not based on asymptotic expansions, but involves two small parameters β and ϱ with mild constraints (1.9) on their relative smallness. The constraints can be expressed in the form of either certain inequalities or equalities, and a possible simple form of such a constraint can be the power law

$$\beta = C\varrho^{\varkappa},\ \text{where } C > 0 \text{ and } \varkappa > 0 \text{ are arbitrary constants.} \tag{2.11}$$

Of course, general features of wavepacket evolution are independent of particular values of the constant C. In addition to that, some fundamental properties such as wavepacket preservation are also totally independent of the particular choice of the values of $\varkappa$ in (2.11), whereas other properties are independent of $\varkappa$ as it varies in certain intervals. For instance, dispersion effects are dominant for $\varkappa < 1/2$, whereas the wavepacket superposition principle of [**7**] holds for $\varkappa < 1$.

To eliminate unbounded (as $\varrho \to 0$) linear term in (2.1) by replacing it with a highly oscillatory factor, we introduce the *slow variable* $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ by the formula

$$\hat{\mathbf{U}}(\mathbf{k}, \tau) = \mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\mathbf{L}(\mathbf{k})}\hat{\mathbf{u}}(\mathbf{k}, \tau) \tag{2.12}$$

and get the following equation for $\hat{\mathbf{u}}(\mathbf{k}, \tau)$:

$$\partial_\tau \hat{\mathbf{u}} = \mathrm{e}^{\frac{\mathrm{i}\tau}{\varrho}\mathbf{L}}\hat{\mathbf{F}}(\mathrm{e}^{\frac{-\mathrm{i}\tau}{\varrho}\mathbf{L}}\hat{\mathbf{u}}), \;\; \hat{\mathbf{u}}|_{\tau=0} = \hat{\mathbf{h}}, \tag{2.13}$$

which, in turn, can be transformed by time integration into the integral form

$$\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}, \;\mathcal{F}(\hat{\mathbf{u}}) = \int_0^\tau \mathrm{e}^{\frac{\mathrm{i}\tau'}{\varrho}\mathbf{L}}\hat{\mathbf{F}}(\mathrm{e}^{\frac{-\mathrm{i}\tau'}{\varrho}\mathbf{L}}\hat{\mathbf{u}}(\tau'))\mathrm{d}\tau' \tag{2.14}$$

with an explicitly defined nonlinear polynomial integral operator $\mathcal{F} = \mathcal{F}(\varrho)$. This operator is bounded uniformly with respect to ϱ in the Banach space $E = C([0, \tau_*], L^1)$. This space has functions $\hat{\mathbf{v}}(\mathbf{k}, \tau)$, $0 \leqslant \tau \leqslant \tau_*$, as elements and has the norm

$$\|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_E = \|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_{C([0,\tau_*],L^1)} = \sup_{0\leqslant\tau\leqslant\tau_*} \int_{\mathbb{R}^d} |\hat{\mathbf{v}}(\mathbf{k}, \tau)|\,\mathrm{d}\mathbf{k}, \tag{2.15}$$

where L^1 is the Lebesgue space of functions $\hat{\mathbf{v}}(\mathbf{k})$ with the standard norm

$$\|\hat{\mathbf{v}}(\cdot)\|_{L^1} = \int_{\mathbb{R}^d} |\hat{\mathbf{v}}(\mathbf{k})|\,\mathrm{d}\mathbf{k}. \tag{2.16}$$

Sometimes, we use more general weighted spaces $L^{1,a}$ with the norm

$$\|\hat{\mathbf{v}}\|_{L^{1,a}} = \int_{\mathbb{R}^d} (1 + |\mathbf{k}|)^a |\hat{\mathbf{v}}(\mathbf{k})|\,\mathrm{d}\mathbf{k}, \;\; a \geqslant 0. \tag{2.17}$$

The space $C([0, \tau_*], L^{1,a})$ with the norm

$$\|\hat{\mathbf{v}}(\mathbf{k}, \tau)\|_{E_a} = \sup_{0\leqslant\tau\leqslant\tau_*} \int_{\mathbb{R}^d} (1 + |\mathbf{k}|)^a |\hat{\mathbf{v}}(\mathbf{k}, \tau)|\mathrm{d}\mathbf{k} \tag{2.18}$$

is denoted by E_a, and, obviously, $E_0 = E$.

A rather elementary existence and uniqueness theorem (Theorem 4.8) implies that if $\hat{\mathbf{h}} \in L^{1,a}$, then for a small and, importantly, independent of

ϱ constant $\tau_* > 0$ this equation has a unique solution

$$\hat{\mathbf{u}}(\tau) = \mathcal{G}(\mathcal{F}(\varrho), \hat{\mathbf{h}})(\tau),\ \tau \in [0, \tau_*],\ \hat{\mathbf{u}} \in C^1([0, \tau_*], L^{1,a}), \tag{2.19}$$

where $\mathcal{G}$ denotes the *solution operator* for Equation (2.14). If $\hat{\mathbf{u}}(\mathbf{k}, \tau)$ is a solution to Equation (2.14), we call the function $\mathbf{U}(\mathbf{r}, \tau)$ defined by (2.12), (2.2) an *F-solution* to Equation (1.1). We denote by $\hat{L}^1$ the space of functions $\mathbf{V}(\mathbf{r})$ such that their Fourier transform $\hat{\mathbf{V}}(\mathbf{k})$ belongs to L^1 and define $\|\mathbf{V}\|_{\hat{L}^1} = \|\hat{\mathbf{V}}\|_{L^1}$. Since

$$\|\mathbf{V}\|_{L^\infty} \leqslant (2\pi)^{-d} \|\hat{\mathbf{V}}\|_{L^1} \text{ and } \hat{L}^1 \subset L^\infty, \tag{2.20}$$

F-solutions to (1.1) belong to $C^1([0, \tau_*], \hat{L}^1) \subset C^1([0, \tau_*], L^\infty)$.

We would like to define wavepackets in a form which explicitly allows them to be real valued. This is accomplished based on the symmetry (2.5) of the dispersion relations, which allows us to introduce a *doublet wavepacket*

$$\begin{aligned}\mathbf{w}(\beta; \mathbf{r}) = {} & \Phi_+(\beta(\mathbf{r} - \mathbf{r}_*))\mathrm{e}^{\mathrm{i}\mathbf{k}_*(\mathbf{r}-\mathbf{r}_*)}\mathbf{g}_{n,+}(\mathbf{k}_*) \\ & + \Phi_-(\beta(\mathbf{r} - \mathbf{r}_*))\mathrm{e}^{-\mathrm{i}\mathbf{k}_*(\mathbf{r}-\mathbf{r}_*)}\mathbf{g}_{n,-}(-\mathbf{k}_*).\end{aligned} \tag{2.21}$$

Such a wavepacket is real if $\Phi_-(\mathbf{r})$, $\mathbf{g}_{n,-}(-\mathbf{k}_*)$ are complex conjugate respectively to $\Phi_+(\mathbf{r})$, $\mathbf{g}_{n,+}(\mathbf{k}_*)$, i.e., if

$$\Phi_-(\mathbf{r}) = \Phi_+^*(\mathbf{r}),\ \mathbf{g}_{n,+}(\mathbf{k}_*) = \mathbf{g}_{n,-}(-\mathbf{k}_*)^*. \tag{2.22}$$

Usually, considering wavepackets with nk-pair $(n, \mathbf{k}_*)$, we mean doublet ones as in (2.21), but sometimes we use the term wavepacket also for an elementary one as defined by (2.10). Note that the latter use is consistent with the former one since it is possible to take one of two terms in (2.21) to be zero.

Below we give a precise definition of a wavepacket. To identify characteristic properties of a wavepacket suitable for our needs, let us look at the Fourier transform $\hat{\mathbf{w}}(\beta; \mathbf{k})$ of an elementary wavepacket $\mathbf{w}(\beta; \mathbf{r})$ defined by (2.10), i.e.,

$$\hat{\mathbf{w}}(\beta; \mathbf{k}) = \beta^{-d}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_*}\hat{\Phi}(\beta^{-1}(\mathbf{k} - \mathbf{k}_*))\mathbf{g}_{n,\zeta}(\mathbf{k}_*). \tag{2.23}$$

We call such $\hat{\mathbf{w}}(\beta; \mathbf{k})$ a wavepacket too, and assume that it possesses the following properties: (i) its L^1 norm is bounded (in fact, constant) uniformly in $\beta \to 0$; (ii) for every $\varepsilon > 0$ the value $\hat{\mathbf{w}}(\beta; \mathbf{k}) \to 0$ for every $\mathbf{k}$ outside a $\beta^{1-\varepsilon}$-neighborhood of $\mathbf{k}_*$, and the convergence is faster than any power of β if Φ is a Schwarz function. To explicitly interpret the last property, we introduce a *cutoff function* $\Psi(\eta)$ which is infinitely smooth and such that

$$\Psi(\eta) \geqslant 0,\ \Psi(\eta) = 1 \text{ for } |\eta| \leqslant 1/2,\ \Psi(\eta) = 0 \text{ for } |\eta| \geqslant 1, \tag{2.24}$$

and its shifted/rescaled modification

$$\Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \mathbf{k}_*)). \tag{2.25}$$

If an elementary wavepacket $\mathbf{w}(\beta; \mathbf{r})$ is defined by (2.23) with $\Phi(\mathbf{r})$ being a Schwarz function, then

$$\|(1 - \Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \cdot))\hat{\mathbf{w}}(\beta; \cdot)\| \leqslant C_{\varepsilon,s}\beta^s, \ 0 < \beta \leqslant 1, \tag{2.26}$$

and the inequality holds for arbitrarily small $\varepsilon > 0$ and arbitrarily large $s > 0$. Based on the above discussion, we give the following definition of a wavepacket which is a minor variation of [**8**, Definiton 8].

Definition 2.1 (single-band wavepacket)**.** Let ε be a fixed number, $0 < \varepsilon < 1$. For a given band number $n \in \{1, \ldots, J\}$ and a principal wavevector $\mathbf{k}_* \in \mathbb{R}^d$ a function $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is called a *wavepacket with nk-pair* $(n, \mathbf{k}_*)$ and the degree of regularity $s > 0$ if for small $\beta < \beta_0$ with some $\beta_0 > 0$ it satisfies the following conditions: (i) $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is L^1-bounded uniformly in β, i.e.,

$$\|\hat{\mathbf{h}}(\beta; \cdot)\|_{L^1} \leqslant C, \ 0 < \beta < \beta_0 \text{ for some } C > 0; \tag{2.27}$$

(ii) $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is composed essentially of two functions $\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k})$, $\zeta = \pm$, which take values in the nth band eigenspace of $\mathbf{L}(\mathbf{k})$ and are localized near $\zeta\mathbf{k}_*$, namely

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \hat{\mathbf{h}}_-(\beta; \mathbf{k}) + \hat{\mathbf{h}}_+(\beta; \mathbf{k}) + D_h, \ 0 < \beta < \beta_0, \tag{2.28}$$

where the components $\hat{\mathbf{h}}_\pm(\beta; \mathbf{k})$ satisfy the condition

$$\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k}) = \Psi(\beta^{1-\varepsilon}/2, \zeta\mathbf{k}_*; \mathbf{k})\Pi_{n,\zeta}(\mathbf{k})\hat{\mathbf{h}}_\zeta(\beta; \mathbf{k}), \ \zeta = \pm, \tag{2.29}$$

where $\Psi(\cdot, \zeta\mathbf{k}_*, \beta^{1-\varepsilon})$ is defined by (2.25) and D_h is small, namely it satisfies the inequality

$$\|D_h\|_{L^1} \leqslant C'\beta^s, \ 0 < \beta < \beta_0, \text{ for some } C' > 0. \tag{2.30}$$

The inverse Fourier transform $\mathbf{h}(\beta; \mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta; \mathbf{k})$ is also called a wavepacket.

Evidently, if a wavepacket has the degree of regularity s, it also has a smaller degree of regularity $s' \leqslant s$ with the same ε. Observe that the degree of regularity s is related to the smoothness of $\Phi_\zeta(\mathbf{r})$ as in (2.10) so that the higher is the smoothness, the higher s/ε can be taken. Namely, if $\hat{\Phi}_\zeta \in L^{1,a}$, then one can take in (2.30) any $s < a\varepsilon$ according to the following inequality:

$$\int |(1 - \Psi(\beta^\varepsilon\eta))\hat{\Phi}_\zeta(\eta)| d\eta \leqslant \beta^{a\varepsilon}\|\hat{\Phi}_\zeta\|_{L^{1,a}} \leqslant C\beta^s. \tag{2.31}$$

For example, if we define $\hat{\mathbf{h}}_\zeta$ similarly to (2.29) and (2.23) by the formula

$$\hat{\mathbf{h}}_\zeta(\beta;\mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k}-\mathbf{k}_*))\beta^{-d}\hat{\Phi}_\zeta(\beta^{-1}(\mathbf{k}-\mathbf{k}_*))\Pi_{n,\zeta}(\mathbf{k})\mathbf{g}, \qquad (2.32)$$

where $\hat{\Phi}_\zeta(\mathbf{k})$ is a scalar Schwarz function and $\mathbf{g}$ is a vector, then, according to (2.31), the estimate (2.30) holds and $\hat{\mathbf{h}}_\zeta(\beta;\mathbf{k})$ is a wavepacket with arbitrarily large degree of regularity s for any given ε such that $0<\varepsilon<1$.

Now let us define a particle-like wavepacket following to the ideas indicated in the Introduction.

Now let us define a particle-like wavepacket following to the ideas indicated in the Introduction.

Definition 2.2 (single-band particle-like wavepacket)**.** We call a function $\hat{\mathbf{h}}(\beta;\mathbf{k}) = \hat{\mathbf{h}}(\beta,\mathbf{r}_*;\mathbf{k})$, $\mathbf{r}_* \in \mathbb{R}^d$, a *particle-like wavepacket with the position* $\mathbf{r}_*$, *nk*-pair $(n,\mathbf{k}_*)$ and the degree of regularity $s>0$ if (i) for every $\mathbf{r}_*$ it is a wavepacket with the degree of regularity s in the sense of the above Definition 2.1 with constants C, C' independent of $\mathbf{r}_* \in \mathbb{R}^d$; (ii) $\hat{\mathbf{h}}_\zeta$ in (2.28) satisfy the inequalities

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i\mathbf{r}_*\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta,\mathbf{r}_*;\mathbf{k}))|d\mathbf{k} \leqslant C_1\beta^{-1-\varepsilon},\ \zeta=\pm,\ \mathbf{r}_*\in\mathbb{R}^d, \qquad (2.33)$$

where $C_1>0$ is an independent of β and $\mathbf{r}_*$ constant, ε is the same as in Definition 2.1. The inverse Fourier transform $\mathbf{h}(\beta;\mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta;\mathbf{k})$ is also called a particle-like wavepacket with the position $\mathbf{r}_*$. We also introduce the quantity

$$a(\mathbf{r}'_*,\hat{\mathbf{h}}_\zeta(\mathbf{r}_*)) = \|\nabla_{\mathbf{k}}(e^{i\mathbf{r}'_*\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta,\mathbf{r}_*;\mathbf{k}))\|_{L^1} \qquad (2.34)$$

which we refer to as the *position detection function* for the wavepacket $\hat{\mathbf{h}}(\beta,\mathbf{r}_*;\mathbf{k})$.

Note that the left-hand side of (2.33) coincides with $a(\mathbf{r}_*,\hat{\mathbf{h}}_\zeta(\mathbf{r}_*))$.

Remark 2.3. If $\hat{\mathbf{h}}(\beta;\mathbf{k})=\hat{\mathbf{h}}(\beta,\mathbf{r}_*;\mathbf{k})$ is a particle-like wavepacket with a position $\mathbf{r}_*$, then, applying the inverse Fourier transform to $\hat{\mathbf{h}}_\zeta(\beta,\mathbf{r}_*;\mathbf{k})$ and $\nabla_{\mathbf{k}}\hat{\mathbf{h}}_\zeta(\beta,\mathbf{r}_*;\mathbf{k})$ as in (2.2), we obtain a function $\mathbf{h}(\beta,\mathbf{r}_*;\mathbf{r})$ which satisfies

$$|\mathbf{r}-\mathbf{r}_*||\mathbf{h}_\zeta(\beta,\mathbf{r}_*;\mathbf{r})| \leqslant (2\pi)^{-d}a(\mathbf{r}_*,\hat{\mathbf{h}}_\zeta) \qquad (2.35)$$

implying that $|\mathbf{h}_\zeta(\beta;\mathbf{r})| \leqslant a(\mathbf{r}_*,\hat{\mathbf{h}}_\zeta)|\mathbf{r}-\mathbf{r}_*|^{-1}$. This inequality is useful for large $|\mathbf{r}-\mathbf{r}_*|$, whereas for bounded $|\mathbf{r}-\mathbf{r}_*|$ (2.27) implies the simpler inequality

$$|\mathbf{h}_\zeta(\beta,\mathbf{r}_*;\mathbf{r})| \leqslant (2\pi)^{-d}\|\hat{\mathbf{h}}\|_{L^1} \leqslant C. \qquad (2.36)$$

The inequalities (2.35) and (2.33) suggest that the quantity $a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*))$ can be interpreted as a size of the particle-like wavepacket $\hat{\mathbf{h}}_\zeta(\beta, \mathbf{r}_*; \mathbf{k})$.

Evidently a particle-like wavepacket is a wave and not a point. Hence the above definition of its position has a degree of uncertainty, allowing, for example, to replace $\mathbf{r}_*$ by $\mathbf{r}_* + \mathbf{a}$ with a *fixed* vector $\mathbf{a}$ (but not allowing unbounded values of $\mathbf{a}$). The above definition of a particle-like wavepacket position was crafted to meet the following requirements: (i) a system of particle-like wavepackets remains to be such a system under the nonlinear evolution; (ii) it is possible (in an appropriate scale) to describe the trajectories traced out by the positions of a system of particle-wavepackets.

Remark 2.4. Typical dependence of the inverse Fourier transform $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ of a wavepacket $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ on $\mathbf{r}_*$ is provided by spatial shifts by $\mathbf{r}_*$ as in (2.21), namely

$$\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r}) = \Phi(\beta(\mathbf{r} - \mathbf{r}_*))\mathrm{e}^{\mathbf{i}\mathbf{k}_* \cdot (\mathbf{r} - \mathbf{r}_*)}\mathbf{g}$$

with a constant $\mathbf{g}$. For such a function $\mathbf{h}$ and for any $\mathbf{r}'_* \in \mathbb{R}^d$

$$\begin{aligned} a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) &= \|\nabla_{\mathbf{k}}(\beta^{-d}\mathrm{e}^{\mathbf{i}\mathbf{k}\mathbf{r}'_*}\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k}))\|_{L^1} \\ &= \|\nabla_{\mathbf{k}}(\beta^{-d}\mathrm{e}^{\mathbf{i}\mathbf{k}\mathbf{r}'_*}\mathrm{e}^{-\mathbf{i}\mathbf{k}\mathbf{r}_*}\hat{\Phi}(\mathbf{k}))\|_{L^1}\|\mathbf{g}\| \\ &= \|\mathbf{g}\| \int |\mathrm{i}(\mathbf{r}'_* - \mathbf{r}_*)\hat{\Phi}(\mathbf{k}') + \frac{1}{\beta}\nabla_{\mathbf{k}'}\hat{\Phi}(\mathbf{k}')|d\mathbf{k}'. \end{aligned}$$

Hence, taking for simplicity $\|\mathbf{g}\| = 1$, we obtain

$$\begin{aligned} |\mathbf{r}'_* - \mathbf{r}_*|\|\hat{\Phi}\|_{L_1} + \frac{1}{\beta}\|\nabla\hat{\Phi}\|_{L_1} &\geqslant a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) \\ &\geqslant ||\mathbf{r}'_* - \mathbf{r}_*|\|\hat{\Phi}\|_{L_1} - \frac{1}{\beta}\|\nabla\hat{\Phi}\|_{L_1}|. \end{aligned} \tag{2.37}$$

For small $|\mathbf{r}'_* - \mathbf{r}_*| \ll \frac{1}{\beta}$ we see that the position detection function $a(\mathbf{r}'_*, \hat{\mathbf{h}})$ is of order $O(\beta^{-1})$, which is in the agreement with (2.33). For large $|\mathbf{r}'_* - \mathbf{r}_*| \gg \frac{1}{\beta}$ the $a(\mathbf{r}'_*, \hat{\mathbf{h}})$ is approximately proportional to $|\mathbf{r}'_* - \mathbf{r}_*|$. Therefore, if we know $a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*))$ as a function of $\mathbf{r}'_*$, we can recover the value of $\mathbf{r}_*$ with the accuracy of order $O(\beta^{-1-\varepsilon})$ with arbitrary small ε. Namely, let us take arbitrary small $\varepsilon > 0$ and some $C > 0$ and consider the set

$$B(\beta) = \{\mathbf{r}'_* : a(\mathbf{r}'_*, \hat{\mathbf{h}}(\mathbf{r}_*)) \leqslant C\beta^{-1-\varepsilon}\} \subset \mathbb{R}^d, \tag{2.38}$$

which should provide an approximate location of $\mathbf{r}_*$. According to (2.37), $\mathbf{r}_*$ lies in this set for small β. If $\mathbf{r}_*'$ lies in this set, then

$$C\beta^{-1-\varepsilon} \geqslant a(\mathbf{r}_*', \hat{\mathbf{h}}(\mathbf{r}_*)) \geqslant ||\mathbf{r}_*' - \mathbf{r}_*|\|\hat{\Phi}\|_{L_1} - \frac{1}{\beta}\|\nabla\hat{\Phi}\|_{L_1}|$$

and $|\mathbf{r}_*' - \mathbf{r}_*| \leqslant C_1\beta^{-1-\varepsilon} + C_2\beta^{-1}$. Hence the diameter of the $B(\beta)$ is of order $O(\beta^{-1-\varepsilon})$. Observe, taking into account Remark 2.3, that the accuracy of the wavepacket location obviously cannot be better than its size $a(\mathbf{r}_*, \hat{\mathbf{h}}_\zeta(\mathbf{r}_*)) \sim \beta^{-1}$. The above analysis suggests that the function $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ can be viewed as *pseudoshifts* of the function $\mathbf{h}(\beta, \mathbf{0}; \mathbf{r})$ by vectors $\mathbf{r}_* \in \mathbb{R}^d$ in the sense that the regular spatial shift by $\mathbf{r}_*$ is combined with a variation of the shape of $\mathbf{h}(\beta, \mathbf{0}; \mathbf{r})$ which is limited by the fundamental condition (2.33). In other words, according Definition 2.2, as a wavepacket moves from $\mathbf{0}$ to $\mathbf{r}_*$ by the corresponding spatial shift, it is allowed to change its shape subject to the fundamental condition (2.33). The later is instrumental for capturing nonlinear evolution of particle-like wavepackets governed by an equation of the form (1.1).

Remark 2.5. The set $B(\beta)$ defined by (2.38) gives an approximate location of the support of the function $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ not only in the special case considered in Remark 2.4, but also when $\mathbf{h}(\beta, \mathbf{r}_*; \mathbf{r})$ is a general particle-like wavepacket. One can apply with obvious modifications the above argument for $e^{\mathbf{i}\mathbf{k}\mathbf{r}_*}\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ in place of $\hat{\Phi}(\mathbf{k})$ using (2.33). Here we give an alternative argument based on (2.35). Notice that the condition $a(\mathbf{r}_{*0}, \hat{\mathbf{h}}(\mathbf{r}_*)) \leqslant C\beta^{-1-\varepsilon}$ can be obviously satisfied not only by $\mathbf{r}_{*0} = \mathbf{r}_*$. But one can show that the diameter of the set of such $\mathbf{r}_{*0}$ is estimated by $O(\beta^{-1-\varepsilon})$. Indeed, assume that a given function $\mathbf{h}(\beta, \mathbf{r})$ does not vanish at a given point $\mathbf{r}_0$, i.e., $|\mathbf{h}(\beta, \mathbf{r}_0)| \geqslant c_0 > 0$ for all $\beta \leqslant \beta_0$. The fulfillment of (2.33) for the function $\mathbf{h}(\beta, \mathbf{r})$ with two different values of $\mathbf{r}_*$, namely $\mathbf{r}_* = \mathbf{r}_*'$ and $\mathbf{r}_* = \mathbf{r}_*''$ implies that

$$a(\mathbf{r}_*', \hat{\mathbf{h}}) \leqslant C_1\beta^{-1-\varepsilon}, \ a(\mathbf{r}_*'', \hat{\mathbf{h}}) \leqslant C_2\beta^{-1-\varepsilon},$$

and, according to (2.35), for all $\mathbf{r}$

$$|\mathbf{r} - \mathbf{r}_*'||\mathbf{h}(\beta, \mathbf{r})| \leqslant (2\pi)^{-d}C_1\beta^{-1-\varepsilon}, \ |\mathbf{r} - \mathbf{r}_*''||\mathbf{h}(\beta, \mathbf{r})| \leqslant (2\pi)^{-d}C_2\beta^{-1-\varepsilon}.$$

Hence

$$|\mathbf{r}_0 - \mathbf{r}_*'| \leqslant \frac{(2\pi)^{-d}C_1\beta^{-1-\varepsilon}}{c_0}, \ |\mathbf{r}_0 - \mathbf{r}_*''| \leqslant \frac{(2\pi)^{-d}C_1\beta^{-1-\varepsilon}}{c_0},$$

and

$$|\mathbf{r}_*' - \mathbf{r}_*''| \leqslant C_3\beta^{-1-\varepsilon}.$$

Note that if we rescale variables $\mathbf{r}$ and $\mathbf{r}_*$ as in Example 2.13, namely $\varrho\mathbf{r} = \mathbf{y}$ and $\varrho\mathbf{r}_* = \mathbf{y}_*$ with $\varrho = \beta^2$, the diameter of the set $B(\beta)$ in the y-coordinates is of order $\beta^{1-\varepsilon} \ll 1$, and hence this set gives a good approximation for the location of the particle-like wavepacket as $\beta \to 0$. It is important to notice that our method to locate the support of wavepackets is applicable to very general wavepackets and does not use their specific form. This flexibility allows us to prove that particle-like wavepackets and their positions are well defined during nonlinear dynamics of generic equations with rather general initial data which form infinite-dimensional function spaces. Another approaches to describe dynamics of waves are applied to situations, where solutions under considerations can be parametrized by a finite number of parameters and the dynamics of parameters describes the dynamics of the solutions. See for example [**25**], [**20**], [**21**], where dynamics of centers of solutions is described.

Remark 2.6. Note that for a single wavepacket initial data $\mathbf{h}(\beta, \mathbf{r} - \mathbf{r}'_*)$ one can make a change of variables to a moving frame $(\mathbf{x}, \tau)$, namely $(\mathbf{r}, \tau) = (\mathbf{x} + \mathbf{v}\tau, \tau)$, where $\mathbf{v} = \frac{1}{\varrho}\nabla\omega(\mathbf{k}_*)$ is the group velocity; this change of variables makes the group velocity zero. Often it is possible to prove that dynamics preserves functions which decay at infinity, namely if the initial data $\mathbf{h}(\beta, \mathbf{x})$ decays at the spatial infinity, then the solution $\mathbf{U}(\beta, \mathbf{x}, \tau)$ also decays at infinity (though the corresponding proofs can be rather technical). This property can be reformulated in rescaled $\mathbf{y}$ variables as follows: if initial data are localized about zero, then the solution is localized about zero as well. Then, using the fact that the equation has constant coefficients, we observe that the solution $\mathbf{U}(\beta, \mathbf{y} - \mathbf{y}'_*, \tau)$ corresponding to $\mathbf{h}(\beta, \mathbf{y} - \mathbf{y}'_*)$ is localized about $\mathbf{y}'_*$ provided that $\mathbf{h}(\beta, \mathbf{y})$ was localized about the origin. Note that, in this paper, we consider the much more complicated case of multiple wavepackets. Even in the simplest case of the initial multi-wavepacket which involves only two components, namely the wavepacket $\mathbf{h}(\beta, \mathbf{r}) = \mathbf{h}_1(\beta, \mathbf{r} - \mathbf{r}'_*) + \mathbf{h}_2(\beta, \mathbf{r} - \mathbf{r}''_*)$ with two principal wave vectors $\mathbf{k}_{1*} \neq \mathbf{k}_{2*}$, it is evident that one cannot use the above considerations based on the change of variables and the translational invariance. Using other arguments developed in this paper, we prove that systems of particle-like wavepackets remain localized in the process of the nonlinear evolution.

Note that similarly to (1.2) and (1.4) a function of the form

$$\beta^{-d}(\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{r}_{*1}} + \mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{r}_{*2}})\left[\hat{h}\left(\frac{\mathbf{k} - \mathbf{k}_*}{\beta}\right)\right]\mathbf{g}_n(\mathbf{k}_*),$$

defined for any pair of $\mathbf{r}_{*1}$ and $\mathbf{r}_{*2}$, where $\hat{h}$ is a Schwarz function and all constants in Definition 2.1 are independent of $\mathbf{r}_{*1}, \mathbf{r}_{*2} \in \mathbb{R}^d$, is not a single particle-like wavepacket since it does not have a single wavepacket position $\mathbf{r}_*$, but rather it is a sum of two particle-like wavepackets with two positions $\mathbf{r}_{*1}$ and $\mathbf{r}_{*2}$.

We want to emphasize once more that a particle-like wavepacket is defined as the family $\hat{\mathbf{h}}(\beta, \mathbf{r}_*; \mathbf{k})$ with $\mathbf{r}_*$ being an independent variable running the entire space $\mathbb{R}^d$, see, for example, (1.2), (1.3), and (2.21). In particular, we can choose a dependence of $\mathbf{r}_*$ on β and ϱ. An interesting type of such a dependence is $\mathbf{r}_* = \mathbf{r}_*^0/\varrho$, where ϱ satisfies (2.11) as we discuss below in Example 2.13.

Our special interest is in the waves that are finite sums of wavepackets which we refer to as *multi-wavepackets*.

Definition 2.7 (multi-wavepacket)**.** Let S be a set of nk-pairs:

$$\begin{gathered} S = \{(n_l, \mathbf{k}_{*l}),\ l = 1, \ldots, N\} \subset \Sigma = \{1, \ldots, J\} \times \mathbb{R}^d, \\ (n_l, \mathbf{k}_{*l}) \neq (n_{l'}, \mathbf{k}_{*l'}) \text{ for } l \neq l', \end{gathered} \tag{2.39}$$

and let $N = |S|$ be their number. Let K_S be a set consisting of all different wavevectors $\mathbf{k}_{*l}$ involved in S with $|K_S| \leqslant N$ being the number of its elements. K_S is called a *wavepacket k-spectrum* and, without loss of generality, we assume the indexing of elements $(n_l, \mathbf{k}_{*l})$ in S to be such that

$$K_S = \{\mathbf{k}_{*i}, i = 1, \ldots, |K_S|\}, \text{ i.e., } l = i \text{ for } 1 \leqslant i \leqslant |K_S|. \tag{2.40}$$

A function $\hat{\mathbf{h}}(\beta) = \hat{\mathbf{h}}(\beta; \mathbf{k})$ is called a *multi-wavepacket* with *nk-spectrum S* if it is a finite sum of wavepackets, namely

$$\hat{\mathbf{h}}(\beta; \mathbf{k}) = \sum_{l=1}^{N} \hat{\mathbf{h}}_l(\beta; \mathbf{k}),\ 0 < \beta < \beta_0 \text{ for some } \beta_0 > 0, \tag{2.41}$$

where $\hat{\mathbf{h}}_l$, $l = 1, \ldots, N$, is a wavepacket with nk-pair $(\mathbf{k}_{*l}, n_l) \in S$ as in Definition 2.1. If all the wavepackets $\hat{\mathbf{h}}_l(\beta; \mathbf{k}) = \hat{\mathbf{h}}_l(\beta, \mathbf{r}_{*l}; \mathbf{k})$ are particle-like ones with respective positions $\mathbf{r}_{*l}$, then the multi-wavepacket is called *multi-particle wavepacket* and we refer to $(\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*N})$ as its position vector.

Note that if $\hat{\mathbf{h}}(\beta;\mathbf{k})$ is a wavepacket, then $\hat{\mathbf{h}}(\beta;\mathbf{k})+O(\beta^s)$ is also a wavepacket with the same nk-spectrum, and the same is true for multi-wavepackets. Hence we can introduce multi-wavepackets equivalence relation "$\simeq$" of the degree s by

$$\begin{gathered}\hat{\mathbf{h}}_1(\beta;\mathbf{k})\simeq\hat{\mathbf{h}}_2(\beta;\mathbf{k}) \text{ if } \|\hat{\mathbf{h}}_1(\beta;\mathbf{k})-\hat{\mathbf{h}}_2(\beta;\mathbf{k})\|_{L^1}\leqslant C\beta^s \\ \text{for some constant } \ C>0.\end{gathered} \tag{2.42}$$

Note that the condition (2.33) does not impose restrictions on the term D_h in (2.28). Therefore, this equivalence can be applied to particle wavepackets.

Let us turn now to the abstract nonlinear problem (2.14), where (i) $\mathcal{F}=\mathcal{F}(\varrho)$ depends on ϱ and (ii) the initial data $\hat{\mathbf{h}}=\hat{\mathbf{h}}(\beta)$ is a multi-wavepacket depending on β. We would like to state our first theorem on multi-wavepacket preservation under the evolution (2.14) as $\beta,\varrho\to 0$, which holds provided its nk-spectrum S satisfies a natural condition called *resonance invariance*. This condition is intimately related to the so-called *phase* and *frequency matching* conditions for stronger nonlinear interactions, and its concise formulation is as follows. We define for given dispersion relations $\{\omega_n(\mathbf{k})\}$ and any finite set $S\subset\{1,\ldots,J\}\times\mathbb{R}^d$ another finite set $\mathcal{R}(S)\subset\{1,\ldots,J\}\times\mathbb{R}^d$, where $\mathcal{R}$ is a certain algebraic operation described in Definition 3.8 below. It turns out that for any S always $S\subseteq\mathcal{R}(S)$, but *if* $\mathcal{R}(S)=S$ *we call* S *resonance invariant*. The condition of resonance invariance is instrumental for the multi-wavepacket preservation, and there are examples showing that if it fails, i.e., $\mathcal{R}(S)\neq S$, the wavepacket preservation does not hold. Importantly, the resonance invariance $R(S)=S$ allows resonances inside the multi-wavepacket, that includes, in particular, resonances associated with the second and third harmonic generations, resonant four-wave interaction, etc. In this paper, we use basic results on wavepacket preservation obtained in [**7**], and we formulate theorems from [**7**] we need here. Since we use constructions from [**7**], for completeness we provide also their proofs in the following subsections. The following two theorems are proved in [**7**].

Theorem 2.8 (multi-wavepacket preservation)**.** *Suppose that the nonlinear evolution is governed by* (2.14) *and the initial data* $\hat{\mathbf{h}}=\hat{\mathbf{h}}(\beta;\mathbf{k})$ *is a multi-wavepacket with nk-spectrum S and the regularity degree s. Assume that S tis resonance invariant in the sense of Definition* 3.8 *below. Let* $\rho(\beta)$ *be any function satisfying*

$$0<\rho(\beta)\leqslant C\beta^s \textit{ for some constant } C>0, \tag{2.43}$$

and let us set $\varrho = \rho(\beta)$. *Then the solution* $\hat{\mathbf{u}}(\tau,\beta) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ *to* (2.14) *for any* $\tau \in [0, \tau_*]$ *is a multi-wavepacket with nk-spectrum* S *and the regularity degree* s, *i.e.,*

$$\hat{\mathbf{u}}(\tau,\beta;\mathbf{k}) = \sum_{l=1}^{N} \hat{\mathbf{u}}_l(\tau,\beta;\mathbf{k}), \tag{2.44}$$

where $\hat{\mathbf{u}}_l$ *is a wavepacket with nk-pair* $(n_l, \mathbf{k}_{*l}) \in S$.

The time interval length $\tau_* > 0$ *depends only on the* L^1*-norms of* $\hat{\mathbf{h}}_l(\beta;\mathbf{k})$ *and* N. *The presentation* (2.44) *is unique up to the equivalence* (2.42) *of degree* s.

The above statement can be interpreted as follows. Modes in *nk*-spectrum S are always resonance coupled with modes in $\mathcal{R}(S)$ through the nonlinear interactions, but if $\mathcal{R}(S) = S$, then (i) all resonance interactions occur inside S and (ii) only small vicinity of S is involved in nonlinear interactions leading to the multi-wavepacket preservation.

The statement of Theorems 2.8 directly follows from the following general theorem proved in [**7**].

Theorem 2.9 (multi-wavepacket approximation)**.** *Let the initial data* $\hat{\mathbf{h}}$ *in the integral equation* (2.14) *be a multi-wavepacket* $\hat{\mathbf{h}}(\beta;\mathbf{k})$ *with nk-spectrum* S *as in* (2.39), *regularity degree* s, *and parameter* $\varepsilon > 0$ *as in Definition* 2.1. *Assume that* S *is resonance invariant in the sense of Definition* 3.8 *below. Let the cutoff function* $\Psi(\beta^{1-\varepsilon}, \mathbf{k}_*; \mathbf{k})$ *and the eigenvector projectors* $\Pi_{n,\pm}(\mathbf{k})$ *be defined by* (2.25) *and* (2.9) *respectively. For a solution* $\hat{\mathbf{u}}$ *of* (2.14) *we set*

$$\hat{\mathbf{u}}_l(\beta;\tau,\mathbf{k}) = \Big[\sum_{\zeta=\pm} \Psi(C\beta^{1-\varepsilon}, \zeta\mathbf{k}_{*l}; \mathbf{k})\Pi_{n_l,\zeta}(\mathbf{k})\Big]\hat{\mathbf{u}}(\beta;\tau,\mathbf{k}), \quad l = 1,\ldots,N. \tag{2.45}$$

Then every such $\hat{\mathbf{u}}_l(\beta;\tau,\mathbf{k})$ *is a wavepacket and*

$$\sup_{0\leqslant\tau\leqslant\tau_*} \Big\|\hat{\mathbf{u}}(\beta;\tau,\mathbf{k}) - \sum_{l=1}^{N} \hat{\mathbf{u}}_l(\beta;\tau,\mathbf{k})\Big\|_{L^1} \leqslant C_1\varrho + C_2\beta^s, \tag{2.46}$$

where the constants C, C_1 *do not depend on* ε, s, ρ, *and* β *and the constant* C_2 *does not depend on* ρ *and* β.

We would like to point out also that Theorem 2.8 allows us to take values $\hat{\mathbf{u}}(\tau_*)$ as new wavepacket initial data for (1.1) and extend the wavepacket invariance of a solution to the next time interval $\tau_* \leqslant \tau \leqslant \tau_{*1}$. This observation allows us to extend the wavepacket invariance to larger values of τ (up to blow-up time or infinity) if some additional information about solutions with wavepacket initial data is available, see [**7**].

Note that the wavepacket form of solutions can be used to obtain long-time estimates of solutions. Namely, very often the behavior of every single wavepacket is well approximated by its own nonlinear Schrödinger equation (NLS), see [**15, 30, 16, 22, 26, 27, 34, 36, 37, 38**] and references therein, see also Section 6. Many features of the dynamics governed by NLS-type equations are well understood, see [**13, 14, 31, 35, 39, 40**] and references therein. These results can be used to obtain long-time estimates for every single wavepacket (as, for example, in [**27**]) and, with the help of the superposition principle, for the multi-wavepacket solution.

2.2. Formulation of new results on particle wavepackets.

In this paper, we prove the following refinement of Theorem 2.8 for the case of multi-particle wavepackets.

Theorem 2.10 (multi-particle wavepacket preservation)**.** *Assume that the conditions of Theorem* 2.9 *hold and, in addition to that, the initial data* $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta; \mathbf{k})$ *is a multi-particle wavepacket of degree* s *with positions* $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*N}$ *and the multi-particle wavepacket is universally resonance invariant in the sense of Definition* 3.8*. Assume also that*

$$\rho(\beta) \leqslant C\beta^{s_0}, s_0 > 0. \tag{2.47}$$

Then the solution $\hat{\mathbf{u}}(\beta; \tau) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ *to* (2.14) *for any* $\tau \in [0, \tau_*]$ *is a multi-particle wavepacket with the same nk-spectrum* S *and the same positions* $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*N}$*. Namely,* (2.46) *holds, where* $\hat{\mathbf{u}}_l$ *is a wavepacket with nk-pair* $(n_l, \mathbf{k}_{*l}) \in S$ *defined by* (2.45)*, the constants* C, C_1, C_2 *do not depend on* $\mathbf{r}_{*l}$*, and every* $\hat{\mathbf{u}}_l$ *is equivalent in the sense of the equivalence* (2.42) *of degree* $s_1 = \min(s, s_0)$ *to a particle wavepacket with the position* $\mathbf{r}_{*l}$*.*

Remark 2.11. Note that in the statement of the above theorem the positions $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*N}$ of wavepackets which compose the solution $\hat{\mathbf{u}}(\beta; \tau, \mathbf{k})$ of (2.13) and (2.14) do not depend on τ and, hence, do not move. Note also that the solution $\hat{\mathbf{U}}(\beta; \tau, \mathbf{k})$ of the original equation (2.1), related to $\hat{\mathbf{u}}(\beta; \tau, \mathbf{k})$ by the change of variables (2.12), is composed of wavepackets $\mathbf{U}_l(\beta; \tau, \mathbf{r})$ corresponding to $\mathbf{u}_l(\beta; \tau, \mathbf{r})$, have their positions moving with respective constant velocities $\nabla_k \omega(\mathbf{k}_{*l})$ (see for details Remark 4.1, see also the following corollary).

Using Proposition 4.2, we obtain from Theorem 2.10 the following corollary.

Corollary 2.12. *Let the conditions of Theorem* 2.10 *hold, and let* $\hat{\mathbf{U}}(\beta;\tau,\mathbf{k})$ *be defined by* (2.12) *in terms of* $\hat{\mathbf{u}}(\beta;\tau,\mathbf{k})$. *Let*

$$\frac{\beta^2}{\varrho} \leqslant C, \text{ with some } C, \ 0<\beta\leqslant\frac{1}{2}, \ 0<\varrho\leqslant\frac{1}{2}. \tag{2.48}$$

Then $\hat{\mathbf{U}}(\beta;\tau,\mathbf{k})$ *is for every* $\tau \in [0,\tau_*]$ *a particle multi-wavepacket in the sense of Definition* 2.2 *with the same nk-spectrum* S, *regularity degree* s_1, *and* τ*-dependent positions* $\mathbf{r}_{*l} + \frac{\tau}{\varrho}\nabla_k \omega_n(\mathbf{k}_{*l})$.

In the following example, we consider the case, where spatial positions of wavepackets have a specific dependence on parameter ϱ, namely $\mathbf{r}_* = \mathbf{r}_*^0/\varrho$.

Example 2.13 (wavepacket trajectories and collisions)**.** Let us rescale the coordinates in the physical space as follows:

$$\varrho\mathbf{r} = \mathbf{y} \tag{2.49}$$

with the consequent rescaling of the wavevector variable (dual with respect to Fourier transform) $\mathbf{k} = \varrho\eta$. It follows then that under the evolution (1.1) the group velocity of a wavepacket with a wavevector $\mathbf{k}_*$ in the new coordinates $\mathbf{y}$ becomes $\nabla_k\omega(\mathbf{k}_*)$ and, evidently, is of order one. If we set the positions $\mathbf{r}_{*l} = \mathbf{r}_{*l}^0/\varrho$ with fixed $\mathbf{r}_{*l}^0$, then, according to (2.35), the wavepackets $|\mathbf{h}(\beta;\mathbf{r})|$ in $\mathbf{y}$-variables have characteristic spatial scale $\mathbf{y} - \mathbf{r}_{*l}^0 \sim \varrho a(\mathbf{r}_{*l},\hat{\mathbf{h}}) \sim \varrho\beta^{-1}$ which is small if ϱ/β is small. The positions of particle-like wavepackets (quasiparticles) $\hat{\mathbf{U}}(\mathbf{y}/\varrho,\tau)$ are initially located at $\mathbf{y}_l = \mathbf{r}_{*l}^0$ and propagate with the group velocities $\nabla_k\omega(\mathbf{k}_{*l})$. Their trajectories are straight lines in the space $\mathbb{R}^d$ described by

$$\mathbf{y} = \tau\nabla_k\omega(\mathbf{k}_{*l}) + \mathbf{r}_{*l}^0, \ 0 \leqslant \tau \leqslant \tau_*,$$

(compare with (1.5)). The trajectories may intersect, indicating "collisions" of quasiparticles. Our results (Theorem 2.10) show that if a multi-particle wavepacket initially was universally resonance invariant, then the involved particle-like wavepackets preserve their identity in spite of collisions and the fact that the nonlinear interactions with other wavepackets (quasiparticles) are not small; in fact, they are of order one. Note that $\mathbf{r}_{*l}^0$ can be chosen arbitrarily implying that up to $N(N-1)$ collisions can occur on the time interval $[0, \tau_*]$ on which we study the system evolution.

To formulate the approximate superposition principle for multi-particle wavepackets, we introduce now the *solution operator* $\mathcal{G}$ mapping the initial data $\hat{\mathbf{h}}$ into the solution $\hat{\mathbf{U}} = \mathcal{G}(\hat{\mathbf{h}})$ of the modal evolution equation (2.14). This operator is defined for $\|\hat{\mathbf{h}}\| \leqslant R$ according to the existence and uniqueness Theorem 4.7. The main result of this paper is the following statement.

Theorem 2.14 (superposition principle)**.** *Suppose that the initial data* $\hat{\mathbf{h}}$ *of* (2.14) *is a multi-particle wavepacket of the form*

$$\hat{\mathbf{h}} = \sum_{l=1}^{N} \hat{\mathbf{h}}_l, \ \ N \max_l \|\hat{\mathbf{h}}_l\|_{L^1} \leqslant R, \tag{2.50}$$

satisfying Definition 2.7 *and its nk-spectrum is universally resonance invariant in the sense of Definition* 3.8*. Suppose also that the group velocities of wavepackets are different, namely*

$$\nabla_{\mathbf{k}} \omega_{n_{l_1}}(\mathbf{k}_{*l_1}) \neq \nabla_{\mathbf{k}} \omega_{n_{l_2}}(\mathbf{k}_{*l_2}) \ \textit{if}\ l_1 \neq l_2 \tag{2.51}$$

and that (2.48) *holds. Then the solution* $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ *to the evolution equation* (2.14) *satisfies the following approximate superposition principle*:

$$\mathcal{G}\Big(\sum_{l=1}^{N_h} \hat{\mathbf{h}}_l\Big) = \sum_{l=1}^{N_h} \mathcal{G}(\hat{\mathbf{h}}_l) + \tilde{\mathbf{D}}, \tag{2.52}$$

with a small remainder $\tilde{\mathbf{D}}(\tau)$ *such that*

$$\sup_{0 \leqslant \tau \leqslant \tau_*} \|\tilde{\mathbf{D}}(\tau)\|_{L^1} \leqslant C_\varepsilon \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta|, \tag{2.53}$$

where (i) ε *is the same as in Definition* 2.1 *and can be arbitrary small*; (ii) τ_* *does not depend on* β, ϱ, $\mathbf{r}_{*l}$, *and* ε; (iii) C_ε *does not depend on* β, ϱ, *and positions* $\mathbf{r}_{*l}$*.*

A particular case of the above theorem in which there was no dependence on $\mathbf{r}_{*l}$ was proved in [**8**] by a different method based on the theory of analytic operators in Banach spaces. The condition (2.51) can be relaxed if the initial positions of involved particle-like wavepackets are far apart, and the corresponding results are formulated in the theorem below and in Example 2.13.

Theorem 2.15 (superposition principle)**.** *Suppose that the initial data* $\hat{\mathbf{h}}$ *of* (2.14) *is a multi-particle wavepacket of the form* (2.50) *with a* *universally resonance invariant nk-spectrum in the sense of Definition* 3.8 *and* (2.48) *holds. Suppose also that either the group velocities of wavepackets are different, namely* (2.51) *holds, or the positions* $\mathbf{r}_{*l}$ *satisfy the inequality*

$$\begin{gathered}\tau_* |\mathbf{r}_{*l_1} - \mathbf{r}_{*l_2}|^{-1} \leqslant \frac{\varrho}{2C_{\omega,2}\beta^{1-\varepsilon}} \\ \textit{if } \nabla_{\mathbf{k}}\omega_{n_{l_1}}(\mathbf{k}_{*l_1}) = \nabla_{\mathbf{k}}\omega_{n_{l_2}}(\mathbf{k}_{*l_2}),\ l_1 \neq l_2,\end{gathered} \tag{2.54}$$

where the constant $C_{\omega,2}$ *is the same as in* (3.2). *Then the solution* $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ *to the evolution equation* (2.14) *satisfies the approximate superposition principle* (2.52), (2.53).

We prove in this paper further generalizations of the particle-like wavepacket preservation and the superposition principle to the cases, where the nk-spectrum of a multi-wavepacket is not universal resonance invariant such as the cases of multi-wavepackets involving the second and third harmonic generation. In particular, we prove Theorem 7.5 showing that many (but, may be, not all) components of involved wavepackets remain spatially localized. Another Theorem 7.7 extends the superposition principle to the case, where resonance interactions between components of a multi-wavepackets can occur.

3. Conditions and Definitions

In this section, we formulate and discuss all definitions and conditions under which we study the nonlinear evolutionary system (1.1) through its modal, Fourier form (2.1). Most of the conditions and definitions are naturally formulated for the modal form (2.1), and this is one of the reasons we use it as the basic one.

3.1. Linear part.

The basic properties of the linear part $\mathbf{L}(\mathbf{k})$ of the system (2.1), which is a $2J \times 2J$ Hermitian matrix with eigenvalues $\omega_{n,\zeta}(\mathbf{k})$, has been already discussed in the Introduction. To account for all needed properties of $\mathbf{L}(\mathbf{k})$, we define the singular set of points $\mathbf{k}$.

Definition 3.1 (band-crossing points)**.** We call $\mathbf{k}_0$ a *band-crossing point* for $\mathbf{L}(\mathbf{k})$ if $\omega_{n+1,\zeta}(\mathbf{k}_0) = \omega_{n,\zeta}(\mathbf{k}_0)$ for some n, ζ or $\mathbf{L}(\mathbf{k})$ is not continuous at $\mathbf{k}_0$ or if $\omega_{1,\pm}(\mathbf{k}_0) = 0$. The set of such points is denoted by σ_{bc}.

In the next condition, we collect all constraints imposed on the linear operator $\mathbf{L}(\mathbf{k})$.

Condition 3.2 (linear part)**.** *The linear part* $\mathbf{L}(\mathbf{k})$ *of the system* (2.1) *is a* $2J \times 2J$ *Hermitian matrix with eigenvalues* $\omega_{n,\zeta}(\mathbf{k})$ *and corresponding eigenvectors* $\mathbf{g}_{n,\zeta}(\mathbf{k})$ *satisfying for* $\mathbf{k} \notin \sigma_{\mathrm{bc}}$ *the basic relations* (2.3)-(2.5)*. In addition to that, we assume:*

(i) *the set of band-crossing points* σ_{bc} *is a closed, nowhere dense set in* $\mathbb{R}^d$ *and has zero Lebesgue measure*;

(ii) *the entries of the Hermitian matrix* $\mathbf{L}(\mathbf{k})$ *are infinitely differentiable in* $\mathbf{k}$ *for all* $\mathbf{k} \notin \sigma_{\mathrm{bc}}$ *that readily implies via the spectral theory,* [**28**]*, infinite differentiability of all eigenvalues* $\omega_n(\mathbf{k})$ *in* $\mathbf{k}$ *for all* $\mathbf{k} \notin \sigma_{\mathrm{bc}}$;

(iii) $\mathbf{L}(\mathbf{k})$ *satisfies the polynomial bound*

$$\|\mathbf{L}(\mathbf{k})\| \leqslant C(1 + |\mathbf{k}|^p), \ \mathbf{k} \in \mathbb{R}^d, \ \textit{for some } C > 0 \textit{ and } p > 0. \tag{3.1}$$

Note that since $\omega_{n,\zeta}(\mathbf{k})$ are smooth if $\mathbf{k} \notin \sigma_{\mathrm{bc}}$, the following relations hold for any (n,k)-spectrum S:

$$\begin{aligned} \max_{|\mathbf{k}\pm\mathbf{k}_{*l}|\leqslant\pi_0,\ l=1,\dots,N,} |\nabla_{\mathbf{k}}\omega_{n_l,\zeta}| &\leqslant C_{\omega,1}, \\ \max_{|\mathbf{k}\pm\mathbf{k}_{*l}|\leqslant\pi_0,\ l=1,\dots,N,} |\nabla_{\mathbf{k}}^2\omega_{n_l,\zeta}| &\leqslant C_{\omega,2}, \end{aligned} \tag{3.2}$$

where $C_{\omega,1}$ and $C_{\omega,2}$ are positive constants and

$$\pi_0 = \frac{1}{2} \min_{l=1,\dots,N} \min(\mathrm{dist}\{\pm\mathbf{k}_{*l}, \sigma_{\mathrm{bc}}\}, 1). \tag{3.3}$$

Remark 3.3 (dispersion relations symmetry)**.** The symmetry condition (2.5) on the dispersion relations naturally arise in many physical problems, for example, the Maxwell equations in periodic media, see [**1**]–[**3**], [**5**], or when $\mathbf{L}(\mathbf{k})$ originates from a Hamiltonian. We would like to stress that this symmetry conditions are not imposed to simplify studies, but rather to take into account fundamental symmetries of physical media. The symmetry causes resonant nonlinear interactions, which create nontrivial effects. Interestingly, many problems without symmetries can be put into the framework with the symmetry by a certain extension, [**7**].

Remark 3.4 (band-crossing points)**.** Band-crossing points are discussed in more detail in [**1**, Section 5.4], [**2**, Sections 4.1, 4.2]. In particular, generically the set σ_{bc} of band-crossing point is a manifold of dimension $d-2$. Notice also that there is a natural ambiguity in the definition of a normalized eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$ of $\mathbf{L}(\mathbf{k})$ which is defined up to a complex number ξ with $|\xi| = 1$. This ambiguity may not allow an eigenvector $\mathbf{g}_{n,\zeta}(\mathbf{k})$ which can be a locally smooth function in $\mathbf{k}$ to be a uniquely defined continuous function in $\mathbf{k}$ globally for all $\mathbf{k} \notin \sigma_{\mathrm{bc}}$ because of a possibility of branching. But, importantly, the orthogonal projector $\Pi_{n,\zeta}(\mathbf{k})$ on $\mathbf{g}_{n,\zeta}(\mathbf{k})$ as defined by (2.9) is uniquely defined and, consequently, infinitely differentiable in $\mathbf{k}$ via the spectral theory, [**28**], for all $\mathbf{k} \notin \sigma_{\mathrm{bc}}$. Since we consider $\hat{\mathbf{U}}(\mathbf{k})$ as an element of the space L^1 and σ_{bc} is of zero Lebesgue measure, considering $\mathbf{k} \notin \sigma_{\mathrm{bc}}$ is sufficient for us.

We introduce for vectors $\hat{\mathbf{u}} \in \mathbb{C}^{2J}$ their expansion with respect to the orthonormal basis$\{\mathbf{g}_{n,\zeta}(\mathbf{k})\}$:

$$\begin{aligned}\hat{\mathbf{u}}(\mathbf{k}) &= \sum_{n=1}^{J} \sum_{\zeta=\pm} \hat{u}_{n,\zeta}(\mathbf{k}) \mathbf{g}_{n,\zeta}(\mathbf{k}) \\ &= \sum_{n=1}^{J} \sum_{\zeta=\pm} \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}), \ \ \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k}) = \Pi_{n,\zeta}(\mathbf{k}) \hat{\mathbf{u}}(\mathbf{k}) \end{aligned} \tag{3.4}$$

and we refer to it as the *modal decomposition* of $\hat{\mathbf{u}}(\mathbf{k})$ and to $\hat{u}_{n,\zeta}(\mathbf{k})$ as the *modal coefficients* of $\hat{\mathbf{u}}(\mathbf{k})$. Evidently,

$$\sum_{n=1}^{j} \sum_{\zeta=\pm} \Pi_{n,\zeta}(\mathbf{k}) = I_{2J}, \tag{3.5}$$

where I_{2J} is the $2J \times 2J$ identity matrix.

Notice that we can define the action of the operator $\mathbf{L}(-\mathrm{i}\nabla_{\mathbf{r}})$ on any Schwarz function $\mathbf{Y}(\mathbf{r})$ by the formula

$$\mathbf{L}(\widehat{-\mathrm{i}\nabla_{\mathbf{r}})\mathbf{Y}}(\mathbf{k}) = \mathbf{L}(\mathbf{k})\hat{\mathbf{Y}}(\mathbf{k}), \tag{3.6}$$

where, in view of the polynomial bound (3.1), the order of $\mathbf{L}$ does not exceed p. In a special case, where all the entries of $\mathbf{L}(\mathbf{k})$ are polynomials, (3.6) turns into the action of the differential operator with constant coefficients.

3.2. Nonlinear part.

The nonlinear term $\hat{F}$ in (2.1) is assumed to be a general functional polynomial of the form

$$\hat{F}(\hat{\mathbf{U}}) = \sum_{m \in \mathfrak{M}_F} \hat{F}^{(m)}(\hat{\mathbf{U}}^m), \tag{3.7}$$

where $\hat{F}^{(m)}$ is an m-homogeneous polylinear operator,

$$\mathfrak{M}_F = \{m_1, \dots, m_p\} \subset \{2, 3, \dots\} \text{ is a finite set,} \\ \text{and } m_F = \max\{m : m \in \mathfrak{M}_F\}. \tag{3.8}$$

The integer m_F in (3.8) is called the *degree of the functional polynomial* $\hat{F}$. For instance, if $\mathfrak{M}_F = \{2\}$ or $\mathfrak{M}_F = \{3\}$, the polynomial $\hat{F}$ is respectively homogeneous quadratic or cubic. Every m-linear operator $\hat{F}^{(m)}$ in (3.7) is assumed to be of the form of a convolution

$$\begin{aligned} &\hat{F}^{(m)}(\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_m)(\mathbf{k}, \tau) \\ &= \int_{\mathbb{D}_m} \chi^{(m)}(\mathbf{k}, \vec{k}) \hat{\mathbf{U}}_1(\mathbf{k}') \dots \hat{\mathbf{U}}_m(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))\, \tilde{\mathrm{d}}^{(m-1)d}\vec{k}, \end{aligned} \tag{3.9}$$

$$\text{where } \mathbb{D}_m = \mathbb{R}^{(m-1)d},\ \tilde{\mathrm{d}}^{(m-1)d}\vec{k} = \frac{\mathrm{d}\mathbf{k}' \dots \mathrm{d}\mathbf{k}^{(m-1)}}{(2\pi)^{(m-1)d}},$$

$$\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}) = \mathbf{k} - \mathbf{k}' - \dots - \mathbf{k}^{(m-1)},\ \vec{k} = (\mathbf{k}', \dots, \mathbf{k}^{(m)}). \tag{3.10}$$

indicating that the nonlinear operator $F^{(m)}(\mathbf{U}_1, \dots, \mathbf{U}_m)$ is translation invariant (it may be local or nonlocal). The quantities $\chi^{(m)}$ in (3.9) are called *susceptibilities*. For numerous examples of nonlinearities of the form similar to (3.7), (3.9) see [**1**]–[**7**] and references therein. In what follows, the nonlinear term $\hat{F}$ in (2.1) will satisfy the following conditions.

Condition 3.5 (nonlinearity)**.** *The nonlinearity* $\hat{F}(\hat{\mathbf{U}})$ *is assumed to be of the form* (3.7)*–*(3.9)*. The susceptibility* $\chi^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})$ *is infinitely differentiable for all* $\mathbf{k}$ *and* $\mathbf{k}^{(j)}$ *which are not band-crossing points, and is bounded, namely for some constant* C_χ

$$\begin{aligned} \|\chi^{(m)}\| = (2\pi)^{-(m-1)d} &\sup_{\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)} \in \mathbb{R}^d \setminus \sigma_{\mathrm{bc}}} |\chi^{(m)}(\mathbf{k}, \mathbf{k}', \dots, \mathbf{k}^{(m)})| \\ &\leqslant C_\chi,\ m \in \mathfrak{M}_F, \end{aligned} \tag{3.11}$$

where the norm $|\chi^{(m)}(\mathbf{k}, \vec{k})|$ *of the m-linear tensor* $\chi^{(m)} : (\mathbb{C}^{2J})^m \to (\mathbb{C}^{2J})^m$ *for fixed* $\mathbf{k}, \vec{k}$ *is defined by*

$$|\chi^{(m)}(\mathbf{k}, \vec{k})| = \sup_{|\mathbf{x}_j| \leqslant 1} |\chi^{(m)}(\mathbf{k}, \vec{k})(\mathbf{x}_1, \dots, \mathbf{x}_m)|, \\ \text{where } |\mathbf{x}| \text{ is the Euclidean norm.} \tag{3.12}$$

Since $\chi^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\mathbf{k}',\ldots,\mathbf{k}^{(m)})$ are smooth if $\mathbf{k} \notin \sigma_{\mathrm{bc}}$, the following relation holds:

$$\max_{|\mathbf{k}\pm\mathbf{k}_{*l}|\leqslant\pi_0,\ l=1,\ldots,N} |\nabla\chi^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\mathbf{k}',\ldots,\mathbf{k}^{(m)})| \leqslant C'_\chi \tag{3.13}$$

if $\mathbf{k}_{*l} \notin \sigma_{\mathrm{bc}}$, π_0 is defined by (3.3), gradient is with respect to $\mathbf{k}$. The case, where $\chi^{(m)}(\mathbf{k},\vec{k})$ depend on small ϱ or, more generally, on ϱ^q, $q>0$, can be treated similarly, see [**7**].

3.3. Resonance invariant *nk*-spectrum.

In this section, being given the dispersion relations $\omega_n(\mathbf{k}) \geqslant 0$, $n \in \{1,\ldots,J\}$, we consider resonance properties of nk-spectra S and the corresponding k-spectra K_S as defined in Definition 2.7, i.e.,

$$\begin{aligned} S &= \{(n_l,\mathbf{k}_{*l}),\ l=1,\ldots,N\} \subset \Sigma = \{1,\ldots,J\}\times\mathbb{R}^d, \\ K_S &= \{\mathbf{k}_{*_l},\ l=1,\ldots,|K_S|\}. \end{aligned} \tag{3.14}$$

We precede the formal description of the *resonance invariance* (see Definition 3.8) with the following guiding physical picture. Initially, at $\tau=0$, the wave is a multi-wavepacket composed of modes from a small vicinity of the nk-spectrum S. As the wave evolves according to (2.1) the polynomial nonlinearity inevitably involves a larger set of modes $[S]_{\mathrm{out}} \supseteq S$, but not all modes in $[S]_{\mathrm{out}}$ are “equal” in developing significant amplitudes. The qualitative picture is that whenever certain interaction phase function (see (4.23) below) is not zero, the fast time oscillations weaken effective nonlinear mode interaction, and the energy transfer from the original modes in S to relevant modes from $[S]_{\mathrm{out}}$, keeping their magnitudes vanishingly small as $\beta,\varrho\to 0$. There is a smaller set of modes $[S]^{\mathrm{res}}_{\mathrm{out}}$ which can interact with modes from S rather effectively and develop significant amplitudes. Now,

$$\text{if } [S]^{\mathrm{res}}_{\mathrm{out}} \subseteq S, \text{ then } S \text{ is called resonance invariant.} \tag{3.15}$$

In simpler situations, the resonance invariance conditions turns into the well-known in nonlinear optics phase and frequency matching conditions. For instance, if S contains $(n_0,\mathbf{k}_{*l_0})$ and the dispersion relations allow for the second harmonic generation in another band n_1 so that $2\omega_{n_0}(\mathbf{k}_{*l_0}) = \omega_{n_1}(2\mathbf{k}_{*l_0})$, then for S to be resonance invariant it must contain $(n_1,2\mathbf{k}_{*l_0})$ too.

Let us turn now to the rigorous constructions. First we introduce the necessary notation. Let $m\geqslant 2$ be an integer, $\vec{l}=(l_1,\ldots,l_m)$, $l_j\in\{1,\ldots,N\}$ be an integer vector from $\{1,\ldots,N\}^m$ and $\vec{\zeta}=(\zeta^{(1)},\ldots,\zeta^{(m)})$,

$\zeta^{(j)} \in \{+1,-1\}$ be a binary vector from $\{+1,-1\}^m$. Note that a pair $(\vec{\zeta},\vec{l})$ naturally labels a sample string of the length m composed of elements $(\zeta^{(j)}, n_{l_j}, \mathbf{k}_{*l_j})$ from the set $\{+1,-1\} \times S$. Let us introduce the sets

$$\Lambda = \{(\zeta, l) : l \in \{1,\dots,N\},\ \zeta \in \{+1,-1\}\},\tag{3.16}$$
$$\Lambda^m = \{\vec{\lambda} = (\lambda_1,\dots,\lambda_m),\ \lambda_j \in \Lambda,\ j = 1,\dots,m\}.$$

There is a natural one-to-one correspondence between Λ^m and $\{-1,1\}^m \times \{1,\dots,N\}^m$, and we write, exploiting this correspondence,

$$\begin{gathered}\vec{\lambda} = ((\zeta', l_1),\dots,(\zeta^{(m)}, l_m)) = (\vec{\zeta},\vec{l}),\ \vec{\vartheta} \in \{-1,1\}^m,\\ \vec{l} \in \{1,\dots,N\}^m \text{ for } \vec{\lambda} \in \Lambda^m.\end{gathered}\tag{3.17}$$

Let us introduce the linear combination

$$\varkappa_m(\vec{\lambda}) = \varkappa_m(\vec{\zeta},\vec{l}) = \sum_{j=1}^m \zeta^{(j)} \mathbf{k}_{*l_j} \text{ with } \zeta^{(j)} \in \{+1,-1\},\tag{3.18}$$

and let $[S]_{K,\mathrm{out}}$ be the set of all its values as $\mathbf{k}_{*l_j} \in K_S$, $\vec{\lambda} \in \Lambda^m$, namely

$$[S]_{K,\mathrm{out}} = \bigcup_{m\in\mathfrak{M}_F} \bigcup_{\vec{\lambda}\in\Lambda^m} \{\varkappa_m(\vec{\lambda})\}.\tag{3.19}$$

We call $[S]_{K,\mathrm{out}}$ *output k-spectrum* of K_S and assume that

$$[S]_{K,\mathrm{out}} \bigcap \sigma_{\mathrm{bc}} = \varnothing.\tag{3.20}$$

We also define the *output nk-spectrum* of S by

$$[S]_{\mathrm{out}} = \{(n,\mathbf{k}) \in \{1,\dots,J\} \times \mathbb{R}^d : n \in \{1,\dots,J\},\ \mathbf{k} \in [S]_{K,\mathrm{out}}\}.\tag{3.21}$$

We introduce the following functions:

$$\begin{gathered}\Omega_{1,m}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^m \zeta^{(j)} \omega_{l_j}(\mathbf{k}_{*l_j}),\\ \vec{k}_* = (\mathbf{k}_{*1},\dots,\mathbf{k}_{*|K_S|}),\ \text{where } \mathbf{k}_{*l_j} \in K_S,\end{gathered}\tag{3.22}$$

$$\Omega(\zeta, n, \vec{\lambda})(\mathbf{k}_{**}, \vec{k}_*) = -\zeta\omega_n(\mathbf{k}_{**}) + \Omega_{1,m}(\vec{\lambda})(\vec{k}_*),\tag{3.23}$$

where $\zeta = \pm 1$, $m \in \mathfrak{M}_F$ as in (3.7). We introduce these functions to apply later to phase functions (4.23).

Now we introduce the *resonance equation*

$$\Omega(\zeta, n, \vec{\lambda})(\zeta\varkappa_m(\vec{\lambda}), \vec{k}_*) = 0,\ \vec{l} \in \{1,\dots,N\}^m,\ \vec{\zeta} \in \{-1,1\}^m,\tag{3.24}$$

denoting by $P(S)$ the set of its solutions $(m, \zeta, n, \vec{\lambda})$. Such a solution is called S-*internal* if

$$(n, \zeta \varkappa_m(\vec{\lambda})) \in S, \text{i.e., } n = n_{l_0},\ \zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_{*l_0},\ l_0 \in \{1, \ldots, N\}, \tag{3.25}$$

and we denote the corresponding $l_0 = I(\vec{\lambda})$. We also denote by $P_{\text{int}}(S) \subset P(S)$ the set of all S-internal solutions to (3.24).

Now we consider the simplest solutions to (3.24) which play an important role. Keeping in mind that the string $\vec{l}$ can contain several copies of a single value l, we can recast the sum in (3.22) as follows:

$$\Omega_{1,m}(\vec{\lambda}) = \Omega_{1,m}(\vec{\zeta}, \vec{l}) = \sum_{l=1}^{N} \delta_l \omega_l(\mathbf{k}_{*l}),$$
$$\text{where } \delta_l = \begin{cases} \sum\limits_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} & \text{if } \quad \vec{l}^{-1}(l) \neq \varnothing, \\ \qquad 0 & \text{if } \quad \vec{l}^{-1}(l) = \varnothing, \end{cases} \tag{3.26}$$
$$\vec{l}^{-1}(l) = \{j \in \{1, \ldots, m\} : l_j = l, \},\ \vec{l} = (l_1, \ldots, l_m),\ 1 \leqslant l \leqslant N.$$

Definition 3.6 (universal solutions)**.** We call a solution $(m, \zeta, n, \vec{\lambda}) \in P(S)$ of (3.24) *universal* if it has the following properties: (i) only a single coefficient out of all δ_l in (3.26) is nonzero, namely for some I_0 we have $\delta_{I_0} = \pm 1$ and $\delta_l = 0$ for $l \neq I_0$; (ii) $n = n_{I_0}$ and $\zeta = \delta_{I_0}$.

We denote the set of universal solutions to (3.24) by $P_{\text{univ}}(S)$. *A justification for calling such a solution universal comes from the fact that if a solution is a universal solution for one $\vec{k}_*$ it is a solution for any other $\vec{k}_*$.* Note that a universal solution is an S-internal solution with $I(\vec{\lambda}) = I_0$ implying

$$P_{\text{univ}}(S) \subseteq P_{\text{int}}(S). \tag{3.27}$$

Indeed, observe that for δ_l as in (3.26)

$$\varkappa_m(\vec{\lambda}) = \varkappa_m(\vec{\zeta}, \vec{l}) = \sum_{j=1}^{m} \zeta^{(j)} \mathbf{k}_{*l_j} = \sum_{l=1}^{N} \delta_l \mathbf{k}_{*l} \tag{3.28}$$

implying $\varkappa_m(\vec{\lambda}) = \delta_{I_0} \mathbf{k}_{*I_0}$ and $\zeta \varkappa_m(\vec{\lambda}) = \delta_{I_0}^2 \mathbf{k}_{*I_0} = \mathbf{k}_{*I_0}$. Then Equation (3.24) is obviously satisfied and $(n, \zeta \varkappa_m(\vec{\lambda})) = (n_{I_0}, \mathbf{k}_{*I_0}) \in S$.

Example 3.7 (universal solutions)**.** Suppose there is just a single band, i.e., $J = 1$, a symmetric dispersion relation $\omega_1(-\mathbf{k}) = \omega_1(\mathbf{k})$, a cubic nonlinearity F with $\mathfrak{M}_F = \{3\}$. We take the nk-spectrum $S =$

$\{(1, \mathbf{k}_*), (1, -\mathbf{k}_*)\}$, i.e., $N = 2$ and $\mathbf{k}_{*1} = \mathbf{k}_*, \mathbf{k}_{*2} = -\mathbf{k}_*$. This example is typical for two counterpropagating waves. Then

$$\Omega_{1,3}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^{3} \zeta^{(j)} \omega_{l_j}(\mathbf{k}_{*l_j}) = (\delta_1 + \delta_2)\omega_1(\mathbf{k}_*)$$

and

$$\varkappa_m(\vec{\lambda}) = \sum_{j=1}^{m} \zeta^{(j)} \mathbf{k}_{*l_j} = \delta_1 \mathbf{k}_{*1} + \delta_2 \mathbf{k}_{*2} = (\delta_1 - \delta_2)\mathbf{k}_*,$$

where we use the notation (3.26). The universal solution set has the form $P_{\text{univ}}(S) = \{(3, \zeta, 1, \vec{\lambda}) : \vec{\lambda} \in \Lambda_\zeta, \zeta = \pm\}$, where Λ_+ consists of vectors $(\lambda_1, \lambda_2, \lambda_3)$ of the form $((+,1),(-,1),(+,1))$, $((+,1),(-,1),(+,2))$, $((+,2),(-,2),(+,1))$, $((+,2),(-,2),(+,2))$, and vectors obtained from the listed ones by permutations of coordinates $\lambda_1, \lambda_2, \lambda_3$. The solutions from $P_{\text{int}}(S)$ have to satisfy $|\delta_1 - \delta_2| = 1$ and $|\delta_1 + \delta_2| = 1$, which is possible only if $\delta_1\delta_2 = 0$. Since $\zeta = \delta_1 + \delta_2$, we have $\zeta\varkappa_m(\vec{\lambda}) = (\delta_1^2 - \delta_2^2)\mathbf{k}_*$ and $\zeta\varkappa_m(\vec{\lambda}) = \mathbf{k}_{*1}$ if $|\delta_1| = 1$ or $\zeta\varkappa_m(\vec{\lambda}) = \mathbf{k}_{*2}$ if $|\delta_2| = 1$. Hence $P_{\text{int}}(S) = P_{\text{univ}}(S)$ in this case. Note that if we set $S_1 = \{(1, \mathbf{k}_*)\}$, $S_2 = \{(1, -\mathbf{k}_*)\}$, then $S = S_1 \cup S_2$, but $P_{\text{int}}(S)$ is larger than $P_{\text{int}}(S_1) \cup P_{\text{int}}(S_2)$. This can be interpreted as follows. When only modes from S_1 are excited, the modes from S_2 remain nonexcited. But when both S_1 and S_2 are excited, there is a resonance effect of S_1 onto S_2, represented, for example, by $\vec{\lambda} = ((+,1),(-,1),(+,2))$, which involves the mode $\zeta\varkappa_m(\vec{\lambda}) = \mathbf{k}_{*2}$.

Now we are ready to define resonance invariant spectra. First, we introduce a subset $[S]_{\text{out}}^{\text{res}}$ of $[S]_{\text{out}}$ by the formula

$$[S]_{\text{out}}^{\text{res}} = \Big\{(n, \mathbf{k}_{**}) \in [S]_{\text{out}} : \mathbf{k}_{**} = \zeta^{(0)}\varkappa_m(\vec{\lambda}),\ m \in \mathfrak{M}_F, \\ \text{where } (m, \zeta, n, \vec{\lambda}) \text{ is a solution of (3.24)}\Big\}, \tag{3.29}$$

calling it *resonant output spectrum* of S, and then we define

$$\text{the resonance selection operation } \mathcal{R}(S) = S \cup [S]_{\text{out}}^{\text{res}}. \tag{3.30}$$

Definition 3.8 (resonance invariant nk-spectrum)**.** The nk-spectrum S is called *resonance invariant* if $\mathcal{R}(S) = S$ or, equivalently, $[S]_{\text{out}}^{\text{res}} \subseteq S$. The nk-spectrum S is called *universally resonance invariant* if $\mathcal{R}(S) = S$ and $P_{\text{univ}}(S) = P_{\text{int}}(S)$.

Obviously, an nk-spectrum S is resonance invariant if and only if all solutions of (3.24) are internal, i.e., $P_{\text{int}}(S) = P(S)$.

It is worth noticing that even when an nk-spectrum is not resonance invariant, often it can be easily extended to a resonance invariant one. Namely, if $\mathcal{R}^j(S) \cap \sigma_{\mathrm{bc}} = \varnothing$ for all j, then the set

$$\mathcal{R}^{\infty}(S) = \bigcup_{j=1}^{\infty} \mathcal{R}^j(S) \subset \Sigma = \{1, \ldots, J\} \times \mathbb{R}^d$$

is resonance invariant. In addition to that, $\mathcal{R}^{\infty}(S)$ is always at most countable. Usually it is finite, i.e., $\mathcal{R}^{\infty}(S) = \mathcal{R}^p(S)$ for a finite p, see examples below; also $\mathcal{R}^{\infty}(S) = S$ for generic K_S.

Example 3.9 (resonance invariant nk-spectra for quadratic nonlinearity)**.** Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F, i.e., $\mathfrak{M}_F = \{2\}$. Let us assume that $\mathbf{k}_* \neq 0$, $\mathbf{k}_*, 2\mathbf{k}_*, \mathbf{0}$ are not band-crossing points and look at two examples. First, suppose that $2\omega_1(\mathbf{k}_*) \neq \omega_1(2\mathbf{k}_*)$ (no second harmonic generation) and $\omega_1(\mathbf{0}) \neq 0$. Let the nk-spectrum be the set $S_1 = \{(1, \mathbf{k}_*)\}$. Then S_1 is resonance invariant. Indeed, $K_{S_1} = \{\mathbf{k}_*\}$, $[S_1]_{K,\mathrm{out}} = \{\mathbf{0}, 2\mathbf{k}_*, -2\mathbf{k}_*\}$, $[S_1]_{\mathrm{out}} = \{(1, \mathbf{0}), (1, 2\mathbf{k}_*), (1, -2\mathbf{k}_*)\}$ and an elementary examination shows that $[S_1]_{\mathrm{out}}^{\mathrm{res}} = \varnothing \subset S_1$ implying $\mathcal{R}(S_1) = S_1$. For the second example let us assume $\omega_1(\mathbf{0}) \neq 0$ and $2\omega_1(\mathbf{k}_*) = \omega_1(2\mathbf{k}_*)$, i.e., the second harmonic generation is present. Here $[S_1]_{\mathrm{out}}^{\mathrm{res}} = \{(1, 2\mathbf{k}_*)\}$ and $\mathcal{R}(S_1) = \{(1, \mathbf{k}_*), (1, 2\mathbf{k}_*)\}$ implying $\mathcal{R}(S_1) \neq S_1$ and hence S_1 is not resonance invariant. Suppose now that $4\mathbf{k}_*, 3\mathbf{k}_* \notin \sigma_{\mathrm{bc}}$ and $\omega_1(\mathbf{0}) \neq 0$, $\omega_1(4\mathbf{k}_*) \neq 2\omega_1(2\mathbf{k}_*)$, $\omega_1(3\mathbf{k}_*) \neq \omega_1(\mathbf{k}_*) + \omega_1(2\mathbf{k}_*)$, and let us set $S_2 = \{(1, \mathbf{k}_*), (1, 2\mathbf{k}_*)\}$. An elementary examination shows that S_2 is resonance invariant. Note that S_2 can be obtained by iterating the resonance selection operator, namely $S_2 = \mathcal{R}(\mathcal{R}(S_1))$. Note also that $P_{\mathrm{univ}}(S_2) \neq P_{\mathrm{int}}(S_2)$. Notice that $\omega_1(\mathbf{0}) = 0$ is a special case since $\mathbf{k} = \mathbf{0}$ is a band-crossing point, and it requires a special treatment.

Example 3.10 (resonance invariant nk-spectra for cubic nonlinearity)**.** Let us consider the one-band case with a symmetric dispersion relation and a cubic nonlinearity that is $\mathfrak{M}_F = \{3\}$. First we take $S_1 = \{(1, \mathbf{k}_*)\}$ and assume that $\mathbf{k}_*, 3\mathbf{k}_*$ are not band-crossing points, implying $[S_1]_{K,\mathrm{out}} = \{\mathbf{k}_*, -\mathbf{k}_*, 3\mathbf{k}_*, -3\mathbf{k}_*\}$. We have $\Omega_{1,3}(\vec{\lambda})(\vec{k}_*) = \sum_{j=1}^{3} \zeta^{(j)} \omega_1(\mathbf{k}_*) = \delta_1 \omega_1(\mathbf{k}_*)$ and $\varkappa_m(\vec{\lambda}) = \delta_1 \mathbf{k}_*$, where we use the notation (3.26), δ_1 takes the values $1, -1, 3, -3$. If $3\omega_1(\mathbf{k}_*) \neq \omega_1(3\mathbf{k}_*)$, then (3.24) has a solution only if $|\delta_1| = 1$ and $\delta_1 = \zeta$. Hence $\zeta \varkappa_m(\vec{\lambda}) = \mathbf{k}_*$ and every solution is internal. Hence $[S_1]_{\mathrm{out}}^{\mathrm{res}} = \varnothing$ and $\mathcal{R}(S_1) = S_1$. Now consider the case associated with the third harmonic generation, namely $3\omega_1(\mathbf{k}_*) = \omega_1(3\mathbf{k}_*)$ and assume that $\omega_1(3\mathbf{k}_*) + 2\omega_1(\mathbf{k}_*) \neq \omega_1(5\mathbf{k}_*)$, $3\omega_1(3\mathbf{k}_*) \neq \omega_1(9\mathbf{k}_*)$, $2\omega_1(3\mathbf{k}_*) + \omega_1(\mathbf{k}_*) \neq \omega_1(7\mathbf{k}_*)$, $2\omega_1(3\mathbf{k}_*) - \omega_1(\mathbf{k}_*) \neq \omega_1(5\mathbf{k}_*)$. An elementary examination shows that the set $S_4 = \{(1, 3\mathbf{k}_*), (1, \mathbf{k}_*), (1, -\mathbf{k}_*)(1, -3\mathbf{k}_*)\}$ satisfies $\mathcal{R}(S_4) = S_4$. Consequently, a multi-wavepacket having S_4 as its resonance invariant nk-spectrum involves the third harmonic generation and, according to Theorem 2.8, it is preserved under nonlinear evolution.

The above examples indicate that, in simple cases, the conditions on $\mathbf{k}_*$ which can make S noninvariant with respect to $\mathcal{R}$ have a form of several algebraic equations, Hence for almost all $\mathbf{k}_*$ such spectra S are resonance invariant. The examples also show that if we fix S and dispersion relations, then we can include S in a larger spectrum $S' = \mathcal{R}^p(S)$ using repeated application of the operation $\mathcal{R}$ to S, and often the resulting extended nk-spectrum S' is resonance invariant. We show in the following section that an nk-spectrum S with generic K_S is universally resonance invariant.

Note that the concept of a resonance invariant nk-spectrum gives a mathematical description of such fundamental concepts of nonlinear optics as phase matching, frequency matching, four wave interaction in cubic media, and three wave interaction in quadratic media. If a multi-wavepacket has a resonance invariant spectrum, all these phenomena may take place in the internal dynamics of the multi-wavepacket, but do not lead to resonant interactions with continuum of all remaining modes.

3.4. Genericity of the nk-spectrum invariance condition.

In simpler situations, where the number of bands J and wavepackets N are not too large, the resonance invariance of an nk- spectrum can be easily verified as above in Examples 3.9, 3.10, but what one can say if J or N are large, or if the dispersion relations are not explicitly given? We show below that, in properly defined nondegenerate cases, a small variation of K_S makes S universally resonance invariant, i.e., the resonance invariance is a generic phenomenon.

Assume that the dispersion relations $\omega_n(\mathbf{k}) \geqslant 0$, $n \in \{1, \ldots, J\}$ are given. Observe then that $\Omega_m(\zeta, n, \vec{\lambda}) = \Omega_m(\zeta, n, \vec{\lambda})(\mathbf{k}_{*1}, \ldots, \mathbf{k}_{*|K_S|})$ defined by (3.23) is a continuous function of $\mathbf{k}_{*l} \notin \sigma_{\mathrm{bc}}$ for every $m, \zeta, n, \vec{\lambda}$.

Definition 3.11 (ω-degenerate dispersion relations)**.** We call dispersion relations $\omega_n(\mathbf{k})$, $n = 1, \ldots, J$, ω-degenerate if there exists such a point

$\mathbf{k}_* \in \mathbb{R}^d \setminus \sigma_{\mathrm{bc}}$ that for all $\mathbf{k}$ in a neighborhood of $\mathbf{k}_*$ at least one of the following four conditions holds: (i) the relations are linearly dependent, namely $\sum_{n=0}^{J} C_n \omega_n(\mathbf{k}) = c_0$, where all C_n are integers, one of which is nonzero, and the c_0 is a constant; (ii) at least one of $\omega_n(\mathbf{k})$ is a linear function; (iii) at least one of $\omega_n(\mathbf{k})$ satisfies the equation $C\omega_n(\mathbf{k}) = \omega_n(C\mathbf{k})$ with some n and integer $C \neq \pm 1$; (iv) at least one of $\omega_n(\mathbf{k})$ satisfies the equation $\omega_n(\mathbf{k}) = \omega_{n'}(-\mathbf{k})$, where $n' \neq n$.

Note that the fulfillment of any of four conditions in Definition 3.11 makes impossible turning some non resonance invariant sets into resonance invariant ones by a variation of $\mathbf{k}_{*l}$. For instance, if $\mathfrak{M}_F = \{2\}$ as in Example 3.9 and $2\omega_1(\mathbf{k}) = \omega_1(2\mathbf{k})$ for all $\mathbf{k}$ in an open set G, then the set $\{(1, \mathbf{k}_*)\}$ with $\mathbf{k}_* \in G$ cannot be made resonance invariant by a small variation of $\mathbf{k}_*$. Below we formulate two theorems which show that if dispersion relations are not ω-degenerate, then a small variation of $\mathbf{k}_{*l}$ turns non resonance invariant sets into resonance invariant; the proofs of the theorems are given in [**7**]

Theorem 3.12. *If $\Omega_m(\zeta, n_0, \vec{\lambda})(\mathbf{k}'_{*1}, \ldots, \mathbf{k}'_{*|K_S|}) = 0$ on a cylinder G in $(\mathbb{R}^d \setminus \sigma_{\mathrm{bc}})^{|K_S|}$ which is a product of small balls $G_i \subset (\mathbb{R}^d \setminus \sigma_{\mathrm{bc}})$ then either $(m, \zeta, n_0, \vec{\lambda}) \in P_{univ}(S)$ or dispersion relations $\omega_n(\mathbf{k})$ are ω-degenerate as in Definition* 3.11.

Theorem 3.13 (genericity of resonance invariance)**.** *Assume that dispersion relations $\omega_n(\mathbf{k})$ are continuous and not ω-degenerate as in Definition* 3.11*. Let $\mathcal{K}_{rinv}$ be a set of points $(\mathbf{k}_{*1}, \ldots, \mathbf{k}_{*|K_S|})$ such that there exists a universally resonance invariant nk-spectrum S for which its k-spectrum $K_S = \left\{\mathbf{k}_{*1}, \ldots, \mathbf{k}_{*|K_S|}\right\}$. Then $\mathcal{K}_{rinv}$ is open and everywhere dense set in $(\mathbb{R}^d \setminus \sigma_{\mathrm{bc}})^{|K_S|}$.*

4. Integrated Evolution Equation

Using the variation of constants formula, we recast the modal evolution equation (2.1) into the following equivalent integral form:

$$\hat{\mathbf{U}}(\mathbf{k}, \tau) = \int_0^\tau \mathrm{e}^{\frac{-\mathrm{i}(\tau-\tau')}{\varrho}\mathbf{L}(\mathbf{k})} \hat{F}(\hat{\mathbf{U}})(\mathbf{k}, \tau)\, \mathrm{d}\tau' + \mathrm{e}^{\frac{-\mathrm{i}\zeta\tau}{\varrho}\mathbf{L}(\mathbf{k})} \hat{\mathbf{h}}(\mathbf{k}),\ \tau \geqslant 0. \quad (4.1)$$

Then we factor $\hat{\mathbf{U}}(\mathbf{k},\tau)$ into the slow variable $\hat{\mathbf{u}}(\mathbf{k},\tau)$ and the fast oscillatory term as in (2.12), namely

$$\hat{\mathbf{U}}(\mathbf{k},\tau) = \mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\mathbf{L}(\mathbf{k})}\hat{\mathbf{u}}(\mathbf{k},\tau),\ \hat{\mathbf{U}}_{n,\zeta}(\mathbf{k},\tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\zeta\omega_n(\mathbf{k})}, \tag{4.2}$$

where $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ are the modal components of $\hat{\mathbf{u}}(\mathbf{k},\tau)$ as in (3.4). Notice that $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ in (4.2) may depend on ϱ and (4.2) is just a change of variables and not an assumption.

Remark 4.1. Note that if $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ is a wavepacket, it is localized near its principal wavevector $\mathbf{k}_*$. The expansion of $\zeta\omega_n(\mathbf{k})$ near the principal wavevector $\zeta\mathbf{k}_*$ (we take $\zeta = 1$ for brevity) takes the form

$$\omega_n(\mathbf{k}) = \omega(\mathbf{k}_*) + \nabla_k\omega_n(\mathbf{k}_*)(\mathbf{k}-\mathbf{k}_*) + \frac{1}{2}\nabla_k^2\omega(\mathbf{k}_*)(\mathbf{k}-\mathbf{k}_*)^2 + \dots .$$

To discuss the impact of the change of variables (4.2), we make the change of variables $\mathbf{k}-\mathbf{k}_* = \xi$. The change of variables (4.2)

$$\begin{aligned}
&\hat{\mathbf{U}}_{n,+}(\mathbf{k},\tau) = \hat{\mathbf{u}}_{n,+}(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k})}\\
&= \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k}_*)}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*)(\mathbf{k}-\mathbf{k}_*)}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}(\frac{1}{2}\nabla_k^2\omega_n(\mathbf{k}_*)(\mathbf{k}-\mathbf{k}_*)^2+\dots)}\\
&= \hat{\mathbf{u}}_{n,+}(\mathbf{k}_*+\xi,\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\zeta\omega_n(\mathbf{k}_*)}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*)\xi}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}R(\xi)},
\end{aligned} \tag{4.3}$$

$$R(\xi)=\omega_n(\mathbf{k}) - \omega_n(\mathbf{k}_*) - \nabla_k\omega_n(\mathbf{k}_*)(\mathbf{k}-\mathbf{k}_*) = \frac{1}{2}\nabla_k^2\omega_n(\mathbf{k}_*)(\xi)^2 + \dots \tag{4.4}$$

has the first factor $\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k}_*)}$ responsible for fast time oscillations of $\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k},\tau)$ and $\mathbf{U}_{n,\zeta}(\mathbf{r},\tau)$. The second factor $\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*)\xi}$ is responsible for the spatial shifts of the inverse Fourier transform by $\frac{\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*)$. Since the shifts are time dependent, they cause the rectilinear movement of the wavepacket $\mathbf{U}_{n,\zeta}(\mathbf{r},\tau)$ with the group velocity $\frac{1}{\varrho}\nabla_k\omega_n(\mathbf{k}_*)$, the third factor is responsible for dispersion effects. Hence the change of variables (4.2) effectively introduces the moving coordinate frame for $\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k},\tau)$ for every $\mathbf{k}$ and in this coordinate frame $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ has zero group velocity and does not have high-frequency time oscillations. The following proposition shows that if $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ is a wavepacket with a constant position, $\hat{\mathbf{U}}_{n,+}(\mathbf{k},\tau)$ is a particle wavepacket in the sense of Definition with position which moves with a constant velocity.

Proposition 4.2. *Let $\hat{\mathbf{u}}_l(\mathbf{k},\tau)$ be for every $\tau \in [0,\tau_*]$ a particle wavepacket in the sense of Definition* 2.2 *with nk-pair $(n,\mathbf{k}_*)$, regularity s, and position $\mathbf{r}_* \in \mathbb{R}^d$ which does not depend on τ. Assume also that the constants C_1 in* (2.33) *and C, C' in* (2.27) *and* (2.30) *do not depend on τ. Let $\hat{\mathbf{U}}_l(\mathbf{k},\tau)$ be defined in terms of $\hat{\mathbf{u}}_l(\mathbf{k},\tau)$ by* (4.2)*. Assume that* (2.48) *holds. Then* $\hat{\mathbf{U}}_l(\mathbf{k},\tau)$ *for every* $\tau \in [0,\tau_*]$ *is a particle wavepacket in the sense of Definition* 2.2 *with nk-pair* $(n,\mathbf{k}_*)$, *regularity s, and* τ*-dependent position* $\mathbf{r}_* + \frac{\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*) \in \mathbb{R}^d$.

Proof. The wavepacket $\hat{\mathbf{u}}_l(\mathbf{k},\tau)$ involves two components $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$, $\zeta = \pm 1$ for which (2.29) holds

$$\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau) = \Psi(\beta^{1-\varepsilon}/2,\zeta\mathbf{k}_*;\mathbf{k})\Pi_{n,\zeta}(\mathbf{k})\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau), \tag{4.5}$$

By (4.2),

$$\hat{\mathbf{U}}_{n,\zeta}(\mathbf{k},\tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\zeta\omega_n(\mathbf{k})}.$$

According to Definition 2.1, the multiplication by a scalar bounded continuous function $\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\zeta\omega_n(\mathbf{k})}$ may only change the constant C' in (2.30). Therefore, it transforms wavepackets into wavepackets. To check that $\hat{\mathbf{U}}_l(\mathbf{k},\tau)$ is a particle-like wavepacket, we consider (2.33) with $\hat{\mathbf{h}}_\zeta(\beta,\mathbf{r}_*;\mathbf{k})$ replaced by $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\zeta\omega_n(\mathbf{k})}$ and $\mathbf{r}_*$ replaced by $\mathbf{r}_* + \frac{\tau}{\rho}\nabla_k\omega_n(\mathbf{k}_*)$. We consider for brevity $\hat{\mathbf{u}}_n(\mathbf{k},\tau) = \hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ with $\zeta = 1$, the case $\zeta = -1$ is similar,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i(\mathbf{r}_*+\frac{\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*))\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k})})|d\mathbf{k}\\
&= \int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i(\mathbf{r}_*+\frac{\tau}{\varrho}\nabla_k\omega_n(\mathbf{k}_*))\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k})}\mathrm{e}^{\frac{\mathrm{i}\tau}{\varrho}\omega_n(\mathbf{k}_*)})|d\mathbf{k}\\
&= \int_{\mathbb{R}^d} |\nabla_{\mathbf{k}}(e^{i\mathbf{r}_*\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau)\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}R(\mathbf{k}-\mathbf{k}_*)})|d\mathbf{k} \leqslant I_1 + I_2,
\end{aligned}$$

where $R(\xi)$ is defined by (4.4),

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} |\,\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}R(\mathbf{k}-\mathbf{k}_*)}\nabla_{\mathbf{k}}(e^{i\mathbf{r}_*\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau))|d\mathbf{k},\\
I_2 &= \int_{\mathbb{R}^d} |\,(e^{i\mathbf{r}_*\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau))\nabla_{\mathbf{k}}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}R(\mathbf{k}-\mathbf{k}_*)}|d\mathbf{k}.
\end{aligned}$$

The integral I_1 is bounded uniformly in $\mathbf{r}_*$ by $C_1'\beta^{-1-\varepsilon}$ since $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ satisfies (2.33). Note that

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} |(e^{i\mathbf{r}_*\mathbf{k}}\hat{\mathbf{u}}_n(\mathbf{k},\tau))\nabla_{\mathbf{k}}\mathrm{e}^{-\frac{\mathrm{i}\tau}{\varrho}R(\mathbf{k}-\mathbf{k}_*)}|d\mathbf{k}\\
&\leqslant \int_{\mathbb{R}^d} |\hat{\mathbf{u}}_n(\mathbf{k},\tau)|\frac{\tau}{\varrho}|\nabla_{\mathbf{k}}R(\mathbf{k}-\mathbf{k}_*)|d\mathbf{k}.
\end{aligned}$$

Note that, according to (4.5) and (2.25), $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau) \neq 0$ only if $|\mathbf{k}-\mathbf{k}_*| \leqslant 2\beta^{1-\varepsilon}$, and for such $\mathbf{k}-\mathbf{k}_*$ we have the Taylor remainder estimate

$$|\nabla_{\mathbf{k}} R(\mathbf{k}-\mathbf{k}_*)| \leqslant C\beta^{1-\varepsilon}.$$

Therefore, $I_2 \leqslant C'\beta^{1-\varepsilon}/\varrho$ and

$$I_1 + I_2 \leqslant C'\beta^{1-\varepsilon}/\varrho.$$

Using (2.48), we conclude that this inequality implies (2.33) for $\hat{\mathbf{U}}_l(\mathbf{k},\tau)$. Therefore, it is a particle-like wavepacket. □

From (4.1) and (4.2) we obtain the following *integrated evolution equation* for $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\mathbf{k},\tau)$, $\tau \geqslant 0$:

$$\hat{\mathbf{u}}(\mathbf{k},\tau) = \mathcal{F}(\hat{\mathbf{u}})(\mathbf{k},\tau) + \hat{\mathbf{h}}(\mathbf{k}),\ \mathcal{F}(\hat{\mathbf{u}}) = \sum_{m\in\mathfrak{M}_F} \mathcal{F}^{(m)}(\hat{\mathbf{u}}^m(\mathbf{k},\tau)), \tag{4.6}$$

$$\mathcal{F}^{(m)}(\hat{\mathbf{u}}^m)(\mathbf{k},\tau) = \int_0^\tau \mathrm{e}^{\frac{\mathrm{i}\tau'}{\varrho}\mathbf{L}(\mathbf{k})} \hat{F}_m((\mathrm{e}^{\frac{-\mathrm{i}\tau'}{\varrho}\mathbf{L}(\cdot)}\hat{\mathbf{u}})^m)(\mathbf{k},\tau')\,\mathrm{d}\tau', \tag{4.7}$$

where $\hat{F}_m$ are defined by (3.7) and (3.9) in terms of the susceptibilities $\chi^{(m)}$, and $\mathcal{F}^{(m)}$ are bounded as in the following lemma.

Recall that the spaces $L^{1,a}$ are defined by formula (2.17). Below we formulate basic properties of these spaces. Recall the Young inequality

$$\|\hat{\mathbf{u}} * \hat{\mathbf{v}}\|_{L^1} \leqslant \|\hat{\mathbf{u}}\|_{L^1}\|\hat{\mathbf{v}}\|_{L^1}. \tag{4.8}$$

This inequality implies the boundedness of convolution in $L^{1,a}$, namely the following lemma holds.

Lemma 4.3. *Let $\hat{H}_1, \hat{H}_2 \in L^{1,a}$ be two scalar functions, $a \geqslant 0$. Let*

$$\hat{H}_3(\mathbf{k}) = \int_{\mathbb{R}^d} \hat{H}_1(\mathbf{k}-\mathbf{k}')\hat{H}_2(\mathbf{k}')\mathrm{d}\mathbf{k}'.$$

Then

$$\|\hat{H}_3(\mathbf{k})\|_{L^{1,a}} \leqslant \|\hat{H}_1(\mathbf{k})\|_{L^{1,a}}\|\hat{H}_1(\mathbf{k})\|_{L^{1,a}}. \tag{4.9}$$

Proof. We have

$$(1+|\mathbf{k}|)^a|\hat{H}_3(\mathbf{k})| \leqslant \sup_{\mathbf{k}',\mathbf{k}''} \frac{(1+|\mathbf{k}'+\mathbf{k}''|)^a}{(1+|\mathbf{k}'|)^a(1+|\mathbf{k}''|)^a}$$
$$\times \int_{\mathbb{R}^d} (1+|\mathbf{k}-\mathbf{k}'|)^a|\hat{H}_1(\mathbf{k}-\mathbf{k}')|(1+|\mathbf{k}'|)^a|\hat{H}_2(\mathbf{k}')|\mathrm{d}\mathbf{k}'.$$

Obviously,

$$\frac{1+|\mathbf{k}'+\mathbf{k}''|}{(1+|\mathbf{k}'|)(1+|\mathbf{k}''|)} \leqslant \frac{(1+|\mathbf{k}'|+|\mathbf{k}''|)}{(1+|\mathbf{k}'|)(1+|\mathbf{k}''|)} \leqslant 1.$$

Applying the Young inequality (4.8), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} (1+|\mathbf{k}|)^a \left|\hat{H}_3(\mathbf{k})\right| \mathrm{d}\mathbf{k} \\ &\leqslant \int_{\mathbb{R}^d} (1+|\mathbf{k}'|) \left|\hat{H}_1(\mathbf{k}')\right| \mathrm{d}\mathbf{k}' \int_{\mathbb{R}^d} (1+|\mathbf{k}''|) \left|\hat{H}_2(\mathbf{k}'')\right| \mathrm{d}\mathbf{k}''.\end{aligned}$$

Using (2.18), we obtain (4.9). □

Using Lemma 4.3, we derive the boundedness of integral operators $\mathcal{F}^{(m)}$.

Lemma 4.4 (boundedness of multilinear operators)**.** *Operator* $\mathcal{F}^{(m)}$ *defined by* (3.9), (4.7) *is bounded from* $E_a = C([0,\tau_*], L^{1,a})$ *into* $C^1([0,\tau_*], L^{1,a})$, $a \geqslant 0$, *and*

$$\|\mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_m)\|_{E_a} \leqslant \tau_* \|\chi^{(m)}\| \prod_{j=1}^{m} \|\hat{\mathbf{u}}_j\|_{E_a}, \tag{4.10}$$

$$\|\partial_\tau \mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_m)\|_{E_a} \leqslant \|\chi^{(m)}\| \prod_{j} \|\hat{\mathbf{u}}_j\|_{E_a}. \tag{4.11}$$

Proof. Notice that since $\mathbf{L}(\mathbf{k})$ is Hermitian, $\left\|\exp\left\{-\mathrm{i}\mathbf{L}(\mathbf{k})\frac{\tau_1}{\varrho}\right\}\right\| = 1$. Using the inequality (4.9) together with (3.9), (4.7), we obtain

$$\begin{aligned}&\|\mathcal{F}^{(m)}(\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_m)(\cdot,\tau)\|_{L^{1,a}} \\ &\leqslant \sup_{\mathbf{k},\vec{k}} |\chi^{(m)}(\mathbf{k},\vec{k})| \int_{\mathbb{R}^d} \int_0^\tau \int_{\mathbb{D}_m} |(1+|\mathbf{k}'|)^a \hat{\mathbf{u}}_1(\mathbf{k}')| \ldots \\ &\times \left|(1+|\mathbf{k}^{(m)}|)^a \hat{\mathbf{u}}_m(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\right| \mathrm{d}\mathbf{k}' \ldots \mathrm{d}\mathbf{k}^{(m-1)} \mathrm{d}\tau_1 \mathrm{d}\mathbf{k} \\ &\leqslant \|\chi^{(m)}\| \int_0^\tau \|\hat{\mathbf{u}}_1(\tau_1)\|_{L^{1,a}} \ldots \|\hat{\mathbf{u}}_m(\tau_1)\|_{L^{1,a}} \mathrm{d}\tau_1 \\ &\leqslant \tau_* \|\chi^{(m)}\| \|\hat{\mathbf{u}}_1\|_{E_a} \ldots \|\hat{\mathbf{u}}_m\|_{E_a},\end{aligned}$$

proving (4.10). A similar estimate produces (4.11). □

Equation (4.6) can be recast as the following abstract equation in a Banach space:

$$\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}, \;\; \hat{\mathbf{u}}, \hat{\mathbf{h}} \in E_a, \tag{4.12}$$

and it readily follows from Lemma 4.4 that $\mathcal{F}(\hat{\mathbf{u}})$ has the following properties.

Lemma 4.5. *The operator* $\mathcal{F}(\hat{\mathbf{u}})$ *defined by* (4.6), (4.7) *satisfies the Lipschitz condition*

$$\|\mathcal{F}(\hat{\mathbf{u}}_1) - \mathcal{F}(\hat{\mathbf{u}}_2)\|_{E_a} \leqslant \tau_* C_F \|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2\|_{E_a} \tag{4.13}$$

where $C_F \leqslant C_\chi C(R)$, $C(R)$ *depends only on* m_F *and* R, *if* $\|\hat{\mathbf{u}}_1\|_{E_a}, \|\hat{\mathbf{u}}_2\|_{E_a} \leqslant 2R$, *with* C_χ *as in* (3.11).

We also use the following form of the contraction principle.

Lemma 4.6 (contraction principle)**.** *Consider the equation*

$$\mathbf{x} = \mathcal{F}(\mathbf{x}) + \mathbf{h}, \;\; \mathbf{x}, \mathbf{h} \in B, \tag{4.14}$$

where B *is a Banach space,* $\mathcal{F}$ *is an operator in* B. *Suppose that for some constants* $R_0 > 0$ *and* $0 < q < 1$

$$\|\mathbf{h}\| \leqslant R_0, \;\; \|\mathcal{F}(\mathbf{x})\| \leqslant R_0 \text{ if } \|\mathbf{x}\| \leqslant 2R_0, \tag{4.15}$$

$$\|\mathcal{F}(\mathbf{x}_1) - \mathcal{F}(\mathbf{x}_2)\| \leqslant q\|\mathbf{x}_1 - \mathbf{x}_2\| \text{ if } \|\mathbf{x}_1\|, \|\mathbf{x}_2\| \leqslant 2R_0. \tag{4.16}$$

Then there exists a unique solution $\mathbf{x}$ *to Equation* (4.14) *such that* $\|\mathbf{x}\| \leqslant 2R_0$. *Let* $\|\mathbf{h}_1\|, \|\mathbf{h}_2\| \leqslant R_0$. *Then two corresponding solutions* $\mathbf{x}_1, \mathbf{x}_2$ *satisfy*

$$\|\mathbf{x}_1\|, \|\mathbf{x}_2\| \leqslant 2R_0, \;\; \|\mathbf{x}_1 - \mathbf{x}_2\| \leqslant (1-q)^{-1}\|\mathbf{h}_1 - \mathbf{h}_2\|. \tag{4.17}$$

Let $\mathbf{x}_1, \mathbf{x}_2$ *be two solutions of correspondingly two equations of the form* (4.14) *with* $\mathcal{F}_1$, $\mathbf{h}_1$ *and* $\mathcal{F}_2$, $\mathbf{h}_2$. *Assume that* $\mathcal{F}_1(\mathbf{u})$ *satisfies* (4.15), (4.16) *with a Lipschitz constant* $q < 1$ *and that* $\|\mathcal{F}_1(\mathbf{x}) - \mathcal{F}_2(\mathbf{x})\| \leqslant \delta$ *for* $\|\mathbf{x}\| \leqslant 2R_0$. *Then*

$$\|\mathbf{x}_1 - \mathbf{x}_2\| \leqslant (1-q)^{-1}(\delta + \|\mathbf{h}_1 - \mathbf{h}_2\|). \tag{4.18}$$

Lemma 4.5 and the contraction principle as in Lemma 4.6 imply the following existence and uniqueness theorem.

Theorem 4.7. *Let* $\|\hat{\mathbf{h}}\|_{E_a} \leqslant R$, *and let* $\tau_* < 1/C_F$, *where* C_F *is a constant from Lemma* 4.5. *Then Equation* (4.6) *has a solution* $\hat{\mathbf{u}} \in E_a = C([0,\tau_*], L^{1,a})$ *which satisfies* $\|\hat{\mathbf{u}}\|_{E_a} \leqslant 2R$, *and such a solution is unique. Hence the solution operator* $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ *is defined on the ball* $\|\hat{\mathbf{h}}\|_{E_a} \leqslant R$.

The following existence and uniqueness theorem is a consequence of Theorem 4.7.

Theorem 4.8. *Let* $a \geqslant 0$, (2.1) *satisfy* (3.11) *and* $\hat{\mathbf{h}} \in L^{1,a}(\mathbb{R}^d)$, $\|\hat{\mathbf{h}}\|_{L^{1,a}} \leqslant R$. *Then there exists a unique solution* $\hat{\mathbf{u}}$ *to the modal evolution equation* (2.1) *in the function space* $C^1([0,\tau_*], L^{1,a})$, $\|\hat{\mathbf{u}}\|_{E_a} + \|\partial\tau\hat{\mathbf{u}}\|_{E_a} \leqslant R_1(R)$. *The number* τ_* *depends on* R *and* C_χ.

Using (2.20) and applying the inverse Fourier transform, we readily obtain the existence of an $F-$solution of (1.1) in $C^1([0,\tau_*], L^\infty(\mathbb{R}^d))$ from the existence of the solution of Equation (2.1) in $C^1([0,\tau_*], L^1)$. The existence of F-solutions with $[a]$ bounded spatial derivatives ($[a]$ being an integer part of a) follows from the solvability in $C^1([0,\tau_*], L^{1,a})$.

Let us recast now the system (4.6), (4.7) into modal components using the projections $\Pi_{n,\zeta}(\mathbf{k})$ as in (2.9). The first step to introduce modal susceptibilities $\chi^{(m)}_{n,\zeta,\vec{\xi}}$ having one-dimensional range in $\mathbb{C}^{2J}$ and vanishing if one of its arguments $\hat{\mathbf{u}}_j$ belongs to a $(2J-1)$-dimensional linear subspace in $\mathbb{C}^{2J}$ (the jth null-space of $\chi^{(m)}_{n,\zeta,\vec{\xi}}$) as follows.

Definition 4.9 (elementary susceptibilities)**.** Let

$$\vec{\xi} = (\vec{n},\vec{\zeta}) \in \{1,\dots,J\}^m \times \{-1,1\}^m = \Xi^m, (n,\zeta) \in \Xi, \tag{4.19}$$

and let $\chi^{(m)}(\mathbf{k},\vec{k})\left[\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\hat{\mathbf{u}}_m(\mathbf{k}^{(m)})\right]$ be m-linear symmetric tensor (susceptibility) as in (3.9).

We introduce *elementary susceptibilities* $\chi^{(m)}_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k}) : (\mathbb{C}^{2J})^m \to \mathbb{C}^{2J}$ as m-linear tensors defined for almost all $\mathbf{k}$ and $\vec{k} = (\mathbf{k}',\dots,\mathbf{k}^{(m)})$ by the following formula:

$$\begin{aligned}
&\chi^{(m)}_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})[\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\hat{\mathbf{u}}_m(\mathbf{k}^{(m)})]\\
&= \chi^{(m)}_{n,\zeta,\vec{n},\vec{\zeta}}(\mathbf{k},\vec{k})[\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\hat{\mathbf{u}}_m(\mathbf{k}^{(m)})] = \Pi_{n,\zeta}(\mathbf{k})\chi^{(m)}(\mathbf{k},\vec{k})\\
&\times [(\Pi_{n_1,\zeta'}(\mathbf{k}')\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\Pi_{n_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\hat{\mathbf{u}}_m(\mathbf{k}^{(m)}))].
\end{aligned} \tag{4.20}$$

Then using (3.5) and the *elementary susceptibilities* (4.20), we get

$$\begin{aligned}
&\chi^{(m)}(\mathbf{k},\vec{k})[\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\hat{\mathbf{u}}_m(\mathbf{k}^{(m)})]\\
&= \sum_{n,\zeta}\sum_{\vec{\xi}} \chi^{(m)}_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})[\hat{\mathbf{u}}_1(\mathbf{k}'),\dots,\hat{\mathbf{u}}_m(\mathbf{k}^{(m)})].
\end{aligned} \tag{4.21}$$

Consequently, the modal components $\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}$ of the operators $\mathcal{F}^{(m)}$ in (4.7) are m-linear *oscillatory integral operators* defined in terms of the elementary susceptibilities (4.21) as follows.

Definition 4.10 (interaction phase)**.** Using the notation from (3.9), we introduce for $\vec{\xi} = (\vec{n}, \vec{\zeta}) \in \Xi^m$ the operator

$$\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_m)(\mathbf{k},\tau) = \int_0^\tau \int_{\mathbb{D}_m} \exp\left\{ \mathrm{i}\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k}) \frac{\tau_1}{\varrho} \right\}$$
$$\times \chi^{(m)}_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k}) \left[\hat{\mathbf{u}}_1(\mathbf{k}',\tau_1), \ldots, \hat{\mathbf{u}}_m(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),\tau_1) \right] \tilde{\mathrm{d}}^{(m-1)d}\vec{k}\mathrm{d}\tau_1 \qquad (4.22)$$

with the *interaction phase function* φ defined by

$$\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k}) = \varphi_{n,\zeta,\vec{n},\vec{\zeta}}(\mathbf{k},\vec{k})$$
$$= \zeta\omega_n(\zeta\mathbf{k}) - \zeta'\omega_{n_1}(\zeta'\mathbf{k}') - \ldots - \zeta^{(m)}\omega_{n_m}(\zeta^{(m)}\mathbf{k}^{(m)}), \qquad (4.23)$$
$$\mathbf{k}^{(m)} = \mathbf{k}^{(m)}(\mathbf{k},\vec{k}),$$

where $\mathbf{k}^{(m)}(\mathbf{k},\vec{k})$ is defined by (3.10).

Using $\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}$ in (4.22), we recast $\mathcal{F}^{(m)}(\mathbf{u}^m)$ in the system (4.6)-(4.7) as follows:

$$\mathcal{F}^{(m)}\left[\hat{\mathbf{u}}_1 \ldots, \hat{\mathbf{u}}_m\right](\mathbf{k},\tau) = \sum_{n,\zeta,\vec{\xi}} \mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}\left[\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_m\right](\mathbf{k},\tau) \qquad (4.24)$$

yielding the following system for the modal components $\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau)$ as in (2.9):

$$\hat{\mathbf{u}}_{n,\zeta}(\mathbf{k},\tau) = \sum_{m\in\mathfrak{M}_F} \sum_{\vec{\xi}\in\Xi^m} \mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\hat{\mathbf{u}}^m)(\mathbf{k},\tau) + \hat{\mathbf{h}}_{n,\zeta}(\mathbf{k}),\ (n,\zeta)\in\Xi. \qquad (4.25)$$

5. Wavepacket Interaction System

The wavepacket preservation property of the nonlinear evolutionary system in any of its forms (1.1), (2.1), (4.6), (4.12), (4.25) is not easy to see directly. It turns out though that dynamics of wavepackets is well described by a system in a larger space E^{2N} based on the original equation (4.6) in the space E. We call it *wavepacket interaction system*, which is useful in three ways: (i) the wavepacket preservation is quite easy to see and verify; (ii) it can be used to prove the wavepacket preservation for the original nonlinear problem; (iii) it can be used to study more subtle properties of the original problem, such as the NLS approximation. We start with the system (4.6), where $\hat{\mathbf{h}}(\mathbf{k})$ is a multi-wavepacket with a given nk-spectrum $S = \{(\mathbf{k}_{*l}, n_l),\ l = 1, \ldots, N\}$ as in (2.39) and a k-spectrum $K_S = \{\mathbf{k}_{*i},\ i = 1, \ldots, |K_S|\}$ as in (2.40). Obviously, for any l $(\mathbf{k}_{*l}, n_l) = (\mathbf{k}_{*i_l}, n_l)$ with $i_l \leqslant |K_S|$ and indexing $i_l = l$ for $l \leqslant |K_S|$ according to (2.40).

When constructing the wavepacket interaction system it is convenient to have relevant functions to be explicitly localized about the k-spectrum K_S of the initial data. We implement that by making up the following cutoff functions based on (2.24), (2.25):

$$\begin{gathered}\Psi_{i,\vartheta}(\mathbf{k}) = \Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i}, \beta^{1-\varepsilon}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \vartheta\mathbf{k}_{*i})),\\ \mathbf{k}_{*i} \in K_S,\ i = 1, \ldots, |K_S|\,,\ \vartheta = \pm,\end{gathered} \tag{5.1}$$

with ε as in Definition 2.1 and $\beta > 0$ small enough to satisfy

$$\beta^{1/2} \leqslant \pi_0,\ \text{where } \pi_0 = \pi_0(S) = \frac{1}{2} \min_{\mathbf{k}_{*i} \in K_S} \operatorname{dist}\{\mathbf{k}_{*i}, \sigma_{\mathrm{bc}}\}\,. \tag{5.2}$$

In what follows, we use the notation from (3.16) and

$$\begin{gathered}\vec{l} = (l_1, \ldots, l_m) \in \{1, \ldots, N\}^m\,,\\ \vec{\vartheta} = (\vartheta', \ldots, \vartheta^{(m)}) \in \{-1, 1\}^m,\ \vec{\lambda} = (\vec{l}, \vec{\vartheta}) \in \Lambda^m,\end{gathered} \tag{5.3}$$

$$\vec{n} = (n_1, \ldots, n_m) \in \{1, \ldots, J\}^m,\ \vec{\zeta} \in \{-1, 1\}^m, \tag{5.4}$$

$$\vec{\xi} = (\vec{n}\,\vec{\zeta}) \in \Xi^m\ \vec{k} = (\mathbf{k}', \ldots, \mathbf{k}^{(m)}) \in \mathbb{R}^m,\ \text{where } \Xi^m \text{ as in (4.19).}$$

Based on the above, we introduce now the *wavepacket interaction system*

$$\begin{aligned}\hat{\mathbf{w}}_{l,\vartheta}(\cdot) &= \Psi(\cdot, \vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}(\cdot)\mathcal{F}\Big(\sum_{(l',\vartheta') \in \Lambda} \hat{\mathbf{w}}_{l',\vartheta'}\Big)\\ &\quad + \Psi(\cdot, \vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}(\cdot)\hat{\mathbf{h}},\ (l, \vartheta) \in \Lambda,\end{aligned} \tag{5.5}$$

$$\tilde{\mathbf{w}} = (\hat{\mathbf{w}}_{1,+}, \hat{\mathbf{w}}_{1,-}, \ldots, \hat{\mathbf{w}}_{N,+}, \hat{\mathbf{w}}_{N,-}) \in E^{2N},\ \hat{\mathbf{w}}_{l,\vartheta} \in E,$$

with $\Psi(\cdot, \vartheta\mathbf{k}_{*i}), \Pi_{n,\vartheta}$ being as in (5.1), (2.9), $\mathcal{F}$ defined by (4.6), and the norm in E^{2N} defined based on (2.15) by the formula

$$\|\tilde{\mathbf{w}}\|_{E^{2N}} = \sum_{l,\vartheta} \|\hat{\mathbf{w}}_{l,\vartheta}\|_E,\ E = C([0, \tau_*], L^1). \tag{5.6}$$

We also use the following concise form of the wave interaction system (5.5):

$$\tilde{\mathbf{w}} = \mathcal{F}_{\Psi}(\tilde{\mathbf{w}}) + \tilde{\mathbf{h}}_{\Psi},\ \text{where} \tag{5.7}$$

$$\tilde{\mathbf{h}}_{\Psi} = (\Psi_{i_1,+}\Pi_{n_1,+}\hat{\mathbf{h}}, \Psi_{i_1,-}\Pi_{n_1,-}\hat{\mathbf{h}}, \ldots, \Psi_{i_N,+}\Pi_{n_N,+}\hat{\mathbf{h}}, \Psi_{i_N,-}\Pi_{n_N,-}\hat{\mathbf{h}}) \in E^{2N}.$$

The following lemma is analogous to Lemmas 4.4 and 4.5.

Lemma 5.1. *The polynomial operator $\mathcal{F}_{\Psi}(\tilde{\mathbf{w}})$ is bounded in E^{2N}, $\mathcal{F}_{\Psi}(\mathbf{0}) = \mathbf{0}$, and satisfies Lipschitz condition*

$$\|\mathcal{F}_{\Psi}(\tilde{\mathbf{w}}_1) - \mathcal{F}_{\Psi}(\tilde{\mathbf{w}}_2)\|_{E^{2N}} \leqslant C\tau_* \|\tilde{\mathbf{w}}_1 - \tilde{\mathbf{w}}_2\|_{E^{2N}}, \tag{5.8}$$

where C depends only on C_χ as in (3.11)*, on the degree of $\mathcal{F}$, and on $\|\tilde{\mathbf{w}}_1\|_{E^{2N}} + \|\tilde{\mathbf{w}}_2\|_{E^{2N}}$, and it does not depend on β and ϱ.*

Proof. We consider any operator $\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\tilde{\mathbf{w}})$ defined by (4.22) and prove its boundedness and the Lipschitz property as in Lemma 4.4 using the inequality $\left|\exp\left\{\mathrm{i}\varphi_{n,\zeta,\vec{\xi}}\frac{\tau_1}{\varrho}\right\}\right| \leqslant 1$ and inequalities (2.24), (3.11). Note that the integration in τ_1 yields the factor τ_* and consequent summation with respect to $n, \zeta, \vec{\xi}$ yields (5.8). □

Lemma 5.1, the contraction principle as in Lemma 4.6, and the estimate (4.11) for the time derivative yield the following statement.

Theorem 5.2. *Let $\|\tilde{\mathbf{h}}_{\Psi}\|_{E^{2N}} \leqslant R$. Then there exists $\tau_* > 0$ and $R_1(R)$ such that Equation* (5.5) *has a solution $\tilde{\mathbf{w}} \in E^{2N}$ which satisfies*

$$\|\tilde{\mathbf{w}}\|_{E^{2N}} + \|\partial_\tau \tilde{\mathbf{w}}\|_{E^{2N}} \leqslant R_1(R), \tag{5.9}$$

and such a solution is unique.

Lemma 5.3. *Every function $\hat{\mathbf{w}}_{l,\zeta}(\mathbf{k},\tau)$ corresponding to the solution of* (5.7) *from E^{2N} is a wavepacket with nk-pair $(\mathbf{k}_{*l}, n_l)$ with the degree of regularity which can be any $s > 0$.*

Proof. Note that, according to (5.1) and (5.7), the function

$$\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau) = \Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i_l}, \beta^{1-\varepsilon})\Pi_{n_l,\vartheta}\mathcal{F}(\mathbf{k},\tau),\ \ \|\mathcal{F}(\tau)\|_{L^1} \leqslant C,\ \ 0 \leqslant \tau \leqslant \tau_*$$

involves the factor $\Psi_{l,\vartheta}(\mathbf{k}) = \Psi(\beta^{-(1-\varepsilon)}(\mathbf{k} - \vartheta\mathbf{k}_{*l}))$, where ε is as in Definition 2.1. Hence

$$\Pi_{n,\vartheta'}\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau) = 0 \text{ if } n \neq n_l \text{ or } \vartheta' \neq \vartheta, \tag{5.10}$$

$$\begin{gathered}\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau) = \Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i_l}, \beta^{1-\varepsilon})\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau), \\ \hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau) = 0 \text{ if } |\mathbf{k} - \vartheta\mathbf{k}_{*l}| \geqslant \beta^{1-\varepsilon},\end{gathered} \tag{5.11}$$

Since

$$\Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i_l}, \beta^{1-\varepsilon})\Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i_l}, \beta^{1-\varepsilon}/2) = \Psi(\mathbf{k}, \vartheta\mathbf{k}_{*i_l}, \beta^{1-\varepsilon}), \tag{5.12}$$

Definition 2.1 for $\hat{\mathbf{w}}_{l,\vartheta}$ is satisfied with $D_h = 0$ for any $s > 0$ and $C' = 0$ in (2.30). □

Now we would like to show that if $\hat{\mathbf{h}}$ is a multi-wavepacket, then the function

$$\hat{\mathbf{w}}(\mathbf{k},\tau) = \sum_{(l,\vartheta)\in\Lambda} \hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau) = \sum_{\lambda\in\Lambda} \hat{\mathbf{w}}_{\lambda}(\mathbf{k},\tau) \tag{5.13}$$

constructed from a solution of (5.7) is an approximate solution of Equation (4.12) (see the notation (3.16)). We follow here the lines of [**7**]. We introduce

$$\Psi_\infty(\mathbf{k}) = 1 - \sum_{\vartheta=\pm}\sum_{i=1}^{|K_S|} \Psi(\mathbf{k},\vartheta\mathbf{k}_{*i}) = 1 - \sum_{\vartheta=\pm}\sum_{\mathbf{k}_{*i}\in K_S} \Psi\Big(\frac{\mathbf{k}-\vartheta\mathbf{k}_{*i}}{\beta^{1-\varepsilon}}\Big). \tag{5.14}$$

Expanding the m-linear operator $\mathcal{F}^{(m)}\Big(\Big(\sum_{l,\vartheta}\hat{\mathbf{w}}_{l,\vartheta}\Big)^m\Big)$ and using the notation (3.16), (3.17), we get

$$\mathcal{F}^{(m)}\Big(\Big(\sum_{l,\vartheta}\hat{\mathbf{w}}_{l,\vartheta}\Big)^m\Big) = \sum_{\vec{\lambda}\in\Lambda^m} \mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \text{ where} \tag{5.15}$$

$$\tilde{\mathbf{w}}_{\vec{\lambda}} = \hat{\mathbf{w}}_{\lambda_1}\dots\hat{\mathbf{w}}_{\lambda_m}, \ \vec{\lambda} = (\lambda_1,\dots,\lambda_m)\in\Lambda^m. \tag{5.16}$$

The next statement shows that (5.13) defines an approximate solution to the integrated evolution equation (4.6).

Theorem 5.4. *Let* $\hat{\mathbf{h}}$ *be a multi-wavepacket with resonance invariant nk-spectrum* S *and regularity degree* s*, let* $\tilde{\mathbf{w}}$ *be a solution of* (5.7)*, and let* $\hat{\mathbf{w}}(\mathbf{k},\tau)$ *be defined by* (5.13)*. Let*

$$\mathbf{D}(\hat{\mathbf{w}}) = \hat{\mathbf{w}} - \mathcal{F}(\hat{\mathbf{w}}) - \hat{\mathbf{h}}. \tag{5.17}$$

Then there exists $\beta_0 > 0$ *such that*

$$\|\mathbf{D}(\hat{\mathbf{w}})\|_E \leqslant C\varrho + C\beta^s, \ \text{if } 0<\varrho\leqslant 1, \ \beta\leqslant\beta_0. \tag{5.18}$$

Proof. Let

$$\begin{aligned}\mathcal{F}^-(\hat{\mathbf{w}}) &= \Big(1-\sum_{l,\vartheta}\Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\Big)\mathcal{F}(\hat{\mathbf{w}}),\\ \hat{\mathbf{h}}^- &= \hat{\mathbf{h}} - \sum_{l,\vartheta}\Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\hat{\mathbf{h}}.\end{aligned} \tag{5.19}$$

Summation of (5.5) with respect to l,ϑ yields

$$\hat{\mathbf{w}} = \sum_{l,\vartheta}\Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\mathcal{F}(\hat{\mathbf{w}}) + \sum_{l,\vartheta}\Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\hat{\mathbf{h}}.$$

Hence from (5.5) and (5.17) we obtain

$$\mathbf{D}(\hat{\mathbf{w}}) = \hat{\mathbf{h}}^- - \mathcal{F}^-(\hat{\mathbf{w}}). \tag{5.20}$$

Using (2.28) and (2.30), we consequently obtain

$$\begin{gathered} \|\Pi_{n_l,\vartheta}\hat{\mathbf{h}}_i\|_{L^1} \leqslant C\beta^s \text{ if } n_l \neq n_i; \\ \|\Psi_{i_l,\vartheta}\hat{\mathbf{h}}_i\|_{L^1} \leqslant C\beta^s \text{ if } \mathbf{k}_{*i_l} \neq \mathbf{k}_{*i}, \\ \|\hat{\mathbf{h}}^-\|_E \leqslant C_1\beta^s. \end{gathered} \tag{5.21}$$

To show (5.18), it suffices to prove that

$$\|\mathcal{F}^-(\hat{\mathbf{w}})\|_E \leqslant C_2\varrho. \tag{5.22}$$

Obviously,

$$\mathcal{F}^-(\hat{\mathbf{w}}) = \Big(1 - \sum_{l,\vartheta} \Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\Big) \sum_m \mathcal{F}^{(m)}(\hat{\mathbf{w}}^m). \tag{5.23}$$

Note that

$$\sum_{l,\vartheta} \Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta} = \sum_{\vartheta=\pm} \sum_{(n,k_*)\in S} \Psi(\cdot,\vartheta\mathbf{k}_*)\Pi_{n,\vartheta}. \tag{5.24}$$

Using (3.5) and (5.14), we consequently obtain

$$\sum_{\vartheta=\pm} \sum_{(n,k_*)\in\Sigma} \Psi(\cdot,\vartheta\mathbf{k}_*)\Pi_{n,\vartheta} + \Psi_\infty = 1, \tag{5.25}$$

$$\Big(1 - \sum_{l,\vartheta} \Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\Big) = \Psi_\infty + \sum_{\vartheta=\pm} \sum_{(n,k_*)\in\Sigma\setminus S} \Psi(\cdot,\vartheta\mathbf{k}_*)\Pi_{n,\vartheta} \tag{5.26}$$

with the set Σ defined in (3.14). Let us expand now $\mathcal{F}^{(m)}(\hat{\mathbf{w}}^m)$ using (5.15). According to (5.23) and (5.26), to prove (5.22) it suffices to prove that for every string $\vec{\lambda} \in \Lambda^m$ the following inequalities hold:

$$\|\Psi_\infty\Pi_{n,\vartheta}\mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})\| \leqslant C_3\varrho \ \text{ for } (n,\vartheta) \in \Lambda, \tag{5.27}$$

$$\|\Psi(\cdot,\vartheta\mathbf{k}_*)\Pi_{n,\vartheta}\mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})\| \leqslant C_3\varrho \ \text{ if } (n,\mathbf{k}_*) \in \Sigma \setminus S. \tag{5.28}$$

We use (5.10) and (5.11) to obtain the above estimates. According to (4.24),

$$\mathcal{F}^{(m)}\left[\tilde{\mathbf{w}}_{\vec{\lambda}}\right](\mathbf{k},\tau) = \sum_{n,\zeta} \sum_{\vec{\xi}} \mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}[\hat{\mathbf{w}}_{\lambda_1} \ldots \hat{\mathbf{w}}_{\lambda_m}](\mathbf{k},\tau). \tag{5.29}$$

Note that, according to (5.10), if $\lambda_i = (l,\vartheta')$, then

$$\hat{\mathbf{w}}_{\lambda_i} = \Pi_{n,\vartheta}\hat{\mathbf{w}}_{\lambda_i} \text{ if } n = n_l \text{ and } \vartheta' = \vartheta. \tag{5.30}$$

Let us introduce the notation

$$\vec{n}(\vec{l}) = (n_{l_1},\ldots,n_{l_m}),\ \vec{\xi}(\vec{\lambda}) = (\vec{n}(\vec{l}),\vec{\vartheta}) \text{ for } \vec{\lambda} = (\vec{l},\vec{\vartheta}) \in \Lambda^m. \tag{5.31}$$

Since

$$\Pi_{n',\vartheta}\Pi_{n,\vartheta'} = 0 \text{ if } n \neq n' \text{ or } \vartheta' \neq \vartheta, \tag{5.32}$$

(5.30) implies

$$\begin{gathered}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}[\hat{\mathbf{w}}_{\lambda_1}\dots\hat{\mathbf{w}}_{\lambda_m}]=0 \text{ if } \vec{\xi}=(\vec{n},\vec{\zeta})\neq\vec{\xi}(\vec{\lambda}), \text{ and hence}\\ \mathcal{F}^{(m)}[\tilde{\mathbf{w}}_{\vec{\lambda}}](\mathbf{k},\tau)=\sum_{n,\zeta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}(\vec{\lambda})}[\hat{\mathbf{w}}_{\lambda_1}\dots\hat{\mathbf{w}}_{\lambda_m}](\mathbf{k},\tau),\end{gathered} \tag{5.33}$$

where we used the notation (3.17), (5.31). Note also that

$$\Pi_{n',\vartheta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}=0 \text{ if } n'\neq n \text{ or } \vartheta\neq\zeta, \tag{5.34}$$

and hence we have nonzero $\Pi_{n',\vartheta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ only if

$$\vec{\xi}=\vec{\xi}(\vec{\lambda}),\ n'=n,\ \vartheta=\zeta. \tag{5.35}$$

By (4.22),

$$\begin{aligned}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau)&=\int_0^\tau\int_{\mathbb{D}_m}\exp\left\{\mathrm{i}\varphi_{n,\zeta,\vec{\xi}(\vec{\lambda})}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}\right\}\chi^{(m)}_{n,\zeta,\vec{\xi}(\vec{\lambda})}(\mathbf{k},\vec{k})\\ &\times[\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}',\tau_1),\dots,\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),\tau_1)]\tilde{\mathrm{d}}^{(m-1)d}\vec{k}\mathrm{d}\tau_1.\end{aligned} \tag{5.36}$$

Now we use (5.11) and notice that, according to the convolution identity in (3.9),

$$\begin{gathered}|\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}',\tau_1)|\cdot\ldots\cdot|\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),\tau_1)|=0\\ \text{if }\left|\mathbf{k}-\sum_i\vartheta_i\mathbf{k}_{*l_i}\right|\geqslant m\beta^{1-\varepsilon}.\end{gathered} \tag{5.37}$$

Hence the integral (5.36) is nonzero only if $(\mathbf{k},\vec{k})$ belongs to the set

$$\begin{aligned}B_\beta=\Big\{(\mathbf{k},\vec{k}):|\mathbf{k}^{(i)}-\vartheta_i\mathbf{k}_{*l_i}|\leqslant\beta^{1-\varepsilon},\ i=1,\dots,m,\\ \left|\mathbf{k}-\sum_i\vartheta_i\mathbf{k}_{*l_i}\right|\leqslant m\beta^{1-\varepsilon}\Big\}.\end{aligned} \tag{5.38}$$

We prove now that if $(n,\mathbf{k}_{*i})\notin S$, then for small β the following alternative holds:

$$\text{either } \Psi(\cdot,\vartheta\mathbf{k}_{*i})\Pi_{n',\vartheta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\tilde{\mathbf{w}}_{\vec{\lambda}})=0 \tag{5.39}$$

$$\text{or (5.35) holds and } |\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})|\geqslant c>0 \text{ for } (\mathbf{k},\vec{k})\in B_\beta. \tag{5.40}$$

Since $\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})$ is smooth, in the notation (3.18) we get

$$\begin{gathered}|\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})-\varphi_{n',\zeta,\vec{\xi}}(\mathbf{k}_{**},\vec{k}_*)|\leqslant C\beta^{1-\varepsilon} \text{ for } (\mathbf{k},\vec{k})\in B_\beta,\\ \vec{\vartheta}=(\vartheta_1,\dots,\vartheta_m),\ \mathbf{k}_{**}=\zeta\sum_i\vartheta_i\mathbf{k}_{*l_i}=\zeta\varkappa_m(\vec{\vartheta},\vec{l}),\end{gathered} \tag{5.41}$$

Hence (5.40) holds if

$$\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}_{**},\vec{k}_*)\neq 0, \tag{5.42}$$

and, consequently, it suffices to prove that either (5.39) or (5.42) holds. Combining (5.38) with $\Psi(\mathbf{k},\vartheta\mathbf{k}_{*i})=0$ for $|\mathbf{k}-\vartheta\mathbf{k}_{*i}|\geqslant\beta^{1-\varepsilon}$, we find that $\Psi_{i,\vartheta}\mathcal{F}^{(m)}\left[\tilde{\mathbf{w}}_{\vec{\lambda}}\right]$ can be nonzero for small β only in a small neighborhood of a point $\zeta\varkappa_m(\vec{\vartheta},\vec{l})\in[S]_{K,\mathrm{out}}$, and that is possible only if

$$\mathbf{k}_{**}=\zeta\varkappa_m(\vec{\vartheta},\vec{l})=\vartheta\mathbf{k}_{*i},\ \mathbf{k}_{*i}\in K_S. \tag{5.43}$$

Let us show that the equality

$$\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k}_{**},\vec{k}_*)=0 \tag{5.44}$$

is impossible for $\mathbf{k}_{**}$ as in (5.43) and $n'=n$ as in (5.34), keeping in mind that $(n,\mathbf{k}_{*i})\notin S$. From (3.23) and (4.23) it follows that Equation (5.44) has the form of the resonance equation (3.24). Since the nk-spectrum S is resonance invariant, in view of Definition 3.8 the resonance equation (5.44) may have a solution only if $\mathbf{k}_{**}=\mathbf{k}_{*i}$, $i=i_l$, $n=n_l$, with $(n_l,\mathbf{k}_{*i_l})\in S$. Since $(n,\mathbf{k}_{*i})\notin S$, that implies (5.44) does not have a solution. Hence (5.42) holds when $(n,\mathbf{k}_{*i})\notin S$. Notice that (5.9) yields the following bounds

$$\|\hat{\mathbf{w}}_{\lambda_i}\|_E\leqslant R_1,\ \|\partial_\tau\hat{\mathbf{w}}_{\lambda_i}\|_E\leqslant C. \tag{5.45}$$

These bounds combined with Lemma 5.5, proved below, imply that if (5.42) holds, then (5.28) holds. Now let us turn to (5.27). According to (5.14) and (5.37), the term $\Psi_\infty\Pi_{n',\vartheta}\mathcal{F}^{(m)}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ can be nonzero only if $\zeta\varkappa_m(\vec{\lambda})=\mathbf{k}_{**}\notin K_S$. Since the nk-spectrum S is resonance invariant we conclude as above that the inequality (5.42) holds in this case as well. The fact that the set of all $\varkappa_m(\vec{\lambda})$ is finite, combined with the inequality (5.42), imply (5.40) for sufficiently small β. Using Lemma 5.5, as above we derive (5.27). Hence all terms in the expansion (5.23) are either zero or satisfy (5.27) or (5.28) implying consequently (5.22) and (5.18). □

Here is the lemma used in the above proof.

Lemma 5.5. *Let assume that*

$$\begin{gathered}|\Psi_{i,\vartheta'}\Pi_{n',\zeta}\chi^{(m)}_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})[\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}',\tau_1),\dots,\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),\tau_1)]|=0\\ \text{for } (\mathbf{k},\vec{k})\in B_\beta \text{ and}\\ |\varphi_{n,\zeta,\vec{\xi}}(\mathbf{k},\vec{k})|\geqslant\omega_*>0 \text{ for } (\mathbf{k},\vec{k})\notin B_\beta,\ \text{where } B_\beta \text{ as in } (5.38).\end{gathered} \tag{5.46}$$

Then

$$\|\Psi(\cdot,\vartheta'\mathbf{k}_{*i})\Pi_{n',\zeta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\tilde{\mathbf{w}}_{\vec{\lambda}})\|_E \leqslant \frac{4\varrho}{\omega_*}\|\chi^{(m)}\|\prod_j\|\hat{\mathbf{w}}_{\lambda_j}\|_E$$
$$+\frac{2\varrho\tau_*}{\omega_*}\|\chi^{(m)}\|\sum_i\|\partial_\tau\hat{\mathbf{w}}_{\lambda_i}\|_E\prod_{j\neq i}\|\hat{\mathbf{w}}_{\lambda_j}\|_E. \tag{5.47}$$

PROOF. Notice that the oscillatory factor in (4.22) is equal to

$$\exp\left\{\mathrm{i}\varphi(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}\right\}=\frac{\varrho}{\mathrm{i}\varphi(\mathbf{k},\vec{k})}\partial_{\tau_1}\exp\left\{\mathrm{i}\varphi(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}\right\}.$$

Denoting $\varphi_{n,\zeta,\vec{\xi}}=\varphi$, $\Psi_{i,\vartheta'}\Pi_{n',\zeta}\chi^{(m)}_{n,\zeta,\vec{\xi}}=\chi^{(m)}_{\vec{\eta}}$ and integrating (4.22) by parts with respect to τ_1, we obtain

$$\Psi(\mathbf{k},\vartheta'\mathbf{k}_{*i})\Pi_{n',\zeta}\mathcal{F}^{(m)}_{n,\zeta,\vec{\xi}}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau)$$
$$=\int_B\Psi(\mathbf{k},\vartheta'\mathbf{k}_{*i})\frac{\varrho e^{\mathrm{i}\varphi(\mathbf{k},\vec{k})\frac{\tau}{\varrho}}}{\mathrm{i}\varphi(\mathbf{k},\vec{k})}\chi^{(m)}_{\vec{\eta}}(\mathbf{k},\vec{k})$$
$$\times\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}',\tau)\ldots\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),\tau)\,\tilde{\mathrm{d}}^{(m-1)d}\vec{k}$$
$$-\int_B\Psi(\mathbf{k},\vartheta'\mathbf{k}_{*i})\frac{\varrho}{\mathrm{i}\varphi(\mathbf{k},\vec{k})}\chi^{(m)}_{\vec{\eta}}(\mathbf{k},\vec{k})$$
$$\times\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}',0)\ldots\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}),0)\,\tilde{\mathrm{d}}^{(m-1)d}\vec{k}$$
$$-\int_0^\tau\int_B\Psi(\mathbf{k},\vartheta'\mathbf{k}_{*i})\frac{\varrho e^{\mathrm{i}\varphi(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}}}{\mathrm{i}\varphi(\mathbf{k},\vec{k})}\chi^{(m)}_{\vec{\eta}}(\mathbf{k},\vec{k})$$
$$\times\partial_{\tau_1}[\hat{\mathbf{w}}_{\lambda_1}(\mathbf{k}')\ldots\hat{\mathbf{w}}_{\lambda_m}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))]\,\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1, \tag{5.48}$$

where B is the set of $\mathbf{k}^{(i)}$ for which (5.38) holds. The relations (3.11) and (2.24) imply $|\chi^{(m)}_{\vec{\eta}}(\mathbf{k},\vec{k})|\leqslant\|\chi^{(m)}\|$. Using then (5.46), the Leibnitz formula, (5.9) and (4.8), we obtain (5.47). □

The main result of this subsection is the next theorem which, combined with Lemma 5.3, implies the wavepacket preservation, namely that the solution $\hat{\mathbf{u}}_{n,\vartheta}(\mathbf{k},\tau)$ of (4.25) is a multi-wavepacket for all $\tau\in[0,\tau_*]$.

Theorem 5.6. *Assume that the conditions of Theorem* 5.4 *are fulfilled. Let* $\hat{\mathbf{u}}_{n,\vartheta}(\mathbf{k},\tau)$ *for* $n=n_l$*, let* $\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k},\tau)$ *be solutions to the respective* *systems* (4.25) *and* (5.5), *and let* $\hat{\mathbf{w}}$ *be defined by* (5.13). *Then for sufficiently small* $\beta_0 > 0$

$$\|\hat{\mathbf{u}}_{n_l,\vartheta} - \Pi_{n_l,\vartheta}\hat{\mathbf{w}}\|_E \leqslant C\varrho + C'\beta^s,\ 0 < \beta \leqslant \beta_0,\ l = 1, \ldots, N. \tag{5.49}$$

PROOF. Note that $\hat{\mathbf{u}}_{n,\vartheta} = \Pi_{n,\vartheta}\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is a solution of (4.6) and, according to Theorem 4.7, $\|\hat{\mathbf{u}}\|_E \leqslant 2R$. Comparing Equations (4.6) and (5.17), which are $\hat{\mathbf{u}} = \mathcal{F}(\hat{\mathbf{u}}) + \hat{\mathbf{h}}$ and $\hat{\mathbf{w}} = \mathcal{F}(\hat{\mathbf{w}}) + \hat{\mathbf{h}} + \mathbf{D}(\hat{\mathbf{w}})$, we find that Lemma 4.6 can be applied. Then we notice that, by Lemma 4.5, $\mathcal{F}$ has the Lipschitz constant $C_F\tau_*$ for such $\hat{\mathbf{u}}$. Taking $C_F\tau_* < 1$ as in Theorem 4.7, we obtain (5.49) from (4.17). □

Notice that Theorem 2.9 is a direct corollary of Theorem 5.6 and Lemma 5.3.

An analogous assertion is proved in [**7**] for parameter-dependent equations of the form (2.1) with $\hat{\mathbf{F}}(\hat{\mathbf{U}}) = \hat{\mathbf{F}}(\hat{\mathbf{U}}, \varrho)$.

The following theorem shows that any multi-wavepacket solution to (4.6) yields a solution to the wavepacket interaction system (5.5).

Theorem 5.7. *Let* $\hat{\mathbf{u}}(\mathbf{k},\tau)$ *be a solution of* (4.6). *Assume that* $\hat{\mathbf{u}}(\mathbf{k},\tau)$ *and* $\hat{\mathbf{h}}(\mathbf{k})$ *are multi-wavepackets with nk-spectrum* $S = \{(n_l, \mathbf{k}_{*l}),\ l = 1, \ldots, N\}$ *and regularity degree s. Let also* $\Psi_{i_l,\vartheta} = \Psi_{i_l,\vartheta}$ *be defined by* (5.1). *Then* $\hat{\mathbf{w}}'_{l,\vartheta}(\mathbf{k},\tau) = \Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\hat{\mathbf{u}}(\mathbf{k},\tau)$ *is a solution to the system* (5.5) *with* $\hat{\mathbf{h}}(\mathbf{k})$ *replaced by* $\hat{\mathbf{h}}'(\mathbf{k},\tau)$ *satisfying*

$$\|\hat{\mathbf{h}}(\mathbf{k}) - \hat{\mathbf{h}}'(\mathbf{k},\tau)\|_{L^1} \leqslant C\beta^s,\ 0 \leqslant \tau \leqslant \tau_*, \tag{5.50}$$

and, if $\hat{\mathbf{w}}_{l,\vartheta}$ *are solutions of* (5.5) *with original* $\hat{\mathbf{h}}(\mathbf{k})$, *then*

$$\|\hat{\mathbf{w}}'_{l,\vartheta}(\mathbf{k},\tau) - \hat{\mathbf{w}}_{l,\vartheta}\|_{L^1} \leqslant C\beta^s,\ 0 \leqslant \tau \leqslant \tau_*. \tag{5.51}$$

PROOF. Multiplying (4.6) by $\Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}$, we get

$$\begin{gathered}\hat{\mathbf{w}}'_{l,\vartheta} = \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\mathcal{F}(\hat{\mathbf{u}})(\mathbf{k},\tau) + \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\hat{\mathbf{h}}(\mathbf{k}),\\ \hat{\mathbf{w}}'_{l,\vartheta} = \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\hat{\mathbf{u}}.\end{gathered} \tag{5.52}$$

Since $\hat{\mathbf{u}}(\mathbf{k},\tau)$ is a multi-wavepacket with regularity s, we have

$$\begin{gathered}\|\hat{\mathbf{u}}(\cdot,\tau) - \hat{\mathbf{w}}'(\cdot,\tau)\|_{L^1} \leqslant C_\varepsilon\beta^s,\\ \text{where}\ \ \hat{\mathbf{w}}'(\cdot,\tau) = \sum_{l,\vartheta}\Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\hat{\mathbf{u}}(\cdot,\tau).\end{gathered} \tag{5.53}$$

Let us recast (5.52) in the form

$$\begin{aligned}\hat{\mathbf{w}}'_{l,\vartheta} &= \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\mathcal{F}(\hat{\mathbf{w}}')(\mathbf{k},\tau)\\ &\quad + \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}[\hat{\mathbf{h}}(\mathbf{k})+\hat{\mathbf{h}}''(\mathbf{k},\tau)],\\ \hat{\mathbf{h}}''(\mathbf{k},\tau) &= [\mathcal{F}(\hat{\mathbf{u}})-\mathcal{F}(\hat{\mathbf{w}}')](\mathbf{k},\tau).\end{aligned} \tag{5.54}$$

Denoting $\hat{\mathbf{h}}(\mathbf{k})+\hat{\mathbf{h}}''(\mathbf{k},\tau) = \hat{\mathbf{h}}'(\mathbf{k},\tau)$, we observe that (5.54) has the form of (5.5) with $\hat{\mathbf{h}}(\mathbf{k})$ replaced by $\hat{\mathbf{h}}'(\mathbf{k},\tau)$. The inequality (5.50) follows then from (5.53) and (4.13). Using Lemma 4.6, we obtain (5.51). □

6. Reduction of Wavepacket Interaction System to an Averaged Interaction System

Our goal in this section is to substitute the wavepacket interaction system (5.5) with a simpler *averaged interaction system* which describes the evolution of wavepackets with the same accuracy, but has a simpler nonlinearity, and we follow here the approach developed in [**7**]. The reduction is a generalization of the classical averaging principle to the case of continuous spectrum, see [**7**] for a discussion and further simplification of the averaged interaction system. In the present paper, we do not need the further simplification to a minimal interaction system leading to a system of NLS-type equations which is done in [**7**].

6.1. Time averaged wavepacket interaction system.

Here we modify the wavepacket interaction system (5.5), substituting its nonlinearity with another one obtained by the time averaging, and prove that this substitution produces a small error of order ϱ. As the first step, we recast (5.5) in a slightly different form by using the expansions (5.15), (5.29) together with (5.33) and (5.34) and writing the nonlinearity in Equation (5.5) in the form

$$\begin{aligned}&\Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\mathcal{F}(\cdot,\tau)\\ &\qquad = \sum_{m\in\mathfrak{M}_F}\sum_{\vec{\lambda}\in\Lambda^m}\Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}),\quad \vec{\lambda}=(\vec{l},\vec{\zeta}),\end{aligned} \tag{6.1}$$

$$\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau) = \mathcal{F}^{(m)}_{n,\zeta,\vec{n},\vec{\zeta}}[\hat{\mathbf{w}}_{\lambda_1}\ldots\hat{\mathbf{w}}_{\lambda_m}](\mathbf{k},\tau)\big|_{\vec{n}=\vec{n}(\vec{l}),\ (n,\zeta)=(n_l,\vartheta)}, \tag{6.2}$$

with $\mathcal{F}^{(m)}_{n,\zeta,\vec{n},\vec{\zeta}}$ as in (4.22) and $\vec{n}(\vec{l})$ as in (5.31), and we call $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ a *decorated monomial* $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}$ evaluated at $\tilde{\mathbf{w}}_{\vec{\lambda}}$. Consequently, the wavepacket interaction system (5.5) can be written in the equivalent form

$$\hat{\mathbf{w}}_{l,\vartheta} = \sum_{m\in\mathfrak{M}_F}\sum_{\vec{\lambda}\in\Lambda^m}\Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}) + \Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\hat{\mathbf{h}}, \quad l = 1,\ldots N,\ \vartheta = \pm. \tag{6.3}$$

The construction of the above-mentioned time averaged equation reduces to discarding certain terms in the original system (6.3). First we introduce the following sets of indices related to the resonance equation (3.24) and Ω_m defined by (3.23):

$$\Lambda^m_{n_l,\vartheta} = \left\{\vec{\lambda} = (\vec{l},\vec{\zeta}) \in \Lambda^m : \Omega_m(\vartheta, n_l, \vec{\lambda}) = 0\right\}, \tag{6.4}$$

and then the *time-averaged nonlinearity* $\mathcal{F}_{\mathrm{av}}$ *by*

$$\mathcal{F}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{w}}) = \sum_{m\in\mathfrak{M}_F}\mathcal{F}^{(m)}_{n_l,\vartheta},\ \mathcal{F}^{(m)}_{n_l,\vartheta} = \sum_{\vec{\lambda}\in\Lambda^m_{n_l,\vartheta}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \tag{6.5}$$

where $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}$ are defined in (6.2).

Remark 6.1. Note that the nonlinearity $\mathcal{F}^{(m)}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{w}})$ can be obtained from $\mathcal{F}^{(m)}_{n_l,\vartheta}$ by an averaging formula using an *averaging operator* A_T acting on polynomial functions $F : (\mathbb{C}^2)^N \to (\mathbb{C}^2)^N$ as follows:

$$(A_T F)_{j,\zeta} = \frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{i}\zeta\varphi_j t} \times F_{j,\zeta}(\mathrm{e}^{\mathrm{i}\varphi_1 t}u_{1,+}, \mathrm{e}^{-\mathrm{i}\varphi_1 t}u_{1,-},\ldots,\mathrm{e}^{\mathrm{i}\varphi_N t}u_{N,+}, \mathrm{e}^{-\mathrm{i}\varphi_N t}u_{N,-})\mathrm{d}t. \tag{6.6}$$

Using this averaging, we define for any polynomial nonlinearity $G : (\mathbb{C}^2)^N \to (\mathbb{C}^2)^N$ the averaged polynomial

$$G_{\mathrm{av},j,\zeta}(\vec{u}) = \lim_{T\to\infty}(A_T G)_{j,\zeta}(\vec{u}). \tag{6.7}$$

If the frequencies φ_j in (6.6) are generic, $G_{\mathrm{av},j,\zeta}(\vec{u})$ is always a universal nonlinearity. Note that $\mathcal{F}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{w}})$ defined by (6.5) can be obtained by formula (6.7), where A_T is defined by formula (6.6) with frequencies $\varphi_j = \omega_{n_j}(\mathbf{k}_{*i_j})$ (it may be conditionally universal if the frequencies φ_j are subjected to a condition of the form (6.22), see the following subsection for details, in particular for definitions of universal and conditionally universal nonlinearities).

Finally, we introduce the wave interaction system with time-averaged nonlinearity as follows:

$$\hat{\mathbf{v}}_{l,\vartheta} = \Psi(\cdot, \vartheta\mathbf{k}_{*i_l})\mathcal{F}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{v}}) + \Psi(\cdot, \vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\hat{\mathbf{h}},\ l = 1, \ldots N, \vartheta = \pm. \tag{6.8}$$

Similarly to (5.7) we recast this system concisely as

$$\tilde{\mathbf{v}} = \mathcal{F}_{\mathrm{av},\Psi}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \tag{6.9}$$

The following lemma is analogous to Lemmas 5.1, 4.5.

Lemma 6.2. *The operator $\mathcal{F}_{\mathrm{av},\Psi}(\tilde{\mathbf{v}})$ is bounded for bounded $\tilde{\mathbf{v}} \in E^{2N}$, $\mathcal{F}_{\mathrm{av},\Psi}(\mathbf{0}) = \mathbf{0}$. The polynomial operator $\mathcal{F}_{\mathrm{av},\Psi}(\tilde{\mathbf{v}})$ satisfies the Lipschitz condition*

$$\|\mathcal{F}_{\mathrm{av},\Psi}(\tilde{\mathbf{v}}_1) - \mathcal{F}_{\mathrm{av},\Psi}(\tilde{\mathbf{v}}_2)\|_{E^{2N}} \leqslant C\tau_* \|\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2\|_{E^{2N}} \tag{6.10}$$

where C depends only on C_χ in (3.11), on the power of $\mathcal{F}$ and on $\|\tilde{\mathbf{v}}_1\|_{E^{2N}} + \|\tilde{\mathbf{v}}_2\|_{E^{2N}}$, and, in particular, it does not depend on β, ϱ.

From Lemma 6.2 and the contraction principle we obtain the following theorem similar to Theorem 5.2.

Theorem 6.3. *Let $\|\tilde{\mathbf{h}}_\Psi\|_{E^{2N}} \leqslant R$. Then there exists $R_1 > 0$ and $\tau_* > 0$ such that Equation (6.9) has a solution $\tilde{\mathbf{v}} \in E^{2N}$ satisfying $\|\tilde{\mathbf{v}}\|_{E^{2N}} \leqslant R_1$, and such a solution is unique.*

The following theorem shows that the averaged interaction system introduced above provides a good approximation for the wave interaction system.

Theorem 6.4. *Let $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ be the solution of (6.8), and let $\hat{\mathbf{w}}_{l,\vartheta}(\mathbf{k}, \tau)$ be the solution of (5.5). Then for sufficiently small β $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ is a wavepacket satisfying (5.10), (5.11) with $\hat{\mathbf{w}}$ replaced by $\hat{\mathbf{v}}$. In addition to that, there exists $\beta_0 > 0$ such that*

$$\begin{gathered} \|\hat{\mathbf{v}}_{l,\vartheta} - \hat{\mathbf{w}}_{l,\vartheta}\|_E \leqslant C\varrho,\ l = 1, \ldots, N,\ \vartheta = \pm, \\ \textit{for } 0 < \varrho \leqslant 1,\ 0 < \beta \leqslant \beta_0. \end{gathered} \tag{6.11}$$

Proof. Formulas (5.10) and (5.11) for $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ follow from (6.8). We note that $\tilde{\mathbf{w}}$ is an approximate solution of (6.8), namely we have an estimate for $\mathbf{D}_{\mathrm{av}}(\hat{\mathbf{w}}) = \hat{\mathbf{w}} - \mathcal{F}_{\mathrm{av},\Psi} - \hat{\mathbf{h}}_\Psi$ which is similar to (5.17), (5.18):

$$\|\mathbf{D}_{\mathrm{av}}(\hat{\mathbf{w}})\| = \|\hat{\mathbf{w}} - \mathcal{F}_{\mathrm{av},\Psi} - \hat{\mathbf{h}}\|_{E^{2N}} \leqslant C\varrho \ \text{ if } 0 < \varrho \leqslant 1, \beta \leqslant \beta_0. \tag{6.12}$$

The proof of (6.12) is similar to the proof of (5.22) with minor simplifications thanks to the absence of terms with Ψ_∞. Using (6.12), we apply Lemma 4.6 and obtain (6.11). □

6.1.1. Properties of averaged nonlinearities. In this section, we discuss elementary properties of nonlinearities obtained by formula (6.5). A key property of such nonlinearities $F_{j,\zeta}$ is the following homogeneity-like property:

$$\begin{aligned} &F_{j,\zeta}(\mathrm{e}^{\mathrm{i}\varphi_1 t}u_{1,+},\mathrm{e}^{-\mathrm{i}\varphi_1 t}u_{1,-},\ldots,\mathrm{e}^{\mathrm{i}\varphi_N t}u_{N,+},\mathrm{e}^{-\mathrm{i}\varphi_N t}u_{N,-}) \\ &\qquad = \mathrm{e}^{\mathrm{i}\zeta\varphi_j t}F_{j,\zeta}(u_{1+},u_{1-},\ldots,u_{N+},u_{N-}). \end{aligned} \tag{6.13}$$

The values of φ_i, $i = 1,\ldots N$, for which this formula holds depend on the resonance properties of the set S which enters (6.5) through the index set $\Lambda^m_{n_l,\vartheta}$. First, let us consider the simplest case, where φ_i are arbitrary. An example of such a nonlinearity is the function

$$F_{2,\zeta}(u_{1,+},u_{1,-},u_{2,+},u_{2,-}) = u_{1,+}u_{1,-}u_{2,+}.$$

We call a nonlinearity which is obtained by formula (6.5) with a universal resonance invariant set S a *universal nonlinearity.*

Proposition 6.5. *If $F_{j,\zeta}$ is a universal nonlinearity, then* (6.13) *holds for arbitrary set of values φ_i, $i = 1,\ldots,N$.*

PROOF. Note that the definition (6.5) of the averaged nonlinearity essentially is based on the selection of vectors $\vec{\lambda} = ((\zeta',l'),\ldots,(\zeta^{(m)},l_m)) \in \Lambda^m_{n_l,\vartheta}$ as in (6.4), which is equivalent to the resonance equation (3.24) with $n = n_l$, $\zeta = \vartheta$. This equation has the form

$$-\zeta\omega_n(\mathbf{k}_{**}) + \sum_{l=1}^{N}\delta_l\omega_l(\mathbf{k}_{*l}) = 0 \tag{6.14}$$

with

$$\mathbf{k}_{**} = -\zeta\sum_{l=1}^{N}\delta_l\mathbf{k}_{*l}, \tag{6.15}$$

where δ_l are the same as in (3.26). If $\vec{\lambda} \in \Lambda^m_{n_l,\vartheta}$ and

$$\begin{aligned} \tilde{\mathbf{w}}_{\vec{\lambda}} &= (\hat{\mathbf{w}}_{\lambda_1}\ldots\hat{\mathbf{w}}_{\lambda_m}) = (\hat{\mathbf{w}}_{\zeta',l_1}\ldots\hat{\mathbf{w}}_{\zeta^{(m)},l_m}) \\ &\qquad = (e^{-i\zeta'\varphi_{l_1}}\hat{\mathbf{v}}_{\zeta',l_1}\ldots e^{-i\zeta^{(m)}\varphi_{l_m}}\hat{\mathbf{v}}_{\zeta^{(m)},l_m}), \end{aligned}$$

then, using (6.2) and the multi-linearity of $\mathcal{F}^{(m)}$, we get

$$\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}) = \mathrm{e}^{-\mathrm{i}\sum\zeta^{(j)}\varphi_{l_j}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{v}}_{\vec{\lambda}})$$

and

$$\sum_{j=1}^{m}\zeta^{(j)}\varphi_{l_j} = \sum_{l=1}^{N}\delta_l\varphi_l, \tag{6.16}$$

where δ_l are the same as in (3.26). If we have a universal solution of (6.14), all coefficients at every $\omega_l(\mathbf{k}_{*l})$ cancel out ($\omega_n(\mathbf{k}_{**})$ also equals one of $\omega_l(\mathbf{k}_{*l})$, namely $\omega_n(\mathbf{k}_{**}) = \omega_{n_{I_0}}(\mathbf{k}_{*I_0})$). Using the notation (3.26), we see that a universal solution is determined by the system of equations on binary indices

$$\delta_l = \sum_{j\in\vec{l}^{-1}(l)} \zeta^{(j)} = 0,\ l \neq I_0,\ \delta_{I_0} = \sum_{j\in\vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta. \tag{6.17}$$

Obviously, the above condition does not involve values of ω_l and hence if δ_l, ζ correspond to a universal solution of (6.14), then we have the identity

$$-\zeta\varphi_{I_0} + \sum_{l=1}^{N} \delta_l \varphi_l = 0, \tag{6.18}$$

which holds for any $(\varphi_1, \ldots, \varphi_N) \in \mathbb{C}^N$. $\square$

Consider now the case, where the nk-spectrum S is resonance invariant, but may be not universal resonance invariant. Definition 3.8 of a resonance invariant nk-spectrum implies that the set $P(S)$ of all the solutions of (3.24) coincides with the set $P_{\text{int}}(S)$ of internal solutions. Hence all solutions of (6.14), (6.15)) are internal, in particular $\mathbf{k}_{**} = \mathbf{k}_{*I_0}, \omega_n(\mathbf{k}_{**}) = \omega_{n_{I_0}}(\mathbf{k}_{*I_0})$ with some I_0.

If we have a nonuniversal internal solution of (6.14), $\omega_l(\mathbf{k}_{*l})$ satisfy the following linear equation:

$$\zeta\omega_{n_{I_0}}(\mathbf{k}_{*I_0}) + \sum_{l=1}^{N} \delta_l\omega_l(\mathbf{k}_{*l}) = 0, \quad \zeta\mathbf{k}_{*I_0} + \sum_{l=1}^{N} \delta_l \mathbf{k}_{*l} = 0 \tag{6.19}$$

where at least one of b_j is nonzero. Note that if (6.19) is satisfied, we have additional (nonuniversal) solutions of (3.24) defined by

$$\sum_{j\in\vec{l}^{-1}(l)} \zeta^{(j)} = \delta_l,\ l \neq I_0,\ \sum_{j\in\vec{l}^{-1}(I_0)} \zeta^{(j)} = \zeta + \delta_{I_0}. \tag{6.20}$$

Now let us briefly discuss properties of Equations (6.20). The right-hand sides of the above system form a vector $\vec{b} = (b_1, \ldots, b_N)$ with $b_l = \delta_l$, $l \neq I_0$, and $b_{I_0} = \zeta + \delta_{I_0}$. Note that $\vec{l} = (l_1, \ldots, l_m)$ is uniquely defined by its level sets $\vec{l}^{-1}(l)$. For every l the number δ_{+l} of positive $\zeta^{(j)}$ and the number δ_{-l} of negative $\zeta^{(j)}$ with $j \in \vec{l}^{-1}(l)$ in (6.20) satisfy the equations

$$\delta_{+l} - \delta_{-l} = \delta_l,\ \delta_{+l} + \delta_{-l} = |\vec{l}^{-1}(l)|, \tag{6.21}$$

where $|\vec{l}^{-1}(l)| = c_l$ is the cardinality (number of elements) of $\vec{l}^{-1}(l)$. Hence δ_{+l}, δ_{-l} are uniquely defined by $\delta_l, |\vec{l}^{-1}(l)|$. Hence the set of binary solutions $\vec{\zeta}$ of (6.20) with a given $\vec{b}$ and a given $\vec{l} = (l_1, \ldots, l_m)$ is determined by subsets of $\vec{l}^{-1}(l)$ with the cardinality δ_{+l} elements. Hence every solution with a given $\vec{b}$ and a given $\vec{l}$ can be obtained from one solution by permutations of indices j inside every level set $\vec{l}^{-1}(l)$. If $\vec{b}$ is given and the cardinalities $|\vec{l}^{-1}(l)| = c_l$ are given, we can obtain different $\vec{l}$ which satisfy (6.20) by choosing different decomposition of $\{1, \ldots, m\}$ into subsets with given cardinalities c_l. For given $\vec{b}$ and $\vec{c} = (c_1, \ldots, c_m)$ we obtain this way the set (may be empty for some $\vec{b}$, $\vec{c}$) of all solutions of (6.20). Solutions with the same $\vec{b}$ and $\vec{c}$ we call equivalent.

When for a given wavepacket there are several nonequivalent nonuniversal solutions, the number of which is denoted by N_c, we obtain from (6.19) a system of equations with integer coefficients

$$\sum_{l=1}^{N} b_{l,i}\omega_l(\mathbf{k}_{*l}) = 0, \; i = 1, \ldots, N_c, \tag{6.22}$$

and solutions to (3.24) can be found from

$$\sum_{j \in \vec{l}^{-1}(l)} \zeta^{(j)} = b_{l,i}, \text{ for some } i, \; 0 \leqslant i \leqslant N_c, \tag{6.23}$$

where to include universal solutions, we set $b_{l,0} = 0$.

Hence when a wavepacket is universally resonance invariant, we conclude that all terms in (6.5) satisfy (6.17). Since (6.18) holds, we get (6.13) for arbitrary $(\varphi_1, \ldots, \varphi_N) \in \mathbb{C}^N$. If the wavepacket is conditionally universal with conditions (6.23), then,using (6.16) and (6.23), we conclude that (6.18) and (6.13) hold if $(\varphi_1, \ldots, \varphi_N)$ satisfy the system of equations

$$\sum_{l=1}^{N} b_{l,i}\varphi_l = 0, \; i = 1, \ldots, N_c. \tag{6.24}$$

Now we wold like to describe a special class of solutions of averaged equations. The evolution equation with an averaged nonlinearity has the form

$$\begin{gathered}
\partial_\tau U_{j,+} = \frac{-\mathrm{i}}{\varrho}\mathcal{L}_j(-\mathrm{i}\nabla)U_{j,+} + F_{j,+}(U_{1,+}, U_{1,-}, \ldots, U_{N,+}, U_{N,-}), \\
\partial_\tau U_{j,-} = \frac{\mathrm{i}}{\varrho}\mathcal{L}_j(\mathrm{i}\nabla)U_{j,+} + F_{j,-}(U_{1,+}, U_{1,-}, \ldots, U_{N,+}, U_{N,-}), \\
j = 1, \ldots, N,
\end{gathered} \tag{6.25}$$

where $\mathcal{L}(-\mathrm{i}\nabla)$ is a linear scalar differential operator with constant coefficients. The characteristic property (6.13) implies that such a system admits special solutions of the form

$$U_{j,\zeta}(\mathbf{r},\tau) = \mathrm{e}^{-\mathrm{i}\varphi_j\tau/\varrho}V_{j,\zeta}(\mathbf{r}), \tag{6.26}$$

where $V_{1,\zeta}(\mathbf{r})$ solve the time-independent *nonlinear eigenvalue problem*

$$\begin{aligned} -\mathrm{i}\varphi_j V_{j,+} &= -\mathrm{i}\mathcal{L}_j(-\mathrm{i}\nabla)V_{j,+} + \varrho F_{j,+}(V_{1,+}, V_{1,-}, \ldots, V_{N,+}, V_{N,-}),\\ \mathrm{i}\varphi_j V_{j,-} &= \mathrm{i}\mathcal{L}_j(\mathrm{i}\nabla)V_{j,+} + \varrho F_{j,-}(V_{1,+}, V_{1,-}, \ldots, V_{N,+}, V_{N,-}),\\ j &= 1,\ldots,N. \end{aligned} \tag{6.27}$$

6.1.2. Examples of universal and conditionally universal nonlinearities. Here we give a few examples of equations with averaged nonlinearities. When the multi-wavepacket is universal resonance invariant, the averaged wave interaction system involves NLS-type equations.

Example 6.6. The simplest example of (6.25) for one wavepacket $(N = 1)$ and one spatial dimension $(d = 1)$ is the nonlinear Schrödinger equation

$$\begin{aligned} \partial_\tau U_{1,+} &= -\frac{\mathrm{i}}{\varrho}a_2\partial_x^2 U_{1,+} - \frac{\mathrm{i}}{\varrho}a_0 U_{1,+} + a_1\partial_x U_{1,+} - \mathrm{i}qU_{1,-}U_{1,+}^2,\\ \partial_\tau U_{1,-} &= \frac{\mathrm{i}}{\varrho}a_2\partial_x^2 U_{j,-} + \frac{\mathrm{i}}{\varrho}a_0 U_{1,-} + a_1\partial_x U_{1,-} + \mathrm{i}qU_{1,+}U_{1,-}^2. \end{aligned} \tag{6.28}$$

Note that, by setting $y = x + a_1\tau/\varrho$, we can make $a_1 = 0$. Obviously, the nonlinearity

$$F_\zeta(U) = -\mathrm{i}\zeta qU_{1,-\zeta}U_{1,\zeta}^2$$

satisfies (6.13):

$$i\zeta qe^{-i\zeta\varphi_1}U_{1,-\zeta}(e^{i\zeta\varphi_1}U_{1,\zeta})^2 = e^{i\zeta\varphi_1}i\zeta qU_{1,-\zeta}(U_{1,\zeta})^2.$$

The eigenvalue problem in this case takes the form

$$\begin{aligned} \mathrm{i}\varphi_1 V_{1,+} &= -\mathrm{i}a_2\partial_x^2 V_{1,+} - \mathrm{i}a_0 V_{1,+} + a_1\partial_x V_{1,+} - \mathrm{i}\varrho qV_{1,-}V_{1,+}^2,\\ -\mathrm{i}\varphi_1 V_{1,-} &= \mathrm{i}a_2\partial_x^2 V_{j,-} + \mathrm{i}a_0 V_{1,-} + a_1\partial_x V_{1,-} + \mathrm{i}\varrho qV_{1,+}V_{1,-}^2. \end{aligned} \tag{6.29}$$

If $a_1 = 0$ and we consider real-valued $V_{1,+} = V_{1,-}$, we obtain the equation

$$(\varphi_1 + a_0)V_{1,+} = -a_2\partial_x^2 V_{1,+} - \varrho qV_{1,+}^3$$

or, equivalently,

$$\frac{(\varphi_1 + a_0)}{\varrho q}V_{1,+} + \frac{a_2}{\varrho q}\partial_x^2 V_{1,+} + V_{1,+}^3 = 0.$$

If

$$c^2 = \frac{a_2}{\varrho q} > 0, \frac{(\varphi_1 + a_0)}{\varrho q} = -b^2 < 0, \tag{6.30}$$

the last equation takes the form

$$-b^2 V_{1,+} + c^2 \partial_x^2 V_{1,+} + V_{1,+}^3 = 0$$

with the family of classical soliton solutions

$$V_{1,+} = 2^{1/2} \frac{b}{\cosh(b(x - x_0)/c)}.$$

Note that the norm of the Fourier transform $\|\hat{V}_{1,+}\|_{L^1} = Cb$, where C is an absolute constant. Hence to have $\hat{V}_{1,+}$ bounded in L^1 uniformly in small ϱ according to (6.30), we should take $\varphi_1 = -a_0 - b^2 \varrho q$ with bounded b.

If the universal resonance invariant multi-wavepacket involves two wavepackets ($N = 2$) and the nonlinearity F is cubic, i.e., $\mathfrak{M}_F = \{3\}$, the semilinear system PDE with averaged nonlinearity has the form

$$\begin{aligned}
\partial_t U_{2,+} &= -iL_2(i\nabla)U_{2,+} + U_{2,+}(Q_{2,1,+}U_{1,+}U_{1,-} + Q_{2,2,+}U_{2,+}U_{2,-}),\\
\partial_t U_{2,-} &= iL_2(-i\nabla)U_{2,-} + U_{2,-}(Q_{2,1,-}U_{1,+}U_{1,-} + Q_{2,2,-}U_{2,+}U_{2,-}),\\
\partial_t U_{1,+} &= -iL_1(i\nabla)U_{1,+} + U_{1,+}(Q_{1,1,+}U_{1,+}U_{1,-} + Q_{1,1,+}U_{2,+}U_{2,-}),\\
\partial_t U_{1,-} &= iL_1(-i\nabla)U_{1,-} + U_{1,-}(Q_{1,1,-}U_{1,+}U_{1,-} + Q_{1,1,-}U_{2,+}U_{2,-}).
\end{aligned}$$

Obviously, (6.13) holds with arbitrary φ_1, φ_2.

Now let us consider quadratic nonlinearities. In particular, let us concider the one-band symmetric case $\omega_n(\mathbf{k}) = \omega_1(\mathbf{k}) = \omega_1(-\mathbf{k})$, i.e., $J = 1$, $\mathfrak{M}_F = \{2\}$, and $m = 2$. Suppose that there is a multi-wavepacket involving two wavepackets with wavevectors $\mathbf{k}_{*1}, \mathbf{k}_{*2}$, i.e., $N = 2$. The resonance equation (3.24) takes now the form

$$-\zeta\omega_1(\zeta'\mathbf{k}_{*l_1} + \zeta''\mathbf{k}_{*l_2}) + \zeta'\omega_1(\mathbf{k}_{*l_1}) + \zeta''\omega_1(\mathbf{k}_{*l_2}) = 0, \tag{6.31}$$

where $l_1, l_2 \in \{1, 2\}$, $\zeta, \zeta', \zeta'' \in \{-1, 1\}$. All possible cases, and there are exactly four of them, correspond to the four well-known effects in the nonlinear optics: (i) $l_1 = l_2$, $\zeta' = \zeta''$ and $\zeta' = -\zeta''$ correspond respectively to second harmonic generation and nonlinear optical rectification; (ii) $l_1 \neq l_2$, $\zeta' = \zeta''$and $\zeta' = -\zeta''$ correspond respectively to sum-frequency and difference-frequency interactions.

Let us suppose now that $\mathbf{k}_{*1}, \mathbf{k}_{*2} \neq 0$ and $\omega_1(\mathbf{k}_{*1}) \neq 0$, $\omega_1(\mathbf{k}_{*2}) \neq 0$, where the last conditions exclude the optical rectification, and that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}$, $\mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Consider first the case, where the wavepacket is universally resonance invariant.

Example 6.7. Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F, i.e., $\mathfrak{M}_F = \{2\}$. Let us pick two points $\mathbf{k}_{*1}$ and $\mathbf{k}_{*2} \neq \pm\mathbf{k}_{*1}$ and assume that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}, \pm\mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Assume also that (i) $2\omega_1(\mathbf{k}_{*i}) \neq \omega_1(2\mathbf{k}_{*i})$, $i, j, l = 1, 2$, so there is no second harmonic generation; (ii) $\omega_1(\mathbf{k}_{*1}) \pm \omega_1(\mathbf{k}_{*2}) \neq \omega_1(\mathbf{k}_{*1} \pm \mathbf{k}_{*2})$, (no sum/difference-frequency interactions); (iii) $\omega_1(\mathbf{0}) \neq 0$, $\omega_j(\mathbf{k}_{*1}) \pm \omega_l(\mathbf{k}_{*2}) \neq 0$. Let set the nk-spectrum be the set $S_1 = \{(1, \mathbf{k}_{*1}), (1, \mathbf{k}_{*2})\}$. Then S_1 is resonance invariant.

In this case, (6.31) does not have solutions. Hence $\Lambda^m_{n_l,\vartheta} = \varnothing$ and the averaged nonlinearity equals zero.

Now let us consider the case, where the wavepacket is not universal resonance invariant, but conditionally universal resonance invariant. In the following example, the conditionally resonance invariant spectrum allows for the second harmonic generation in the averaged system.

Example 6.8. Suppose there is a single band, i.e., $J = 1$, with a symmetric dispersion relation, and a quadratic nonlinearity F, i.e., $\mathfrak{M}_F = \{2\}$. Let us pick two points $\mathbf{k}_{*1}$ and $\mathbf{k}_{*2}$ such that $\mathbf{k}_{*2} = 2\mathbf{k}_{*1}$ and assume that $\mathbf{k}_{*i} \neq 0$ and $\mathbf{k}_{*i}, 2\mathbf{k}_{*i}, \mathbf{0}, \pm\mathbf{k}_{*1} \pm \mathbf{k}_{*2}$ are not band-crossing points. Assume also that (i) $2\omega_1(\mathbf{k}_{*1}) = \omega_1(2\mathbf{k}_{*1})$ (second harmonic generation); (ii) $\omega_i(\mathbf{k}_{*1}) \pm \omega_j(\mathbf{k}_{*2}) \neq \omega_l(\mathbf{k}_{*1} \pm \mathbf{k}_{*2})$, $i, j, l = 1, 2$ (no sum-/difference-frequencies interaction); (iii) $\omega_1(\mathbf{0}) \neq 0$, $\omega_j(\mathbf{k}_{*1}) \pm \omega_l(\mathbf{k}_{*2}) \neq 0$. Let set the nk-spectrum be the set $S = \{(1, \mathbf{k}_{*1}), (1, \mathbf{k}_{*2})\}$. Then S is resonance invariant. The condition (6.19) is takes here the form

$$2\omega_1(\mathbf{k}_{*1}) - \omega_1(\mathbf{k}_{*2}) = 0,\ 2\mathbf{k}_{*1} - \mathbf{k}_{*2} = 0,$$

and the condition (6.24) turns into

$$2\omega_1(\mathbf{k}_{*1}) - \omega_1(\mathbf{k}_{*2}) = 0.$$

The wavepacket interaction system for such a multi-wavepacket has the form

$$\begin{aligned}
\partial_t U_{2,+} &= -iL_2(i\nabla)U_{2,+} + Q_{2,2,+}U_{1,+}U_{1,+},\\
\partial_t U_{2,-} &= iL_2(-i\nabla)U_{2,-} + Q_{2,2,-}U_{1,-}U_{1,-},\\
\partial_t U_{1,+} &= -iL_1(i\nabla)U_{1,+} + Q_{1,2,+}U_{2,+}U_{1,-},\\
\partial_t U_{1,-} &= iL_1(-i\nabla)U_{1,-} + Q_{1,2,-}U_{2,-}U_{1,+}.
\end{aligned}$$

6.2. Invariance of multi-particle wavepackets.

The following lemma shows that particle wavepackets are preserved under action of certain types of nonlinearities with elementary susceptibilities as in (4.20). In the following section, we show, in particular, that universal nonlinearities are composed of such terms.

Lemma 6.9. *Let the components* $\hat{\mathbf{w}}_{l_i,\zeta} = \hat{\mathbf{w}}_{\lambda_i}$ *of* $\tilde{\mathbf{w}}_{\vec{\lambda}} = \hat{\mathbf{w}}_{\lambda_1} \ldots \hat{\mathbf{w}}_{\lambda_m}$ *be particle-like wavepackets in the sense of Definition 2.2, and let* $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ *be as in (6.5). Assume that*

$$\hat{\mathbf{w}}_{l_i,\zeta}(\mathbf{k},\beta) = 0 \text{ if } |\mathbf{k} - \zeta \mathbf{k}_{*li}| \geqslant \beta^{1-\varepsilon},\ \zeta = \pm,\ i = 1,\ldots,m. \tag{6.32}$$

Assume that the vector index $\vec{\lambda} \in \Lambda^m_{n_l,\vartheta}$ *is such a vector which has at least one component* $\lambda_j = (\zeta_j, l_j)$ *such that*

$$\nabla\omega_{n_l}(\mathbf{k}_{*l}) = \nabla\omega_{n_{l_j}}(\mathbf{k}_{*l_j}). \tag{6.33}$$

Then for any $\mathbf{r}_* \in \mathbb{R}^d$

$$\begin{aligned} &\|\nabla_{\mathbf{k}}(\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}}\Psi(\cdot,\mathbf{k}_{*l},\beta^{1-\varepsilon})\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}))\|_E \\ &\leqslant C\tau_*\|\nabla_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}^{(j)}}\mathbf{w}_{l_j}\|_E \prod_{i\neq j}\|\mathbf{w}_{l_j,\zeta_j}\|_E \\ &+ C\tau_*\left(\beta^{-1+\varepsilon} + \frac{\beta^{1-\varepsilon}}{\varrho}\right)\prod_{j=1}^m \|\mathbf{w}_{l_j,\zeta_j}\|_E, \end{aligned} \tag{6.34}$$

where C *does not depend on* $\mathbf{r}_*$ *and small* β, ϱ.

Proof. Note that

$$\mathbf{r}_*\mathbf{k} = \mathbf{r}_*(\mathbf{k}' + \ldots + \mathbf{k}^{(m)}).$$

We have by (4.22)

$$\begin{aligned} &\nabla_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau) = \nabla_{\mathbf{k}}\int_0^\tau \int_{[-\pi,\pi]^{2d}} \exp\left\{\mathrm{i}\varphi_{\ \theta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}\right\} \\ &\times \Psi\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}}\chi^{(m)}_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\mathbf{w}_{l_1,\zeta'}(\mathbf{k}')\ldots\mathbf{w}_{l_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\,\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1. \end{aligned} \tag{6.35}$$

Without loss of generality, we assume that in (6.33) $l_j = l_m$ (the general case is reduced to this one by a re-enumeration of variables of integration). By the Leibnitz formula,

$$\nabla_{\mathbf{k}}[\Psi\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})](\mathbf{k},\tau) = I_1 + I_2 + I_3, \tag{6.36}$$

where

$$I_1 = \int_0^\tau \int_{[-\pi,\pi]^{(m-1)d}} \nabla_{\mathbf{k}} \exp\left\{ \mathrm{i}\varphi_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho} - \mathrm{i}\mathbf{r}_*\mathbf{k} \right\} \Psi \chi^{(m)}_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}'}$$
$$\times \mathbf{w}_{l_1,\zeta'}(\mathbf{k}') \ldots \mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}^{(m)}} \mathbf{w}_{l_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1,$$

$$I_2 = \int_0^\tau \int_{[-\pi,\pi]^{(m-1)d}} \Psi \exp\left\{ \mathrm{i}\varphi_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho} - \mathrm{i}\mathbf{r}_*\mathbf{k} \right\}$$
$$\times \left[\nabla_{\mathbf{k}}(\Psi(\mathbf{k},\mathbf{k}_{*l},\beta^{1-\varepsilon})\chi^{(m)}_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k}))\right]\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}'}\mathbf{w}_{l_1,\zeta'}(\mathbf{k}') \ldots \mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}^{(m)}}$$
$$\times \mathbf{w}_{l_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1,$$

$$I_3 = \int_0^\tau \int_{[-\pi,\pi]^{(m-1)d}} \exp\left\{ \mathrm{i}\varphi_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho} - \mathrm{i}\mathbf{r}_*\mathbf{k} \right\} \Psi \chi^{(m)}_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}'}$$
$$\times \mathbf{w}_{l_1,\zeta'}(\mathbf{k}') \ldots \nabla_{\mathbf{k}}(\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}^{(m)}(\mathbf{k},\vec{k})} \mathbf{w}_{l_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k})))\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1.$$

Since $\mathbf{w}_{j,\zeta}$ are bounded, we have

$$\|\mathrm{e}^{-\mathrm{i}\mathbf{r}_{*j}\mathbf{k}^{(j)}}\mathbf{w}_{l_j,\zeta^{(j)}}(\mathbf{k}^{(j)})\|_{L^1} \leqslant \|\mathbf{w}_{l_j,\zeta^{(j)}}(\mathbf{k}^{(j)})\|_{L^1} \leqslant C_1, \ j=1,\ldots,m. \tag{6.37}$$

Using (4.8) and (6.37), we get

$$|I_3| \leqslant \|\chi^{(m)}\| \prod_{j=1}^{m-1} \|\mathbf{w}_{l_j,\zeta^{(j)}}\|_E \int_0^\tau \|\nabla_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}^{(m)}(\mathbf{k},\vec{k})}\mathbf{w}_{l_m,\zeta^{(m)}}\|_E d\tau_1. \tag{6.38}$$

From (6.37), (2.25), (3.13) and the smoothness of $\Psi(\mathbf{k},\mathbf{k}_{*l},\beta^{1-\varepsilon})$ we get

$$|I_2| \leqslant C_2\beta^{-1+\varepsilon} \prod_{j=1}^{m} \|\mathbf{w}_{l_j,\zeta_j}\|_E. \tag{6.39}$$

Now let us estimate I_1. Using (4.23), we obtain

$$I_1 = \int_0^\tau \int_{[-\pi,\pi]^{(m-1)d}} \left[\exp\left\{ \mathrm{i}\varphi_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho} \right\} \right]$$
$$\times \frac{\tau_1}{\varrho}\left[-\theta\nabla_{\mathbf{k}}\omega_{n_l}(\mathbf{k}) + \zeta^{(m)}\nabla_{\mathbf{k}}\omega_{n_{l_m}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\right]$$
$$\times \chi^{(m)}_{\theta,\vec{\zeta}}(\mathbf{k},\vec{k})\mathbf{w}_{l_1,\zeta'}(\mathbf{k}') \ldots \mathbf{w}_{l_m,\zeta^{(m)}}(\mathbf{k}^{(m)}(\mathbf{k},\vec{k}))\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1. \tag{6.40}$$

The difficulty in the estimation of the integral I_1 comes from the factor τ_1/ϱ since ϱ is small. Since (6.32) holds, it is sufficient to estimate I_1 if

$$|\mathbf{k}^{(j)} - \zeta^{(j)}\mathbf{k}_{*n_j}| \leqslant \beta^{1-\varepsilon} \text{ for all } j. \tag{6.41}$$

According to (3.18), since $\vec{\lambda} \in \Lambda^m_{n_l,\vartheta}$, we have

$$\mathbf{k}^{(m)}(\mathbf{k}_{*n_l}, \vec{k}_*) = \mathbf{k}_{*n_{l_m}}.$$

Hence, using (6.33) and (4.23), we obtain

$$\nabla_{\mathbf{k}}\varphi_{\theta,\vec{\zeta}}(\mathbf{k}_{*n_l}, \vec{k}_*) = [-\theta\nabla_{\mathbf{k}}\omega_{n_l}(\mathbf{k}_{*n_l}) + \zeta^{(m)}\nabla_{\mathbf{k}}\omega_{n_{l_m}}((\mathbf{k}^{(m)}(\mathbf{k}_{*n_l}, \vec{k}_*)))] = 0. \tag{6.42}$$

Using (3.2), we conclude that, in a vicinity of $\vec{k}_*$ defined by (6.41), we have

$$t|[-\theta\nabla_{\mathbf{k}}\omega(\mathbf{k}) + \zeta^{(m)}\nabla_{\mathbf{k}}\omega(\mathbf{k}^{(m)}(\mathbf{k}, \vec{k}))]| \leqslant 2(m+1)C_{\omega,2}\beta^{1-\varepsilon}.$$

This yields the estimate

$$|I_1| \leqslant C_3\beta^{1-\varepsilon}/\varrho. \tag{6.43}$$

Combining (6.43), (6.39) and (6.38), we obtain (6.48). □

We introduce a β-dependent Banach space E^1 of differentiable functions of variable $\mathbf{k}$ by the formula

$$\|\mathbf{w}\|_{E^1(\mathbf{r}_*)} = \beta^{1+\varepsilon}\|\nabla_{\mathbf{k}}(\mathrm{e}^{-\mathrm{i}\mathbf{r}_*\mathbf{k}}\mathbf{w})\|_E + \|\mathbf{w}\|_E. \tag{6.44}$$

We use for $2N$-component vectors with elements $\mathbf{w}_i(\mathbf{k}) \in E^2$ the following notation:

$$\begin{gathered}\tilde{\mathbf{w}}(\mathbf{k}) = (\mathbf{w}_1(\mathbf{k}), \ldots, \mathbf{w}_N(\mathbf{k})),\\ \tilde{\mathbf{r}}_* = (\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*N}), \quad \mathbf{w}_i(\mathbf{k}) = (\mathbf{w}_{i,+}(\mathbf{k}), \mathbf{w}_{i,-}(\mathbf{k})),\\ \mathrm{e}^{-\mathrm{i}\tilde{\mathbf{r}}_*\mathbf{k}}\tilde{\mathbf{w}}(\mathbf{k}) = (\mathrm{e}^{-\mathrm{i}\mathbf{r}_{*1}\mathbf{k}}\mathbf{w}_1(\mathbf{k}), \ldots, \mathrm{e}^{-\mathrm{i}\mathbf{r}_{*N}\mathbf{k}}\mathbf{w}_N(\mathbf{k})),\end{gathered} \tag{6.45}$$

Similarly to (5.6) we introduce the space $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ with the norm

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} = \sum_{l,\vartheta} \|\hat{\mathbf{w}}_{l,\vartheta}\|_{E^1(\mathbf{r}_{*l})}. \tag{6.46}$$

The following proposition is obtained by comparing (6.44) and (2.33).

Proposition 6.10. *A multi-wavepacket $\tilde{\mathbf{w}}$ is a multi-particle one with positions $\mathbf{r}_{*1},\ldots,\mathbf{r}_{*N}$ if and only if*

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leqslant C,$$

where the constant C does not depend on β, $0 < \beta \leqslant 1/2$, and $\tilde{\mathbf{r}}_*$.

In view of the above, we call $E^1(\mathbf{r}_*)$ and $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ particle spaces. We also use the notation

$$\Psi_2\tilde{\mathbf{w}}_{\vec{\lambda}} = (\Psi(\cdot,\mathbf{k}_{*l_1},\beta^{1-\varepsilon}/2)\mathbf{w}_{\lambda_1},\ldots,\Psi(\cdot,\mathbf{k}_{*l_m},\beta^{1-\varepsilon}/2)\mathbf{w}_{\lambda_m}),$$

$$\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{w}}^m) = \Psi(\cdot,\mathbf{k}_{*l},\beta^{1-\varepsilon})\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\Psi_2\tilde{\mathbf{w}}_{\vec{\lambda}}).$$

Lemma 6.11. *Let* $\tilde{\mathbf{w}},\tilde{\mathbf{v}}\in(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ *and* $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ *be as in* (6.5). *Assume that the vector index* $\vec{\lambda}\in\Lambda^m_{n_l,\vartheta}$ *is such a vector which has at least one component* $\lambda_j=(\zeta_j,l_j)$ *with* $l_j=l$. *Assume that* (1.9) *holds and* $\Psi(\cdot,\mathbf{k}_*,\beta^{1-\varepsilon})$ *is defined in* (2.25). *Let* $\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)}\leqslant 2R$. *Then*

$$\|\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{w}})\|_{E^1(\mathbf{r}_{*l})}\leqslant C\tau_*\|\tilde{\mathbf{w}}\|^{m-1}_{(E)^{2N}}\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \tag{6.47}$$

where C *does not depend on* β, $0<\beta\leqslant 1/2$, *and on* $\tilde{\mathbf{r}}_*$, $\mathbf{r}_{*l}$ *and* $\tilde{\mathbf{r}}_*$ *is defined by* (6.45). *If* $\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)}\leqslant 2R$ *the following Lipschitz inequality holds*:

$$\|\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{w}})-\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{v}})\|_{E^1(\mathbf{r}_{*l})}\leqslant C\tau_*\|\tilde{\mathbf{w}}-\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)}. \tag{6.48}$$

where C *does not depend on* β, $0<\beta\leqslant 1/2$, *and on* $\tilde{\mathbf{r}}_*$, $\mathbf{r}_{*l}$.

Proof. Note that $\Psi_2\tilde{\mathbf{w}}_{\vec{\lambda}}$ and $\Psi_2\tilde{\mathbf{v}}_{\vec{\lambda}}$ are wavepackets in the sense of Definition 2.2. To obtain (6.47), we apply the inequality (6.34) and use (1.9); for the part of the E^1-norm without $\mathbf{k}$-derivatives we use (4.10). Using multilinearity of $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}$, we observe that

$$\begin{aligned}&\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{w}})-\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{v}})\\&=\sum_{j=1}^{m}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\mathbf{w}_{\lambda_1},\ldots,\mathbf{w}_{\lambda_j}-\mathbf{v}_{\lambda_j},\mathbf{v}_{\lambda_{j+1}},\ldots,\mathbf{v}_{\lambda_m}).\end{aligned} \tag{6.49}$$

We can apply to every term the inequality (6.34). Multiplying (6.34) by $\beta^{1+\varepsilon}$ and using (1.9), we deduce (6.48). $\square$

Now we consider a system similar to (6.8),

$$\hat{\mathbf{v}}_{l,\vartheta}=\mathcal{F}_{\mathrm{av},\Psi_2,n_l,\vartheta}(\tilde{\mathbf{v}})+\Psi(\cdot,\vartheta\mathbf{k}_{*i_l})\Pi_{n_l,\vartheta}\hat{\mathbf{h}},\ l=1,\ldots N,\vartheta=\pm, \tag{6.50}$$

where $\mathcal{F}_{\mathrm{av},\Psi,n_l,\vartheta}$ is defined by a formula similar to(6.5):

$$\mathcal{F}_{\mathrm{av},\Psi_2,n_l,\vartheta}(\tilde{\mathbf{v}})=\sum_{m\in\mathfrak{M}_F}\mathcal{F}^{(m)}_{n_l,\vartheta},\ \mathcal{F}^{(m)}_{n_l,\vartheta}=\sum_{\vec{\lambda}\in\Lambda^m_{n_l,\vartheta}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi}(\tilde{\mathbf{v}}). \tag{6.51}$$

The system (6.50) can be written in the form similar to (6.9)

$$\tilde{\mathbf{v}} = \mathcal{F}_{\mathrm{av},\Psi_2}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \tag{6.52}$$

Theorem 6.12 (solvability in particle spaces)**.** *Let the initial data* $\tilde{\mathbf{h}}$ *in the averaged wavepacket interaction system* (6.52) *be a multi-particle wavepacket* $\hat{\mathbf{h}}(\beta, \mathbf{k})$ *with the nk-spectrum* S *as in* (2.39)*, regularity degree* s*, and positions* $\mathbf{r}_{*l}$*,* $l = 1, \dots, N$*. Let* $\|\tilde{\mathbf{h}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leqslant R$*. Assume that* S *is universally resonance invariant in the sense of Definition* 3.8*. Then there exists* $\tau_{**} > 0$ *which does not depend on* $\tilde{\mathbf{r}}_*$*,* β*, and* ϱ *such that if* $\tau_* \leqslant \tau_{**}$*, Equation* (6.52) *has a unique solution* $\tilde{\mathbf{v}}$ *in* $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ *such that*

$$\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leqslant 2R, \tag{6.53}$$

where R *does not depend on* $\varrho, \beta,$ *and* $\tilde{\mathbf{r}}_*$*. This solution is a multi-particle wavepacket with positions* $\mathbf{r}_{*l}$*.*

PROOF. Since S is universally resonance invariant, every vector index $\vec{\lambda} \in \Lambda^m_{n_l,\vartheta}$ has at least one component $\lambda_j = (\zeta_j, l_j)$ with $l_j = l$. Hence Lemma 6.11 is applicable and, according to (6.48), the operator $\mathcal{F}_{\mathrm{av},\Psi_2}$ defined by (6.51) is Lipschitz in the ball $\|\tilde{\mathbf{v}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leqslant 2R$ with a Lipschitz constant $C'\tau_*$, where C' does not depend on $\varrho, \beta,$, and $\tilde{\mathbf{r}}_*$. We choose τ_{**} so that $C'\tau_{**} \leqslant 1/2$ and use Lemma 4.6. According to this lemma, Equation (6.52) has a solution $\tilde{\mathbf{v}}$ which satisfies (6.53). This solution is a multi-particle wavepacket according to Proposition 6.10. □

Theorem 6.13 (particle wavepacket approximation)**.** *Let the initial data* $\hat{\mathbf{h}}$ *in the integral equation* (2.14) *with solution* $\hat{\mathbf{u}}(\tau, \beta; \mathbf{k})$ *be an multi-particle wavepacket* $\hat{\mathbf{h}}(\beta, \mathbf{k})$ *with the nk-spectrum* S *as in* (2.39)*, regularity degree* s*, and positions* $\mathbf{r}_{*l}$ $l = 1, \dots, N$*, and let the components of* $\hat{\mathbf{h}}(\beta, \mathbf{k})$ *satisfy the inequality* $\|\tilde{\mathbf{h}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*)} \leqslant R$*. Let* $\tau_* \leqslant \tau_{**}$*. Assume that* S *is universally resonance invariant in the sense of Definition* 3.8*. We define* $\hat{\mathbf{v}}(\tau, \beta; \mathbf{k})$ *by the formula*

$$\hat{\mathbf{v}}(\tau, \beta; \mathbf{k}) = \sum_{l=1}^{N} \sum_{\zeta=\pm} \hat{\mathbf{v}}_{l,\vartheta}(\tau, \beta; \mathbf{k}), \; l = 1, \dots, N, \tag{6.54}$$

where $\hat{\mathbf{v}}_{l,\vartheta}(\tau, \beta; \mathbf{k})$ *is a solution of* (6.8)*. Then every such* $\hat{\mathbf{v}}_l(\mathbf{k}; \tau, \beta)$ *is a particle-like wavepacket with position* $\mathbf{r}_{*l}$ *and*

$$\sup_{0 \leqslant \tau \leqslant \tau_*} \|\hat{\mathbf{u}}(\tau, \beta; \mathbf{k}) - \hat{\mathbf{v}}(\tau, \beta; \mathbf{k})\|_{L^1} \leqslant C_1 \varrho + C_2 \beta^s, \tag{6.55}$$

where the constant C_1 *does not depend on* ϱ*,* s*, and* β *and the constant* C_2 *does not depend on* ϱ, β*.*

Proof. Let $\tilde{\mathbf{v}} \in (E^1)^{2N}(\tilde{\mathbf{r}}_*)$ be the solution of Equation (6.52) which exists by Theorem 6.12. It is a particle-like wavepacket. Note that

$$\Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon}/2)\Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon}) = \Psi(\cdot, \mathbf{k}_{*l_1}, \beta^{1-\varepsilon})$$

and a solution of (6.52) has the form $\hat{\mathbf{v}}_{l,\vartheta}(\tau, \beta; \mathbf{k}) = \Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon})[\ldots]$. Consequently, for such solutions $\Psi_2\tilde{\mathbf{v}}_{\vec{\lambda}} = \tilde{\mathbf{v}}_{\vec{\lambda}}$ the nonlinearity $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\lambda},\Psi_2}(\tilde{\mathbf{v}})$ coincides with the equation $\Psi(\cdot, \mathbf{k}_{*l}, \beta^{1-\varepsilon})\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{v}}_{\vec{\lambda}})$ and Equation (6.50) coincides with (6.8). Hence $\tilde{\mathbf{v}}$ is a solution of (6.8). The estimate (6.55) follows from the estimates (6.11) and (5.49). □

Now, we are able to prove Theorem 2.10.

Corollary 6.14 (proof of Theorem 2.10)**.** *If the conditions of Theorem* 2.10 *are satisfied, the statement of Theorem* 2.10 *holds.*

Proof. Note that the functions $\hat{\mathbf{w}}'_{l,\vartheta}(\mathbf{k}, \tau) = \Psi_{i_l,\vartheta}\Pi_{n_l,\vartheta}\hat{\mathbf{u}}(\mathbf{k}, \tau)$, $\theta = \pm$, in Theorem 5.7 are two components of $\hat{\mathbf{u}}_l(\tau, \beta; \mathbf{k})$ in (2.45). Hence (5.51) implies that

$$\|\hat{\mathbf{u}}_l - \hat{\mathbf{w}}_{l,+} - \hat{\mathbf{w}}_{l,-}\|_E \leqslant C'\beta^s,\ 0 < \beta \leqslant \beta_0, \tag{6.56}$$

where $\hat{\mathbf{w}}_{l,\vartheta}$ are solutions to (5.5). According to (6.11), if $\hat{\mathbf{v}}_{l,\vartheta}(\mathbf{k}, \tau)$ is the solution of (6.8), we have

$$\|\hat{\mathbf{v}}_{l,\vartheta} - \hat{\mathbf{w}}_{l,\vartheta}\|_E \leqslant C\varrho,\ l = 1, \ldots, N;\ \vartheta = \pm. \tag{6.57}$$

Hence

$$\|\hat{\mathbf{u}}_l - \hat{\mathbf{v}}_{l,+} - \hat{\mathbf{v}}_{l,-}\|_E \leqslant C\varrho + C'\beta^s,\ 0 < \beta \leqslant \beta_0. \tag{6.58}$$

This inequality implies (2.46). We have proved that $\hat{\mathbf{v}}_{l,\vartheta}$ is a particle-like wavepacket as in Theorem 6.13. The estimate (6.58) implies that $\hat{\mathbf{u}}_l$ is equivalent to $\hat{\mathbf{v}}_l = \hat{\mathbf{v}}_{l,+} + \hat{\mathbf{v}}_{l,-}$ in the sense of (2.42) of degree $s_1 = \min(s, s_0)$. □

7. Superposition Principle and Decoupling of the Wavepacket Interaction System

In this section, we give the proof of the superposition principle of [**8**] which is based on the study of the wavepacket interaction system (6.8). We show that when we omit cross-terms in the averaged wavepacket interaction system, the resulting error is estimated by $\frac{\varrho}{\beta^{1+\varepsilon}}\left|\ln\beta\right|$, i.e., component wavepackets evolve essentially independently and the time averaged wavepacket interaction system almost decouples.

Let $\mathcal{F}_{\mathrm{av},n_l,\vartheta}$ be defined by (6.5), and let a decoupled nonlinearity $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{diag}}$ be defined by

$$\begin{aligned}\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{diag}}(\tilde{\mathbf{w}}) &= \sum_{m\in\mathfrak{M}_F}\mathcal{F}^{(m)}_{n_l,\vartheta},\ \mathcal{F}^{(m)}_{n_l,\vartheta,\mathrm{diag}}(\tilde{\mathbf{w}})\\ &= \sum_{\vec{\lambda}\in\Lambda^{m,\mathrm{diag}}_{n_l,\vartheta}}\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}),\end{aligned} \tag{7.1}$$

where the set of indices $\Lambda^{m,\mathrm{diag}}_{n_l,\vartheta}$ is defined by the formula

$$\Lambda^{m,\mathrm{diag}}_{n_l,\vartheta} = \left\{\vec{\lambda} = (\vec{l},\vec{\zeta}) \in \Lambda^{m,}_{n_l,\vartheta} : l_j = l,\ j = 1,\ldots,m\right\}. \tag{7.2}$$

Note that $\mathcal{F}^{(m)}_{n_l,\vartheta,\mathrm{diag}}$ in (7.1) depends only on $\mathbf{w}_{l,+}$ and $\mathbf{w}_{l,-}$:

$$\mathcal{F}^{(m)}_{n_l,\vartheta,\mathrm{diag}}(\tilde{\mathbf{w}}) = \mathcal{F}^{(m)}_{\vartheta,\mathrm{diag},l}(\mathbf{w}_l),\ \ \mathbf{w}_l = (\mathbf{w}_{l,+},\mathbf{w}_{l,-}). \tag{7.3}$$

The coupling between different variables $\mathbf{v}_l$ in (6.8) is caused by nondiagonal terms

$$\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}(\tilde{\mathbf{w}}) = \mathcal{F}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{w}}) - \mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{diag}}(\tilde{\mathbf{w}}). \tag{7.4}$$

Obviously, Equation (6.9) can be written in the form

$$\tilde{\mathbf{v}} = \mathcal{F}_{\mathrm{av},\Psi,\mathrm{diag}}(\tilde{\mathbf{v}}) + \mathcal{F}_{\mathrm{av},\Psi,\mathrm{coup}}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \tag{7.5}$$

The system of decoupled equations has the form

$$\tilde{\mathbf{v}}_{\mathrm{diag}} = \mathcal{F}_{\mathrm{av},\Psi,\mathrm{diag}}(\tilde{\mathbf{v}}_{\mathrm{diag}}) + \tilde{\mathbf{h}}_{\Psi} \tag{7.6}$$

or, when written in components,

$$\mathbf{v}_{\mathrm{diag},l} = \mathcal{F}^{(m)}_{\mathrm{av},\Psi,\mathrm{diag},l}(\mathbf{v}_{\mathrm{diag},l}) + \mathbf{h}_{\Psi,l},\ l = 1,\ldots,N. \tag{7.7}$$

We prove that the contribution of $\mathcal{F}_{\mathrm{av},\Psi,\mathrm{coup}}$ in (7.5) is small. The proof is based on the following lemma.

Lemma 7.1 (small coupling terms). *Let* $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})$ *be as in* (6.5), *let all the components* $\mathbf{w}_{\lambda_i}$ *of* $\tilde{\mathbf{w}}_{\vec{\lambda}}$ *satisfy* (6.32) *and be wavepackets in the sense of Definition* 2.1, *and let* (1.9) *hold. Assume also that*: (i) *the vector index* $\vec{\lambda}$ *has at least two components* $\lambda_i = (\zeta_i, l_i)$ *and* $\lambda_j = (\zeta_j, l_j)$ *with*

$l_i \neq l_j$; (ii) *both* $\mathbf{w}_{\lambda_i}$ *and* $\mathbf{w}_{\lambda_j}$ *are particle wavepackets in the sense of Definition* 2.2; (iii) *either* (2.51) *or* (2.54) *holds. Then for small* β *and* ϱ

$$\|\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})\|_{E^N} \leqslant C\frac{\varrho}{\beta^{1+\varepsilon}}\left|\ln\beta\right|. \tag{7.8}$$

Proof. Since $\mathbf{k}_{*l}$ are not band-crossing points, according to Definition 3.1 and Condition 3.2 the inequalities (3.2) and (3.13) hold. According to the assumption of the theorem, at least two $\hat{\mathbf{w}}_{l_j}$ are different for different j. Let us assume that $l_{j_1} = l_1$, $l_{j_2} = l_m$, $l_1 \neq l_m$ (the general case can be easily reduced to this one by relabeling variables). Since $\hat{\mathbf{w}}_{l_1}$ and $\hat{\mathbf{w}}_{l_m}$ are particle wavepackets, they satisfy (2.33) with $\mathbf{r}$ replaced by $\mathbf{r}_{l_1}$ and $\mathbf{r}_{l_m}$ respectively. Let us rewrite the integral with respect to τ_1 in (4.22) as

$$\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau) = \int_0^\tau \int_{\mathbb{D}_m} \exp\left\{\mathrm{i}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho}\right\} A^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1, \tag{7.9}$$

where

$$A^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k}) = \chi^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\mathbf{w}_{l_1}(\mathbf{k}')\ldots\mathbf{w}_{l_m}(\mathbf{k}^{(m)}), \tag{7.10}$$

and then rewrite (7.9) in the form

$$\begin{aligned}
&\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}})(\mathbf{k},\tau) = \mathcal{F}^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{w}_{l_1}\ldots\mathbf{w}_{l_m})(\mathbf{k},\tau)\\
&= \int_0^\tau \int_{\mathbb{D}_m} \exp_\varphi(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1,
\end{aligned} \tag{7.11}$$

where

$$\begin{aligned}
&\exp_\varphi(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m}) = \exp\left\{\mathrm{i}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\frac{\tau_1}{\varrho} - \mathrm{i}\mathbf{r}_{l_1}\mathbf{k}' - \mathrm{i}\mathbf{r}_{l_m}\mathbf{k}^{(m)}\right\},\\
&A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m}) = \mathrm{e}^{\mathrm{i}\mathbf{r}_{l_1}\mathbf{k}'}\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_m}\mathbf{k}^{(m)}}A^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k}).
\end{aligned} \tag{7.12}$$

According to (3.10), $\mathbf{k}^{(m)}(\mathbf{k},\vec{k}) = \mathbf{k} - \mathbf{k}' - \ldots - \mathbf{k}^{(m-1)}$. Hence, picking a vector $\mathbf{p}$ with a unit length, we obtain the formula

$$\exp_\varphi(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m}) = \frac{\varrho\mathbf{p}\cdot\nabla_{\mathbf{k}'}\exp_\varphi(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})}{\mathrm{i}[\mathbf{p}\cdot\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\tau_1 - \varrho\mathbf{p}\cdot(\mathbf{r}_{l_1}-\mathbf{r}_{l_m})]}. \tag{7.13}$$

If we set

$$\begin{gathered}
\varphi' = \nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k}_{*l},\vec{k}_*) = \nabla_{\mathbf{k}'}\omega(\zeta'\mathbf{k}'_*) - \nabla_{\mathbf{k}^{(m)}}\omega(\zeta^{(m)}\mathbf{k}^{(m)}_*),\\
c_p = \mathbf{p}\cdot\varphi', q_p = \varrho\mathbf{p}\cdot(\mathbf{r}_{l_1}-\mathbf{r}_{l_m}),
\end{gathered} \tag{7.14}$$

$$\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1)=\frac{(c_p\tau_1-q_p)}{[\mathbf{p}\cdot\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})\tau_1-\mathbf{p}\cdot(\mathbf{r}_{l_1}-\mathbf{r}_{l_m})]},\tag{7.15}$$

then (7.13) can be recast as

$$\exp_{\varphi}(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})=\frac{\varrho\mathbf{p}\cdot\nabla_{\mathbf{k}'}\exp_{\varphi}(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})}{\mathrm{i}(c_p\tau_1-q_p)}\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1).\tag{7.16}$$

If (2.51) holds, then $\varphi'\neq 0$, and to get $|c_p|\neq 0$, we can take

$$\mathbf{p}=|\varphi'|^{-1}\cdot\varphi',\ \ |c_p|=p_0>0.\tag{7.17}$$

If (2.54) holds, we have $\varphi'=0$, and we set

$$\mathbf{p}=|(\mathbf{r}_{l_1}-\mathbf{r}_{l_m})|^{-1}\cdot(\mathbf{r}_{l_1}-\mathbf{r}_{l_m}).\tag{7.18}$$

Let consider first the case, where (2.51) holds. Notice that the denominator in (7.16) vanishes for

$$\tau_{10}=\frac{q_p}{c_p}.\tag{7.19}$$

We split the integral with respect to τ_1 in (7.11) into the sum of two integrals, namely

$$\mathcal{F}^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{w}_{l_1}\ldots\mathbf{w}_{l_m})(\mathbf{k},\tau)=F_1+F_2,\tag{7.20}$$

$$F_1=\int\limits_{|\tau_{10}-\tau_1|\geqslant c_0\beta^{1-\varepsilon}|\ln\beta|}\ \int\limits_{\mathbb{D}_m}\exp_{\varphi}(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})\times A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1,$$

$$F_2=\int\limits_{|\tau_{10}-\tau_1|<c_0\beta^{1-\varepsilon}|\ln\beta|}\ \int\limits_{\mathbb{D}_m}\exp_{\varphi}(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})\times A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})\tilde{\mathrm{d}}^{(m-1)d}\vec{k}d\tau_1,$$

where c_0 is a large enough constant which we estimate below in (7.28). Since $\mathbf{w}_j$ are bounded in E and (2.48) holds, we obtain similarly to (4.10) the estimate

$$\|F_2\|_{L^1}\leqslant Cc_0\beta^{1-\varepsilon}|\ln\beta|\prod_{j=1}^{m}\|\mathbf{w}_{l_j}\|_E\leqslant C_1(R)\frac{\varrho|\ln\beta|}{\beta^{1+\varepsilon}}.\tag{7.21}$$

To estimate the norm of F_1, we use (7.13) and integrate by parts the integral in (7.20) with respect to $\mathbf{k}'$. We obtain

$$F_1 = \int\limits_{|\tau_{10}-\tau_1|\geqslant\beta^{1-\varepsilon}|\ln\beta|} I(\mathbf{k},\tau_1)d\tau_1, \tag{7.22}$$

$$I(\mathbf{k},\tau_1) = -\int\limits_{\mathbb{D}_m} \frac{\varrho\exp_{\varphi}(\mathbf{k},\vec{k},\tau_1,\varrho,\mathbf{r}_{l_1},\mathbf{r}_{l_m})}{\mathrm{i}(c_p\tau_1-\varrho q_p)} \\ \times \mathbf{p}\cdot\nabla_{\mathbf{k}'}[\theta_0 A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})]\,\tilde{\mathrm{d}}^{(m-1)d}\vec{k}.$$

According to (7.10) and (3.10), the expansion of the gradient $\nabla_{\mathbf{k}'}$ in the above formula involves the derivatives of χ,θ_0, $\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_1}\mathbf{k}'}\mathbf{w}_{l_1}$ and $\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_m}\mathbf{k}^{(m)}}\mathbf{w}_{l_m}$. To estimate θ_0 and $\nabla\theta_0$, we note that

$$\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1) = \frac{(\mathbf{p}\cdot\varphi'\tau_1-q_p)}{(\mathbf{p}\cdot\varphi'\tau_1-q_p)+\tau_1\mathbf{p}\cdot[\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi']} \\ = \frac{1}{1+\tau_1\mathbf{p}\cdot[\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi']/(c_p\tau_1-q_p)}. \tag{7.23}$$

Since $|\tau_{10}-\tau_1|\geqslant c_0\beta^{1-\varepsilon}|\ln\beta|$, from (7.19) we infer

$$|c_p\tau_1-q_p|\geqslant c_pc_0\beta^{1-\varepsilon}|\ln\beta|. \tag{7.24}$$

From (6.32) we see that in the integral (7.22) the integrands are nonzero only if

$$|\mathbf{k}^{(j)}-\zeta^{(j)}\mathbf{k}_*^{(j)}|\leqslant\pi_0\beta^{1-\varepsilon}, |\mathbf{k}-\zeta\mathbf{k}_*|\leqslant m\pi_0\beta^{1-\varepsilon}, \tag{7.25}$$

where $\pi_0\leqslant 1$. Using the Taylor remainder estimate for $\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}$ at $\vec{k}_*$, we obtain the inequality

$$|\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi'|\leqslant 2mC_{\omega,2}\beta^{1-\varepsilon}. \tag{7.26}$$

Hence in (7.23)

$$|\tau_1\mathbf{p}\cdot[\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi']/(c_p\tau_1-q_p)|\leqslant 2m\tau_*C_{\omega,2}/(c_pc_0|\ln\beta|). \tag{7.27}$$

Suppose that $\beta\leqslant 1/2$ is small and c_0 satisfies

$$\frac{m\tau_*C_{\omega,2}}{|\ln\beta|}\leqslant\frac{m\tau_*C_{\omega,2}}{\ln 2}\leqslant\frac{1}{4}|c_p|c_0. \tag{7.28}$$

Then it follows from (7.23) with the help of (7.28), (3.2), (7.24) and (7.27) that

$$|\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1)|\leqslant 2. \tag{7.29}$$

Obviously,

$$\nabla_{\mathbf{k}'}\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1)=\frac{-\tau_1\nabla_{\mathbf{k}'}[\mathbf{p}\cdot(\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi')]}{(c_p\tau_1-q_p)[1+\tau_1\mathbf{p}\cdot[\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi']/(c_p\tau_1-q_p)]^2}. \tag{7.30}$$

Using (7.27), (7.28), and (3.2), we obtain

$$|\nabla_{\mathbf{k}'}\theta_0(\mathbf{k},\vec{k},\varrho,\tau_1)|\leqslant\frac{4\tau_*}{|c_p\tau_1-q_p|}|\nabla_{\mathbf{k}'}[\mathbf{p}\cdot(\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k})-\varphi')]|\leqslant\frac{4\tau_*C_{\omega,2}}{|c_p\tau_1-q_p|}. \tag{7.31}$$

To estimate $\nabla_{\mathbf{k}'}\chi$, we use (3.13). We conclude that the absolute value of the integral (7.22) is not greater than

$$\begin{aligned}|I(\mathbf{k},\tau_1)|\leqslant{}&\frac{4\varrho\tau_*C_{\omega,2}}{|\tau_1c_p-q_p|^2}\int\limits_{\mathbb{D}_m}|A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})|\tilde{\mathrm{d}}^{(m-1)d}\vec{k}\\&+\frac{2\varrho\tau_*}{|\tau_1c_p-q_p|}\int\limits_{\mathbb{D}_m}\big[|\nabla_{\mathbf{k}'}A(\mathbf{k},\vec{k},\mathbf{r}_{l_1},\mathbf{r}_{l_m})|\big]\tilde{\mathrm{d}}^{(m-1)d}\vec{k}\\\leqslant{}&\Big[\frac{4C_{\omega,2}\varrho\tau_*}{|\tau_1c_p-q_p|^2}\|\chi^{(m)}(\mathbf{k},\cdot)\|+\frac{2\varrho\tau_*}{|\tau_1c_p-q_p|}\|(\nabla_{k'}-\nabla_{k^{(m)}})\chi^{(m)}(\mathbf{k},\cdot)\|\Big]\\&\times\prod_{j=1}^m\|\mathbf{w}_j\|_{L^1}+\frac{2\varrho\tau_*\|\chi^{(m)}(\mathbf{k},\cdot)\|}{|\tau_1c_p-q_p|}\Big[\prod_{j=2}^m\|\mathbf{w}_{l_j}\|_{L^1}\|\nabla_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_1}\mathbf{k}'}\mathbf{w}_{l_1}\|_{L^1}\\&+\prod_{j=1}^{m-1}\|\mathbf{w}_j\|_{L^1}\|\nabla_{\mathbf{k}^{(m)}}\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_m}\mathbf{k}^{(m)}}\mathbf{w}_m\|_{L^1}\Big].\end{aligned} \tag{7.32}$$

Note that $\|\mathbf{w}_j\|_{L^1}$ are bounded according to (2.27) and $\nabla_{\mathbf{k}^{(m)}}\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_m}\mathbf{k}^{(m)}}\mathbf{w}_{l_m}$, $\nabla_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}\mathbf{r}_{l_1}\mathbf{k}'}\mathbf{w}_{l_1}$ by (2.33). Hence we obtain

$$|I(\mathbf{k},\tau_1)|\leqslant\frac{C_2\varrho\beta^{-1-\varepsilon}}{\tau_1c_p-q_p}+\frac{\varrho C_2}{|\tau_1c_p-q_p|^2}. \tag{7.33}$$

Obviously,

$$\begin{aligned}\int\limits_{|\tau_1-q_p/c_p|\geqslant c_0\beta^{1-\varepsilon}|\ln\beta|}|\tau_1c_p-q_p|^{-1}d\tau_1&=\frac{1}{c_p}\int\limits_{c_0\beta^{1-\varepsilon}|\ln\beta|}^{\tau_*-q_p/c_p}\frac{d\tau_1}{\tau_1}\\&=\frac{1}{c_p}\ln\frac{\tau_*-q_p/c_p}{c_0\beta^{1-\varepsilon}|\ln\beta|}\leqslant\frac{1}{c_p}(C+|\ln[\beta^{1-\varepsilon}|\ln\beta|]|)\\&\leqslant\frac{1}{c_p}[C+|\ln\beta|+|\ln|\ln\beta||]\leqslant\frac{1}{c_p}[C+2|\ln\beta|].\end{aligned}$$

Similarly, using (2.48), we get

$$\int\limits_{|\tau_1 - q_p/c_p| \geqslant c_0\beta^{1-\varepsilon}|\ln\beta|} |\tau_1 c_p - q_p|^{-2} d\tau_1$$

$$= \frac{1}{c_p} \int\limits_{c_0\beta^{1-\varepsilon}|\ln\beta|}^{\tau_* - q_p/c_p} \frac{d\tau_1}{\tau_1^2} = \frac{1}{c_p}\left[\frac{1}{c_0\beta^{1-\varepsilon}|\ln\beta|} - \frac{1}{\tau_* - q_p/c_p}\right]$$

$$\leqslant \frac{1}{c_p c_0 \beta^{1-\varepsilon}|\ln\beta|} \leqslant \frac{C_3\varrho}{\beta^{-1-\varepsilon}|\ln\beta|}.$$

Hence we obtain for small β

$$\|\mathcal{F}^{(m)}_{\zeta,\vec{\zeta}}(\mathbf{w}_1 \dots \mathbf{w}_m)(\mathbf{k},\tau)\|_E \leqslant C_4 \frac{\varrho}{\beta^{1+\varepsilon}}|\ln\beta|. \tag{7.34}$$

Now let us consider the case, where (2.54) holds, $\varphi' = 0$ and $\mathbf{p}$ is defined by (7.18). Turning to expression (7.23), we notice that

$$c_p\tau_1 - q_p = -\varrho|\mathbf{r}_{l_1} - \mathbf{r}_{l_m}|, \ \tau_*|c_p\tau_1 - q_p|^{-1} \leqslant \frac{1}{\beta^{1+\varepsilon}},$$

and, according to (7.26),

$$|\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k}) - \varphi'| \leqslant C_{\omega,2}\beta^{1-\varepsilon}.$$

Then we estimate the denominator in (7.23) and (7.30) using (2.54):

$$|\tau_1\mathbf{p}\cdot[\nabla_{\mathbf{k}'}\varphi_{\zeta,\vec{\zeta}}(\mathbf{k},\vec{k}) - \varphi']/(c_p\tau_1 - q_p)| \leqslant \tau_* C_{\omega,2}\beta^{1-\varepsilon}/(\varrho|\mathbf{r}_{l_1} - \mathbf{r}_{l_m}|) \leqslant \frac{1}{2}.$$

If β is so small that (7.28) holds, we again get (7.29) and (7.31). Hence we obtain (7.34) in this case as well (in fact, in this case, the logarithmic factor can be omitted). Finally, we obtain (7.35) from (7.34) after summing up over all $\vec{\lambda}, \vec{\zeta}$. □

Lemma 7.2. *Let the nk-spectrum S be universally resonance invariant. Let the operators* $\mathcal{F}_{\mathrm{av},n_l,\vartheta}(\tilde{\mathbf{w}})$, $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{diag}}(\tilde{\mathbf{v}})$, *and* $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}$ *be defined respectively by* (6.5), (4.7), *and* (7.4). *Let* $\tilde{\mathbf{v}}$, $\|\tilde{\mathbf{v}}\|_{E^N} \leqslant 2R$, *be a multi-wavepacket solution of* (6.9) *with the nk-spectrum S. Then for small β and ϱ*

$$\|\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}(\tilde{\mathbf{v}})\|_{E^N} \leqslant C\frac{\varrho}{\beta^{1+\varepsilon}}|\ln\beta|. \tag{7.35}$$

Proof. According to (6.5), (7.1), and (7.4), $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}$ involves only terms with $\vec{\lambda} \in \Lambda^{m,}_{n_l,\vartheta} \setminus \Lambda^{m,\mathrm{diag}}_{n_l,\vartheta}$ and it is sufficient to prove the estimate (7.8) for indices $\vec{\lambda} \in \Lambda^{m,}_{n_l,\vartheta} \setminus \Lambda^{m,\mathrm{diag}}_{n_l,\vartheta}$. Such indices involve at least two components $\lambda_i = (\zeta_i, l_i)$ and $\lambda_j = (\zeta_j, l_j)$ with $l_i \neq l_j$ since the nk-spectrum is universally invariant, see (3.26). According to Theorem 6.13, the solution $\tilde{\mathbf{v}}$ is a particle-like wavepacket. Therefore, all the components of $\tilde{\mathbf{v}}_{\vec{\lambda}}$ are particle-like; (6.32) holds according to (5.11). Hence all the conditions of Lemma 7.1 are fulfilled and (7.35) follows from (7.8). □

Note now that every equation (7.7) is an approximation of Equation (4.6) with single-wavepacket initial data $\hat{\mathbf{h}}_l$, namely

$$\hat{\mathbf{u}}_l(\mathbf{k}, \tau) = \mathcal{F}(\hat{\mathbf{u}}_l)(\mathbf{k}, \tau) + \hat{\mathbf{h}}_l(\mathbf{k}). \tag{7.36}$$

One can apply to this equation Theorems 5.6 and 6.4 formally restricted to the case $N = 1$ of a single wavepacket. Based on this observation and above lemma, we prove the following theorem which implies previously formulated Theorems 2.14 and 2.15.

Theorem 7.3. *Assume that the multi-wavepacket $\tilde{\mathbf{h}} = \sum \hat{\mathbf{h}}_l$ is particle-like and its nk-spectrum is universally resonance invariant. Assume also that either* (2.51) *or* (2.54) *holds. Let $\hat{\mathbf{u}}$ be asolution of Equation* (4.6)*. Let $\hat{\mathbf{u}}_l$ be solutions of* (7.36)*. Then the superposition principle holds, namely*

$$\left\| \hat{\mathbf{u}} - \sum_{l=1}^{N} \hat{\mathbf{u}}_l \right\| \leqslant C \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C\beta^s. \tag{7.37}$$

Proof. Let $\mathbf{v}_{\mathrm{diag},l}$ be a solution of the decoupled system (7.7). We compare the systems (7.5) and (7.6). The difference between the systems is the term $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}(\tilde{\mathbf{v}})$. According to Theorem 6.12, the solution $\tilde{\mathbf{v}}$ is a particle-like wavepacket and we can apply Lemma 7.2. According to this lemma, (7.35) holds. Applying Lemma 4.6 to Equations (7.5) and (7.6) and using (7.35), we conclude that the difference of their solutions satisfies the inequality

$$\|\mathbf{v}_l - \mathbf{v}_{\mathrm{diag},l}\|_E \leqslant C' \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C'\beta^s. \tag{7.38}$$

According to Theorem 6.13, the inequality (6.55) holds, where $\tilde{\mathbf{v}}$ is a solution of (6.9) which can be rewritten in the form of (7.5). From (6.55) and (7.38) we infer

$$\left\| \hat{\mathbf{u}} - \sum_{l=1}^{N} \mathbf{v}_{\mathrm{diag},l} \right\| \leqslant C_1 \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C_1\beta^s. \tag{7.39}$$

Note that Equation (7.7) for $\mathbf{v}_{\mathrm{diag},l}$ coincides with the averaged equation (6.9) obtained for the wave interaction system derived for (7.36). Therefore, applying Theorems 5.6 and 6.4 to the case $N = 1$ and $\hat{\mathbf{h}} = \hat{\mathbf{h}}_l$, we deduce from (5.49) and (6.11) the estimate

$$\|\hat{\mathbf{u}}_l - \mathbf{v}_{\mathrm{diag},l}\|_E \leqslant C_2\varrho + C_2'\beta^s. \tag{7.40}$$

Finally, from (7.39) and (7.40) we infer (7.37). □

7.1. Generalizations.

In this section, we show that the particle-like wavepacket invariance can be extended to the case, where nk-spectra S are not universally resonance invariant. So, suppose that an nk-spectrum S is resonance invariant and consider nonlinearities of the form similar to (6.5)

$$\mathcal{F}_{\mathrm{res},n_l,\vartheta}(\tilde{\mathbf{w}}) = \sum_{m\in\mathfrak{M}_F} \mathcal{F}^{(m)}_{n_l,\vartheta},\ \mathcal{F}^{(m)}_{n_l,\vartheta} = \sum_{\vec{\lambda}\in\Lambda'_{n_l,\vartheta}} \mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}), \tag{7.41}$$

where $\Lambda'_{n_l,\vartheta} \subseteq \Lambda^m_{n_l,\vartheta}$ is a given subset of Λ^m. Obviously, $\mathcal{F}_{\mathrm{av}}$ defined by (6.5) has the form of (7.41) with $\Lambda'_{n_l,\vartheta} = \Lambda^m_{n_l,\vartheta}$. Let us introduce a multi-wavepacket

$$\tilde{\mathbf{w}} = (\mathbf{w}_{n_1,+}, \mathbf{w}_{n_1,-}, \ldots, \mathbf{w}_{n_N,+}, \mathbf{w}_{n_N,-}) \tag{7.42}$$

with the nk-spectrum $S = \{(n_l,\theta),\ l = 1,\ldots,N; \theta = \pm\}$.

We call a subset $S' \subset S$ *sign-invariant* if when it has (n_l,θ) as an element, then $(n_l,-\theta)$ is also its element. Suppose that $S' \subset S$ is sign-invariant. It is easy to see that if a set $S' \subset S$ is sign-invariant, then it is uniquely defined by a subset of indices $I' = I'(S') \subset I = \{1,\ldots,N\}$, namely

$$S' = \{(n_l,\theta) : l \in I'(S'),\ \theta = \pm\}.$$

Definition 7.4. We call an index pair $(n_l, \mathbf{k}_{*l})$ *Group Velocity Matched* (GVM) with $\mathcal{F}_{\mathrm{res},n_l,\vartheta}$ if every nonzero term $\mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}$ in the sum (7.41) has an index $\vec{\lambda}$ such that for at least one component $\lambda_j = (\zeta^{(j)}, l_j)$ of this index the following equality holds:

$$\nabla\omega_{n_l}(\mathbf{k}_{*l}) = \nabla\omega_{n_{l_j}}(\mathbf{k}_{*l_j}). \tag{7.43}$$

We call S' a GVM *set* with respect to the nonlinearity $\mathcal{F}_{\mathrm{res}}$ defined by (7.41) if $S' \subset S$ is sign-invariant and every $(n_l,\mathbf{k}_{*l}) \in S'$ is GVM.

Obviously, if S is universally resonance invariant and $\Lambda'_{n_l,\vartheta} = \Lambda^m_{n_l,\vartheta}$ as in (6.5), then S is a GVM set and, in this case, $l_j = I_0$ as in Definition 3.6. If $S' \subset S$ is sign-invariant, we call a multi-wavepacket $\tilde{\mathbf{w}}$ as in (7.42) with the

nk-spectrum $S = \{(n_l, \theta), l = 1, \ldots, N; \theta = \pm\}$ *partially S'-localized multi-wavepacket* if for every $(n_l, \theta) \in S'$ the wavepacket $\mathbf{w}_{n_1,\theta}$ is a spatially localized with position $\mathbf{r}_{*l}$. Note that, according to Definition 2.7, if $S' = S$ and $\mathbf{w}$ is a partially S'-localized multi-wavepacket, then it is a multi-particle wavepacket.

Theorem 2.10 on the particle-like wavepacket preservation can be generalized as follows.

Theorem 7.5 (preservation of spatially localized wavepackets)**.** *Assume that the conditions of Theorem* 2.9 *hold, in particular the initial datum* $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta, \mathbf{k})$ *is a multi-wavepacket with an nk-spectrum S. Assume also that* $S' \subset S$ *is a* GVM *set,* $\hat{\mathbf{h}} = \hat{\mathbf{h}}(\beta, \mathbf{k})$ *is partially S'-localized wavepacket with positions* $\mathbf{r}_{*l}$, $l \in I'(S')$, *and* (2.47) *holds. Then the solution* $\hat{\mathbf{u}}(\tau, \beta) = \mathcal{G}(\mathcal{F}(\rho(\beta)), \hat{\mathbf{h}}(\beta))(\tau)$ *to* (2.14) *for any* $\tau \in [0, \tau_*]$ *is a multi-wavepacket with the nk-spectrum S and it is an S'-localized wavepacket with positions* $\mathbf{r}_{*l}$, $l \in I'(S')$. *Namely,* (2.46) *holds, where* $\hat{\mathbf{u}}_l$ *is a wavepacket with the nk-pair* $(n_l, \mathbf{k}_{*l}) \in S'$ *defined by* (2.45)*, the constants* C, C_1, C_2 *do not depend on* $\mathbf{r}_{*l}$*, and every* $\hat{\mathbf{u}}_l$, $l \in I'(S')$, *is equivalent in the sense of the equivalence* (2.42) *of degree* $s_1 = \min(s, s_0)$ *to a spatially localized wavepacket with position* $\mathbf{r}_{*l}$.

Proof. The proof of the theorem is the same as the proof of Theorem 2.10 since it used only the fact that a universally resonance invariant set is a GVM one, that allows us to apply Lemma 6.9. One also have to use the space $(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')$ with the norm defined by the formula similar to (6.46):

$$\|\tilde{\mathbf{w}}\|_{(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')} = \sum_{l,\vartheta} \|\hat{\mathbf{w}}_{l,\vartheta}\|_E + \beta^{1+\varepsilon} \sum_{\vartheta=\pm} \sum_{l \in I'(S')} \|\nabla_{\mathbf{k}}(\mathrm{e}^{-\mathrm{i}\mathbf{r}_{*l}\mathbf{k}} \hat{\mathbf{w}}_{l,\vartheta})\|_E. \tag{7.44}$$

After replacing $(E^1)^{2N}(\tilde{\mathbf{r}}_*)$ with $(E^1)^{2N}(\tilde{\mathbf{r}}_*, S')$ we can literally repeat all the steps of the proof of Theorem 2.10 and obtain the statement of Theorem 7.5. $\square$

Below we prove that the superposition principle can hold not only for universal resonance invariant multi-wavepackets, but for other cases allowing resonant processes such as the second and third harmonic generations, three-wave interaction, etc. Here we prove a theorem applicable to such situations, which is more general than Theorem 2.14.

Let us consider a multi-wavepacket with a resonance invariant nk-spectrum

$$S = \{(n_l, \mathbf{k}_{*l}),\ l = 1, \dots, N\}$$

as in (3.14), and assume that is the union of spectra S_p:

$$S = S_1 \cup \ldots \cup S_K,\ \ S_p \cap S_q = \varnothing \ \text{ if } \ p \neq q. \tag{7.45}$$

Recall that resonance interactions are defined in terms of vectors $\vec{\lambda} \in \Lambda^m$ (see (3.16), (3.17)). We call a vector $\vec{\lambda} = ((\zeta', l_1), \dots, (\zeta^{(m)}, l_m)) \in \Lambda^m$ a *cross-interacting (CI)* if there exist at least two indices $(\zeta^{(i)}, l_i)$ and $(\zeta^{(j)}, l_j)$ such that $(\zeta^{(i)}, l_i) \in S_{p_i}, (\zeta^{(j)}, l_j) \in S_{p_j}$ with $p_i \neq p_j$.

Definition 7.6 (partially GVM decomposition)**.** We call the decomposition (7.45) *partially* GVM with respect to $\mathcal{F}_{\mathrm{res}}$ defined by (7.41) if the following two conditions are satisfied: (i) every spectrum S_j, $j = 1, \dots, K$, is resonance invariant; (ii) a solution $(m, \zeta, n, \vec{\lambda}) \in P(S)$ of the resonance equation (3.24) with CI vector $\vec{\lambda} = ((\zeta', l_1), \dots, (\zeta^{(m)}, l_m))$ has at least two indices $(\zeta^{(i)}, l_i) \in S_{p_i}$ and $(\zeta^{(j)}, l_j) \in S_{p_j}$ with $p_i \neq p_j$ such that both l_i and l_j are GVM with respect to $\mathcal{F}_{\mathrm{res}}$ and

$$|\nabla_{\mathbf{k}} \omega_{n_{l_i}}(\mathbf{k}_{*l_i}) - \nabla_{\mathbf{k}} \omega_{n_{l_j}}(\mathbf{k}_{*l_j})| \neq 0. \tag{7.46}$$

Now we use Lemma 7.1 for small coupling. Being given a partially GVM decomposition (7.45), we introduce the set of coupling terms between S_{p_i} and S_{p_j} as follows:

$$\Lambda^{m,\mathrm{coup}}_{n_l,\vartheta} = \left\{ \vec{\lambda} = (\vec{l}, \vec{\zeta}) \in \Lambda^{m,}_{n_l,\vartheta} : \exists\ i \neq j \text{ such that } l_i \in S_{p_i}, l_j \in S_{p_j} \right\}, \tag{7.47}$$

We also introduce a set of interactions reducible to every S_p (block-diagonal) which is similar to (7.2):

$$\Lambda^{m,\mathrm{red}}_{n_l,\vartheta} = \Lambda^{m}_{n_l,\vartheta} \setminus \Lambda^{m,\mathrm{coup}}_{n_l,\vartheta}, \tag{7.48}$$

and the reduced operator

$$\begin{aligned}\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{red}}(\tilde{\mathbf{w}}) &= \sum_{m \in \mathfrak{M}_F} \mathcal{F}^{(m)}_{n_l,\vartheta,\mathrm{red}}(\tilde{\mathbf{w}}),\ \mathcal{F}^{(m)}_{n_l,\vartheta,\mathrm{red}}(\tilde{\mathbf{w}}) \\ &= \sum_{\vec{\lambda} \in \Lambda^{m,\mathrm{red}}_{n_l,\vartheta}} \mathcal{F}^{(m)}_{n_l,\vartheta,\vec{\xi}(\vec{\lambda})}(\tilde{\mathbf{w}}_{\vec{\lambda}}),\end{aligned} \tag{7.49}$$

where $\Lambda^{m,\mathrm{red}}_{n_l,\vartheta}$ is defined by (7.48). Note that if the set S is universal resonance invariant and every S_{p_i} is a two-point set $\{(+, l_i), (+, l_i)\}$, then

$\Lambda_{n_l,\vartheta}^{m,\mathrm{red}} = \Lambda_{n_l,\vartheta}^{m,\mathrm{diag}}$. We introduce also a partially decoupled, reduced system similar to (7.6)

$$\tilde{\mathbf{v}}_{\mathrm{red}} = \mathcal{F}_{\mathrm{av},\Psi,\mathrm{red}}(\tilde{\mathbf{v}}_{\mathrm{red}}) + \tilde{\mathbf{h}}_{\Psi}, \tag{7.50}$$

which can be rewritten in the decoupled form similar to (7.7):

$$\mathbf{v}_{\mathrm{red},p} = \mathcal{F}^{(m)}_{\mathrm{av},\Psi,\mathrm{red},p}(\mathbf{v}_{\mathrm{red},p}) + \mathbf{h}_{\mathrm{red},\Psi,p},\ p = 1,\ldots K. \tag{7.51}$$

Now $\mathbf{v}_{\mathrm{red},p}$ may include more than one wavepacket, namely

$$\mathbf{v}_{\mathrm{red},p} = \sum_{(n_l,\theta)\in S_p} (\tilde{\mathbf{v}}_{\mathrm{red}})_{n_l,\theta},\ \mathbf{h}_{\mathrm{red},\Psi,p} = \sum_{(n_l,\theta)\in S_p} (\tilde{\mathbf{h}}_{\Psi})_{n_l,\theta},\ p = 1,\ldots K. \tag{7.52}$$

The following theorem is a generalization of Theorem 2.14 on the superposition.

Theorem 7.7 (general superposition principle)**.** *Suppose that the initial data* $\hat{\mathbf{h}}$ *of* (2.14) *is a multi-wavepacket of the form*

$$\hat{\mathbf{h}} = \sum_{p=1}^{K} \hat{\mathbf{h}}_{\mathrm{red},p}, \tag{7.53}$$

where $\hat{\mathbf{h}}$ *is a multi-wavepacket in the sense of Definition* 3.8 *with a resonance invariant nk-spectrum* S*,* $\hat{\mathbf{h}}_{\mathrm{red},p}$ *is a multi-wavepacket with a resonance invariant nk-spectrum* S_p*, and the decomposition* (7.45) *is a partially* GVM *in the sense of Definition* 7.6 *with respect to the nonlinearity* $\mathcal{F}_{av}$ *defined by* (6.5)*. Suppose also that* (2.48) *holds. Then the solution* $\hat{\mathbf{u}} = \mathcal{G}(\hat{\mathbf{h}})$ *to the evolution equation* (2.14) *satisfies the approximate superposition principle*

$$\mathcal{G}\Big(\sum_{p=1}^{K} \hat{\mathbf{h}}_{\mathrm{red},p}\Big) = \sum_{p=1}^{K} \mathcal{G}(\hat{\mathbf{h}}_{\mathrm{red},p}) + \tilde{\mathbf{D}}, \tag{7.54}$$

with a small remainder $\tilde{\mathbf{D}}(\tau)$ *satisfying the following estimate:*

$$\sup_{0\leqslant\tau\leqslant\tau_*} \|\tilde{\mathbf{D}}(\tau)\|_{L^1} \leqslant C_\varepsilon \frac{\varrho}{\beta^{1+\varepsilon}} |\ln\beta|, \tag{7.55}$$

where ε *is the same as in Definition* 2.1 *and can be arbitrary small,* τ_* *does not depend on* $\beta, \varrho,$ *and* ε*.*

Proof. The proof of Theorem 7.7 is similar to the proof of Theorem 7.3. The averaged system (6.9) can be written similarly to (7.5) in the form

$$\tilde{\mathbf{v}} = \mathcal{F}_{\mathrm{av},\Psi,\mathrm{red}}(\tilde{\mathbf{v}}) + \mathcal{F}_{\mathrm{av},\Psi,\mathrm{coup}}(\tilde{\mathbf{v}}) + \tilde{\mathbf{h}}_{\Psi}. \tag{7.56}$$

Comparing now the systems (7.56) and (7.50), we find that the difference between them is the term $\mathcal{F}_{\mathrm{av},n_l,\vartheta,\mathrm{coup}}(\tilde{\mathbf{v}})$. According to Theorem 7.5, the solution $\tilde{\mathbf{v}}$ is a spatially localized wavepacket and hence we can apply Lemma 7.2 getting the inequality (7.35). Applying Lemma 4.6 to Equations (7.56) and (7.50) and using (7.35), we conclude that the difference of their solutions satisfies the inequality

$$\|\mathbf{v}_p - \mathbf{v}_{\mathrm{red},p}\|_E \leqslant C' \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C' \beta^s, p = 1, \dots, K \tag{7.57}$$

According to Theorem 6.13, the inequality (6.55) holds, where $\tilde{\mathbf{v}}$ is a solution of (7.56), and we infer from (7.57)

$$\left\| \hat{\mathbf{u}} - \sum_{p=1}^{K} \mathbf{v}_{\mathrm{red},p} \right\| \leqslant C_1 \frac{\varrho}{\beta^{1+\varepsilon}} |\ln \beta| + C_1 \beta^s. \tag{7.58}$$

Similarly to (7.36) we introduce equation for $\hat{\mathbf{u}}_{\mathrm{red},p} = \mathcal{G}(\hat{\mathbf{h}}_{\mathrm{red},p})$

$$\hat{\mathbf{u}}_{\mathrm{red},p}(\mathbf{k}, \tau) = \mathcal{F}(\hat{\mathbf{u}}_{\mathrm{red},p})(\mathbf{k}, \tau) + \hat{\mathbf{h}}_{\mathrm{red},p}(\mathbf{k}). \tag{7.59}$$

Applying Theorems 5.6 and 6.4, we infer similarly to (7.40) the inequality

$$\|\hat{\mathbf{u}}_{\mathrm{red},p} - \mathbf{v}_{\mathrm{red},p}\|_E \leqslant C_2 \varrho + C_2' \beta^s. \tag{7.60}$$

Finally, from (7.58) and (7.60) we infer (7.55). □

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Attractors for Nonautonomous Navier–Stokes System and Other Partial Differential Equations

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General methods for constructing and studying global attractors of nonautonomous evolution partial differential equations are presented. The nonautonomous 2D Navier–Stokes system with time-dependent external force serves as the main example. The Kolmogorov ϵ-entropy and fractal dimension of global attractors are considered for this system and other important equations in mathematical psychics. The convergence of global attractors of nonautotnomous equations with singularly oscillating terms to attractors of the corresponding "limit" equations is also established. Bibliography: 136 titles.

Introduction

One of the major mathematical aspect in the study of evolution equations arising in different areas of mechanics and physics is the study of the final

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behavior of solutions when time is large or tends to infinity. The related important question concerns the stability of solutions as $t \to +\infty$ or the nature of instability if a solution is unstable in some sense. Over the last decades, considerable progress has been achieved in the study of autonomous partial differential equations. For many basic autonomous evolution equations in mathematical physics it was shown that the long time behavior of solutions is characterized by finite-dimensional global attractors (see, for example, [**119, 91, 9, 68, 40, 115**] and the references therein).

Nonautonomous evolution partial differential equations and their global attractors are less studied. However, in the last decade, a notable advance was made in this perspective area of mathematical researches. In particular, the global attractor was constructed and studied for the nonautonomous 2D Navier–Stokes system with external force depending on time t. We note that the process $\{U(t,\tau)\} := \{U(t,\tau) \mid t \geqslant \tau;\ t,\tau \in \mathbb{R}\}$ corresponds to this system. The mapping $U(t,\tau)$ acts by the formula $u(\tau) \longmapsto U(t,\tau)u(\tau) := u(t)$, where $u(t)$ is a solution of the Navier–Stokes system with initial data $u(\tau)$. The process $\{U(t,\tau)\}$ is a two-parameter family of mappings acting in the phase space of the evolution equation. Therefore, the study of the behavior of solutions $u(t)$ of the nonautonomous evolution equation as $t \to +\infty$ is equivalent to the study of the corresponding process $\{U(t,\tau)\}$ as $t \to +\infty$. Thus, in the study of solutions $u(t)$ of nonautonomous equations, the processes $\{U(t,\tau)\}$ play the same role as the semigroups $\{S(t), t \geqslant 0\}$ in the study of solutions $u(t)$ of autonomous equations as $t \to +\infty$.

In this paper, we deal with nonautonomous partial differential equations and the corresponding processes $\{U(t,\tau)\}$. Particular emphasis is placed to the study of the global attractor of the nonautonomous 2D Navier–Stokes system.

In Section 1, we sketch out the general theory of global attractors of semigroups and consider some basic autonomous equations in mathematical physics. We also consider questions related to the dimension and ε-entropy of invariant sets and present upper estimates for the fractal dimension and the ε-entropy of global attractors of autonomous equations. We derive such estimates for the 2D Navier–Stokes system, the dissipative wave equation, and the complex Ginzburg–Landau equation.

In Section 2, we study the uniform global attractors of general processes and nonautonomous equations. We note that, studying global attractors of such equations, there is a good reason to introduce a notion of the time symbol $\sigma(t)$. The time symbol of a nonautonomous equation is the collection of all time-dependent terms of this equation. Along with solution dynamics, we study the symbol dynamics as $t \to +\infty$.

In Section 2, we formulate results concerning the existence of the uniform global attractor $\mathcal{A}$ of the process $\{U_\sigma(t,\tau)\}$ corresponding to a nonautonomous equation with translation compact symbol $\sigma(t)$. We also present a theorem on the structure of the set $\mathcal{A}$. Then we consider the uniform global attractor $\mathcal{A}$ of the 2D Navier–Stokes system with time-dependent external force that is the symbol of this system. We study in detail the case, where the system has a unique bounded complete solution $\{z(t), t \in \mathbb{R}\}$ attracting all other solutions $\{u(t), t \geqslant \tau\}$ of the 2D Navier–Stokes system as $t \to +\infty$ with exponential rate. Similar problems for a nonautonomous dissipative wave equation and the nonautonomous Ginzburg–Landau equation are also considered.

Many important questions related to the global attractors of nonautonomous equations and the corresponding processes were discussed, for example, in [**73, 68, 34, 115**] (see also the references therein), and in many papers cited in the Bibliography to this paper.

As is known, the fractal dimension of the global attractor of a general nonautonomous partial differential equation can be infinite (see, the example at the end of Section 2). However, the ε-entropy of the global attractor is always finite since the attractor is a compact set.

In Section 3, we present estimates for the ε-entropy of global attractors of nonautonomous equations with translation compact symbols. We also consider applications to the nonautonomous 2D Navier–Stokes system and some other equations in mathematical physics. A particular attention is devoted to the case, where, for example, the external force of the 2D Navier–Stokes system is a quasiperiodic function in time with k rationally independent frequencies. In this case, the global attractor has finite fractal dimension and the upper estimate for its dimension has a summand k. This means that the fractal dimension can grow with no limit as $k \to \infty$. The corresponding examples are given.

In Section 4, we study the global attractor $\mathcal{A}_\varepsilon$ of the 2D Navier–Stokes system with singularly oscillating external force of the form

$$g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t), \quad 0 \leqslant \rho \leqslant 1, \quad 0 < \varepsilon \leqslant 1.$$

The behavior of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0+$ is discussed. A similar problem is studied in Section 5 for the nonautonomous complex Ginzburg–Landau equation.

1. Attractors of Autonomous Equations

In this section, we briefly present fundamental results concerning the global attractors of semigroups corresponding to autonomous evolution equations. Details can be found in many books on infinite-dimensional dynamical systems and attractors (see, for example, [**74, 119, 68, 9, 91, 122, 50, 38, 61, 112, 115, 34**]).

1.1. Semigroups and global attractors.

We consider a general (nonlinear) semigroup $\{S(t)\}$ acting on E, where E is a complete metric space or a Banach space. In particular, E can be a closed subset of a Banach space.

Definition 1.1. A family of mappings $S(t) : E \to E$ depending on the real parameter (time) $t \geqslant 0$ is called a *semigroup* acting on E and is denoted by $\{S(t)\}$ if it satisfies the semigroup identity

$$S(t_1)S(t_2) = S(t_1 + t_2) \quad \forall t_1, t_2 \geqslant 0 \tag{1.1}$$

and

$$S(0) = \mathrm{Id}. \tag{1.2}$$

Hereinafter, Id denotes the identity operator. If $S(t)$ is defined for any real t and the identity (1.1) holds for any $t_1, t_2 \in \mathbb{R}$, the $\{S(t)\}$ is called a *group*.

Assume that a semigroup $\{S(t)\}$ acts in a complete metric space or a Banach space E. Let $\mathcal{B}(E)$ be the collection of all bounded sets in E with respect to the metric in E.

A semigroup $\{S(t)\}$ is said to be (E, E)-*bounded* if $S(t)B \in \mathcal{B}(E)$ for all $B \in \mathcal{B}(E)$ and $t \geqslant 0$. A semigroup $\{S(t)\}$ is said to be *uniformly* (E, E)-*bounded* if for every $B \in \mathcal{B}(E)$ there exists $B_1 \in \mathcal{B}(E)$ such that $S(t)B \subset B_1$ for all $t \geqslant 0$.

We will consider *dissipative* dynamical systems. In application to general semigroups, the dissipation property means the existence of bounded or compact absorbing or attracting sets.

A set $B_0 \subset E$ is said to be *absorbing* for a semigroup $\{S(t)\}$ if for every $B \in \mathcal{B}(E)$ there exists $T = T(B) > 0$ such that $S(t)B \subset B_0$ for all $t \geqslant T$. A set $P \subset E$ is said to be *attracting* for $\{S(t)\}$ if for any $B \in \mathcal{B}(E)$

we have $\mathrm{dist}_E(S(t)B, P) \to 0$ as $t \to +\infty$, where

$$\mathrm{dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|y - x\|_E, \ X, Y \subseteq E, \tag{1.3}$$

is the *Hausdorff* (nonsymmetric) *distance* between sets X and Y. It is clear that any absorbing set is attracting.

A semigroup $\{S(t)\}$ is said to be *compact* if there exists a compact absorbing set $P \Subset E$ for $\{S(t)\}$ and *asymptotically compact* if there exists a compact attracting set $K \Subset E$. These notions reflect the dissipativity of dynamical systems under consideration.

A semigroup $\{S(t)\}$ is (E, E)-*continuous* if every mapping $S(t)$, $t \geqslant 0$, is continuous from E into E.

The behavior of a semigroup $\{S(t)\}$ as $t \to +\infty$ can be described in terms of global attractors.

Definition 1.2. A set $\mathcal{A} \in \mathcal{B}(E)$ is called a *global attractor* for $\{S(t)\}$ if it possesses the following properties:

1) $\mathcal{A}$ is compact in E ($\mathcal{A} \Subset E$),

2) $\mathcal{A}$ is an attracting set for $\{S(t)\}$, i.e., $\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$ for every $B \in \mathcal{B}(E)$,

3) $\mathcal{A}$ is strictly invariant with respect to $\{S(t)\}$, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$.

As was shown in [**9**], the global attractor $\mathcal{A}$ for $\{S(t)\}$ is the maximal bounded invariant set for $\{S(t)\}$ (see also [**88, 89, 91**]). This means the following: if $Y \in \mathcal{B}(E)$ and $S(t)Y = Y$ for all $t \geqslant 0$, then $Y \subset \mathcal{A}$. This implies, in particular, the uniqueness of a global attractor for $\{S(t)\}$.

Definition 1.3. For a bounded set $B \in \mathcal{B}(E)$ the set

$$\omega(B) = \bigcap_{h \geqslant 0} \Big[\bigcup_{t \geqslant h} S(t)B \Big]_E \tag{1.4}$$

is called an ω-*limit set* for B. Here, $[\,\cdot\,]_E$ denotes the closure in E.

We formulate the classical attractor existence theorem.

Theorem 1.1. *Let $\{S(t)\}$ be a continuous semigroup in a complete metric space E, and let $\{S(t)\}$ have a compact attracting set $K \Subset E$. Then $\{S(t)\}$ has a global attractor $\mathcal{A}$ ($\mathcal{A} \subseteq K$). The attractor $\mathcal{A}$ coincides with $\omega(K) : \mathcal{A} = \omega(K)$. (If E is a Banach space, then the set $\mathcal{A}$ is connected).*

The proof can be found, for example, in [**9, 119**].

We need one more notion to describe the general structure of a global attractor. A curve $u(s)$, $s \in \mathbb{R}$, is called a *complete trajectory* of a semigroup $\{S(t)\}$ if

$$S(t)u(s) = u(t+s) \quad \forall s \in \mathbb{R},\, t \in \mathbb{R}_+. \tag{1.5}$$

Definition 1.4. The *kernel* $\mathcal{K}$ of a semigroup $\{S(t)\}$ consists of all bounded complete trajectories of $\{S(t)\}$:

$$\mathcal{K} = \{u(\cdot) \mid u(s) \text{ satisfies (1.5) and } \|u(s)\|_E \leqslant C_u \text{ for } s \in \mathbb{R}\}.$$

Definition 1.5. The *kernel section* at time $s \in \mathbb{R}$ is the set in E defined by the formula $\mathcal{K}(s) = \{u(s) \mid u \in \mathcal{K}\}$.

Remark 1.1. The kernel $\mathcal{K}$ of $\{S(t)\}$ corresponding to an autonomous equation (see Section 1.2) consists of all solutions $u(t)$ that are determined on the entire time-axis $\{t \in \mathbb{R}\}$ that are bounded in E. The kernel includes equilibrium points, as well as periodic, quasiperiodic, and almost periodic orbits. Heteroclinic and homoclinic orbits belong to $\mathcal{K}$ and, in general, the structure of $\mathcal{K}$ can be extremely complicated even with chaotic behavior of its elements.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, the global attractor $\mathcal{A}$ of the semigroup $\{S(t)\}$ coincides with the kernel section*

$$\mathcal{A} = \mathcal{K}(0), \tag{1.6}$$

where 0 *can be replaced with any $s \in \mathbb{R}$.*

The proof can be found, for example, in [**9**].

In the following sections, we apply Theorems 1.1 and 1.2 to different semigroups $\{S(t)\}$ corresponding to different partial differential equations in mathematical physics.

1.2. Cauchy problem and corresponding semigroup.

For the sake of simplicity, we suppose that E is a Banach space. Let $\{S(t)\}$ act on the entire Banach space E. Such semigroups are usually generated by evolution equations of the form

$$\partial_t u = A(u), \tag{1.7}$$

where A is a (nonlinear) operator defined on a Banach space E_1 and A maps E_1 into a Banach space E_0. We suppose that $E_1 \subseteq E \subseteq E_0$, where all the embeddings are dense. We now construct the semigroup $\{S(t)\}$ corresponding to Equation (1.7) and acting on E.

Assume that for arbitrary $v_0 \in E$ Equation (1.7) with initial data

$$u|_{t=0} = u_0 \tag{1.8}$$

has a unique solution $u(t)$, $t \geqslant 0$, such that $u(t) \in E$ for all $t \geqslant 0$. The meaning of the expression "$u(t)$ is a solution of the Cauchy problem (1.7), (1.8)" should be clarified in each particular case. Usually, for every fixed $T > 0$ the solutions $u(t)$, $0 \leqslant t \leqslant T$, of (1.7) are taken from the class $\mathcal{F}_T$ of functions such that $u(\cdot) \in L_\infty(0, T; E)$ and $u(\cdot) \in L_p(0, T; E_1)$, where E_1 is the Banach space on which the operator A is defined and $1 < p \leqslant \infty$. Moreover, $A(u(\cdot)) \in L_q(0, T; E_0)$ for some $1 < q < \infty$ and $\partial_t u(\cdot) \in L_q(0, T; E_0)$ (the derivative is taken in the sense of distributions). In this case, Equation (1.7) is understood as equality in $L_q(0, T; E_0)$. Thus, $u(t)$ satisfies (1.7) in the sense of distributions in $\mathcal{D}'(]0, T[; E_0)$ (see [**96, 9**] for details). Using embedding theorems (see, for example, [**95, 117**]), one can show that $u(t) \in C_w([0, T]; E)$ and even $u(t) \in C([0, T]; E)$ and (1.8) makes sense: $u(t) \to u_0$ weakly or strongly in E as $t \to 0+$. Moreover, $u(t) \in E$ for every $t \in [0, T]$. In special cases, it is convenient to take the space E_0 sufficiently large since the extension of E does not cause any difficulties, but facilitates the verification of the conditions $A(u) \in E_0$ and $\partial_t u \in E_0$.

The operators $S(t) : E \to E$ generated by Equation (1.7) are usually defined as follows. For arbitrary $u_0 \in E$ we consider the solution $u(t)$, $t \geqslant 0$, of the problem (1.7), (1.8). For all $\tau \geqslant 0$ the element $u(\tau)$ of the space E is uniquely determined. Therefore, the formula

$$S(\tau) : u_0 = u|_{t=0} \mapsto u|_{t=\tau} \tag{1.9}$$

defines the family of mappings $\{S(\tau),\ \tau \geqslant 0\}$, $S(\tau) : E \to E$. These mappings form a semigroup. Indeed, suppose that $v_0 \in E$, $v_1 = S(t_1)v_0$, $t_1 > 0$, and $v_2 = S(t_2 + t_1)v_0$, $t_2 > 0$. It is obvious that v_0, v_1 and v_2 are the values of the solution $u(\cdot) \in \mathcal{F}_{t_2+t_1}$ at $t = 0$, $t = t_1$, and $t = t_2 + t_1$ respectively. Consider the function $u_1(t) = u(t + t_1)$, $t \in [0, t_2]$. Since $u(\cdot) \in \mathcal{F}_{t_2+t_1}$, it follows that $u_1(\cdot) \in \mathcal{F}_{t_2}$. It is clear that $u_1(t)$ is a solution of Equation (1.7). It is obvious that $u_1|_{t=0} = v_1$ and $u_1|_{t=t_2} = v_2$, i.e., $v_2 = S(t)v_1$ by the definition of $\{S(t)\}$. Hence $S(t_2)S(t_1)v_0 = S(t_2 + t_1)v_0$ for all $v_0 \in E$ and the semigroup identity (1.1) is proved.

Below, for particular equations of the form (1.7) we only formulate the existence and uniqueness theorems and specify a space or a set where the semigroup $\{S(t)\}$ acts. We assume that operators $S(t)$ are defined by formula (1.9).

1.3. Global attractors for autonomous equations.

1.3.1. 2D Navier–Stokes system. The Navier–Stokes system is probably the most popular example of a partial differential equation having a global attractor. A considerable part of the theory of infinite-dimensional dynamical systems has been developed from this example.

We consider the autonomous 2D Navier–Stokes system in a bounded domain $\Omega \Subset \mathbb{R}^2$

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{2} u^i \partial_{x_i} u = \nu \Delta u - \nabla p + g(x), \\ &(\nabla, u) = 0, \ u|_{\partial\Omega} = 0, \quad (x_1, x_2) \in \Omega, \end{aligned} \tag{1.10}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity vector, the scalar function $p = p(x,t)$ is the pressure, ν is the kinematic viscosity coefficient, and $g = g(x) = (g^1(x), g^2(x))$ is the forcing term.

We denote by H and $V = H^1$ the closure of the set $\mathcal{V} = \{v \mid v \in (C_0^\infty(\Omega))^2,\ (\nabla, v) = 0\}$ in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $(L_2(\Omega))^2$ and $(H_0^1(\Omega))^2$ respectively. Recall that

$$\|u\|^2 = |\nabla u|^2 = \sum_{i=1}^{2} \int_\Omega |\nabla u^i(x)|^2 dx.$$

We denote by P the orthogonal projection from $(L_2(\Omega))^2$ onto H or an extension of H.

Excluding the pressure, we can write the system (1.10) in the form

$$\partial_t u + \nu L u + B(u,u) = g_0(x), \tag{1.11}$$

where $L = -P\Delta$, $B(u,v) = P \sum_{i=1}^{2} u^i \partial_{x_i} v$, $g_0 = Pg$.

Denote by $V' = V^*$ the dual of V. The Stokes operator L, considered as an operator on $V \cap (H^2(\Omega))^2$, is positive and selfadjoint. The minimal eigenvalue λ_1 of L is positive. Suppose that $g(\cdot) \in H$. The initial conditions are posed at $t = 0$:

$$u|_{t=0} = u_0(x), \quad u_0 \in H. \tag{1.12}$$

The operator L is a bounded operator from V into V'.

Consider the trilinear continuous on V form

$$b(u,v,w) = (B(u,v),w) = \int\limits_\Omega \sum_{i,j=1}^{2} u^i \partial_{x_i} v^j w^j \, dx,$$

where the operator B maps $V \times V$ into V'. The form b satisfies the identities

$$b(u,v,v) = 0, \; b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in V \tag{1.13}$$

and the estimate (see [**87, 117**])

$$|b(u,u,v)| \leqslant c_0^2 |u| \|u\| \|v\| \quad \forall u,v \in V, \tag{1.14}$$

where the constant c_0 can be taken from the inequality

$$\|f\|_{L_4(\Omega)} \leqslant c|f|^{1/2} |\nabla f|^{1/2}, \; f \in H_0^1(\Omega), \; c_0 = c. \tag{1.15}$$

The constant c (and c_0) is independent of Ω. In particular, from (1.14) it follows that

$$|B(u,u)|_{V'} \leqslant c_0^2 \|u\| |u|.$$

Thus, if $u \in L_2(0,T;V) \cap L_\infty(0,T;H)$, then $-\nu L u - B(u,u) + g(x) \in L_2(0,T;V')$, Equation (1.11) can be considered in the sense of distributions in the space $\mathcal{D}'(0,T;V')$, and $\partial_t u \in L_2(0,T;V')$.

Proposition 1.1. *The problem* (1.11), (1.12) *has a unique solution* $u(t) \in C(\mathbb{R}_+;H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+;V)$, $\partial_t u \in L_2^{\mathrm{loc}}(\mathbb{R}_+;V')$, *and the following estimates hold*:

$$|u(t)|^2 \leqslant |u(0)|^2 e^{-\nu\lambda t} + \nu^{-2}\lambda^{-2}|g|^2, \tag{1.16}$$

$$|u(t)|^2 + \nu \int\limits_0^t \|u(s)\|^2 ds \leqslant |u(0)|^2 + t\nu^{-1}\lambda^{-1}|g|^2, \tag{1.17}$$

$$t\|u(t)\|^2 \leqslant C(t, |u(0)|^2), \tag{1.18}$$

where $\lambda = \lambda_1$ is the first eigenvalue of the Stokes operator L and $C(z,R)$ is a monotone continuous functions of $z = t$ and R.

The existence and uniqueness theorem is a classical result. A detailed proof can be found in [**87, 96, 117, 9, 40**].

Thus, there exists a semigroup $\{S(t)\}$ acting in H, i.e., $S(t) : H \to H$ for $t \geqslant 0$, and corresponding to the problem (1.11), (1.12), i.e., $S(t)u_0 = u(t)$, where $u(t)$ is a solution of (1.11), (1.12).

Proposition 1.2. *The semigroup $\{S(t)\}$ corresponding to the problem* (1.11), (1.12) *is uniformly (H,H)-bounded, compact, and (H,H)-continuous.*

A detailed proof can be found, for example, in [**9, 119**]. The existence of a bounded absorbing set follows from (1.16) (see also Section 2.6.1 below, where a nonautonomous system is considered). By Propositions 1.1 and 1.2, the semigroup $\{S(t)\}$ satisfies all the assumptions of Theorem 1.2.

Theorem 1.3. *The semigroup* $\{S(t)\}$ *corresponding to the problem* (1.11), (1.12) *has a global attractor* $\mathcal{A}$ *which is compact in* H *and coincides with the kernel section, i.e.,* $\mathcal{A} = \mathcal{K}(0)$.

Introduce a dimensionless number, called the (generalized) *Grashof number*, by the formula

$$G = \frac{|g|}{\nu^2 \lambda_1}.$$

It plays an important role in the analysis of the structure of $\mathcal{A}$.

Proposition 1.3. *Suppose that*

$$G < 1/c_0^2, \tag{1.19}$$

where c_0 *is the constant from the inequality* (1.14). *Then Equation* (1.11) *has a unique stationary solution* $z \in V$, *and this solution is globally asymptotically stable, i.e.,* $\mathcal{A} = \{z\}$.

Proof. It is well known that Equation (1.11) has a stationary solution z (see, for example [**117**]), $\nu L z + B(z, z) = g$. By (1.17),

$$\|z\|^2 = |\nabla z|^2 \leqslant \frac{|g|^2}{\nu^2 \lambda_1}. \tag{1.20}$$

Every solution $u(t)$ of Equation (1.11) can be written as $u(t) = z + v(t)$, where $v(t)$ satisfies the equation

$$\partial_t v + \nu L v + B(v, v) + B(v, z) + B(z, v) = 0.$$

Multiplying by v and using (1.14), (1.13), the inequality $|v| \leqslant \lambda_1^{-1/2} \|v\|$, and (1.20), we find

$$\begin{aligned} \partial_t |v|^2 + 2\nu \|v\|^2 &= 2b(v, v, z) \leqslant 2c_0^2 |v| \|v\| \|z\| \\ &\leqslant 2c_0^2 \lambda_1^{-1/2} \|v\|^2 \|z\| \leqslant 2c_0^2 \lambda_1^{-1} \nu^{-1} |g| \|v\|^2. \end{aligned}$$

Finally,

$$\partial_t |v|^2 + 2(\nu - c_0^2 \lambda_1^{-1} \nu^{-1} |g|) \|v\|^2 \leqslant 0.$$

Hence

$$\partial_t |v(t)|^2 + \alpha |v(t)|^2 \leqslant 0,$$

where $\alpha = 2(\nu - c_0^2\lambda_1^{-1}\nu^{-1}|g|)\lambda_1^{-1} > 0$ since $\dfrac{|g|}{\nu^2\lambda_1} = G < c_0^{-2}$. This implies

$$|v(t)|^2 = |u(t) - z|^2 \leqslant |u(0) - z|^2 e^{-\alpha t}.$$

Consequently, the stationary solution is unique and asymptotically stable, and $\mathcal{A} = \{z\}$. □

Remark 1.2. The inequality (1.15) was originally proved with $c \leqslant 2^{1/4}$ in [**87**]. It is known [**109**] that $c < \left(\dfrac{16}{27\pi}\right)^{1/4}$. As was shown in [**16**], the constant c_0^2 in (1.14) can be taken as $c_0^2 = \dfrac{c^2}{\sqrt{2}} = \left(\dfrac{8}{27\pi}\right)^{1/2}$. Therefore, the attractor $\mathcal{A}$ is trivial if $G < 3.2562$.

If the Grashof number $G = \dfrac{|g|}{\nu^2\lambda_1}$ is large, the solutions of the Navier–Stokes system can tend, as $t \to +\infty$, to an attracting set much more complicated than a stationary solution. Such a situation is plausible by the physical evidence and simulation results. Respectively, the structure of the global attractor $\mathcal{A}$ can be very complicated and, possibly, chaotic (see, for example, [**58, 59, 60**]). In Section 1.4.2, we study upper bounds for the dimension of the global attractors of the Navier–Stokes equations which depend of the Grashof numbers. Roughly speaking, flows can be described by a finite (possibly, very large) number of parameters, despite the fact that the system is infinite-dimensional.

1.3.2. Wave equation with dissipation. We consider the hyperbolic equation

$$\partial_t^2 u + \gamma\partial_t u = \Delta u - f(u) + g(x), \ \ u|_{\partial\Omega} = 0, \, x \in \Omega \Subset \mathbb{R}^n, \tag{1.21}$$

with the damping (dissipation) term $\gamma\partial_t u$, $\gamma > 0$. We assume that $g \in L_2(\Omega)$ and the nonlinear function $f(v) \in C^1(\mathbb{R})$ satisfies the conditions

$$F(v) \geqslant -mv^2 - C_m, \ F(v) = \int_0^v f(w)dw, \tag{1.22}$$

$$f(v)v - \gamma_1 F(v) + mv^2 \geqslant -C_m \quad \forall\, v \in \mathbb{R}, \tag{1.23}$$

where $m > 0$, $\gamma_1 > 0$, and m is sufficiently small ($m < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with zero boundary conditions).

Remark 1.3. The conditions (1.22) and (1.23) are satisfied, for example, if

$$\liminf_{|v|\to\infty} \frac{F(v)}{v^2} \geqslant 0, \quad ; \liminf_{|v|\to\infty} \frac{f(v)v - \gamma_1 F(v)}{v^2} \geqslant 0. \tag{1.24}$$

Assume that ρ is positive and $\rho < 2/(n-2)$ for $n \geqslant 3$ and ρ is arbitrary for $n = 1, 2$. We also assume that

$$|f'(v)| \leqslant C_0(1 + |u|^\rho). \tag{1.25}$$

The case $\rho < 2/(n-2)$ for Equation (1.21) was studied in [**71, 64**] and other works. The case $\rho = 2/(n-2)$ was considered in [**9, 90, 2**] (see also [**55, 66, 111**]). Here, we discuss the case $\rho < 2/(n-2)$.

Remark 1.4. Nonlinear hyperbolic equations of type (1.21) appear in many branches of physics. For example, the dynamics of a Josephson junction driven by a current source is simulated by the sine–Gordon equation of the form (1.21) with

$$f(u) = \beta \sin u.$$

It is clear that the inequality (1.24) holds. Another important example is encountered in relativistic quantum mechanics with the nonlinear term

$$f(u) = |u|^\rho u.$$

In this case, $F(u) = |u|^{\rho+2}/(\rho+2)$ and the inequality (1.24) holds with $\gamma_1 = 1/(\rho+2)$ (see [**119**] and the references therein).

From (1.25) it follows that

$$|f(v)| \leqslant C_1(1 + |u|^{\rho+1}). \tag{1.26}$$

By the Sobolev embedding theorem,

$$H_0^1(\Omega) \subset L_{2(\rho+1)}(\Omega). \tag{1.27}$$

For $n = 1, 2$ it is valid for any ρ. For $n \geqslant 3$, by the above assumptions, $2(\rho+1) < 2n/(n-2)$, where $2n/(n-2)$ is the critical exponent in the Sobolev embedding theorem.

Suppose that $u \in L_\infty(0, T; H_0^1(\Omega))$ and $\partial_t u \in L_\infty(0, T; L_2(\Omega))$. Then $\Delta u \in L_\infty(0, T; H^{-1}(\Omega))$ and $f(u) \in L_\infty(0, T; L_2(\Omega))$ in view of (1.27). Therefore, $-\gamma\partial_t u + \Delta u - f(u) + g(x) \in L_\infty(0, T; H^{-1}(\Omega))$ and Equation (1.21) can be considered in the sense of distributions in the space $\mathcal{D}'(0, T; H^{-1}(\Omega))$. In particular, $\partial_t^2 u \in L_\infty(0, T; H^{-1}(\Omega))$ (see [**96**]).

The initial conditions are posed at $t = 0$:

$$u|_{t=0} = u_0(x),\ \partial_t u|_{t=0} = p_0(x). \tag{1.28}$$

Proposition 1.4. *If $u_0 \in H_0^1(\Omega)$ and $p_0 \in L_2(\Omega)$, then, under the above assumptions, the problem* (1.21), (1.28) *has a unique solution $u(t) \in C(\mathbb{R}_+; H_0^1(\Omega))$, $\partial_t u(t) \in C(\mathbb{R}_+; L_2(\Omega))$, $\partial_t^2 u(t) \in L_\infty(\mathbb{R}_+; H^{-1}(\Omega))$.*

We write $y(t) = (u(t),\ \partial_t u(t)) = (u(t), p(t)),\ y_0 = (u_0, p_0) = y(0)$ for brevity. We denote by E the space of vector-valued functions $y(x) = (u(x),\ p(x))$ with finite energy norm $\|y\|_E^2 = |\nabla u|^2 + |p|^2$ in $E = H_0^1(\Omega) \times L_2(\Omega)$. Then $y(t) \in E$ for every $t \geqslant 0$.

The unique solvability of the problem (1.21), (1.28) in the energy space E and properties of solutions are established in [**96, 9, 119, 68**] (see also [**34**] for more general cases).

The problem (1.21), (1.28) is equivalent to the problem

$$\begin{aligned} &\partial_t u = p, \\ &\partial_t p = -\gamma p + \Delta u - f(u) + g, \\ &u|_{t=0} = u_0, \quad p|_{t=0} = p_0, \end{aligned}$$

which can be written in the short form as

$$\partial_t y = A(y),\ y|_{t=0} = y_0. \tag{1.29}$$

Thus, if $y_0 \in E$, then the problem (1.21), (1.28) has a unique solution $y(t) \in C_b(\mathbb{R}_+; E)$. This means that the semigroup $\{S(t)\}$, $S(t)y_0 = y(t)$, is defined in E.

Proposition 1.5. *The semigroup $\{S(t)\}$ corresponding to the problem* (1.21), (1.28) *is bounded, asymptotically compact, and (E, E)-continuous.*

We will come back to this assertion in Section 2.6.2, where more general nonautonomous hyperbolic equations are considered.

Theorem 1.2 and Proposition 1.5 imply the following assertion.

Theorem 1.4. *The semigroup $\{S(t)\}$ corresponding to the problem* (1.21), (1.28) *has a global attractor $\mathcal{A}$ which is compact in E and coincides with the kernel section, i.e., $\mathcal{A} = \mathcal{K}(0)$.*

1.3.3. Ginzburg–Landau equation. This equation serves as a model in many areas of physics and mechanics [**84, 86**], for example, in the theory of superconductivity. The complex Ginzburg–Landau equation has the form

$$\partial_t u = (1 + \alpha i)\Delta u + Ru - (1 + i\beta)|u|^2 u,\ \ x \in \Omega \Subset \mathbb{R}^n. \tag{1.30}$$

We consider the case of periodic boundary conditions in $\Omega =]0, 2\pi[^n$ or zero boundary conditions $u|_{\partial\Omega} = 0$ in an arbitrary domain $\Omega \Subset \mathbb{R}^n$. In (1.30), $u = u^1 + iu^2$, $\alpha, \beta \in \mathbb{R}$ are the dispersion parameters, and $R > 0$ is the instability parameter. For $\mathbf{u} = (u^1, u^2)^\top$ we have

$$\begin{aligned} &\partial_t u^1 = \Delta u^1 - \alpha\Delta u^2 + Ru^1 - (|u^1|^2 + |u^1|^2)(u^1 - \beta u^2), \\ &\partial_t u^2 = \alpha\Delta u^1 + \Delta u^2 + Ru^2 - (|u^1|^2 + |u^1|^2)(\beta u^1 + u^2) \end{aligned} \tag{1.31}$$

or, shortly,

$$\partial_t \mathbf{u} = a\Delta\mathbf{u} + R\mathbf{u} - \mathbf{f}(\mathbf{u}), \tag{1.32}$$

where $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$ and $\mathbf{f}(\mathbf{u}) = |\mathbf{u}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{u}$. Consider the Jacobi matrix of $\mathbf{f}(\mathbf{u})$

$$\mathbf{f}_{\mathbf{u}}(\mathbf{u}) \\ = \begin{pmatrix} 3(u^1)^2 - 2\beta(u^1)(u^2) + (u^2)^2 & -\beta(u^1)^2 + 2(u^2)(u^1) - 3\beta(u^2)^2 \\ 3\beta(u^1)^2 + 2(u^2)(u^1) + \beta(u^2)^2 & (u^1)^2 + 2\beta(u^1)(u^2) + 3(u^2)^2 \end{pmatrix}. \tag{1.33}$$

Denote by B the matrix of the bilinear form corresponding to the matrix on the right-hand side of (1.33):

$$B = \begin{pmatrix} 3(u^1)^2 - 2\beta(u^1)(u^2) + (u^2)^2 & \beta(u^1)^2 + 2(u^2)(u^1) - \beta(u^2)^2 \\ \beta(u^1)^2 + 2(u^2)(u^1) - \beta(u^2)^2 & (u^1)^2 + 2\beta(u^1)(u^2) + 3(u^2)^2 \end{pmatrix}.$$

The diagonal elements of B are positive if $|\beta| \leqslant \sqrt{3}$. Moreover,

$$\det B = (3 - \beta^2)((u^1)^2 + (u^2)^2) = (3 - \beta^2)|\mathbf{u}|^4$$

is also positive. Thus, the matrix B is positive definite. Therefore,

$$\mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{v} \cdot \mathbf{v} \geqslant 0 \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^2 \tag{1.34}$$

if $|\beta| \leqslant \sqrt{3}$.

We use the spaces $\mathbf{H} = L_2(\Omega; \mathbb{C})$, $\mathbf{V} = H_0^1(\Omega; \mathbb{C})$, and $\mathbf{L}_4 = L_4(\Omega; \mathbb{C})$. The Cauchy problem for Equation (1.32) with initial data

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x), \ \mathbf{u}_0(\cdot) \in \mathbf{H}, \tag{1.35}$$

has a unique weak solution $\mathbf{u}(t) := \mathbf{u}(x, t)$ such that

$$\mathbf{u}(\cdot) \in C(\mathbb{R}_+; \mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap L_4^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_4), \tag{1.36}$$

and $\mathbf{u}(t)$ satisfies Equation (1.32) in the sense of distributions in the space $\mathcal{D}'(\mathbb{R}_+; \mathbf{H}^{-r})$, where $\mathbf{H}^{-r} = H^{-r}(\Omega; \mathbb{C})$ and $r = \max\{1, n/4\}$ (recall that $n = \dim(\Omega)$). In particular, $\partial_t \mathbf{u}(\cdot) \in L_2(0, M; \mathbf{H}^{-1}) + L_{4/3}(0, M; \mathbf{L}_{4/3})$ for any $M > 0$. The existence of such a solution $\mathbf{u}(t)$ can be proved, for example, by the Galerkin approximation method (see, for example, [**119, 9, 34**]). The proof of the uniqueness theorem is also standard and relies on the inequality (1.34). If (1.34) fails, the uniqueness theorem for $n \geqslant 3$ and arbitrary values of the dispersion parameters α and β was not proved yet (see [**101, 102, 136**] for known uniqueness results).

Any solution $\mathbf{u}(t), t \geqslant 0$, of (1.32) satisfies the differential identity

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|^2 + \|\nabla \mathbf{u}(t)\|^2 + \|\mathbf{u}(t)\|^4_{\mathbf{L}_4} - R\|\mathbf{u}(t)\|^2 = 0 \quad \forall\, t \geqslant 0, \tag{1.37}$$

where the real-valued function $\|\mathbf{u}(t)\|^2$ is absolutely continuous for $t \geqslant 0$. Here, $\|\cdot\|$ denotes the L_2-norm in $\mathbf{H}$.

The proof of (1.37) is similar to that of the identity for weak solutions of the reaction–diffusion systems considered in [**32, 34, 129**].

Equation (1.32) generates a semigroup $\{S(t)\}$ in $\mathbf{H}$ which is $(\mathbf{H}, \mathbf{H})$-continuous and compact (see, for example, [**119, 34**]). By Theorem 1.1, there exists a global attractor $\mathcal{A}$ of this semigroup. It describes the long time behavior of solutions of the Ginzburg–Landau equation. As is known, the dynamics of this system is chaotic for certain values of parameters, for example, $\alpha\beta < 0$ (see [**10, 46**]). However, in Section 1.4.2, we show that the dimension of the global attractor of the Ginzburg–Landau equation is finite.

We consider the case $|\beta| > \sqrt{3}$, where (1.34) is not longer valid. If $n = 1, 2$, it is still possible to construct a semigroup in $\mathbf{H} = (L_2(\Omega))^2$ with a compact global attractor (see [**63, 119**]). If $n \geqslant 3$, it is possible to prove the existence of a global attractor in $\mathbf{L}_p = (L_p(\Omega))^2$, $p > n$, provided that $(\alpha, \beta) \in \mathcal{P}(n)$, where $\mathcal{P}(n)$ is a subset of $\mathbb{C}$ (see [**46, 47, 103, 101**] for details).

Thus, we see that if (1.34) fails and $|\beta| \leqslant \sqrt{3}$, there is an obstacle for constructing a semigroup and studying a global attractor. Fortunately, this obstacle can be removed by using another approach based on the so-called trajectory attractors (see [**34, 129**]). In particular, this method works for the Ginzburg–Landau equation with arbitrary n, α, and β.

The inhomogeneous Ginzburg–Landau equation

$$\partial_t u = (1 + \alpha i)\Delta u + Ru - (1 + i\beta)|u|^2 u + g(x), \ g \in L_2(\Omega; \mathbb{C}),$$

is also considered in applications, where, for example, $g(x) = \delta \exp(ik \cdot x)$, $k \in \mathbb{Z}^n$, $\delta > 0$. This equation generates a semigroup, and Theorem 1.1 is applicable.

1.4. Dimension of global attractors.

In this section, we present some known results concerning the dimension of global attractors of autonomous evolution equations. Upper and lower dimension estimates are discussed in detail in [**119**] and [**9**] (see also [**37, 3**]).

1.4.1. Dimension of invariant sets. We define the Kolmogorov ε-entropy of a compact set X in a Hilbert (Banach) space E. We denote by $N_\varepsilon(X, E) = N_\varepsilon(X)$ the minimum number of open balls in E with radius ε which is necessary to cover X :

$$N_\varepsilon(X) := \Big\{ \min N \mid X \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon) \Big\}.$$

Here, $B(x_i, \varepsilon) = \{x \in E \mid \|x - x_i\|_E < \varepsilon\}$ is the ball in E with center x_i and radius ε. Since the set X is compact, $N_\varepsilon(X) < +\infty$ for any $\varepsilon > 0$.

Definition 1.6. The *Kolmogorov ε-entropy* of a set X in the space E is the number

$$\mathbf{H}_\varepsilon(X, E) := \mathbf{H}_\varepsilon(X) := \log_2 N_\varepsilon(X). \tag{1.38}$$

For particular sets X, the problem is to study the asymptotic behavior of $\mathbf{H}_\varepsilon(X)$ as $\varepsilon \to 0+$. This characteristic of compact sets was originally introduced by Kolmogorov and was studied in [**83**], where the ε-entropy was considered for different classes of functions. An important notion of the *entropy dimension* of a compact set was also introduced there. This dimension is often referred to as the *fractal dimension*.

Definition 1.7. The (upper) *fractal dimension* of a compact set X in E is the number

$$\mathbf{d}_F(X, E) := \mathbf{d}_F(X) := \limsup_{\varepsilon \to 0+} \frac{\mathbf{H}_\varepsilon(X)}{\log_2(1/\varepsilon)}. \tag{1.39}$$

The fractal dimension of a compact set in an infinite-dimensional space can be infinite. However, if $0 < \mathbf{d}_F(X) < +\infty$, then $\mathbf{H}_\varepsilon(X) \approx \mathbf{d}_F(X) \log_2(1/\varepsilon)$. Therefore, in this case, $N_\varepsilon(X) \approx (1/\varepsilon)^{\mathbf{d}_F(X)}$ points are required for approximating the set X with accuracy ε.

Another important characteristic of a compact set X is the Hausdorff dimension

$$\mathbf{d}_H(X) := \inf\{d \mid \mu(X, d) = 0\},$$

where $\mu(X, d) = \inf \sum r_i^d$ and the infimum is taken over all coverings of the set X by balls $B(x_i, r_i)$ with radii $r_i \leqslant \varepsilon$ (see [**120**]). Apparently, $\mathbf{d}_H(X) \leqslant \mathbf{d}_F(X)$, and there are examples of sets such that $\mathbf{d}_H(X) = 0$ but $\mathbf{d}_F(X) = +\infty$.

In this paper, we deal only with the fractal dimension because it is closely connected with the ε-entropy of compact sets.

Remark 1.5. The fractal and Hausdorff dimensions are very useful for studying the structure of "nonsmooth" sets, for example, the selfsimilar sets or fractals. The simplest example of such a set is the Cantor set K on the segment $[0,1]$ for which $\mathbf{d}_F(K) = \mathbf{d}_H(K) = \log_3 2 < 1$. For a compact smooth manifold the fractal (and Hausdorff) dimension is equal to the usual dimension and thereby is an integer. The example of the Cantor set shows that the fractal dimension is not necessarily integer.

Consider the ε-entropy and fractal dimension of strictly invariant sets and global attractors of autonomous evolution equations of the form (1.7). Let the Cauchy problem (1.7), (1.8) generate a semigroup $\{S(t)\}$ acting in a Hilbert space E (see Section 1.1). Consider a compact set $X \Subset E$. Let the set X be strictly invariant with respect to $\{S(t)\}$, i.e., $S(t)X = X$ for all $t \geqslant 0$. (For example, $X = \mathcal{A}$, where $\mathcal{A}$ is the global attractor.) We assume that the semigroup $\{S(t)\}$ is uniformly quasidifferentiable on X in the following sense: for any $t \geqslant 0$, $u \in X$ there is a linear bounded operator (*quasidifferential*) $L(t,u) : E \to E$ such that

$$\|S(t)v_1 - S(t)v - L(t,u)(v_1 - v)\|_E \leqslant \gamma(\|v_1 - v\|_E, t)\|v_1 - v\|_E \tag{1.40}$$

for all $v, v_1 \in X$ and $\gamma = \gamma(\xi, t) \to 0+$ as $\xi \to 0+$ for every fixed $t \geqslant 0$. Assume that the linear operators $L(t,u)$ are generated by the variational equation for (1.7) which we write in the form

$$\partial_t v = A_u(u(t))v, \quad v|_{t=0} = v_0 \in E, \tag{1.41}$$

where $u(t) = S(t)u_0$, $u_0 \in X$, $A_u(\cdot)$ is the formal derivative in u of the operator $A(\cdot)$ in (1.7) and the domain E_1 of the operator $A_u(u(t))$ is dense in E. We also assume that for every $u_0 \in X$ the linear problem (1.41) is uniquely solvable for all $v_0 \in E$. By assumption, the quasidifferentials $L(t,u_0)$ in (1.40) act on a vector v_0 by the rule $L(t,u_0)v_0 = v(t)$, where $v(t)$ is a solution of Equation (1.41) with initial data v_0.

Let $j \in \mathbb{N}$, and let $L : E_1 \to E$ be a linear (possibly, unbounded) operator. The j-*trace* of the operator L is the number

$$\mathrm{Tr}_j L := \sup_{\{\varphi_i\}_{i=1}^{j}} \sum_{i=1}^{j} (L\varphi_i, \varphi_i), \tag{1.42}$$

where the infimum is taken over all orthonormal in E families of vectors $\{\varphi_i\}_{i=1,\dots,j}$ belonging to E_1 and (ψ, φ) denotes the inner product of ψ and φ in E.

Definition 1.8. We set

$$\widetilde{q}_j := \limsup_{T\to+\infty} \sup_{u_0\in X} \frac{1}{T}\int_0^T \mathrm{Tr}_j A_u(u(t))dt,\ j=1,2,\dots, \tag{1.43}$$

where $u(t) = S(t)u_0$.

Theorem 1.5. *Suppose that a semigroup $\{S(t)\}$ acting in E has a compact strictly invariant set X and is uniformly quasidifferentiable on X. Let $\widetilde{q}_j \leqslant q_j$, $j = 1, 2, 3, \dots$, where $\widetilde{q}_j$ are defined in* (1.43)*. Suppose that q_j is concave in j (like $\cap$). Let m be the smallest integer such that $q_{m+1} < 0$ (then, clearly, $q_m \geqslant 0$). Let*

$$d = m + \frac{q_m}{q_m - q_{m+1}}. \tag{1.44}$$

Then X has the finite fractal dimension and

$$\mathbf{d}_F(X) \leqslant d. \tag{1.45}$$

Furthermore, for every $\delta > 0$ there exist real numbers $\eta \in (0,1)$ and $\varepsilon_0 > 0$ such that for the ε-entropy $\mathbf{H}_\varepsilon(X)$ of X the following estimate holds:

$$\mathbf{H}_\varepsilon(X) \leqslant (d+\delta)\log_2(\varepsilon_0/\eta\varepsilon) + \mathbf{H}_{\varepsilon_0}(X) \quad \forall\varepsilon < \varepsilon_0. \tag{1.46}$$

This theorem is proved in [**34**]. The proof is based on the study of the volume contraction properties under the action of the quasidifferentials of semigroup operators. Estimates, similar to (1.45), for the Hausdorff dimension of invariant sets were first obtained [**48**] for a finite-dimensional space E and then were generalized [**41, 119**] for an infinite-dimensional space E (see also [**76, 4, 9**]).

We note that the estimate (1.46) for the ε-entropy of $\mathcal{A}$ follows from (1.45) and, in general, does not give any new information about global attractors. However, in the study of nonautonomous equations (see Section 3), the estimates for the ε-entropy of global attractors become more informal and constitutive. This explains why the estimate (1.46) is included into this key theorem.

Remark 1.6. In applications, the numbers q_j are usually used in the form $q_j = \varphi(j)$, where $\varphi = \varphi(x)$, $x \geqslant 0$, is a smooth concave function. Consider the root d^* of φ, i.e., $\varphi(d^*) = 0$. It is obvious that $d \leqslant d^*$ since φ is concave. For large d the root d^* is very close to d defined by (1.44). Since d^* is sometimes expressed in a simpler way than d, we will use d^* instead of d as the upper bound in (1.45) for the fractal dimension of attractors. In this case, in (1.46), we can set $\delta = d^* - d$ if it is positive.

Recently [**16**], the estimates (1.46) and (1.45) were proved for the exact values $q_j = \widetilde{q}_j$ without the concavity assumption on $\widetilde{q}_j$ in j. The so-called (*global*) *Lyapunov dimension* of a set X (see [**82, 49**]) is defined by the formula

$$d_L := m + \frac{\widetilde{q}_m}{\widetilde{q}_m - \widetilde{q}_{m+1}}.$$

The inequality $d_H(X) \leqslant d_L(X)$ was proved in [**48, 39, 119**]. As was shown in [**16**], $d_F(X) \leqslant d_L(X)$. A similar result was obtained earlier in [**12**]; namely, it was proved that if $\widetilde{q}_m < 0$ for some $m \in \mathbb{N}$, then $\mathbf{d}_F(X) \leqslant m$ (see also [**75**]).

Many examples of evolution equations in mathematical physics and mechanics are described in [**9, 119, 68**], where global attractors are also constructed and upper estimates were proved for the Hausdorff dimension and the fractal dimension of these attractors.

Further, we discuss fractal dimensions estimates for global attractors of autonomous equations considered in Section 1.3.

1.4.2. Dimension estimates for autonomous equations.

2D Navier–Stokes system

We consider the 2D Navier–Stokes system

$$\partial_t u = -\nu L u - B(u,u) + g, \ (\nabla, u) = 0, \ u|_{\partial\Omega} = 0, \tag{1.47}$$

$$u|_{t=0} = u_0, \ u_0 \in H, \tag{1.48}$$

where $g \in H$. The problem (1.47), (1.48) defines a semigroup $\{S(t)\}$ acting in H (see Section 1.3.1). By Theorem 1.3, the semigroup $\{S(t)\}$ has a global attractor $\mathcal{A}$ which is bounded in V and is compact in H.

Theorem 1.6. *The fractal dimension of the global attractor $\mathcal{A}$ of the problem* (1.47), (1.48) *satisfies the estimate*

$$\mathbf{d}_F\,\mathcal{A} \leqslant c\frac{|g||\Omega|}{\nu^2}, \tag{1.49}$$

where c depends on the shape of Ω ($c(\lambda\Omega) = c(\Omega)$ for all $\lambda > 0$).

The Kolmogorov ε-entropy of $\mathcal{A}$ satisfies the estimate

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant c\frac{|g||\Omega|}{\nu^2}\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \quad \forall \varepsilon < \varepsilon_0, \tag{1.50}$$

where η and ε_0 are small positive numbers.

Proof. The semigroup $\{S(t)\}$ is uniformly quasidifferentiable on $\mathcal{A}$ in H and the quasidifferential of $\{S(t)\}$ is the operator $L(t, u_0)v_0 = v(t)$, $v_0 \in H$, where $v(t)$ is a solution of the variation equation

$$\partial_t v = -\nu L - B(u(t), v) - B(v, u(t)) := A_u(u(t))v, \quad v|_{t=0} = v_0$$

(see [**4, 9**]). We need to estimate the j-trace of $A_u(u(t))$. Note that for all $v \in V$

$$(A_u(u(t))v, v) = \nu\|v\|^2 - (B(v, u(t)), v) \tag{1.51}$$

since $(B(u, v), v) = 0$ for $u, v \in V$.

Let $\varphi_1, \dots, \varphi_j \in V$ be an arbitrary orthonormal family in H. Using (1.51), we find

$$\begin{aligned}\sum_{i=1}^{j}(A_u(u(t))\varphi_i, \varphi_i) &= -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 - \sum_{i=1}^{j}(B(\varphi_i, u(t)), \varphi_i)\\ &= -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 - \int\limits_{\Omega}\sum_{i=1}^{j}\sum_{k,l=1}^{2}\varphi_i^k\partial_{x_k}u^l(t)\varphi_i^l dx\\ &\leqslant -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 + \int\limits_{\Omega}\rho(x)|\nabla u(t)|dx\\ &\leqslant -\nu\sum_{i=1}^{j}|\nabla\varphi_i|^2 + |\rho||\nabla u(t)|,\end{aligned} \tag{1.52}$$

where $\rho(x) = \sum\limits_{i=1}^{j}|\varphi_i(x)|^2$ (see [**41, 119**]).

Since functions in V vanish on $\partial\Omega$ we can extend them by zero outside Ω. Then we obtain functions $\varphi_i(x)$, $x \in \mathbb{R}^2$, in $(H^1(\mathbb{R}^2))^2$ that are orthonormal in $(L_2(\mathbb{R}^2))^2$. The following result [**94**] is extremely important.

Lemma 1.1 (Lieb–Thirring inequality)**.** *Let* $\varphi_1, \dots, \varphi_j \in (H^1(\mathbb{R}^n))^m$ *be an orthonormal family of vectors in* $(L_2(\mathbb{R}^n))^m$. *Then for*

$$\rho(x) = \sum_{i=1}^{j}|\varphi_i(x)|^2$$

the following estimate holds:

$$\int\limits_{\mathbb{R}^n}(\rho(x))^{1+2/n}dx \leqslant C_{m,n}\sum_{i=1}^{j}\int\limits_{\mathbb{R}^n}|\nabla\varphi_i|^2 dx, \tag{1.53}$$

where $C_{m,n}$ *depends only on* m *and* n.

Remark 1.7. As was proved in [**77**], for $m = 2$, $n = 2$ one has $C_{2,2} \leqslant 2$ if $\operatorname{div} \varphi_i = 0$.

By the variational principle,

$$\sum_{i=1}^{j} |\nabla \varphi_i|^2 \geqslant \lambda_1 + \lambda_2 + \ldots + \lambda_j, \tag{1.54}$$

where $\lambda_1, \lambda_2, \ldots$ are the ordered eigenvalues of the operator L. It is known that $\lambda_i \geqslant C_0 |\Omega|^{-1} i$. Therefore,

$$\lambda_1 + \lambda_2 + \ldots + \lambda_j \geqslant C_2 \frac{j^2}{|\Omega|}, \ \lambda_1 \geqslant \frac{C_1}{|\Omega|}, \tag{1.55}$$

where C_0, C_1, and C_2 are dimensionless constants depending on the shape of Ω (see, for example, [**100**]). Using (1.53) with $C_{2,2} = 2$, (1.54), and (1.55), from (1.52) we find

$$\begin{aligned} & -\nu \sum_{i=1}^{j} |\nabla \varphi_i|^2 + \Big(2 \sum_{i=1}^{j} |\nabla \varphi_i|^2 \Big)^{1/2} |\nabla u(t)| \\ \leqslant \quad & -\frac{\nu}{2} \sum_{i=1}^{j} |\nabla \varphi_i|^2 + \frac{1}{\nu} |\nabla u(t)|^2 \leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{1}{\nu} |\nabla u(t)|^2. \end{aligned}$$

Thus,

$$\operatorname{Tr}_j (A_u(u(t)) \leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{1}{\nu} |\nabla u(t)|^2.$$

Using the estimate

$$\int_0^t \|u(s)\|^2 ds \leqslant \frac{|u(0)|^2}{\nu} + \frac{|g|^2}{\nu^2 \lambda_1} t$$

(see (1.17)), we find

$$\begin{aligned} \widetilde{q}_j &= \limsup_{T \to \infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{T} \int_0^T \operatorname{Tr}_j (A_u(u(t)) dt \\ &\leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \lim_{T \to \infty} \frac{1}{\nu^2 T} \sup_{u_0 \in \mathcal{A}} |u_0|^2 + \frac{|g|^2}{\nu^3 \lambda_1}. \end{aligned}$$

Since $\sup\limits_{u_0 \in \mathcal{A}} |u_0|^2 \leqslant C_3$, we have

$$\widetilde{q}_j \leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|g|^2}{\nu^3 \lambda_1}. \tag{1.56}$$

Using the second estimate for λ_1 in (1.55), we find

$$\widetilde{q}_j \leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|g|^2|\Omega|}{\nu^3 C_1} =: \varphi(j) = q_j.$$

We note that the function $\varphi(j)$ is concave in j (like $\cap$). Looking for the root d^* of the equation $\varphi(d) = 0$, we find $d^* = \sqrt{\dfrac{2}{C_1 C_2}\dfrac{|g||\Omega|}{\nu^2}}$. Hence (1.50) and (1.49) immediately follow from Theorem 1.5 with $c = \sqrt{\frac{2}{C_1 C_2}}$ (see also Remark 1.6). □

Remark 1.8. By (1.56), the estimate (1.49) takes the form

$$\mathbf{d}_F\,\mathcal{A} \leqslant c'G, \tag{1.57}$$

where $G = \dfrac{|g|}{\nu^2\lambda_1}$ is the Grashof number and $c' = 2\sqrt{|\Omega|\lambda_1}/C_2$ depends on the shape of Ω. This estimate was proved in [**39, 41**] (see also [**119**]).

Remark 1.9. As was proved in [**78**], $C_1 \geqslant 2\pi,\ C_2 \geqslant \pi$ in any domain Ω of finite measure. Therefore, the constant c in (1.49) satisfies $c \leqslant 1/\pi$ and for c' in (1.57) we have $c' \leqslant 2\sqrt{|\Omega|\lambda_1}/\pi$. These estimates were improved in [**16**] as follows: $c \leqslant (2\pi^{3/2})^{-1}$ and $c' \leqslant \sqrt{|\Omega|\lambda_1}/(\sqrt{2}\pi)$.

Corollary 1.1. *Let $g \in H$. Then*

$$\mathbf{d}_F\,\mathcal{A} \leqslant \frac{1}{\sqrt{2}\pi}(|\Omega|\lambda_1)^{1/2}\frac{|g|}{\nu^2\lambda_1} \leqslant \frac{1}{2\pi^{3/2}}\frac{|g||\Omega|}{\nu^2}. \tag{1.58}$$

We note that the last estimate in (1.58) contains only the explicit physical parameters of system (1.47) and the estimate $c \leqslant (2\pi^{3/2})^{-1}$ seems the best up-to-date.

Remark 1.10. According to the proof of Proposition 1.3, $\mathcal{A} = \{z\}$ and, consequently, $\mathbf{d}_F\,\mathcal{A} = 0$ if $G = \dfrac{|g|}{\nu^2\lambda_1} < \dfrac{1}{c_0^2}$. Since $\lambda_1 \geqslant \dfrac{2\pi}{|\Omega|}$, the last inequality holds provided that $\dfrac{|g||\Omega|}{\nu^2} < \dfrac{2\pi}{c_0^2}$. Using the expression $c_0^2 = \left(\dfrac{8}{27\pi}\right)^{1/2}$ (see Remark 1.2), we see that $\mathcal{A} = \{z\}$ and $\mathbf{d}_F\,\mathcal{A} = 0$ provided that $\dfrac{|g||\Omega|}{\nu^2} < \left(\dfrac{27\pi^3}{2}\right)^{1/2} \approx 20.46$.

Remark 1.11. The estimates (1.58) and (1.49) hold for the 2D Navier–Stokes systems in unbounded domains with finite measure [**78**].

Remark 1.12. For the 2D Navier–Stokes system (1.47) in $\Omega = [0, 2\pi]^2$ with periodic boundary conditions the estimate (1.57) was improved in [**42**] (see also [**119**]). As was shown,

$$\mathbf{d}_F\, \mathcal{A} \leqslant c'' G^{2/3}(1 + \log G), \tag{1.59}$$

where $G = \dfrac{|g|}{\lambda_1 \nu^2}$ (note that $\lambda_1 = 1$ in this case). The estimate (1.59) is optimal in a sense (see [**97, 135**]).

Dissipative wave equation

Consider the equation

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g(x), \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbb{R}^3, \tag{1.60}$$

where $\gamma > 0$ (see Section 1.3.2). For brevity, we consider the case $n = 3$. We assume that $g(\,\cdot\,) \in L_2(\Omega)$, $f(v) \in C^2(\mathbb{R};\mathbb{R})$, and f satisfies (1.22), (1.23), and (1.25) with $\rho < 2$. We also assume that

$$|f'(v_1) - f'(v_2)| \leqslant C(|v_1|^{2-\delta} + |u_2|^{2-\delta} + 1)|v_1 - v_2|^{\delta},\ 0 \leqslant \delta \leqslant 1. \tag{1.61}$$

The Hilbert space $E = H_0^1(\Omega) \times L_2(\Omega)$ is the phase space for this equation. We introduce the space $E_1 = H^2(\Omega) \times H_0^1(\Omega)$ endowed with the norm $\|y\|_{E_1} = (\|u\|_2^2 + \|p\|_1^2)^{1/2}$. We consider the semigroup $\{S(t)\}$ in E generated by Equation (1.61). By Theorem 1.4, this semigroup has the global attractor $\mathcal{A} \Subset E$. As was proved in [**9, 119**], the set $\mathcal{A}$ is bounded in E_1 :

$$\|w\|_{E_1} \leqslant M \quad \forall\, w \in \mathcal{A},$$

where the constant M is independent of w. By the Sobolev embedding theorem,

$$\|u(\cdot)\|_{C(\overline{\Omega})} \leqslant M_1 \quad \forall\, w = (u(\cdot), p(\cdot)) = w(\cdot) \in \mathcal{A}. \tag{1.62}$$

We estimate $\mathbf{d}_F\mathcal{A}$ using Theorem 1.5 and the technique described in [**64**] (see also [**119, 34**]).

Theorem 1.7. *For the fractal dimension of the global attractor $\mathcal{A}$ of Equation* (1.60) *the following estimate holds*:

$$\mathbf{d}_F\mathcal{A} \leqslant \frac{C}{\alpha^3}, \tag{1.63}$$

where $\alpha = \min\{\gamma/4, \lambda_1/(2\gamma)\}$ *and* $C = C(M_1)$ (*see* (1.62)).

For the ε-entropy of $\mathcal{A}$ the following estimate holds:

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant \frac{C(M_1)}{\alpha^3} \log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \quad \forall\, \varepsilon < \varepsilon_0, \tag{1.64}$$

where η, ε_0 *are some positive numbers.*

PROOF. As in [**64, 119**], introduce the new variables $w = (u,v) = R_\alpha y = (u, u_t + \alpha u)$ and $u_t = \partial_t u$, $\alpha = \min\{\gamma/4, \lambda_1/(2\gamma)\}$, where λ_1 is the first eigenvalue of the operator $-\Delta u$, $u|_{\partial\Omega} = 0$. In these variables, Equation (1.60) takes the form

$$\partial_t w = L_\alpha w - G(w) =: A_\alpha w, \quad w|_{t=0} = w_0, \tag{1.65}$$

where $w_0 \in E$,

$$L_\alpha = \begin{pmatrix} -\alpha I & I \\ \Delta + \alpha(\gamma - \alpha) & -(\gamma-\alpha)I \end{pmatrix}, \quad G(w) = (0, f(u) - g(x)). \tag{1.66}$$

By (1.61), the operators $\{S(t)\}$ are uniformly quasidifferentiable on $\mathcal{A}$ and the quasidifferentials $L(t, w_0)z_0 = z(t)$ satisfy the variation equation of the problem (1.65):

$$\partial_t z = L_\alpha z - G_w(w)z =: A_{\alpha w}(w(t))z, \quad z|_{t=0} = z_0, \tag{1.67}$$

where $z = (r,q)$ and $G_w(w(t))z = (0, f'(u(t))r)$ (see, for example, [**119**]). Let us estimate the sum

$$\sum_{i=1}^{j} (A_{\alpha w}(w(t))\zeta_i, \zeta_i)_E, \tag{1.68}$$

where $\zeta_i = (r_i, q_i)$ is an arbitrary orthonormal family in E. We have

$$\begin{aligned} (A_{\alpha w}(w(t))\zeta_i, \zeta_i)_E &= (L_\alpha \zeta_i, \zeta_i) - (f'(u)r_i, q_i) \leqslant -(\alpha/2)\|\zeta_i\|_E^2 \\ +C_0(M_1)\|r_i\|_0\|q_i\|_0 &\leqslant -\alpha/4(\|r_i\|_1^2 + \|q_i\|_0^2) + (C_1(M_1)/\alpha)\|r_i\|_0^2. \end{aligned} \tag{1.69}$$

The parameter α is chosen in such a way that the operator L_α is negative:

$$(L_\alpha \zeta_i, \zeta_i) \leqslant -\alpha/2\|\zeta_i\|_E^2.$$

Observe that it was essential that

$$\sup\{\|f'(u(t))\|_{C_b} \mid (u(\cdot), \partial_t u(\cdot)) = w(\cdot) \in \mathcal{A} \ \ t \in \mathbb{R}\} \leqslant C_0(M_1) \tag{1.70}$$

(see (1.62)). Since the system ζ_i is orthonormal in E, from (1.69) it follows that

$$\begin{aligned} \sum_{i=1}^{j} (A_{\alpha w}(w(t))\zeta_i, \zeta_i)_E &\leqslant -(\alpha/4)j + (C_0^2(M_1)/\alpha)\sum_{i=1}^{j} \|r_i\|_0^2 \\ &\leqslant -(\alpha/4)j + (C_0^2(M_1)/\alpha)\sum_{i=1}^{j} \lambda_i^{-1} \\ &\leqslant -(\alpha/4)j + (C_1(M_1)/\alpha)j^{1/3}, \end{aligned} \tag{1.71}$$

where $C_1(M_1) = c_1 C_0^2(M_1)$ and λ_i, $i = 1, \dots, j$, are the first j eigenvalues of the operator $-\Delta u$, $u|_{\partial\Omega} = 0$, written in nondecreasing order. It is known that $\lambda_i \geqslant c_0 i^{2/3}$. Therefore, $\sum\limits_{i=1}^{j} \lambda_i^{-1} \leqslant c_1 j^{1/3}$. In the second inequality of (1.71), we used the inequality $\sum\limits_{i=1}^{j} \|r_i\|_0^2 \leqslant \sum\limits_{i=1}^{j} \lambda_i^{-1}$ proved in [**119**]. Thus, $\mathrm{Tr}_j A_{\alpha w}(w(t)) \leqslant \varphi(j) = -(\alpha/4)j + (C_1(M_1)/\alpha) j^{1/3}$, where the function $\varphi(x)$ is concave. The root of φ is expressed as follows:

$$d^* = \frac{8C_1(M_1)^{3/2}}{\alpha^3} = \frac{C(M_1)}{\alpha^3},$$

where $C(M) = 8C_1(M_1)^{3/2}$. Finally, we obtain (1.64) and (1.63) from Theorem 1.5 and Remark 1.6. $\square$

We consider the sine-Gordon equation with $f(u) = \beta \sin(u)$. It is clear that $C_0(M_1) = \beta$ in (1.70). Therefore, $C_1(M_1) = c_1 \beta^2$, i.e., $C(M_1) = 8c_1^{3/2}\beta^3 = c\beta^3$. Thus, the estimates (1.64) and (1.63) for the sine-Gordon equation have the form

$$\begin{aligned} \mathbf{d}_F(\mathcal{A}) &\leqslant c\frac{\beta^3}{\alpha^3}, \\ \mathbf{H}_\varepsilon(\mathcal{A}) &\leqslant c\frac{\beta^3}{\alpha^3}\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \quad \forall\, \varepsilon < \varepsilon_0, \end{aligned} \tag{1.72}$$

where the constant c depends on Ω.

Ginzburg–Landau equation

We consider an inhomogeneous equation similar to (1.30)

$$\partial_t u = \nu(1 + \alpha i)\Delta u + Ru - (1 + i\beta)|u|^2 u + g(x),\ x \in]0, 2\pi[^3 =: \mathbb{T}^3, \tag{1.73}$$

with periodic boundary conditions in $\mathbb{T}^3$ and $g(x) = g^1(x) + ig^2(x) \in L_2(\mathbb{T}^3; \mathbb{C})$. Here, ν is a positive parameter. For the sake of simplicity, we take $n = 3$. We assume that $|\beta| \leqslant \sqrt{3}$. Then Equation (1.73) generates the semigroup $\{S(t)\}$ acting in $\mathbf{H} = (L_2(\mathbb{T}^3))^2$ and having a global attractor $\mathcal{A}$ which is compact in $\mathbf{H}$ (see [**119, 34**]).

We write Equation (1.73) in the vector form (1.32)

$$\partial_t \mathbf{u} = \nu a \Delta \mathbf{u} + R\mathbf{v} - \mathbf{f}(\mathbf{u}) + \mathbf{g}(x), \tag{1.74}$$

where $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$, $\mathbf{f}(\mathbf{v}) = |\mathbf{v}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{v}$, $\mathbf{g}(x) = (g^1(x), g^2(x))^\top$. As was proved in [**4**], the semigroup $\{S(t)\}$ is uniformly quasidifferentiable on

$\mathcal{A}$ and the corresponding variational equation reads

$$\partial_t \mathbf{v} = \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f_u}(\mathbf{u})\mathbf{v}, \ \mathbf{v}|_{t=0} = \mathbf{v}_0 \in \mathbf{H}, \tag{1.75}$$

where the matrix $\mathbf{f_u}(\mathbf{u})$ is defined by (1.33). By (1.34), we have

$$\begin{aligned} &\langle \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f_u}(\mathbf{u})\mathbf{v}, \mathbf{v}\rangle = -\nu\|\nabla \mathbf{v}\|^2 + R\|\mathbf{v}\|^2 - \langle \mathbf{f_u}(\mathbf{u})\mathbf{v}, \mathbf{v}\rangle \\ &\leqslant -\nu\|\nabla \mathbf{v}\|^2 + R\|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{H}^2. \end{aligned} \tag{1.76}$$

In order to use Theorem 1.5 and to estimate $\mathbf{d}_F(\mathcal{A})$, we need to study the j-trace of the operator on the right-hand side of (1.75). By (1.76), we have

$$\begin{aligned} &\sum_{i=1}^{j} (A_u(u(t))\varphi_i, \varphi_i) = \sum_{i=1}^{j} -\nu\|\nabla \varphi_i\|^2 + R\|\varphi_i\|^2 - \langle \mathbf{f_u}(\mathbf{u})\varphi_i, \varphi_i\rangle \\ &\leqslant \sum_{i=1}^{j} -\nu\|\nabla \varphi_i\|^2 + R\|\varphi_i\|^2 = -\nu \sum_{i=1}^{j} \|\nabla \varphi_i\|^2 + Rj, \end{aligned} \tag{1.77}$$

where $\{\varphi_i, \ i = 1, \ldots, j\}$ is an arbitrary set of functions from $\mathbf{V} = (H^1(\mathbb{T}^3))^2$, orthonormal in $\mathbf{H}$. By the variational principle,

$$\sum_{i=1}^{j} |\nabla \varphi_i|^2 \geqslant \lambda_1 + \lambda_2 + \ldots + \lambda_j, \tag{1.78}$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the operator $-\Delta$ in $\mathbf{H}$. It is well known that the eigenvalues of this operator have the form $k_1^2 + k_2^2 + k_3^2$, where $(k_1, k_2, k_3) \in (\mathbb{Z}_+)^3$. Therefore, $\lambda_i \geqslant C_0 i^{2/3}$ and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_j \geqslant C_1 j^{5/3} \tag{1.79}$$

with some constants C_0 and C_1. Using (1.78) and (1.79) in (1.77), we obtain

$$Tr_j A_u(u(t)) \leqslant -\nu C_1 j^{5/3} + Rj = \varphi(j) \quad \forall j = 1, 2, \ldots \tag{1.80}$$

The function $\varphi(x) = -\nu C_1 x^{5/3} + Rx$ is concave and has the root $d^* = (R/(C_1\nu))^{3/2}$. Thus, we have proved the following assertion.

Theorem 1.8. *The fractal dimension of the global attractor $\mathcal{A}$ of Equation* (1.73) *admits the estimate*

$$\mathbf{d}_F(\mathcal{A}) \leqslant \Big(\frac{R}{C_1 \nu}\Big)^{3/2}, \tag{1.81}$$

where C_1 is an absolute constant taken from (1.79) *and can be estimated explicitly* (*see, for example,* [**93, 34**]).

The ε-entropy of $\mathcal{A}$ satisfies the inequality

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant \Big(\frac{R}{C_1\nu}\Big)^{3/2} \log_2\Big(\frac{\varepsilon_0}{\eta\varepsilon}\Big) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \quad \forall \varepsilon < \varepsilon_0, \tag{1.82}$$

where η and ε_0 are some small positive numbers.

2. Attractors of Nonautonomous Equations

In this section, we consider general processes and their global attractors. The notion of a process is used for describing the behavior of nonautonomous dynamical systems. A process is a generalization of the notion of a semigroup which plays a key role in the study of autonomous dynamical systems. Nonautonomous dynamical systems and their global attractors are discussed in [**73, 34**] (see also [**14**]).

In Section 2.1, we study processes $\{U(t,\tau), t \geqslant \tau\}$ and their uniform global attractors. Recall that the processes are generated by nonautonomous evolution equations if, for example, an external force or some other terms of the equation depend explicitly on time t. If the Cauchy problem for this equation is well-posed, the process $\{U(t,\tau)\}$ sends the value of the solution $u(\tau)$ at time $\tau \in \mathbb{R}$ to the value of $u(t)$ at time $t \geqslant \tau$.

Below, we give a definition of a general process $\{U(t,\tau)\}$ and introduce notions of uniformly absorbing and attracting sets of a process. We study the main properties of ω-limit sets for bounded sets. Then we define the uniform global attractor $\mathcal{A}$ of a process $\{U(t,\tau)\}$. We prove the theorem on the existence of a uniform global attractor of a process using the notion of the ω-limit set. We also define the kernel $\mathcal{K}$ of a process and study its properties.

In Section 2.2, we consider uniform and nonuniform global attractors of a process and compare their properties. In particular, we present an example of a nonautonomous equation, given by Haraux. This example shows that the uniform global attractor can be larger than the nonuniform one. We also study periodic processes for which uniform and nonuniform global attractors always coincide.

In Section 2.4, we introduce the notion of the time symbol $\{\sigma(t), t \in \mathbb{R}\}$ of a nonautonomous equation. Roughly speaking, the time symbol is the collection of all time-dependent terms of the equation. We define the hull $\mathcal{H}(\sigma)$ of σ. We also define a translation compact function. We mostly study nonautonomous equations having translation compact symbols $\sigma(t)$.

We present translation compactness criteria in different topological spaces, which will be used in the sequel.

In Section 2.5, we formulate the main theorem about the existence and structure of the uniform global attractor of a process $\{(U_\sigma(t,\tau\}$ of a nonautonomous equation with translation compact symbol $\sigma(t)$.

In Section 2.6.1, we study the uniform global attractor of the nonautonomous 2D Navier–Stokes system with translation compact external force. A special attention is given to the case, where the system has a unique bounded complete solution attracting any other solution as $t \to +\infty$ with exponential rate. In Sections 2.6.2 and 2.6.3, we consider analogous problems for the nonautonomous dissipative hyperbolic equation and for the nonautonomous complex Ginzburg–Landau equation with translation compact terms.

2.1. Processes and their uniform global attractors.

Let E be a complete metric space or a Banach space. Consider a two-parameter family of operators $\{U(t,\tau), \tau \in \mathbb{R}, t \geqslant \tau\}$, $U(t,\tau) : E \to E$.

Definition 2.1. A family of mappings $\{U(t,\tau)\} := \{U(t,\tau), \tau \in \mathbb{R}, t \geqslant \tau\}$ in E is called a *process* if

1) $U(\tau,\tau) = \mathrm{Id}$ for all $\tau \in \mathbb{R}$, where Id is the identity operator,

2) $U(t,s) \circ U(s,\tau) = U(t,\tau)$ for all $t \geqslant s \geqslant \tau,\ \tau \in \mathbb{R}$.

As in Section 1, we denote by $\mathcal{B}(E)$ the family of all bounded (in the norm of E) sets in E. A process $\{U(t,\tau)\}$ is said to be (E,E)-*bounded* if $U(t,\tau)B \in \mathcal{B}(E)$ for all $B \in \mathcal{B}(E)$, $\tau \in \mathbb{R}$, $t \geqslant \tau$. A process $\{U(t,\tau)\}$ is said to be *uniformly* (E,E)-*bounded* if for every $B \in \mathcal{B}(E)$ there exists $B_1 \in \mathcal{B}(E)$ such that $U(t,\tau)B \subset B_1$ for all $\tau \in \mathbb{R}$, $t \geqslant \tau$.

The following two notions describe the dissipativity properties of nonautonomous dynamical systems. A set $B_0 \subset E$ is said to be *uniformly* (with respect to $\tau \in \mathbb{R}$) *absorbing* for a process $\{U(t,\tau)\}$ if for any set $B \in \mathcal{B}(E)$ there is a number $h = h(B)$ such that

$$U(t,\tau)B \subseteq B_0 \ \forall t,\tau,\ t-\tau \geqslant h. \tag{2.1}$$

A set $P \subset E$ is said to be *uniformly* (with respect to $\tau \in \mathbb{R}$) *attracting* for a process $\{U(t,\tau)\}$ if for every $\varepsilon > 0$ the set $\mathcal{O}_\varepsilon(P)$ is uniformly absorbing for $\{U(t,\tau)\}$ (hereinafter, $\mathcal{O}_\varepsilon(M)$ denotes an ε-neighborhood of a set M in the space E), i.e., for every bounded set $B \in \mathcal{B}(E)$ there exists a number

$h = h(\varepsilon, P)$ such that

$$U(t,\tau)B \subseteq \mathcal{O}_\varepsilon(P) \quad \forall t,\tau,\ t-\tau \geqslant h. \tag{2.2}$$

The property (2.2) can be formulated in the following form: for every set $B \in \mathcal{B}(E)$

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(\tau+h,\tau)B, P) \to 0 \quad \text{as } h \to +\infty, \tag{2.3}$$

where $\operatorname{dist}_E(X,Y)$ denotes the Hausdorff distance between sets X and Y in the space E (see (1.3)).

A process having a compact uniformly absorbing set is called *uniformly compact* and a process having a compact uniformly attracting set is called *uniformly asymptotically compact.*

Now, we define the *uniform global attractor* $\mathcal{A}$ of a process $\{U(t,\tau)\}$.

Definition 2.2. A set $\mathcal{A} \subset E$ is called a *uniform* (with respect to $\tau \in \mathbb{R}$) *global attractor* of the process $\{U(t,\tau)\}$ if it is closed in E, is uniformly attracting for $\{U(t,\tau)\}$, and satisfies the following *minimality* condition: $\mathcal{A}$ belongs to any closed uniformly attracting set of the process.

It is easy to see that any process has at most one uniform global attractor. A uniform global attractor was introduced in [**73**] (see also [**18, 23, 25, 34**]).

For an arbitrary set $B \in \mathcal{B}(E)$, we define the uniform ω-limit set $\omega(B)$ by the formula

$$\omega(B) = \bigcap_{h\geqslant 0} \Big[\bigcup_{t-\tau\geqslant h} U(t,\tau)B \Big]_E, \tag{2.4}$$

where $[\,\cdot\,]_E$ denotes the closure in the space E and the union is taken for all t, τ such that $\tau \in \mathbb{R}$ and $t \geqslant \tau + h$ (see (1.4)).

Proposition 2.1. *If a process $\{U(t,\tau)\}$ in E has a compact uniformly attracting set P, then for any $B \in \mathcal{B}(E)$*

(i) $\omega(B) \neq \varnothing$, *$\omega(B)$ is compact in E, and $\omega(B) \subseteq P$,*

(ii) $\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(h+\tau,\tau)B, \omega(B)) \to 0$ $(h \to +\infty)$,

(iii) *if Y is closed and* $\sup_{\tau\in\mathbb{R}} \operatorname{dist}_E(U(h+\tau,\tau)B, Y)$ $(h \to +\infty)$, *then* $\omega(B) \subseteq Y$.

Proof. From the definition (2.4) of $\omega(B)$ it follows that

$$y \in \omega(B) \Leftrightarrow \begin{cases} \text{there are } \{x_n\} \subseteq B,\ \{\tau_n\} \subseteq \mathbb{R}, \{h_n\} \subset \mathbb{R}_+ : \\ h_n \to +\infty,\ U(\tau_n + h_n, \tau_n)x_n \to y \ (n \to \infty). \end{cases} \tag{2.5}$$

(i) We show that $\omega(B) \neq \varnothing$. For any fixed $\tau \in \mathbb{R}$ and $x \in B$ we consider an arbitrary positive sequence $\{h_n\}$, $h_n \to +\infty$ as $n \to \infty$. According to the uniformly attracting property (2.3), $\mathrm{dist}_E(U(\tau + h_n, \tau)x, P) \to 0$ $(n \to \infty)$, i.e., for some sequence $\{y_n\} \subseteq P$

$$\|U(\tau + h_n, \tau)x - y_n\|_E \to 0 \quad \text{as } n \to \infty).$$

Since the set P is compact, we can extract a subsequence $\{y_{n'}\}$ of $\{y_n\}$ converging to $y \in P$. Hence $U(\tau + h_{n'}, \tau)x \to y$ $(n' \to \infty)$. By (2.5), $y \in \omega(B)$, i.e., $\omega(B) \neq \varnothing$. Let us verify that $\omega(B) \subseteq P$. Let $y \in \omega(B)$ and let $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ be sequences defined in (2.5). By the uniform attracting property of P (see (2.3)), we have

$$\mathrm{dist}_E(U(\tau_n + h_n, \tau_n)x_n, P) \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\mathrm{dist}_E(y, P) = 0$. The set P is closed, i.e., $y \in P$ for all $y \in \omega(B)$ and $\omega(B) \subseteq P$. This implies that $\omega(B)$ is compact since $\omega(B)$ is closed by definition (see (2.4)).

(ii) Assume the contrary: for some $B \in \mathcal{B}(E)$

$$\sup_{\tau \in \mathbb{R}} \mathrm{dist}_E(U(\tau + h, \tau)B, \omega(B)) \not\to 0 \quad \text{as } n \to \infty,$$

i.e., for some sequences $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ $(h_n \to +\infty)$

$$\mathrm{dist}_E(U(\tau_n + h_n, \tau_n)x_n, \omega(B)) \geqslant \delta > 0 \quad \forall\, n \in \mathbb{N}. \tag{2.6}$$

By the uniform attracting property of P,

$$\mathrm{dist}_E(U(\tau_n + h_n, \tau_n)x_n, P) \to 0 \quad \text{as } n \to \infty.$$

So once again, we find a sequence $\{y_n\} \subset P$ such that

$$\|U(\tau_n + h_n, \tau_n)x_n - y_n\|_E \to 0 \quad \text{as } n \to \infty.$$

The set P is compact, and we may assume by refining that $y_n \to y$ as $n \to \infty$ for some $y \in P$, i.e., $U(\tau_n + h_n, \tau_n)x_n \to y$ as $n \to \infty$. From (2.5) it follows that $y \in \omega(B)$. However, (2.6) implies that $\mathrm{dist}_E(y, \omega(B)) \geqslant \delta > 0$, which leads to a contradiction.

(iii) Let Y be a closed uniformly attracting set of the process $\{U(t, \tau)\}$. If $y \in \omega(B)$, then, in view of (2.5), for some sequences $\{x_n\} \subseteq B$, $\{\tau_n\} \subseteq \mathbb{R}$, $\{h_n\} \subset \mathbb{R}_+$ we have $U(\tau_n + h_n, \tau_n)x_n \to y$ as $h_n \to \infty$. Since Y is a uniformly attracting set, it follows that $\mathrm{dist}_E(U(\tau_n + h_n, \tau_n)x_n, Y) \to 0$ as $n \to \infty$ and, consequently, $\mathrm{dist}(y, Y) = 0$, i.e., $y \in Y$ for all $y \in \omega(B)$. Hence $\omega(B) \subseteq Y$. $\square$

Using Proposition 2.1, we formulate the following important assertion.

Theorem 2.1. *If a process $\{U(t,\tau)\}$ is uniformly asymptotically compact, then it has a compact* (*in E*) *uniform global attractor $\mathcal{A}$.*

Proof. We show that the set

$$\mathcal{A} = \Big[\bigcup_{n\in\mathbb{N}} \omega(B_n)\Big]_E, \tag{2.7}$$

where $B_n = \{x \in E \mid \|x\|_E \leqslant n\}$, is the required uniform global attractor. Indeed, for the set $\mathcal{A}$ defined in (2.7) we have $\mathcal{A} \subseteq P$ (see Proposition 2.1,(i)). Moreover, if $B \subseteq \mathcal{B}(E)$, then $B \subseteq B_n$ for some $n \in \mathbb{N}$ and, consequently, $\omega(B) \subseteq \omega(B_n) \subseteq \mathcal{A}$, i.e., $\mathcal{A}$ uniformly attracts $U_\sigma(t,\tau)B$ (see Proposition 2.1,(ii)). However, by Proposition 2.1,(iii), the set $\omega(B_n)$ belongs to every closed uniformly attracting set. Therefore, the minimality property is valid for $\mathcal{A}$ defined in (2.7). □

Remark 2.1. We cannot assert that $\mathcal{A} = \omega(P)$, where P is an arbitrary compact uniformly attracting set for $\{U(t,\tau)\}$. It is obvious that $\omega(P) \subseteq \mathcal{A}$ since $P \subseteq B_N$ for large N, so $\omega(P) \subseteq \omega(B_N)$. Therefore, $\omega(P) \subseteq \mathcal{A}$. However, it is not clear if the inverse inclusion holds since we do not know whether $\omega(B) \subseteq \omega(P)$ for any $B \subseteq \mathcal{B}(E)$. However, if B_0 is a compact uniformly absorbing set, then apparently

$$\mathcal{A} = \omega(B_0) = \bigcap_{h\geqslant 0} \Big[\bigcup_{t-\tau\geqslant h} U(t,\tau)B_0\Big]_E.$$

For a compact uniformly attracting set P the equality $\mathcal{A} = \omega(P)$ can be also proved under some additional assumptions of continuity of the process $\{U(t,\tau)\}$ (see Theorem 1.1 for the autonomous case and [**34**] for the nonautonomous cases).

Remark 2.2. In Theorem 2.1, we do not assume that the process $\{U(t,\tau)\}$ is continuous in E. (This assumption was essential in the existence theorems for global attractors of semigroups corresponding to autonomous evolution equations.) The reason is that we use only the minimality property in the definition of a global attractor.

To describe a general structure of the uniform global attractor of a process, we need the notion of the *kernel* of a process which generalizes the notion of the kernel of a semigroup.

A function $u(s)$, $s \in \mathbb{R}$, with values in E is called a *complete trajectory* of a process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t) \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R}. \tag{2.8}$$

A complete trajectory $u(s)$ is said to be *bounded* if the set $\{u(s),\ s \in \mathbb{R}\}$ is bounded in E.

Definition 2.3. The *kernel* $\mathcal{K}$ of a process $\{U(t,\tau)\}$ is the family of all bounded complete trajectories of $\{U(t,\tau)\}$:

$$\mathcal{K} = \{u(\cdot) \mid u \text{ satisfies (2.8) and } \|u(s)\|_E \leqslant C_u \ \forall\ s \in \mathbb{R}\}.$$

The set $\mathcal{K}(t) = \{u(t) \mid u(\cdot) \in \mathcal{K}\} \subset E,\ t \in \mathbb{R}$, is called the *kernel section* at time t.

It is easy to prove the following assertion.

Proposition 2.2. *If the process $\{U(t,\tau)\}$ has the global attractor $\mathcal{A}$, then*

$$\bigcup_{t\in\mathbb{R}} \mathcal{K}(t) \subseteq \mathcal{A}. \tag{2.9}$$

Comparing (2.9) with identity (1.6) in the autonomous case, we see that, in the nonautonomous case, $\mathcal{K}(t)$ may depend on time t and the inclusion in (2.9) can be strict, i.e., in order to describe the structure of the global attractor $\mathcal{A}$ of a process $\{U(t,\tau)\}$ it is not sufficient to know only the structure of $\mathcal{K}$. This question will be discussed in Section 2.5.

2.2. On nonuniform global attractors of processes and the Haraux example.

Following Haraux [**72, 73**], we define a (nonuniform) global attractor of a process $\{U(t,\tau)\}$ acting in E. A set P_0 is called a (nonuniform) attracting set of $\{U(t,\tau)\}$ if for any bounded set $B \in \mathcal{B}(E)$ and *fixed* $\tau \in \mathbb{R}$

$$\operatorname{dist}_E(U(t,\tau)B, P_0) \to 0 \quad \text{as } t \to +\infty, \tag{2.10}$$

i.e., for any $\varepsilon > 0$ there exists $T = T(\tau, B, \varepsilon) \geqslant \tau$ such that

$$U(t,\tau)B \subseteq \mathcal{O}_\varepsilon(P_0) \quad \forall\ t \geqslant T. \tag{2.11}$$

A process having a compact attracting set is called an *asymptotically compact* process.

Definition 2.4. A set $\mathcal{A}_0 \subset E$ is called the (nonuniform) global attractor of a process $\{U(t,\tau)\}$ if it is closed in E, is attracting for the process $\{U(t,\tau)\}$, and satisfies the property of minimality: $\mathcal{A}_0$ belongs to any closed attracting set of the process.

Theorem 2.2. *If a process* $\{U(t,\tau)\}$ *is asymptotically compact, then it has a compact* (*nonuniform*) *global attractor* $\mathcal{A}_0$.

It is obvious that a uniformly asymptotically compact process $\{U(t,\tau)\}$ is (nonuniformly) asymptotically compact as well and, thereby, $\mathcal{A}_0 \subseteq \mathcal{A}$. However, as was pointed out by Haraux, this inclusion can be strict, i.e., the uniform global attractor can be larger than the nonuniform one. We describe the example from [**72, 73**]. Consider the nonautonomous ordinary differential equation in $\mathbb{R}$

$$d_t u + a(t)u + u^3 = 0 \ (d_t = d/dt) \tag{2.12}$$

with initial data

$$u|_{t=\tau} = u_\tau, \ u_\tau \in \mathbb{R}, \tag{2.13}$$

where

$$a(t) = \sum_{n=1}^{\infty} n^{-2} \sin(2n^{-4}t). \tag{2.14}$$

The function $a(t)$ is almost periodic (see Example 2.1) since it is the uniform limit of almost periodic (and even quasiperiodic) functions. Equation (2.12) generates a process $\{U(t,\tau)\}$ in $\mathbb{R} : U(t,\tau)u_\tau = u(t)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where $u(t)$ is a solution of the problem (2.12), (2.13) with initial data u_τ. We set

$$A(t) = \int\limits_0^t a(s)ds = \sum_{n=1}^{\infty} n^2 \sin^2(n^{-4}t), \ t \in \mathbb{R}. \tag{2.15}$$

We find a (*nonuniform*) global attractor of the process $\{U(t,\tau)\}$. From (2.12) it follows that

$$d_t u^2 = -2a(t)u^2 - 2u^4 \leqslant -2a(t)u^2. \tag{2.16}$$

Therefore,

$$u^2(t) \leqslant u^2(\tau)\exp(2A(\tau))\exp(-2A(t)) \quad \forall t \geqslant \tau.$$

Setting $n = [|t|^{1/4}] + 1$ in (2.15), we obtain

$$A(t) \geqslant c|t|^{1/2} \quad \forall \ t \in \mathbb{R} \tag{2.17}$$

for some $c > 0$. Hence $u(t) \to 0$ as $t \to +\infty$; moreover, $U(t,\tau)B \to 0$ as $t \to \infty$ for each fixed $\tau \in \mathbb{R}$ and any bounded set $B \in \mathcal{B}(\mathbb{R})$. We conclude that the process $\{U(t,\tau)\}$ has a (nonuniform) global attractor $\mathcal{A}_0 = \{0\}$, i.e., a single point.

Let us study the *uniform* global attractor of a process $\{U(t,\tau)\}$. First of all, we note that the process is uniformly compact, i.e., it has a compact (bounded in $\mathbb{R}$) uniformly absorbing set. Indeed, since $a(t)$ is bounded, we have

$$-2a(t)u^2 - 2u^4 \leqslant 2Ru^2 - 2u^4 \leqslant -\gamma u^2 + C$$

for suitable positive R, γ, and C. By (2.16),

$$d_t u^2 \leqslant -\gamma u^2 + C, \quad u^2(t) \leqslant u^2(\tau)\exp(-\gamma(t-\tau)) + C/\gamma.$$

Hence the set $B_0 = \{|u|^2 \leqslant 2C/\gamma\}$ is uniformly absorbing for the process $\{U(t,\tau)\}$. The set B_0 is compact, and the uniform global attractor $\mathcal{A}$ exists in view of Theorem 2.1. It is clear that $\{0\} = \mathcal{A}_0 \subseteq \mathcal{A}$. We claim that $\mathcal{A} \neq \{0\}$.

It suffices to prove that there exists a nonzero *bounded* solution $\widetilde{u}(t)$ of Equation (2.12) defined for all $t \in \mathbb{R}$. Such a solution belongs to the kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$, and from (2.9) it follows that $\Big\{\bigcup\limits_{t\in\mathbb{R}} \widetilde{u}(t)\Big\} \subseteq \mathcal{A}$. Hence $\mathcal{A}$ is larger than $\mathcal{A}_0 = \{0\}$.

Integrating (2.16), we obtain

$$d_t(u^2 e^{2A(t)}) + 2u^4 e^{2A(t)} = 0, \quad d_t(v) + 2v^2 e^{-2A(t)} = 0,$$

where $v(t) = u^2(t)e^{2A(t)}$. Integrating again, we obtain

$$\frac{1}{v(t)} = \frac{1}{v(0)} + 2\int\limits_0^t e^{-2A(s)}ds.$$

Note that $e^{-2A(s)} \in L_1(\mathbb{R};\mathbb{R}_+)$ due to (2.17). Finally,

$$\widetilde{u}(t) = \pm\left(e^{-2A(t)} \Big/ \Big(\frac{1}{|u_0|^2} + 2\int\limits_0^t e^{-2A(s)}ds\Big)\right)^{1/2}, \ t \in \mathbb{R},$$

is the desired solution of (2.12) if

$$\frac{1}{|u_0|^2} > 2\int\limits_{-\infty}^0 e^{-2A(s)}ds.$$

The sign of $\widetilde{u}$ coincides with that of u_0. Indeed, $\widetilde{u}$ satisfies Equation (2.12) for all $t \in \mathbb{R}$ and is bounded in $\mathbb{R}$.

Note that for a periodic process the uniform global attractor coincides with the nonuniform global attractor (see [**124, 26**] for details). Now, we present a simple result on periodic processes.

A process $\{U(t,\tau)\}$ is said to be *periodic* with period p if

$$U(t+p,\tau+p) = U(t,\tau) \quad \forall t \geqslant \tau, \ \tau \in \mathbb{R}. \tag{2.18}$$

For a periodic process $\{U(t,\tau)\}$, in order to prove that a set P is uniformly attracting for $\{U(t,\tau)\}$, it suffices to show the following limit relation instead of (2.3):

$$\sup_{\tau\in[0,p)} \operatorname{dist}_E(U(\tau+h,\tau)B, P) \to 0 \quad \text{as } h \to +\infty. \tag{2.19}$$

Indeed, for arbitrary $\tau \in \mathbb{R}$ we have $\tau = \tau' + np$, where $\tau' \in [0,p)$ and $n \in \mathbb{Z}$. By periodicity, $U(h+\tau,\tau)B = U(h+\tau'+np, \tau'+np)B = U(h+\tau', \tau')B$ and (2.19) implies (2.3).

By the above arguments, the following assertion holds.

Theorem 2.3. *If a periodic process $\{U(t,\tau)\}$ is uniformly bounded and has a compact (nonuniformly) attracting set, then it is uniformly asymptotically compact. In particular, the process $\{U(t,\tau)\}$ has both uniform and nonuniform global attractors $\mathcal{A}$ and $\mathcal{A}_0$, and $\mathcal{A} = \mathcal{A}_0$.*

PROOF. Let $P_0 \Subset E$ be a compact attracting set of a periodic process $\{U(t,\tau)\}$ with period p. By Theorem 2.2, the process has a (nonuniform) global attractor $\mathcal{A}_0$.

Consider an arbitrary bounded set $B \in \mathcal{B}(E)$. Since $\{U(t,\tau)\}$ is uniformly bounded, we have

$$\widetilde{B} = \bigcup_{\tau\in[0,p)} U(p,\tau)B \in \mathcal{B}(E).$$

Since P_0 is (nonuniformly) attracting, for $\tau = p$ we have

$$\operatorname{dist}_E(U(t,p)\widetilde{B}, P_0) \to 0 \quad \text{as } t \to +\infty. \tag{2.20}$$

Note that for all $\tau \in [0,p)$

$$U(t,\tau)B = U(t,p)U(p,\tau)B \subseteq U(t,p)\widetilde{B} \quad \forall t \geqslant p.$$

Then from (2.20) it follows that

$$\sup_{\tau\in[0,p)} \operatorname{dist}_E(U(\tau+h,\tau)B, P_0) \leqslant \operatorname{dist}_E(U(t,p)\widetilde{B}, P_0) \to 0 \quad \text{as } t \to +\infty,$$

and the relation (2.19) is proved for the set P_0. Therefore, the process $\{U(t,\tau)\}$ is uniformly asymptotically compact. Repeating the above argument for $\mathcal{A}_0$ instead of P_0, we conclude that the set $\mathcal{A}_0$ is uniformly attracting. At the same time, $\mathcal{A}_0$ is the minimal uniformly attracting set since it is minimal (nonuniformly) attracting. Thus, $\mathcal{A}_0 = \mathcal{A}$ is the uniform global attractor of the periodic process $\{U(t,\tau)\}$. $\square$

In this paper, we study mostly uniform global attractors of processes corresponding to nonautonomous evolution equations.

2.3. Cauchy problem and corresponding process.

We explain how to construct a process corresponding to a nonautonomous evolution equation of the form

$$\partial_t u = A(u,t),\ t \geqslant \tau,\ \tau \in \mathbb{R}, \tag{2.21}$$

where $A(u,t)$ is a nonlinear operator $A(\cdot,t) : E_1 \to E_0$ for every $t \in \mathbb{R}$, E_1 and E_0 are Banach spaces, $E_1 \subseteq E_0$. We study solutions $u(t)$ for all $t \geqslant \tau$. For $t = \tau$ we consider the initial condition

$$u(\tau) = u|_{t=\tau} = u_\tau,\ u_\tau \in E, \tag{2.22}$$

where E is a Banach space such that $E_1 \subseteq E \subseteq E_0$. We assume that for all $\tau \in \mathbb{R}$ and $u_\tau \in E$ the Cauchy problem (2.21), (2.22) has a unique solution $u(t)$ such that $u(t) \in E$ for all $t \geqslant \tau$. The meaning of the expression "the function $u(t)$ is a solution of the problem (2.21), (2.22)" should be clarified for each particular example. As in the case of the solution of the autonomous equation (1.7), the solutions $u(t)$, $\tau \leqslant t \leqslant T$, of (2.21) are considered in the class $\mathcal{F}_{\tau,T}$ of functions such that $u \in L_\infty(\tau,T;E)$ and $u \in L_p(\tau,T;E_1)$. We assume that $A(u,t) \in L_q(\tau,T;E_0)$ for some q, $1 < q < \infty$, and $\partial_t u \in L_q(\tau,T;E_0)$. The equality (2.21) holds in the space $L_q(\tau,T;E_0)$. Thus, a function $u(t)$ in $\mathcal{F}_{\tau,T}$ should satisfy (2.21) in the sense of distributions in the space $\mathcal{D}'(]\tau,T[;E_0)$ (see [**96, 9, 34**] for details). To interpret the initial condition (2.22), we could use embedding theorems (see, for example, [**95, 117**]).

We study the following two-parametric family of operators $\{U(t,\tau)\}$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, generated by the problem (2.21), (2.22) and acting in E in accordance with the formula

$$U(t,\tau)u_\tau = u(t),\ t \geqslant \tau,\ \tau \in \mathbb{R}, \tag{2.23}$$

where $u(t)$ is a solution of the problem (2.21), (2.22) with initial data $u_\tau \in E$. Since the Cauchy problem (2.21), (2.22) is uniquely solvable, the family $\{U(t,\tau)\}$ satisfies the properties from Definition 2.1. Thus, $\{U(t,\tau)\}$ is referred to as the process corresponding to the problem (2.21), (2.22).

Below, we study global attractors of the processes corresponding to different nonautonomous dissipative evolution equations in mathematical physics.

2.4. Time symbols of nonautonomous equations.

Theorem 2.1 is applicable to processes generated by nonautonomous evolution equations. However, it provides a little information about the structure of uniform global attractors, and we need to study some extra properties of processes. For this purpose, the notion of the kernel of a process turns out to be very useful (see Definition 2.3). Recall that the kernel of Equation (2.21) is the union of all bounded complete solutions $u(t)$, $t \in \mathbb{R}$, of (2.21) determined on the entire time-axis $\{t \in \mathbb{R}\}$.

For the global attractor $\mathcal{A}$ of the nonautonomous equation (2.21) we always have the inclusion (2.9). However, in the general case, the inclusion can be strict, i.e., there exist points of the global attractor $\mathcal{A}$ that are not values of bounded complete trajectories of the original equation (2.21) (see Remark 2.7). Nevertheless, we can show that such points lie on the complete trajectories of "contiguous" equations. To describe "contiguous" equations, we introduce the notion of the *time symbol* of the equation under consideration. Speaking informally, the time symbol reflects the time-dependence of the right-hand side of the nonautonomous equation under consideration. We assume that all the terms of Equation (2.21) depending explicitly on time t can be presented by a function $\sigma(t)$, $t \in \mathbb{R}$, with values in an appropriate Banach space Ψ. We write Equation (2.21) in the form

$$\partial_t u = A_{\sigma(t)}(u), \ t \geqslant \tau, \ \tau \in \mathbb{R}. \tag{2.24}$$

The function $\sigma(t)$ is called the *time symbol* of the equation. In applications, $\sigma(t)$ consists of the coefficients and terms of the equation depending on time. For example, for the nonautonomous Navier–Stokes system $\partial_t u + \nu L u + B(u,u) = g(x,t)$ with time-dependent external force $g(x,t) \in C_{\mathrm{b}}(\mathbb{R}; H)$ the time symbol is $\sigma(t) = g(x,t)$. (This example will be considered in Section 2.6.1 in detail.)

We assume that the symbol $\sigma(t)$, regarded a function of t, belongs to the enveloping space

$$\Xi := \{\xi(t), t \in \mathbb{R} \mid \xi(t) \in \Psi \text{ for almost all } t \in \mathbb{R}\}$$

endowed with the Hausdorff topology. In the case of the 2D Navier–Stokes system, $\Psi = H$ and $\Xi = C_{\mathrm{b}}(\mathbb{R}; H)$ can be taken for the enveloped space. Recall that $g(x,t) \in C_{\mathrm{b}}(\mathbb{R}; H)$ if

$$\|g(\cdot,\cdot)\|_{C_{\mathrm{b}}(\mathbb{R};H)} := \sup\{\|g(\cdot,t)\|_H, \ t \in \mathbb{R}\} < +\infty.$$

We assume that the translation group $\{T(h), h \in \mathbb{R}\}$ acting by the formula $T(h)\xi(t) = \xi(h+t)$ is continuous in Ξ. This assumption is satisfied for $\Xi = C_{\mathrm{b}}(\mathbb{R}; H)$.

The symbol of the original equation (2.21) is denoted by $\sigma_0(t)$. We also consider Equation (2.24) with symbol $\sigma_h(t) = \sigma_0(t+h)$ for any $h \in \mathbb{R}$ and equations with symbols $\sigma(t)$ that are the limits of $\sigma_{h_n}(t) = \sigma_0(t+h_n)$ as $n \to \infty$ in Ξ. The resulting family of symbols is the *hull* $\mathcal{H}(\sigma_0)$ of the original symbol $\sigma_0(t)$ in Ξ.

Definition 2.5. The *hull* $\mathcal{H}(\sigma)$ of $\sigma(t)$ in the space Ξ is defined by the formula

$$\mathcal{H}(\sigma_0) := [\{\sigma(t+h) \mid h \in \mathbb{R}\}]_\Xi, \tag{2.25}$$

where $[\cdot]_\Xi$ denotes the closure in the topological space Ξ.

We will study equations of the form (2.21) and (2.24) whose symbols $\sigma(t)$ are *translation compact* functions in Ξ (see [**27, 28, 29, 34**]).

Definition 2.6. A function $\sigma(t) \in \Xi$ is called a *translation compact* function in Ξ if the hull $H(\sigma)$ is compact in Ξ.

Consider the main examples of translation compact functions which will be used in this paper.

Example 2.1. Let $\Xi = C_b(\mathbb{R}; \mathcal{M})$, where $\mathcal{M}$ is a complete metric space. Let $\sigma_0(s)$ be an almost periodic function with values in $\mathcal{M}$. By the Bochner–Amerio criterion, an almost periodic function $\sigma_0(s)$ possesses the following characteristic property: the set of all translations $\{\sigma_0(s + h) = T(h)\sigma_0(s) \mid h \in \mathbb{R}\}$ is precompact in $C_b(\mathbb{R}; \mathcal{M})$ (see, for example, [**1, 92**]). The closure in $C_b(\mathbb{R}; \mathcal{M})$ of this set is called the *hull* $\mathcal{H}(\sigma_0)$ of $\sigma_0(s)$ (see (2.25)). By Definition 2.6, $\sigma_0(s)$ is a translation compact function in $C_b(\mathbb{R}; \mathcal{M})$. If $\sigma_0(s)$ is almost periodic, then any function $\sigma(s) \in \mathcal{H}(\sigma_0)$ is almost periodic. It is obvious that the time translation group $\{T(h) \mid h \in \mathbb{R}\}$ is continuous in $C_b(\mathbb{R}; \mathcal{M})$.

Example 2.2. Let $\Xi = L_p^{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$, where $p \geqslant 1$ and $\mathcal{E}$ is a Banach space. The space $L_p^{\mathrm{loc}}(\mathbb{R}; \mathcal{E})$ consists of functions $\xi(t), t \in \mathbb{R}$ with values in $\mathcal{E}$ that are p-power locally integrable in the Bochner sense, i.e.,

$$\int_{t_1}^{t_2} \|\xi(t)\|_{\mathcal{E}}^p dt < +\infty \ \forall [t_1, t_2] \subset \mathbb{R}.$$

We consider the following convergence topology in the space $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$. By definition, $\xi_n(t) \to \xi(t)$ as $n \to \infty$ in $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ if

$$\int\limits_{t_1}^{t_2} \|\xi_n(t) - \xi(t)\|_{\mathcal{E}}^p dt \to 0 \quad \text{as } n \to \infty$$

for every interval $[t_1, t_2] \subset \mathbb{R}$. The space $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is countably normable, metrizable, and complete.

Consider translation compact functions in the space $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$. The following criterion holds (see, for example, [**34**]):

$\sigma_0(t)$ is a translation compact function in $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ if and only if

(i) for any $h \geqslant 0$ the set $\left\{ \int\limits_t^{t+h} \sigma_0(s)ds \mid t \in \mathbb{R} \right\}$ is precompact in $\mathcal{E}$,

(ii) there exists a positive function $\beta(s) \to 0$ $(s \to 0+)$ such that

$$\int\limits_t^{t+1} \|\sigma_0(s) - \sigma_0(s+l)\|_{\mathcal{E}}^p ds \leqslant \beta(|l|) \quad \forall t \in \mathbb{R}.$$

From this criterion it follows that

$$\sup_{t\in\mathbb{R}} \int\limits_t^{t+1} \|\sigma_0(s)\|_{\mathcal{E}}^p ds < +\infty \quad \forall\, t \in \mathbb{R}, \tag{2.26}$$

for any translation compact function in $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$.

It is obvious that $\{T(h) \mid h \in \mathbb{R}\}$ is continuous in $L_p^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$.

Example 2.3. Similarly, we can define translation compact functions in the space $C^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ of continuous functions $\xi(t), t \in \mathbb{R}$ with values in $\mathcal{E}$. The space $C^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is endowed with the local uniform convergence topology on every interval $[t_1, t_2] \subset \mathbb{R}$ (see [**34**]). By the Arzelá–Ascoli theorem, we obtain the following criterion (see [**34**] for details):

$\sigma_0(t)$ is a translation compact function in $C^{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ if and only if

(i) the set $\{\sigma_0(h) \mid h \in \mathbb{R}\}$ is precompact in $\mathcal{E}$,

(ii) $\sigma_0(t)$ is uniformly continuous on $\mathbb{R}$, i.e., there exists a positive function $\alpha(s) \to 0+$ $(s \to 0+)$ such that

$$\|\sigma_0(t_1) - \sigma_0(t_2)\|_{\mathcal{E}} \leqslant \alpha(|t_1 - t_2|) \quad \forall t_1, t_2 \in \mathbb{R}.$$

In particular, any translation compact function in $C^{\,\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is bounded in $\mathcal{E}$. The translation group $\{T(h) \mid h \in \mathbb{R}\}$ is continuous in $C^{\,\mathrm{loc}}(\mathbb{R};\mathcal{E})$.

Example 2.4. Almost periodic functions with values in $\mathcal{E}$, i.e., translation compact functions in $C_b(\mathbb{R};\mathcal{E})$, are translation compact functions in $C^{\,\mathrm{loc}}(\mathbb{R};\mathcal{E})$.

Example 2.5. In the class of almost periodic functions, we extract the subclass of quasiperiodic functions. A function $\sigma_0(t) \in C(R;E)$ is said to be *quasiperiodic* if

$$\sigma_0(t) = \varphi(\alpha_1 t, \alpha_2 t, \dots, \alpha_k t) = \varphi(\bar{\alpha} t), \tag{2.27}$$

where the function $\varphi(\bar{\omega}) = \varphi(\omega_1, \omega_2, \dots, \omega_k)$ is continuous and 2π-periodic with respect to each variable $\omega_i \in \mathbb{R} : \varphi(\omega_1, \dots, \omega_i + 2\pi, \dots, \omega_k) = \varphi(\omega_1, \dots, \omega_i, \dots, \omega_k)$, $i = 1, \dots, k$. Denote by $\mathbb{T}^k = [\mathbb{R} \bmod 2\pi]^k$ the k-dimensional torus. Then $\varphi \in C(\mathbb{T}^k;\mathcal{E})$. We assume that the real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ in (2.27) are rationally independent (otherwise, we can reduce the number of independent variables ω_i in (2.27)). It follows that the hull of the quasiperiodic function $\sigma_0(t)$ in $C(\mathbb{R};\mathcal{E})$ is the set

$$\{\varphi(\bar{\alpha} t + \bar{\omega}_1) \mid \bar{\omega}_1 \in \mathbb{T}^k\} = \mathcal{H}(\sigma_0),\ \bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k). \tag{2.28}$$

Consequently, the set $\mathcal{H}(\sigma_0)$ is the continuous image of the k-dimensional torus $\mathbb{T}^k$. For $k = 1$ we obtain the periodic function $\sigma_0(t + 2\pi) = \sigma_0(t)$.

In [**34**], there are other examples of translation compact functions in $C(\mathbb{R};\mathcal{E})$ that are not almost periodic or quasiperiodic.

2.5. On the structure of uniform global attractors.

Consider a family of equations of type (2.24) with symbols $\sigma(t)$ from the hull $\mathcal{H}(\sigma_0)$, where $\sigma_0(t)$ is the symbol of the original equation,

$$\partial_t u = A_{\sigma(t)}(u),\ \sigma \in \mathcal{H}(\sigma_0), \tag{2.29}$$

with initial data

$$u|_{t=\tau} = u_\tau. \tag{2.30}$$

We assume that $\sigma_0(t)$ is a translation compact function in the topological space Ξ. For the sake of simplicity, we assume that $\mathcal{H}(\sigma_0)$ is a complete metric space. In the above examples, this assumption was satisfied. Suppose that for every symbol $\sigma \in \mathcal{H}(\sigma_0)$ the Cauchy problem (2.29), (2.30) has a unique solution for any $\tau \in \mathbb{R}$ and initial condition $u_\tau \in E$. Thus, we have the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in the space E.

The family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, is said to be $(E \times \mathcal{H}(\sigma_0), E)$–*continuous* if for any t and $\tau, t \geqslant \tau$ the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous from $E \times \mathcal{H}(\sigma_0)$ into E.

Proposition 2.3. *If the process $\{U_{\sigma_0}(t,\tau)\}$ has a compact uniformly attracting set P and the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, corresponding to* (2.29) *is $(E \times \mathcal{H}(\sigma_0), E)$-continuous, then for every $\sigma \in \mathcal{H}(\sigma_0)$ the set P is also uniformly attracting for $\{U_\sigma(t,\tau)\}$. Moreover, $\mathcal{A}_\sigma \subseteq \mathcal{A} = \mathcal{A}_{\sigma_0}$, where $\mathcal{A}_\sigma$ is the uniform global attractor of the process $\{U_\sigma(t,\tau)\}$* (*the inclusion $\mathcal{A}_\sigma \subseteq \mathcal{A}_{\sigma_0}$ can be strict*).

The proof can be found in [**25, 34**].

Remark 2.3. A translation compact function σ_0 in Ξ is said to be *recurrent* if $\mathcal{H}(\sigma) = \mathcal{H}(\sigma_0)$ for every $\sigma \in \mathcal{H}(\sigma_0)$. Any almost periodic function is recurrent. If, in Proposition 2.3, the translation compact symbol σ_0 is recurrent (for example, almost periodic), then $\mathcal{A}_\sigma = \mathcal{A}_{\sigma_0} = \mathcal{A}$ for every $\sigma \in \mathcal{H}(\sigma_0)$. In this case, the uniform global attractor $\mathcal{A}$ describes the limit behavior of solutions of the entire family of Equations (2.29).

The following *translation identity* holds for the family of processes corresponding to (2.29):

$$U_{T(h)\sigma}(t,\tau) = U_\sigma(t+h,\tau+h) \quad \forall h \geqslant 0, t \geqslant \tau, \tau \in \mathbb{R}, \tag{2.31}$$

where $T(h)\sigma(t) = \sigma(t+h)$. This identity directly follows from the uniqueness of a solution $u(t)$ of the problem (2.29), (2.30). To prove (2.31), we replace $\sigma(s)$ in (2.29) with $T(h)\sigma(s) = \sigma(s+h)$ and make the change of variable $t + h = t_1$. The identity (2.31) means that the shift by h of the argument of the symbol $\sigma(s)$ in the problem (2.29), (2.30) is equivalent to solving Equation (2.29) with symbol $\sigma(s)$ at time $t+h$ with initial data $u|_{t=\tau+h} = u_\tau$.

Consider a special case of the symbol $\sigma_0(t)$ of Equation (2.29) such that the translation semigroup $\{T(h) \mid h \geqslant 0\}$ maps it into itself: $T(h)\sigma_0(t) = \sigma_0(t+h) \equiv \sigma_0(t)$ for all $h \geqslant 0$. In other words, $\sigma_0(t)$ is independent of t: $\sigma_0(t) = \sigma_0$ for any $s \in \mathbb{R}$, where $\sigma_0 \in \Psi$. Then, by (2.31), the corresponding process $\{U_{\sigma_0}(t,\tau)\}$ satisfies the equality $U_{\sigma_0}(t,\tau) = U_{\sigma_0}(t+h,\tau+h) = U_{\sigma_0}(t-\tau,0)$ for all $h \geqslant 0$, $t \geqslant \tau$, $\tau \in \mathbb{R}$. Thus, the process $\{U_{\sigma_0}(t,\tau)\}$ is completely described by the set of one-parameter mappings $S(t) = U_{\sigma_0}(t,0)$, $t \geqslant 0$. It is evident that $\{S(t)\}$ forms the semigroup corresponding to the autonomous equation with the constant symbol $\sigma(t) = \sigma_0$. Such equations were treated in Section 1. We conclude that the semigroups generated by autonomous evolution equations are special cases of processes generated by nonautonomous equations.

Having the family of nonautonomous equations (2.29), we consider the extended phase space $E \times \mathcal{H}(\sigma_0)$. Using the identity (2.31), we construct the semigroup $\{\mathcal{S}(h), h \geqslant 0\}$ acting in the space $E \times \mathcal{H}(\sigma_0)$ by the formula

$$\mathcal{S}(h)(u,\sigma) = (U_\sigma(h,0)u, T(h)\sigma), h \geqslant 0. \tag{2.32}$$

We prove that the family $\{\mathcal{S}(h)\}$ forms a semigroup in $E \times \mathcal{H}(\sigma_0)$. For this purpose, it suffices to verify the semigroup relation

$$\begin{aligned}
\mathcal{S}(h_1+h_2)(u,\sigma) &= (U_\sigma(h_1+h_2,0)u, T(h_1+h_2)\sigma) \\
&= (U_\sigma(h_1+h_2,h_2)U_\sigma(h_2,0)u, T(h_1)T(h_2)\sigma) \\
&= (U_{T(h_2)\sigma}(h_1,0)U_\sigma(h_2,0)u, T(h_1)(T(h_2)\sigma)) \\
&= \mathcal{S}(h_1)(U_\sigma(h_2,0)u, T(h_2)\sigma) = \mathcal{S}(h_1)\mathcal{S}(h_2)(u,\sigma).
\end{aligned}$$

Here, the property 2 of Definition 2.1 and the translation identity (2.31) were used. It is also obvious that $\mathcal{S}(0) = \mathrm{Id}$.

We denote by Π_1 and Π_2 the projections operators acting from $E \times \mathcal{H}(\sigma_0)$ onto E and $\mathcal{H}(\sigma_0)$ by the formula

$$\Pi_1(u,\sigma) = u, \quad \Pi_2(u,\sigma) = \sigma.$$

We now formulate the main theorem about the structure of the global attractor of Equation (2.21) with translation compact symbol $\sigma_0(t)$. Denote by $\{U_{\sigma_0}(t,\tau)\}$ the corresponding original process with symbol σ_0.

Theorem 2.4. *Suppose that $\sigma_0(t)$ is a translation compact function in Ξ. Let the process $\{U_{\sigma_0}(t,\tau)\}$ be asymptotically compact, and let the corresponding family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, be $(E\times\mathcal{H}(\sigma_0), E)$-continuous. Then the semigroup $\{\mathcal{S}(h)\}$ acting in $E \times \mathcal{H}(\sigma_0)$ by formula (2.32) has the global attractor $\mathfrak{A}$, $\mathcal{S}(h)\mathfrak{A} = \mathfrak{A}$ for all $h \geqslant 0$. Moreover, the following assertions hold*:

(i) $\Pi_2\mathfrak{A} = \mathcal{H}(\sigma_0)$,

(ii) $\Pi_1\mathfrak{A} = \mathcal{A}$ *is the global attractor of the process* $\{U_{\sigma_0}(t,\tau)\}$,

(iii) *the global attractor $\mathcal{A}$ admits the representation*

$$\mathcal{A} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0) = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(t), \tag{2.33}$$

where $\mathcal{K}_\sigma$ is the kernel of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \mathcal{H}(\sigma_0)$, t is any fixed number, the kernel $\mathcal{K}_\sigma$ is nonempty for every $\sigma \in \mathcal{H}(\sigma_0)$.

A detailed proof of Theorem 2.4 can be found in [**25, 34**]. The existence of the global attractor $\mathfrak{A}$ follows from Theorem 1.1. To prove Theorem 1.1, we need to check whether the conditions of asymptotic compactness and continuity hold for the semigroup $\{\mathcal{S}(h)\}$ acting in $E \times \mathcal{H}(\sigma_0)$ by formula (2.32). Let P be a compact uniformly (with respect to $\sigma \in \mathcal{H}(\sigma_0)$) attracting set for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$. It is obvious that the set $P \times \mathcal{H}(\sigma_0)$ is a compact (in $E \times \Xi$) attracting set for the extended semigroup $\{\mathcal{S}(h), h \geqslant 0\}$. It is clear that the semigroup $\{\mathcal{S}(h)\}$ is continuous since the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ is $(E \times \mathcal{H}(\sigma_0), E)$-continuous and the translation semigroup $\{T(h)\}$ is continuous by assumption. Therefore, by Theorem 1.1, the set

$$\mathfrak{A} = \omega(P \times \mathcal{H}(\sigma_0)) = \bigcap_{h \geqslant 0} \Big[\bigcup_{\eta \geqslant h} \mathcal{S}(\eta)(P \times \mathcal{H}(\sigma_0)) \Big]_{E \times \Xi} \tag{2.34}$$

is the global attractor of the semigroup $\{\mathcal{S}(h)\}$ and the first assertion of Theorem 2.4 is proved.

The remaining assertions of Theorem 2.4 are proved (see [**34**] for details) with the help of the representation (see (1.6) in Theorem 1.2)

$$\mathfrak{A} = \{\gamma(0) \mid \gamma(\cdot) \text{ is a complete bounded trajectory of} \{\mathcal{S}(h)\}\}. \tag{2.35}$$

Remark 2.4. Using (2.33), it is easy to show that $\mathcal{A} = \omega(P)$, where P is an arbitrary compact uniformly attracting set of the process $\{U_{\sigma_0}(t,\tau)\}$ (see Remark 2.1).

Remark 2.5. If the time symbol $\sigma_0(t)$ is periodic with period p, $\sigma_0(t+p) = \sigma_0(t)$, then the corresponding process $\{U_{\sigma_0}(t,\tau)\}$ is also periodic with period p. In this case, the uniform and nonuniform attractors coincide, $\mathcal{A}_0 = \mathcal{A}$ (see Theorem 2.3 and [**124, 26**]). Moreover, the hull $\mathcal{H}(\sigma_0) = \{\sigma_0(t+h) \mid h \in [0,p)\}$ and formula (2.33) can be written in a simpler form $\mathcal{A} = \bigcup_{h \in [0,p)} \mathcal{K}_{\sigma_0}(h)$, where $\mathcal{K}_{\sigma_0}$ is the kernel of the original periodic process $\{U_{\sigma_0}(t,\tau)\}$ (see (2.9)).

2.6. Uniform global attractors for nonautonomous equations.

In this section, we apply the general theory of uniform global attractors of processes corresponding to abstract nonautonomous equations (2.21) and (2.24) to some important evolution equation in mathematical physics.

2.6.1. 2D Navier–Stokes system with time-dependent force. We consider the nonautonomous 2D Navier–Stokes system with time-dependent external force

$$\begin{aligned} &\partial_t u = -\nu L u - B(u,u) + g_0(x,t),\ (\nabla, u) = 0,\\ &u|_{\partial\Omega} = 0,\ x = (x_1,x_2) \in \Omega \Subset \mathbb{R}^2. \end{aligned} \tag{2.36}$$

We use the notation from Section 1.3.1, where the autonomous 2D Navier–Stokes system (1.11) is considered with time-independent external force $g_0(x)$.

We assume that $g_0(\cdot,t) \in H$ for almost every $t \in \mathbb{R}$ and g_0 has finite norm in the space $L_2^b(\mathbb{R}; H)$, i.e.,

$$\|g_0\|^2_{L_2^b(\mathbb{R};H)} = \|g_0\|^2_{L_2^b} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |g_0(\cdot,s)|^2 ds < +\infty. \tag{2.37}$$

We consider (2.36) with initial conditions

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in H,\ \tau \in \mathbb{R}. \tag{2.38}$$

The problem (2.36), (2.38) has a unique solution $u(t) \in C(\mathbb{R}_\tau; H) \cap L_2^b(\mathbb{R}_\tau; V)$ such that $\partial_t u \in L_2^b(\mathbb{R}_\tau; V')$, $\mathbb{R}_\tau = [\tau, +\infty)$ (see [**96, 87, 119, 9, 34**]). The solution $u(t)$ in this space satisfies Equation (2.36) in the sense of distributions in the space $\mathcal{D}'(\mathbb{R}_\tau; V')$. Moreover, the following estimates hold:

$$|u(t)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda(t-\tau)} + \lambda^{-1}(1 + (\nu\lambda)^{-1})\|g_0\|^2_{L_2^b}, \tag{2.39}$$

$$|u(t)|^2 + \nu \int_\tau^t \|u(s)\|^2 ds \leqslant |u(\tau)|^2 + (\nu\lambda)^{-1} \int_\tau^t |g_0(s)|^2 ds, \tag{2.40}$$

$$(t-\tau)\|u(t)\|^2 \leqslant C\left(t-\tau, |u(\tau)|^2, \int_\tau^t |g_0(s)|^2 ds\right) \tag{2.41}$$

where $\lambda = \lambda_1$ is the first eigenvalue of the Stokes operator L and $C(z,R,R_1)$ is a monotone continuous functions of $z = t - \tau$, R, R_1 (see [**34**]).

Consequently, the problem (2.36), (2.38) generates a process $\{U_{g_0}(t,\tau)\}$ acting in H by the formula $U_{g_0}(t,\tau)u_\tau = u(t)$, where $u(t)$ is a solution of (2.36), (2.38).

From (2.39) it follows that the process $\{U_{g_0}(t,\tau)\}$ has the uniformly absorbing set $B_0 = \{u \in H \mid |u| \leqslant 2R_0\}$, $R_0^2 = (\nu\lambda)^{-1}(1 + (\nu\lambda)^{-1})\|g_0\|^2_{L_2^b}$, By the inequality (2.41), the set

$$B_1 = \bigcup_{\tau \in \mathbb{R}} U_{g_0}(\tau + 1, \tau) B_0 \tag{2.42}$$

is also uniformly absorbing. Moreover, B_1 is bounded in V and, consequently, is compact in H (see [**96, 34**]). Thus, the process $\{U_{g_0}(t,\tau)\}$ is uniformly compact in H. By Theorem 2.1, we conclude that the process $\{U_{g_0}(t,\tau)\}$ has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is bounded in V. Using Remark 2.1, we observe that the global attractor $\mathcal{A}$ can be constructed by the formula

$$\mathcal{A} = \omega(B_0) = \bigcap_{h \geqslant 0} \Big[\bigcup_{t-\tau \geqslant h} U_{g_0}(t,\tau) B_0 \Big]_H.$$

We now assume that $g_0(\cdot, t) =: g_0(t)$ is a translation compact function in $L_2^{\rm loc}(\mathbb{R}; H)$. The corresponding necessary and sufficient conditions are given in Section 2.4. We indicate another sufficient condition: $g_0(t)$ is a translation compact function in $L_2^{\rm loc}(\mathbb{R}; H)$ if $g_0 \in L_2^b(\mathbb{R}; V)$ and $\partial_t g_0 \in L_2^b(\mathbb{R}; V')$, i.e.,

$$\|g_0\|^2_{L_2^b(\mathbb{R};V)} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_0(\cdot, s)\|^2 ds \leqslant M_1 < +\infty,$$

$$\|\partial_t g_0\|^2_{L_2^b(\mathbb{R};V')} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\partial_t g_0(\cdot, s)\|^2_{V'} ds \leqslant M_{-1} < +\infty$$

(see [**34**]). We denote by $\mathcal{H}(g_0)$ the hull of g_0 in the space $L_2^{\rm loc}(\mathbb{R}; H)$. It is clear that

$$\|g\|^2_{L_2^b} \leqslant \|g_0\|^2_{L_2^b} \leqslant M \tag{2.43}$$

for every $g \in \mathcal{H}(g_0)$.

The symbol of Equation (2.36) is $g_0(t) = \sigma_0(t)$. For every symbol $g \in \mathcal{H}(g_0)$ the corresponding problem (2.36), (2.38) (with external force g instead of g_0) is uniquely solvable and the solution $u_g(t)$ satisfies the inequalities (2.39)–(2.41). Hence the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, acting on H is defined. As is proved in [**34**], this family is $(H \times \mathcal{H}(g_0))$-continuous. Therefore, from Theorem 2.4 it follows that

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0), \tag{2.44}$$

where $\mathcal{K}_g$ is the kernel of the process $\{U_g(t,\tau)\}$ consisting of all the bounded complete solutions $u_g(t), t \in \mathbb{R}$, of the 2D Navier–Stokes system with external force $g(t)$. The kernel $\mathcal{K}_g$ is nonempty for every $g \in \mathcal{H}(g_0)$. Note that

$$\mathcal{A} \subset B_0 = B_{R_0}(0),\ R_0^2 = (\nu\lambda)^{-1}(1+(\nu\lambda)^{-1})\|g_0\|^2_{L_2^b}, \tag{2.45}$$

$$\mathcal{A} \subset B_1,\ B_1 = \{u \in V \mid \|v\| \leqslant R'\}, \tag{2.46}$$

where R' depends on ν, λ, and $\|g_0\|^2_{L_2^b}$. In particular, from (2.44) it follows that

$$\|u(t)\| \leqslant R' \quad \forall t \in \mathbb{R} \tag{2.47}$$

for every function $u_g(\cdot) \in \mathcal{K}_g$, $g \in \mathcal{H}(g_0)$.

Consider an important special case of the system (2.36). As in the autonomous case, we introduce the Grashof number G for the nonautonomous 2D Navier–Stokes system by the formula

$$G := \frac{\|g_0\|_{L_2^b}}{\lambda\nu^2}.$$

Proposition 2.4. *Suppose that G satisfies the inequality*

$$G < 1/c_0^2, \tag{2.48}$$

where the constant c_0 is taken from the inequality (1.14) (*see* (1.19)). *Then for every $g \in \mathcal{H}(g_0)$ the Navier–Stokes system*

$$\partial_t u = -\nu Lu - B(u,u) + g(t) \tag{2.49}$$

has a unique solution $z_g(t)$, $t \in \mathbb{R}$, bounded in H, i.e., the kernel $\mathcal{K}_g$ consists of a single trajectory $z_g(t)$. This solution $z_g(t)$ is exponentially stable, i.e., for every solution $u_g(t)$ of (2.49)

$$|u_g(t) - z_g(t)| \leqslant C_0|u_\tau - z_g(\tau)|e^{-\beta(t-\tau)} \quad \forall t \geqslant \tau, \tag{2.50}$$

where $u_g(t) = U_g(t,\tau)u_\tau$, and the constants C_0, β are independent of u_τ and τ.

Proof. By (2.44), there exists at least one bounded solution $z_g(t) := z(t)$. Let $u_g(t) := u(t)$ be an arbitrary solution of (2.49). The function $w(t) = u(t) - z(t)$ satisfies the equation

$$\partial_t w + \nu Lw + B(w, w+z) + B(z,w) = 0.$$

Multiplying this equation by w and using the identities $(B(z,w),w) = (B(w,w),w) = 0$ (see (1.13)) and the inequality (1.14), we find

$$\partial_t|w|^2 + 2\nu\|w\|^2 = 2(B(w,z),w) \leqslant 2c_0^2|w|\|w\|\|z\| \leqslant \nu\|w\|^2 + c_0^4\nu^{-1}|w|^2\|z\|^2.$$

Since $\lambda|w|^2 \leqslant \|w\|^2$, we have

$$\partial_t|w|^2 + \nu\lambda|w|^2 \leqslant \partial_t|w|^2 + \nu\|w\|^2 \leqslant c_0^4\nu^{-1}|w|^2\|z\|^2. \tag{2.51}$$

Consequently,

$$\partial_t|w|^2 + (\nu\lambda - c_0^4\nu^{-1}\|z(t)\|^2)|w|^2 \leqslant 0. \tag{2.52}$$

Multiplying this inequality by $\exp\Big\{\int\limits_\tau^t(\nu\lambda - c_0^4\nu^{-1}\|z(s)\|^2)ds\Big\}$ and integrating over $[\tau, t]$, we obtain

$$\begin{aligned}|w(t)|^2 &\leqslant |w(\tau)|^2\exp\Big\{\int\limits_\tau^t(-\nu\lambda + c_0^4\nu^{-1}\|z(s)\|^2)ds\Big\}\\ &= |w(\tau)|^2\exp\Big\{-\nu\lambda(t-\tau) + c_0^4\nu^{-1}\int\limits_\tau^t\|z(s)\|^2ds\Big\}.\end{aligned} \tag{2.53}$$

By (2.40), we have

$$\begin{aligned}\int\limits_\tau^t\|z(s)\|^2ds &\leqslant \nu^{-1}|z(\tau)|^2 + (\nu^2\lambda)^{-1}\int\limits_\tau^t|g(s)|^2ds\\ &\leqslant \nu^{-1}|z(\tau)|^2 + (\nu^2\lambda)^{-1}(t-\tau+1)\|g\|^2_{L_2^b}\\ &\leqslant \nu^{-1}|z(\tau)|^2 + (\nu^2\lambda)^{-1}(t-\tau+1)\|g_0\|^2_{L_2^b}.\end{aligned}$$

Since $z(\tau) \in \mathcal{A}_{\mathcal{H}(g_0)}$, from (2.45) it follows that

$$|z(\tau)|^2 \leqslant (\nu\lambda)^{-1}(1 + (\nu\lambda)^{-1})\|g_0\|^2_{L_2^b} = R_0^2.$$

Hence

$$\begin{aligned}\int\limits_\tau^t\|z(s)\|^2ds &\leqslant (\nu^{-1}R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L_2^b}) + (\nu^2\lambda)^{-1}(t-\tau)\|g_0\|^2_{L_2^b}\\ &= R_1^2 + (\nu^2\lambda)^{-1}(t-\tau)\|g_0\|^2_{L_2^b},\end{aligned}$$

where $R_1^2 = \nu^{-1}R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L_2^b}$. Substituting this estimate into (2.53), we obtain the inequality

$$|w(t)|^2 \leqslant |w(\tau)|^2C_0\exp(-\beta(t-\tau)),$$

where $\beta = \nu\lambda - c_0^4(\nu^3\lambda)^{-1}\|g_0\|^2_{L_2^b}$ and $C_0 = \exp(c_0^4\nu^{-1}R_1^2)$. Note that

$$\nu^{-4}\lambda^{-2}\|g_0\|^2_{L_2^b} = G^2 < 1/c_0^4$$

and, consequently, $\beta = \nu\lambda c_0^4(c_0^{-4} - \nu^{-4}\lambda^{-2}\|g_0\|^2_{L_2^b}) > 0$. This implies

$$|w(t)|^2 = |u(t) - z(t)|^2 \leqslant |u(\tau) - z(\tau)|^2 C_0 e^{-\beta(t-\tau)}.$$

The inequality (2.50) is proved.

Now, we show that such a function $z(t)$ is unique. If there are two bounded complete solutions $z_1(t)$ and $z_2(t)$, $t \in \mathbb{R}$, then

$$|z_1(t) - z_2(t)|^2 \leqslant |z_1(\tau) - z_2(\tau)|^2 C_0 e^{-\beta(t-\tau)} \leqslant C_1 C_0 e^{-\beta(t-\tau)}$$

in view of (2.50). Fixing t and letting $\tau \to -\infty$, we obtain $|z_1(t) - z_2(t)|^2 = 0$ for all $t \in \mathbb{R}$. □

The properties (2.50) and (2.44) imply that the set

$$\mathcal{A} = [\{z_{g_0}(t) \mid t \in \mathbb{R}\}]_H = \bigcup_{g \in \mathcal{H}(g_0)} \{z_g(0)\} \tag{2.54}$$

is the global attractor of (2.36) under the condition (2.48).

Remark 2.6. In [**16**] it is shown that $c_0^2 < (8/(27\pi))^{1/2}$ (see also Remark 1.2). Therefore, formula (2.54) holds for $G < 3.2562$.

Remark 2.7. It is easy to construct examples of functions $g_0(x,t)$ satisfying (2.48) such that the set $\{z_{g_0}(t) \mid t \in \mathbb{R}\}$ is not closed in H. Nevertheless, the set $\mathcal{A}$ is always closed, and to describe $\mathcal{A}$, we need to consider all the functions $z_g(t)$ in the kernels of equations with external forces $g \in \mathcal{H}(g_0)$.

Remark 2.8. The inequality (2.50) implies that, under the condition (2.48), the global attractor $\mathcal{A}$ of the system (2.36) is exponential, i.e., $\mathcal{A}$ attracts bounded sets of initial data with exponential rate.

Consider some special cases of the function $g \in \mathcal{H}(g_0)$.

Corollary 2.1. *Let $g(t)$ in (2.49) be periodic with period p. Then $z_g(t)$ has period p.*

Proof. Consider the corresponding bounded complete trajectory $z_g(t)$. It is obvious that $z_g(t+p)$ is a bounded complete trajectory of (2.49) with external force $g(t+p) \equiv g(t)$. Therefore, belongs to the kernel $\mathcal{K}_g$ consisting of the single trajectory $z_g(t)$. Hence $z_g(t+p) \equiv z_g(t)$. □

Corollary 2.2. *If $g(t) \in \mathcal{H}(g_0)$ is almost periodic, then $z_g(t)$ is also almost periodic.*

PROOF. Consider the function $w(t) = z(t) - z(t+p)$, where $z(t) := z_g(t)$ and p is an arbitrary fixed number. As in the case of (2.52), we obtain the inequality

$$\partial_t |w|^2 + (\nu\lambda - c_0^4 \nu^{-1} \|z(t)\|^2)|w|^2 \leqslant 2|w| \cdot |g(t) - g(t+p)|,$$

which implies

$$\partial_t |w|^2 + (\nu\lambda - c_0^4 \nu^{-1} \|z(t)\|^2 - \delta)|w|^2 \leqslant \delta^{-1} |g(t) - g(t+p)|^2, \qquad (2.55)$$

where δ is a fixed positive number which will be specified later. From the inequality (2.40) it follows that

$$\begin{aligned}\nu \int\limits_\tau^t \|z(s)\|^2 ds &\leqslant |z(\tau)|^2 + (\nu\lambda)^{-1} \int\limits_\tau^t |g(s)|^2 ds \\ &\leqslant |z(\tau)|^2 + (\nu\lambda)^{-1}(t - \tau + 1)\|g\|^2_{L_2^b} \\ &\leqslant |z(\tau)|^2 + (\nu\lambda)^{-1}(t - \tau + 1)\|g_0\|^2_{L_2^b}. \qquad (2.56)\end{aligned}$$

Since $z(\tau) \in \mathcal{A}$, from (2.45) it follows that

$$|z(\tau)|^2 \leqslant (\nu\lambda)^{-1}(1 + (\nu\lambda)^{-1})\|g_0\|^2_{L_2^b} = R_0^2.$$

By (2.56), we have

$$\begin{aligned}\int\limits_\tau^t \|z(s)\|^2 ds &\leqslant (\nu^{-1} R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L_2^b}) + (\nu^2\lambda)^{-1}(t - \tau)\|g_0\|^2_{L_2^b} \\ &= R_1^2 + (\nu^2\lambda)^{-1}(t - \tau)\|g_0\|^2_{L_2^b}, \qquad (2.57)\end{aligned}$$

where $R_1^2 = \nu^{-1} R_0^2 + (\nu^2\lambda)^{-1}\|g_0\|^2_{L_2^b}$. We set $\alpha(t) = \nu\lambda - c_0^4 \nu^{-1} \|z(t)\|^2 - \delta$. Multiplying (2.55) by $\exp\Big\{ \int\limits_\tau^t \alpha(s) ds \Big\}$ and integrating over $[\tau, t]$, we find

$$|w(t)|^2 \leqslant |w(\tau)|^2 e^{-\int\limits_\tau^t \alpha(s)ds} + \frac{1}{\delta} \int\limits_\tau^t |g(\theta) - g(\theta + p)|^2 e^{-\int\limits_\theta^t \alpha(s)ds} d\theta. \qquad (2.58)$$

Using (2.57), we find

$$-\int_\theta^t \alpha(s)ds \leqslant c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L_2^b}(t-\theta) - (\nu\lambda-\delta)(t-\theta) + c_0^4\nu^{-1}R_1^2$$
$$= -(\nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L_2^b} - \delta)(t-\theta) + R_2^2$$
$$= -(\beta-\delta)(t-\theta) + R_2^2, \tag{2.59}$$

where $R_2^2 = c_0^4\nu^{-1}R_1^2$ and $\beta = \nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L_2^b}$. We note that

$$\nu^{-4}\lambda^{-2}\|g_0\|^2_{L_2^b} = G^2 < c_0^{-4}$$

(see (2.48)). Therefore,

$$\beta = \nu\lambda - c_0^4\nu^{-3}\lambda^{-1}\|g_0\|^2_{L_2^b} > 0.$$

We set $\delta = \beta/2$. Then (2.58) implies that

$$|w(t)|^2 \leqslant |w(\tau)|^2 e^{R_2^2} e^{-\beta(t-\tau)/2} + \frac{2}{\beta} e^{R_2^2} \int_\tau^t |g(\theta) - g(\theta+p)|^2 e^{-\beta(t-\theta)/2} d\theta. \tag{2.60}$$

Let p be an ε-period of g, i.e., $|g(\theta) - g(\theta+p)| \leqslant \varepsilon$ for all $\theta \in \mathbb{R}$. By (2.60), we have

$$|w(t)|^2 \leqslant |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2 \frac{2}{\beta}\varepsilon^2 \int_\tau^t e^{-\beta(t-\theta)/2} d\theta$$
$$\leqslant |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2((2\varepsilon)/\beta)^2(1 - e^{-\beta(t-\tau)/2})$$
$$\leqslant |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2} + C_2((2\varepsilon)/\beta)^2, \tag{2.61}$$

where $C_2 = e^{R_2^2}$. Note that $|w(\tau)| \leqslant C'$ for all $\tau \in \mathbb{R}$. Using (2.61) and letting $\tau \to -\infty$, we obtain the inequality

$$|w(t)| = |z(t) - z(t+p)| \leqslant \varepsilon \frac{2\sqrt{C_2}}{\beta}. \tag{2.62}$$

Hence p is an $\varepsilon \dfrac{2\sqrt{C_2}}{\beta}$-period of the function $z(t)$. Hence $z(t)$ is almost periodic. $\square$

Consider the case, where $g_0(t)$ is quasiperiodic, i.e.,

$$g_0(x,t) = \varphi(x, \alpha_1 t, \ldots, \alpha_k t) = \varphi(x, \bar{\alpha} t), \tag{2.63}$$

$\varphi(\cdot,\bar{\omega}) \in C^{\mathrm{Lip}}(\mathbb{T}^k;H)$, $\bar{\omega}=(\omega_1,\dots,\omega_k)$, and real numbers $(\alpha_1,\dots,\alpha_k)=\bar{\alpha}$ are rationally independent (see Example 2.5).

Proposition 2.5. *Let the condition* (2.48) *hold, and let the function $g_0(t)$ be quasiperiodic. Then the corresponding function $z_0(t)=z_{g_0}(t)$ (unique by Theorem* 2.4) *is also quasiperiodic, i.e., there exists a function $\Phi(x,\bar{\omega}) \in C^{\mathrm{Lip}}(\mathbb{T}^k;H)$ such that $z_0(x,t)=\Phi(x,\alpha_1 t,\dots,\alpha_k t)$ and the frequencies $(\alpha_1,\dots,\alpha_k)$ are the same as those for the function $g_0(x,t)$.*

PROOF. Consider the external force $g_{\bar{\omega}}(x,t)=\varphi(x,\bar{\alpha}t+\bar{\omega})$, where $\bar{\omega}\in\mathbb{T}^k$. It is obvious that $g_{\bar{\omega}}\in\mathcal{H}(g_0)$ (see (2.28)). By (2.48), with each such an external force $g_{\bar{\omega}}$ we can associate a unique bounded complete trajectory $z_{\bar{\omega}}(x,t)$ of the Navier–Stokes equation with external force $g_{\bar{\omega}}(x,t)$ which satisfies (2.50). We set

$$\Phi(x,\bar{\omega})=z_{\bar{\omega}}(x,0) \tag{2.64}$$

and prove that Φ is the desired function. First of all, we note that

$$z_{\bar{\omega}}(x,t+h)=z_{\bar{\alpha}h+\bar{\omega}}(x,t) \tag{2.65}$$

because of the uniqueness of the bounded complete trajectory $z_{\bar{\alpha}h+\bar{\omega}}(x,t)$ corresponding to $g_{\bar{\alpha}h+\bar{\omega}}(x,t)$. It is easy to see that the function $z_{\bar{\omega}}(x,t+h)$ satisfies the Navier–Stokes system with external force $\varphi(x,\bar{\alpha}(t+h)+\overline{\omega})=g_{\bar{\alpha}h+\bar{\omega}}(x,t)$. By (2.64), we conclude that

$$z_{\bar{\omega}}(x,h)=\Phi(x,\bar{\alpha}h+\bar{\omega}),$$

i.e., $z_{\bar{\omega}}(x,t)=\Phi(x,\bar{\alpha}t+\bar{\omega})$ for all $t\in\mathbb{R}$.

We show that $\Phi(x,\bar{\omega})=\Phi(x,\omega_1,\dots,\omega_k)$ has period 2π with respect to each variable ω_i. This property follows from the uniqueness of bounded complete trajectories because

$$\Phi(x,\bar{\omega}+2\pi\bar{e}_i)=z_{\bar{\omega}+2\pi\bar{e}_i}(x,0)=z_{\bar{\omega}}(x,0)=\Phi(x,\bar{\omega}),$$

where $\{\bar{e}_i, i=1,\dots,k\}$ is the standard basis for $\mathbb{R}^k$. It remains to verify the Lipschitz condition with respect to $\bar{\omega}\in\mathbb{T}^k$ for the function Φ. We set $w(t)=z_{\bar{\omega}_1}(t)-z_{\bar{\omega}_2}(t)$. As in the case of (2.60), we prove the inequality

$$|w(t)|^2 \leqslant |w(\tau)|^2 C_2 e^{-\beta(t-\tau)/2}+\frac{2}{\beta}C_2\int\limits_\tau^t |g_{\bar{\omega}_1}(\theta)-g_{\bar{\omega}_2}(\theta)|^2 e^{-\beta(t-\theta)/2}d\theta. \tag{2.66}$$

The function φ satisfies the inequality

$$|\varphi(\bar{\omega}_1)-\varphi(\bar{\omega}_2)| \leqslant \varkappa|\bar{\omega}_1-\bar{\omega}_2| \quad \forall\, \bar{\omega}_1,\bar{\omega}_2\in\mathbb{T}^k.$$

Therefore,

$$|g_{\bar{\omega}_1}(\theta) - g_{\bar{\omega}_2}(\theta)| \leqslant \varkappa|\bar{\omega}_1 - \bar{\omega}_2|. \tag{2.67}$$

From (2.66) and (2.67), as in the case of (2.61) and (2.62), we find

$$|w(t)| = |z_{\bar{\omega}_1}(t) - z_{\bar{\omega}_2}(t)| \leqslant \varkappa\frac{2\sqrt{C_2}}{\beta}|\bar{\omega}_1 - \bar{\omega}_2|.$$

Finally, by (2.64), we have

$$|\Phi(\cdot, \bar{\omega}_1) - \Phi(\cdot, \bar{\omega}_2)| = |z_{\bar{\omega}_1}(0) - z_{\bar{\omega}_2}(0)| \leqslant \varkappa\frac{2\sqrt{C_2}}{\beta}|\bar{\omega}_1 - \bar{\omega}_2|, \tag{2.68}$$

i.e., $\Phi(x, \bar{\omega}) \in C^{\,\mathrm{Lip}}(\mathbb{T}^k; H)$. □

Corollary 2.3. *Under the assumptions of Theorem* 2.5, *the global attractor $\mathcal{A}$ of the Navier–Stokes system is the Lipschitz continuous image of the k-dimensional torus*:

$$\mathcal{A} = \Phi(\mathbb{T}^k) \tag{2.69}$$

and the set $\mathcal{A}$ attracts solutions of the equation with exponential rate (*see* (2.50)).

Recall that $\Phi(\cdot, \bar{\omega}) = \Phi(\cdot, \bar{\alpha}t + \bar{\omega})|_{t=0} = z_{\bar{\omega}}(x, t)|_{t=0}$, $\bar{\omega} \in \mathbb{T}^k$.

Remark 2.9. By (2.69), the uniform global attractor $\mathcal{A}$ of the Navier–Stokes system with quasiperiodic external force g_0 satisfying (2.48) and (2.63) is finite-dimensional, and $\mathbf{d}_F(\mathcal{A}) \leqslant k$, where $d_F(\mathcal{A})$ is the fractal dimension of $\mathcal{A}$ (see Section 1.4.1). It is easy to construct examples of external forces satisfying (2.48) and (2.63) such that $\mathbf{d}_F(\mathcal{A}) = k$ (see, for example, [**25**]). Thus, the dimension of global attractors $\mathcal{A}$ of nonautonomous Navier–Stokes systems may grow to infinity as $k \to \infty$, while the Grashof numbers (or Reynolds numbers) remain bounded. Moreover, there are almost periodic external forces such that $\mathbf{d}_F(\mathcal{A}) = \infty$ (see Section 2.7). Such phenomena do not occur in the autonomous case, where the dimension of the global attractor is always less than the multiple of the Grashof number (see Theorem 1.6 and (1.57)). In Section 3, we will consider the Kolmogorov ε-entropy and the fractal dimension of uniform global attractors of nonautonomous equations in detail.

2.6.2. Nonautonomous damped wave equations. Consider the nonautonomous wave equation with damping

$$\partial_t^2 u + \gamma\partial_t u = \Delta u - f_0(u, t) + g_0(x, t),\ u|_{\partial\Omega} = 0,\ x \in \Omega \Subset \mathbb{R}^n, \tag{2.70}$$

where $\gamma\partial_t u$ is the dissipation term ($\gamma > 0$). The autonomous case was considered in Section 1.3.2. We assume that $f_0(v,t) \in C^1(\mathbb{R}\times\mathbb{R};\mathbb{R})$ and

$$F_0(v,t) \geqslant -mv^2 - C_m,\ F_0(v,t) := \int\limits_0^v f_0(w,t)dw, \tag{2.71}$$

$$f_0(v,t)v - \gamma_1 F_0(v,t) + mv^2 \geqslant -C_m \quad \forall\,(v,t) \in \mathbb{R}\times\mathbb{R}, \tag{2.72}$$

where $m > 0$ is sufficiently small and $\gamma_1 > 0$.

Assume that ρ is a positive number such that $\rho < 2/(n-2)$ for $n \geqslant 3$ and is arbitrarily large for $n = 1, 2$. Let

$$|\partial_v f_0(v,t)| \leqslant C_0(1+|v|^\rho),\ |\partial_t f_0(v,t)| \leqslant C_0(1+|v|^{\rho+1}), \tag{2.73}$$

$$\partial_t F_0(v,t) \leqslant \delta^2 F_0(v,t) + C_1 \quad \forall\ (v,t) \in \mathbb{R}\times\mathbb{R}, \tag{2.74}$$

where δ is sufficiently small.

Remark 2.10. Let $f_0(v,t) = f(v)\varphi(t)$, where, for example, $f(v) = |v|^\rho v$ or $f(v) = R + \beta\sin(v)$, $|\beta| < R$, and $\varphi(s)$ is a positive bounded continuous function such that $\varphi'(t) \leqslant \delta^2\varphi(t)$ for all $t \in \mathbb{R}$. Then $f_0(v,s)$ satisfies (2.71)–(2.74).

From (2.73) it follows that

$$|f_0(v,t)| \leqslant C_0'(1+|v|^{\rho+1}), \quad |F_0(v,s)| \leqslant C_0'(1+|v|^{\rho+2}). \tag{2.75}$$

Assume that $g_0 \in L_2^b(\mathbb{R}; L_2(\Omega))$.

The initial conditions are posed at $t = \tau$:

$$u|_{t=\tau} = u_\tau(x), \quad \partial_t u|_{t=\tau} = p_\tau(x),\ \tau \in \mathbb{R}. \tag{2.76}$$

Proposition 2.6. *If $u_\tau \in H_0^1(\Omega)$ and $p_\tau \in L_2(\Omega)$, then the problem* (2.70), (2.76) *has a unique solution $u(t) \in C(\mathbb{R}_\tau; H_0^1(\Omega))$ such that $\partial_t u(t) \in C(\mathbb{R}_\tau; L_2(\Omega))$ and $\partial_t^2 u(t) \in L_2^{\mathrm{loc}}(\mathbb{R}_\tau; H^{-1}(\Omega))$.*

The proof can be found in [**119, 68, 9, 34**].

We set $y(t) = (u(t), \partial_t u(t)) = (u(t), p(t))$ and $y_\tau = (u_\tau, p_\tau) = y(\tau)$ for brevity. Denote by E the space of vector-valued functions $y(x) = (u(x), p(x))$ with finite energy norm

$$\|y\|_E^2 = \|(u,p)\|_E^2 = |\nabla u|^2 + |p|^2$$

in the space $E = H_0^1(\Omega) \times L_2(\Omega)$. Recall that $|\cdot|$ denotes the norm in $L_2(\Omega)$. By Proposition 2.6, $y(t) \in E$ for all $t \geqslant 0$.

The problem (2.70), (2.76) is equivalent to the system

$$\begin{cases} \partial_t u = p \\ \partial_t u = -\gamma p + \Delta u - f_0(u,t) + g_0(x,t), \end{cases} \qquad \begin{cases} u|_{t=\tau} = u_\tau \\ p|_{t=\tau} = p_\tau, \end{cases}$$

which can be rewritten in the operator form

$$\partial_t y = A_{\sigma_0(t)}(y), \quad y|_{t=\tau} = y_\tau, \tag{2.77}$$

for an appropriate operator $A_{\sigma_0(t)}(\,\cdot\,)$, where $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ is the symbol of Equation (2.77) (see Section 2.4). If $y_\tau \in E$ then, by Proposition 2.6, the problem (2.77) has a unique solution $y(t) \in C_b(\mathbb{R}_\tau; E)$. This implies that the process $\{U_{\sigma_0}(t,\tau)\}$ given by the formula $U_{\sigma_0}(t,\tau)y_\tau = y(t)$ is defined in E.

Proposition 2.7. *The process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to the problem* (2.77) *is uniformly bounded, and the following estimate holds*:

$$\|y(t)\|_E^2 \leqslant C_1 \|y_\tau\|_E^{\rho+2} \exp(-\beta(t-\tau)) + C_2, \ \beta > 0, \tag{2.78}$$

where $y(t) = U_{\sigma_0}(t,\tau)y_\tau$ and the constants C_1, C_2 are independent of y_τ.

The proof can be found in [**34**].

By Proposition 2.7, the process $\{U_{\sigma_0}(t,\tau)\}$ has a bounded (in E) uniformly absorbing set $B_0 = \{y = (u,p) \mid \|y\|_E^2 \leqslant 2C_2\}$, i.e., $U_{\sigma_0}(t,\tau)B \subseteq B_0$, $t-\tau \geqslant h(B)$, for every $B \in \mathcal{B}(E)$. The following result is more complicated (see the proof in [**34**]).

Proposition 2.8. *The process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to the problem* (2.77) *is uniformly asymptotically compact in E.*

By Theorem 2.1 and Proposition 2.8, the process $\{U_{\sigma_0}(t,\tau)\}$ has the global attractor $\mathcal{A}$, and the set $\mathcal{A}$ is compact in E.

Now, we introduce the enveloped space Ξ for the symbol $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ of Equation (2.77). Suppose that $g_0(x,t)$ is a translation compact function in $L_2^{\text{loc}}(\mathbb{R}; L_2(\Omega))$, the function $f_0(v,t)$ satisfies (2.71)–(2.74), and $(f_0(v,t), \partial_t f_0(v,t))$ is a translation compact function in $C(\mathbb{R}; \mathcal{M})$. Here, $\mathcal{M}$ is the space of functions $\{(\psi(v), \psi_1(v)),\ v \in \mathbb{R} \mid (\psi, \psi_1) \in C(\mathbb{R}; \mathbb{R}^2)\}$ endowed with the norm

$$\|(\psi, \psi_1)\|_{\mathcal{M}} = \sup_{v \in \mathbb{R}} \left\{ \frac{|\psi(v)| + |\psi_1(v)|}{|v|^{\rho+1} + 1} + \frac{|\psi'(v)|}{|v|^\rho + 1} \right\}. \tag{2.79}$$

It is obvious that $\mathcal{M}$ is a Banach space and $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ is a translation compact function in $\Xi = C(\mathbb{R}\mathcal{M}) \times L_2^{\text{loc}}(\mathbb{R}; H)$.

Consider the hull $\mathcal{H}(\sigma_0)$ of the symbol σ_0 in the space Ξ. It is easy to show that for any $\sigma(t) = (f(v,t), g(x,t)) \in \mathcal{H}(\sigma_0)$, the function $f(v,t)$ satisfies the inequalities (2.71)–(2.74) with the same constants as those for $f_0(v,t)$. Thus, the problem (2.77) is well posed for all $\sigma \in \mathcal{H}(\sigma_0)$ and generates a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in E. The following assertion is proved in [**34**].

Proposition 2.9. *The family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, *corresponding to the problem* (2.77) *is* $(E \times \mathcal{H}(\sigma_0), E)$*-continuous.*

Using Theorem 2.4, we obtain the following assertion.

Theorem 2.5. *If* $\sigma_0(t) = (f_0(v,t), g_0(x,t))$ *is a translation compact function in* $\Xi = C(\mathbb{R};\mathcal{M}) \times L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$, *then the process* $\{U_{\sigma_0}(t,\tau)\}$ *corresponding to the problem* (2.77) *has the uniform global attractor*

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ *is the kernel of the process* $\{U_\sigma(t,\tau)\}$ *with symbol* $\sigma \in \mathcal{H}(\sigma_0)$. *The kernel* $\mathcal{K}_\sigma$ *is nonempty for all* $\sigma \in \mathcal{H}(\sigma_0)$; *moreover,*

$$\mathcal{A} = \omega(B_0) = \bigcap_{h \geqslant 0} \Big[\bigcup_{t-\tau \geqslant h} U(t,\tau)B_0 \Big]_E.$$

We consider a special case of (2.70): the sine-Gordon type equation with dissipation

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - f(u) + g_0(x,t), \ u|_{\partial\Omega} = 0, \ x \in \Omega, \tag{2.80}$$

where $\Omega \Subset \mathbb{R}^n$, $\gamma > 0$, $f \in C(\mathbb{R})$, $g_0(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$. Assume that $f(u)$ satisfies the inequalities

$$|f(v)| \leqslant C \quad \forall v \in \mathbb{R}, \tag{2.81}$$

$$|f(v_1) - f(v_2)| \leqslant K|v_1 - v_2| \quad \forall v_1, v_2 \in \mathbb{R}. \tag{2.82}$$

Remark 2.11. For $f(u) = K\sin(u)$ Equation (2.80) is the sine-Gordon equation with dissipation (see [**119**]).

We assume that the external force $g(x,t)$ satisfies the condition

$$\|g_0\|^2_{L_2^b} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g_0(s)\|^2_{L_2(\Omega)} ds < +\infty. \tag{2.83}$$

As above, we consider the Cauchy problem for Equation (2.80) with initial conditions

$$u|_{t=\tau} = u_\tau \in H_0^1(\Omega), \quad \partial_t u|_{t=\tau} = p_\tau \in L_2(\Omega). \tag{2.84}$$

As in Proposition 2.6, we prove that for any $u_\tau(x) \in H_0^1(\Omega)$ and $p_\tau(x) \in L_2(\Omega)$ the problem (2.80), (2.84) has a unique solution $u(t) \in C(\mathbb{R}_\tau; H_0^1(\Omega))$ such that $\partial_t u(t) \in C(\mathbb{R}_\tau; L_2(\Omega))$ and $\partial_t^2 u(t) \in L_2^{\text{loc}}(\mathbb{R}_\tau; H^{-1}(\Omega))$ (see, for example, [**119, 68, 9, 34**]). Denoting $y(t) = (u(t), p(t)) = (u(t), \partial_t u(t))$ and $y_\tau = (u_\tau, p_\tau)$, we see that $y(t) \in C(\mathbb{R}_\tau; E)$, $y(\tau) = y_\tau$. Then the problem (2.80), (2.84) has the form of an evolution equation

$$\begin{cases} \partial_t u = p \\ \partial_t p = -\gamma p + \Delta u - f(u) + g_0(x,t), \end{cases}, \begin{cases} u|_{t=\tau} = u_\tau, \\ p|_{t=\tau} = p_\tau \end{cases} \tag{2.85}$$

(see (2.77)). The time symbol of this system is a one-component function $\sigma_0(t) = g_0(\cdot, t)$ with values in $L_2(\Omega)$. Since (2.85) has a unique solution, it defines via $y(t) = U_{g_0}(t,\tau)y_\tau$ a process $\{U_{g_0}(t,\tau)\}$ acting in E. Propositions 2.7, 2.8, and 2.9 hold for the process $\{U_{g_0}(t,\tau)\}$ with $\rho = 0$. Consider the uniform global attractor $\mathcal{A}$ of this process.

Proposition 2.10. *Under the conditions* (2.81), (2.82), (2.83), *the problem* (2.85) *has a global attractor $\mathcal{A}$, and the set $\mathcal{A}$ is compact in E.*

We refer to [**34, 25, 36**]. We note that the process $\{U_{g_0}(t,\tau)\}$ is not uniformly compact, but only uniformly *asymptotically* compact.

For studying the structure of the global attractor $\mathcal{A}$, we assume that $g_0(x,t)$ is a translation compact function in $L_2^{\text{loc}}(\mathbb{R}; L_2(\Omega))$. Consider the hull $\mathcal{H}(g_0)$. For any symbol $g \in \mathcal{H}(g_0)$ the problem (2.85) with g instead of g_0 generates the process $\{U_g(t,\tau)\}$ in E. As was proved in [**34**], the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, is $(E \times \mathcal{H}(g_0), E)$-continuous. Using Theorem 2.4, we obtain the following assertion.

Proposition 2.11. *Let $g_0(x,t)$ be a translation compact function in $L_2^{\text{loc}}(\mathbb{R}; L_2(\Omega))$. Then the global attractor $\mathcal{A}$ of the process $\{U_{g_0}(t,\tau)\}$ can be represented as*

$$\mathcal{A} = \bigcup_{g \in \mathcal{H}(g_0)} \mathcal{K}_g(0), \tag{2.86}$$

where $\mathcal{K}_g$ is the kernel of Equation (2.85) *with symbol $g \in \mathcal{H}(g_0)$. The kernel $\mathcal{K}_g$ is nonempty for every g.*

We now specify the case, where the global attractor $\mathcal{A}$ has a simple structure and is exponentially attracting. We denote by λ the first eigenvalue of the Laplacian on $H_0^1(\Omega)$. We have the following

Theorem 2.6. *Let the Lipschitz constant K in* (2.82) *satisfy the inequality*

$$K < \lambda. \tag{2.87}$$

and let the dissipation rate γ in (2.80) *satisfy the condition*

$$\gamma^2 > \gamma_0^2 := 2(\lambda - \sqrt{\lambda^2 - K^2}). \tag{2.88}$$

Then for every $g \in \mathcal{H}(g_0)$ Equation (2.85) *with external force g has a unique bounded* (*in E*) *solution $z(t) = (w(t), \partial_t w(t))$ for all $t \in \mathbb{R}$. Moreover, for any solution $y(t) = U_g(t,\tau)y_\tau$ of Equation* (2.85), *the following inequality holds*:

$$\|y(t) - z(t)\|_E \leqslant C\|y_\tau - z(\tau)\|_E e^{-\beta(t-\tau)}, \tag{2.89}$$

where $C > 0$ and $\beta > 0$ are independent of y_τ.

Proof. We repeat the arguments of [**36**]. The relations below can be justified with the help of the Galerkin approximation method (see [**96, 119, 9**]). Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (2.80) with external force $g \in \mathcal{H}(g_0)$. Then the difference $w(x,t) := u_1(x,t) - u_2(x,t)$ is a solution of the problem

$$\partial_t^2 w + \gamma \partial_t w = \Delta w - (f(u_1) - f(u_2)) \ \text{ in } \Omega \ \text{ and } w|_{\partial\Omega} = 0, \tag{2.90}$$

The equation in (2.90) can be written in the form

$$\begin{aligned}&\partial_t(\partial_t w + \alpha w) + (\gamma - \alpha)(\partial_t w + \alpha w) - \Delta w - \alpha(\gamma-\alpha)w \\ &= -(f(u_1) - f(u_2)),\end{aligned} \tag{2.91}$$

where α is a suitable parameter which will be chosen later. Multiplying Equation (2.91) by $v = \partial_t w + \alpha w$, integrating over Ω, integrating by parts, and using the condition (2.82), we arrive at the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|v|^2 + |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2) + (\gamma-\alpha)|v|^2 \\ &+ \alpha(|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2) = -(f(u_1) - f(u_2), v) \leqslant K|w||v|.\end{aligned} \tag{2.92}$$

We choose $\alpha > 0$ such that

$$\alpha(\gamma - \alpha) < \lambda. \tag{2.93}$$

Using the Poincaré inequality $\lambda|w|^2 \leqslant |\nabla w|^2$, we find

$$\lambda|w|^2 - \alpha(\gamma-\alpha)|w|^2 \leqslant |\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2,$$

i.e.,

$$|w|^2 \leqslant \frac{|\nabla w|^2 - \alpha(\gamma-\alpha)|w|^2}{\lambda - \alpha(\gamma-\alpha)}. \tag{2.94}$$

By (2.94) and (2.92), we have

$$\frac{1}{2}\frac{d}{dt}(X^2+Y^2)+\left\{(\gamma-\alpha)X^2+\alpha Y^2-\frac{K}{\sqrt{\lambda-\alpha(\gamma-\alpha)}}XY\right\}<0, \quad (2.95)$$

where $X^2=|v|^2=|\partial_t w+\alpha w|^2$ and $Y^2=|\nabla w|^2-\alpha(\gamma-\alpha)|w|^2$.

The quadratic form $\{\dots\}$ in (2.95) is positive definite provided that $\alpha>0$, $\gamma-\alpha>0$, and

$$\alpha(\gamma-\alpha)-\frac{K}{4(\lambda-\alpha(\gamma-\alpha))}>0. \quad (2.96)$$

We set $\varrho=\alpha(\gamma-\alpha)$. The inequality (2.96) is equivalent to the inequality

$$\varrho^2-\lambda\varrho+\frac{K^2}{4}<0. \quad (2.97)$$

Since $K<\lambda$, the quadratic inequality (2.97) is satisfied by any ϱ such that

$$\frac{\lambda-\sqrt{\lambda^2-K^2}}{2}<\varrho<\frac{\lambda+\sqrt{\lambda^2-K^2}}{2}. \quad (2.98)$$

From (2.98) it follows that $\varrho<\lambda$, i.e., $\alpha(\gamma-\alpha)<\lambda$ and the condition (2.93) is satisfied. Thus, we need to find $\alpha>0$ such that

$$\frac{\lambda-\sqrt{\lambda^2-K^2}}{2}<\alpha(\gamma-\alpha)<\frac{\lambda+\sqrt{\lambda^2-K^2}}{2}. \quad (2.99)$$

Note that such α always exists if the maximum of $\alpha(\gamma-\alpha)$ with respect to α is greater than the left bound in (2.99), i.e., if

$$\frac{\gamma^2}{4}>\frac{\lambda-\sqrt{\lambda^2-K^2}}{2}. \quad (2.100)$$

This inequality coincides with the assumption (2.88). Consequently, taking α that satisfies both inequalities in (2.99), we see that the quadratic form $\{\dots\}$ in (2.95) is positive definite and

$$(\gamma-\alpha)X^2+\alpha Y^2-\frac{K}{\sqrt{\lambda-\alpha(\gamma-\alpha)}}XY\geqslant\beta(X^2+Y^2),\ \beta>0, \quad (2.101)$$

where β explicitly depends on γ, λ, K. Then (2.95) takes the form

$$\frac{1}{2}\frac{d}{dt}(X^2+Y^2)+\beta(X^2+Y^2)<0$$

and the Gronwall inequality yields

$$X^2(t)+Y^2(t)\leqslant(X^2(\tau)+Y^2(\tau))e^{-2\beta(t-\tau)}. \quad (2.102)$$

We see that the expression $X^2+Y^2 = |\partial_t w+\alpha w|^2+|\nabla w|^2-\alpha(\gamma-\alpha)|w|^2$ is equivalent to the norm $\|y_1-y_2\|_E^2 = |\partial_t w|^2+|\nabla w|^2$. Hence (2.102) implies the inequality

$$\|y_1(t)-y_2(t)\|_E^2 \leqslant C^2\|y_1(\tau)-y_2(\tau)\|_E^2 e^{-2\beta(t-\tau)} \quad \forall t \geqslant \tau, \tag{2.103}$$

with some constant $C = C(\gamma,\lambda,\alpha)$.

By Proposition 2.11, the kernel $\mathcal{K}_g$ of Equation (2.85) is nonempty, i.e., there is a bounded (in E) solution $z(t) = z_g(t)$, $t \in \mathbb{R}$, of the system (2.85). Substituting $z(t)$ into (2.103), for any other solution $y(t) = U_g(t,\tau)y_\tau$ we obtain the estimate

$$\|y(t)-z(t)\|_E \leqslant C\|y_\tau - z(\tau)\|_E e^{-\beta(t-\tau)} \quad \forall t \geqslant \tau, \tag{2.104}$$

which means that $z(t)$ is the unique bounded complete trajectory of the process $\{U_g(t,\tau)\}$ corresponding to (2.85). □

Now, we formulate consequences of Theorem 2.6 which can be proved in the same way as the corresponding assertions for the 2D Navier–Stokes system in Section 2.6.1 (see Corollaries 2.1–2.3 and Proposition 2.5).

Corollary 2.4. *Under the assumptions* (2.87) *and* (2.88), *the global attractor of Equation* (2.85) *has the form*

$$\mathcal{A} = [\{z_{g_0}(t) \mid t \in \mathbb{R}\}]_E = \bigcup_{g\in\mathcal{H}(g_0)} \{z_g(0)\}. \tag{2.105}$$

Corollary 2.5. *The constructed global attractor $\mathcal{A}$ is exponential, i.e., for every bounded set $B \subset E$*

$$\mathrm{dist}_E(U_{g_0}(t,\tau)B,\mathcal{A}) \leqslant C\|B\|_E e^{-\beta(t-\tau)} \quad \forall\, t \geqslant \tau, \tag{2.106}$$

where $\|B\|_E = \sup\{\|y\|_E \mid y \in B\}$.

Corollary 2.6. *If $g(t)$ is periodic with period p, then $z_g(t)$ is also periodic with period p.*

Corollary 2.7. *If $g(t)$ is almost periodic, then $z_g(t)$ is also almost periodic.*

PROOF. As in the case of (2.102), we show that $w(t) = z_g(t)-z_g(t+p)$ satisfies the inequality

$$\frac{d}{dt}(X^2+Y^2)+2\beta(X^2+Y^2) \leqslant 2|g(t)-g(t+p)||v|, \tag{2.107}$$

where $X^2 = |v(t)|^2 = |\partial_t w(t)+\alpha w(t)|^2$ and $Y^2 = |\nabla w(t)|^2 - \alpha(\gamma-\alpha)|w(t)|^2$. Using the estimate

$$2|g(t) - g(t+p)||v| \leqslant \beta X^2 + \beta^{-1}|g(t) - g(t+p)|^2, \tag{2.108}$$

we find that

$$\frac{d}{dt}(X^2 + Y^2) + \beta(X^2 + Y^2) \leqslant \beta^{-1}|g(t) - g(t+p)|^2. \tag{2.109}$$

If p is an ε-period of g, i.e., $|g(t) - g(t+p)| < \varepsilon$ for all $t \in \mathbb{R}$, then from (2.109) it follows that

$$X^2(t) + Y^2(t) \leqslant (X^2(\tau) + Y^2(\tau))e^{-\beta(t-\tau)} + \varepsilon^2/\beta^2.$$

Fixing t and letting $\tau \to -\infty$, we find

$$\|z_g(t) - z_g(t+p)\|_E^2 \leqslant C(X^2(t) + Y^2(t)) \leqslant C\frac{\varepsilon^2}{\beta^2} \quad \forall t \in \mathbb{R}, \tag{2.110}$$

i.e., p is an $\varepsilon\sqrt{C}/\beta$-period of the function z_g and thereby $z_g(t)$ is almost periodic. $\square$

We now assume that $g_0(t)$ is quasiperiodic and has k rationally independent frequencies, i.e.,

$$g_0(t) = \varphi(x, \alpha_1 t, \ldots, \alpha_k t) = \varphi(x, \bar{\alpha}t), \tag{2.111}$$

where $\varphi \in C^{\,\mathrm{Lip}}(\mathbb{T}^k; L_2(\Omega))$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}$ (see Example 2.5).

Proposition 2.12. *If $g_0(t)$ is a quasiperiodic function of the form* (2.111)*, then the corresponding function $z_{g_0}(t)$ is also quasiperiodic. Thus, there exists $\Phi \in C^{\,\mathrm{Lip}}(\mathbb{T}^k; L_2(\Omega))$ such that $z_{g_0}(t) = \Phi(x, \alpha_1 t, \ldots, \alpha_k t)$.*

The proof is similar to that of Proposition 2.5.

Corollary 2.8. *If $g_0(t)$ has the form* (2.111)*, then the global attractor $\mathcal{A}$ of Equation* (2.80) *is the Lipschitz continuous image of the k-dimensional torus $\mathbb{T}^k : \mathcal{A} = \Phi(\mathbb{T}^k)$ and $\mathbf{d}_F(\mathcal{A}) \leqslant k$.*

Remark 2.12. It is easy to construct external forces $g_0(t)$ of the form (2.111) such that $\mathbf{d}_F(\mathcal{A}) = k$. Moreover, there exist almost periodic external forces such that $\mathbf{d}_F(\mathcal{A}) = \infty$ (see Section 2.7).

Remark 2.13. Making the change of the time variable $t = t'/\gamma$ in (2.80), we obtain the problem

$$\varepsilon\partial_t^2 u + \partial_t u = \Delta u - f(u) + g_0(x,t), \; u|_{\partial\Omega} = 0,$$

where $\varepsilon = \gamma^{-2}$. The above results are applicable provided that $|f'(u)| < \lambda$ for all $u \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0 := 2^{-1}(\lambda - \sqrt{\lambda^2 - k^2})^{-1}$.

2.6.3. Nonautonomous Ginzburg–Landau equation. Consider the following nonautonomous generalization of the Ginzburg–Landau equation (see Section 1.3.3) with zero boundary conditions (periodic boundary conditions can be treated in a similar way):

$$\begin{gathered} \partial_t u = (1 + i\alpha_0(t))\Delta u + R_0(t)u - (1 + i\beta_0(t))|u|^2 u + g_0(x,t), \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{2.112}$$

where $u = u^1(x,t) + iu^2(x,t)$ is the unknown complex function and $x \in \Omega \Subset \mathbb{R}^n$. The coefficients $\alpha_0(t)$, $\beta_0(t)$, and $R_0(t)$ are given real-valued functions in $C_b(\mathbb{R})$. We assume that

$$|\beta_0(t)| \leqslant \sqrt{3} \quad \forall t \in \mathbb{R}. \tag{2.113}$$

The phase space for (2.112) is $\mathbf{H} = L_2(\Omega;\mathbb{C})$. The norm in $\mathbf{H}$ is denoted by $\|\cdot\|$. We also introduce the notation $\mathbf{V} = H_0^1(\Omega;\mathbb{C})$ and $\mathbf{L}_4 = L_4(\Omega;\mathbb{C})$. Assume that $g_0(x,t) = g_0^1(x,t) + ig_0^2(x,t)$ belongs to $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})$, i.e.,

$$\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_0(\cdot,s)\|^2 ds. \tag{2.114}$$

Recall that Equation (2.112) is equivalent to the following system relative to the vector-valued function $\mathbf{u} = (u^1, u^2)^\top$:

$$\partial_t \mathbf{u} = \begin{pmatrix} 1 & -\alpha_0(t) \\ \alpha_0(t) & 1 \end{pmatrix} \Delta\mathbf{u} + R_0(t)\mathbf{u} - \begin{pmatrix} 1 & -\beta_0(t) \\ \beta_0(t) & 1 \end{pmatrix} |\mathbf{u}|^2\mathbf{u} + \mathbf{g}_0(x,t),$$

where $\mathbf{g}_0 = (g_0^1, g_0^2)^\top$. Under the above assumptions, the Cauchy problem for Equation (2.112) with initial data

$$u|_{t=\tau} = u_\tau(x), \quad u_\tau(\cdot) \in \mathbf{H},\ \tau \in \mathbb{R}, \tag{2.115}$$

has a unique weak solution $u(t) := u(x,t)$ such that

$$u(\cdot) \in C_{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{H}) \cap L_2^{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{V}) \cap L_4^{\mathrm{b}}(\mathbb{R}_\tau;\mathbf{L}_4)$$

and (2.112) is satisfied by $u(t)$ in the sense of distributions (see [**119, 9, 34**]).

Any solution $u(t), t \geqslant \tau$, of (2.112) satisfies the differential identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^4_{\mathbf{L}_4} - R(t)\|u(t)\|^2 \\ &= \langle g_0(t), u(t)\rangle \quad \forall t \geqslant \tau. \end{aligned} \tag{2.116}$$

The function $\|u(t)\|^2$ is absolutely continuous for $t \geqslant \tau$. We note that the parameters $\alpha_0(t)$ and $\beta_0(t)$ are omitted in this identity. The proof of (2.116) is similar to that of the corresponding identities for weak solutions of general reaction–diffusion systems [**34, 32**] (see also [**129**]).

Using standard transformations and the Gronwall lemma, from (2.116) we deduce that any weak solution $u(t)$ of (2.112) satisfies the inequality

$$\|u(t)\|^2 \leqslant \|u(\tau)\|^2 e^{-2\lambda(t-\tau)} + C_0^2 \qquad \forall t \geqslant \tau, \tag{2.117}$$

where λ is the first eigenvalue of the operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$ and the constant C_0 depends on $\|R_0\|_{C_b} = \sup\limits_{t\in\mathbb{R}} |R(t)|$ and $\|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}$.

Let $\{U(t,\tau)\}$ be the process corresponding to the problem (2.112), (2.115) and acting in the space $\mathbf{H}$. Recall that the mappings $U(t,\tau) : \mathbf{H} \to \mathbf{H}$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, are defined by the formula

$$U(t,\tau)u_\tau = u(t) \qquad \forall u_\tau \in \mathbf{H}, \tag{2.118}$$

where $u(t)$, $t \geqslant \tau$, is a solution of Equation (2.112) with initial data $u|_{t=\tau} = u_\tau$. By the estimates (2.117), the process $\{U(t,\tau)\}$ has the uniformly absorbing set

$$B_0 = \{v \in \mathbf{H} \mid \|v\| \leqslant 2C_0\} \tag{2.119}$$

which is bounded in $\mathbf{H}$.

The process $\{U(t,\tau)\}$ has a *compact* in $\mathbf{H}$ uniformly absorbing set

$$B_1 = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \leqslant C_0'\} \tag{2.120}$$

for an appropriate C_0'. For the proof of this assertion we refer to [**34, 129**] and Section 5.1. The set B_1 is bounded in $\mathbf{V}$ and compact in $\mathbf{H}$ since the embedding $\mathbf{V} \Subset \mathbf{H}$ is compact. Thus, the process $\{U(t,\tau)\}$ corresponding to (2.112) is uniformly compact.

Applying Theorem 2.1, we conclude that the process $\{U(t,\tau)\}$ has the global attractor $\mathcal{A}$ and the set $\mathcal{A}$ is compact in $\mathbf{H}$, bounded in $\mathbf{V}$, and can be constructed by the formula

$$\mathcal{A} = \omega(B_0) = \bigcap_{h\geqslant 0} \Big[\bigcup_{t-\tau\geqslant h} U(t,\tau)B_0 \Big]_{\mathbf{H}}.$$

The time symbol of Equation (2.112) is the function

$$\sigma_0(t) = (\alpha_0(t), \beta_0(t), R_0(t), g_0(x,t)),\ t \in \mathbb{R},$$

with values in $\Psi = \mathbb{R}^3 \times \mathbf{H}$. We assume that $\beta_0(t)$ satisfies (2.113).

Let $\alpha_0(t)$, $\beta_0(t)$, and $R_0(t)$ be translation compact functions in $C^{\,\mathrm{loc}}(\mathbb{R})$, and let $g_0(x,t)$ be a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$. Then $\sigma_0(t)$ is a translation compact function in $\Xi = C^{\,\mathrm{loc}}(\mathbb{R};\mathbb{R}^3) \times L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$. Consider the hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(t)$ in the space $C^{\,\mathrm{loc}}(\mathbb{R};\mathbb{R}^3) \times L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$.

Along with Equation (2.112), we consider the family of equations

$$\partial_t u = (1+i\alpha(t))\Delta u + R(t)u - (1+i\beta(t))|u|^2 u + g(x,t),\ \sigma \in \mathcal{H}(\sigma_0) \quad (2.121)$$

with symbols $\sigma(t) = (\alpha(t),\ \beta(t),\ R(t),\ g(x,t))$, where $\sigma \in \mathcal{H}(\sigma_0)$. We note that for every $\sigma = (\alpha, \beta, R, g) \in \mathcal{H}(\sigma_0)$ the function $\beta(t)$ satisfies (2.113) and $g(x,t)$ satisfies (2.114). Therefore, Equations (2.121) generates the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in $\mathbf{H}$ (see [**34, 129**]). Recall that $\{U(t,\tau)\} = \{U_{\sigma_0}(t,\tau)\}$ is the process corresponding to the Ginzburg–Landau equation (2.112). Consider the kernels $\mathcal{K}_\sigma$, $\sigma \in \mathcal{H}(\sigma_0)$, of Equations (2.121). As is proved in [**34, 129**], the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, is $(\mathbf{H}\times\mathcal{H}(\sigma_0); \mathbf{H})$-continuous. Then, by Theorem 2.4,

$$\mathcal{A} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0), \quad (2.122)$$

where the kernel $\mathcal{K}_\sigma$ of Equation (2.121) is nonempty for all $\sigma \in \mathcal{H}(\sigma_0)$.

Now, we describe an example of the Ginzburg–Landau equation having a simple global attractor.

Proposition 2.13. *Let $\beta_0(t)$ satisfy* (2.113)*, and let $R_0(t)$ satisfy the inequality*

$$R_0(t) \leqslant \lambda - \delta \quad \forall t \in \mathbb{R}, \quad 0 < \delta < \lambda. \quad (2.123)$$

Then for any $\sigma \in \mathcal{H}(\sigma_0)$ the kernel $\mathcal{K}_\sigma$ of Equation (2.121) *consists of a single element $\{z_\sigma(t),\ t \in \mathbb{R}\}$; moreover, $\{z_\sigma(t), t \in \mathbb{R}\}$ exponentially attracts any solution $\{u_\sigma(t), t \geqslant \tau\}$ of Equation* (2.121):

$$\|u_\sigma(t) - z_\sigma(t)\| \leqslant e^{-\delta(t-\tau)}\|u_\sigma(\tau) - z_\sigma(\tau)\| \quad \forall t \geqslant \tau. \quad (2.124)$$

Proof. Since the kernel $\mathcal{K}_\sigma$ of Equation (2.121) is nonempty, there exists a bounded complete solution $z_\sigma(t), t \in \mathbb{R}$, of (2.121). Consider any other solution $\{u_\sigma(t), t \geqslant \tau\}$ of Equation (2.121). The difference $w(t) = u_\sigma(t) - z_\sigma(t)$ satisfies the equation

$$\begin{aligned}\partial_t w(t) &= (1 + i\alpha(t))\Delta w(t) + R(t)w(t)\\ &\quad - (1 + i\beta(t))(|u(t)|^2 u(t) - |z(t)|^2 z(t)).\end{aligned} \quad (2.125)$$

We set $A(t)v = (1 + i\alpha(t))\Delta v + R(t)v$ and $f(t,v) = (1 + i\beta(t))|v|^2 v$. Using (2.123), we find

$$\begin{aligned}\langle A(t)w, w\rangle &= -\langle(1 + i\alpha(t))\nabla w, \nabla w\rangle + \langle R(t)w, w\rangle\\ &= -\langle\nabla w, \nabla w\rangle + \langle R(t)w, w\rangle\\ &\leqslant -\lambda\|w\|^2 + R(t)\|w\|^2 \leqslant -\delta\|w\|^2.\end{aligned} \quad (2.126)$$

By (2.113), the function $f(t,u)$ is monotone with respect to u :

$$\begin{aligned}\langle f(t,u)-f(t,z),u-z\rangle &= \langle f'_u(t,v)(u-z),u-z\rangle \\ &= \langle f'_u(t,v)w,w\rangle \geqslant 0, \end{aligned} \tag{2.127}$$

where $v = z + \theta(u-z)$, $0 \leqslant \theta(x,t) \leqslant 1$ (see (1.34) and [**34**] for details).

Multiplying Equation (2.125) by w, integrating over Ω, and using (2.126) and (2.127), we obtain the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 &= \langle A(t)w,w\rangle - \langle f(t,u)-f(t,z),w\rangle \\ &\leqslant -\delta\|w\|^2 - \langle f'_u(t,v)w,w\rangle \leqslant -\delta\|w\|^2, \end{aligned} \tag{2.128}$$

which implies

$$\begin{aligned}\|u(t)-z(t)\|^2 = \|w(t)\|^2 &\leqslant e^{-2\delta(t-\tau)}\|w(\tau)\|^2 \\ &= e^{-2\delta(t-\tau)}\|u(\tau)-z(\tau)\|^2 \quad \forall t \geqslant \tau.\end{aligned}$$

Thus, the inequality (2.124) is proved for any function $z_\sigma(t)$ from the kernel $\mathcal{K}_\sigma$ of Equation (2.121).

From (2.124) it follows that $\{z_\sigma(t),\, t \in \mathbb{R}\}$ is a unique element of the kernel $\mathcal{K}_\sigma$ of Equation (2.121). $\square$

Remark 2.14. The property (2.124) expressing the exponential attraction by the unique trajectory $\{z_\sigma(x,t), t \in \mathbb{R}\}$ of all solutions $\{u_\sigma(x,t),$ $t \geqslant \tau\}$ of Equation (2.121) is a nonautonomous analog of the exponential stability of the unique stationary point $\{z(x)\}$ of the autonomous equation (2.21) for $R < \lambda$ and $|\beta| \leqslant \sqrt{3}$.

Finally, we formulate natural consequences of Proposition 2.13.

Corollary 2.9. *Under the assumptions of Proposition* 2.13*, the global attractor* $\mathcal{A}$ *of Equation* (2.112) *has the form*

$$\mathcal{A} = \Big[\bigcup\nolimits_{t\in\mathbb{R}}\{z_{\sigma_0}(t)\}\Big]_{\mathbf{H}} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)}\{z_\sigma(0)\}; \tag{2.129}$$

moreover, the global attractor $\mathcal{A}$ *is exponential, i.e., for every bounded set* $B \subset \mathbf{H}$

$$\operatorname{dist}_{\mathbf{H}}(U_{\sigma_0}(t,\tau)B,\mathcal{A}) \leqslant C(\|B\|)e^{-\delta(t-\tau)} \quad \forall\, t \geqslant \tau, \tag{2.130}$$

where $\|B\| = \sup\{\|y\| \mid y \in B\}$.

Corollary 2.10. *If the symbol $\sigma(t)$ is periodic, then $z_\sigma(t)$ is also periodic. If $\sigma(t)$ is almost periodic, then $z_\sigma(t)$ is almost periodic as well. If the initial symbol $\sigma_0(t)$ is quasiperiodic of the form*

$$\sigma_0(t) = \varphi(\alpha_1 t, \ldots, \alpha_k t) = \varphi(\bar{\alpha} t),$$

where $\varphi \in C^{\mathrm{Lip}}(\mathbb{T}^k; \mathbb{R}^3 \times \mathbf{H})$ and the numbers $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ are rationally independent, then $z_{\sigma_0}(t)$ is also quasiperiodic, i.e., there exists a function $\Phi \in C^{\mathrm{Lip}}(\mathbb{T}^k; \mathbf{H})$ such that $z_{\sigma_0}(t) = \Phi(\alpha_1 t, \ldots, \alpha_k t)$. Moreover, the global attractor $\mathcal{A}$ is the Lipschitz continuous image of a k-dimensional torus $\mathbb{T}^k$: $\mathcal{A} = \Phi(\mathbb{T}^k)$ and$\mathbf{d}_F(\mathcal{A}) \leqslant k$.

The proof is similar to that of Corollaries 2.1–2.3 and 2.4–2.8.

Remark 2.15. There are symbols $\sigma_0(t)$ satisfying (2.113) and (2.123) such that $\mathbf{d}_F(\mathcal{A}) = k$. Moreover, it is easy to construct almost periodic symbols such that $\mathbf{d}_F(\mathcal{A}) = \infty$.

2.7. On the dimension of global attractors of processes.

Studying nonautonomous evolution equations, we see that the dimension of the uniform global attractors depends on the dimension of the symbol hulls. For example, for evolution equations with quasiperiodic time symbols the fractal dimension of global attractors depends on the number of rationally independent frequencies of the symbols (see Remarks 2.9, 2.12, and 2.15).

Let us show that the uniform global attractors of processes corresponding to general nonautonomous evolution equation can have infinite fractal dimension.

Consider a process $\{U(t,\tau)\}$ acting in a Hilbert (or Banach) space E and assume that it is uniformly asymptotically compact. By Theorem 2.1, $\{U(t,\tau)\}$ has a global attractor $\mathcal{A}$. Consider the kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$. By Proposition 2.2, the set $\mathbf{K} = \bigcup_{\tau \in \mathbb{R}} \mathcal{K}(\tau)$ of all values of all complete trajectories $u \in \mathcal{K}$ of the process belongs to $\mathcal{A}$. Moreover, the closure $\overline{\mathbf{K}}$ of this set in E also belongs to $\mathcal{A}$ since the global attractor is a closed set.

We claim that the set $\overline{\mathbf{K}}$ can have infinite dimension

$$\mathbf{d}_F \overline{\mathbf{K}} = +\infty \tag{2.131}$$

for all the problems discussed in Section 2.6. For example, for the Navier–Stokes system we set

$$u_0(x,t) = \sum_{j=1}^{\infty} a_j(x)\cos(\mu_j t) + b_j(x)\sin(\mu_j t), \tag{2.132}$$

where $a_j(x) = (a_j^1(x), a_j^2(x))$, $b_j(x) = (b_j^1(x), b_j^2(x))$ are smooth linearly independent vector-valued functions such that $a_j|_{\partial\Omega} = 0$, $(\nabla, a_j) = 0$, $b_j|_{\partial\Omega} = 0$, $(\nabla, b_j) = 0$. We assume that the series (2.132) and its derivatives with respect to x and t converge rapidly. We also assume that the frequencies μ_j, $j = 1, 2, \ldots$, are rationally independent real numbers. Setting

$$g_0(x,t) = \partial_t u_0(x,t) + \nu L u_0(x,t) + B(u_0(x,t), u_0(x,t)), \tag{2.133}$$

and we see that $g_0(\cdot) \in C_b(\mathbb{R}; H)$. The system (2.36) with external force $g_0(x,t)$ generates a process $\{U(t,\tau)\}$ in H having the compact attractor $\mathcal{A}$ (see Section 2.6.1). The process $\{U(t,\tau)\}$ has at least one complete bounded solution; namely $u_0(t)$. Thus, the kernel $\mathcal{K}$ is nonempty and $u_0 \in \mathcal{K}$. It is easy to show that the projection $u_0^N(x,t)$ of $u_0(x,t)$ onto the $2N$-dimensional space spanned by the vector-valued functions $\{(a_j(x), b_j(x)) \mid j = 1, \ldots, N\}$ provides a dense winding of the N-dimensional torus $\mathbb{T}^N \subset H$ (the rational independence of $\{\mu_j\}$ was used). Therefore, the set $\overline{\operatorname{Im} u_0} = \overline{\{u_0(\cdot, t) : t \in \mathbb{R}\}}$ has the fractal dimension greater than N : $\mathbf{d}_F \overline{\operatorname{Im} u_0} \geqslant N$ for each $N \in \mathbb{N}$, i.e., $\mathbf{d}_F \overline{\operatorname{Im} u_0} = \infty$. It is evident that $\overline{\operatorname{Im} u_0} \subseteq \overline{\mathbf{K}}$. Hence $\mathbf{d}_F \overline{\mathbf{K}} = +\infty$. We recall that $\overline{\mathbf{K}} \subseteq \mathcal{A}$, and thereby $\mathbf{d}_F \mathcal{A} = +\infty$.

3. Kolmogorov ε-Entropy of Global Attractors

As was shown at the end of Section 2, the fractal dimension of the global attractor $\mathcal{A}$ of a nonautonomous evolution equation can be infinite. At the same time, the global attractors are always compact sets in the corresponding phase spaces. Therefore, it is reasonable to study the Kolmogorov ε-entropy because it is finite for every ε.

Here, we derive upper estimates for the Kolmogorov ε-entropy of the global attractors of nonautonomous evolution equations with translation compact symbols. These estimates are optimal in a sense and generalize estimates for the ε-entropy of the finite-dimensional global attractors of the corresponding autonomous equations considered in Section 1.4.

In Section 3.1, we present a general upper estimate for the ε-entropy of the uniform global attractor $\mathcal{A}$ of the process $\{U_\sigma(t,\tau)\}$ corresponding to the nonautonomous equation $\partial_t u = A_{\sigma(t)}(u)$ with translation compact symbol $\sigma(t)$.

In Section 3.2, we consider the case, where the fractal dimension $\mathbf{d}_F\mathcal{A}$ of the uniform global attractor $\mathcal{A}$ is finite. This property holds if, for example, the time symbol $\sigma(t)$ is a quasiperiodic function in t with k rationally independent frequencies. Then we show that $\mathbf{d}_F\mathcal{A} \leqslant d + k$ for some d depending on the problem under consideration. This means that the dimension $\mathbf{d}_F\mathcal{A}$ can grow to infinity as $k \to +\infty$.

In Section 3.3, the above-mentioned results are applied to the estimates of the ε-entropy and the fractal dimension of the uniform global attractor of some nonautonomous equations in mathematical physics; namely, the 2D Navier–Stokes system with translation compact external force, the damped wave equation with translation compact terms, and the nonautonomous complex Ginzburg–Landau equation.

We emphasize the fundamental role of the paper [**83**] in the study of the ε-entropy of compact sets in Hilbert or Banach spaces.

3.1. Estimates for ε-entropy.

We use the notation from Section 2. Consider the family of the Cauchy problems for nonautonomous equations

$$\partial_t u = A_{\sigma(t)}(u), \ u|_{t=\tau} = u_\tau, \quad u_\tau \in E, \tag{3.1}$$

with symbols $\sigma(t) \in \mathcal{H}(\sigma_0(t))$. Here, E is a Hilbert space. We assume that the symbol $\sigma_0(t)$ of the original equation (2.21) is a translation compact function in the space Ξ. We assume that the topological space Ξ is a Hausdorff space. In applications, $\Xi = C(\mathbb{R};\Psi)$ or $\Xi = L_p^{\mathrm{loc}}(\mathbb{R};\Psi)$, $p \geqslant 1$, where Ψ is a Banach space, or the product of such spaces. The space Ξ is endowed with the local uniform convergence topology on every bounded segment in $\mathbb{R}$. By definition, a sequence $\{\sigma_n(\cdot)\}$ converges to $\sigma(\cdot)$ as $n \to \infty$ in Ξ if $\|\Pi_{t_1,t_2}(\sigma_n(\cdot) - \sigma(\cdot))\|_{\Xi_{t_1,t_2}} \to 0$ as $n \to \infty$ for every closed interval $[t_1,t_2] \subset \mathbb{R}$. Here, Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1,t_2]$, Ξ_{t_1,t_2} is the family of Banach spaces generating Ξ, and $\|\xi\|_{\Xi_{t_1,t_2}}$ is the norm of ξ in Ξ_{t_1,t_2}. For example, if $\Xi = C(\mathbb{R};\Psi)$, then $\Xi_{t_1,t_2} = C([t_1,t_2];\Psi)$ and $\sigma_n(\cdot) \to \sigma(\cdot)$ as $n \to \infty$ in $C(\mathbb{R};\Psi)$ provided that

$$\max_{s\in[t_1,t_2]} \|\sigma_n(s) - \sigma(s)\|_\Psi \to 0 \quad \text{as } n \to \infty \tag{3.2}$$

for every $[t_1, t_2] \subset \mathbb{R}$. Similarly, $\sigma_n(\cdot) \to \sigma(\cdot)$ as $n \to \infty$ in $\Xi = L_p^{\mathrm{loc}}(\mathbb{R}; \Psi)$ if

$$\int_{t_1}^{t_2} \|\sigma_n(s) - \sigma(s)\|_\Psi^p ds \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

for all $[t_1, t_2] \subset \mathbb{R}$ (see [**34**] for details). In addition, we assume that the norms in Ξ_{t_1,t_2} satisfy the following condition:

$$\|\Pi_{t_1',t_2'}\xi\|_{\Xi_{t_1',t_2'}} \leqslant \|\Pi_{t_1,t_2}\xi\|_{\Xi_{t_1,t_2}} \quad \forall\, [t_1', t_2'] \subset [t_1, t_2]. \tag{3.4}$$

It is clear that (3.4) is valid for the spaces $C([t_1, t_2]; \Psi)$ and $L_p^{\mathrm{loc}}(t_1, t_2; \Psi)$.

Suppose that for every $\sigma \in \mathcal{H}(\sigma_0)$ the Cauchy problem (3.1) generates the process $\{U_\sigma(t, \tau)\}$ acting in E by the formula $U_\sigma(t, \tau)u_\tau = u(t)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where $u(t)$ is a solution of the Cauchy problem (3.1) with initial data $u_\tau \in E$. Let the assumptions of Theorem 2.4 hold. Then the process $\{U_{\sigma_0}(t, \tau)\}$ has a global attractor $\mathcal{A}$ of the form (2.33).

Our goal is to study the ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A}) = \mathbf{H}_\varepsilon(\mathcal{A}, E)$ of the global attractor $\mathcal{A}$ in the space E (see Definition 1.6). We intend to estimate $\mathbf{H}_\varepsilon(\mathcal{A})$ by using an information about the behavior of the ε-entropy of the sets $\Pi_{0,l}\mathcal{H}(\sigma_0)$ in the space $\Xi_{0,l}$ (where, for example, $\Xi_{0,l} = C([0, l]; \Psi)$ or $\Xi_{0,l} = L_p^{\mathrm{loc}}(0, l; \Psi)$). It is assumed that the behavior of the ε-entropy is known as $l \to +\infty$ and $\varepsilon \to 0+$. Here, $\Pi_{0,l}$ denotes the restriction operator on the segment $[0, l]$.

To formulate the main theorem, we need to introduce some notions and conditions on the process $\{U_{\sigma_0}(t, \tau)\}$. First of all, we must generalize the quasidifferentiability property (1.40) introduced in Section 1.4.1 for semigroups.

Let $\{U(t, \tau)\}$ be a process in E. Consider the kernel $\mathcal{K}$ of $\{U(t, \tau)\}$ (see Definition 2.3). It is clear that the kernel sections satisfy the following invariance property:

$$U(t, \tau)\mathcal{K}(t) = \mathcal{K}(\tau) \quad \forall t \geqslant \tau, \ \ \tau \in \mathbb{R}. \tag{3.5}$$

Definition 3.1. A process $\{U(t, \tau)\}$ in E is *uniformly quasidifferentiable* on $\mathcal{K}$ if there exists a family of linear bounded operators $\{L(t, \tau, u)\}$, $u \in \mathcal{K}(\tau)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, such that

$$\begin{aligned} &\|U(t, \tau)u_1 - U(t, \tau)u - L(t, \tau, u)(u_1 - u)\|_E \\ &\leqslant \gamma(\|u_1 - u\|_E, t - \tau)\|u_1 - u\|_E \end{aligned} \tag{3.6}$$

for all $u, u_1 \in \mathcal{K}(\tau)$, $\tau \in \mathbb{R}$, where $\gamma = \gamma(\xi, s) \to 0+$ as $\xi \to 0+$ for each fixed $s \geqslant 0$.

Assume that the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to (3.1) is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$ and the quasidifferentials of this process are generated by the variational equation

$$\partial_t v = A_{\sigma_0 u}(u(t))v, \quad v|_{t=\tau} = v_\tau, \quad v_\tau \in E, \tag{3.7}$$

where $u(t) = U_{\sigma_0}(t,\tau)u_\tau$, $u_\tau \in \mathcal{K}_{\sigma_0}(\tau)$, i.e., $L(t,\tau,u_\tau)v_\tau = v(t)$, where $v(t)$ is a solution of (3.7) with initial data v_τ. We assume that the Cauchy problem is uniquely solvable for all $u_\tau \in \mathcal{K}_{\sigma_0}(\tau)$ and $v_\tau \in E$.

As in the case of (1.43), we introduce the numbers

$$\widetilde{q}_j := \limsup_{T\to+\infty} \sup_{\tau\in\mathbb{R}} \sup_{u_\tau\in\mathcal{K}(\tau)} \frac{1}{T}\int_\tau^{\tau+T} \mathrm{Tr}_j A_{\sigma_0 u}(u(t))dt, \tag{3.8}$$

where $u(t) = U_{\sigma_0}(t,\tau)u_\tau$ and the j-trace $\mathrm{Tr}_j(L)$ of a linear operator L in a Hilbert space E is defined in (1.42).

Assume that the following Lipschitz condition holds for the processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ corresponding to (3.1):

$$\|U_{\sigma_1}(h,0)u_0 - U_{\sigma_2}(h,0)u_0\|_E \leqslant C(h)\|\sigma_1 - \sigma_2\|_{\Xi_{0,h}} \tag{3.9}$$

for all $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, $u_0 \in \mathcal{A}$, $h \geqslant 0$.

From (3.9) it follows that

$$|U_{\sigma_1}(t,\tau)u_\tau - U_{\sigma_2}(t,\tau)u_\tau| \leqslant C(|t-\tau|)\|\sigma_1 - \sigma_2\|_{\Xi_{\tau,t}}$$

for all $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, $u_\tau \in \mathcal{A}$, $t > \tau$, $\tau \in \mathbb{R}$.

Now, we are ready to formulate the main theorem of this section.

Theorem 3.1. *Let the assumptions of Theorem* 2.4 *hold. Suppose that the original process* $\{U_{\sigma_0}(t,\tau)\}$ *is uniformly quasidifferentiable on* $\mathcal{K}_{\sigma_0}$*, the quasidifferentials of this process are generated by the variational equation* (3.7)*, and* $\widetilde{q}_j$ *defined by* (3.8) *satisfy the inequalities*

$$\widetilde{q}_j \leqslant q_j, \quad j = 1, 2, 3, \ldots . \tag{3.10}$$

Assume that the Lipschitz condition (3.9) *holds for the family of processes* $\{U_\sigma(t,\tau)\}$*,* $\sigma \in \mathcal{H}(\sigma_0)$*, and the function* q_j *is concave in* j *(like* $\cap$*). Let* m *be the smallest number such that* $q_{m+1} < 0$ *(then* $q_m \geqslant 0$*), and let*

$$d = m + \frac{q_m}{(q_m - q_{m+1})}. \tag{3.11}$$

Then for every $\delta > 0$ there exist $\eta \in (0,1)$, $\varepsilon_0 > 0$, and $h > 0$ such that

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \\ + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\Pi_{0,h\log_{1/\eta}(\varepsilon_0/\eta\varepsilon)}\mathcal{H}(\sigma_0)) \tag{3.12}$$

for all $\varepsilon < \varepsilon_0$, where $C(h)$ is the Lipschitz constant from (3.9).

Recall that $\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(\sigma_0))$ on the right-hand side of (3.12) denotes the ϵ-entropy of the set $\mathcal{H}(\sigma_0)$ restricted to the interval $[0,l]$ and the ϵ-entropy is measured in the space $\Xi_{0,l}$ (for example, in $C([0,l];\Psi)$ or $L_p^{\mathrm{loc}}(0,l;\Psi)$).

The proof of Theorem 3.1 is contained in [**24, 34**].

Remark 3.1. Comparing the inequality (3.12) with the estimate (1.46) in the autonomous case, we observe that the term $(d+\delta)\log_2(\varepsilon_0/\eta\varepsilon)$ corresponds to the upper estimate for the ε-entropy of the kernel sections $\mathcal{K}(\tau)$ and, in particular, $d_F\mathcal{K}(\tau) \leqslant d$ for all $\tau \in \mathbb{R}$ (see [**34**]).

Remark 3.2. If δ is small, the inequality (3.12) is optimal with respect to the estimate of the ε-entropy of the kernel sections. However, another important parameter h in (3.12) tends to infinity as $\delta \to 0+$. The parameter h controls the denominator in $\epsilon = \dfrac{\varepsilon\eta}{4C(h)}$, where $C(h)$ is the Lipschitz constant in (3.9) which usually grows exponentially as $h \to \infty$. Thus, if the hull $\mathcal{H}(\sigma_0)$ is infinite-dimensional, then the ϵ-entropy of $\mathcal{H}(\sigma_0)$ can grow rapidly as $\varepsilon \to 0+$ and faster than $D\log(1/\epsilon)$ for arbitrary D. Thus, it is reasonable to optimize the estimate (3.12) with respect to small values of h. The following assertion presents a result in this direction. The proof can be found in [**34**].

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *hold, and let $\widetilde{q}_j \leqslant q_j$, $j = 1,2,\ldots$. Assume that*

$$\frac{q_j}{j} \to -\infty \quad \textit{as } j \to \infty. \tag{3.13}$$

Then for any $h > 0$ there exist $D > 0$ and $\varepsilon_0 > 0$ such that

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant D\log_2((2\varepsilon_0)/\varepsilon) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) \\ + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}(\Pi_{0,h\log_2(2\varepsilon_0/\varepsilon)}\mathcal{H}(\sigma_0)) \tag{3.14}$$

for all $\varepsilon \leqslant \varepsilon_0$. (*In applications, $C(h)$ usually approaches* 1 *as $h \to +0$.*)

We consider a particular case, where $\sigma_0(t)$ is an almost periodic function, i.e., the hull $\mathcal{H}(\sigma_0)$ is compact in $C_b(\mathbb{R};\Psi)$ with respect to the topology of uniform convergence on $\mathbb{R}$. The norm in $C_b(\mathbb{R};\Psi)$ is defined by the formula

$$\|\xi\|_{C_b(\mathbb{R};\Psi)} := \sup_{t\in\mathbb{R}} \|\xi(t)\|_\Psi.$$

Since

$$\|\xi\|_{C([0,l];\Psi)} \leqslant \|\xi\|_{C_b(\mathbb{R};\Psi)} \quad \forall l > 0,$$

we have

$$\begin{aligned}\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(\sigma_0); C([0,l];\Psi)) &\leqslant \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0); C_b(\mathbb{R};\Psi)) \\ &= \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) \quad \forall l > 0,\end{aligned} \tag{3.15}$$

and Theorems 3.1 and 3.2 imply the following assertion.

Corollary 3.1. *Let $\sigma_0(t)$ be almost periodic, and let the assumptions of Theorem* 3.1 *hold. Then*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant (d+\delta)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\mathcal{H}(\sigma_0)) \ \forall\ \varepsilon < \varepsilon_0, \tag{3.16}$$

where $\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0))$ is the ϵ-entropy of the hull $\mathcal{H}(\sigma_0)$ in the space $C_b(\mathbb{R};\Psi)$.

Corollary 3.2. *Let the assumptions of Theorem 3.2 hold, and let $\mathcal{H}(\sigma_0) \Subset C_b(\mathbb{R};\Psi)$. Then*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant D\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}(\mathcal{H}(\sigma_0)) \quad \forall\varepsilon < \varepsilon_0. \tag{3.17}$$

Remark 3.3. If it is known that $\mathcal{H}(\sigma_0) \Subset L_p^b(\mathbb{R};\Psi)$, i.e., $\sigma_0(t)$ is an almost periodic functions in the Stepanov sense, then the estimates (3.16) and (3.17) hold. In this case, $\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0))$ denotes the ϵ-entropy of $\mathcal{H}(\sigma_0)$ in the space $L_p^b(\mathbb{R};\Psi)$ measured in the norm

$$\|f\|_{L_p^b(\mathbb{R};\Psi)} := \left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|_\Psi^p ds\right)^{1/p}.$$

The estimate (3.16) shows that for a general almost periodic function $\sigma_0(t)$ having infinitely many rationally independent frequencies, the main contribution to the estimate for the ε-entropy of the global attractor $\mathcal{A}$ is made by the εL-entropy of the hull $\mathcal{H}(\sigma_0)$, where $L = (4C(h))/\eta$. However, if $\sigma_0(t)$ has finitely many frequencies, i.e., it is quasiperiodic, then the contribution of this quantity is comparable with that of the term $d\log_2(\varepsilon_0/(\alpha\eta))$. This means that the global attractor of the nonautonomous equation has finite dimension. We discuss this question later.

We consider two important characteristics of a compact set X in E, introduced in [**83**]. The number

$$\mathbf{df}(X,E) = \mathbf{df}(X) := \limsup_{\epsilon\to 0+} \frac{\log_2(\mathbf{H}_\varepsilon(X))}{\log_2\log_2(1/\varepsilon)} \tag{3.18}$$

is called the *functional dimension* of X in E, and the number

$$\mathbf{q}(X,E) = \mathbf{q}(X) := \limsup_{\epsilon\to 0+} \frac{\log_2(\mathbf{H}_\varepsilon(X))}{\log_2(1/\varepsilon)} \tag{3.19}$$

is called the *metric order* of X in E. It is easy to see that $\mathbf{df}(X) = 1$ and $\mathbf{q}(X) = 0$ if $\mathbf{d}_F(X) < +\infty$. Thus, the values $\mathbf{df}(X)$ and $\mathbf{q}(X)$ characterize infinite-dimensional sets. Some examples of calculations of these values are given in [**83**] (see also [**125, 127**]).

Using Corollaries 3.1 and 3.2, we obtain the following assertion.

Corollary 3.3. *Let $\sigma_0(t)$ be an almost periodic function. Then*

$$\mathbf{df}(\mathcal{A},E) \leqslant \mathbf{df}(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)), \tag{3.20}$$

$$\mathbf{q}(\mathcal{A},E) \leqslant \mathbf{q}(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)). \tag{3.21}$$

3.2. Finite fractal dimension of global attractor.

In this section, we study the fractal dimension of the uniform global attractor $\mathcal{A}$ of the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to (2.21) and its dependence on the fractal dimension of the hull $\mathcal{H}(\sigma_0)$.

We start with a very important example of a quasiperiodic symbol $\sigma_0(t)$ (see Example 2.5): $\sigma_0(t) = \varphi(\alpha_1 t, \alpha_2 t, \dots, \alpha_k t) = \varphi(\bar\alpha s)$, where $\varphi(\bar\omega), \bar\omega = (\omega_1,\dots,\omega_k)$, is a 2π-periodic function in each variable ω_i, $i = \overline{1,k}$, $\bar\alpha = (\alpha_1,\alpha_2,\dots,\alpha_k), \alpha_i \in \mathbb{R}$, $\{\alpha_i\}$ are rationally independent numbers. We assume that $\varphi(\bar\omega)$ is a Lipschitz continuous function on the k-dimensional torus $\mathbb{T}^k = [\mathbb{R} \bmod 2\pi]^k$ with values in a Banach space Ψ, $\varphi \in C^{\mathrm{lip}}(\mathbb{T}^k;\Psi)$, i.e.,

$$\|\varphi(\bar\omega_1) - \varphi(\bar\omega_2)\|_\Psi \leqslant L|\bar\omega_1 - \bar\omega_2|_{\mathbb{T}^k} \quad \forall\, \bar\omega_1, \bar\omega_2 \in \mathbb{T}^k, \tag{3.22}$$

where $|\cdot|_{\mathbb{T}^k}$ denotes the usual Euclidean norm in $\mathbb{R}^k$. By (2.28), the hull $\mathcal{H}(\sigma_0)$ of the function $\sigma_0(t)$ in the space $C_b(\mathbb{R};\Psi)$ coincides with

$$\{\varphi(\bar\alpha s + \bar\theta) \mid \bar\theta \in \mathbb{T}^k\} = \mathcal{H}(\sigma_0). \tag{3.23}$$

Proposition 3.1. *If $\sigma_0(t)$ is a quasiperiodic function, then*

$$\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) := \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0), C_b(\mathbb{R};\Psi)) \leqslant \mathbf{H}_{L\epsilon}(\mathbb{T}^k) \leqslant k\log_2\left(\frac{2}{L\epsilon}\right) \tag{3.24}$$

for all $\epsilon < L^{-1}$ *and*

$$\mathbf{d}_F(\mathcal{H}(\sigma_0)) = \mathbf{d}_F(\mathcal{H}(\sigma_0), C_b(\mathbb{R}; \Psi)) \leqslant k.$$

PROOF. If $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, then $\sigma_i = \varphi(\bar{\alpha}s + \bar{\theta}_i)$ for some $\bar{\theta}_i \in \mathbb{T}^k$, $i = 1, 2$, by (3.23) and

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{C_b(\mathbb{R};\Psi)} &:= \sup_{t\in\mathbb{R}} \|\sigma_1(t) - \sigma_2(t)\|_\Psi \\ &= \sup_{t\in\mathbb{R}} \|\varphi(\bar{\alpha}t + \bar{\theta}_1) - \varphi(\bar{\alpha}t + \bar{\theta}_2)\|_\Psi \leqslant L|\bar{\theta}_1 - \bar{\theta}_2|_{\mathbb{T}^k}\end{aligned}$$

by (3.22). Therefore,

$$N_\epsilon(\mathcal{H}(\sigma_0)) \leqslant N_{L\epsilon}(\mathbb{T}^k).$$

It is known that the torus $\mathbb{T}^k$ endowed with the Euclidean metric can be covered by at most $(2/\varepsilon)^k$ balls of radius $\varepsilon < 1$ (see, for example, [**43**]). Hence

$$N_\epsilon(\mathcal{H}(\sigma_0)) \leqslant (2/(L\epsilon))^k, \quad \mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)) \leqslant k \log_2(2/(L\epsilon)) \quad \forall \epsilon < L^{-1}$$

and, consequently,

$$\mathbf{d}_F(\mathcal{H}(\sigma_0)) := \limsup_{\epsilon\to 0+} \frac{\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0))}{\log_\epsilon(1/\varepsilon)} \leqslant k,$$

which completes the proof. □

Remark 3.4. In the general case, $\mathcal{H}(\sigma_0)$ is a Lipschitz continuous manifold in $C_b(\mathbb{R}; \Psi)$, isometric to the torus $\mathbb{T}^k$. Hence $\mathbf{d}_F(\mathcal{H}(\sigma_0)) = k$.

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *hold, and let* $\sigma_0(t)$ *be a quasiperiodic function of the form* $\sigma_0(t) = \varphi(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t) = \varphi(\bar{\alpha}t)$, *where* $\varphi(\omega_1, \omega_2, \ldots, \omega_k) = \varphi(\bar{\omega}) \in C^{\mathrm{Lip}}(\mathbb{T}^k; \Psi)$. *Then the estimate* (3.16) *takes the form*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant (d + \delta) \log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + k \log_2\left(\frac{8C(h)}{L\eta\varepsilon}\right) \quad \forall\ \varepsilon < \varepsilon_0, \tag{3.25}$$

where L *is the Lipschitz constant from the inequality* (3.22). *Moreover,*

$$\mathbf{d}_F(\mathcal{A}) \leqslant d + k. \tag{3.26}$$

PROOF. Indeed, the inequality (3.16), together with (3.24), yields

$$\begin{aligned}\mathbf{H}_\varepsilon(\mathcal{A}) &\leqslant (d + \delta) \log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\mathcal{H}(\sigma_0)) \\ &\leqslant (d + \delta) \log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + k \log_2\left(\frac{8C(h)}{L\eta\varepsilon}\right).\end{aligned}$$

Passing to the limit in the ratio $\mathbf{H}_\varepsilon(\mathcal{A})/\log_2(1/\varepsilon)$ as $\varepsilon \to 0+$, we find $\mathbf{d}_F(\mathcal{A}) \leqslant d+\delta+k$. Since δ is arbitrarily small, we obtain (3.26). $\square$

Recall that, in the autonomous case $k=0$, the estimate (1.45) is an analog of the estimate (3.26), where $X=\mathcal{A}:\ \mathbf{d}_F(\mathcal{A}) \leqslant d$.

We generalize Theorem 3.3 to the case of more general symbols $\sigma_0(t)$ that are not almost periodic, but the dimension of the corresponding global attractors $\mathcal{A}$ is finite.

As above, let $\sigma_0(t)$ be a translation compact function in Ξ and thereby the hull $\mathcal{H}(\sigma_0)$ is compact in Ξ. (For example, for Ξ one can take $C(\mathbb{R};\Psi)$ or $\Xi = L_p^{\mathrm{loc}}(\mathbb{R};\Psi)$.) As is proved in [**34**],

$$\limsup_{\epsilon\to 0+} \mathbf{H}_\epsilon(\Pi_{0,l\log_2(K/\epsilon)}\Sigma)\big/\log_2(1/\epsilon) \tag{3.27}$$

is independent of $K>0$ for any compact subset $\Sigma \Subset \Xi$. For Σ we introduce the number

$$\mathbf{d}_F^{\mathrm{loc}}(\Sigma,l) := \limsup_{\epsilon\to 0+} \mathbf{H}_\epsilon(\Pi_{0,l\log_2(1/\epsilon)}\Sigma)\big/\log_2(1/\epsilon) \tag{3.28}$$

depending on the positive parameter l.

Remark 3.5. If $\Sigma = \mathcal{H}(\sigma_0)$, where σ_0 is a smooth quasiperiodic function with k independent frequencies, then $\mathbf{d}_F^{\mathrm{loc}}(\Sigma,l) \leqslant k$ for any l because $\mathcal{H}(\sigma_0)$ is the Lipschitz continuous image of the k-dimensional torus $\mathbb{T}^k$ (see Proposition 3.1).

If for some l we have $\mathbf{d}_F^{\mathrm{loc}}(\Sigma,l) < +\infty$. then we say that Σ has the local *fractal dimension* $\mathbf{d}_F^{\mathrm{loc}}(\Sigma,l)$ in the topological space $C(\mathbb{R};\Psi)$.

Theorem 3.4. *Let the assumptions of Theorem* 3.1 *hold, and let*

$$\mathbf{d}_F^{\mathrm{loc}}(\mathcal{H}(\sigma_0),h_1) < +\infty,$$

where $h_1 = h(\delta)/\log_2(1/\eta)$. *Then for any* $\delta>0$

$$\mathbf{d}_F(\mathcal{A}) \leqslant d+\delta+\mathbf{d}_F^{\mathrm{loc}}(\mathcal{H}(\sigma_0),h_1). \tag{3.29}$$

Moreover, if $\mathbf{d}_F^{\mathrm{loc}}(\mathcal{H}(\sigma_0),h) \leqslant k$ *for all* $h>0$, *then* $\mathbf{d}_F(\mathcal{A}) \leqslant d+k$.

Indeed, dividing (3.12) by $\log_2(1/\varepsilon)$ and making the change of variables $\epsilon = \dfrac{\eta}{4C(h)}\varepsilon$, we find

$$\mathbf{d}_F(\mathcal{A}) \leqslant (d+\delta) + \limsup_{\epsilon\to 0+} \mathbf{H}_\epsilon \frac{\Pi_{0,h\log_{1/\alpha}(\frac{\varepsilon_0}{4C(h)\epsilon})}\mathcal{H}(\sigma_0)}{\log_2(1/\epsilon)+\log_2\frac{\eta}{4C(h)}}$$

$$= (d+\delta) + \limsup_{\epsilon\to 0+} \frac{\mathbf{H}_\epsilon(\Pi_{0,h_1\log_2(K/\epsilon)}\mathcal{H}(\sigma_0))}{\log_2(1/\epsilon)}$$
$$= (d+\delta) + \mathbf{d}_F^{\mathrm{loc}}(\mathcal{H}(\sigma_0), h_1),$$

where $K = \varepsilon_0/(4C(h))$ and we used the fact that the expression (3.27) is independent of K.

3.3. Applications to nonautonomous equations.

3.3.1. 2D Navier–Stokes system. Consider the family of the Cauchy problems

$$\begin{aligned} &\partial_t u + \nu L u + B(u,u) = g(x,t), \\ &u|_{t=\tau} = u_\tau, \quad u_\tau \in H \end{aligned} \tag{3.30}$$

(see Section 2.6.1) with external forces $g \in \mathcal{H}(g_0)$. We assume that the original external force $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; H) =: \Xi$. The space $L_2^{\mathrm{loc}}(\mathbb{R}; H)$ is endowed with the topology of strong convergence on every $[t_1, t_2] \subset \mathbb{R}$. Then $g_0 \in L_2^b(\mathbb{R}; H)$ and

$$\|g\|^2_{L_2^b} \leqslant \|g_0\|^2_{L_2^b} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |g_0(s)|^2 ds < \infty \tag{3.31}$$

for every function $g \in \mathcal{H}(g_0)$ (see (2.37) and (2.43)).

Consider the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, corresponding to the family of Cauche problems (3.30) and acting in H. As was proved in Section 2.6.1, the process $\{U_{g_0}(t,\tau)\}$ has the uniform global attractor $\mathcal{A} \Subset H$ and the set $\mathcal{A}$ has the form

$$\mathcal{A} = \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{K}_g(0), \tag{3.32}$$

where $\mathcal{K}_g$ is the kernel of $\{U_g(t,\tau)\}$ with external force $g \in \mathcal{H}(g_0)$.

Consider the Kolmogorov ε-entropy $\mathbf{H}_\varepsilon(\mathcal{A})$ of the set $\mathcal{A}$ in H.

In [**34**], it is proved that the family $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, satisfies the Lipschitz condition (3.9); namely,

$$|U_{g_1}(h,0)u_0 - U_{g_2}(h,0)u_0| \leqslant C(h)\|g_1 - g_2\|_{L_2(0,h;H)} \tag{3.33}$$

for all $g_1, g_2 \in \mathcal{H}(g_0)$, $u_0 \in \mathcal{A}$, where the Lipschitz constant $C(h)$ depends on ν, λ_1, $\|g_0\|^2_{L_2^b}$ and exponentially grows in h.

Consider the quasidifferentiability property in detail. As is proved in [**34**], the process $\{U_{g_0}(t,\tau)\}$ is uniformly quasidifferentiable on $\mathcal{K}_{g_0}$ and the corresponding variation equation has the form

$$\begin{aligned}\partial_t v &= -\nu L v - B(u(t), v) - B(v, u(t)) \\ &=: A_{g_0 u}(u(t), t)v, \quad v|_{t=\tau} = v_\tau,\end{aligned} \tag{3.34}$$

where $u(t) = U_{g_0}(t, \tau)u_\tau$ and $u_\tau \in \mathcal{K}_{g_0}(\tau)$ (the proof is based on the methods from [**9**] and [**119**]). Thus, the quasidifferentials are the mappings $L(t, \tau; u_\tau) : H \to H$ and $L(t, \tau; u_\tau)v_\tau = v(t)$, where $v(t)$ is a solution of (3.34).

Following the scheme described in Section 3.1, we set

$$\widetilde{q}_j := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{u_\tau\in\mathcal{K}_{g_0}(\tau)} \Big(\frac{1}{T}\int_\tau^{\tau+T} \mathrm{Tr}_j A_{g_0 u}(u(s))ds\Big), \quad j \in \mathbb{N},$$

where $u(t) = U_{g_0}(t, \tau)u_\tau$ and Tr_j denotes the j-dimensional trace of an operator. As in the autonomous case (see the proof of Theorem 1.6), we obtain the estimate

$$\int_\tau^t \mathrm{Tr}_j A_{g_0 u}(u(s))ds \leqslant -\frac{\nu C_2 j^2}{2|\Omega|}(t-\tau) + \frac{1}{\nu^2}|u_\tau|^2 + \frac{1}{\lambda_1 \nu^3}\int_\tau^t |g_0(s)|^2 ds.$$

Therefore,

$$\widetilde{q}_j \leqslant -\frac{\nu C_2 j^2}{2|\Omega|} + \frac{|\Omega|}{C_1 \nu^3} M(|g_0|^2) =: \varphi(j) = q_j, \quad j = 1, 2, \dots, \tag{3.35}$$

where

$$M(|g_0|^2) := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \Big(\frac{1}{T}\int_\tau^{\tau+T} |g_0(t)|^2 dt\Big) \leqslant \|g_0\|^2_{L_2^b} < \infty$$

and the dimensionless constants C_1 and C_2 are taken from (1.55) (see also Remark 1.9). The function $\varphi(j)$ in (3.35) is concave in j.

Let m be the smallest integer such that $q_{m+1} = \varphi(m+1) < 0$ (see Theorem 3.1). We set

$$d = m + \frac{q_m}{q_m - q_{m+1}}.$$

Let d^* be the root of the equation $\varphi(x) = 0$, i.e.,

$$d^* = c\frac{M(|g_0|^2)^{1/2}|\Omega|}{\nu^2}, \; c = \Big(\tfrac{2}{C_1 C_2}\Big)^{1/2}. \tag{3.36}$$

Then

$$d^* \leqslant c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2}$$

since $M(|g_0|^2) \leqslant \|g_0\|^2_{L_2^b}$. It is obvious that

$$d \leqslant d^* \leqslant c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2} \tag{3.37}$$

because the function φ is concave (see Remark 1.6).

Hence Theorem 3.1 is applicable, and we get the following assertion.

Theorem 3.5. *For any $\delta > 0$ there exist $h > 0$, $\varepsilon_0 > 0$, and $\eta < 1$ such that*

$$\begin{aligned}\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant \Big(c\frac{\|g_0\|_{L_2^b}|\Omega|}{\nu^2} + \delta\Big)\log_2\Big(\frac{\varepsilon_0}{\eta\varepsilon}\Big) + \mathbf{H}_{\varepsilon_0}(\mathcal{A})\\ + \mathbf{H}_{(\varepsilon\eta)/(4C(h))}(\Pi_{0,h\log_{1/\eta}(\varepsilon_0/(\eta\varepsilon))}\mathcal{H}(g_0))\end{aligned} \tag{3.38}$$

for all $\varepsilon \leqslant \varepsilon_0$, where $C(h)$ is taken from (3.33) *and $\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(g_0))$ denotes the ϵ-entropy of the set $\Pi_{0,l}\mathcal{H}(g_0)$ in the space $L_2(0,l;H)$.*

Remark 3.6. The best up-to-date estimate for the constant c in (3.38) is as follows (see Remark 1.9 and [**16**]):

$$c \leqslant \frac{1}{2\pi^{3/2}}.$$

Note that $\varphi(j)/j \to -\infty$ as $j \to \infty$ (see (3.35)). Thus, using Theorem 3.2, we obtain the following assertion.

Theorem 3.6. *For any $h > 0$ there are $D > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant D\log_2\Big(\frac{2\varepsilon_0}{\varepsilon}\Big) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}(\Pi_{0,h\log_2((2\varepsilon_0)/\varepsilon)}\mathcal{H}(g_0)) \tag{3.39}$$

for all $\varepsilon \leqslant \varepsilon_0$.

Consider a special case, where $g_0(x,t)$ is a quasiperiodic function, i.e., $g_0(x,t) = G(x,\alpha_1 t,\alpha_2 t,\dots,\alpha_k t) = G(x,\bar\alpha t)$, where $G(\cdot) \in C^{\mathrm{Lip}}(\mathbb{T}^k;H)$ and the numbers $\bar\alpha = (\alpha_1,\alpha_2,\dots,\alpha_k)$ are rationally independent (see Section 3.2). Thus, $\mathcal{H}(g_0) = \{G(x,\bar\alpha t+\bar\theta) \mid \bar\theta \in \mathbb{T}^k\}$.

By the Kronecker–Weyl theorem (see, for example, [**85**]),

$$\begin{aligned} M(|g_0|^2) &:= \lim_{T\to\infty}\sup_{\bar\theta\in\mathbb{T}^k}\Big(\frac{1}{T}\int_0^T |G(\cdot,\bar\theta+\bar\alpha t)|^2 dt\Big)\\ &= \frac{1}{|2\pi|^k}\int\cdots\int_{\mathbb{T}^k} |G(\cdot,\omega_1,\dots,\omega_k)|^2 d\omega_1\cdots d\omega_k =: \Gamma^2.\end{aligned}$$

Then from (3.36) it follows that

$$d \leqslant d^* = c\frac{M(|\mathcal{H}(g_0)|^2)^{1/2}|\Omega|}{\nu^2} = c\frac{\Gamma|\Omega|}{\nu^2}.$$

Using Theorem 3.3, we obtain the following assertion.

Theorem 3.7. *The fractal dimension of the uniform attractor* $\mathcal{A}$ *of the* $2D$ *Navier–Stokes system with quasiperiodic external force* $g_0(x,s) = G(x,\bar{\alpha}s)$, $G \in C(\mathbb{T}^k; H)$ *satisfies the estimate*

$$\mathbf{d}_F\mathcal{A} \leqslant c\frac{\Gamma|\Omega|}{\nu^2} + k, \tag{3.40}$$

where the dimensionless constant c *depends on the shape of* Ω $(c(\Omega) = c(\lambda\Omega))$ *and admits the following absolute upper bound*:

$$c < \frac{1}{2\pi^{3/2}}.$$

Remark 3.7. In the autonomous case $k = 0$, the estimate (3.40) becomes the upper bound (1.49) for the fractal dimension of the attractor of the autonomous Navier–Stokes system (where $\Gamma = |g_0|$, $g_0 = g_0(x)$). In the nonautonomous case, the estimate (3.40) contains also the term $k = \dim\mathbb{T}^k$, i.e., the dimension of the hull $\mathcal{H}(g_0) = \{G(x,\bar{\alpha}s+\bar{\theta}) \mid \bar{\theta} \in \mathbb{T}^k\}$, where k is the number of rationally independent frequencies of the quasiperiodic external force $g_0(x,t)$.

Remark 3.8. As was proved in [**34**],

$$\mathbf{d}_F\mathcal{K}_g(t) \leqslant c\frac{\Gamma|\Omega|}{\nu^2} \quad \forall t \in \mathbb{R}$$

and, since $\mathbf{d}_F\mathcal{H}(g_0) \leqslant \dim\mathbb{T}^k = k$, we conclude that the estimate (3.40) well agrees with the representation (3.32).

Remark 3.9. Assume that $G_k(x,\omega_1,\dots,\omega_k) = G_k(x,\bar{\omega}^k)$, $\bar{\omega}^k \in \mathbb{T}^k$, $k = 1,2,\dots$, are such that

$$\Gamma_k = \Big(\frac{1}{|2\pi|^k}\int_{\mathbb{T}^k} |G_k(\,.\,,\bar{\omega}^k)|^2 d\bar{\omega}^k\Big)^{1/2} \leqslant R \quad \forall\, k \in \mathbb{N}.$$

Assume also that $1/\nu \leqslant R_1$. Consider the global attractors $\{\mathcal{A}^k\}$ of the 2D Navier–Stokes systems with external forces

$$g_{0k}(x,t) = G_k(x,\alpha_1 t,\alpha_2 t,\dots,\alpha_k t),$$

where the sequence $\{\alpha_i\}$ consists of rationally independent numbers. From (3.40) it follows that

$$\mathbf{d}_F\mathcal{A}^k \leqslant k + D \quad \forall\, k \in \mathbb{N}, \tag{3.41}$$

where $D = D(R, R_1)$. Therefore, the right-hand side of (3.41) tends to infinity as $k \to \infty$, while the nonautonomous analogs of the Reynolds number Re and the Grashof number Gr depending on R, $1/\nu$, and $|\Omega|$ remain bounded.

Let us present an example of the external forces $\{\widehat{G}_k(x, \bar{\omega}^k)\}$ satisfying the conditions of Remark 3.9 such that

$$\mathbf{d}_F \mathcal{A}_{\mathbb{T}^k} \geqslant k. \tag{3.42}$$

Consider the function

$$\widehat{u}(x,t) = \sum_{i=1}^{k} (a_i(x)\cos(\alpha_i t) + a_{i+k}(x)\sin(\alpha_i t)), \tag{3.43}$$

where $a_i(x)$, $i = 1, \ldots, 2k, \ldots$, are linearly independent vector-valued functions, $a_i(x) = (a_i^1(x), a_i^2(x))$, satisfying the following conditions: $a_i(x) \in (C^2(\bar{\Omega}))^2$, $(\nabla, a_i(x)) = 0$, $a_i|_{\partial\Omega} = 0$. We assume that the frequencies $(\alpha_1, \ldots, \alpha_k, \ldots)$ are rationally independent. We set

$$\widehat{g}_k(x, \bar{\alpha}t) = \partial_t \widehat{u} + \nu L \widehat{u} + B(\widehat{u}, \widehat{u}), \tag{3.44}$$

where $\widehat{u}(x,t)$ is defined by formula (3.43). It is obvious that $\widehat{g}_k(x, \overline{\alpha}t)$ is quasiperiodic. The function $\widehat{u}(x,t)$ is a complete bounded trajectory of the Navier–Stokes system with external force $\widehat{g}_k$. If the coefficients $a_i(x)$ in (3.43) decay rapidly, then $\Gamma_k \leqslant R$ for all $k \in \mathbb{N}$. We note that $\widehat{u}(\cdot, t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. It is easy to see that the trajectory $\widehat{u}(\cdot, t)$ provides an everywhere dense winding of the k-dimensional torus $\widetilde{\mathbb{T}}^k \subset H$. Therefore, the closure in $H:\ \overline{\{\widehat{u}(t) \mid t \in \mathbb{R}\}} = \widetilde{\mathbb{T}}^k$ belongs to $\mathcal{A}$. Hence

$$\mathbf{d}_F \widetilde{\mathbb{T}}^k = k \leqslant \mathbf{d}_F \mathcal{A}.$$

This example shows that the main term k in (3.41) is precise.

3.3.2. Wave equation with dissipation. We consider the nonautonomous wave equation from Section 2.6.2:

$$\begin{aligned} &\partial_t^2 u + \gamma \partial_t u = \Delta u - f_0(u,t) + g_0(x,t), \quad u|_{\partial\Omega} = 0, \\ &u|_{t=\tau} = u_\tau, \quad \partial_t u|_{t=\tau} = p_\tau, \quad u_\tau \in H_0^1(\Omega), p_\tau \in L_2(\Omega), \end{aligned} \tag{3.45}$$

where $x \in \Omega \Subset \mathbb{R}^3$. The function $f_0(v,t) \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfies the conditions (2.71)–(2.74) and the following inequality, similar to (1.61):

$$|f_v(v_1,t) - f_v(v_2,t)| \leqslant C(|v_1|^{2-\delta} + |u_2|^{2-\delta} + 1)|v_1 - v_2|^\delta \tag{3.46}$$

for all $v_1, v_2 \in \mathbb{R}$, $t \in \mathbb{R}$, where $0 < \delta \leqslant 1$. Moreover, we assume that $(f_0(v,t), f_{0t}(v,t))$ is a translation compact function in $C(\mathbb{R}; \mathcal{M}_2)$ and $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$. The norm in the Banach space $\mathcal{M}_2$ is defined bu formula (2.79). The symbol of the problem (3.45) is

$\sigma_0(t) = (f_0(v,t), g_0(x,t))$. It is clear that $\sigma_0(t)$ is a translation compact function in $\Xi = C(\mathbb{R}; \mathcal{M}_2) \times L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega))$. As usual, $\mathcal{H}(\sigma_0)$ denotes the hull of $\sigma_0(t)$ in Ξ. Consider (3.45) with symbols $\sigma(t) = (f(v,t), g(x,t)) \in \mathcal{H}(\sigma_0)$. By Proposition 2.6, the family of problems (3.45) generates the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, $U_\sigma(t,\tau) : E \to E$, acting in the energy space $E = H_0^1(\Omega) \times L_2(\Omega)$. By Propositions 2.8 and 2.9, the process $\{U_{\sigma_0}(t,\tau)\}$ is uniformly asymptotically compact and the family $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, is $(E \times \mathcal{H}(\sigma_0), E)$-continuous. Proposition 2.5 implies that the process $\{U_{\sigma_0}(t,\tau)\}$ has the uniform global attractor

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ is the kernel of $\{U_\sigma(t,\tau)\}$. The set $\mathcal{A}$ is compact in E.

As is proved in [**34**], $\mathcal{A}$ is bounded in $E_1 = H^2(\Omega) \times H_0^1(\Omega)$ (recall that $\Omega \Subset \mathbb{R}^3$),

$$\|y\|_{E_1} \leqslant M \quad \forall\, y \in \mathcal{A},$$

where the constant M is independent of y. By the Sobolev embedding theorem,

$$\|u(\cdot)\|_{C(\overline{\Omega})} \leqslant M_1 \quad \forall\, y = (u(\cdot), p(\cdot)) \in \mathcal{A}. \tag{3.47}$$

We study the ε-entropy of the global attractor $\mathcal{A}$ in E. As was proved in [**34**], the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, corresponding to the family of problems (3.45) satisfies the Lipschitz condition (3.6): for any $h > 0$

$$|U_{\sigma_1}(h,0)y - U_{\sigma_2}(h,0)y| \leqslant C(h)\|\sigma_1 - \sigma_2\|_{\Xi_{0,h}} \tag{3.48}$$

for all $\sigma_1, \sigma_2 \in \mathcal{H}(\sigma_0)$, $y \in \mathcal{A}$; $\Xi_{0,h} = C([0,h]; \mathcal{M}_2) \times L_2(0,h; L_2(\Omega))$. Moreover, there is an explicit formula for the Lipschitz constant $C(h)$ in [**34**].

As in the autonomous case (see the proof of Theorem 1.7), we write the problem (3.45) in the form

$$\partial_t w = A(w) = L_\alpha w - G_{\sigma_0(t)}(w), \;\; w|_{t=\tau} = w_\tau, \tag{3.49}$$

where $w = (u,v) = (u, p + \alpha u)$, the operator L_α is defined in (1.66), and $G_{\sigma_0(t)}(w) = (0, f_0(u,t) - g_0(x,t))$. Here, α is a real parameter to be chosen later.

The variational equation for (3.49) has the form

$$\begin{aligned} &\partial_t z = L_\alpha z - G_{\sigma_0 w}(w(t))z := A_{\sigma_0 w}(w(t))z, \\ &z|_{t=\tau} = z_\tau, \quad z = (r,q), \end{aligned} \tag{3.50}$$

where $G_{\sigma_0 w}(w(t))z = (0, f_u(u(t),t)r)$. As in the autonomous case (see [**9**]), we prove that the process $\{U_{\sigma_0}(t,\tau)\}$ of the problem (3.49) is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$ and the quasidifferentials are generated by the system (3.50). We set

$$\widetilde{q}_j := \limsup_{T\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{w_\tau\in\mathcal{K}_{\sigma_0}(\tau)} \Big(\frac{1}{T}\int\limits_{\tau}^{\tau+T} \mathrm{Tr}_j A_{\sigma_0 w}(w(t))dt\Big),\ j=1,2,\ldots,$$

where $w(t) = U_{\sigma_0}(t,\tau)w_\tau$. Arguing in the same way as in the proof of Theorem 1.7, we obtain the following estimate for the numbers $\widetilde{q}_j$:

$$\widetilde{q}_j \leqslant q_j = -(\alpha/4)j + (C(M_1)/\alpha)j^{1/3} =: \varphi(j) \quad \forall j\in\mathbb{N}, \tag{3.51}$$

where, owing to the inequality (see (3.47)), M_1 is such that

$$\sup\{\|u(\cdot,t)\|_{C(\bar{\Omega})} \mid t\in\mathbb{R},\ (u(\cdot),\partial_t u(\cdot))\in\mathcal{K}_{\sigma_0}\} \leqslant M_1$$

The function $\varphi(x)$, $x\geqslant 0$, in (3.51) is concave and the root of φ is $d^* = 8C_1(M_1)^{3/2}\alpha^{-3} =: C(M_1)\alpha^{-3}$. All the assumptions of Theorem 3.1 are verified. Thus, we have the following assertion.

Theorem 3.8. *For any $\delta>0$ there exist $h>0$, $\varepsilon_0>0$, and $\eta<1$ such that*

$$\begin{aligned}\mathbf{H}_\varepsilon(\mathcal{A}_0) \leqslant \Big(\frac{C}{\alpha^3}+\delta\Big)\log_2\Big(\frac{\varepsilon_0}{\eta\varepsilon}\Big) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}_0)&\\ + \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\Pi_{0,h\log_{1/\eta}(\varepsilon_0/(\eta\varepsilon))}\mathcal{H}(\sigma_0))&\end{aligned} \tag{3.52}$$

for all $\varepsilon\leqslant\varepsilon_0$, where $\alpha=\min\{\gamma/4,\lambda_1/(2\gamma)\}$ and $C=C(M_1)$ (see (3.51)). Here, $\mathbf{H}_\epsilon(\Pi_{0,l}\mathcal{H}(\sigma_0))$ denotes the ϵ-entropy of the set $\mathcal{H}(\sigma_0)$ measured in the space $\Xi_{0,l}=C([0,l];\mathcal{M}_2)\times L_2^{\mathrm{loc}}(0,l;L_2(\Omega))$.

Remark 3.10. We cannot apply Theorem 3.2 to the hyperbolic equation (3.45) because the function $\varphi(j)$ in (3.51) does not satisfy (3.13).

Consider a hyperbolic equation with quasiperiodic terms. Let

$$\begin{aligned}f_0(v,t) &= \Phi(v,\alpha_1 t,\alpha_2 t,\ldots,\alpha_k t) = \Phi(v,\bar{\alpha}t),\\ g_0(x,t) &= G(x,\alpha_1 t,\alpha_2 t,\ldots,\alpha_k t) = G(x,\bar{\alpha}t),\end{aligned}$$

where $\Phi(v,\bar{\omega})\in C^{\mathrm{Lip}}(\mathbb{T}^k;\mathcal{M}_2)$ and $G(x,\bar{\omega})\in C^{\mathrm{Lip}}(\mathbb{T}^k;L_2(\Omega))$. To obtain the inequality (2.74), we assume that $|\bar{\alpha}|\leqslant\varkappa\ll 1$, where $\varkappa=\varkappa(\delta)$. Now, if $\Phi(v,\bar{\omega})$ satisfies the inequality

$$|\Phi_{\bar{\omega}}(v,\bar{\omega})| \leqslant \delta_1^2\Phi(v,\bar{\omega}) + C_1 \quad \forall\,(v,\bar{\omega})\in\mathbb{R}\times\mathbb{T}^k,$$

then (2.74) is also valid for a small $\varkappa$. Then

$$\mathcal{H}(\sigma_0) = \{(\Phi(v,\bar{\alpha}t+\bar{\theta}),G(x,\bar{\alpha}s+\bar{\theta})) \mid \bar{\theta}\in\mathbb{T}^k\}$$

and hence

$$\mathbf{d}_F(\mathcal{H}(\sigma_0), C_b(\mathbb{R}; \mathcal{M}_2 \times L_2(\Omega)) = \mathbf{d}_F \mathcal{H}(\sigma_0) \leqslant k.$$

Using Theorem 3.3 we obtain the following assertion.

Theorem 3.9. *The fractal dimension of the uniform global attractor $\mathcal{A}$ of the hyperbolic equation* (3.45) *with quasiperiodic symbol $\sigma_0(t) = (\Phi(v, \bar{\alpha}t), G(x, \bar{\alpha}t))$ satisfies the estimate*

$$\mathbf{d}_F \mathcal{A} \leqslant \frac{C}{\alpha^3} + k. \tag{3.53}$$

To illustrate Theorem 3.9, we consider the dissipative sine-Gordon equation with quasiperiodic forcing term

$$\partial_t^2 u + \gamma \partial_t u = \Delta u - \beta \sin(u) + \psi(\bar{\alpha}t) g(x),\ u|_{\partial\Omega} = 0,\ \Omega \Subset \mathbb{R}^3, \tag{3.54}$$

where $\psi \in C^1(\mathbb{T}^k; \mathbb{R})$ and $g \in L_2(\Omega)$. Observe that the constant C in (3.52) and (3.53) does not exceed $c\beta^3$, where c depends on Ω (see (1.69) and (1.70)). For the global attractor $\mathcal{A}$ of the problem (3.54) we have the estimate

$$\mathbf{d}_F \mathcal{A} \leqslant c\frac{\beta^3}{\alpha^3} + k. \tag{3.55}$$

Remark 3.11. In the autonomous case $k = 0$, the estimates (3.53) and (3.55) coincide with (1.63) and (1.72) respectively.

3.3.3. Ginzburg–Landau equation. We continue to study the nonautonomous Ginzburg–Landau equation (2.112) from Section 2.6.3. Consider the family of problems with periodic boundary conditions

$$\begin{aligned} &\partial_t u = \nu(1 + i\alpha)\Delta u + Ru - (1 + i\beta(t))|u|^2 u + g(x,t),\ x \in \mathbb{T}^3, \\ &u|_{t=\tau} = u_\tau(x), \quad u_\tau \in \mathbf{H} = L_2(\mathbb{T}^3; \mathbb{C}). \end{aligned} \tag{3.56}$$

For the sake of simplicity, we assume that the coefficients α and R are independent of time. The symbol $\sigma(t) = (\beta(t), g(x,t))$ of (3.56) belongs to the hull $\mathcal{H}(\sigma_0)$ of the original symbol $\sigma_0(t) = (\beta_0(t), g_0(x,t))$. We assume that $\sigma_0(t)$ is a translation compact function in $C^{\,\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}) \times L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}) =: \Xi$ and the parameter $\beta_0(t)$ satisfies the inequality (2.113).

As in the autonomous case (see Section 1.4.2), we write the problem (3.56) in the vector form

$$\partial_t \mathbf{u} = \nu a \Delta \mathbf{u} + R\mathbf{u} - \mathbf{f}(\mathbf{u}, \beta(t)) + \mathbf{g}(x,t),\ \mathbf{u}|_{t=\tau} = \mathbf{u}_\tau,\ \mathbf{u}_\tau \in \mathbf{H}. \tag{3.57}$$

where $a = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$, $\mathbf{f}(\mathbf{v}, \beta) = |\mathbf{v}|^2 \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} \mathbf{v}$, $\mathbf{g}(x) = (g_1(x), g_2(x))^\top$.

We know (see Section 2.6.3) that for every $\sigma \in \mathcal{H}(\sigma_0)$ the problem (3.56) has a unique solution $u \in C(\mathbb{R}_\tau; \mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_\tau; \mathbf{V}) \cap L_4^{\mathrm{loc}}(\mathbb{R}_\tau; \mathbf{L}_4)$ (see also [**9, 25, 31**]). Thus, for a given symbol $\sigma_0(t)$, the problem (3.56) generates the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$, acting in $\mathbf{H}$. It is proved that the process $\{U_{\sigma_0}(t,\tau)\}$ has the uniform global attractor $\mathcal{A}$ and

$$\mathcal{A} = \bigcup_{\sigma \in \mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0),$$

where $\mathcal{K}_\sigma$ is the kernel of $\{U_\sigma(t,\tau)\}$. The set $\mathcal{A}$ is bounded in $\mathbf{V}$.

In [**34**], the Lipschitz condition is established for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$:

$$\begin{aligned} &\|U_{\sigma_1}(h,0)\mathbf{u}_0 - U_{\sigma_2}(h,0)\mathbf{u}_0\|_{\mathbf{H}} \\ &\leqslant C(h)(\|\beta_1 - \beta_2\|_{C([0,h])} + \|g_1 - g_2\|_{L_2(0,h;H)}), \\ &\forall \sigma_1 = (\beta_1, g_1) \in \mathcal{H}(\sigma_0), \sigma_2 = (\beta_2, g_2) \in \mathcal{H}(\sigma_0),\ \mathbf{u}_0 \in \mathcal{A}. \end{aligned} \tag{3.58}$$

In order to use Theorem 3.1, we need to check that the process $\{U_{\sigma_0}(t,\tau)\}$ corresponding to the problem (3.57) with the original symbol $\sigma_0(t)$ is uniformly quasidifferentiable on the kernel $\mathcal{K}_{\sigma_0}$. This fact is proved in [**34**]. Recall that the variational equation for (3.57) is as follows:

$$\begin{aligned} &\partial_t \mathbf{v} = \nu a \Delta \mathbf{v} + R\mathbf{v} - \mathbf{f}_{\mathbf{u}}(\mathbf{u}(t), \beta(t))\mathbf{v} =: A_{\sigma_0 \mathbf{u}}(\mathbf{u}(t))\mathbf{v}, \\ &\mathbf{v}|_{t=\tau} = \mathbf{v}_\tau \in \mathbf{H}, \end{aligned} \tag{3.59}$$

where the Jacobi matrix $\mathbf{f}_{\mathbf{u}}(\mathbf{u}, \beta)$ is defined in (1.33). As in the autonomous case, we prove that

$$\begin{aligned} \widetilde{q}_j &= \limsup_{T\to\infty} \sup_{\tau \in \mathbb{R}} \sup_{\mathbf{u}_\tau \in \mathcal{K}_{\sigma_0}(\tau)} \Big(\frac{1}{T} \int_\tau^{\tau+T} \mathrm{Tr}_j (A_{\sigma_0 \mathbf{u}}(\mathbf{u}(t)) dt \Big) \\ &\leqslant -\nu C_1 j^{5/3} + Rj =: \varphi(j) = q_j, \quad j = 1, 2, \ldots, \end{aligned}$$

where $\mathbf{u}(t) = U_{\sigma_0}(t,\tau)\mathbf{u}_\tau$. Finally (see (3.11)),

$$d \leqslant d^* = \Big(\frac{R}{C_1 \nu} \Big)^{3/2},$$

where d^* is the root of the equation $\varphi(x) = 0$ and C_1 was defined in (1.79). Hence Theorem 3.1 is applicable to the problem (3.57) and the following assertion holds.

Theorem 3.10. *For any $\delta > 0$ there exist $h > 0$, $\varepsilon_0 > 0$, and $\eta < 1$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant \left(\left(\frac{R}{C_1\nu}\right)^{3/2} + \delta\right)\log_2\left(\frac{\varepsilon_0}{\eta\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A})$$
$$+ \mathbf{H}_{\frac{\varepsilon\eta}{4C(h)}}(\Pi_{0,h\log_{1/\eta}(\varepsilon_0/(\eta\varepsilon))}\mathcal{H}(\sigma_0))$$

for all $\varepsilon \leqslant \varepsilon_0$, where $C(h)$ is taken from (3.58) *and $\mathbf{H}_\epsilon(\mathcal{H}(\sigma_0)_{0,l})$ denotes the ϵ-entropy of $\mathcal{H}(\sigma_0)$ in $C([0,l]) \times L_2(0,l;\mathbf{H})$.*

Theorem 3.2 implies the following assertion.

Theorem 3.11. *For any $h > 0$ there are $D > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathbf{H}_\varepsilon(\mathcal{A}) \leqslant D\log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) + \mathbf{H}_{\varepsilon_0}(\mathcal{A}) + \mathbf{H}_{\frac{\varepsilon}{8C(h)}}(\Pi_{0,h\log_2(2\varepsilon_0/\varepsilon)}\mathcal{H}(\sigma_0))$$

for all $\varepsilon \leqslant \varepsilon_0$.

Consider the Ginzburg–Landau equation with quasiperiodic terms

$$\beta_0(t) = B(\alpha_1 t, \alpha_2 t, \ldots, \alpha_k t) = B(\bar{\alpha}t),$$
$$g_0(x,t) = G(x, \alpha_1 t, \alpha_2 t, \ldots, \alpha_k t) = G(x, \bar{\alpha}t),$$

where $B(\bar{\omega}) \in C^{\,\mathrm{Lip}}(\mathbb{T}^k;\mathbb{R})$, $|B| \leqslant \sqrt{3}$, and $G(x,\bar{\omega}) \in C^{\,\mathrm{Lip}}(\mathbb{T}^k;\mathbf{H})$. Asume that the numbers $(\alpha_1, \alpha_2, \ldots, \alpha_k) =: \bar{\alpha}$ are rationally independent. As we know, $\mathcal{H}(\sigma_0) = \{(B(\bar{\alpha}t + \bar{\theta}), G(x, \bar{\alpha}t + \bar{\theta})) \mid \bar{\theta} \in \mathbb{T}^k\}$ and

$$\mathbf{d}_F(\mathcal{H}(\sigma_0), C_b(\mathbb{R}) \times L_2(\mathbb{R};\mathbf{H})) = \mathbf{d}_F\mathcal{H}(\sigma_0) \leqslant k$$

(see Section 3.2). Using Theorem 3.3 we obtain the following assertion.

Theorem 3.12. *The fractal dimension of the global attractor $\mathcal{A}$ of the Ginzburg–Landau equation with quasiperiodic symbol $\sigma(s) = (B(\bar{\alpha}t), G(x,\bar{\alpha}t))$ satisfies the estimate*

$$\mathbf{d}_F\mathcal{A} \leqslant \left(\frac{R}{C_1\nu}\right)^{3/2} + k. \tag{3.60}$$

As in the case of the Navier–Stokes system, we consider the sequence of functions $B_k(\bar{\omega}^k)$ and $G_k(x,\bar{\omega}^k)$ satisfying the above conditions. Denote by $\mathcal{A}(k)$ the corresponding uniform global attractors. The inequality (3.60) implies

$$\mathbf{d}_F\mathcal{A}(k) \leqslant k + D, \tag{3.61}$$

where the constant D is independent of k.

As at the end of Section 3.3.1, we can construct examples of Ginzburg–Landau equations with terms $B_k(\bar{\omega}^k)$ and $G_k(x,\bar{\omega}^k)$ and the uniform global attractors $\mathcal{A}(k)$ such that

$$k \leqslant \mathbf{d}_F\mathcal{A}(k).$$

Therefore, the main term k in the estimate (3.61) is precise.

4. Nonautonomous 2D Navier–Stokes System with Singularly Oscillating External Force

We study the global attractor $\mathcal{A}^\varepsilon$ of the nonautonomous 2D Navier–Stokes system with singularly oscillating external force of the form

$$g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t), \quad x \in \Omega \Subset \mathbb{R}^2,\ t \in \mathbb{R},\ 0 < \rho \leqslant 1.$$

If $g_0(x,t)$ and $g_1(z,t)$ are translation bounded functions in the corresponding spaces, then the global attractor $\mathcal{A}^\varepsilon$ is bounded in the space H (see Section 2.6.1). However, the norm $\|\mathcal{A}^\varepsilon\|_H$, regarded as a function of ε, can be unbounded as $\varepsilon \to 0+$ since the magnitude of the external force is growing.

Assuming that $g_1(z,t)$ admits the divergence representation

$$g_1(z,t) = \partial_{z_1} G_1(z,t) + \partial_{z_2} G_2(z,t), \quad z = (z_1, z_2) \in \mathbb{R}^2,$$

where $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ (see Section 4.2), we prove that the global attractors $\mathcal{A}^\varepsilon$ of the Navier–Stokes system are uniformly bounded:

$$\|\mathcal{A}^\varepsilon\|_H \leqslant C \quad \forall\, 0 < \varepsilon \leqslant 1.$$

We also consider the "limiting" 2D Navier–Stokes system with external force $g_0(x,t)$. We derive an explicit estimate for the deviation of the solution $u^\varepsilon(x,t)$ of the original Navier–Stokes system from the solution $u^0(x,t)$ of the "limiting" Navier–Stokes system with the same initial data. If $g_1(z,t)$ admits the divergence representation and $g_0(x,t)$, $g_1(z,t)$ are translation compact functiosn in the corresponding spaces, then we prove that the global attractors $\mathcal{A}^\varepsilon$ converge to the global attractor $\mathcal{A}^0$ of the "limiting" system as $\varepsilon \to 0+$ in the norm of H. In Section 4.5, we present the following explicit estimate for the Hausdorff deviation of $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho}$$

in the case, where the global attractor $\mathcal{A}^0$ is exponential (providing that the Grashof number of the "limiting" 2D Navier–Stokes system is small).

Some problems related to homogenization and averaging of the global attractors for the Navier–Stokes systems and for other evolution equations in mathematical physics with rapidly (nonsingularly) oscillating coefficients and terms were studied in [**70, 79, 80, 126, 131, 128, 53, 36, 17**].

4.1. 2D Navier–Stokes system with singularly oscillating force.

We consider the nonautonomous 2D Navier–Stokes system

$$\begin{aligned} &\partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u = \nu\Delta u - \nabla p + g_0(x,t) + \frac{1}{\varepsilon^\rho}g_1(x/\varepsilon,t),\\ &\partial_{x_1}u_1 + \partial_{x_2}u_2 = 0, \quad u|_{\partial\Omega} = 0,\ x := (x_1,x_2)\in\Omega,\ \Omega \Subset \mathbb{R}^2, \end{aligned} \tag{4.1}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity vector field, $p = p(x,t)$ is the pressure, and ν is the kinematic viscosity. In (4.1), ε is a small parameter, $0 < \varepsilon \leqslant 1$, and ρ is fixed, $0 \leqslant \rho \leqslant 1$. We assume that $0 \in \Omega$.

The vector-valued functions $g_0(x,t) = (g_{01}(x,t), g_{02}(x,t))$, $x \in \Omega$, $t \in \mathbb{R}$, and $g_1(z,t) = (g_{11}(z,t), g_{12}(z,t))$, $z \in \mathbb{R}^2$, $t \in \mathbb{R}$ are given. The function $g_0(x,t) + \frac{1}{\varepsilon^\rho}g_1(x/\varepsilon,t)$ is called the *external force.* For every fixed ε the external force is assumed to belong to $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ (we clarify this assumption later). Under this condition, the Cauchy problem for (4.1) is well studied (see, [**96, 87, 117, 40, 9, 34**] and Section 2.6.1).

As usual, we denote by H and $V = H^1$ funciton spaces which are the closures of the set

$$\mathcal{V}_0 := \{v \in (C_0^\infty(\Omega))^2 \mid \partial_{x_1}v_1(x) + \partial_{x_2}v_2(x) = 0 \quad \forall x \in \Omega\}$$

in the norms $|\cdot|$ and $\|\cdot\|$ of the spaces $L_2(\Omega)^2$ and $H_0^1(\Omega)^2$ respectively. We recall that

$$\|v\|^2 = |\nabla v|^2 = \int_\Omega (|\partial_{x_1}v^1(x)|^2 + |\partial_{x_2}v^1(x)|^2 + |\partial_{x_1}v^2(x)|^2 + |\partial_{x_2}v^2(x)|^2)dx.$$

The space $V' = V^*$ is dual to V. We denote by P the orthogonal projection from $L_2(\Omega)^2$ onto H (see Section 1.3.1) and set

$$g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho}Pg_1\Big(\frac{x}{\varepsilon},t\Big).$$

Applying the operator P to both sides of the first equation in (4.1), we exclude the pressure $p(x,t)$ and obtain the following equation for the velocity vector field $u(x,t)$:

$$\partial_t u + \nu L u + B(u,u) = g^\varepsilon(x,t), \tag{4.2}$$

where $L = -P\Delta$ is the Stokes operator, $B(u,v) = P[u^1\partial_{x_1}v + u^2\partial_{x_2}v]$ and $g^\varepsilon(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$. The Stokes operator L is selfadjoint and the minimal eigenvalue λ_1 of L is positive.

We assume that the function $g_0(\cdot, t)$ belongs to $L_2(\Omega)^2$ for almost all $t \in \mathbb{R}$ and has finite norm in the space $L_2^b(\mathbb{R}; L_2(\Omega)^2)$, i.e.,

$$\|g_0\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} = \|g_0\|^2_{L_2^b} := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} (\|g_0(\cdot, s)\|^2_{L_2(\Omega)^2}) ds < +\infty. \qquad (4.3)$$

To describe the vector-valued function $g_1(z, t)$, $z = (z_1, z_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$, we use the space $Z = L_2^b(\mathbb{R}_z^2; \mathbb{R}^2)$. By definition,

$$\varphi(z) = (\varphi_1(z_1, z_2), \varphi_2(z_1, z_2)) \in Z$$

if

$$\|\varphi(\cdot)\|^2_Z = \|\varphi(\cdot)\|^2_{L_2^b(\mathbb{R}_z^2;\mathbb{R}^2)} := \sup_{(z_1,z_2)\in\mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |\varphi(\zeta_1, \zeta_2)|^2 d\zeta_1 d\zeta_2 < +\infty.$$

We assume that $g_1(\cdot, t) \in Z$ for almost all $t \in \mathbb{R}$ and has finite norm in the space $L_2^b(\mathbb{R}; Z)$, i.e.,

$$\begin{aligned} \|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} &:= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} (\|g_1(\cdot, s)\|^2_Z) ds \\ &= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \left(\sup_{(z_1,z_2)\in\mathbb{R}^2} \int_{z_1}^{z_1+1} \int_{z_2}^{z_2+1} |g_1(\zeta_1, \zeta_2, s)|^2 d\zeta_1 d\zeta_2 \right) ds < +\infty. \end{aligned} \qquad (4.4)$$

For (4.1) the initial data are imposed at arbitrary $\tau \in \mathbb{R}$:

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in H. \qquad (4.5)$$

For fixed $\varepsilon > 0$ the Cauchy problem (4.1), (4.5) has a unique solution $u(t) := u(x, t)$ in a weak sense, i.e., $u(t) \in C(\mathbb{R}_\tau; H) \cap L_2^{\text{loc}}(\mathbb{R}_\tau; V)$, $\partial_t u \in L_2^{\text{loc}}(\mathbb{R}_\tau; V')$, and $u(t)$ satisfies (4.1) in the sense of distributions in the space $\mathcal{D}'(\mathbb{R}_\tau; V')$, where $\mathbb{R}_\tau = [\tau, +\infty)$ (see [**96, 87, 40, 9, 34, 119**] and Sections 1.3.1, 2.6.1).

Recall that every weak solution $u(t)$ of the problem (4.1) satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 = \langle u(t), g^\varepsilon(t)\rangle \quad \forall t \geqslant \tau, \qquad (4.6)$$

where the function $|u(t)|^2$ is absolutely continuous in t (see Section 1.3.1).

We need the following lemma proved in [**34**].

Lemma 4.1. *Let a real-valued function $y(t)$, $t \geqslant 0$, be uniformly continuous and satisfy the inequality*

$$y'(t) + \gamma y(t) \leqslant f(t) \quad \forall\, t \geqslant 0, \tag{4.7}$$

where $\gamma > 0$, $f(t) \geqslant 0$ for all $t \geqslant 0$, and $f \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$. Suppose that

$$\int\limits_t^{t+1} f(s)ds \leqslant M \qquad \forall t \geqslant 0. \tag{4.8}$$

Then

$$y(t) \leqslant y(0)e^{-\gamma t} + M(1+\gamma^{-1}) \qquad \forall t \geqslant 0. \tag{4.9}$$

Using standard transformations and the Poincaré inequality, from (4.6) we obtain the differential inequalities

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \quad \leqslant \quad (\nu\lambda_1)^{-1}|g(t)|^2, \tag{4.10}$$

$$\Downarrow$$

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 \quad \leqslant \quad (\nu\lambda_1)^{-1}|g(t)|^2. \tag{4.11}$$

Applying Lemma 4.1 to (4.11) with

$$y(t) = |u(t+\tau)|^2, \quad f(t+\tau) = (\nu\lambda_1)^{-1}|g^\varepsilon(t)|^2,$$
$$\gamma = \nu\lambda_1, \quad M = (\nu\lambda_1)^{-1}\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)},$$

we obtain the following main a priori estimate for a weak solution $u(t)$ of the problem (4.1):

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + D\|g^\varepsilon\|^2_{L_2^b(\mathbb{R};H)}, \tag{4.12}$$

where $D = (\nu\lambda_1)^{-1}(1+(\nu\lambda_1)^{-1})$. The inequality (4.10) implies

$$|u(t)|^2 + \nu\int\limits_\tau^t \|u(s)\|^2 ds \leqslant |u(\tau)|^2 + (\nu\lambda_1)^{-1}\int\limits_\tau^t |g^\varepsilon(s)|^2 ds. \tag{4.13}$$

Lemma 4.2. *If $\varphi(z) \in Z = L_2^b(\mathbb{R}^2_z\mathbb{R}^2)$, then $\varphi(x/\varepsilon) \in L_2(\Omega)^2$ for all $\varepsilon > 0$ and*

$$\Big\|\varphi\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L_2(\Omega_x)^2} \leqslant C\|\varphi(\cdot)\|_{L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)}, \tag{4.14}$$

where the constant C is independent of ε and φ.

Proof. Indeed, making the change of variables $x/\varepsilon = z$, $dx = \varepsilon^2 dz$, we find

$$\Big\|\varphi\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|^2_{L_2(\Omega)^2} = \int\limits_{\Omega} \Big|\varphi\Big(\frac{x}{\varepsilon}\Big)\Big|^2 dx = \varepsilon^2 \int\limits_{\varepsilon^{-1}\Omega} |\varphi(z)|^2 dz$$

$$\leqslant C^2\varepsilon^{-2} \sup_{(z_1,z_2)\in\mathbb{R}^2} \varepsilon^2 \int\limits_{z_1}^{z_1+1} \int\limits_{z_2}^{z_2+1} |\varphi(\zeta_1,\zeta_2)|^2 d\zeta_1 d\zeta_2$$

$$= C^2\|\varphi(\cdot)\|^2_{L_2^b(\mathbb{R}^2_z;\mathbb{R}^2)}.$$

In the last inequality, we used the fact that the domain $\varepsilon^{-1}\Omega$ can be covered by at most $C^2\varepsilon^{-2}$ unit squares of the form $[z_1, z_1+1] \times [z_2, z_2+1]$, where C depends only on the area of the domain Ω. □

Corollary 4.1. *If $g_0(x,t) \in L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^b(\mathbb{R}; Z)$, where $Z = L_2^b(\mathbb{R}^2_z; \mathbb{R}^2)$, then the external force*

$$g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1(x/\varepsilon, t)$$

belongs to the space $L_2^b(\mathbb{R}; H)$ and

$$\|g^\varepsilon\|_{L_2^b(\mathbb{R};H)} \leqslant \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} + \frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)}, \tag{4.15}$$

where the constant C is independent of ε.

The inequality (4.15) directly follows from Lemma 4.2 and formulas (4.3) and (4.4) for the norms in $L_2^b(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^b(\mathbb{R}; Z)$.

Using the inequality (4.15) in (4.12), we find

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + C_0^2 + \varepsilon^{-2\rho} C_1^2, \tag{4.16}$$

where C_0 and C_1 are constants depending on ν, λ_1, and $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$, $\|g_1\|_{L_2^b(\mathbb{R};Z)}$ respectively.

We consider the process $\{U_\varepsilon(t,\tau)\} := \{U_\varepsilon(t,\tau), t \geqslant \tau, \tau \in \mathbb{R}\}$ corresponding to the problem (4.2), (4.5) and acting in the space H (see Section 2.6.1). Recall that the mapping $U_\varepsilon(t,\tau) : H \to H$ is defined by the formula

$$U_\varepsilon(t,\tau)u_\tau = u(t) \qquad \forall u_\tau \in H,\ t \geqslant \tau,\ \tau \in \mathbb{R}, \tag{4.17}$$

where $u(t)$ is the solution of the problem (4.2), (4.5).

By the estimate (4.16), for every $0 < \varepsilon \leqslant 1$ the process $\{U_\varepsilon(t,\tau)\}$ has the *uniformly* (with respect to $\tau \in \mathbb{R}$) *absorbing set*

$$B_{0,\varepsilon} = \{v \in H \mid |v| \leqslant 2(C_0 + C_1\varepsilon^{-\rho})\} \tag{4.18}$$

which is bounded in H for fixed ε, i.e., for any bounded (in H) set B there exists $t' = t'(B)$ such that $U(t+\tau,\tau)B \subseteq B_{0,\varepsilon}$ for all $t \geqslant t(B)$ and $\tau \in \mathbb{R}$.

Arguing in a standard way, we prove that the process $\{U_\varepsilon(t,\tau)\}$ has the compact (in H) uniformly absorbing set

$$B_{1,\varepsilon} = \{v \in V \mid \|v\| \leqslant C_2(\nu, \lambda_1, C_0 + C_1\varepsilon^{-\rho})\}, \tag{4.19}$$

where $C_2(y_1, y_2, y_3)$ is a positive increasing function in each $y_j, j = 1, 2, 3$ (see (2.41)). Thus, the process $\{U_\varepsilon(t,\tau)\}$ corresponding to the problem (4.1), (4.5) is uniformly compact and has the compact uniformly absorbing set $B_{1,\varepsilon}$ (bounded in V) defined by formula (4.19). Consequently, the process $\{U_\varepsilon(t,\tau)\}$ has the *uniform global attractor* $\mathcal{A}^\varepsilon$ (see Section 2.6.1) and $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$.

Since $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon}$, from (4.16) and (4.18) if follows that

$$\|\mathcal{A}^\varepsilon\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}). \tag{4.20}$$

Remark 4.1. For $\rho > 0$ the norm in H of the uniform global attractor A^ε of the 2D Navier–Stokes system (4.1) may grow up as $\varepsilon \to 0+$. In the next sections, we present conditions providing the uniform boundedness of $\mathcal{A}^\varepsilon$ in H with respect to ε. We also study the convergence of $\mathcal{A}^\varepsilon$ to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" equation as $\varepsilon \to 0+$.

Along with the original Navier–Stokes system (4.1), we consider the "limiting" system

$$\begin{aligned} &\partial_t u + u^1\partial_{x_1}u + u^2\partial_{x_2}u = \nu\Delta u - \nabla p + g_0(x,t), \\ &\partial_{x_1}u_1 + \partial_{x_2}u_2 = 0, \quad u|_{\partial\Omega} = 0, \end{aligned} \tag{4.21}$$

without the term depending on ε. Excluding the pressure, we obtain the equivalent equation

$$\partial_t u + \nu L u + B(u,u) = Pg_0(x,t), \tag{4.22}$$

where $Pg_0(x,t) \in L_2^b(\mathbb{R}; H)$. Then the Cauchy problem for (4.22) has a unique solution $u(t) := u(x,t)$ in the sense of distributions. Hence there exists the "limiting" process $\{U_0(t,\tau)\}$ acting in H : $U_0(t,\tau)u_\tau = u(t)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where $u(t)$ is the solution of the problem (4.22), (4.5). As in the case of (4.12) and (4.13), we have

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + D\|Pg_0\|^2_{L_2^b(\mathbb{R};H)}, \tag{4.23}$$

$$|u(t)|^2 + \nu\int_\tau^t \|u(s)\|^2 ds \leqslant |u(\tau)|^2 + (\nu\lambda_1)^{-1}\int_\tau^t |Pg_0(s)|^2 ds. \tag{4.24}$$

From (4.16) it follows that

$$|u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + C_0^2, \tag{4.25}$$

which implies that the set

$$B_{0,0} = \{v \in H \mid |v| \leqslant 2C_0\} \tag{4.26}$$

is uniformly absorbing for the process $\{U_0(t,\tau)\}$. (The constant C_0 is the same as in (4.16).) Moreover, this process has the compact (in H) absorbing set

$$B_{1,0} = \{v \in V \mid \|v\| \leqslant C_2(\nu, \lambda_1, C_0)\}. \tag{4.27}$$

Therefore, the process $\{U_0(t,\tau)\}$ is uniformly compact and has the compact global attractor $\mathcal{A}^0$ such that $\mathcal{A}^0 \subset B_{0,0} \cap B_{1,0}$ and

$$\|\mathcal{A}^0\|_H \leqslant C_0. \tag{4.28}$$

4.2. Divergence condition and properties of global attractors $\mathcal{A}^\varepsilon$.

We consider the nonautonomous 2D Navier–Stokes system (4.2) with external force

$$g^\varepsilon(x,t) = Pg_0(x,t) + \frac{1}{\varepsilon^\rho} Pg_1(x/\varepsilon, t).$$

We assume that the function $g_0(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$, satisfies (4.3), i.e., $\|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} < +\infty$ and the function $g_1(z,t)$, $z \in \mathbb{R}^2$, $t \in \mathbb{R}$, satisfies (4.4), i.e., $\|g_1(\cdot)\|^2_{L_2^b(\mathbb{R};Z)} < +\infty$, where $Z = L_2^b(\mathbb{R}_z^2; \mathbb{R}^2)$.

- *Divergence condition.* There exist vector-valued functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$, $j = 1, 2$, such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ and

$$\partial_{z_1} G_1(z_1, z_2, t) + \partial_{z_2} G_2(z_1, z_2, t) = g_1(z_1, z_2, t) \tag{4.29}$$

for all $(z_1, z_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$.

Theorem 4.1. *If $g_1(z,t)$ satisfies the divergence condition* (4.29), *then for every $0 \leqslant \rho \leqslant 1$ the global attractors $\mathcal{A}^\varepsilon$ of the 2D Navier–Stokes system are uniformly (with respect to $\varepsilon \in]0,1]$) bounded in H, i.e.,*

$$\|\mathcal{A}^\varepsilon\|_H \leqslant C_2 \quad \forall\, \varepsilon \in]0,1], \tag{4.30}$$

where C_2 is independent of ε.

Proof. Taking the inner product of Equation (4.2) and $u(t)$ in H, we obtain the equality (4.6), i.e.,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2=\langle u(t),g^\varepsilon(t)\rangle\\&=(g_0(\cdot,t),u(\cdot,t))+\varepsilon^{-\rho}(g_1(\cdot/\varepsilon,t),u(\cdot,t)).\end{aligned}\tag{4.31}$$

The first term on the right-hand side of (4.31) satisfies the inequality

$$(g_0(\cdot,t),u(\cdot,t))\leqslant\frac{1}{4}\nu\|u(t)\|^2+\frac{1}{\nu\lambda_1}|g_0(t)|^2.\tag{4.32}$$

By (4.29), for the second term on the right-hand side of (4.31) we have

$$\begin{aligned}\varepsilon^{-\rho}\Big(g_1\Big(\frac{\cdot}{\varepsilon},t\Big),u(\cdot,t)\Big)&=\varepsilon^{-\rho}\sum_{j=1}^{2}\int_\Omega\Big(\partial_{z_j}G_j\Big(\frac{x}{\varepsilon},t\Big),u(x,t)\Big)dx\\&=\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega\Big(\partial_{x_j}G_j\Big(\frac{x}{\varepsilon},t\Big),u(x,t)\Big)dx\\&=-\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega\Big(G_j\Big(\frac{x}{\varepsilon},t\Big),\partial_{x_j}u(x,t)\Big)dx\\&\leqslant\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2dx+\frac{1}{4}\nu\|u(t)\|^2.\end{aligned}\tag{4.33}$$

In the third equality, we integrated by parts with respect to x taking into account the zero boundary condition in (4.1). Substituting (4.33) and (4.32) into (4.31), we find

$$\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2\leqslant\frac{2}{\nu\lambda_1}|g_0(t)|^2+2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2dx.$$

By the Poincaré inequality,

$$\frac{d}{dt}|u(t)|^2+\nu\lambda_1|u(t)|^2\leqslant h(t),\tag{4.34}$$

where

$$h(t)=\frac{2}{\nu\lambda_1}|g_0(t)|^2+2\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2dx.$$

By assumption,

$$\int_t^{t+1} |g_0(t)|^2 ds \leqslant \|g_0(\cdot)\|^2_{L_2^b(\mathbb{R};L_2(\Omega)^2)} = M_0 \quad \forall t \in \mathbb{R}. \tag{4.35}$$

By Lemma 4.2,

$$\int_t^{t+1} \int_\Omega \Big|G_j\Big(\frac{x}{\varepsilon}, t\Big)\Big|^2 dx \leqslant C\|G_j(\cdot)\|_{L_2^b(\mathbb{R};Z)} = M_j \quad \forall t \in \mathbb{R},\ j = 1, 2, \tag{4.36}$$

where C is independent of ε.

Using Lemma 4.1 with $y(t) = |u(t+\tau)|^2$, $\gamma = \nu\lambda_1$, $M = 2(\nu\lambda_1)^{-1}M_0 + 2\varepsilon^{2(1-\rho)}\nu^{-1}(M_1 + M_2)$, we obtain the following main estimate for the function $u(t)$:

$$\begin{aligned} |u(t+\tau)|^2 \leqslant |u(\tau)|^2 e^{-\nu\lambda_1 t} + [2(\nu\lambda_1)^{-1}M_0 \\ + 2\varepsilon^{2(1-\rho)}\nu^{-1}(M_1 + M_2)]D_1, \end{aligned} \tag{4.37}$$

where $D_1 = (1 + (\nu\lambda_1)^{-1})$.

Since $0 \leqslant \rho \leqslant 1$ and $0 < \varepsilon \leqslant 1$, the inequality (4.37) implies that the process $\{U_\varepsilon(t,\tau)\}$ corresponding to (4.1) has the uniformly absorbing set

$$\widetilde{B} = \{v \in H \mid |v| \leqslant C_2\}, \tag{4.38}$$

where $C_2^2 = 2[2(\nu\lambda_1)^{-1}M_0 + 2\nu^{-1}(M_1 + M_2)]D_1$. It is clear that the global attractor $\mathcal{A}^\varepsilon$ belongs to any absorbing set, i.e.,

$$\|\mathcal{A}^\varepsilon\|_H \leqslant C_2 \quad \forall\ 0 < \varepsilon \leqslant 1, \tag{4.39}$$

provided that the divergence condition (4.29) is satisfied. □

We now estimate the deviation of the solution of the original 2D Navier–Stokes system (4.2) from the solution of the "limiting" system (4.22). We supplement (4.2) and (4.22) with the same initial data at $t = \tau$:

$$u|_{t=\tau} = u_\tau, \quad u^0|_{t=\tau} = u_\tau, \quad u_\tau \in \widetilde{B}, \tag{4.40}$$

where the absorbing ball $\widetilde{B}$ is defined by formula (4.38). Recall that the set $\widetilde{B}$ is independent of $0 \leqslant \rho \leqslant 1$ and $0 < \varepsilon \leqslant 1$.

Let $u(x,t)$ and $u^0(x,t)$ be solutions of Equations (4.2) and (4.22) respectively with the same initial data (4.40) taken from the ball $\widetilde{B}$. Let us estimate the deviation of $u(x,t)$ from $u^0(x,t)$ for $t \geqslant \tau$. Let $w(x,t) =$

$u(x,t) - u^0(x,t)$. For the sake of simplicity, we set $\tau = 0$. The function $w(x,t)$ satisfies the equation

$$\partial_t w + \nu L w + B(u,u) - B(u^0,u^0) = \frac{1}{\varepsilon^\rho} P g_1\Big(\frac{x}{\varepsilon}, t\Big) \tag{4.41}$$

and zero initial data

$$w|_{t=0} = 0. \tag{4.42}$$

We note that

$$B(u,u) - B(u^0,u^0) = B(w,u^0) + B(u^0,w) + B(w,w).$$

Taking the inner product of Equation (4.41) and w in H, we find

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \langle B(w,u^0), w\rangle$$

$$+ \langle B(u^0,w), w\rangle + \langle B(w,w), w\rangle = \frac{1}{\varepsilon^\rho}\Big\langle g_1\Big(\frac{\cdot}{\varepsilon}, t\Big), w\Big\rangle. \tag{4.43}$$

From (1.13) it follows that $\langle B(u^0,w), w\rangle = 0$ and $\langle B(w,w), w\rangle = 0$. Therefore,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 + \langle B(w,u^0(t)), w\rangle = \frac{1}{\varepsilon^\rho}\Big\langle g_1\Big(\frac{\cdot}{\varepsilon}, t\Big), w\Big\rangle. \tag{4.44}$$

Using the divergence condition, similarly to (4.33) we find

$$\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{\cdot}{\varepsilon}, t\Big), w\Big\rangle = -\varepsilon^{1-\rho}\sum_{j=1}^{2}\int_\Omega \Big(G_j\Big(\frac{x}{\varepsilon}, t\Big), \partial_{x_j} u(x,t)\Big)dx$$

$$\leqslant \frac{1}{2}\varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \Big|G_j\Big(\frac{x}{\varepsilon}, t\Big)\Big|^2 dx + \frac{1}{2}\nu\|u(t)\|^2. \tag{4.45}$$

From (1.13) and (1.14) it follows that

$$|\langle B(w,u^0), w\rangle| = |\langle B(w,w), u^0\rangle| \leqslant c_0^2|w|\|w\|\|u^0\|. \tag{4.46}$$

Then

$$|\langle B(w,u^0), w\rangle| \leqslant c_0^2|w|\|u^0\|\|w\| \leqslant \frac{1}{2}\nu\|w\|^2 + \frac{1}{2}\frac{c_0^4}{\nu}|w|^2\|u^0\|^2. \tag{4.47}$$

Combining (4.45) and (4.47) in (4.44), we find

$$\frac{d}{dt}|w(t)|^2 \leqslant \frac{c_0^4}{\nu}|w(t)|^2\|u^0(t)\|^2 + \varepsilon^{2(1-\rho)}\nu^{-1}\sum_{j=1}^{2}\int_\Omega \Big|G_j\Big(\frac{x}{\varepsilon}, t\Big)\Big|^2 dx.$$

We set

$$z(t) = |w(t)|^2, \quad \gamma(t) = c_0^4 \nu^{-1} \|u^0(t)\|^2,$$
$$b(t) = \varepsilon^{2(1-\rho)} \nu^{-1} \sum_{j=1}^{2} \int_\Omega \left| G_j\left(\frac{x}{\varepsilon}, t\right)\right|^2 dx.$$

Then we obtain the differential inequality

$$z'(t) \leqslant b(t) + \gamma(t) z(t), \quad z(0) = 0. \tag{4.48}$$

Using the Gronwall lemma, we find

$$z(t) \leqslant \int_0^t b(s) \exp\Big(\int_s^t \gamma(\theta) d\theta\Big) ds \leqslant \Big(\int_0^t b(s) ds\Big) \exp\Big(\int_0^t \gamma(s) ds\Big). \tag{4.49}$$

Recall that $u^0(t)$ satisfies (4.24) and $u_0 \in \widetilde{B}$, i.e.,

$$\begin{aligned}\int_0^t \gamma(s) ds = c_0^4 \nu^{-1} \int_0^t &\|u^0(t)\|^2 ds\\
&\leqslant c_0^2 \nu^{-2} (|u_0|^2 + \Big(\nu\lambda_1)^{-1} \int_0^t |g_0(s)|^2 ds\Big)\\
&\leqslant c_0^4 \nu^{-2} (C_2^2 + (\nu\lambda_1)^{-1} (t+1) \|g_0(\cdot)\|^2_{L_2^b(\mathbb{R}; L_2(\Omega)^2)})\\
&\leqslant C_3(t+1).\end{aligned} \tag{4.50}$$

By (4.36),

$$\begin{aligned}\int_0^t b(s) ds &= \varepsilon^{2(1-\rho)} \nu^{-1} \sum_{j=1}^{2} \int_0^t \int_\Omega \left| G_j\left(\frac{x}{\varepsilon}, s\right)\right|^2 dx ds\\
&\leqslant \varepsilon^{2(1-\rho)} \nu^{-1} C(t+1) \sum_{j=1}^{2} \|G_j(\cdot)\|_{L_2^b}\\
&\leqslant \varepsilon^{2(1-\rho)} \nu^{-1} (t+1)(M_1' + M_2').\end{aligned} \tag{4.51}$$

Replacing (4.50) and (4.51) with (4.49), we find

$$\begin{aligned}|w(t)|^2 &\leqslant \varepsilon^{2(1-\rho)} \nu^{-1} (t+1)(M_1' + M_2') e^{C_3(t+1)}\\
&= \varepsilon^{2(1-\rho)} \nu^{-1} (M_1' + M_2') \varepsilon^t e^{C_3(t+1)} = \varepsilon^{2(1-\rho)} C_4^2 e^{2rt},\end{aligned} \tag{4.52}$$

where $C_4^2 = \nu^{-1}(M_1' + M_2')e^{C_3}$, $2r = C_3 + 1$. The constants C_4 and r are independent of ε. The inequality (4.52) holds for all $0 \leqslant \rho \leqslant 1$. Thus, we proved the following assertion.

Theorem 4.2. *Let $g_1(z,t)$ satisfy the divergence condition* (4.29). *Then for every initial data $u_\tau \in \widetilde{B}$ (see* (4.38)*) the difference $w(x,t) = u(x,t) - u^0(x,t)$ of the solutions of the Navier–Stokes equations* (4.2) *and* (4.22) *respectively with the initial data* (4.40) *taken from the ball $\widetilde{B}$ satisfies the inequality*

$$|w(t)| = |u(t) - u^0(t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)} \quad \forall \varepsilon, 0 < \varepsilon \leqslant 1, \tag{4.53}$$

where the constants C_4 and r are independent of ε, $u_\tau \in \widetilde{B}$, $0 \leqslant \rho \leqslant 1$.

In Section 4.4 below, using Theorems 4.1 and 4.2, we prove that the global attractors $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ in the norm of H as $\varepsilon \to 0+$.

4.3. On the structure of global attractors $\mathcal{A}^\varepsilon$.

We start by considering translation compact functions with values in the spaces $L_2(\Omega)^2$ and Z. The definition of a translation compact function in $\Xi = L_p^{\mathrm{loc}}(\mathbb{R}; E)$ with values in a Banach space E is given in Section 2.4 (see Example 2.2). Below, we consider translation compact functions in the case, where $\Xi = L_p^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $\Xi = L_p^{\mathrm{loc}}(\mathbb{R}; Z)$.

Consider vector-valued functions $g_0(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$, and $g_1(z,t)$, $z \in \mathbb{R}^2$, $t \in \mathbb{R}$, that appear on the right-hand side of the 2D Navier–Stokes system. We assume that $g_0(x,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^{\mathrm{loc}}(\mathbb{R}; Z)$.

Proposition 4.1. *If $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, then for every fixed $0 < \varepsilon \leqslant 1$ the $g_1(x/\varepsilon, t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$, $\Omega \Subset \mathbb{R}^2$.*

Proof. We need to establish that the set $\{g_1(x/\varepsilon, t+h) \mid h \in \mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Let $\{h_n,\ n = 1, 2, \ldots\}$ be an arbitrary sequence of real numbers. Since $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, there is a subsequence $\{h_{n'}\} \subset \{h_n\}$ such that $g_1(z, t + h_{n'})$ converge to a function $\widehat{g}_1(z,t)$ as $n' \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$, i.e., for every interval $[t_1, t_2] \subset \mathbb{R}$

$$\int_{t_1}^{t_2} \|g_1(\cdot, s + h_{n'}) - \widehat{g}_1(\cdot, s)\|_Z^2 ds \to 0 \quad \text{as } n' \to \infty.$$

Using the inequality (4.14), we conclude that

$$\int_{t_1}^{t_2} \Big\| g_1\Big(\frac{\cdot}{\varepsilon}, s+h_{n'}\Big) - \widehat{g}_1\Big(\frac{\cdot}{\varepsilon}, s\Big)\Big\|^2_{L_2(\Omega)^2} ds$$

$$\leqslant C^2 \int_{t_1}^{t_2} \|g_1(\cdot, s+h_{n'}) - \widehat{g}_1(\cdot, s)\|^2_Z ds,$$

i.e., $g_1(x/\varepsilon, t+h_{n'})$ converge to $\widehat{g}_1(x/\varepsilon, t)$ as $n' \to \infty$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Thus, $\{g_1(x/\varepsilon, t+h) \mid h \in \mathbb{R}\}$ is precompact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. $\square$

Proposition 4.2. *Suppose that $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. Consider the function*

$$g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$$

as an element of the space $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$. Then g^ε is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and the hull $\mathcal{H}(g^\varepsilon(x,t))$ (in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$) consists of (translation compact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$) functions $\widehat{g}^\varepsilon(x,t)$ of the form

$$\widehat{g}^\varepsilon(x,t) = \widehat{g}_0(x,t) + \varepsilon^{-\rho} \widehat{g}_1(x/\varepsilon, t)$$

with some $\widehat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$, $\widehat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of $g_0(x,t)$ and $g_1(z,t)$ respectively.

Proof. By Proposition 4.1, for fixed $\varepsilon \in (0,1]$ the function $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is translation compact in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ (as the sum of two translation compact functions). Let $\widehat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$, i.e., there is a sequence $\{h_n\}$ such that $g^\varepsilon(x, t+h_n) = g_0(x, t+h_n) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_n) \to \widehat{g}^\varepsilon(x,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ as $n \to \infty$. Since $g_0(x,t)$ and $g_1(z,t)$ are translation compact functions in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$ respectively, we can assume, passing to a subsequence $\{h_{n'}\} \subset \{h_n\}$ if necesary, that $g_0(x, t+h_{n'}) \to \widehat{g}_0(x,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z, t+h_{n'}) \to \widehat{g}_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$ as $n' \to \infty$. Therefore, $g^\varepsilon(x, t+h_{n'}) = g_0(x, t+h_{n'}) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_{n'}) \to \widehat{g}_0(x,t) + \varepsilon^{-\rho} \widehat{g}_1(x/\varepsilon, t)$ in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ as $n' \to \infty$. Therefore,

$$\begin{aligned}
\widehat{g}^\varepsilon(x,t) &= \lim_{n\to\infty} [g_0(x, t+h_n) + \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_n)] \\
&= \lim_{n'\to\infty} g_0(x, t+h_{n'}) + \lim_{n'\to\infty} \varepsilon^{-\rho} g_1(x/\varepsilon, t+h_{n'}) \\
&= \widehat{g}_0(x,t) + \varepsilon^{-\rho} \widehat{g}_1(x/\varepsilon, t).
\end{aligned}$$

Thus, every function $\widehat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ has the form

$$\widehat{g}^\varepsilon(x,t) = \widehat{g}_0(x,t) + \varepsilon^{-\rho}\widehat{g}_1(x/\varepsilon,t)$$

for some $\widehat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\widehat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$. □

Consider Equation (4.2)

$$\partial_t u + \nu L u + B(u,u) = g^\varepsilon(x,t), \tag{4.54}$$

where $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho}Pg_1(x/\varepsilon,t)$ and ε is fixed. Assume that $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};Z)$. In particular, $g_0(x,t) \in L_2^b(\mathbb{R};L_2(\Omega)^2)$ and $g_1(z,t) \in L_2^b(\mathbb{R};Z)$.

Let $\mathcal{H}(g^\varepsilon)$ be the hull of the function $g^\varepsilon(x,t)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$:

$$\mathcal{H}(g^\varepsilon) = [\{g^\varepsilon(\cdot,t+h) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R};H)}. \tag{4.55}$$

Recall that $\mathcal{H}(g^\varepsilon)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R};H)$ and, by Proposition 4.2, each element $\widehat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ can be written in the form

$$\widehat{g}^\varepsilon(x,t) = P\widehat{g}_0(x,t) + \varepsilon^{-\rho}P\widehat{g}_1(x/\varepsilon,t) \tag{4.56}$$

with some functions $\widehat{g}_0(x,t) \in \mathcal{H}(g_0(x,t))$ and $\widehat{g}_1(z,t) \in \mathcal{H}(g_1(z,t))$, where $\mathcal{H}(g_0(x,t))$ and $\mathcal{H}(g_1(z,t))$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R};Z)$ respectively.

We note that

$$\|\widehat{g}_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} \leqslant \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} \quad \forall \widehat{g}_0 \in \mathcal{H}(g_0),$$
$$\|\widehat{g}_1\|_{L_2^b(\mathbb{R};Z)} \leqslant \|g_1\|_{L_2^b(\mathbb{R};Z)} \quad \forall \widehat{g}_1 \in \mathcal{H}(g_1).$$

By Corollary 4.1,

$$\|\widehat{g}^\varepsilon\|_{L_2^b(\mathbb{R};H)} \leqslant \|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)} + \frac{C}{\varepsilon^\rho}\|g_1\|_{L_2^b(\mathbb{R};Z)} \ \forall\, g^\varepsilon \in \mathcal{H}(g^\varepsilon), \tag{4.57}$$

where the constant C is independent of g_0, g_1, ρ, and ε (see (4.14) and (4.15)).

It was shown in Section 4.1 that the process $\{U_\varepsilon(t,\tau)\} := \{U_{g^\varepsilon}(t,\tau)\}$ corresponding to Equation (4.54) has the uniform global attractor $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$, (see (4.18) and (4.19)) and

$$\|\mathcal{A}^\varepsilon\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}), \tag{4.58}$$

where the constants C_0 and C_1 depend on $\|g_0\|_{L_2^b(\mathbb{R};L_2(\Omega)^2)}$ and $\|g_1\|_{L_2^b(\mathbb{R};Z)}$ respectively.

Now, we describe the structure of the attractor $\mathcal{A}^\varepsilon$. Along with Equation (4.54), we consider the family of equations

$$\partial_t \widehat{u} + \nu L\widehat{u} + B(\widehat{u}, \widehat{u}) = \widehat{g}^\varepsilon(x,t) \tag{4.59}$$

with external forces $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. It is clear that for every $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ Equation (4.59) generates the process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$ acting in H. We note that the processes $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$ possess properties similar to the properties of the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to the 2D Navier–Stokes system (4.54) with original external force $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho}Pg_1(x/\varepsilon, t)$. In particular, the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing for every process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}, \widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ (see (4.57)). Moreover, every process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$ has a uniform global attractor $\mathcal{A}_{\widehat{g}^\varepsilon}$ which belongs to the global attractor $\mathcal{A}^\varepsilon = \mathcal{A}_{g^\varepsilon}$ of the 2D Navier–Stokes system (4.54) with initial external force $g^\varepsilon(x,t)$, $\mathcal{A}_{\widehat{g}^\varepsilon} \subseteq \mathcal{A}_{g^\varepsilon}$, where the inclusion can be strict (see Proposition 2.3).

Proposition 4.3. *Suppose that $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$. Then for any fixed $0 < \varepsilon \leqslant 1$ the family of processes $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$, $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, corresponding to Equation (4.59) has an absorbing set $B_{1,\varepsilon}$ which is bounded in H and V and satisfies the inequality*

$$\|B_{1,\varepsilon}\|_H \leqslant (C_0 + C_1\varepsilon^{-\rho}). \tag{4.60}$$

The family $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$, $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(H \times \mathcal{H}(g^\varepsilon); H)$-continuous, i.e.,

$$\begin{aligned} &\widehat{g}^\varepsilon_n \to \widehat{g}^\varepsilon \quad \text{in } L_2^{\mathrm{loc}}(\mathbb{R}; H) \text{ as } n \to \infty, \\ &u_{\tau n} \to u_\tau \quad \text{in } H \text{ as } n \to \infty, \end{aligned} \tag{4.61}$$

implies

$$U_{\widehat{g}^\varepsilon_n}(t,\tau)u_{\tau n} \to U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau \quad \text{in } H \text{ as } n \to \infty. \tag{4.62}$$

The proof is similar to that of the corresponding assertions in [**34**] in the case of a nonoscillating translation compact external force in $L_2^{\mathrm{loc}}(\mathbb{R}; H)$).

We denote by $\mathcal{K}_{\widehat{g}^\varepsilon}$ the *kernel* of Equation (4.59) (and of the process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$) with external force $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. Recall that the kernel $\mathcal{K}_{\widehat{g}^\varepsilon}$ is the family of all complete solutions $\widehat{u}(t)$, $t \in \mathbb{R}$, of (4.59) which are bounded in the norm of H :

$$|\widehat{u}(t)| \leqslant M_{\widehat{u}} \quad \forall t \in \mathbb{R}. \tag{4.63}$$

The set $\mathcal{K}_{\widehat{g}^\varepsilon}(s) = \{\widehat{u}(s) \mid \widehat{u} \in \mathcal{K}_{\widehat{g}^\varepsilon}\}$, $s \in \mathbb{R}$, in H is called the *kernel section* at time $t = s$.

We formulate the theorem (see the proof in [**34**]) about the structure of the uniform global attractor $\mathcal{A}^\varepsilon$ of the 2D Navier-Stokes system (4.54) (see also (2.44)).

Theorem 4.3. *If $g^\varepsilon(x,t)$ is a translation compact function in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$, then the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to Equation* (4.59) *has the uniform global attractor $\mathcal{A}^\varepsilon$ and the following equality holds*:

$$\mathcal{A}^\varepsilon = \bigcup_{\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\widehat{g}^\varepsilon}(0). \tag{4.64}$$

Moreover, the kernel $\mathcal{K}_{\widehat{g}^\varepsilon}$ is nonempty for all $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

We note that the attractor $\mathcal{A}^\varepsilon$ is given by the formula

$$\mathcal{A}^\varepsilon = \omega(B_0) = \bigcap_{h\geqslant 0} \Big[\bigcup_{t-\tau\geqslant h} U_{g^\varepsilon}(t,\tau)B_0 \Big]_H,$$

which means that for constructing the attractor $\mathcal{A}^\varepsilon$ of the entire family of processes $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$, $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, it is possible to use only the process $\{U_{g^\varepsilon}(t,\tau)\}$ of the original equation (4.54) with external force

$$g^\varepsilon = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon, t).$$

All the above-mentioned results remain valid for the "limiting" 2D Navier–Stokes system (4.22)

$$\partial_t u + \nu L u + B(u,u) = g^0(x,t) \tag{4.65}$$

with translation compact external force $g^0(t) := Pg_0(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R};H)$. Equation (4.65) generates the "limiting" process $\{U_0(t,\tau)\} = \{U_{g^0}(t,\tau)\}$ which has the uniform global attractor $\mathcal{A}^0$ (see Section 4.1).

Consider the family of equations

$$\partial_t \widehat{u} + \nu L \widehat{u} + B(\widehat{u},\widehat{u}) = \widehat{g}^0(x,t) \tag{4.66}$$

with external forces $\widehat{g}^0 \in \mathcal{H}(g^0)$ (the hull $\mathcal{H}(g^0)$ is taken in the space $L_2^{\mathrm{loc}}(\mathbb{R};H)$) and the corresponding family of processes $\{U_{\widehat{g}^0}(t,\tau)\}$, $\widehat{g}^0 \in \mathcal{H}(g^0)$.

We can directly apply Proposition 4.3 and Theorem 4.3 to (4.65) and (4.66) by setting $g_1(z,t) \equiv 0$. Therefore, the family $\{U_{\widehat{g}^0}(t,\tau)\}$, $\widehat{g}^0 \in \mathcal{H}(g^0)$, has a uniformly absorbing set $B_{1,0}$ (bounded in V),

$$\|B_{1,0}\|_H \leqslant C_0, \tag{4.67}$$

and the family $\{U_{\widehat{g}^0}(t,\tau)\}$, $\widehat{g}^0 \in \mathcal{H}(g^0)$, is $(H \times \mathcal{H}(g^0); H)$-continuous. Moreover, the attractor $\mathcal{A}^0$ of the "limiting" equation (4.65) has the form

$$\mathcal{A}^0 = \bigcup_{\widehat{g}^0 \in \mathcal{H}(g^0)} \mathcal{K}_{\widehat{g}^0}(0), \tag{4.68}$$

where $\mathcal{K}_{\widehat{g}^0}$ is the kernel of Equation (4.66) with external force $\widehat{g}^0 \in \mathcal{H}(g^0)$.

Formulas (4.64) and (4.68) will be used in the following section, where we study the strong convergence of $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ as $\varepsilon \to 0+$.

4.4. Convergence of the global attractors $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$.

Consider Equations (4.54) and (4.65), where $g_0(x,t)$ and $g_1(z,t)$ are translation compact functions in $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ and $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$ respectively. Assume that the function $g_1(z,t)$ satisfies the divergence condition (4.29). Then, by Theorem 4.1, the uniform global attractors $\mathcal{A}^\varepsilon$ of Equations (4.54) with external forces $g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon t)$ are uniformly bounded in H with respect to ε :

$$\|\mathcal{A}^\varepsilon\|_H \leqslant C_2 \quad \forall\, 0 < \varepsilon \leqslant 1, \tag{4.69}$$

where the constant C_2 is independent of ε. We also consider the global attractor $\mathcal{A}^0$ of the "limiting" equation (4.65) with external force $g^0(t) = Pg_0(\cdot, t)$. It is clear that the set $\mathcal{A}^0$ is bounded in H (see (4.67)).

We need a generalization of Theorem 4.2 which can be applied to the solutions of the entire families of equations (4.59) and (4.66).

We choose an arbitrary element $u_\tau \in \widetilde{B}$. Let $\widehat{u}(\cdot,t) = U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau$, $t \geqslant \tau$, be the solution of (4.59) with external force $\widehat{g}^\varepsilon = P\widehat{g}_0 + \varepsilon^{-\rho} P\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$, and let $\widetilde{u}^0(\cdot,t) = U_{\widetilde{g}^0}(t,\tau)u_\tau$, $t \geqslant \tau$, be the solution of (4.66) with external force $\widetilde{g}^0 \in \mathcal{H}(g^0)$. We assume that the initial data at $t = \tau$ for both solutions are the same: $\widehat{u}(\cdot,\tau) = \widetilde{u}^0(\cdot,\tau) = u_0$, $u_0 \in \widetilde{B}$, where the absorbing ball $\widetilde{B}$ is defined in (4.38). (Note that $\widetilde{g}^0$ can be different from the term $\widehat{g}^0 = P\widehat{g}_0$, the first summand in the representation $\widehat{g}^\varepsilon = P\widehat{g}_0 + \varepsilon^{-\rho} P\widehat{g}_1$.) Consider the difference

$$\widehat{w}(x,t) = \widehat{u}(x,t) - \widetilde{u}^0(x,t), \quad t \geqslant \tau.$$

Proposition 4.4. *Let the original functions* $g_0(x,t)$ *and* $g_1(z,t)$ *in* (4.1) *be translation compact functions in* $L_2^{\mathrm{loc}}(\mathbb{R}; L_2(\Omega)^2)$ *and* $L_2^{\mathrm{loc}}(\mathbb{R}; Z)$ *respectively. Let* $g_1(z,t)$ *satisfy the divergence condition* (4.29)*. Let*

$$g^\varepsilon(x,t) = Pg_0(x,t) + \varepsilon^{-\rho} Pg_1(x/\varepsilon, t), \quad g^0(x,t) = Pg_0(x,t).$$

Then for every external force $\widehat{g}^\varepsilon = P\widehat{g}_0 + \varepsilon^{-\rho} P\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$ *there exists an external force* $\widetilde{g}^0 \in \mathcal{H}(g^0)$ *such that for every initial data* $u_\tau \in \widetilde{B}$ *(see* (4.38)*) the difference*

$$\widehat{w}(t) = \widehat{u}(t) - \widetilde{u}^0(t) = U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau - U_{\widetilde{g}^0}(t,\tau)u_\tau$$

of the solutions of the 2*D Navier–Stokes systems* (4.59) *and* (4.66) *with external forces* $\widehat{g}^\varepsilon(x,t) = P\widehat{g}_0(x,t) + \varepsilon^{-\rho} P\widehat{g}_1(x/\varepsilon, t)$ *and* $\widetilde{g}^0(x,t)$ *respectively* *and with the same initial data u_τ satisfies the inequality*

$$|\widehat{w}(t)| = |\widehat{u}(t) - \widetilde{u}^0(t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)} \quad \forall\, 0 < \varepsilon \leqslant 1, \tag{4.70}$$

where the constant C_4 and r are the same as in Theorem 4.2 *and are independent of ε and $0 \leqslant \rho \leqslant 1$.*

Proof. Consider the functions

$$u(t) = U_{g^\varepsilon}(t,\tau)u_\tau, \quad u^0(t) = U_{g^0}(t,\tau)u_\tau \ \forall t \geqslant \tau, \tag{4.71}$$

where $g^\varepsilon(t) = Pg_0(t) + \varepsilon^{-\rho} Pg_1(t)$ and $g^0(t) = Pg_0(t)$ are the original external forces. Using (4.71), we write the inequality (4.53) in the form

$$|U_{g^\varepsilon}(t,\tau)u_\tau - U_{g^0}(t,\tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}. \tag{4.72}$$

By Theorem 4.2, the inequality (4.72) holds for all $u_\tau \in \widetilde{B}$. We claim that (4.72) also holds for the time-shifted external forces

$$g_h^\varepsilon(t) = g^\varepsilon(t+h) = Pg_0(t+h) + \varepsilon^{-\rho} Pg_1(t+h),$$
$$g_h^0(t) = g^0(t+h) = Pg_0(t+h)$$

with arbitrary $h \in \mathbb{R}$, i.e.,

$$|U_{g_h^\varepsilon}(t,\tau)u_\tau - U_{g_h^0}(t,\tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}, \tag{4.73}$$

where the constants C_4 and r are independent of h. Indeed, for every $h \in \mathbb{R}$ the time-shifted function $g_{1h}(z,t) = g_1(z,t+h)$ apparently satisfies the divergence condition (4.29) for the time-shifted functions $G_j^h(z,t) = G_j(z,t+h) \in L_2^b(\mathbb{R}; Z)$, $j = 1, 2$. Thus, (4.73) directly follows from Theorem 4.2.

We recall that the family of processes $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$, $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, is $(H \times \mathcal{H}(g^\varepsilon); H)$-continuous. In particular (see (4.61) and (4.62)), for fixed $u_\tau \in \widetilde{B}$

$$\widehat{g}_n^\varepsilon \to \widehat{g}^\varepsilon \quad \text{in } L_2^{\text{loc}}(\mathbb{R}; H) \text{ as } n \to \infty$$

implies

$$U_{\widehat{g}_n^\varepsilon}(t,\tau)u_\tau \to U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau \quad \text{in } H \text{ as } n \to \infty \tag{4.74}$$

and, similarly,

$$U_{\widehat{g}_n^0}(t,\tau)u_\tau \to U_{\widetilde{g}^0}(t,\tau)u_\tau \quad \text{in } H \text{ as } n \to \infty \tag{4.75}$$

if $\widehat{g}_n^0 \to \widetilde{g}^0$ as $n \to \infty$ in $L_2^{\text{loc}}(\mathbb{R}; H)$ for some $\widetilde{g}^0 \in \mathcal{H}(g^0)$.

We now fix an external force $\widehat{g}^\varepsilon = P\widehat{g}_0 + \varepsilon^{-\rho} P\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$. Since $\widehat{g}^\varepsilon(t)$ is a translation compact function in $L_2^{\text{loc}}(\mathbb{R}; H)$, there exists a sequence $\{h_i\} \subset \mathbb{R}$ such that

$$g_{h_i}^\varepsilon \to \widehat{g}^\varepsilon \quad \text{in } L_2^{\text{loc}}(\mathbb{R}; H) \text{ as } n \to \infty, \tag{4.76}$$

where $g^\varepsilon_{h_i}(t) = g^\varepsilon(t+h_i)$. Consider a sequence of external forces $g^0_{h_i} = g^0(t+h_i)$. Since $g^0(t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};H)$, there exists a function $\widetilde{g}^0 \in \mathcal{H}(g^0)$ such that

$$g^0_{h_i} \to \widetilde{g}^0 \quad \text{in } L_2^{\mathrm{loc}}(\mathbb{R};H) \text{ as } n \to \infty \tag{4.77}$$

(where we can pass to a subsequence of h_i, if necessary). From (4.73) it follows that

$$|U_{g^\varepsilon_{h_i}}(t,\tau)u_\tau - U_{g^0_{h_i}}(t,\tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)} \quad \forall i \in \mathbb{N}. \tag{4.78}$$

Using (4.76) and (4.77) in (4.74) and (4.75), we pass to the limit in (4.78) as $i \to \infty$ and obtain the required inequality:

$$|U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau - U_{\widetilde{g}^0}(t,\tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)}. \tag{4.79}$$

Thus, the inequality (4.70) is proved. □

We formulate the main result of this section.

Theorem 4.4. *Assume that* $0 \leqslant \rho < 1$. *Let* $g_0(x,t)$ *and* $g_1(z,t)$ *in* (4.1) *be translation compact functions in the* $L_2^{\mathrm{loc}}(\mathbb{R};L_2(\Omega)^2)$ *and* $L_2^{\mathrm{loc}}(\mathbb{R};Z)$ *respectively, and let* $g_1(z,t)$ *satisfy the divergence condition* (4.29). *Then the global attractors* $\mathcal{A}^\varepsilon$ *of Equations* (4.54) *converge to the global attractor* $\mathcal{A}^0$ *of the "limiting" equation* (4.65) *in the norm of* H *as* $\varepsilon \to 0+$, *i.e.*,

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \quad \textit{as } \varepsilon \to 0+. \tag{4.80}$$

Proof. Denote by u^ε an arbitrary element of $\mathcal{A}^\varepsilon$. By (4.64), there exists a bounded complete solution $\widehat{u}^\varepsilon(t)$, $t \in \mathbb{R}$, of Equation (4.59) with some external force $\widehat{g}^\varepsilon = P\widehat{g}_0 + \varepsilon^{-\rho}P\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$, $\widehat{g}_0 \in \mathcal{H}(g_0)$, $\widehat{g}_1 \in \mathcal{H}(g_1)$, such that

$$u^\varepsilon = \widehat{u}^\varepsilon(0). \tag{4.81}$$

Consider the point $\widehat{u}^\varepsilon(-R)$ which clearly belongs to $\mathcal{A}^\varepsilon$ and hence

$$\widehat{u}^\varepsilon(-R) \in \widetilde{B} \tag{4.82}$$

(see (4.38)). Recall that $\widetilde{B}$ is an absorbing set and the global attractor $\mathcal{A}^\varepsilon$ belongs to $\widetilde{B}$. The number R will be chosen later.

For the constructed external force $\widehat{g}^\varepsilon$ we apply Proposition 4.4: there is a "limiting" external force $\widetilde{g}^0 \in \mathcal{H}(g^0)$ such that for any $\tau \in \mathbb{R}$ and $u_\tau \in \widetilde{B}$ the following inequality holds:

$$|U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau - U_{\widetilde{g}^0}(t,\tau)u_\tau| \leqslant \varepsilon^{(1-\rho)} C_4 e^{r(t-\tau)} \quad \forall t \geqslant \tau. \tag{4.83}$$

Consider the "limiting" equation (4.65) with the chosen "limiting" external force $\widetilde{g}^0$. We set $\tau = -R$. Let $\widetilde{u}^0(t)$, $t \geqslant -R$, be the solution of this equation with initial data

$$\widetilde{u}^0|_{t=-R} = \widehat{u}^\varepsilon(-R). \tag{4.84}$$

Taking $-R$ in place of τ and $-R+t$ in place of t, from (4.83) (see also (4.82)) we find

$$|\widehat{u}^\varepsilon(-R+t) - \widetilde{u}^0(-R+t)| \leqslant \varepsilon^{(1-\rho)} C_4 e^{rt} \quad \forall t \geqslant 0, \tag{4.85}$$

where $\widehat{u}^\varepsilon(-R+t) = U_{\widehat{g}^\varepsilon}(-R+t,-R)\widehat{u}^\varepsilon(-R)$ and $\widetilde{u}^0(-R+t) = U_{\widetilde{g}^0}(-R+t,-R)\widehat{u}^\varepsilon(-R)$.

The set $\mathcal{A}^0$ attracts $U_{\widehat{g}^0}(t+\tau,\tau)\widetilde{B}$ in H as $t \to +\infty$ (uniformly with respect to $\tau \in \mathbb{R}$ and $\widehat{g}^0 \in \mathcal{H}(g^0)$, see [**34**]). Therefore, for any $\delta > 0$ there exists a number $T = T(\delta)$ such that

$$\operatorname{dist}_H(U_{\widehat{g}^0}(t+\tau,\tau)\widetilde{B}, \mathcal{A}^0) \leqslant \frac{\delta}{2} \quad \forall \tau \in \mathbb{R}, \widehat{g}^0 \in \mathcal{H}(g^0), t \geqslant T(\delta).$$

Hence for $\tau = -R$ and $\widehat{u}^\varepsilon(-R) \in \widetilde{B}$

$$\operatorname{dist}_H(U_{\widehat{g}^0}(-R+t,-R)\widehat{u}^\varepsilon(-R), \mathcal{A}^0) \leqslant \frac{\delta}{2} \quad \forall \widehat{g}^0 \in \mathcal{H}(g^0), t \geqslant T(\delta).$$

In particular, for $\widetilde{g}^0$ specified above we have

$$\begin{aligned} &\operatorname{dist}_H(\widetilde{u}^0(-R+t), \mathcal{A}^0) = \operatorname{dist}_H(U_{\widetilde{g}^0}(-R+t,-R)\widehat{u}^\varepsilon(-R), \mathcal{A}^0) \\ &\leqslant \delta/2 \quad \forall t \geqslant T(\delta). \end{aligned} \tag{4.86}$$

Recall that $T(\delta)$ is independent of $u^\varepsilon \in \mathcal{A}^\varepsilon$.

From (4.86) and (4.85) it follows that

$$\begin{aligned} &\operatorname{dist}_H(\widehat{u}^\varepsilon(-R+t), \mathcal{A}^0) \\ &\leqslant |\widehat{u}^\varepsilon(-R+t) - \widetilde{u}^0(-R+t)| + \operatorname{dist}_H(\widetilde{u}^0(-R+t), \mathcal{A}^0) \\ &\leqslant \varepsilon^{(1-\rho)} C_4 e^{rt} + \frac{\delta}{2} \quad \forall t \geqslant T(\delta). \end{aligned} \tag{4.87}$$

We set $t = R = T(\delta)$ in (4.87). Since $\widehat{u}^\varepsilon(0) = u^\varepsilon$, we have

$$\operatorname{dist}_H(u^\varepsilon, \mathcal{A}^0) = \operatorname{dist}_H(\widehat{u}^\varepsilon(0), \mathcal{A}^0) \leqslant \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2} \quad \forall u^\varepsilon \in \mathcal{A}^\varepsilon.$$

Consequently,

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant \varepsilon^{(1-\rho)} C_4 e^{rT(\delta)} + \frac{\delta}{2} \quad \forall\, \delta > 0. \tag{4.88}$$

Finally, for arbitrary $\delta > 0$ we take $\varepsilon_0 = \varepsilon_0(\delta)$ such that

$$\varepsilon_0^{(1-\rho)} C_4 e^{rT(\delta)} = \delta/2.$$

Thus, if $\varepsilon \leqslant \varepsilon_0(\delta) = \left(\frac{\delta}{2C_4 e^{rT(\delta)}}\right)^{\frac{1}{1-\rho}}$, then $\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant \delta$. Therefore, $\operatorname{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0$ as $\varepsilon \to 0+$. □

4.5. Estimate for the distance from $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$.

Consider the 2D Navier–Stokes system (4.54) in the case, where the Grashof number of the corresponding "limiting" Navier–Stokes system (4.65) is small. In this case, the global attractor $\mathcal{A}^0$ is exponential, i.e., $\mathcal{A}^0$ attracts bounded sets of initial data with exponential rate as time tends to infinity. This property allows us to estimate explicitly the distance from $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$.

We consider the "limiting" system (4.65) with external force $g^0(t) := Pg_0(\cdot, t) \in L_2^{\mathrm{loc}}(\mathbb{R}; H)$. Let the Grashof number G of this 2D Navier–Stokes system satisfy the inequality

$$G := \frac{\|g^0\|_{L_2^b}}{\lambda_1 \nu^2} < \frac{1}{c_0^2}, \tag{4.89}$$

where the constant c_0^2 is taken from the inequality (1.14).

Then, by Proposition 2.4, Equation (4.65) has a unique solution $z_{g^0}(t)$, $t \in \mathbb{R}$, bounded in H, i.e., the kernel $\mathcal{K}_{g^0}$ consists of a single trajectory $z_{g^0}(t)$. This solution $z_{g^0}(t)$ is exponentially stable, i.e., for every solution $u_{g^0}(t)$ of Equation (4.65)

$$|u_{g^0}(t+\tau) - z_{g^0}(t+\tau)| \leqslant C_0 |u_\tau - z_{g^0}(\tau)| e^{-\beta t} \quad \forall t \geqslant 0, \tag{4.90}$$

where $u_{g^0}(t+\tau) = U_{g^0}(t+\tau, \tau) u_\tau$ and C_0, β are independent of u_τ, τ.

The property (4.90) implies that the set

$$\mathcal{A}^0 = [\{z_{g^0}(t) \mid t \in \mathbb{R}\}]_H = \bigcup_{g \in \mathcal{H}(g^0)} \{z_g(0)\} \tag{4.91}$$

is the global attractor of Equation (4.65) under the condition (4.89) (see (2.54)).

Remark 4.2. As was shown in [**16**], the inequality (1.14) holds with $c_0^2 = \left(\frac{8}{27\pi}\right)^{1/2} = 0.3071\ldots$. Based on the numerical result from [**134**], it was also shown in [**16**] that $c_0^2 = 0.2924\ldots$. This value is possibly the best one for the inequality (1.14). Thus, (4.90) and (4.91) are valid if $G < 3.42$.

Remark 4.3. The inequality (4.90) implies that the global attractor $\mathcal{A}^0$ of the system (4.65) is exponential under the condition (4.89), i.e., for any bounded set B in H

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_H(U_{g^0}(t+\tau,\tau)B,\mathcal{A}^0) \leqslant C_1(|B|)e^{-\beta t}, \tag{4.92}$$

where C_1 depends on the norm of B in H

The following assertion concerns the distance from $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$.

Theorem 4.5. *Let the assumptions of Theorem* 4.4 *be satisfied. Suppose that the Grashof number G of the "limiting"* 2*D Navier–Stokes system satisfies* (4.89)*. Then the Hausdorff distance* (*in H*) *from the global attractor $\mathcal{A}^\varepsilon$ of the original* 2*D Navier–Stokes system* (4.54) *to the global attractor $\mathcal{A}^0$ of the corresponding "limiting" system* (4.65) *satisfies the inequality*

$$\operatorname{dist}_H(\mathcal{A}^\varepsilon,\mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho} \quad \forall\, 0<\varepsilon\leqslant 1,$$

where $0\leqslant\rho<1$ and $C(\rho)>0$ depends on ν, $\|g_0\|_{L_2^b}$, and $\|g_1\|_{L_2^b}$.

The proof of Theorem 4.5 is similar to that of the corresponding assertion concerning the complex Ginzburg–Landau equation with singularly oscillating terms (see Section 5.4).

Remark 4.4. In this section, we consider nonautonomous 2D Navier–Stokes systems with singularly oscillating external forces and prove results concerning the behavior of their global attractors. Similar assertions holds for other nonautonomous evolution equations in mathematical physics with singularly oscillating terms, for example, for the damped wave equation

$$\partial_t^2 u+\gamma\partial_t u=\Delta u-f(u)+g_0(x,t)+\varepsilon^{-\rho}g_1(x,t/\varepsilon),\ u|_{\partial\Omega}=0,$$

where $\gamma>0$, $0\leqslant\rho\leqslant\rho_0$, $0<\varepsilon\leqslant 1$, $t\in\mathbb{R}$, $x\in\Omega\Subset\mathbb{R}^n$, and $g_0(x,t)$, $g_1(x,t)$ are translation compact functions in the corresponding spaces (see [**130**])).

5. Uniform Global Attractor of Ginzburg–Landau Equation with Singularly Oscillating Terms

In this section, we study the global attractor $\mathcal{A}^\varepsilon$ of the nonautonomous complex Ginzburg–Landau equation with constant dispersion parameters α, β and singularly oscillating external force of the form $g_0(x,t)+\varepsilon^{-\rho}g_1(x/\varepsilon,t)$, $x\in\Omega\Subset\mathbb{R}^n$, $n\geqslant 3$, $0<\rho\leqslant 1$. We assume that $|\beta|\leqslant\sqrt{3}$. In this case, the Cauchy problem for the Ginzburg–Landau equation has a unique solution and the corresponding process $\{U_\varepsilon(t,\tau)\}$ acting in the space $\mathbf{H} = L_2(\Omega;\mathbb{C})$ has the global attractor $\mathcal{A}^\varepsilon$ (see Sections 1.3.3 and 2.6.3). Along with the Ginzburg–Landau equation, we consider the "limiting" equation with external force $g_0(x,t)$. We assume that the function $g_1(z,t)$ admits the divergence presentation

$$g_1(z,t) = \sum_{i=1}^{n} \partial_{z_1} G_i(z,t), \quad z = (z_1,\ldots,z_n) \in \mathbb{R}^n_z,$$

where the norms of $G_i(z,t)$ are bounded in $L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})$, $\mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n_z;\mathbb{C})$ (see Section 5.1).

We estimate the deviation (in $\mathbf{H}$) of the solutions of the original Ginzburg–Landau equation from the solution of the corresponding "limiting" equation with the same initial data.

If $g_1(z,t)$ admits the divergence representation and $g_0(x,t)$ and $g_1(z,t)$ are translation compact functions in the corresponding spaces, we prove that the global attractors $\mathcal{A}^\varepsilon$ converge to the global attractor $\mathcal{A}^0$ of the "limiting" system as $\varepsilon \to 0+$ in the norm of $\mathbf{H}$. We also study the case, where the global attractor $\mathcal{A}^0$ of the "limiting" Ginzburg–Landau equation is exponential. In such a situation, we obtain the following estimate for the deviation of the global attractor $\mathcal{A}^\varepsilon$ from $\mathcal{A}^0$:

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon,\mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho} \quad \forall\, 0 < \varepsilon \leqslant 1,$$

where the constant $C(\rho)$ is independent of ε.

5.1. Ginzburg–Landau equation with singularly oscillating external force.

We consider the nonautonomous Ginzburg–Landau equation

$$\begin{aligned} \partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u + g_0(x,t) \\ + \frac{1}{\varepsilon^\rho} g_1\Big(\frac{x}{\varepsilon}, t\Big), \quad u|_{\partial\Omega} = 0, \end{aligned} \tag{5.1}$$

where $u = u_1(x,t) + iu_2(x,t)$ is the unknown complex function of $x \in \Omega \Subset \mathbb{R}^n$ and $t \in \mathbb{R}$ (see Sections 1.3.3 and 2.6.3). We assume that $0 \in \Omega$ and

$$|\beta| \leqslant \sqrt{3}. \tag{5.2}$$

In (5.1), $0 \leqslant \rho \leqslant 1$ and ε is a small positive parameter. Let $\mathbf{H} = L_2(\Omega;\mathbb{C})$ and $\mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}^n;\mathbb{C})$. The norm in $\mathbf{H}$ is denoted by $\|\cdot\|_{\mathbf{H}}$. A function $f(z)$

belongs to $\mathbf{Z} = L_2^{\mathrm{b}}(\mathbb{R}_z^n; \mathbb{C})$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, if

$$\|f(\cdot)\|_{\mathbf{Z}}^2 = \|f(\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R}_z^n;\mathbb{C})}^2$$

$$:= \sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |f(\zeta_1, \ldots, \zeta_n)|^2 d\zeta_1 \cdots d\zeta_n < +\infty. \tag{5.3}$$

We assume that $g_0(x,t) = g_{01}(x,t) + ig_{02}(x,t)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, belongs to the space $L_2^{\mathrm{b}}(\mathbb{R}; \mathbf{H})$ and $g_1(z,t) = g_{11}(z,t) + ig_{12}(z,t)$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$, belongs to the space $L_2^{\mathrm{b}}(\mathbb{R}; \mathbf{Z})$, i.e., these functions have finite norms

$$\begin{aligned} \|g_0(\cdot,\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}^2 \quad &:= \quad \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_0(\cdot,s)\|_{\mathbf{H}}^2 ds \\ &= \quad \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \Big(\int_\Omega |g_0(x,s)|^2 dx \Big) ds < +\infty, \end{aligned} \tag{5.4}$$

$$\|g_1(\cdot,\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}^2 := \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|g_1(\cdot,s)\|_{\mathbf{Z}}^2 ds \tag{5.5}$$

$$= \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \Big(\sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |g_1(\zeta_1, \ldots, \zeta_n, s)|^2 d\zeta_1 \cdots d\zeta_n \Big) ds < +\infty,$$

where $z = (z_1, z_2, \ldots, z_n)$.

Equation (5.1) is equivalent to the following system of two equations for the real vector-valued function $\mathbf{u} = (u_1, u_2)^\top$:

$$\begin{aligned} \partial_t \mathbf{u} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \Delta \mathbf{u} + R\mathbf{u} - \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix} |\mathbf{u}|^2 \mathbf{u} \\ + \mathbf{g}_0(x,t) + \frac{1}{\varepsilon^\rho} \mathbf{g}_1\Big(\frac{x}{\varepsilon}, t\Big), \end{aligned} \tag{5.6}$$

where $\mathbf{g}_0 = (g_{01}, g_{02})^\top$ and $\mathbf{g}_1 = (g_{11}, g_{12})^\top$.

Under the above assumption, for every fixed $0 < \varepsilon \leqslant 1$ the Cauchy problem for (5.1) with initial data

$$u|_{t=\tau} = u_\tau(x), \quad u_\tau(\cdot) \in \mathbf{H}, \tag{5.7}$$

where τ is arbitrary and fixed, has a unique solution $u(t) := u(x,t)$ such that

$$\begin{aligned} &u(\cdot) \in C(\mathbb{R}_\tau; \mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_\tau; \mathbf{V}) \cap L_4^{\mathrm{loc}}(\mathbb{R}_\tau; \mathbf{L}_4), \\ &\mathbf{V} = H_0^1(\Omega; \mathbb{C}), \quad \mathbf{L}_4 = L_4(\Omega; \mathbb{C}), \quad \mathbb{R}_\tau = [\tau, +\infty). \end{aligned} \tag{5.8}$$

and $u(t)$ satisfies (5.1) in the sense of distributions in the space $\mathcal{D}'(\mathbb{R}_\tau; \mathbf{H}^{-r})$, where $\mathbf{H}^{-r} = H^{-r}(\Omega; \mathbb{C})$ and $r = \max\{1, n/4\}$ (recall that $n = \dim(\Omega)$). In particular,

$$\partial_t u(\cdot) \in L_2(\tau, T; \mathbf{H}^{-1}) + L_{4/3}(\tau, T; \mathbf{L}_{4/3}) \quad \forall T > \tau.$$

The proof of the existence of such a solution $u(t)$ is based on the Galerkin approximation method (see, for example, [**119, 9, 34**]). The proof of the uniqueness uses the inequality (5.2) (see, for example, [**34**]).

We recall that if (5.2) fails, for $n \geqslant 3$ and arbitrary values of the dispersion parameters α and β the uniqueness is not proved yet (see [**101, 102, 136**] for known uniqueness theorems).

We set $\|\cdot\| := \|\cdot\|_{\mathbf{H}}$ for brevity. Any solution $u(t)$, $t \geqslant \tau$, of (5.1) satisfies the differential identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{\mathbf{L}_4}^4 - R\|u(t)\|^2 = \langle g^\varepsilon(t), u(t)\rangle \tag{5.9}$$

for all $t \geqslant \tau$, where $g^\varepsilon(t) := g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$. The function $\|u(t)\|^2$ is absolutely continuous for $t \geqslant \tau$. The proof of (5.9) is similar to that of the corresponding identity for weak solutions of the reaction-diffusion systems considered in [**34, 32**] (see also [**129**]).

Using standard transformations and the Gronwall lemma, from (5.9) we deduce that any solution $u(t)$ of (5.1) satisfies the inequality

$$\|u(t+\tau)\|^2 \leqslant \|u(\tau)\|^2 e^{-2\lambda_1 t} + C_0^2 + C_1^2 \varepsilon^{-2\rho} \quad \forall t \geqslant 0,\ \tau \in \mathbb{R}, \tag{5.10}$$

where λ_1 is the first eigenvalue of the operator $\{-\Delta u,\ u|_{\partial\Omega} = 0\}$, the constant C_0 depends on R and $\|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}$, and the constant C_1 depends on $\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}$ (see (5.4) and (5.5)). We also have the inequality

$$\int_\tau^t \int_\Omega \left| g_1\left(\frac{x}{\varepsilon}, s\right)\right|^2 e^{-\lambda_1(t-s)}\,dxds \leqslant C\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}^2 \quad \forall t \geqslant \tau, \tau \in \mathbb{R}, \tag{5.11}$$

where C is independent of ε. Indeed,

$$\int_\tau^t \int_\Omega \Big| g_1\Big(\frac{x}{\varepsilon}, s\Big)\Big|^2 e^{-\lambda_1(t-s)} dx ds = \int_\tau^t e^{-\lambda_1(t-s)} \Big(\varepsilon^n \int_{\varepsilon^{-1}\Omega} |g_1\Big(z, s)|^2 dz\Big) ds$$

$$\leqslant C' \int_\tau^t e^{-\lambda_1(t-s)} \Big(\sup_{z\in\mathbb{R}^n} \int_{z_1}^{z_1+1} \cdots \int_{z_n}^{z_n+1} |g_1(\zeta_1, \ldots, \zeta_n, s)|^2 d\zeta_1 \cdots d\zeta_n \Big) ds$$

$$\leqslant C''(\lambda_1) \|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}$$

since the domain $\varepsilon^{-1}\Omega$ can be covered by $C'\varepsilon^{-n}$ unit boxes (see the proof of Lemma 4.2). Hence (5.11) is true.

Integrating (5.9) with respect to time from τ to $\tau+t$ and using (5.10), we find (see (5.4) and (5.5))

$$\frac{1}{2}\|u(\tau+t)\|^2 + \int_\tau^{\tau+t} (\|\nabla u(s)\|^2 + \|u(s)\|^4_{\mathbf{L}_4}) ds$$

$$\leqslant \frac{1}{2}\|u(\tau)\|^2 + R \int_\tau^{\tau+t} \|u(s)\|^2 ds + \int_\tau^{\tau+t} \|g^\varepsilon(s)\| \cdot \|u(s)\| ds,$$

$$\int_\tau^t (\|\nabla u(s)\|^2 + \|u(s)\|^4_{\mathbf{L}_4}) ds \tag{5.12}$$

$$\leqslant \frac{1}{2}\|u(\tau)\|^2 + C_2(t+1) + C_3\big(\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + \varepsilon^{-2\rho}\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}\big)t.$$

We consider the process $\{U_\varepsilon(t,\tau)\} := \{U_\varepsilon(t,\tau) \mid t \geqslant \tau, \tau \in \mathbb{R}\}$ corresponding to the problem (5.1), (5.7) and acting in the space $\mathbf{H}$ (see formula (2.118)). By (5.10), the process $\{U_\varepsilon(t,\tau)\}$ has the uniformly absorbing set

$$B_{0,\varepsilon} = \{v \in \mathbf{H} \mid \|v\| \leqslant 2C_0 + C_1\varepsilon^{-\rho})\} \tag{5.13}$$

which is bounded in $\mathbf{H}$ for every fixed $\varepsilon > 0$.

We now prove that the process $\{U_\varepsilon(t,\tau)\}$ has the compact (in $\mathbf{H}$) uniformly absorbing set

$$B_{1,\varepsilon} = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \leqslant C_0' + C_1'\varepsilon^{-\rho}\}. \tag{5.14}$$

For this purpose, we take the inner product of the first equation in (5.1) and $-t\Delta u$ in $\mathbf{H}$. Making standard transformations, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2)-\frac{1}{2}\|\nabla u\|^2+t\|\Delta u\|^2-Rt\|\nabla u\|^2\\&-\langle(1+i\beta)|u|^2u,t\Delta u\rangle=-\langle g_0,t\Delta u\rangle-\varepsilon^{-\rho}\langle g_1(x/\varepsilon),t\Delta u\rangle.\end{aligned}\tag{5.15}$$

Introduce the notation

$$f(\mathbf{v})=|\mathbf{v}|^2\begin{pmatrix}1&-\beta\\\beta&1\end{pmatrix}\mathbf{v},\quad \mathbf{v}=(v_1,v_2).$$

Since $|\beta|\leqslant\sqrt{3}$, the matrix $f'_{\mathbf{v}}(\mathbf{v})$ is positive definite, i.e.,

$$f'_{\mathbf{v}}(\mathbf{v})\mathbf{w}\cdot\mathbf{w}\geqslant 0\quad\forall\mathbf{v}=(v_1,v_2),\ \mathbf{w}=(w_1,w_2),\ t\geqslant 0\tag{5.16}$$

(see (1.34)). Therefore, the term in (5.15) containing β is also positive. Indeed,

$$\begin{aligned}&-\langle(1+i\beta)|u|^2u,t\Delta u\rangle=-\langle f(\mathbf{u}),t\Delta\mathbf{u}\rangle\\&=t\sum_{i=1}^{n}\int\limits_{\Omega}(f'_{\mathbf{u}}(\mathbf{u})\partial_{x_i}\mathbf{u},\partial_{x_i}\mathbf{u})dx\geqslant 0\quad\forall t\geqslant 0.\end{aligned}\tag{5.17}$$

Integrating both sides of the equality (5.15) with respect to t and taking into account (5.17), we find

$$\begin{aligned}&\frac{1}{2}t\|\nabla u(t)\|^2-\frac{1}{2}\int\limits_0^t\|\nabla u(s)\|^2ds+\int\limits_0^t s\|\Delta u(s)\|^2ds-R\int\limits_0^t s\|\nabla u(s)\|^2ds\\&\leqslant-\int\limits_0^t\langle g_0(s),s\Delta u(s)\rangle ds-\varepsilon^{-\rho}\int\limits_0^t\langle g_1(x/\varepsilon,s),s\Delta u(s)\rangle ds.\end{aligned}\tag{5.18}$$

Using (5.12), from (5.18) we obtain the inequality

$$\begin{aligned}&\frac{1}{2}t\|\nabla u(t)\|^2+C_5\int\limits_0^t s\|\Delta u(s)\|^2ds\leqslant R\int\limits_0^t s\|\nabla u(s)\|^2ds\\&+C_6\Big(\int\limits_0^t s\|g_0(s)\|^2ds+\varepsilon^{-2\rho}\int\limits_0^t s\|g_1(x/\varepsilon,s)\|^2ds\Big).\end{aligned}\tag{5.19}$$

Using an inequality similar to (5.11) in (5.19), we find

$$t\|\nabla u(t)\|^2\leqslant C_7(t\|u(0)\|^2+t+1+t\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})}+t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}).$$

Assuming that $u(0) \in B_{0,\varepsilon}$ and setting $t = 1$, we obtain

$$\|\nabla u(1)\| \leqslant C_8(1 + \|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}). \tag{5.20}$$

It is clear that the same inequalities hold if we replace 0 and t with τ and $\tau + t$:

$$t\|\nabla u(\tau+t)\|^2 \leqslant C_7(t\|u(\tau)\|^2 + t + 1 + t\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + t\varepsilon^{-2\rho}\|g_1\|^2_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}),$$

Thus, if $u(\tau) \in B_{0,\varepsilon}$, then

$$\|\nabla u(\tau+1)\| \leqslant C_8(1 + \|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{H})} + \varepsilon^{-\rho}\|g_1\|_{L_2^{\mathrm{b}}(\mathbb{R};\mathbf{Z})}) \quad \forall \tau \geqslant 0. \tag{5.21}$$

By (5.21), the set

$$B_{1,\varepsilon} = \{v \in \mathbf{V} \mid \|v\|_{\mathbf{V}} \leqslant C_8(1 + \|g_0\|_{L_2^{\mathrm{b}}} + \varepsilon^{-\rho}\|g_1\|_{L_2^{\mathrm{b}}})\} \tag{5.22}$$

is uniformly absorbing for the process $\{U_\varepsilon(t,\tau)\}$ corresponding to the Ginzburg–Landau equation (5.1). The set $B_{1,\varepsilon}$ is bounded in $\mathbf{V}$ and compact in $\mathbf{H}$ since the embedding $\mathbf{V} \Subset \mathbf{H}$ is compact. Thus, we have proved the following assertion.

Proposition 5.1. *For any fixed $\varepsilon > 0$ the process $\{U_\varepsilon(t,\tau)\}$ corresponding to Equation* (5.1) *is uniformly compact in the space $\mathbf{H}$ and has the compact uniformly absorbing set $B_{1,\varepsilon}$ defined by formula* (5.22).

Along with the Ginzburg–Landau equation (see (5.1)), we consider the "limiting" equation

$$\partial_t u^0 = (1+i\alpha)\Delta u^0 + Ru^0 - (1+i\beta)|u^0|^2u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0, \tag{5.23}$$

where the coefficients α, β, R and the external force $g_0(x,t)$ are the same as in (5.1). In particular, the conditions (5.2) and (5.4) are satisfied. Therefore, the Cauchy problem for this equation with initial data

$$u^0|_{t=\tau} = u_\tau(x), \quad u_\tau(\cdot) \in \mathbf{H}, \tag{5.24}$$

has a unique solution $u^0(x,t)$ and there exists the corresponding process $\{U_0(t,\tau)\}$ in $\mathbf{H}$: $U_0(t,\tau)u_\tau = u^0(t)$, $t \geqslant \tau \in \mathbb{R}$, where $u^0(t)$, $t \geqslant \tau$, is a solution of (5.23) with initial data $u|_{t=\tau} = u_\tau$. As in the case of (5.10), the main a priory estimate for (5.23) reads

$$\|u^0(\tau+t)\|^2 \leqslant \|u^0(\tau)\|^2 e^{-2\lambda_1 t} + C_0^2. \tag{5.25}$$

Following the above reasoning, we prove that the process $\{U_0(t,\tau)\}$ has the uniformly absorbing set

$$B_{0,0} = \{v \in \mathbf{H} \mid \|v\| \leqslant 2C_0\}. \tag{5.26}$$

(Comparing (5.26) with (5.13), we see that the parameter ε is missing in (5.26) since the term $\varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is missing in Equation (5.23).) Moreover, the process also has the uniformly absorbing set

$$B_{1,0} = \{v \in \mathbf{V} \mid \|\nabla v\|_{\mathbf{V}} \leqslant C_8(1 + \|g_0\|_{L_2^b(\mathbb{R};\mathbf{H})})\} \tag{5.27}$$

which is bounded in $\mathbf{V}$ and is compact in $\mathbf{H}$. Hence the process $\{U_0(t,\tau)\}$ corresponding to the "limiting" equation (5.23) is uniformly compact in $\mathbf{H}$ and Proposition 5.1 holds for the "limit" case $\varepsilon = 0$.

Based on this result, it is easy to see that the processes $\{U_\varepsilon(t,\tau)\}$, $\varepsilon > 0$, and $\{U_0(t,\tau)\}$ have the uniform global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ respectively (see [**34**] and Section 2.6.3) such that

$$\|\mathcal{A}_\varepsilon\|_{\mathbf{H}} \leqslant C_0 + C_1\varepsilon^{-\rho}, \quad \|\mathcal{A}_0\|_{\mathbf{H}} \leqslant C_0.$$

However, the above conditions on $g_1(z,t)$ are not sufficient for establishing the uniform boundedness of $\mathcal{A}_\varepsilon$ in $\mathbf{H}$ with respect to $\varepsilon > 0$.

Now, we formulate a condition providing the uniform boundedness of the global attractors $\mathcal{A}_\varepsilon$, $0 < \varepsilon \leqslant 1$. Assume that $g_1(z,t)$ satisfies the following condition.

- *Divergence condition.* There exist vector-valued functions $G_j(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$, $j = \overline{1,n}$, such that $\partial_{z_j} G_j(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$ and

$$\sum_{j=1}^{n} \partial_{z_j} G_j(z,t) = g_1(z,t) \quad \forall z \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{5.28}$$

Theorem 5.1. *If $g_1(z,t)$ satisfies the divergence condition* (5.28), *then for every $0 \leqslant \rho \leqslant 1$ the global attractors $\mathcal{A}^\varepsilon$ of the Ginzburg–Landau equations are uniformly* (*with respect to $\varepsilon \in]0,1]$*) *bounded in* $\mathbf{H}$, *i.e.*,

$$\|\mathcal{A}^\varepsilon\|_{\mathbf{H}} \leqslant C_2 \quad \forall\ \varepsilon \in]0,1]. \tag{5.29}$$

The proof is similar to that of Theorem 4.1.

5.2. Deviation of solutions of the Ginzburg–Landau equation.

In this section, we consider the equation (see (5.1))

$$\begin{aligned} &\partial_t u = (1 + i\alpha)\Delta u + Ru - (1 + i\beta)|u|^2 u + g_0(x,t) \\ &+ \frac{1}{\varepsilon^\rho} g_1\Big(\frac{x}{\varepsilon}, t\Big), \quad u|_{\partial\Omega} = 0, \end{aligned} \tag{5.30}$$

where the coefficients satisfy the conditions (5.2)–(5.5) and $0 < \rho \leqslant 1$. The corresponding "limiting" equation has the form

$$\partial_t u^0 = (1 + i\alpha)\Delta u^0 + Ru^0 - (1 + i\beta)|u^0|^2 u^0 + g_0(x,t),\ u^0|_{\partial\Omega} = 0. \quad (5.31)$$

The initial data are imposed at $t = \tau$:

$$u|_{t=\tau} = u_\tau(x), \quad u^0|_{t=\tau} = u_\tau(x), \quad u_\tau(\cdot) \in \mathbf{H}. \quad (5.32)$$

Suppose that $u(x,t)$, $t \geqslant \tau$, and $u^0(x,t)$, $t \geqslant \tau$, are solutions of the problems (5.30), (5.32) and (5.31), (5.32) respectively. We set $w(x,t) = u(x,t) - u^0(x,t)$. The function $w(t) := w(\cdot,t)$ satisfies the equations

$$\begin{aligned} &\partial_t w = (1 + i\alpha)\Delta w + Rw - (1 + i\beta)(|u|^2 u - |u^0|^2 u^0) \\ &+ \frac{1}{\varepsilon^\rho} g_1\Big(\frac{x}{\varepsilon}, t\Big), \quad w|_{\partial\Omega} = 0 \end{aligned} \quad (5.33)$$

with initial data $w(\tau) = 0$.

Theorem 5.2. *Under the divergence condition* (5.28)*, the difference* $w(t) = u(\cdot,t) - u^0(\cdot,t)$ *of the solutions* $u(x,t)$ *and* $u^0(x,t)$ *of the problems* (5.30) *and* (5.31) *respectively with the same initial data* (5.32) *satisfies the inequality*

$$\|w(t)\| = \|u(\cdot,t) - u^0(\cdot,t)\| \leqslant C\varepsilon^{(1-\rho)} e^{r(t-\tau)} \quad \forall t \geqslant \tau, \quad (5.34)$$

where

$$r = \begin{cases} 0, & R < \lambda_1, \\ R - \lambda_1 + \delta, & R \geqslant \lambda_1, \end{cases} \quad (5.35)$$

$\delta > 0$ *is arbitrarily small, and* $C = C(\delta)$ *for* $R \geqslant \lambda_1$.

Proof. For the sake of simplicity, we assume that $\tau = 0$. Taking the inner product of Equation (5.33) and w in $\mathbf{H}$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 - R\|w\|^2 \\ &+ \langle (1 + i\beta)(|u|^2 u - |u^0|^2 u^0), u - u^0 \rangle = \varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle. \end{aligned} \quad (5.36)$$

Since $|\beta| \leqslant \sqrt{3}$, from (5.16) it follows that

$$\langle (1 + i\beta)(|u|^2 u - |u^0|^2 u^0), u - u^0 \rangle \geqslant 0 \quad (5.37)$$

(see also (1.34) and [**34**]). From (5.36) and (5.37) we obtain

$$\frac{d}{dt}\|w\|^2 + 2\|\nabla w\|^2 \leqslant 2R\|w\|^2 + 2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon}, t\Big), w\Big\rangle. \quad (5.38)$$

Using (5.28), we find

$$2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon},t\Big),w\Big\rangle ds = 2\varepsilon^{-\rho}\sum_{j=1}^{n}\Big\langle \partial_{z_j}G_j\Big(\frac{x}{\varepsilon},t\Big),w\Big\rangle$$

$$= 2\varepsilon^{1-\rho}\sum_{j=1}^{n}\Big\langle \partial_{x_j}G_j\Big(\frac{x}{\varepsilon},t\Big),w\Big\rangle = -2\varepsilon^{1-\rho}\sum_{j=1}^{n}\Big\langle G_j\Big(\frac{x}{\varepsilon},t\Big),\partial_{x_j}w\Big\rangle$$

$$\leqslant \frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}\sum_{j=1}^{n}\int\limits_{\Omega}\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2 dx + \frac{2\delta}{\lambda_1}\int\limits_{\Omega}|\nabla w(x,t)|^2 dx,\ \delta>0. \tag{5.39}$$

We claim that

$$\int\limits_{\Omega}\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2 dx = \varepsilon^n\int\limits_{\varepsilon^{-1}\Omega}|G_j(z,t)|^2 dx \leqslant C\|G_j(\cdot,t)\|^2_{\mathbf{Z}}. \tag{5.40}$$

Here, we used an n-dimensional analog of Lemma 4.2. Hence

$$\sum_{j=1}^{n}\int\limits_{\Omega}\Big|G_j\Big(\frac{x}{\varepsilon},t\Big)\Big|^2 dx \leqslant C\sum_{j=1}^{n}\|G_j(\cdot,t)\|^2_{\mathbf{Z}} \quad \forall t\in\mathbb{R}. \tag{5.41}$$

By (5.39) and (5.41), we have

$$2\varepsilon^{-\rho}\Big\langle g_1\Big(\frac{x}{\varepsilon},t\Big),w\Big\rangle \leqslant \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)h(t) + \frac{2\delta}{\lambda_1}\|\nabla w\|^2,\ \delta>0,$$

where $h(t)=\sum\limits_{j=1}^{n}\|G_j(\cdot,t)\|^2_{\mathbf{Z}}$. From (5.38) it follows that

$$\frac{d}{dt}\|w\|^2 + (2-2\delta\lambda_1^{-1})\|\nabla w\|^2 \leqslant 2R\|w\|^2 + \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)h(t). \tag{5.42}$$

Let $\delta<\lambda_1$. By the Poincaré inequality,

$$\frac{d}{dt}\|w\|^2 \leqslant 2(R-\lambda_1+\delta)\|w\|^2 + \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)h(t). \tag{5.43}$$

If $R\geqslant\lambda_1$, then $r=R-\lambda_1+\delta>0$ and, consequently,

$$\frac{d}{dt}\|w(t)\|^2 \leqslant r\|w(t)\|^2 + \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)h(t), \quad \|w(0)\|^2=0.$$

By the Gronwall inequality (see (4.48) and (4.49)),

$$\|w(t)\|^2 \leqslant \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)\int\limits_0^t h(s)e^{r(t-s)}ds. \tag{5.44}$$

Recall that $G_j(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$ since g_1 satisfies the divergence condition. Therefore,

$$\int_t^{t+1} h(s)ds \leqslant \sum_{j=1}^n \|G_j\|^2_{L_2^b(\mathbb{R};\mathbf{Z})} =: M \tag{5.45}$$

and, consequently,

$$\int_0^t h(s)e^{-rs}ds = \int_0^1 h(s)e^{-rs}ds + \int_1^2 h(s)e^{-rs}ds + \ldots + \int_{[t]}^t h(s)e^{-rs}ds$$

$$\begin{aligned}
&\leqslant \int_0^1 h(s)ds + e^{-r}\int_1^2 h(s)ds + \ldots + e^{-[t]}\int_{[t]}^t h(s)ds \\
&\leqslant M(1 + e^{-r} + \ldots + e^{-[t]}) \leqslant M(1 + e^{-r} + \ldots) \\
&= \frac{M}{1 - e^{-r}} \leqslant M(1 + r^{-1}).
\end{aligned}$$

Using this estimate in (5.45), we obtain

$$\|w(t)\|^2 \leqslant \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}CM(1 + r^{-1})\Big)e^{rt} \quad \forall t \geqslant 0, \tag{5.46}$$

i.e.,

$$\|w(t)\| \leqslant C(\delta)\varepsilon^{(1-\rho)}e^{rt},$$

where $r = R - \lambda_1 + \delta$ and $C(\delta) = (\delta^{-1}2^{-1}\lambda_1 CM(1 + r^{-1}))^{1/2}$. If $R < \lambda_1$, then $-r_1 = R - \lambda_1 + \delta < 0$ for sufficiently small $\delta > 0$. Then from (5.43) we deduce

$$\frac{d}{dt}\|w\|^2 \leqslant -r_1\|w\|^2 + \Big(\frac{\lambda_1}{2\delta}\varepsilon^{2(1-\rho)}C\Big)h(t). \tag{5.47}$$

By Lemma 4.1 and (5.45),

$$\|w(t)\|^2 \leqslant \|w(0)\|^2 e^{-r_1 t} + 2^{-1}\delta^{-1}\lambda_1 CM(1 + r_1^{-1})\varepsilon^{2(1-\rho)} \quad \forall t \geqslant 0,$$

and, since $w(0) = 0$,

$$\|w(t)\| \leqslant C(\delta)\varepsilon^{(1-\rho)},$$

where $C(\delta) = (2^{-1}\delta^{-1}\lambda_1 CM(1 + r_1^{-1}))^{1/2}$ and $r_1 = \lambda_1 - R - \delta > 0$. The inequality (5.34) is proved. □

5.3. On the structure of attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$.

Consider the Ginzburg–Landau equation (see (5.30))

$$\partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u + g^\varepsilon(x,t), \quad u|_{\partial\Omega} = 0, \tag{5.48}$$

where ε is fixed and $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ is the *time symbol* (see Section 2.4). Assume that $g_0(x,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $g_1(z,t)$ is a translation compact function in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$. In particular, $g_0(x,t) \in L_2^b(\mathbb{R};\mathbf{H})$ and $g_1(z,t) \in L_2^b(\mathbb{R};\mathbf{Z})$.

Let $\mathcal{H}(g^\varepsilon)$ be the hull of the symbol $g^\varepsilon(x,t)$ in the space $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$:

$$\mathcal{H}(g^\varepsilon) = [\{g^\varepsilon(\cdot, t+h) \mid h \in \mathbb{R}\}]_{L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})}. \tag{5.49}$$

Recall that $\mathcal{H}(g^\varepsilon)$ is compact in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and every element $\widehat{g}^\varepsilon(x,t) \in \mathcal{H}(g^\varepsilon(x,t))$ can be written in the form

$$\widehat{g}^\varepsilon(x,t) = \widehat{g}_0(x,t) + \varepsilon^{-\rho}\widehat{g}_1(x/\varepsilon, t) \tag{5.50}$$

with some functions $\widehat{g}_0 \in \mathcal{H}(g_0)$ and $\widehat{g}_1 \in \mathcal{H}(g_1)$, where $\mathcal{H}(g_0)$ and $\mathcal{H}(g_1)$ are the hulls of the functions $g_0(x,t)$ and $g_1(z,t)$ in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$ respectively (see Proposition 4.2 which remains true for the n-dimensional complex spaces $\mathbf{H}$ and $\mathbf{Z}$).

As was shown in Section 5.1, the process $\{U_\varepsilon(t,\tau)\} := \{U_{g^\varepsilon}(t,\tau)\}$ corresponding to (5.48) has the uniform global attractor $\mathcal{A}^\varepsilon \subseteq B_{0,\varepsilon} \cap B_{1,\varepsilon}$ (see (5.13) and (5.14)) and

$$\|\mathcal{A}^\varepsilon\|_{\mathbf{H}} \leqslant (C_0 + C_1\varepsilon^{-\rho}). \tag{5.51}$$

Now, we describe the structure of the attractor $\mathcal{A}^\varepsilon$. Along with Equation (5.48), we consider the family of equations

$$\partial_t \widehat{u}^\varepsilon = (1+i\alpha)\Delta\widehat{u}^\varepsilon + R\widehat{u}^\varepsilon - (1+i\beta)|\widehat{u}^\varepsilon|^2\widehat{u}^\varepsilon + \widehat{g}^\varepsilon(x,t), \ \widehat{u}^\varepsilon|_{\partial\Omega} = 0 \tag{5.52}$$

with symbols $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$. It is clear that for every $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ Equation (5.52) generates the process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$ acting in $\mathbf{H}$. We note that the processes $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$ possess properties similar to the properties of the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to the Ginzburg–Landau equation (5.48) with original symbol $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho}g_1(x/\varepsilon,t)$. In particular, the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing for each process of the family $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$, $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

We denote by $\mathcal{K}_{\widehat{g}^\varepsilon}$ the *kernel* of the system (5.52) (and of the process $\{U_{\widehat{g}^\varepsilon}(t,\tau)\}$) with symbol $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

We formulate the theorem on the structure of the uniform global attractor $\mathcal{A}^\varepsilon$ of the Ginzburg–Landau equation (5.48) (see Section 2.6.3 and (2.122)).

Theorem 5.3. *If $g^\varepsilon(x,t)$ is a translation compact function in the space $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$, then the process $\{U_{g^\varepsilon}(t,\tau)\}$ corresponding to* (5.52) *has the uniform global attractor $\mathcal{A}^\varepsilon$ and*

$$\mathcal{A}^\varepsilon = \bigcup_{\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{\widehat{g}^\varepsilon}(0); \tag{5.53}$$

moreover, the kernel $\mathcal{K}_{\widehat{g}^\varepsilon}$ is nonempty for every $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$.

All the above results are valid for the "limiting" Ginzburg–Landau equation (see (5.31))

$$\partial_t u^0 = (1+i\alpha)\Delta u^0 + Ru^0 - (1+i\beta)|u^0|^2u^0 + g^0(x,t),\ u^0|_{\partial\Omega} = 0, \tag{5.54}$$

with translation compact symbol $g^0(t) := g_0(\cdot,t) \in L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$. Equation (5.54) generates the "limiting" process $\{U_0(t,\tau)\} := \{U_{g^0}(t,\tau)\}$ which has the uniform global attractor $\mathcal{A}^0$ (see Section 5.2).

Consider the family of equations

$$\partial_t \widehat{u}^0 = (1+i\alpha)\Delta \widehat{u}^0 + R\widehat{u}^0 - (1+i\beta)|\widehat{u}^0|^2\widehat{u}^0 + \widehat{g}^0(x,t), \widehat{u}^0|_{\partial\Omega} = 0, \tag{5.55}$$

with symbols $\widehat{g}^0 \in \mathcal{H}(g^0)$ and the family of processes $\{U_{\widehat{g}^0}(t,\tau)\}$, $\widehat{g}^0 \in \mathcal{H}(g^0)$. Note that we can apply Theorem 5.3 directly to (5.54) and (5.55) by setting $g_1(z,t) \equiv 0$. Therefore, the attractor $\mathcal{A}^0$ of the "limiting" equation (5.54) has the form

$$\mathcal{A}^0 = \bigcup_{\widehat{g}^0 \in \mathcal{H}(g^0)} \mathcal{K}_{\widehat{g}^0}(0), \tag{5.56}$$

where $\mathcal{K}_{\widehat{g}^0}$ is the kernel of (5.55) with symbol $\widehat{g}^0 \in \mathcal{H}(g^0)$.

5.4. Convergence of $\mathcal{A}^\varepsilon$ to $\mathcal{A}^0$ and estimate for deviation.

All the results of Sections 4.3 and 4.4 can be also established for the Ginzburg–Landau equation. We consider (5.48) and (5.54), where $g_0(x,t)$ and $g_1(z,t)$ are translation compact functions in $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R};\mathbf{Z})$ respectively. Assume that $g_1(z,t)$ satisfies the divergence condition (5.28). Then, by Theorem 5.1, the uniform global attractors $\mathcal{A}^\varepsilon$ of (5.48) with external forces $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho}g_1(x/\varepsilon,t)$ are uniformly (with respect to ε) bounded in $\mathbf{H}$:

$$\|\mathcal{A}^\varepsilon\|_{\mathbf{H}} \leqslant C_2 \quad \forall\, 0 < \varepsilon \leqslant 1. \tag{5.57}$$

We also consider the global attractor $\mathcal{A}^0$ of the "limiting" equation (5.54) with external force $g^0(t) = g_0(\cdot, t)$.

We need to generalize Theorem 5.2 in order to apply the estimate (5.34) to the families of equations (5.52) and (5.55).

Consider an arbitrary initial data $u_\tau \in \mathbf{H}$. Let $\widehat{u}^\varepsilon(\cdot, t) = U_{\widehat{g}^\varepsilon}(t, \tau)u_\tau$, $t \geqslant \tau$, be the solution of (5.52) with symbol $\widehat{g}^\varepsilon = \widehat{g}_0 + \varepsilon^{-\rho}\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$, and let $\widetilde{u}^0(\cdot, t) = U_{\widetilde{g}^0}(t, \tau)u_\tau$, $t \geqslant \tau$, be the solution of (5.55) with symbol $\widetilde{g}^0 \in \mathcal{H}(g^0)$ and the same initial data. We note that the symbol $\widetilde{g}^0$ can be different from the function $\widehat{g}^0 = \widehat{g}_0$ in the representation $\widehat{g}^\varepsilon = \widehat{g}_0 + \varepsilon^{-\rho}\widehat{g}_1$. Consider the difference

$$\widehat{w}(x, t) = \widehat{u}^\varepsilon(x, t) - \widetilde{u}^0(x, t), \quad t \geqslant \tau.$$

Proposition 5.2. *Let $g_0(x, t)$ and $g_1(z, t)$ in (5.1) be translation compact functions in the spaces $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{Z})$ respectively, and let $g_1(z, t)$ satisfy the divergence condition (5.28). Let*

$$g^\varepsilon(x, t) = g_0(x, t) + \varepsilon^{-\rho}g_1(x/\varepsilon, t), \quad g^0(x, t) = g_0(x, t).$$

Then for every symbol $\widehat{g}^\varepsilon = \widehat{g}_0 + \varepsilon^{-\rho}\widehat{g}_1 \in \mathcal{H}(g^\varepsilon)$ there exists a symbol $\widetilde{g}^0 \in \mathcal{H}(g^0)$ such that for every initial data $u_\tau \in \mathbf{H}$ the difference

$$\widehat{w}(t) = \widehat{u}^\varepsilon(t) - \widetilde{u}^0(t) = U_{\widehat{g}^\varepsilon}(t, \tau)u_\tau - U_{\widetilde{g}^0}(t, \tau)u_\tau$$

of the solutions of the Ginzburg–Landau equations (5.52) and (5.55) with symbols $\widehat{g}^\varepsilon(x, t) = \widehat{g}_0(x, t) + \varepsilon^{-\rho}\widehat{g}_1(x/\varepsilon, t)$ and $\widetilde{g}^0(x, t)$ respectively and the same initial data u_τ satisfies the inequality

$$\|\widehat{w}(t)\| = \|\widehat{u}^\varepsilon(\cdot, t) - \widetilde{u}^0(\cdot, t)\| \leqslant C\varepsilon^{(1-\rho)}e^{r(t-\tau)} \quad \forall t \geqslant \tau, \tag{5.58}$$

where the constants C and r are the same as in Theorem 5.2 and are independent of ε and $0 \leqslant \rho \leqslant 1$.

The proof is similar to that of Proposition 4.4.

We formulate an analog of Theorem 4.4 about the strong convergence of the global attractors $\mathcal{A}_\varepsilon$ of the Ginzburg–Landau equation (5.30) to the global attractor $\mathcal{A}_0$ of the "limiting" equation (5.31) as $\varepsilon \to 0+$.

Theorem 5.4. *Assume that $0 \leqslant \rho < 1$. Let $g_0(x, t)$ and $g_1(z, t)$ in (5.30) be translation compact functions in the spaces $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{H})$ and $L_2^{\mathrm{loc}}(\mathbb{R}; \mathbf{Z})$ respectively, and let $g_1(z, t), z \in \mathbb{R}^n$, satisfy the divergence condition (5.28). Then the global attractors $\mathcal{A}^\varepsilon$ of (5.30) converge to the global attractor $\mathcal{A}^0$ of the "limiting" equation (5.31) in the norm of $\mathbf{H}$ as $\varepsilon \to 0+$, i.e.,*

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0 \quad \text{as } \varepsilon \to 0+. \tag{5.59}$$

The proof is similar to that of Theorem 4.4.

Using Proposition 2.13, we estimate $\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0)$ explicitly under the assumption that the global attractor $\mathcal{A}^0$ is exponential. Let

$$R \leqslant \lambda_1 - \varkappa \quad \forall t \in \mathbb{R}, \tag{5.60}$$

where $\varkappa > 0$ and λ_1 is the first eigenvalue of the operator $\{-\Delta,\ u|_{\partial\Omega} = 0\}$. Then the global attractor has simple structure. We reformulate the corresponding results from Section 2.6.3.

Proposition 5.3. *Let the assumptions of Theorem* 5.4 *hold, and let* R *satisfy the inequality* (5.60). *Then the following assertions hold.*

(i) *For every* $\widehat{g}^0 \in \mathcal{H}(g^0)$ *there exists a unique bounded* (*in* $\mathbf{H}$) *complete solution* $z_{\widehat{g}^0}(t)$, $t \in \mathbb{R}$, *of* (5.55) *with symbol* $\widehat{g}^0$, *i.e., the kernel* $\mathcal{K}_{\widehat{g}^0}$ *consists of a single element* $z_{\widehat{g}^0}$ *and, in this case, formula* (5.56) *for the global attractor* $\mathcal{A}^0$ *has the form*

$$\mathcal{A}^0 = \bigcup_{\widehat{g}^0 \in \mathcal{H}(g^0)} \{z_{\widehat{g}^0}(0)\}. \tag{5.61}$$

(ii) *The complete solution* $z_{\widehat{g}^0}(t)$, $t \in \mathbb{R}$, *attracts any solution* $\widehat{u}_{\widehat{g}^0}(t) = U_{\widehat{g}^0}(t,\tau)u_\tau$, $t \geqslant \tau$, *with exponential rate*:

$$\|\widehat{u}_{\widehat{g}^0}(t) - z_{\widehat{g}^0}(t)\| \leqslant \|\widehat{u}_{\widehat{g}^0}(\tau) - z_{\widehat{g}^0}(\tau)\| e^{-\varkappa(t-\tau)} \quad \forall t \geqslant \tau,\ \tau \in R, \tag{5.62}$$

and, consequently, the global attractor $\mathcal{A}^0$ *is exponential, i.e.,*

$$\sup_{\widehat{g}^0 \in \mathcal{H}(g^0)} \mathrm{dist}_{\mathbf{H}}(U_{\widehat{g}^0}(t,\tau)B, \mathcal{A}) \leqslant C e^{-\varkappa(t-\tau)}, \quad C = C(\|B\|_{\mathbf{H}}), \tag{5.63}$$

where B *is a bounded* (*in* $\mathbf{H}$) *set of initial data and* $\varkappa$ *is taken from the condition* (5.60).

From Propositions 5.2 and 5.3, we obtain the following assertion.

Theorem 5.5. *Let* $0 < \rho < 1$. *Suppose that the assumptions of Theorem* 5.4 *and the condition* (5.60) *are satisfied. Then the Hausdorff distance* (*in* $\mathbf{H}$) *from the global attractor* $\mathcal{A}^\varepsilon$ *to the "limiting" global attractor* $\mathcal{A}^0$ *satisfies the inequality*

$$\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C(\rho)\varepsilon^{1-\rho} \quad \forall 0 < \varepsilon \leqslant 1. \tag{5.64}$$

Proof. We fix ε. Let u^ε be an arbitrary element of $\mathcal{A}^\varepsilon$. By (5.53), there exists a bounded complete solution $\widehat{u}^\varepsilon(t)$, $t \in \mathbb{R}$, of (5.48) with some symbol $\widehat{g}^\varepsilon = \widehat{g}_0(x,t) + \varepsilon^{-\rho}\widehat{g}_1(x/\varepsilon, t) \in \mathcal{H}(g^\varepsilon)$ such that

$$\widehat{u}^\varepsilon(0) = u^\varepsilon. \tag{5.65}$$

Consider the point $\widehat{u}^\varepsilon(-T)$ which clearly belongs to $\mathcal{A}^\varepsilon$. From (5.57) it follows that

$$\|\widehat{u}^\varepsilon(-T)\| \leqslant C_2, \tag{5.66}$$

where C_2 is independent of ε and T.

We apply Proposition 5.2 for the constructed external force $\widehat{g}^\varepsilon$: there is a "limiting" external force $\widetilde{g}^0 \in \mathcal{H}(g^0)$ such that for any $\tau \in \mathbb{R}$ and $u_\tau \in \mathbf{H}$

$$\|U_{\widehat{g}^\varepsilon}(t+\tau,\tau)u_\tau - U_{\widetilde{g}^0}(t+\tau,\tau)u_\tau\| \leqslant C\varepsilon^{(1-\rho)} \quad \forall t \geqslant 0, \tag{5.67}$$

where $r = 0$ since $R < \lambda_1$ (see (5.35)). Here, C is independent of u_τ.

Consider the "limiting" equation (5.55) with the chosen "limiting" external force $\widetilde{g}^0$. We set $\tau = -R$. Let $\widetilde{u}^0(t)$, $t \geqslant -T$, be the solution of this equation with initial data

$$\widetilde{u}^0|_{t=-T} = \widehat{u}^\varepsilon(-T). \tag{5.68}$$

By Proposition 5.3, there exists a unique bounded complete solution $z^0(t)$, $t \in \mathbb{R}$, of (5.55) with symbol $\widetilde{g}^0$ such that

$$\|\widetilde{u}^0(-T+t) - z^0(-T+t)\| \leqslant \|\widetilde{u}^0(-T) - z^0(-T)\|e^{-\varkappa t} \quad \forall t \geqslant 0. \tag{5.69}$$

Recall that $z^0(t) \in \mathcal{A}^0$ for all $t \in \mathbb{R}$. Therefore,

$$\|z^0(-T)\| \leqslant \|\mathcal{A}^0\| \leqslant C', \tag{5.70}$$

where C' is independent of z^0 and T. By (5.68) and (5.66),

$$\|\widetilde{u}^0(-T)\| = \|\widehat{u}^\varepsilon(-T)\| \leqslant C_2. \tag{5.71}$$

From (5.69), (5.70), and (5.71) it follows that

$$\|\widetilde{u}^0(-T+t) - z^0(-T+t)\| \leqslant C''e^{-\varkappa t} \quad \forall t \geqslant 0, \tag{5.72}$$

where $C'' = C' + C_2$.

Setting $\tau = -T$ in (5.67), we have

$$\begin{aligned}
&\|\widehat{u}^\varepsilon(-T+t) - \widetilde{u}^0(-T+t)\| \\
&= \|U_{\widehat{g}^\varepsilon}(t+\tau,\tau)u_\tau - U_{\widetilde{g}^0}(t+\tau,\tau)u_\tau\| \leqslant C\varepsilon^{(1-\rho)} \quad \forall t \geqslant 0.
\end{aligned} \tag{5.73}$$

Using (5.72) and (5.73), we find

$$\begin{aligned}
&\|\widehat{u}^\varepsilon(-T+t) - z^0(-T+t)\| \\
&\leqslant \|\widehat{u}^\varepsilon(-T+t) - \widetilde{u}^0(-T+t)\| + \|\widetilde{u}^0(-T+t) - z^0(-T+t)\| \\
&\leqslant C\varepsilon^{(1-\rho)} + C''e^{-\varkappa t}.
\end{aligned} \tag{5.74}$$

We choose T from the equation $\varepsilon^{(1-\rho)} = e^{-\varkappa T}$, i.e., $T = \dfrac{1-\rho}{\varkappa}\log(1/\varepsilon)$ and substitute $t = T$ in (5.74). Then

$$\|\widehat{u}^\varepsilon(0) - z^0(0)\| \leqslant (C + C'')\varepsilon^{(1-\rho)}$$

and, consequently,

$$\mathrm{dist}_{\mathbf{H}}(u^\varepsilon, \mathcal{A}^0) \leqslant \|u^\varepsilon - z^0(0)\| = \|\widehat{u}^\varepsilon(0) - z^0(0)\| \leqslant C(\rho)\varepsilon^{(1-\rho)},$$

where $C(\rho) = (C + C'')$. Since u^ε is an arbitrary point of $\mathcal{A}^\varepsilon$, we have $\mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C(\rho)\varepsilon^{(1-\rho)}$. □

Remark 5.1. In the case $R < \lambda_1$, Proposition 5.3 holds for (5.48) with symbols $g^\varepsilon(x,t) = g_0(x,t) + \varepsilon^{-\rho} g_1(x/\varepsilon, t)$ and for the family of equations (5.52) with symbols $\widehat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ (see Proposition 2.13 and Corollary 2.9). In particular, the global attractor $\mathcal{A}^\varepsilon$ of (5.48) is exponential, as well as the global attractor $\mathcal{A}^0$, and the attraction rate is the same.

Remark 5.2. In fact, the inequality (5.64) holds (with some other constant C) for the symmetric distance $\mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) = \mathrm{dist}_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) + \mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^0, \mathcal{A}^\varepsilon)$:

$$\mathrm{dist}^s_{\mathbf{H}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leqslant C_1(\rho)\varepsilon^{1-\rho} \quad \forall\, 0 < \varepsilon \leqslant 1.$$

This result relies on the property of the exponential attraction of solutions to the global attractor $\mathcal{A}^\varepsilon$, mentioned in Remark 5.1.

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Recent Results in Large Amplitude Monophase Nonlinear Geometric Optics

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For quasilinear first order systems the standard regime is weakly nonlinear geometric optics which considers near some background state perturbations of amplitude ε with wave length $\varepsilon \in]0,1]$ ($\varepsilon \to 0$). However, when the oscillations are associated to a linearly degenerate mode, stronger waves can also be considered. The question of the existence, propagation, and interaction of such larger amplitude waves is the matter of supercritical Wentzel–Kramers–Brillouin analysis. Some recent results in this direction and, in particular, the case of incompressible Euler equations are described. Bibliography: 22 titles.

1. Introduction

We start with a general presentation.

1.1. Background results in nonlinear geometric optics.

Many works are devoted to the study of high frequency oscillatory waves

$$\mathbf{u}^\varepsilon(t,x) \sim \mathbf{u}_a^\varepsilon(t,x) := \sum_{j=0}^{\infty} \varepsilon^{j/l} U_j(t,x,\varphi^\varepsilon(t,x)/\varepsilon), \quad \varepsilon \to 0, \tag{1.1}$$

which satisfy quasilinear first order systems

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(t,x,\mathbf{u})\partial_{x_j}\mathbf{u} + \mathbf{f}(t,x,\mathbf{u}) = 0. \tag{1.2}$$

Here, $\varepsilon \in]0,1]$ is a small parameter (going to 0), the profiles $U_j(t,x,\theta)$ are periodic in the θ variable, the phase $\varphi^\varepsilon(t,x)$ is a real function, l and d are positive fixed integers. We will work with oscillations in the *space* variable, assuming that

$$\varphi^\varepsilon(t,x) = \sum_{j=0}^{l-1} \varepsilon^{j/l}\varphi_j(t,x), \quad \nabla_x\varphi_0 \not\equiv 0. \tag{1.3}$$

In the notation

$$\mathbf{u}_a^\varepsilon(t,x) = \mathbf{U}_a^\varepsilon(t,x,\varphi^\varepsilon(t,x)/\varepsilon), \quad \mathbf{U}_a^\varepsilon(t,x,\theta) := \sum_{j=0}^{\infty} \varepsilon^{j/l} U_j(t,x,\theta),$$

the expression $\mathbf{u}_a^\varepsilon$ is interpreted as a monophase oscillation. Since φ^ε can depend on $\varepsilon \in]0,1]$ through (1.3), we have

$$\mathbf{u}_a^\varepsilon(t,x) = \sum_{j=0}^{\infty} \varepsilon^{j/l}\tilde{U}_j\Big(t,x,\frac{\varphi_0(t,x)}{\varepsilon},\frac{\varphi_1(t,x)}{\varepsilon^{1-(1/l)}},\cdots,\frac{\varphi_{l-1}(t,x)}{\varepsilon^{1/l}}\Big) \tag{1.4}$$

with

$$\tilde{U}_j(t,x,\theta_0,\theta_1,\cdots,\theta_{l-1}) := U_j(t,x,\theta_0+\theta_1+\cdots+\theta_{l-1}) \quad \forall j \in \mathbb{N}. \tag{1.5}$$

We see in (1.4) that multiphase and multiscale features are also present. Of course, due to (1.5), they are organized in a very particular manner.

The goal is to construct families $\{\mathbf{u}^\varepsilon\}_\varepsilon$ which are solutions to (1.2) on some open domain $\Omega \subset \mathbf{R} \times \mathbf{R}^d$ independent of $\varepsilon \in]0,1]$ and which satisfy the asymptotic behavior (1.1). This requires to identify the terms U_j and φ_j in order to build approximate solutions $\mathbf{u}_a^\varepsilon$ meaning that

$$\mathbf{f}_a^\varepsilon := \partial_t\mathbf{u}_a^\varepsilon + \sum_{j=1}^{d} A_j(t,x,\mathbf{u}_a^\varepsilon)\partial_{x_j}\mathbf{u}_a^\varepsilon + \mathbf{f}(t,x,\mathbf{u}_a^\varepsilon) = O(\varepsilon^N)$$

for some $N \gg 1$. This includes also to study the validity of the nonlinear geometric optics approximation $\mathbf{u}_a^\varepsilon$. We want to know if there exists some solution $\mathbf{u}^\varepsilon$ of (1.2) corresponding to $\mathbf{u}_a^\varepsilon$.

Looking at oscillations such as $\mathbf{u}_a^\varepsilon$ is a way to point some special mechanisms of nonlinear interaction. These mechanisms can be hidden in the original full set of Equations (1.2). On the other hand, they can be visible at the level of the transport equations giving rise to the profiles U_j or even at the level of the eikonal equations yielding the phases φ_j.

The use of (1.1) implies that the smallest wavelength in $\mathbf{u}^\varepsilon$ is fixed: it is ε. Then the analysis depends crucially on the amplitude of the oscillation.

(i) If

$$\exists j \in \{0, \cdots, l-1\}; \quad \partial_\theta U_j \not\equiv 0, \tag{1.6}$$

the regime is called *supercritical*. The problem may as well be ill-posed. Due, for instance, to the formation of shocks, it could be not possible to find smooth solutions $\mathbf{u}^\varepsilon$ of (1.2) satisfying (1.1) on some open set $\Omega \subset \mathbf{R} \times \mathbf{R}^d$ with Ω independent of $\varepsilon \in]0,1]$.

(ii) If

$$\partial_\theta U_j \equiv 0 \quad \forall j \in \{0, \cdots, l\}, \tag{1.7}$$

the analysis is of reduced interest: the transport equations for all U_j are linear and expansions similar to (1.1) are easily justified.

(ii) If

$$\partial_\theta U_l \not\equiv 0, \quad \partial_\theta U_j \equiv 0 \quad \forall j \in \{0, \cdots, l-1\}, \tag{1.8}$$

the regime is called *critical*. This situation is more interesting. It is the matter of *weakly nonlinear geometric optics*, a theory which seems mainly achieved (see [**16, 17, 13, 14, 21**] and the related references).

However, the above general picture, insisting on the relevance of (1.8), is proving to be not convenient in many physical situations. This happens when the transport equation for U_l is linear instead of being nonlinear, meaning that some interaction coefficients are trivial. Then to exhibit nonlinear phenomena, waves of larger amplitude must be involved. The supercritical regime becomes the situation to deal with. This typically occurs when the wave $\mathbf{u}^\varepsilon$ is associated to a linearly degenerate mode.

1.2. Propagation of oscillations on a linearly degenerate field.

All linearly degenerate modes do not share the same properties. They can be classified according to the *transparency* conditions which they induce [**7**]. Consider, for instance, the model of entropic gas dynamics

$$\begin{aligned}
&\partial_t \varrho + (\mathbf{u} \cdot \nabla_x)\varrho + \varrho \operatorname{div}_x \mathbf{u} = 0, \\
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \varrho^{-1} \nabla_x \mathbf{p} = 0, \\
&\partial_t \mathbf{s} + (\mathbf{u} \cdot \nabla_x)\mathbf{s} = 0,
\end{aligned} \tag{1.9}$$

where the pressure $\mathbf{p}$ is given by a state relation $\mathbf{p} = P(\varrho, \mathbf{s})$. In (1.9), entropy waves (carried by the component $\mathbf{s}$) must be distinguished from speed waves (related to some well polarized components of $\mathbf{u}$).

Concerning entropy waves, a complete discussion is accessible [**8**]. We can face (1.9) in the case (1.6) even if we deal with large amplitude oscillations ($\partial_\theta S_0 \not\equiv 0$). This includes some sort of stability results.

On the contrary, the study of speed waves can lead to violent instability phenomena. Fix $l = 2$. Suppose that $U_0(t,x,\theta) = \mathbf{u}_0(t,x)$ is a given solution of (1.2). Seek U_1 with $\partial_\theta U_1 \not\equiv 0$. Select any $m \in \mathbb{N}_*$. Then it is possible to find two families $\{\mathbf{u}^\varepsilon\}_\varepsilon$ and $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$ of solutions to (1.2) adjusted in such a way that

$$\parallel \mathbf{u}^\varepsilon(0,\cdot) - \tilde{\mathbf{u}}^\varepsilon(0,\cdot) \parallel_{L^2} = O(\varepsilon^m), \tag{1.10}$$

whereas, at the time $t^\varepsilon = -m\varepsilon \ln \varepsilon$, we find

$$\parallel \mathbf{u}^\varepsilon(t^\varepsilon,\cdot) - \tilde{\mathbf{u}}^\varepsilon(t^\varepsilon,\cdot) \parallel_{L^2} \neq o(\varepsilon). \tag{1.11}$$

In fact, the linearized equations of (1.2) along $\mathbf{u}_a^\varepsilon$ give rise to an amplification factor of size $O(e^{ct/\varepsilon})$ with $c > 0$. Small $O(\varepsilon^m)$ error terms can therefore be multiplied by $e^{ct/\varepsilon}$ yielding an $O(1)$ modification at the time t^ε. In [**7**], this linear mechanism is shown to pass to the nonlinear framework (1.2).

The amplification phenomenon (1.10), (1.11) can be due to various reasons. The structure of the background solution $\mathbf{u}_0$ can suffice to engage it. Even if $\mathbf{u}_0$ is some constant basic state, the arbitrary oscillations contained in the small remainder $\mathbf{f}_a^\varepsilon$ (especially the oscillations according to phases which are transversal to φ_0) can interact with $\mathbf{u}_a^\varepsilon$ in a way to affect at the time t^ε the leading order term in the expansion $\mathbf{u}_a^\varepsilon$. Such phenomena have motivated many recent contributions, all issued from the pioneering works [**11, 15, 20**]. The situation is still far to be completely understood.

A common idea in science texts is that partial differential equations depend on parameters which are only known approximately. Therefore, an infinitely accurate approximation is for all practical purposes as good as an exact solution. Following this remark, the justification of nonlinear geometric optics is often regarded as working only towards mathematical ends.

The instability results alluded above seem to go in the opposite direction. Their interpretation is that, in supercritical WKB regimes, the replies given by nonlinear systems are very sensitive to the selection of the parameters or initial data which are involved. This would incline to stop from making determinist predictions.

For all that, these considerations do not mean that exact or even formal supercritical WKB expansions have no signification. Certainly, nothing guarantees that the behavior coded in $\mathbf{u}^\varepsilon$ or $\mathbf{u}_a^\varepsilon$ is physically selected. Yet, the nonlinear phenomena which intervene in the construction of $\mathbf{u}^\varepsilon$ or $\mathbf{u}_a^\varepsilon$ are susceptible of occurring. This is the reason why they have not only theoretical consequences, but also practical interests.

2. Case of Incompressible Euler Equations

The incompressible Euler equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \nabla_x \mathbf{p} = 0, \quad \mathrm{div}_x\, \mathbf{u} = 0 \tag{2.1}$$

do not fall exactly within the scope (1.2). Yet, the evolution of $\mathbf{u}$ in (2.1) can inherit some analogies with the evolution of the speed waves alluded above. Up to certain extent, we can say that the divergence free condition forces the wave $\mathbf{u}$ to have the behavior of a linearly degenerate speed mode (even if this mechanism is distorted by the influence of the pressure term). This fact is clear below when examining (2.1) in the case (1.8).

2.1. Weakly nonlinear geometric optics.

Weakly nonlinear geometric optics for (2.1) has not attracted many attention, probably because this is just an adaptation of general results stated about (1.2). Yet, let us recall briefly what happens when doing formal computations. Look at expansions $\mathbf{u}^\varepsilon(t,x)$ and $\mathbf{p}^\varepsilon(t,x)$ having the form (1.1) in the case (1.8). In other words, $\mathbf{u}^\varepsilon(t,x) \sim \mathbf{U}^\varepsilon(t,x,\varphi^\varepsilon(t,x)/\varepsilon)$ and $\mathbf{p}^\varepsilon(t,x) \sim \mathbb{P}^\varepsilon(t,x,\varphi^\varepsilon(t,x)/\varepsilon)$ as $\varepsilon \to 0$ with

$$\begin{aligned} &\mathbf{U}^\varepsilon(t,x,\theta) = \mathbf{u}_0(t,x) + \varepsilon U_l(t,x,\theta) + O(\varepsilon^{(l+1)/l}), \\ &\mathbb{P}^\varepsilon(t,x,\theta) = \mathbf{p}_0(t,x) + \varepsilon \mathbf{p}_l(t,x) + \varepsilon^2 P_{2l}(t,x,\theta) + O(\varepsilon^{(2l+1)/l}). \end{aligned} \tag{2.2}$$

We want to adjust the various ingredients composing $\mathbf{u}^\varepsilon$ and $\mathbf{p}^\varepsilon$ so that they furnish a solution of (2.1). Select some background solution of (2.1) made of $\mathbf{u}_0(t,x)$ and $\mathbf{p}_0(t,x)$. Keep in mind to impose the eikonal equation

$$\partial_t \varphi^\varepsilon + (\bar{\mathbf{U}}^\varepsilon \cdot \nabla_x)\varphi^\varepsilon = 0, \tag{2.3}$$

where $\bar{\mathbf{U}}^\varepsilon$ is the mean value of the profile $\mathbf{U}^\varepsilon$, i.e.,

$$\bar{\mathbf{U}}^\varepsilon(t,x) = \int_{\mathbb{T}} \mathbf{U}^\varepsilon(t,x,\theta)\,d\theta, \quad \mathbf{U}^{\varepsilon *}(t,x,\theta) := \mathbf{U}^\varepsilon(t,x,\theta) - \bar{\mathbf{U}}^\varepsilon(t,x).$$

Observe that (2.3) contains

$$\partial_t \varphi_0 + (\mathbf{u}_0 \cdot \nabla_x)\varphi_0 = 0. \tag{2.4}$$

Now, formal computations indicate that the divergence free condition implies

$$\mathrm{div}_x\, \bar{U}_l = 0, \quad \nabla_x \varphi_0 \cdot \partial_\theta U_l^* = 0. \tag{2.5}$$

Plug $\mathbf{u}^\varepsilon$ and $\mathbf{p}^\varepsilon$ as above in (2.1). Expand with respect to the powers of $\varepsilon \in]0,1]$. Use (2.3), (2.4), and (2.5) to simplify. The contribution which remains with ε in factor is

$$\partial_t U_l + (\mathbf{u}_0 \cdot \nabla_x) U_l + (U_l \cdot \nabla_x)\mathbf{u}_0 + \nabla_x \mathbf{p}_l + \partial_\theta P_{2l} \nabla_x \varphi_0 = 0. \tag{2.6}$$

There is no difficulty to solve the system (2.5), (2.6). Start by extracting $\bar{U}_l$ and $\mathbf{p}_l$ from

$$\partial_t \bar{U}_l + (\mathbf{u}_0 \cdot \nabla_x)\bar{U}_l + (\bar{U}_l \cdot \nabla_x)\mathbf{u}_0 + \nabla_x \mathbf{p}_l = 0, \quad \mathrm{div}_x\, \bar{U}_l = 0. \tag{2.7}$$

Then, noting that $\Pi_0(t,x)$ is the orthogonal projector onto the hyperplane $\nabla_x \varphi_0(t,x)^\perp := \{\mathbf{v} \in \mathbf{R}^d; \nabla_x \varphi_0(t,x) \cdot \mathbf{v} = 0\}$, it suffices to identify $\Pi_0 U_l^*$ through the transport equation

$$(\partial_t + \mathbf{u}_0 \cdot \nabla_x)\Pi_0 U_l^* + (\Pi_0 U_l^* \cdot \nabla_x)\mathbf{u}_0 - (\partial_t \Pi_0 + (\mathbf{u}_0 \cdot \nabla_x)\Pi_0)\Pi_0 U_l^* = 0.$$

By (2.5), we have $U_l^* = \Pi_0 U_l^*$.

Observe that these manipulations involve only linear equations (to identify the main profile U_l). This indicates that the regime (1.8) is not optimal within the framework (2.1). Again, this brings to consider stronger waves (i.e., waves of larger amplitudes). Keeping in mind the specific structure of (2.1), some special supercritical WKB analysis is needed to do that.

Precisely, the purpose of the next two sections is to review recent results in this direction, revealing in particular new nonlinear effects. From now on, the task is to construct expressions $\mathbf{u}^\varepsilon$ which are given by (1.1) in the case (1.6) and which are solutions to (2.1).

2.2. Creation of new scales by nonlinear interaction.

The hypothesis (1.6) can be separated in situations corresponding to growing difficulties. The first case to appear is when

$$l = 2, \quad \partial_\theta U_0 \equiv 0, \quad \partial_\theta U_1 \not\equiv 0. \tag{2.8}$$

This regime (2.8) is the one of *strong oscillations*. The WKB construction can still be achieved in full generality (see [**7**]). However, phenomena of amplification like (1.10), (1.11) do occur and they prevent to show by usual methods the existence of exact solutions $\mathbf{u}^\varepsilon$ close to $\mathbf{u}_a^\varepsilon$.

A way to get round this difficulty consists in adding some well adjusted anisotropic vanishing viscosity. On one hand, the viscosity is small enough in the direction $\nabla_x \varphi^\varepsilon(t,x)$ to allow us the propagation of oscillations like (1.1). On the other hand, it is large enough in all the other directions to kill by dissipation the transversal oscillations. It follows that it becomes possible to justify the nonlinear geometric optics. This is basically this argument which is exploited in [**3**, Theorem 1] and [**4**, Theorem 5.1].

Consider again (2.1), but now in the case

$$l \geqslant 3, \quad \partial_\theta U_0 \equiv 0, \quad \partial_\theta U_1 \not\equiv 0. \tag{2.9}$$

The situation (2.9) is more captivating for two main reasons:

i) Mathematically, WKB constructions involving (2.9) used to be incomplete. For instance, the transport equations derived in [**2, 19**] rely on some heuristic hypothesis which is not rigorously justified. The underlying difficulty is related to *closure* problems.

ii) Physically, expressions $\mathbf{u}_a^\varepsilon$ satisfying (2.9) give rise to characteristic rates of eddy dissipation which do not vanish when $\varepsilon \to 0$. Thereby, as it is explained in [**2, 19**], the description is concerned with *turbulent* flows. It must be connected to the general discussion of [**1**].

An analysis taking into account (2.9) within the framework (2.1) is proposed in the recent article [**4**]. We observe that:

i) To get round the mathematical difficulty (the closure problems), it is necessary to perform the WKB calculus with a phase including more terms than in (1.3). More precisely, we do not plug in (2.1) an expression $\mathbf{u}_a^\varepsilon$ with φ^ε as in (1.3). Instead, we appeal to

$$\mathbf{u}_a^\varepsilon(t,x) := \sum_{j=0}^{\infty} \varepsilon^{j/l} \tilde{U}_j(t,x,\tilde{\varphi}^\varepsilon(t,x)/\varepsilon), \tag{2.10}$$

where $\tilde{\varphi}^\varepsilon$ is given by some complete expansion

$$\tilde{\varphi}^\varepsilon(t,x) = \varphi^\varepsilon(t,x) + \sum_{j=l}^{\infty} \varepsilon^{j/l} \tilde{\varphi}_j(t,x). \tag{2.11}$$

The supplementary terms $\tilde{\varphi}_j$, $j \geqslant l$, are called *adjusting* phases. As was explained in [**5**], they are crucial to put the system of formal equations in a triangular form. They are the key to obtain an algorithm which allows us to compute the profiles $\tilde{U}_j$ step by step.

Of course, once $\mathbf{u}_a^\varepsilon$ as in (2.10) has been identified, the adjusting phases can be removed from $\mathbf{u}_a^\varepsilon$, just by performing Taylor expansions with respect to $\varepsilon \in [0,1]$ in the expressions

$$\tilde{U}_j\big(t,x,\varphi^\varepsilon(t,x)/\varepsilon + \tilde{\varphi}_l(t,x) + \sum_{j=1}^{\infty} \varepsilon^{j/l} \tilde{\varphi}_{l+j}(t,x)\big), \quad j \in \mathbb{N}_*,$$

in order to recover the form

$$\begin{aligned} \mathbf{u}^\varepsilon(t,x) \sim \mathbf{u}_0(t,x) + \sum_{j=1}^{\infty} \varepsilon^{j/l} U_j\Big(t,x,\frac{\varphi_0(t,x)}{\varepsilon} + \frac{\varphi_1(t,x)}{\varepsilon^{(l-1)/l}} \\ + \cdots + \frac{\varphi_{l-2}(t,x)}{\varepsilon^{2/l}} + \frac{\varphi_{l-1}(t,x)}{\varepsilon^{1/l}}\Big), \quad \varepsilon \to 0, \end{aligned} \tag{2.12}$$

Briefly, the construction of infinite accurate approximate solutions $\mathbf{u}_a^\varepsilon$ and $\mathbf{p}_a^\varepsilon$ of (2.1) satisfying

$$\partial_t \mathbf{u}_a^\varepsilon + (\mathbf{u}_a^\varepsilon \cdot \nabla_x)\mathbf{u}_a^\varepsilon + \nabla_x \mathbf{p}_a^\varepsilon = O(\varepsilon^\infty), \quad \operatorname{div}_x \mathbf{u}_a^\varepsilon = 0 \tag{2.13}$$

with $\mathbf{u}_a^\varepsilon$ as in (1.1) can be completed (see [**4**, Theorem 2.1]).

ii) The main physical phenomenon is the following: The scales associated with the phase shifts φ_j, $j \in \{2,\cdots,l-1\}$, can be created by nonlinear interaction. For instance (case $l=3$), initial data like

$$\mathbf{u}^\varepsilon(0,x) \sim \mathbf{u}_0(0,x) + \varepsilon^{1/3} U_1\Big(0,x,\frac{\varphi_0(0,x)}{\varepsilon}\Big) + O(\varepsilon^{2/3}) \quad \varepsilon \to 0, \tag{2.14}$$

can become at a time $t > 0$

$$\mathbf{u}^\varepsilon(t,x) \sim \mathbf{u}_0(t,x) + \varepsilon^{1/3} U_1\Big(t,x,\frac{\varphi_0(t,x)}{\varepsilon} + \frac{\varphi_2(t,x)}{\varepsilon^{1/3}}\Big) + O(\varepsilon^{2/3}), \quad \varepsilon \to 0, \tag{2.15}$$

The condition

$$\nabla_x \varphi_0(0,\cdot) \cdot P \operatorname{div}_x \langle U_1^*(0,\cdot) \otimes U_1^*(0,\cdot)\rangle \not\equiv 0 \tag{2.16}$$

is necessary and sufficient to see a nontrivial phase shift φ_2 appearing. This is explained in Remark 4.3.6 of [**4**]. Above, the notation P designates the Leray projector, whereas the symbol $\langle\cdot\rangle$ (as for $\bar{\cdot}$) means that we extract the mean value of a profile.

3. Large Amplitude Waves

Consider the oscillating problem made of (2.1) and the initial data

$$\mathbf{u}^\varepsilon(0,x) = \mathbf{u}_a^\varepsilon(0,x) = \sum_{j=0}^{\infty} \varepsilon^j U_j^0(x, \psi(x)/\varepsilon), \quad \varepsilon \in]0,1]. \tag{3.1}$$

We work here in the (supercritical) regime of large amplitude high frequency waves, meaning that $\partial_\theta U_0^0 \not\equiv 0$. The first approach would be to seek the corresponding solution $\mathbf{u}^\varepsilon$ in the form (1.1) with

$$l = 1, \quad \partial_\theta U_0 \not\equiv 0. \tag{3.2}$$

But, in general, this comes to nothing. We explain why in Section 3.1 by looking at some links between the situations (2.9) and (3.2).

3.1. Preliminaries.

First look at (2.1) under the condition (2.9). Suppose that $U_j(0,\cdot) \equiv 0$ for all $j \notin \{1 + lp, p \in \mathbb{N}\}$, $\varphi_1(0,\cdot) \equiv \varphi_2(0,\cdot) \equiv \cdots \equiv \varphi_{l-1}(0,\cdot) \equiv 0$. In the notation $\psi(\cdot) := \varphi_0(0,\cdot)$, $U_p^0(\cdot) := U_{1+lp}(0,\cdot)$ forall $p \in \mathbb{N}_*$, this means to start with

$$\mathbf{u}^\varepsilon(0,x) = \mathbf{u}_a^\varepsilon(0,x) = \varepsilon^{1/l} \sum_{j=0}^{\infty} \varepsilon^j U_j^0(x, \psi(x)/\varepsilon), \quad \varepsilon \in]0,1]. \tag{3.3}$$

According to Section 2, whatever the data ψ and U_p^0 with $p \in \mathbb{N}$ are, we can construct supercritical WKB expansions

$$\mathbf{u}_a^\varepsilon(t,x) := \sum_{j=1}^{\infty} \varepsilon^{j/l} U_j(t,x,\varphi^\varepsilon(t,x)/\varepsilon), \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^d, \tag{3.4}$$

which satisfy

$$\mathbf{u}_a^\varepsilon(0,x) = \sum_{j=1}^{\infty} \varepsilon^{j/l} U_j(0,x,\varphi_0(0,x)/\varepsilon) = \varepsilon^{1/l} \sum_{j=0}^{\infty} \varepsilon^j U_j^0(x,\psi(x)/\varepsilon)$$

and which are infinite accurate approximate solutions of (2.1). More precisely, the expression $\mathbf{u}_a^\varepsilon$ is divergence free ($\operatorname{div}_x \mathbf{u}_a^\varepsilon \equiv 0$) and furnishes the source term $\mathbf{f}_a^\varepsilon := \partial_t \mathbf{u}_a^\varepsilon + (\mathbf{u}_a^\varepsilon \cdot \nabla_x)\mathbf{u}_a^\varepsilon + \nabla_x \mathbf{p}_a^\varepsilon$ which, for all $T \in]0,+\infty[$, $s \in \mathbf{R}$, and $N \in \mathbb{N}$ is subjected to

$$\exists C(T,s,N); \quad \sup_{t\in[0,T]} \| \mathbf{f}_a^\varepsilon(t,\cdot) \|_{H^s(\mathbf{R}^d)} \leqslant C(T,s,N)\varepsilon^N. \tag{3.5}$$

Now, the incompressible Euler equations (2.1) are invariant under the change $\mathbf{u}(t,x)/\lambda\mathbf{u}(\lambda t,x)$, $\mathbf{p}(t,x)/\lambda^2\mathbf{p}(\lambda t,x)$, $\lambda > 0$. If we take $\lambda = \varepsilon^{-1/l}$, formula (3.4) is transformed into

$$\tilde{\mathbf{u}}_a^\varepsilon(t,x) = \sum_{j=0}^{\infty} \varepsilon^{j/l}\tilde{U}_j\Big(\varepsilon^{-1/l}t, x, \frac{\varphi_0(\varepsilon^{-1/l}t,x)}{\varepsilon} + \frac{\varphi_1(\varepsilon^{-1/l}t,x)}{\varepsilon^{(l-1)/l}} + \cdots + \frac{\varphi_{l-2}(\varepsilon^{-1/l}t,x)}{\varepsilon^{2/l}} + \frac{\varphi_{l-1}(\varepsilon^{-1/l}t,x)}{\varepsilon^{1/l}}\Big). \tag{3.6}$$

In (3.6), we have $\tilde{U}_j = U_{j+1}$ for all $j \geqslant 1$. In particular, at the initial time $t = 0$, we recover the large amplitude oscillation (3.1). The expression $\tilde{\mathbf{u}}_a^\varepsilon(t,x)$ gives rise to the error term $\tilde{\mathbf{f}}_a^\varepsilon := \partial_t\tilde{\mathbf{u}}_a^\varepsilon + (\tilde{\mathbf{u}}_a^\varepsilon \cdot \nabla_x)\tilde{\mathbf{u}}_a^\varepsilon + \nabla_x\tilde{\mathbf{p}}_a^\varepsilon$. Due to the change of time scale, the bound (3.5) becomes

$$\exists C(T,s,N); \qquad \sup_{t\in[0,\varepsilon^{1/l}T]} \parallel \tilde{\mathbf{f}}_a^\varepsilon(t,\cdot) \parallel_{H^s(\mathbf{R}^d)} \leqslant C(T,s,N)\varepsilon^N. \tag{3.7}$$

In other words, the preceding manipulations allow us to convert $\mathbf{u}_a^\varepsilon(t,x)$ into some large amplitude oscillation $\tilde{\mathbf{u}}_a^\varepsilon(t,x)$ which is proved to be an approximate solution of (2.1) on the time interval $[0,\varepsilon^{1/l}T]$. By this way, they bring informations about the oscillating Cauchy problem made of (2.1) associated with the initial data (3.1). Indeed, select any $\alpha \in]0,1]$. To seek a WKB expansion which is issued from (2.1), (3.1) and which makes sense on the time interval $[0,\varepsilon^\alpha]$, it suffices to select $l \geqslant 1/\alpha$ and to proceed as above, i.e., to use $\tilde{\mathbf{u}}_a^\varepsilon(t,x)$.

Note that the structure of $\tilde{\mathbf{u}}_a^\varepsilon$ becomes more and more complicated when l is increasing. Gradually, all the adjusting phases φ_j, $1 \leqslant j \leqslant \infty$, play a part at the level of the leading oscillating term. This confirms that all the terms φ_j, $1 \leqslant j \leqslant \infty$, have straight off some intrinsic sense.

Observe also that times $O(1)$ are not reached by this method. On one hand, we do not know how the terms φ_j and $\tilde{U}_j$ go together as $l \to +\infty$. On the other hand, even if $\tilde{\mathbf{u}}_a^\varepsilon$ is globally defined (and therefore is a good candidate to deal with), no control on the size (the smallness as $\varepsilon \to 0$) of $C(\varepsilon^{-1/l}T,s,N)\varepsilon^N$ has yet been obtained. Therefore, the pertinence of $\tilde{\mathbf{u}}_a^\varepsilon$ for times $O(1)$ is not sure to hold.

In short, concerning (2.1), general large amplitude WKB computations based on the standard formula

$$\mathbf{u}^\varepsilon(t,x) \sim \mathbf{u}_a^\varepsilon(t,x) := \sum_{j=0}^{\infty} \varepsilon^j U_j(t,x,\varphi_0(t,x)/\varepsilon), \quad \varepsilon \to 0, \tag{3.8}$$

do not work. A way to proceed is to pass as above through (2.9) in order to reach times $O(\varepsilon^{1/l})$ for any $l \in \mathbb{N}_*$. What happens for times $O(1)$ is not yet clear. By (3.6), structures more complicated than (3.8) can spontaneously appear.

In what follows, we still work with (3.8), but we take into account only special situations. The purpose now is indeed t o prepare ψ and the initial profiles U_j^0 in a way to be sure that the incoming wave (3.8) can be pertinent. To understand the underlying matter, we advise the reader to refer to the recent work [**6**]. Our goal here is only to illustrate through specific examples, in a way as simple as possible, a few ideas already contained in [**6**].

3.2. Special oscillating initial data.

To simplify, we work in space dimension two ($d = 2$). This is a much more easier case since the global in time existence is then guaranteed by standard results. Note however that, due to (1.6), we have $\| \operatorname{curl} \mathbf{u}^\varepsilon(0, \cdot) \|_p = O(1/\varepsilon)$ for all L^p norms $\| \cdot \|_p$. Therefore, the situations under study get out (as $\varepsilon \to 0$) the context of [**9**].

We will moreover limit our study to very special data. Select two arbitrary scalar functions $f \in C^\infty(\mathbf{R};\mathbf{R})$ and $g \in C^\infty(\mathbf{R};\mathbf{R})$. Choose a C^1 initial phase ψ which is defined on some open set $\omega \subset \mathbf{R}^2$, is bounded on ω with the bounded derivatives

$$\sup_{x\in\omega} |\psi(x)| < \infty, \quad \sup_{x\in\omega} |\nabla_x \psi(x)| < \infty,$$

and is such that

$$\partial_1 \psi(x) = f(\psi(x)) \partial_2 \psi(x) \quad \forall x \in \omega. \tag{3.9}$$

For instance, we can impose

$$\psi(0, x_2) = \psi_0(x_2) \quad \forall x_2 \in \mathbf{R}, \quad \psi_0 \in C_b^1(\mathbb{R}) \tag{3.10}$$

and obtain $\psi(x)$ by solving (3.9), (3.10) on the strip $\omega =]-X_1, X_1[\times \mathbf{R}$ for some suitable $X_1 > 0$.

Consider also scalar profiles $p^\varepsilon(r,\theta) \in C^\infty(\mathbf{R}\times\mathbb{T};\mathbf{R})$, $q^\varepsilon(r,\theta) \in C^\infty(\mathbf{R}\times\mathbb{T};\mathbf{R})$, $\varepsilon \in [0,1]$, which are smooth with respect to the parameter $\varepsilon \in [0,1]$. Note that $p^\varepsilon = p^0 + \varepsilon p^1 + O(\varepsilon^2)$ and $q^\varepsilon = q^0 + \varepsilon q^1 + O(\varepsilon^2)$.

Suppose that $\partial_\theta p^0 \not\equiv 0$ and p^ε and q^ε are linked together by the relation

$$\begin{aligned} g' - g f f'/(1+f^2) + f' p^\varepsilon + (1+f^2)\partial_\theta q^\varepsilon \\ + \varepsilon(1+f^2)\partial_r q^\varepsilon + \varepsilon f f' q^\varepsilon = 0. \end{aligned} \tag{3.11}$$

At the initial time $t = 0$, we impose

$$\mathbf{u}^\varepsilon(0,x) = \frac{g(\psi(x))}{1+f(\psi(x))^2}\begin{pmatrix} f(\psi(x)) \\ 1 \end{pmatrix} + p^\varepsilon(\psi(x),\psi(x)/\varepsilon)\begin{pmatrix} -1 \\ f(\psi(x)) \end{pmatrix} + \varepsilon q^\varepsilon(\psi(x),\psi(x)/\varepsilon)\begin{pmatrix} f(\psi(x)) \\ 1 \end{pmatrix} \quad \forall x \in \omega. \tag{3.12}$$

We recover the form (3.1) with, in particular,

$$U_0^0(x,\theta) = \frac{g(\psi(x))}{1+f(\psi(x))^2}\begin{pmatrix} f(\psi(x)) \\ 1 \end{pmatrix} + p^0(\psi(x),\theta)\begin{pmatrix} -1 \\ f(\psi(x)) \end{pmatrix},$$

$$U_1^0(x,\theta) = q^0(\psi(x),\theta)\begin{pmatrix} f(\psi(x)) \\ 1 \end{pmatrix} + p^1(\psi(x),\theta)\begin{pmatrix} -1 \\ f(\psi(x)) \end{pmatrix}.$$

By (3.9), the relation (3.11) is exactly what is needed to guarantee the divergence free condition

$$\mathrm{div}_x\, \mathbf{u}^\varepsilon(0,x) = 0 \quad \forall x \in \omega. \tag{3.13}$$

Example 3.2.1 (linear phases)**.** The choices $f \equiv a \in \mathbf{R}$ and $g \equiv b \in \mathbf{R}$ are compatible with the selection of

$$\psi(x) = \chi(ax_1 + x_2), \quad \chi \in C^\infty(\mathbf{R};\mathbf{R}). \tag{3.14}$$

Then, if we take $q^\varepsilon \equiv 0$, we can choose any profile p^ε without any contradiction with (3.11). It remains the oscillating initial data

$$\hat{\mathbf{u}}^\varepsilon(0,x) = \hat{\mathbf{U}}_0^\varepsilon(\chi(ax_1+x_2), \chi(ax_1+x_2)/\varepsilon) \tag{3.15}$$

with

$$\hat{\mathbf{U}}_0^\varepsilon(r,\theta) = \frac{b}{1+a^2}\begin{pmatrix} a \\ 1 \end{pmatrix} + p^\varepsilon(r,\theta)\begin{pmatrix} -1 \\ a \end{pmatrix}.$$

The Cauchy problem (2.1), (3.15) is easy to solve. The solution is explicit. It is the simple wave $\hat{\mathbf{u}}^\varepsilon(t,x) = \hat{\mathbf{U}}_0^\varepsilon(\chi(ax_1+x_2-bt))$, $\hat{\mathbf{p}}^\varepsilon(t,x) = 0$. Observe that the weak limit of the family $\{\hat{\mathbf{u}}^\varepsilon\}_\varepsilon$ is

$$\langle \hat{\mathbf{u}} \rangle(t,x) := \frac{b}{1+a^2}\begin{pmatrix} a \\ 1 \end{pmatrix} + \bar{p}^0(\chi(ax_1+x_2-bt))\begin{pmatrix} -1 \\ a \end{pmatrix}$$

which is obviously still a solution of (2.1) with $\mathbf{p} \equiv 0$.

Remark 3.2.1 (nonlinear phases)**.** By (3.9), linear phases such as (3.14) are possible only if $f' \equiv 0$. Equation (3.9) allows us to take into account functions f with $f' \not\equiv 0$, Therefore, it contains many generalizations of (3.15). The relation (3.9) means that ψ is constant on pieces of lines. The geometrical interpretation of the condition $f' \not\equiv 0$ is that ψ is not constant on *parallel* lines. When $f' \not\equiv 0$, because of the formation of shocks, smooth solutions of (3.9) can exist only locally, on some open domain ω strictly included in $\mathbf{R}^2$. Moreover, if f' is nowhere zero, the function p^ε can be deduced from q^ε through (3.11).

Remark 3.2.2 (no creation of phase shifts)**.** It is interesting to test the condition (2.16) in the case (3.12). This means to replace $\varphi_0(0,\cdot)$ by $\psi(\cdot)$ and $U_1^*(0,\cdot)$ by $U_0^{0*}(\cdot)$. First, use (3.9) to obtain

$$\operatorname{div}_x \langle U_0^{0*} \otimes U_0^{0*} \rangle = \begin{pmatrix} -\partial_2[\chi(\psi)] \\ \partial_1[\chi(\psi)] \end{pmatrix},$$

where $\chi(r)$ is a function such that $\chi'(r) = f'(r)\langle p^{0*}(r,\cdot)^2\rangle$ for all $r \in \mathbf{R}$. It is obvious that $P \operatorname{div}_x \langle U_0^{0*} \otimes U_0^{0*} \rangle = \operatorname{div}_x \langle U_0^{0*} \otimes U_0^{0*} \rangle$, so that it remains

$$\nabla_x \psi \cdot P \operatorname{div}_x \langle U_0^{0*} \otimes U_0^{0*} \rangle = \chi'(\psi)(\partial_1\psi, \partial_2\psi) \cdot \begin{pmatrix} -\partial_2\psi \\ \partial_1\psi \end{pmatrix} \equiv 0.$$

The conclusion is that data like (3.12) do not give rise to the phenomenon of cascade of phase shifts quoted in Section 2. This is already an indication that the monophase large amplitude structure (3.1) can be preserved when it is issued from data as in (3.12).

3.3. Special local solutions.

Introduce the functions $s^\varepsilon(r) := g(r) + \varepsilon(1 + f(r)^2) q^\varepsilon(r, r/\varepsilon)$, $\varepsilon \in]0,1]$, and compute

$$\begin{aligned} \partial_2[s^\varepsilon(\psi)] &= g'(\psi)\partial_2\psi + (1 + f(\psi)^2)\partial_\theta q^\varepsilon(\psi, \psi/\varepsilon)\partial_2\psi \\ &\quad + \varepsilon(1 + f(\psi)^2)\partial_r q^\varepsilon(\psi, \psi/\varepsilon)\partial_2\psi + 2\varepsilon f(\psi) f'(\psi) q^\varepsilon(\psi, \psi/\varepsilon)\partial_2\psi. \end{aligned}$$

By the above assumptions on f, g, and ψ, we have

$$M_1 := \sup_{\varepsilon \in]0,1]} \sup_{x \in \omega} |s^\varepsilon(\psi(x))| < \infty, \quad M_2 := \sup_{\varepsilon \in]0,1]} \sup_{x \in \omega} |\partial_2[s^\varepsilon(\psi(x))]| < \infty.$$

It follows that the Cauchy problem

$$\partial_t \varphi^\varepsilon + s^\varepsilon(\varphi^\varepsilon)\partial_2\varphi^\varepsilon = 0, \quad \varphi^\varepsilon(0,x) = \psi(x) \quad \forall x \in \omega, \tag{3.16}$$

can be solved on the domain of determinacy $\Omega := \{(t,x) \in \mathbf{R}^+ \times \mathbf{R}^2; 0 \leqslant t \leqslant M_2^{-1}, (x_1, x_2 + sM_1) \in \omega \ \forall s \in [-t,t]\}$. It is obvious that Ω is independent of $\varepsilon \in]0,1]$ and is such that $\Omega \cap (\{0\} \times \mathbf{R}^2) = \omega$, $\Omega \cap (\mathbf{R}_*^+ \times \mathbf{R}^2) \neq \varnothing$. With (3.16), we find $[\partial_t + s^\varepsilon(\varphi^\varepsilon)\partial_2](\partial_1\varphi^\varepsilon - f(\varphi^\varepsilon)\partial_2\varphi^\varepsilon) = -\partial_2[s^\varepsilon(\varphi^\varepsilon)](\partial_1\varphi^\varepsilon - f(\varphi^\varepsilon)\,\partial_2\varphi^\varepsilon)$. This implies that the relation (3.9) is preserved during the evolution

$$\partial_1\varphi^\varepsilon - f(\varphi^\varepsilon)\partial_2\varphi^\varepsilon = 0 \quad \forall(\varepsilon, t, x) \in]0,1] \times \Omega. \tag{3.17}$$

Consider the expression

$$\mathbf{u}^\varepsilon(t,x) = \tilde{\mathbf{u}}^\varepsilon(\varphi^\varepsilon(t,x)) = \tilde{\mathbf{U}}^\varepsilon(\varphi^\varepsilon(t,x), \varphi^\varepsilon(t,x)/\varepsilon) \tag{3.18}$$

with

$$\tilde{\mathbf{U}}^\varepsilon(r,\theta) = \frac{g(r)}{1+f(r)^2}\begin{pmatrix} f(r) \\ 1 \end{pmatrix} + p^\varepsilon(r,\theta)\begin{pmatrix} -1 \\ f(r) \end{pmatrix} + \varepsilon q^\varepsilon(r,\theta)\begin{pmatrix} f(r) \\ 1 \end{pmatrix}. \tag{3.19}$$

Consider

$$\begin{aligned} \partial_t\varphi^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla_x\varphi^\varepsilon &= \partial_t\varphi^\varepsilon + (1+f(\varphi^\varepsilon)^2)^{-1} g(\varphi^\varepsilon)(f(\varphi^\varepsilon)\partial_1\varphi^\varepsilon + \partial_2\varphi^\varepsilon) \\ &+ p^\varepsilon(\varphi^\varepsilon, \varphi^\varepsilon/\varepsilon)(-\partial_1\varphi^\varepsilon + f(\varphi^\varepsilon)\partial_2\varphi^\varepsilon) + \varepsilon q^\varepsilon(\varphi^\varepsilon, \varphi^\varepsilon/\varepsilon)(f(\varphi^\varepsilon)\partial_1\varphi^\varepsilon + \partial_2\varphi^\varepsilon). \end{aligned}$$

Using (3.16) and (3.17), simplify it as follows:

$$\partial_t\varphi^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla_x\varphi^\varepsilon = \partial_t\varphi^\varepsilon + s^\varepsilon(\varphi^\varepsilon)\partial_2\varphi^\varepsilon = 0.$$

Therefore, for all $(t,x) \in \Omega$ we have

$$\partial_t\mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla_x\mathbf{u}^\varepsilon = (\partial_t\varphi^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla_x\varphi^\varepsilon)(\partial_r\tilde{\mathbf{u}}^\varepsilon)(\varphi^\varepsilon) = 0 \tag{3.20}$$

and, exploiting again (3.11), we find

$$\operatorname{div}_x \mathbf{u}^\varepsilon(t,x) = 0 \quad \forall(t,x) \in \Omega. \tag{3.21}$$

In other words, the expressions $\mathbf{u}^\varepsilon$ are on ω pressureless solutions of (2.1). Note that the functions $\mathbf{u}^\varepsilon$ are uniformly bounded:

$$\sup_{\varepsilon\in]0,1]} \sup_{(t,x)\in\Omega} |\mathbf{u}^\varepsilon(t,x)| \leqslant M < \infty.$$

On one hand, this majoration implies that the speed of propagation is uniformly bounded. On the other hand, following [**10**], it gives rise to a Young measure

$$\nu\,\Omega \longrightarrow ProbM(\mathbf{R}^2) \qquad (t,x) \longmapsto \nu_{(t,x)}(u)$$

which is a (locally) measure-valued solution of (2.1). In other words, the following equation is satisfied in the weak sense:

$$\partial_t\langle\nu, u\rangle + \operatorname{div}_x\langle\nu, u\otimes u\rangle + \nabla_x p = 0, \quad \operatorname{div}_x\langle\nu, u\rangle = 0. \tag{3.22}$$

In the next section, in order to capture ν, we study more precisely the family $\{\mathbf{u}^\varepsilon\}_{\varepsilon\in]0,1]}$ as $\varepsilon \to 0$.

3.4. The asymptotic behavior of the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$.

Consider the expression $\varphi_0 \in C^1(\Omega; \mathbf{R})$ which is obtained by solving the Cauchy problem

$$\partial_t \varphi_0 + g(\varphi_0)\partial_2 \varphi_0 = 0, \qquad \varphi_0(0,x) = \psi(x) \quad \forall x \in \omega. \tag{3.23}$$

Either directly from (3.9), (3.23) or from (3.17) we can extract

$$\partial_1 \varphi_0 - f(\varphi_0)\partial_2 \varphi_0 = 0 \quad \forall (t,x) \in \Omega. \tag{3.24}$$

Then decompose φ^ε into $\varphi^\varepsilon(t,x) = \varphi_0(t,x) + \varepsilon \Phi_1^\varepsilon(t,x,\varphi_0(t,x)/\varepsilon)$. By (3.16), the profile $\Phi_1^\varepsilon(t,x,\theta)$ must satisfy

$$\Phi_1^\varepsilon(0,x,\theta) = 0 \quad \forall (\varepsilon,x,\theta) \in]0,1] \times \omega \times \mathbb{T}. \tag{3.25}$$

Plug φ^ε as above in (3.16). Use (3.23) to make simplifications. It remains to consider the equation

$$\begin{aligned}
&\partial_t \Phi_1^\varepsilon + g(\varphi_0 + \varepsilon \Phi_1^\varepsilon)\partial_2 \Phi_1^\varepsilon \\
&+ \varepsilon(1 + f(\varphi_0 + \varepsilon\Phi_1^\varepsilon)^2) q^\varepsilon(\varphi_0 + \varepsilon\Phi_1^\varepsilon, \theta + \Phi_1^\varepsilon)\partial_2 \Phi_1^\varepsilon \\
&+ w^\varepsilon(t,x,\theta,\Phi_1^\varepsilon)\partial_\theta \Phi_1^\varepsilon + w^\varepsilon(t,x,\theta,\Phi_1^\varepsilon) = 0
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
w^\varepsilon(t,x,\theta,\lambda) :=& \varepsilon^{-1}[\partial_t \varphi_0 + s^\varepsilon(\varphi_0 + \varepsilon\lambda)\partial_2\varphi_0] \\
&= \left(\int_0^1 g'(\varphi_0 + \varepsilon s \lambda)ds \right) \partial_2 \varphi_0 \lambda \\
&\qquad + (1 + f(\varphi_0 + \varepsilon\lambda)^2) q^\varepsilon(\varphi_0 + \varepsilon\lambda, \theta + \lambda)\partial_2 \varphi_0.
\end{aligned} \tag{3.27}$$

The function w^ε is smooth with respect to $(t,x,\theta,\lambda) \in \Omega \times \mathbb{T} \times \mathbf{R}$ and also $\varepsilon \in [0,1]$. Therefore, the solution Φ_1^ε of (3.25)–(3.26) is smooth with respect to the same variables. In particular, we can get a complete expansion of Φ_1^ε in powers of ε:

$$\Phi_1^\varepsilon(t,x,\theta) = \sum_{j=0}^{N} \varepsilon^j \Phi_1^j(t,x,\theta) + O(\varepsilon^{N+1}), \quad N \gg 1.$$

In particular, the first contribution $\Phi_1^0(t,x,\theta)$ is subjected to the scalar conservation law

$$\partial_t \Phi_1^0 + g(\varphi_0)\partial_2 \Phi_1^0 + w^0(t,x,\theta,\Phi_1^0)\partial_\theta \Phi_1^0 + w^0(t,x,\theta,\Phi_1^0) = 0, \tag{3.28}$$

where $w^0(t,x,\theta,\lambda) = g'(\varphi_0)\partial_2\varphi_0\lambda + (1 + f(\varphi_0)^2) q^0(\varphi_0, \theta + \lambda)\partial_2\varphi_0$.

By (3.25), the evolution equation (3.28) must be completed with the following initial data:

$$\Phi_1^0(0,x,\theta)=0 \quad \forall (x,\theta)\in \omega\times\mathbb{T}. \tag{3.29}$$

Coming back to (3.18), we can associate to $\{\mathbf{u}^\varepsilon\}_\varepsilon$ the following monophase description: $\mathbf{u}^\varepsilon(t,x)=\mathbf{U}^0(t,x,\varphi_0(t,x)/\varepsilon)+O(\varepsilon)$, where the main profile $\mathbf{U}^0$ is defined according to

$$\begin{aligned}\mathbf{U}^0(t,x,\theta)=&\ \frac{g(\varphi_0(t,x))}{1+f(\varphi_0(t,x))^2}\begin{pmatrix} f(\varphi_0(t,x)) \\ 1\end{pmatrix}\\ &+p^0(\varphi_0(t,x),\theta+\Phi_1^0(t,x,\theta))\begin{pmatrix} -1 \\ f(\varphi_0(t,x))\end{pmatrix}.\end{aligned} \tag{3.30}$$

Of course, at the initial time $t=0$, we recover $\mathbf{U}^0(0,x,\theta)=U_0^0(x,\theta)$ for all $(x,\theta)\in\omega\times\mathbb{T}$. On the other hand, for all $g\in C^0(\mathbf{R}^2;\mathbf{R}^2)$ and for all $\varphi\in C_0(\Omega)$, we have

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0}\iint_{\Omega} g(\mathbf{u}^\varepsilon(t,x))\varphi(t,x)dtdx &= \iint_{\Omega}\langle \nu_{(t,x)},g(u)\rangle\varphi(t,x)dtdx\\ &=\iiint_{\Omega\times\mathbb{T}} g(\mathbf{U}^0(t,x,\theta))\varphi(t,x)dtdxd\theta.\end{aligned}$$

By construction, the Young measure ν is a measure valued solution of (2.1). But is it possible to use it for defining a solution $\mathbf{u}(t,x)$ of (2.1) in the classical weak sense? This question is discussed in Section 3.5 and in other issues concerning $\mathbf{U}^0$.

3.5. Conclusion and remarks.

The Navier–Stokes equations (with Reynolds number ε^{-1}) are given by

$$\partial_t\mathbf{v}^\varepsilon+(\mathbf{v}^\varepsilon\cdot\nabla_x)\mathbf{v}^\varepsilon+\nabla_x\mathbf{p}^\varepsilon=\varepsilon\Delta_x\mathbf{v}^\varepsilon,\quad \operatorname{div}_x\mathbf{v}^\varepsilon=0. \tag{3.31}$$

Fix initial data

$$\mathbf{v}^\varepsilon(0,x)=\mathbf{v}_0(x). \tag{3.32}$$

The structure of $\mathbf{v}^\varepsilon(t,x)$ as $\varepsilon\to 0$ is a problem of wide current interest [**1, 9, 10, 18**]. The same comment applies to other approximations of the Euler equations (2.1). If $\mathbf{v}_0$ is smooth, say $\mathbf{v}_0\in C_0^\infty(\mathbf{R}^d)$, there exists a fixed interval of time $[0,T]$, $T>0$, where the Navier–Stokes solutions $\mathbf{v}^\varepsilon$ converge strongly in L^2. Moreover, the limiting fields $\mathbf{v}$ are on the strip $[0,T]\times\mathbf{R}^d$ conventional solutions of (2.1).

Now, the complexity of the flow can increase as time evolves. After the time T, the solutions $\mathbf{v}^\varepsilon$ may converge weakly in L^2 (instead of converging strongly in L^2) due to the development of oscillations or concentrations. The following majoration

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \| \mathbf{v}^\varepsilon(t, \cdot) \|_{L^2(\mathbf{R}^2)} < \infty \tag{3.33}$$

is the only control which is known to be uniform in $\varepsilon \in]0,1]$. Of course, it suffices to extract a Young measure ν (see [**10**]) and, in particular, to isolate a weak limit $\mathbf{v}(t,x) \in L^2$. But is $\mathbf{v}$ still a weak solution of (2.1)?

Our goal is to show that the following *local* version of (3.33)

$$\sup_{\varepsilon \in]0,1]} \sup_{(t,x) \in \Omega} \| \mathbf{u}^\varepsilon \|_{L^2(\Omega)} < \infty, \quad \Omega \text{ is an open domain of } \mathbf{R} \times \mathbf{R}^2, \tag{3.34}$$

is not sufficient to deduce that $\mathbf{v}$ is still a weak solution of (2.1).

We will not deal directly with (3.31), (3.32). Instead, we consider Equation (2.1). We want to model the situation which is alluded above (after the time T). For this purpose,, instead of fixing the initial data (as in (3.32)), we look at a family $\{\mathbf{u}^\varepsilon(0,\cdot)\}_{\varepsilon \in]0,1]}$ of initial data such that $\mathbf{u}^\varepsilon(0,\cdot)$ converges (as $\varepsilon \to 0$) weakly (but not strongly) in L^2.

In fact, we consider the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$ constructed in the previous sections. The functions $\mathbf{u}^\varepsilon$ satisfy (2.1) and (3.34). The corresponding weak limit has been identified. It is

$$\mathbf{u}(t,x) = \langle \nu_{(t,x)}, u \rangle = \bar{\mathbf{U}}^0(t,x) = \int_{\mathbb{T}} \mathbf{U}^0(t,x,\theta) d\theta.$$

Therefore, if it would be possible (through some kind of compensated compactness argument) to pass to the limit as $\varepsilon \to 0$ from (3.31) to (2.1) *by using only* Equation (3.31) and estimates like (3.34), then both $\mathbf{v}(t,x)$ and $\mathbf{u}(t,x)$ should be weak solutions of (2.1).

However, this is not always the case. Objections can come from the presence of oscillations as these contained in $\{\mathbf{u}^\varepsilon\}_\varepsilon$. To see this, we have to compute

$$\mathbf{f}(t,x) := \partial_t \mathbf{u}(t,x) + [(\mathbf{u} \cdot \nabla_x)\mathbf{u}](t,x). \tag{3.35}$$

Because of the explicit formula (3.30) and Equations (3.23) and (3.24), the function $\mathbf{f}$ can be reduced to

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix} := \langle \partial_\theta p^0(\varphi_0, \theta + \Phi_1^0)(\partial_t \Phi_1^0 + (\mathbf{u} \cdot \nabla_x)\Phi_1^0) \rangle \begin{pmatrix} -1 \\ f(\varphi_0) \end{pmatrix}.$$

By (3.28) and (3.29), we have $\partial_t \Phi_1^0(0,x,\theta) = -w^0(0,x,\theta,0)$ and $\mathbf{u}\cdot\nabla_x)\Phi_1^0(0,x,\theta) = 0$. It follows that

$$\mathbf{f}(0,x) := -(1+f(\psi)^2)\langle(\partial_\theta p^0 q^0)(\psi,\theta)\rangle \partial_2\psi \begin{pmatrix} -1 \\ f(\psi) \end{pmatrix}. \tag{3.36}$$

The constraint (3.11) implies that

$$g' - gff'/(1+f^2) + f'p^0 + (1+f^2)\partial_\theta q^0 = 0. \tag{3.37}$$

This relation (3.37) can be split in two conditions

$$g' - gff'/(1+f^2) + f'\bar{p}^0 = 0, \tag{3.38}$$

$$f'p^{0*} + (1+f^2)\partial_\theta q^0 = 0. \tag{3.39}$$

Equation (3.38) means that $\bar{p}^0$ can be determined from f and g. Equation (3.39) imposes a link between p^{0*} and $\partial_\theta q^0$. Since $\langle\partial_\theta p^0 q^0\rangle = -\langle p^{0*}\partial_\theta q^0\rangle$, we have, in fact, to deal with

$$\mathbf{f}(0,x) := -f'(\psi)\langle p^{0*}(\psi,\theta)^2\rangle \partial_2\psi \begin{pmatrix} -1 \\ f(\psi) \end{pmatrix}. \tag{3.40}$$

Introduce a function $K \in C^1(\mathbf{R};\mathbf{R})$ such that $K'(r) = -f'(r)\langle p^{0*}(r,\theta)^2\rangle$ for all $r \in \mathbf{R}$. Use (3.9) to interpret (3.40) according to

$$\mathbf{f}(0,x) := \begin{pmatrix} -\partial_2[K(\psi)] \\ \partial_1[K(\psi)] \end{pmatrix}. \tag{3.41}$$

Both function $\mathbf{u}(t,x)$ and source term $\mathbf{f}(t,x)$ are smooth (at least, of class C^1). Therefore, we can state that the weak limit $\mathbf{u}(t,x)$ is not a solution of (2.1) if and only if there exists no scalar function $\mathbf{p}$ such that $\mathbf{f} = \nabla_x \mathbf{p}$, or if and only if

$$\operatorname{curl}\mathbf{f} := \partial_1 \mathbf{f}_2 - \partial_2 \mathbf{f}_1 = \Delta_x[K(\psi)] \not\equiv 0. \tag{3.42}$$

It remains to check this condition on formula (3.40). We find

$$\operatorname{curl}\mathbf{f}(0,x) = \partial_2\{[1+f(\psi(x))^2]\partial_2[K(\psi(x))]\}.$$

Recall that the data f, g, p^{0*} and also $\psi_0 \equiv \psi_{|x_1=0}$ can be chosen arbitrarily. In particular, they can be adjusted so that there exists $x_2 \in \mathbf{R}$ such that $\operatorname{curl}\mathbf{f}(0,0,x_2) \neq 0$, showing that the weak limit $\mathbf{u}$ is not necessarily a solution of (2.1).

The preceding reasoning underlies a result which is pushed forward in [**6**]. For the sake of completeness, we recall it below.

Theorem 1.1 (see [**6**])**.** *There is a bounded open domain $\Omega \subset \mathbb{R}\times\mathbb{R}^2$ and a family of functions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ such that*

(i) $\mathbf{u}^\varepsilon \in C^1(\Omega)$, $\sup\{\| \mathbf{u}^\varepsilon \|_{L^\infty(\Omega)} ; \varepsilon \in]0,1]\} < \infty$,

(ii) $\mathbf{u}^\varepsilon$ *is a solution of* (2.1) *on* Ω,

(iii) $\mathbf{u}^\varepsilon$ *converges weakly to* $\mathbf{u}^0 \in C^1(\Omega)$ *as* $\varepsilon \to 0$.

But $\mathbf{u}^0$ *is not a solution of* (2.1).

This result must be connected with the question raised in [**18**, p. 479] since it produces local obstructions to the concentration–cancellation property. We refer to [**6**] for details. In [**6**], the goal is to put in place a nonlinear geometric optics under constraint for Burger type equations. Many new phenomena, including the creation of $O(\varepsilon^{-2})$ scales by interaction of $O(\varepsilon^{-1})$ transversal oscillations, are revealed in [**6**].

Remark 3.5.1 (on nonlinearity of f)**.** Note that if $f' \equiv 0$, we have $\mathbf{f}(0, \cdot) \equiv 0$. This is the reason why the preceding arguments do not apply when appealing only to simple waves involving linear phases, like in (3.15). In fact [**6**], weak limits of the families $\{\hat{\mathbf{u}}^\varepsilon\}_\varepsilon$ are always still solutions of (2.1) (see Example 3.2.1).

Consider a general sequence $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$ uniformly bounded in L^2 and made of approximate or exact solutions $\tilde{\mathbf{u}}^\varepsilon$ of (2.1). It would be very helpful to find a criterion allowing us to decide if the weak extracted limits are still solutions of (2.1) or not. For instance, this could be applied when passing to the limit by vanishing viscosity ($\varepsilon \to 0$) in the Navier–Stokes equations (3.31). The above discussion does not furnish such a criterion. The constraint (3.42) can be expressed explicitly only in the case of the families $\{\mathbf{u}^\varepsilon\}_\varepsilon$ under consideration. Yet, it indicates that, in order to have (3.42), it is necessary to impose the nonlinearity of the phase (induced here by the condition $f' \not\equiv 0$) that is rapid variations in moving directions.

Remark 3.5.2 (on interdependence between $O(1)$ and $O(\varepsilon)$ terms)**.** The above construction also attracts attention to another more subtle implemented effect which is important to notice. Seeking an equation on $\mathbf{u}$ like (3.35) or, more generally, writing some closed set of equations in order to deduce $\mathbf{U}^0(t, \cdot)$ from $U_0^0(\cdot)$ (as it is proposed, for instance, in [**22**]) means implicitly that the time evolution respects the hierarchy between the first order contributions (i.e., of size ε^0) and the lower order terms (say, of size ε^1 or less): the first ones can be determined before the second ones.

However, in supercritical regimes such a separation between $O(1)$ and $O(\varepsilon)$ contributions turns to be artificial. A similar observation has already been made in the context of *time* oscillations [**12**]. Let us explain how it can be put in a specific form when dealing with *space* oscillations. To do this, we examine more carefully what tells the study of $\mathbf{U}^0$.

The preliminary remark is concerned with the preparation of the initial data $\mathbf{U}^0(0,\cdot) \equiv U_0^0$ (see Section 3.2). The definition of U_0^0 can already not be dissociated from what happens at the level of the $O(\varepsilon)$ remainders. Indeed, do not forget (3.11) and its consequence (3.37): the profiles U_0^0 and U_1^0 cannot be fixed independently.

Otherwise, the definition of $\mathbf{U}^0(t,\cdot)$ involves $\Phi_1^0(t,\cdot)$ which itself depends on q^0 through (3.28). To verify that the influence of q^0 on $\mathbf{U}^0(t,\cdot)$ actually occurs, it suffices to measure it for small times. From (3.26) deduce that $\Phi_1^0(t,x,\theta) = -(1+f(\psi(x))^2)q^0(\psi(x),\theta)t + O(t^2)$. It follows that

$$\begin{aligned} &p^0(\varphi_0(t,x),\theta+\Phi_1^0(t,x,\theta)) = p^0 l(\varphi_0(t,x),\theta) \\ &- t(1+f(\psi(x))^2)\partial_\theta p^0(\varphi_0(t,x),\theta)q^0(\psi(x),\theta) + O(t^2). \end{aligned}$$

Then it suffices to plug this time expansion inside (3.30). In particular, if $g \equiv 0$, we have $\varphi_0(t,\cdot) \equiv \psi(\cdot)$ for all $t \in [0,T]$ and it remains

$$\begin{aligned} \mathbf{U}^0(t,x,\theta) &= p^0(\psi(x),\theta) \begin{pmatrix} -1 \\ f(\psi(x)) \end{pmatrix} \\ &- t(1+f(\psi(x))^2)(\partial_\theta p^0 q^0)(\psi(x),\theta) \begin{pmatrix} -1 \\ f(\psi(x)) \end{pmatrix} + O(t^2). \end{aligned} \tag{3.43}$$

We recover here, in factor of t, the expression $\partial_\theta p^0 q^0$ which has already been observed at the level of (3.36). This product $\partial_\theta p^0 q^0$ combines the term p^{0*} with ε^0 in factor at the level of $\mathbf{u}^\varepsilon(0,\cdot)$ and the term q^0 with ε^1 in factor at the level of $\mathbf{u}^\varepsilon(0,\cdot)$.

This mixing during the time evolution between an $O(1)$ term and an $O(\varepsilon)$ term was partly responsible for (3.42). It shows definitely that *$O(\varepsilon)$ perturbations at the initial time $t=0$ can have some nontrivial $O(1)$ influence at a further time $t>0$.* This expresses a very strong instability.

Of course, this interpretation could be contested in view of the relation (3.37). Indeed, the constraint (3.37) indicates that the preceding distinction between the orders of p^{0*} and p^{0*} is debatable: in practice, we cannot modify q^{0*} without touching p^{0*}.

But it is still possible to make (at $t=0$) arbitrary perturbations of $\bar{q}^0$ with p^0 (and, therefore, U_0^0 fixed). By (3.43), this modify (at a time $t>0$) the expression $\mathbf{U}^0(t,\cdot)$ and, therefore, the Young measure ν. This allows us to give a certain sense to what is said above in italics.

Remark 3.5.3 (the weak limit is a more rigid object)**.** Recall that the weak limit of $\{\mathbf{u}^\varepsilon\}_\varepsilon$ is as follows:

$$\mathbf{u}(t,x) = \frac{g(\varphi_0)}{1+f(\varphi_0)^2}\begin{pmatrix} f(\varphi_0) \\ 1 \end{pmatrix} + \chi(t,x)\begin{pmatrix} -1 \\ f(\varphi_0) \end{pmatrix}$$

with $\chi(t,x) := \int_{\mathbb{T}} p^0(\varphi_0(t,x), \theta + \Phi_1^0(t,x,\theta))d\theta.$

Suppose that $g \equiv 0$. Simple computations indicate that the function χ satisfies the equation

$$\partial_t\chi = -f'(\psi)\partial_2\psi\langle p^{0*}(\psi,\theta)^2\rangle,$$

where the influence of $\bar{q}^0$ is removed. Therefore, the preceding construction does not imply that $O(\varepsilon)$ perturbations can modify the weak limit. It only means that the profile, the Young measure and other quantities (like the energy) can be changed by this way.

In conclusion, we have made a review of recent progresses [**3**]–[**8**] in large amplitude nonlinear geometric optics. We have also lay stress on the construction (directly extracted from [**6**]) of extensions $\mathbf{u}^\varepsilon$ of the classical simple waves $\hat{\mathbf{u}}^\varepsilon$. On this occasion, we have observed new phenomena which indicate that the study of weak solutions of (2.1) should require both microlocal and nonlinear tools within the framework of a *supercritical* WKB *analysis*. Applications could be a better understanding of turbulent flows.

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Existence Theorems for the 3D–Navier–Stokes System Having as Initial Conditions Sums of Plane Waves

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We study the 3-dimensional Navier–Stokes system on $\mathbb{R}^3$ without external forcing and prove local and global existence theorems for initial conditions which are the Fourier transforms of finite linear combinations of δ–functions. Bibliography: 8 titles.

1. Formulation of the result. We consider the $3D$ Navier–Stokes system on $\mathbb{R}^3$ for incompressible fluids. The viscosity is taken to be 1, and no external forcing is assumed. After the Fourier transform the system takes the form

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$$v(k,t) = \exp\{-t|k|^2\}v(k,0)$$
$$+ i\int_0^t \exp\{-(t-s)|k|^2\}ds \int_{\mathbb{R}^3} \langle v(k-k',s),k\rangle P_k v(k',s)d^3k', \tag{1}$$

where $v(k,0)$ is the initial condition, P_k is the orthogonal projection to the subspace orthogonal to k, i.e., $P_k v = v - \frac{\langle v,k\rangle}{\langle k,k\rangle}k$. The values $v(k,s) \in \mathbb{C}^3$, the incompressibility means that $\langle v(k,t),k\rangle = 0$. If $v(k,t)$ is the Fourier transform of a real-valued function $u(x,t)$, then

$$v(-k,t) = \overline{v(k,t)}. \tag{2}$$

However, in this paper, we consider arbitrary $v(k,t)$ not assuming (2) (see also [**5**]).

Beginning with the works of Leray [**1**], Hopf [**3**], Kato [**4**], people considered the problem of local and global existence of solutions of (1) with initial conditions having finite energy

$$E(0) = \int_{\mathbb{R}^3} \langle v(k,0), \overline{v(k,0)}\rangle d^3k < \infty.$$

In this paper, we deal with another class of initial conditions having infinite energy. Namely, we consider the initial condition of the form

$$v(k,0) = \sum_{j=1}^{n} B_j \delta(k-k_j), \tag{3}$$

where the sum is taken over a finite set $\{k_j\}$ of points $k_j \neq 0$, $\langle B_j,k_j\rangle = 0$ for any j, and $\delta(k-k_j)$ is the delta-function concentrated at k_j. Since $\delta(k-k_j) \notin L^2(\mathbb{R}^3)$, the initial condition $v(k,0)$ has infinite energy.

Denote by $G(k_1,\dots,k_n)$ the semigroup generated by a finite set $\{k_j\}$, i.e., $k \in G(k_1,\dots,k_n)$ if and only if $k = \sum_{j=1}^{n} p_j k_j$ for some nonnegative integers p_j. The main results of this paper are the following theorems.

Theorem 1. *Let* $v(k,0) = \sum_{j=1}^{n} B_j\delta(k-k_j)$. *There exists* $T > 0$ *depending on* $\{k_j\}$, $\{B_j\}$ *such that there exists a solution* $v(k,t)$ *of* (1) *on the interval* $0 \leqslant t \leqslant T$ *which can be written as a signed measure*

$$v(k,t) = \sum_{g \in G(k_1,\ldots,k_n),\ g \neq 0} B_g(t)\delta(k-g). \tag{4}$$

The coefficients $B_g(t)$ *satisfy the inequalities:*

$$|B_g(t)| < t^{\frac{p(g)}{2}} C^{p(g)} \left(\sum_{j=1}^{n} |B_j|\right)^{p(g)} \left(1 - t^{\frac{1}{2}} C \sum_{j=1}^{n} |B_j|\right)^{-1}, \tag{5}$$

where $p(g) = \min\limits_{\sum p_j k_j = g} \sum\limits_{i=1}^{n} p_i$, C *depends only on* k_j.

Theorem 2. *Let* k_j, $1 \leqslant j \leqslant n$, *belong to some cone with angle less than* π, *i.e., the angle between any* k_{j_1} *and* k_{j_2} *is less than* π. *Then for a sufficiently small* B *and an initial condition* $v(k,0)$ *such that* $\sum\limits_{j=1}^{n} |B_j| < B$ *there exists a global solution of* (1) *having the form* (4) *for which*

$$\sum_{g \in G(k_1,\ldots,k_n),\ g \neq 0} |B_g(t)| < \infty.$$

Some existence results for a similar class of initial conditions can be found in [**2, 8**].

2. Proof of Theorems 1 and 2. The proofs of both theorems are based on the method of power series introduced in [**6, 7**]. We consider a one-parameter family of initial conditions $Av(k,0) = v_A(k,0)$ and write down the solution of (1) as a power series with respect to the complex parameter A:

$$v_A(k,t) = Ah_1(k,t) + \sum_{p>1} A^p \int_0^t \exp\{-(t-s)|k|^2\} h_p(k,s)ds, \tag{6}$$

where $h_1(k,t) = \exp\{-t|k|^2\}v(k,0)$. Substituting (6) into (1), we obtain the following system of recurrent relations between the functions $h_p(k,t)$:

$$h_2(k,t) = i \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k v(k',0) \exp\{-t|k-k'|^2 - t|k'|^2\} d^3k'$$

and for $p > 2$

$$h_p(k,t) = i \int_0^t ds_2 \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k h_{p-1}(k',s_2)$$

$$\times \exp\{-t|k-k'|^2-(t-s_2)|k'|^2\}d^3k'$$

$$+\,i\sum_{\substack{p_1+p_2=p\\ p_1,\,p_2>1}}\int\limits_0^t ds_1\int\limits_0^t ds_2\int\limits_{\mathbb{R}^3}\langle h_{p_1}(k-k',s_1),k\rangle P_k h_{p_2}(k',s_2)$$

$$\times \exp\{-(t-s_1)|k-k'|^2-(t-s_2)|k'|^2\}d^3k'$$

$$+\,i\int\limits_0^t ds_1\int\limits_{\mathbb{R}^3}\langle h_{p-1}(k-k',s_1),k\rangle P_k v(k',0)$$

$$\times \exp\{-(t-s_1)|k-k'|^2-t|k'|^2\}d^3k'. \tag{7}$$

In our case of solutions of type (4), the convolutions given by the integrals in (7) are well defined, and we can write

$$h_1(k,t)=\sum_{j=1}^{n}\exp\{-t|k_j|^2\}B_j\delta(k-k_j).$$

Recall that B_j are 3–dimensional vectors $\langle B_j,k_j\rangle=0$. Further,

$$h_2(k,t)=i\int\limits_{\mathbb{R}^3}\sum_{j_1,j_2=1}\langle B_{j_1},k\rangle\left(B_{j_2}-\frac{\langle B_{j_2},k\rangle k}{\langle k,k\rangle}\right)$$

$$\times \exp\{-t|k-k'|^2-t|k'|^2\}\delta(k-k'-k_{j_1})\delta(k'-k_{j_2})d^3k'$$

$$=\sum_{j_1,j_2=1}^{n}B_{j_1,j_2}(t)\delta(k-(k_{j_1}+k_{j_2}))$$

with

$$B_{j_1,j_2}(t)=i\langle B_{j_1},k_{j_2}\rangle\exp\{-t|k_{j_1}|^2-t|k_{j_2}|^2\}$$
$$\times\left(B_{j_2}-\frac{\langle B_{j_2},(k_{j_1}+k_{j_2})\rangle(k_{j_1}+k_{j_2})}{\langle(k_{j_1}+k_{j_2}),(k_{j_1}+k_{j_2})\rangle}\right).$$

If $k_{j_1}+k_{j_2}=0$, then the corresponding term in the last sum is zero.

From the last formula it easily follows that

$$|B_{j_1,j_2}(t)|\leqslant C_1|B_{j_1}||B_{j_2}|, \tag{8}$$

$C_1=\max\limits_{1\leqslant j\leqslant n}|k_j|$, and

$$\sum_{j_1,j_2=1}^{n}|B_{j_1,j_2}(t)|\leqslant C_1\left(\sum_{j=1}^{n}|B_j|\right)^2. \tag{9}$$

The proof of Theorem 1 is based on the following assertion.

Lemma 1. *Assume that for any initial condition* (3) *and* $0 < t < 1$ *the functions* $h_q(k,t)$, $2 \leqslant q < p$, *can be written in the form*

$$h_q(k,t) = \sum_{1 \leqslant j_1,\dots,j_q \leqslant n} B_{j_1\dots j_q}(t)\delta\Big(k - \sum_{j=1}^{q} k_j\Big), \tag{10}$$

where $B_{j_1\dots j_q}(t)$ *are continuous functions of* t *and*

$$|B_{j_1\dots j_q}(t)| \leqslant C_2^{q-1} t^{\frac{q-2}{2}} \prod_{l=1}^{q} |B_{j_l}|, \tag{11}$$

C_2 *is another constant which depends only on the vectors* k_j, $1 \leqslant j \leqslant n$. *Then* (8) *and* (9) *are valid for* $q = p$.

Using Lemma 1, we derive Theorem 1. We have

$$\begin{aligned} v(k,t) &= \sum_{j=1}^{n} B_j \exp\{-t|k|^2\}\delta(k - k_j) \\ &\quad + \sum_{p>1} \sum_{j_1,\dots,j_p=1}^{n} \tilde{B}_{j_1\dots j_p}(t)\delta\left(k - \sum_{l=1}^{p} k_{j_l}\right) \end{aligned} \tag{12}$$

and

$$\tilde{B}_{j_1\dots j_p}(t) = \int_0^t \exp\{-(t-s)|k|^2\} B_{j_1\dots j_p}(s)ds.$$

If $B_{j_1\dots j_p}(t)$ satisfies (8), then from Lemma 1 it follows that

$$|\tilde{B}_{j_1\dots j_p}(t)| \leqslant \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \prod_{l=1}^{p} |B_{j_l}|$$

and

$$\sum_{j_1,\dots,j_p=1}^{n} |\tilde{B}_{j_1\dots j_p}(t)| \leqslant \frac{2}{p} C_2^{p-1} t^{\frac{p}{2}} \left(\sum_{j=1}^{n} |B_j|\right)^p. \tag{13}$$

Therefore, the series $\sum\limits_{p>1} \sum\limits_{j_1,\dots,j_p=1}^{n} |\tilde{B}_{j_1\dots j_p}(t)|$ converges absolutely if

$$t^{\frac{1}{2}} < \min\Big(1, \Big(C_2 \sum_{j=1}^{n} |B_j|\Big)^{-1}\Big).$$

Taking together all terms with the same value of the sum $k_{j_1} + \cdots + k_{j_p}$, we get the representation of the solution $v(k,t)$ as

$$\sum_{g \in G(k_{j_1},\dots,k_{j_p})} B_g(t)\delta(k-g),$$

where for each $B_g(t)$ we have the estimate (5). Theorem 1 is proved. $\square$

Proof of Lemma 1. For $p = 2$ the statement of the lemma is already proved (see (8), (9)). For $p \geqslant 3$ we use the recurrent relation (7). Denote by $h_p^{(p_1,p_2)}(k,t)$ the term in (7) which corresponds to p_1, p_2. Consider the case p_1, $p_2 \geqslant 2$. We have

$$\begin{aligned}
&h_p^{(p_1,p_2)}(k,t) = i\int_0^t ds_1 \int_0^t ds_2 \int_{\mathbb{R}^3} \langle h_{p_1}(k-k',s_1),k\rangle \\
&\times P_k h_{p_2}(k',s_2)\exp\{-(t-s_1)|k-k'|^2-(t-s_2)|k'|^2\}d^3k' \\
&= i\int_0^t ds_1 \int_0^t ds_2 \sum_{j_1\dots j_p=1}^{n} \langle B_{j_1\dots j_{p_1}}(s_1),k\rangle \\
&\times \left(B_{j_{p_1+1}\dots j_p}(s_2) - \frac{\langle B_{j_{p_1+1}\dots j_p}(s_2),k\rangle k}{|k|^2}\right) \\
&\times \left[\left(\exp\{-(t-s_1)|k|^2\}\delta\Big(k-\sum_{l=1}^{p_1} k_{j_l}\Big)\right)\right. \\
&\circledast \left.\left(\exp\{-(t-s_2)|k|^2\}\delta\Big(k-\sum_{l=p_1+1}^{p} k_{j_l}\Big)\right)\right] \\
&= \sum_{j_1\dots j_p}^{n} B_{j_1\dots j_p}^{(p_1,p_2)}(s)\delta\Big(k-\sum_{l=1}^{p} k_{j_l}\Big),
\end{aligned}$$

where $\circledast$ is the convolution and

$$\begin{aligned}
B_{j_1\dots j_p}^{(p_1,p_2)}(t) = i\int_0^t ds_1 \int_0^t ds_2 &\Big\langle B_{j_1\dots j_{p_1}}(s_1), \sum_{l=p_1+1}^{p} k_{j_l}\Big\rangle \\
&\times \exp\Big\{-(t-s_1)\Big|\sum_{l=1}^{p_1} k_{j_l}\Big|^2 - (t-s_2)\Big|\sum_{l=p_1+1}^{p} k_{j_l}\Big|^2\Big\}
\end{aligned}$$

$$\times \left(B_{j_{p_1+1}\dots j_p}(s_2) - \frac{\langle B_{j_{p_1+1}\dots j_p}(s_2), \sum\limits_{l=1}^{p} k_{j_l}\rangle \sum\limits_{l=1}^{p} k_{j_l}}{|\sum\limits_{l=1}^{p} k_{j_l}|^2} \right).$$

Here, we used the incompressibility condition

$$\Big\langle B_{j_1\dots j_{p_1}}(s_1), \sum_{l=1}^{p_1} k_{j_l} \Big\rangle = 0.$$

It is clear that $B_p^{(p_1,p_2)}(t)$ is continuous as a function of $t \in [0,1]$. By the assumption of Lemma 1,

$$\Big| B_{j_1\dots j_p}^{(p_1,p_2)}(s) \Big| \leqslant \int\limits_0^s s_1^{\frac{p_1-2}{2}} \exp\Big\{-(s-s_1)\Big|\sum_{l=1}^{p_1} k_{j_l}\Big|^2\Big\} ds_1$$

$$\times \int\limits_0^s s_2^{\frac{p_2-2}{2}} \exp\Big\{-(s-s_2)\Big|\sum_{l=p_1+1}^{p} k_{j_l}\Big|^2\Big\}\Big|\sum_{l=p_1+1}^{p} k_{j_l}\Big| ds_2$$

$$\times\, C^{p_1-1}\prod_{l=1}^{p_1}|B_{j_l}| C^{p_2-1}\prod_{l=p_1+1}^{p}|B_{j_l}|.$$

In the integral with respect to s_1, we replace the exponent by 1, and we estimate the integral with respect to s_2 with the help of the Cauchy–Schwarz inequality:

$$\int\limits_0^s s_2^{\frac{p_2-2}{2}} \exp\{-(s-s_2)|\overline{k}|^2\}|\overline{k}| ds_2$$

$$\leqslant \Big(|; \int\limits_0^s s_2^{p_2-2} ds_2 \int\limits_0^s \exp\{-2(s-s_2)|\overline{k}|^2\}|\overline{k}|^2 ds_2 \Big)^{\frac{1}{2}}.$$

Thus, we get

$$|B_{j_1\dots j_p}^{(p_1,p_2)}(s)| \leqslant \frac{s^{\frac{p_1}{2}}}{p_1} \frac{s^{\frac{p_2-1}{2}}}{\sqrt{\frac{p_2-1}{2}}} C^{p-2}\prod_{l=1}^{p}|B_{j_l}| \leqslant \frac{s^{\frac{p-2}{2}}}{p_1\sqrt{\frac{p_2-1}{2}}} C^{p-2}\prod_{l=1}^{p}|B_{j_l}|.$$

At the last step, we used the fact that $0 \leqslant s \leqslant 1$. The first and last terms in (7) are estimated in a similar way:

$$|B^{(1,p-1)}_{j_1\ldots j_p}(s)| \leqslant \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2} \prod_{l=1}^{p} |B_{j_l}|$$

and

$$|B^{(p-1,1)}_{j_1\ldots j_p}(s)| \leqslant \frac{s^{\frac{p-2}{2}}}{\sqrt{\frac{p-2}{2}}} C^{p-2} \prod_{l=1}^{p} |B_{j_l}|.$$

For $h_p(k,s)$ we have the representation:

$$h_p(k,s) = \sum_{p_1=1}^{p-1} h_p^{(p_1,p-p_1)}(s) = \sum_{p_1=1}^{p-1} \sum_{j_1\ldots j_p=1}^{n} B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s)\delta\Bigl(k - \sum_{l=1}^{n} k_{j_l}\Bigr)$$

$$= \sum_{j_1\ldots j_p=1}^{n} B_{j_1\ldots j_p}(s)\delta\Bigl(k - \sum_{l=1}^{n} k_{j_l}\Bigr),$$

where

$$B_{j_1\ldots j_p}(s) = \sum_{p_1=1}^{p-1} B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s).$$

For $B_{j_1\ldots j_p}(s)$ we have the estimate

$$|B_{j_1\ldots j_p}(s)| \leqslant \sum_{p_1=1}^{p-1} |B^{(p_1,p-p_1)}_{j_1\ldots j_p}(s)|$$

$$\leqslant s^{\frac{p-2}{2}} C^{p-2} \prod_{l=1}^{p} |B_{j_l}| \Bigl(\sum_{p_1=2}^{p-2} \frac{\sqrt{2}}{p_1\sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}} \Bigr).$$

It is easy to see that for $p \geqslant 3$ the sum

$$\sum_{p_1=2}^{p-2} \frac{\sqrt{2}}{p_1\sqrt{p-p_1}} + \frac{2\sqrt{2}}{\sqrt{p-2}}$$

is bounded by some constant C_1. Taking $C_2 = \max(C_1, \max_{1\leqslant j\leqslant n} |k_j|)$, we get the required inequality for $q = p$. The lemma is proved. $\square$

Proof of Theorem 2. We use the following lemma.

Lemma 2. *Let $\{k_j\}$ be contained in a cone whose angle is less than π. Assume that for $2 \leqslant q < p$ the functions $h_q(k,s)$ have the representation*

(10) *and for some constant D depending on initial conditions*

$$|B_{j_1\dots j_q}(s)| \leqslant D^{q-1}\prod_{l=1}^{q}|B_{j_l}|.$$

Then the same representation is valid for $q=p$.

Proof. Using the notation from Lemma 1, we can write

$$|B^{(p_1,p_2)}_{j_1\dots j_p}(s)| \leqslant \int\limits_0^s ds_1\int\limits_0^s ds_2\Big|\sum_{l=p_1}^{p}k_{j_l}\Big|D^{p_1-1}\prod_{l=1}^{p_1}|B_{j_l}|D^{p_2-1}\prod_{l=p_1+1}^{p}|B_{j_l}|$$

$$\times\exp\Big\{-(s-s_1)\Big|\sum_{l=1}^{p_1}k_{j_l}\Big|^2-(s-s_2)\Big|\sum_{l=p_1+1}^{p}k_{j_l}\Big|^2\Big\}$$

$$< D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{\Big|\sum\limits_{l=p_1+1}^{p}k_{j_l}\Big|}{\Big|\sum\limits_{l=1}^{p_1}k_{j_l}\Big|^2\Big|\sum\limits_{l=p_1+1}^{p}k_{j_l}\Big|^2}.$$

By the assumption of the lemma, $\Big|\sum\limits_j\alpha_jk_j\Big|\geqslant d\sum\alpha_j$, where $\alpha_j\geqslant 0$ and $d>0$ is some constant depending on the initial vectors k_j. Indeed, under some rotation of $\mathbb{R}^3$, the vectors k_j can be brought inside a cone of angle less than π belonging to the subspace $k^{(1)}>0$. Then

$$\Big|\sum_j\alpha_jk_j\Big|=\Big|\sum_j\alpha_j\overline{k}_j\Big|\geqslant\sum_j\alpha_j\min_j\overline{k}^{(1)}_j,$$

where $\overline{k}_j$ is the image of k_j under the above-mentioned rotation.

Put $d=\min\limits_j|\overline{k}^{(1)}_j|$. Now, we can write

$$|B^{(p_1,p_2)}_{j_1\dots j_p}(s)|\leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{p_2\max\limits_j|k_j|}{p_1^2p_2^2d^4}.$$

Similarly, for $p_1=1$

$$|B^{(1,p-1)}_{j_1\dots j_p}(s)|\leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{(p-1)\max\limits_j|k_j|}{(p-1)^2d^2}$$

and for $p_2=1$

$$|B^{(p-1,1)}_{j_1\dots j_p}(s)| \leqslant D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\frac{(p-1)\max\limits_{j}|k_j|}{(p-1)^2d^2}.$$

As in the proof of Lemma 1, we have

$$\begin{aligned}|B_{j_1\dots j_p}(s)| &\leqslant \sum_{p_1=1}^{p-1}|B^{(p_1,p-p_1)}_{j_1\dots j_p}(s)|\\ &< D^{p-2}\prod_{l=1}^{p}|B_{j_l}|\max_{1\leqslant j\leqslant n}|k_j|\left[\frac{2}{(p-1)d^2}+\sum_{p_1=2}^{p-2}\frac{1}{p_1^2(p-p_1)d^4}\right].\end{aligned}$$

The expression in the square brackets is bounded by some constant D_0 depending on initial conditions. Taking $D \geqslant D_0 \max\limits_{1\leqslant j\leqslant n}|k_j|$, we get the required inequality for $q = p$. The lemma is proved. □

Now, we derive Theorem 2 from Lemma 2. Using the notation from the proof of Theorem 1, we can write

$$|\tilde{B}_{j_1\dots j_p}(t)| < D^{p-1}\frac{\prod_{l=1}^{p}|B_{j_l}|}{\left|\sum\limits_{l=1}^{p}k_{j_l}\right|} < \frac{D^{p-1}\prod_{l=1}^{p}|B_{j_l}|}{(pd)^2}$$

and

$$\sum_{j_1\dots j_p=1}^{n}|\tilde{B}_{j_1\dots j_p}(t)| < \frac{D^{p-1}}{(pd)^2}\left(\sum_{j=1}^{n}|B_j|\right)^p.$$

Therefore, if

$$\sum_{j=1}^{n}|B_j| < D^{-1},$$

then the series

$$\sum_{p>0}\sum_{j_1\dots j_p=1}^{n}|B_{j_1\dots j_p}(t)|$$

converges absolutely for all t. The end of the proof is the same as in Theorem 1. □

3. Example. We consider $n = 2$ and the initial condition

$$v(k,0) = B_1\delta(k-k_1) + B_2\delta(k-k_2), \tag{14}$$

where the vectors k_1 and k_2 are linearly independent and B_1 is orthogonal to k_1 and k_2.

Theorem 3. *For the initial condition* (14) *the system* (1) *has a global solution.*

Proof. In this case,

$$h_1(k,s) = \sum_{j=1}^{2} \exp\{-s|k_j|^2\} B_j \delta(k-k_j),$$

$$\begin{aligned} h_2(k,s) &= i\sum_{j_1,j_2=1}^{2} \langle k, B_{j_1}\rangle \left(B_{j_2} - \frac{\langle k, B_{j_2}\rangle k}{|k|^2} \right) \\ &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k-(k_{j_1}+k_{j_2}) \\ &= i \sum_{j_1,j_2=1}^{2} \langle k_{j_1}+k_{j_2}, B_{j_1}\rangle \left(B_{j_2} - \frac{\langle k_{j_1}+k_{j_2}, B_{j_2}\rangle (k_{j_1}+k_{j_2})}{|k_{j_1}+k_{j_2}|^2} \right) \\ &\quad \times \exp\{-s|k_{j_1}|^2 - s|k_{j_2}|^2\} \delta(k-(k_{j_1}+k_{j_2}). \end{aligned}$$

It is easy to see that only the term with $j_1 = 2$, $j_1 = 1$ is different from zero. Therefore,

$$h_2(k,s) = \langle k_1, B_2\rangle B_2 \exp\{-s|k_1|^2 - s|k_2|^2\} \delta(k-(k_1+k_2)).$$

From the recurrent relation it easily follows that, in the sum (7), only the term with $p_1 = p-1$ is different from zero. If

$$h_q(k,s) = B^{(q)}(s)\delta(k-(k_1+(q-1)k_2)$$

and

$$|B^{(q)}(s)| < \frac{C^{q-1}}{\sqrt{(q-2)!}} |B_1||B_2|^{q-1},$$

then the same inequality is valid for $q = p$ (see the proof of Lemma 1). Theorem 3 is proved. □

Remark. Theorem 3 is equivalent to the estimates of simple diagrams in [**7**].

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Bursting Dynamics of the 3*D* Euler Equations in Cylindrical Domains

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A class of three-dimensional initial data characterized by uniformly large vorticity is considered for the 3D incompressible Euler equations in bounded cylindrical domains. The fast singular oscillating limits of the 3D Euler equations are investigated for parametrically resonant cylinders. Resonances of fast oscillating swirling Beltrami waves deplete the Euler nonlinearity. These waves are exact solutions of the 3D Euler equations. We construct the 3D resonant Euler systems; the latter are countable uncoupled and coupled $SO(3;\mathbf{C})$ and $SO(3;\mathbf{R})$ rigid body systems. They conserve both energy and helicity. The 3D resonant Euler systems are vested

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with bursting dynamics, where the ratio of the enstrophy at time $t = t^*$ to the enstrophy at $t = 0$ of some remarkable orbits becomes very large for very small times t^*; similarly for higher norms $\mathbf{H}^s$, $s \geqslant 2$. These orbits are topologically close to homoclinic cycles. For the time intervals, where $\mathbf{H}^s$ norms, $s \geqslant 7/2$, of the limit resonant orbits do not blow up, we prove that the full 3D Euler equations possess smooth solutions close to the resonant orbits uniformly in strong norms. Bibliography: 41 titles.

1. Introduction

The issues of blowup of smooth solutions and finite time singularities of the vorticity field for 3D incompressible Euler equations are still a major open question. The Cauchy problem in 3D bounded axisymmetric cylindrical domains is attracting considerable attention: with bounded smooth *non*-axisymmetric 3D initial data, under the constraints of conservation of *bounded* energy, can the vorticity field blow up in *finite* time? Outstanding numerical claims for this were recently disproved [**25, 14, 23**]. The classical analytical criterion of Beale–Kato–Majda [**8**] for non-blow up in finite time requires the time integrability of the L^∞ norm of the vorticity. DiPerna and Lions [**27**] gave examples of global weak solutions of the 3D Euler equations which are smooth (hence unique) if the initial conditions are smooth (specifically in $\mathbf{W}^{1,p}(D)$, $p > 1$). However, these flows are really 2-dimensional in x_1, x_2, 3-components flows, independent from the third coordinate x_3. Their examples [**15**] show that solutions (even smooth ones) of the 3D Euler equations cannot be estimated in $\mathbf{W}^{1,p}$ for $1 < p < \infty$ on any time interval $(0, T)$ if the initial data are only assumed to be bounded in $\mathbf{W}^{1,p}$. Classical local existence theorems in 3D bounded or periodic domains by Kato [**24**], Bourguignon–Brézis [**11**] and Yudovich [**38, 39**] require some minimal smoothness for the initial conditions (IC), for example, in $\mathbf{H}^s(D)$, $s > \frac{5}{2}$.

The classical formulation for the Euler equations is as follows:

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p, \ \nabla \cdot \mathbf{V} = 0, \tag{1.1}$$

$$\mathbf{V} \cdot \mathbf{N} = 0 \text{ on } \partial D, \tag{1.2}$$

where ∂D is the boundary of a bounded connected domain D, $\mathbf{N}$ the normal to ∂D, $\mathbf{V}(t, y) = (V_1, V_2, V_3)$ the velocity field, $y = (y_1, y_2, y_3)$, and p is the pressure.

The equivalent Lamé form [**3**]

$$\partial_t \mathbf{V} + \text{curl } \mathbf{V} \times \mathbf{V} + \nabla \left(p + \frac{1}{2}|\mathbf{V}|^2 \right) = 0, \tag{1.3}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{1.4}$$

$$\partial_t \boldsymbol{\omega} + \text{curl}\,(\boldsymbol{\omega} \times \mathbf{V}) = 0, \tag{1.5a}$$

$$\boldsymbol{\omega} = \text{curl } \mathbf{V}, \tag{1.5b}$$

implies conservation of energy

$$E(t) = \frac{1}{2} \int_D |\mathbf{V}(t, y)|^2 \, dy. \tag{1.6}$$

The helicity Hel$\,(t)$ [**3, 33**], is conserved:

$$\text{Hel}\,(t) = \int_D \mathbf{V} \cdot \boldsymbol{\omega} \, dy, \tag{1.7}$$

for $D = \mathbf{R}^3$ and when D is a periodic lattice. Helicity is also conserved for cylindrical domains, provided that $\boldsymbol{\omega} \cdot \mathbf{N} = 0$ on the cylinder lateral boundary at $t = 0$ (see [**29**]).

From the theoretical point of view, the principal difficulty in the analysis of 3D Euler equations is due to the presence of the vortex stretching term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{V}$ in the vorticity equation (1.5a). Equations (1.3) and (1.5a) are equivalent to the following:

$$\partial_t \boldsymbol{\omega} + [\boldsymbol{\omega}, \mathbf{V}] = 0, \tag{1.8}$$

where $[a, b] = \text{curl}\,(\mathbf{a} \times \mathbf{b})$ is the commutator in the infinite dimensional Lie algebra of divergence-free vector fields [**3**]. This point of view has led to celebrated developments in topological methods in hydrodynamics [**3, 33**]. The striking analogy between the Euler equations for hydrodynamics and the Euler equations for a rigid body (the latter associated to the Lie algebra of the Lie group $SO(3, \mathbf{R})$) was already pointed out by Moreau [**31**]; Moreau was the first to demonstrate conservation of helicity (1961) [**32**]. This has led to extensive speculations to what extent/in what cases are the solutions of the 3D Euler equations "close" to those of coupled 3D rigid body equations in some asymptotic sense. Recall that the Euler equations for a rigid body in $\mathbf{R}^3$ is as follows:

$$\mathbf{m}_t + \boldsymbol{\omega} \times \mathbf{m} = 0, \ \mathbf{m} = A\boldsymbol{\omega}, \tag{1.9a}$$

$$\mathbf{m}_t + [\boldsymbol{\omega}, \mathbf{m}] = 0, \tag{1.9b}$$

where $\mathbf{m}$ is the vector of angular momentum relative to the body, $\boldsymbol{\omega}$ the angular velocity in the body, and A the inertia operator [**1, 3**].

The Russian school of Gledzer, Dolzhanskij, Obukhov [**20**] and Visik [**36**] has extensively investigated dynamical systems of hydrodynamic type and their applications. They considered hydrodynamical models built upon generalized rigid body systems in $SO(n, \mathbf{R})$, following Manakov [**30**]. Inspired by turbulence physics, they investigated "shell" dynamical systems modeling turbulence cascades; albeit such systems are flawed as they only preserve energy, not helicity. To address this, they constructed and studied in depth n-dimensional dynamical systems with quadratic homogeneous nonlinearities and **two** quadratic first integrals F_1, F_2. Such systems can be written using sums of Poisson brackets:

$$\frac{dx^{i_1}}{dt} = \frac{1}{2} \sum_{i_2,\ldots,i_n} \epsilon^{i_1 i_2 \ldots i_n} p_{i_4 \ldots i_n} \left(\frac{\partial F_1}{\partial x^{i_2}} \frac{\partial F_2}{\partial x^{i_3}} - \frac{\partial F_1}{\partial x^{i_3}} \frac{\partial F_2}{\partial x^{i_2}} \right), \tag{1.10}$$

where constants $p_{i_4 \ldots i_n}$ are antisymmetric in $i_4, \ldots, i_n$.

A simple version of such a quadratic hydrodynamic system was introduced by Gledzer [**19**] in 1973. A deep open issue of the work by the Gledzer–Obukhov school is whether there exist indeed classes of I.C. for the 3D Cauchy Euler problem (1.1) for which solutions are actually asymptotically close in strong norm, on arbitrary large time intervals to solutions of such hydrodynamic systems, with conservation of both energy and helicity. Another unresolved issue is the blowup or global regularity for the "enstrophy" of such systems when their dimension $n \to \infty$.

This article reviews some current new results of a research program in the spirit of the Gledzer–Obukhov school; this program builds-up on the results of [**29**] for 3D Euler in bounded cylindrical domains. Following the original approach of [**4**]–[**7**] in periodic domains, [**29**] prove the non blowup of the 3D incompressible Euler equations for a class of three-dimensional initial data characterized by uniformly large vorticity in bounded cylindrical domains. There are no conditional assumptions on the properties of solutions at later times, nor are the global solutions close to some 2D manifold. The initial vortex stretching is large. The approach of proving regularity is based on investigation of fast singular oscillating limits and nonlinear averaging methods in the context of almost periodic functions [**10, 9, 13**]. Harmonic analysis tools based on curl eigenfunctions and eigenvalues are crucial. One establishes the global regularity of the 3D limit resonant Euler equations without any restriction on the size of 3D initial data. The resonant Euler equations are characterized by a depleted nonlinearity. After establishing strong convergence to the limit resonant equations, one bootstraps this into the regularity on arbitrary large time intervals of the solutions of 3D Euler Equations with weakly aligned uniformly large vorticity at $t = 0$. The theorems in [**29**] hold for generic cylindrical domains, for a set of height/radius ratios of full Lebesgue measure. For such cylinders the 3D limit resonant Euler equations are restricted to two-wave resonances of the vorticity waves and are vested with an infinite countable number of new conservation laws. The latter are adiabatic invariants for the original 3D Euler equations.

Three-wave resonances exist for a nonempty countable set of h/R (h height, R radius of the cylinder) and moreover accumulate in the limit of vanishingly small vertical (axial) scales. This is akin to Arnold tongues [**2**] for the Mathieu–Hill equations and raises nontrivial issues of possible singularities/lack thereof for dynamics ruled by infinitely many resonant triads at vanishingly small axial scales. In such a context, the 3D resonant Euler equations do conserve the energy and helicity of the field.

In this review, we consider cylindrical domains with parametric resonances in h/R and investigate in depth the structure and dynamics of 3D resonant Euler systems. These parametric resonances in h/R are proved to be non-empty. Solutions to Euler equations with uniformly large initial vorticity are expanded along a full *complete* basis of elementary swirling waves ($\mathbf{T}^2$ in time). Each such quasiperiodic, dispersive vorticity wave is a quasiperiodic Beltrami flow; these are exact solutions of 3D Euler equations with vorticity parallel to velocity. There are no Galerkin-like truncations in the decomposition of the full 3D Euler field. The Euler equations, restricted to resonant triplets of these dispersive Beltrami waves, determine the "resonant Euler systems." The basic "building block" of these (a priori ∞-dimensional) systems are proved to be $SO(3;\mathbf{C})$ and $SO(3;\mathbf{R})$ rigid body systems

$$\begin{aligned}
\dot{U}_k + (\lambda_m - \lambda_n)U_m U_n &= 0,\\
\dot{U}_m + (\lambda_n - \lambda_k)U_n U_k &= 0,\\
\dot{U}_n + (\lambda_k - \lambda_m)U_k U_m &= 0.
\end{aligned} \tag{1.11}$$

These λ's are *eigenvalues of the curl operator* in the cylinder, curl $\mathbf{\Phi}_n^{\pm} = \pm\lambda_n\mathbf{\Phi}_n^{\pm}$; the curl eigenfunctions are steady elementary Beltrami flows, and the dispersive Beltrami waves oscillate with the frequencies $\pm\dfrac{h}{2\pi\epsilon}\dfrac{n_3}{\lambda_n}$, n_3 vertical wave number (vertical shear), $0 < \epsilon < 1$. Physicists [**12**] computationally demonstrated the physical impact of the polarization of Beltrami modes $\mathbf{\Phi}^{\pm}$ on intermittency in the joint cascade of energy and helicity in turbulence.

Another "building block" for resonant Euler systems is a pair of $SO(3;\mathbf{C})$ or $SO(3;\mathbf{R})$ rigid bodies coupled via a common principal axis of inertia/moment of inertia:

$$\dot{a}_k = (\lambda_m - \lambda_n)\Gamma a_m a_n, \tag{1.12a}$$

$$\dot{a}_m = (\lambda_n - \lambda_k)\Gamma a_n a_k, \tag{1.12b}$$

$$\dot{a}_n = (\lambda_k - \lambda_m)\Gamma a_k a_m + (\lambda_{\tilde{k}} - \lambda_{\tilde{m}})\tilde{\Gamma} a_{\tilde{k}} a_{\tilde{m}}, \tag{1.12c}$$

$$\dot{a}_{\tilde{m}} = (\lambda_n - \lambda_{\tilde{k}})\tilde{\Gamma} a_n a_{\tilde{k}}, \tag{1.12d}$$

$$\dot{a}_{\tilde{k}} = (\lambda_{\tilde{m}} - \lambda_n)\tilde{\Gamma} a_{\tilde{m}} a_n, \tag{1.12e}$$

where Γ and $\tilde{\Gamma}$ are parameters in $\mathbf{R}$ defined in Theorem 4.10. Both resonant systems (1.11) and (1.12) conserve energy and helicity. We prove that the dynamics of these resonant systems admit equivariant families of homoclinic cycles connecting hyperbolic critical points. We demonstrate bursting dynamics: the ratio

$$||\mathbf{u}(t)||^2_{H^s}/||\mathbf{u}(0)||^2_{H^s},\ s \geqslant 1,$$

can burst arbitrarily large on arbitrarily small times, for properly chosen parametric domain resonances h/R. Here,

$$||\mathbf{u}(t)||^2_{H^s} = \sum_n (\lambda_n)^{2s} |u_n(t)|^2\,. \tag{1.13}$$

The case $s = 1$ is the enstrophy. The "bursting" orbits are topologically close to the homoclinic cycles.

Are such dynamics for the resonant systems relevant to the full 3D Euler equations (1.1)–(1.8)? The answer lies in the following crucial "shadowing" Theorem 2.10. Given the same initial conditions, given the maximal time interval $0 \leqslant t < T_m$ where the resonant orbits of the resonant Euler equations do **not** blow up, then the *strong* norm $\mathbf{H}^s$ of the difference between the exact Euler orbit and the resonant orbit is uniformly small on $0 \leqslant t < T_m$, provided that the vorticity of the I.C. is large enough. Paradoxically, the larger the vortex stretching of the I.C., the better the uniform approximation. This deep result is based on cancellation of fast oscillations in *strong* norms, in the context of almost periodic functions of time with values in Banach spaces [**29**, Sec. 4]. It includes uniform approximation in the spaces $\mathbf{H}^s$, $s > 5/2$. For instance, given a quasiperiodic orbit on some time torus $\mathbf{T}^l$ for the resonant Euler systems, the exact solutions to the

Euler equations will remain ϵ-close to the resonant quasiperiodic orbit on a time interval $0 \leqslant t \leqslant \max T_i$, $1 \leqslant i \leqslant l$, T_i elementary periods, for large enough initial vorticity. If orbits of the resonant Euler systems admit bursting dynamics in the strong norms $\mathbf{H}^s$, $s \geqslant 7/2$, so do *some exact solutions of the full 3D Euler equations*, for properly chosen parametrically resonant cylinders.

2. Vorticity Waves and Resonances of Elementary Swirling Flows

We study the initial value problem for the three-dimensional Euler equations with initial data characterized by uniformly large vorticity

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p, \ \nabla \cdot \mathbf{V} = 0, \tag{2.1}$$

$$\mathbf{V}(t, y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2}\mathbf{e}_3 \times y, \tag{2.2}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t, y) = (V_1, V_2, V_3)$ is the velocity field, and p is the pressure. In Equations (1.1), $\mathbf{e}_3$ denotes the vertical unit vector and Ω is a constant parameter. The field $\tilde{\mathbf{V}}_0(y)$ depends on three variables y_1, y_2, and y_3. Since $\text{curl}\,(\frac{\Omega}{2}\mathbf{e}_3 \times y) = \Omega \mathbf{e}_3$, the vorticity vector at initial time $t = 0$ is

$$\text{curl } \mathbf{V}(0, y) = \text{curl } \tilde{\mathbf{V}}_0(y) + \Omega \mathbf{e}_3, \tag{2.3}$$

and the initial vorticity has a large component weakly aligned along $\mathbf{e}_3$, when $\Omega >> 1$. These are fully three-dimensional large initial data with large initial 3D vortex stretching. We denote by $\mathbf{H}^s_\sigma$ the usual Sobolev space of solenoidal vector fields.

The base flow

$$\mathbf{V}_s(y) = \frac{\Omega}{2}\mathbf{e}_3 \times y, \text{curl } \mathbf{V}_s(y) = \Omega \mathbf{e}_3 \tag{2.4}$$

is called a steady swirling flow and is a steady state solution (1.1)–(1.4), as $\text{curl}\,(\Omega \mathbf{e}_3 \times \mathbf{V}_s(y)) = 0$. In (2.2) and (2.3), we consider I.C. which are an arbitrary (***not small***) perturbation of the base swirling flow $\mathbf{V}_s(y)$ and introduce

$$\mathbf{V}(t, y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}(t, y), \tag{2.5}$$

$$\text{curl } \mathbf{V}(t, y) = \Omega \mathbf{e}_3 + \text{curl } \tilde{\mathbf{V}}(t, y), \tag{2.6}$$

$$\partial_t \tilde{\mathbf{V}} + \text{curl } \tilde{\mathbf{V}} \times \tilde{\mathbf{V}} + \Omega \mathbf{e}_3 \times \tilde{\mathbf{V}} + \text{curl } \tilde{\mathbf{V}} \times \mathbf{V}_s(y) + \nabla p' = 0, \ \nabla \cdot \tilde{\mathbf{V}} = 0, \tag{2.7}$$

$$\tilde{\mathbf{V}}(t, y)|_{t=0} = \tilde{\mathbf{V}}_0(y). \tag{2.8}$$

Equations (2.1) and (2.7) are studied in cylindrical domains

$$\mathbf{C} = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha,\ y_1^2 + y_2^2 < R^2\}, \tag{2.9}$$

where α and R are positive real numbers. If h is the height of the cylinder, $\alpha = 2\pi/h$. Let

$$\Gamma = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : 0 < y_3 < 2\pi/\alpha,\ y_1^2 + y_2^2 = R^2\}. \tag{2.10}$$

Without loss of generality, we can assume that $R = 1$. Equations (2.1) are considered with periodic boundary conditions in y_3

$$\mathbf{V}(y_1, y_2, y_3) = \mathbf{V}(y_1, y_2, y_3 + 2\pi/\alpha) \tag{2.11}$$

and vanishing normal component of velocity on Γ

$$\mathbf{V} \cdot \mathbf{N} = \tilde{\mathbf{V}} \cdot \mathbf{N} = 0 \text{ on } \Gamma, \tag{2.12}$$

where $\mathbf{N}$ is the normal vector to Γ. From the invariance of 3D Euler equations under the symmetry $y_3 \to -y_3$, $V_1 \to V_1$, $V_2 \to V_2$, $V_3 \to -V_3$, all results in this article extend to cylindrical domains bounded by two horizontal plates. Then the boundary conditions in the vertical direction are zero flux on the vertical boundaries (zero vertical velocity on the plates). One only needs to restrict vector fields to be even in y_3 for V_1, V_2 and odd in y_3 for V_3, and double the cylindrical domain to $-h \leqslant y_3 \leqslant +h$.

We choose $\tilde{\mathbf{V}}_0(y)$ in $\mathbf{H}^s(\mathbf{C})$, $s > 5/2$. In [**29**], for the case of "non-resonant cylinders," i.e., non-resonant $\alpha = 2\pi/h$, we have established regularity for arbitrarily large finite times for the 3D Euler solutions for Ω large, but *finite*. Our solutions are not close in any sense to those of the 2D or "quasi 2D" Euler and they are characterized by fast oscillations in the $\mathbf{e}_3$ direction, together with a large vortex stretching term

$$\boldsymbol{\omega}(t, y) \cdot \nabla \mathbf{V}(t, y) = \omega_1 \frac{\partial \mathbf{V}}{\partial y_1} + \omega_2 \frac{\partial \mathbf{V}}{\partial y_2} + \omega_3 \frac{\partial \mathbf{V}}{\partial y_3}, \quad t \geqslant 0,$$

with leading component $\left|\Omega \dfrac{\partial}{\partial y_3} \mathbf{V}(t, y)\right| \gg 1$. There are no assumptions on oscillations in y_1, y_2 for our solutions (nor for the initial condition $\tilde{\mathbf{V}}_0(y)$).

Our approach is entirely based on fast singular oscillating limits of Equations (1.1)–(1.5a), nonlinear averaging, and cancelation of oscillations in the nonlinear interactions for the vorticity field for large Ω. This was developed in [**5, 6, 7**] for the cases of periodic lattice domains and the infinite space $\mathbf{R}^3$.

It is well known that fully three-dimensional initial conditions with uniformly large vorticity excite fast Poincaré vorticity waves [**5, 6, 7, 34**].

Since individual Poincaré wave modes are related to the eigenfunctions of the curl operator, they are exact time-dependent solutions of the full nonlinear 3D Euler equations. Of course, their linear superposition does not preserve this property. Expanding solutions of (2.1)–(2.8) along such vorticity waves demonstrates potential nonlinear resonances of such waves. First recall spectral properties of the curl operator in bounded connected domains.

Proposition 2.1 ([**29**]). *The curl operator admits a self-adjoint extension under the zero flux boundary conditions, with a discrete real spectrum $\lambda_n = \pm|\lambda_n|$, $|\lambda_n| > 0$ for every n and $|\lambda_n| \to +\infty$ as $|n| \to \infty$. The corresponding eigenfunctions $\mathbf{\Phi}_n^{\pm}$*

$$\text{curl } \mathbf{\Phi}_n^{\pm} = \pm|\lambda_n|\mathbf{\Phi}_n^{\pm} \tag{2.13}$$

are complete in the space

$$\boldsymbol{J}^0 = \Big\{ \boldsymbol{U} \in L^2(D) : \nabla \cdot \mathbf{U} = 0, \mathbf{U} \cdot \mathbf{N}|_{\partial D} = 0, \int_o^h \mathbf{U}\, dz = 0 \Big\}. \tag{2.14}$$

Remark 2.2. In cylindrical domains, with cylindrical coordinates (r, θ, z), the eigenfunctions admit the representation

$$\mathbf{\Phi}_{n_1,n_2,n_3} = (\Phi_{r,n_1,n_2,n_3}(r), \Phi_{\theta,n_1,n_2,n_3}(r), \Phi_{z,n_1,n_2,n_3}(r))\, e^{in_2\theta} e^{i\alpha n_3 z} \tag{2.15}$$

with $n_2 = 0, \pm 1, \pm 2, \ldots$, $n_3 = \pm 1, \pm 2, \ldots$, and $n_1 = 0, 1, 2, \ldots$. Here, n_1 indexes the eigenvalues of the equivalent Sturm–Liouville problem in the radial coordinates and $n = (n_1, n_2, n_3)$ (see [**29**] for technical details). From now on, we use the generic variable z for any vertical (axial) coordinate y_3 or x_3. For $n_3 = 0$ (vertical averaging along the axis of the cylinder) 2-dimensional 3-component solenoidal fields must be expanded along a complete basis for fields derived from 2D stream functions:

$$\begin{aligned} &\mathbf{\Phi}_n = \big(\big(\operatorname{curl}(\varphi_n \mathbf{e}_3),\, \varphi_n \mathbf{e}_3\big)\big),\ \varphi_n = \varphi_n(r, \theta), \\ &-\triangle \varphi_n = \mu_n \varphi_n,\ \varphi_n|_{\partial\Gamma} = 0, \\ &\text{curl } \mathbf{\Phi}_n = \big(\big(\operatorname{curl}(\varphi_n \mathbf{e}_3),\, \mu_n \varphi_n \mathbf{e}_3\big)\big). \end{aligned}$$

Here, $\big(\big(\mathbf{a},\, b\mathbf{e}_3\big)\big)$ denotes a 3-component vector whose horizontal projection is $\mathbf{a}$ and vertical projection is $b\mathbf{e}_3$.

Let us explicit elementary swirling wave flows which are *exact* solutions to (2.1) and (2.7).

Lemma 2.3. *For every* $n = (n_1, n_2, n_3)$ *the following quasiperiodic* ($\mathbf{T}^2$ *in time*) *solenoidal fields are exact solution of the full* 3*D nonlinear Euler equations* (2.1)

$$\mathbf{V}(t,y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \exp\Big(\frac{\Omega}{2}\mathbf{J}t\Big)\mathbf{\Phi}_n\Big(\exp\Big(-\frac{\Omega}{2}\mathbf{J}t\Big)y\Big)\exp\Big(\pm i\frac{n_3}{|\lambda_n|}\alpha\Omega t\Big), \tag{2.16}$$

n_3 *is the vertical wave number of* $\mathbf{\Phi}_n$ *and* $\exp(\frac{\Omega}{2}\mathbf{J}t)$ *the unitary group of rigid body rotations*:

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{\Omega \mathbf{J}t/2} = \begin{pmatrix} \cos(\frac{\Omega t}{2}) & -\sin(\frac{\Omega t}{2}) & 0 \\ \sin(\frac{\Omega t}{2}) & \cos(\frac{\Omega t}{2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.17}$$

Remark 2.4. These fields are exact quasiperiodic, nonaxisymmetric swirling flow solutions of the 3D Euler equations. For $n_3 \neq 0$ their second components

$$\tilde{\mathbf{V}}(t,y) = \exp\Big(\frac{\Omega}{2}\mathbf{J}t\Big)\mathbf{\Phi}_n\Big(\exp\Big(-\frac{\Omega}{2}\mathbf{J}t\Big)y\Big)\exp\Big(\pm \frac{in_3}{|\lambda_n|}\alpha\Omega t\Big) \tag{2.18}$$

are Beltrami flows (curl $\tilde{\mathbf{V}} \times \tilde{\mathbf{V}} \equiv 0$) exact solutions of (2.7) with $\tilde{\mathbf{V}}(t = 0, y) = \mathbf{\Phi}_n(y)$.

In Equation (2.18), $\tilde{\mathbf{V}}(t,y)$ are dispersive waves with frequencies $\dfrac{\Omega}{2}$ and $\dfrac{n_3\alpha}{|\lambda_n|}\Omega$, $\alpha = \dfrac{2\pi}{h}$. Moreover, each $\tilde{\mathbf{V}}(t,y)$ is a *traveling wave* along the cylinder axis since it contains the factor $\exp\Big(i\alpha n_3(\pm z \pm \dfrac{\Omega}{|\lambda_n|}t)\Big)$. Note that n_3 large corresponds to small axial (vertical) scales, albeit $0 \leqslant \alpha|n_3/\lambda_n| \leqslant 1$.

Proof of Lemma 2.3. Through the canonical rigid body transformation for both the field $\mathbf{V}(t,y)$ and the space coordinates $y = (y_1, y_2, y_3)$:

$$\mathbf{V}(t,y) = e^{+\Omega\mathbf{J}t/2}\mathbf{U}(t, e^{-\Omega\mathbf{J}t/2}y) + \frac{\Omega}{2}\mathbf{J}y, \; x = e^{-\Omega\mathbf{J}t/2}y, \tag{2.19}$$

the 3D Euler equations (2.1), (2.2) transform into

$$\partial_t\mathbf{U} + (\text{curl }\mathbf{U} + \Omega\mathbf{e}_3) \times \mathbf{U} = -\nabla\left(p - \frac{\Omega^2}{4}(|x_1|^2 + |x_2|^2) + \frac{|\mathbf{U}|^2}{2}\right), \tag{2.20}$$

$$\nabla \cdot \mathbf{U} = 0, \quad \mathbf{U}(t,x)|_{t=0} = \mathbf{U}(0) = \tilde{\mathbf{V}}_0(x), \tag{2.21}$$

For Beltrami flows such that curl $\mathbf{U} \times \mathbf{U} \equiv 0$ these Euler equations (2.20), (2.21) in a rotating frame reduce to

$$\partial_t \mathbf{U} + \Omega \mathbf{e}_3 \times \mathbf{U} + \nabla \pi = 0, \ \nabla \cdot \mathbf{U} = 0,$$

which are identical to the Poincaré–Sobolev nonlocal wave equations in the cylinder [**29, 34, 35, 3**]

$$\partial_t \boldsymbol{\Psi} + \Omega \mathbf{e}_3 \times \boldsymbol{\Psi} + \nabla \pi = 0, \ \nabla \cdot \boldsymbol{\Psi} = 0, \tag{2.22}$$

$$\frac{\partial^2}{\partial t^2} \operatorname{curl}^2 \boldsymbol{\Psi} - \boldsymbol{\Omega}^2 \frac{\partial^2}{\partial x_3^2} \boldsymbol{\Psi} = 0, \ \boldsymbol{\Psi} \cdot \mathbf{N}|_{\partial D} = 0. \tag{2.23}$$

It suffices to verify that the Beltrami flows

$$\boldsymbol{\Psi}_n(t,x) = \boldsymbol{\Phi}_n(x) \exp\Big(\pm i \frac{\alpha n_3}{|\lambda_n|} \Omega t \Big),$$

where $\boldsymbol{\Phi}_n^{\pm}(x)$ and $\pm|\lambda_n|$ are curl eigenfunctions and eigenvalues, are exact solutions to the Poincaré–Sobolev wave equation, in such a rotating frame of reference.

Remark 2.5. The frequency spectrum of the Poincaré vorticity waves (solutions to (2.22)) is exactly $\pm i \dfrac{\alpha n_3}{|\lambda_n|} \Omega$, $n = (n_1, n_2, n_3)$, indexing the spectrum of curl. Note that $n_3 = 0$ (zero frequency of rotating waves) corresponds to 2-dimensional 3-components solenoidal vector fields.

We now transform the Cauchy problem for the 3D Euler equations (2.1), (2.2) into an infinite dimensional nonlinear dynamical system by expanding $\mathbf{V}(t,y)$ along the swirling wave flows (2.16)–(2.18)

$$\begin{aligned} \mathbf{V}(t,y) &= \frac{\Omega}{2} \mathbf{e}_3 \times y + \exp\left(\frac{\Omega}{2} \mathbf{J} t \right) \\ &\times \Big\{ \sum_{n=(n_1,n_2,n_3)} \mathbf{u}_n(t) \exp\left(\pm i \frac{\alpha n_3}{|\lambda_n|} \Omega t \right) \boldsymbol{\Phi}_n \left(\exp\left(-\frac{\Omega}{2} \mathbf{J} t \right) y \right) \Big\}, \end{aligned} \tag{2.24a}$$

$$\mathbf{V}(t=0,y) = \frac{\Omega}{2} \mathbf{e}_3 \times y + \tilde{\mathbf{V}}_0(y) \tag{2.24b}$$

$$\tilde{\mathbf{V}}_0(y) = \sum_{n=(n_1,n_2,n_3)} \mathbf{u}_n(0) \boldsymbol{\Phi}_n(y), \tag{2.24c}$$

where $\boldsymbol{\Phi}_n$ denotes the curl eigenfunctions of Proposition 2.1 if $n_3 \neq 0$, and $\boldsymbol{\Phi}_n = \big(\big(\operatorname{curl}(\varphi_n \mathbf{e_3}), \, \varphi_n \mathbf{e}_3 \big) \big)$ if $n_3 = 0$ (2D case, Remark 2.2).

As we focus on the case, where helicity is conserved for (2.1), (2.2), we consider the class of initial data $\tilde{\mathbf{V}}_0$ such that (see [**29**]) curl $\tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0$ on Γ, where Γ is the lateral boundary of the cylinder.

The infinite dimensional dynamical system is then equivalent to the 3D Euler equations (2.1), (2.2) in the cylinder, with $n = (n_1, n_2, n_3)$ ranging over the whole spectrum of curl, i.e.,

$$\begin{aligned}\frac{d\mathbf{u}_n}{dt} = - \sum_{\substack{k,m \\ k_3+m_3=n_3 \\ k_2+m_2=n_2}} &\exp\left(i\left(\pm\frac{n_3}{|\lambda_n|} \pm \frac{k_3}{|\lambda_k|} \pm \frac{m_3}{|\lambda_m|}\right)\alpha\Omega t\right) \\ &\times \langle \text{curl } \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n \rangle \mathbf{u}_k(t)\mathbf{u}_m(t). \end{aligned} \tag{2.25}$$

Here,

$$\begin{aligned}&\text{curl } \mathbf{\Phi}_k^{\pm} = \pm\lambda_k \mathbf{\Phi}_k^{\pm} \text{ if } \ k_3 \neq 0, \\ &\text{curl } \mathbf{\Phi}_k = \big(\big(\text{curl}\,(\varphi_k \mathbf{e}_3),\ \mu_k \varphi_k \mathbf{e}_3\big)\big) \text{ if } \ k_3 = 0\end{aligned}$$

(2D, 3-components, Remark 2.2), similarly for $m_3 = 0$ and $n_3 = 0$. The inner product $\langle\ ,\ \rangle$ denotes the L^2 complex-valued inner product in D.

This is an infinite dimensional system of coupled equations with quadratic nonlinearities, which conserve both the energy

$$E(t) = \sum_n |\mathbf{u}_n(t)|^2$$

and the helicity

$$\text{Hel}\,(t) = \sum_n \pm|\lambda_n|\,|\mathbf{u}_n^{\pm}(t)|^2.$$

The quadratic nonlinearities split into *resonant terms*, where the exponential oscillating phase factor in (2.25) reduces to unity and fast oscillating non-resonant terms ($\Omega >> 1$). The resonant set K is defined in terms of vertical wave numbers k_3, m_3, n_3 and eigenvalues $\pm\lambda_k$, $\pm\lambda_m$, $\pm\lambda_n$ of curl:

$$K = \Big\{ \pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, n_3 = k_3 + m_3, n_2 = k_2 + m_2 \Big\}. \tag{2.27}$$

Here, k_2, m_2, n_2 are azimuthal wave numbers.

We call the “resonant Euler equations” the following ∞-dimensional dynamical system restricted to $(k, m, n) \in K$:

$$\frac{d\mathbf{u}_n}{dt} + \sum_{(k,m,n)\in K} \langle \text{curl } \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n \rangle \mathbf{u}_k \mathbf{u}_m = 0, \tag{2.28a}$$

$$\mathbf{u}_n(0) \equiv \langle \tilde{\mathbf{V}}_0, \mathbf{\Phi}_n \rangle. \tag{2.28b}$$

Here, curl $\mathbf{\Phi}_k^{\pm} = \pm\lambda_k \mathbf{\Phi}_k^{\pm}$ if $k_3 \neq 0$, curl $\mathbf{\Phi}_k = \big(\big(\operatorname{curl}(\varphi_k \mathbf{e}_3),\, \mu_k \varphi_k \mathbf{e}_3\big)\big)$ if $k_3 = 0$; similarly for $m_3 = 0$ and $n_3 = 0$ (2D components, Remark 2.2). If there are no terms in (2.28a) satisfying the resonance conditions, then there will be some modes for which $\dfrac{d\mathbf{u}_j}{dt} = 0$.

Lemma 2.6. *The resonant* 3D *Euler equations* (2.28) *conserve both energy* $E(t)$ *and helicity* Hel(t). *The energy and helicity are identical to that of the full exact* 3D *Euler equations* (2.1), (2.2).

The set of resonances K is studied in depth in [**29**]. To summarize, K splits into

(i) 0-wave resonances, with $n_3 = k_3 = m_3 = 0$; the corresponding resonant equations are identical to the 2-dimensional 3-components Euler equations with I.C.

$$\frac{1}{h}\int_0^h \tilde{\mathbf{V}}_0(y_1, y_2, y_3)\, dy_3.$$

(ii) Two-wave resonances, with $k_3 m_3 n_3 = 0$, but two of them are not null; the corresponding resonant equations (called "catalytic equations") are proved to possess an infinite countable set of new conservation laws [**29**].

(iii) Strict three-wave resonances for a subset $K^* \subset K$.

Definition 2.7. The set K^* of strict 3 wave resonances is:

$$K^* = \left\{ \pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, k_3 m_3 n_3 \neq 0, n_3 = k_3 + m_3, n_2 = k_2 + m_2 \right\}. \tag{2.29}$$

Note that K^* is parameterized by h/R since $\alpha = \dfrac{2\pi}{h}$ parameterizes the eigenvalues $\lambda_n, \lambda_k, \lambda_m$ of the curl operator.

Proposition 2.8. *There exists a countable nonempty set of parameters* $\frac{h}{R}$ *for which* $K^* \neq \varnothing$.

Proof. The technical details, together with a more precise statement, are postponed to the proof of Lemma 3.7. Concrete examples of resonant axisymmetric and helical waves are discussed in [**28**] (see Fig. 2 in [**28**]). □

Corollary 2.9. *Let*

$$\int_0^h \tilde{\mathbf{V}}_0(y_1, y_2, y_3)\, dy_3 = 0,$$

i.e., zero vertical mean for the I.C. $\tilde{\mathbf{V}}_0(y)$ *in* (2.2), (2.8), (2.24c), *and* (2.28b). *Then the resonant* 3*D Euler equations are invariant on* K^* :

$$\frac{d\mathbf{u}_n}{dt} + \sum_{(k,m,n)\in K^*} \lambda_k \langle \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n \rangle \mathbf{u}_k \mathbf{u}_m = 0,\ k_3 m_3 n_3 \neq 0, \tag{2.30a}$$

$$\mathbf{u}_n(0) = \langle \tilde{\mathbf{V}}_0, \mathbf{\Phi}_n \rangle, \tag{2.30b}$$

where $\tilde{\mathbf{V}}_0$ *has spectrum restricted to* $n_3 \neq 0$.

Proof. This is an immediate consequence of the "operator splitting" theorem (see [**29**, Theorem 3.2]). □

We call the above dynamical systems the "*strictly resonant Euler system.*" This is an ∞-dimensional Riccati system which conserves energy and helicity. It corresponds to *nonlinear interactions depleted* on K^*.

How do dynamics of the resonant Euler equations (2.28) or (2.30) approximate *exact* solutions of the Cauchy problem for the full Euler equations in strong norms? This is answered by the following theorem proved in [**29**, Sec. 4].

Theorem 2.10. *Consider the initial value problem*

$$\mathbf{V}(t=0, y) = \frac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}_0(y),\ \tilde{\mathbf{V}}_0 \in \mathbf{H}^s_\sigma,\ s > 7/2,$$

for the full 3D Euler equations, with $||\tilde{\mathbf{V}}_0||_{\mathbf{H}^s_\sigma} \leqslant M^0_s$ *and* curl $\tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0$ *on* Γ.

- *Let* $\mathbf{V}(t,y) = \dfrac{\Omega}{2}\mathbf{e}_3 \times y + \tilde{\mathbf{V}}(t,y)$ *denote the solution to the exact Euler equations.*
- *Let* $\mathbf{w}(t,x)$ *denote the solution to the resonant 3D Euler equations with the initial condition* $\mathbf{w}(0,x) \equiv \mathbf{w}(0,y) = \tilde{\mathbf{V}}_0(y)$.
- *Let* $||\mathbf{w}(t,y)||_{\mathbf{H}^s_\sigma} \leqslant M_s(T_M, M^0_s)$ *on* $0 \leqslant t \leqslant T_M$, $s > 7/2$.

Then for all $\epsilon > 0$ *there exists* $\Omega^*(T_M, M^0_s, \epsilon)$ *such that for all* $\Omega \geqslant \Omega^*$

$$\Bigg|\Bigg|\tilde{\mathbf{V}}(t,y) - \exp\left(\frac{\mathbf{J}\Omega t}{2}\right)\left\{\sum_n \mathbf{u}_n(t) e^{-i\frac{n_3}{\lambda_n}\alpha\Omega t}\mathbf{\Phi}_n(e^{-\frac{\mathbf{J}\Omega t}{2}}y)\right\}\Bigg|\Bigg|_{H^\beta} \leqslant \epsilon$$

on $0 \leqslant t \leqslant T_M$ *for all* $\beta \geqslant 1$, $\beta \leqslant s-2$. *Here,* $||\cdot||_{H^\beta}$ *is defined in* (1.13).

The 3D Euler flow preserves the condition curl $\tilde{\mathbf{V}}_0 \cdot \mathbf{N} = 0$ on Γ, i.e., curl $\mathbf{V}(t, y) \cdot \mathbf{N} = 0$ on Γ, for every $t \geqslant 0$ [**29**]. The proof of this "error-shadowing" theorem is delicate, beyond the usual Gronwall differential inequalities and involves estimates of oscillating integrals of almost periodic functions of time with values in Banach spaces. Its importance lies in that solutions of the resonant Euler equations (2.28) and/or (2.30) are uniformly close in strong norms to those of the exact Euler equations (2.1), (2.2), on any time interval of existence of smooth solutions of the *resonant* system. The infinite dimensional Riccati systems (2.28) and (2.30) are not just hydrodynamic models, but *exact asymptotic limit systems* for $\Omega \gg 1$. This is in contrast to all previous literature on conservative 3D hydrodynamic models such as in [**20**].

3. The Strictly Resonant Euler Systems: the SO(3) Case

We investigate the structure and the dynamics of the "strictly resonant Euler systems" (2.30). Recall that the set of 3-wave resonances is:

$$K^* = \Big\{(k, m, n) : \pm\frac{k_3}{\lambda_k} \pm \frac{m_3}{\lambda_m} \pm \frac{n_3}{\lambda_n} = 0, k_3 m_3 n_3 \neq 0,$$
$$n_3 = k_3 + m_3, n_2 = k_2 + m_2\Big\}. \tag{3.1}$$

From the symmetries of the curl eigenfunctions $\mathbf{\Phi}_n$ and eigenvalues λ_n in the cylinder, we have

if

$$n_2 \to -n_2,\ n_3 \to -n_3, \tag{3.2a}$$

then

$$\mathbf{\Phi}(n_1, -n_2, -n_3) = \mathbf{\Phi}^*(n_1, n_2, n_3) \tag{3.2b}$$

and

$$\lambda(n_1, -n_2, -n_3) = \lambda(n_1, n_2, n_3), \tag{3.2c}$$

where * designates the complex conjugate (see [**29**, Sec. 3] for details). The eigenfunctions $\mathbf{\Phi}(n_1, n_2, n_3)$ involve the radial functions $J_{n_2}(\beta(n_1, n_2, \alpha n_3) r)$ and $J'_{n_2}(\beta(n_1, n_2, \alpha n_3) r)$, with

$$\lambda^2(n_1, n_2, n_3) = \beta^2(n_1, n_2, \alpha n_3) + \alpha^2 n_3^2;$$

$\beta(n_1, n_2, \alpha n_3)$ are discrete, countable roots of Equation (3.30) in [**29**], obtained via an equivalent Sturm–Liouville radial problem. Since the curl eigenfunctions are even in $r \to -r$, $n_1 \to -n_1$, we extend the indices $n_1 = 1, 2, \ldots, +\infty$ to $-n_1 = -1, -2, \ldots$ with the above radial symmetry in mind.

Corollary 3.1. *The* 3*-wave resonance set* K^* *is invariant under the symmetries* σ_j*,* $j = 0, 1, 2, 3$*, where*

$$\sigma_0(n_1, n_2, n_3) = (n_1, n_2, n_3), \quad \sigma_1(n_1, n_2, n_3) = (-n_1, n_2, n_3),$$
$$\sigma_2(n_1, n_2, n_3) = (n_1, -n_2, n_3), \quad \sigma_3(n_1, n_2, n_3) = (n_1, n_2, -n_3).$$

Remark 3.2. For $0 < i \leqslant 3$, $0 < j \leqslant 3$, $0 < l \leqslant 3$, $\sigma_j^2 = Id$, $\sigma_i\sigma_j = -\sigma_l$, if $i \neq j$ and $\sigma_i\sigma_j\sigma_l = -Id$, for $i \neq j \neq l$. The σ_j do preserve the convolution conditions in K^*.

We choose an α for which the set K^* is not empty. We further take the hypothesis of a single triple wave resonance (k, m, n), modulo the symmetries σ_j.

Hypothesis 3.3. K^* is such that there exists a single triple wave number resonance (n, k, m), modulo the symmetries σ_j, $j = 1, 2, 3$, and $\sigma_j(k) \neq k$, $\sigma_j(m) \neq m$, $\sigma_j(n) \neq n$ for $j = 2$ and $j = 3$.

Under the above assumption, one can demonstrate that the strictly resonant Euler system splits into three uncoupled systems in $\mathbf{C}^3$:

Theorem 3.4. *Under the hypothesis* 3.3*, the resonant Euler system reduces to three uncoupled rigid body systems in* $\mathbf{C}^3$

$$\frac{dU_n}{dt} + i(\lambda_k - \lambda_m)C_{kmn}U_kU_m = 0, \tag{3.3a}$$

$$\frac{dU_k}{dt} - i(\lambda_m - \lambda_n)C_{kmn}U_nU_m^* = 0, \tag{3.3b}$$

$$\frac{dU_m}{dt} - i(\lambda_n - \lambda_k)C_{kmn}U_nU_k^* = 0, \tag{3.3c}$$

where $C_{kmn} = i\langle \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* \rangle$*,* C_{kmn} *real and the other two uncoupled systems obtained with the symmetries* $\sigma_2(k, m, n)$ *and* $\sigma_3(k, m, n)$*. The energy and the helicity of each subsystem are conserved*:

$$\frac{d}{dt}(U_kU_k^* + U_mU_m^* + U_nU_n^*) = 0,$$

$$\frac{d}{dt}(\lambda_kU_kU_k^* + \lambda_mU_mU_m^* + \lambda_nU_nU_n^*) = 0.$$

PROOF. From $U_{-k} = U_k^*$ it follows that $\lambda(-k) = \lambda(+k)$, similarly for m and n; and in a very essential way from the antisymmetry of $\langle \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* \rangle$, together with curl $\mathbf{\Phi}_k = \lambda_k \mathbf{\Phi}_k$. That C_{kmn} is real follows from the eigenfunctions detailed in [**29**, Sec. 3]. □

Remark 3.5. This deep structure, i.e., $SO(3; \mathbf{C})$ rigid body systems in $\mathbf{C}^3$ is a direct consequence of the Lamé form of the full 3D Euler equations, see Equations (1.3) and (2.7), and the nonlinearity curl $\mathbf{V} \times \mathbf{V}$.

The system (3.3) is equivariant with respect to the symmetry operators

$$(z_1, z_2, z_3) \to (z_1^*, z_2^*, z_3^*),\ (z_1, z_2, z_3) \to (\exp(i\chi_1)z_1, \exp(i\chi_2)z_2, \exp(i\chi_3)z_3)\,,$$

provided $\chi_1 = \chi_2 + \chi_3$. It admits other integrals known as the Manley–Rowe relations (see, for instance [**37**]). It differs from the usual 3-wave resonance systems investigated in the literature such as in [**40, 41, 22**] in that

(1) helicity is conserved,
(2) dynamics of these resonant systems rigorously "shadow" those of the exact 3D Euler equations, see Theorem 2.10.

Real forms of the system (3.3) are found in [**20**], corresponding to the exact invariant manifold $U_k \in i\mathbf{R}$, $U_m \in \mathbf{R}, U_n \in \mathbf{R}$, albeit without any rigorous asymptotic justification. The $\mathbf{C}^3$ systems (3.3) with *helicity* conservation laws are not discussed in [**20**].

The only nontrivial Manley–Rowe conservation laws for the resonant system (3.3), rigid body $SO(3; \mathbf{C})$, which are independent from energy and helicity, are as follows:

$$\frac{d}{dt}\left(r_k r_m r_n \sin(\theta_n - \theta_k - \theta_m)\right) = 0,$$

where $U_j = r_j \exp(i\theta_j)$, $j = k, m, n$, and

$$\mathcal{E}_1 = (\lambda_k - \lambda_m) r_n^2 - (\lambda_m - \lambda_n) r_k^2,$$
$$\mathcal{E}_2 = (\lambda_m - \lambda_n) r_k^2 - (\lambda_n - \lambda_k) r_m^2.$$

The resonant system (3.3) is well known to possess hyperbolic equilibria and heteroclinic/homoclinic orbits on the energy surface. We are interested in rigorously proving arbitrary large bursts of enstrophy and higher norms on arbitrarily small time intervals, for properly chosen h/R. To simplify the presentation, we establish the results for the simpler invariant manifold $U_k \in i\mathbf{R}$, and $U_m, U_n \in \mathbf{R}$.

Rescale time as $t \to t/C_{kmn}$. Start from the system

$$\begin{aligned} \dot{U}_n + i(\lambda_k - \lambda_m)U_k U_m &= 0, \\ \dot{U}_k - i(\lambda_m - \lambda_n)U_n U_m^* &= 0, \\ \dot{U}_m - i(\lambda_n - \lambda_k)U_n U_k^* &= 0. \end{aligned} \tag{3.4}$$

Assume that $U_k \in i\mathbf{R}$ and $U_m, U_n \in \mathbf{R}$. Set $p = iU_k$, $q = U_m$, and $r = U_n$, as well as $\lambda_k = \lambda$, $\lambda_m = \mu$, and $\lambda_n = \nu$. Then

$$\begin{aligned} \dot{p} + (\mu - \nu)qr &= 0, \\ \dot{q} + (\nu - \lambda)rp &= 0, \\ \dot{r} + (\lambda - \mu)pq &= 0. \end{aligned} \tag{3.5}$$

This system admits two first integrals

$$\begin{aligned} E &= p^2 + q^2 + r^2 \qquad &\text{(energy)} \\ H &= \lambda p^2 + \mu q^2 + \nu r^2 \qquad &\text{(helicity)} \end{aligned} \tag{3.6}$$

The system (3.5) is exactly the $SO(3,\mathbf{R})$ rigid body dynamics Euler equations, with inertia momenta $I_j = \dfrac{1}{|\lambda_j|}$, $j = k, m, n$ [**1**].

Lemma 3.6 ([**1, 20**]). *With the ordering $\lambda_k > \lambda_m > \lambda_n$, i.e., $\lambda > \mu > \nu$, the equilibria $(0, \pm 1, 0)$ are hyperbolic saddles on the unit energy sphere, and the equilibria $(\pm 1, 0, 0)$, $(0, 0, \pm 1)$ are centers. There exist equivariant families of heteroclinic connections between $(0, +1, 0)$ and $(0, -1, 0)$. Each pair of such connections correspond to equivariant homoclinic cycles at $(0, 1, 0)$ and $(0, -1, 0)$.*

We investigate bursting dynamics along orbits with large periods, with initial conditions close to the hyperbolic point $(0, E(0), 0)$ on the energy sphere E. We choose resonant triads such that $\lambda_k > 0, \lambda_n < 0, \lambda_k \sim |\lambda_n|$, $|\lambda_m| \ll \lambda_k$, equivalently:

$$\lambda > \mu > \nu,\ \lambda\nu < 0,\ |\mu| \ll \lambda \text{ and } \lambda \sim |\nu|. \tag{3.7}$$

Lemma 3.7. *There exists h/R with $K^* \neq \varnothing$ such that*

$$\lambda_k > \lambda_m > \lambda_n,\ \lambda_k\lambda_n < 0,\ |\lambda_m| \ll \lambda_k, \quad \lambda_k \sim |\lambda_n|.$$

Remark 3.8. Together with the polarity $\pm$ of the curl eigenvalues, these are 3-wave resonances where two of the eigenvalues are much larger in moduli than the third one. In the limit $|k|,\ |m|, |n| \gg 1, \lambda_k \sim \pm|k|,\ \lambda_m \sim \pm|m|,\ \lambda_n \sim \pm|n|$, the eigenfunctions $\mathbf{\Phi}$ have leading asymptotic terms which involve cosines and sines periodic in r (see [**29**, Sec. 3]). In the strictly resonant equations (2.30), the summation over the quadratic terms becomes an asymptotic convolution in $n_1 = k_1 + n_1$. The resonant three waves in Lemma 3.7 are equivalent to Fourier triads $k + m = n$, with $|k| \sim |n|$ and $|m| \ll |k|, |n|$, in periodic lattices. In the physics of spectral theory of turbulence [**17, 26**], these are exactly the triads responsible from transfer of energy between large scales and small scales. These are the triads which have hampered mathematical efforts at proving the global regularity of the Cauchy problem for 3D Navier–Stokes equations in periodic lattices [**16**].

Proof of Lemma 3.7 (see [**29**]). The transcendental dispersion law for 3-waves in K^* for cylindrical domains, is a polynomial of degree four in $\vartheta_3 = 1/h^2$:

$$\tilde{P}(\vartheta_3) = \tilde{P}_4\vartheta_3^4 + \tilde{P}_3\vartheta_3^3 + \tilde{P}_2\vartheta_3^2 + \tilde{P}_1\vartheta_3 + \tilde{P}_0 = 0, \tag{3.8}$$

with $n_2 = k_2 + m_2$ and $n_3 = k_3 + m_3$.

Then with

$$h_k = \frac{\beta^2(k_1, k_2, \alpha k_3)}{k_3^2}, \; h_m = \frac{\beta^2(m_1, m_2, \alpha m_3)}{m_3^2}, \; h_n = \frac{\beta^2(n_1, n_2, \alpha n_3)}{n_3^2}$$

(see the radial Sturm–Liouville problem in [**29**, Sec. 3], the coefficients of $\tilde{P}(\vartheta_3)$ are given by the formula

$$\begin{aligned}
\tilde{P}_4 &= -3,\\
\tilde{P}_3 &= -4(h_k + h_m + h_n),\\
\tilde{P}_2 &= -6(h_k h_m + h_k h_n + h_m h_n),\\
\tilde{P}_1 &= -12 h_k h_m h_n,\\
\tilde{P}_0 &= h_m^2 h_n^2 + h_k^2 h_n^2 + h_m^2 h_k^2 - 2(h_k h_m h_n^2 + h_k h_n h_m^2 + h_m h_n h_k^2).
\end{aligned}$$

Similar formulas for the periodic lattice domain were first derived in [**5, 6, 7**]. In cylindrical domains, the resonance condition for K^* is identical to

$$\pm\frac{1}{\sqrt{\vartheta_3 + h_k}} \pm \frac{1}{\sqrt{\vartheta_3 + h_m}} \pm \frac{1}{\sqrt{\vartheta_3 + h_n}} = 0,$$

with $\vartheta_3 = 1/h^2$, $h_k = \beta^2(k)/k_3^2$, $h_m = \beta^2(m)/m_3^2$, $h_n = \beta^2(n)/n_3^2$; Equation (3.8) is the equivalent rational form.

From the asymptotic formula (3.44) in [**29**] for large β

$$\beta(n_1, n_2, n_3) \sim n_1\pi + n_2\frac{\pi}{2} + \frac{\pi}{4} + \psi, \tag{3.9}$$

where $\psi = 0$ if $\lim \dfrac{m_2}{m_3}\dfrac{h}{2\pi} = 0$ (for example, h fixed, $\dfrac{m_2}{m_3} \to 0$) and $\psi = \pm\dfrac{\pi}{2}$ if $\lim \dfrac{m_2}{m_3}\dfrac{h}{2\pi} = \pm\infty$ (for example, $\dfrac{m_2}{m_3}$ fixed, $h \to \infty$). The proof is completed by taking leading terms $\tilde{P}_0 + \vartheta_3\tilde{P}_1$ in (3.8), $\vartheta_3 = \dfrac{1}{h^2} \ll 1$, and $m_2 = 0$, $k_2 = \mathcal{O}(1)$, $n_2 = \mathcal{O}(1)$. $\square$

We now state a theorem for bursting of the $\mathbf{H}^3$ norm in arbitrarily small times, for initial data close to the hyperbolic point $(0, E(0), 0)$.

Theorem 3.9 (bursting dynamics in $\mathbf{H}^3$)**.** *Suppose that $\lambda > \mu > \nu$, $\lambda\nu < 0$, $|\mu| \ll \lambda$, and $\lambda \sim |\nu|$. Let $W(t) = \lambda^6 p(t)^2 + \mu^6 q(t)^2 + \nu^6 r(t)^2$ be the $\mathbf{H}^3$-norm squared of an orbit of* (3.5)*. Choose initial data such that $W(0) = \lambda^6 p(0)^2 + \mu^6 q(0)^2$ with $\lambda^6 p(0)^2 \sim \frac{1}{2}W(0)$ and $\mu^6 q(0)^2 \sim \frac{1}{2}W(0)$. Then there exists $t^* > 0$ such that*

$$W(t) \geqslant \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^6 W(0),$$

where $t^* \leqslant \dfrac{6}{\sqrt{W(0)}}\mu^2 Ln(\lambda/|\mu|)(\lambda/|\mu|)^{-1}$.

Remark 3.10. Under the conditions of Lemma 3.7, $(\lambda/\mu)^6 \gg 1$, whereas $\mu^2(Ln(\lambda/|\mu|))(\lambda/|\mu|)^{-1} \ll 1$. Therefore, over a small time interval of length $O(\mu^2(Ln(\lambda/|\mu|))(\lambda/|\mu|)^{-1}) \ll 1$, the ratio $||\mathbf{U}(t)||_{H^3}/||\mathbf{U}(0)||_{H^3}$ grows to reach a maximal value $O\left((\lambda/|\mu|)^3\right) \gg 1$. Since the orbit is periodic, the $\mathbf{H}^3$ semi-norm eventually relaxes to its initial state after some time (this being a manifestation of the time-reversibility of the Euler flow on the energy sphere). The "shadowing" Theorem 2.10 with $s > 7/2$ ensures that the full, original 3D Euler dynamics, with the same initial conditions, will undergo the same type of burst. Notice that, with the definition (1.13) of $\|\cdot\|_{H^s}$, one has

$$||\Omega\mathbf{e}_3 \times y||_{H^3} = ||\operatorname{curl}^3(\Omega\mathbf{e}_3 \times y)||_{L^2} = 0\,.$$

Hence the solid rotation part of the original 3D Euler solution does not contribute to the ratio $||\mathbf{V}(t)||_{H^3}/||\mathbf{V}(0)||_{H^3}$.

Theorem 3.11 (bursting dynamics of the enstrophy)**.** *Under the same conditions for the 3-wave resonance, let $\Xi(t) = \lambda^2 p(t)^2 + \mu^2 q(t)^2 + \nu^2 r(t)^2$ be the enstrophy. Choose initial data such that $\Xi(0) = \lambda^2 p(0)^2 + \mu^2 q(0)^2 + \nu^2 r(0)^2$ with $\lambda^2 p(0)^2 \sim \frac{1}{2}\Xi(0)$, $\mu^2 q(0)^2 \sim \frac{1}{2}\Xi(0)$. Then there exists $t^{**} > 0$ such that*

$$\Xi(t^{**}) \geqslant (\lambda/\mu)^2,$$

where $t^{**} \leqslant \dfrac{1}{\sqrt{2}} \dfrac{1}{\sqrt{\Xi(0)}} Ln(\lambda/|\mu|)(\lambda/|\mu|)^{-1}$.

Remark 3.12. It is interesting to compare this mechanism for bursts with earlier results in the same direction obtained by DiPerna and Lions. Indeed, for each $p \in (1, \infty)$, each $\delta \in (0, 1)$ and each $t > 0$, DiPerna and Lions [**15**] constructed examples of 2D-3 components solutions to Euler equations such that

$$||\mathbf{V}(0)||_{W^{1,p}} \leqslant \epsilon \quad \text{while} \quad ||\mathbf{V}(t)||_{W^{1,p}} \geqslant 1/\delta\,.$$

Their examples essentially correspond to shear flows of the form

$$\mathbf{V}(t, x_1, x_2) = \begin{pmatrix} u(x_2) \\ 0 \\ w(x_1 - tu(x_2), x_2) \end{pmatrix},$$

where $u \in W^{1,p}_{x_2}$ while $w \in W^{1,p}_{x_2}$. It is obvious that

$$\text{curl } \mathbf{V}(t, x_1, x_2) = \begin{pmatrix} (\partial_2 - tu'(x_2)\partial_1)w(x_1 - tu(x_2), x_2) \\ -\partial_1 w(x_1 - tu(x_2), x_2) \\ -u'(x_2) \end{pmatrix}.$$

Thus, all the components in curl $\mathbf{V}(t, x_1, x_2)$ belong to L^p_{loc}, except for the term $-tu'(x_2)\partial_1 w(x_1 - tu(x_2), x_2)$. For each $t > 0$ this term belongs to L^p for any choice of $u \in W^{1,p}_{x_2}$ and $w \in W^{1,p}_{x_1,x_2}$ if and only if $p = \infty$. Whenever $p < \infty$, DiPerna and Lions construct their examples as some smooth approximation of the situation above in the strong $W^{1,p}$ topology.

In other words, the DiPerna–Lions construction works only in cases where the initial vorticity does not belong to an algebra — specifically to L^p, which is not an algebra unless $p = \infty$.

The type of burst obtained in our construction above is different: in that case, the original vorticity belongs to the Sobolev space H^2, which is an algebra in space dimension 3. Similar phenomena are observed in all Sobolev spaces H^β with $\beta \geqslant 2$ — which are also algebras in space dimension 3.

In other words, our results complement those of DiPerna–Lions on bursts in higher order Sobolev spaces, however at the expense of using more intricate dynamics.

We proceed to the proofs of Theorems 3.9 and 3.11. We are interested in the evolution of

$$\Xi = \lambda^2 p^2 + \mu^2 q^2 + \nu^2 r^2 \qquad \text{(enstrophy)} \tag{3.10}$$

Compute

$$\dot{\Xi} = -2\left(\lambda^2(\mu-\nu) + \mu^2(\nu-\lambda) + \nu^2(\lambda-\mu)\right) pqr. \tag{3.11}$$

Then

$$\dot{(pqr)} = -(\mu-\nu)q^2r^2 - (\nu-\lambda)r^2p^2 - (\lambda-\mu)p^2q^2. \tag{3.12}$$

Using the first integrals above, one has

$$Van \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = \begin{pmatrix} E \\ H \\ \Xi \end{pmatrix}, \tag{3.13}$$

where Van is the Vandermonde matrix

$$Van = \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \end{pmatrix}.$$

For $\lambda \neq \mu \neq \nu \neq \lambda$ this matrix is invertible and

$$Van^{-1} = \begin{pmatrix} \frac{\mu\nu}{(\lambda-\mu)(\lambda-\nu)} & \frac{-(\mu+\nu)}{(\lambda-\mu)(\lambda-\nu)} & \frac{1}{(\lambda-\mu)(\lambda-\nu)} \\ \frac{\nu\lambda}{(\mu-\nu)(\mu-\lambda)} & \frac{-(\nu+\lambda)}{(\mu-\nu)(\mu-\lambda)} & \frac{1}{(\mu-\nu)(\mu-\lambda)} \\ \frac{\lambda\mu}{(\nu-\lambda)(\nu-\mu)} & \frac{-(\lambda+\mu)}{(\nu-\lambda)(\nu-\mu)} & \frac{1}{(\nu-\lambda)(\nu-\mu)} \end{pmatrix}.$$

Hence

$$\begin{aligned} p^2 &= \frac{1}{(\lambda-\mu)(\lambda-\nu)}\left(\Xi - (\mu+\nu)H + \mu\nu E\right), \\ q^2 &= \frac{1}{(\mu-\nu)(\mu-\lambda)}\left(\Xi - (\nu+\lambda)H + \nu\lambda E\right), \\ r^2 &= \frac{1}{(\nu-\lambda)(\nu-\mu)}\left(\Xi - (\lambda+\mu)H + \lambda\mu E\right), \end{aligned} \tag{3.14}$$

so that

$$(\mu-\nu)q^2r^2 = -\frac{(\Xi-(\nu+\lambda)H+\nu\lambda E)\,(\Xi-(\lambda+\mu)H+\lambda\mu E)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)},$$

$$(\nu-\lambda)r^2p^2 = -\frac{(\Xi-(\lambda+\mu)H+\lambda\mu E)\,(\Xi-(\mu+\nu)H+\mu\nu E)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)},$$

$$(\lambda-\mu)p^2q^2 = -\frac{(\Xi-(\mu+\nu)H+\mu\nu E)\,(\Xi-(\nu+\lambda)H+\nu\lambda E)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.$$

Later on, we use the notation

$$\begin{aligned} x_-(\lambda,\mu,\nu) &= (\mu+\nu)H-\mu\nu E,\\ x_0\,(\lambda,\mu,\nu) &= (\mu+\lambda)H-\mu\lambda E,\\ x_+(\lambda,\mu,\nu) &= (\lambda+\nu)H-\lambda\nu E. \end{aligned} \tag{3.15}$$

Therefore, we find that Ξ satisfies the second order ODE

$$\begin{aligned} \ddot{\Xi} = &-2K_{\lambda,\mu,\nu}\,((\Xi-x_-(\lambda,\mu,\nu))(\Xi-x_0(\lambda,\mu,\nu))\\ &+(\Xi-x_0(\lambda,\mu,\nu))(\Xi-x_+(\lambda,\mu,\nu))+(\Xi-x_+(\lambda,\mu,\nu))(\Xi-x_0(\lambda,\mu,\nu))) \end{aligned}$$

which can be put in the form

$$\ddot{\Xi} = -2K_{\lambda,\mu,\nu}P'_{\lambda,\mu,\nu}(\Xi), \tag{3.16}$$

where $P_{\lambda,\mu,\nu}$ is the cubic

$$P_{\lambda,\mu,\nu}(X) = (X-x_-(\lambda,\mu,\nu))(X-x_0(\lambda,\mu,\nu))(X-x_+(\lambda,\mu,\nu)) \tag{3.17}$$

and

$$K_{\lambda,\mu,\nu} = \frac{\lambda^2(\mu-\nu)+\mu^2(\nu-\lambda)+\nu^2(\lambda-\mu)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}. \tag{3.18}$$

In the sequel, we assume that the initial data for (p,q,r) are such that $r(0)=0$, $p(0)(q(0)\neq 0$. Let us compute

$$\begin{aligned} x_-(\lambda,\mu,\nu) &= \lambda\nu p(0)^2+\mu^2q(0)^2+\mu(\lambda-\nu)p(0)^2.\\ x_0\,(\lambda,\mu,\nu) &= \lambda^2p(0)^2+\mu^2q(0)^2,\\ x_+(\lambda,\mu,\nu) &= \lambda^2p(0)^2+\left(\frac{\nu+\lambda}{\mu}-\frac{\nu\lambda}{\mu^2}\right)\mu^2q(0)^2. \end{aligned} \tag{3.19}$$

We also assume that

$$\lambda>\mu>\nu\,,\qquad \lambda\nu<0\,,\qquad |\mu|\ll\lambda,\quad \lambda\sim|\nu|. \tag{3.20}$$

Then $K_{\lambda,\mu,\nu} > 0$. In fact, $K_{\lambda,\mu,\nu} \sim 2$, and Ξ is a periodic function of t such that

$$\inf_{t\in\mathbf{R}} \Xi(t) = x_0(\lambda,\mu,\nu)\,, \quad \sup_{t\in\mathbf{R}} \Xi(t) = x_+(\lambda,\mu,\nu) \tag{3.21}$$

with half-period

$$T_{\lambda,\mu,\nu} = \frac{1}{2\sqrt{K_{\lambda,\mu,\nu}}} \int_{x_0(\lambda,\mu,\nu)}^{x_+(\lambda,\mu,\nu)} \frac{dx}{\sqrt{-P_{\lambda,\mu,\nu}(x)}}. \tag{3.22}$$

We are interested in the growth of the (squared) $\mathbf{H}^3$ norm

$$W(t) = \lambda^6 p(t)^2 + \mu^6 q(t)^2 + \nu^6 r(t)^2. \tag{3.23}$$

Expressing p^2, q^2, and r^2 in terms of E, H, and Ξ, we find

$$W = \frac{\lambda^6(\Xi - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)} + \frac{\mu^6(\Xi - x_+(\lambda,\mu,\nu))}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^6(\Xi - x_0(\lambda,\mu,\nu))}{(\nu-\lambda)(\nu-\mu)}. \tag{3.24}$$

Hence, if $\Xi = x_+(\lambda,\mu,\nu)$, then

$$\begin{aligned} W &= \frac{\lambda^6(x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)} + \frac{\nu^6(x_+(\lambda,\mu\,\nu) - x_0(\lambda,\mu,\nu))}{(\nu-\lambda)(\nu-\mu)} \\ &\geqslant \frac{\lambda^6(x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu))}{(\lambda-\mu)(\lambda-\nu)}. \end{aligned}$$

Let us compute

$$\begin{aligned} x_+(\lambda,\mu\,\nu) - x_-(\lambda,\mu,\nu) &= (\lambda-\mu)(\lambda-\nu)p(0)^2 \\ &+ \Big(\frac{\nu+\lambda}{\mu} - \frac{\nu\lambda}{\mu} - 1\Big)\mu^2 q(0)^2 \gtrsim -\nu\lambda q(0)^2 \sim \lambda^2 q(0)^2. \end{aligned} \tag{3.25}$$

We pick the initial data such that

$$W(0) = \lambda^6 p(0)^6 + \mu^6 q(0)^6 \text{ with } \lambda^6 p(0)^2 \sim \tfrac{1}{2}W(0) \text{ and } \mu^6 q(0)^2 \sim \tfrac{1}{2}W(0). \tag{3.26}$$

When Ξ reaches $x_+(\lambda,\mu,\nu)$, we have

$$W \gtrsim \frac{\lambda^8 q(0)^2}{(\lambda-\mu)(\lambda-\nu)} \sim \tfrac{1}{2}\frac{\lambda^8}{\mu^6(\lambda-\mu)(\lambda-\nu)}W(0) \sim \tfrac{1}{4}\frac{\lambda^6}{\mu^6}W(0)\,. \tag{3.27}$$

Hence W jumps from $W(0)$ to $\sim \frac{1}{4}\frac{\lambda^6}{\mu^6}W(0)$ in an interval of time that does not exceed one period of the Ξ motion, i.e., $2T_{\lambda,\mu,\nu}$. Let us estimate this interval of time. We recall the asymptotic equivalent for the period of an elliptic integral in the modulus 1 limit.

Lemma 3.13. *Assume that $x_- < x_0 < x_+$. Then*

$$\int_{x_0}^{x_+} \frac{dx}{\sqrt{(x-x_-)(x-x_0)(x_+-x)}} \sim \frac{1}{\sqrt{x_+-x_-}} \ln\left(\frac{1}{1-\sqrt{\frac{x_+-x_0}{x_+-x_-}}}\right)$$

uniformly in x_-, x_0, and x_+ as $\dfrac{x_+-x_0}{x_+-x_-} \to 1$.

Here,

$$\frac{1}{\sqrt{x_+(\lambda,\mu,\nu)-x_-(\lambda,\mu,\nu)}} \lesssim \frac{1}{\sqrt{\lambda^2 q(0)^2}} \sim \frac{|\mu|^3}{\lambda}\sqrt{\frac{2}{W(0)}}.$$

Further,

$$x_0(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 \tag{3.28}$$

so that

$$\frac{1}{1-\sqrt{\frac{x_+-x_0}{x_+-x_-}}} \sim \frac{1}{1-\sqrt{1-\frac{(\lambda-\mu)(\lambda-\nu)p(0)^2}{(\lambda-\mu)(\lambda-\nu)p(0)^2+(\mu(\nu+\lambda)-\nu\lambda-\mu^2)q(0)^2}}}$$

$$\sim \frac{(\lambda-\mu)(\lambda-\nu)p(0)^2+(\mu(\nu+\lambda)-\nu\lambda-\mu^2)q(0)^2}{2(\lambda-\mu)(\lambda-\nu)p(0)^2}$$

$$\sim \frac{q(0)^2}{2p(0)^2} \sim \tfrac{1}{2}\frac{W(0)/2\mu^6}{W(0)/2\lambda^6} = \frac{\lambda^6}{2\mu^6}.$$

Hence

$$2T_{\lambda,\mu,\nu} \lesssim \frac{2}{\sqrt{W(0)}}\frac{\mu^3}{\lambda}\ln\left(\frac{\lambda^6}{2\mu^6}\right) \leqslant \frac{12}{\sqrt{W(0)}}\frac{|\mu|^3}{\lambda}\ln\left(\frac{\lambda}{\mu}\right). \tag{3.29}$$

Conclusion: collecting (3.26), (3.27), and (3.29), we see that the squared $\mathbf{H}^3$ norm W varies from $W(0)$ to $\sim \rho^6 W(0)$ in an interval of time $\lesssim \dfrac{12}{\sqrt{W(0)}}\dfrac{\mu^2\ln\rho}{\rho}$. (Here, $\rho = \lambda/\mu$.)

We now proceed by obtaining similar bursting estimates for the enstrophy. We return to (3.21) and (3.22). Pick the initial data so that

$$\Xi(0) = \lambda^2 p(0)^2 + \mu^2 q(0)^2 \text{ with } \lambda^2 p(0)^2 \sim \frac{1}{2}\Xi(0) \text{ and } \mu^2 q(0)^2 \sim \frac{1}{2}\Xi(0).$$

Then

$$x_+(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 + \left(\frac{\nu+\lambda}{\mu} - \frac{\nu\lambda}{\mu^2} - 1\right)\mu^2 q(0)^2$$
$$\sim 2\lambda^2 p(0)^2 + \lambda^2 q(0)^2 \sim \frac{\lambda^2}{\mu^2}\Xi(0),$$

while $x_0(\lambda,\mu,\nu) - x_-(\lambda,\mu,\nu) = (\lambda-\mu)(\lambda-\nu)p(0)^2 \sim 2\lambda^2 p(0)^2 \sim \Xi(0)$. Hence, in the limit as $\rho = \lambda/|\mu| \to +\infty$, one has

$$\begin{aligned} 2T_{\lambda,\mu,\nu} &\sim \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{\rho^2\Xi(0)}}\ln\frac{1}{1-\sqrt{1-\frac{\Xi(0)}{\frac{1}{2}\rho^2\Xi(0)}}} \\ &= \frac{1}{2\sqrt{2\Xi(0)}}\frac{1}{\rho}\ln\frac{1}{1-\sqrt{1-2\rho^{-2}}} \sim \frac{1}{\sqrt{2\Xi(0)}}\frac{\ln\rho}{\rho}, \end{aligned}$$

and Ξ varies from $x_0(\lambda,\mu,\nu) = \Xi(0)$ to $x_+(\lambda,\mu,\nu) \sim \rho^2\Xi(0)$ on an interval of time of length $T_{\lambda,\mu,\nu}$. □

4. Strictly Resonant Euler Systems: the Case of 3-Wave Resonances on Small-Scales

4.1. Infinite dimensional uncoupled $SO(3)$ systems.

In this section, we consider the 3-wave resonant set K^* when $|k|^2$, $|m|^2$, $|n|^2 \geqslant 1/\eta^2$, $0 < \eta \ll 1$, i.e., 3-wave resonances on small scales; here $|k|^2 = k_1^2+k_2^2+k_3^2$, where (k_1,k_2,k_3) index the curl eigenvalues, and similarly for $|m|^2$, $|n|^2$. Recall that $k_2+m_2 = n_2$, $k_3+m_3 = n_3$ (exact convolutions), but that the summation on k_1, m_1 on the right-hand side of Equations (2.30) is not a convolution. However, for $|k|^2$, $|m|^2$, $|n|^2 \geqslant 1/\eta^2$, the summation in k_1, m_1 becomes an asymptotic convolution.

Proposition 4.1. *The set K^* restricted to $|k|^2, |m|^2, |n|^2 \geqslant 1/\eta^2$, for all $0 < \eta \ll 1$ is not empty: there exist at least one h/R with resonant three waves satisfying the above small scales condition.*

Proof. We follow the algebra of the exact transcendental dispersion law (3.8) derived in the proof of Lemma 3.7. Note that $\tilde{P}(\vartheta_3) < 0$ for $\vartheta_3 = 1/h^2$ large enough. We can choose $h_m = \dfrac{\beta^2(m_1,m_2,\alpha m_3)}{m_3^2} = 0$, say in the specific limit $\dfrac{h}{2\pi m_3} \to 0$, and $\beta(m_1,m_2,\alpha m_3) \sim m_1\pi + m_2\dfrac{\pi}{2} + \dfrac{\pi}{4}$.

Then $\tilde{P}_0 = h_k^2 h_n^2 > 0$ and $\tilde{P}(\vartheta_3)$ must possess at least one (transcendental) root $\vartheta_3 = 1/h^2$. $\square$

In the above context, the radial components of the curl eigenfunctions involve cosines and sines in $\beta r/R$ (see [**29**, Sec. 3]) and the summation in k_1, m_1 on the right-hand side of the resonant Euler equations (2.30) becomes an *asymptotic convolution*. The rigorous asymptotic convolution estimates are highly technical and detailed in [**18**]. The 3-wave resonant systems for $|k|^2,\ |m|^2,\ |n|^2 \geqslant 1/\eta^2$ are equivalent to those of an equivalent periodic lattice $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi h]$, $\vartheta_3 = 1/h^2$; the resonant three wave relation becomes:

$$\pm\left(\vartheta_3 + \vartheta_1 \frac{n_1^2}{n_3^2} + \vartheta_2 \frac{n_2^2}{n_3^2}\right)^{-\frac{1}{2}} \pm \left(\vartheta_3 + \vartheta_1 \frac{k_1^2}{k_3^2} + \vartheta_2 \frac{k_2^2}{k_3^2}\right)^{-\frac{1}{2}}$$
$$\pm\left(\vartheta_3 + \vartheta_1 \frac{m_1^2}{m_3^2} + \vartheta_2 \frac{m_2^2}{m_3^2}\right)^{-\frac{1}{2}} = 0, \tag{4.1a}$$
$$k + m = n,\ \ k_3 m_3 n_3 \neq 0. \tag{4.1b}$$

The algebraic geometry of these rational 3-wave resonance equations has been investigated in depth in [**6**] and [**7**]. Here, $\vartheta_1,\ \vartheta_2,\ \vartheta_3$ are periodic lattice parameters; in the small-scales cylindrical case, $\vartheta_1 = \vartheta_2 = 1$ (after rescaling of $n_2,\ k_2,\ m_2$), $\vartheta_3 = 1/h^2$, h height. Based on the algebraic geometry of "resonance curves" in [**6, 7**], we investigate the resonant 3D Euler equations (2.30) in the equivalent *periodic lattices*.

First, triplets (k, m, n) solution of (4.1) are invariant under the reflection symmetries $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ defined in Corollary 3.1 and Remark 3.2: $\sigma_0 = Id$, $\sigma_j(k) = (\epsilon_{i,j} k_i)$, $1 \leqslant i \leqslant 3$, $\epsilon_{i,j} = +1$ if $i \neq j$, $\epsilon_{i,j} = -1$ if $i = j$, $1 \leqslant j \leqslant 3$. Second, the set K^* in (4.1) is invariant under the homothetic transformations:

$$(k, m, n) \to (\gamma k, \gamma m, \gamma n),\ \ \gamma \text{ rational.} \tag{4.2}$$

The resonant triplets lie on projective lines in the wave number space, with equivariance under σ_j, $0 \leqslant j \leqslant 3$ and γ-rescaling. For every given equivariant family of such projective lines the resonant curve is the graph of $\frac{\vartheta_3}{\vartheta_1}$ versus $\frac{\vartheta_2}{\vartheta_1}$, for parametric domain resonances in $\vartheta_1,\ \vartheta_2,\ \vartheta_3$.

Lemma 4.2 ([**7**, p. 17])**.** *For every equivariant (k, m, n) the resonant curve in the quadrant $\vartheta_1 > 0, \vartheta_2 > 0, \vartheta_3 > 0$ is the graph of a smooth function $\vartheta_3/\vartheta_1 \equiv F(\vartheta_2/\vartheta_1)$ intersected with the quadrant.*

Theorem 4.3 ([**7**, p. 19]). *A resonant curve in the quadrant $\vartheta_3/\vartheta_1 > 0$, $\vartheta_2/\vartheta_1 > 0$ is called irreducible if*

$$det \begin{pmatrix} k_3^2 & k_2^2 & k_1^2 \\ m_3^2 & m_2^2 & m_1^2 \\ n_3^2 & n_2^2 & n_1^2 \end{pmatrix} \neq 0. \tag{4.3}$$

An irreducible resonant curve is uniquely characterized by six nonnegative algebraic invariants $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{S}_1$, and $\mathcal{S}_2$ such that

$$\left\{\frac{n_1^2}{n_3^2}, \frac{n_2^2}{n_3^2}\right\} = \left\{\mathcal{P}_1^2, \mathcal{P}_2^2\right\},$$

$$\left\{\frac{k_1^2}{k_3^2}, \frac{k_2^2}{k_3^2}\right\} = \left\{\mathcal{R}_1^2, \mathcal{R}_2^2\right\},$$

$$\left\{\frac{m_1^2}{m_3^2}, \frac{m_2^2}{m_3^2}\right\} = \left\{\mathcal{S}_1^2, \mathcal{S}_2^2\right\},$$

and permutations thereof.

Lemma 4.4 ([**7**, p. 25]). *For resonant triplets (k, m, n) associated to a given irreducible resonant curve, i.e., verifying Equation (4.3), consider the convolution equation $n = k+m$. Let $\sigma_i(n) \neq n$ for all $1 \leqslant i \leqslant 3$. Then there are at most two solutions (k, m) and (m, k) for a given n provided that six non-degeneracy conditions (3.39)–(3.44) in [**7**] for the algebraic invariants of the irreducible curve are verified.*

For more details on the technical non-degeneracy conditions see Appendix below. An exhaustive algebraic geometric investigation of all solutions to $n = k+m$ on irreducible resonant curves is found in [**7**]. The essence of the above lemma lies in that given such an irreducible "non-degenerate" triplet (k, m, n) on K^*, all other triplets on the same irreducible resonant curves are exhaustively given by the equivariant projective lines:

$$(k, m, n) \quad \to \quad (\gamma k, \gamma m, \gamma n), \quad \text{for some } \gamma \text{ rational}, \tag{4.4}$$

$$(k, m, n) \quad \to \quad (\sigma_j k, \sigma_j m, \sigma_j n), \;\; j = 1, 2, 3, \tag{4.5}$$

and permutations of k and m in the above. Of course, the homothety γ and the σ_j symmetries preserve the convolution. This context of irreducible "non-degenerate" resonant curves yields an infinite dimensional uncoupled system of rigid body $SO(3; \mathbf{R})$ and $SO(3; \mathbf{C})$ dynamics for the 3D resonant Euler equations (2.30).

Theorem 4.5. *For any irreducible triplet* (k, m, n) *satisfying Theorem* 4.3 *and under the "non-degeneracy" conditions of Lemma* 4.4 (*csee Appendix*), *the resonant Euler equations split into the infinite countable sequence of uncoupled* $SO(3; \mathbf{R})$ *systems*

$$\dot{a}_k = \Gamma_{kmn}(\lambda_m - \lambda_n)a_m a_n, \tag{4.6a}$$

$$\dot{a}_m = \Gamma_{kmn}(\lambda_n - \lambda_k)a_n a_k, \tag{4.6b}$$

$$\dot{a}_n = \Gamma_{kmn}(\lambda_k - \lambda_m)a_k a_m \tag{4.6c}$$

$$\begin{gathered} \textit{for all } (k, m, n) = \gamma(\sigma_j(k^*), \sigma_j(m^*), \sigma_j(n^*)), \\ \gamma = \pm 1, \pm 2, \pm 3 \ldots, 0 \leqslant j \leqslant 3. \end{gathered} \tag{4.7}$$

k^*, m^*, n^* *are some relatively prime integer vectors in* $\mathbf{Z}^3$ *characterizing the equivariant family of projective lines* (k, m, n); $\Gamma_{kmn} = i\langle \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* \rangle$, Γ_{kmn} *real.*

Theorem 4.5 is a simpler version for invariant manifolds of more general $SO(3; \mathbf{C})$ systems. It is a straightforward consequence of Propositions 3.2 and 3.3 and Theorems 3.3–3.5 in [**7**]. The latter article did not explicit the resonant equations and did not use the curl-helicity algebra fundamentally underlying this present work. Rigorously asymptotic infinite countable sequences of uncoupled $SO(3; \mathbf{R})$, $SO(3; \mathbf{C})$ systems are not derived via the usual harmonic analysis tools of Fourier modes, in the 3D Euler context. Polarization of curl eigenvalues and eigenfunctions and helicity play an essential role.

Corollary 4.6. *Under the conditions* $\lambda_{n^*} - \lambda_{k^*} > 0$, $\lambda_{k^*} - \lambda_{m^*} > 0$, *the resonant Euler systems* (4.6) *admit a disjoint, countable family of homoclinic cycles. Moreover, under the conditions* $\lambda_{n^*} \gg +1$, $\lambda_{m^*} \ll -1$, $|\lambda_{k^*}| \ll \lambda_{n^*}$, *each subsystem* (4.6) *possesses orbits whose* $\mathbf{H}^s$ *norms,* $s \geqslant 1$, *burst arbitrarily large in arbitrarily small times.*

Remark 4.7. One can prove that there exists some $0 < \Gamma_{\max} < \infty$, such that $|\Gamma_{kmn}| < \Gamma_{max}$ for all (k, m, n) on the equivariant projective lines defined by (4.7). The systems (4.6) "freeze" cascades of energy; their total enstrophy

$$\Xi(t) = \sum_{(k,m,n)} (\lambda_k^2 a_k^2(t) + \lambda_m^2 a_m^2(t) + \lambda_n^2 a_n^2(t))$$

remains bounded, albeit with large bursts of $\Xi(t)/\Xi(0)$, on the reversible orbits topologically close to the homoclinic cycles.

4.2. Coupled $SO(3)$ rigid body resonant systems.

We now derive a new resonant Euler system which couples **two** $SO(3;\mathbf{R})$ rigid bodies via a common principle axis of inertia and a common moment of inertia. This 5-dimensional system conserves energy, helicity, and is rather interesting in that dynamics on its homoclinic manifolds show bursting cascades of enstrophy to the smallest scale in the resonant set. We consider the equivalent periodic lattice geometry under the assumptions of Proposition 4.1.

In Appendix, we prove that for an "irreducible" 3-wave resonant set which now satisfies the algebraic "degeneracy" (A-4), there exist exactly two "primitive" resonant triplets (k,m,n) and $(\tilde{k},\tilde{m},n)$, where $k,\ m,\ \tilde{k},\ \tilde{m}$ are relative prime integer valued vectors in $\mathbf{Z}^3$.

Lemma 4.8. *Under the algebraic degeneracy condition* (A-4)*, the irreducible equivariant family of projective lines in K^* is exactly generated by the following two "primitive" triplets*:

$$n = k + m, \quad k = a\overline{k}, \quad m = b\overline{m}, \tag{4.8a}$$

$$n = \tilde{k} + \tilde{m}, \quad \tilde{k} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}), \tag{4.8b}$$

i.e.,

$$n = a\overline{k} + b\overline{m}, \tag{4.8c}$$

$$n = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}), \tag{4.8d}$$

where $\sigma_i \neq \sigma_j$ are some reflection symmetries, a, b, a', b' are relatively prime integers, positive or negative, and $\overline{k}$, $\overline{m}$ are relatively prime integer valued vectors in $\mathbf{Z}^3$, i.e., $(a,a') = (b,b') = (a,b) = (a',b') = 1$, $(\overline{k},\overline{m}) = 1$, where $(\ )$ denotes the greatest common denominator of two integers. All other resonant wave number triplets are generated by the group actions σ_l, $l = 1,2,3$, and homothetic rescalings

$$(k,m,n) \to \gamma(k,m,n), \quad (\tilde{k},\tilde{m},n) \to \gamma(\tilde{k},\tilde{m},n) \quad (\gamma \in \mathbf{Z})$$

of the "primitive" triplets.

Remark 4.9. It can be proved that the set of such coupled "primitive" triplets is not empty on the periodic lattice. The algebraic irreducibility condition of Lemma 4.2 implies that $\pm k_3/|k| = \pm\tilde{k}_3/|\tilde{k}|$ and $\pm m_3/|m| = \pm\tilde{m}_3/|\tilde{m}|$, which is obviously verified in Equations (4.8).

Theorem 4.10. *Under the assumptions of Lemma* 4.8*, the resonant Euler system reduces to a system of two rigid bodies coupled via $a_n(t)$:*

$$\dot{a}_k = (\lambda_m - \lambda_n)\Gamma a_m a_n, \tag{4.9a}$$
$$\dot{a}_m = (\lambda_n - \lambda_k)\Gamma a_n a_k, \tag{4.9b}$$
$$\dot{a}_n = (\lambda_k - \lambda_m)\Gamma a_k a_m + (\lambda_{\tilde{k}} - \lambda_{\tilde{m}})\tilde{\Gamma} a_{\tilde{k}} a_{\tilde{m}}, \tag{4.9c}$$
$$\dot{a}_{\tilde{m}} = (\lambda_n - \lambda_{\tilde{k}})\tilde{\Gamma} a_n a_{\tilde{k}}, \tag{4.9d}$$
$$\dot{a}_{\tilde{k}} = (\lambda_{\tilde{m}} - \lambda_n)\tilde{\Gamma} a_{\tilde{m}} a_n, \tag{4.9e}$$

where $\Gamma = i\langle \mathbf{\Phi}_k \times \mathbf{\Phi}_m, \mathbf{\Phi}_n^* \rangle$, $\tilde{\Gamma} = i\langle \mathbf{\Phi}_{\tilde{k}} \times \mathbf{\Phi}_{\tilde{m}}, \mathbf{\Phi}_n^* \rangle$. *Energy and helicity are conserved.*

Theorem 4.11. *The resonant system* (4.9) *possesses three independent conservation laws*

$$\mathcal{E}_1 = a_k^2 + (1-\alpha)a_m^2, \tag{4.10a}$$
$$\mathcal{E}_2 = a_n^2 + \alpha a_m^2 + (1-\tilde{\alpha})a_{\tilde{m}}^2, \tag{4.10b}$$
$$\mathcal{E}_3 = a_{\tilde{k}}^2 + \tilde{\alpha} a_{\tilde{m}}^2, \tag{4.10c}$$

where

$$\alpha = (\lambda_m - \lambda_k)/(\lambda_n - \lambda_k), \tag{4.11a}$$
$$\tilde{\alpha} = (\lambda_{\tilde{m}} - \lambda_n)/(\lambda_{\tilde{k}} - \lambda_n). \tag{4.11b}$$

Theorem 4.12. *Under the conditions*

$$\lambda_m < \lambda_k < \lambda_n, \tag{4.12a}$$
$$\lambda_{\tilde{m}} < \lambda_n < \lambda_{\tilde{k}}, \tag{4.12b}$$

which imply $\alpha < 0$, $\tilde{\alpha} < 0$, *the equilibria* $(\pm a_k(0), 0, 0, 0, \pm a_{\tilde{k}}(0))$ *are hyperbolic for* $|a_{\tilde{k}}(0)|$ *small enough with respect to* $|a_k(0)|$. *The unstable manifolds of these equilibria are one dimensional, and the nonlinear dynamics of the system* (4.9) *are constrained on the ellipse* $\mathcal{E}_1$ (4.10a) *for* $a_k(t)$, $a_m(t)$, *the hyperbola* $\mathcal{E}_3$ (4.10c) *for* $a_{\tilde{k}}(t)$, $a_{\tilde{m}}(t)$, *and the hyperboloid* $\mathcal{E}_2$ (4.10b) *for* $a_m(t)$, $a_{\tilde{m}}(t)$, $a_n(t)$.

Theorem 4.13. *Let the* 2*-manifold* $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ *be coordinatized by* $(a_m, a_{\tilde{m}})$. *On this* 2*-manifold, the resonant system* (4.9) *is Hamiltonian, and therefore integrable. Its* (*multi-valued*) *Hamiltonian is defined by the closed* 1*-form*

$$\mathbf{h} \;=\; \Gamma(\lambda_n - \lambda_k)\frac{da_{\tilde{m}}}{a_{\tilde{k}}} - \tilde{\Gamma}(\lambda_n - \lambda_{\tilde{k}})\frac{da_m}{a_k}, \tag{4.13}$$

while the symplectic 2-form is

$$\omega = \frac{da_m \wedge da_{\tilde{m}}}{a_k a_n a_{\tilde{k}}}\,. \tag{4.14}$$

PROOF. Eliminating $a_k(t)$ via $\mathcal{E}_1$, $a_n(t)$ via $\mathcal{E}_2$, $a_{\tilde{k}}(t)$ via $\mathcal{E}_3$, the resonant system (4.9) reduces to

$$\begin{aligned}
\dot{a}_m &= \pm\Gamma(\lambda_n-\lambda_k)(\mathcal{E}_1-(1-\alpha)a_m^2)^{\frac{1}{2}}(\mathcal{E}_2-\alpha a_m^2+(\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}},\\
\dot{a}_{\tilde{m}} &= \pm\tilde{\Gamma}(\lambda_n-\lambda_{\tilde{k}})(\mathcal{E}_2-\alpha a_m^2+(\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}}(\mathcal{E}_3-\tilde{\alpha}a_{\tilde{m}}^2)^{\frac{1}{2}};
\end{aligned}$$

after changing the time variable into

$$t \to \int_0^t (\mathcal{E}_1-(1-\alpha)a_m^2)^{\frac{1}{2}}(\mathcal{E}_2-\alpha a_m^2+(\tilde{\alpha}-1)a_{\tilde{m}}^2)^{\frac{1}{2}}(\mathcal{E}_3-\tilde{\alpha}a_{\tilde{m}}^2)^{\frac{1}{2}}\,ds\,.$$

On each component of the manifold $\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$, the following functionals are conserved:

$$\begin{aligned}
\mathcal{H}(a_m,a_{\tilde{m}}) = &\pm\tilde{\Gamma}(\lambda_n-\lambda_{\tilde{k}})\int\frac{da_m}{(\mathcal{E}_1-(1-\alpha)a_m^2)^{1/2}}\\
&\pm\Gamma(\lambda_n-\lambda_k)\int\frac{da_{\tilde{m}}}{(\mathcal{E}_3-\tilde{\alpha}a_{\tilde{m}}^2)^{1/2}}.
\end{aligned}$$ □

We note that the system of two coupled rigid bodies (4.9) does not seem to admit a simple Lie–Poisson bracket in the original variables $(a_k, a_m, a_n, a_{\tilde{m}}, a_{\tilde{k}})$. Yet, when restricted to the 2-manifold $\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$ that is invariant under the flow of (4.9), it is Hamiltonian and therefore integrable.

This raises the following interesting issue: according to the shadowing Theorem 2.10, the Euler dynamics remains asymptotically close to that of chains of coupled $SO(3;\mathbf{R})$ and $SO(3;\mathbf{C})$ rigid body systems. Perhaps, some new information could be obtained in this way. We are currently investigating this question and will report on it in a forthcoming publication [**21**].

Already the simple 5-dimensional system (4.9) has interesting dynamical properties, which we could not find in the existing literature on systems related to spinning tops.

Consider, for instance, the dynamics of the resonant system (4.9) with I.C. topologically close to the hyperbola equilibria $(\pm a_k(0),0,0,0,\pm a_{\tilde{k}}(0))$. Under the conditions of (4.12) and with the help of the integrability theorem (see Theorem 4.13), it is easy to construct equivariant families of homoclinic cycles at these hyperbolic critical points.

Corollary 4.14. *The hyperbolic critical points* $(\pm a_k(0), 0, 0, 0, \pm a_{\tilde{k}}(0))$ *possess* 1*-dimensional homoclinic cycles on the cones*

$$a_n^2 + (1 - \tilde{\alpha})a_{\tilde{m}}^2 = -\alpha a_m^2 \quad \text{with } \alpha < 0,\ \tilde{\alpha} < 0. \tag{4.15}$$

Note that these are genuine homoclinic cycles, NOT sums of heteroclinic connections. Initial conditions for the resonant system (4.9) are now chosen in a small neighborhood of these hyperbolic critical points, the corresponding orbits are topologically close to these cycles. With the ordering:

$$\lambda_m < \lambda_k < \lambda_n, \tag{4.16a}$$

$$|\lambda_k| \ll |\lambda_m|,\ |\lambda_k| \ll \lambda_n, \tag{4.16b}$$

$$\lambda_{\tilde{m}} < \lambda_n < \lambda_{\tilde{k}}, \tag{4.16c}$$

$$|\lambda_{\tilde{m}}| \ll \lambda_{\tilde{k}}, \tag{4.16d}$$

$$\lambda_{\tilde{k}} \gg \lambda_n, \tag{4.16e}$$

which can be realized with $|a'/a| \gg 1$ and $|b'/b| \ll 1$ in the resonant triplets (4.8), we can demonstrate bursting dynamics akin to Theorems 3.9 and 3.11 for enstrophy and $\mathbf{H}^s$ norms, $s \geqslant 2$. The interesting feature is the maximization of $|a_{\tilde{k}}(t)|$ near the turning points of the homoclinic cycles on the cones (4.15). This corresponds to transfer of energy to the smallest scale $\tilde{k}$, $\lambda_{\tilde{k}}$.

In the forthcoming publication [**21**], we investigate infinite systems of the coupled rigid bodies equations (4.9).

Appendix

We focus on a resonant wave number triplet $(n, k, m) \in (\mathbf{Z}^*)^3$ verifying

- the convolution relation

$$n = k + m, \tag{A-1}$$

- the resonant 3-wave resonance relation

$$\pm \frac{n_3}{\sqrt{\vartheta_1 n_1^2 + \vartheta_2 n_2^2 + \vartheta_3 n_3^2}} \pm \frac{k_3}{\sqrt{\vartheta_1 k_1^2 + \vartheta_2 k_2^2 + \vartheta_3 k_3^2}} \pm \frac{m_3}{\sqrt{\vartheta_1 m_1^2 + \vartheta_2 m_2^2 + \vartheta_3 m_3^2}} = 0, \tag{A-2}$$

- the condition of "non-catalyticity"

$$k_3 m_3 n_3 \neq 0, \tag{A-3}$$

- and the degeneracy condition of [**7**, p. 26]

$$G^{ir}_{i,j}(k,m) = k_i n_j m_l + k_l m_j n_i = 0, \tag{A-4}$$

where (i,j,l) is a permutation of $(1,2,3)$.

As is known [**7**, Lemma 3.5 (2)], the system of equations (A-3)–(A-4) for unknown k and m, given a vector n, admits exactly 4 solutions in $\mathbf{Z}^3 \times \mathbf{Z}^3$:

$$(k,m),\ (m,k),\ (\tilde{k},\tilde{m}),\ (\tilde{m},\tilde{k}).$$

Here, k and m are two vectors of the original resonant triplet, whereas $\tilde{k} = \alpha\sigma_i(k)$, $\tilde{m} = \beta\sigma_j(m)$, where

$$\alpha = \frac{m_i k_l - m_l k_i}{m_i k_l + m_l k_i} \notin \{0, \pm 1\}, \quad \beta = \frac{m_l k_j - m_j k_l}{m_l k_j + m_j k_l} \notin \{0, \pm 1\},$$

and the symmetries σ_i and σ_j are defined by

$$\sigma_i : u = (u_l)_{l=1,2,3} \to \left((-1)^{\delta_{il}} u_l\right)_{l=1,2,3}.$$

One verifies that $\sigma_i^2 = \sigma_j^2 = Id$, $\sigma_i\sigma_j = \sigma_j\sigma_i = -\sigma_l$, i.e., the group generated by σ_i and σ_j is the Klein group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

Let us write irrational numbers α and β under the irreducible representation $\alpha = a'/a$, $\beta = b'/b$, with $a, a', b, b' \in \mathbf{Z}^*$ and $(a,a') = (b,b') = 1$, where ( ) denotes the greatest common denominator of the integer pair. From $\tilde{k} \in \mathbf{Z}^3$ it follows that $a|a'k$. However, since $(a,a') = 1$, the Euclid lemma yields that $a|k$. Similarly, $b|m$. We set $\overline{k} = \frac{1}{a}k \in \mathbf{Z}^3$, $\overline{m} = \frac{1}{b}m \in \mathbf{Z}^3$. Hence the integer vector n admits two decompositions

$$n = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}).$$

Since the function $z \longmapsto \dfrac{z_3}{\sqrt{\vartheta_1 z_1^2 + \vartheta_2 z_2^2 + \vartheta_3 z_3^2}}$ is homogeneous of degree 0, we see that, within the resonance condition (A-2), we can replace each vector k, m and n by any collinear vectors, integer or not. Suppose that there exists a positive integer $d \neq 1$ such that $d|\overline{k}$. Then $d|n$, so that by setting $n_0 = \frac{1}{d}n$, $k_0 = \frac{1}{d}\overline{k}$, $m_0 = \frac{1}{d}\overline{m}$, we finally obtain

$$n_0 = ak_0 + bk_0 = a'\sigma_i(k_0) + b'\sigma_j(m_0).$$

The triplets (n_0, ak_0, bm_0) and $(n_0, a'\sigma_i(k_0), b'\sigma_j(m_0))$ verify from the above remark, the convolution relation (A-1), and the resonance relation (A-2). Hence without loss of generality we can assume that the only positive integer d such that $d|\overline{k}$ and $d|\overline{m}$ is 1; which we denote by $(\overline{k}, \overline{m}) = 1$. Equivalently,

$$\overline{k}_1\mathbf{Z} + \overline{k}_2\mathbf{Z} + \overline{k}_3\mathbf{Z} + \overline{m}_1\mathbf{Z} + \overline{m}_2\mathbf{Z} + \overline{m}_3\mathbf{Z} + = \mathbf{Z}.$$

Finally, suppose that there exists a positive integer $d \neq 1$ such that $d|a$ and $d|b$. Then $d|n$. We set $n_0 = \frac{1}{d}n$, $a_0 = \frac{1}{d}a$, $b_0 = \frac{1}{d}b$. Observe that

$$G^{ir}_{i,j}(a_0\overline{k}, b_0\overline{m}) = \frac{1}{d^3}G^{ir}_{i,j}(a\overline{k}, b\overline{m}) = 0.$$

From [**7**, Lemma 3.5 (2)] it follows that the vector n_0 of the resonant triplet $(n_0, a_0\overline{k}, b_0\overline{m})$ can also be written as

$$n_0 = \hat{k} + \hat{m} \text{ with } (n_0, \hat{k}, \hat{m}) \text{ verifying (A-2).}$$

But then $n = dn_0 = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m}) = d\hat{k} + d\hat{m}$. By [**7**, Lemma 3.5 (2)], $(d\hat{k}, d\hat{m})$ must coincide with one of the pairs $(a'\sigma_i(\overline{k}), b'\sigma_j(\overline{m}))$, $(b'\sigma_j(\overline{m}), a'\sigma_i(\overline{k}))$. In particular, $d|a'\overline{k}$ and $d|b'\overline{m}$. Since $d|a$ and $(a, a') = 1$, we have (d, a') similarly $(d, b') = 1$. But then the Euclid lemma yields $d|\overline{k}$ and $d|\overline{m}$, which contradicts the fact that $(\overline{k}, \overline{m}) = 1$. Hence we have proved that $(a, b) = 1$. In a similar way, one can show that $(a', b') = 1$.

Conclusion: From the above consideration it follows that $n \in \mathbf{Z}^*$ admits two decompositions

$$n = a\overline{k} + b\overline{m} = a'\sigma_i(\overline{k}) + b'\sigma_j(\overline{m})$$

with $(a, a') = (b, b') = (a, b) = (a', b') = 1$, $(\overline{k}, \overline{m}) = 1$.

The triplets $(n, a\overline{k}, b\overline{m})$ and $(n, a'\sigma_i(\overline{k}), b'\sigma_j(\overline{m}))$ verify the resonant condition (A-2) (from the homogeneity of this condition) and the condition of non-catalyticity (A-3). Indeed, $aba'b' \neq 0$ and the condition (A-3) on the initial triplet (n, k, m) imply that the reduced triplet $(n, \overline{k}, \overline{m})$ also verifies (A-3)). Finally, the degeneracy condition (A-4) $G^{ir}_{i,j}(a\overline{k}, b\overline{m}) = 0$ is verified.

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Increased Stability in the Cauchy Problem for Some Elliptic Equations

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We derive some bounds which can be viewed as an evidence of increasing stability in the Cauchy problem for the Helmholtz equation with lower order terms when frequency is growing. These bounds hold under certain (pseudo-)convexity properties of the surface, where the Cauchy data are given, and of variable zero order coefficient of the Helmholtz equation. Proofs use Carleman estimates, the theory of elliptic and hyperbolic boundary value problems in Sobolev spaces, and Fourier analysis. We outline open problems and possible future developments. Bibliography: 12 titles.

1. Introduction

Uniqueness and stability in the Cauchy problem for partial differential equations is an issue of fundamental theoretical and applied importance. In particular, it is quite important for control theory and inverse problems. Uniqueness implies approximate controllability, and the Lipschitz stability estimates lead to exact controllability. The Cauchy problem plays a crucial role in recovery properties of media or obstacles from remote sensing.

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Theory of the Cauchy problem has a long history, starting with the classical Holmgren–John theorem about the uniqueness of continuation across a noncharacteristic initial surface Γ for equations and systems with analytic coefficients.

In 1938, Carleman used weighted energy estimates to handle nonanalytic coefficients. His method generated a variety of results, mainly for scalar equations, published in hundreds of research papers and monographs. This method works under the so-called pseudo-convexity condition on the weight function. If this condition is not satisfied, there are examples of nonunique continuation across a noncharacteristic surface. With an exception of the hyperbolic equations and space like initial surfaces Γ, the Cauchy problem is not well posed, in particular, there exists no solution in classical function spaces. If a solution is unique, then one can claim some stability provided that solutions are bounded in some standard norms. As is well known [**9**], for general analytic equations the best possible stability is stability of logarithmic type. This is quite pessimistic for the numerical solution of the continuation problem and therefore for various applications. The Carleman method implies much better Hölder type stability estimates and, in some interesting cases, even the best possible Lipschitz type estimates. For brief history and references see [**4, 5**].

Needs of prospecting by acoustical, elastic, and electromagnetic waves stimulate the study of this problem for the Helmholtz equation

$$(\Delta + b \cdot \nabla + a_0^2 k^2)u = f \text{ in } \Omega,\ u \in H_{(1)}(\Omega), \tag{1.1}$$

with the Cauchy data

$$u = u_0,\ \partial_\nu u = u_1 \text{ on } \Gamma, \tag{1.2}$$

where Ω is a domain in $\mathbf{R^n}$, $\Gamma \in C^1$ is an (open) part of its boundary $\partial\Omega$, and ν is the exterior unit normal to $\partial\Omega$. For fixed k we have a conditional Hölder stability estimate [**5**, Sec. 3.3], however the constants in this estimate may depend on k. Due to the celebrated results of John [**9**], in the general case, these constants blow up as k goes to ∞, and one can expect only a quite weak logarithmic k-independent stability estimate. However, in several important practical examples (for examples, in computations for inverse scattering [**1**] and in near field acoustical holography [**2, 8**]), it was observed that stability (and, as a consequence, resolution) in the Cauchy problem and in some inverse problems is increasing with k. In [**3**], the authors found new stability estimates explaining this phenomenon for constant a_0 and illustrated it by the numerical solution of some important applied problems. In [**7**], the author extended results of [**3**] to variable a_0 and $b = 0$. The goal of this paper is to show that the addition of regular b does not change results of [**7**]. This is achieved by more careful and complicated analysis using the general scheme of [**7**]. In particular, we again employ hyperbolic energy inequalities in the low frequency zone, use Carleman estimates for the time dependent wave equation to get k-independent Carleman estimates for Equation (1.1), and "freeze" b, a_0 at certain points in a special way. Again, we have to impose some (nontrapping) condition on a_0. Proofs are getting more complicated because, in addition to difficulties with the (tangential) Fourier transform for variable coefficients, we have to handle "non-selfadjoint" terms resulting from the Fourier transform of $b \cdot \nabla u$.

This paper is organized as follows. In Section 1, we describe the current state of the problem, our main results and adjust to the Helmholtz equation the famous counterexample of Fritz John for the wave equation. In Section 2, we give energy estimates in the low frequency zone for constant and variable coefficients. An important ingredient of the proof of the k-independent stability for the Cauchy problem is a Carleman type estimate for (1.1) which does not depend on k. In Section 3, we derive this estimate from a known estimate for hyperbolic equations exactly as in [**7**]. Using the results of Sections 2 and 3, in Section 4 we give (standard and similar to [**7**]) proofs of the main results.

We write $x = (x', x_n) \in \mathbf{R}^n, 2 \leqslant n$. Let Ω be an open subset of the cylinder $\{0 < x_n < h, |x'| < r\}$ with the Lipschitz boundary $\partial\Omega$, $\bar{\Omega} \subset \{x_n < h\}$, and let Γ be the part of $\partial\Omega$ contained in the layer $\{0 < x_n < h\}$. Suppose that $\Omega(d) = \Omega \cap \{d < x_n\}$ and $\Omega^*(d) = \mathbf{R}^{n-1} \times (d, h)$, $0 \leqslant d$. Let $e_n = (0, \dots, 0, 1)$. We denote by C and $\varkappa$ constants that depend only on Ω, S, a_0, b, d. Any other dependence is indicated. We denote by $\|u\|_{(l)}(\Omega)$ the standard norm in the Sobolev space $H_{(l)}(\Omega)$ and write $\|u\|(\Omega) = \|u\|_{(0)}(\Omega)$. We set $M_1 = \|u\|_{(1)}(\Omega)$, $F = \|f\|(\Omega) + \|u\|(\Gamma) + \|\nabla u\|(\Gamma)$, and $F(k) = F + k\|u\|(\Gamma)$. Denote by $V(\xi, x_n)$ the (partial) Fourier transform $\mathcal{F}v(\xi, x_n)$ of a function $v(x)$ with respect to x'.

Since we are interested in increasing wave numbers k, for the sake of simplicity, we assume that

$$1 \leqslant k. \tag{1.3}$$

Theorem 1.1. *Assume that* $b, a_0 \in C^1(\overline{\Omega}), 0 < a_0$ *on* $\overline{\Omega}$, *and*

$$0 < a_0 + \nabla a_0 \cdot x + \beta_n \partial_n a_0, \; 0 \leqslant \partial_n a_0 \;\; on \; \overline{\Omega} \tag{1.4}$$

for some positive β_n. *Let* $0 < d$. *Then for any* ε *there are* $C, C(\varepsilon), \varkappa(d) \in (0, 1)$ *such that*

$$\|u\|_{(0)}(\Omega(d)) \leqslant C\Big(F + \varepsilon\|u\|_{(1)}(\Omega) + C(\varepsilon)\frac{M_1^{1-\varkappa}F(k)^{\varkappa} + F}{k}\Big) \tag{1.5}$$

for all u *solving* (1.1), (1.2).

As is known, $C(\varepsilon)$ indeed depends on d and blows up as $d \to 0$ [**3**],[**5**, Ch. 3].

Theorem 1.1 allows us to consider more general domains Ω. Let S be a compact subset of Ω. We denote by $P(\nu; d)$ the half-space of $\mathbf{R}^n$ with the exterior normal ν which has the distance d from S. We denote by γ all ν such that $P(\nu; d) \cap \partial\Omega$ is contained in Γ. Let $\Omega(\nu; \Gamma, d)$ be $P(\nu; d) \cap \Omega$, and, finally, let $\Omega(\Gamma, d)$ be the union of all such $\Omega(\nu; \Gamma, d)$ over $\nu \in \gamma$. If $\Gamma = \partial\Omega$, then $\Omega(\Gamma, 0)$ is the difference of Ω and the convex hull of S and $\Omega(\Gamma, d)$ is the collection of points of $\Omega(\Gamma, 0)$ which are at distance d from S. As in [**3**], applying Theorem 1.1 to any $\Omega = \Omega(\nu; \Gamma, d), \nu \in \gamma$ and using an appropriate partition of unity, we obtain the following assertion.

Corollary 1.1. *Let the condition* (1.4) *be satisfied in any* $\Omega(\nu; \Gamma, d)$, $\nu \in \gamma$, *with the* x_n*-direction replaced by* ν. *Then the bound* (1.5) *with* $\Omega(\Gamma, d)$ *instead of* $\Omega(d)$ *is valid.*

There is an important particular case, where the norm of the data does not explicitly depend on k. Let us keep the notation of Corollary1.1. Let ω be an open subset of Ω with $\Gamma \subset \partial\omega$ (boundary layer) such that Γ is at the distance d_0 from $\partial\omega \cap \Omega$. Let $F_\omega = \|f\|(\Omega) + \|u\|_{(1)}(\omega)$.

Corollary 1.2. *Under the assumptions of Corollary* 1.1, *there are constants* $C, C(d_0), C(\varepsilon)$ *such that for any solution* u *to the Cauchy problem* (1.1), (1.2)

$$\|u\|(\Omega(\Gamma, d)) \leqslant C\Big(d_0)(F_\omega + \varepsilon\|u\|_{(1)}(\Omega) + C(\varepsilon)\frac{M_1^{1-\varkappa}F_\omega^{\varkappa} + F_\omega}{k}\Big). \tag{1.6}$$

To derive Corollary 1.2 from Corollary 1.1, we let χ to be a cut off function that is equal to 1 on $\Omega \backslash \omega$ and vanishes near Γ. Applying Corollary 1.1 to χu instead of u and using that

$$\begin{aligned}(\Delta + b\cdot\nabla + k^2 a_0^2)(\chi u) &= \chi((\Delta + k^2)u) + 2\nabla\chi\cdot\nabla u + (\Delta\chi + b\cdot\nabla\chi)u \\ &= \chi f + 2\nabla\chi\cdot\nabla u + (\Delta\chi + b\cdot\nabla\chi)u\end{aligned}$$

and the function χu has zero Cauchy data on Γ, we obtain (1.6) from Corollary 1.1.

Theorem 1.1 and its corollaries show an improved stability in the Cauchy problem (1.1), (1.2) when one continues the solution of the differential equation inside the convex hull of Γ. Due to the results of John [**9**], this is impossible when one continues to the outside of a convex Γ.

Our proof of Theorem 1.1 is based on the following assertion.

Theorem 1.2. *Let the condition* (1.4) *be satisfied. Then there are constants* $C, \varkappa(d) \in (0,1)$ *such that for any solution* u *to the Cauchy problem* (1.1), (1.2)

$$\|u\|_{(1)}(\Omega(d)) \leqslant C(F + (M_1)^{1-\varkappa} F(k)^{\varkappa}), \tag{1.7}$$

which is of its own interest since C and $\varkappa$ are independent of k.

Due to the above-mentioned results of John, the stability estimates of Theorems 1.1, 1.2 seem to be optimal. We remind the remarkable argument from [**9**, p. 569-571].

Let $n = 2, r = |x|, x_1 = r\cos\theta, x_2 = r\sin\theta$. The functions

$$u_k(x) = k^{-\frac{2}{3}} J_k(kr) e^{ik\theta}$$

solve Equation (1.1) (with $b = 0, a_0 = 1$) in $\mathbf{R}^2$. Let Ω be the annular domain$\{\frac{1}{2} < |x| < 2\}$ and $\Gamma = \{|x| = \frac{1}{2}\}$. John showed that

$$|J_k(kr)| \leqslant q^k \text{ when } \frac{1}{3} < r < \frac{2}{3}$$

for some $0 < q < 1$, and, on the other hand,

$$|u_k| = J_k(k) \geqslant C_0 k^{-\frac{1}{2}} \text{ on } \{|x| = 1\}.$$

From the first bound and known recurrent relations for Bessel functions we have a similar inequality for J_k', and hence

$$k\|u_k\|(\Gamma) + \|\nabla u_k\|(\Gamma) \leqslant q^k \text{ for some } q \in (0,1), C < k. \tag{1.8}$$

Moreover, from [**9**, p. 570]

$$\|u_k\|_{(1)}(\Omega) \leqslant C \tag{1.9}$$

and

$$J_k^2(kr) = \frac{2}{\pi k}(r^2-1)^{-\frac{1}{2}} \cos^2\left(-\frac{\pi}{4} + k\left((r^2-1)^{\frac{1}{2}} - \cos^{-1}\frac{1}{r}\right)\right) + o\left(\frac{1}{k}\right), \tag{1.10}$$

where o is uniform on $(3/2, 2)$. We have

$$\int_{\frac{3}{2}}^{2} \cos^2\left(-\frac{\pi}{4} + k\left((r^2-1)^{\frac{1}{2}} - \cos^{-1}\frac{1}{r}\right)\right) dr \geqslant \frac{1}{C}\int_{\alpha}^{\beta} \cos^2\left(-\frac{\pi}{4} + ks\right) ds$$

$$= \frac{\beta - \alpha}{C} - \frac{\sin(-\frac{\pi}{2} + 2k\alpha) - \sin(-\frac{\pi}{2} + 2k\beta)}{4Ck} \geqslant \frac{1}{C} \tag{1.11}$$

provided that $k > C$. Here, we used the substitution $s = (r^2 - 1)^{\frac{1}{2}} - \cos^{-1} \frac{1}{r}$ and observed that

$$\frac{1}{C} < \frac{ds}{dr} = (1 - r^{-2})^{\frac{1}{2}} < C \text{ when } \frac{3}{2} < r < 2.$$

Using (1.10) and (1.11), we yield

$$\|u_k\|^2(\Omega)$$
$$\geqslant \frac{1}{Ck} \int\limits_{\frac{3}{2}}^{2} (r^2 - 1)^{-\frac{1}{2}} \cos^2 \Big(-\frac{\pi}{4} + k\Big((r^2 - 1)^{\frac{1}{2}} - \operatorname{arc}\cos \frac{1}{r}\Big)\Big) r dr + o\Big(\frac{1}{k^2}\Big)$$
$$\geqslant \frac{1}{Ck} \int\limits_{\frac{3}{2}}^{2} \cos^2 \Big(-\frac{\pi}{4} + k\Big((r^2 - 1)^{\frac{1}{2}} - \operatorname{arc}\cos \frac{1}{r}\Big)\Big) dr + o\Big(\frac{1}{k^2}\Big) \geqslant \frac{1}{Ck}.$$

This inequality and bound (1.9) demonstrate that for different geometries (when Ω is not in the convex hull of Γ) or without the condition (1.4) Theorem 1.1 and its corollaries are wrong. Moreover, this example shows that the constants in the bound (1.5) (which holds at fixed k, [**5**, Secs. 3.2, 3.3]) blow up when k grows. So, without convexity type conditions, the stability in the Cauchy problem for the Helmholtz equation is not improving, but on the contrary it is deteriorating.

2. Energy Type Estimates in Low Frequency Zone

We obtain some auxiliary results imitating the standard energy estimate for hyperbolic initial value problems.

Lemma 2.1. *Let $a(n), b(n) \in C^1([0, h])$ depend only on x_n, and let $v \in C^2(\bar{\Omega}^*)$ solve the initial value problem*

$$\begin{gathered}(\Delta + b(n) \cdot \nabla + a(n)^2 k^2) v_j = \partial_j f_j \text{ in } \Omega^*(d),\ j = 1, \dots, n-1, \\ v_j = 0 \text{ on } \Omega^*(h_1)\end{gathered} \tag{2.1}$$

for some $h_1 \in (d, h)$, $f_j \in C^\infty(\bar{\Omega}^*(d))$, $f_j = 0$ on $\Omega^*(h_1)$, and

$$V_j(\xi, x_n) = 0 \text{ when } \frac{a^2(x_n; n)}{2} k^2 < |\xi|^2. \tag{2.2}$$

Then there is a constant C depending only on h, $\sup(|b(n)| + |\partial_n b(n)| + |a(n)| + |\partial_n a(n)|)$, $\sup a^{-1}(n)$ *over* $(0, h)$ *such that*

$$\|v_j\|(\Omega^*(d)) \leqslant C\|f_j\|(\Omega^*(d)). \tag{2.3}$$

PROOF. By the Parseval identity, it suffices to show that the solution to the initial value problem

$$\begin{aligned}\partial_n^2 V_j + b_n(n)\partial_n V_j + (a(n)^2k^2 - |\xi|^2)V_j\\ - ib(n)\cdot\xi V_j = -i\xi_j F_j \text{ on } (d, h), \quad j = 1, \dots, n-1,\end{aligned} \tag{2.4}$$

with zero final conditions

$$V_j = 0,\ F_j = 0 \text{ on } (h_1, h) \tag{2.5}$$

satisfies the bound

$$\int_d^h |V_j|^2(\xi, s)ds \leqslant C\int_d^h |F_j|^2(\xi, s)ds,\ j = 1, \dots, n-1. \tag{2.6}$$

Multiplying both sides of (2.4) by $\partial_n \bar{V}_j$, taking the complex conjugate, and adding results, we yield

$$\begin{aligned}&(\partial_n^2 V_j)\partial_n\bar{V}_j + (\partial_n^2\bar{V}_j)\partial_n V_j + 2b_n(n)|\partial_n V_j|^2\\ &+ (a(n)^2k^2 - |\xi|^2)(V_j\partial_n\bar{V}_j + \bar{V}_j\partial_n V_j) - ib(n)\cdot\xi(V_j\partial_n\bar{V}_j - \bar{V}_j\partial_n V_j)\\ &= i\xi_j(F_j\partial_n\bar{V}_j - \bar{F}_j\partial_n V_j).\end{aligned}$$

Observing that $\partial_n|V|^2 = V\partial_n\bar{V} + \partial_n V\bar{V}$ and multiplying by -$e^{\tau x_n}$, we obtain

$$\begin{aligned}&-(\partial_n|\partial_n V|^2)e^{\tau x_n} - b_n(n)2|\partial_n V_j|^2e^{\tau x_n}\\ &-(a(n)^2k^2 - |\xi|^2)\partial_n|V_j|^2e^{\tau x_n} + ib(n)\cdot\xi(V_j\partial_n\bar{V}_j - \bar{V}_j\partial_n V_j)e^{\tau x_n}\\ &= -i\xi_j(F_j\partial_n\bar{V}_j - \bar{F}_j\partial_n V_j)e^{\tau x_n}.\end{aligned}$$

Integrating by parts over the interval (x_n, h) with the use of (2.5), we obtain

$$\begin{aligned}|\partial_n V_j|^2(x_n)e^{\tau x_n} + (a^2(n)k^2 - |\xi|^2)|V_j|^2(x_n)e^{\tau x_n}\\ + \int_{x_n}^h (\tau - 2b_n(n))|\partial_n V_j|^2(s)e^{\tau s}ds\end{aligned}$$

$$+\int_{x_n}^{h}(\tau(a_n^2(n)k^2-|\xi|^2)+2a(n)\partial_n a(n)k^2)|V_j|^2(s)e^{\tau s}ds$$
$$+\,i\int_{x_n}^{h} b(n)\cdot\xi(V_j\partial_n\bar{V}_j-\bar{V}_j\partial_n V_j)(s)e^{\tau s}ds$$
$$=-i\xi_j\int_{x_n}^{h}(F_j\partial_n\bar{V}_j-\bar{F}_j\partial_n V_j)(s)e^{\tau s}ds. \tag{2.7}$$

By elementary inequalities,

$$\left|\int_{x_n}^{h} b(n)\cdot\xi(V_j\partial_n\bar{V}_j-\bar{V}_j\partial_n V_j)(s)e^{\tau s}ds\right|$$
$$\leqslant\int_{x_n}^{h}|b|^2|\partial_n V_j|^2(s)e^{\tau s}ds+\int_{x_n}^{h}|\xi|^2|V_j|^2(s)e^{\tau s}ds$$

and

$$\left|\xi_j\int_{x_n}^{h}(F_j\partial_n\bar{V}_j-\bar{F}_j\partial_n V_j)(s)e^{\tau s}ds\right|$$
$$\leqslant\int_{x_n}^{h}|\xi|^2|F_j|^2(s)e^{\tau s}ds+\int_{x_n}^{h}|\partial_n V_j|^2(s)e^{\tau s}ds,$$

so, using the condition (2.2) and dropping the first two terms on the left-hand side of (2.7), we yield

$$\int_{x_n}^{h}(\tau-2b_n(n)-|b'(n)|^2-1))|\partial_n V_j|^2(s)e^{\tau s}ds$$
$$+\int_{x_n}^{h}(\tau(a_n^2(n)k^2-|\xi|^2)+2a(n)\partial_n a(n)k^2-|\xi|^2)|V_j|^2(s)e^{\tau s}ds$$
$$\leqslant\int_{x_n}^{h}|\xi|^2|F_j|^2(s)e^{\tau s}ds. \tag{2.8}$$

Choosing

$$\tau = \max\Big(\sup\,(2b_n(n) + |b'(n)|^2) + 1, 4\sup\Big(-\frac{\partial_n a(n)}{a(n)}\Big) + 2\Big),$$

where the supremum is taken over $(0, h)$, we guarantee that

$$0 \leqslant \tau - 2b_n(n) - |b'(n)|^2 - 1,$$
$$\frac{a(n)^2k^2}{2} \leqslant \frac{\tau}{2}a(n)^2k^2 + 2a(n)\partial_n a(n)k^2 - |\xi|^2.$$

Hence from (2.8) we derive

$$\int_{x_n}^{h} \frac{a(n)^2k^2}{2}|V_j|^2(s)e^{\tau s}ds \leqslant \int_{x_n}^{h} |\xi|^2|F_j|^2(s)e^{\tau s}ds$$

and, using (2.2), we arrive at (2.3). The proof is complete. □

Lemma 2.2. *Let $a(n) \in C^1([0,h])$ depend only on x_n, and let $v_n \in C^2(\bar{\Omega}^*)$ solve the initial value problem*

$$\begin{gathered}(\Delta + b(n)\cdot\nabla + a(n)^2k^2)v_n = \partial_n f_n \text{ in } \Omega^*(d),\\ v_n = 0 \text{ on } \Omega^*(h_1)\end{gathered} \tag{2.9}$$

for some $h_1 \in (d, h)$, $f_n \in C^\infty(\bar{\Omega}^*(d))$, $f_n = 0$ on $\Omega^*(h_1)$, and

$$V_n(\xi, x_n) = 0 \text{ when } \frac{a^2(x_n, n)}{2}k^2 < |\xi|^2. \tag{2.10}$$

Then there is a constant C depending only on $h, \sup(|b(n)| + |\partial_n b(n)| + |a(n)| + |\partial_n a(n)|)$, $\sup a^{-1}(n)$ over $(0, h)$ such that

$$\|v_j\|(\Omega^*(d)) \leqslant C\|f_n\|(\Omega^*(d)). \tag{2.11}$$

Proof. By the Parseval identity, it suffices to show that solutions to the initial value problem

$$\partial_n^2 V_n + b_n(n)\partial_n V_n + (a(n)^2k^2 - |\xi|^2 - ib(n)\cdot\xi)V_n = \partial_n F_n \text{ on } (d,h) \tag{2.12}$$

with zero final conditions

$$V_n = 0,\ F_n = 0 \text{ on } (h_1, h) \tag{2.13}$$

satisfy the bound

$$\int_d^h |V_n|^2(\xi, s)ds \leqslant C\int_d^h |F_n|^2(\xi, s)ds. \tag{2.14}$$

Integrating Equation (2.12) over (x_n, h) and using the final conditions (2.13), we obtain

$$
\begin{aligned}
&-\partial_n V_n(x_n) - b_n(n)V_n(x_n) - \int_{x_n}^{h} (\partial_n b_n(n))V_n(s)ds \\
&+ \int_{x_n}^{h} (a(n)^2(s)k^2 - |\xi|^2 - ib(n)\cdot\xi)V_n(s)ds = -F_n(x_n). \qquad (2.15)
\end{aligned}
$$

Multiplying (2.15) by $\bar{V}_n(x_n)e^{\tau x_n}$, taking the complex conjugate, and adding, we yield

$$
\begin{aligned}
&-(\partial_n|V_n|^2)(x_n)e^{\tau x_n} - 2b_n(n)|V_n|^2(x_n)e^{\tau x_n} \\
&-\Big(\Big(\int_{x_n}^{h} \partial_n b_n(n)V_n(s)ds\Big)\bar{V}_n(x_n) + \Big(\int_{x_n}^{h} \partial_n b_n(n)\bar{V}_n(s)ds\Big)V_n(x_n)\Big)e^{\tau x_n} \\
&+\Big(\int_{x_n}^{h} (a(n)^2k^2 - |\xi|^2 - ib(n)\cdot\xi)V_n(s)ds\Big)\bar{V}_n(x_n)e^{\tau x_n} \\
&+\Big(\int_{x_n}^{h} (a(n)^2k^2 - |\xi|^2 + ib(n)\cdot\xi)\bar{V}_n(s)ds\Big)V_n(x_n)e^{\tau x_n} \\
&= -(F_n\bar{V}_n + \bar{F}_nV_n)(x_n)e^{\tau x_n}. \qquad (2.16)
\end{aligned}
$$

Setting for brevity

$$
A(x_n,\xi) = a(n)^2(x_n)k^2 - |\xi|^2, \quad B(x_n,\xi) = b(n)(x_n)\cdot\xi, \qquad (2.17)
$$

observing that

$$
\begin{aligned}
&\Big(\int_{x_n}^{h} (A - iB)V_n(s)ds\Big)\bar{V}_n(x_n) + \Big(\int_{x_n}^{h} (A + iB)\bar{V}_n(s)ds\Big)V_n(x_n) \\
&= -\frac{1}{A(x_n) + iB(x_n))}\partial_n\Big|\int_{x_n}^{h} (A - iB)V_n(s)ds\Big|^2 \\
&+ \Big(1 - \frac{A(x_n) - iB(x_n)}{A_(x_n) + iB(x_n)}\Big)\Big(\int_{x_n}^{h} (A + iB)\bar{V}_n(s)ds\Big)V_n(x_n),
\end{aligned}
$$

and integrating (by parts) (2.16) over (x_n, h), we arrive at

$$
\begin{aligned}
&|V_n|^2(x_n)e^{\tau x_n} + \int_{x_n}^{h} (\tau - 2b_n(n))|V_n|^2(s)e^{\tau s}ds \\
&- 2\,\mathrm{Re} \int_{x_n}^{h} \Big(\int_{t}^{h} \partial_n b(s,n)V_n(s)ds\Big)\bar{V}_n(t))e^{\tau t}dt \\
&+ \frac{1}{A(x_n) + iB(x_n)}\Big|\int_{x_n}^{h} (A - iB)(s)V_n(s)ds\Big|^2 e^{\tau x_n} \\
&+ \int_{x_n}^{h} \Big(\Big(\frac{\tau}{A+iB} - \frac{\partial_n(A-iB)}{(A+iB)^2}(t)\Big|\int_{t}^{h}(A-iB)(s)V_n(s)ds\Big|^2\Big)e^{\tau t}dt\Big) \\
&+ \int_{x_n}^{h} \Big(\frac{2iB}{A+iB}\Big)(t)\Big(\int_{t}^{h}(A+iB)(s)\bar{V}_n(s)ds\Big)V_n(t)e^{\tau t}dt \\
&= - \int_{x_n}^{h} (F_n\bar{V}_n + \bar{F}_nV_n)(s)e^{\tau s}ds \\
&\leqslant \int_{x_n}^{h} |F_n|^2(s)e^{\tau s}ds + \int_{x_n}^{h} |V_n|^2(s)e^{\tau s}ds
\end{aligned}
\tag{2.18}
$$

by the elementary inequality $2ab \leqslant a^2 + b^2$.

We will now bound some terms in (2.18).

We have

$$
\begin{aligned}
&\Big|2\,\mathrm{Re} \int_{x_n}^{h} \Big(\int_{t}^{h} \partial_n b(s,n)V_n(s)ds\Big)\bar{V}_n(t))e^{\tau t}dt\Big| \\
&\leqslant C \int_{x_n}^{h} \Big(\int_{t}^{h} |V_n(s)||V_n(t)|ds\Big)e^{\tau t}dt \\
&\leqslant C \int_{x_n}^{h} \Big(\int_{t}^{h} (|V_n(s)|^2 + |V_n(t)|^2)ds\Big)e^{\tau t}dt
\end{aligned}
$$

$$\leqslant C\Big(h\int_{x_n}^{h}|V_n(t)|^2e^{\tau t}dt+\int_{x_n}^{h}\Big(\int_{x_n}^{h}|V_n|^2(s)ds\Big)e^{\tau t}dt\Big)$$

$$\leqslant Ch\int_{x_n}^{h}|V_n(s)|^2e^{\tau s}ds. \tag{2.19}$$

To bound the next term, we observe that

$$\operatorname{Re}\frac{1}{A+iB}=\frac{A}{A^2+B^2}\geqslant A^{-1}\geqslant C^{-1}k^{-2}$$

by the definition (2.17) of A and the condition (2.10), so the real part of this term is nonnegative.

Similarly,

$$\left|\frac{\partial_n(A-iB)}{(A+iB)^2}\right|\leqslant Ck^{-2},$$

and hence

$$\operatorname{Re}\Big(\frac{\tau}{A+iB}-\frac{\partial_n(A-iB)}{(A+iB)^2}\Big)\geqslant\frac{\tau}{Ck^2}. \tag{2.20}$$

Finally, using again (2.17) and (2.10), we yield

$$\left|\frac{2iB}{A+iB}\right|\leqslant\frac{C}{k},$$

and hence

$$\left|\int_{x_n}^{h}\Big(\frac{2iB}{A+iB}\Big)(t)\Big(\int_{t}^{h}(A+iB)(s)\bar{V}_n(s)ds\Big)V_n(t)e^{\tau t}dt\right|$$

$$\leqslant\frac{C}{k}\int_{x_n}^{h}\left|\int_{t}^{h}(A-iB)(s)V_n(s)ds\right||V_n|(t)e^{\tau t}dt$$

$$\leqslant C\Big(\frac{1}{k^2}\int_{x_n}^{h}\left|\int_{t}^{h}(A-iB)(s)V_n(s)ds\right|^2e^{\tau t}dt+\int_{x_n}^{h}|V_n(t)|^2e^{\tau t}dt\Big). \tag{2.21}$$

Dropping the first and fourth terms in (2.18) and using (2.19), (2.20), (2.21), we yield

$$\int_{x_n}^{h}(\tau - 2b_n(n) - Ch)|V_n|^2(s)ds$$
$$+\Big(\frac{\tau}{Ck^2} - C\frac{1}{k^2}\Big)\int_{x_n}^{h}\Big|\int_{t}^{h}(A-iB)(s)V_n(s)ds\Big|^2 e^{\tau t}dt$$
$$\leqslant \int_{x_n}^{h}|F_n(s)|^2 e^{\tau s}ds + \int_{x_n}^{h}|V_n(s)|^2 e^{\tau s}ds.$$

Now, we choose

$$\tau = \max(2\sup b_n(n) + Ch + 2, C^2),\ \sup\ \text{over}\ x_n \in (0,h). \tag{2.22}$$

Due to this choice of τ, the last inequality implies

$$\int_{x_n}^{h}|V_n(s)|^2 e^{\tau s}ds \leqslant \int_{x_n}^{h}|F_n(s)|^2 e^{\tau s}ds.$$

So we obtain (2.14) with $C = e^{\tau h}$ and τ defined by (2.22).

The proof is complete. □

Lemma 2.3. *Let $a(n) \in C^1([0,h])$ depend only on x_n, and let $v_{n+1} \in C^2(\bar{\Omega}^*)$ solve the initial value problem*

$$\begin{gathered}(\Delta + b(n)\cdot\nabla + a(n)^2k^2)v_{n+1} = kf_{n+1}\ in\ \Omega^*(d),\\ v_{n+1} = 0\ on\ \Omega^*(h_1)\end{gathered} \tag{2.23}$$

for some $h_1 \in (d,h)$, $f_n \in C^\infty(\bar{\Omega}^*(d))$, $f_{n+1} = 0$ on $\Omega^*(h_1)$, and

$$V_{n+1}(\xi, x_n) = 0\ when\ \frac{a^2(x_n,n)}{2}k^2 < |\xi|^2. \tag{2.24}$$

Then there is a constant C depending only on $h, \sup(|b(n)| + |\partial_n b(n)| + |a(n)| + |\partial_n a(n)|), \sup a^{-1}(n)$ over $(0,h)$ such that

$$\|v_{n+1}\|(\Omega^*(d)) \leqslant C\|f_{n+1}\|(\Omega^*(d)). \tag{2.25}$$

The proof of Lemma 2.3 is similar to that of Lemma 2.1.

Lemma 2.4. *Let $a(n) \in C^1([0,h])$ depend only on x_n. Let $v_0 \in C^2(\bar{\Omega}^*)$ solve the initial value problem*

$$
\begin{gathered}
(\Delta + b(n)\cdot\nabla + a(n)^2k^2)v_0 = k^2 f_0 \ in\ \Omega^*(d),\\
v_0 = 0 \ on\ \Omega^*(h_1)
\end{gathered}
\tag{2.26}
$$

for some $h_1 \in (d,h)$, $f_n \in C^\infty(\bar\Omega^*(d))$, $f_{n+1} = 0$ *on* $\Omega^*(h_1)$, *and*

$$
V_0(\xi, x_n) = 0 \ when\ \frac{a^2(x_n,n)}{2}k^2 < |\xi|^2. \tag{2.27}
$$

Then there is a constant C *depending only on* $h, \sup(|b(n)| + |\partial_n b(n)| + |a(n)| + |\partial_n a(n)|), \sup a^{-1}(n)$ *over* $(0,h)$ *such that*

$$
\|v_0\|(\Omega^*(d)) \leqslant C(\|f_0\|(\Omega^*(d) + \|\partial_n f_0\|(\Omega^*(d))). \tag{2.28}
$$

Proof. We need to modify slightly the previous argument. Indeed, as in the bounds for V_j,

$$
\begin{aligned}
&|\partial_n V_0|^2(x_n)e^{\tau x_n} + (a_n^2(x_n)k^2 - |\xi|^2)|V_0|^2(x_n)e^{\tau x_n}\\
&+ \int_{x_n}^{h} (\tau - 2b_n(n))|\partial_n V_0|^2(s)e^{\tau s}ds\\
&+ \int_{x_n}^{h} (\tau(a(s,n)^2(s)k^2 - |\xi|^2) + 2k^2 a(s,n)\partial_n a(s,n))|V_0|^2(s)e^{\tau s}ds\\
&+ i\int_{x_n}^{h} b(s,n)\cdot\xi(V_0\partial_n\bar V_0 - \bar V_0\partial_n V_0)(s)e^{\tau s}ds\\
&= -k^2\int_{x_n}^{h} \mathrm{Re}\,(F_0\partial_n\bar V_0)(s)e^{\tau s}ds\\
&= k^2\,\mathrm{Re}\,(F_0\bar V_0)(x_n)e^{\tau x_n} + k^2\int_{x_n}^{h}\mathrm{Re}\,((\partial_n F_0 + \tau F_0)\bar V_0)(s)e^{\tau s}ds.
\end{aligned}
\tag{2.29}
$$

To bound the last integral, we observe that

$$
\begin{aligned}
&\left|\int_{x_n}^{h} b(s,n)\cdot\xi(V_0\partial_n\bar V_0 - \bar V_0\partial_n V_0)(s)e^{\tau s}ds\right|\\
&\qquad \leqslant \int_{x_n}^{h} (|b(s,n)|^2|\xi|^2|V_0|^2 + |\partial_n\bar V_0|^2)(s)e^{\tau s}ds
\end{aligned}
$$

$$\leqslant Ck^2 \int_{x_n}^{h} |V_0|^2(s)e^{\tau s}ds + \int_{x_n}^{h} |\partial_n \bar{V}_0|^2(s)e^{\tau s}ds$$

because $|\xi| \leqslant Ck$ due to the condition (2.27). Dropping the first term on the left-hand side of (2.29), using the elementary inequalities

$$|F_0 V_0| \leqslant \Big(\frac{1}{a(n)^2}|F_0|^2 + \frac{a(n)^2}{4}|V_0|^2\Big),$$

$$|(\partial_n F_0 + \tau F_0)V_0| \leqslant 2(|\partial_n F_0|^2 + \tau^2|F_0|^2) + |V_0|^2$$

and the assumption (2.27), we yield

$$\frac{a(n)^2(x_n)k^2}{2}|V_0|^2(x_n)e^{\tau x_n} + \int_{x_n}^{h}(\tau - 2b_n(n) - 1)|\partial_n V_0|^2(s)e^{\tau s}ds$$

$$+ \int_{x_n}^{h}\Big(\frac{\tau}{2}a^2(n)k^2 - Ck^2\Big)|V_0|^2(s)e^{\tau s}ds$$

$$\leqslant \frac{a^2(x_n,n)k^2}{4}|V_0|^2(x_n) + \frac{k^2}{a(x_n,n)^2}|F_0|^2(x_n)$$

$$+ k^2\int_{x_n}^{h}|V_0|^2(s)e^{\tau s}ds + k^2\int_{x_n}^{h}4(|\partial_n F_0|^2 + \tau^2|F_0|^2)(s)e^{\tau s}ds.$$

Choosing

$$\tau = \max(2\sup b_n(n) + 1, 2\sup(C+1)a(n)^{-2}),\ \sup\ \text{over}\ (0,h),$$

we guarantee the positivity of the second integral. So, we can absorb the first integral on the right-hand side by the last integral on the left-hand side to arrive at

$$a(n)^2(x_n)|V_0|^2(x_n)e^{\tau x_n}$$

$$\leqslant \frac{4}{a(n)^2(x_n)}|F_0|^2(x_n)e^{\tau x_n} + 8\int_{d}^{h}(|\partial_n F_0|^2 + \tau^2|F_0|^2)(s)e^{\tau s}ds$$

or

$$|V_0|^2(x_n)e^{\tau x_n} \leqslant C(|F_0|^2(x_n)e^{\tau x_n} + \int_{d}^{h}(|\partial_n F_0|^2 + \tau^2|F_0|^2)(s)e^{\tau s}ds.$$

Integrating with respect to x_n over (d,h) and replacing $e^{\tau x_n}$ on the left-hand side by 1 and on the right-hand side by $e^{\tau h}$, we arrive at (2.28).

The proof is complete. □

Now, using Lemmas 2.1–2.4, freezing the coefficient with respect to x', and partitioning unity, we obtain energy type estimates for general variable a_0, b.

Let $\varepsilon > 0$. We denote by $X(j)$ points with integer coordinates. Let $x(j)$, $j = 1, \ldots, J$, be points $\varepsilon X(j)$ that are contained in $\Omega' = \{x' : x \in \Omega\}$. It is clear that $J \leqslant C\varepsilon^{-n}$. The balls $B'(x(j);\varepsilon)$ form an open covering of $\overline{\Omega'}$. We define $\Omega_j = B'(x(j);\varepsilon) \times (0,h)$. Let $\chi(x';j)$ be a partition of unity subordinated to this covering. We can assume that

$$0 \leqslant \chi(;j) \leqslant 1,\ |\nabla\chi(;j)| \leqslant C\varepsilon^{-1},\ |\Delta\chi(;j)| \leqslant C\varepsilon^{-2}. \tag{2.30}$$

We introduce a "low frequency" projection $v_1 = Pv$ of a function v. Introduce a function $\chi \in C^\infty(\mathbf{R})$ such that $\chi = 1$ on $(0,1/2)$, $\chi = 0$ on $(3/4,\infty)$, $0 \leqslant \chi \leqslant 1$. Let $\chi_j(x_n;\xi) = \chi(k^{-1}a_0^{-1}(x(j),x_n)|\xi|)$. We define

$$v(;j) = \chi(;j)v,\ P_j v(;j) = \mathcal{F}^{-1}\chi_j\mathcal{F}v(;j),\ v_1 = \sum_{j=1}^{J} P_j v(;j). \tag{2.31}$$

For brevity we set $\|v\| = \|v\|_{(0)}(\Omega^*(d))$.

Lemma 2.5. *Let $v \in C^2(\bar\Omega^*(d))$ solve the initial value problem*

$$\begin{gathered}(\Delta + b\cdot\nabla + a_0^2k^2)v = \partial_1 f_1 + \ldots + \partial_n f_n + kf_{n+1} + k^2 f_0 \ \mathit{in}\ \Omega^*(d),\\ v = 0\ \mathit{on}\ \Omega^*(h_1)\end{gathered} \tag{2.32}$$

for some $h_1 < h$. Then there is a constant C such that

$$\begin{aligned}\|v\| \leqslant C((1+\varepsilon^{-1-n/2}k^{-1})(\|f_1\| + \ldots + \|f_n\|) + \|f_{n+1}\|\\ + \|f_0\| + \|\partial_n f_0\| + \varepsilon^{-2}k^{-1}\|v\|_{(1)}(\Omega^*(d)) + \varepsilon(\|v\| + \|\partial_n v\|)).\end{aligned} \tag{2.33}$$

Proof. From (2.31) and the Leibniz formula we have

$$\begin{aligned}&\Delta v(;j) + b\cdot\nabla v(;j) + k^2a_0^2 v(;j)\\ &= \chi(;j)(\partial_1 f_1 + \ldots + \partial_n f_n + kf_{n+1} + k^2 f_0)\\ &+ 2\nabla\chi(;j)\cdot\nabla v + (b\cdot\nabla\chi(;j) + \Delta\chi(;j))v,\end{aligned}$$

so

$$\begin{aligned}&\Delta v(;j) + b(x'(j),x_n)\cdot\nabla v(;j) + k^2a_0^2(x'(j),x_n)v(;j)\\ &= \partial_1(\chi(;j)f_1) + \ldots + \partial_n(\chi(;j)f_n) - \partial_1\chi(;j)f_1 - \ldots - \partial_{n-1}\chi(;j)f_{n-1}\end{aligned}$$

$$+ k\chi(;j)f_{n+1} + k^2\chi(;j)f_0 + 2\nabla\chi(;j)\cdot\nabla v + (b\cdot\nabla\chi(;j) + \Delta\chi(;j))v$$
$$+ (b(x'(j),x_n) - b(x))\cdot\nabla v(;j) + k^2((a_0^2(x'(j),x_n) - a_0^2(x))v(;j).$$

Applying the low frequency projection P_j to both sides, we yield

$$\begin{aligned}
&\Delta P_j v(;j) + b(;j,n)\cdot\nabla P_j v(;j) + k^2 a^2(;n,j)P_j v(;j)\\
&= \mathcal{F}^{-1}\partial_n^2\chi_j\mathcal{F}v(;j) + 2\mathcal{F}^{-1}\partial_n\chi_j\mathcal{F}\partial_n v(;j) + b_n(;j)\mathcal{F}^{-1}\partial_n\chi_j\mathcal{F}v(;j)\\
&+ \partial_1 P_j(\chi(;j)f_1) + \ldots + \partial_n P_j(\chi(;j)f_n) - P_{j,n}(\chi(;j)f_n)\\
&- P_j((\partial_1\chi(;j))f_1) - \ldots - P_j((\partial_{n-1}\chi(;j))f_{n-1})\\
&+ kP_j(\chi(;j)f_{n+1}) + k^2 P_j(\chi(;j)f_0)\\
&+ P_j(2\nabla'\chi(j)\cdot\nabla v) + P_j((\Delta\chi(;j) + b\cdot\nabla\chi(;j))v)\\
&+ P_j(b(;j) - b)\cdot\nabla v(;j)) + k^2 P_j((a^2(;n,j) - a_0^2)v(;j)),
\end{aligned}$$

where $P_{j,n}(f) = \mathcal{F}^{-1}\partial_n\chi_j\mathcal{F}f$, $b(;j) = b(x'(j),)$, and $a(;n,j) = a_0(x'(j),)$. Observing that

$$|(a^2(;n,j) - a_0^2)| + |\partial_n(a^2(;n,j) - a_0^2)| \leqslant C\varepsilon$$

on the support of $v(;j)$ and $\|P_j f\| \leqslant \|f\|$, using (2.30), and applying Lemmas 2.1–2.4, we obtain

$$\begin{aligned}
\|P_j v(;j)\|^2 \leqslant C(&\|\chi(;j)f_1\|^2 + \ldots + \|\chi(;j)f_n\|^2 + \varepsilon^{-2}k^{-2}(\|f_1\|^2 + \ldots\\
&+ \|f_n\|^2) + \|\chi(;j)f_{n+1}\|^2 + \|\chi(;j)f_0\|^2\\
&+ \|\chi(;j)\partial_n f_0\|^2 + \varepsilon^{-2}k^{-2}\|\nabla v\|^2(\Omega_j)\\
&+ \varepsilon^{-4}k^{-2}\|v\|^2(\Omega_j) + \varepsilon^2(\|v\|^2(\Omega_j) + \|\partial_n v\|^2(\Omega_j)).
\end{aligned} \tag{2.34}$$

Now, summing the local estimates (2.34), we obtain a bound for v_1 given by (2.31). The support of $v(;j)$ intersects at most 2^n supports of other $v(;k)$, but this is not true for $P_j v(;j)$. To make certain constants be ε independent (as in (1.5)), we use that $(I - P_j)v(;j)$ is a high frequency component of $v(;j)$ as defined by (2.31). Hence

$$\|(I - P_j)v(;j)\|^2 \leqslant Ck^{-2}\|v(;j)\|^2_{(1)}$$

and

$$\|v(;j)\|^2 = \|P_j v(;j)\|^2 + \|(I - P_j)v(;j)\|^2 \leqslant \|P_j v(;j)\|^2 + Ck^{-2}\|v(;j)\|^2_{(1)}.$$

Using that the multiplicity of covering Ω_j is at most 2^n and summing (2.34) over $j = 1,\ldots,J$, we yield

$$\|v\|^2 \leqslant C\sum_{j=1}^{J}\|v(;j)\|^2 \leqslant C\Big(\sum_{j=1}^{J}\|\chi(;j)f_0\|^2 + \ldots + \sum_{j=1}^{J}\|\chi(;j)f_{n+1}\|^2$$

$$+\sum_{j=1}^{J}\|\chi(;j)\partial_n f_0\|^2+\varepsilon^{-n-2}k^{-2}(\|f_1\|^2+\ldots+\|f_n\|^2)$$
$$+\varepsilon^{-4}k^{-2}\|v\|^2_{(1)}(\Omega^*(d))+\varepsilon^2(\|v\|^2+\|\nabla v\|^2)\Big).$$

Using that $\chi^2(;1)+\ldots+\chi^2(;J)\leqslant 1$, we obtain (2.33) and complete the proof of Lemma 2.5. $\square$

3. Some Carleman Estimates

Let

$$w(x;\tau)=\int_{-1}^{1} exp(2\tau e^{\sigma(|x-\beta|^2-\theta^2t^2)})dt, \tag{3.1}$$

where $\beta=(0,\ldots,0,\beta_n)$ is a vector to be chosen later. We remind a result from [**7**]. We will give its short proof.

Lemma 3.1. *Let the condition* (1.4) *be satisfied. Then there is a constant C such that*

$$\int_\Omega((\tau^3+\tau k^2)|u|^2+\tau|\nabla u|^2)w(,\tau)$$
$$\leqslant C\Big(\int_\Omega|(\Delta+a_0^2k^2)u|^2w(,\tau)+\int_{\partial\Omega}((\tau^3+\tau k^2)|u|^2+\tau|\nabla u|^2)w(,\tau)\Big) \tag{3.2}$$

for all $u\in H^2(\Omega_1)$ and $\tau>C$.

Proof. As is known [**5, 12**], under the condition (1.4), there are positive σ,θ depending on Ω,a_0,β such that with $\varphi(x,t)=e^{\sigma(|x-\beta|^2-\theta^2t^2)}$ we have the following Carleman estimate for the wave operator:

$$\int_{\Omega\times(-1,1)}(\tau^3|U|^2+\tau|\nabla U|^2+\tau|\partial_tU|^2)e^{2\tau\varphi}$$
$$\leqslant C\Big(\int_{\Omega\times(-1,1)}|(\Delta-a_0^2\partial_t^2)U|^2e^{2\tau\varphi}+\int_{\partial\Omega\times(-1,1)}(\tau^3|U|^2+\tau|\nabla U|^2+\tau|\partial_tU|^2)e^{2\tau\varphi}$$
$$+\int_{\Omega\times\{-1,1\}}(\tau^3|U|^2+\tau|\nabla U|^2+\tau|\partial_tU|^2)e^{2\tau\varphi}\Big). \tag{3.3}$$

We apply (3.3) to the function

$$U(x,t) = u(x)e^{ikt}, \tag{3.4}$$

choose large τ to absorb the integral over $\Omega \times \{-1, 1\}$ by the left-hand side of(3.3), and integrate with respect to t to obtain the weight function w.

From the definition (3.4)

$$\nabla U(x,t) = \nabla u(x)e^{ikt}, \quad \partial_t U(x,t) = iku(x)e^{ikt}$$

and

$$(\Delta - a_0^2(x)\partial_t^2)U(x,t) = (\Delta u(x) + a_0^2(x)u(x))e^{ikt}.$$

Hence the Carleman estimate (3.3) implies that

$$\int_\Omega (\tau^3|u|^2(x) + \tau|\nabla u(x)|^2 + \tau k^2|u(x)|^2)\Big(\int_{-1}^{1} e^{2\tau\varphi(x,t)}dt\Big)dx$$

$$\leqslant C\Big(\int_\Omega |(\Delta + a_0^2(x))u(x)|^2\Big(\int_{-1}^{1} e^{2\tau\varphi(x,t)}dt\Big)dx$$

$$+ \int_{\partial\Omega} ((\tau^3 + \tau k^2)|u|^2(x) + \tau|\nabla u(x)|^2)\Big(\int_{-1}^{1} e^{2\tau\varphi(x,t)}dt\Big)dx$$

$$+ \int_\Omega ((\tau^3 + \tau k^2)|u|^2(x) + \tau|\nabla u(x)|^2)e^{2\tau\varphi(x,1)}dx\Big). \tag{3.5}$$

Now, choosing τ large and using different growth rate of the weight function on the left-hand side of (3.5), we eliminate the last term on the right-hand side. Indeed, let $E > 0$. By definition,

$$\varphi(x,t) - \varphi(x,1) = e^{\sigma|x-\beta|^2}(e^{-\theta^2 t^2} - e^{-\theta^2}) > \varepsilon_1(\theta)$$

when $|t| < 1/2, x \in \Omega$. Hence there is $C(E)$ such that

$$E < \int_{-1/2}^{1/2} e^{2\tau(\varphi(x,t)-\varphi(x,1))}dt < \int_{-1}^{1} e^{2\tau(\varphi(x,t)-\varphi(x,1))}dt$$

when $C(E) < \tau$. Then

$$Ee^{2\tau\varphi(x,1)} < \int_{-1}^{1} e^{2\tau\varphi(x,t)}dt$$

provided that $C(E) < \tau$. Setting $E = 2C$, we can absorb the last term on the right-hand side of (3.5) by the left-hand side. The proof is complete. $\square$

4. Proofs of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.2. We choose

$$\beta_n = -\Big(\frac{2r^2}{d} - \frac{3}{8}d\Big), \quad \beta = (0, \dots, 0, \beta_n).$$

Introduce the notation $\Omega_d = \Omega \cap \{(d-\beta_n)^2 < |x-\beta|^2\}$. We assume that $3d^2 < 16r^2$, so that $\beta_n < 0$. Using the choice of β and considering the intersection of level surface

$$|x-\beta|^2 = \Big(\frac{1}{2}d - \beta_n\Big)^2$$

with the lateral wall $\{|x'| = r\}$ of the cylindrical domain, one can be convinced that the boundary layer $\{x_n < \frac{1}{4}d\} \cap \Omega$ does not intersect $\Omega_{\frac{d}{2}}$. Indeed, if (x', x_n^*) is a point of the intersection of this cylindrical domain and of the boundary of $\Omega_{\frac{d}{2}}$, then $r^2 + (x_n^* - \beta_n)^2 = (d-\beta_n)^2 = \Big(\frac{d}{8} + \frac{2r^2}{d}\Big)^2$, $(x_n^* - \beta_n)^2 = \Big(\frac{d}{8} - \frac{2r^2}{d}\Big)^2$, and $x_n^* - \beta_n = \frac{2r^2}{d} - \frac{d}{8}$, which gives $x_n^* = \frac{d}{4}$. Hence there is a cut-off function χ that is equal to 1 on $\Omega_{\frac{d}{2}}$, vanishes near $\partial\Omega \cap \{x_n = 0\}$, and satisfies the bounds $|\nabla\chi| \leqslant Cd^{-1}$, $|\Delta\chi| \leqslant Cd^{-2}$.

Writing Equation (1.1) as $(\Delta + k^2 a_0^2)u = f - b \cdot \nabla u$, applying Lemma 3.1 to χu instead of u, and shrinking the domain in the norms on the left-hand side of (3.2), we get

$$\begin{aligned}
&\int\limits_{\Omega_d} ((\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2) w(;\tau) \\
&\leqslant C\Big(\int\limits_{\Omega} |f|^2 w(;\tau) + \int\limits_{\Omega} |\nabla u|^2 w(;\tau) + \int\limits_{\Omega\setminus\Omega_{\frac{d}{2}}} |\nabla\chi \cdot \nabla u + (\Delta\chi)u|^2 w(;\tau) \\
&+ \int\limits_{\Gamma} ((\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 + \tau|\nabla\chi u|^2) w(;\tau)\Big),
\end{aligned} \tag{4.1}$$

where we used that $\chi = 1$ on $\Omega_{\frac{d}{2}}$ and the triangle inequality. Choosing $\tau > 2C$, we absorb the second integral on the right-hand side by the left-hand side.

Let $b = e^{\sigma X^2}$, $b_1 = e^{\sigma|d-\beta_n|^2}$, $b_2 = e^{\sigma|\frac{d}{2}-\beta_n|^2}$, where $X = \sup|x-\beta|$ over $x \in \Omega$,

$$W(\tau) = \int_{-1}^{1} e^{2\tau b e^{-\sigma t^2}}\, dt,\ w_1(\tau) = \int_{-1}^{1} e^{2\tau b_1 e^{-\sigma t^2}}\, dt,\ w_2(\tau) = \int_{-1}^{1} e^{2\tau b_2 e^{-\sigma t^2}}\, dt.$$

Observing that $w_1 \leqslant w$ on Ω_d, $w \leqslant W$ on Ω, and $w \leqslant w_2$ on $\Omega \setminus \Omega_{\frac{d}{2}}$ and replacing w by its minimal value on the left-hand side and by maximal values on the right-hand side of (4.1), we yield

$$\begin{aligned}&\tau^3 w_1(\tau)\|u\|^2(\Omega_d)) + \tau w_1(\tau)\|\nabla u\|^2(\Omega_d)\\ &\leqslant C(W(\tau)(\|f\|^2(\Omega) + (\tau^3 + \tau(k^2 + d^{-2}))\|u\|^2(\Gamma) + \tau\|\nabla u\|^2(\Gamma))\\ &+ d^{-4} w_2(\tau)(\|\nabla u\|^2(\Omega) + \|u\|^2(\Omega))).\end{aligned}$$

Dividing both sides of this inequality by w_1, we obtain

$$\begin{aligned}\tau^3\|u\|^2(\Omega_d) + \tau\|\nabla u\|^2(\Omega_d) \leqslant C(W(\tau)w_1^{-1}(\tau)(\|f\|^2(\Omega)\\ + (\tau^3 + \tau(k^2 + d^{-2}))\|u\|^2(\Gamma) + \tau\|\nabla u|^2(\Gamma))\\ + d^{-4} w_2(\tau) w_1^{-1}(\tau)(\|\nabla u\|^2(\Omega) + \|u\|^2(\Omega))).\end{aligned} \tag{4.2}$$

It is obvious that $W(\tau)w_1^{-1}(\tau) \leqslant Ce^{C\tau}$. A crucial observation is that

$$w_2(\tau)w_1^{-1}(\tau) \leqslant Ce^{-\frac{\tau}{C}}.$$

Indeed, by the definition of b_j and β and elementary calculations,

$$b_1 - b_2 = e^{\sigma(2r^2 - \frac{d^2}{8} + (\frac{2r^2}{d} - \frac{3d}{8})^2)}(e^{\sigma(\frac{3d^2}{8} + 2r^2)} - 1) \geqslant C^{-1}.$$

Therefore,

$$w_1(\tau) \geqslant \int_{-1}^{1} e^{2\tau b_2 e^{-\theta^2 t^2}} e^{2\tau(b_1 - b_2)e^{-\theta^2}}\, dt \geqslant w_2(\tau)e^{2\tau/C}.$$

Hence from (4.2) we have

$$\|u\|^2(\Omega_d) + \|\nabla u\|^2(\Omega_d) \leqslant C(e^{C\tau}\tau^3 F^2(k) + e^{-\tau/C}\tau^3 M_1^2) \text{ when } C < \tau. \tag{4.3}$$

By increasing C, we can eliminate τ^3 on the right-hand side.

To use (4.3), we need τ to be large. If $M_1 \leqslant CF(k)$ for some C, then we have the Lipschitz bound (1.7). Otherwise, we can equalize two terms in (4.3) by setting

$$\tau = \frac{C^2}{C^2+1} 2ln \frac{M_1}{d^2 F(k,d)}.$$

Then the right-hand side of (4.3) is

$$CF(k)^{2\varkappa} M_1^{2(1-\varkappa)}, \ \varkappa = \frac{1}{C^2+1}$$

and, using that $\Omega(d) \subset \Omega_d$, we obtain (1.7). The proof is complete. □

Proof of Theorem 1.1. Since Γ is Lipschitz, by known extension theorems, there is a function u^* such that $u = u^*, \nabla u = \nabla u^*$ on Γ and

$$\|u^*\|_{(1)}(\Omega^*(0)) \leqslant C(\|u\|(\Gamma) + \|\nabla u\|(\Gamma)) \leqslant CF, \tag{4.4}$$

where we used the definition of F. Let $v = u - u^*$ on Ω and $v = 0$ on $\Omega^*(0) \setminus \Omega$. It suffices to obtain (1.5) for v instead of u. Observe that

$$\Delta v + b \cdot \nabla v + a_0^2 k^2 v = f + f^* - b \cdot \nabla u^* - a_0^2 k^2 u^* \text{ in } \Omega^*(0), \tag{4.5}$$

where $f^* = -\operatorname{div}(\nabla u^*)$. Since v vanishes outside some cylinder, by using known results about the H^1-approximation of energy solutions by H^2-solutions, we can assume that $v \in H^2(\mathbf{R}^{n-1} \times (0,h))$ and hence $f^* = \partial_1 f_1 + \ldots + \partial_n f_n + f_{n+1}$ with $\|f_j\| \leqslant CF$. By (4.5) and Lemma 2.5,

$$\begin{aligned} \|v\|(\mathbf{R}^{n-1} \times (d,h)) &\leqslant C((1+\varepsilon^{-n/2-1}k^{-1})F + k^{-1}F + \|u^*\| + \|\partial_n u^*\| \\ &+ \varepsilon^{-2}k^{-1}(\|v\|_{(1)}(\Omega(d)) + \varepsilon(\|v\| + \|\partial_n v\|)) \\ &\leqslant C(F + C(\varepsilon)k^{-1}F + C(\varepsilon)k^{-1}\|u\|_{(1)}(\Omega(d)) + \varepsilon(\|u\|_{(1)}(\Omega) + F)), \end{aligned}$$

where we used that $\|v\|_{(1)} \leqslant \|u\|_{(1)} + F$ due to (4.4). From this bound and (1.7) we obtain the needed bound (1.5) for v. The proof is complete. □

Conclusion. It is clear that difficulties in theory and applications of many important inverse problems are due to their notorious (exponential) instability. In practical situations, logarithmic stability permits, as a rule, to find only 10–20 Fourier coefficients of a solution at distance from Γ, which results in very poor resolution and disappointment of engineers or scientists expecting effective mathematical processing of experimental data whose acquisition is often very laborious and expensive. So, any way to increase stability and to increase resolution is indeed valuable. While increasing stability with the wave number is observed experimentally in several basic inverse problems, before there was no theoretical explanation. Moreover, there is a belief that stability always grows with frequency. As was shown, it is true only under some (convexity) type conditions. Otherwise, stability might deteriorate.

One of the next natural questions is to trace the dependence of constants on distance d and to study stability in the whole domain Ω. For example, we expect that $C \leqslant C_0 d^{-4}$, where C_0 does not depend on d. To demonstrate it, we need more detailed Carleman type estimates. We expect that this increased stability is more dramatic in the three-dimensional case, when the data are given at a larger distance, when singularities of the solution are distributed over $\partial\Omega \setminus \Gamma$, and certainly for large frequencies. Accordingly, the most stable solution (for the same space geometry as in Section 1) is anticipated in the time domain (i.e., when the Helmholtz equation is replaced by the wave equation) provided that the initial data are zero. In near future we plan to study this issue theoretically and to link it to the increased stability for the Helmholtz equation and to the (largely open) problem of the exact controllabity in a subdomain. Observe that the exact controllability in the whole domain is relatively well understood [**6, 10**]. The present paper outlines a possible way to study increasing stability of the continuation for the equation

$$\varepsilon\Delta u + b \cdot \nabla u + a_0^2 u = 0.$$

Large k corresponds to smaller viscosity ε and has natural links to standard smoothing regularization technique. The analysis of Section 2 carries through, however at present we do not know how to derive appropriate Carleman type estimates, like in Section 3. These estimates help to handle the high frequencies zone. As follows from John's example, this high frequency zone might interact with the low frequency zone (where the solution is stable disregard of any (pseudo-)convexity conditions) and damage overall stability. Similar results are expected for continuation from a lateral wall of solutions to parabolic and hyperbolic equations

$$(\partial_t - \Delta - k^2 a_0^2)u = 0,\ (\partial_t^2 - \Delta - k^2 a_0^2)u = 0$$

and for more general equations and systems.

The author already showed the increased stability of recovery a potential in the Schrödinger equation $(-\Delta - k^2 + c(x))u = 0$ from its Dirichlet-to-Neumann map. The results were presented at the international conferences "Applied Inverse Problems 2005" in Cirencester, England, and "Inverse Problems and Applications," Banff, Canada, in 2006. The paper with complete proofs using complex geometerical optics technique and some sharp estimates of regular fundamental solutions of operators with constant coefficients [**4, 5**] is in preparation. Probably, it is harder to show increased stability for the coefficient a_0 in the equation $(-\Delta - k^2 a_0^2(x))u = 0$. At present, there are only some preliminary results (in the low frequency zone) [**11**], methods of (complex) geometrical optics do not look promising, and we do not know a good alternative. The next step is to obtain similar estimates for the inverse scattering problems by obstacles and by the medium.

In particular, it is still an open question whether stability of recovery of near filed from far field pattern is improving with growing frequency. It is clear that one has to impose some (pseudo)convexity condition on unknown coefficients or obstacles.

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