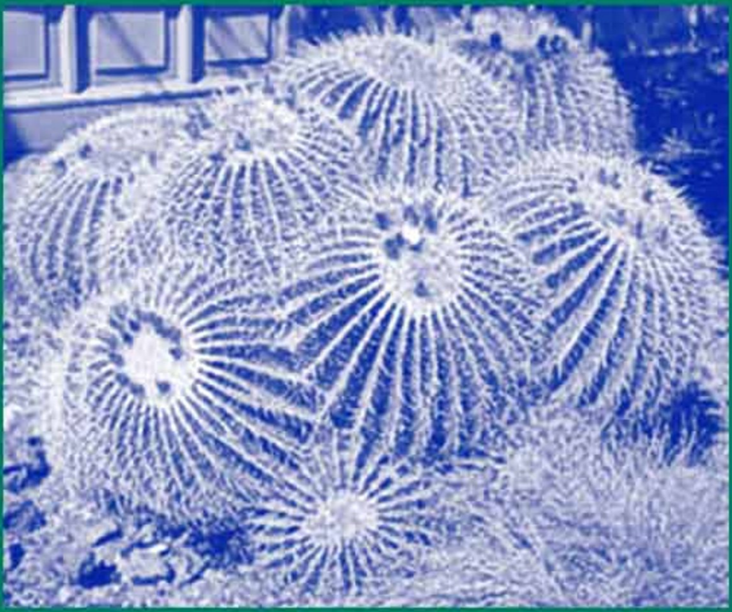
Progress in Nonlinear Differential Equations and Their Applications

Takashi Suzuki

Free Energy and Self-Interacting Particles



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Free Energy and Self-Interacting Particles

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Cover illustration: A profile of self-organization—Some patterns of self-organization, sealed in the closed system, are revealed in the activities of life, and the motion of the mean field of many particles.

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To Mitsu, Jun, and Ai

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Preface

This book examines a system of parabolic-elliptic partial differential equations proposed in mathematical biology, statistical mechanics, and chemical kinetics.

In the context of biology, this system of equations describes the chemotactic feature of cellular slime molds and also the capillary formation of blood vessels in angiogenesis. There are several methods to derive this system. One is the biased random walk of the individual, and another is the reinforced random walk of one particle modelled on the cellular automaton.

In the context of statistical mechanics or chemical kinetics, this system of equations describes the motion of a mean field of many particles, interacting under the gravitational inner force or the chemical reaction, and therefore this system is affiliated with a hierarchy of equations: Langevin, Fokker–Planck, Liouville–Gel'fand, and the gradient flow. All of the equations are subject to the second law of thermodynamics — the decrease of free energy. The mathematical principle of this hierarchy, on the other hand, is referred to as the quantized blowup mechanism; the blowup solution of our system develops delta function singularities with the quantized mass.

The aim of this book is to prove the original result — Theorem 1.2 stated in the first chapter — but several motivations are also described in detail, because they are quite important in creating mathematical techniques, and furthermore, are obtained from statistical mechanics, field theory, nonequilibrium thermodynamics, system biology, and so forth. We have made every effort to keep the discussion self-contained, and any special knowledge concerning recent mathematical research is not assumed; for example, we have provided the complete proof of our previous result, Theorem 1.1. We have also described several mathematically open problems concerning this system.

The main result, Theorem 1.2, assures that if the solution to this system blows up in finite time, then it develops delta function singularities with the quantized mass, called collapses, as the measure-theoretical singular part. If such a collapse has an envelope, the region containing the whole blowup mechanism in space-time, then mass and entropy are exchanged at the wedge of this envelope.

In the context of biology, this means the birth of the quantized "clean" self, while technical motivations result from the above-mentioned hierarchy of the mean field of particles: the quantized blowup mechanism of the stationary state and the existence of the weak solution of the associated kinetic equation global in time.

Thus, this quantization of the nonstationary blowup state is a consequence of the nonlinear quantum mechanics; more precisely, it comes from the quantization of both mass and location of the singularity that appears in the singular limit of the stationary solution, with the stationary state realized as a nonlinear elliptic eigenvalue problem with nonlocal terms, where total mass acts as the principal parameter because it is preserved in the nonstationary state. Mass quantization of the collapse, on the other hand, is proved if the nonstationary solution under consideration has a post-blowup continuation, while the weak solution always exists globally in time in the kinetic equation.

However, all of these facts are just support for the proof of Theorem 1.2, and we have to prove it rigorously within this level of hierarchy, that is, our system.

We prove the above-mentioned quantized blowup mechanism, describing the following: physical principles and derivation of a series of affiliated equations, biological modelling based on the random walk, mathematical study of the stationary state via the variational structure, blowup analysis of the stationary problem using the symmetry of the Green's function, existence and nonexistence of the nonstationary solution globally in time, and details of the quantized blowup mechanism of nonstationary solutions and their proofs by several analytical tools — localization, symmetrization, and rescaling.

Chapters 1 and 2 are the summary and description of modelling, respectively. The proof of Theorem 1.2 is carried out in Chapter 15, using the boundary behavior of the Green's function and the generation of the weak solution established in Chapters 5 and 13, respectively.

Chapters 12 and 14 describe technical motivations — the quantization of the collapse formed in infinite time and that of subcollapses formed in the scaled limit of the blowup solution in finite time, respectively. Thus, the reader can skip these chapters to follow the proof. Theorem 1.1, on the other hand, is a preparatory result of our previous work, the formation of collapses and the estimate of their masses from below, and the proof is described in Chapter 11 for completeness.

Chapters 3–5 are devoted to the classical theory for this system. First, the fundamental theorem, the unique existence of the solution locally in time, is proved in Chapter 3. Then the threshold for the existence of the solution globally in time, which is eventually explained in a unified way by the quantized blowup mechanism, is established in Chapters 4 and 5. The results described in these three chapters are more or less known, but we adopt a general setting and new arguments.

Chapters 6–10 are devoted to the stationary problem. We have provided them to describe the quantized blowup mechanism of the nonstationary state. Thus, the reader can skip these chapters, but this methodology of classifying stationary solutions is quite efficient in the study of nonlinear problems. We mention also that this stationary problem arises in many areas: semiconductors, gauge theory, turbulence, astrophysics, chemical kinetics, combustion, geometry, and so forth.

First, we describe a survey on this stationary problem in Chapter 6. Then we show more or less known results in Chapters 7 and 8, but here we use a new argument, called symmetrization, motivated by the study of nonstationary problems. Chapter 9 is original and describes the effect of these (unstable) stationary solutions to the local dynamics. Chapter 10 is the application obtained by this study to the global dynamics. Chapters 11–15 are devoted to the proof of Theorems 1.1 and 1.2 as we mentioned.

Finally, in Chapter 16 we develop the abstract theory of dual variation, and the results of Chapter 6 are extended in the context of convex analysis. We show that this stationary problem has two equivalent variational formulations, where the cost functionals are associated with the particle density and the field distribution. These two functionals have the duality through the Legendre transformation, and furthermore, are combined with a functional, called the Lagrange function. We also add a result concerning the stability of the stationary solution.

This story of dual variation is widely observed in mean field theories, particularly in the ones associated with the particles having self-interaction with their creating field. This book, we hope, will provide a new point of view for these theories.

Recently, nonequilibrium statistical mechanics has been discussed in the context of self-organization, particularly the formation of the cascade of cyclic reactions against the increase of global entropy. In this system, as we have described, the blowup mechanism is controlled by the stationary solution, and in this sense our study is referred to as that of simple systems. However, the total set of stationary solutions, particularly the unstable ones, is associated with this phenomenon of quantization, and in this sense our study is strongly influenced by the theory of complex systems.

We hope that this book provides some theoretical inspiration and technical suggestions to researchers or Ph.D. students in mathematics and applied mathematics who are interested in statistical physics, physical chemistry, mathematical biology, nonlinear partial differential equations, or variational methods. We also hope this book helps the researcher in other fields such as physics, chemistry, biology, engineering, and medical science, to realize the benefit and necessity of the methods of mathematical science, analytical techniques, physical principles, and so forth, in studying their own problems.

In conclusion, we caution the reader that in spite of these wide applications, the mathematical analysis in this book is limited to a very special system of equations. In addition, we employ numerous mathematical models using the sensitivity function, activator-inhibitor factors, cross-diffusion, and so forth, that involve significant mathematical analysis as well. This book is far from being a complete overview, but fortunately we can mention the survey of Horstmann [67, 68] concerning the biologically motivated study, where several mathematical models of chemotaxis, mathematical studies, and their relations are discussed.

We also mention some other monographs to access related concepts in this book. First, breaking down the construction or continuation of the solution follows from the blowup of approximated solutions in nonlinear problems. Control of its blowup mechanism can then guarantee the existence of an actual solution. Several examples of this new argument are described in Evans [44]. The quantized blowup mechanism in variational problems and their mathematical treatments are mentioned in Struwe [159]. Physical, chemical, and biological motivations are obtained by Risken [138], Doi and Edwards [40], and Murray [105], respectively. Finally, many suggestions are obtained from Tanaka [170] on system biology.

Takashi Suzuki
Osaka, Japan
October 30, 2004

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First of all, I express my gratitude to Professor Haim Brezis who proposed that I write this book and encouraged me frequently. Thanks are also due to Takasi Senba without whom I could not have realized some of my ideas in mathematics. I also appreciate Shinji Odanaka who drew my attention to statistical mechanics and gave me a chance to touch on the whole story behind the hierarchy of systems, and also Hayato Nawa for stimulating discussions on his beautiful work concerning the nonlinear Schrödinger equation. I appreciate Professors Toshitaka Nagai and Masayasu Mimura for their interest and comments from both technical and biological points of view. I also thank Professors Piotr Biler, Dongho Chae, Hua Chen, Miguel A. Herrero, Dirk Horstmann, Chan-Shou Lin, Juan J.L. Velázquez, Gershon Wolansky, Hatem Zaag, and Boris Zaltzman for several personal communications, comments, and suggestions. I also thank Ann Kostant of Birkhäuser for her kind help during my writing of this book.

Finally, I express my sincere and hearty thanks to Professor Hiroshi Fujita who brought me to the community of mathematicians as a researcher.

1 Summary

Our study is concerned with the system of elliptic-parabolic partial differential equations arising in mathematical biology and statistical mechanics. A typical example is

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u\nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0, T),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$
$$u\big|_{t=0} = u_0(x) \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, and ν is the outer unit vector on $\partial\Omega$. This system was proposed by Nagai [106] in the context of chemotaxis in mathematical biology. Here, $u = u(x, t)$ and $v = v(x, t)$ stand for the density of *cellular slime molds* and the concentration of chemical substances secreted by themselves, respectively, at the position $x \in \Omega$ and the time $t > 0$.

In this case, the first equation is equivalent to the equality

$$\frac{d}{dt}\int_\omega u\, dx = -\int_{\partial\omega} j \cdot \nu dS$$

for any subdomain $\omega \Subset \Omega$ with smooth boundary $\partial\omega$, where dS denotes the surface element. Namely, it describes the conservation of mass, where the flux of u is given by $j = -\nabla u + u\nabla v$.

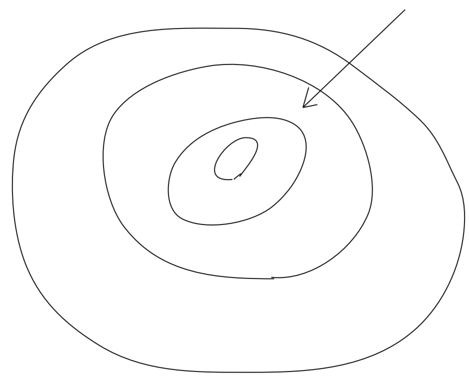


Figure 1.1.

The first term $-\nabla u$ of j is the vector field with the direction parallel to the one where u decreases mostly, and with the rate equal to its derivative toward that direction, that is, u is involved by the process of diffusion.

The second term $u\nabla v$ of j, on the other hand, indicates that u is carried by the vector field ∇v toward the direction where v increases mostly, with the rate equal to its derivative to that direction. In other words, this term represents the chemotactic aggregation of slime molds with v acting as a carrier, and in this way, the effect of *diffusion* $-\nabla u$ and that of *chemotaxis* $u\nabla v$ are competing for u to vary. See Figure 1.1.

Actually, this system describes the motion of cellular slime molds. They are a kind of amoeba usually, but when foods become rare, they begin to secrete chemical substances on their own and create a chemical gradient attracting themselves. Eventually spores are formed by this process. The actual mechanism is much more complicated, involving chemical and biological actions and reactions, and the derivation of (1.1) from the *biased random walk* has been done [3, 128].

In 1970, Keller and Segel [81] proposed a system of parabolic equations to describe such a phenomenon. If the second equation is replaced by

$$\tau v_t = \Delta v - av + u \quad \text{in} \quad \Omega \times (0, T) \tag{1.2}$$

in (1.1), then a closer form to that original system is obtained, where $\tau > 0$ is a small constant. In this system, the density of the chemical substances, v, is subject to the linear diffusion equation, provided with the dissipative term $-av$ and the growth term u. Thus, v diffuses and is destroyed with the rate $a > 0$ by itself, and is created proportionally to u.

Writing τv_t as $\partial v/\partial(\tau^{-1}t)$, we can see that the parameter τ represents the relaxation time, that is, the rate of the time scale of v relative to u. Since the biological time scale is much slower than the chemical one, the assumption $0 < \tau \ll 1$ is reasonable. Then, putting $\tau = 0$ gives (1.1), where the initial layer takes a role in this process of singular perturbation and the initial condition of v is lost in (1.1).

This form, system (1.1) with the second equation replaced by (1.2), is called the *full system* in this book. We have two parameters, a and τ, and actually, we can regard this system as a normal form, where other possible constant coefficients have been reduced to one by suitable transformations of variables.

In the context of statistical mechanics, the bounded domain Ω under consideration is sometimes replaced by the whole space $\mathbf{R}^n$. In this case, boundary conditions are replaced by the requirement that $u(\cdot, t)$ and $v(\cdot, t)$ are in appropriate function spaces, and the second equation of (1.1) takes the form

$$v(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y)u(y,t)\,dy, \tag{1.3}$$

with

$$\Gamma(x) = \begin{cases} \frac{1}{2}\,|x| & (n=1), \\ \frac{1}{2\pi}\log\frac{1}{|x|} & (n=2), \\ \frac{1}{4\pi|x|} & (n=3), \end{cases} \tag{1.4}$$

standing for (-1) times potential driven by gravitational force. This system of equations is concerned with the motion of a *mean field* of many particles, subject to the *self-interaction* caused by the gravitational force of their own, namely, while the first equation of (1.1) describes mass conservation of many particles, the second equation replaced by (1.3) comes from the formation of the gravitational field made by these particles. Here, the diffusion term Δu of the right-hand side of the first equation is to be noted, and it comes from the fluctuation of particles as we shall see.

Equation (1.3) is regarded as a form of the second equation of (1.1), naturally extended to the whole space $\mathbf{R}^n$. In fact, the latter is equivalent to

$$v(x,t) = \int_{\Omega} G(x,x')u(x',t)\,dx', \tag{1.5}$$

where $G = G(x,x')$ denotes *the Green's function* in Ω of the differential operator $-\Delta + a$ under the Neumann boundary condition, and it holds that

$$G(x,x') = K(x,x') + \begin{cases} \Gamma(x-x') & (x' \in \Omega), \\ 2\Gamma(x-x') & (x' \in \partial\Omega), \end{cases} \tag{1.6}$$

with $K = K(x, x')$ standing for the regular part of $G(x, x')$.

Coupled with

$$u_t = \nabla \cdot (\nabla u + u\nabla v) \quad \text{in} \quad \Omega \times (0, T)$$

and

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

equation (1.5) can cast the semiconductor device equation in the DD (drift-diffusion) model, of which mathematical study has been done by several authors. See [15, 76] and the references therein. In this case the system is dissipative, and the long-term behavior of the solution is quite different from the one described in this book, equation (1.1) for instance.

Other forms of the second equation to (1.1) are also proposed [38, 73]. Each of them is written as

$$\tau \frac{dv}{dt} + Av = u \quad \text{in} \quad L^2(\Omega) \tag{1.7}$$

in an abstract manner, where A is a positive definite self-adjoint operator with the compact resolvent and $\tau \geq 0$. Following the earlier terminology, we call (1.1) with the second equation replaced by (1.7) for $\tau > 0$ the full system also. There, similarly, the additional initial condition $v|_{t=0} = v_0(x)$ is imposed. If $\tau = 0$, the initial value is provided only for u as in (1.1). This case is called the *simplified system* in this book. Thus, (1.1) is regarded as a simplified system of chemotaxis.

As might be suspected from the above description, the field generated by particles is *physical* in the simplified system. This means that it is formed at once from each particle. The relaxation time τ of the full system, on the other hand, suggests that some intermediate process, such as the chemical reaction inside the biological media, is involved when the field is created from particles. Thinking thus that the full system describes such an intermediate process in a very simple way, we say that (1.2) indicates the formation of the *chemical* in this book.

In the other case, the second and third equations of (1.1) are replaced by the ordinary differential equation

$$\tau \frac{\partial v}{\partial t} = u \quad \text{in} \quad \Omega \times (0, T) \tag{1.8}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.9}$$

respectively. It is derived from the statistical model of the *reinforced random walk* developed on the lattice of a cellular automaton, where the effect of transmissive action of control species is restricted to each cell. We call this case the *biological* field in this book, although the interaction particle is restricted to adjacent cells and the field is not formed in the classical sense. Recently, much attention has been paid to the final form in medical science, in the context of self-organization such as *angiogenesis*, especially for the growth of tumors [34, 158, 188].

Thus, we have distinguished three kinds of fields — physical, chemical, and biological — which make the second equation of (1.1) appear in slightly different forms. Although several variations of the first and second equations of (1.1) are proposed — using the sensitivity function, activator-inhibitor factors, cross-diffusion, and so forth [67, 68, 127] — this slight difference in the formation of the field from particles makes the solution quite different. Our study is mostly concerned with the first case. Some results are still valid for the second case.

The long-term behavior of the solution to (1.1), on the other hand, depends on the space dimension n, and most of our study of the blowup mechanism is restricted to the case $n = 2$. Then the blowup solution develops a delta function singularity with the quantized mass, while quite different profiles of the blowup solution are obtained for $n = 3$ [62, 63].

We think that these features are related to the structures of the set of stationary solutions. In fact, this set is also sensitive to the space dimension, and only for $n = 2$, is the quantized blowup mechanism observed. Furthermore, several suggestions are obtained from this set concerning the long-term behavior of nonstationary solutions.

Before describing this any further, we confirm that the classical solution to (1.1) exists locally in time if the initial value is smooth. It becomes positive if the initial value is nonnegative and not identically zero. These fundamental theorems were established by [14, 187]. See also [152].

To control the long-term behavior, it is important to determine whether invariant quantities or *Lyapunov functions* exist or not. However, this system of (1.1) is provided with the total mass conservation and the decrease of free energy, following physical requirements to describe the motion of the mean field of many particles. Thus, we have one invariant and one Lyapunov function. The *total mass* is given by

$$\lambda = \|u(t)\|_1 ,$$

and the free energy is the total energy minus entropy so that is equal to

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)\, dx - \frac{1}{2} \iint_{\Omega\times\Omega} G(x, x')u \otimes u\, dx\, dx'. \qquad (1.10)$$

Here and henceforth, $u \otimes u$ stands for $u(x)u(x')$ and the standard L^p norm is denoted by $\| \cdot \|_p$ for $p \in [1, \infty]$.

These facts are valid even for the full system and we can confirm them mathematically. In fact, first, positivity of the solution is preserved. Precisely,

$$u_0(x) \geq 0 \quad \text{and} \qquad u_0(x) \not\equiv 0$$

imply $u(x, t) > 0$ for $(x, t) \in \overline{\Omega} \times (0, T)$ by the strong maximum principle. This gives the total mass conservation,

$$\|u(t)\|_1 = \|u_0\|_1 \equiv \lambda, \qquad (1.11)$$

by

$$\begin{aligned} \frac{d}{dt} \int_\Omega u\, dx &= \int_\Omega u_t\, dx = \int_\Omega \nabla \cdot (\nabla u - u \nabla v)\ dx \\ &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) dS = 0. \qquad (1.12) \end{aligned}$$

Next, in the full and simplified systems, the Lyapunov function is provided by

$$\mathcal{W}(u, v) = \int_\Omega \left(u(\log u - 1) - uv + \frac{1}{2} |\nabla v|^2 + \frac{a}{2} v^2 \right) dx. \qquad (1.13)$$

In fact, writing the first equation of (1.1) as

$$u_t = \nabla \cdot u \nabla \left(\log u - v \right),$$

we have

$$\int_\Omega u_t \left(\log u - v \right)\ dx = - \int_\Omega u \left| \nabla \left(\log u - v \right) \right|^2\ dx$$

from the boundary condition. Here, the left-hand side is equal to

$$\frac{d}{dt} \int_\Omega \left(u(\log u - 1) - uv \right)\ dx + \int_\Omega u v_t\, dx,$$

and (1.2) is applicable to the second term. This gives

$$\begin{aligned} \int_\Omega u v_t\, dx &= \int_\Omega (\tau v_t - \Delta v + a v)\, v_t\, dx \\ &= \tau \|v_t\|_2^2 + \frac{1}{2} \frac{d}{dt} \left(\|\nabla v\|_2^2 + a \|v\|_2^2 \right). \end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}\mathcal{W}(u,v)+\tau\|v_t\|_2^2+\int_\Omega u\left|\nabla(\log u-v)\right|^2\,dx=0, \tag{1.14}$$

and hence $\mathcal{W}(u,v)$ is nonincreasing:

$$\frac{d}{dt}\mathcal{W}(u,v)\le 0.$$

This argument is also applicable to the biological field, that is, system (1.1) with the second and the third equations replaced by (1.8) and (1.9), respectively:

$$\frac{d}{dt}\int_\Omega (u(\log u-1)-uv)\,dx+\tau\|v_t\|_2^2 + \int_\Omega u\left|\nabla(\log u-v)\right|^2\,dx=0.$$

In the simplified system (1.1), the Lyapunov function $\mathcal{W}(u,v)$ defined by (1.13) is equal to $\mathcal{F}(u)$ of (1.10):

$$\mathcal{F}(u)=\int_\Omega (u(\log u-1))\,dx-\frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\,dx\,dx'.$$

In fact, by (1.5) it holds that

$$\int_\Omega uv\,dx=\int_\Omega\int_\Omega G(x,x')u(x)u(x')\,dx\,dx',$$

while we also have

$$\int_\Omega\left(|\nabla v|^2+av^2\right)\,dx=\int_\Omega(-\Delta v+av)v\,dx=\int_\Omega uv\,dx,$$

and therefore from (1.13) it follows that

$$\mathcal{W}(u,v)=\mathcal{F}(u)$$

in this case. Thus, system (1.1) is subject to the second law of thermodynamics, the decrease of free energy, as well as of mass conservation.

In 1981, Childress and Percus [33] tried a semianalysis of the full system of chemotaxis, and conjectured that in the case of $n=2$ there is a threshold in the L^1 norm of the initial value for the blowup of the solution to occur.

To make the description simple, let $T_{\max} \in (0, +\infty]$ be the *blowup time*, that is, the supremum of the existence time of the solution. Thus, the solution exists globally in time if $T_{\max} = +\infty$, while $T_{\max} < +\infty$ means the *blowup* of the solution. Under this notation, their conjecture is that in the case of $n = 2$, $\|u_0\|_1 < 8\pi$ implies $T_{\max} = +\infty$, while $T_{\max} < +\infty$ can happen if $\|u_0\|_1 > 8\pi$. Their other conjecture is that such a threshold for the blowup of the solution does not exist if the space dimension is not two. This work is actually based on the method referred to as *nonlinear quantum mechanics* in this book, of which details will be described later.

It should be noted that Childress and Percus were motivated also by Nanjundiah [119], who had conjectured that u will develop a delta function singularity in finite time. This singularity was expected to explain the formation of spores of cellular slime molds when their food become rare, and is called the (chemotactic) *collapse*. See Figure 1.2.

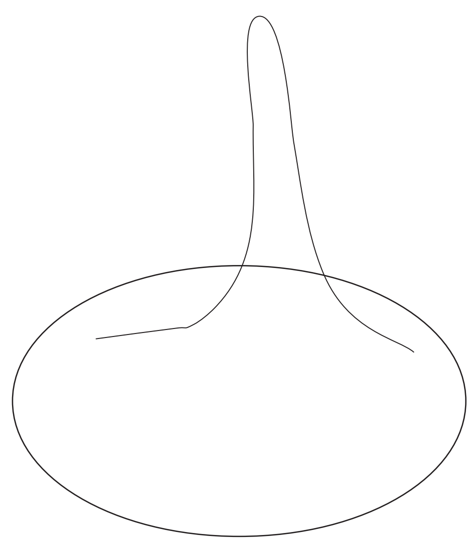


Figure 1.2.

To examine this conjecture of [119], [33] first applied the *dimensional analysis*, under the observation that if u is concentrated in $\mathbf{R}^n$ in a narrow area of radius $\delta > 0$, then it holds that $\nabla \sim \delta^{-1}$ and the magnitude of u is $O(\delta^{-n})$ because of the conservation of total mass (1.11). Sometimes, such $\delta > 0$ is called the *dimension*.

If we put $v = O(\delta^{1-(n/2)})$ and $t = O(\delta^{1+(n/2)})$, then the dimensions of

$$u_t - \nabla \cdot (u \nabla v) - \Delta u = 0$$

and

$$\tau v_t - u + av - \Delta v = 0 \tag{1.15}$$

balance as

$$\delta^{-(1+3n/2)} \left(\delta^0, \delta^0, \delta^{n/2-1} \right) = 0 \tag{1.16}$$

and

$$\delta^{-n} \left(\delta^0, \delta^0, \cdot, \delta^{n/2-1} \right) = 0,$$

where the term av is neglected in (1.15). From these relations, we can observe that according to $n = 1$ and $n = 3$, the diffusion, compared with the chemotaxis, dominates and is negligible for $0 < \delta \ll 1$, respectively. In the former case, collapses may not be formed, while in the latter case the concentration will be enforced. This suggests a stronger singularity than the delta-function in the case $n = 3$ and also the nonexistence of the collapse for $n = 1$. It also suggests that the actual blowup of the solution is not controlled by the total mass in the case of $n = 3$. Figure 1.3 illustrates such a situation, which is an overview of the radially symmetric blowup solution of $n = 3$ constructed by [62].

In this case of $n = 3$, there is also a self-similar blowup, distinguished from the above profile, such as

$$u(r, T_{\max}) \sim (8\pi + \delta)(4\pi r^2)^{-1}$$

for $r \downarrow 0$ [63]. In contrast with these cases of $n = 1$ and $n = 3$, the effect of diffusion and that of chemotaxis compete in the case of $n = 2$, and the formation of collapses [119] is expected only for $n = 2$.

Then how did [33] get the idea of a *threshold* total mass for the blowup in the case of $n = 2$, particularly the threshold value 8π? Actually, this is done by the study of the stationary problem of (1.1):

$$\begin{aligned} \nabla \cdot (\nabla u - u \nabla v) = 0 \quad &\text{in} \quad \Omega, \\ \Delta v - av + u = 0 \quad &\text{in} \quad \Omega, \\ \partial u / \partial \nu = \partial v / \partial \nu = 0 \quad &\text{on} \quad \partial \Omega, \end{aligned} \tag{1.17}$$

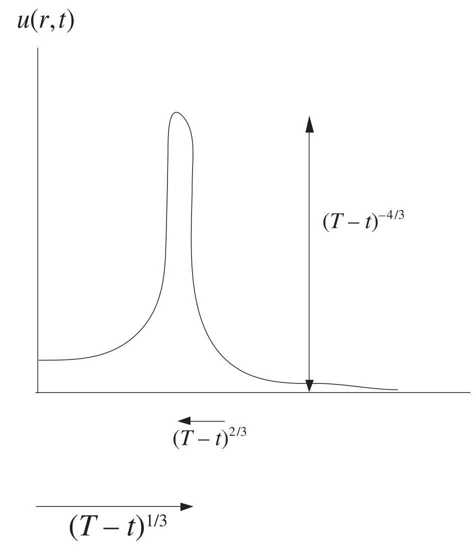


Figure 1.3.

and we call this part of their study nonlinear quantum mechanics.

First, we introduce the equilibrium state in the following way. In fact, the principal system to u is written as

$$\Delta u = \nabla v \cdot \nabla u + (\Delta v)u \quad \text{in} \quad \Omega, \qquad \partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega.$$

Since we are interested in the nontrivial case of $u \geq 0$ and $u \not\equiv 0$, it follows that $u > 0$ on $\overline{\Omega}$ from the strong maximum principle again. Now, writing the first equation of (1.17) as

$$\nabla \cdot u \nabla (\log u - v) = 0,$$

we have

$$\int_\Omega u \left| \nabla (\log u - v) \right|^2 \, dx = 0$$

similarly to (1.14), and hence

$$\log u - v = \log \sigma \tag{1.18}$$

follows in Ω with a constant $\sigma > 0$. This unknown constant σ can be prescribed if we take into account the equality (1.11) valid for the nonstationary problem (1.1). Namely, in terms of $\lambda = \|u\|_1$, relation (1.18) implies

$$u = \frac{\lambda e^v}{\int_\Omega e^v \, dx}.$$

Substituting this into the second equation of (1.1), we reach the elliptic eigenvalue problem with the nonlocal term,

$$\begin{aligned} -\Delta v + av &= \frac{\lambda e^v}{\int_\Omega e^v \, dx} \quad \text{in} \quad \Omega \\ \frac{\partial v}{\partial \nu} &= 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.19}$$

Here, the parameter $\lambda = \|u\|_1 > 0$ is regarded as an eigenvalue.

The stationary problem, (1.17), admits a constant solution $(u, v) = (\lambda/\,|\Omega|\,, \lambda/(a\,|\Omega|)$ with $\|u\|_1 = \lambda$. This trivial solution generates the branch of (constant) solutions to (1.19),

$$\mathcal{C}_c = \left\{ (\lambda, \lambda/\,(a\,|\Omega|)) \;\middle|\; \lambda > 0 \right\}$$

in $\lambda - v$ space, which bifurcates nontrivial solutions to (1.19).

Before proceeding to this bifurcation analysis, we want to mention a different problem very close to (1.19), that is,

$$\begin{aligned} -\Delta v &= \frac{\lambda e^v}{\int_\Omega e^v \, dx} \quad \text{in} \quad \Omega \\ v &= 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.20}$$

This arises in the theory of combustion [51] for $n = 3$ and in statistical mechanics for vortex points [20, 21, 82] for $n = 2$. Analogous problems are also found in self-dual gauge theory [189]. For $n \geq 3$, the structure of radially symmetric solutions is studied by [45, 51, 75, 117]. Two-dimensional radially symmetric solutions, on the other hand, are easier to handle because they are given explicitly. These elliptic problems are related also to the complex function theory and the theory of surfaces, and our previous monograph [166] is mostly devoted to these elliptic problems. There are other monographs [5, 9, 96], from the viewpoints of geometry, combustion, and hydrodynamics. In many cases, if the problem is involved with the exponential nonlinearity against the two-dimensional diffusion, then we obtain the quantized blowup mechanism.

Taking these works into account, [33] studied (1.19) as follows. First, they applied the bifurcation analysis to the branch of trivial solutions $\mathcal{C}_c$ by restricting the problem to the radially symmetric case, namely, Ω is put to be the unit disc,

$$D = \left\{ x \in \mathbf{R}^2 \;\middle|\; |x| < 1 \right\},$$

and the solution v is supposed to be radially symmetric as $v = v(|x|)$. Then, (1.19) is reduced to the two-point boundary value problem of an ordinary differential equation.

Note that problem (1.20) can be treated similarly. Although the constant solution does not exist in this case, all radially symmetric solutions to this problem are given explicitly and form a branch

$$\mathcal{C}_{rd} = \left\{ (\lambda, v_\lambda(x)) \;\middle|\; \lambda \in (0, 8\pi) \right\}$$

satisfying $\lim_{\lambda \downarrow 0} v_\lambda(x) = 0$ and

$$\lim_{\lambda \uparrow 8\pi} v_\lambda(x) = 4 \log \frac{1}{|x|}, \tag{1.21}$$

and, moreover, any $\lambda \in (0, 8\pi)$ admits a unique solution $v_\lambda = v_\lambda(x)$, and there is no solution for $\lambda \geq 8\pi$. See Figure 1.4.

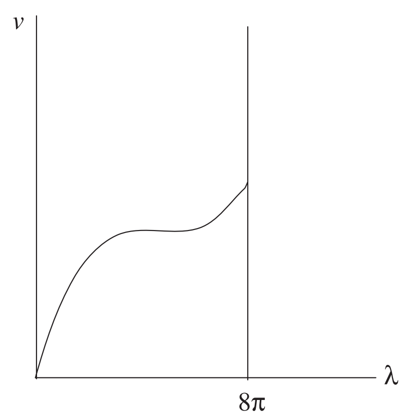


Figure 1.4.

In (1.17), on the contrary, a branch $\mathcal{C}_r$ of radially symmetric solutions bifurcates from that of constant solutions, $\mathcal{C}_c$, at some $\lambda = \lambda_* > 8\pi$. From the bifurcation theory it is confirmed also that $\mathcal{C}_r$ is in $\lambda < \lambda_*$ near the bifurcation point. Then, it is natural to ask whether or not $\mathcal{C}_r$ reaches or exceeds $\lambda = 8\pi$.

Under these conditions, [33] tried using the numerical computation to follow $\mathcal{C}_r$ toward the direction where λ decreases. What they observed is that it does not exist beyond $\lambda = 8\pi$, which led them to another conjecture concerning (1.17) when Ω is a disc, that is, only a constant stationary solution exists if λ is in $0 < \lambda < 8\pi$, while a very spiky stationary solution exists for each λ in $0 < \lambda - 8\pi \ll 1$. See Figure 1.5. (Later, [144] showed that this extra conjecture holds in the affirmative.)

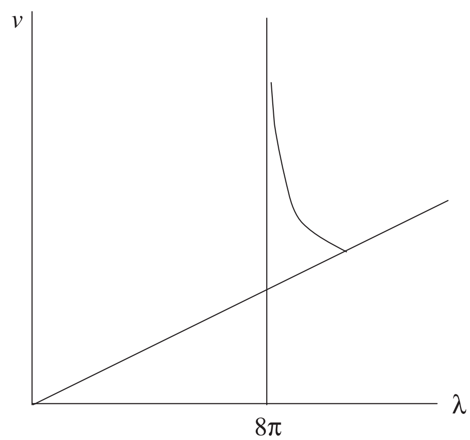


Figure 1.5.

Taking into account that the nonstationary solution lies in the manifold $\|u\|_1 = \lambda$ of function spaces, [33] arrived at the original conjecture by this extra conjecture — that is, in (1.1), with $n = 2$, $\|u_0\|_1 < 8\pi$ will imply $T_{\max} = +\infty$, while $T_{\max} < +\infty$ can occur if $\|u_0\|_1 > 8\pi$, because the "blowup solution should have the radially symmetric profile around the blowup point."

It is interesting but difficult to approach these conjectures rigorously. However, in 1992 Jäger and Luckhaus [73] introduced a simplified system as the limiting state of $\tau \sim a \downarrow 0$. There, the second equation of (1.1) is replaced by

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u\,dx \quad \text{in} \quad \Omega \tag{1.22}$$

with the side condition

$$\int_\Omega v\,dx = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.23}$$

For this system [73] showed that if $n = 2$, then the condition $\|u_0\|_1 \ll 1$ implies $T_{\max} = +\infty$, and $T_{\max} < +\infty$ occurs if $u_0(x)$ is sufficiently concentrated on a point in Ω and $\|u_0\|_1 \gg 1$. Then, in 1995, Nagai [106] showed that the threshold conjecture holds exactly true for *radially symmetric solutions* to (1.1). Thus, $\|u_0\|_1 = 8\pi$ is the actual threshold of the blowup of the solution in this case. However, this was not the end of the story.

Meanwhile, several tools to treat these systems were proposed mathematically. They are summarized as the use of the Lyapunov function, the *Trudinger–Moser inequality*, and the *second moment* of total mass. Based on these methods, it was proven that $\|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$ in (1.1) for the general case [14, 50, 110]. (Here, we confirm that the bounded domain $\Omega \subset \mathbf{R}^2$ is supposed to have the smooth boundary, consisting of a finite number of smooth Jordan curves. In fact, if $\partial\Omega$ has corners, then this constant 4π must be reduced more. We shall see that the quantized blowup mechanism explains completely what this means.)

Thus, around 1997, we were very close to a complete the proof of the conjecture [33], except for the discrepancy of the constant, 4π and 8π. This difference is due to *Moser–Onofri-type inequalities* in $H_0^1(\Omega)$ and $H^1(\Omega)$, but is not technical. In fact, the structure of the set of solutions to (1.20) and (1.17) are rather different.

Before describing this difference, we point out that the above-mentioned inequalities of real analysis were discovered in the study of *Nirenberg's problem* in differential geometry. Actually, these inequalities are associated with a variational functional. A key tool to approach our problem, the 4π-8π discrepancy, called the *concentration lemma*, was proposed by [22, 23]. It focuses on the behavior of a family of functions defined on S^2, and using them we can show that if the blowup occurs in (1.1) with $4\pi < \|u_0\|_1 < 8\pi$, then the solution must concentrate on the boundary [108].

This means that we have to take into account the nonradially symmetric stationary solution to (1.17) to understand the whole blowup mechanism of (1.1). This is nothing but the methodology of [33], that is, the singular limit of stationary solutions controls the blowup mechanism of nonstationary solutions. However, we added a new motivation for unifying these two phenomena, threshold and collapse, by the blowup mechanism in this book. Another new viewpoint is the spectral equivalence of the variational structures of the equilibrium state, induced by the free energy and the standard elliptic theory. We call this part the *theory of dual variation*.

Now we describe what is known for (1.20). First, the blowup can occur only at quantized values of λ [114, 115], namely, if

$$\left\{ (\lambda_k, v_k(x)) \right\}$$

is a sequence of solutions to (1.20) with $\lambda = \lambda_k$ and $v(x) = v_k(x)$, satisfying $\lambda_k \to \lambda_0 \in [0, \infty)$ and $\|v_k\|_\infty \to +\infty$, then it holds that $\lambda_0 \in 8\pi\mathbf{N}$. Furthermore, $\lambda_0/(8\pi)$ coincides with the number of blowup points of $\{u_k\}$ and we have

$$-\Delta v_k(x)\, dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} 8\pi \delta_{x_0}(dx),$$

$*$-weakly in the sense of measures on $\overline{\Omega}$, where $\mathcal{S}$ denotes the blowup set with the location controlled by the Green's function for $-\Delta$ in Ω under the Dirichlet boundary condition. This phenomenon of concentration is already described by (1.21) in the radially symmetric case.

To approach (1.17) in comparison with (1.20), examining the role of symmetry is essential. More precisely, the general theory of Gidas et al. [52] guarantees that any classical positive solution to a semilinear elliptic problem

$$-\Delta u = f(u) \quad \text{in} \quad \Omega \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with Ω equal to the unit ball,

$$B = \left\{ x \in \mathbf{R}^n \;\middle|\; |x| < 1 \right\},$$

must be radially symmetric, provided that $f : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous. This theorem is applicable to (1.20), and the set of solutions for $\Omega = D$, two-dimensional unit ball, coincides with the branch $\mathcal{C}_{rd}$ mentioned before. (Bandle [5] gave a different proof of the radial symmetry of the solution to (1.20) on the two-dimensional disc.)

Now, we describe how (1.17) is different from (1.20). Actually, there are nonradially symmetric solutions to (1.17) even for $\Omega = D$ [144], and these solutions play essential roles in the long-term behavior of the nonstationary solution. For instance, a family of solutions to (1.17) blows up only at $\lambda_0 \in 4\pi\mathbf{N}$ and this λ_0 is equal to 8π times the number of interior blowup points plus 4π times that of boundary blowup points. In this connection, it is worth mentioning that any family of solutions to (1.20) takes no boundary blowup point at all. See Figure 1.6. Thus, the 4π-8π discrepancy is a consequence of

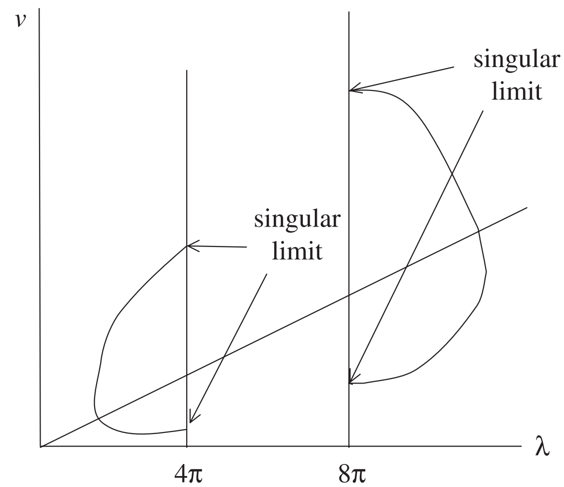


Figure 1.6.

the boundary blowup point in the equilibrium state, and this is actually the case in the nonequilibrium state.

Here is a theorem [145] concerning the formation of collapses in the blowup solution to (1.1). There, $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightharpoonup$ the $*$-weak convergence, and

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega)\,, \\ 4\pi & (x_0 \in \partial\Omega)\,. \end{cases} \tag{1.24}$$

Theorem 1.1 *If $T_{\max} < +\infty$ in (1.1), then there exists a finite set*

$$\mathcal{S} \subset \overline{\Omega}$$

and a nonnegative $f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ such that

$$u(x,t)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx \tag{1.25}$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T_{\max}$ with

$$m(x_0) \geq m_*(x_0) \qquad (x_0 \in \mathcal{S})\,. \tag{1.26}$$

We have also

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = +\infty \tag{1.27}$$

and $\mathcal{S}$ coincides with the blowup set of u, namely, $x_0 \in \mathcal{S}$ if and only if there exist $x_k \to x_0$ and $t_k \uparrow T_{\max}$ such that $u(x_k, t_k) \to +\infty$. Since

$$\|u(t)\|_1 = \|u_0\|_1$$

holds for $t \in [0, T_{\max})$, we thus obtain

$$2 \cdot \sharp (\Omega \cap \mathcal{S}) + \sharp (\partial\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi) \tag{1.28}$$

by (1.25) and (1.26). This inequality is regarded as a refinement of the previous work concerning the criterion of $T_{\max} = +\infty$, as it assures that $\|u_0\|_1 < 4\pi$ implies $\mathcal{S} = \emptyset$ and therefore $T_{\max} = +\infty$.

Each collapse

$$m(x_0)\delta_{x_0}(dx)$$

stands for the spores made by cellular slime molds, and if the equality holds in (1.24), then the quantized value of the mass of collapses on the boundary is counted as half of the one in the interior.

Inequality (1.26) indicates that the mass of collapses made by the blowup solution cannot be under the fundamental level. If this estimate is optimal, then conjecture [33] follows with the value 8π replaced by 4π. More precisely, any $\lambda > 4\pi$ admits $u_0(x) \geq 0$ such that $\|u_0\|_1 = \lambda$ and $T_{\max} < +\infty$. This is actually the case, and if

$$\|u_0\|_1 > 4\pi$$

and $u_0(x)$ is sufficiently concentrated at a point on the boundary, then we obtain $T_{\max} < +\infty$ [107, 146]. Thus, conjecture [33] holds in the affirmative if the threshold value is reduced to a half of the expected one.

This sharp blowup criterion, on the contrary, means the existence of a family of blowup solutions to (1.1), each of which has one blowup point on the boundary of their own with the collapse mass as close as possible to 4π. The next question is the mass quantization of each collapse, $m(x_0) = m_*(x_0)$ in (1.25). Actually, Herrero and Velázquez [64] constructed a family of radially symmetric solutions to the system studied by [73], satisfying

$$u(x,t)\,dx \rightharpoonup 8\pi\delta_0(dx) + f(x)\,dx \tag{1.29}$$

as $t \uparrow T_{\max} < +\infty$ with nonnegative $f = f(x) \in L \log L(D)$, where $D \subset \mathbf{R}^2$ is the unit disc and $L \log L$ denotes the Zygmund space. Thus, the equality holds in (1.26) in this example, and we have also the stability of this blowup pattern [181].

What we call *mass quantization of collapses* in this book is the equality $m(x_0) = m_*(x_0)$ in (1.25). This is actually the case if the solution is continued after the blowup time as a *weak solution* [147], and if the solution *blows up in infinite time* [148]. In this context, we know that the *Fokker–Planck equation* admits a weak solution globally in time, provided that the initial value has a finite second moment, is bounded, and is summable [182]. This Fokker–Planck equation is derived from the *Langevin equation*, and describes the *kinetic mean field* of many *self-interacting particles*. Since system (1.1) arises as its *adiabatic limit*, we can expect that (1.1) admits a weak solution globally in time, and this implies the equality in (1.26).

Unfortunately, this question of post blowup continuation is open; maybe it will not be true, because the Fokker–Planck equation is valid only when the distribution of particles is thin, and its physical scale is different from that of (1.1). However, we can prove the mass quantization using the scaling argument, the success of which may be promised by the dimensional analysis stated above; that is, the concentration of the rescaled mass, called *subcollapse*, dominates the aggregation of the residual term, excluding the possibility of the formation of *multicollapses* [149].

In more detail, this question of mass quantization of collapses is related to the control of the blowup rate of the solution. Actually, the asymptotics (1.29) of Herrero–Velázquez's solution are derived from its local behavior,

$$u(x,t) = \frac{1}{r(t)^2}\overline{u}\Big(\frac{x}{r(t)}\Big)\{1+o(1)\} + O\Big(\frac{e^{-\sqrt{2}|\log(T-t)|^{1/2}}}{|x|^2}\cdot 1_{\{|x|\geq r(t)\}}\Big) \tag{1.30}$$

as $t \uparrow T = T_{\max}$ uniformly in $|x| \leq C(T-t)^{1/2}$, where

$$r(t) = (T-t)^{1/2}\cdot e^{-\sqrt{2}/2\ |\log(T-t)|^{1/2}}\big|\log(T-t)\big|^{\frac{1}{4}\log^{-1/2}(T-t)-\frac{1}{4}}\big(1+o(1)\big)$$

and $\overline{u}(y) = 8\cdot\big(1+|y|^2\big)^{-2}$. This $\overline{u}(y)$ is nothing but the stationary solution on the whole space $\mathbf{R}^2$, but the rate $r(t)$ is also important.

In fact, this asymptotic behavior of the solution is quite different from the one derived from the subcritical nonlinearity, say,

$$u_t - \Delta u = u^p, \quad u \geq 0 \quad \text{in} \quad \Omega\times(0,T)$$

with $u|_{\partial\Omega} = 0$ for $1 < p < (n+2)/(n-2)_+$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. In this case the local profile of the solution in the *parabolic region* is controlled by the ODE part $\dot{u} = u^p$, and, for example, if Ω is convex and $x_0 \in \Omega$ is a blowup point, then it holds that

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}} \{1 + o(1)\} \tag{1.31}$$

as $t \uparrow T = T_{\max} < +\infty$ uniformly in $|x - x_0| \leq C\,(T-t)^{1/2}$ [54, 55, 56]. Thus, the concentration is so slow that $u(x,t)$ becomes flat in the parabolic region in this case, and consequently, the total blowup mechanism is not enveloped there.

On the contrary, we have $0 < r(t) \ll R(t) \equiv (T-t)^{1/2}$ in (1.30), and therefore the standard *backward self-similar transformation*

$$z(y,s) = (T-t)u(x,t) \tag{1.32}$$

with

$$\begin{aligned} y &= (x - x_0)/(T-t)^{1/2}, \\ s &= -\log(T-t), \end{aligned} \tag{1.33}$$

reproduces a collapse again, which we call the *subcollapse*. More precisely,

$$z(y,s)\,dy \quad \rightharpoonup \quad 8\pi\delta_0(dy)$$

holds as $s \to +\infty$ in the sense of measures in $\mathbf{R}^2$, which suggests that the nonlinearity is supercritical in the case of (1.1), and the concentration, relative to the aggregation, is so rapid that the whole blowup mechanism is enveloped in the parabolic region. Here, the *aggregation* and the *concentration* indicate the growth of the local L^1 norm and that of the L^∞ norm of the solution, respectively. See Figure 1.7.

From these considerations, it is natural to classify the blowup point by the blowup rate in accordance with the standard backward self-similar transformation [165]. Namely, we say that $x_0 \in \mathcal{S}$ is *type (I)* if

$$\limsup_{t \to T} \sup_{x \in \Omega,\ |x - x_0| \leq CR(t)} R(t)^2 u(x,t) < +\infty$$

holds for any $C > 0$ and it is *type (II)* for the other case:

$$\limsup_{t \to T} \sup_{x \in \Omega,\ |x - x_0| \leq CR(t)} R(t)^2 u(x,t) = +\infty$$

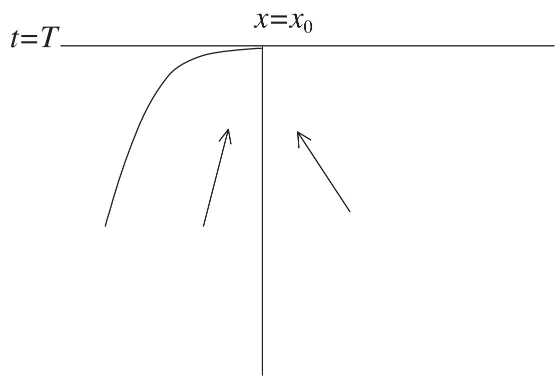


Figure 1.7.

with some $C > 0$, where

$$T = T_{\max} < +\infty \quad \text{and} \qquad R(t) = (T - t)^{1/2}.$$

Then, we can show the following theorem.

Theorem 1.2 *In any case, mass quantization $m(x_0) = m_*(x_0)$ holds in (1.25) for each blowup point $x_0 \in \mathcal{S}$. If it is type (II) and*

$$\lim_{n\to+\infty} \sup_{x\in\Omega,\ |x-x_0|\le CR(t_n)} R(t_n)^2 u(x, t_n) = +\infty$$

for $t_n \to T$, then we obtain

$$z(y, s_n + \cdot)\, dy \ \rightharpoonup\ m_*(x_0)\delta_0(dy)$$

in $C_*\left((-\infty, +\infty), \mathcal{M}(\mathbf{R}^2)\right)$, where $s_n = -\log(T - t_n)$. Here, zero extension of $z = z(y, z)$ is taken in the region where it is not defined by (1.32) and (1.33). If $x_0 \in \mathcal{S}$ is type (I), on the other hand, then it holds that

$$\lim_{t\to T_{\max}} \mathcal{F}_{bR(t)}(u(t)) = +\infty$$

for any $b > 0$, where $\mathcal{F}_R(t)$ is the local free energy defined by

$$\mathcal{F}_R(u) = \int_\Omega \psi_{x_0,R,2R} u(\log u - 1)\, dx \\ - \frac{1}{2}\int_\Omega\int_\Omega \psi_{x_0,R,2R}(x)\psi_{x_0,R,2R}(x')G(x,x')u\otimes u\, dx\, dx'$$

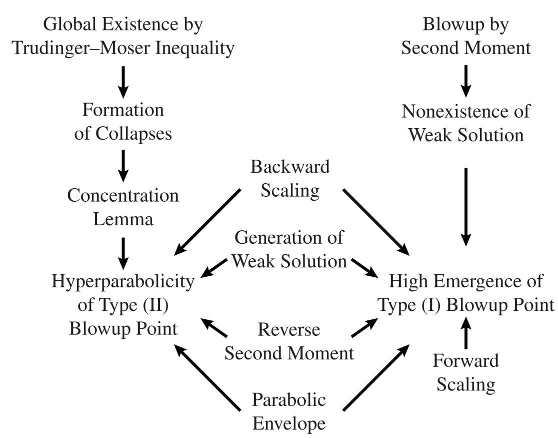
for smooth function $\psi = \psi_{x_0,R,2R}$ satisfying $0 \le \psi \le 1$, $\psi = 1$ in $\Omega \cap B(x_0, R)$, $\psi = 0$ in $\Omega \setminus B(x_0, 2R)$, and $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$.

The above theorem says that the type (II) blowup point is *hyperparabolic*, which means that the whole blowup mechanism is included in an infinitely small parabolic region called the *hyperparabola*, associated with the standard self-similar transformation. Obviously, Hererro–Velázquez's solution has such a profile. Around this type of blowup point, a collapse with the quantized mass is formed by the concentration of particles, asymptotically radially symmetrically. However, this theorem says also that even around the type (I) blowup point, the whole blowup mechanism is enveloped by an infinitely large parabolic region called the *parabolic envelope*. If such a blowup point exists, then it arises from the wedge of the parabolic region, not necessarily radially symmetric, and possibly moving. Another profile of the solution around the type (I) blowup point is that the entropy is swept away to the wedge of the parabolic envelope, which has been shown to have a similarity of *emergence* by Kauffman [78]. Although the actual existence of a type (I) blowup point is open to discussion, even a type (II) blowup point can take such a profile in the other space-time rescaling; for example, Herrero–Velazquez's solution satisfies

$$\lim_{T \to T_{\max}} \mathcal{F}_{br(t)}\,(u(t)) = +\infty$$

for any $b > 0$.

This quantized blowup mechanism, however, is not extended to the system associated with the chemical field as it is. In fact, in the full system some spiral movements are added to the blowup mechanism because of the time lag for the creation of the field from particles, and this may make it possible for the blowup set to be a continuum. In this way, each hierarchy of the mean field equations has is own mathematical principle, besides the physical principle to derive them.



We have described that the system of self-interacting particles is subject to the story of nonlinear quantum mechanics. First, the stationary state is realized as a nonlinear elliptic eigenvalue problem with nonlocal terms. Next, quantization of the singular limit of the stationary state induces that of the nonequilibrium state — its dynamics and the blowup mechanism. This hierarchy is derived from the physical principle of mass conservation and the decrease of the free energy. An important structure is the self-interaction associated with the symmetric kernel, which is to be called *compensated compactness via symmetrization*. Finally, the equilibrium state is subject to the dual variation associated with the particle density and the field distribution being dynamically equivalent to each other, and transformed through the Legendre transformation.

The above table indicates how the proofs of Theorems 1.1 and 1.2 are completed. In spite of several technical ingredients, it will be observed that mass quantization is a consequence of the L^1 threshold in the blowup criterion.

Now, we discuss the structure of the present monograph. The book is divided into five parts. Chapter 2 is the introduction and presents the hierarchy of the system of equations, one of the main themes of this monograph.

Chapters 3, 4, and 5 present the classical theory for the time-dependent problem. First, Chapter 3 establishes the unique existence of the classical solution locally in time, and also the general criterion for the blowup of the solution. Then the existence and the nonexistence of the classical solution globally in time are discussed in Chapters 4 and 5, respectively.

The third part — Chapters 6, 7, 8, 9, and 10 — deals with the stationary problem. Chapter 6 formulates the problem, and then Chapter 7 shows quantization of the mass and location of the singularities of the singular limit of the solution. This phenomenon has been already treated in our previous monograph [166], but the mathematical tool adopted here is quite different. That is, we make use of a delicate profile of the Green's function, rather than the complex function theory or the theory of surfaces. This analysis is applied to show the existence of the nontrivial stationary solution using the variational method in Chapter 8. Then, Chapter 9 describes the second theme, unfolding the Legendre duality via the Lagrange function. This spectral theory is realized as the dynamical and the stability equivalences. Then, in Chapter 10, the quantized blowup mechanism is suggested from observations of the stability and the instability of stationary solutions.

The main theme, the quantized blowup mechanism, is proven in the fourth part, Chapters 11, 12, 13, 14, and 15. First, we establish the formation of collapses by the space localization of the method of Chapter 4 concerning the existence of the solution globally in time (Chapter 11). This consequence is discretized in the time variable to study the blowup in infinite time (Chapter 12). The method of Chapter 5 for the nonexistence of the solution globally in time, on the other hand, is also localized in the space variable, and it is shown that the mass quantization of the collapse occurs if the solution continues after the blowup time as a weak solution (Chapter 13). In Chapter 14 it is suggested that the actual quantization is related to the blowup rate, and finally Chapter 15 establishes the mass quantization of collapses using the parabolic envelope and the blowup criterion of the weak solution on the whole space generated by the limiting process for the rescaled solution. We use also the reverse second moment and the forward self-similar transformation for this purpose. It is proven also that the type (I) blowup point shows the profile of emergence and the type (II) blowup point is hyperparabolic using the concentration lemma.

The final part, Chapter 16, is an epilogue, where the general variational principle for the study of the mean field equations is presented, including the unfolding of the Legendre duality. It is an abstract theory based on the material discussed in the third part in the context of convex analysis, and applications to other systems are proposed.

2
Background

Nobody has ignored living things who
has thought of entropy seriously.
— H. Tanaka

This chapter is a short description of mathematical modelling of the problem. First, we describe the physical motivation. In fact, parabolic-elliptic systems with drift terms are found in several areas of science involved with the transport theory; statistical mechanics, quantum mechanics, physical chemistry, and so forth. Here, we mention two of them, the *semiconductor device equation* and the *vortex equation*.

The first system, referred to as the *DD model*, is written as

$$\left.\begin{aligned} n_t &= \nabla \cdot (\nabla n - n\nabla\varphi) \\ p_t &= \nabla \cdot (\nabla p + p\nabla\varphi) \\ \Delta\varphi &= n - p \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0, T),$$
$$\left.\begin{aligned} \frac{\partial n}{\partial \nu} - n\frac{\partial \varphi}{\partial \nu} &= 0 \\ \frac{\partial p}{\partial \nu} + p\frac{\partial p}{\partial \nu} &= 0 \\ \varphi &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2.1}$$

where $n = n(x,t)$ and $p = p(x,t)$ denote the densities of the electron and the positron, respectively, and $\varphi = \varphi(x,t)$ is the electric charge field derived from those particles. As is described in the previous chapter, in the case of $n = 0$, this system is reduced to (1.1) with the opposite sign in the second term of the right-hand side of the first equation, if the second equation is replaced by (1.5), where $G = G(x,x')$ denotes the Green's function for $-\Delta$ in Ω under the Dirichlet boundary condition. Here, the formation of the electric charge field provides the self-repulsive force to the electrons. Positrons are similar, and therefore this system is *dissipative*. Its physical modelling is described in [152], and several variants are mentioned in [6].

The second system is given by

$$\left.\begin{aligned} \omega_t &= \nabla\cdot\left(\nabla\omega - \omega\nabla^{\perp}\psi\right) \\ -\Delta\psi &= \omega \end{aligned}\right\} \quad \text{in} \quad \mathbf{R}^2\times(0,T), \tag{2.2}$$

where

$$\nabla^{\perp} = \begin{pmatrix} -\partial/\partial x_2 \\ \partial/\partial x_1 \end{pmatrix}$$

for $x = (x_1, x_2)$ denotes the *antigradient*. It comes from the *Navier–Stokes system*

$$\left.\begin{aligned} u_t - \Delta u + u\cdot\nabla u = \nabla p \\ \nabla\cdot u = 0 \end{aligned}\right\} \quad \text{in} \quad \mathbf{R}^3\times(0,T),$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad \nabla = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}$$

denote the velocity and the gradient operator, respectively, and p is the pressure. If we take the two-dimensional model of $x = (x_1, x_2, 0)$ and $u_3 = 0$, then it holds that

$$\nabla\times u = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad \text{for} \quad \omega = \omega(x_1, x_2).$$

This system is also dissipative, but some underlying chaotic profiles are observed.

The direction of the self-interacting force in these systems — chemotaxis, semiconductor devices, and vortices — is different, that is, the particles create the field to be attractive, repulsive, and perpendicular to themselves, respectively. However, some common structures are noticed, and in particular, the first two systems share the physical principle of the *second law of thermodynamics*, that is, the decrease of the *free energy*. For instance, an equilibrium state is stable if it is a local minimum of the free energy, while the transient dynamics is controlled by the unstable equilibrium states.

We note that the free energy is given by the *inner energy* minus *entropy*. If $\rho = \rho(x) \geq 0$ denotes the density of particles, then the entropy on the domain $\Omega \subset \mathbf{R}^n$ in consideration is defined by

$$-\int_\Omega \rho(\log\rho - 1)\,dx.$$

On the other hand, the inner energy is composed of the *kinetic* and *potential energies* and therefore it is defined by

$$-\frac{1}{2}\iint_{\Omega\times\Omega} H(x,x')\rho\otimes\rho\,dx\,dx' + \int_\Omega \rho V\,dx, \tag{2.3}$$

where $-H(x,x')$ and $V(x)$ denote the potential of the self-interaction and that of the external force, respectively. Here, from *Newton's third law* we have

$$H(x,x') = H(x',x),$$

and the modulus a half of the first term of (2.3) is a consequence of self-interaction. If the self-interaction is due to the gravitational force, then we have $H(x,x') = \Gamma(x-x')$ for $\Gamma = \Gamma(x)$ given by (1.4). In any case, the system of equations describing these self-interactions is required to be provided with the property of the decrease of free energy,

$$\mathcal{F}(\rho) = \int_\Omega \rho(\log\rho - 1)\,dx - \frac{1}{2}\iint_{\Omega\times\Omega} H(x,x')\rho\otimes\rho\,dx\,dx' + \int_\Omega \rho V\,dx.$$

There is an approach by the inner friction and fluctuations of particles [8, 185, 186]. We shall illustrate the classical theory before following the argument. In both cases, the complete mathematical justification is hard, and here we draw the stories only. See the above-mentioned papers and the references therein for the actual rigorous proof.

First, the classical theory is based on the *Newton equation*. Thus, if m and N denote the mass and number of each particle, respectively, then it holds that

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \\ m\frac{dv_i}{dt} &= \nabla_{x_i}\Big\{ -mV(x_i) + m^2 \sum_{j\neq i} H(x_j, x_i)\Big\} \end{aligned} \tag{2.4}$$

for $i = 1, \ldots, N$. Making $N \to \infty$ with $M = mN$ preserved, we obtain the limiting distribution $f(x, v, t)\, dx\, dv$:

$$\begin{aligned} \mu^N(dx, dv, t) &= m \sum \delta_{x_i(t)}(dx) \otimes \delta_{v_i(t)}(dv) \\ &\rightharpoonup f(x, v, t)\, dx\, dv. \end{aligned}$$

This $f(x, v, t)$ is subject to the *kinetic model*, referred to as the *Jeans–Vlasov equation*. In the normal form, it is given as

$$\begin{aligned} f_t &= -\nabla_x \cdot (vf) + \gamma \nabla_v \cdot [f \nabla_x (V - U)], \\ &U(x, t) \end{aligned}$$

with a constant $\gamma > 0$.

In the next process of $(dv_i)/(dt) \to 0$, comparable to make $\gamma \to +\infty$, the distribution function $f(x, v, t)$ is replaced by the *Maxwellian* $\omega(x, t)\pi^{-n/2} \cdot e^{-v^2/2}$. This is called the *adiabatic limit*. If $n = 2$, then $\omega(x, t)$ is subject to the *vorticity equation* derived from the *Euler equation*,

$$\begin{aligned} -\Delta\psi &= \omega \\ \omega_t &= -\nabla \cdot \left(\omega \nabla^{\perp} (\psi + V)\right). \end{aligned}$$

In the stationary state of this system, $\omega = \omega(x)$ is associated with the elliptic problem

$$-\Delta\psi = g(\psi + V)$$

with the nonlinearity g unknown [179, 180]. If the particles are spatially concentrated as

$$\omega(x, t) = \sum_{i=1}^{N} \delta_{x_i(t)}(dx), \tag{2.5}$$

on the other hand, then the concentration spots are subject to the *Hamiltonian system*

$$\frac{dx_i}{dt} = \nabla_{x_i}^{\perp} \mathcal{H}(x_1, x_2, \ldots, x_N) \tag{2.6}$$

for $i = 1, 2, \ldots, N$ with the *Hamiltonian*

$$\mathcal{H}(x_1, x_2, \ldots, x_N) = -\sum_i V(x_i) + \sum_{j<i} H(x_i, x_j).$$

If the gravitational force acts as the self-interaction, then $H = H(x, x')$ is given by $H(x, x') = \Gamma(x - x')$. If this $H(x, x')$ is replaced by $G(x, x')$, the Green's function of $-\Delta$ provided with the Dirichlet boundary condition, then

$$\frac{1}{2}\sum_i R(x_i)$$

is added to the right-hand side of the Hamiltonian, where $G(x, x')$ and $R(x)$ denote the *Green's function* and the *Robin function*, respectively; that is, $R(x) = K(x, x)$ with $K = K(x, x')$ defined by

$$K(x, x') = G(x, x') + \frac{1}{2\pi} \log \left| x - x' \right|.$$

However, this hierarchy of the system of equations is not subject to the second law of thermodynamics, the decrease of free energy. Actually, it is governed by three laws of conservation; *mass*, *momentum*, and *energy*. As a consequence, the solution has the profile of *chaotic* motion.

Contrasted with this classical theory, the other approach assumes *friction* and *random fluctuations* of particles, and replaces the Newton equation by the *Langevin equation*:

$$\begin{aligned} dx_i &= v_i \, dt, \\ m \, dv_i &= \nabla_{x_i} \Big(-mV(x_i) + m^2 \sum_{j \neq i} H(x_j, x_i) \Big) - \beta v dt + (2\beta k T)^{1/2} \, dW_t^i. \end{aligned}$$

Here, k, T, and β are the *Boltzmann constant*, temperature, and friction coefficient, respectively, and $\left(W_t^i\right)$ denotes the white noise. Its kinetic model becomes the *Fokker–Planck equation*, in the form of

$$\begin{aligned} f_t &= -\nabla_x \cdot (vf) + \nabla_v \cdot [f \nabla_x (V - U)] + \beta k T \nabla_v \cdot (vf + \nabla_v f) \\ U(x, t) &= \iint_{\Omega \times \mathbf{R}^n} H(x, x') f(x', v, t) \, dx' dv, \end{aligned} \tag{2.7}$$

where

$$\rho(x,t) = \int_{\mathbf{R}^n} f(x,v,t)\,dv,$$

is the density and therefore

$$\lambda = \int_\Omega \rho(x,t)\,dx$$

stands for the total mass. Then the *adiabatic limit* of this kinetic model is obtained by $\beta \to +\infty$;

$$\rho_t = \nabla \cdot (\rho \nabla (V - U)) + \Delta \rho.$$

If $V = 0$ and the kernel $H = H(x,x')$ is replaced by the Green's function $G = G(x,x')$ with $-\Delta + a$ under the Neumann boundary condition, the above equation is nothing but the simplified system of chemotaxis, (1.1).

As was described in the previous chapter, the equilibrium state of this system is given by the semilinear elliptic eigenvalue problem, (1.19), with the nonlinearity prescribed as the exponential function. The spatially localized solution (2.5), on the other hand, is subject to the gradient flow with $\nabla^\perp$ replaced by ∇ in (2.6):

$$\frac{dx_i}{dt} = \nabla_{x_i} \mathcal{H}(x_1, x_2, \ldots, x_N),$$

where $i = 1, 2, \ldots, N$. If $H = H(x,x')$ is replaced by the Green's function of $-\Delta + a$ with the Neumann boundary condition, denoted by $G = G(x,x')$, then the spatially localized particles are in the interior or on the boundary exclusively, and the gradient flow is defined by

$$\chi(x_i)\frac{dx_i}{dt} = \nabla_{x_i}\Big(-\chi(x_i)V(x_i) + \sum_{j \neq i} \chi(x_j) G(x_i, x_j) + \frac{\chi(x_i)}{2} R(x_i)\Big),$$

where $R(x)$ is the Robin function and

$$\chi(x) = \begin{cases} 1 & (x \in \Omega), \\ 1/2 & (x \in \partial\Omega). \end{cases}$$

Furthermore, location of the N blowup points of the singular limit of the stationary solution, (1.17), forms a stationary point of this ODE system [166].

The Fokker–Planck equation (2.7), on the other hand, is provided with the free energy,

$$\hat{\mathcal{F}}(f) = \iint_{\Omega\times\mathbf{R}^n} f(\log f - 1)\,dx\,dv - \frac{1}{2}\iint_{\Omega^2\times\mathbf{R}^{2n}} H(x,x')f(x,v,t)f(x',v',t)\,dx\,dx'dv\,dv',$$

and this new hierarchy is derived from the physical principle of the second law of thermodynamics. These two hierarchies are summarized in the following table. We are not concerned with their mathematical justification. Our interest is in the quantized blowup mechanism observed in the second hierarchy.

ODE	Newton	Langevin
Kinetic	Jeans–Vlasov	Fokker–Planck
PDE	Euler	Keller–Segel
Time-localized	Elliptic eigenvalue	Liouville–Gel'fand
Space-localized	Hamiltonian	Gradient
Physics	Conservation laws	Free energy
Mathematics	Chaos	Quantization

Part of the biological background to (1.1), on the other hand, is the microscopic derivation made from the biased random walk [3, 128]. Another is the reinforced random walk [127]. The underlying structure is the movement of many particles controlled by the other species. In this chapter we discuss the argument for the latter. We discuss mostly the one-dimensional lattice $\mathcal{L}$, but the n-dimensional lattice $\mathcal{L}^n$ is treated similarly. See also [152] for more details.

First, we identify $\mathcal{L}$ with

$$\mathcal{Z} = \left\{\ldots, -n-1, -n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n, n+1, \ldots\right\}.$$

If $p_n(t) \in [0,1]$ denotes the conditional probability that the walker stayed on site $n = 0$ at time $t = 0$, and is on site $n = n$ at time $t = t$, then we obtain the *master equation*:

$$\frac{\partial p_n}{\partial t} = \hat{T}^+_{n-1} p_{n-1} + \hat{T}^-_{n+1} p_{n+1} - (\hat{T}^+_n + \hat{T}^-_n) p_n, \tag{2.8}$$

where $\hat{T}^\pm_n$ denotes the transition rates that the walker staying on site n jumps to site $n \pm 1$ in the unit time. We consider the case that these transient rates $\hat{T}^\pm_n$ are controlled by the other species living in the sublattice $\hat{\mathcal{L}}$, of which the mesh size is half that of $\mathcal{L}$. Let the density of that species be

$$w = \big(\ldots, w_{-n-1/2}, w_{-n}, w_{-n+1/2}, \ldots, w_{-1/2}, \\ w_0, w_{1/2}, \ldots, w_{n-1/2}, w_n, w_{n+1/2}, \ldots\big).$$

If the transition probabilities depend only on the density of the control species at that site, then it holds that $\hat{T}^\pm_n = \hat{T}(w_n)$ and hence (2.8) is written as

$$\frac{\partial p_n}{\partial t} = \hat{T}(w_{n-1}) p_{n-1} + \hat{T}(w_{n+1}) p_{n+1} - 2\hat{T}(w_n) p_n. \tag{2.9}$$

Therefore, writing $x = nh$ by the mesh size h of the lattice, we obtain

$$\frac{\partial p}{\partial t} = h^2 \frac{\partial^2}{\partial x^2}\big(\hat{T}(w)p\big) + O(h^4). \tag{2.10}$$

If we have the scaling $t' = \lambda t$, then we can take $\hat{T}(w) = \lambda T(w)$. Under the assumption $\lim_{h\downarrow 0} \lambda h^2 = D > 0$, it follows formally that

$$\frac{\partial p}{\partial t} = D \frac{\partial^2}{\partial x^2}\left(T(w)p\right)$$

by putting t for t'. Thus, the response function $T(w)$ represents the microscopic mechanism of the jump process. On the other hand, the variables p and w are coupled, and w is subject to another equation involving p.

In the *barrier model*, the transient rate at site n is determined by the densities of the control species at site $n \pm 1/2$. Thus, the control species which governs the jump process makes a barrier to the particle. We have

$$\hat{T}^\pm_n(w) = \hat{T}(w_{n\pm 1/2})$$

and the master equation (2.9) is now reduced to

$$\frac{\partial p_n}{\partial t} = \hat{T}(w_{n-1/2}) p_{n-1} + \hat{T}(w_{n+1/2}) p_{n+1} \\ - \big(\hat{T}(w_{n+1/2}) + \hat{T}(w_{n-1/2})\big) p_n.$$

Here, the right-hand side is equal to

$$\begin{aligned}
&\hat{T}(w_{n+1/2})(p_{n+1} - p_n) + \hat{T}(w_{n-1/2})(p_{n-1} - p_n) \\
&\qquad = h\Big(\hat{T}(w_{n+1/2}) - \hat{T}(w_{n-1/2})\Big)\Big(\frac{\partial p}{\partial x} + o(1)\Big) \\
&\qquad = h^2\Big\{\frac{\partial}{\partial x}\Big(\hat{T}(w)\frac{\partial p}{\partial x}\Big) + o(1)\Big\}
\end{aligned}$$

and we obtain

$$\frac{\partial p}{\partial t} = D\frac{\partial}{\partial x}\Big(T(w)\frac{\partial p}{\partial x}\Big) \tag{2.11}$$

under the same scaling $\lim_{h\downarrow 0} \lambda h^2 = D > 0$.

We note that the mean waiting time of the particle at site n is given by $(\hat{T}_n^+ + \hat{T}_n^-)^{-1}$ in (2.9). In the case that it is independent of w and n, it holds that

$$\hat{T}_n^+(w) + \hat{T}_n^-(w) = 2\lambda,$$

where $\lambda > 0$ is a constant. If the barrier model is adopted here, then $\hat{T}_n^\pm(w) = \hat{T}(w_{n\pm 1/2})$ follows. These relations imply

$$\hat{T}_n^\pm(w) = 2\lambda \cdot \frac{\hat{T}(w_{n\pm 1/2})}{\hat{T}(w_{n+1/2}) + \hat{T}(w_{n-1/2})}.$$

The *renormalization* is the procedure of introducing a new jump process by replacing the right-hand side as

$$\hat{T}_n^\pm(w) = 2\lambda \cdot \frac{T(w_{n\pm 1/2})}{T(w_{n+1/2}) + T(w_{n-1/2})}$$

with some $T(w)$. Writing

$$\begin{aligned}
N^+(w_{n+1/2}, w_{n-1/2}) &= \frac{T(w_{n+1/2})}{T(w_{n+1/2}) + T(w_{n-1/2})}, \\
N^-(w_{n-1/2}, w_{n+1/2}) &= \frac{T(w_{n-1/2})}{T(w_{n-1/2}) + T(w_{n+1/2})},
\end{aligned}$$

we have

$$\hat{T}_n^\pm(w) = \begin{cases} 2\lambda N^+(w_{n+1/2}, w_{n-1/2}), \\ 2\lambda N^-(w_{n-1/2}, w_{n+1/2}), \end{cases}$$

and the master equation (2.9) is reduced to

$$\frac{1}{2\lambda}\frac{\partial p_n}{\partial t} = N^+(w_{n-1/2}, w_{n-3/2})p_{n-1} + N^-(w_{n+1/2}, w_{n+3/2})p_{n+1} - \left\{N^+(w_{n+1/2}, w_{n-1/2}) + N^-(w_{n-1/2}, w_{n+1/2})\right\}p_n. \quad (2.12)$$

In this model, the sublattice is assumed to be homogeneous so that $N^\pm$ is independent of n. Letting $N(u, v) = N^+(u, v)$, we have $N^-(v, u) = 1 - N(u, v)$ and

$$N(u, v) = \frac{T(u)}{T(u) + T(v)}. \quad (2.13)$$

Putting $x = nh$, we see that (2.12) has the form

$$\frac{1}{\lambda}\frac{\partial p}{\partial t} = h^2\frac{\partial}{\partial x}\Big(\frac{\partial p}{\partial x} - 2p(N_u - N_v)\frac{\partial w}{\partial x}\Big) + o(h^2).$$

Therefore, under the scaling $\lim_{h\downarrow 0}\lambda h^2 = D > 0$ we obtain

$$\frac{\partial p}{\partial t} = D\frac{\partial}{\partial x}\Big(\frac{\partial p}{\partial x} - 2p\left(N_u - N_v\right)\frac{\partial w}{\partial x}\Big). \quad (2.14)$$

Here, we have

$$N_u(w, w) = \frac{T(v)T'(u)}{\big(T(u) + T(v)\big)^2}\bigg|_{u=v=w} = \frac{1}{4}\big(\log T(w)\big)'$$

and $N_v(w, w) = -N_u(w, w)$ by (2.13) so that equation (2.14) is written as

$$\frac{\partial p}{\partial t} = D\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x} - p\frac{\partial}{\partial x}\log T(w)\right).$$

In the n space dimensions, we have

$$\frac{\partial p}{\partial t} = D\nabla\cdot\left(\nabla p - p\nabla\log T(w)\right).$$

This is the form of the first equation of (1.1), and there the chemotactic sensitivity function and the average particle velocity are given by

$$\chi(w) = D\big(\log T(w)\big)'$$

and

$$\boldsymbol{v} = -D\nabla\log p + D\big(\log T(w)\big)'\nabla w,$$

respectively.

3
Fundamental Theorem

We study the system of chemotaxis, or the adiabatic limit of the Fokker–Planck equation, and thus, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $V = \log W$ stands for the potential of the outer force, where $W = W(x) > 0$ is a smooth function of $x \in \overline{\Omega}$.

This system is also involved with the parameter $\tau \geq 0$ and the self-adjoint operator $A > 0$ in $L^2(\Omega)$ with the compact resolvent, defining the relaxation time and the field formation, respectively. Then, this system is given by

$$\begin{aligned} u_t = \nabla \cdot \big(\nabla u - u\nabla\left(v + \log W\right)\big) \quad &\text{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu} u - u\frac{\partial}{\partial \nu}\left(v + \log W\right) = 0 \quad &\text{on} \quad \partial\Omega \times (0, T), \\ \tau \frac{d}{dt} v + Av = u \quad &\text{for} \quad t \in (0, T), \end{aligned} \tag{3.1}$$

where $u = u(x, t)$ and $v = v(x, t)$ are unknown functions of $(x, t) \in \overline{\Omega} \times [0, T)$. The initial value is provided with

$$u|_{t=0} = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{3.2}$$

and if we take the full system of $\tau > 0$, then the additional initial value

$$v|_{t=0} = v_0(x) \quad \text{in} \quad \Omega \tag{3.3}$$

is also prescribed.

The operator A can be $-\Delta+a$ with the Neumann boundary condition, where $a > 0$ is a constant. It may be $-\Delta$ with the Neumann boundary condition under the constraint $\int_\Omega \cdot \, dx = 0$, that is, $Av = u$ if and only if

$$\begin{aligned} -\Delta v &= u - \frac{1}{|\Omega|}\int_\Omega u\,dx && \text{in} \quad \Omega \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on} \quad \partial\Omega, \quad \int_\Omega v\,dx = 0. \end{aligned} \tag{3.4}$$

In the third case, it is $-\Delta$ with the Dirichlet boundary condition.

These cases are studied by [38, 73, 106], and are referred to as the (N), (JL), and (D) fields, respectively, in this book. Excluding the boundary blowup of the solution to the (D) field is open. Except for it, we do not have any essential differences in these fields in the study of the quantized blowup mechanism of the simplified systems.

Unique solvability of (3.1) with (3.2) and (3.3) locally in time is more or less known [152]. In this chapter we follow the method of [150] and give a scheme valid for the semiabstract system, (3.1)–(3.3). Because the simplified system is easier to handle, we will concentrate on the full system.

We propose for A to be regarded as an operator in $L^p(\Omega)$ for $p \in (1,\infty)$, satisfying

$$\left\| A^{-1} \right\|_{L^p(\Omega),W^{2,p}(\Omega)} \leq A(p) \tag{3.5}$$

with a constant $A(p) > 0$. Here and henceforth, $W^{m,p}(\Omega)$ denotes the usual Sobolev space composed of the functions defined on Ω whose derivatives up to mth order belong to $L^p(\Omega)$. The identity operator is denoted by I if it is necessary to indicate explicitly. Putting

$$A_\beta = A - \beta I$$

and $\Sigma_\omega = \left\{ z \in \mathbf{C} \,\middle|\, 0 \leq |\arg z| \leq \omega \right\}$, we suppose the existence of $\beta > 0$, $\omega \in (0,\pi/2)$, and $M \geq 1$ such that A_β is of type (ω, M) in $L^p(\Omega)$, that is to say, $\mathbf{C} \setminus \Sigma_\omega \subset \rho(A_\beta)$ holds with

$$\left\| z\left(zI - A_\beta\right)^{-1} \right\|_{L^p(\Omega),L^p(\Omega)} \leq M$$

for $z \in \mathbf{C} \setminus \Sigma_\omega$, and each $\varepsilon > 0$ admits $M_\varepsilon \geq M$ satisfying

$$\left\| z\left(zI - A_\beta\right)^{-1} \right\|_{L^p(\Omega),L^p(\Omega)} \leq M_\varepsilon$$

for $z \in \mathbf{C} \setminus \Sigma_{\omega+\varepsilon}$. Here and henceforth, $\rho(A_\beta) \equiv \mathbf{C} \setminus \sigma(A_\beta)$ denotes the resolvent set of A_β, so that $\sigma(A_\beta)$ indicates its spectrum.

These operator-theoretic assumptions guarantee the generation of the *analytic semigroup*

$$\left\{e^{-tA}\right\}_{t\geq 0}$$

in $L^p(\Omega)$, and the (N), (JL), and (D) fields actually satisfy them. (See Tanabe [168, 169].) For these concrete cases, on the other hand, the method of Sobolev and Morrey spaces has been proposed to show the unique solvability of (3.1) locally in time [14, 187]. This method requires more real analytic profiles than the assumed kernel $G(x, x')$ of A^{-1}, but reduces the assumption to initial values. Here, we use the operator-theoretic approach, because we are interested in the classical solution mostly, and we can reduce the real analytic assumption to A under the cost of the regularity of the initial value by this method. Thus, we show the following theorem.

Theorem 3.1 *Let* (3.5) *hold and* $A_\beta = A - \beta I$ *be of type* (ω, M) *in* $L^p(\Omega)$, *where* $p > \max\{2, n\}$, $\beta > 0$, $\omega \in (0, \pi/2)$, *and* $M \geq 1$. *Then, if the initial value is taken as*

$$(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}(L^p(\Omega)),$$

system (3.1) *with* $\tau > 0$ *admits a unique classical solution* (u, v) *locally in time, that is,* $u = u(x, t) \geq 0$ *is continuous on* $\overline{\Omega} \times [0, T]$ *and* $C^{2,1}$ *in* $\overline{\Omega} \times (0, T]$, $v = v(\cdot, t)$ *belongs to* $C^1\big([0, T], L^p(\Omega)\big)$ *and to* $C\big([0, T], A^{-1}(L^p(\Omega))\big)$ *for some* $T > 0$, *and* (u, v) *solves* (3.1) *with* (3.2) *and* (3.3). *Furthermore,* $u(x, t) > 0$ *holds for* $(x, t) \in \overline{\Omega} \times (0, T]$ *if* $u_0 \not\equiv 0$.

Proof: We put $\tau = 1$ for simplicity. Then, by

$$U = u \cdot \exp\left(-v - \log W\right),$$

system (3.1) is transformed into

$$\begin{aligned} U_t = \Delta U + \nabla\left(v + \log W\right) \cdot \nabla U - v_t \cdot U \quad &\text{in} \quad \Omega \times (0, T), \\ \frac{\partial U}{\partial \nu} = 0 \quad &\text{on} \quad \partial\Omega \times (0, T), \\ \frac{dv}{dt} + Av = U \cdot \exp\left(v + \log W\right) \quad &\text{for} \quad t \in (0, T) \end{aligned} \tag{3.6}$$

with the initial value

$$U|_{t=0} = U_0(x) \quad \text{and} \quad v|_{t=0} = v_0(x) \quad \text{in} \quad \Omega, \tag{3.7}$$

where $U_0 = u_0 \cdot \exp(-v_0 - \log W)$. If Δ_N denotes the differential operator Δ provided with the Neumann boundary condition, then they are reduced to the system of integral equations

$$U(t) = e^{t\Delta_N} U_0 + \int_0^t e^{(t-s)\Delta_N} \left[\nabla (v(s) + \log W) \cdot \nabla U(s) + v_t(s) \cdot U(s)\right] ds$$
$$v(t) = e^{-tA} v_0 + \int_0^t e^{-(t-s)A} \left[U(s) \cdot \exp(v(s) + \log W)\right] ds. \tag{3.8}$$

Here and henceforth, $\{e^{t\Delta_N}\}_{t\geq 0}$ and $\{e^{-tA}\}_{t\geq 0}$ denote the semigroups generated by Δ_N and $-A$, respectively.

In what follows, $p > \max\{2, n\}$ is fixed. Then, Sobolev's imbedding theorem guarantees $0 \leq u_0 \in W^{1,p}(\Omega) \subset C(\overline{\Omega})$ and $v_0 \in A^{-1}(L^p(\Omega)) \subset W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$. This implies $0 \leq U_0 \in W^{1,p}(\Omega)$.

To get the solution by the contraction mapping principle, we take

$$\begin{aligned} B(L,T) = \Big\{ &(U,v) \in C\left([0,T], L^p(\Omega) \times L^p(\Omega)\right) \;\Big| \\ &\int_0^T \|U_t(t)\|_2^2 \, dt \leq L^2, \quad \sup_{t\in[0,T]} \|Av(t)\|_p \leq L, \\ &\sup_{t\in[0,T]} \|U(t)\|_{W^{1,p}(\Omega)} \leq L, \quad U(0) = U_0, \\ &v(0) = v_0, \quad \int_0^T \|v_t(t)\|_p^p \, dt \leq L^p \Big\} \end{aligned}$$

and set $\mathcal{F}(U,v) = (\mathcal{F}_1(U,v), \mathcal{F}_2(U,v))$, where $T, L > 0$ are constants and

$$\begin{aligned} \mathcal{F}_1(U,v)(t) &= e^{t\Delta_N} U_0 \\ &\quad + \int_0^t e^{(t-s)\Delta_N} \left[\nabla (v(s) + \log W) \cdot \nabla U(s) + v_t(s) \cdot U(s)\right] ds \\ \mathcal{F}_2(U,v)(t) &= e^{-tA} v_0 \\ &\quad + \int_0^t e^{-(t-s)A} \left[U(s) \cdot \exp(v(s) + \log W)\right] ds. \end{aligned}$$

Then the fixed point of $\mathcal{F}$ in $B(L,T)$ is obtained by the following lemma.

Lemma 3.1 *We have $L, T > 0$ satisfying*

$$\mathcal{F}(B(L,T)) \subset B(L,T) \tag{3.9}$$

and

$$\|\mathcal{F}(U_1, v_1) - \mathcal{F}(U_2, v_2)\|_{X(T)} \leq \frac{1}{2} \|(U_1, v_1) - (U_2, v_2)\|_{X(T)} \tag{3.10}$$

for $(U_1, v_1), (U_2, v_2) \in B(L, T)$*, where*

$$\|(U, v)\|_{X(T)} = \sup_{t\in[0,T]} \|U(t)\|_{W^{1,p}(\Omega)} + \sup_{t\in[0,T]} \|Av(t)\|_p + \left\{ \int_0^T \left\|U_t(t)\right\|_2^2 dt \right\}^{1/2} + \left\{ \int_0^T \left\|v_t(t)\right\|_p^p dt \right\}^{1/p}.$$

Proof: We can show that (3.9) and (3.10) are satisfied for $L \geq 1$, arbitrarily if $T > 0$ is taken to be sufficiently small. The proof of these relations is similar, and we only show the former.

The operator-theoretic features of $-\Delta_N$ necessary for the proof of this lemma are well known. Thus, there is a constant $M_1 > 0$ such that

$$\left\|(-\Delta_N + 1)^{1/2} e^{t\Delta_N}\right\|_{L^p(\Omega), L^p(\Omega)} \leq M_1 t^{-1/2}$$

for $t > 0$, and furthermore,

$$(-\Delta_N + 1)^{-1/2} \left(L^p(\Omega)\right) = W^{1,p}(\Omega)$$

and

$$\left\|e^{t\Delta_N}\right\|_{L^p(\Omega), L^p(\Omega)} \leq 1$$

hold true. On the other hand, the relation

$$A^{-1/2} \left(L^p(\Omega)\right) \subset W^{1,p}(\Omega) \subset C(\overline{\Omega}) \tag{3.11}$$

is obtained by (3.5) and the interpolation theory.

First, we note

$$\begin{aligned} \left\| (-\Delta_N + 1)^{1/2} \mathcal{F}_1(U, v)(t)\right\|_p \leq{}& \left\|e^{t\Delta_N} (-\Delta_N + 1)^{1/2} U_0\right\|_p \\ &+ \int_0^t \Big\| (-\Delta_N + 1)^{1/2} e^{(t-s)\Delta_N} \\ &\cdot \left[\nabla \left(v(s) + \log W\right) \cdot \nabla U(s) + v_t(s) \cdot U(s)\right] \Big\|_p ds. \end{aligned}$$

The first and the second terms of the right-hand side are estimated from above by

$$\left\| (-\Delta_N + 1)^{1/2} U_0\right\|_p$$

and

$$M_1 \int_0^t (t-s)^{-1/2} \left\| \nabla\,(v(s)+\log W)\cdot \nabla U(s) + v_t(s)\cdot U(s) \right\|_p ds,$$

respectively.

We have constants $K_i > 0$ $(i = 1, 2)$ determined by Ω and p, that is,

$$\left\| U \right\|_\infty \le K_1 \left\| U \right\|_{W^{1,p}(\Omega)}$$

and

$$\left\| \,(-\Delta_N + 1)^{1/2}\, U \right\|_p \le K_2 \left\| U \right\|_{W^{1,p}(\Omega)}.$$

From the first inequality we have

$$\|\nabla v\|_\infty \le K_1 A(p)\, \|Av\|_p\,.$$

Therefore it holds that

$$\begin{aligned}
&\left\| \nabla\big(v(s)+\log W\big)\cdot \nabla U(s) + v_t(s)\cdot U(s) \right\|_p \\
&\qquad \le \left\| \nabla\big(v(s)+\log W\big) \right\|_\infty \cdot \left\| \nabla U(s) \right\|_p + \left\| v_t(s) \right\|_p \cdot \left\| U(s) \right\|_\infty \\
&\qquad \le K_1\big(A(p)\left\| Av(s) \right\|_p + \left\| \nabla \log W \right\|_\infty + \left\| v_t(s) \right\|_p\big) \left\| U(s) \right\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

For $(U, v) \in B(L, T)$ this implies

$$\begin{aligned}
\left\| (-\Delta_N + 1)^{1/2}\, \mathcal{F}_1(U,v)(t) \right\|_p \le{} & \left\| (-\Delta_N + 1)^{1/2}\, U_0 \right\|_p \\
& + 2T M_1 K_1 \left(A(p) L + \|\nabla \log W\|_\infty \right) L \\
& + M_1 K_1 L \int_0^t (t-s)^{-1/2}\, \|v_t(s)\|_p\, ds.
\end{aligned}$$

Here, the last term of the right-hand side is estimated from above by

$$M_1 K_1 L^2 \cdot \left(\int_0^t (t-s)^{-\frac{1}{2}\cdot\frac{p}{p-1}}\, ds \right)^{\frac{p-1}{p}},$$

and we end up with

$$\begin{aligned}
\|\mathcal{F}_1(U,v)(t)\|_{W^{1,p}(\Omega)} \le{} & K_2\, \|U_0\|_{W^{1,p}(\Omega)} \\
& + 2T M_1 K_1 \left(A(p) L + \|\nabla \log W\|_\infty \right) L \\
& + \frac{2(p-1)}{p-2} \cdot T^{\frac{1}{2}-\frac{1}{p}} M_1 K_1 L^2
\end{aligned}$$

for $t \in [0, T]$. Therefore, if we take $L > 0$ as large as

$$K_2 \|U_0\|_{W^{1,p}(\Omega)} \le L/2,$$

and then take $T > 0$ as small as

$$2TM_1K_1\big(A(p)L + \|\nabla \log W\|_\infty\big)L + \frac{2(p-1)}{p-2} \cdot T^{\frac{1}{2}-\frac{1}{p}} M_1K_1L^2 \le L/2,$$

it holds that

$$\sup_{t\in[0,T]} \|\mathcal{F}_1(U,v)(t)\|_{W^{1,p}(\Omega)} \le L.$$

The function $W = \mathcal{F}_1(U, v)$ solves

$$W_t = \Delta W + \nabla (v + \log W) \cdot \nabla U - v_t \cdot U \quad \text{in} \quad \Omega \times (0,T),$$
$$\frac{\partial W}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

Therefore, testing W_t, we get

$$\begin{aligned}\int_0^T \|W_t\|_2^2 dt &\le \frac{1}{2}\|\nabla U_0\|_2^2 \\ &\quad + \int_0^T \big(\|\nabla (v + \log W) \cdot \nabla U\|_2 + \|v_t \cdot U\|_2\big) \cdot \|W_t\|_2 dt \\ &\le \frac{1}{2}\|\nabla U_0\|_2^2 + \frac{1}{2}\int_0^T \|W_t\|_2^2 dt \\ &\quad + \frac{1}{2}\int_0^T \Big(\|\nabla (v + \log W) \cdot \nabla U\|_2^2 + \|v_t \cdot U\|_2^2\Big)\, dt.\end{aligned}$$

This implies

$$\begin{aligned}\int_0^T \|W_t\|_2^2 dt &\le \|\nabla U_0\|_2^2 \\ &\quad + \int_0^T \Big(\|\nabla (v + \log W) \cdot \nabla U\|_2^2 + \|v_t \cdot U\|_2^2\Big)\, dt \\ &\le \|\nabla U_0\|_2^2 + T\Big\{\sup_{t\in[0,T]} \|\nabla v(t)\|_\infty + \|\nabla \log W\|_\infty\Big\}^2 \\ &\quad \cdot \sup_{t\in[0,T]} \|\nabla U(t)\|_2^2 \\ &\quad + \sup_{t\in[0,T]} \|U(t)\|_\infty^2 \cdot \left(\int_0^T \|v_t\|_2^p dt\right)^{2/p} \cdot T^{p/(p-2)},\end{aligned}$$

and hence

$$\int_0^T \|W_t\|_2^2 dt \leq \|\nabla U_0\|_2^2$$
$$+T\left(K_1 A(p)L + \|\nabla \log W\|_\infty\right)^2 K_1^2 L^2 + T^{p/(p-2)} K_1^2 L^4$$

follows. Again, taking $L >$ as large as $\|\nabla U_0\|_2^2 \leq L/2$ and then $T >$ as small as

$$T\big(K_1 A(p)L + \|\nabla \log W\|_\infty\big)^2 K_1^2 L^2 + T^{p/(p-2)} K_1^2 L^4 \leq L/2,$$

we have

$$\int_0^T \|W_t\|_2^2 dt \leq L$$

for any $(U, v) \in B(L, T)$.

We turn to the estimates for $Z = \mathcal{F}_2(U, v)$. First, we note

$$\begin{aligned} \|AZ(t)\|_p &\leq \|e^{-tA} A v_0\|_p \\ &\quad + \left\| \int_0^t A e^{-(t-s)A} \left[U(t) \cdot e^{v(t)+\log W}\right] ds \right\|_p \\ &\quad + \int_0^T \|A e^{-(t-s)A}\|_{L^p(\Omega), L^p(\Omega)} \|W\|_\infty \\ &\quad \cdot \left\| U(t) e^{v(t)} - U(s) e^{v(s)} \right\|_p ds. \end{aligned} \tag{3.12}$$

Here, the second term of the right-hand side is equal to

$$\big\|\big(e^{-tA} - I\big)\big[W \cdot U(t) e^{v(t)}\big]\big\|_p.$$

We have

$$\|e^{-tA}\|_{L^p(\Omega), L^p(\Omega)} \leq M_2$$

for $t > 0$ with a constant $M_2 > 0$, and this term is estimated from above by

$$(M_2 + 1) \cdot \|W\|_\infty \cdot \big(\|U_0 e^{v_0}\|_p + \|U(t) e^{v(t)} - U_0 e^{v_0}\|_p\big).$$

For the third term, we apply the smoothing property,

$$\|A e^{-tA}\|_{L^p(\Omega), L^p(\Omega)} \leq M_3 t^{-1} \tag{3.13}$$

for $t > 0$, where $M_3 > 0$ is a constant. Then, it is estimated from above by

$$M_3 \cdot \|W\|_\infty \cdot \int_0^t (t-s)^{-1} \left\| U(t)e^{v(t)} - U(s)e^{v(s)} \right\|_p ds.$$

Now, we note

$$\begin{aligned} \left\| U(t)e^{v(t)} - U(s)e^{v(s)} \right\|_p &\le \left\| U(t) - U(s) \right\|_p \cdot \exp\left(\|v(t)\|_\infty \right) \\ &+ \|U(s)\|_\infty \cdot \exp\left\{ \sup_{t\in[0,T]} \|v(t)\|_\infty \right\} \cdot \left\| \int_s^t v_t(\xi) d\xi \right\|_p. \end{aligned}$$

From $p > 2$, the first term of the right-hand side is estimated from above by

$$\begin{aligned} &\|U(t) - U(s)\|_\infty^{(p-2)/p} \cdot \|U(t) - U(s)\|_2^{2/p} \cdot \exp\left(\|v(t)\|_\infty \right) \\ &\le \left(\|U(t)\|_\infty + \|U(s)\|_\infty \right)^{(p-2)/p} \left| \int_s^t \|U_t(\xi)\|_2 d\xi \right|^{2/p} \exp\left(\|v(t)\|_\infty \right). \end{aligned}$$

This implies

$$\begin{aligned} &\left\| U(t)e^{v(t)} - U(s)e^{v(s)} \right\|_p \\ &\quad \le \left(\|U(t)\|_\infty + \|U(s)\|_\infty \right)^{(p-2)/p} \cdot \left\{ \int_0^T \|U_t\|_2^2 dt \right\}^{1/p} \\ &\quad \cdot |t-s|^{1/p} \cdot \exp\left\{ \sup_{t\in[0,T]} K_1 A(p) \|Av(t)\|_p \right\} \\ &\quad + \|U(s)\|_\infty \cdot \exp\left\{ K_1 A(p) \sup_{t\in[0,T]} \|Av(t)\|_p \right\} \\ &\quad \cdot \left\{ \int_0^T \|v_t\|_p^p dt \right\}^{1/p} \cdot |t-s|^{1-1/p} \end{aligned}$$

and therefore for $(U, v) \in B(L, T)$, we have

$$\begin{aligned} &\left\| U(t)e^{v(t)} - U(s)e^{v(s)} \right\|_p \\ &\quad \le (2K_1 L)^{(p-2)/p} |t-s|^{1/p} L^{2/p} \cdot \exp\left(K_1 A(p) L \right) \\ &\quad + K_1 L \cdot \exp\left(K_1 A(p) L \right) \cdot L \cdot |t-s|^{1-1/p}. \end{aligned}$$

These estimates are summarized by

$$\begin{aligned} &\left\| U(t)e^{v(t)} - U(s)e^{v(s)} \right\|_p \\ &\quad \le a_1(L)(t-s)^{1/p} + a_2(L)(t-s)^{1-1/p}, \end{aligned}$$

where $a_i(L) > 0$ is a function of L for $i = 1, 2$.

Returning to (3.12), we obtain

$$\begin{aligned}\|AZ(t)\|_p &\le M_2\|Av_0\|_p + (M_2+1)\,\|W\cdot e^{v_0}U_0\|_p \\ &\quad + (M_2+1)\,\|W\|_\infty\cdot\left\{a_1(L)T^{1/p} + a_2(L)T^{1-1/p}\right\} \\ &\quad + M_3\|W\|_\infty\left\{pa_1(L)T^{1/p} + \frac{p}{p-1}a_2(L)T^{1-1/p}\right\}.\end{aligned}$$

Therefore, the condition

$$\sup_{t\in[0,T]}\|A\mathcal{F}_2(U,v)(t)\|_p \le L$$

holds if $L > 0$ is as large as

$$M_2\|Av_0\|_p + (M_2+1)\,\|W\cdot e^{v_0}U_0\|_p \le L/2,$$

and $T > 0$ is as small as

$$\begin{aligned}&(M_2+1)\,\|W\|_\infty\left\{a_1(L)T^{1/p} + a_2(L)T^{1-1/p}\right\} \\ &\qquad + M_3\|W\|_\infty\left\{pa_1(L)T^{1/p} + \frac{p}{p-1}a_2(L)T^{1-1/p}\right\} \le L/2.\end{aligned}$$

Finally, using

$$\frac{dZ}{dt} + AZ = e^{v+\log W}U,$$

we obtain

$$\begin{aligned}\left\{\int_0^T \|Z_t\|_p^p dt\right\}^{1/p} &\le \left\{\int_0^T \|AZ\|_p^p dt\right\}^{1/p} + \left\{\int_0^T \|W\cdot e^v U\|_p^p dt\right\}^{1/p} \\ &\le T^{1/p}\cdot\sup_{t\in[0,T]}\|AZ(t)\|_p + T^{1/p}\cdot\|W\|_\infty \\ &\qquad \cdot\exp\left\{\sup_{t\in[0,T]}\|v(t)\|_\infty\right\}\cdot\sup_{t\in[0,T]}\|U(t)\|_p \\ &\le T^{1/p}L + T^{1/p}\cdot\|W\|_\infty\cdot\exp(K_1L)\cdot K_1L.\end{aligned}$$

Thus, we have

$$\int_0^T \|\mathcal{F}_2(U,v)\|_p^p dt \le L^p$$

for $(U, v) \in B(L, T)$ for $T > 0$ sufficiently small, and the proof is complete. □

Now we complete the following proof.

Proof of Theorem 3.1: From Lemma 3.1, we have a solution to (3.8) in $(U, v) \in B(L, T)$ for $T > 0$ sufficiently small. It becomes a classical solution to (3.6) with (3.7) by $U_0 \in W^{1,p}(\Omega)$, $v_0 \in A^{-1}(L^p(\Omega))$, and $p > n$, and consequently, the solution to (3.1) with (3.2) and (3.3) is obtained.

The positivity of $u(x, t)$ follows from the strong maximum principle applied to the first equation of (3.6). Finally, uniqueness of the solution to (3.8) is a consequence of the proof of Lemma 3.1.

In fact, if (U_1, v_1), $(U_2, v_2) \in B(L, T)$ are fixed point of $\mathcal{F}$ for some $L, T > 0$, then $(U_1, v_1)(t) = (U_2, v_2)(t)$ follows for $t \in [0, T_1]$ with some $T_1 \in (0, T)$. Now, the continuation argument gives $(U_1, v_1)(t) = (U_2, v_2)(t)$ for $t \in [0, T]$. This implies the uniqueness of the solution to (3.1), because any solution with the required regularity is transformed into a fixed point of $\mathcal{F}$ on $B(L, T)$ for some $L, T > 0$. The proof is complete. □

We proceed to the second topic of this chapter. That is, system (3.1) satisfies the standard criterion for the blowup of the solution, and hence $T_{\max} < +\infty$ implies $\lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = +\infty$. Moreover, it is shown that there is $p > 1$ such that if

$$\left\|u(t)\right\|_p \leq C$$

holds with a constant $C > 0$ independent of $t \in [0, T_{\max})$, then we have

$$T_{\max} = +\infty \quad \text{and} \quad \sup_{t \geq 0} \left\|u(t)\right\|_\infty < +\infty.$$

Furthermore, we show that this uniform estimate assures the compactness of the semiorbit

$$\mathcal{O} = \left\{ (u(t), v(t)) \right\}_{t \geq 1}$$

in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. The proof for these facts is given by [14, 50, 110] based on *Moser's iteration scheme* [2], but here we make use of the *maximal regularity theorem* of Dore and Venni [41].

In fact, as a special case, this theorem assures that if the *maximal accretive operator A* in $L^p(\Omega)$ admits constants $B_p > 0$ and $\gamma \in (0, \pi/2)$ such that

$$\left\|A^{\iota s}\right\|_{L^p(\Omega), L^p(\Omega)} \leq B_p e^{\gamma|s|} \tag{3.14}$$

for $s \in \mathbf{R}$, and if $u = u(t) \in C([0, T), L^p(\Omega))$ solves

$$\frac{du}{dt} + Au = f$$

for $t \in [0, T)$ and

$$u(0) = 0,$$

then it holds that

$$\int_0^T \|u_t(t)\|_p^p dt + \int_0^T \|Au(t)\|_p^p dt \leq C(T, p) \int_0^T \|f(t)\|_p^p dt,$$

where $C(T, p) > 0$ is a constant determined by $T > 0$ and $p \in (1, \infty)$. (This constant $C(p, T)$ can be taken independently of T [57, 132], but this refined version is not necessary in later arguments.)

We also suppose

$$A^{-\alpha}(L^q(\Omega)) = W^{2\alpha,q}(\Omega) \tag{3.15}$$

for $q \in (1, \infty)$and $\alpha \in (0, 1/4)$. These conditions (3.14) and (3.15) are actually satisfied for the (N), (JL), and (D) fields [61].

Henceforth, $\| \cdot \|_{C^{m+\theta}(\Omega)}$ denotes the standard Schauder norm, where $m = 0, 1, \ldots$ and $\theta \in (0, 1)$. Then, we can show the following theorem.

Theorem 3.2 *Assume the hypotheses of Theorem 3.1 with $p > n + 2$, and let $T_{\max}$ be the supremum of the existence time $T > 0$ of the solution to* (3.1). *Suppose, furthermore, that* (3.14) *and* (3.15) *hold with $p > n+2$, $q \in (1, \infty)$, and $\alpha \in (0, 1/4)$. Finally, take*

$$(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}\left(L^p(\Omega)\right)$$

for $p > \max\{2, n\}$. Then, if $T_{\max} < +\infty$, we have

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_\infty = +\infty.$$

Conversely, if $T_{\max} = +\infty$ and $\limsup_{t\uparrow+\infty} \|u(t)\|_\infty < +\infty$, then it holds that

$$\sup_{t\geq 1} \left\{ \|u(t)\|_{C^{2+\theta}(\Omega)} + \|v(t)\|_{C^{2+\theta}(\Omega)} \right\} < +\infty, \tag{3.16}$$

where $\theta \in \left(0, \min\left\{\frac{1}{2}, 1 - \frac{n+2}{p}\right\}\right)$. On the other hand, if

$$\|u(t)\|_p \leq C$$

holds with $p > \max\{2, n\}$ and $C > 0$ independent of $t \in [0, T_{\max})$, then we have $T_{\max} = +\infty$ and $\sup_{t\geq 0} \|u(t)\|_\infty < +\infty$.

Proof: We take the case $\tau = 1$ without loss of generality. We shall show that if $T < T_{\max}$,

$$\ell = \sup_{t \in [0,T]} \left\| u(t) \right\|_\infty, \tag{3.17}$$

and $\delta \in (0, T)$, then there is a constant $C(\delta, \ell) > 0$ independent of T such that

$$\sup_{t \in (\delta, T)} \left\{ \left\| u(t) \right\|_{C^{2+\theta}(\Omega)} + \left\| v(t) \right\|_{C^{2+\theta}(\Omega)} + \left\| Av(t) \right\|_p \right\} \leq C'(\delta, \ell). \tag{3.18}$$

This implies

$$\left\| u \right\|_{C^{2+\theta, 1+\theta/2}(\Omega \times (\delta, T))} + \left\| v \right\|_{C^{2+\theta, 1+\theta/2}(\Omega \times (\delta, T))} \leq C(\delta, \ell)$$

from the standard theory [85].

Since the blowup time of the solution assured by Theorem 3.1 is estimated from below by $\|u_0\|_{W^{1,p}(\Omega)} + \|Av_0\|_p$, inequality (3.18) implies the first assertion of the theorem. In fact, taking a sequence $t_k \uparrow T_{\max}$ as that of initial times, from the above criterion we see that $T_{\max} < +\infty$ cannot occur in the case of

$$\sup_k \left\| u(t_k) \right\|_\infty < +\infty.$$

In other words, we can exclude the possibility

$$\liminf_{t \uparrow T_{\max}} \left\| u(t) \right\|_\infty < +\infty \quad \text{with} \quad T_{\max} < +\infty.$$

The second assertion of the theorem also follows from (3.18). In fact, if

$$T_{\max} = +\infty$$

and $\limsup_{t \uparrow +\infty} \left\| u(t) \right\|_\infty < +\infty$, then inequality (3.16) follows as a consequence of (3.18). The last assertion of the theorem is obtained by the proof of (3.18). In fact, it shows that ℓ of (3.17) is replaced by

$$\sup_{t \in [0,T]} \left\| u(t) \right\|_p$$

for $p > \max\{2, n\}$ in this inequality.

Thus, the main part of the above theorem is reduced to (3.18), which we prove by several lemmas. For the moment, the exponent p is taken in $p > n+2$. First, we show the following lemma.

Lemma 3.2 *The inequality*

$$\sup_{t\in[0,T]} \|v(t)\|_\infty \le C_0 \tag{3.19}$$

holds with a constant $C_0 > 0$ determined by ℓ.

Proof: The assumptions to A imply

$$\|A^{1/2}e^{-tA}\|_{L^p(\Omega),L^p(\Omega)} \le M_4 t^{-1/2}e^{-\beta_1 t}$$

with some $M_4 > 0$ and $\beta_1 \in (0, \beta)$. From the third equation of (3.1) this gives

$$\begin{aligned}\|A^{1/2}v(t)\|_p &\le M_2\|A^{1/2}v_0\|_p + M_4\int_0^t (t-s)^{-1/2}e^{-\beta_1(t-s)}\|u(s)\|_p ds \\ &\le M_2\|A^{1/2}v_0\|_p + M_4\int_0^\infty s^{1/2}e^{-\beta_1 s}ds\, |\Omega|^{1/p}\,\ell,\end{aligned}$$

and hence

$$\sup_{t\in[0,T]} \|A^{1/2}v(t)\|_p \le C_1 \tag{3.20}$$

follows with a constant $C_1 > 0$ determined by ℓ. This gives (3.19) by (3.11), and the proof is complete. □

Inequality (3.19) implies

$$\sup_{t\in[0,T]} \|U(t)\|_\infty \le \ell\cdot\exp\left(C_0 + \|\log W\|_\infty\right) \tag{3.21}$$

for $U = u\cdot\exp(-v-\log W)$. Now, we show the following lemma.

Lemma 3.3 *The inequality*

$$\sup_{t\in[\delta,T]} \|\nabla U(t)\|_p \le C_2 \tag{3.22}$$

holds with a constant $C_2 > 0$ determined by ℓ and $\delta \in (0, T)$.

Proof: We may suppose $T > 1$ and $\delta \in (0, T-1)$. Then, we shall prove that

$$\sup_{t \in [t_0, t_0+1]} \left\| \nabla U(t) \right\|_p \leq C_2 \tag{3.23}$$

for $t_0 \in [\delta, T-1]$. Here $C_2 > 0$ is a constant determined by ℓ and δ, and are independent of T and t_0. So are $C_i > 0$ ($i = 3, \ldots, 6$) prescribed later, and thus inequality (3.22) follows from (3.23).

First, we see that

$$v_1(t) = \int_{t_0-\delta}^{t} e^{-(t-s)A} u(s) ds$$

satisfies

$$\frac{dv_1}{dt} + Av_1 = u$$

for $t \in [t_0 - \delta, t_0 + 1]$ with $v_1(t_0 - \delta) = 0$, and therefore the maximal regularity theorem guarantees

$$\int_{t_0-\delta}^{t_0+1} \left\| \frac{dv_1}{dt} \right\|_p^p dt + \int_{t_0-\delta}^{t_0+1} \left\| Av_1 \right\|_p^p dt \leq C_3.$$

On the other hand, we have

$$v(t) = v_1(t) + v_2(t)$$

for $t \in [t_0 - \delta, T_{\max})$, where

$$v_2(t) = e^{-(t-t_0+\delta)A} v(t_0 - \delta).$$

Inequality (3.13) implies

$$\left\| \frac{dv_2}{dt} \right\|_p = \left\| Av_2(t) \right\|_p \leq M_3 \cdot 2\delta^{-1} \cdot |\Omega|^{1/p} C_0$$

for $t \in [t_0 - \delta/2, T]$, and hence

$$\int_{t_0-\delta/2}^{t_0+1} \left\| v_t \right\|_p^p dt + \int_{t_0-\delta/2}^{t_0+1} \left\| Av \right\|_p^p dt \leq C_4 \tag{3.24}$$

follows. This gives also

$$\int_{t_0-\delta/2}^{t_0+1} \left\| \nabla v(t) \right\|_\infty^p dt \leq \left(K_1 A(p) \right)^p C_4. \tag{3.25}$$

Let $t_1 = t_0 - \delta/2$. Then, from the first and the second equations of (3.6), for $t \in [0, 1 + \delta/2]$ we have

$$\begin{aligned}
\big\|\big(-\Delta_N + I\big)^{1/2} U(t+t_1)\big\|_p &\le \big\|\big(-\Delta_N + I\big)^{1/2} e^{t\Delta_N} U(t_1)\big\|_p \\
&+ \int_0^t \big\|(-\Delta_N + I)^{1/2} e^{(t-s)\Delta_N}\big[\nabla\big(v(s+t_1) + \log W\big) \\
&\qquad \cdot \nabla U(s+t_1) + v_t(s+t_1)\cdot U(s+t_1)\big]\big\|_p\, ds \\
&\le M_1 t^{-1/2} \cdot |\Omega|^{1/p} \cdot \ell \cdot \exp\big(C_0 + \|\log W\|_\infty\big) \\
&+ M_1 \int_0^t (t-s)^{-1/2} \|\nabla\,(v(s+t_1) + \log W)\,\|_\infty \cdot\ \big\|\nabla U(s+t_1)\big\|_p\, ds \\
&+ M_1 \int_0^t (t-s)^{-1/2} \big\|U(s+t_1)\big\|_\infty\ \big\|v_t(s+t_1)\big\|_p ds.
\end{aligned}$$

Here, by (3.25) we have

$$\begin{aligned}
&\int_0^t (t-s)^{-1/2} \|\nabla\,(v(s+t_1) + \log W)\,\|_\infty \big\|\nabla U(s+t_1)\big\|_p ds \\
&\le \left\{ \int_{t_0-\delta/2}^{t_0+1} \big(\|\nabla v(t)\|_\infty + \|\nabla \log W\|_\infty\big)^p\, dt \right\}^{1/p} \\
&\quad \cdot \left\{ \int_0^t (t-s)^{-\frac{p}{2(p-1)}} \big\|\nabla U(s+t_1)\big\|_p^{\frac{p}{p-1}} ds \right\}^{(p-1)/p} \\
&\le \Big(K_1 A(p) C_4^{1/p} + (1+\delta/2)^{1/p}\, \|\nabla \log W\|_\infty\Big) \\
&\quad \cdot \left\{ \int_0^t (t-s)^{-\frac{p}{2(p-1)}} \big\|\nabla U(s+t_1)\big\|_p^{\frac{p}{p-1}} ds \right\}^{(p-1)/p}
\end{aligned}$$

and by (3.21) and (3.24)

$$\begin{aligned}
&\int_0^t (t-s)^{-1/2} \big\|U(s+t_1)\big\|_\infty \big\|v_t(s+t_1)\big\|_p ds \\
&\quad \le \ell \cdot \exp\big(C_0 + \|\log W\|_\infty\big) \cdot \left\{ \int_0^t (t-s)^{-\frac{p}{2(p-1)}} ds \right\}^{(p-1)/p} \cdot C_4^{1/p}.
\end{aligned}$$

Furthermore, it holds that

$$\phi(t) \equiv \big\|\nabla U(t+t_1)\big\|_p \le K_2 \big\|\,(-\Delta_N + I)^{1/2}\, U(t+t_1)\big\|_p.$$

Therefore, for $\alpha = p/2(p-1) \in (0, 1)$ we obtain

$$\phi(t) \le C_5 \left\{ t^{-1/2} + \int_0^t (t-s)^{-\alpha} \phi(s)^{2\alpha} ds \right\}^{1/(2\alpha)}.$$

Then, Gronwall's lemma implies

$$\phi(t) \leq C_6 \left(t^{-1/2} + 1\right)$$

for $t \in (0, 1+\delta/2]$. Restricting t in $[\delta/2, 1+\delta/2]$, we get (3.23), and the proof is complete. □

Now, we show the following lemma.

Lemma 3.4 *It holds that*

$$\sup_{t\in[\delta,T]} \left\{ \left\| v(t) \right\|_{C^{2+\theta}(\Omega)} + \left\| Av(t) \right\|_{C^{\theta}(\Omega)} \right\} \leq C_7 \tag{3.26}$$

with a constant $C_7 > 0$ determined by $\theta \in \left(0, \min\{\frac{1}{2}, 1 - \frac{n}{p}\}\right)$, ℓ, and $\delta \in (0, T)$.

Proof: Similarly to the proof of the previous lemma, we suppose $T > 1$, take $\delta \in (0, T-1)$, and show

$$\sup_{t\in[t_0,t_0+1]} \left\| Av(t) \right\|_{C^{\theta}(\Omega)} \leq C_7 \tag{3.27}$$

for $t_0 \in [\delta, T-1]$. The constants $C_i > 0$ $(i = 8, \ldots, 13)$ given below are determined similarly by ℓ and δ, and are independent of t_0 and T. Therefore, (3.26) will follow.

First, for $\theta_1 \in \left(0, \min\{\frac{1}{2}, 1 - \frac{n}{p}\}\right)$ we have

$$\sup_{t\in[t_0-\delta/2,t_0+1]} \left\{ \left\| U(t) \right\|_{C^{\theta_1}(\Omega)} + \left\| v(t) \right\|_{C^{\theta_1}(\Omega)} \right\} \leq C_8$$

by (3.20)–(3.22) with δ replaced by $\delta/2$. This implies

$$\sup_{t\in[t_0-\delta/2,t_0+1]} \left\| u(t) \right\|_{C^{\theta_1}(\Omega)} \leq C_9$$

and hence

$$\sup_{t\in[t_0-\delta/2,t_0+1]} \left\| u(t) \right\|_{W^{\theta_1,q}(\Omega)} \leq C_{10}$$

follows for $q > 1$. For $\alpha = \theta_1/2 \in (0, 1/4)$ we have

$$\sup_{t\in[t_0-\delta/2,t_0+1]} \left\| A^{\alpha} u(t) \right\|_q \leq C_{11}$$

by (3.15). Therefore, writing

$$A^{1+\gamma} v(t+t_1) = A^{1+\gamma} e^{-tA} v(t_1) + \int_0^t A^{1-\alpha+\gamma} e^{-(t-s)A} \cdot A^\alpha u(s+t_1)\, ds \tag{3.28}$$

with $t_1 = t_0 - \delta/2$, we get

$$\sup_{t\in[t_0,t_0+1]} \left\| A^{1+\gamma} v(t) \right\|_q \leq C_{12} \tag{3.29}$$

for any $\gamma \in (0, \alpha)$ because

$$\left\| A^\beta e^{-tA} \right\|_{L^q(\Omega), L^q(\Omega)} \leq M_\beta t^{-\beta}$$

holds for $\beta \geq 0$ with $M_\beta > 0$.

We have

$$A^{-\gamma}\left(L^q(\Omega)\right) \subset W^{2\gamma,q}(\Omega) \subset C^\theta(\Omega)$$

and

$$A^{-1-\gamma}\left(L^q(\Omega)\right) \subset W^{2+2\gamma,q}(\Omega) \subset C^{2+\theta}(\Omega)$$

if $\theta = 2\gamma - \frac{n}{q} > 0$. This condition holds for large q, and then

$$\sup_{t\in[t_0,t_0+1]} \left\{ \left\| Av(t) \right\|_{C^\theta(\Omega)} + \left\| v(t) \right\|_{C^\theta(\Omega)} \right\} \leq C_{13}$$

follows from (3.29). The exponent θ can be taken arbitrarily in $\theta \in (0, 2\alpha)$, and the proof of (3.26) is complete. □

Now, we proceed to the following lemma.

Lemma 3.5 *We have*

$$\sup_{t\in[\delta,T]} \left\| U(t) \right\|_{C^{1+\theta}(\Omega)} \leq C_{14} \tag{3.30}$$

for $\theta = 1 - \frac{n+2}{p}$ *with a constant* $C_{14} > 0$ *determined by* ℓ *and* $\delta \in (0, T)$.

Proof: Similarly to the proof of the previous lemma, we suppose $T > 1$, take $\delta \in (0, T-1)$, and show

$$\sup_{t\in[t_0,t_0+1]} \left\| U(t) \right\|_{C^\theta(\Omega)} \leq C_{14} \tag{3.31}$$

for $t_0 \in [\delta, T-1]$.

Letting $w = \nabla(v + \log W) \cdot \nabla U - v_t \cdot U$, we have

$$\int_{t_0-\delta/2}^{t_0} \|w(t)\|_p^p dt \leq C_{15}$$

by (3.26), (3.22) with δ replaced by $\delta/2$, (3.24), and (3.21). Similarly to (3.28), we have

$$(-\Delta_N + I)^\gamma U(t) = (-\Delta_N + I)^\gamma e^{t\Delta_N} U(t_1) + \int_0^t (-\Delta_N + I)^\gamma e^{(t-s)\Delta_N} w(s+t_1)\, ds \quad (3.32)$$

for $t_1 = t_0 - \delta/2$. The second term of the right-hand side is estimated as

$$\begin{aligned} &\left\| \int_0^t (-\Delta_N + I)^\gamma e^{(t-s)\Delta_N} w(s+t_1)\, ds \right\|_p \\ &\quad \leq \hat{M}_\gamma \int_0^t (t-s)^{-\gamma} \|w(s+t_1)\|_p\, ds \\ &\quad \leq \hat{M}_\gamma \left\{ \int_0^t (t-s)^{-\gamma \cdot \frac{p}{p-1}}\, ds \right\}^{(p-1)/p} \cdot C_{15}^{1/p} \end{aligned}$$

with $\hat{M}_\gamma > 0$ and hence

$$\sup_{t \in [t_0, t_0+1]} \|(-\Delta_N + I)^\gamma U(t)\|_p \leq C_{16}$$

follows for $\gamma \in \left[0, \frac{p-1}{p}\right)$. This gives

$$\sup_{t \in [\delta, T]} \|(-\Delta_N + I)^\gamma U(t)\|_p \leq C_{16} \quad (3.33)$$

and then (3.30), because $W^{2\gamma,p}(\Omega) \subset C^{1+\theta}(\Omega)$ holds for

$$\theta = \frac{2(p-1)}{p} - 1 - \frac{n}{p} = 1 - \frac{n+2}{p} > 0. \qquad \square$$

Now, we complete the proof of Theorem 3.2. First, we confirm that inequality (3.18) holds. In fact, it is a consequence of

$$\sup_{t\in[\delta,T]} \left\| U(t) \right\|_{C^{2+\theta}(\Omega)} \leq C_{17} \tag{3.34}$$

by (3.26).

To show (3.34), we note that

$$\sup_{t\in[\delta,T]} \left\| u(t) \right\|_{C^{1+\theta_2}(\Omega)} \leq C_{18}$$

holds for $\theta_2 \in \big(\min\{\frac{1}{2}, 1 - \frac{n+2}{p}\}\big)$ by (3.26) and (3.30). This implies

$$\sup_{t\in[\delta,T]} \left\| v_t(t) \right\|_{C^{\theta_2}(\Omega)} \leq C_{19}$$

by (3.26) and the third equation of (3.1). Therefore,

$$\sup_{t\in[\delta,T]} \left\| w(t) \right\|_{W^{\theta,q}(\Omega)} \leq C_{20}$$

follows for $q > 1$, where

$$w = \nabla (v + \log W) \cdot \nabla U - v_t \cdot U.$$

We have

$$\sup_{t\in[\delta,T]} \left\| (-\Delta_N + I)^{\alpha}\, w(t) \right\|_q \leq C_{21}$$

by $\alpha = \theta_2/2 < 1/4$. Then, the second term of the right-hand side of (3.33) is estimated as

$$\left\| \int_0^t (-\Delta_N + I)^{\gamma}\, e^{(t-s)\Delta_N} w(s + t_1)\, ds \right\|_q \leq \hat{M}_{\gamma-\alpha} \int_0^t (t-s)^{-\gamma+\alpha}\, ds \cdot C_{21}$$

with $\hat{M}_{\gamma-\alpha} > 0$, and hence

$$\sup_{t\in[\delta,T]} \left\| (-\Delta_N + I)^{\gamma}\, U(t) \right\|_q \leq C_{22}$$

follows for $\gamma \in [0, 1 + \alpha)$. We have

$$W^{2\gamma,q}(\Omega) \subset C^{2+\theta_3}(\Omega)$$

for $\theta_3 = 2\alpha - \frac{n}{q} = \theta_2 - \frac{n}{q}$. Taking large q, we have (3.34) for each $\theta \in \left(0, \min\{\frac{1}{2}, 1 - \frac{n+2}{p}\}\right)$. Thus, inequality (3.18) is proven.

To confirm the final part of Theorem 3.2, let us note that Lemma 3.4 holds under the assumption of Theorem 3.1. Thus, the solution (u, v) constructed in the previous theorem is in

$$(u(t), v(t)) \in C^2\left(\overline{\Omega}\right) \times A^{-1}\left(C^\theta(\Omega)\right)$$

for $t \in (0, T_{\max})$. In other words, the condition $p > n + 2$ is only necessary to confirm (3.14) in Theorem 3.2, and the initial values can be taken in

$$(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}\left(L^p(\Omega)\right)$$

for $p > \max\{2, n\}$. Furthermore, then, Lemma 3.2 holds if ℓ is replaced by

$$\sup_{t \in [0,T]} \left\| u(t) \right\|_p \tag{3.35}$$

for $p > n$, and the proof of Lemma 3.3 is valid similarly for $p > 2$. Thus, we get the final assertion of the theorem because the value given by (3.35) for $p > \max\{2, n\}$ can take place of ℓ. □

So far, we have confirmed the local well-posedness and the standard blowup criterion for (3.1). We conclude this chapter by a note on the convergence of the full system to the simplified system. We study this problem in an abstract setting, which is applicable to the concrete problem, say, (3.1) for the appropriately regular initial value u_0, v_0. Thus, if $-L$ and $-H$ are generators of the holomorphic semigroups on a Banach space X, denoted by $\{e^{-tL}\}_{t \geq 0}$ and $\{e^{-tH}\}_{t \geq 0}$, respectively, then this full system in the abstract form is indicated by

$$\begin{aligned} u_t + Lu &= N(u, v), \quad & u(0) &= u_0, \\ \tau v_t + Hu &= u, & v(0) &= v_0, \end{aligned}$$

where $\tau > 0$ and $N : X \times X \to X$ is a locally Lipschitz-continuous mapping.

Let $u^\tau = u^\tau(t),\ v^\tau = v^\tau(t) \in C\left([0, T], X\right)$ be its solution, that is,

$$\begin{aligned} u^\tau(t) &= e^{-tL} u_0 + \int_0^t e^{-(t-s)L} N\left(u^\tau(s), v^\tau(s)\right) ds, \\ v^\tau(t) &= e^{-\tau^{-1} tH} v_0 + \int_0^t e^{-\tau^{-1}(t-s)H} \tau^{-1} u^\tau(s)\, ds, \end{aligned} \tag{3.36}$$

for $t \in [0, T]$. Let $C > 0$ be such that

$$\max \left\{ \sup_{t \in [0,T]} \left\| u^\tau(t) \right\|, \sup_{t \in [0,T]} \left\| v^\tau(t) \right\| \right\} \leq C$$

for $0 < \tau \ll 1$, and take $L > 0$ in

$$\left\| N(u, v) - N(u', v') \right\| \leq L \left(\left\| u - v \right\| + \left\| u' - v' \right\| \right)$$

for $u, v, u', v' \in B(0, C) \subset X$. We also suppose the existence of $M > 0$ and $\delta > 0$ such that

$$\left\| e^{-tL} \right\| \leq M \quad \text{and} \quad \left\| e^{-tH} \right\| \leq M e^{-\delta t}$$

for $t \in [0, T]$ and $t \geq 0$, respectively. Finally, we assume the existence of $u = u(t), v = v(t) \in C\left([0, T], X\right)$ in

$$\max \left\{ \sup_{t \in [0,T]} \left\| u(t) \right\|, \sup_{t \in [0,T]} \left\| v(t) \right\| \right\} \leq C,$$

satisfying the simplified system in the sense that

$$\begin{aligned} u(t) &= e^{-tL} u_0 + \int_0^t e^{-(t-s)L} N\left(u(s), v(s)\right) ds, \\ v(t) &= H^{-1} u(t). \end{aligned} \tag{3.37}$$

Then, from the first equations of (3.36) and (3.36) we have

$$\left\| u^\tau(t) - u(t) \right\| \leq ML \cdot \int_0^t \left\{ \left\| u^\tau(s) - u(s) \right\| + \left\| v^\tau(s) - v(s) \right\| \right\} ds.$$

Next, the second equations of (3.36) and (3.37) are written as

$$\begin{aligned} v^\tau(t) &= e^{-\tau^{-1} t H} v_0 + \int_0^{\tau^{-1} t} e^{-sH} u^\tau(t - \tau s)\, ds, \\ v(t) &= \int_0^\infty e^{-sH} u(t)\, ds. \end{aligned}$$

Then, it holds that

$$\begin{aligned} v^\tau(t) - v(t) = e^{-\tau^{-1} t H} v_0 &+ \int_0^{\tau^{-1} t} \left\{ u^\tau(t - \tau s) - u(t - \tau s) \right\} ds \\ &+ \int_0^{\tau^{-1} t} e^{-sH} \left\{ u(t - \tau s) - u(t) \right\} ds + \int_{\tau^{-1} t}^\infty e^{-sH} u(t)\, ds, \end{aligned}$$

and each term of the right-hand side is estimated as follows:

$$\left\| e^{-\tau^{-1}tH} v_0 \right\| \le M e^{-\tau^{-1}t\delta} \left\| v_0 \right\|,$$

$$\left\| \int_0^{\tau^{-1}t} e^{-sH} \{u(t-\tau s) - u(t)\}\, ds \right\| \le M \int_0^{\tau^{-1}t} e^{-\delta s} \left\| u(t-\tau s) - u(t) \right\| ds,$$

$$\left\| \int_0^{\tau^{-1}t} e^{-sH} \left\{ u^\tau(t-\tau s) - u(t-\tau s) \right\} ds \right\| \le M \int_0^t \left\| u^\tau(s) - u(s) \right\| ds,$$

$$\left\| \int_{\tau^{-1}t}^{\infty} e^{-sH} u(t)\, ds \right\| \le C \int_{\tau^{-1}t}^{\infty} e^{-s\delta}\, ds = C\delta^{-1} e^{-\tau^{-1}\delta t}.$$

Thus, we obtain

$$\begin{aligned} &\left\| u^\tau(t) - u(t) \right\| + \left\| v^\tau(t) - v(t) \right\| \\ &\le M(L+1) \int_0^t \left\{ \left\| u^\tau(s) - u(s) \right\| + \left\| v^\tau(s) - v(s) \right\| \right\} ds + B^\tau(t) \end{aligned} \tag{3.38}$$

with

$$B^\tau(t) = M e^{-\tau^{-1}\delta t} \left\| v_0 \right\| + \delta^{-1} C e^{-\tau^{-1}\delta t} + M \int_0^{\tau^{-1}t} e^{-\delta s} \left\| u(t-\tau s) - u(t) \right\| ds.$$

Here, the first two terms of $B^\tau(t)$ converges to zero locally uniformly in $t \in (0, T]$, while the last term is estimated from above by

$$M \int_0^{\tau^{-1}T} e^{-\delta s} \sup_{t \in [0,T]} \left\| u(t-\tau s) - u(t) \right\| ds.$$

It also converges to zero by the uniform continuity of $u = u(t) \in C([0, T], X)$ and the dominated convergence theorem.

On the other hand, (3.38) is written as

$$g_\tau(t) \le a \int_0^t g_\tau(s)\, ds + B^\tau(s)$$

with $a = M(L+1)$ and $g_\tau(t) = \|u^\tau(t) - u(t)\| + \|v^\tau(t) - v(t)\|$, and therefore Gronwall's lemma guarantees

$$g_\tau(t) \leq a \int_0^t e^{-(t-s)a} B^\tau(s)\, ds + B^\tau(t)$$

for $t \in [0, T]$. We have $\sup_{t\in[0,T]} B^\tau(t) \leq C'$ with a constant $C' > 0$ independent of $0 < \tau \ll 1$, on the other hand, and therefore the dominated convergence theorem guarantees again

$$\lim_{\tau\downarrow 0} \|u^\tau(t) - u(t)\| = \lim_{\tau\downarrow 0} \|v^\tau(t) - v(t)\| = 0 \tag{3.39}$$

locally uniformly in $t \in (0, T]$.

Concluding the present chapter, we note that similarly, convergence of the generators $\{A^k\}_{k=1,2,\ldots}$ implies that of solutions. An interesting example is the convergence of the (N) filed to (JL) field as $a \downarrow 0$.

4
Trudinger–Moser Inequality

This chapter studies the existence of the solution to (3.1) globally in time:

$$
\begin{aligned}
u_t = \nabla \cdot \big(\nabla u - u \nabla (v + \log W) \big) \quad & \text{in} \quad \Omega \times (0, T), \\
\frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu} (v + \log W) = 0 \quad & \text{on} \quad \partial\Omega \times (0, T), \\
\tau \frac{d}{dt} v + A v = u \quad & \text{for} \quad t \in (0, T), \\
u\big|_{t=0} = u_0(x) \quad & \text{in} \quad \Omega,
\end{aligned}
\tag{4.1}
$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $W = W(x) > 0$ is a smooth positive function defined on $\overline{\Omega}$. The initial value $u_0 = u_0(x)$ is a nonnegative function not identically equal to 0, and in the case of $\tau > 0$, the additional initial condition $v|_{t=0} = v_0(x)$ is imposed. Then, we shall show that $\lambda = \|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$ in the case of $n = 2$.

More precisely, following Theorem 3.1, we suppose sufficient regularity to the initial value for unique existence of the classical solution locally in time. Then, $T_{\max} > 0$ denotes the blowup time, that is, the supremum of its existence time. Also, sometimes $\|\cdot\|$ and $(\ , \ )$ are written in place of the L^2 norm $\|\cdot\|_2$ and the L^2 inner product, respectively:

$$
\|v\| = \left\{ \int_\Omega v^2 \, dx \right\}^{1/2}, \quad (v, w) = \int_\Omega v w \, dx.
$$

We confirm that

$$\left\| u(t) \right\|_1 = \left\| u_0 \right\|_1 \qquad (t \in [0, T_{\max}))$$

follows from

$$\int_\Omega u_t \, dx = 0$$

and the positivity of the solution $u = u(x,t)$ to (4.1), $u(x,t) > 0$ holds for $(x,t) \in \overline{\Omega} \times (0, T_{\max})$. Furthermore, the Lyapunov function is given as

$$\mathcal{W}(u, v) = \int_\Omega \left(u(\log u - 1) - u \log W - uv \right) \, dx + \frac{1}{2} \left\| A^{1/2} v \right\|_2^2.$$

In fact, we can apply the argument of Chapter 1, and writing the first equation of (3.1) as

$$u_t = \nabla \cdot u \nabla \left(\log u - v - \log W \right),$$

we obtain

$$\frac{d}{dt} \mathcal{W}(u, v) + \tau \left\| v_t \right\|_2^2 + \int_\Omega u \left| \nabla \left(\log u - v - \log W \right) \right|^2 dx = 0 \tag{4.2}$$

for $t \in (0, T_{\max})$. Thus, $\mathcal{W}(u(t), v(t))$ is a nonincreasing function of t.

Based on the description of Chapter 1, we study the case $n = 2$ mostly. Results stated in this chapter are valid for both simplified and full systems. Remember that in the (D) field, A is $-\Delta$ with the Dirichlet boundary condition. In this case, if $\left\| u_0 \right\|_1 < 8\pi$ then

$$T_{\max} = +\infty \quad \text{and} \quad \sup_{t \geq 0} \left\| u(t) \right\|_\infty < +\infty \tag{4.3}$$

can be proven. On the other hand, in the (N) field, A is $-\Delta + a$ with the Neumann boundary condition, where $a > 0$ is a constant. Then, $\left\| u_0 \right\|_1 < 4\pi$ implies the same conclusion (4.3). This is also the case of the (JL) field, and in the next chapter we show that these results are optimal.

We make use of several versions of the *Trudinger–Moser inequality*. In fact, the result on the (N) and (JL) fields is associated with *Chang–Yang's inequality* [23],

$$\log \left(\frac{1}{|\Omega|} \int_\Omega e^v \, dx \right) \leq \frac{1}{8\pi} \left\| \nabla v \right\|_2^2 + \frac{1}{|\Omega|} \int_\Omega v \, dx + K \tag{4.4}$$

valid for $v \in H^1(\Omega)$, where K is a constant determined by $\Omega \subset \mathbf{R}^2$, a bounded domain with smooth boundary. On the other hand that on the (D) field is a consequence of the *Moser–Onofri inequality* [102],

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^v\,dx\right) \leq \frac{1}{16\pi}\|\nabla v\|_2^2 + 1 \tag{4.5}$$

valid for $v \in H_0^1(\Omega)$. This form (4.5) holds for any bounded domain $\Omega \subset \mathbf{R}^2$, and its optimality is shown by [108]. In this connection, we expect that $K = 2$ is optimal in (4.4). That is, (4.4) will hold with $K = 2$ if $\Omega \subset \mathbf{R}^2$ is bounded with smooth boundary $\partial\Omega$, and this value will not be improved. It is also worth mentioning that the constant 8π in the right-hand side of (4.4) must be reduced if Ω has corners. See [23] for more details.

Here, we shall spend a couple of pages for the background of the Trudinger–Moser inequality. Namely, we have

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega) & (1 \leq p < n), \\ C^{1-\frac{n}{p}}(\Omega) & (p > n) \end{cases}$$

for each open set $\Omega \subset \mathbf{R}^n$, where $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The former and the latter cases are referred to as the *Sobolev* and the *Morrey* imbeddings, respectively. In the critical case $p = n$, $W_0^{1,p}(\Omega)$ is imbedded into the Orlicz space, which was shown by Pohozaev [130] and Trudinger [177] independently. Moser [102] gave a sharp form, one of which is stated as follows. That is, there is a constant $C > 0$ such that if w is a smooth function defined on the two-dimensional unit sphere S^2, then

$$\int_{S^2} w\,dS = 0 \quad \text{and} \quad \|\nabla w\|_2 \leq 1$$

imply

$$\int_{S^2} e^{4\pi w^2} dS \leq C.$$

If $v \not\equiv 0$ satisfies $\int_{S^2} v\,dS = 0$, then for $w = v/\|\nabla v\|_2$ we have

$$\int_{S^2} e^{4\pi w^2} dS \leq C.$$

Here, we have

$$v = w\|\nabla v\|_2 \leq 4\pi w^2 + \frac{1}{16\pi}\|\nabla v\|_2^2$$

and hence it follows that

$$\int_{S^2} e^v \, dS \le C \cdot \exp\left(\frac{1}{16\pi}\left\|\nabla v\right\|_2^2\right).$$

Letting $K = \log(C/(4\pi))$, we have

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^v \, dS\right) \le \frac{1}{16\pi}\left\|\nabla v\right\|_2^2 + K. \tag{4.6}$$

As for general $v \in H^1(S^2)$, we take

$$v - \frac{1}{4\pi}\int_{S^2} v \, dS$$

for v in (4.6), and obtain that

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^v \, dS\right) \le \frac{1}{16\pi}\left\|\nabla v\right\|_2^2 + \frac{1}{4\pi}\int_{S^2} v \, dS + K. \tag{4.7}$$

Later, the best constant of this K is obtained as $K = 0$ by Onofri [126] and Hong [66], and that is the exact form of the original Moser–Onofri inequality:

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^v \, dS\right) \le \frac{1}{16\pi}\left\|\nabla v\right\|_2^2 + \frac{1}{4\pi}\int_{S^2} v \, dS,$$

valid for any $v \in H^1(S^2)$.

Moser [103] noticed that the term $\left\|\nabla v\right\|_2^2/(16\pi)$ in the right-hand side of (4.7) is replaced by $\left\|\nabla v\right\|_2^2/(32\pi)$ in the projective space $\mathcal{P}^2$, and proved the existence of the solution to Nirenberg's problem in this case. This problem has been studied extensively, and Chang–Yang's inequality (4.4) was presented in that context.

The term $\left\|\nabla v\right\|_2^2/(8\pi)$ in the right-hand side of this inequality is directly associated with the criterion, that $\left\|u_0\right\|_1 < 4\pi$ implies $T_{\max} = +\infty$ in (1.1). Its discrepancy to the conjecture $\left\|u_0\right\|_1 < 8\pi$ of [33] for $T_{\max} = +\infty$ actually occurs by the concentration toward the boundary of the solution. More precisely, this discrepancy is a consequence of the formation of the boundary collapse, and in this context smoothness of the boundary $\partial\Omega$ is essential in (4.4). Namely, the constant 8π in the right-hand side of this inequality must be reduced if $\partial\Omega$ has corners, and so is $\left\|u_0\right\|_1 < 4\pi$ accordingly in (1.1) for $T_{\max} = +\infty$.

Coming back to system (4.1), now we describe how these Trudinger–Moser inequalities are applied to its study. In fact, use of the free energy as a Lyapunov function in the study of the long-term behavior of the solution was adopted first for the semiconductor device equation (2.1) by [11, 15, 49, 99]. However, since (4.1) is not dissipative, the free energy is not always bounded from below in (4.1). Its boundedness is actually achieved by Chang–Yang's inequality (4.4) and Jensen's inequality under the constraint that

$$\|u_0\|_1 < 4\pi.$$

This fact was the starting point of [14, 50, 110] to establish $T_{\max} = +\infty$ in the case of $\|u_0\|_1 < 4\pi$. For the simplified system, we have an alternative proof using Brezis–Merle's inequality [18] and Young's inequality. This method was proposed by [108], and was adopted to the full system by [60]. In the final chapter of this book, we shall present the third proof, based on the dual form of the Trudinger–Moser inequality.

In this chapter, we describe the former argument for the simplified system of the (N) field, and show the following theorem.

Theorem 4.1 *If* $\lambda = \|u_0\|_1 < 4\pi$ *holds in*

$$\begin{aligned} u_t = \nabla \cdot \big(\nabla u - u \nabla (v + \log W)\big) \quad & in \quad \Omega \times (0, T), \\ 0 = \Delta v - av + u \quad & in \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu} (v + \log W) = 0 \quad & on \quad \partial\Omega \times (0, T), \\ \frac{\partial v}{\partial \nu} = 0 \quad & on \quad \partial\Omega \times (0, T), \\ u|_{t=0} = u_0(x) \quad & in \quad \Omega, \end{aligned} \tag{4.8}$$

then (4.3) *follows, where* $\Omega \subset \mathbf{R}^2$ *is a bounded domain with smooth boundary* $\partial\Omega$*, and* $W = W(x) > 0$ *and* $u_0 = u_0(x) \geq 0$ *are smooth functions defined on* $\overline{\Omega}$.

For the proof to perform, we set $a = 1$ for simplicity. Then, it holds that

$$\mathcal{W}(u, v) = \int_\Omega (u(\log u - 1 - \log W) - uv)\ dx + \frac{1}{2}\|v\|_{H^1}^2$$

for

$$\|v\|_{H^1} = \sqrt{\|\nabla v\|_2^2 + \|v\|_2^2}.$$

We put

$$\lambda = \|u_0\|_1 = \|u(t)\|_1 \qquad \big(t \in [0, T_{\max})\big) \tag{4.9}$$

and apply the L^1 *estimate* of Brezis and Strauss [19] to $Au = v$, where A denotes $-\Delta + 1$ with the Neumann boundary condition. See also Stampacchia [156, 157] for this type of elliptic estimate. In this case of two space dimensions, for each $q \in [1, 2)$ we have

$$\left\| v(t) \right\|_{W^{1,q}} \leq C_q \tag{4.10}$$

with a constant $C_q > 0$ independent of $t \in [0, T_{\max})$, where

$$\left\| v \right\|_{W^{1,q}} = \left(\left\| \nabla v \right\|_q^q + \left\| v \right\|_q^q \right)^{1/q}.$$

It holds also that

$$\mathcal{W}(u(t), v(t)) \leq \mathcal{W}(u_0, v_0) \tag{4.11}$$

for $0 \leq t < T_{\max}$, where $v_0 = A^{-1} u_0$.

We take

$$\mu = \mu(t) = \int_\Omega e^{bv}\, dx$$

with the constant $b > 0$ prescribed later. Using

$$\int_\Omega \frac{u}{\lambda}\, dx = 1,$$

we apply *Jensen's inequality* as

$$\begin{aligned} 0 &= -\log\left(\mu^{-1} \int_\Omega e^{bv}\, dx \right) = -\log\left(\int_\Omega \frac{\lambda}{u} \cdot \frac{e^{bv}}{\mu} \cdot \frac{u}{\lambda}\, dx \right) \\ &\leq \int_\Omega \left[-\log\left(\frac{\lambda}{u} \cdot \frac{e^{bv}}{\mu} \right) \right] \frac{u}{\lambda}\, dx \\ &= \frac{1}{\lambda} \int_\Omega \left(u \log u - u \log \lambda - buv + u \log \mu \right) dx. \end{aligned}$$

This means

$$0 \leq \int_\Omega (u \log u - buv)\, dx - \lambda \log \lambda + \lambda \log \mu. \tag{4.12}$$

On the other hand, we have (4.4) and hence it holds that

$$\begin{aligned} \log \mu &= \log\left(\frac{1}{|\Omega|} \int_\Omega e^{bv}\, dx \right) + \log |\Omega| \\ &\leq \frac{b^2}{8\pi} \left\| \nabla v \right\|_2^2 + \frac{b}{|\Omega|} \left\| v \right\|_1 + K + \log |\Omega| \\ &= \frac{b^2}{8\pi} \left\| \nabla v \right\|_2^2 + O(1) \end{aligned}$$

by (4.10). Combining this with (4.12), we obtain

$$\begin{aligned} 0 &\le \frac{b^2}{8\pi}\|\nabla v\|_2^2 \cdot \lambda + \int_\Omega \big(u(\log u - 1 - \log W) - buv\big)\,dx + O(1) \\ &= W(u,v) + (1-b)\int_\Omega uv\,dx - \Big(\frac{1}{2} - \frac{b^2}{8\pi}\lambda\Big)\|\nabla v\|_2^2 \\ &\quad - \frac{1}{2}\|v\|_2^2 + O(1). \end{aligned}$$

Henceforth, $C_i > 0$ $(i = 1, 2, \dots)$ denotes a sequence of constants independent of $t \in [0, T_{\max})$. Then, again by (4.11) we have

$$\Big(\frac{1}{2} - \frac{b^2}{8\pi}\lambda\Big)\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 + (b-1)\int_\Omega uv\,dx \le C_1.$$

In the case that

$$\lambda \equiv \|u_0\|_1 = \|u(t)\|_1 < 4\pi \qquad (0 \le t < T_{\max}),$$

we can take $b > 0$ in

$$b > 1 \quad \text{and} \quad \frac{1}{2} - \frac{b^2}{8\pi}\lambda > 0.$$

Then, we have

$$\|v(t)\|_{H^1} \le C_2, \qquad \int_\Omega (uv)(t)\,dx \le C_2,$$

and

$$\int_\Omega (u\log u)(t)\,dx \le W\big(u(t), v(t)\big) + \big(1 + \|\log W\|_\infty\big)\lambda + \int_\Omega (uv)(t)\,dx.$$

Hence it holds that

$$\int_\Omega (u\log u)(t)\,dx \le C_3 \quad \text{for} \quad t \in [0, T_{\max}). \tag{4.13}$$

Writing $u = Ww$ in (4.8), we have

$$\begin{aligned} w_t &= \nabla\cdot j + b\cdot j \quad &&\text{in} \quad \Omega\times(0,T), \\ 0 &= \Delta v - v + Ww \quad &&\text{in} \quad \Omega\times(0,T), \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad &&\text{on} \quad \partial\Omega\times(0,T), \\ u\big|_{t=0} &= u_0(x) \quad &&\text{in} \quad \Omega, \end{aligned} \tag{4.14}$$

with

$$b = \nabla \log W \quad \text{and} \quad j = \nabla w - w\nabla v.$$

In fact, we have

$$\frac{\partial}{\partial \nu} \log u - \frac{\partial}{\partial \nu} \log W = 0 \quad \text{on} \quad \partial\Omega$$

and hence $(\partial w)/(\partial \nu) = 0$ holds on the boundary. On the other hand, the first equation of (4.8) is equal to

$$\begin{aligned} W w_t &= \nabla \cdot (\nabla u - u\nabla v - w\nabla W) \\ &= \nabla \cdot (W\nabla w - Ww\nabla v) \\ &= W\nabla \cdot (\nabla w - w\nabla v) + \nabla W \cdot (\nabla w - w\nabla v) \end{aligned}$$

and hence (4.14) follows.

We have

$$\begin{aligned} \int_\Omega (\nabla \cdot j) w\, dx &= -\int_\Omega j \cdot \nabla w\, dx \\ &= -\|\nabla w\|_2^2 + \int_\Omega w\nabla v \cdot \nabla w\, dx \\ &= -\|\nabla w\|_2^2 - \frac{1}{2}\int_\Omega w^2 \Delta v\, dx \\ &= -\|\nabla w\|_2^2 - \frac{1}{2}\int_\Omega w^2 (v - Ww)\, dx \\ &\le -\|\nabla w\|_2^2 + \frac{1}{2}\|w\|_3^2 \|v\|_3 + \frac{1}{2}\|W\|_\infty \|w\|_3^3 \end{aligned}$$

and

$$\begin{aligned} \int_\Omega (b \cdot j) w\, dx &= \int_\Omega b \cdot (w\nabla w - w^2 \nabla v)\, dx \\ &= \int_\Omega \left(\frac{1}{2} b \cdot \nabla w^2 - w^2 b \cdot \nabla v\right) dx \\ &= -\int_\Omega \left(\frac{1}{2} w^2 \nabla \cdot b + w^2 b \cdot \nabla v\right) dx \\ &\le \|w\|_3^2 \left(\frac{1}{2}\|\nabla \cdot b\|_3 + \|b\|_\infty \|\nabla v\|_3\right). \end{aligned}$$

Now, we apply Sobolev's imbedding $W^{1,6/5}(\Omega) \subset L^3(\Omega)$ and the elliptic estimate. Then we obtain

$$\begin{aligned}\|v\|_{W^{1,3}} &= \left(\|\nabla v\|_3^3 + \|v\|_3^3\right)^{1/3} \le C_4\|v\|_{W^{2,6/5}} \\ &\le C_5\|Ww\|_{6/5} \le C_5\|W\|_\infty\|w\|_{6/5}.\end{aligned}$$

Therefore, multiplying w to the first equation of (4.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \|\nabla w\|_2^2 \le C_6\left(\|w\|_3^3 + 1\right). \tag{4.15}$$

We make use of the following lemma [15]. It is derived from *Gagliardo–Nirenberg's inequality* [48, 120]. In fact, we have $W^{1,1}(\Omega) \subset L^2(\Omega)$ if $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. From this fact, it is proven that each $1 \le q \le p < +\infty$ admits a constant $C_{p,q} > 0$ satisfying

$$\|w\|_p \le C_{p,q}\|w\|_q^{1-a} \cdot \|w\|_{H^1}^a \tag{4.16}$$

for any $w \in H^1(\Omega)$, where $a = 1 - \frac{q}{p}$. Then, inequality (4.16) implies the following lemma.

Lemma 4.1 *Any $\varepsilon > 0$ admits $C_\varepsilon > 0$ such that*

$$\|w\|_3^3 \le \varepsilon\|w\|_{H^1}^2\|w\log|w|\,\|_1 + C_\varepsilon\|w\|_1 \tag{4.17}$$

for $w \in H^1(\Omega)$.

Proof: We take $N > 1$ and $\chi \in C^\infty(\mathbf{R})$ satisfying

$$\chi(s) = \begin{cases} 0 & (|s| \le N), \\ 2 & (|s| - N), \\ |s| & (|s| \ge 2N). \end{cases}$$

By means of $|\chi(s)| \le |s|$, we have

$$\|\,|w| - \chi(w)\|_p^p \le 2^p \int_{\{|w|\le 2N\}} |w|^p \le 2^p \cdot (2N)^{p-1} \cdot \|w\|_1.$$

On the other hand, we have $|\chi(s)| \le 2$ and hence it holds that

$$\begin{aligned}\|\chi(w)\|_{H^1}^2 &= \|\nabla\chi(w)\|_2^2 + \|\chi(w)\|_2^2 \\ &= \|\chi'(w)\nabla w\|_2^2 + \|\chi(w)\|_2^2 \\ &= \|\chi'(w)\nabla w\|_2^2 + \|\chi(w)\|_2^2 \\ &\le 4\|\nabla w\|_2^2 + \|w\|_2^2 \le 4\|w\|_{H^1}^2.\end{aligned}$$

This means

$$\|\chi(w)\|_{H^1} \leq 2\|w\|_{H^1},$$

and by (4.16) we have

$$\begin{aligned}\|\chi(w)\|_p^p &\leq C_7\|\chi(w)\|_{H^1}^{p-1}\|\chi(w)\|_1 \\ &\leq C_7 2^{p-1}\|w\|_{H^1}^{p-1}(\log N)^{-1}\|w\log|w|\,\|_1.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|w\|_p^p &\leq 2^p\left\{\|\,|w|-\chi(w)\|_p^p+\|\chi(w)\|_p^p\right\} \\ &\leq C_8\left\{N^{p-1}\|w\|_1+(\log N)^{-1}\|w\|_{H^1}^{p-1}\|w\log|w|\,\|_1\right\}.\end{aligned}$$

Then, inequality (4.17) is obtained by making N large for the case of $p = 3$. □

We have *Poincaré–Wirtinger's inequality*

$$\|\nabla w\|_2^2 \geq \mu_2\left\|w-\frac{1}{|\Omega|}\int_\Omega w\,dx\right\|_2^2 \tag{4.18}$$

with $\mu_2 > 0$ being the second eigenvalue of $-\Delta$ in Ω under the Neumann boundary condition. This implies

$$\begin{aligned}\|w\|_2 &\leq \mu_2^{-1/2}\|\nabla w\|_2+|\Omega|^{-1}\,\|w\|_1 \\ &\leq \mu_2^{-1/2}\|\nabla w\|_2+|\Omega|^{-1}\,\|W^{-1}\|_\infty\lambda.\end{aligned}$$

By (4.13) it holds that

$$\int_\Omega (w\log w)(t)\,dx \leq C_9 \quad \text{for} \quad t\in[0,T_{\max}).$$

Therefore, using (4.17), we see that any $\varepsilon > 0$ admits $C_\varepsilon > 0$ satisfying

$$\|w\|_3^3 \leq \varepsilon\|\nabla w\|_2^2 + C_\varepsilon,$$

and it follows that

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2+\|\nabla w\|_2^2 \leq \frac{1}{2}\|\nabla w\|_2^2 + C_{10}$$

by (4.15). Again by (4.18) we have

$$\frac{d}{dt}\|w\|_2^2+\mu_2\|w\|_2^2 \leq C_{11}$$

and hence

$$\left\| w(t) \right\|_2 \leq C_{12} \qquad (0 \leq t < T_{\max}) \tag{4.19}$$

holds true.

Now, we multiply w^2 to the first equation of (4.14). We have

$$\begin{aligned}\int_\Omega (\nabla \cdot j) w^2 \, dx &= -\int_\Omega j \cdot \nabla w^2 \, dx \\ &= -2 \int_\Omega w \left| \nabla w \right|^2 \, dx + 2 \int_\Omega w^2 \nabla v \cdot \nabla w \, dx\end{aligned}$$

and writing $w_1 = w^{3/2}$, we obtain

$$\frac{1}{3} \frac{d}{dt} \int_\Omega w_1^2 \, dx + \frac{8}{9} \int_\Omega \left| \nabla w_1 \right|^2 \, dx = \frac{4}{3} \int_\Omega w_1 \nabla v \cdot \nabla w_1 \, dx + \int_\Omega (b \cdot j) w \, dx.$$

Here, the first term of the right-hand side is treated similarly as before, and we obtain

$$\begin{aligned}\int_\Omega w_1 \nabla v \cdot \nabla w_1 \, dx &= \frac{1}{2} \int_\Omega \nabla v \cdot \nabla w_1^2 \, dx \\ &= -\frac{1}{2} \int_\Omega \Delta v \cdot w_1^2 \, dx = \frac{1}{2} \int_\Omega (Ww - v) w_1^2 \, dx \\ &\leq \left\| W \right\|_\infty \frac{1}{2} \left\| w_1 \right\|_{8/3}^{8/3} + \frac{1}{2} \left\| w_1 \right\|_3^2 \left\| v \right\|_3 \\ &\leq \frac{1}{2} \left\| W \right\|_\infty \left\| \Omega \right\|^{1/9} \left\| w_1 \right\|_3^{8/3} + \frac{1}{2} \left\| w_1 \right\|_3^2 \left\| v \right\|_3.\end{aligned}$$

Similarly, for the second term we have

$$\begin{aligned}\int_\Omega (b \cdot j) w^2 \, dx &= \int_\Omega b \cdot \left(w^2 \nabla w - w^3 \nabla v \right) \, dx \\ &= \int_\Omega b \cdot \left(\frac{1}{3} \nabla w^3 - w^3 \nabla v \right) \, dx \\ &= -\int_\Omega w^3 \left(\frac{1}{3} \nabla \cdot b + b \cdot \nabla v \right) \, dx \\ &= -\int_\Omega w_1^2 \left(\frac{1}{3} \nabla \cdot b + b \cdot \nabla v \right) \, dx \\ &\leq \left\| w_1 \right\|_3^2 \left(\frac{1}{3} \left\| \nabla b \right\|_3 + \left\| b \right\|_\infty \left\| \nabla v \right\|_3 \right).\end{aligned}$$

Then, we obtain

$$\frac{1}{3}\frac{d}{dt}\left\|w_1\right\|_2^2+\frac{8}{9}\left\|\nabla w_1\right\|_2^2\leq C_{13}\left(\left\|w_1\right\|_3^3+1\right)$$

similarly to (4.15).

Here, we recall that (4.19) is established, and hence it holds that

$$\left\|w_1(t)\right\|_{4/3}\leq C_{13}.$$

This implies

$$\int_\Omega (w_1\log w_1)(t)\,dx\leq C_{14}\quad\text{and}\quad\left\|w_1(t)\right\|_1\leq C_{14},$$

and we can argue similarly. Thus, we obtain $\left\|w_1(t)\right\|_2\leq C_{15}$, or equivalently,

$$\left\|w(t)\right\|_3\leq C_{16}\quad\text{for}\quad t\in[0,T_{\max}). \tag{4.20}$$

Now, Theorem 3.2 guarantees (4.3), that is, $T_{\max}=+\infty$ and

$$\sup_{t\geq 0}\left\|u(t)\right\|_\infty<+\infty,$$

and the proof of Theorem 4.1 is complete. □

Having proven the main result of this chapter, we mention some other technical devices. We have the following facts for the other cases than (4.8). First, simplified system of the (JL) field is treated similarly:

$$\begin{aligned}
u_t=\nabla\cdot(\nabla u-u\nabla(v+\log W)) &\quad\text{in}\quad \Omega\times(0,T),\\
-\Delta v=u-\tfrac{1}{|\Omega|}\textstyle\int_\Omega u\,dx &\quad\text{in}\quad \Omega\times(0,T),\\
\textstyle\int_\Omega v\,dx=0 &\quad\text{for}\quad t\in(0,T),\\
\tfrac{\partial}{\partial\nu}u-u\tfrac{\partial}{\partial\nu}(v+\log W)=0 &\quad\text{on}\quad \partial\Omega\times(0,T),\\
\tfrac{\partial v}{\partial\nu}=0 &\quad\text{on}\quad \partial\Omega\times(0,T),\\
u|_{t=0}=u_0(x) &\quad\text{in}\quad \Omega.
\end{aligned} \tag{4.21}$$

If $\left\|u_0\right\|_1<4\pi$ holds in (4.21), then it follows that

$$T_{\max}=+\infty\quad\text{and}\quad\sup_{t\geq 0}\left\|u(t)\right\|_\infty<+\infty,$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $W=W(x)>0$ is a smooth function defined on $\overline{\Omega}$.

On the other hand, inequality (4.5) is available for the simplified system of the (D) field,

$$\begin{aligned} u_t = \nabla \cdot (\nabla u - u \nabla (v + \log W)) \quad & \text{in} \quad \Omega \times (0, T), \\ -\Delta v = u \quad & \text{in} \quad \Omega \times (0, T), \\ \tfrac{\partial}{\partial \nu} u - u \tfrac{\partial}{\partial \nu} (v + \log W) = 0 \quad & \text{on} \quad \partial\Omega \times (0, T) \\ v = 0 \quad & \text{on} \quad \partial\Omega \times (0, T), \\ u|_{t=0} = u_0(x) \quad & \text{in} \quad \Omega. \end{aligned} \tag{4.22}$$

Consequently, if $\|u_0\|_1 < 8\pi$, then we obtain

$$\int_\Omega (u \log u)(t)\, dx \le C_{17} \quad \text{for} \quad t \in [0, T_{\max}).$$

In this case we have $v > 0$ in $\overline{\Omega} \times (0, T_{\max})$, and therefore it holds that

$$\frac{\partial v}{\partial \nu} \le 0 \quad \text{on} \quad \partial\Omega.$$

Under the assumption

$$\frac{\partial}{\partial \nu} \log W \le 0 \quad \text{on} \quad \partial\Omega, \tag{4.23}$$

we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 &= \int_\Omega u \nabla (v + \log W) \cdot \nabla u\, dx \\ &= \frac{1}{2}\int_\Omega \nabla (v + \log W) \cdot \nabla u^2\, dx \\ &\le -\frac{1}{2}\int_\Omega \Delta (v + \log W) \cdot u^2\, dx \\ &\le \frac{1}{2}\|u\|_3^3 + \frac{1}{2}\|\log W\|_\infty \|u\|_2^2 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{3}\frac{d}{dt}\|u\|_3^3 + 2\int_\Omega u\, |\nabla u|^2\, dx &= \frac{2}{3}\int_\Omega \nabla (v + \log w) \cdot \nabla u^3\, dx \\ &\le -\frac{2}{3}\int_\Omega \Delta (v + \log W) \cdot u^3\, dx \\ &\le \frac{2}{3}\|u\|_4^4 + \frac{2}{3}\|\Delta \log W\|_\infty \|u\|_3^3. \end{aligned}$$

Then, we can argue similarly as in the (N) field. Thus, if

$$\|u_0\|_1 < 8\pi,$$

then $T_{\max} = +\infty$ and $\sup_{t\geq 0} \|u(t)\|_\infty < +\infty$ hold in (4.22), where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $W = W(x) > 0$ is a smooth function defined on $\overline{\Omega}$ satisfying (4.23).

The above results for a global existence of the solution are valid even to the full system. To treat this system we make use of the *parabolic L^1 estimate* instead of the analogous elliptic estimate. In two space dimensions, by

$$\tau \frac{dv}{dt} + Av = u \qquad \text{with} \qquad v|_{t=0} = v_0$$

and (4.9) we have

$$\sup_{t\geq 0} \|v(t)\|_{W^{1,q}} \leq C_{q,\varepsilon}\left(\|A^{1/2+\varepsilon} v_0\|_q + \|u_0\|_1\right) \tag{4.24}$$

for each $q \in (1, 2)$ and $\varepsilon > 0$. Therefore, we still have (4.10) for each $q \in [1, 2)$ under the assumptions on the initial value stated in Theorem 3.1. Then, we obtain (4.13) even in this case, using the Trudinger–Moser inequality and the Lyapunov function $W(u, v)$ similarly. On the other hand, deriving (4.20) from energy method, we obtain

$$\int_0^t \|v_t(t)\|_2^2 dt \leq C_{18} \quad \text{for} \qquad t \in [0, T_{\max}).$$

Consequently, we obtain the same conclusion that the conditions $\|u_0\|_1 < 4\pi$ for the (N), (JL) fields, and $\|u_0\|_1 < 8\pi$ for the (D) field to the full system, respectively, imply (4.3). See [14, 50, 110] to confirm details.

Other methods to show (4.3) are the following. First, we can avoid the use of the maximal regularity theorem, or the final part of the conclusion of Theorem 3.1, to guarantee the results stated in this chapter. In other words, it is possible to derive (4.3) only from the *energy method*, if we apply Moser's iteration scheme to our system. This argument can be localized, and then we can obtain the first step of the proof for the formation of collapses, Theorem 1.1. This kind of localization is efficient also for the study of the full system [109]. Next, use of the Trudinger–Moser inequality and Jensen's inequality to derive (4.13) can take the place of the Brezis–Merle type inequality and Young's inequality. This argument was adopted by [108] for (1.1), but here we shall describe it for (4.22), the simplified system of the the (D) field.

Brezis–Merle's inequality for

$$-\Delta v = u \quad \text{in} \quad \Omega \quad \text{with} \quad v = 0 \quad \text{in} \quad \partial\Omega$$

is indicated as

$$\int_\Omega \exp\Big(\frac{4\pi - \delta}{\|u\|_1}\,|v(x)|\Big)\,dx \le \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2, \tag{4.25}$$

where Ω is a two-dimensional domain and $\delta \in (0, 4\pi)$. On the other hand, the Lyapunov function $\mathcal{W} = \mathcal{W}(u, v)$ is reduced to the free energy,

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1 - \log W)\,dx - \frac{1}{2}\int_\Omega uv\,dx,$$

in this case, (4.22) of the simplified system. Here, *Young's inequality* is applicable as

$$auv \le u\log u + \frac{1}{e}e^{av} \qquad (u, v \ge 0),$$

where $a > 0$ is a constant. Namely, from $\mathcal{F}(u(t)) \le \mathcal{F}(u_0)$ we obtain

$$\Big(a - \frac{1}{2}\Big)\int_\Omega uv\,dx \le \frac{1}{e}\int_\Omega e^{av}\,dx + O(1).$$

In the case of $\lambda = \|u\|_1 < 8\pi$, we have also

$$\int_\Omega e^{av}\,dx = O(1)$$

for $0 < a - \frac{1}{2} \ll 1$ by (4.25) and hence it holds that

$$\int_\Omega uv\,dx = O(1) \quad \text{and} \quad \int_\Omega u\log u\,dx = O(1).$$

This means (4.13), and again we can show that $\|u_0\|_1 < 8\pi$ implies (4.3) in (4.22) with (4.23). (Let us confirm that for the (D) field the threshold of $\lambda = \|u_0\|_1$ for $T_{\max} = +\infty$ is 8π.)

We have several other versions of (4.25) applicable to the (N) and (JL) fields. By them, we can show similarly that if $\partial\Omega$ is smooth then $\|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$ in simplified and full systems of the (N) and (JL) fields. Use of the Brezis–Merle inequality is also efficient to show the concentration toward boundary of the solution to the simplified system of the (N) and (JL) fields in the case of

$$T_{\max} < +\infty \quad \text{and} \quad 4\pi < \|u_0\|_1 < 8\pi.$$

Inequality (4.25) also has a parabolic version and such a phenomenon, with concentration toward the boundary, is examined even in the full system. See [60, 108, 109].

The fact that

$$T = T_{\max} < +\infty \quad \Rightarrow \quad \lim_{t \to T} \|u(t)\|_\infty = +\infty \tag{4.26}$$

indicated in Theorem 3.1, is also proven by the energy method. In fact, if $u = u(x,t)$ denotes the solution to (4.1), the standard theory [85] guarantees that $T \leq T_{\max}$ and

$$\limsup_{t \to T} \|u(t)\|_\infty < +\infty \quad \Rightarrow \quad T_{\max} > T. \tag{4.27}$$

On the other hand, from the proof of Theorem 4.1 we see

$$\limsup_{t \to T} \int_\Omega (u \log u)(t)\, dx < +\infty \Rightarrow \limsup_{t \to T} \|u(t)\|_\infty < +\infty. \tag{4.28}$$

Now, we shall show

$$\liminf_{t \to T} \int_\Omega (u \log u)(t)\, dx < +\infty$$
$$\Rightarrow \quad \limsup_{t \to T} \int_\Omega (u \log u)(t)\, dx < +\infty. \tag{4.29}$$

Actually, relations (4.27)–(4.29) imply

$$T = T_{\max} < +\infty \quad \Rightarrow \quad \lim_{t \to T} \int_\Omega (u \log u)(t)\, dx = +\infty, \tag{4.30}$$

and (4.26) follows in particular.

First, we take the case $W \equiv 1$ for (4.29) to prove. For this purpose we make use of the following fact comparable to Lemma 4.1. It will be applied also in the proof of the formation of collapses.

Lemma 4.2 *We have a constant $K = K(\Omega) > 0$ determined by the bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$, satisfying*

$$\int_\Omega u^2\, dx \leq \frac{2K^2}{\log s} \int_\Omega \left(u \log u + e^{-1}\right) dx \cdot \int_\Omega u^{-1} |\nabla u|^2\, dx$$
$$+ 2K^2 \|u\|_1^2 + 3s^2 |\Omega| \tag{4.31}$$

for any $s > 1$.

Proof: We have $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ and there is $K = K(\Omega) > 0$ that admits the estimate

$$\|w\|_2 \leq K^2 \left(\|\nabla w\|_1^2 + \|w\|_1^2 \right) \tag{4.32}$$

for any $w \in W^{1,1}(\Omega)$. Letting $w = (u-s)_+$, we have $\|w\|_1 \leq \|u\|_1$,

$$\begin{aligned} \|w\|_2^2 &= \int_{\{u>s\}} (u-s)^2\,dx \geq \int_{\{u>s\}} \left(\frac{1}{2}u^2 - s^2\right) dx \\ &\geq \int_\Omega \frac{1}{2}u^2\,dx - \int_{\{u\leq s\}} \frac{1}{2}u^2\,dx - s^2\,|\Omega| \\ &\geq \frac{1}{2}\int_\Omega u^2\,dx - \frac{3}{2}s^2\,|\Omega|, \end{aligned}$$

and

$$\begin{aligned} \|\nabla w\|_1^2 &\leq \left\{ \int_{\{u>s\}} |\nabla u|\ dx \right\}^2 \\ &\leq \int_{\{u>s\}} u\,dx \cdot \int_{\{u>s\}} u^{-1}\,|\nabla u|^2\ dx \\ &\leq \frac{1}{\log s} \int_\Omega \left(u\log u + e^{-1}\right) dx \cdot \int_\Omega u^{-1}\,|\nabla u|^2\ dx, \end{aligned}$$

where $s > 1$ and $u \log u \geq -e^{-1}$ for $u \geq 0$ are made use of. These relations imply (4.31) and the proof is complete. □

Dealing with the case $W \equiv 1$ of (4.8), i.e., (1.1), we multiply $\log u$ to the first equation. This implies

$$\frac{d}{dt}\int_\Omega u\log u\,dx + \int_\Omega u^{-1}\,|\nabla u|^2\ dx + \int_\Omega uv\,dx = \int_\Omega u^2.$$

We also have $v > 0$ in $\overline{\Omega} \times (0, T)$ in this case, and hence obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u\log u\,dx + \left(1 - \frac{2K^2}{\log s}\int_\Omega \left(u\log u + e^{-1}\right) dx\right) \\ &\qquad \cdot \int_\Omega u^{-1}\,|\nabla u|^2\ dx \leq 2K^2 \|u_0\|_1^2 + 3s^2\,|\Omega| \end{aligned}$$

by (4.31). Then, taking

$$s = s(t) = \exp\left(2K^2 \int_\Omega \left(u\log u + e^{-1}\right)\ dx\right)$$

and

$$J = J(t) = \int_\Omega \left(u \log u + e^{-1} \right) dx,$$

we have

$$\frac{dJ}{dt} \leq 2K^2 \|u_0\|_1 + 3\,|\Omega| \exp\left(4K^2 J\right).$$

From this differential inequality, we can conclude that (4.29) holds true.

To treat the general case (4.1), we take $w = W^{-1}u$ and transform it to (4.14). Then, multiplying $\log w$ to the first equation, we obtain

$$\int_\Omega w_t \log w \, dx = \int_\Omega (\nabla \cdot j + b \cdot j) \log w \, dx.$$

The left-hand side and the first term of the right-hand side are treated similarly, and we obtain

$$\begin{aligned} \frac{d}{dt} \int_\Omega w(\log w - 1)\, dx &+ \int_\Omega w^{-1} |\nabla w|^2 \, dx + \int_\Omega wv \, dx \\ &= \int_\Omega W w^2 \, dx + \int_\Omega (b \cdot j) \log w \, dx. \end{aligned}$$

The second term of the right-hand side of the above equality is equal to

$$\begin{aligned} \int_\Omega (b \cdot j) \log w \, dx &= \int_\Omega b \cdot (\log w \, (\nabla w - w \nabla v)) \, dx \\ &= \int_\Omega (b \cdot \nabla (w(\log w - 1)) - (w \log w) b \cdot \nabla v) \, dx \\ &= -\int_\Omega (w(\log w - 1)\nabla \cdot b + (w \log w) b \cdot \nabla v) \, dx, \end{aligned} \tag{4.33}$$

and the first term of the right-hand side of this equality is treated similarly. For the second term, we make use of another linear theory valid for the second equation of (4.14). That is, if

$$-\Delta v + v = f \quad \text{in} \quad \Omega \quad \text{with} \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

holds for the two-dimensional bounded domain Ω with smooth boundary $\partial\Omega$, then it follows that

$$\|\nabla v\|_2 \leq C\, [f]_{L \log L},$$

where

$$[f]_{L\log L} = \int_\Omega |f| \log\Big(e + \frac{|f(x)|}{\|f\|_1}\Big)\, dx$$

denotes the *Zygmund norm* of Iwaniec and Verde [72]. We also make use of the inequality

$$[f]_{L\log L} \le \int_\Omega \Big(|f| \log |f| + e^{-1}\Big)\, dx + (e - e^{-1})\,|\Omega|$$

proven in Chapter 9. Then, the second term of (4.33) is estimated from above by

$$\|w \log w\|_2 \cdot \Big(\int_\Omega w \log w \, dx + O(1)\Big).$$

Now, we take $s > 0$ and put $(w \log w)\chi_{\{w>s\}}$ for w in (4.32):

$$\|(w\log w)\chi_{\{w>s\}}\|_2^2 \le K^2\Big(\|\nabla\big((w\log w)\chi_{\{w>s\}}\big)\|_1^2 + \|(w\log w)\chi_{\{w>s\}}\|_1^2\Big).$$

The left-hand side is estimated from below by

$$\int_\Omega (w\log w)^2\, dx - \big(s\log s + e^{-1}\big)^2\,|\Omega|\,,$$

while the second and the first terms of the right-hand side are estimated from above by

$$\int_\Omega \big(w \log w + e^{-1}\big)\, dx$$

and

$$\begin{aligned}
&\Big\{\int_{\{w>s\}} |\log w + 1|\,|\nabla w|\, dx\Big\}^2 \\
&\qquad \le \int_{\{w>s\}} w\,|\log w + 1|^2 \cdot \int_{\{w>s\}} w^{-1}\,|\nabla w|^2\, dx \\
&\qquad \le \frac{1}{s}\int_\Omega (w(\log w + 1))^2\, dx \cdot \int_\Omega w^{-1}\,|\nabla w|^2\, dx,
\end{aligned}$$

respectively. Based on these estimates, we can argue similarly to the previous case, and conclude that (4.29) holds true. Thus, we obtain (4.30) to (4.1):

$$T = T_{\max} < +\infty \quad \Rightarrow \quad \lim_{t\to T}\int_\Omega (u\log u)(t)\, dx = +\infty.$$

5
The Green's Function

The criterion $\left\| u_0 \right\|_1 < 4\pi$ is sharp for $T_{\max} = +\infty$ in the simplified system of the (N) field,

$$\begin{aligned}
u_t = \nabla \cdot \left(\nabla u - u \nabla \left(v + \log W \right) \right) & \quad \text{in} \quad \Omega \times (0, T), \\
0 = \Delta v - av + u & \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu} \left(v + \log W \right) = 0 & \quad \text{on} \quad \partial\Omega \times (0, T), \\
\frac{\partial v}{\partial \nu} = 0 & \quad \text{on} \quad \partial\Omega \times (0, T), \\
u\big|_{t=0} = u_0(x) & \quad \text{in} \quad \Omega,
\end{aligned} \tag{5.1}$$

and in this chapter we prove the following theorem [146].

Theorem 5.1 *If $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and*

$$\int_{\Omega \cap B(x_0, R)} u_0(x)\, dx > 4\pi$$

holds for $x_0 \in \partial\Omega$ and $R > 0$ in (5.1), then there exists $\eta > 0$ determined by $\lambda = \|u_0\|_1$ and $\|u_0\|_{L^1(\Omega \cap B(x_0, R))}$ such that

$$\frac{1}{R^2} \int_{\Omega \cap B(x_0, 4R)} |x - x_0|^2 u_0(x)\, dx < \eta \tag{5.2}$$

implies $T_{\max} < +\infty$.

Here and henceforth, $B(x_0, R)$ denotes the open disc with the center x_0 and the radius $R > 0$:

$$B(x_0, R) = \left\{ x \in \mathbf{R}^2 \;\middle|\; |x - x_0| < R \right\},$$

and the requirements of the above theorem are satisfied if

$$\lambda = \left\| u_0 \right\|_1 > 4\pi$$

and $u_0(x)\,dx$ is sufficiently concentrated at $x_0 \in \partial\Omega$. Therefore, we can conclude that $\lambda = 4\pi$ is the threshold for the blowup of the solution to (5.1). This is also the case of the simplified system of the (JL) field, and is proven similarly.

On the other hand, if $\|u_0\|_1 > 8\pi$ and $u_0(x)\,dx$ is sufficiently concentrated on an interior point, then $T_{\max} < +\infty$ follows. This blowup criterion is valid even to the (D) field, and therefore 8π is the threshold for blowup of the solution in the simplified system of the (D) field.

We emphasize that these criteria on $T_{\max} < +\infty$ are established only to the simplified system. Actually, we have a different kind of condition for the blowup of the solution to the full system [69, 146]. This criterion is given in terms of the value $\mathcal{W}(u_0, v_0)$ in accordance with $\inf\left\{ \mathcal{F}(u) \;\middle|\; u \in \mathcal{S}_\lambda \right\}$, where $\mathcal{S}_\lambda$ denotes the set of stationary solutions for $\lambda = \|u_0\|_1$. There, the possibility of the blowup in infinite time is included but we expect that only the blowup in finite time occurs in this case, and also that $\lambda = 4\pi$ and $\lambda = 8\pi$ are thresholds of the blowup of the solution in finite time even in full systems of the (N), (JL), and (D) fields, respectively. We further expect that the blowup in infinite time occurs only when the total mass of the initial value $\lambda = \|u_0\|_1$ is quantized, such as $\lambda \in 4\pi\mathcal{N}$ and $\lambda \in 8\pi\mathcal{N}$ for the (N), (JL) fields and the (D) field, respectively, and the solution converges to a singular limit of the stationary solution in infinite time.

We note that Theorem 1.1 guarantees that if

$$T_{\max} < +\infty$$

occurs with $\lambda = \|u_0\|_1 \in (4\pi, 8\pi)$ to (1.1), then exactly one blowup point of the solution lies on $\partial\Omega$. This is also the case of (5.1), and in this connection, we expect that the blowup point never arises on the boundary for the simplified system of the (D) field, if the environment function $W = W(x)$ satisfies (4.23).

Now, we come back to (5.1). For the proof of Theorem 5.1, we make use of a remarkable structure of the simplified system. It may be referred to as the *compensated compactness* via the *symmetrization*. The threshold values

4π and 8π are technically associated with this process. On the other hand, the left-hand side of (5.2) indicates the *local second moment* of $u(x,t)\,dx$. The *total moments* are virial quantities, and their use in the study of the blowup of the solution was adopted by [16, 106] for radially symmetric cases. The Green's function fits in naturally with that process, and in that context Biler [13, 14] introduced the method of symmetrization mentioned above. Actually, this technique of symmetrization for the double integral operator associated with the Green's function has been used in the study of the weak solution to Euler and Navier–Stokes systems in the vorticity formulation, such as (2.2) [37, 143]. There are several studies concerning the formation of singularities, using the second moment [95, 141, 178].

To explain the idea, here, we describe the following argument [17], where the system

$$
\begin{aligned}
u_t &= \nabla \cdot (\nabla u - u\nabla v) \quad \text{in} \quad \Omega \times (0,T),\\
\frac{\partial u}{\partial \nu}u - u\frac{\partial v}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \times (0,T),\\
u\big|_{t=0} &= u_0(x) \quad \text{in} \quad \Omega,
\end{aligned}
$$

with

$$
v(x,t) = \int_\Omega G_0(x,x')u(x',t)\,dx'
$$

for

$$
G_0(x,x') = \Gamma(x-x') \equiv \frac{1}{2\pi}\log\frac{1}{|x-x'|}
$$

is proposed to describe the motion of the mean field of many self-interacting particles under the gravitational force. In this system, if $\Omega \subset \mathbf{R}^2$ is star-shaped with respect to the origin, then we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega |x|^2\,u(x,t)\,dx &= \int_\Omega |x|^2\,u_t(x,t)\,dx\\
&= -\int_\Omega 2x\cdot(\nabla u - u\nabla v)\ dx\\
&= -\int_{\partial\Omega} 2(x\cdot\nu)u\,dx + \int_\Omega 4u\,dx + \int_\Omega 2ux\cdot\nabla v\,dx\\
&\le 4\lambda + \int_\Omega\int_\Omega 2u(x,t)x\cdot\nabla_x G_0(x,x')u(x',t)dy\,dx\\
&= 4\lambda + \frac{1}{2}\int_\Omega\int_\Omega \rho(x,x')u(x,t)u(x',t)\,dx\,dx',
\end{aligned}
$$

where $\lambda = \|u_0\|_1$ and

$$\rho(x, x') = 2x \cdot \nabla_x G_0(x, x') + 2x' \cdot \nabla_y G_0(x, x') = -\frac{1}{\pi}.$$

Thus, it holds that

$$\frac{d}{dt} I(t) \le 4\lambda - \frac{1}{2\pi}\lambda^2$$

for

$$I(t) = \int_\Omega |x|^2 u(x, t)$$

and therefore, if $\lambda = \|u_0\|_1 > 8\pi$ and $T_{\max} = +\infty$, then $I(t)$ becomes negative in a finite time. This is a contradiction and $T_{\max} < +\infty$ follows from $\|u_0\|_1 > 8\pi$.

In the above case, the concentration to the origin of the initial mass is not necessary to infer $T_{\max} < +\infty$. Actually, this condition of concentration fits in when the kernel has the additional regular part. Besides, we have to localize these arguments near the blowup point in our case, and this process is justified again by the method of symmetrization. Namely, it assures that the local L^1 norm has a bounded variation in time, and the strong concentration implies a contradiction before this concentration is broken. These arguments are valid for the general simplified system, and for (5.1) we have the following lemma.

Lemma 5.1 *Given the simplified system*

$$\begin{aligned} u_t = \nabla \cdot \big(\nabla u - u\nabla(v + \log W)\big) \quad &\textit{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu}(v + \log W) = 0 \quad &\textit{on} \quad \partial\Omega \times (0, T), \\ Av = u \quad &\textit{for} \quad t \in (0, T), \end{aligned}$$

let A^{-1} be provided with the integral kernel $G = G(x, x')$:

$$(A^{-1}u)(x) = \int_\Omega G(x, x')u(x')\,dx',$$

satisfying

$$\int_\Omega \int_\Omega \left|\nabla_x G(x, x')\right|\,dx\,dx' < +\infty. \tag{5.3}$$

Then any C^2 function ψ defined on $\overline{\Omega}$ satisfying

$$\left.\frac{\partial \psi}{\partial \nu}\right|_{\partial\Omega} = 0$$

admits the estimate

$$\left|\frac{d}{dt}\int_\Omega u\psi\, dx\right| \le \frac{\lambda^2}{2}\|\rho_\psi\|_{L^\infty(\Omega\times\Omega)} + \lambda\left(\|\Delta\psi\|_\infty + \|\nabla \log W\|_\infty \cdot \|\nabla\psi\|_\infty\right) \tag{5.4}$$

for $t \in [0, T_{\max})$, where

$$\rho_\psi(x, x') = \nabla\psi(x)\cdot\nabla_x G(x, x') + \nabla\psi(x')\cdot\nabla_{x'}G(x, x') \tag{5.5}$$

and $\lambda = \|u_0\|_1$.

Proof: Since A is self-adjoint, the kernel $G = G(x, x')$ is symmetric:

$$G(x, x') = G(x', x). \tag{5.6}$$

Furthermore, the second equation of (5.4) is replaced by (1.5):

$$v(x, t) = \int_\Omega G(x, x')u(x', t)\, dx'. \tag{5.7}$$

Testing $\psi \in C^2(\overline{\Omega})$ with $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega} = 0$ to the first equation of (5.1), we obtain the weak formulation,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega & u(x,t)\psi(x)\, dx - \int_\Omega u(x,t)\Delta\psi(x)\, dx \\
&= \int_\Omega u(x,t)\nabla v(x,t)\cdot\nabla\psi(x)\, dx + \int_\Omega u\nabla\log W\cdot\nabla\psi\, dx \\
&= \iint_{\Omega\times\Omega}\left[\nabla\psi(x)\cdot\nabla_x G(x,x')\right]u(x,t)u(x',t)\, dx\, dx' \\
&\qquad + \int_\Omega u\nabla\log W\cdot\nabla\psi\, dx \\
&= \frac{1}{2}\iint_{\Omega\times\Omega}\rho_\psi(x,x')u(x,t)u(x',t)\, dx\, dx' \\
&\qquad + \int_\Omega u\nabla\log W\cdot\nabla\psi\, dx,
\end{aligned} \tag{5.8}$$

by (5.3), (5.6), and (5.7), with $\rho_\psi(x, y)$ defined by (5.5). Then, (5.4) follows from (1.11):

$$\|u(t)\|_1 = \|u_0\|_1 \equiv \lambda, \tag{5.9}$$

and the proof is complete. □

In the standard fields of the (N), (JL), and (D), it holds that

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x')$$

with $K \in C^{1+\theta}_{\text{loc}}(\Omega \times \Omega)$ for $\theta \in (0, 1)$. This implies

$$\rho_\psi(x, x') = -\frac{\left(\nabla\psi(x) - \nabla\psi(x')\right) \cdot (x - x')}{2\pi\, |x - x'|^2} + C^{\theta}_{\text{loc}}(\Omega \times \Omega).$$

Here, the first term of the right-hand side belongs to $L^\infty(\Omega \times \Omega)$, although it is not continuous. More delicate analysis is necessary on $\partial\Omega$, but the estimates proven below induce that the local L^1 norm of u has a bounded variation in $t \in [0, T_{\max})$. This is also a key fact to prove the finiteness of blowup points.

Namely, we show the following lemma. Inequality (5.3) for $G = G(x, x')$ in consideration is assured also in the proof.

Lemma 5.2 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and let $G = G(x, x')$ be the Green's function of $-\Delta + a$ in Ω under the Neumann boundary condition, where $a > 0$ is a constant. Then, the function*

$$\rho_\psi(x, x') = \nabla\psi(x) \cdot \nabla_x G(x, x') + \nabla\psi(x') \cdot \nabla_{x'} G(x, x')$$

belongs to $L^\infty(\Omega \times \Omega)$ if $\psi \in C^2(\overline{\Omega})$ satisfies

$$\left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0,$$

and it holds that

$$\|\rho_\psi\|_{L^\infty(\Omega\times\Omega)} \leq K \|\nabla\psi\|_{C^1(\overline{\Omega})}, \tag{5.10}$$

where $K = K(\Omega) > 0$ is a constant determined by Ω and $a > 0$. In particular, we have

$$\left| \frac{d}{dt} \int_\Omega \psi(x) u(x, t)\, dx \right| \leq \frac{K\lambda^2}{2} \|\nabla\psi\|_{C^1(\overline{\Omega})} + \lambda \left(\|\Delta\psi\|_\infty + \|\nabla \log W\|_\infty \|\nabla\psi\|_\infty \right) \tag{5.11}$$

for $t \in [0, T_{\max})$.

Proof: Using the decomposition of the unity and the compactness of $\overline{\Omega}$, the assertion is reduced to the cases that supp $\psi \subset \Omega$ and supp $\psi \subset \overline{\Omega} \cap B(x_0, R)$ with $x_0 \in \partial\Omega$ and $0 < R \ll 1$.

To treat the first case, we take the fundamental solution $e_0(|x|)$ of $-\Delta + a$:

$$(-\Delta + a)e_0 = \delta_0(dx).$$

Actually, it is given by the series

$$e_0(r) = e_0(r; a) = \frac{1}{2\pi} \log \frac{1}{r} + \frac{a}{2\pi} \cdot \frac{r^2}{4} \log \frac{1}{r} + \frac{a}{2\pi} \cdot \frac{a}{4} \cdot \frac{r^4}{16} \log \frac{1}{r} + \frac{a}{2\pi} \cdot \frac{a}{4} \cdot \frac{a}{16} \cdot \frac{r^6}{36} \log \frac{1}{r} + \cdots, \quad (5.12)$$

and we have

$$e_0(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|} + C^{1+\theta}_{\text{loc}}(\mathbf{R}^2) \quad (5.13)$$

for $\theta \in (0, 1)$.

Given $x \in \Omega$, we define $K_0 = K_0(x, x')$ by

$$G(x, x') = e_0\left(\left|x - x'\right|\right) + K_0(x, x').$$

It holds that

$$(-\Delta_{x'} + a)\, K_0 = 0 \qquad \left(x' \in \Omega\right)$$

and

$$\frac{\partial}{\partial \nu_{x'}} K_0 = -\frac{\partial}{\partial \nu_{x'}} e_0\left(\left|x - x'\right|\right) \qquad \left(x' \in \partial\Omega\right).$$

Therefore, we have $K_0 \in C^{2+\theta}_{\text{loc}}\left(\Omega \times \overline{\Omega}\right)$ from the elliptic regularity. This, combined with (5.13), implies

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x')$$

with $K \in C^{1+\theta}_{\text{loc}}\left(\Omega \times \overline{\Omega}\right)$.

We also have $K \in C^{1+\theta}_{\text{loc}}\left(\overline{\Omega} \times \Omega\right)$ by (5.6), and hence

$$\rho_\psi(x, x') = -\frac{(x - x') \cdot \left(\nabla\psi(x) - \nabla\psi(x')\right)}{2\pi\, |x - x'|^2} + \nabla\psi(x) \cdot \nabla_x K(x, x') + \nabla\psi(x') \cdot \nabla_{x'} K(x, x')$$

belongs to $L^\infty(\Omega \times \Omega)$ because supp $\psi \subset \Omega$ is assumed.

To treat the second case of supp $\psi \subset \overline{\Omega} \cap B(x_0, R)$ with $x_0 \in \partial\Omega$ and $0 < R \ll 1$, first we assume that Ω is simply connected.

In this case we have a smooth conformal mapping

$$X : \overline{\Omega} \to \mathbf{R}^2,$$

satisfying

$$\begin{aligned} x_0 &\mapsto 0, \\ X(\Omega) &= \mathbf{R}^2_+ = \left\{(x_1, x_2) \,\middle|\, x_2 > 0\right\}, \\ X(\partial\Omega) &= \partial\mathbf{R}^2_+ = \left\{(x_1, x_2) \,\middle|\, x_2 = 0\right\}. \end{aligned} \tag{5.14}$$

We also take a smooth extension, denoted by $\hat{c}$, to the whole space $\mathbf{R}^2$ of

$$c = a\left|X'\right|^2 \circ X^{-1}$$

defined on $\overline{\mathbf{R}^2_+}$. Then, for

$$e_1(\xi, \xi') = e_0\left(\left|\xi' - \xi\right|, \hat{c}(\xi)\right) \tag{5.15}$$

with $e_0(r, a)$ given by (5.12), we have

$$\left(-\Delta_{\xi'} + \hat{c}(\xi)\right) e_1(\xi, \xi') = \delta_\xi(d\xi')$$

for $\xi', \xi \in \mathbf{R}^2$.

Now, we take the cut-off function

$$\zeta = \zeta(|y|) \in C_0^\infty(\mathbf{R}^2)$$

satisfying

$$0 \le \zeta(|y|) \le 1, \qquad \zeta(|y|) = \begin{cases} 1 & \left(y \in B(0, 1/2)\right), \\ 0 & \left(y \in \mathbf{R}^2 \setminus B(0,1)\right). \end{cases} \tag{5.16}$$

Then we apply the elliptic theory and take

$$e_2 = e_2(\xi, \xi') \in C^{\theta, 2+\theta}(\mathbf{R}^2 \times \mathbf{R}^2)$$

satisfying

$$\left(-\Delta_{\xi'} + \hat{c}(\xi')\right) e_2(\xi, \xi') = \zeta(\left|\xi' - \xi\right|) \cdot \left(\hat{c}(\xi) - \hat{c}(\xi')\right) e_1(\xi, \xi')$$

for $\xi, \xi' \in \mathbf{R}^2$. In this case,

$$e(\xi, \xi') = e_1(\xi, \xi') + e_2(\xi, \xi')$$

solves

$$\left(-\Delta_{\xi'} + \hat{c}(\xi')\right) e(\xi, \xi') = \delta_\xi(d\xi') + \tilde{\varphi}(\xi, \xi') \qquad (\xi, \xi' \in \mathbf{R}^2)$$

for

$$\tilde{\varphi} = \tilde{\varphi}(\xi, \xi') = \left\{\zeta\left(\left|\xi' - \xi\right|\right) - 1\right\}\left(\hat{c}(\xi) - \hat{c}(\xi')\right) e_1(\xi, \xi') \in C^\infty(\mathbf{R}^2 \times \mathbf{R}^2).$$

Let

$$E(\xi, \xi') = e(\xi, \xi') + e(\xi, \xi'_*)$$

with $\xi'_* = (\xi'_1, -\xi'_2)$ for $\xi' = (\xi'_1, \xi'_2)$. Then, given $\xi \in \mathbf{R}^2_+$, we have

$$\begin{aligned} \left(-\Delta_{\xi'} + c(\xi')\right) E(\xi, \xi') &= \delta_\xi(d\xi') + \hat{\Phi}(\xi, \xi') && \left(\xi' \in \mathbf{R}^2_+\right), \\ \frac{\partial}{\partial \nu_{\xi'}} E(\xi, \xi') &= 0 && \left(\xi' \in \partial\mathbf{R}^2_+\right), \end{aligned}$$

for

$$\begin{aligned} \hat{\Phi} = \hat{\Phi}(\xi, \xi') &= \tilde{\varphi}(\xi, \xi') + \left(-\Delta_{\xi'} + c(\xi')\right) e(\xi, \xi'_*) \\ &= \tilde{\varphi}(\xi, \xi') + \left(-\Delta_{\xi'} + c(\xi')\right) e_2(\xi, \xi'_*) \\ &\quad + \left(c(\xi') - \hat{c}(\xi^*)\right) e_0\left(\left|\xi_* - \eta\right|, \hat{c}(\xi_*)\right) \in C^\theta(\mathbf{R}^2 \times \mathbf{R}^2), \end{aligned}$$

because $\delta_{\xi_*}(d\xi') = 0$ holds in $\mathbf{R}^2_+$ if $\xi \in \mathbf{R}^2_+$. Here, X is conformal, and the above relation induces

$$\begin{aligned} (-\Delta_{x'} + a)\, E\left(X(x), X(x')\right) &= \delta_x(dx') + \varphi(x, x') && \left(x' \in \Omega\right), \\ \frac{\partial}{\partial \nu_{x'}} E\left(X(x), X(x')\right) &= 0 && \left(x' \in \partial\Omega\right), \end{aligned}$$

for each $x \in \Omega$, where

$$\varphi = \varphi(x, x') = \left|X'(x')\right|^2 \hat{\Phi}\left(X(x), X(x')\right) \in C^\theta(\overline{\Omega} \times \overline{\Omega}).$$

Therefore,

$$K_1(x, x') = G(x, x') - E\left(X(x), X(x')\right)$$

satisfies

$$\begin{aligned} (-\Delta_{x'} + a)\, K_1(x, x') &= \varphi(x, x') && (x' \in \Omega), \\ \frac{\partial}{\partial \nu_{x'}} K_1(x, x') &= 0 && (x' \in \partial\Omega), \end{aligned} \tag{5.17}$$

for each $x \in \Omega$, and from the elliptic regularity $K_1(x, x')$ is extended to an element in $C^{\theta,2+\theta}(\overline{\Omega} \times \overline{\Omega})$.

By (5.12), these relations,

$$G(x, x') = E\left(X(x), X(x')\right) + K_1(x, x'),$$

$K_1 \in C^{\theta,2+\theta}(\overline{\Omega} \times \overline{\Omega})$, $E = e_1 + e_2$, $e_2 \in C^{\theta,2+\theta}(\mathbf{R}^2 \times \mathbf{R}^2)$, and (5.15), imply

$$\begin{aligned} G(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')|} \\ + \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')_*|} + K_2(x, x') \end{aligned} \tag{5.18}$$

with $K_2 \in C^{\theta,1+\theta}\left(\overline{\Omega} \times \overline{\Omega}\right)$. We also have

$$K_2(x, x') = K_2(x', x)$$

by (5.6) and it follows that

$$K_2 \in C^{\theta,1+\theta}\left(\overline{\Omega} \times \overline{\Omega}\right) \cap C^{1+\theta,\theta}\left(\overline{\Omega} \times \overline{\Omega}\right). \tag{5.19}$$

In this way, we obtain

$$\nabla\psi(x) \cdot \nabla_x K_2(x, x') + \nabla\psi(x') \cdot \nabla_{x'} K_2(x, x') \in C^{\theta}(\overline{\Omega} \times \overline{\Omega}).$$

Now, we recall that $\psi \in C^2(\overline{\Omega})$ satisfies

$$\left.\frac{\partial \psi}{\partial \nu}\right|_{\partial\Omega} = 0 \quad \text{and} \quad \operatorname{supp} \psi \subset \overline{\Omega} \cap B(x_0, R) \tag{5.20}$$

for $x_0 \in \partial\Omega$ and $0 < R \ll 1$. To examine the first term of the right-hand side of (5.18), we set

$$G_1(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')|}.$$

Writing

$$\Psi = \psi \circ X^{-1} \quad \text{and} \quad g(\xi, \xi') = \frac{1}{2\pi} \log \frac{1}{|\xi - \xi'|},$$

we have

$$\begin{aligned}
&\nabla\psi(x)\cdot\nabla_x G_1(x,x') + \nabla\psi(y)\cdot\nabla_{x'} G_1(x,x') \\
&\quad = c(\xi)\nabla\Psi(\xi)\cdot\nabla_\xi g(\xi,\xi') + c(\xi')\nabla\Psi(\xi')\cdot\nabla_{\xi'} g(\xi,\xi') \\
&\quad = -\frac{(\xi-\xi')\cdot\left(c(\xi)\nabla\Psi(\xi) - c(\eta)\nabla\Psi(\xi')\right)}{2\pi\,|\xi-\xi'|^2} \\
&\quad \in L^\infty(\Omega\times\Omega).
\end{aligned}$$

To treat the second term of the right-hand side of (5.18),

$$G_2(x,x') = \frac{1}{2\pi}\log\frac{1}{|X(x)-X(x')_*|},$$

we make use of $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega} = 0$. In fact, this condition gives

$$\left.\frac{\partial\Psi}{\partial\xi_2}\right|_{\xi_2=0} = 0$$

and hence

$$\begin{aligned}
&\nabla\psi(x)\cdot\nabla_x G_2(x,x') + \nabla\psi(x')\cdot\nabla_y G_2(x,x') \\
&\quad = c(\xi)\nabla\Psi(\xi)\cdot\nabla_\xi g(\xi,\xi'_*) + c(\xi')\nabla\Psi(\xi')\cdot\nabla_\eta g(\xi,\xi'_*) \\
&\quad = -\frac{\left\{c(\xi)\Psi_{\xi_1}(\xi) - c(\xi')\Psi_{\xi'_1}(\xi')\right\}(\xi_1-\xi'_1)}{2\pi\left|\xi-\xi'_*\right|^2} \\
&\qquad - \frac{\left\{c(\xi)\Psi_{\xi_2}(\xi) + c(\xi')\Psi_{\xi'_2}(\xi)\right\}(\xi_2+\xi'_2)}{2\pi\left|\xi-\xi'_*\right|^2} \\
&\quad \in L^\infty(\Omega\times\Omega)
\end{aligned}$$

follows. These relations are summarized as

$$\rho \in L^\infty(\Omega\times\Omega).$$

We now proceed to the general case of Ω. In fact, it is multiply connected, and given $\psi \in C^2(\overline{\Omega})$ with (5.20) for $x_0 \in \partial\Omega$ and $0 < R \ll 1$, we can take a Jordan curve $\gamma \subset \partial\Omega$ containing x_0 , and obtain the domain $\hat{\Omega}$ satisfying $\partial\hat{\Omega} = \gamma$ and $\Omega \subset \hat{\Omega}$. Either $\hat{\Omega}$ or $\mathbf{R}^2 \setminus \hat{\Omega}$ is simply connected in this case.

Replacing Ω by this $\hat{\Omega}$, we repeat the previous argument. We obtain, instead of (5.17),

$$(-\Delta_{x'} + a)\, K_1(x, x') = \varphi(x, x') \qquad (x' \in \Omega).$$
$$\frac{\partial}{\partial \nu_{x'}} K_1(x, x') = h(x, x') \qquad (x' \in \partial\Omega),$$

with $\varphi \in C^\theta(\overline{\Omega} \times \overline{\Omega})$ and $h \in C^\infty(\Omega \times \partial\Omega)$. This h satisfies

$$h(x, x') = 0$$

for $(x, x') \in \Omega \times \gamma$. On the other hand, $G = G(x, x')$ is extended smoothly for $(x, x') \in (\Omega \cup \gamma) \times (\partial\Omega \setminus \gamma)$ by the elliptic regularity. This is also the case of $E\left(X(x), X(x')\right)$, and we can assume

$$h \in C^\infty\left((\Omega \cup \gamma) \times \partial\Omega\right).$$

Therefore, from the elliptic regularity again,

$$K_1 = K_1(x, x')$$

is extended to an element in $C^{\theta, 2+\theta}\left((\Omega \cup \gamma) \times \overline{\Omega}\right)$, and then (5.19) is replaced by

$$K_2 \in C^{\theta, 1+\theta}\left((\Omega \cup \gamma) \times \overline{\Omega}\right) \cap C^{1+\theta, \theta}\left(\overline{\Omega} \times (\Omega \cup \gamma)\right).$$

From this relation, the conclusion

$$\rho_\psi \in L^\infty(\Omega \times \Omega)$$

follows similarly. □

From the proof the above lemma, we see that (5.10) is refined as

$$\left\| \rho_\psi \right\|_{L^\infty(\Omega \times \Omega)} \leq \frac{1}{2\pi} \left\| \nabla \psi \right\|_{C^{0,1}(\Omega)} + C(\Omega) \left\| \nabla \psi \right\|_\infty, \qquad (5.21)$$

where $C(\Omega) > 0$ is a constant determined by Ω. Inequality (5.21) controls the rate of variation in time of the local mass of $u(x, t)\, dx$ through (5.11), and this provides the principal motivation for the proof of mass quantization.

In what follows, we make use of the specific cut-off function, denoted by

$$\varphi = \varphi_{x_0, R', R},$$

where $x_0 \in \overline{\Omega}$ and $0 < R' < R \ll 1$. First, if $x_0 \in \Omega$, we assume that $R > 0$ is as small as $B(x_0, 2R) \subset \Omega$, and take $\varphi \in C_0^\infty(\mathbf{R}^2)$ satisfying $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1 & \big(x \in B(x_0, R')\big) \\ 0 & (x \in B(x_0, R)) \,. \end{cases} \tag{5.22}$$

Given $x_0 \in \partial\Omega$, we take the conformal mapping $X : \hat{\Omega} \to \mathbf{R}^2$ described in the previous lemma. This X satisfies, similarly to (5.14),

$$\begin{gathered} x_0 \mapsto 0, \\ X\big(B(x_0, 2R) \cap \Omega\big) \subset \mathbf{R}^2_+ = \big\{(x_1, x_2) \bigm| x_2 > 0\big\}, \\ X\big(B(x_0, 2R) \cap \partial\Omega\big) \subset \partial\mathbf{R}^2_+ = \big\{(x_1, x_2) \bigm| x_2 = 0\big\} \end{gathered}$$

for $0 < R \ll 1$. Given $R' \in (0, R)$, we can furthermore impose

$$\begin{gathered} X\left(B(x_0, R') \cap \Omega\right) \subset B(0, r'), \\ X\big((B(x_0, 2R) \cap \Omega) \setminus B(x_0, R)\big) \subset \mathbf{R}^2 \setminus B(0, r), \end{gathered}$$

with $0 < r \ll 1$ and $r' \in (0, r)$ proportional to R and R', respectively. Then, we take $\zeta_{r',r} = \zeta_{r',r}(|y|) \in C_0^\infty(\mathbf{R}^2)$, satisfying, similarly to (5.16),

$$0 \le \zeta_{r',r}(|y|) \le 1, \qquad \zeta_{r',r}(|y|) = \begin{cases} 1 & \big(y \in B(0, r')\big) \\ 0 & \big(y \in \mathbf{R}^2 \setminus B(0, r)\big), \end{cases}$$

and put $\varphi(x) = \zeta_{r',r}\,(X(x))$. Then, it holds that

$$\frac{\partial}{\partial\zeta}\zeta \circ X = \frac{\partial X}{\partial\nu} \cdot (\nabla\zeta \circ X) = 0$$

on $\partial\Omega$ because X is conformal and ζ is a function of $|y|$. Thus, we have $\varphi = \varphi_{x_0,R',R}(x)$ satisfying

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0 \tag{5.23}$$

besides (5.22) in the case of $x_0 \in \partial\Omega$. This $\varphi_{x_0,R',R}$ satisfies

$$\left\| D^\alpha \varphi_{x_0,R',R} \right\|_\infty = O\big((R - R')^{-|\alpha|}\big)$$

uniformly in $x_0 \in \overline{\Omega}$ for each multi-index α, because of

$$\left\| D^\alpha X \right\|_\infty = O(1). \tag{5.24}$$

Returning to the proof of Theorem 5.1, we recall that x_0 is given on $\partial\Omega$. This is because the blowup near the threshold value $\|u_0\|_1 = 4\pi$ occurs on the boundary, and the second moment to be used for the proof of this theorem must be localized around the boundary point $x_0 \in \partial\Omega$.

Thus, for $R > 0$ sufficiently small, we take

$$\phi_i = \varphi_{x_0, 4^{i-1}R, 2\cdot 4^{i-1}R}$$

and set

$$\psi_i = \phi_i^4 \quad \text{and} \quad m(x) = \frac{|X(x)|^2}{\left|\frac{\partial X}{\partial x}(x_0)\right|}.$$

for $i = 1, 2$. Then, it holds that

$$|\nabla\psi_i| = 4\psi_i^{3/4}\,|\nabla\varphi_i| = O(R^{-1})\psi_i^{3/4}. \tag{5.25}$$

We also have

$$m(x) = O\big(|x - x_0|^2\big), \qquad \nabla m(x) = O\left(|x - x_0|\right) \tag{5.26}$$

by (5.24).

Regarding $m(x)$ as the weight function, we define the moment

$$I_1(t) = \int_\Omega u(x,t)m(x)\psi_1(x)\,dx$$

localized around $x_0 \in \partial\Omega$. Eliminating $v = v(x)$ of the right-hand side of

$$\frac{dI_1}{dt} = \int_\Omega u_t m\psi_1\,dx$$

using the second equation of (5.1), and then making the symmetrization and cut-off process, we can show the following lemma.

Lemma 5.3 *Under the assumptions of the previous lemma, if*

$$\begin{aligned}\rho(x,x') = &\left[\nabla(m\psi_1)(x)\cdot\nabla_x G(x,x')\right]\psi_2(x') \\ &+ \left[\nabla(m\psi_1)(x')\cdot\nabla_{x'} G(x,x')\right]\psi_2(x),\end{aligned} \tag{5.27}$$

then it holds that

$$\begin{aligned}\left|\rho(x,x') + \frac{2}{\pi}\psi_1(x)\psi_2(x')\right| \le{}& CR^{-1}\left(|x - x_0| + \left|x' - x_0\right|\right) \\ &\cdot \psi_1(x)^{1/2}\psi_2(x') + CR^{-1}\left|x' - x_0\right|\psi_2(x')^{1/2}\end{aligned} \tag{5.28}$$

with a constant $C > 0$.

Proof: From the proof of Lemma 5.2, if Ω is simply connected, then relation (5.18) holds with (5.19), where $(\eta_1, \eta_2)_* = (\eta_1, -\eta_2)$. Even if it is not the case, these relations are localized to each component of $\partial\Omega$, and generally we have

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')|} + \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')_*|} + K(x, x') \quad (5.29)$$

for $x, x' \in \overline{B(x_0, 16R) \cap \Omega}$ with

$$K \in \left(C^{\theta,1+\theta} \cap C^{1+\theta,\theta}\right)\left(\overline{B(x_0, 16R) \cap \Omega} \times \overline{B(x_0, 16R) \cap \Omega}\right). \quad (5.30)$$

First, we take the term associated with

$$G_1(x, x') = g(\xi, \xi') = \frac{1}{2\pi} \log \frac{1}{|\xi - \xi'|}$$

for $\xi = X(x)$ and $\xi' = X(x')$. Since X is conformal, it holds that

$$\left(\frac{\partial X}{\partial x}\right) \cdot {}^t\!\left(\frac{\partial X}{\partial x}\right) = \left|\frac{\partial X}{\partial x}\right| \cdot Id.$$

Defining

$$c(\xi) = \frac{\left|\frac{\partial X}{\partial x}\right|}{\left|\frac{\partial X}{\partial x}(x_0)\right|} \quad \text{and} \quad \Psi_i(\xi) = \psi_i(x),$$

we have

$$\begin{aligned}
\rho_1(x, x') &\equiv \left[\nabla_x (m\psi_1)(x) \cdot \nabla_x G_1(x, x')\right] \psi_2(x') \\
&\quad + \left[\nabla_{x'} (m\psi_1)(x') \cdot \nabla_y G_1(x, x')\right] \psi_2(x) \\
&= c(\xi)\Psi_2(\xi')\nabla_\xi \left(|\xi|^2\, \Psi_1(\xi)\right) \cdot \nabla_\xi g(\xi, \xi') \\
&\quad + c(\xi')\Psi_2(\xi)\nabla_{\xi'} \left(\left|\xi'\right|^2\, \Psi_1(\xi')\right) \cdot \nabla_{\xi'} g(\xi, \xi') \\
&= -\frac{\xi - \xi'}{2\pi\, |\xi - \xi'|^2} \cdot \Big\{c(\xi)\Psi_2(\xi')(2\xi\Psi_1(\xi) + |\xi|^2\, \nabla_\xi \Psi_1(\xi)) \\
&\quad - c(\xi')\Psi_2(\xi)(2\xi'\Psi_1(\xi') + \left|\xi'\right|^2 \nabla_{\xi'} \Psi_1(\xi'))\Big\}.
\end{aligned}$$

This implies $\rho_1 = -(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V})$ with

$$
\begin{aligned}
\mathrm{I} &= \frac{1}{\pi} c(\xi)\Psi_1(\xi)\Psi_2(\xi') \\
\mathrm{II} &= \frac{(\xi - \xi')\cdot \xi'}{\pi\,|\xi - \xi'|^2} \left\{ c(\xi)\Psi_2(\xi')\Psi_1(\xi) - c(\xi')\Psi_2(\xi)\Psi_1(\xi') \right\} \\
\mathrm{III} &= \frac{(\xi - \xi')}{\pi\,|\xi - \xi'|^2} \cdot \nabla_\xi \Psi_1(\xi) c(\xi)\Psi_2(\xi') \left(|\xi|^2 - \left|\xi'\right|^2 \right) \\
\mathrm{IV} &= \frac{(\xi - \xi')}{2\pi\,|\xi - \xi'|^2} \cdot \left(\nabla_\xi \Psi_1(\xi) - \nabla_{\xi'} \Psi_1(\xi') \right) c(\xi)\Psi_2(\xi') \left|\xi'\right|^2 \\
\mathrm{V} &= \frac{(\xi - \xi')}{2\pi\,|\xi - \xi'|^2} \cdot \nabla'_\xi \Psi_1(\xi')(c(\xi)\Psi_2(\xi') - c(\xi')\Psi_2(\xi)) \left|\xi'\right|^2 .
\end{aligned}
$$

We have $c(\xi) = 1 + O\,(|x - x_0|)$ and hence

$$
\mathrm{I} = \frac{1}{\pi} \{1 + \mathrm{O}\,(|\mathrm{x} - \mathrm{x}_0|\})\, \psi_1(\mathrm{x})\psi_2(\mathrm{x}').
$$

Similarly, we obtain

$$
\begin{aligned}
\mathrm{II} &= \frac{(\xi - \xi')\cdot \xi'}{\pi\,|\xi - \xi'|^2} \{ (c(\xi) - c(\xi'))\Psi_2(\xi')\Psi_1(\xi) \\
&\qquad + c(\xi')\Psi_2(\xi')(\Psi_1(\xi) - \Psi_1(\xi')) \\
&\qquad + c(\xi')(\Psi_2(\xi') - \Psi_2(\xi))\Psi_1(\xi') \} \\
&= O\left(\left|\xi'\right|\right)\Psi_2(\xi')\Psi_1(\xi) + O\left(\left|\xi'\right|\right) O(R^{-1})\Psi_2(\xi') \\
&\qquad + O\left(\left|\xi'\right|\right) O(R^{-1})\Psi_1(\xi') \\
&= O\left(\left|x' - x_0\right|\right) \{ \psi_2(x')\psi_1(x) + O(R^{-1})\psi_2(x') \\
&\qquad + O(R^{-1})\psi_1(x') \} \\
\mathrm{III} &= \frac{(\xi - \xi')}{\pi\,|\xi - \xi'|^2} \cdot \nabla_\xi \Psi_1(\xi) c(\xi)\Psi_2(\xi')(\xi - \xi')\cdot(\xi + \xi') \\
&= O\left(|x - x_0| + \left|x' - x_0\right|\right) O(R^{-1})\psi_1(x)^{1/2}\psi_2(x') \\
\mathrm{IV} &= O\left(\left|\xi'\right|^2\right) O(R^{-2})\Psi_2(\xi') = O\left(\left|x' - x_0\right| R^{-1}\right)\psi_2(x') \\
\mathrm{V} &= O\left(\left|\xi'\right|^2\right) O(R^{-1}) \left|\nabla_{\xi'}\Psi_1(\xi')\right| \\
&= O\left(\left|x' - x_0\right| R^{-1}\right) \psi_1(x')^{1/2},
\end{aligned}
$$

where (5.25) is used. These relations are summarized as

$$\begin{aligned}
&\left|\rho_1(x,x') + \frac{1}{\pi}\psi_1(x)\psi_2(x')\right| \\
&\qquad + \leq \frac{C}{R}\left(|x-x_0| + \left|x'-x_0\right|\right)\psi_1(x)^{1/2}\psi_2(x') \\
&\qquad + \frac{C}{R}\left|x'-x_0\right|\psi_2(x')^{1/2}
\end{aligned}$$

by $\psi_2 \geq \psi_1$.

We turn to the term associated with

$$G_2(x,y) = g^*(\xi,\xi') = \frac{1}{2\pi}\log\frac{1}{\left|\xi-\xi'_*\right|}.$$

Using $\frac{\partial\psi_i}{\partial\nu}\big|_{\partial\Omega} = 0$, we obtain

$$\left.\frac{\partial\Psi_i}{\partial\xi_2}\right|_{\xi_2=0} = 0$$

for $i = 1, 2$. Writing $\Phi_i(\xi) = \phi_i(x)$, we also have $\frac{\partial\Phi_i}{\partial\xi_2}\big|_{\xi_2=0} = 0$, which implies $\left|\frac{\partial\Phi_i}{\partial\xi_2}\right| = O\left(R^{-2}\xi_2\right)$. Using $\Psi_i = \Phi_i^4$, we have

$$\left|\frac{\partial\Psi_i}{\partial\xi_2}\right| = O\left(\Psi_i^{3/4}R^{-2}\xi_2\right) \tag{5.31}$$

similarly to (5.25).

Now, we have

$$\begin{aligned}
\rho_2(x,x') &\equiv \left[\nabla_x(m\psi_1)(x)\cdot\nabla_x G_2(x,x')\right]\psi_2(x') \\
&\qquad + \left[\nabla_x(m\psi_1)(x')\cdot\nabla_{x'}G_2(x,x')\right]\psi_2(x) \\
&= c(\xi)\Psi_2(\xi')\nabla_\xi(|\xi|^2\Psi_1(\xi))\cdot\nabla_\xi g^*(\xi,\xi') \\
&\qquad + c(\xi')\Psi_2(\xi)\nabla_{\xi'}(\left|\xi'\right|^2\Psi_1(\xi'))\cdot\nabla_{\xi'}g^*(\xi,\xi') \\
&= -\left(\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX}\right)
\end{aligned}$$

with

$$\mathrm{VI} = \frac{(\xi_1 - \xi_1')}{\pi|\xi - \xi_*'|^2}\Big\{c(\xi)\xi_1\Psi_1(\xi)\Psi_2(\xi') - c(\xi')\xi_1'\Psi_1(\xi')\Psi_2(\xi)\Big\}$$

$$\mathrm{VII} = \frac{(\xi_1 - \xi_1')}{2\pi|\xi - \xi_*|^2}\Big\{c(\xi)|\xi|^2\Psi_{1\xi_1}(\xi)\Psi_2(\xi') - c(\xi')|\xi'|^2\Psi_{1\xi_1'}(\xi')\Psi_2(\xi)\Big\}$$

$$\mathrm{VIII} = \frac{(\xi_2 + \xi_2')}{\pi|\xi - \xi_*'|^2}\Big\{c(\xi)\xi_2\Psi_1(\xi)\Psi_2(\xi') + c(\xi')\xi_2'\Psi_1(\xi')\Psi_2(\xi)\Big\}$$

$$\mathrm{IX} = \frac{(\xi_2 + \xi_2')}{2\pi|\xi - \xi_*'|^2}\Big\{c(\xi)|\xi|^2\Psi_{1\xi_2}(\xi)\Psi_2(\xi') + c(\xi')|\eta|^2\Psi_{1\xi_2'}(\xi')\Psi_2(\xi)\Big\}.$$

Similarly to $G_1(x, x')$, the estimate

$$\Big|\mathrm{VI} + \mathrm{VII} + \frac{(\xi_1 - \xi_1')^2}{\pi|\xi - \xi_*'|^2}\Psi_1(\xi)\Psi_2(\xi')\Big|$$

$$\leq CR^{-1}\big(|x - x_0| + |x' - x_0|\big)\psi_1(x)^{1/2}\psi_2(x') + CR^{-1}|x' - x_0|\psi_2(x')^{1/2}$$

holds. On the other hand, we have

$$\begin{aligned}\mathrm{VIII} &= \frac{(\xi_2 + \xi_2')^2}{\pi|\xi - \xi_*'|^2}c(\xi)\Psi_1(\xi)\Psi_2(\xi') \\ &\quad - \frac{(\xi_2 + \xi_2')\xi_2'}{\pi|\xi - \xi_*'|^2}\Big\{(c(\xi) - c(\xi'))\Psi_1(\xi)\Psi_2(\xi') + c(\xi')(\Psi_1(\xi) - \Psi_1(\xi')) \\ &\qquad \cdot \Psi_2(\xi') + c\big(\xi'\big)\Psi_1(\xi')(\Psi_2(\xi') - \Psi_2(\xi))\Big\} \\ &= \frac{(\xi_2 + \xi_2')^2}{\pi|\xi - \xi_*'|^2}c(\xi)\Psi_1(\xi)\Psi_2(\xi') + \frac{(\xi_2 + \xi_2')\xi_2'}{\pi|\xi - \xi_*'|^2} \\ &\quad \cdot O\big(|\xi - \xi'|\big)\big(\Psi_1(\xi)\Psi_2(\xi') + O(R^{-1})\Psi_2(\xi') + O(R^{-1})\Psi_1(\xi')\big).\end{aligned}$$

Now, estimate (5.31) gives

$$\Psi_{1\xi_2}(\xi) + \Psi_{1\xi_2'}(\xi') = O\Big(\big(\xi_2\Psi_1(\xi)^{3/4} + \xi_2'\Psi(\xi')^{3/4}\big)R^{-2}\Big)$$

and therefore from $\xi_2, \xi_2' \geq 0$ we have

$$
\begin{aligned}
\mathrm{IX} = \frac{(\xi_2 + \xi_2')}{2\pi|\xi - \xi_*'|^2} \Big\{ & (c(\xi) - c(\xi'))|\xi|^2 \Psi_{1\xi_2}(\xi)\Psi_2(\xi') \\
& + c(\xi')(|\xi|^2 - |\xi'|^2)\Psi_{1\xi_2}(\xi)\Psi_2(\xi') \\
& + c(\xi')|\xi'|^2(\Psi_{1\xi_2}(\xi) + \Psi_{1\xi_2'}(\xi'))\Psi_2(\xi') \\
& - c(\xi')|\xi'|^2\Psi_{1\xi_2'}(\xi')(\Psi_2(\xi') - \Psi_2(\xi)) \Big\} \\
= \frac{(\xi_2 + \xi_2')}{2\pi|\xi - \xi_*'|^2} \Big\{ & O\left(|\xi - \xi'|\right)|\xi|^2 \Psi_{1\xi_2}(\xi)\Psi_2(\xi') \\
& + c(\xi') O\big(|\xi - \xi'|(|\xi| + |\xi'|)\big)\Psi_{1\xi_2}(\xi)\Psi_2(\xi') \\
& + c(\xi')|\xi'|^2 O\Big(\big(\xi_2\Psi_1(\xi)^{3/4} + \xi_2'\Psi_1(\xi')^{3/4}\big)R^{-2}\Big)\Psi_2(\xi') \\
& + c(\xi')|\xi'|^2\Psi_{1\xi_2'}(\xi') O\big(R^{-1}|\xi - \xi'|\big) \Big\} \\
= {} & O(R^{-1})|\xi|^2\Psi_1^{1/2}(\xi)\Psi_2(\xi') + O\big(R^{-1}\big(|\xi| + |\xi'|\big)\big) \\
& \cdot \Psi_1^{1/2}(\xi)\Psi_2(\xi') + O(R^{-2})|\xi'|^2\Psi_1^{1/2}(\xi').
\end{aligned}
$$

These relations are summarized as

$$
\begin{aligned}
& \left|\rho_2(x, x') + \frac{1}{\pi}\psi_1(x)\psi_2(x')\right| \\
& \quad \leq \frac{C}{R}\left(|x - x_0| + |x' - x_0|\right)\psi_1(x)^{1/2}\psi_2(x') \\
& \qquad + \frac{C}{R}|x' - x_0|\psi_2(x')^{1/2}.
\end{aligned}
$$

Finally, from (5.30), (5.25), and (5.26) we have

$$
\begin{aligned}
& \big[\nabla(m\psi_1)(x) \cdot \nabla_x K(x, x')\big]\psi_2(x') + \big[\nabla(m\psi_1)(x') \cdot \nabla_y K(x, x')\big]\psi_2(x') \\
& \quad = \psi_2(x')\Big(O(1)|x - x_0|\psi_1(x) + O(R^{-1})|x - x_0|^2\psi_1^{1/2}(x)\Big) \\
& \qquad + \psi_2(x)\Big(O(1)|x' - x_0|\psi_1(x') + O(R^{-1})|x' - x_0|^2\psi_1^{1/2}(x')\Big) \\
& \quad = O(1)\Big(|x - x_0|\psi_1(x)\psi_2(x') + |x - x_0|\psi_1(x)^{1/2}\psi_2(x')\Big) \\
& \qquad + O(1)\Big(|x' - x_0|\psi_1(x')\psi_2(x) + |x' - x_0|\psi_1(x')^{1/2}\psi_2(x)\Big).
\end{aligned}
$$

The proof is complete. □

Now, we give the following proof.

Proof of Theorem 5.1: We have

$$\left|\frac{\partial X}{\partial x}(x_0)\right|\frac{\partial m}{\partial \nu} = \nabla_\xi |\xi|^2 \cdot \frac{\partial X}{\partial \nu} = 2\frac{\partial X}{\partial \nu}\cdot X$$
$$= 2\frac{\partial X_2}{\partial \nu}\cdot X_2 = 0 \quad \text{on} \quad \partial\Omega.$$

Also, we have the following estimates similar to (5.26), where $x_0 = (x_{01}, x_{02})$:

$$m(x) = |x-x_0|^2 + O\big(|x-x_0|^3\big),$$
$$m_{x_i} = 2\,(x_i - x_{i0}) + O\big(|x-x_0|^2\big),$$
$$m_{x_i x_j} = 2\delta_{ij} + O\big(|x-x_0|\big).$$

Defining

$$I_i(t) = \int_\Omega u(x,t)m(x)\psi_i(x)\,dx,$$

we obtain

$$\frac{d}{dt}I_1 = \int_\Omega u_t m\psi_1\,dx = -\int_\Omega (\nabla u - u\nabla v)\cdot\nabla(m\psi_1)\,dx$$
$$= \int_\Omega u\Delta(m\psi_1)\,dx + \int_\Omega u\nabla v\cdot\nabla(m\psi_1)\,dx$$
$$+ \int_\Omega u\nabla\log W\cdot\nabla(m\psi_1)\,dx = \mathrm{I}+\mathrm{II}+\mathrm{III}$$

from the first equation of (5.1). Since

$$|\nabla\psi_i| \le CR^{-1}\psi_i^{1/2} \quad \text{and} \quad |\Delta\psi_i| \le CR^{-2}\psi_i^{1/2}$$

hold similarly to (5.25), we obtain

$$\mathrm{I} = \int_\Omega u\big\{\psi_1\Delta m + 4\nabla m\cdot\nabla\psi_1 + m\Delta\psi_1\big\}\,dx$$
$$\le 4\int_\Omega u\psi_1\,dx + CR^{-1}\int_\Omega |x-x_0|\,\psi_1^{1/2}u\,dx$$
$$\le 4M_1 + CR^{-1}\lambda^{1/2}I_1^{1/2},$$

where $M_i(t) = \int_\Omega \psi_i(x)u(x,t)\,dx$. We also have

$$\begin{aligned}
\mathrm{III} &= \int_\Omega u\big(\psi_1 \nabla \log W \cdot \nabla m + m \nabla \log W \cdot \nabla \psi_1\big)\,dx \\
&\le CR^{-1}\int_\Omega |x-x_0|\,\psi_1 u\,dx + CR^{-1}\int_\Omega |x-x_0|^2\,\psi_1^{1/2} u\,dx \\
&\le CR^{-1}\int_\Omega |x-x_0|\,\psi_1^{1/2} u\,dx \le CR^{-1}\lambda^{1/2} I_1^{1/2}.
\end{aligned}$$

Now, we apply the second equation of (5.1) to II and obtain

$$\begin{aligned}
\mathrm{II} &= \int_\Omega\int_\Omega u(x,t)\nabla_x G(x,x')\cdot\nabla_x(m\psi_1)(x)u(x',t)\,dx\,dx' \\
&= \int_\Omega\int_\Omega u(x,t)\psi_2(x')\nabla_x G(x,x')\cdot\nabla_x(m\psi_1)(x)u(x',t)\,dx\,dx' \\
&\quad + \int_\Omega\int_\Omega u(x,t)(1-\psi_2(x'))\nabla_x G(x,x') \\
&\qquad \cdot \nabla_x(m\psi_1)(x)u(x',t)\,dx\,dx' = \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

Here,

$$\int_\Omega\int_\Omega \cdots\ dx\,dx'$$

of V is reduced to

$$\iint_{|x'-x_0|>16R,\ |x-x_0|<8R} \cdots\ dx\,dx'$$

and estimated from above by

$$\iint_{|x-x'|>8R} |\cdots|\ dx\,dx'.$$

Therefore, it follows that

$$\begin{aligned}
\mathrm{V} &\le CR^{-1}\int_\Omega\int_\Omega |x-x_0|\,\psi_1(x)^{1/2}u(x,t)u(x',t)\,dx\,dx' \\
&= CR^{-1}\lambda\int_\Omega |x-x_0|\,\psi_1(x)^{1/2}u(x,t)\,dx \le CR^{-1}\lambda^{3/2}I_1^{1/2}.
\end{aligned}$$

On the other hand, we have

$$\mathrm{IV} = \frac{1}{2}\int_\Omega\int_\Omega \mathrm{u(x,t)}\rho(\mathrm{x,x'})\mathrm{u(x',t)\,dx\,dx'}$$

by the symmetrization, and then Lemma 5.3 implies

$$\begin{aligned}
\left| \text{III} + \frac{1}{\pi} M_1 M_2 \right| &\leq \int_\Omega \int_\Omega u(x,t) \\
&\quad \cdot \frac{1}{2} \left| \rho(x,y) + \frac{2}{\pi} \psi_1(x) \psi_2(y) \right| u(y,t)\, dx dy \\
&\leq C R^{-1} \lambda \int_\Omega |x - x_0|\, \psi_1(x)^{1/2} u(x,t)\, dx \\
&\quad + C R^{-1} \lambda \int_\Omega |y - x_0|\, \psi_2(y) u(y,t)\, dx \\
&= C R^{-1} \lambda \int_\Omega |x - x_0|\, (\psi_1(x)^{1/2} + \psi_1(x)) u(x,t)\, dx \\
&\quad + C R^{-1} \lambda \int_\Omega |x - x_0|\, (\psi_2(x) - \psi_1(x)) u(x,t)\, dx \\
&\leq C R^{-1} \lambda^{3/2} I_1^{1/2} + C \lambda \int_\Omega (\psi_2(x) - \psi_1(x)) u(x,t)\, dx.
\end{aligned}$$

Thus, we have

$$\text{IV} \leq -\frac{1}{\pi} M_1^2 + C R^{-1} \lambda^{3/2} I_1^{1/2} + C \lambda \int_\Omega (\psi_2(x) - \psi_1(x)) u(x,t)\, dx.$$

These relations are summarized as

$$\frac{d}{dt} I_1 \leq 4 M_1 - \frac{1}{\pi} M_1^2 + C_* R^{-1} \left(\lambda^{3/2} + \lambda^{1/2} \right) I_1^{1/2} + C_* \lambda (M_2 - M_1) \tag{5.32}$$

for $t \in [0, T_{\max})$ with a constant $C_* > 0$ independent of $R > 0$ sufficiently small, say $0 < R \leq 1$.

Here, we have

$$\begin{aligned}
M_2(t) - M_1(t) &\leq \int_{R/2 < |x - x_0| < 8R} \psi_2(x) u(x,t)\, dx \\
&\leq \frac{2}{R} \int_\Omega |x - x_0|\, \psi_2(x) u(x,t)\, dx \leq 2 \lambda^{1/2} R^{-1} I_2(t)^{1/2}
\end{aligned}$$

and hence

$$\frac{dI_1}{dt} \leq 4 M_1 - \frac{M_1^2}{\pi} + C_1 R^{-1} \left(\lambda^{3/2} + \lambda^{1/2} \right) I_2^{1/2}$$

follows. We also have

$$\begin{aligned}
I_2(t) &= I_1(t) + \int_\Omega |x - x_0|^2\, (\psi_2(x) - \psi_1(x))\, u(x,t)\, dx \\
&\leq I_1(t) + 16 R^2 \int_\Omega (\psi_2(x) - \psi_1(x))\, u(x,t)\, dx
\end{aligned}$$

and therefore

$$\frac{dI_1}{dt} \le 4M_1 - \frac{M_1^2}{\pi} + C_2 R^{-1}\left(\lambda^{3/2} + \lambda^{1/2}\right) I_1^{1/2}$$
$$+ 4C_2\big(\lambda^{3/2} + \lambda^{1/2}\big)\bigg\{\int_\Omega (\psi_2(x) - \psi_1(x))\, u(x,t)\, dx\bigg\}^{1/2}$$

is obtained.

Here, from (5.11) we deduce

$$\left|\frac{d}{dt}\Big(4M_1 - \frac{M_1^2}{\pi}\Big)\right| \le \Big(4 + \frac{\lambda}{\pi}\Big)\cdot\big(\lambda + \lambda^2\big)C_* R^{-2}$$

and

$$\left|\frac{d}{dt}\int_\Omega \Big(\psi_2(x) - \psi_1(x)\Big)u(x,t)\, dx\right| \le C_*\big(\lambda + \lambda^2\big)R^{-2},$$

noting $0 < R \le 1$. Therefore, we obtain

$$\frac{dI_1}{dt} \le 4M_1(0) - \frac{M_1(0)^2}{\pi} + C_2 R^{-1}\big(\lambda^{3/2} + \lambda^{1/2}\big)I_1^{1/2}$$
$$+ 4C_3\big(\lambda^{2/3} + \lambda^{1/2}\big)\bigg\{\int_\Omega (\psi_2(x) - \psi_1(x))\, u_0(x)\, dx\bigg\}^{1/2}$$
$$+ C_3\big(\lambda^{1/2} + \lambda^3\big)\big(R^{-2}t + R^{-1}t^{1/2}\big).$$

Finally, we have

$$\int_\Omega (\psi_2(x) - \psi_1(x))\, u_0(x)\, dx \le \int_{B(x_0,4R)\setminus B(x_0,R/2)} u_0(x)\, dx$$
$$\le 4R^{-2} I_2(0).$$

Thus, writing

$$B = C_2\big(\lambda^{3/2} + \lambda^{1/2}\big),$$
$$a(s) = C_3\big(\lambda^{1/2} + \lambda^3\big)(s^2 + s),$$
$$J(t) = 4M_1(t) - \frac{M_1(t)^2}{\pi} + 8BR^{-1}I_2(t)^{1/2},$$

and $I(t) = I_1(t)$, we obtain

$$\frac{dI}{dt} \le J(0) + a\big(R^{-1}t^{1/2}\big) + BR^{-1}I(t)^{1/2}. \tag{5.33}$$

Under the assumption of $M_1(0) > 4\pi$, we can take $\eta > 0$ so small that inequality (5.2) implies the existence of $T >$ satisfying

$$J(0) + a\left(R^{-1}T^{1/2}\right) + BR^{-1}I(0)^{1/2} = -2\delta < 0$$

and

$$I(0) - 2\delta T < 0.$$

Then, the standard continuation argument guarantees that

$$\frac{dI}{dt} \leq 0 \quad \text{and} \quad I(t) \leq I(0)$$

for $t \in [0, T)$. This implies

$$\frac{dI}{dt} \leq -2\delta \qquad (0 \leq t < T)$$

and

$$I(T) \leq I(0) - 2\delta T < 0,$$

in turn. Then it is a contradiction and the proof is complete. □

The case $x_0 \in \Omega$ is treated similarly, where the second moments $I_i(t)$ $(i = 1, 2)$ can be localized without using the conformal mapping. Namely, we replace the weight function by

$$m(x) = |x - x_0|^2 .$$

Then, the second term of the right-hand side of (5.29) is not involved in the proof of the analogous fact to Lemma 5.3. Therefore, the inequality

$$\begin{aligned} \left|\rho(x, x') + \frac{1}{\pi}\psi_1(x)\psi_2(x')\right| &\leq CR^{-1}\left(|x - x_0| + \left|x' - x_0\right|\right) \\ &\quad \cdot \psi_1(x)^{1/2}\psi_2(x') + CR^{-1}\left|x' - x_0\right|\psi_2(x')^{1/2} \end{aligned}$$

takes the place of (5.28) with $\rho(x, y)$ kept in the right-hand side of (5.27). Consequently, inequality (5.32) is replaced by

$$\frac{dI_1}{dt} \leq 4M_1 - \frac{M_1^2}{2\pi} + C_* R^{-1}\left(\lambda^{3/2} + \lambda^{1/2}\right)I_1^{1/2} + C_*\lambda\left(M_2 - M_1\right), \tag{5.34}$$

and the following theorem is obtained.

Theorem 5.2 *If* $\Omega \subset \mathbf{R}^2$ *is a bounded domain with smooth boundary* $\partial\Omega$ *and*

$$\int_{B(x_0,R)} u_0(x)\,dx > 8\pi$$

holds for $x_0 \in \Omega$ *and* $0 < R \ll 1$ *in (5.1), then there exists* $\eta > 0$ *determined by* $\lambda = \|u_0\|_1$ *and* $\|u_0\|_{L^1(B(x_0,R))}$ *such that*

$$\frac{1}{R^2}\int_{B(x_0,4R)} |x - x_0|^2\, u_0(x)\,dx < \eta$$

implies $T_{\max} < +\infty$.

Inequality (5.33) has a general form including the case of $x_0 \in \Omega$. In details, taking

$$m(x) = \begin{cases} |x - x_0|^2 & (x_0 \in \Omega) \\ |X(x)|^2 \,/\, \left|\frac{\partial X}{\partial x}(x_0)\right| & (x_0 \in \partial\Omega) \end{cases}$$

and $\psi_R = \varphi^4_{x_0,R,2R}$, we put

$$\begin{aligned} I_R(t) &= \int_\Omega m(x)u(x,t)\psi_R(x)\,dx \\ M_R(t) &= \int_\Omega u(x,t)\psi_R(x)\,dx \\ J_R(t) &= 4M_R(t) - \frac{4M_R(t)^2}{m_*(x_0)} + 8BR^{-1}I_{4R}(t)^{1/2} \end{aligned}$$

for $0 < R \ll 1$. Then it holds that

$$\frac{dI_R}{dt}(t) \le J_R(0) + a\big(R^{-1}t^{1/2}\big) + BR^{-1}I_R(t)^{1/2} \tag{5.35}$$

for $t \in [0, T_{\max})$, where

$$B = C_*\big(\lambda^{3/2} + \lambda^{1/2}\big)$$

and

$$a(s) = C_*\big(\lambda^{1/2} + \lambda^3\big)(s^2 + s)$$

with $C_* > 0$ determined by Ω.

Inequality (5.35) will provide a motivation for the proof of Theorem 1.2.

6
Equilibrium States

Self-assembly is the beginning of the selfness of life.
— H. Tanaka

In this chapter, we begin the study of the stationary problem to (3.1):

$$\begin{aligned} u_t = \nabla \cdot \left(\nabla u - u \nabla \left(v + \log W \right) \right) \quad & \text{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu} \left(v + \log W \right) = 0 \quad & \text{on} \quad \partial\Omega \times (0, T), \\ \tau \frac{d}{dt} v + A v = u \quad & \text{for} \quad t \in (0, T), \end{aligned} \tag{6.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $W = W(x) > 0$ is a smooth function of $x \in \overline{\Omega}$, and $A > 0$ is a self-adjoint operator in $L^2(\Omega)$ with compact resolvent. This study provides several heuristic supports Theorems 1.1 and 1.2, although it does not bring any rigorous proof or tool to them.

In the nontrivial case $u_0(x) \not\equiv 0$ of

$$u|_{t=0} = u_0(x) \geq 0,$$

we have $u(x, t) > 0$ for $(x, t) \in \overline{\Omega} \times (0, T_{\max})$, and system (6.1) is provided with the mass conservation

$$\left\| u(t) \right\|_1 = \left\| u_0 \right\|_1 \tag{6.2}$$

and decrease of the Lyapunov function,

$$\mathcal{W}(u, v) = \int_\Omega (u \log u - u \log W - uv)\ dx + \frac{1}{2}\left\|A^{1/2}v\right\|^2 \tag{6.3}$$

satisfying (4.2):

$$\frac{d}{dt}\mathcal{W}(u, v) + \tau\left\|v_t\right\|^2 + \int_\Omega u\left|\nabla\left(\log u - v - \log W\right)\right|^2\ dx = 0. \tag{6.4}$$

We recall that $T_{\max} > 0$ denotes the supremum of the existence time of the solution and $\|\ \cdot\ \|$ and $(\ ,\ )$ indicate $\|\ \cdot\ \|_2$ and L^2 inner product, respectively:

$$\|v\| = \left\{\int_\Omega v^2\, dx\right\}^{1/2}, \qquad (v, w) = \int_\Omega vw\, dx.$$

If

$$\frac{d}{dt}\mathcal{W}\left(u(t), v(t)\right) = 0$$

holds at some $t = t_0 \in (0, T_{\max})$, then it follows that

$$\log u - v - \log W = \text{constant} \quad \text{in} \quad \Omega$$

for $u = u(t_0)$ and $v = v(t_0)$ from the third term of the left-hand side of (6.4). Thus, we obtain

$$u = \frac{\lambda W e^v}{\int_\Omega W e^v\, dx} \tag{6.5}$$

for $\lambda = \|u_0\|_1$. In the case of $\tau > 0$, (6.4) implies also $v_t(t_0) = 0$. Therefore, in both cases of $\tau > 0$ and $\tau = 0$, we have

$$u = Av$$

by (6.1), and hence

$$v \in \operatorname{dom}(A) \quad \text{and} \quad Av = \frac{\lambda W e^v}{\int_\Omega W e^v\, dx} \tag{6.6}$$

follow, where $\operatorname{dom}(A)$ denotes the domain of A. If $v = v(x)$ solves (6.6) conversely, then (u, v) with $u = u(x)$ defined by (6.5) is a stationary solution to (6.1) satisfying (6.2). Thus, the stationary problem for (6.1) with $\|u_0\|_1 = \lambda$ is formulated as (6.6), which is equivalent to

$$\log u - A^{-1}u - \log W = \text{constant}, \qquad \left\| u \right\|_1 = \lambda \tag{6.7}$$

in terms of $u = u(x) > 0$.

Problem (6.6) with $n = 2$ arises in several areas — geometry, fluid dynamics, field theory, combustion theory, chemical reaction theory, and so on — and we can use the method of complex variables, spectral analysis combined with isoperimetric inequalities on surfaces, the bubbled Harnack principle, and so forth [166]. Another device is the method of rescaling, often called the *blowup analysis* [88].

An example of (6.6) is

$$\begin{aligned} -\Delta v &= \frac{\lambda W e^v}{\int_\Omega W e^v \, dx} && \text{in} \quad \Omega, \\ v &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{6.8}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $W = W(x) > 0$ is a smooth function defined on $\overline{\Omega}$. This problem is related to the *complex function theory* and the *theory of surfaces*, but if $W \equiv 1$ it arises also in statistical mechanics as the *mean field equation* of many vortex points in *Onsagar's formulation* [82, 20, 21].

Another example is the prescribed *Gaussian curvature equation* studied by Kazdan and Warner [80],

$$-\Delta_g v = \lambda \Big(\frac{W e^v}{\int_{\mathcal{M}} W e^v \, dv_g} - V \Big) \quad \text{on} \quad \mathcal{M}, \tag{6.9}$$

where $(\mathcal{M}, g)$ is a compact Riemannian surface, and $V = V(x)$ and $W = W(x)$ are smooth functions on $\mathcal{M}$ satisfying $W(x) > 0$ somewhere and

$$\int_{\mathcal{M}} V \, dv_g = 1.$$

This problem is relative to the stationary problem of chemotaxis associated with the (JL) field, and here, Ω is replaced by the two-dimensional Riemannian surface $\mathcal{M}$ without boundary and Δ_g and dv_g indicate the *Laplace–Beltrami operator* and *volume element*, respectively.

The constant case of $V = V(x)$ often arises in (6.9):

$$-\Delta_g v = \lambda \Big(\frac{W e^v}{\int_{\mathcal{M}} W e^v dv_g} - \frac{1}{\text{vol}\,(\mathcal{M})} \Big) \quad \text{on} \quad \mathcal{M}. \tag{6.10}$$

In fact, the solution of Tarantello [171], describing the limiting state in the relativistic Abelian *Chern-Simons gauge theory* concerning superconductivity in high temperature, solves (6.10), where $(\mathcal{M}, g)$ is a flat torus, $\lambda = 4\pi$, and $W = \exp u_0$ with $u_0 = u_0(x)$ satisfying

$$-\Delta u_0 = \frac{4\pi N}{v_g(\mathcal{M})} - 4\pi \sum_{j=1}^{N} \delta_{p_j}(dx) \quad \text{on} \quad \mathcal{M}$$

and

$$\int_{\mathcal{M}} u_0 \, dv_g = 0$$

for $p_1, \ldots, p_N \in \mathcal{M}$. On the other hand, *Nirenberg's problem* is nothing but (6.10) in the case of $\mathcal{M} = S^2$ and $\lambda = 8\pi$ [4, 22, 26, 86, 103].

Finally, we note that (6.6) in the (N) field is described as

$$\begin{aligned} -\Delta v + av &= \frac{\lambda W e^v}{\int_\Omega W e^v \, dx} && \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{in} \quad \partial\Omega, \end{aligned} \tag{6.11}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a positive constant, $W = W(x) > 0$ is a smooth function defined on $\overline{\Omega}$, and ν is the outer unit normal vector on $\partial\Omega$.

Variational structures of these problems have their own real analytic profiles. For instance, $v = v(x)$ is a solution to (6.8) with $W(x) \equiv 1$ if and only if it is a critical point of the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log\Big(\int_\Omega e^v \, dx\Big)$$

defined for $v \in H_0^1(\Omega)$. Then, the Trudinger–Moser inequality [102] assures its global minimizer for $\lambda \in (0, 8\pi)$, and hence the solution to (6.8). If $\Omega \subset \mathbf{R}^2$ is simply connected and $\lambda \in (0, 8\pi)$, then this problem has a unique solution [113, 162]. It is proven by the bifurcation theory, spectral analysis, and an isoperimetric inequality on surfaces [5]. If Ω has genus $g \geq 1$, on the contrary, then the *mini-max principle* is applicable, and there is a solution for $\lambda \in (8\pi, 16\pi)$ [39]. It is proven by the concentration behavior of blowup functions associated with the Trudinger–Moser inequality. (See [25] concerning the nontriviality of this type of solution applied to the mean field equation (6.9) with constants $W(x)$ and $V(x)$.) If $W(x)$ is not constant in (6.8), on the other hand, then the associated functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega W(x)e^v\,dx\right)$$

defined for $v \in H_0^1(\Omega)$ may have the minimizer for $\lambda = 8\pi$ [121].

If $W = W(x)$ is constant, then problems (6.10) and (6.11) admit constant solutions. Nonconstant solutions to these problems are obtained by the *mountain pass lemma* [161, 144], where the blowup analysis is applied. In more detail, the blowup of a family of solutions can occur only at the quantized value of λ in $8\pi\mathbf{N}$ or $4\pi\mathbf{N}$ [18, 88], and therefore, first, existence of the solution is assured for almost every λ by Struwe's argument [74], and then the solution for the nonquantized value of λ is obtained by this quantization.

In general, the quantized blowup mechanism for the family of solutions plays a fundamental role in the study of this kind of elliptic problem [155]. This quantization was observed first in (6.8) with $W(x) \equiv 1$ using complex variables [114, 115]. Then, [18, 88] introduced the method of the *Green's function* and that of the *blowup analysis* using sup + inf inequality [140], and after that, some refinements were done [87, 94, 190]. If the boundary condition is not imposed, then the multiblowup points can occur to the solution sequence [18, 32, 88]. On the contrary, extra constraints on the family such as the boundary condition make any blowup point simple, and furthermore, their locations are controlled by the Green's function [87, 94, 115, 190].

The quantized blowup mechanism stated above induces a related, but different approach to (6.6) from the variational method, that is, the *topological degree* [87]. The advantage of this method is its stability under rough perturbations, and we shall describe this situation in more detail for the Gaussian curvature equation (6.9). A related topic is the reverse theory of the classification of the singular limit as $\lambda \to \lambda_0 \in 8\pi\mathbf{N}$, that is, *singular perturbation*, constructing classical solutions close to each singular limit for (6.8) with $W(x) \equiv 1$ [100, 184]. After several refinements and generalizations [101, 163, 183], Baraket and Pacard [7] showed that the classification of [115] is optimal; the generic singular limit of [115] generates a family of classical solutions converging to it.

Describing the topological degree approach, we recall that the blowup of the solution sequence occurs only at the quantized values of λ, and therefore we have local uniform boundedness of the solution associated with λ in each connected component of $\mathbf{R} \setminus 8\pi\mathbf{N}$. In particular, the *total degree* of the solution, denoted by $d(\lambda)$, is constant in each connected component of

$\lambda \in [0, \infty) \setminus 8\pi\mathbf{N}$. The quantized blowup mechanism admits even continuous perturbation of $W = W(x) > 0$ in the uniform norm. In more details, $d(\lambda)$ does not depend on the continuous function $W = W(x) > 0$, nor on the conformal deformation of $\mathcal{M}$, and therefore we can take suitable representatives in calculating $d(\lambda)$. In a serious papers, C.-C. Chen and C.-S. Lin used this structure, and applied the methods of moving plane and moving sphere [27, 28, 29, 89, 90, 91]. Then, using the blowup analysis, they succeeded in calculating $d(\lambda)$ precisely, and the total degree $d(\lambda)$ is determined by the topology of $\mathcal{M}$ [30, 31]. In more detail, by the study of the linearized operator using the scaling argument, they calculated the discrepancy of $d(\lambda)$ at each quantized value $\lambda \in 8\pi\mathcal{N}$.

Actually, for (6.10) we have

$$d(\lambda) = \binom{m - \chi(\mathcal{M})}{m}$$

if $\lambda \in (8\pi m, 8\pi(m+1))$ with $m = 0, 1, \ldots,$ where $\chi(\mathcal{M})$ denotes the *Euler characteristics* of $\mathcal{M}$ and

$$\binom{m_1}{m_2} = \begin{cases} \dfrac{m_2(m_2-1)\cdots(m_2-m_1+1)}{m_1!} & (m_1 > 0), \\ 1 & (m_1 = 0). \end{cases}$$

If g denotes the *genus* of $\mathcal{M}$, then it holds that $\chi(\mathcal{M}) = 2 - 2g$. In particular, $\chi(S^2) = 2$ and hence we have

$$d(\lambda) = \frac{(m-2)(m-3)\cdots(-1)}{m!} = \begin{pmatrix} 0 & (m \geq 2) \\ -1 & (m = 1), \\ 1 & (m = 0), \end{pmatrix} \tag{6.12}$$

for $\lambda \in (8\pi m, 8\pi(m+1))$ whenever $\mathcal{M}$ is homeomorphic to S^2. In the case that $\mathcal{M}$ is homeomorphic to the torus T^2, we have $\chi(\mathcal{M}) = 0$, and it holds that

$$d(\lambda) = 1 \tag{6.13}$$

for any $\lambda \in [0, \infty) \setminus 8\pi\mathbf{N}$.

In the (JL) field defined by (3.4), if $\Omega \subset \mathbf{R}^2$ is simply connected, then the domain Ω is conformally equivalent to the chemisphere. Attaching two copies of them, we obtain (6.9) on S^2 by the boundary condition of (9.5). Therefore, the latter problem is transformed into the former problem with even symmetry. Similarly, if Ω is doubly connected, then we reach the problem on the torus.

Therefore, if we can find $W = W(x)$ such that any solution to (6.10) has such a symmetry, then formula (6.12) is transformed into a form valid for this problem, that is, λ replaced by 2λ. (So far, this argument is not justified, although there is $K = K(x)$ such that any solution to (6.10) for $\mathcal{M} = S^2$ is rotation invariant [89].)

Concerning (6.8) with $0 < W = W(x) \in C^1(\overline{\Omega})$, we have

$$d(\lambda) = \binom{m+1-g}{m}$$

for $\lambda \in (8\pi m, 8\pi(m+1))$, where g denotes the genus of Ω, and in particular, if Ω is not simply connected, then any $\lambda \notin 8\pi\mathbf{N}$ admits a solution [31]. Using the quantized blowup mechanism again, this existence result is valid for all $\lambda \notin 8\pi\mathbf{N}$ and $0 < W = W(x) \in C(\overline{\Omega})$, and in this way we have an extension of the result of [39] concerning the existence of the solution in the case of $g \geq 1$.

In this book, problem (6.6) is regarded as a stationary state of (6.1), and the effect that these profiles of the stationary problem suggest for the dynamics of the nonstationary problem is studied. In fact, we call $\tau > 0$ and $\tau = 0$ the full and simplified systems, respectively. In the simplified system, we have $v = A^{-1}u$, and therefore the Lyapunov function $\mathcal{W}(u, v)$ defined by (6.3),

$$\mathcal{W}(u, v) = \int_\Omega (u \log u - u \log W - uv)\ dx + \frac{1}{2}\left\| A^{1/2} v \right\|^2,$$

is reduced to the free energy:

$$\begin{aligned}\mathcal{F}(u) &= \mathcal{W}(u, A^{-1}u) \\ &= \int_\Omega (u \log u - u \log W)\ dx - \frac{1}{2}(A^{-1}u, u).\end{aligned}$$

Since the stationary state is regarded as a critical point of the free energy, this $\mathcal{F}(u)$ induces a variational structure to (6.1), that is, in the stationary state, $u = u(x) > 0$ is a critical function of the functional $\mathcal{F}$ defined on

$$\mathcal{P}_\lambda = \left\{ u : \text{measurable} \;\middle|\; u \geq 0 \text{ a.e.}, \quad \left\| u \right\|_1 = \lambda \right\}, \tag{6.14}$$

and a similar variational structure is adopted for (6.8) [20, 21].

On the other hand, problem (6.6) has a variational structure of its own; it is not hard to see, at least formally, that $v = v(x)$ is a solution to (6.6) if and only if it is a critical point of the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\left\| A^{1/2} v \right\|^2 - \lambda \log \left(\int_\Omega W e^v\, dx \right)$$

defined for $v \in V \equiv D(A^{1/2})$. Furthermore, the linearized operator around the stationary solution $v = v(x)$ is associated with the bilinear form defined on $V \times V$,

$$\mathcal{A}(\varphi, \varphi) = \left\| A^{1/2}\varphi \right\|^2 - \int_\Omega p\varphi^2 \, dx + \frac{1}{\lambda}\left\{ \int_\Omega p\varphi \, dx \right\}^2,$$

where

$$p = \frac{\lambda W e^v}{\int_\Omega W e^v \, dx},$$

and the abstract theory of Friedrichs-Kato [77] concerning the *self-adjoint operator* and *associated bilinear form* is applicable.

In this way we have two structures of variation to the stationary problem of (6.1). Actually, they are equivalent to each other. Here is the key identity for the proof:

$$\mathcal{W}\Big(\frac{\lambda W e^v}{\int_\Omega W e^v \, dx}, v \Big) = \mathcal{J}_\lambda(v) + \lambda \log \lambda. \tag{6.15}$$

This means that the functionals $\mathcal{F}(u)$ and $\mathcal{J}_\lambda(v) + \lambda \log \lambda$ are nothing but the restrictions of $\mathcal{W}(u, v)$ to the manifolds

$$\mathcal{M} = \Big\{ (u, v) \mid v = A^{-1}u, \ \left\| u \right\|_1 = \lambda \Big\}$$

and

$$\mathcal{N} = \Big\{ (u, v) \mid u = \frac{\lambda W e^v}{\int_\Omega W e^v \, dx} \Big\},$$

respectively. Then, we can see that the intersection of these manifolds coincides with the set of stationary solutions. However, these manifolds $\mathcal{M}$ and $\mathcal{N}$ meet transversally, and the spectral equivalence described above is never trivial. Here, an algebraic property of $\mathcal{W}(u, v)$ takes a role and we have the vanishing of $\mathcal{W}_v$ and $\mathcal{W}_u$ on the tangential bundles of $\mathcal{M}$ and $\mathcal{N}$, respectively. Nevertheless, this spectral equivalence is still reasonable, because $\mathcal{F}(u)$, $\mathcal{W}(u, v)$, and $\mathcal{J}_\lambda(v) + \lambda \log \lambda$ are regarded as the free energies for (6.1) with $\tau = 0$, $0 < \tau < +\infty$, and $\tau = +\infty$, respectively.

Actually, the Lyapunov function $\mathcal{W}(u, v)$ defined by (6.3) has a remarkable structure. The first term,

$$\int_\Omega u(\log u - 1) \, dx,$$

indicates physically (-1) times entropy, but is associated with the Zygmund norm mathematically. This structure is described at the end of Chapter 4; the Orlicz space $L \log L(\Omega)$ is provided with the norm

$$[f]_{L\log L} = \int_\Omega |f| \log\left(e + \frac{|f|}{\|f\|_1}\right) dx.$$

We note that $L \log L(\Omega)$ and $\exp(\Omega)$ form a duality, which is regarded as a local version of the one between Hardy and BMO spaces valid in a two-space dimension [135]. In this context, the third term of $\mathcal{W}(u, v)$,

$$\int_\Omega uv\, dx,$$

is nothing but the paring of this duality, and the fourth term,

$$\frac{1}{2}\left\|A^{1/2}v\right\|^2$$

is usually equivalent to the square of the H^1 norm. Actually, in the concrete cases of the (N), (JL), and (D) fields, we have $V = D(A^{1/2})$ for $V = H^1(\Omega)$, $V = H^1(\Omega) \cap L_0^2(\Omega)$, and $V = H_0^1(\Omega)$, respectively, where

$$L_0^2(\Omega) = \left\{ v \in L^2(\Omega) \mid \int_\Omega v\, dx = 0 \right\}.$$

In the case of $n = 2$, furthermore, we have a fine real analytic structure of $H^1 \subset \mathrm{BMO}$, which guarantees

$$\mathcal{W}(u, v) \quad \approx \quad [u]_{L\log L} + \|v\|_V^2 - \langle v, u\rangle_{L\log L, \exp}$$

provided with the inclusion and the duality

$$V \hookrightarrow \exp \cong (L \log L)'.$$

These structures are useful for the construction of the local dynamical theory around the equilibrium point of (6.1).

Equivalence of these variations can be extended to the general system possessing the Lyapunov function. The underlying structure is nothing but the *Toland duality* in a convex analysis, where the Lyapunov function acts as the *Lagrange function*. This formulation covers many mathematical models proposed in mean field theories, and especially their local dynamics around the equilibrium point is described in a unified way. See the final chapter.

7

Blowup Analysis for Stationary Solutions

This chapter studies the quantized blowup mechanism of the stationary system of chemotaxis. Actually, Ma and Wei [94] took

$$\begin{aligned} -\Delta v &= \frac{\lambda W(x)e^v}{\int_\Omega W(x)e^v\,dx} \quad \text{in} \quad \Omega, \\ v &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \end{aligned} \tag{7.1}$$

and showed the following theorem, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $W = W(x) > 0$ is a C^1 function of $x \in \overline{\Omega}$.

Theorem 7.1 *If $\{(\lambda_k, v_k)\}$ is a family of solutions to (7.1) satisfying*

$$\lambda_k \to \lambda_0 \in [0, +\infty) \quad \textit{and} \quad \left\| v_k \right\|_\infty \to +\infty,$$

then $\lambda_0 \in 8\pi\mathbf{N}$. Furthermore, passing through a subsequence, we have

$$\lambda_0 = 8\pi \cdot \sharp\,\mathcal{S}$$

with $\mathcal{S}$ being the blowup set:

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid v_k(x_k) \to +\infty,\ x_k \to x_0 \textit{ for some } \{x_k\} \subset \Omega \right\}.$$

We have $\mathcal{S} \subset \Omega$ and

$$v_k(x) \ \to \ \sum_{x_0 \in \mathcal{S}} 8\pi G(x, x_0) \tag{7.2}$$

locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$, and the location of blowup points is controlled by

$$\nabla_x \Big(8\pi K(x, x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} 8\pi G(x, x_0')\Big)\Big|_{x=x_0} + \nabla_x \log W(x)\Big|_{x=x_0} = 0, \quad (7.3)$$

which is valid for any $x_0 \in \mathcal{S}$. Here, $G = G(x, x')$ denotes the Green's function for $-\Delta$ under the Dirichlet boundary condition, and

$$K(x, x') = G(x, x') + \frac{1}{2\pi} \log \left| x - x' \right|$$

is its regular part.

This theorem is a generalization of [115] for the case $W(x) = 1$, where the method of complex variables is applied. The idea is roughly described as follows [166].

First, we take

$$z = x_1 + \imath x_2 \quad \text{and} \quad \overline{z} = x_1 - \imath x_2$$

for $x = (x_1, x_2) \in \Omega$, and put

$$s = v_{zz} - \frac{1}{2} v_z^2 \quad (7.4)$$

for the solution $v = v(x)$ to (7.1). Then, the equation

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v \, dx}$$

reads

$$v_{z\overline{z}} = -\frac{\sigma}{4} e^v$$

for $\sigma = \lambda / \int_\Omega e^v \, dx$, and hence it follows that

$$s_{\overline{z}} = v_{zz\overline{z}} - v_z v_{z\overline{z}} = 0,$$

and therefore $s = s(z)$ is a *holomorphic function* of z. On the other hand, (7.4) is regarded as the *Riccati equation* of v and hence $\phi = e^{-v/2}$ satisfies

$$\phi_{zz} + \frac{1}{2} s\phi = 0. \quad (7.5)$$

Now, we take the fundamental system of solutions to (7.5), denoted by $\{\phi_1, \phi_2\}$, satisfying

$$\phi_1\big|_{z=z^*} = \frac{\partial}{\partial z}\phi_2\big|_{z=z^*} = 1$$

and

$$\frac{\partial}{\partial z}\phi_1\big|_{z=z^*} = \phi_2\big|_{z=z^*} = 0,$$

where $z^* = x_1^* + \imath x_2^*$ is obtained by the maximum point $x^* = (x_1^*, x_2^*)$ of $v(x)$. Then it holds that

$$e^{-v/2} = \overline{f_1}(\overline{z})\phi_1(z) + \overline{f_2}(\overline{z})\phi_2(z)$$

with some functions $\overline{f_1}$ and $\overline{f_2}$ of $\overline{z}$. Here, we have

$$\begin{aligned} f_1(z) = C_1\phi_1(z), &\quad f_2(z) = C_2\phi_2(z) \\ C_1 = e^{-v/2}\big|_{x=x^*}, &\quad C_2 = \frac{\sigma}{8}e^{v/2}\big|_{x=x^*}, \end{aligned}$$

where $f_1(z) = \overline{\overline{f}_1(\overline{z})}$ and $f_2(z) = \overline{\overline{f}_1(\overline{z})}$, and therefore

$$e^{-v/2} = C_1\,|\phi_1|^2 + C_2\,|\phi_2|^2\,. \tag{7.6}$$

Next, given a family $\big\{\big(\lambda_k, v_k(x)\big)\big\}$ satisfying

$$-\Delta v_k = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k}\,dx} \quad \text{in} \quad \Omega, \qquad v_k = 0 \quad \text{on} \quad \partial\Omega \tag{7.7}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0, +\infty), \qquad \lim_{k\to\infty} \left\|v_k\right\|_\infty = +\infty,$$

we apply the arguments of *reflection* and the *boundary estimate* [36, 52], and show that $\{v_k(x)\}$ is uniformly bounded near the boundary up to its derivatives of any order. Thus, we obtain $\mathcal{S} \subset \Omega$, and hence the uniform boundedness of the holomorphic functions $\{s_k(z)\}$ near $\partial\Omega$. This implies $\|s_k\|_\infty = O(1)$ by the maximum principle, and then *Montel's theorem* assures the existence of a subsequence, still denoted by $\{s_k(z)\}$, and that of a holomorphic function $s_0 = s_0(z)$ of $z \in \Omega$ such that

$$s_k(z) \;\to\; s_0(z) \tag{7.8}$$

locally uniformly in Ω.

The holomorphic function $s = s_k(z)$, on the other hand, induces the analytic functions $\phi_1 = \phi_{k1}(z)$ and $\phi_2 = \phi_{k2}(z)$ of $z \in \Omega$ similarly, and the above convergence (7.8) implies the existence of their limiting functions denoted by $\phi_{01}(z)$ and $\phi_{02}(z)$, that is,

$$\lim_{k\to\infty} \phi_{k1}(z) = \phi_{01}(z), \qquad \lim_{k\to\infty} \phi_{k2}(z) = \phi_{02}(z),$$

locally uniformly in $z \in \Omega$. Then, making $k \to \infty$ in (7.6) for $(v, \phi_1, \phi_2) = (v_k, \phi_{k1}, \phi_{k2})$, we obtain

$$e^{-v_0/2} = C_{02}\, |\phi_{02}|^2$$

with $C_{02} \in [0, +\infty]$, because $C_{k1} = e^{-\|v_k\|_\infty/2} \to 0$. Here, $v_0(x)$ is the limiting function of $\{v_k(x)\}$ in the weak topology of $W^{1,q}(\Omega)$ for $q \in (1, 2)$, which is assured by the L^1 estimate [19]. We can show also that $\lambda_0 < +\infty$ implies $C_{02} < +\infty$, and that the blowup set of $\{v_k\}$, denoted by $\mathcal{S}$, coincides with the set of zeros of ϕ_{02}. Therefore, $\mathcal{S}$ is discrete in Ω from the *theorem of identity*, because the analytic function ϕ_{02} cannot be identically zero. This implies the finiteness of $\mathcal{S}$ by $\mathcal{S} \subset \Omega$. We can show also

$$\int_\Omega e^{v_k}\, dx \;\to\; +\infty \tag{7.9}$$

and therefore $v_0 = v_0(x)$ is a harmonic function of $x \in \Omega \setminus \mathcal{S}$. However, each $x_0 \in \mathcal{S}$ is a removable isolated singular point of

$$s_0 = v_{0zz} - \frac{1}{2} v_{0z}^2,$$

and then the *residue analysis* of vanishing coefficients of poles of the first and the second orders guarantees the mass quantization in the form of

$$-\Delta v_0(x)\, dx = \sum_{x_0 \in \mathcal{S}} 8\pi\, \delta_{x_0}(dx),$$

and also the control of the location of the blowup point (7.3), respectively.

In this argument, relation (7.9) takes an important role. In this connection, we mention that first, the above problem was formulated by

$$-\Delta v_k = \sigma_k e^{v_k} \quad \text{in} \quad \Omega, \qquad v_k = 0 \quad \text{in} \quad \partial\Omega,$$

with $\sigma_k \downarrow 0$ [115]. This formulation includes the case

$$\lambda_k = \sigma_k \int_\Omega e^{v_k}\, dx \;\to\; +\infty,$$

where $v_k(x) \to +\infty$ occurs for any $x \in \Omega$. (Radially symmetric solutions on annulus have such a profile. See [92, 111, 116].)

To treat the general nonlinearity, [115] used the estimate of [53] derived from *Obata's identity* [122], and consequently some technical conditions about the nonlinearity were assumed. Then, using another argument, [190] eliminated these assumptions, and thus the asymptotic behavior of the solution presented in Theorem 7.1 is valid for any exponentially dominated nonlinearity, homogeneous in space. On the other hand, [7] constructed a family of solutions for the homogeneous case $W(x) = 1$ under the nondegeneracy of the blowup points. In fact, (7.3) in this case means that $(x_1^*, \ldots, x_m^*)$ is a critical point of

$$\mathcal{H}(x_1, \ldots, x_m) = \frac{1}{2}\sum_i R(x_i) + \sum_{i \neq j} G(x_i, x_j),$$

where $\mathcal{S} = \{x_1^*, \ldots, x_m^*\}$. If this critical point is nondegenerate, then there is a family of solutions $\{(\lambda, v(x))\}$ for (7.1) satisfying (7.2) locally uniformly on $\overline{\Omega} \setminus \mathcal{S}$. In this connection, we have the previous work [155, 183] on the rectangular domain. See also [166] for their background and motivations.

To treat the inhomogeneous case of (7.1), [18] made use of a rough estimate to derive (7.9). Later, [125] proposed the method of symmetrization stated below. It assures mass quantization first, and then derives (7.9) as its consequence. Actually, [18] proposed a real analytic argument based on the linear theory. More precisely, we can show the following theorem, using Brezis–Merle's inequality (4.25), although the proof is not provided here.

Theorem 7.2 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, and suppose that $v_k = v_k(x) \in C^2(\Omega)$ solves*

$$-\Delta v_k = V_k(x)e^{v_k} \quad in \quad \Omega$$

with $V_k(x)$ satisfying

$$0 \leq V_k(x) \leq b \quad in \quad \Omega \quad and \quad \int_\Omega e^{v_k}\, dx = O(1),$$

where $b > 0$ is a constant independent of $k = 1, 2, \ldots$. Then, passing through a subsequence, we have the following alternatives.

1. *$\{v_k(x)\}$ is locally uniformly bounded in Ω.*
2. *For any compact set $K \subset \Omega$, it holds that*

$$sup_K v_k \ \to \ -\infty.$$

3. *There is a set of finite points* $\{y_1, \ldots, y_p\}$ *and sequences* $\{x_k^i\}$ *for* $i = 1, \ldots, p$ *in* Ω *satisfying*

$$x_k^i \to y_i \quad \text{and} \quad v_k(x_k^i) \to +\infty.$$

Furthermore, for any compact set $K \subset \Omega \setminus \{y_1, \ldots, y_p\}$ *we have*

$$sup_K v_k \ \to \ -\infty$$

and

$$V_k(x)e^{v_k(x)}\,dx \ \rightharpoonup \ \sum_{i=1}^{p} \hat{c}_i \delta_{y_i}(dx)$$

in $\mathcal{M}(\Omega)$ *with some* $\hat{c}_i \geq 4\pi$.

Li and Shafrir [88] then introduced the blowup analysis to this problem and refined the above theorem as follows.

Theorem 7.3 *If* $\{V_k(x)\}$ *converges locally uniformly in* Ω*, then in the third alternative of the above theorem it holds that* $\hat{c}_i \in 8\pi\mathcal{N}$ *for* $i = 1, \ldots, p$.

The first fundamental technique of the blowup analysis is the scaling. More precisely, if

$$-\Delta v = V(x)e^{v} \tag{7.10}$$

holds, then $\tilde{v}(x) = v(\delta x + x_0) + 2\log\delta$ with $\delta = e^{-v(x_0)/2}$ satisfies

$$-\Delta\tilde{v} = V(\delta x + x_0)e^{\tilde{v}} \tag{7.11}$$

and therefore any result valid for (7.10) is applicable to (7.11). The next technique is to “envelope” the blowup mechanism inside this transformation, and to show the vanishing of the residual term other than the collapses. For this purpose, sup + inf inequality or asymptotic symmetry is used [87, 88]. Later, we shall follow this story for the nonstationary problem (1.1), by different tools and structures.

As described above, generally, the multiblowup of $\hat{c}_i \neq 8\pi$ can occur, although it is not the case if the boundary condition is provided. Actually, it is indicated in [88] that the following theorem is obtained by G. Wolansky, and we can see the proof in [87, 118].

Theorem 7.4 *In addition to the assumptions of the previous theorem, if* $\{V_k(x)\}$ *are* C^1 *functions satisfying*

$$\left\| \nabla V_k \right\|_\infty \leq C_1$$

and it holds that

$$\max_k v_k - \min_k v_k \leq C_2$$

with the constants C_1*,* C_2 *independent of k, then in the third alternative of Theorem 7.2 we have* $\hat{c}_i = 8\pi$ *for* $i = 1, \ldots, p$.

Now, we return to [94]. In fact, problem (7.1) reads:

$$\begin{aligned} -\Delta v &= V(x)e^v \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

with

$$V(x) = \frac{\lambda W(x)}{\int_\Omega W(x)e^v\, dx}.$$

First, we show that the blowup set is contained in Ω by the argument of [52]. Namely, given $x_0 \in \partial\Omega$, we take $B(x_1, r) \subset \Omega^c$ such that $\overline{B(x_1, r)} \cap \overline{\Omega} = \{x_0\}$. Then, using the Kelvin transformation

$$y = r^2 \frac{x - x_1}{|x - x_1|^2} \quad \text{and} \quad w(y) = v(x)$$

we obtain

$$\begin{aligned} -\Delta w &= f(y, w) \quad \text{in} \quad \Omega', \\ w &= 0 \quad \text{on} \quad \partial\Omega', \end{aligned}$$

where $x \in \Omega \mapsto y \in \Omega' \subset B(0, r)$ and

$$f(y, w) = \frac{r^4}{|y|^4} V\Big(x_1 + r^2 \frac{y}{|y|^2}\Big) e^w.$$

Now, we take the outer normal derivative from Ω' at

$$y_0 = x_0 - x_1 \in \partial\Omega',$$

putting $y = \rho(x_0 - x_1)/r$ with $\rho = |y| \in (0, r)$:

$$\frac{\partial}{\partial\rho} f(y, w)\Big|_{\rho=r} = \frac{\partial}{\partial\rho}\left\{\frac{r^4}{\rho^4} V\Big(x_1 + \frac{r}{\rho}(x_0 - x_1)\Big)\right\}\Big|_{\rho=r} \cdot e^w$$

$$= \Big\{-4r^{-1}V(x_0) + \nabla V(x_0)\cdot \nu_{x_0}\Big\}\cdot e^w. \tag{7.12}$$

Here,

$$\nu_{x_0} = \frac{x_1 - x_0}{r}$$

denotes the outer unit normal vector on $\partial\Omega$ at x_0. The right-hand side of (7.12) is negative for $0 < r \ll 1$ by

$$\nabla \log V = \nabla \log W$$

and $W(x_0) > 0$, and therefore the moving plane method [52] is applicable. Consequently, we have $r > 0$ and $\delta \in (0, 1)$ such that for any $x_0 \in \partial\Omega$ and the unit vector ξ satisfying $\left|\xi \cdot \nu_{x_0}\right| < 1 - \delta$, it holds that

$$\frac{d}{ds} v(x_0 + s\xi) < 0$$

for $s \in (-r, 0)$. The L^1 estimate [19] guarantees

$$\left\| v_k \right\|_{W^{1,q}(\Omega)} = O(1)$$

for $q \in [1, 2)$, and therefore we can show that the blowup set $\mathcal{S}$ of $\{v_k\}$ is contained in Ω by the method of [36]. On the other hand, we have $v_k(x) > 0$ from the maximum principle and also $\|v_k\|_\infty \to +\infty$ from the assumption. Therefore, the first and the second alternatives of Theorem 7.2 are excluded, and now Theorem 7.4 implies

$$V_k(x)e^{v_k}\,dx \rightharpoonup 8\pi \sum_{x_0\in\mathcal{S}} \delta_{x_0}(dx) \tag{7.13}$$

in $\mathcal{M}(\overline{\Omega})$. Then, convergence (7.2) in $W^{1,q}(\Omega)$ follows from the L^1 estimate for $q \in [1, 2)$, and also it is locally uniform on $\overline{\Omega} \setminus \mathcal{S}$ by the elliptic regularity.

To show (7.3), we make use of the Pohozaev-type identity [79, 80].

Lemma 7.1 *If $D \subset \mathbf{R}^2$ is a domain with C^3 boundary ∂D, and $v \in C^3(\overline{D})$ and $V \in C^1(\overline{D})$ satisfy*

$$-\Delta v = V(x)e^v \quad in \quad D,$$

then it holds that

$$\int_D \nabla V(x)e^v\, dx = \int_{\partial D} \Big(\frac{\partial v}{\partial \nu}\nabla v - \frac{1}{2}\,|\nabla v|^2\,\nu + V(x)e^v\nu\Big) d\sigma, \qquad (7.14)$$

where ν denotes the outer unit normal vector on ∂D.

Proof: The proof is elementary. In fact, we have

$$\begin{aligned} -\int_D \Delta v \nabla v\, dx &= \int_D V(x)e^v \nabla v\, dx \\ &= \int_{\partial D} V(x)e^v \nu d\sigma - \int_D \nabla V(x)e^v\, dx, \end{aligned}$$

which means

$$\int_D \nabla V(x)e^v\, dx = \int_{\partial D} V(x)e^v\nu d\sigma + \int_D \Delta v \nabla v.$$

Here, we have

$$\begin{aligned} I_{ij} = \int_D v_{ii}v_j &= \int_{\partial D} \nu_i v_i v_j d\sigma - \int_D v_i v_{ij}\, dx \\ &= \int_{\partial D} \big(\nu_i v_i v_j - \nu_i v v_{ij}\big)\, d\sigma + \int_D v v_{iij}\, dx \\ &= \int_{\partial D} \big(\nu_i v_i v_j - \nu_i v v_{ij} + \nu_j v v_{ii}\big)\, d\sigma - I_{ij} \end{aligned}$$

and hence

$$I_{ij} = \frac{1}{2}\int_{\partial D} \big(\nu_i v_i v_j - \nu_i v v_{ij} + \nu_j v v_{ii}\big)\, d\sigma$$

follows. This implies

$$\begin{aligned} \int_D \Delta v \nabla v\, dx &= \frac{1}{2}\int_{\partial D} \left(\frac{\partial v}{\partial \nu}\nabla v - v\frac{\partial}{\partial \nu}\,(\nabla v) + \nu v \Delta v\right) d\sigma \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu}\nabla v d\sigma + \frac{1}{2}\int_{\partial D} \left(-\frac{\partial}{\partial \nu}(v\nabla v) + \nu v\Delta v\right) d\sigma \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu}\nabla v d\sigma - \frac{1}{2}\int_D \left(\Delta(v\nabla v) - \nabla(v\Delta v)\right)\, dx \\ &= \int_{\partial D} \frac{\partial v}{\partial \nu}\nabla v - \frac{1}{2}\int_D \nabla\,|\nabla v|^2\, dx \\ &= \int_{\partial D} \left(\frac{\partial v}{\partial \nu}\nabla v - \frac{1}{2}\,|\nabla v|^2\,\nu\right) d\sigma, \end{aligned}$$

and then (7.14) is obtained. □

Now, we complete the following proof.

Proof of Theorem 7.1: Taking $x_0 \in \mathcal{S}$ and small $r > 0$, we apply (7.14) for

$$V_k(x) = \frac{\lambda W(x)}{\int_\Omega W(x) e^{v_k}\, dx}$$

and $D = B(x_0, r)$. From

$$-\Delta v_k = V_k(x) e^{v_k} \quad \text{in} \quad D,$$

this implies

$$\begin{aligned}\int_D \left[\nabla \log W\right] V_k(x) e^{v_k}\, dx &= \int_D \left[\nabla \log V_k\right] V_k e^{v_k}\, dx \\ = \int_D \nabla V_k e^{v_k}\, dx &= \int_{\partial D} \left\{ \frac{\partial v_k}{\partial \nu} \nabla v_k - \frac{1}{2} \left| \nabla v_k \right|^2 + V_k e^{v_k} \nu \right\} d\sigma,\end{aligned}$$

where the left-hand side converges to

$$8\pi \nabla \log W(x_0)$$

by (7.13).

To treat the right-hand side, we make use of convergence (7.2), which holds locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$, and the elliptic regularity. We also apply (7.13) for the third term, and in this way we see that the right-hand side converges to

$$\int_{\partial D} \left(\frac{\partial v_0}{\partial \nu} \nabla v_0 - \frac{1}{2} \left| \nabla v_0 \right|^2 \nu \right) d\sigma,$$

where $v_0(x) = 8\pi \sum_{x_0' \in \mathcal{S}} G(x, x_0')$. Then, letting

$$v_1(x) = 8\pi G(x, x_0)$$

and

$$v_2(x) = 8\pi \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} G(x, x_0),$$

we obtain

$$\lim_{r \downarrow 0} \int_{\partial D} \left(\frac{\partial v_0}{\partial \nu} \nabla v_0 - \frac{1}{2} \left| \nabla v_0 \right|^2 \nu \right) d\sigma = \lim_{r \downarrow 0} \left(\mathrm{I} + \mathrm{II} \right)$$

with

$$\mathrm{I} = \int_{\partial D} \left(\frac{\partial v_1}{\partial \nu} \nabla v_1 - \frac{1}{2} |\nabla v_1|^2 \nu \right) d\sigma$$

and

$$\mathrm{II} = \int_{\partial D} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_1}{\partial \nu} \nabla v_2 - \nabla v_1 \cdot \nabla v_2 \nu \right) d\sigma.$$

Furthermore, we have

$$v_1(x) = v_{11}(x) + v_{12}(x)$$

with

$$v_{11}(x) = 8\pi \cdot \frac{1}{2\pi} \log \frac{1}{|x - x_0|}$$

and

$$v_{12}(x) = K(x, x_0),$$

and hence it follows that

$$\lim_{r \downarrow 0} \mathrm{II} = \lim_{r \downarrow 0} \int_{\partial D} \left(\frac{\partial v_{11}}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_{11} - \nabla v_{11} \cdot \nabla v_2 \nu \right) d\sigma.$$

Here, we have

$$\begin{aligned}
\lim_{r \downarrow 0} \int_{\partial D} \frac{\partial}{\partial \nu} \left(\frac{1}{2\pi} \log \frac{1}{|x - x_0|} \right) f(x)\, d\sigma_x &= -f(x_0), \\
\lim_{r \downarrow 0} \int_{\partial D} \nabla \left(\frac{1}{2\pi} \log \frac{1}{|x - x_0|} \right) f(x)\, d\sigma_x &= 0, \\
\lim_{r \downarrow 0} \int_{\partial D} \nabla \left(\frac{1}{2\pi} \log \frac{1}{|x - x_0|} \right) j(x) \nu \, d\sigma_x &= 0,
\end{aligned}$$

for continuous $f(x)$ and $j(x)$. Thus, we obtain

$$\begin{aligned}
\lim_{r \downarrow 0} \mathrm{II} = \lim_{r \downarrow 0} \int_{\partial D} \frac{\partial v_{11}}{\partial \nu} \nabla v_2 d\sigma &= -8\pi \nabla v_2(x_0) \\
&= -64\pi^2 \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} \nabla_x G(x_0, x_0').
\end{aligned}$$

On the other hand, we have

$$\lim_{r\downarrow 0} I = \lim_{r\downarrow 0}(\mathrm{III}+\mathrm{IV})$$

with

$$\mathrm{III} = \int_{\partial D}\left(\frac{\partial v_{11}}{\partial \nu}\nabla v_{11} - \frac{1}{2}\left|\nabla v_{11}\right|^2 \nu\right) d\sigma$$

and

$$\mathrm{IV} = \int_{\partial D}\left(\frac{\partial v_{11}}{\partial \nu}\nabla v_{12} + \frac{\partial v_{12}}{\partial \nu}\nabla v_{11} - \nabla v_{11}\cdot\nabla v_{12}\nu\right) d\sigma.$$

Similarly to the case of II, we obtain

$$\lim_{r\downarrow 0} \mathrm{IV} = -64\pi^2 \nabla_{\mathrm{x}} \mathrm{K}(\mathrm{x}_0, \mathrm{x}_0).$$

Finally, we have

$$\begin{aligned}
&\int_{\partial D} \frac{\partial}{\partial \nu}\log|x-x_0|\,\nabla \log|x-x_0| - \frac{1}{2}\left|\nabla \log|x-x_0|\right|^2 \nu\, d\sigma \\
&\quad = \int_{\partial D}\left(\frac{1}{|x-x_0|}\cdot\frac{x-x_0}{|x-x_0|} - \frac{1}{2}\frac{x-x_0}{|x-x_0|^2}\right) d\sigma \\
&\quad = \frac{1}{2}\int_{\partial D}\frac{x-x_0}{|x-x_0|^2}\, d\sigma \to 0
\end{aligned}$$

as $r \downarrow 0$. Then, (7.3) follows and the proof is complete. □

Problem (7.1) is the stationary state for the (D) field. For the (N) and (JL) fields, there arise boundary blowup points, which are hard to control by the above-stated methods. However, the method of symmetrization is efficient even for this problem, and the following theorem is a special case of [124]. Thus, several techniques are available for this kind of problem: Brezis–Merle's inequality, blowup analysis, sup + inf inequality, Kazdan–Warner's inequality, complex variables, symmetrization, and so forth.

The rest of this chapter is devoted to the following theorem [124, 125].

Theorem 7.5 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, $W = W(x) > 0$ be a C^1 function on $\overline{\Omega}$, and the C^2 function $v_k = v_k(x)$ solve (6.11) for $\lambda = \lambda_k \geq 0$:*

$$-\Delta v_k + a v_k = \frac{\lambda_k W(x) e^{v_k}}{\int_\Omega W(x)e^{v_k}\,dx} \quad \text{in} \quad \Omega,$$

$$\frac{\partial v_k}{\partial \nu} = 0 \qquad \text{on} \quad \partial\Omega. \tag{7.15}$$

Suppose $\lambda_k \to \lambda_0 \in [0, +\infty)$ *and set*

$$\mu_k(dx) = \frac{\lambda_k W(x) e^{v_k}}{\int_\Omega W(x) e^{v_k}\, dx}.$$

Since $\{\mu_k(dx)\}$ *is bounded in* $\mathcal{M}(\overline{\Omega})$, *we may assume without loss of generality that*

$$\mu_k(dx) \rightharpoonup \mu(dx) \tag{7.16}$$

in $\mathcal{M}(\overline{\Omega})$ *for some* $\mu(dx) \in \mathcal{M}(\overline{\Omega})$. *Then the following alternatives hold:*

1. *There exist smooth* $v = v(x)$ *and further subsequences of* $\{v_k\}$, *still denoted by the same symbol, such that* $v_k \to v$ *uniformly on* $\overline{\Omega}$ *and*

$$\mu(dx) = \frac{\lambda_0 W(x) e^{v}}{\int_\Omega W(x) e^{v}\, dx}\, dx.$$

   We note that this $v(x)$ *is always a solution for (7.15).*

2. *We have* $\lambda_0 \in 4\pi \mathbf{N}$ *and there is a nonempty* $\mathcal{S} \subset \overline{\Omega}$ *satisfying*

$$2 \cdot \sharp\,(\mathcal{S} \cap \Omega) + \sharp\,(\mathcal{S} \cap \partial\Omega) = \lambda_0/(4\pi).$$

   For this $\mathcal{S}$, *it holds that*

$$\mu(dx) = \sum_{x_0 \in \mathcal{S}} m_*(x_0) \delta_{x_0}(dx)$$

   with $m_*(x_0)$ *defined by (1.24):*

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega,) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

   Furthermore, we have

$$\nabla_x \Big(m_*(x_0) K(x, x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} m_*(x_0') G(x, x_0') \Big)\Big|_{x=x_0} + \nabla_x \log W(x)\Big|_{x=x_0} = 0 \tag{7.17}$$

   for each $x_0 \in \mathcal{S}$, *where* $G(x, y)$ *denotes the Green's function of* $-\Delta + a$ *under the Neumann boundary condition and*

$$K(x, y) = G(x, y) + \begin{cases} \frac{1}{2\pi} \log|x - y| & (y \in \Omega), \\ \frac{1}{\pi} \log|x - y| & (y \in \partial\Omega). \end{cases}$$

   In (7.17), ∇_x *takes only the tangential derivative in the case of* $x_0 \in \partial\Omega$.

Let us note that only tangential derivatives are taken for the boundary blowup point to control, and system (7.17) is well-posed under this agreement.

For the proof, we employ the argument of symmetrization, introducing $u_k(x)$ by $u_k(x)\,dx = \mu_k(dx)$. Then it holds that

$$-\Delta v_k + a v_k = u_k \quad \text{in} \quad \Omega, \qquad \frac{\partial v_k}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

and

$$\Delta u_k = \nabla \cdot (u_k \nabla (v_k + \log W)) \quad \text{in} \quad \Omega$$

with

$$\frac{\partial}{\partial \nu} u_k - u_k \frac{\partial}{\partial \nu} (v_k + \log W) = 0 \quad \text{on} \quad \partial\Omega.$$

Testing $\psi \in C^2(\overline{\Omega})$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$, we obtain the weak formulation

$$-\int_\Omega \Delta\psi(x)\mu_k(dx) = \frac{1}{2}\iint_{\Omega\times\Omega} \rho_\psi(x,x')\mu_k \otimes \mu_k(\,dx\,dx') + \int_\Omega \nabla \log W(x) \cdot \nabla\psi(x)\mu_k(dx) \quad (7.18)$$

with

$$\rho_\psi(x,x') = \nabla\psi(x) \cdot \nabla_x G(x,x') + \nabla\psi(x') \cdot \nabla_{x'} G(x,x').$$

First, we show that the limit measure $\mu(dx)$ is a finite sum of delta functions. Then, we take the second moment of $\mu_k(dx)$ to control their masses and locations. These processes are called the *rough* and *fine* estimates, respectively.

First, rough estimate is a consequence of the following lemma.

Lemma 7.2 *If the first case does not occur in the previous theorem, then the second case holds with $m_*(x_0)$ replaced by $m(x_0)$ in $m(x_0) \ge m_*(x_0)/2$.*

For the proof, we take $w_k = v_k + a\Delta_D^{-1} v_k$ with $-\Delta_D$ being the Laplace operator in Ω under the Dirichlet boundary condition. Then, it holds that

$$-\Delta w_k = V_k(x)e^{w_k} \quad \text{in} \quad \Omega \quad (7.19)$$

for

$$V_k(x) = \frac{\lambda_k W(x) e^{-a\Delta_D^{-1} v_k}}{\int_\Omega W(x)e^{v_k}\,dx}.$$

At this moment, direct application of [88] has an obstruction for the boundary blowup point, and we adopt a different approach based on Brezis–Merle's inequality. This argument is valid for rougher cases than the ones that [88] treats, such as the lack of uniform convergence of $\{V_k(x)\}$. Actually, Corollary 4 of [18] is stated as follows.

Lemma 7.3 *Let $\{w_k\}$ be a sequence of solutions to (7.19), and suppose the existence of C_1, C_2, and ε_0 independent of k such that*

$$\left\| V_k \right\|_p \leq C_1, \qquad \left\| w_{k+} \right\|_1 \leq C_2,$$

and

$$\int_\Omega |V_k| \, e^{w_k} \, dx \leq \varepsilon_0 < 4\pi/p',$$

where $p \in (1, \infty]$, and $(1/p)+(1/p') = 1$. Then $\{w_{k+}\}$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. Here and henceforth, $w_+ = \max\{0, w\}$.

Now, we give the following proof.

Proof of Lemma 7.2: Given $\{(\lambda_k, v_k)\}$ as in Theorem 7.5, we apply the L^1 estimate of [19] to (7.15). This implies

$$\left\| v_k \right\|_{W^{1,q}(\Omega)} = O(1)$$

for each $q \in [1, 2)$, and especially

$$\left\| v_k \right\|_p = O(1) \tag{7.20}$$

follows for each $p \in [1, \infty)$ by Sobolev's embedding theorem.

The maximum principle guarantees $v_k(x) \geq 0$, and hence it follows that

$$\int_\Omega W(x) e^{v_k} \, dx \geq \int_\Omega W(x) \, dx. \tag{7.21}$$

The right-hand side of the first equation of (7.15) is represented as $V_k(x) e^{v_k^1}$, where

$$V_k(x) = \lambda_k W(x)$$

and

$$v_k^1(x) = v_k(x) - \log \left(\int_\Omega W(x) e^{v_k} \, dx \right).$$

Under this notation, we have

$$-\Delta v_k^1 + a v_k = V_k(x) e^{v_k^1} \quad \text{in} \quad \Omega, \qquad \frac{\partial v_k^1}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{7.22}$$

with

$$\left\| V_k \right\|_p = O(1) \tag{7.23}$$

for each $p \in [1, \infty)$. Actually, (7.23) is valid even for $p = +\infty$. Now, we claim the following lemma.

Lemma 7.4 *If*

$$\limsup_{k\to\infty} \int_{B(x_0,2R)} V_k(x) e^{v_k^1}\, dx < 4\pi \tag{7.24}$$

holds with $\overline{B(x_0, 2R)} \subset \Omega$*, then it follows that*

$$\left\| v_{k+}^1 \right\|_{L^\infty(B(x_0,R))} = O(1).$$

Lemma 7.5 *For each* $x_0 \in \partial\Omega$ *and each sufficiently small* $R > 0$*, there exists* $\sigma \in (0, 2)$ *satisfying the following properties: if*

$$\limsup_{k\to\infty} \int_{B(x_0,2R)} V_k(x) e^{v_k^1}\, dx < 2\pi,$$

then it holds that

$$\left\| v_{k+}^1 \right\|_{L^\infty(B(x_0,\sigma R))} = O(1).$$

Proof of Lemma 7.4: We take v_k^2 by

$$-\Delta v_k^2 + a v_k = 0 \quad \text{in} \quad B(x_0, 2R), \qquad v_k^2 = 0 \quad \text{on} \quad \partial B(x_0, 2R).$$

Then $\|v_k^2\|_{W^{2,p}(B(x_0,2R))} = O(1)$ follows from (7.20). Especially, we obtain

$$\left\| v_k^2 \right\|_{L^\infty(B(x_0,2R))} = O(1). \tag{7.25}$$

Regarding

$$-\Delta(v_k^1 - v_k^2) = V_k(x) e^{v_k^2} e^{v_k^1 - v_k^2},$$

we shall apply Lemma 7.3 to $w_k = v_k^1 - v_k^2$ with $V_k(x)$ replaced by $V_k e^{v_k^2}$.

In fact, first, the relation

$$\left\| V_k e^{v_k^2} \right\|_{L^p(B(x_0,2R))} = O(1)$$

holds for every $p \in [1, \infty)$ by (7.23) and (7.25). Next, we have

$$\begin{aligned} v_k^1 &= v_k - \log\Big(\int_\Omega W(x) e^{v_k}\, dx \Big) \\ &\le v_k - \log\Big(\int_\Omega W(x)\, dx \Big) \end{aligned}$$

by (7.21), and hence

$$\begin{aligned} \left\| w_{k+} \right\|_{L^1(B(x_0,2R))} &\le \left\| v_{k+}^1 \right\|_{L^1(B(x_0,2R))} + \left\| v_k^2 \right\|_{L^1(B(x_0,2R))} \\ &= O(1) \end{aligned}$$

follows from (7.20) and (7.25). Finally, from the assumption (7.24) there are $p \in (1, \infty)$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned} \int_{B(x_0,2R)} \left| V_k e^{v_k^2} \right| e^{w_k}\, dx &= \int_{B(x_0,2R)} V_k e^{v_k^1}\, dx \\ &\le \varepsilon_0 < \frac{4\pi}{p'} < 4\pi. \end{aligned}$$

Therefore, the desired conclusion follows from Lemma 7.3. □

Proof of Lemma 7.5: We make use of the argument of Chapter 5 and extend v_k^1 outside Ω by reflection. More precisely, we take the conformal mapping

$$X : B(x_0, 2R) \cap \overline{\Omega} \to \mathbf{R}^2$$

satisfying the following properties:

$$\begin{aligned} &X\big(B(x_0, 2R) \cap \Omega\big) \subset \mathbf{R}_+^2 := \{(X_1, X_2) | X_2 > 0\}, \\ &X\big(B(x_0, 2R) \cap \partial\Omega\big) \subset \partial\mathbf{R}_+^2, \\ &X\big(B(x_0, 2R) \cap \Omega\big) \supset B(0, 1) \cap \mathbf{R}_+^2, \\ &X\big(B(x_0, \sigma R) \cap \Omega\big) \subset B(0, 1/2) \cap \mathbf{R}_+^2, \end{aligned} \tag{7.26}$$

where $\sigma \in (0, 2)$.

Henceforth, we will write $f^* = f \circ X^{-1}$ for each function f defined on $B(x_0, 2R) \cap \overline{\Omega}$. Then we obtain

$$-\Delta_X v_k^{1*} + a\left|g'\right|^2 v_k^* = \left|g'\right|^2 V_k^* e^{v_k^{1*}} \quad \text{in} \quad B(0,1) \cap \mathbf{R}_+^2,$$
$$\frac{\partial v_k^{1*}}{\partial X_2} = 0 \qquad \text{on} \quad B(0,1) \cap \partial \mathbf{R}_+^2,$$

where g' is the derivative of g as a function of the complex variable $X_1 + iX_2$. Furthermore, $\hat{f}$ denotes the even extension of a function f defined on $B(0,1) \cap \partial \mathbf{R}_+^2$:

$$\hat{f}(X_1, X_2) = \begin{cases} f(X_1, X_2) & \left(X \in B(0,1) \cap \mathbf{R}_+^2\right), \\ f(X_1, -X_2) & \left(X \in B(0,1) \backslash \mathbf{R}_+^2\right). \end{cases}$$

To simplify the writing, we abbreviate $\hat{f}^*$ to $\hat{f}$. From the Neumann condition of v_k^{1*} on $B(0,1) \cap \partial \mathbf{R}_+^2$, we see that $\hat{v}_k^1$ satisfies

$$-\Delta_X \hat{v}_k^1 + a\widehat{|g'|^2}\hat{v}_k = \widehat{|g'|^2}\hat{V}_k e^{\hat{v}_k^1} \quad \text{in} \quad B(0,1).$$

Here we have $\widehat{|g'|^2} \in L^\infty(B(0,1))$, and therefore similarly to Lemma 7.4 the condition

$$\limsup_{k\to\infty} \int_{B(0,1)} \widehat{|g'|^2}\hat{V}_k e^{\hat{v}_k^1}\, dX < 4\pi$$

implies

$$\left\| \hat{v}_k^1 \right\|_{L^\infty(B(0,1/2))} = O(1).$$

This means

$$\limsup_{k\to\infty} \int_{B(x_0,2R)} V_k e^{v_k^1}\, dx < 2\pi$$

implies

$$\left\| v_k^1 \right\|_{L^\infty(B(x_0,\sigma R))} = O(1)$$

and the proof is complete. □

Now, we give the following proof.

Proof of Lemma 7.2: Recalling

$$\mu_k(dx) = \frac{\lambda_k W(x) e^{v_k}}{\int_\Omega W(x) e^{v_k}\, dx}\, dx = V_k(x) e^{v_k^1}\, dx \quad \rightharpoonup \mu(dx)$$

in $\mathcal{M}(\overline{\Omega})$, we put

$$\mathcal{S} = \left\{ x_0 \in \Omega \;\middle|\; \mu(\{x_0\}) \geq 4\pi \right\} \cup \left\{ x_0 \in \partial\Omega \;\middle|\; \mu(\{x_0\}) \geq 2\pi \right\}.$$

We have $\sharp\mathcal{S} < \infty$ by $\mu(\overline{\Omega}) = \lambda_0 < \infty$. Now, we divide the proof into the following two steps:

1. If $\mathcal{S} = \emptyset$, then the first case of Theorem 7.5 holds.
2. If $\mathcal{S} \neq \emptyset$, then the second case of Theorem 7.5 holds with $m_*(x_0)$ replaced by $m(x_0) \geq m_*(x_0)/2$.

Moreover, we divide the second step into the following two substeps:

1. The case when $\mathcal{S} \cap \Omega \neq \emptyset$.
2. The case when $\mathcal{S} \cap \Omega = \emptyset$.

Step 1. ($\mathcal{S} = \emptyset$): In this case we have $\|v_{k+}^1\|_{L^\infty(\Omega)} = O(1)$ by Lemmas 7.4 and 7.5. Especially it holds that

$$\left\| V_k(x) e^{v_k^1} \right\|_p \leq \left\| V_k(x) e^{v_{k+}^1} \right\|_p = O(1)$$

for each $p \in (1, \infty)$ from (7.23). Combining this estimate, (7.20), and the standard elliptic estimate to (7.22), we obtain

$$\left\| v_k \right\|_{W2,p(\Omega)} = O(1).$$

Thus, we obtain the first case of Theorem 7.5 by Morrey's theorem.

Step 2. ($\mathcal{S} \neq \emptyset$): In this case, we have $\lambda_0 \geq 2\pi$. This step is reduced to the proof of

$$\int_\Omega W(x) e^{v_k}\, dx \to +\infty. \tag{7.27}$$

In fact, similarly to the case $\mathcal{S} = \emptyset$, we have from Lemmas 7.4 and 7.5 that

$$\left\| v_{k+}^1 \right\|_{L^\infty(\omega)} = O(1) \tag{7.28}$$

for each subdomain ω in $\overline{\omega} \subset \overline{\Omega} \setminus \mathcal{S}$. Taking smaller ω, denoted by the same symbol, we obtain

$$\left\| v_k \right\|_{L^\infty(\omega)} = O(1)$$

from (7.20) and the local elliptic estimate to (7.22). Therefore, if (7.27) holds, then (7.23) gives

$$\frac{\lambda_k W(x) e^{v_k}}{\int_\Omega W(x) e^{v_k}\, dx} = \frac{V_k(x) e^{v_k}}{\int_\Omega W(x) e^{v_k}\, dx} \;\to\; 0 \tag{7.29}$$

in $L^p_{\mathrm{loc}}(\overline{\Omega} \setminus \mathcal{S})$ for every $p \in (1, \infty)$. This implies $\operatorname{supp} \mu = \mathcal{S}$ and then the conclusion of Lemma 7.2 follows.

For (7.27) to prove, we distinguish two cases.

Step 2.1. ($\mathcal{S} \cap \Omega \neq \emptyset$): In this case, we take $x_0 \in \mathcal{S} \cap \Omega$ and $R > 0$ satisfying $\overline{B(x_0, 2R)} \subset \Omega$ and $B(x_0, 2R) \cap \mathcal{S} = \{x_0\}$. Given $\varepsilon \in (0, 2R)$, we set

$$\omega_\varepsilon = B(x_0, 2R) \setminus \overline{B(x_0, \varepsilon)}.$$

Let v_k^3 be the solution to

$$\begin{aligned} -\Delta v_k^3 + a v_k &= V_k(x) e^{v_k^1} && \text{in} \quad \omega_\varepsilon \\ v_k^3 &= 0 && \text{on} \quad \partial\omega_\varepsilon. \end{aligned}$$

Then, $v_k^1 - v_k^3$ is harmonic:

$$-\Delta \left(v_k^1 - v_k^3 \right) = 0 \quad \text{in} \quad \omega_\varepsilon.$$

On the other hand, from Lemma 7.4, Lemma 7.5, (7.28), and (7.23), we have

$$\left\| V_k(x) e^{v_k^1} \right\|_{L^p(\omega_\varepsilon)} \le \left\| V_k(x) e^{v_{k+}^1} \right\|_{L^p(\omega_\varepsilon)} = O(1)$$

for each $p \in (1, \infty)$. Therefore, by (7.20) we have

$$\left\| v_k^3 \right\|_{W^{2,p}(\omega_\varepsilon)} = O(1),$$

and hence it follows that

$$\left\| v_k^3 \right\|_{L^\infty(\omega_\varepsilon)} = O(1). \tag{7.30}$$

Combining this estimate with (7.28), we see that

$$\left\{ v_k^1 - v_k^3 \right\}$$

is a sequence of harmonic functions uniformly bounded from above. Therefore, the classical Harnack principle guarantees either

(a) $v_k^1 \to -\infty$ locally uniformly in ω_ε as $n \to \infty$, or

(b) there exists a subsequence of $\{v_k^1\}$ that is locally uniformly bounded in ω_ε.

The case (a) implies (7.27) and we finish the proof of Lemma 7.2. Indeed, in this case we have

$$\begin{aligned} v_k^1(x) &= v_k(x) - \log\left(\int_\Omega K(x)e^{v_k}\,dx\right) \\ &\geq -\log\left(\int_\Omega K(x)e^{v_k}\,dx\right) \to -\infty \end{aligned}$$

for any $x \in \omega_\varepsilon$ by $v_k(x) \geq 0$. On the other hand, if (b) holds for any $\varepsilon \in (0, 2R)$, then we obtain a contradiction. In fact, if this is the case, each subdomain ω in $\overline{\omega} \subset B(x_0, 2R) \setminus \{x_0\}$ admits a subsequence of $\left\{v_k^1\right\}$, denoted by the same symbol, such that $\|v_k^1\|_{L^\infty(\omega)} = O(1)$. We now apply the interior elliptic estimate to (7.22) similarly to Step 1, and obtain for smaller ω, denoted by the same symbol,

$$\left\|v_k^1\right\|_{W^{2,p}(\omega)} = O(1)$$

for each $p \in (1, \infty)$. The standard diagonal argument now guarantees the existence of $v_2 \in C(B(x_0, 2R) \setminus \{x_0\})$ and that of a subsequence of $\{v_k^1\}$, denoted by the same symbol, satisfying

$$v_k^1 \to v_1 \tag{7.31}$$

locally uniformly in $B(x_0, 2R) \setminus \{x_0\}$. In particular, we have

$$v_k^1 \geq -C_4 \quad \text{on} \quad \partial B(x_0, R) \tag{7.32}$$

with a constant C_4 independent of k.

Let

$$N_k(x) = \int_{B(x_0, R/2)} \frac{1}{2\pi} \log \frac{1}{|x-y|} \cdot V_k(y)e^{v_k^1(y)}dy.$$

Then, there exists a constant C_5 such that

$$v_k^1 - N_k \geq -C_5 \quad \text{in} \quad B(x_0, R). \tag{7.33}$$

In fact, we have

$$N_k(x) \leq \frac{1}{2\pi}\left(\log\frac{2}{R}\right)\int_{B(x_0, R/2)} V_k(y)e^{v_k^1(y)}dy = O(1)$$

for every $x \in \partial B(x_0, R)$, and it holds that

$$\begin{aligned} -\Delta(v_k^1 - N_k) + a v_k \geq 0 &\quad \text{in} \quad B(x_0, R), \\ v_k^1 - N_k \geq -C_6 &\quad \text{on} \quad \partial B(x_0, R) \end{aligned}$$

with a constant C_6. If N_k^1 denotes the solution to

$$\begin{aligned} -\Delta N_k^1 + a v_k = 0 &\quad \text{in} \quad B(x_0, R), \\ N_k^1 = -C_6 &\quad \text{on} \quad \partial B(x_0, R), \end{aligned}$$

then we have

$$\begin{aligned} -\Delta(v_k^1 - N_k - N_k^1) \geq 0 &\quad \text{in} \quad B(x_0, R), \\ v_k^1 - N_k - N_k^1 \geq 0 &\quad \text{on} \quad \partial B(x_0, R). \end{aligned} \tag{7.34}$$

Since

$$\left\| N_k^1 \right\|_{L^\infty(B(x_0,R))} = O(1)$$

follows from (7.20), we obtain (7.33) from the maximum principle for (7.34).

Taking nonnegative $\varphi \in C(\overline{\Omega})$ in $\varphi(x_0) = 1$, we have

$$\begin{aligned} v_k^1(x) &\geq N_k(x) - C_5 \\ &\geq \int_{B(x_0,R/2)} \min\left\{ t\varphi(y), \frac{1}{2\pi} \log \frac{1}{|x-y|} \right\} V_k(y) e^{v_k^1(y)} dy - C_5 \end{aligned}$$

for any $x \in B(x_0, R/2)$ and $t > 0$. Since

$$y \in \overline{\Omega} \quad \mapsto \quad \min\left\{ t\varphi(y), \frac{1}{2\pi} \log \frac{1}{|x-y|} \right\}$$

is continuous, from (7.31) we get

$$v_1(x) \geq 4\pi \cdot \min\left\{ t\varphi(x_0), \frac{1}{2\pi} \log \frac{1}{|x-x_0|} \right\} - C_5$$

for any $x \in B(x_0, R) \setminus \{x_0\}$. Making $t \to \infty$, we obtain

$$e^{v_1(x)} \geq \frac{e^{-C_5}}{|x-x_0|^2}. \tag{7.35}$$

On the other hand, we have

$$V_k(x) = \lambda_k W(x) \to \lambda_0 W(x)$$

in $L^1(\Omega)$ (actually uniformly), and therefore it holds that

$$\lambda_k \geq \int_{B(x_0,R)\setminus B(x_0,\varepsilon)} V_k(x)e^{v_k^1}\,dx \;\to\; \int_{B(x_0,R)\setminus B(x_0,\varepsilon)} \lambda_0 K(x)e^{v_1}\,dx$$

for each $\varepsilon \in (0, R)$. Since $\lambda_0 \geq 2\pi$ and (7.35) holds for any $x \in B(x_0, R) \setminus \{x_0\}$, this implies

$$\left\{\min_{x\in\overline{\Omega}} W(x)\right\}^{-1} \geq e^{-C_5}\int_{B(x_0,R)\setminus B(x_0,\varepsilon)} \frac{dx}{|x-x_0|^2}.$$

However, this gives a contradiction when $\varepsilon \downarrow 0$.

In this way, (a) holds with some $\varepsilon \in (0, 2R)$, and we have (7.27) in the case of $\mathcal{S} \cap \Omega \neq \emptyset$.

Step 2.2. ($\mathcal{S} \neq \emptyset$ and $\mathcal{S} \cap \Omega = \emptyset$): We take $x_0 \in \mathcal{S} \subset \partial\Omega$ and apply the reflection argument as in the proof of Lemma 7.5. Thus, we have

$$-\Delta_X \hat{v}_k^1 + a\widehat{|g'|^2}\hat{v}_k = \widehat{|g'|^2}\hat{V}_k e^{\hat{v}_k^1} \quad \text{in} \quad B(0,1).$$

Similarly to Step 2.1, relation (7.27) is proven if we get a contradiction, assuming a subsequence of $\{\hat{v}_k^1\}$, denoted by the same symbol, locally uniformly bounded in $B(0,1) \setminus \overline{B(0,\varepsilon)}$ for every $\varepsilon \in (0, 1)$. As we did in Step 2.1, if this is the case, we may assume furthermore that there is $\hat{v}_1 \in C(B(0,1) \setminus \{0\})$ satisfying

$$\hat{v}_k^1 \;\to\; \hat{v}_1$$

locally uniformly in $B(0,1) \setminus \{0\}$. This implies

$$\hat{v}_k^1 \geq -C_7 \quad \text{on} \quad \partial B(0, 1/2) \tag{7.36}$$

with a constant C_7.

This time, we have

$$\widehat{|g'|^2}\hat{V}_k e^{\hat{v}_k^1}\,dX \;\rightharpoonup\; 2m(x_0)\delta_0(dX)$$

in $\mathcal{M}(\overline{B(0,1)})$, and hence it holds that

$$e^{\hat{v}_1(X)} \geq \frac{e^{-C_8}}{|X|^2}$$

for any $X \in B(0, 1/2) \setminus \{0\}$. This implies a contradiction similarly to Step 2.1, and the proof is complete. $\square$

Now, we give the following proof.

Proof of Theorem 7.5: We divide the proof into two cases:

1. $x_0 \in \mathcal{S} \cap \Omega$,
2. $x_0 \in \mathcal{S} \cap \partial\Omega$.

Case 1. ($x_0 \in \mathcal{S} \cap \Omega$): For

$$u_k = \frac{\lambda_k W(x) e^{v_k}}{\int_\Omega W(x) e^{v_k}\, dx} \tag{7.37}$$

we have proven

$$u_k(x)\, dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx) \tag{7.38}$$

in $\mathcal{M}(\overline{\Omega})$ with $m(x_0) \geq m_*(x_0)/2$. We have proven also

$$u_k \ \to \ 0 \tag{7.39}$$

in $L^p_{\mathrm{loc}}(\overline{\Omega} \setminus \mathcal{S})$ for every $p \in [1, \infty)$. See (7.29). Actually, this convergence is locally uniform because $W(x)$ is continuous.

The left-hand side of (7.15) is denoted by u_k and hence we have

$$v_k(x) = \int_\Omega G(x, x') u_k(x')\, dx'$$

with $G = G(x, x')$ being the Green's function of $-\Delta + a$ in Ω with Neumann boundary condition. From (7.37) we have

$$\nabla u_k = u_k \left(\nabla v_k + \nabla \log W \right), \tag{7.40}$$

and hence it follows that

$$\Delta u_k = \nabla \cdot (u_k \nabla v_k) + \nabla \cdot (u_k \nabla \log W).$$

Testing $\psi \in C^2(\overline{\Omega})$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$, we have

$$\begin{aligned} -\int_\Omega u_k \Delta\psi\, dx = \int_\Omega u_k \nabla \log W \cdot \nabla \psi\, dx \\ + \int_\Omega \int_\Omega \nabla_x G(x, x') \cdot \nabla \psi(x) u_k(x) u_k(x')\, dx\, dx'. \end{aligned} \tag{7.41}$$

Now, we take the limit $k \to \infty$ in this equality. For this purpose, we take a small $R > 0$ satisfying $\overline{B(x_0, 2R)} \subset \Omega$ and $B(x_0, 2R) \cap \mathcal{S} = \{x_0\}$ and always assume that the test function ψ satisfies

$$\text{supp}\,\psi \subset B(x_0, R).$$

First, from (7.38) we have

$$\begin{aligned}&\int_\Omega u_k \Delta\psi\, dx \to m(x_0)\Delta\psi(x_0),\\ &\int_\Omega u_k \nabla \log W \cdot \nabla\psi\, dx \to m(x_0)\nabla \log W(x_0) \cdot \nabla\psi(x_0),\end{aligned}$$

as $k \to \infty$. Let $\xi \in C^2(\overline{\Omega})$ be a cut-off function around x_0 satisfying $0 \le \xi(x) \le 1$, $\xi(x) = 1$ in $B(x_0, R)$, and $\text{supp}\,\xi \subset B(x_0, 2R)$. Then it holds that

$$\psi = \xi\psi \quad \text{and} \quad \nabla\psi = \xi\nabla\psi.$$

We also have

$$\begin{aligned} u_k^0\, dx \equiv \xi(x)u_k(x)\, dx \ &\rightharpoonup\ m(x_0)\delta_{x_0}(dx),\\ (1-\xi(x))u_k(x)\, dx \ &\rightharpoonup \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')\delta_{x_0'}(dx),\end{aligned}$$

in $\mathcal{M}(\overline{\Omega})$, and

$$\begin{aligned}&\int_\Omega\int_\Omega \nabla_x G(x,x') \cdot \nabla\psi(x)u_k(x)u_k(x')\, dx\, dx'\\ &\quad= \int_\Omega\int_\Omega \nabla_x G(x,x') \cdot \nabla\psi(x)u_k^0(x)u_k(x')\, dx\, dx'\\ &\quad= \int_\Omega\int_\Omega \nabla_x G(x,x') \cdot \nabla\psi(x)u_k^0(x)u_k^0(x')\, dx\, dx',\\ &\qquad + \int_\Omega\int_\Omega \nabla_x G(x,x') \cdot \nabla\psi(x)u_k^0(x)(1-\xi(x'))u_k(x')\, dx\, dx'\\ &\quad= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{7.42}$$

Here, we have

$$\mathrm{II} \to \mathrm{m}(\mathrm{x}_0) \sum_{\mathrm{x}_0' \in \mathcal{S}\setminus\{\mathrm{x}_0\}} \mathrm{m}(\mathrm{x}_0')\nabla\psi(\mathrm{x}_0) \cdot \nabla_{\mathrm{x}}\mathrm{G}(\mathrm{x}_0, \mathrm{x}_0') \tag{7.43}$$

because $G(x, x')$ is smooth on $\text{supp}\,\psi \times \text{supp}\,(1-\xi)$.

In Chapter 5, we have proven

$$G(x,x') = \frac{1}{2\pi}\log\frac{1}{|x-x'|} + K(x,x')$$

with $K(x,y) \in C^{1+\theta}(\Omega\times\Omega)$ for every $\theta \in (0,1)$. This implies

$$\begin{aligned} \mathrm{I} &= \int_\Omega\int_\Omega \nabla_x\Big(\frac{1}{2\pi}\log\frac{1}{|x-x'|}\Big)\cdot\nabla\psi(x)u_k^0(x)u_k^0(x')\,dx\,dx' \\ &\quad + \int_\Omega\int_\Omega \nabla_x K(x,x')\cdot\nabla\psi(x)u_k^0(x)u_k^0(x')\,dx\,dx' \\ &= I_1 + I_2 \end{aligned} \tag{7.44}$$

with

$$I_2 \to m(x_0)^2\nabla\psi(x_0)\cdot\nabla_x K(x_0,x_0). \tag{7.45}$$

To treat I_1, we apply the symmetrization technique developed in Chapter 5. Thus, we have

$$I_1 = \frac{1}{2}\int_\Omega\int_\Omega \rho_\psi^0(x,x')u_k^0(x)u_k^0(x')\,dx\,dx'$$

with

$$\begin{aligned} \rho_\psi^0(x,x') &= \nabla_x\Big(\frac{1}{2\pi}\log\frac{1}{|x-x'|}\Big)\cdot\nabla\psi(x) \\ &\quad + \nabla_{x'}\Big(\frac{1}{2\pi}\log\frac{1}{|x-x'|}\Big)\cdot\nabla\psi(x') \\ &= -\frac{1}{2\pi}\cdot\frac{\big(\nabla\psi(x)-\nabla\psi(x')\big)\cdot(x-x')}{|x-x'|^2}. \end{aligned}$$

It is easy to see that $\rho_\psi^0(x,x')$ is continuous in $\overline{\Omega}\times\overline{\Omega}\setminus\{x=x'\}$ and $\rho_\psi^0 = \rho_\psi^0(x,x') \in L^\infty(\Omega\times\Omega)$. For general ψ, we do not know what (I_1) converges, but taking appropriate ψ makes it possible.

Namely, we take

$$\psi(x) = \varphi(x)\,|x-\boldsymbol{a}|^2$$

for $\boldsymbol{a} \in \mathbf{R}^2$ and $\varphi \in C_0^2(\Omega)$, satisfying supp $\varphi \subset B(x_0,R)$ and $\varphi = 1$ on $B(x_0,R/2)$. Then it holds that

$$\left.\begin{aligned} \nabla\psi(x) &= 2(x-\boldsymbol{a}) \\ \Delta\psi(x) &= 4 \end{aligned}\right\} \quad \text{in} \quad B(x_0,R/2), \tag{7.46}$$

and hence

$$\rho_\psi^0(x, x') = -\frac{1}{\pi} \tag{7.47}$$

follows for $(x, x') \in B(x_0, R/2) \times B(x_0, R/2)$. Here, we have

$$\begin{aligned} I_1 = &\frac{1}{2}\int_\Omega\int_\Omega \rho_\psi^0(x, x')\varphi(x)u_k^0(x)\varphi(x')u_k^0(x')\,dx\,dx' \\ &+ \frac{1}{2}\int_\Omega\int_\Omega \rho_\psi^0(x, x')\big(1-\varphi(x)\big)u_k^0(x)\varphi(x')u_k^0(x')\,dx\,dx' \\ &+ \frac{1}{2}\int_\Omega\int_\Omega \rho_\psi^0(x, x')u_k^0(x)\big(1-\varphi(x')\big)u_k^0(x')\,dx\,dx' \\ = &I_{1,1} + I_{1,2} + I_{1,3}. \end{aligned} \tag{7.48}$$

First, from (7.47) we obtain

$$I_{1,1} \to -\frac{1}{2\pi}m(x_0)^2.$$

Next, we have

$$\begin{aligned} \left|I_{1,2} + I_{1,3}\right| &\le \lambda_k \left\|\rho_\psi^0\right\|_{L^\infty(\Omega\times\Omega)}\left\|(1-\varphi)u_k^0\right\|_{L^1(\Omega)} \\ &\le \lambda_k \left\|\rho_\psi^0\right\|_{L^\infty(\Omega\times\Omega)}\left\|u_k\right\|_{L^1(B(x_0,2R)\setminus B(x_0,R/2))} \to 0 \end{aligned}$$

by (7.39). Consequently, it holds that

$$I_1 \to -\frac{1}{2\pi}m(x_0)^2. \tag{7.49}$$

Combining (7.41)–(7.46) and (7.49), we get

$$\begin{aligned} 0 = 4m(x_0) - \frac{1}{2\pi}m(x_0)^2 + 2m(x_0)\,(x_0 - \boldsymbol{a}) \\ \cdot \Big(m(x_0)\nabla_x K(x_0, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0, x_0')\Big) \\ + 2m(x_0)\,(x_0 - \boldsymbol{a}) \cdot \nabla_x \log W(x_0) \end{aligned}$$

for every $\boldsymbol{a} \in \mathbf{R}^2$.

First, letting $\boldsymbol{a} = x_0$, we have

$$m(x_0) = 8\pi$$

because Lemma 7.2 assures $m(x_0) > 0$. Then, taking

$$\boldsymbol{a} = x_0 - m(x_0)\nabla_x K(x_0, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m(x_0')\nabla_x G(x_0, x_0') + \nabla_x \log W(x_0),$$

we get (7.17).

Case 2. ($x_0 \in \mathcal{S} \cap \partial\Omega$): We apply the reflection argument near x_0 as in the proof of Lemma 7.5. First, we take $R > 0$ sufficiently small so that $B(x_0, R/2) \cap \mathcal{S} = \{x_0\}$ and the conformal mapping X satisfies (7.26). Letting

$$N_{x_0} = X^{-1}\left(B(0,1) \cap \mathbf{R}^2_+\right) \quad \text{and} \quad f_k = v_k\big|_{\partial N_{x_0} \cap \Omega},$$

we have

$$\begin{aligned} -\Delta_X \hat{v}_k + a\widehat{|g'|^2}\hat{v}_k &= \widehat{|g'|^2}\frac{\lambda_k \hat{W} e^{\hat{v}_k}}{\int_\Omega W e^{v_k}\,dx} &&\text{in} \quad B(0,1), \\ \hat{v}_k &= \hat{f}_k &&\text{on} \quad \partial B(0,1), \end{aligned}$$

under the notation of the proof of Lemma 7.5.

Using the Neumann boundary condition of v_k, we get

$$\left\|\hat{f}_k\right\|_{W^{2-\frac{1}{p},p}(\partial B(0,1))} = O(1)$$

for each $p \in (1, \infty)$ from (7.20), (7.29), the local elliptic estimate, and the trace theorem of Sobolev functions. If $\hat{F}_k$ denotes the solution of

$$\begin{aligned} -\Delta_X \hat{F}_k + a\widehat{|g'|^2}\hat{F}_k &= 0 &&\text{in} \quad B(0,1), \\ \hat{F}_k &= \hat{f}_k &&\text{on} \quad \partial B(0,1), \end{aligned}$$

then we obtain

$$\left\|\hat{F}_k\right\|_{W^{2,p}(B(0,1))} = O(1)$$

for each $p \in (1, \infty)$ from the elliptic estimate. We also have

$$\hat{F}_k(X_1, X_2) = \hat{F}_k(X_1, -X_2)$$

for every $X = (X_1, X_2) \in \overline{B(0,1)}$ from the uniqueness of $\hat{F}_k$. We also have

$$\begin{aligned} -\Delta_X(\hat{v}_k - \hat{F}_k) + a\widehat{|g'|^2}(\hat{v}_k - \hat{F}_k) &= \frac{\widehat{|g'|^2}\lambda_k \hat{W} e^{\hat{v}_k}}{\int_\Omega W e^{v_k}} &&\text{in} \quad B(0,1), \\ \hat{v}_k - \hat{F}_k &= 0 &&\text{on} \quad \partial B(0,1). \end{aligned}$$

Letting

$$\hat{u}_k = \widehat{|g'|^2}\frac{\lambda_k \hat{W} e^{\hat{v}_k}}{\int_\Omega W e^{v_k}\,dx},$$

we have

$$\hat{u}_k(X)\,dX \rightharpoonup 2m(x_0)\delta_0(dX)$$

in $\mathcal{M}(\overline{B(0,1)})$ and

$$\hat{u}_k \to 0 \quad \text{in} \quad L^p_{\mathrm{loc}}(B(0,1)\setminus\{0\})$$

for every $p \in (1,\infty)$ from the proof of Lemma 7.2. We also have

$$\hat{v}_k = (-\Delta_X + a\widehat{|g'|^2})^{-1}_D \hat{u}_k + \hat{F}_k,$$

where $(-\Delta_X + a\widehat{|g'|^2})^{-1}_D$ is the inverse operator of $-\Delta + a\widehat{|g'|^2}$ under the Dirichlet boundary condition on $\partial B(0,1)$. Without loss of generality, we may assume

$$\hat{F}_k \rightharpoonup \hat{F} \quad \text{weakly in} \quad W^{2,p}(B(0,1)$$

as $k \to \infty$ for some $\hat{F} \in W^{2,p}(B(0,1))$. This implies

$$\hat{F}_k \to \hat{F} \quad \text{in} \quad C^1\left(B(0,1)\right).$$

We have

$$\nabla \hat{u}_k = \hat{u}_k \nabla\left(\hat{v}_k + \log(\widehat{|g'|^2}\hat{K})\right).$$

Similarly to Case 1, we obtain

$$\begin{aligned} -\int_{B(0,1)} \hat{u}_k \nabla\big(\hat{F}_k + \log(\widehat{|g'|^2}\hat{K})\big)\cdot\nabla\psi\,dX &= \int_{B(0,1)} \hat{u}_k \Delta_X \psi\,dX \\ &+ \int_{B(0,1)}\int_{B(0,1)} \nabla_X G_B(X,X')\cdot\nabla\psi(X)\hat{u}_k(X)\hat{u}_k(X')\,dX\,dX' \end{aligned}$$

for each test function $\psi \in C^2_0(B(0,1))$, where $G_B(X,X')$ denotes the Green's function for $-\Delta + a\widehat{|g'|^2}$ in $B(0,1)$ under the Dirichlet boundary condition.

Here, we have

$$G_B(X,X') = \frac{1}{2\pi}\log\frac{1}{|X-X'|} + K_B(X,X'),$$

with $K_B(X,X') \in C^{1+\theta}(B(0,1)\times B(0,1))$ for every $\theta \in (0,1)$. If $f \in W^{1,p}\left(B(0,1)\cap \mathbf{R}^2_+\right)$, then it holds that $\hat{f} \in W^{1,p}\left(B(0,1)\right)$ and

$$2\frac{\partial \hat{f}}{\partial X_1} = \widehat{\frac{\partial f}{\partial X_1}},$$

$$\frac{\partial \hat{f}}{\partial X_2} = \begin{cases} \frac{\partial f}{\partial X_2}(X_1,X_2) & (X_2 > 0)\,, \\ -\frac{\partial f}{\partial X_2}(X_1,-X_2) & (X_2 < 0)\,, \end{cases}$$

as distributions. Thus, for every test function $\psi \in C_0^2(B(0,1))$, we have

$$\int_{B(0,1)} \hat{u}_k \nabla \hat{f} \cdot \nabla \psi \, dX$$
$$= \int_{B(0,1)\cap \mathbf{R}_+^2} \hat{u}_k \left[\frac{\partial f}{\partial X_1} \left\{ \frac{\partial \psi}{\partial X_1}(X_1, X_2) + \frac{\partial \psi}{\partial X_1}(X_1, -X_2) \right\} + \frac{\partial f}{\partial X_2} \left\{ \frac{\partial \psi}{\partial X_2}(X_1, X_2) - \frac{\partial \psi}{\partial X_2}(X_1, -X_2) \right\} \right] dX.$$

Consequently, if $f \in C^1\left(\overline{B(0,1) \cap \mathbf{R}_+^2}\right)$, we obtain

$$\int_{B(0,1)} \hat{u}_k \nabla \hat{f} \cdot \nabla \psi \, dX \to 2m(x_0) \frac{\partial f}{\partial X_1}(0) \cdot \frac{\partial \psi}{\partial X_1}(0)$$

as $k \to \infty$.

Using these calculations and a similar argument to Case 1, we obtain the limiting equation

$$-2m(x_0) \cdot 2(-a_1) \cdot \left\{ \frac{\partial}{\partial X_1} \hat{F}(X) + \frac{\partial}{\partial X_1} \log |g'(X)|^2 + \frac{\partial}{\partial X_1} \log \hat{W}(X) \right\}\Big|_{X=0}$$
$$= 4 \cdot 2m(x_0) - \frac{1}{2\pi} \left\{ 2m(x_0) \right\}^2 + 2m(x_0) \cdot 2(-\boldsymbol{a}) \cdot \nabla_X \left\{ 2m(x_0) K_B(X,0) \right\}\Big|_{X=0}$$

for every $\boldsymbol{a} = (a_1, a_2) \in \mathbf{R}^2$. Here we have $K_B(X_1, -X_2; Y) = K_B(X_1, X_2; Y)$, and hence

$$\frac{\partial}{\partial X_2} K_B(X,0)\Big|_{X=0} = 0$$

follows. This implies $m(x_0) = 4\pi$ and

$$\frac{\partial}{\partial X_1} \left\{ \hat{F}(X) + \log |g'(X)|^2 + \log \hat{W}(X) + 2m(x_0) K_B(X,0) \right\}\Big|_{X=0} = 0.$$

We have

$$v_k \to m(x_0) G(\cdot, x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} m(x_0') G(\cdot, x_0')$$
$$= F_{0,1} + F_{0,2}$$

in $W^{2,p}(\omega)$ for every subdomain ω in $\overline{\omega} \subset \overline{\Omega} \setminus \mathcal{S}$ with $F_{0,1} = F_{0,1}(x)$ and $F_{0,2} = F_{0.2}(x)$ satisfying

$$-\Delta_X \hat{F}_{0,1} + a \widehat{|g'|^2} \hat{F}_{0,1} = 2m(x_0) \delta_0(dX)$$

as distributions in $B(0,1)$ and

$$-\Delta_X \hat{F}_{0,2} + a\widehat{|g'|^2}\hat{F}_{0,2} = 0 \quad \text{in} \quad B(0,1), \tag{7.50}$$

$$\hat{F}_{0,1} + \hat{F}_{0,2} = \hat{F} \quad \text{on} \quad \partial B(0,1). \tag{7.51}$$

Here, we have

$$\hat{F}_{0,1}(X) = m(x_0)\Big(\frac{1}{\pi}\log|X|^{-1} + \hat{K}(X,0)\Big)$$

with $\hat{K}(\cdot\,,\cdot) \in C^{1+\theta}(B(0,1)\times B(0,1))$ satisfying

$$\hat{K}(X_1, -X_2; 0) = \hat{K}(X_1, X_2; 0)$$

and hence it follows that

$$\begin{aligned}\hat{F}(X) &= \hat{F}_{0,1}(X) - 2m(x_0)G_B(X,0) + \hat{F}_{0,2}(X)\\ &= -2m(x_0)K_B(X,0) + m(x_0)\hat{K}(X,0) + \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')\hat{G}(X,x_0'),\end{aligned}$$

where $\hat{G}\left(X(x), x'\right) = G(x,x')$. This means

$$\frac{\partial}{\partial\tau}\Big[m(x_0)\hat{K}(X(x),0) + \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')G(x,x_0') + \log\left|X'(x)\right|^{-2} + \log W(x)\Big]\Big|_{x=x_0} = 0,$$

where τ denotes the unit tangential vector on $\partial\Omega$.

We have

$$\begin{aligned}\hat{K}(X(x),0) &= G(x,x_0) - \frac{1}{\pi}\log\frac{1}{|X(x)|}\\ &= K(x,x_0) + \frac{1}{\pi}\log\frac{1}{|x-x_0|} - \frac{1}{\pi}\log\frac{1}{|X(x)|}\end{aligned}$$

and

$$\nabla\Big[\log\frac{1}{|X(x)|} - \log\frac{1}{|x-x_0|}\Big]_{x=x_0} = \frac{1}{4}\nabla\log\frac{1}{|X'(x)|^2}\Big|_{x=x_0}.$$

Since $m(x_0) = 4\pi$, we obtain

$$\frac{\partial}{\partial\tau_x}\Big(m(x_0)K(x,x_0) + \sum_{x_0'\in\mathcal{S}\setminus\{x_0\}} m(x_0')G(x,x_0')\Big)\Big|_{x=x_0}\frac{\partial}{\partial\tau_x}\log W(x)\Big|_{x=x_0} = 0.$$

Relation (7.17) holds with ∇_x replaced by $\partial/\partial\tau_x$ for $x_0 \in \partial\Omega$, and the proof is complete. □

8
Multiple Existence

In this chapter we study the existence of the solution for

$$-\Delta v + av = \frac{\lambda e^v}{\int_\Omega e^v \, dx} \quad \text{in} \quad \Omega,$$
$$\frac{\partial v}{\partial \nu} = 0 \qquad\qquad \text{on} \quad \partial\Omega, \tag{8.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. It is the stationary problem of (3.1), that is, (7.15) with $W(x) = 1$, and we will obtain several suggestions to the nonstationary problem. In this special case of $W(x)$, problem (8.1) admits a constant solution, and this trivial solution generates nontrivial ones.

As is described in Chapter 1, [33] studied radially symmetric solutions to (8.1) for

$$\Omega = B \equiv \left\{ x \in \mathbf{R}^2 \;\middle|\; |x| < 1 \right\},$$

and obtained the conjecture that problem (1.1) will admit the solution global in time if $n = 2$ and $\|u_0\|_1 < 8\pi$. More precisely, from the branch of the constant solution to (8.1) denoted by

$$\mathcal{C}_c = \left\{ (\lambda, \lambda/(a\pi)) \;\middle|\; 0 < \lambda < +\infty \right\},$$

another branch of nonconstant radially symmetric solutions, denoted by $\mathcal{C}_r$, bifurcates from $\mathcal{C}_c$ in $\lambda - v$ space. Then, it was suspected from the numerical computation that $\mathcal{C}_r$ is absorbed into the hyperplane $\lambda = 8\pi$. This fact was proven later by [144] using a lemma of [162]. On the other hand, the discrepancy of the threshold value for $T_{\max} = +\infty$ in (1.1) between that of the conjecture, $\|u_0\|_1 = 8\pi$, and that actually proven, $\|u_0\|_1 = 4\pi$, led us to recognize the role of the boundary blowup point [108]. More precisely, the boundary blowup is observed even in the stationary problem, and the structure of the set of solutions to (8.1) on disc has more varieties than the one suspected by [33].

The following theorem is concerned with the (multiple) existence of the solution, where $\left\{\mu_j^*\right\}_{j=1}^{\infty}$ denotes the set of eigenvalues of $-\Delta$ on Ω under the Neumann boundary condition. Here, we recall that the isoperimetric inequality of Polyá, Szegö, and Weinberger is indicated as

$$|\Omega|\mu_2^* \leq \ell^2\pi,$$

where ℓ denotes the first zero point of J_1', with J_ν being the Bessel function of the first kind [5]. We have $\ell = 1.841\ldots$ and hence $\lambda_1 < 4\pi$ holds for $0 < a \ll 1$, where

$$\lambda_1 = |\Omega|\left(a + \mu_2^*\right).$$

On the other hand, it is obvious that $\lambda_1 > 4\pi$ holds for $a \gg 1$.

Theorem 8.1 *If $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary, then the following facts hold for (8.1).*

1. *If $0 < \lambda \ll 1$, then any nonconstant solution does not exist.*
2. *If $\lambda_1 < 4\pi$, then any $\lambda \in (\lambda_1, 4\pi)$ admits a nonconstant solution.*
3. *If $\lambda_1 > 4\pi$, then any $\lambda \in (4\pi, \lambda_1)\backslash 4\pi\mathcal{N}$ admits a nonconstant solution.*

The next theorem deals with the case $0 < a \ll 1$.

Theorem 8.2 *If $\Omega \subset \mathbf{R}^2$ is a simply connected bounded domain with smooth boundary, then there exists a constant $\delta > 0$ with the following property. Namely, given $\lambda \in (4\pi, 4\pi + \delta)$, any solution to (8.1) is linearized unstable, if $a > 0$ is small.*

In spite of the nonlocal term, the linearized operator around the stationary solution to (8.1) is realized as a self-adjoint operator in $L^2(\Omega)$. The linearized instability mentioned above indicates the negativity of its first eigenvalue. It will be shown that this means the dynamical instability as a stationary solution to (1.1). Therefore, Theorem 8.2 suggests that when $0 < \lambda - 4\pi \ll 1$ and $0 < a \ll 1$, the set of solutions to (1.1) is very simple or otherwise the dynamics of (1.11), including the possible blowup of the solution, is rather complicated.

In this connection, the result on topological degree described in Chapter 6 is worth remembering. In fact, the (JL) field is realized as the limit case $a = 0$ of the (N) field, and the quantized blowup mechanism is kept for all $a > 0$. Therefore, the (total) topological degree to these systems is constant in each component of $[0, +\infty)\backslash 4\pi\mathcal{N}$. If what we expected in Chapter 6 is correct, then the total degree $d(\lambda)$ of the solution to (8.1) will be equal to -1 if $\lambda \in (4\pi, 8\pi)$ and Ω is simply connected. Our new conjecture is that $\lambda \in (4\pi, 8\pi)$ admits only constant solutions if Ω is simply connected and

$$a \in \big(0, (4 - \ell^2)\pi\big).$$

Namely, in this case, we expect that the dynamics of (1.1) is quite simple and the solutions blow up generically in finite time with one blowup point on the boundary, and that the other exceptional solutions exist globally in time and converge to the constant stationary solution.

Beginning the proof of the above theorems, first, we confirm the following fundamental facts.

Lemma 8.1 *A function $v = v(x)$ is a solution to (8.1) if and only if it is a critical point of $\mathcal{J}_\lambda$ defined on $H^1(\Omega)$ given by*

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{a}{2}\|v\|_2^2 - \lambda \log\Big(\int_\Omega e^v\,dx\Big),$$

which means that $v \in H^1(\Omega)$ satisfies

$$\big\langle \mathcal{J}_\lambda'(v), \varphi\big\rangle \equiv \frac{d}{ds}\mathcal{J}_\lambda(v + s\varphi)\Big|_{s=0} = 0$$

for any $\varphi \in H^1(\Omega)$, where $\langle\ ,\ \rangle$ denotes the pairing between V' and $V = H^1(\Omega)$.

Lemma 8.2 *The linearized operator around a solution $v = v(x)$ to* (8.1) *is given as*

$$L\phi = -\Delta\phi + a\phi - \lambda\left(\frac{e^v}{\int_\Omega e^v\,dx}\phi - \frac{e^v}{\left(\int_\Omega e^v\,dx\right)^2}\int_\Omega e^v\phi\,dx\right), \quad \text{in} \quad \Omega$$

$$\frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on} \quad \partial\Omega. \tag{8.2}$$

This means that if v is a critical function of $\mathcal{J}_\lambda$ in $H^1(\Omega)$, then the self-adjoint operator L defined above in $L^2(\Omega)$ is associated with the bilinear form

$$\mathcal{A}(\varphi,\varphi) = \frac{d^2}{ds^2}\mathcal{J}_\lambda(v+s\varphi)\Big|_{s=0}$$

defined for $\varphi \in H^1(\Omega)$ through the relation

$$(L\varphi,\psi) = \mathcal{A}(\varphi,\psi)$$

for $\varphi \in \mathrm{dom}(L) \subset H^1(\Omega)$ and $\psi \in H^1(\Omega)$, where $(\ ,\ )$ denotes the L^2 inner product.

In particular, if $(\lambda, v(x)) = (0,0)$, then L is $-\Delta + a$ in Ω under the Neumann boundary condition by Lemma 8.2, where this operator is invertible. Now, we show the following proof.

Proof of Theorem 8.1 for the first case: By Theorem 7.5, we have the following fact:

Any $\varepsilon \in (0, 4\pi)$ admits a constant $C_\varepsilon > 0$ such that for any solution $v = v(x)$ of (8.1) with $0 \le \lambda < 4\pi - \varepsilon$ it holds that $\|v\|_\infty \le C_\varepsilon$.

On the other hand the linearized operator is invertible around $v(x) \equiv 0$ for $\lambda = 0$. Therefore, the implicit function theorem assures local uniqueness of the solution near from $(\lambda, v(x)) = (0,0)$ in $\mathbf{R} \times C(\overline{\Omega})$. Then the assertion follows from the standard argument. More precisely, there is a branch of constant solutions denoted by $\mathcal{C}_c$ in $\lambda - v$ space starting from $(\lambda, v(x)) = (0,0)$. If $0 < \lambda \ll 1$ then no other solutions exist near $\mathcal{C}_c$. Suppose that there is a family of solutions denoted by $\mathcal{S} = \{(\lambda_k, v_k(x))\}$, satisfying $(\lambda_k, v_k(x)) \notin \mathcal{C}_c$ and $\lambda_k \downarrow 0$. Then, the above L^∞ estimate applied to $\{v_k(x)\}$ gives compactness of $\mathcal{S} = \{(\lambda_k, v_k(x))\}$ in $\lambda - v$ space from the standard elliptic estimate.

Since $v(x) \equiv 0$ is the only solution for (8.1) with $\lambda = 0$, $v_k(x)$ converges uniformly to 0. Therefore, $(\lambda_k, v_k(x))$ becomes close to $\mathcal{C}_c$ and this contradicts to the property of the solution on $\mathcal{C}_c$ with $0 < \lambda \ll 1$, that is, its local uniqueness. □

The above proof guarantees again that

$$d(\lambda) = 1$$

for $\lambda \in (0, 4\pi)$, where $d(\lambda)$ denotes the total degree of the solution for (8.1). Now, given the solution $v(x)$ for (8.1), we study the linearized operator in more details. In fact, putting

$$p = \frac{\lambda e^v}{\int_\Omega e^v \, dx},$$

we see that the linearized operator around $v(x)$ is given by $L = L_* + a$, where

$$L_*\phi = -\Delta\phi - p\phi + \frac{p}{\lambda} \int_\Omega p\phi \, dx \quad \text{in} \quad \Omega$$

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{8.3}$$

Provided with the domain

$$\mathrm{dom}(L_*) = \left\{ \phi \in H^2(\Omega) \;\middle|\; \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega \right\}, \tag{8.4}$$

this L_* is realized as a self-adjoint operator in $L^2(\Omega)$. It is associated with the bilinear form $\mathcal{A}_*$ in $H^1(\Omega) \times H^1(\Omega)$ defined by

$$\mathcal{A}_*(\phi, \phi) = \int_\Omega \left(|\nabla\phi|^2 - p\phi^2 \right) dx + \frac{1}{\lambda} \left(\int_\Omega p\phi \, dx \right)^2. \tag{8.5}$$

Here, we note that 0 is always the eigenvalue of L_* corresponding to the constant eigenfunction.

Bifurcation of nonconstant solutions and their stability from the branch of constant solutions, especially when the space dimension is one, are studied by Schaff [142] including the other cases of the nonlinearity. Here, we can confirm that the following facts are valid for this problem, where $\left\{\mu_j^*\right\}_{j=1}^\infty$ denotes the set of eigenvalues of $-\Delta$ in Ω under the Neumann boundary condition.

1. Each $\lambda > 0$ admits a unique constant solution for (1.1) denoted by $v = \boldsymbol{s} \equiv \lambda / a|\Omega|$.

2. For this $\boldsymbol{s}$, the linearized operator L is given by $L = L_* + a$ with

$$L_*\phi = -\Delta\phi - p\phi + p \frac{1}{|\Omega|} \int_\Omega \phi \, dx,$$

where $p = \lambda / |\Omega|$.

3. If ϕ_j^* denotes the L^2 normalized eigenfunction of $-\Delta$ in Ω under the Neumann boundary condition corresponding to the eigenvalue μ_j^*, then we have

$$L_*\phi_j^* = \big(\mu_j^* - p\big)\phi_j^* \quad \text{in} \quad \Omega$$

with

$$\frac{\partial \phi_j^*}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

for $j \geq 2$, where $p = \lambda/|\Omega|$.

4. We have $L_*\phi_1^* = 0$, on the other hand, and therefore $\{\phi_j^*\}_{j=1}^{\infty}$ forms a complete system of eigenfunctions of L_*.

Consequently, we have the following lemma.

Lemma 8.3 *The set of eigenvalues of the linearized operator L around the constant solution $v = \mathbf{s}\ (= \lambda/a|\Omega|)$ is given by*

$$a + \big\{0, \mu_j^* - \lambda/|\Omega| \;\big|\; j \geq 2\big\}.$$

The eigenfunction corresponding to $a + \mu_j^* - \lambda/|\Omega|$ for $j \geq 2$ is ϕ_j^*, and that to a is ϕ_1^*.

This lemma is applicable to examine the linearized stability of the constant stationary solution. In fact, for $p = \lambda/|\Omega|$ and $\phi \in H^1(\Omega)$ we have

$$\begin{aligned}\left(\frac{d}{ds}\right)^2 \mathcal{J}_\lambda(\mathbf{s} + s\phi)\Big|_{s=0} &= \mathcal{L}_*(\phi, \phi) + a\|\phi\|_2^2 \\ &= \int_\Omega \big(|\nabla\phi|^2 - p\phi^2\big) + \frac{1}{\lambda}\Big(\int_\Omega p\phi\, dx\Big)^2 + a\big\|\phi\big\|_2^2 \\ &= \int_\Omega |\nabla\phi|^2 - \frac{\lambda}{|\Omega|}\int_\Omega \phi^2\, dx + \frac{\lambda}{|\Omega|}\Big(\frac{1}{|\Omega|^{1/2}}\int_\Omega \phi\Big)^2 + a\big\|\phi\big\|_2^2.\end{aligned}$$

Writing

$$\phi = \sum_{j=1}^{\infty} \big(\phi, \phi_j^*\big)\phi_j^*$$

with the standard L^2 inner product (,), we have

$$\left(\frac{d}{ds}\right)^2 \mathcal{J}_\lambda(\mathbf{s} + s\phi)\Big|_{s=0} = \sum_{j=2}^{\infty}\Big(\mu_j^* - \frac{\lambda}{|\Omega|} + a\Big)\big|\big(\phi, \phi_j^*\big)\big|^2 + a\big|(\phi, \phi_1^*)\big|^2. \quad (8.6)$$

Thus, we obtain the following lemma.

Lemma 8.4 *If $\lambda < \lambda_1 = |\Omega|\left(a + \mu_2^*\right)$, then the constant solution $v = s$ is a strict local minimum. On the other hand, if $\lambda > \lambda_1$ then it is not a local minimum of $\mathcal{J}_\lambda$ on $H^1(\Omega)$.*

Now, we show the following proof.

Proof of Theorem 8.1 for the second case: We recall that if $0 < a \ll 1$ then

$$\lambda_1 \equiv |\Omega|\left(a + \mu_2^*\right) < 4\pi$$

holds. Therefore, this case actually arises. Since $\Omega \subset \mathbf{R}^2$ has smooth boundary, Chang–Yang's inequality (4.4) holds true:

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^w\,dx\right) \leq \frac{1}{8\pi}\int_\Omega |\nabla w|^2\,dx + \frac{1}{|\Omega|}\int_\Omega w\,dx + K, \tag{8.7}$$

where K is a constant determined by Ω, and $w \in H^1(\Omega)$ is arbitrary. Using

$$\left|\frac{1}{|\Omega|}\int_\Omega w\,dx\right| \leq \frac{\|w\|_2}{|\Omega|^{1/2}},$$

we have a constant $C > 0$ satisfying

$$\mathcal{J}_\lambda(v) \geq \left(\frac{1}{2} - \frac{\lambda}{8\pi}\right)\left\{\|\nabla v\|_2^2 + a\|v\|_2^2\right\} - C \tag{8.8}$$

for any $v \in H^1(\Omega)$. Actually, we can take

$$C = \lambda\left(\frac{2\pi}{a\,|\Omega|} + K + \log|\Omega|\right).$$

This means that $\mathcal{J}_\lambda$ is bounded from below on $H^1(\Omega)$ in the case of $\lambda \in (0, 4\pi)$, and we can take the minimizing sequence $\{v_k\} \subset H^1(\Omega)$ as

$$\mathcal{J}_\lambda(v_k) \;\to\; j_\lambda \equiv \inf_{v \in H^1(\Omega)} \mathcal{J}_\lambda(v) \geq -C.$$

Furthermore, this $\{v_k\}$ is bounded in $H^1(\Omega)$ again by (8.8).

Namely, passing through a subsequence (denoted by the same symbol), we have the weak convergence

$$v_k \;\rightharpoonup\; v$$

in $H^1(\Omega)$. If it is shown that $\mathcal{J}_\lambda$ is weakly lower semicontinuous, then the standard argument guarantees the existence of a global minimum of $\mathcal{J}_\lambda$ on $H^1(\Omega)$

for $\lambda \in (0, 4\pi)$. Actually, if this is the case, v attains the global minimum of $\mathcal{J}_\lambda$ in $H^1(\Omega)$ by

$$j_\lambda = \liminf_{k\to+\infty} \mathcal{J}_\lambda(v_k) \geq \mathcal{J}_\lambda(v) \geq j_\lambda.$$

Furthermore, this $v = v(x)$ is not a constant if $\lambda \in (\lambda_1, 4\pi)$, because the constant solution is not any local minimum in this case.

Now, to prove the weak lower semicontinuity of $\mathcal{J}_\lambda$ on $H^1(\Omega)$, we have only to confirm that

$$\int_\Omega e^{v_k}\,dx \;\to\; \int_\Omega e^{v}\,dx \tag{8.9}$$

follows from $v_k \rightharpoonup v$ in $H^1(\Omega)$. In fact, if $v_k \rightharpoonup v$ in $H^1(\Omega)$, then,

$$\left\|\nabla v_k\right\|_2 \leq L \quad \text{and} \quad \left\|v_k\right\|_1 \leq C$$

holds for $k = 0, 1, 2, \ldots$ with some constants $L > 0$ and $C > 0$. Therefore, we have

$$\begin{aligned}
\left|\int_\Omega e^{v_k}\,dx - \int_\Omega e^{v}\,dx\right| &= \left|\int_\Omega \int_0^1 e^{sv_k+(1-s)v}(v_k - v)ds\,dx\right| \\
&\leq \left\{\int_\Omega (v_k - v)^2\,dx\right\}^{1/2}\left\{\int_0^1 ds \int_\Omega e^{2sv_k+2(1-s)v}\,dx\right\}^{1/2} \\
&\leq \left\|v_k - v\right\|_2 \exp\left(\frac{L^2}{4\pi} + \frac{C}{|\Omega|} + \frac{K}{2} + \frac{\log|\Omega|}{2}\right)
\end{aligned}$$

using (8.7). This implies (8.9) because $v_k \to v$ in $L^2(\Omega)$ holds by Rellich's theorem, and the proof is complete. □

From the above result, if $\lambda \in (\lambda_1, 4\pi)$, then there is a nonconstant solution. More precisely, $\mathcal{J}_\lambda$ is bounded from below on $H^1(\Omega)$ for $\lambda < 4\pi$, and the constant solution $v = \boldsymbol{s}$ is not a local minimum for $\lambda > \lambda_1$. This nonconstant solution, denoted by $\underline{v} = \underline{v}(x)$, is a global minimum of $\mathcal{J}_\lambda$ on $H^1(\Omega)$, and therefore another local minimum of $\mathcal{J}_\lambda$ can be expected.

In fact, the total degree $d(\lambda)$ is equal to $+1$ for $\lambda \in (0, 4\pi)$, while the local degree of $\boldsymbol{s}$ is -1 between the first and the second degeneracies of L, that is, for $\lambda \in (\lambda_1, \min\{4\pi, \lambda_2\})$, where

$$\lambda_2 = |\Omega|\,(a + \mu_3^*).$$

On the other hand, an abstract theorem of Rabinowitz [133] says that the local degree of local minimum is $+1$ if it is isolated, and therefore we have at least three solutions (including s and $\underline{v}$) in the case of $\lambda \in (\lambda_1, \min\{4\pi, \lambda_2\})$. If $\lambda_2 > 4\pi$ holds furthermore, then it is expected that these two nonconstant stationary solutions blow up, developing singular points on $\partial\Omega$ as $\lambda \uparrow 4\pi$. Furthermore, their blowup points will be the maximum and minimum of the Robin function $R(x) = K(x, x)$ restricted to $x \in \partial\Omega$, respectively. In this way, we can expect, more generally, that generic converses of Theorems 7.5 and 7.1 are true.

We have discussed the case $0 < a \ll 1$ such that $\lambda_1 < 4\pi$. If $a > 0$ becomes as large as $\lambda_1 > 4\pi$, then the structure of the bifurcation from $\mathcal{C}_c$ changes drastically, while the quantized blowup mechanism of the family of stationary solutions is kept. What we have examined in this case is the following. First, if $\lambda > 4\pi$, then $\mathcal{J}_\lambda$ is not bounded from below on $H^1(\Omega)$ any more. Next, if $\lambda < \lambda_1$, then the constant solution $v = s$ is a local minimum. Thus, we can expect the mountain pass type critical point for $\lambda \in (4\pi, \lambda_1)$. Furthermore, if Ω is simply connected and what we expected is correct, then the total degree will still be equal to $+1$ for $\lambda \in (4\pi, 8\pi)$. If this is the case, then a theorem of Hoffer [65] concerning the mountain pass critical point can take place of that of [133] on the local minimum. Namely, the local degree of the mountain pass critical point is -1 if it is isolated, and therefore we can expect at least two nonconstant solutions for $\lambda \in (4\pi, \lambda_1)$ in this case: $\lambda_1 > 4\pi$ and simply connected Ω. We expect also that these solutions blow up as $\lambda \downarrow 4\pi$ with the blowup points equal to the maximum and the minimum of the Robin function restricted to $\partial\Omega$, respectively.

We begin with the following lemma for this case of $\lambda_1 > 4\pi$.

Lemma 8.5 *If $\lambda > 4\pi$, then there is $v_0 \in H^1(\Omega)$ satisfying*

$$\mathcal{J}_\lambda(v_0) < \mathcal{J}_\lambda(s). \tag{8.10}$$

Proof: We note that the Moser–Onofri inequality (4.5),

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^v\,dx\right) \leq \frac{1}{16\pi}\|\nabla v\|_2^2 + 1$$

is sharp, and the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\left(\int_\Omega e^v\,dx\right)$$

defined for $v \in H_0^1(\Omega)$ does not attain the minimum if $\lambda = 8\pi$. Since it is attained for $\lambda \in (0, 8\pi)$, the singular limit

$$4 \log \frac{1}{|x|}$$

of radially symmetric solutions to (7.1) as $\lambda \uparrow 8\pi$, will generate an unbounded sequence to the above $\mathcal{J}_\lambda$ for $\lambda > 8\pi$. Here we are taking the functional associated with Chang–Yang's inequality, and therefore this singular limit, shifted to the boundary, will play the same role.

More precisely, taking $x_0 \in \partial\Omega$ and $\mu > 0$, we put

$$w_\mu(x) = 2 \log \left(\frac{1+\mu}{|x - x_0|^2 + \mu} \right)$$

and show

$$\lim_{\mu \downarrow 0} \mathcal{J}_\lambda \left(w_\mu \right) = -\infty \tag{8.11}$$

for $\lambda > 4\pi$. In fact, we have

$$\begin{aligned} \mathcal{J}_\lambda \left(w_\mu \right) &= \frac{1}{2} \int_\Omega \left| \nabla w_\mu \right|^2 dx - \lambda \log \left(\int_\Omega e^{w_\mu} dx \right) + O(1) \\ &= \frac{1}{2} \int_{\Omega \cap D_\delta} \left| \nabla 2 \log \left(\frac{1+\mu}{|x-x_0|^2 + \mu} \right) \right|^2 dx \\ &\quad - \lambda \log \left(\int_{\Omega \cap D_\delta} \left(\frac{1+\mu}{|x-x_0|^2+\mu} \right)^2 dx \right) + O(1), \end{aligned}$$

where $D_\delta = \left\{ x \in \mathbf{R}^2 \;\middle|\; |x - x_0| < \delta \right\}$ for $\delta > 0$. We take a cone with vertex x_0 and angle $\pi(1 - 2\varepsilon)$ for $\varepsilon \in (0, \pi)$, denoted by K, satisfying $K \subset \Omega^c$. Then, the first term of the right-hand side is estimated from above by

$$\begin{aligned} \frac{1}{2} \int_{K^c \cap D_\delta} \left| \nabla w_\mu \right|^2 &= \frac{1}{2} \int_{-\pi\varepsilon}^{\pi(1+\varepsilon)} \int_0^\delta \frac{16 r^2}{\left(\mu + r^2 \right)^2} r \, dr \, d\theta \\ &= 4\pi(1 + 2\varepsilon) \log \frac{1}{\mu} + O(1). \end{aligned}$$

Taking similar cone K^* satisfying $D_\delta \cap K^* \subset \Omega$, we can estimate the second term from below as

$$\begin{aligned} \lambda \log \left(\frac{1}{|\Omega|} \int_{K^* \cap D_\delta} e^{w_\mu} \right) &= \lambda \log \left(\int_{\pi\varepsilon}^{\pi(1-\varepsilon)} \int_0^\delta \left(\frac{1+\mu}{r^2+\mu} \right)^2 r \, dr \, d\theta \right) \\ &= \lambda \log \frac{1}{\mu} + O(1). \end{aligned}$$

This means

$$\mathcal{J}_\lambda\left(w_\mu\right) \leq (4\pi(1+2\varepsilon)-\lambda)\log\frac{1}{\mu}+O(1)$$

as $\mu \downarrow 0$. Hence (8.11) holds for $\lambda > 4\pi$. □

In the case of

$$4\pi < \lambda < \lambda_1,$$

we put

$$\Gamma = \left\{\gamma \in C\big([0,1], H^1(\Omega)\big) \mid \gamma(0)=s, \quad \gamma(1)=v_0\right\}$$

and

$$j_\lambda = \inf_{\gamma\in\Gamma}\max_{\gamma}\mathcal{J}_\lambda, \tag{8.12}$$

where $v_0 = v_0(x)$ is the function given by Lemma 8.5. Since s is a strict local minimum and (8.10) is satisfied, it holds that $j > \mathcal{J}_\lambda(s)$. Then, Ekeland's variational principle [42] assures the existence of a Palais–Smale sequence, denoted by $\{v_k\}$, satisfying $\mathcal{J}_\lambda'(v_k) \to 0$ and $\mathcal{J}_\lambda(v_k) \to j$.

The problem arising here is its H^1 boundedness. If this is the case, then $\{v_k\} \subset H^1(\Omega)$ is precompact, because $v_k \rightharpoonup v$ implies (8.9). (Details are described below in the proof of the third case of Theorem 8.1.) One of the motivations of [124] is to examine the quantized blowup mechanism for the Palais–Smale sequence relative to $\mathcal{J}_\lambda$ defined on $H^1(\Omega)$. We have a generalization of Theorem 7.5 in this direction, which, however, is not sufficient to control the general Palais–Smale sequence. Here, we take the different argument of Struwe [160, 161], which guarantees the boundedness of the Palais–Smale sequence obtained by the mini-max variational formulation for almost every λ. Then, we apply the quantized blowup mechanism for the family of solutions to extend the existence to any nonquantized value of λ.

The former part of the above argument has a sophisticated abstract version. In fact, we have

$$\mathcal{J}_\lambda = A_\lambda - \lambda B$$

with

$$A_\lambda v = \frac{1}{2}\|v\|_{H^1}^2 \quad \text{and} \quad Bv = \log\left(\int_\Omega e^v\,dx\right),$$

and it holds that $\mathcal{J}_\lambda \in C^1\big(H^1(\Omega), \mathbf{R}\big)$. Here and henceforth, we set

$$\|v\|_{H^1} = \left(\|\nabla v\|_2^2 + a\|v\|_2^2\right)^{1/2}$$

for simplicity. Now, we confirm the following lemma.

Lemma 8.6 *If* $\{(\lambda_k, v_k)\} \subset H^1(\Omega) \times (4\pi, \lambda_1)$ *satisfies*

$$\lambda_k \uparrow \lambda_0 \in (4\pi, \lambda_1), \quad \mathcal{J}_{\lambda_k}(v_k) \le C, \quad \text{and} \quad \mathcal{J}_{\lambda_0}(v_k) \ge -C,$$

then we have the following, where $C > 0$ *is a constant independent of* $k = 1, 2, \dots$.

1. *In the case of* $\|v_k\|_{H^1} \to +\infty$*, it holds that* $B(v_k) \to +\infty$.
2. *If* $\left\{\|v_k\|_{H^1}\right\}$ *is bounded, then we have*
$$\liminf_{k\to+\infty} B(v_k) > -\infty.$$
3. *There is* $L > 0$ *such that*
$$A_{\lambda_0}(v_k) - A_{\lambda_k}(v_k) \le L\,(\lambda_0 - \lambda_k)$$
for any k.

Proof: We have

$$\begin{aligned}\frac{1}{2}\|v_k\|_{H^1}^2 &= \mathcal{J}_{\lambda_k}(v_k) + \lambda_k\left\{\log\left(\frac{1}{|\Omega|}\int_\Omega e^{v_k}\,dx\right) + \log|\Omega|\right\} \\ &\le C + \lambda_0 \log\left(\frac{1}{|\Omega|}\int_\Omega e^{v_k}\,dx\right) + \lambda_k \log|\Omega|\end{aligned}$$

and this implies the first assertion. The second assertion is obtained by

$$\begin{aligned}\lambda_0 \log\left(\int_\Omega e^{v_k}\,dx\right) &= -\mathcal{J}_{\lambda_0}(v_k) + \frac{1}{2}\|v_k\|_{H^1}^2 \\ &\le C + \tfrac{1}{2}\|v_k\|_{H^1}^2.\end{aligned}$$

The third assertion is obvious, because $Av = \frac{1}{2}\|v\|_{H^1}^2$ is independent of λ. □

Now, we can apply an abstract theorem [74] stated as follows. Henceforth, X denotes a Banach space over $\mathbf{R}$, and $\langle\ ,\ \rangle$ and $\|\cdot\|_*$ are the pairing between X' and X and the dual norm in X', respectively.

Theorem 8.3 *Let X and I be a Banach space over $\mathbf{R}$ and a nonempty interval, respectively. Suppose that $\mathcal{J}_\lambda \in C^1(X, \mathbf{R})$ is given for each $\lambda \in I$, satisfying the assumptions of Lemma 8.6 for*

$$\mathcal{J}_\lambda = A_\lambda - \lambda B,$$

and provided with the mountain pass geometry, which means that there exist $v_0, v_1 \in X$ such that

$$j_\lambda \equiv \inf_{\gamma\in\Gamma} \max_{\gamma} \mathcal{J}_\lambda > \max\left\{\mathcal{J}_\lambda(v_0),\ \mathcal{J}_\lambda(v_1)\right\}$$

for any $\lambda \in I$, where

$$\Gamma = \left\{\gamma \in C([0,1], X) \;\middle|\; \gamma(0) = v_0,\ \ \gamma(1) = v_1\right\}.$$

Then, for almost every $\lambda_0 \in I$, $\mathcal{J}_{\lambda_0}$ has a bounded Palais–Smale sequence $\{v_k\}$ of level j_λ, which means that $\{v_k\} \subset X$ is bounded,

$$\mathcal{J}_{\lambda_0}(v_k) \to j_{\lambda_0} \quad \text{and} \qquad \left\|\mathcal{J}'_{\lambda_0}(v_k)\right\|_* \to 0.$$

Actually, using the above theorem we can show the following proof.

Proof of Theorem 8.1 for the third case: We note that v_0 in Lemma 8.5 can be taken locally uniformly in $\lambda \in (4\pi, \lambda_1)$. Therefore, by Lemma 8.6, Theorem 8.3 is applicable. Consequently, we have, for almost every $\lambda \in (4\pi, \lambda_1)$, a bounded Palais–Smale sequence $\{v_k\} \subset H^1(\Omega) = X$ of level j_λ. This means that there is $\{w_k\} \subset H^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta(v_k - w_k) + a(v_k - w_k) &= \frac{\lambda e^{v_k}}{\int_\Omega e^{v_k}\,dx} && \text{in} \quad \Omega, \\ \frac{\partial}{\partial \nu}(v_k - w_k) &= 0 && \text{on} \quad \partial\Omega, \end{aligned}$$

and

$$\left\|w_k\right\|_{H^1} \to 0.$$

Since $\{v_k\}$ is bounded in $H^1(\Omega)$, Chang–Yang's inequality (8.7) guarantees the uniform boundedness of $\lambda e^{v_k} \big/ \int_\Omega e^{v_k}\,dx$ in $L^p(\Omega)$ for each $p \in (1,\infty)$ and hence $\{v_k - w_k\}$ has a subsequence, denoted by the same symbol, converging weakly in $W^{2,p}(\Omega)$, with the limit denoted by $v \in W^{2,p}(\Omega)$. This implies the strong convergence $v_k \to v$ in $H^1(\Omega)$, and then that of

$$\frac{\lambda e^{v_k}}{\int_\Omega e^{v_k}\,dx} \to \frac{\lambda e^{v}}{\int_\Omega e^{v}\,dx}$$

in $L^p(\Omega)$ is proven similarly to (8.9). In particular, v is a solution to (8.1) with $\mathcal{J}_\lambda(v) = j_\lambda > \mathcal{J}_\lambda(s)$, and hence we have a nonconstant solution for almost every $\lambda \in (4\pi, \lambda_1)$.

On the other hand, any $\lambda \in (4\pi, \lambda_1) \setminus 4\pi\mathcal{N}$ admits a sequence $\lambda_k \uparrow \lambda$ provided with a nonconstant solution $v = v_k$ to (8.1) for $\lambda = \lambda_k$ satisfying

$$\mathcal{J}_{\lambda_k}(v_k) = j_{\lambda_k}$$

for each k. Since

$$\lambda \in (4\pi, \lambda_1) \quad \mapsto \quad j_\lambda$$

is nonincreasing, it holds that

$$\mathcal{J}_{\lambda_k}(v_k) \geq j_\lambda.$$

If $\lambda \notin 4\pi\mathcal{N}$, then $\{\|v_k\|_\infty\}$ is bounded by Theorem 7.5. Therefore, from the elliptic estimate we have

$$v_k \to v$$

in $C^2(\overline{\Omega})$, passing through a subsequence. Then

$$\mathcal{J}_\lambda(v) \geq j_\lambda > \mathcal{J}_\lambda(s)$$

follows, and $v = v(x)$ is a nonconstant solution to (8.1). The proof is complete. □

To prove Theorem 8.3, first, we show the following lemma.

Lemma 8.7 *Let X and I be a Banach space over* **R** *and a nonempty interval, respectively. Suppose that $\mathcal{J}_\lambda \in C^1(X, \mathbf{R})$ is given for each $\lambda \in I$, satisfying the assumption of Lemma 8.6 with*

$$\mathcal{J}_\lambda = A_\lambda - \lambda B. \tag{8.13}$$

Then, if $\{(\lambda_k, v_k)\} \subset I \times X$ *satisfies*

$$\lambda \uparrow \lambda_0 \in I, \qquad -\mathcal{J}_{\lambda_0}(v_k) \leq K, \qquad \mathcal{J}_{\lambda_k}(v_k) \leq K,$$

and

$$\frac{\mathcal{J}_{\lambda_k}(v_k) - \mathcal{J}_{\lambda_0}(v_k)}{\lambda_0 - \lambda_k} \leq K$$

with a constant K *independent of* k*, this* $\{v_k\} \subset X$ *is bounded, and any* $\varepsilon > 0$ *admits* N *such that*

$$\mathcal{J}_{\lambda_0}(v_k) \leq \mathcal{J}_{\lambda_k}(v_k) + \varepsilon$$

for any $k \geq N$.

Proof: From (8.13) we have

$$\begin{aligned} \mathcal{J}_{\lambda_k}(v_k) - \mathcal{J}_{\lambda_0}(v_k) &= A_{\lambda_k}(v_k) - \lambda_k B(v_k) - A_{\lambda_0}(v_k) + \lambda_0 B(v_k) \\ &\geq -C(\lambda_0 - \lambda_k) + (\lambda_0 - \lambda_k) B(v_k). \end{aligned} \tag{8.14}$$

This implies

$$\frac{\mathcal{J}_{\lambda_k}(v_k) - \mathcal{J}_{\lambda_0}(v_k)}{\lambda_0 - \lambda_k} \geq -C + B(v_k)$$

and hence

$$\limsup_{k \to +\infty} B(v_k) < +\infty$$

follows. In particular, $\{v_k\}$ is bounded in X from the first assumption of Lemma 8.6, and we have

$$B(v_k) \geq -M$$

with a constant M independent of k from the second assumption. Now, inequality (8.14) implies

$$\mathcal{J}_{\lambda_0}(v_k) - \mathcal{J}_{\lambda_k}(v_k) \leq C(\lambda_0 - \lambda_k) + (\lambda_0 - \lambda_k) \cdot M,$$

and hence the conclusion. □

We shall make use of the theorem of Denjoy–Young–Sacks to show the following lemma. This theorem is concerned with four Dini derivatives of each real-valued function defined on an interval:

$$\Lambda^{\pm} = \limsup_{\pm h \downarrow 0} \frac{f(x+h) - f(x)}{h},$$

$$\lambda^{\pm} = \liminf_{\pm h \downarrow 0} \frac{f(x+h) - f(x)}{h}.$$

We say that two of these numbers are *associated* if they take the same side such as λ^+ and Λ^+, and *opposed* for the other case such as λ^+ and Λ^-. Then, except for x in a set of measure zero, $f = f(x)$ admits the following properties:

1. Two associated derivatives are either equal and finite or unequal with at least one infinite.
2. Two opposed derivatives are either finite and equal or infinite and unequal with the one of higher index, Λ, equal to ∞ and the other, λ, equal to $-\infty$.

See Riesz and Nagy [137] for the proof of the above fact.

Lemma 8.8 *For almost every $\lambda_0 \in I$, there is $\lambda_k \uparrow \lambda_0$ and $M(\lambda_0) < +\infty$ such that*

$$-\frac{j_{\lambda_0} - j_{\lambda_k}}{\lambda_0 - \lambda_k} \le M(\lambda_0). \tag{8.15}$$

Proof: Condition (8.15) is violated if and only if the left Dini derivative of j_λ is $-\infty$ at λ_0. It cannot occur except for a set of measure zero of λ_0 by the theorem of Denjoy–Young–Sacks. □

Lemma 8.9 *Under the assumptions of Theorem 8.3, suppose that $\lambda_0 \in I$ satisfies (8.15). Then, we have a family $\{\gamma_k\} \in \Gamma$ and a constant $K = K(\lambda_0) > 0$ provided with the following properties.*

1. *If*

$$\mathcal{J}_{\lambda_0}(\gamma_k(t)) \ge j_{\lambda_0} - (\lambda_0 - \lambda_k), \tag{8.16}$$

then

$$\|\gamma_k(t)\| \le K$$

holds for $t \in [0, 1]$.

2. *Any* $\varepsilon > 0$ *admits* N *such that*

$$\max_{\gamma_k} \mathcal{J}_{\lambda_0} \le j_{\lambda_0} + \varepsilon$$

for every $n \ge N$.

Proof: By the definition, we have $\{\gamma_k\} \subset \Gamma$ such that

$$\max_{\gamma_k} \mathcal{J}_{\lambda_k} \le j_{\lambda_k} + (\lambda_0 - \lambda_k). \tag{8.17}$$

First, if (8.16) holds, then we have

$$\begin{aligned}\frac{\mathcal{J}_{\lambda_k}(\gamma_k(t)) - \mathcal{J}_{\lambda_0}(\gamma_k(t))}{\lambda_0 - \lambda_k} &\le \frac{j_{\lambda_k} + (\lambda_0 - \lambda_k) - j_{\lambda_0} + (\lambda_0 - \lambda_k)}{\lambda_0 - \lambda_k} \\ &\le M(\lambda_0) + 2.\end{aligned}$$

Next, $\left\{\mathcal{J}_{\lambda_0}(\gamma_k(t))\right\}$ is bounded from below by (8.16). Finally, $\left\{\mathcal{J}_{\lambda_k}(\gamma_k(t))\right\}$ is bounded from above by (8.17) and (8.15). Therefore, $\{\gamma_k(t)\}$ is bounded by Lemma 8.7. This shows the first case of the assertion.

To prove the second case, we take $v_k \in \gamma_k([0, 1])$ satisfying

$$\max_{\gamma_k} \mathcal{J}_{\lambda_0} = \mathcal{J}_{\lambda_0}(v_k) \quad \left(\ge j_{\lambda_0}\right).$$

Then, it holds that

$$\mathcal{J}_{\lambda_0}(v_k) = \max_{\gamma_k} \mathcal{J}_{\lambda_0} > -\infty$$

and

$$\begin{aligned}\mathcal{J}_{\lambda_k}(v_k) \le \max_{\gamma_k} \mathcal{J}_{\lambda_k} &\le j_{\lambda_k} + (\lambda_0 - \lambda_k) \\ &\le j_{\lambda_0} + (M(\lambda_0) + 1)(\lambda_0 - \lambda_k) < +\infty\end{aligned}$$

by (8.17) and (8.15). These inequalities also imply

$$\frac{\mathcal{J}_{\lambda_k}(v_k) - \mathcal{J}_{\lambda_0}(v_k)}{\lambda_0 - \lambda_k} \le M(\lambda_0) + 1$$

and hence Lemma 8.7 is applicable. Given $\varepsilon > 0$, we have

$$d_k \equiv \mathcal{J}_{\lambda_0}(v_k) - \mathcal{J}_{\lambda_k}(v_k) < \frac{\varepsilon}{3}$$

for any $k \geq N$. Furthermore, we have

$$\max_{\gamma_k} \mathcal{J}_{\lambda_0} = \mathcal{J}_{\lambda_0}(v_k) = \mathcal{J}_{\lambda_k}(v_k) + d_k \\ \leq \max_{\gamma_k} \mathcal{J}_{\lambda_k} + d_k \leq j_{\lambda_k} + (\lambda_0 - \lambda_k) + d_k$$

by (8.17). Then, (8.15) gives

$$j_{\lambda_k} - j_{\lambda_0} < \frac{\varepsilon}{3}$$

for k sufficiently large, and the proof is complete. □

Lemma 8.10 *For K in Lemma 8.9, we put*

$$F_a = \left\{ v \in X \;\middle|\; \|v\| \leq K+1, \;\; \left|\mathcal{J}_{\lambda_0}(v) - j_{\lambda_0}\right| \leq a \right\},$$

where $a > 0$. Then, we have

$$\inf \left\{ \left\| \mathcal{J}'_{\lambda_0}(v) \right\|_* \;\middle|\; v \in F_a \right\} = 0$$

if $0 < a \ll 1$.

Proof: If this is not the case, we have

$$\inf_{v \in F_a} \left\| \mathcal{J}'_{\lambda_0}(v) \right\|_* \geq a$$

and

$$a \in \left(0, \frac{1}{2}\left[j_{\lambda_0} - \max\left\{ \mathcal{J}_{\lambda_0}(v_0), \mathcal{J}_{\lambda_0}(v_1) \right\} \right] \right) \tag{8.18}$$

for $0 < a \ll 1$. Then, the deformation theory [134] guarantees the existence of $\alpha \in (0, a)$ and a homeomorphism $\eta : X \to X$ such that

$$\eta(v) = v \qquad \text{if} \quad \left|\mathcal{J}_{\lambda_0}(v) - j_{\lambda_0}\right| \geq a, \tag{8.19}$$

$$\mathcal{J}_{\lambda_0} \circ \eta \leq \mathcal{J}_{\lambda_0} \qquad \text{on} \quad X, \tag{8.20}$$

and

$$\mathcal{J}_{\lambda_0}\left(\eta(v)\right) \leq j_{\lambda_0} - \alpha \tag{8.21}$$

for $v \in X$ in $\|v\| \leq K$ and $\mathcal{J}_{\lambda_0}(v) \leq j_{\lambda_0} + \alpha$.

We take the family $\{\gamma_k\} \subset \Gamma$ of Lemma 8.9. Then, for $k \gg 1$, it holds that

$$\max_{\gamma_k} \mathcal{J}_{\lambda_0} \leq j_{\lambda_0} + \alpha \tag{8.22}$$

and

$$\alpha > \lambda_0 - \lambda_k. \tag{8.23}$$

Here, we have $\eta \circ \gamma_k \in \Gamma$ by (8.18). Let $v \in \gamma_k([0,1])$. Then, in the case of

$$\mathcal{J}_{\lambda_0}(v) \leq j_{\lambda_0} - (\lambda_0 - \lambda_k)$$

we have

$$\mathcal{J}_{\lambda_0}(\eta(v)) \leq j_{\lambda_0} - (\lambda_0 - \lambda_k)$$

by (8.20). In the other case of

$$\mathcal{J}_{\lambda_0} > j_{\lambda_0} - (\lambda_0 - \lambda_k)$$

we have $\|v\| \leq K$ by Lemma 8.9. We can apply (8.21) by (8.22) and hence it follows that

$$\mathcal{J}_{\lambda_0}(v) \leq j_{\lambda_0} - \alpha \leq j_{\lambda_0} - (\lambda_0 - \lambda_k)$$

from (8.23). In any case, we have

$$\max_{\eta \circ \gamma_k} \mathcal{J}_{\lambda_0} \leq j_{\lambda_0} - (\lambda_0 - \lambda_k),$$

a contradiction, and the proof is complete. □

Now, we give the following proof.

Proof of Theorem 8.3: Suppose (8.15) at $\lambda_0 \in I$, and apply Lemma 8.10 for $a = 1/k$ $(k = 1, 2, \ldots)$. Then, there is $v_k \in X$ such that

$$\|v_k\| \leq K + 1, \qquad \mathcal{J}_{\lambda_0}(v_k) \to j_{\lambda_0},$$

and

$$\mathcal{J}'_{\lambda_0}(v_k) \to 0.$$

The proof is complete. □

In the rest of this chapter we prove Theorem 8.2, supposing $0 < a \ll 1$. First, we show that the (JL) field is regarded as the limit case of the (N) field of $a \downarrow 0$ including their linearized operators.

Lemma 8.11 *Let $\lambda \notin 4\pi\mathcal{N}$ be fixed, and $\{v_a\}$ be a family of solutions to (1.1) with $a \downarrow 0$. Then, passing through a subsequence we have*

$$p_a = \frac{\lambda e^{v_a}}{\int_\Omega e^{v_a}\,dx} \quad \longrightarrow \quad p_0 \tag{8.24}$$

in $C^2(\overline{\Omega})$ with a positive function $p_0(x)$ defined on $\overline{\Omega}$.

Proof: We write $v = v_a$ and $p = p_a$ for simplicity. First, (8.1) implies

$$a\|v\|_1 = \lambda$$

because $v > 0$ in Ω, and hence

$$\|v\|_1 \to +\infty$$

follows as $a \downarrow 0$. We take

$$w = v - \frac{1}{|\Omega|}\int_\Omega v\,dx.$$

It satisfies

$$\begin{aligned} -\Delta w &= -av + \lambda\frac{e^v}{\int_\Omega e^v\,dx} \quad \text{in} \quad \Omega, \\ \frac{1}{|\Omega|}\int_\Omega w\,dx &= 0, \\ \frac{\partial w}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{8.25}$$

with

$$\left\| -av + \frac{\lambda e^v}{\int_\Omega e^v\,dx}\right\|_1 \leq 2\lambda.$$

Therefore, the L^1 estimate guarantees

$$\|w\|_{W^{1,q}(\Omega)} \leq C, \tag{8.26}$$

where $C > 0$ denotes a constant independent of a.

Since the first equality of (8.25) is written as

$$-\Delta w = -aw - \frac{\lambda}{|\Omega|} + \frac{\lambda e^w}{\int_\Omega e^w\,dx}, \tag{8.27}$$

we have

$$\begin{aligned} -\Delta w + w &\geq (1-a)w - \frac{\lambda}{|\Omega|} && \text{in} \quad \Omega, \\ \frac{\partial w}{\partial \nu} &= 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Here, (8.26) implies

$$\left\| (1-a)w - \frac{\lambda}{|\Omega|} \right\|_{W^{1,q}} \leq C$$

for $1 < q < 2$. Then, the elliptic regularity, Morrey's inequality, and the comparison theorem assure

$$w(x) \geq -C$$

uniformly. In particular, the blowup set $\mathcal{S}$ of any subsequence $\{w'_a\} \subset \{w_a\}$ coincides with

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \;\middle|\; \text{there exist } a_k \to 0 \text{ and } x_k \to x_0 \text{ satisfying } w'_{a_k}(x_k) \to +\infty \right\}.$$

Now, we can argue similarly as in the previous chapter using (8.26) and (8.27). Thus, $\{w\}$ is uniformly bounded by $\lambda \notin 4\pi\mathcal{N}$. Passing through a subsequence, we have

$$w'_a \longrightarrow w_0$$

in $C^2(\overline{\Omega})$ with some $w_0(x)$ from the elliptic regularity. Therefore, (8.24) holds for

$$p_a = \frac{\lambda e^{w'_a}}{\int_\Omega e^{w'_a}\,dx} \quad \text{and} \quad p_0 = \frac{\lambda e^{w_0}}{\int_\Omega e^{w_0}\,dx},$$

and the proof is complete. □

If $\lambda = 4\pi\ell$ and the blowup actually occurs to $\{w_a\}$, we have

$$\ell = 2\sharp\,(\Omega \cap \mathcal{S}) + \sharp\,(\mathcal{S} \cap \partial\Omega)\,,$$
$$\left(-\Delta w_a' + a w_a'\right)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0)\delta_{x_0}(dx),$$
$$\nabla_x \Big[m_*(x_0) K(x, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m_*(x_0') G(x, x_0') \Big]_{x=x_0} = 0$$

for each $x_0 \in \mathcal{S}$, where only the tangential derivative is taken in the last relation if $x_0 \in \partial\Omega$, and $G = G(x, x')$ denotes the Green's function to the (JL) field:

$$-\Delta_x G(x, x') = \delta_{x'}(dx) \quad (x \in \Omega),$$
$$\int_\Omega G(x, x')\,dx = 0,$$
$$\left.\frac{\partial G}{\partial \nu_x}\right|_{x \in \partial\Omega} = 0.$$

This is proven similarly as in the proof of Theorem 7.5.

We proceed to the spectral analysis of the linearized operator L defined by (8.2). This operator is defined for any positive continuous function $p = p(x)$ on $\overline{\Omega}$ by (8.3) and

$$L = L_* + a.$$

Let $\{\mu_j\}_{j=2}^\infty$ be the set of eigenvalues for L_* of which eigenfunctions are different from the constant, and let $\{\rho_j\}_{j=1}^\infty$ be the set of all eigenvalues of

$$-\Delta - p \quad \text{with} \quad \left.\frac{\partial}{\partial\nu}\;\cdot\;\right|_{\partial\Omega} = 0.$$

We set $X = H^1(\Omega)$ for the moment.

Lemma 8.12 *The values μ_j, ρ_j, and $\sigma_j - 1$ have the same sign for each $j \geq 2$, where*

$$\sigma_j = \sup_{\substack{X_{j-1} \subset X \\ \operatorname{codim}(X_{j-1}) = j-2}} \inf \left\{ \int_\Omega |\nabla\phi|^2\,dx \;\middle|\; \phi \in X_{j-1},\ \int_\Omega p\phi^2\,dx = 1,\ \int_\Omega p\phi\,dx = 0 \right\}. \tag{8.28}$$

Proof: We recall that the bilinear form $\mathcal{A}_*$ associated with L_* is defined on $X \times X$ and is given by (8.5). This means

$$\mathcal{A}_*(\phi, \phi) = \mathcal{B}_p(\psi, \psi)$$

with

$$\mathcal{B}_p(\psi, \psi) = \int_\Omega \left(|\nabla \psi|^2 - p\psi^2 \right) dx$$

and

$$\psi = \phi - \frac{1}{\lambda} \int_\Omega p\phi \, dx.$$

Here,

$$Q : \phi \in X \quad \mapsto \quad \psi \in X$$

is a projection with the range and the kernel given by

$$\mathrm{ran}(Q) = X_0 \quad \text{and} \quad \ker(Q) = \{\text{constant functions}\},$$

respectively, where

$$X_0 = \left\{ \psi \in X \;\middle|\; \int_\Omega p\psi \, dx = 0 \right\}.$$

The bilinear form $\mathcal{B}_p$ defined on $X_0 \times X_0$ is associated with a self-adjoint operator in $L^2(\Omega)$. If its eigenvalues are denoted by $\left\{\hat{\mu}_j\right\}_{j\geq 2}$, then $\hat{\mu}_j$ and μ_j have the same sign. Furthermore, the mini-max principle guarantees

$$\hat{\mu}_j = \sup_{\substack{X_{j-1}\subset X \\ \mathrm{codim}(X_{j-1})=j-2}} \inf \left\{ \int_\Omega \left(|\nabla\psi|^2 - p\psi^2 \right) dx \;\middle|\; \psi \in X_{j-1}, \int_\Omega p\psi \, dx = 0, \int_\Omega \psi^2 \, dx = 1 \right\}$$

and hence $\hat{\mu}_j$ and $\sigma_j - 1$ have the same sign. Consequently, μ_j and $\sigma_j - 1$ have the same sign.

To examine the relation between ρ_j and σ_j, we take

$$\tilde{\sigma}_j = \sup_{\substack{X_j\subset X \\ \mathrm{codim}(X_j)=j-1}} \inf \left\{ \int_\Omega |\nabla\phi|^2 \, dx \;\middle|\; \phi \in X_j, \int_\Omega p\phi^2 \, dx = 1 \right\}$$

for $j = 1, 2, \ldots$. It is the j-th eigenvalue of

$$-\Delta\phi = \tilde{\sigma} p\phi \quad \text{in} \quad \Omega,$$
$$\frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on} \quad \partial\Omega,$$

so that $\tilde{\sigma}_j - 1$ and ρ_j have the same sign for $j \geq 1$. On the other hand we have $\tilde{\sigma}_1 = 0$ with the constant eigenfunction and hence $\sigma_j = \tilde{\sigma}_j$ for $j \geq 2$. Thus, ρ_j and $\sigma_j - 1$ have the same sign for $j \geq 2$. □

We have arrived at the eigenvalue problem

$$-\Delta\phi = \tilde{\sigma} p\phi \quad \text{in} \quad \Omega, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega$$

for

$$p = \frac{\lambda e^v}{\int_\Omega e^v\,dx},$$

with $v = v(x)$ satisfying (8.1). If it holds that

$$-\Delta \log p \leq p \quad \text{in} \quad \Omega,$$

then Bandle's theory [5] on isoperimetric inequality of eigenvalues is applicable. Consequently, we obtain the following lemma.

Lemma 8.13 *If $\Omega \subset \mathbf{R}^2$ is a simply connected domain and $p = p(x)$ is a positive C^2 function on $\overline{\Omega}$ satisfying*

$$-\Delta \log p \leq p \quad in \quad \Omega \quad and \qquad \lambda \equiv \int_\Omega p\,dx < 8\pi, \tag{8.29}$$

then there is $\delta > 0$ such that for $\lambda \in (0, 4\pi + \delta)$ it holds that

$$\sigma_2 \leq \frac{4\pi}{\lambda}. \tag{8.30}$$

The above lemma is concerned with the Neumann boundary condition, and the method of *conformal plantation* is adopted for the proof. However, more essentially, the isoperimetric inequality on surfaces, referred to as the *Alexandroff–Bol inequality*, is applied for the proof. Therefore, it is regarded as a variant of Polyá-Szegö-Weinberger's inequality developed on the round sphere S^2 in $\mathbf{R}^3$. The *associated Legendre equation* arises in this context as the polar decomposition of $-\Delta$ defined on $\mathbf{R}^3$, or equivalently, the *Laplace–Beltrami operator* defined on S^2. This geometric account of Bandle's theory is developed in [162], and the following fact is proven from that point of view.

Lemma 8.14 *If $\Omega \subset \mathbf{R}^2$ is a simply connected bounded domain with smooth boundary and $q = q(x) > 0$ is a smooth function on $\overline{\Omega}$ satisfying*

$$-\Delta \log q \leq q \quad \textit{in} \quad \Omega \quad \textit{and} \qquad \int_\Omega q\, dx < 8\pi,$$

then it holds that

$$\inf \left\{ \|\nabla \psi\|_2^2 \;\middle|\; \psi \in H_c^1(\Omega),\ \int_\Omega q\psi\, dx = 0,\ \int_\Omega q\psi^2\, dx = 1 \right\} > 1.$$

As we have illustrated, these lemmas are involved with the theory of nonlinear partial differential equations, the isoperimetric inequality on surfaces associated with Gaussian curvature, complex function theory, spectral analysis of the linearized operator, spectral geometry on the Laplace–Beltrami operator defined on surfaces, nonlinear functional analysis including the theory of bifurcation, the theory of special functions, particularly that on the associated Legendre equation, and so forth. We refer to [166] for the proof of Lemma 8.14, and now we give the following proof.

Proof of Lemma 8.13: We have only to examine [5] and therefore just sketch the outline of the proof. Note that $\tilde{\sigma}_j = \sigma_j$ holds for $j \geq 2$. First, each $\lambda \in (0, 8\pi)$ determines $R = R(\lambda) \in (-1, 1)$ and the eigenvalue problems in consideration are associated with this value R as

$$\left[\left(1 - \xi^2\right) \Phi_\xi \right]_\xi + 2\tau \Phi = 0 \qquad (-1 < \xi < R) \tag{8.31}$$

with

$$|\Phi(-1)| < +\infty \quad \text{and} \quad \Phi'(R) = 0 \tag{8.32}$$

and

$$\left[\left(1 - \xi^2\right) \Phi_\xi \right]_\xi - \frac{\Phi}{1 - \xi^2} + 2\nu \Phi = 0 \qquad (-1 < \xi < R) \tag{8.33}$$

with

$$\Phi(-1) = 0 \quad \text{and} \quad \Phi'(R) = 0. \tag{8.34}$$

Here, $\lambda \mapsto R(\lambda)$ is monotone increasing and $\lambda = 0, 4\pi, 8\pi$ correspond to $R = -1, 0, +1$, respectively.

If the second eigenvalue of (8.31) with (8.32) is denoted by τ_2, and the first one of (8.33) with (8.34) by ν_1, then the conclusion of Lemma 3.3.10 of [5] holds as far as $\tau_2 > \nu_1$ is preserved. On the other hand, in the proof of Lemma 3.3.10 of [5] it is shown that $\tau_2 > \nu_1$ is satisfied in the case of $R \leq 0$, or equivalently, $\lambda \leq 4\pi$. From the continuity, $\tau_2 > \nu_1$ holds true for $0 < \lambda - 4\pi \ll 1$. Namely, the conclusion of Lemma 3.3.10 of [5] holds even if $0 < \lambda - 4\pi \ll 1$. Next we examine Lemma 3.3.12 of [5]. Let us suppose that $e(\zeta)$ is nonincreasing only in (a, b_*) with some $b_* < b$. Even so, the proof exposed there is valid if $b - b_* > 0$ is sufficiently small so that the conclusion of this lemma holds even in this case.

Thanks to these facts, we can reproduce the argument in the proof of Corollary 3.3.10 of [5] even for $\lambda \in (0, 4\pi + \delta)$ with $\delta > 0$ sufficiently small. Then, we obtain

$$\frac{1}{\sigma_2} + \frac{1}{\sigma_3} \geq \frac{\lambda}{2\pi},$$

and in particular (8.30) follows. □

We complete the following proof.

Proof of Theorem 8.2: We take $\delta > 0$ of Lemma 8.13 and fix $\lambda \in (4\pi, 4\pi + \delta)$. Suppose that there exists a family of solutions (8.1) with $a = a_k \downarrow 0$, denoted by $\{v_k\}$, of which the first eigenvalues of linearized operators are nonnegative. We write $a = a_k$, $v_a = v_{a_k}$, and

$$p_a = \frac{\lambda e^{v_a}}{\int_\Omega e^{v_a}\, dx}$$

for simplicity. By Lemma 8.11, we have a subsequence (denoted by the same symbol) satisfying

$$p_a \to p_0 \tag{8.35}$$

in $C^2(\overline{\Omega})$ with a positive function $p_0(x)$ defined on $\overline{\Omega}$. We note that this p_0 satisfies (8.29).

If φ_a attains

$$\sigma_2(a) \equiv \inf\left\{ \int_\Omega |\nabla\phi|^2\, dx \;\middle|\; \phi \in H^1(\Omega),\ \int_\Omega p_a\phi\, dx = 0,\ \int_\Omega p_a\phi^2\, dx = 1 \right\},$$

then $\{\varphi_a\}$ is compact in $H^1(\Omega)$. It is obvious that $\{\sigma_2(a)\}$ is bounded, and hence we have

$$\varphi_a \to \varphi \quad \text{in} \quad H^1(\Omega) \quad \text{and} \qquad \sigma_2(a) \to \sigma_2^0,$$

passing through a subsequence. This σ_2^0 attains

$$\sigma_2^0 \equiv \inf\left\{\int_\Omega |\nabla\phi|^2\, dx \;\middle|\; \phi \in H^1(\Omega),\ \int_\Omega p_0\phi\, dx = 0,\ \int_\Omega p_0\phi^2\, dx = 1\right\},$$

and therefore Lemma 8.13 assures

$$\sigma_2^0 \le \frac{4\pi}{\lambda} < 1$$

for $\lambda \in (4\pi, 4\pi + \delta)$. This implies $\mu_2^0 < 0$ by Lemma 8.12.

On the other hand, the set of eigenvalues of the linearized operator around $v_a(x)$ is given by

$$a + \left\{0 + \mu_j(a) \;\middle|\; j \ge 2\right\}.$$

Then, (8.35) implies

$$\mu_2(a) \qquad \to \qquad \mu_2^0,$$

with the second eigenvalue μ_2^0 introduced before Lemma 8.12 for $p = p_0$. See Kato [77] for this convergence of eigenvalues. Then,

$$\lim_{a\downarrow 0}(\mu_2(a) + a) = \mu_2^0 < 0$$

follows and hence $\mu_2(a) + a < 0$ holds for $a > 0$ sufficiently small. This contradicts the assumption and the proof is complete. □

From the proof of Theorem 8.2, we see that there is a constant $\delta > 0$ such that for any $\delta_1 \in (0, \delta)$ there exists $a_1 > 0$ such that if $4\pi + \delta_1 < \lambda < 4\pi + \delta$ and $0 < a < a_1$ then any solution of (8.1) is linearized unstable, provided that $\Omega \subset \mathbf{R}^2$ is a simply connected bounded domain with smooth boundary.

9
Dynamical Equivalence

This chapter returns to the general $W(x)$ and describes the fact that the variational structure $\mathcal{F}$ defined on $\mathcal{P}_\lambda$ and that of $\mathcal{J}_\lambda$ on $V = \mathrm{dom}(A^{1/2})$ stated in Chapter 6 are equivalent up to Morse indices. This fact was known concerning the stability in the case that A is equal to $-\Delta$ with the Dirichlet boundary condition, but actually general theory holds true. This structure is not restricted to the Keller–Segel system; it is valid for several mean field theories.

We recall that $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $W = W(x) > 0$ is a smooth function of $x \in \overline{\Omega}$, and $A > 0$ is a self-adjoint operator in $L^2(\Omega)$ with compact resolvent. We set

$$\mathcal{W}(u, v) = \int_\Omega (u \log u - u \log W - uv)\ dx + \frac{1}{2}\left\| A^{1/2} v \right\|^2$$

for $u \in \mathcal{P}_\lambda \cap C(\overline{\Omega})$ and $v \in \mathrm{dom}(A^{1/2})$, where $\lambda > 0$ is a given constant and $\mathcal{P}_\lambda$ is the set defined by (6.14):

$$\mathcal{P}_\lambda = \left\{ u : \text{measurable} \;\middle|\; u \geq 0 \ \text{ a.e.}, \quad \left\| u \right\|_1 = \lambda \right\}.$$

Putting

$$\begin{aligned}\mathcal{F}(u) &= \mathcal{W}(u, A^{-1}u) \\ &= \int_\Omega (u \log u - u \log W)\ dx - \frac{1}{2}(A^{-1}u, u),\end{aligned}$$

we say that $\overline{u} = \overline{u}(x) > 0$ $(x \in \overline{\Omega})$ is a critical function of $\mathcal{F}$ on $\mathcal{P}_\lambda$ if it belongs to $\mathcal{P}_\lambda \cap C(\overline{\Omega})$ and satisfies

$$\frac{d}{ds}\mathcal{F}(\overline{u} + s\varphi)\Big|_{s=0} = 0$$

for any $\varphi \in C(\overline{\Omega})$ in $\int_\Omega \varphi\, dx = 0$. Here, we note that the relation

$$\overline{u} + s\varphi \in \mathcal{P}_\lambda \cap C(\overline{\Omega})$$

is valid for $|s| \ll 1$. In this case, the bilinear form

$$\mathcal{F}''(\overline{u})[\varphi, \varphi] = \frac{d^2}{ds^2}\mathcal{F}(\overline{u} + s\varphi)\Big|_{s=0}$$

defined for $\varphi \in C(\overline{\Omega})$ in $\int_\Omega \varphi\, dx = 0$ is shown to be bounded in $L^2(\Omega)$, and hence is *closable* in $L^2_0(\Omega)$, where

$$L^2_0(\Omega) = \left\{\varphi \in L^2(\Omega) \;\Big|\; \int_\Omega \varphi\, dx = 0\right\}.$$

We call the maximum integer k satisfying

$$\inf\left\{ \sup_{\varphi \in Y\setminus\{0\}} \frac{\mathcal{F}''(\overline{u})[\varphi, \varphi]}{\|\varphi\|^2} \;\Big|\; Y \subset L^2_0(\Omega),\ \dim Y = k \right\} < 0$$

the *Morse index* of $\overline{u} = \overline{u}(x)$.

Next, we take

$$\begin{aligned}\mathcal{J}_\lambda(v) &= \mathcal{W}\Big(\frac{\lambda W e^v}{\int_\Omega W e^v\, dx}, v\Big) - \lambda \log \lambda \\ &= \frac{1}{2}\left\|A^{1/2} v\right\|^2 - \lambda \log\Big(\int_\Omega W e^v\, dx\Big)\end{aligned}$$

defined for $v \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$. We say that such $\overline{v} = \overline{v}(x)$ is a critical function of $\mathcal{J}_\lambda$ on $\mathrm{dom}(A^{1/2})$ if

$$\frac{d}{ds}\mathcal{J}_\lambda(\overline{v} + sw)\Big|_{s=0} = 0$$

holds for any $w \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$. Then, similarly we can show that the bilinear form

$$\mathcal{J}''_\lambda(\overline{v})[w, w] = \frac{d^2}{ds^2}\mathcal{J}_\lambda(\overline{v} + sw)\Big|_{s=0}$$

defined for $w \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$ is semi-bounded in $L^2(\Omega)$. Under the assumption

$$\left[C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})\right]^{L^2(\Omega)} = L^2(\Omega), \tag{9.1}$$

this bilinear form is closable in $L^2(\Omega)$, where [ ] denotes the closure. Then it is associated with a self-adjoint operator in $L^2(\Omega)$ provided with the standard norm. It is denoted by M and the number of negative eigenvalues of M is called the Morse index of $\overline{v} = \overline{v}(x)$.

We impose the following condition:

$$\varphi \in L_0^2(\Omega),\ A^{-1}\varphi = \text{constant} \quad \Rightarrow \quad \varphi = 0. \tag{9.2}$$

In the cases of the (JL) and (D) fields, $1 \not\in \mathrm{dom}(A)$ and this condition is automatically satisfied. In the (N) field, we have $1 \in \mathrm{dom}(A)$, but if $A^{-1}\varphi$ is a constant, then so is φ. This implies $\varphi = 0$ because $\int_\Omega \varphi = 0$ holds for $\varphi \in L_0^2(\Omega)$. Thus, (9.2) is valid for the (N), (JL), and (D) fields.

Under these notations, we can show the following theorem.

Theorem 9.1 *A positive function $\overline{u} = \overline{u}(x) \in \mathcal{P}_\lambda \cap C(\overline{\Omega})$ is a critical function of $\mathcal{F}$ defined on $\mathcal{P}_\lambda$ if and only if it solves (6.7), and this is equivalent to that $\overline{v} = A^{-1}\overline{u}$ is a solution to (6.6):*

$$v \in \mathrm{dom}(A), \qquad Av = \frac{\lambda W e^v}{\int_\Omega W e^v \, dx}. \tag{9.3}$$

Conversely, $\overline{v} = \overline{v}(x) \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$ is a critical function of $\mathcal{J}_\lambda$ on $\mathrm{dom}(A^{1/2})$ if and only if it solves (9.3), and this is equivalent to that $\overline{u} = f_\lambda(\overline{v})$ solves (6.7):

$$\log u - A^{-1}u - \log W = \mathit{constant}, \qquad \|u\|_1 = \lambda, \tag{9.4}$$

where

$$f_\lambda(v) = \frac{\lambda W e^v}{\int_\Omega W e^v \, dx}.$$

Finally, the Morse indices of these $\overline{u}$ and $\overline{v}$ are equal under the assumptions (9.1) and (9.2).

Proof: We can confirm the following identities, where $u \in \mathcal{P}_\lambda \cap C(\overline{\Omega})$ is a positive function on $\overline{\Omega}$, $v \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$, $\varphi \in C(\overline{\Omega})$ satisfies

$\int_\Omega \varphi\, dx = 0$, and $w \in \mathrm{dom}(A^{1/2})$:

$$\begin{aligned}
\mathcal{W}_u(u,v)[\varphi] &= \frac{d}{ds}\mathcal{W}(u+s\varphi, v)\Big|_{s=0} \\
&= \int_\Omega \varphi\,(\log u - \log W - v)\ dx \\
\mathcal{W}_{uu}(u,v)[\varphi,\varphi] &= \frac{d^2}{ds^2}\mathcal{W}(u+s\varphi, v)\Big|_{s=0} \\
&= \int_\Omega u^{-1}\varphi^2\, dx \\
\mathcal{W}_{uv}(u,v)[\varphi, w] &= \frac{\partial^2}{\partial s_1 \partial s_2}\mathcal{W}(u+s_1\varphi, v+s_2 w)\Big|_{(s_1,s_2)=(0,0)} \\
&= -\int_\Omega \varphi w\, dx \\
\mathcal{W}_v(u,v)[w] &= \frac{d}{ds}\mathcal{W}(u, v+sw)\Big|_{s=0} \\
&= -\int_\Omega uw\, dx + \left(A^{1/2}v, A^{1/2}w\right) \\
\mathcal{W}_{vv}(u,v)[w,w] &= \frac{d^2}{ds^2}\mathcal{W}(u, v+sw)\Big|_{s=0} \\
&= \left\| A^{1/2}w \right\|^2.
\end{aligned}$$

First, we examine the variational structure of $\mathcal{F}$ on $\mathcal{P}_\lambda$. Given $\overline{u} \in \mathcal{P}_\lambda \cap C(\overline{\Omega})$ and $\varphi \in C(\overline{\Omega})$ satisfying $u(x) > 0$ for $x \in \overline{\Omega}$ and $\int_\Omega \varphi\, dx = 0$, respectively, we set

$$u(s) = \overline{u} + s\varphi$$

for $|s| \ll 1$. Then, from

$$\mathcal{F}\big(u(s)\big) = \mathcal{W}\big(u(s), A^{-1}u(s)\big)$$

it follows that

$$\frac{d}{ds}\mathcal{F}\big(u(s)\big) = \mathcal{W}_u\big(u(s), A^{-1}u(s)\big)[\varphi] + \mathcal{W}_v\big(u(s), A^{-1}u(s)\big)[A^{-1}\varphi].$$

Here, we have

$$\begin{aligned}
&\mathcal{W}_v\big(u(s), A^{-1}u(s)\big)[A^{-1}\varphi] \\
&\qquad = -\big(u(s), A^{-1}\varphi\big) + \big(A^{1/2}\cdot A^{-1}u(s), A^{1/2}\cdot A^{-1}\varphi\big) = 0
\end{aligned}$$

and hence

$$\frac{d}{ds}\mathcal{F}(\overline{u}+s\varphi) = \mathcal{W}_u\big(\overline{u}+s\varphi,\, A^{-1}\overline{u}+sA^{-1}\varphi\big)[\varphi] \tag{9.5}$$

follows. This implies

$$\begin{aligned}\mathcal{F}'(\overline{u})[\varphi] &= \mathcal{W}_u\big(\overline{u},\, A^{-1}\overline{u}\big)[\varphi] \\ &= \big(\varphi, \log\overline{u} - \log W - A^{-1}\overline{u}\big)\end{aligned}$$

and therefore $\overline{u} = \overline{u}(x)$ is a critical function of $\mathcal{F}$ on $\mathcal{P}_\lambda$ if and only if (9.4) holds. This means that $\overline{v} = A^{-1}\overline{u}$ is a solution to (9.3). In this case, for $u(s) = \overline{u} + s\varphi$ we have

$$\begin{aligned}\mathcal{F}''(\overline{u})[\varphi,\varphi] &= \frac{d^2}{ds^2}\mathcal{F}\left(u(s)\right)\Big|_{s=0} \\ &= \frac{d}{ds}\mathcal{W}_u\left(u(s),\, A^{-1}u(s)\right)[\varphi]\Big|_{s=0} \\ &= \mathcal{W}_{uu}(\overline{u},\overline{v})[\varphi,\varphi] + \mathcal{W}_{uv}(\overline{u},\overline{v})[\varphi, A^{-1}\varphi] \\ &= \int_\Omega \overline{u}^{-1}\varphi^2\,dx - (\varphi, A^{-1}\varphi)\end{aligned} \tag{9.6}$$

by (9.5). If we provide $L_0^2(\Omega)$ with the norm $\|\cdot\|_{\overline{u}^{-1}}$ by

$$\|\varphi\|_{\overline{u}^{-1}} = (\varphi,\varphi)_{\overline{u}^{-1}}^{1/2} \quad \text{and} \quad (\varphi,\psi)_{\overline{u}^{-1}} = \int_\Omega \overline{u}^{-1}\varphi\cdot\psi\,dx,$$

then the bounded symmetric bilinear form

$$\mathcal{L}(\varphi,\psi) = \int_\Omega \overline{u}^{-1}\varphi\psi\,dx - \left(A^{-1/2}\varphi,\, A^{-1/2}\psi\right) \tag{9.7}$$

defined for $\varphi, \psi \in L_0^2(\Omega)$ is associated with

$$L = I - P\overline{u}A^{-1}$$

with respect to $\|\cdot\|_{\overline{u}^{-1}}$, through the relation

$$\mathcal{L}(\varphi,\psi) = (L\varphi,\psi)_{\overline{u}^{-1}}$$

for $\varphi, \psi \in L_0^2(\Omega)$. Here, $\overline{u}$ is identified with the multiplication by itself, and $P : L^2(\Omega) \to L_0^2(\Omega)$ is the orthogonal projection with respect to $\|\cdot\|_{\overline{u}^{-1}}$, that is,

$$Pv = v - \frac{\overline{u}}{\lambda}\int_\Omega v\,dx. \tag{9.8}$$

The operator $T = P\overline{u}A^{-1}$ restricted to $L_0^2(\Omega)$ is compact. From Riesz–Schauder's theory, $\sigma(L) \subset \sigma_p(L) \cup \{1\}$, $\sigma_p(L)$ is discrete and can accumulate only at 1, and each eigenvalue of L except for 1 has finite multiplicity. Here and henceforth, $\sigma(L)$ and $\sigma_p(L) \subset \sigma(L)$ denote the spectrum and the point spectrum of L, respectively.

Furthermore, L is self-adjoint in $L_0^2(\Omega)$ with respect to the norm $\|\cdot\|_{\overline{u}^{-1}}$ and has the spectral decomposition denoted by $L = \int_{-\infty}^{\infty} \lambda dE(\lambda)$. Since $\sigma(L) \setminus \{0\} \subset \mathbf{R} \setminus [-\delta, \delta]$ holds with some $\delta > 0$, we have the orthogonal decomposition

$$L_0^2(\Omega) = X_- \oplus X_0 \oplus X_+,$$

where $X_0 = \ker(L)$, $X_- = E\big((-\infty, -\delta)\big)\left(L_0^2(\Omega)\right)$, and

$$X_+ = E\big((\delta, +\infty)\big)\left(L_0^2(\Omega)\right).$$

We have

$$X_\pm \setminus \{0\} \subset \left\{v \in L_0^2(\Omega) \,\middle|\, \pm\mathcal{L}(v, v) > 0\right\},$$

and $\pm\mathcal{L}(\cdot, \cdot)$ provides an inner product, equivalent to $(\ ,\ )_{\overline{u}^{-1}}$, on $X_\pm$. See [77, 152, 191] for the operator theory used here.

We have $\dim X_- < +\infty$, and X_- coincides with the maximum linear space $Y \subset L_0^2(\Omega)$ satisfying $\mathcal{L}(v, v) < 0$ for any $v \in Y \setminus \{0\}$, namely, $\dim X_-$ is equal to the maximum integer k satisfying

$$\sigma_k \equiv \min\left\{ \max_{v \in Y \setminus \{0\}} \frac{\mathcal{L}(v, v)}{\|v\|_{\overline{u}^{-1}}^2} \,\middle|\, Y \subset L_0^2(\Omega),\ \dim Y = k \right\} < 0 \tag{9.9}$$

by the mini-max principle. Since $\|\cdot\|_{\overline{u}^{-1}}$ is equivalent to $\|\cdot\|$, this maximum integer k is equal to the Morse index of $\overline{u}$. Furthermore, from (9.7) it follows that

$$\sigma_k = 1 - 1\Big/\min\left\{ \max_{v \in Y \setminus \{0\}} \frac{\|v\|_{\overline{u}^{-1}}^2}{\|A^{-1/2}v\|^2} \,\middle|\, Y \subset L_0^2(\Omega),\ \dim Y = k \right\}.$$

In other words, the Morse index of $\overline{u}$ is equal to the maximum integer k satisfying

$$\mu_k \equiv \min\left\{ \max_{v \in Y \setminus \{0\}} \frac{\|v\|_{\overline{u}^{-1}}}{\|A^{-1/2}v\|^2} \,\middle|\, Y \subset L_0^2(\Omega),\ \dim Y = k \right\} < 1.$$

This value μ_k is associated with the eigenvalue problem of finding $\varphi \in L^2_0(\Omega)$ such that

$$\int_\Omega \overline{u}^{-1}\phi\psi\, dx = \mu\big(A^{-1/2}\varphi, A^{-1/2}\psi\big) \tag{9.10}$$

for any $\psi \in L^2_0(\Omega)$. Actually, the operator-theoretic form of this eigenvalue problem takes

$$T\varphi = \mu^{-1}\varphi \qquad \big(\varphi \in L^2_0(\Omega)\big) \tag{9.11}$$

for $T = P\overline{u}A^{-1}$.

Since T is compact and self-adjoint in $L^2_0(\Omega)$, the normalized eigenfunctions of (9.10) form a complete orthonormal system of $\overline{\mathrm{ran}(T)} = \ker(T)^\perp$ [152, 154]. Here and henceforth, $\mathrm{ran}(T)$ and $\ker(T)$ denote the range and the kernel of T, respectively. Here, the eigenvalue μ of (9.10) is always positive, and the normalization and the orthogonality are taken with respect to the norm $\|\cdot\|_{\overline{u}^{-1}}$. If $\varphi \in L^2_0(\Omega)$ satisfies

$$T\varphi = P\overline{u}A^{-1}\varphi = 0,$$

then it holds that

$$A^{-1}\varphi = \frac{1}{\lambda}\int_\Omega \overline{u}\cdot A^{-1}\varphi\, dx$$

by (9.8). This means that $A^{-1}\varphi$ is a constant, and hence $\varphi = 0$ follows from (9.2). Thus, we have

$$\ker(T) = \{0\}$$

and those normalized eigenfunctions $\{\varphi_i\}$ form a complete orthonormal system in $L^2_0(\Omega)$, and therefore the Morse index of $\overline{u}$, that is, the maximum integer k in $\sigma_k < 0$, is equal to the number of eigenvalues of (9.11) in $\mu < 1$, where σ_k is the value defined by (9.9). In terms of $w = A^{-1}\varphi$, problem (9.11) is equivalent to finding $w \in \mathrm{dom}(A)$ such that

$$Aw = \mu\overline{u}\Big(w - \frac{1}{\lambda}\int_\Omega \overline{u}w\, dx\Big). \tag{9.12}$$

Thus, $\dim X_-$, the Morse index of $\overline{u}$, is equal to the number of eigenvalues of (9.12) in $\mu < 1$. Now we shall show that this number is also equal to that of negative eigenvalues of M defined above in terms of $\overline{v} = A^{-1}\overline{u}$.

For this purpose, first, we examine the variational structure of $\mathcal{J}_\lambda$. In fact, given $v \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$, we have

$$\mathcal{J}_\lambda(v) = \mathcal{W}\left(f_\lambda(v), v\right) - \lambda \log \lambda$$

for $f_\lambda(v) = \lambda W e^v / \int_\Omega W e^v \, dx$ by (6.15). In terms of

$$\varphi_s \equiv \frac{\partial}{\partial s} f_\lambda(\overline{v} + sw)$$

this implies

$$\begin{aligned}\frac{d}{ds}\mathcal{J}_\lambda(\overline{v} + sw) = {} & \mathcal{W}_u\left(f_\lambda(\overline{v} + sw), \overline{v} + sw\right)[\varphi_s] \\ & + \mathcal{W}_v\left(f_\lambda(\overline{v} + sw), \overline{v} + sw\right)[w].\end{aligned}$$

By means of $\int_\Omega f_\lambda(\overline{v} + sw)\, dx = \lambda$, it holds that

$$\int_\Omega \varphi_s \, dx = 0.$$

This implies

$$\begin{aligned}\mathcal{W}_u\big(f_s(\overline{v} + sw)\ ,\ \overline{v} + sw\big)[\varphi_s] & \\ &= \int_\Omega \varphi_s \cdot \left\{ \log f_\lambda(\overline{v} + sw) - \log W - (\overline{v} + sw) \right\} dx \\ &= \int_\Omega \varphi_s \cdot \left\{ \log \lambda - \log\left(\int_\Omega W e^{\overline{v} + sw}\, dx \right) \right\} dx = 0\end{aligned}$$

and hence

$$\frac{d}{ds}\mathcal{J}_\lambda(\overline{v} + sw) = \mathcal{W}_v\left(f_\lambda(\overline{v} + sw), \overline{v} + sw\right)[w] \tag{9.13}$$

holds true. In particular, we have

$$\begin{aligned}\mathcal{J}_\lambda'(\overline{v})[w] &= \frac{d}{ds}\mathcal{J}_\lambda(\overline{v} + sw)\Big|_{s=0} \\ &= \mathcal{W}_v\left(f_\lambda(\overline{v}), \overline{v}\right)[w] \\ &= -\left(f_\lambda(\overline{v}), w\right) + \left(A^{1/2}\overline{v}, A^{1/2}w\right).\end{aligned}$$

Using assumption (9.1), we conclude that

$$\overline{v} \in C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$$

is a critical function of $\mathcal{J}_\lambda$ if and only if it satisfies (9.3). This means that the positive function $\overline{u} = f_\lambda(\overline{v})$ is a solution to (9.4).

Let us compute the Morse index of $\overline{v} = \overline{v}(x)$. In fact, putting

$$\varphi \equiv \varphi_s\big|_{s=0} = \frac{\partial}{\partial s} f_\lambda(\overline{v} + sw)\Big|_{s=0},$$

we have

$$\begin{aligned}
\mathcal{J}_\lambda''(\overline{v})[w, w] &= \frac{d^2}{ds^2}\mathcal{J}_\lambda(\overline{v} + sw)\Big|_{s=0} \\
&= \mathcal{W}_{uv}(\overline{u}, \overline{v})[\varphi, w] + \mathcal{W}_{vv}(\overline{u}, \overline{v})[w, w] \\
&= \left\| A^{1/2} w \right\|^2 - (\varphi, w)
\end{aligned}$$

by (9.13). On the other hand, from $\overline{u} = f_\lambda(\overline{v})$ we have

$$\begin{aligned}
\varphi &= \frac{\partial}{\partial s} f_\lambda(\overline{v} + sw)\Big|_{s=0} \\
&= \lambda \left\{ \frac{We^{\overline{v}} w}{\int_\Omega We^{\overline{v}}\, dx} - \frac{We^{\overline{v}} \int_\Omega We^{\overline{v}} w\, dx}{\left(\int_\Omega We^{\overline{v}}\, dx\right)^2} \right\} \\
&= \overline{u}\Big(w - \frac{1}{\lambda} \int_\Omega \overline{u} w\, dx \Big).
\end{aligned}$$

Therefore, we have

$$w = \overline{u}^{-1}\varphi + \frac{1}{\lambda}\int_\Omega \overline{u} w\, dx$$

and hence

$$(\varphi, w) = \int_\Omega \overline{u}^{-1}\varphi^2\, dx = \int_\Omega \overline{u} \left(w - \frac{1}{\lambda}\int_\Omega \overline{u} w\, dx \right)^2 dx$$

follows from $\int_\Omega \varphi\, dx = 0$. We obtain

$$\mathcal{J}_\lambda''(\overline{v})[w, w] = \left\| A^{1/2} w \right\|^2 - \int_\Omega \overline{u} \left(w - \frac{1}{\lambda}\int_\Omega \overline{u} w\, dx \right)^2 dx \tag{9.14}$$

and the linearized operator M stated above is realized as

$$Mw = Aw - \overline{u}\left(w - \frac{1}{\lambda}\int_\Omega \overline{u} w\, dx \right)$$

with $w \in \mathrm{dom}(M) = \mathrm{dom}(A)$, that is,

$$\mathcal{J}_\lambda''(\overline{v})[w, \psi] = (Mw, \psi)$$

for $w \in \mathrm{dom}(M) \subset \mathrm{dom}(A^{1/2})$ and $\psi \in \mathrm{dom}(A^{1/2})$.

Here, we apply the theory of perturbation [77], which assures that M is a self-adjoint operator with respect to the standard L^2 norm, and that its normalized eigenfunctions form a complete orthonormal system of $L^2(\Omega)$. Furthermore, there is a maximum linear space $Y \subset V = \mathrm{dom}(A^{1/2})$, denoted by Y_-, satisfying

$$\mathcal{J}_\lambda''(\overline{v})[w, w] < 0 \tag{9.15}$$

for any $w \in Y \setminus \{0\}$. Its dimension is equal to the number of negative eigenvalues of M, that is, the Morse index of $\overline{v}$, and is finite. Thus, we have only to show that the number of negative eigenvalues of M defined above is equal to that of eigenvalues in $\mu < 1$ of (9.12) for the given positive function $\overline{u} = \overline{u}(x) \in \mathcal{P}_\lambda \in C(\overline{\Omega})$. Here,

$$b(v, w) = \int_\Omega \overline{u}\Big(v - \frac{1}{\lambda}\int_\Omega \overline{u}v\,dx\Big)\cdot\Big(w - \frac{1}{\lambda}\int_\Omega \overline{u}w\,dx\Big)\,dx$$

is a nonnegative bounded bilinear form of $v, w \in L^2(\Omega)$, and $V = \mathrm{dom}(A^{1/2})$ is a Hilbert space provided with the inner product

$$a(v, w) = \big(A^{1/2}v, A^{1/2}w\big)$$

defined for $v, w \in V$. Problem (9.12) is equivalent to finding $w \in V = \mathrm{dom}(A^{1/2})$ such that

$$a(w, v) = \mu b(w, v)$$

for any $v \in V$.

Since $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact, Riesz's representation theorem induces the compact operator $S : L^2(\Omega) \to L^2(\Omega)$ through the relation

$$a(Sv, w) = b(v, w),$$

where $v \in L^2(\Omega)$, $Sv \in \mathrm{dom}(A^{1/2})$, and $w \in \mathrm{dom}(A^{1/2})$. Then, problem (9.12) is written as

$$Sv = \mu^{-1}v$$

with $v \in L^2(\Omega)$, which is equivalent to finding

$$v \in V = \mathrm{dom}(A^{1/2})$$

such that

$$Sv = \mu^{-1} v. \tag{9.16}$$

This S is reformulated as a compact operator on

$$V = \mathrm{dom}(A^{1/2}),$$

because $v \in \mathrm{dom}(A^{1/2})$ implies $Sv \in \mathrm{dom}(A)$. It is symmetric with respect to $a(\ ,\ )$, and therefore its normalized eigenfunctions, denoted by $\{v_i\}$, form a complete orthogonal system of $\overline{\mathrm{ran}(S)} = \ker(S)^{\perp}$ in $V = \mathrm{dom}(A^{1/2})$. Here, normalization and orthogonalization are taken with respect to the norm $\|A^{1/2} \cdot\ \|$.

Given $v \in L^2(\Omega)$, we have $b(v, v) = 0$ if and only if it is a constant. Conversely, we have

$$b(1, v) = 0 \tag{9.17}$$

for any $v \in L^2(\Omega)$. Therefore, in the case of $1 \notin V = \mathrm{dom}(A^{1/2})$, we have $\ker(S) = \{0\}$ and the maximum linear space $Y \subset V$ satisfying (9.15), denoted by Y_-, coincides with the linear space generated by the eigenfunctions of (9.12) with $\mu < 1$. Hence the Morse index of $\overline{u}$ is equal to the number of negative eigenvalues of M, that is, the Morse index of $\overline{v}$. In the other case of $1 \in V = \mathrm{dom}(A^{1/2})$, it holds that $\ker(S) = \{1\}$. However,

$$\int_\Omega \overline{u}\Big(v - \frac{1}{\lambda}\int_\Omega \overline{u} v\, dx\Big)\, dx = 0$$

holds for any $v \in L^2(\Omega)$, which implies

$$a(1, v) = (1, Av) = 0$$

for any eigenfunction v of (9.16) by (9.12). If $\mathcal{V}$ denotes the linear space generated by eigenfunctions of S defined on $V = \mathrm{dom}(A^{1/2})$, this means $\mathcal{V} \perp \{1\}$ with respect to $a(\ ,\ )$, where $\{1\}$ denotes the linear space generated by 1. Hence

$$[\mathcal{V} \oplus \{1\}]^V = V$$

follows.

Using (9.17), we obtain again that Y_-, the maximum linear space $Y \subset \mathrm{dom}\,(A^{1/2})$ satisfying (9.15), coincides with the linear space generated by eigenfunctions of (9.12) with $\mu < 1$. We have the same conclusion in this case also, and the proof is complete. □

From the proof of the above theorem, we have also

$$\dim\ \ker(L) = \dim X_0 = \dim\ \ker\left(\mathcal{J}_\lambda''(\overline{v})\right).$$

We call $\dim X_- + \dim X_0$ the *augmented Morse index* of $\overline{u}$ or $\overline{v}$. Now, we show that the stationary solution to the (N) and (JL) fields takes unstable and center manifolds with the dimensions equal to Morse and augmented Morse indices, respectively, in the simplified system (3.1):

$$\begin{aligned} u_t &= \nabla \cdot \left(\nabla u - u\nabla(v + \log W)\right) \quad \text{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu}(v + \log W) &= 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ Av &= u \quad \text{for} \quad t \in (0, T). \end{aligned} \tag{9.18}$$

We shall show that the Morse index represents the stability of the equilibrium state faithfully even in the full system of $n = 2$. Furthermore, this result is extended to more general systems.

For the moment, we develop a formal argument to take the structure of the problem. Namely, if $\mathcal{F}'(u)$ is identified with

$$\log u - u \log W - A^{-1} u$$

through usual L^2 inner product, then the simplified system is written as

$$u_t = \nabla \cdot \left(u \nabla \mathcal{F}'(u)\right)$$

and its stationary problem is given as (6.7):

$$\log u - A^{-1}u - \log W = \text{constant}, \qquad \left\|u\right\|_1 = \lambda. \tag{9.19}$$

Since the stationary solution $\overline{u}$ is characterized by

$$\mathcal{F}'(\overline{u}) = 0,$$

the linearized system is given by

$$\varphi_t = \nabla \cdot \left(\overline{u} \nabla \mathcal{F}''(\overline{u})\varphi\right),$$

or equivalently,

$$\begin{aligned} \varphi_t &= \nabla \cdot \left(\overline{u}\nabla(\overline{u}^{-1}\varphi - A^{-1}\varphi)\right) \quad \text{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu}\left(\overline{u}^{-1}\varphi - A^{-1}\varphi\right) &= 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ \int_\Omega \varphi \, dx &= 0 \quad \text{for} \quad t \in (0, T). \end{aligned} \tag{9.20}$$

In what follows, these operators S, T and families $\{v_i\}$, $\{\mu_i\}$ are differently denoted from these defined above. On the other hand, the bounded self-adjoint operator L on $L_0^2(\Omega)$ continues to indicate the one associated with the bilinear form

$$\mathcal{L}(\varphi, \psi) = \int_\Omega \overline{u}^{-1} \varphi \psi \, dx - \left(A^{-1/2}\varphi, A^{-1/2}\psi \right)$$

and the norm $\| \cdot \|_{\overline{u}^{-1}}$ in such a way as

$$\mathcal{L}(\varphi, \psi) = (L\varphi, \psi)_{\overline{u}},$$

where $\varphi, \psi \in L_0^2(\Omega)$. Namely, S denotes the self-adjoint operator in $L^2(\Omega)$ associated with the bilinear form

$$\mathcal{S}(v, w) = \int_\Omega \overline{u} \nabla v \cdot \nabla w \, dx$$

defined for $v, w \in H^1(\Omega)$ and usual L^2 norm $\| \cdot \|$ in such a way as

$$\mathcal{S}(v, w) = (Sv, w),$$

where $v \in \mathrm{dom}(S) \subset H^1(\Omega)$ and $w \in H^1(\Omega)$. This implies

$$\mathrm{dom}(S) = \left\{ v \in H^2(\Omega) \mid \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

and

$$Sv = -\nabla \cdot (\overline{u} \nabla v)$$

for $v \in \mathrm{dom}(S)$ under the assumption of $\overline{u} \in C^1(\overline{\Omega})$. Let $H = S\overline{u}^{-1}L$ with

$$\mathrm{dom}(H) = \left\{ \varphi \in L_0^2(\Omega) \mid \overline{u}^{-1} L\varphi \in \mathrm{dom}(S) \right\}.$$

Then, (9.20) is realized as the evolution equation in $L_0^2(\Omega)$,

$$\frac{d\varphi}{dt} + H\varphi = 0. \tag{9.21}$$

If we have $\overline{u} \in C^2(\overline{\Omega})$ furthermore, and

$$\left\| A^{-1} \right\|_{L^2(\Omega), H^2(\Omega)} \leq A(2) \tag{9.22}$$

holds with a constant $A(2) > 0$, then

$$\mathrm{dom}(H) = \left\{ \varphi \in H^2(\Omega) \cap L^2_0(\Omega) \;\middle|\; \frac{\partial}{\partial \nu}\left(\overline{u}^{-1}\varphi - A^{-1}\varphi\right) = 0 \text{ on } \partial\Omega \right\} \quad (9.23)$$

is a closed subspace of $H^2(\Omega) \cap L^2_0(\Omega)$ and

$$H\varphi = -\nabla \cdot \left(\overline{u}\nabla(\overline{u}^{-1}\varphi - A^{-1}\varphi)\right) \quad (9.24)$$

holds for $\varphi \in \mathrm{dom}(H)$. Then, the well-posedness of (9.21) is assured by the following theorem.

Theorem 9.2 *Let $\overline{u} = \overline{u}(x)$ be a stationary solution to (9.18) in $\overline{u} \in C^2(\overline{\Omega})$ and suppose that (9.22) holds. Then, the operator $-H$ defined by (9.23) and (9.24) generates a holomorphic semigroup*

$$\left\{e^{-tH}\right\}_{t \geq 0}$$

in $L^2_0(\Omega)$. Furthermore, $\sigma(H) = \sigma_p(H) \subset \mathbf{R}$ and the numbers of negative and nonpositive eigenvalues of H are equal to the Morse and the augmented Morse indices of $\overline{u} = \overline{u}(x)$, respectively.

Proof: We study the spectrum of H first, and take the eigenvalue problem of finding $\varphi \in \mathrm{dom}(H)$ such that

$$H\varphi = \mu\varphi. \quad (9.25)$$

We note that the inverse operator of S in $L^2(\Omega)$ is realized by $S^{-1} : L^2_0(\Omega) \to L^2_{\overline{u}}(\Omega)$ for

$$L^2_{\overline{u}}(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; (v, \overline{u}) = 0 \right\},$$

where $S^{-1}w = v$ if and only if

$$-\nabla \cdot (\overline{u}\nabla v) = w \quad \text{in} \quad \Omega,$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \qquad \int_\Omega \overline{u}v\, dx = 0.$$

On the other hand, relation (9.25) means

$$S\overline{u}^{-1}L\varphi = \mu\varphi \in L^2_0(\Omega)$$

with $\overline{u}^{-1}L\varphi \in L^2_{\overline{u}}(\Omega)$, and hence it is equivalent to

$$\overline{u}^{-1}L\varphi = \mu S^{-1}\varphi,$$

or finding $\varphi \in L_0^2(\Omega)$ such that

$$\mathcal{L}(\varphi, \psi) = \mu\big(S^{-1}\varphi, \psi\big) \tag{9.26}$$

for any $\psi \in L_0^2(\Omega)$.

Now we introduce $T : L_0^2(\Omega) \to L_0^2(\Omega)$ by $Tw = v$ if and only if

$$\begin{aligned} -\nabla \cdot (\overline{u}\nabla v) &= w \quad \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega, \quad \int_\Omega v\,dx = 0. \end{aligned}$$

Then (9.26) is equivalent to finding $\varphi \in L_0^2(\Omega)$ such that

$$\mathcal{L}(\varphi, \psi) = \mu\,(T\varphi, \psi) \tag{9.27}$$

for any $\psi \in L_0^2(\Omega)$. Here, $T : L_0^2(\Omega) \to L_0^2(\Omega)$ is a compact positive self-adjoint operator satisfying

$$\operatorname{ran}(T^{1/2}) = H^1(\Omega) \cap L_0^2(\Omega).$$

Taking $v = T^{1/2}\varphi$ and $w = T^{1/2}\psi$, we see that (9.27) is equivalent to finding $v \in H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$\mathcal{L}(T^{-1/2}v, T^{-1/2}w) = \mu(v, w) \tag{9.28}$$

for any $w \in H^1(\Omega) \cap L_0^2(\Omega)$.

The bilinear form

$$\hat{\mathcal{L}}(v, w) = \mathcal{L}(T^{-1/2}v, T^{-1/2}w)$$

defined for $v, w \in H^1(\Omega) \cap L_0^2(\Omega)$ splits in

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 - \hat{\mathcal{L}}_1,$$

where

$$\hat{\mathcal{L}}_0(v, w) = \big(\overline{u}^{-1/2}T^{-1/2}v, \overline{u}^{-1/2}T^{-1/2}w\big)$$

and

$$\hat{\mathcal{L}}_1(v, w) = \big(A^{-1/2}T^{-1/2}v, A^{-1/2}T^{-1/2}w\big).$$

Here,

$$\hat{\mathcal{L}}_0(v, v) \approx \big\|T^{-1/2}v\big\|^2$$

for $v \in H^1(\Omega) \cap L_0^2(\Omega)$ and $A^{-1/2} : L^2(\Omega) \to H^1(\Omega)$ is bounded because of (9.22) and the interpolation theory. Also, we have

$$\left\| T^{1/2} A^{1/2} w \right\| \approx \|w\|$$

for $w \in D(A^{1/2})$ with $A^{1/2} w \in L_0^2(\Omega)$, and hence each $\varepsilon > 0$ admits $C'_\varepsilon > 0$ such that

$$\|w\| \leq \varepsilon \left\| A^{1/2} w \right\| + C'_\varepsilon \left\| T^{1/2} A^{1/2} w \right\|$$

for any $w \in A^{-1/2}\left(L_0^2(\Omega)\right)$. Therefore, we have $C_\varepsilon > 0$ satisfying

$$0 \leq \hat{\mathcal{L}}_1(v, v) \leq \varepsilon \hat{\mathcal{L}}_0(v, v) + C_\varepsilon \left\| v \right\|^2$$

for any $v \in H^1(\Omega) \cap L_0^2(\Omega)$.

On the other hand, the self-adjoint operator in $L_0^2(\Omega)$, associated with the bilinear form $\hat{\mathcal{L}}_0$ on $H^1(\Omega) \cap L_0^2(\Omega) \times H^1(\Omega) \cap L_0^2(\Omega)$ and usual L^2 norm, is given by $G_0 = T^{-1/2} \overline{u}^{-1} T^{-1/2}$:

$$\hat{\mathcal{L}}_0(v, w) = (G_0 v, w),$$

where $v \in \mathrm{dom}(G_0) \subset H^1(\Omega) \cap L_0^2(\Omega)$ and $w \in H^1(\Omega) \cap L_0^2(\Omega)$. This implies

$$\mathrm{dom}(G_0) = H^2(\Omega) \cap L_0^2(\Omega) \cap L_{\overline{w}}^2(\Omega)$$

with

$$\overline{w} = T^{-1/2}\Big(\overline{u}^{-1} - \frac{1}{|\Omega|} \int_\Omega \overline{u}^{-1}\, dx\Big)$$

and also that $G_0^{-1} = T^{1/2} \overline{u} T^{1/2}$ is compact in $L_0^2(\Omega)$. Then, the perturbation [77] guarantees that $\hat{\mathcal{L}}$ is associated with a self-adjoint operator with compact resolvent in $L_0^2(\Omega)$. Consequently, problem (9.28) provides a complete orthonormal system of $L_0^2(\Omega)$, denoted by $\{v_i\} \subset H^1(\Omega) \cap L_0^2(\Omega)$. We have

$$\big[\{v_i\}\big]^{L^2(\Omega)} = L_0^2(\Omega), \qquad \big[\{v_i\}\big]^{H^1(\Omega)} = H^1(\Omega) \cap L_0^2(\Omega),$$

and $(v_i, v_j) = \delta_{ij}$.

Here, we recall the decomposition

$$L_0^2(\Omega) = X_- \oplus X_0 \oplus X_+. \tag{9.29}$$

Since $T^{-1/2} : H^1(\Omega) \cap L_0^2(\Omega) \to L_0^2(\Omega)$ is an isomorphism, the numbers of negative and zero eigenvalues of (9.28) are equal to dim X_- and dim X_0, respectively. The system of eigenfunctions of (9.25) is given by $\{\varphi_i\}$ for $\varphi_i = T^{-1/2} v_i \in L_0^2(\Omega)$. It satisfies

$$\left[\{\varphi_i\}\right]^{L^2(\Omega)} = L_0^2(\Omega) \quad \text{and} \quad \mathcal{L}(\varphi_i, \varphi_j) = \mu_i \delta_{ij}$$

by (9.27). This decomposition (9.29) is orthogonal with respect to the norm $\| \cdot \|_{\overline{u}^{-1}}$. However, these subspaces are eigenspaces of L are orthogonal also with respect to $\mathcal{L}(\ ,\ )$. That is, $\mathcal{L}(\phi, \psi) = 0$ if ϕ and ψ belong to the different spaces of X_-, X_0, and X_+. Furthermore, $\pm\mathcal{L}$ is equivalent to $(\ ,\ )_{\overline{u}^{-1}}$ on $X_\pm$, respectively. Therefore, we can introduce an equivalent inner product to $(\ ,\ )_{\mathcal{L}}$ on $L_0^2(\Omega) = X_- \oplus X_0 \oplus X_+$ by

$$(\phi, \psi)_{\mathcal{L}} = \begin{cases} \pm\mathcal{L}(\phi, \psi) & (\phi, \psi \in X_\pm) \\ (\phi, \psi) & (\psi, \phi \in X_0) \end{cases}$$

and

$$(\phi, \psi)_{\mathcal{L}} = (\phi_-, \psi_-)_{\mathcal{L}} + (\psi_0, \phi_0)_{\mathcal{L}} + (\phi_+, \psi_+)_{\mathcal{L}}$$

for

$$\phi = \phi_- + \phi_0 + \phi_+ \in X_- \oplus X_0 \oplus X_+$$

and

$$\psi = \psi_- + \psi_0 + \psi_+ \in X_- \oplus X_0 \oplus X_+.$$

Putting $n = \dim X_-$ and $m = \dim X_0$, we arrange $\sigma(H) = \sigma_p(H) = \{\mu_i\}$ as

$$\mu_1 \leq \cdots \leq \mu_n < 0 = \mu_{n+1} = \cdots = \mu_{n+m+1} < \mu_{m+n+1} \leq \cdots .$$

We take $\hat{\varphi}_i = \frac{\varphi_i}{\sqrt{\pm\mu_i}}$ according to $\pm\mu_i > 0$ and retake $\{\varphi_i\}_{i=n+1}^{m+n}$ to be an orthonormal basis of X_0 with respect to $\| \cdot \|$. In this way, we obtain a complete orthonormal system $\{\hat{\varphi}_i\}$ in $L_0^2(\Omega)$ with respect to $(\ ,\ )_{\mathcal{L}}$, composed of eigenfunctions of H. Then, it holds that

$$\left[\{\hat{\varphi}_i\}_{i=1}^{n}\right]^{L^2(\Omega)} = X_-, \qquad \left[\{\hat{\varphi}_i\}_{i=n+1}^{n+m+1}\right]^{L^2(\Omega)} = X_0,$$

and

$$\left[\left\{\hat{\varphi}_i\right\}_{i=n+m+1}^{\infty}\right]^{L^2(\Omega)} = X_+,$$

and therefore $-H$ generates the holomorphic semigroup

$$\left\{e^{-tH}\right\}_{t\geq 0}$$

in $L_0^2(\Omega)$ defined by

$$e^{-tH} v = \sum_{i=1}^{\infty} e^{-t\mu_i} \left(v, \hat{\varphi}_i\right)_{\mathcal{L}} \hat{\varphi}_i,$$

where the right-hand side converges strongly in $L_0^2(\Omega)$ because of

$$\dim X_- < +\infty$$

and the proof is complete. □

Returning to (9.18), we define the perturbation from the stationary solution, taking the stationary solution $\overline{u} = \overline{u}(x)$, put $u = \overline{u} + \varphi$, which is a system of φ:

$$\begin{aligned}
\varphi_t = \nabla \cdot \left(\overline{u}\nabla(\overline{u}^{-1}\varphi - A^{-1}\varphi)\right) - \nabla \cdot \left(\varphi \nabla A^{-1}\varphi\right) \qquad & \text{in} \quad \Omega \times (0, T), \\
\overline{u}\frac{\partial}{\partial \nu}\left(\overline{u}^{-1}\varphi - A^{-1}\varphi\right) = \varphi \frac{\partial}{\partial \nu} A^{-1}\varphi \qquad & \text{on} \quad \partial\Omega \times (0, T), \\
\int_\Omega \varphi \, dx = 0. \qquad & \qquad (9.30)
\end{aligned}$$

Henceforth, $X = L_0^2(\Omega)$ is provided with the standard L^2 norm. If

$$\operatorname{dom}(A) \subset \left\{v \in H^2(\Omega) \;\middle|\; \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\right\}, \tag{9.31}$$

then the quadratic term vanishes in the boundary condition of (9.30), and

$$D \equiv \operatorname{dom}(H) = \left\{\varphi \in H^2(\Omega) \cap L_0^2(\Omega) \;\middle|\; \frac{\partial}{\partial \nu}(\overline{u}^{-1}\varphi)\Big|_{\partial\Omega} = 0\right\} \tag{9.32}$$

is a closed subspace of $H^2(\Omega)$. We show that the nonlinear mapping $N : D \to X = L_0^2(\Omega)$ is well defined by

$$N(\varphi) = -\nabla \cdot \left(\varphi \nabla A^{-1}\varphi\right)$$

in the case of $n \leq 8$. Then, (9.30) is reduced to the evolution equation

$$\frac{d\varphi}{dt} + H\varphi = N\varphi \quad \text{in} \quad X = L_0^2(\Omega), \tag{9.33}$$

This fact is justified by the next lemma, because it assures

$$N(D) \subset L_0^2(\Omega)$$

by (9.31) and the generalized Green's formula [59]:

$$\int_\Omega N\varphi\, dx = -\left\langle 1, \varphi \frac{\partial}{\partial \nu} A^{-1}\varphi \right\rangle = 0,$$

where $\langle\ ,\ \rangle$ denotes the pairing between

$$H^{1/2}(\partial\Omega) \quad \text{and} \quad H^{-1/2}(\partial\Omega).$$

Now, we show the following lemma.

Lemma 9.1 *Under the assumption of (9.31), the continuous bilinear form*

$$(\varphi, \psi) \in D \times D \quad \mapsto \quad \nabla \cdot \left(\varphi \nabla A^{-1}\psi\right) \in L^2(\Omega) \tag{9.34}$$

is well defined for $n \leq 8$.

Proof: We recall the constant $A(p) > 0$ introduced in (3.5):

$$\left\| A^{-1} \right\|_{L^p(\Omega), W^{2,p}(\Omega)} \leq A(p) \tag{9.35}$$

and the relation

$$\nabla \cdot \left(\varphi \nabla A^{-1}\psi\right) = \nabla\varphi \cdot \nabla A^{-1}\psi + \varphi \Delta A^{-1}\psi.$$

In fact, taking $r, p \in (1, \infty)$ in $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, we have

$$\left\| \varphi \cdot \Delta A^{-1}\psi \right\| \leq \left\| \varphi \right\|_r \cdot A(p) \cdot \left\| \psi \right\|_p.$$

Here, by Sobolev's imbedding theorem we obtain

$$\left\| \varphi \right\|_r \leq K_3 \left\| \varphi \right\|_D \quad \text{and} \quad \left\| \psi \right\|_p \leq K_3 \left\| \varphi \right\|_D$$

for $\frac{1}{2} - \frac{2}{n} \leq \frac{1}{r}$ and $\frac{1}{2} - \frac{2}{n} \leq \frac{1}{p}$, where $K_3 > 0$ is a constant determined by Ω. If $2 \cdot \left(\frac{1}{2} - \frac{2}{n}\right) \leq \frac{1}{2}$, that is, $n \leq 8$, we have such r, p and then

$$\left\| \varphi \cdot \Delta A^{-1}\psi \right\| \leq K_3^2 \cdot A(p) \cdot \left\| \varphi \right\|_D \left\| \psi \right\|_D$$

follows.

Similarly, we have

$$\left\| \nabla \psi \cdot \nabla A^{-1} \psi \right\| \leq \left\| \nabla \varphi \right\|_r \cdot \left\| \nabla A^{-1} \psi \right\|_p$$

for $r,\ p \in (1, \infty)$ in $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$. Using Sobolev's imbedding theorem, we have

$$\left\| \nabla \varphi \right\|_r \leq K_4 \left\| \varphi \right\|_D$$

for $\frac{1}{2} - \frac{1}{n} \leq \frac{1}{r}$ and

$$\left\| \nabla A^{-1} \psi \right\|_p \leq K_4 \cdot A(q) \cdot \left\| \psi \right\|_D$$

for $q \in (1, \infty)$ in $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ and $\frac{1}{q} = \frac{1}{2} - \frac{2}{n}$, where $K_4 > 0$ is a constant determined by Ω. We have such $r,\ p, q \in (1, \infty)$ if $\left(\frac{1}{2} - \frac{1}{n}\right) + \left(\frac{1}{2} - \frac{3}{n}\right) \leq \frac{1}{2}$, or $n \leq 8$ again, and then

$$\left\| \nabla \psi \cdot \nabla A^{-1} \psi \right\| \leq K_4^2 \cdot A(q) \cdot \left\| \varphi \right\|_D \left\| \psi \right\|_D$$

follows. The proof is complete. □

Condition (9.31) is satisfied for the (N) and (JL) fields. If this is not the case, we need the fundamental solution to treat the inhomogeneous boundary condition of (9.30). From this technical difficulty, local dynamical theory around the stationary solution has not been clarified for the (D) field. From now on, we shall concentrate on the case (9.31).

Let $Q : X \to X_-$ be the projection defined by

$$Qv = \sum_{i=1}^{n} \left(v, \hat{\varphi}_i\right)_{\mathcal{L}} \hat{\varphi}_i,$$

and $\hat{X}_+ = (I - Q)D \subset D$. Given a time interval I and an exponent $\alpha \in (0, 1)$, we put

$$\begin{aligned}
\|w\|_{C^\alpha(I,D)} &= \|w\|_{L^\infty(I,D)} + [w]_{C^\alpha(I,D)}, \\
\|w\|_{L^\infty(I,D)} &= \sup_{t \in I} \left\| w(t) \right\|_D, \\
[w]_{C^\alpha(I,D)} &= \sup_{t,s \in I, t \neq s} \frac{\|w(t) - w(s)\|_D}{|t - s|^\alpha}, \\
\|w\|_{C^\alpha_\alpha((0,1],D)} &= \|w\|_{L^\infty((0,1),D)} + \left[t^\alpha w(\cdot)\right]_{C^\alpha((0,1),D)}.
\end{aligned}$$

The sets G_Φ and G_Ψ, stated there, are to be called local unstable and stable manifolds of $\overline{u} = \overline{u}(x)$, respectively. Thus, we have the unstable manifold with the dimension equal to the Morse index, in the case that the stationary solution is not degenerate.

Theorem 9.3 *If* $\overline{u} = \overline{u}(x) \in C^2(\overline{\Omega})$ *is a stationary solution to (9.18) satisfying (9.19),* $X_0 = \{0\}$*,* $n \leq 8$*, (9.35), and (9.31), then we have a neighborhood* B *of* $\varphi = 0$ *in* D *defined by (9.32), and the Lipschitz-continuous mappings*

$$\Phi : B \cap X_- \to \hat{X}_+$$

and

$$\Psi : B \cap \hat{X}_+ \to X_-$$

differentiable at $\varphi = 0$*, satisfying* $\Phi(0) = 0$*,* $\Phi'(0) = 0$*,* $\Psi(0) = 0$*, and* $\Psi'(0) = 0$*. Furthermore, given* $w_0 \in D$*, we have the following items, where* $\alpha \in (0, 1)$ *and* $R_0 > 0$ *is a small constant.*

1. *This* w_0 *is in* $G_\Phi \equiv \left\{(\varphi, \Phi(\varphi)) \,\middle|\, \varphi \in B \cap X_-\right\}$ *if and only if* $Qw_0 \in B$ *and there exists* $w \in C^\alpha((-\infty, 0], D)$ *satisfying*

$$\begin{aligned} \frac{dw}{dt} + Hw = Nw \quad (-\infty < t \leq 0), \\ \|w\|_{C^\alpha((-\infty,0],D)} \leq R_0, \\ w(0) = w_0, \\ \lim_{t\to-\infty} \left\|w(t)\right\|_D = 0. \end{aligned}$$

2. *This* w_0 *is in* $G_\Psi \equiv \left\{(\varphi, \Psi(\varphi)) \mid \varphi \in B \cap \hat{X}_+\right\}$ *if and only if* $(I - Q)w_0 \in B$ *and there exists* $w \in C^\alpha_\alpha\big((0, 1], D\big) \cap C([0, 1], X) \cap C^\alpha([1, +\infty), D\big)$ *satisfying*

$$\begin{aligned} \frac{dw}{dt} + Hw = Nw \quad (0 \leq t < +\infty), \\ \|w\|_{C^\alpha_\alpha((0,1],D)} + \|w\|_{C^\alpha([1,+\infty),D)} \leq R_0, \\ w(0) = w_0, \\ \lim_{t\to+\infty} \left\|w(t)\right\|_D = 0. \end{aligned}$$

Proof: Since the results are obtained by the general theory [93], we have only to show that $N \in C^1(D, X)$ and its derivative N' is locally Lipschitz continuous in D. In fact, we have

$$N'(\varphi)\psi = -\nabla \cdot \left(\varphi \nabla A^{-1}\psi\right) - \nabla \cdot \left(\psi \nabla A^{-1}\varphi\right)$$

and the right-hand side is regarded as a continuous bilinear mapping from $D \times D$ into $L^2_0(\Omega)$ by Lemma 9.1. All requirements are verified and the proof is complete. □

We recall that the first eigenvalue of H is denoted by σ_1. Then, for $a \in (-\infty, \sigma_1)$ we have

$$D_{1/2} \equiv \mathrm{dom}\left((H-a)^{1/2}\right) \subset H^1(\Omega)$$

from the interpolation theory. This implies that

$$(\varphi, \psi) \in D \times D \quad \mapsto \quad \nabla \cdot \left(\varphi \nabla A^{-1} \psi\right) \in D_{1/2}$$

is a continuous mapping if $n \leq 6$. The following theorem is obtained similarly, where

$$Q_0 v = \sum_{i=1}^{n+m} \left(v, \hat{\varphi}_i\right)_{\mathcal{L}} \hat{\varphi}_i,$$

$\hat{X}_+ = (I - Q_0)(D)$, and $\rho : X_- \oplus X_0 \to \mathbf{R}$ is a smooth mapping such that

$$0 \leq \rho(\varphi) \leq 1 \quad \text{and} \quad \rho(\varphi) = \begin{cases} 1 & (\|\varphi\| \leq 1/2), \\ 0 & (\|\varphi\| \geq 1). \end{cases}$$

There,

$$\mathcal{M} = \left\{ (\varphi, \Gamma\varphi) \mid \varphi \in X_- \otimes X_0,\ \|\varphi\| < r/2 \right\}$$

is to be called the local center unstable manifold of $\overline{u} = \overline{u}(x)$. Thus, we have a center unstable manifold with dimensions equal to the augmented Morse index.

Theorem 9.4 *Under the assumptions of the previous theorem except for $X_0 = \{0\}$, if $n \leq 6$ we have $r > 0$ sufficiently small and the bounded Lipschitz-continuous mapping $\Gamma : X_- \otimes X_0 \to \hat{X}_+$ such that if $\psi_0 = \Gamma\varphi_0$, then there exist $\varphi = \varphi(t) \in X_- \otimes X_0$ and $\psi = \psi(t) \in \hat{X}_+$ such that*

$$\frac{d\varphi}{dt} + H\varphi = Q_0 N\left(\rho(\varphi/r)\varphi + \psi\right),$$
$$\frac{d\psi}{dt} + H\psi = (I - Q_0) N\left(\rho(\varphi/r)\varphi + \psi\right)$$

for $t \in (-\infty, +\infty)$, $\varphi\Big|_{t=0} = \varphi_0$, and $\psi|_{t=0} = \psi_0$.

In the full system, on the other hand, the spectral properties of the linearized system around the stationary solution are disturbed by τ. Consequently, the analogous result to Theorem 9.2 will not hold, because this field provides streaming movement to particles.

To confirm these facts, we take the (N) field with $W \equiv 1$:

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) && \text{in} \quad \Omega \times (0, T), \\ \tau v_t &= \Delta v - a v + u && \text{in} \quad \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 && \text{on} \quad \partial\Omega \times (0, T). \end{aligned}$$

In this case, A is equal to $-\Delta + a$ with the Neumann boundary condition with a constant $a > 0$, and system (3.1) takes the stationary constant solution $(\overline{u}, \overline{v}) = (\lambda / |\Omega|, \lambda / (a |\Omega|))$ for each $\lambda = \|u_0\|_1$, which forms the branch of trivial solutions, denoted by $\mathcal{C}_c$ in $\lambda - v$ space. In this case, the linearized system around this $(\overline{u}, \overline{v})$ is given as

$$\begin{aligned} \varphi_t &= \Delta\varphi - \overline{u}\Delta\psi && \text{in} \quad \Omega \times (0, T), \\ \tau \psi_t &= \Delta\psi - a\psi + \varphi && \text{in} \quad \Omega \times (0, T), \\ \frac{\partial \varphi}{\partial \nu} &= \frac{\partial \psi}{\partial \nu} = 0 && \text{on} \quad \partial\Omega \times (0, T), \\ \int_\Omega \varphi \, dx &= 0 && \text{for} \quad t \in (0, T), \end{aligned}$$

and the corresponding eigenvalue problem takes the form of

$$\begin{aligned} \Delta\varphi - \overline{u}\Delta\psi + \eta\varphi &= 0 \quad \text{in} \quad \Omega, \\ \tau^{-1}\varphi + \tau^{-1}(\Delta - a)\,\psi + \eta\psi &= 0 \quad \text{in} \quad \Omega, \\ \frac{\partial \varphi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega, \\ \int_\Omega \varphi \, dx &= 0. \end{aligned} \tag{9.36}$$

Defining the eigenvalues of $-\Delta$ in Ω under the Neumann boundary condition by

$$\mu_0 = 0 < \mu_1 \le \mu_2 \le \cdots,$$

and expanding φ and ψ using the associated eigenfunctions, we see that η is an eigenvalue to (9.36) if and only if it is either that of

$$A_i = \begin{pmatrix} \mu_i & -\overline{u}\mu_i \\ -\tau^{-1} & \tau^{-1}(\mu_i + a) \end{pmatrix}$$

for $i \ge 1$ or that of A_0 with the eigenvector ${}^t(0, 1)$. This means that $\eta = \tau^{-1}a > 0$ for $i = 0$ and η is a root of

$$\eta^2 - \left[(1 + \tau^{-1})\mu_i + \tau^{-1}a\right]\eta + \mu_i\tau^{-1}(\mu_i + a - \overline{u}) = 0 \tag{9.37}$$

for $i \ge 1$ [142].

These eigenvalues of the constant stationary solution are actually involved with τ, but we can show that any eigenvalue η is shown to be real, although it is not known that this is always the case even for the nonconstant stationary solution. On the other hand, the lowest eigenvalue is taken from either $\tau^{-1}a$ or (9.37) with $i = 1$ in this case, and therefore $(\overline{u}, \overline{v})$ is linearized stable if and only if

$$a > \overline{u} - \mu_1 = \frac{\lambda}{|\Omega|} - \mu_1,$$

regardless of the value τ. Actually, the dynamical *stability* in the full system of any stationary solution is controlled by its Morse index.

More precisely, if

$$(\overline{u}, \overline{v}) \in \mathcal{P}_\lambda \cap C(\overline{\Omega}) \times C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$$

is a stationary solution to (3.1):

$$\begin{aligned} u_t &= \nabla \cdot \big(\nabla u - u\nabla(v + \log W)\big) \quad \text{in} \quad \Omega \times (0, T), \\ \frac{\partial}{\partial \nu}u - u\frac{\partial}{\partial \nu}&(v + \log W) = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ \tau \frac{d}{dt}v + Av &= u \quad \text{for} \quad t \in (0, T), \end{aligned} \tag{9.38}$$

then these $\overline{u}$ and $\overline{v}$ are critical functions of $\mathcal{F}$ on $\mathcal{P}_\lambda$ and $\mathcal{J}_\lambda$, respectively, and it holds that $\overline{v} = A^{-1}\overline{u}$ and $\overline{u} = f_\lambda(\overline{v})$. We say that $(\overline{u}, \overline{v})$ is linearized stable if the augmented Morse index of $\overline{u}$, and hence that of $\overline{v}$, is 0, and linearized unstable if the Morse index of $\overline{u}$, and hence that of $\overline{v}$, is not 0.

Remembering that $L \log L(\Omega)$ denotes the Zygmund space on Ω and its norm is provided by

$$[f]_{L \log L} = \int_\Omega |f(x)| \log \left(e + \frac{|f(x)|}{\|f\|_1}\right) dx,$$

we can show the following theorem.

Theorem 9.5 *If $n \leq 2$, the assumptions of Theorem 3.2 hold, the initial values are in*

$$(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}\left(L^p(\Omega)\right)$$

with $p > 2$, the supremum of the existence time $T > 0$ of the solution to the full system (9.38) is denoted by $T_{\max} > 0$, $C^{2+\theta}(\Omega) \cap A^{-1}(L^p(\Omega))$ is compact in $A^{-1}(L^p(\Omega))$ for $\theta \in (0, 1)$ and $p > 1$, and $(\overline{u}, \overline{v})$ is a stationary solution to (9.38) satisfying $\|\overline{u}\|_1 = \lambda$ and $\overline{u} = \overline{u}(x) \in C^2(\overline{\Omega})$, then we have the following:

1. *If* $(\overline{u}, \overline{v})$ *is linearized stable, then there is* $\varepsilon_0 > 0$ *such that*

$$\left\|u_0\right\|_1 = \lambda, \quad [u_0 - \overline{u}]_{L \log L} < \varepsilon_0, \\ \left\|A^{1/2}(v_0 - \overline{v})\right\| < \varepsilon_0 \tag{9.39}$$

implies $T_{\max} = +\infty$ *and*

$$\lim_{t \uparrow +\infty} \left\{ \left\|u(t) - \overline{u}\right\|_\infty + \left\|v(t) - \overline{v}\right\|_\infty \right\} = 0. \tag{9.40}$$

2. *If* $(\overline{u}, \overline{v})$ *is linearized unstable, then any* $\varepsilon > 0$ *admits an initial value* (u_0, v_0) *such that*

$$\left\|u_0\right\|_1 = \lambda, \qquad \left\|u_0 - \overline{u}\right\|_\infty + \left\|v_0 - \overline{v}\right\|_\infty < \varepsilon, \tag{9.41}$$

and

$$\liminf_{t \uparrow +T_{\max}} \left\{ \left\|u(t) - \overline{u}\right\|_{L \log L} + \left\|A^{1/2}(v(t) - \overline{v})\right\| \right\} > 0. \tag{9.42}$$

Here, (9.31) is not necessary to assume.

Proof: First, if

$$(\overline{u}, \overline{v}) \in \mathcal{P}_\lambda \cap C(\overline{\Omega}) \times C(\overline{\Omega}) \cap \mathrm{dom}(A^{1/2})$$

is a stationary solution to (9.38), then we have

$$\overline{v} = A^{-1}(\overline{u}) \in A^{-1}\left(L^p(\Omega)\right) \subset W^{2,p}(\Omega)$$

and

$$\overline{u} = f_\lambda(\overline{v}) \in W^{1,p}(\Omega).$$

Thus, $(\overline{u}, \overline{v})$ belongs to the function space where the well-posedness of (9.38) is assured.

Next, the inequalities

$$\left| f \log |f| - g \log |g| - (f - g) \log |f - g| \right| \le 2 |f - g| \log \left(e + \frac{|f| + |g|}{|f - g|} \right)$$

and

$$\int_\Omega |f - g| \log \left(e + \frac{|f| + |g|}{|f - g|} \right) dx \le \|f - g\|_1 \log \left(e + \frac{\|f\|_1 + \|g\|_1}{\|f - g\|_1} \right)$$

are known [72], and therefore using

$$
\begin{aligned}
&\Big|\int_\Omega |f-g| \log |f-g| \ dx\Big| \\
&\quad = \|f-g\|_1 \int_\Omega \frac{|f-g|}{\|f-g\|_1}\Big|\log \frac{|f-g|}{\|f-g\|_1}\Big|\, dx \\
&\qquad + \|f-g\|_1 \cdot \Big|\log \|f-g\|_1\Big| \\
&\quad \le \|f-g\|_1 \int_\Omega \Big(e^{-1} + \frac{|f-g|}{\|f-g\|_1}\Big) \log\Big(e + \frac{|f-g|}{\|f-g\|_1}\Big)\, dx \\
&\qquad + \|f-g\|_1 \cdot \Big|\log \|f-g\|_1\Big| \\
&\quad = \big[f-g\big]_{L\log L} + \|f-g\|_1 \Big(e^{-1} + e^{-1}|\Omega| + \Big|\log \|f-g\|_1\Big|\Big),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\Big|\int_\Omega f \log |f| - g \log |g| \, dx\Big| \\
&\quad \le 2\|f-g\|_1 \log\Big(e + \frac{\|f\|_1 + \|g\|_1}{\|f-g\|_1}\Big) \\
&\qquad + \|f-g\|_1 \Big(e^{-1} + e^{-1}|\Omega| + \Big|\log \|f-g\|_1\Big|\Big) + \big[f-g\big]_{L\log L}.
\end{aligned}
$$

This implies the continuity of

$$
f \in L \log L(\Omega) \quad \mapsto \quad \int_\Omega f \log |f| \ dx.
$$

On the other hand,

$$
(u, v) \in L \log L(\Omega) \times \mathrm{EXP}(\Omega) \quad \mapsto \quad \int_\Omega uv \, dx
$$

is a continuous bilinear form [135]. Furthermore, $H^1(\Omega) \subset \mathrm{EXP}(\Omega)$ holds if $n \le 2$ by the Trudinger–Moser inequality [164]. Therefore, we have a constant $L > 0$ determined by Ω such that

$$
\Big|\int_\Omega uv\, dx\Big| \le L \, [u]_{L\log L} \, \big\|A^{1/2} v\big\| \tag{9.43}
$$

for $(u, v) \in L \log L(\Omega) \times \mathrm{dom}(A^{1/2})$. In particular,

$$
(u, v) \in L \log L_+(\Omega) \times \mathrm{dom}(A^{1/2}) \quad \mapsto \quad W(u, v) \in \mathbf{R} \tag{9.44}
$$

is continuous, where

$$L \log L_+(\Omega) = \left\{ u \in L \log L(\Omega) \;\middle|\; u \geq 0 \; (a.e.) \right\}.$$

Now, we show the second part of the theorem. In fact, any eigenfunction of $L = I - P\overline{u}A^{-1}$ with the eigenvalue not equal to 1 belongs to $H^2(\Omega)$ by $\overline{u} \in C^2(\overline{\Omega})$ and $u(x) > 0$ for $x \in \overline{\Omega}$. Since $\overline{u}$ is linearized unstable, we have $\varphi_1 \in H^2(\Omega) \subset C(\overline{\Omega})$ satisfying

$$\int_\Omega \varphi_1 \, dx = 0 \quad \text{and} \quad \mathcal{F}''(\overline{u})[\varphi_1, \varphi_1] = -2\delta_1 < 0.$$

Here, we have

$$\mathcal{F}''(u(s))[\varphi_1, \varphi_1] = \int_\Omega u(s)^{-1} \varphi_1^2 \, dx - (\varphi_1, A^{-1}\varphi_1)$$

for $u(s) = \overline{u} + s\varphi_1$ with $|s| \ll 1$ similarly to (9.6). Therefore, there exists $s_1 > 0$ such that

$$\mathcal{F}''(u(s))[\varphi_1, \varphi_1] \leq -\delta_1$$

for $|s| \leq s_1$. By means of $\varphi_2 = s_1\varphi_1 \in H^2(\Omega)$ and $\delta_2 = \delta_1 s_1^2 > 0$, this implies

$$\mathcal{F}''(\overline{u} + s\varphi_2)[\varphi_2, \varphi_2] \leq -\delta_2$$

for $|s| \leq 1$. Therefore, we obtain

$$\begin{aligned} \mathcal{F}(\overline{u} + s\varphi_2) - \mathcal{F}(\overline{u}) &= \frac{1}{2} \int_0^1 (1-t)^2 \mathcal{F}''(\overline{u} + st\varphi_2)[s\varphi_2, s\varphi_2] \, dt \\ &\leq -\frac{s^2\delta_2}{6} \end{aligned} \tag{9.45}$$

for $|s| \leq 1$.

Given $\varepsilon > 0$, we take s in $0 < |s| \ll 1$ such that (9.41) holds for

$$u_0 = \overline{u} + s\varphi_2 \quad \text{and} \quad v_0 = A^{-1}u_0.$$

Then, we obtain

$$(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}\left(L^p(\Omega)\right)$$

and

$$\begin{aligned} W\left(u(t), v(t)\right) &\leq W\left(u_0, v_0\right) = \mathcal{F}(u_0) \\ &\leq \mathcal{F}(\overline{u}) - \frac{s^2\delta_2}{6} = \mathcal{W}(\overline{u}, \overline{v}) - \frac{s^2\delta_2}{6} \end{aligned}$$

for $t \in [0, T_{\max})$, and therefore relation (9.42) follows from the continuity of (9.44). Thus, we obtain the second part.

The first part is proven by the method of [164]. In fact, if $\overline{v} = \overline{v}(x)$ is linearized stable, then we have $\delta_3 > 0$ satisfying

$$\begin{aligned}\mathcal{J}_\lambda''(\overline{v})[w,w] &= \left\|A^{1/2}w\right\|^2 - \int_\Omega \overline{u}\Big(w - \frac{1}{\lambda}\int_\Omega \overline{u}w\,dx\Big)^2 dx \\ &\geq 2\delta_3\left\|A^{1/2}w\right\|^2\end{aligned}$$

for $w \in \mathrm{dom}(A^{1/2})$. On the other hand, for

$$v, w \in \mathrm{dom}(A^{1/2}) \subset H^1(\Omega)$$

we have

$$J_\lambda''(v)[w,w] = \left\|A^{1/2}w\right\|^2 - \int_\Omega u\Big(w - \frac{1}{\lambda}\int_\Omega uw\,dx\Big)^2 dx$$

similarly to (9.14), where $u = f_\lambda(v)$. Therefore, there is $\varepsilon_1 > 0$ such that

$$\mathcal{J}_\lambda''(\overline{v}+\zeta)[w,w] \geq \delta_3\left\|A^{1/2}w\right\|^2 \tag{9.46}$$

for $w, \zeta \in \mathrm{dom}(A^{1/2})$ in $\left\|A^{1/2}\zeta\right\| \leq \varepsilon_1$. This implies

$$\begin{aligned}\mathcal{J}_\lambda(\overline{v}+\zeta) - \mathcal{J}_\lambda(\overline{v}) &= \frac{1}{2}\int_0^1 (1-s)^2 \mathcal{J}_\lambda''(\overline{v}+s\zeta)[\zeta,\zeta]ds \\ &\geq \frac{\delta_3}{6}\left\|A^{1/2}\zeta\right\|^2\end{aligned}$$

for $\zeta \in \mathrm{dom}(A^{1/2})$ in $\|A^{1/2}\zeta\| \leq \varepsilon_1$. Now, we use of the following lemma.

Lemma 9.2 *If $(\overline{u}, \overline{v})$ is linearized stable, then we have*

$$\mathcal{W}(u,v) - \mathcal{W}(\overline{u},\overline{v}) \geq \frac{\delta_3}{6}\left\|A^{1/2}(v-\overline{v})\right\|^2. \tag{9.47}$$

for $(u,v) \in L\log L_+(\Omega) \times \mathrm{dom}(A^{1/2})$ satisfying $u \in \mathcal{P}_\lambda$ and

$$\left\|A^{1/2}(v-\overline{v})\right\| < \varepsilon_1.$$

Proof: We have $\|u\|_1 = \lambda$. Putting

$$\mu = \int_\Omega We^v\,dx \quad \text{and} \quad \psi = \frac{\lambda We^v}{\int_\Omega We^v\,dx},$$

we obtain

$$0 = -\log\Big(\int_\Omega \frac{\psi}{\lambda}\,dx\Big) = -\log\Big(\int_\Omega \frac{\psi}{u}\cdot\frac{u}{\lambda}\,dx\Big) \le \int_\Omega \Big(-\log\frac{\psi}{u}\Big)\cdot\frac{u}{\lambda}\,dx$$

by Jensen's inequality. This means

$$\int_\Omega (\log u - \log\psi)\cdot u\,dx \ge 0,$$

or equivalently,

$$\begin{aligned}\mathcal{W}(u,v) - \mathcal{W}\Big(\frac{\lambda We^v}{\int_\Omega We^v\,dx}, v\Big) &= \int_\Omega (u\log u - u\log W - uv)\;dx \\ &\quad + \lambda\log\Big(\int_\Omega We^v\,dx\Big) - \lambda\log\lambda \ge 0.\end{aligned}$$

We now recall (6.15):

$$\mathcal{W}\Big(\frac{\lambda We^v}{\int_\Omega We^v\,dx}, v\Big) = \mathcal{J}_\lambda(v) + \lambda\log\lambda.$$

Actually, this implies

$$\begin{aligned}\mathcal{W}(u,v) - \mathcal{W}(\overline{u},\overline{v}) &\ge \mathcal{W}\Big(\frac{\lambda We^v}{\int_\Omega We^v\,dx}, v\Big) - \mathcal{W}\Big(\frac{\lambda We^{\overline{v}}}{\int_\Omega We^{\overline{v}}\,dx}, \overline{v}\Big) \\ &= \mathcal{J}_\lambda(v) - \mathcal{J}_\lambda(\overline{v}),\end{aligned}$$

and therefore (9.47) holds for $\|A^{1/2}(v-\overline{v})\| < \varepsilon_1$. The proof is complete. □

To continue proving the first part of the theorem, we put $\varepsilon_2 = \min\{\varepsilon_1, 1/L\}$, where L is the constant in (9.43). Then we take $\delta_4 > 0$ satisfying

$$\mathcal{W}(u,v) - \mathcal{W}(\overline{u},\overline{v}) < \frac{\delta_3\varepsilon_2^2}{24} \tag{9.48}$$

for any $(u,v) \in L\log L_+(\Omega)\times\mathrm{dom}(A^{1/2})$ such that

$$[u-\overline{u}]_{L\log L} < \delta_4 \quad \text{and} \quad \big\|A^{1/2}(v-\overline{v})\big\| < \delta_4,$$

and put $\varepsilon_0 = \min\{\delta_4, \varepsilon_2/2\}$.

If $(u_0, v_0) \in W^{1,p}(\Omega) \times A^{-1}(L^p(\Omega))$ satisfies (9.39), then we have

$$\begin{aligned} \mathcal{W}(u(t), v(t)) - \mathcal{W}(\overline{u}, \overline{v}) &\le \mathcal{W}(u_0, v_0) - \mathcal{W}(\overline{u}, \overline{v}) \\ &< \frac{\delta_3 \varepsilon_2^2}{24} \end{aligned} \tag{9.49}$$

for $t \in [0, T_{\max})$. At $t = 0$, we have

$$\left\| A^{1/2} \left(v(t) - \overline{v} \right) \right\| < \varepsilon_2/2. \tag{9.50}$$

Even if

$$\left\| A^{1/2} \left(v(t_1) - \overline{v} \right) \right\| = \varepsilon_2/2$$

holds for some $t_1 \in (0, T_{\max})$, inequality (9.47) still holds for $(u, v) = \big(u(t_1), v(t_1)\big)$. Therefore,

$$\frac{\delta_3}{6} \left\| A^{1/2} \left(v(t_1) - \overline{v} \right) \right\|^2 < \frac{\delta_3 \varepsilon_2^2}{24}$$

follows from (9.49) with $t = t_1$, but this means

$$\left\| A^{1/2} \left(v(t_1) - \overline{v} \right) \right\| < \varepsilon_2/2,$$

a contradiction. Therefore, inequality (9.50) keeps to hold for $t \in [0, T_{\max})$. In particular, we have

$$\begin{aligned} \left| \int_\Omega u(t) \left(v(t) - \overline{v} \right) \, dx \right| &\le L \left[u(t) \right]_{L \log L} \left\| A^{1/2} \left(v(t) - \overline{v} \right) \right\| \\ &\le \frac{1}{2} \left[u(t) \right]_{L \log L} \end{aligned} \tag{9.51}$$

for $t \in [0, T_{\max})$ by (9.43). Here, we use the following lemma.

Lemma 9.3 *We have*

$$[f]_{L \log L} \le \int_\Omega |f| \log |f| \, dx + e \|f\|_1 \cdot |\Omega| - \|f\|_1 \log \|f\|_1 \tag{9.52}$$

for $f \in L \log L(\Omega)$.

Proof: Writing

$$[f]_{L \log L} = \|f\|_1 \int_\Omega \frac{|f|}{\|f\|_1} \log \left(e + \frac{|f|}{\|f\|_1} \right) dx,$$

we apply $\sup_{t>0} \frac{\log(1+t)}{t} = 1$:

$$s \log(e + s) - s \log s = e \cdot \frac{\log(1 + e s^{-1})}{e s^{-1}} \le e,$$

where $s = |f| / \|f\|_1$. Then (9.52) follows. □

We continue the proof of the theorem. By means of (9.51) and (9.52), we obtain

$$\begin{aligned}\int_\Omega (uv)(t)\,dx &= \int_\Omega u(t)\,(v(t)-\overline{v})\;dx + \int_\Omega u(t)\cdot\overline{v}\,dx \\ &\le \frac{1}{2}\,[u(t)]_{L\log L} + \lambda\cdot\left\|\overline{v}\right\|_\infty \\ &\le \frac{1}{2}\left\{\int_\Omega (u\log u)\,(t)\,dx + e\lambda\,|\Omega| - \lambda\log\lambda\right\} + \lambda\left\|\overline{v}\right\|_\infty .\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega (u\log u - uv)\,(t)\,dx - \lambda\log\left\|V\right\|_\infty &\le \mathcal{W}\,(u(t), v(t)) \\ &\le \mathcal{W}(u_0, v_0).\end{aligned}$$

These inequalities imply

$$\int_\Omega (u\log u)\,(t)\,dx \le C \tag{9.53}$$

with a constant $C > 0$ independent of $t \in [0, T_{\max})$. Then, the argument of Chapter 4 guarantees that $T_{\max} = +\infty$ and

$$\sup_{t\in[0,\infty)} \left\|u(t)\right\|_\infty < +\infty.$$

This implies the compactness of the semiorbit

$$\mathcal{O} = \left\{(u(t), v(t))\right\}_{t\ge 1} \subset W^{1,p}(\Omega)\times A^{-1}(L^p(\Omega))$$

by Theorem 3.2, and then, theory of the infinite-dimensional dynamical system [61] is applicable. Thus, the ω-limit set of $\mathcal{O}$, denoted by $\omega(\mathcal{O})$, is contained in the set of stationary solutions. Therefore, $(u_*, v_*) \in \omega(\mathcal{O})$ implies $\mathcal{J}_\lambda'(v_*) = 0$.

On the other hand,

$$\left\|A^{1/2}\,(v_* - \overline{v})\right\| \le \varepsilon_2/2$$

follows from (9.50). This implies $v_* = \overline{v}$ by (9.46), and hence $u_* = \overline{u}$. This means $\omega(\mathcal{O}) = \{(\overline{u}, \overline{v})\}$ and (9.42). The proof is complete. □

10
Formation of Collapses

In this chapter, we conclude the study of stationary solutions and describe several suggestions obtained by this for the dynamics of (3.1),

$$
\begin{aligned}
u_t = \nabla \cdot \big(\nabla u - u \nabla (v + \log W)\big) \quad & \text{in} \quad \Omega \times (0,T), \\
\frac{\partial}{\partial \nu} u - u \frac{\partial}{\partial \nu}(v + \log W) = 0 \quad & \text{on} \quad \partial\Omega \times (0,T), \\
\tau \frac{d}{dt} v + Av = u \quad & \text{for} \quad t \in (0,T),
\end{aligned} \tag{10.1}
$$

in particular, the formation of collapses of the blowup solution. Thus, we take the case $n = 2$, regarding the conjecture [33] and mathematical work [14, 50, 107, 106, 110, 146]. Therefore, stability of the stationary solution $(\overline{u}, \overline{v})$ is controlled by its Morse index, and this Morse index is equal to the number of eigenvalues in $\mu < 1$ in the eigenvalue problem (9.12):

$$
Aw = \mu \overline{u} \Big(w - \frac{1}{\lambda} \int_\Omega \overline{u} w \, dx \Big). \tag{10.2}
$$

First, we study the (D) field, where A is equal to $-\Delta$ provided with the Dirichlet boundary condition. In this case, (10.2) is equivalent to finding $w \in H_0^1(\Omega)$ such that

$$
(\nabla w, \nabla \psi) = \mu \, (Hw, H\psi)_{\overline{u}} \tag{10.3}
$$

for any $\psi \in H_0^1(\Omega)$, where

$$Hw = w - \frac{1}{\lambda}\int_\Omega \overline{u} w \, dx. \tag{10.4}$$

Here, we have $(Hw, 1)_{\overline{u}} = 0$, and furthermore, if $w \in H_0^1(\Omega)$ is a nonzero solution to (10.3), then

$$\hat{w} = Hw$$

is a nonconstant function belonging to

$$H_c^1(\Omega) = \left\{ \psi \in H^1(\Omega) \mid \psi = \text{constant on } \partial\Omega \right\}.$$

Given $\hat{\psi} \in H_c^1(\Omega)$, we have

$$\psi \equiv \hat{\psi} - \hat{\psi}\big|_{\partial\Omega} \in H_0^1(\Omega)$$

and hence

$$(\nabla \hat{w}, \nabla \hat{\psi}) = (\nabla w, \nabla \psi) = \mu \, (Hw, H\psi)_{\overline{u}}$$

follows. Furthermore, we have

$$(Hw, H\psi)_{\overline{u}} = (Hw, \psi)_{\overline{u}} = (Hw, \hat{\psi})_{\overline{u}} = (\hat{w}, \hat{\psi})_{\overline{u}}$$

by $(Hw, 1)_{\overline{u}} = 0$. Thus, we obtain a nonconstant $\hat{w} \in H_c^1(\Omega)$ from this nonzero $w \in H_0^1(\Omega)$ satisfying

$$(\nabla \hat{w}, \nabla \hat{\psi}) = \mu(\hat{w}, \hat{\psi})_{\overline{u}} \tag{10.5}$$

for any $\hat{\psi} \in H_c^1(\Omega)$.

Conversely, if $\hat{w} \in H_c^1(\Omega)$ is a nonconstant function satisfying (10.5) for any $\hat{\psi} \in H_c^1(\Omega)$, then

$$w \equiv \hat{w} - \hat{w}\big|_{\partial\Omega} \in H_0^1(\Omega)$$

is a nonzero function. Furthermore, given $\psi \in H_0^1(\Omega)$ we have

$$\hat{\psi} = H\psi \in H_c^1(\Omega)$$

and hence

$$(\nabla w, \nabla \psi) = (\nabla \hat{w}, \nabla \hat{\psi}) = \mu(\hat{w}, \hat{\psi})_{\overline{u}}$$

follows. Here, we have

$$(\hat{w}, \hat{\psi})_{\overline{u}} = (Hw, \hat{\psi})_{\overline{u}} = (Hw, H\psi)_{\overline{u}}$$

by $(1, \hat{\psi})_{\overline{u}} = 0$, which implies (10.3). Therefore, the eigenvalue problem (10.5) is equivalent to finding ϕ such that

$$\begin{aligned} -\Delta\phi &= \mu\overline{u}\phi \qquad \text{in} \quad \Omega \\ \phi &= \text{constant} \quad \text{on} \quad \partial\Omega \\ \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} d\sigma &= 0. \end{aligned}$$

The first eigenvalue of the above problem is $\mu = 0$, which is associated with the eigenfunction $\phi = 1$. Therefore, the following lemma is obtained by the mini-max principle.

Lemma 10.1 *If A is equal to $-\Delta$ provided with the Dirichlet boundary condition, then the Morse index of the stationary solution $(\overline{u}, \overline{v})$ to (10.1) is equal to the number of eigenvalues of (10.5) in $\mu < 1$ minus 1. In particular, it is linearized stable (resp. unstable) if and only if $\mu_2 > 1$ (resp. $\mu_2 < 1$), where*

$$\mu_2 = \inf\left\{ \left\|\nabla\psi\right\|_2^2 \;\middle|\; \psi \in H_c^1(\Omega),\ \int_\Omega \overline{u}\psi\, dx = 0,\ \int_\Omega \overline{u}\psi^2\, dx = 1 \right\}.$$

The above lemma is valid even in the case of $n \neq 2$. It was found by [162, 185] independently, where the Morse indices are defined in terms of $\overline{u}$ and $\overline{v}$, respectively. Thus, Theorem 9.1 was first proven in this special case without being recognized explicitly, while a generalized abstract theory for this equivalence is described in the last chapter.

We hereby apply the isoperimetric inequality, Lemma 8.14, for

$$q = \overline{u} = \frac{\lambda W e^{\overline{v}}}{\int_\Omega W e^{\overline{v}}\, dx},$$

where $\overline{v}$ is a solution to (7.1):

$$\begin{aligned} -\Delta\overline{v} &= \frac{\lambda W e^{\overline{v}}}{\int_\Omega W e^{\overline{v}}\, dx} \quad \text{in} \quad \Omega, \\ \overline{v} &= 0 \qquad\qquad\qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{10.6}$$

In more details, if $\lambda = \left\|\overline{u}\right\|_1 \in (0, 8\pi)$, $\Omega \subset \mathbf{R}^2$ is simply connected, and

$$-\Delta \log W \leq 0 \quad \text{in} \quad \Omega, \tag{10.7}$$

then any stationary solution to (10.1) is linearized stable, where A is $-\Delta$ with the Dirichlet boundary condition.

Theorem 7.1 says, on the other hand, that any family of solutions to (10.6) can blowup only at the quantized value of λ in $8\pi\mathbf{N}$. In particular, any $\varepsilon > 0$ admits $C_\varepsilon > 0$ such that any solution $\{\overline{v}\}$ to (10.6) with $\lambda \in (0, 8\pi - \varepsilon)$ takes the estimate

$$\left\|\overline{v}\right\|_\infty \leq C_\varepsilon.$$

In particular,r the argument developed for the proof of Theorem 8.1 assures the unique existence of the solution for $0 < \lambda \ll 1$, which is extended to $\lambda \in (0, 8\pi)$ by Lemma 8.14 and the standard argument of continuation [166]. In other words, similarly to the first case of Theorem 8.1, the set of stationary solutions

$$\left\{(\lambda, \underline{v}_\lambda) \mid \lambda \in (0, 8\pi)\right\}$$

forms a branch in $\lambda - v$ space, and any other solution is not admitted for $\lambda \in (0, 8\pi)$. This implies, in particular, the unique existence of the stationary solution $(\overline{u}, \overline{v})$ to (10.7), satisfying $\|\overline{u}\|_1 = \lambda \in (0, 8\pi)$.

As is described in Chapter 4, on the other hand, nonstationary problem (10.1) admits the solution $u = u(\cdot, t)$ globally in time if $\|u_0\|_1 < 8\pi$ satisfying

$$\sup_{t\geq 0} \left\|u(t)\right\|_\infty < +\infty.$$

Therefore, the following theorem is obtained by the theory of an infinite-dimensional dynamical system, similarly to Theorem 9.5.

Theorem 10.1 *If $\Omega \subset \mathbf{R}^2$ is a simply connected bounded domain with smooth boundary $\partial\Omega$, A is equal to $-\Delta$ provided with the Dirichlet boundary condition, and $W(x) > 0$ is a smooth function of $x \in \overline{\Omega}$ satisfying* (10.7)*, then each $\lambda \in (0, 8\pi)$ admits a unique stationary solution $(\underline{u}_\lambda, \underline{v}_\lambda)$ to* (10.1) *such that $\|\underline{u}_\lambda\|_1 = \lambda$, and if $\|u_0\|_1 = \lambda$, then $T_{\max} = +\infty$ and*

$$\lim_{t\uparrow+\infty} \left\{\left\|u(t) - \underline{u}_\lambda\right\|_\infty + \left\|v(t) - \underline{v}_\lambda\right\|_\infty\right\} = 0,$$

where $(u(t), v(t))$ denotes the nonstationary solution such that $u_0 = u|_{t=0}$, both to the simplified and full systems of (10.1).

Problem (10.6) admits the solutions for $\lambda \geq 8\pi$ even if $W = 1$ and $\Omega \subset \mathbf{R}^2$ is convex. This is actually the case when Ω is thin [20, 21, 24, 98, 167], and then the branch $\{(\lambda, \underline{v}_\lambda) \mid \lambda \in (0, 8\pi)\}$ of stationary solutions to (10.1) does not blow up as $\lambda \uparrow 8\pi$. In any case, if Ω is simply connected, then we have an upper bound $\overline{\lambda} < +\infty$ of λ for the existence of the solution. (This is not the case of multiply connected Ω [92, 112, 116, 166].

If $W = 1$ and Ω is close to a ball, then $\overline{\lambda} = 8\pi$ and $\underline{v}_\lambda(x)$ of Theorem 10.1 takes the singular limit

$$\lim_{\lambda \uparrow 8\pi} \underline{v}_\lambda(x) = 8\pi G(x, x_0) \tag{10.8}$$

in $W^{1,q}(\Omega)$ for $q \in [1, 2)$, with $x_0 \in \Omega$ satisfying

$$\nabla R(x_0) = 0,$$

where $G = G(x, x')$ and

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log \left|x - x'\right|\right]_{x'=x}$$

denote the Green's and the Robin functions to $-\Delta$ provided with the Dirichlet boundary condition, respectively [98, 166, 167].

If Ω is such a domain, then the limit as $t \uparrow +\infty$ of the nonstationary solution $u(\cdot, t)$ such that

$$u\big|_{t=0} = u_0(x) \quad \text{for} \quad \lambda = \left\|u_0\right\|_1 \in (0, 8\pi),$$

denoted by $(\underline{u}_\lambda, \underline{v}_\lambda)$, becomes spiky as $\lambda \uparrow 8\pi$. More precisely, it holds that

$$\underline{u}_\lambda(x)\, dx \rightharpoonup 8\pi \delta_{x_0}(dx)$$

as $\lambda \uparrow 8\pi$. Then, we can expect that the blowup in finite time occurs for $u(\cdot, t)$ if $\|u_0\|_1 = \lambda$ holds with $0 < \lambda - 8\pi \ll 1$, because the branch of stationary solutions formed by $\underline{u}_\lambda(x)$ does not exceed across $\lambda = 8\pi$. Since this branch leaves the singular limit $8\pi\delta_0(dx)$ at $\lambda = 8\pi$, the above nonstationary solution may make a collapse in Ω with the quantized mass. This is nothing but a detailed observation of [33] concerning the threshold of $\lambda = \|u_0\|_1$ for $T_{\max} < +\infty$, regarding the formation of collapses of the blowing-up solution.

The above observed blowup mechanism of the nonstationary solution to (10.1) will not be different from the other cases. In fact, in the (JL) field, the operator A is defined by (3.4); $Av = u$ if and only if

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u\, dx \quad \text{in} \quad \Omega,$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad \int_\Omega v\, dx = 0. \tag{10.9}$$

Therefore, the stationary problem is formulated as

$$-\Delta v = \lambda\Big(\frac{We^v}{\int_\Omega We^v\,dx} - \frac{1}{|\Omega|}\Big) \quad \text{in} \quad \Omega,$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad \int_\Omega v\,dx = 0 \tag{10.10}$$

in terms of v, and the linearized problem (10.2) is equivalent to finding

$$w \in H^1(\Omega) \cap L_0^2(\Omega)$$

such that

$$(\nabla w, \nabla \psi) = \mu\,(Hw, H\psi)_{\overline{u}} \tag{10.11}$$

for any $\psi \in H^1(\Omega) \cap L_0^2(\Omega)$, where H is the operator defined by (10.4):

$$Hw = w - \frac{1}{\lambda}\int_\Omega \overline{u}w\,dx.$$

If $w \in H^1(\Omega) \cap L_0^2(\Omega)$ is a nonzero solution to this linearized problem, and take $\hat{\psi} \in H^1(\Omega)$, then

$$\psi \equiv \hat{\psi} - \int_\Omega \hat{\psi}\,dx$$

belongs to $H^1(\Omega) \cap L_0^2(\Omega)$, and $\hat{w} = Hw \in H^1(\Omega)$ satisfies

$$(\nabla\hat{w}, \nabla\hat{\psi}) = (\nabla w, \nabla\psi) = \mu\,(Hw, H\psi)_{\overline{u}}.$$

Here, we have

$$(Hw, H\psi)_{\overline{u}} = (\hat{w}, \hat{\psi})_{\overline{u}}$$

by $(Hw, 1)_{\overline{u}} = 0$, and therefore $\hat{w} \in H^1(\Omega)$ is a nonconstant solution to

$$(\nabla\hat{w}, \nabla\hat{\psi}) = \mu(\hat{w}, \hat{\psi})_{\overline{u}} \tag{10.12}$$

for any $\hat{\psi} \in H^1(\Omega)$.

If $\hat{w} \in H^1(\Omega)$ is a nonconstant solution to (10.12), conversely, then

$$w \equiv \hat{w} - \frac{1}{|\Omega|}\int_\Omega \hat{w}\,dx \in H^1(\Omega)$$

is a nonconstant solution to (10.11). Here, the eigenvalue problem (10.12) is equivalent to finding ϕ such that

$$-\Delta\phi = \mu\overline{u}\phi \quad \text{in} \quad \Omega, \qquad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega.$$

This problem has $\mu = 0$ as the first eigenvalue, and the associated eigenfunction is $\phi = \hat{w} = 1$. Thus we obtain the following lemma.

Lemma 10.2 *If* (10.1) *is associated with the (JL) field, that is, if*

$$Av = u$$

is equivalent to (10.9)*, then the Morse index of the stationary solution* $(\overline{u}, \overline{v})$ *is equal to the number of negative eigenvalues of* $-\Delta - \overline{u}$ *under the Neumann boundary condition minus* 1*. In particular, it is linearized stable (resp. unstable) if and only if* $\mu_2 > 1$ *(resp.* $\mu_2 < 1$*), where*

$$\mu_2 = \inf\left\{ \|\nabla\psi\|_2^2 \;\middle|\; \psi \in H^1(\Omega),\ \int_\Omega \overline{u}\psi\, dx = 0,\ \int_\Omega \overline{u}\psi^2\, dx = 1 \right\}.$$

Because the stationary solution $\overline{v}$ satisfies (10.10), this lemma can be combined with the following lemma obtained in the proof of Theorem 8.2.

Lemma 10.3 *There exists* $\delta > 0$ *such that if* $\Omega \subset \mathbf{R}^2$ *is a simply connected bounded domain with smooth boundary* $\partial\Omega$ *and* $q = q(x) > 0$ *is a smooth function of* $x \in \overline{\Omega}$ *satisfying*

$$-\Delta \log q \le q \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega q\, dx \in (4\pi, 4\pi + \delta),$$

then it holds that

$$\inf\left\{ \|\nabla\psi\|_2^2 \;\middle|\; \psi \in H^1(\Omega), \int_\Omega q\psi\, dx = 0,\ \int_\Omega q\psi^2\, dx = 1 \right\} < 1.$$

Consequently, we obtain the following theorem.

Theorem 10.2 *If* $\Omega \subset \mathbf{R}^2$ *is a simply connected bounded domain with smooth boundary* $\partial\Omega$*,* $W = W(x) > 0$ *is a smooth function defined on* $\overline{\Omega}$*, A is the (JL) field, that is,* $u = Av$ *if and only if* (3.4)*, and condition* (10.7) *holds for* $W = W(x)$*, then any stationary solution* $(\overline{u}, \overline{v})$ *to* (10.1) *satisfying* $\lambda = \|\overline{u}\|_1 \in (4\pi, 4\pi + \delta)$ *is not asymptotically stable (even if it exits).*

We shall study the (N) field in comparison with the (JL) field. This time A is equal to $-\Delta + a$ provided with the Neumann boundary condition, where $a > 0$ is a constant. Then, from Chapter 4, the condition $\|u_0\|_1 = \lambda < 4\pi$ implies $T_{\max} = +\infty$ and $\sup_{t\geq 0} \|u(t)\|_\infty < +\infty$.

If $W(x) \equiv 1$, we have the constant stationary solution

$$(\overline{u}, \overline{v}) = (\lambda / |\Omega|, \lambda / (a|\Omega|))$$

and its Morse index is calculated in Chapter 8. In particular, it is linearized stable (resp. unstable) if and only if $\lambda < \lambda_1$ (resp. $\lambda > \lambda_1$), where $\lambda_1 = |\Omega| \left(a + \mu_2^*\right)$ with μ_2^* being the second eigenvalue of $-\Delta$ under the Neumann boundary condition. Here, from the isoperimetric inequality of Polyá–Szegö–Weinberger, we have $\lambda_1 < 4\pi$ for $a > 0$ sufficiently small, because the first zero of the Bessel function is less than 4π [5]. If $\lambda_1 < 4\pi$ and $\lambda \in (\lambda_1, 4\pi)$, then $\mathcal{J}_\lambda$ admits a global minimum in $H^1(\Omega)$ with the minimizer denoted by $\overline{v}$. Then by Theorem 9.5, the stationary solution $\overline{u} = f_\lambda(\overline{v})$ is asymptotically stable if it is nondegenerate. By Theorem 8.1, on the other hand, any stationary solution $(\overline{u}, \overline{v})$ is constant if $0 < \lambda = \|\overline{u}\|_1 \ll 1$, and therefore if $\|u_0\|_1 = \lambda \ll 1$ the constant stationary solution is a global attractor of the dynamical system induced by (10.1).

In the (JL) field with $W(x) \equiv 1$, we have the trivial solution $v = 0$ to (10.10). This situation is quite similar to the (N) field with $0 < a \ll 1$. In particular, the generation of nonradially symmetric solutions occurs to (10.1) with Ω equal to the unit disc

$$\Omega = D \equiv \left\{ x \in \mathbf{R}^2 \;\middle|\; |x| < 1 \right\}.$$

Studying the (N) field with $W(x) = 1$ and $\Omega = B$ in more detail [144], we recall that $\left\{\mu_j^*\right\}_{j=1}^\infty$ denotes the set of eigenvalues of

$$-\Delta \quad \text{with} \quad \frac{\partial}{\partial \nu} \cdot \Big|_{\partial\Omega} = 0.$$

Then, Lemma 8.3 clarifies how $\left\{\mu_j^*\right\}_{j\geq 2}$ is committed to the spectrum of a linearized operator around the constant solution to (8.1):

$$\begin{aligned} -\Delta v + av &= \frac{\lambda e^v}{\int_\Omega e^v \, dx} && \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{10.13}$$

that is, in this case of $\Omega = B$, a complete system of eigenfunctions is obtained by the separation of variables. If n denotes the number of zeros in radial direction of the eigenfunction, and m its period with respect to the argument mode, then there are double eigenvalues and $\mu_2^* = \mu_3^*$, $\mu_4^* = \mu_5^*$, and μ_6^* are associated with the eigenfunctions with $(n, m) = (0, 1)$, $(n, m) = (0, 2)$, and $(n, m) = (1, 0)$, respectively, and μ_6^* is simple. Furthermore, we have $\mu_2^* < 4$ and $\mu_4^* > 8$ [5].

The set of solutions to (10.13) on $\Omega = B$ is now illustrated as follows. First, a branch of radial solutions to (8.1), denoted by $\mathcal{C}_c$, bifurcates from that of constant solutions, denoted by $\mathcal{C}_{rd}$, at $\lambda = \lambda_3 \equiv |\Omega| \left(a + \mu_6^*\right)$ in $\lambda - v$ space (Chapter 1). From the local theory of bifurcation, this $\mathcal{C}_{rd}$ is transversal to $\mathcal{C}_c$. Then, the following theorem suggests that $\mathcal{C}_{rd}$ is absorbed into $\lambda = 8\pi$.

Theorem 10.3 *We have the following facts for* (10.13) *with* $\Omega = B$ *and* $v = v(|x|)$*:*

1. *If* $\lambda \in (0, 8\pi)$*, then only constant solutions are admitted.*
2. *If* $\lambda \in (8\pi, \lambda_3)$*, then there exists a nonconstant solution, where* $\lambda_3 = |\Omega| \left(a + \mu_6^*\right)$*.*

The first assertion justifies the numerical computation [33]. Then, the second assertion assures the family of radial solutions

$$\mathcal{C}_{rd}^* = \left\{(\lambda, v_\lambda) \mid 8\pi < \lambda < \lambda_3\right\}$$

absorbed into the hyperplane $\lambda = 8\pi$ with the radial singular limit of Theorem 7.5. Thus, the functions on $\mathcal{C}_{rd}^*$ make one point blow up at the origin as $\lambda \downarrow 8\pi$.

Now, we restrict ourselves to the case of $0 < a \ll 1$. Then the first bifurcation point $\lambda = \lambda_1 \equiv |\Omega| \left(a + \mu_2^*\right)$ of the branch of constant solutions $\mathcal{C}_c$ is less than 4π. At this point of $\lambda = \lambda_1$, the linearized operator takes 0 as an eigenvalue, and the corresponding eigenfunction has the argument mode 1. Since S^1 acts on $B = \Omega$, this bifurcated object forms a two-dimensional manifold in $\lambda - v$ space, denoted by $\mathcal{C}_1$.

Regarding the second case of Theorem 8.1, we can expect that $\mathcal{C}_1$ is composed of the global minima of $\mathcal{J}_\lambda(v)$ defined on $H^1(B)$ with $\lambda \in (\lambda_1, 4\pi)$, and this branch continues up to $\lambda = 4\pi$. Therefore, based on Theorem 8.2, we suspect that $\mathcal{C}_1$ does not exceed across $\lambda = 4\pi$, and $\mathcal{C}_1$ is connected with a family of singular limits denoted by $\mathcal{O}_1$. Here, each element of $\mathcal{O}_1$ has a singular point of its own on the boundary. This $\mathcal{O}_1$ will be located on $\lambda = 4\pi$ in $\lambda - v$ space and will form an object homeomorphic to S^1 because of the S^1 action to B.

A similar profile is expected to the other two-dimensional manifold in $\lambda - v$ space, denoted by $\mathcal{C}_2$, bifurcating from $\mathcal{C}_c$ at $\lambda = \lambda_2 \equiv |\Omega| \left(a + \mu_4^*\right) > 8\pi$. In more details, this $\mathcal{C}_2$ will be connected with a S^1 families of singular limits denoted by $\mathcal{O}_2$ on the hyperplane $\lambda = 8\pi$, of which members have two singular points of their own on the boundary. The bifurcation diagram of $\mathcal{C}_1$ will be drastically changed in $a \gg 1$, and then it bifurcates from $\mathcal{C}_c$ at $\lambda = \lambda_1 > 4\pi$. However, the ultimate state will be similar, absorbed into $\mathcal{O}_1$ as $\lambda \downarrow 4\pi$.

If Ω is slightly perturbed from B, then these $\mathcal{C}_1$ and $\mathcal{C}_2$ will be reduced to curves making pitchfork bifurcations from $\mathcal{C}_c$, and similarly, both $\mathcal{O}_1$ and $\mathcal{O}_2$ will be reduced to two singular limits of their own, but the other structures will be kept (Figure 1.6). After such a profile is obtained for the set of stationary solutions, then the dynamics of (10.1) with $W(x) = 1$, the (N) field, and $0 < a \ll 1$, is suspected as follows.

First, if

$$\lambda = \left\|u_0\right\|_1 \in (0, \lambda_1),$$

then any solution is global in time and converges to the constant stationary solution uniformly in infinite time. Even if

$$\lambda = \left\|u_0\right\|_1 \in (\lambda_1, 4\pi),$$

again any solution to (10.1) is global in time (Chapter 4), but now the constant stationary solution located on $\mathcal{C}_c$ is linearized unstable, while the nonconstant stationary solution on $\mathcal{C}_1$ is the global minimum of $\mathcal{J}_\lambda(v)$. Therefore, generic solutions to (10.1) with $\|u_0\|_1 \in (\lambda_1, 4\pi)$ will have an ω limit set contained in $\mathcal{C}_1$, because the linearized stability implies the dynamical stability (Chapter 9). This means that they have a tendency to concentrate on the boundary with one peak. One can expect that this tendency is kept even in

$$\lambda = \left\|u_0\right\|_1 \in (4\pi, 8\pi).$$

However, only an unstable constant solution will exist as the stationary solution, and therefore except for the initial values on a thin set of stable manifolds of the constant stationary solution, the solution to (10.1) with $\|u_0\|_1 \in (4\pi, 8\pi)$ will blow up in a finite time, concentrating on the boundary with one peak. This profile is quite similar to the case of the (D) field described earlier, but this time the (N) field, the boundary blowup point, is involved in the threshold phenomenon of the blowup.

So far, we have obtained several suggestions for the dynamics of the nonstationary solution from the study of stationary solutions. Now, we give the following proof.

Proof of Theorem 10.3 for the first case: We can apply Lemma 8.14 for $\Omega = B$ and

$$H_r^1(B) = \left\{ v \in H^1(B) \mid v = v(|x|) \right\},$$

because it holds that $H_r^1(B) \subset H_c^1(B)$. Then, we obtain

$$\inf\left\{ \int_B |\nabla v|^2 \, dx \mid v \in H_r^1(B), \int_B q v^2 \, dx = 1, \int_B q v \, dx = 0 \right\} > 1 \tag{10.14}$$

for $q \in C^2(\overline{B})$, satisfying

$$-\Delta \log q \leq q \quad \text{in} \quad B \quad \text{and} \quad \int_B q \, dx < 8\pi.$$

Given a solution $v = v(|x|)$ to (10.13) on $\Omega = B$, we put

$$q = \frac{\lambda e^v}{\int_\Omega e^v \, dx}.$$

Then, the linearized operator is given as $L = L_0 + a$, for L_0 defined by (8.3). Let L_{0r} be the radial part of L_0. Then, any eigenvalue of L_{0r} corresponding to the nonconstant eigenfunction is positive by (10.14) and the proof of Lemma 8.12. Therefore, if $\lambda \in (0, 8\pi)$ and $v = v(|x|)$ is a solution of (10.13) on $\Omega = B$, then $L_{0r} + a$ is invertible.

On the other hand, any compact set

$$\Lambda \subset [0, +\infty) \setminus \{8\pi\}$$

admits a constant $C > 0$ such that any solution $v(|x|)$ of (10.13) on $\Omega = B$ with $\lambda \in \Lambda$ satisfies

$$\|v\|_\infty \leq C \tag{10.15}$$

by Theorem 7.5, and therefore the proof of Theorem 8.1 guarantees the uniqueness of the radial solution for (10.13) for $\lambda \in (0, 8\pi)$. The proof is complete. □

Proof of Theorem 10.3 for the second case: We recall that μ_6^* is the second eigenvalue in (8.6), associated with a radially symmetric eigenfunction of $-\Delta$ under the Neumann boundary condition. Therefore, if $\lambda \in (0, \lambda_3)$, then the constant solution

$$s = \lambda / a|\Omega|$$

is a strict local minimum of $\mathcal{J}_\lambda(v)$ on $H_r^1(B)$, where

$$\lambda_3 = |\Omega| \left(a + \mu_6^*\right).$$

On the other hand, there exists $v_0 \in H_r^1(B)$ such that

$$\mathcal{J}_\lambda(v_0) < \mathcal{J}_\lambda(s)$$

if $\lambda > 8\pi$. In fact, we have only to take $x_0 = 0$ in the proof of Lemma 8.5.

Now, we reproduce the proof of Theorem 8.1 for the third case, replacing the underlying space $H^1(\Omega)$ by $H_r^1(B)$. Then we obtain the conclusion, using (10.15) for $\lambda \in \Lambda$. □

In the case of $W(x) \not\equiv 1$, the calculation of the Morse indices of the stationary solution of the (N) field is not reduced for simpler problems as in the (D) or (JL) field. However, the following theorem is obtained similarly to Theorem 8.2.

Theorem 10.4 *If A is the (N) field: $-\Delta + a$ with the Neumann boundary condition, and the positive smooth function $W(x)$ satisfies* (10.7) *in* (10.1)*, then we have $\delta > 0$ such that any $\lambda \in (4\pi, 4\pi + \delta)$ admits a_0 such that if $a \in (0, a_0)$ then any stationary solution $(\overline{u}, \overline{v})$ in $\|\overline{u}\|_1 = \lambda$ is not asymptotically stable (even if it exists). Furthermore, this a_0 is locally uniform in $\lambda \in (4\pi, 4\pi + \delta)$.*

We have observed that in the simplified system, the Morse index of the stationary solution is reflected faithfully in the dynamics of the nonstationary solution near them, while the stability of the stationary solution is still kept in the full system. If we have a stationary solution of which the linearized system as the full system possesses a nonreal eigenvalue, then some spiral movement is suggested for the nonstationary solution. This problem is open mathematically.

11
Finiteness of Blowup Points

Similarly to life, thermodynamical circuitry enables the organic activity against entropy, by a circular reaction within the gradient of nonequilibrium.
— H. Tanaka

In this chapter we discuss the blowup mechanism of the nonstationary simplified system of chemotaxis. This chapter is devoted to the proof of Theorem 1.1.

The first process is to introduce the localized version of Theorem 4.1. Here, Moser's iteration scheme is applied and the blowup point of the solution is characterized by the blowup of the local Zygmund norm around it. Then, Gagliardo–Nirenberg's inequality provides a universal constant $\varepsilon_0 > 0$ satisfying

$$\limsup_{t \uparrow T_{\max}} \left\| u(t) \right\|_{L^1(B_R(x_0) \cap \Omega)} \geq \varepsilon_0 \tag{11.1}$$

for any blowup point $x_0 \in \overline{\Omega}$ and $R > 0$. The next process is to provide the global profile of this localization by the method of symmetrization described in Chapter 5. More precisely, the function

$$t \in [0, T_{\max}) \mapsto \left\| u(t) \right\|_{L^1(B(x_0,R) \cap \Omega)}$$

has a uniform bounded variation, and consequently (11.1) is improved by

$$\liminf_{t\uparrow T_{\max}} \left\| u(t) \right\|_{L^1(B_R(x_0)\cap\Omega)} \geq \varepsilon_0.$$

This inequality implies the finiteness of blowup points by $\left\|u(t)\right\|_1 = \left\|u_0\right\|_1$ for $t \in [0, T_{\max})$, and therefore each blowup point is isolated, which induces the chemotactic collapse with a sharp estimate of the collapse mass from below, and hence an inequality estimating the number of blowup points from above.

Henceforth, we study the simplified system (3.1), provided with the (N) field and $W(x) \equiv 1$ for the sake of simplicity, that is,

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0, T),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$
$$u\big|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \tag{11.2}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, ν denotes the outer unit normal vector, and $u_0 = u_0(x)$ is a smooth nonnegative function not identically 0 on $\overline{\Omega}$. As before, $T_{\max} \in (0, +\infty]$ denotes the supremum of existence time of the solution, and therefore $T_{\max} < +\infty$ means the blowup of the solution in finite time. Thus, we show that

$$u(x,t)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx \tag{11.3}$$

holds in $\mathcal{M}(\overline{\Omega})$ in the case of $T_{\max} < +\infty$, where $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$, $m(x_0) \geq m_*(x_0)$ with $m_*(x_0) = 8\pi$ and $m_*(x_0) = 4\pi$ according to $x_0 \in \Omega \cap \mathcal{S}$ and $x_0 \in \partial\Omega \cap \mathcal{S}$, respectively, and $\mathcal{S}$ denotes the blowup set of $u(\cdot, t)$ as $t \uparrow T_{\max}$. We confirm that the Dirac measure $\delta_{x_0}(dx) \in \mathcal{M}(\overline{\Omega})$ acts as

$$\left\langle \eta(x), \delta_{x_0}(dx) \right\rangle = \eta(x_0) \qquad \left(x_0 \in \overline{\Omega}\right)$$

for $\eta \in C(\overline{\Omega})$, and consequently, the finiteness of blowup points follows as (1.28):

$$2 \cdot \sharp(\text{interior blowup points}) + \sharp(\text{boundary blowup points}) \leq \left\|u_0\right\|_1 / 4\pi. \tag{11.4}$$

In particular, if $4\pi < \|u_0\|_1 < 8\pi$ and $T_{\max} < +\infty$, then $u(x,t)\,dx$ concentrates to a point on the boundary as $t \uparrow T_{\max}$. An interesting open question is whether one can prescribe the numbers of interior and boundary blowup points independently, that is, whether the quantization in space implies synchronization in time or not. It is open also to exclude the equality of (11.4).

Some results proven for (11.2) are shown to be valid even for the full system, for example, $\|u_0\|_1 < 4\pi$ implies $T_{\max} < +\infty$ and there are chemotactic collapses (11.3) for the radially symmetric case [108]. In fact, any $x_0 \in \mathcal{S}$ and $R > 0$ admit (11.1) even in this case. An important open question is whether any blowup point is isolated in the full system, which we do not think to be true (Chapter 1).

We show the result proven in this chapter more precisely. First, if $T_{\max} < +\infty$ then

$$\lim_{t \uparrow T_{\max}} \left\| u(t) \right\|_\infty = +\infty \tag{11.5}$$

holds (Theorem 3.2), and we define the *blowup set* $\mathcal{S}$ of u (Chapter 1):

$$\mathcal{S} = \Big\{ x_0 \in \overline{\Omega} \ \Big| \quad \text{there exist} \quad t_k \uparrow T_{\max} \quad \text{and} \quad x_k \to x_0 \quad \text{such that} \quad u(x_k, t_k) \to +\infty \quad \text{as} \quad k \to \infty \Big\}.$$

Each $x_0 \in \mathcal{S}$ is called the *blowup point*. The condition $T_{\max} < +\infty$ implies $\mathcal{S} \neq \emptyset$ by (11.5). Then we have the finiteness of blowup points, (11.4), and the formation of chemotactic collapse (11.3) as follows.

Theorem 11.1 *If $T_{\max} < +\infty$, then we have (11.3) in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T_{\max}$ with $m(x_0) \geq m_*(x_0)$ and*

$$0 \leq f = f(x) \in C(\overline{\Omega} \setminus \mathcal{S}) \cap L^1(\Omega), \tag{11.6}$$

where

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

Henceforth, we put $a = 1$ for simplicity, using the Gagliardo–Nirenberg inequality in two-dimensional space indicated as (4.32):

$$\|w\|_2^2 \leq K^2 \big(\|\nabla w\|_1^2 + \|w\|_1^2 \big). \tag{11.7}$$

It is valid for any $w \in W^{1,1}(\Omega)$, where $K > 0$ is a constant determined by Ω. We recall the notation

$$B(x_0, R) = \Big\{ x \in \mathbf{R}^2 \ \Big| \ |x - x_0| < R \Big\}.$$

Given $x_0 \in \overline{\Omega}$, now we take the cut-off function $\varphi = \varphi_{x_0,R',R}$ defined for $0 < R' < R \ll 1$ introduced in Chapter 5. It satisfies $0 \leq \varphi \leq 1$ in Ω, $\varphi(x) = 1$ for $x \in B(x_0, R') \cap \Omega$, $\varphi(x) = 0$ for $x \in B(x_0, R) \cap \Omega$, $\| D^\alpha \varphi \|_\infty = O\big((R-R')^{-|\alpha|}\big)$ for any multi-index α, and $\operatorname{supp} \varphi \subset \Omega$ and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ in the cases of $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively. Then, $\psi = (\varphi_{x_0,R',R})^6$ satisfies

$$\psi(x) = \begin{cases} 1 \big(x \in B(x_0, R') \in \Omega\big) \\ 0\big(x \in \Omega \setminus B(x_0, R)\big) \end{cases}$$

$$0 \leq \psi \leq 1 \quad \text{in} \quad \Omega, \qquad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

$$\left.\begin{array}{l} |\nabla \psi| \leq A \psi^{5/6} \\ |\Delta \psi| \leq B \psi^{2/3} \end{array}\right\} \quad \text{in} \quad \Omega,$$

where $A > 0$ and $B > 0$ are constants determined by $0 < R' < R \ll 1$. We have, more precisely,

$$A = O\big((R-R')^{-1}\big) \quad \text{and} \quad B = \big((R-R')^{-2}\big),$$

but these rates are not used in this chapter.

First, we prove some variants of (4.31):

$$\int_\Omega u^2 \, dx \leq \frac{2K^2}{\log s} \int_\Omega \big(u \log u + e^{-1}\big) \, dx \cdot \int_\Omega u^{-1} |\nabla u|^2 \, dx + 2K^2 \|u\|_1^2 + 3s^2 |\Omega| .$$

Lemma 11.1 *The following inequalities hold for any $s > 1$, where $C > 0$ is a constant determined by A and K:*

$$\int_\Omega u^2 \psi \, dx \leq 2K^2 \int_{B(x_0,R)\cap\Omega} u \, dx \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx + K^2 \Big(\frac{A^2}{2} + 1\Big) \|u\|_1^2. \tag{11.8}$$

$$\int_\Omega u^3 \psi \, dx \leq \frac{72K^2}{\log s} \int_{B(x_0,R)\cap\Omega} \big(u \log u + e^{-1}\big) \, dx \int_\Omega |\nabla u|^2 \psi \, dx + C \|u\|^3_{L^1(B(x_0,R)\cap\Omega)} + 10 |\Omega| s^3. \tag{11.9}$$

Proof: Putting $w = u\psi^{1/2}$, we have

$$\begin{aligned}
\left\{ \int_\Omega |\nabla w| \ dx \right\}^2 &\le 2\left\{ \int_\Omega |\nabla u| \, \psi^{1/2} \, dx \right\}^2 + 2\left\{ \int_\Omega u \left|\nabla \psi^{1/2}\right| dx \right\}^2 \\
&\le 2\left\{ \int_\Omega |\nabla u| \, \psi^{1/2} \, dx \right\}^2 + \frac{A^2}{2} \|u\|_1^2 \\
&\le 2 \int_{B(x_0,R)\cap\Omega} u \, dx \cdot \int_\Omega u^{-1} \, |\nabla u|^2 \, \psi \, dx + \frac{A^2}{2} \|u\|_1^2.
\end{aligned}$$

Hence (11.8) follows from (11.7) and $\|w\|_1 \le \|u\|_1$. To prove (11.9), we apply (11.7) for

$$w = (u-s)_+^{3/2} \psi^{1/2}.$$

First, we get

$$\begin{aligned}
\|w\|_2^2 &= \int_{\{u>s\}} (u-s)_+^3 \psi \, dx \int_{\{u>s\}} \left(\frac{1}{4}u^3 - s^3\right) \psi \, dx \\
&\ge \frac{1}{4} \int_\Omega u^3 \psi \, dx - \frac{5}{4} s^3 \, |\Omega| \, .
\end{aligned}$$

Next, we have

$$|\nabla w| \le \frac{3}{2}(u-s)_+^{1/2} \, |\nabla u| \, \psi^{1/2} + \frac{1}{2} A (u-s)_+^{3/2} \psi^{1/3}$$

and hence it holds that

$$\begin{aligned}
\|\nabla w\|_1^2 \le \frac{9}{2}&\left\{ \int_{\{u>s\}} (u-s)^{1/2} \, |\nabla u| \, \psi^{1/2} \, dx \right\}^2 \\
&+ \frac{A^2}{2} \left\{ \int_{\{u>s\}} (u-s)^{3/2} \psi^{1/3} \, dx \right\}^2.
\end{aligned}$$

Here, we have

$$\begin{aligned}
\left\{ \int_{\{u>s\}} (u-s)^{1/2} |\nabla u| \psi^{1/2} \, dx \right\}^2 &\le \left\{ \int_{\{u>s\}} u^{1/2} \, |\nabla u| \, \psi^{1/2} \, dx \right\}^2 \\
&\le \int_{B(x_0,R)\cap\{u>s\}} u \, dx \cdot \int_{\{u>s\}} |\nabla u|^2 \, \psi \, dx \\
&\le \frac{1}{\log s} \int_{B(x_0,R)\cap\Omega} (u \log u + e^{-1}) \, dx \cdot \int_\Omega |\nabla u|^2 \, \psi \, dx
\end{aligned}$$

and

$$\begin{aligned}\left\{\int_{\{u>s\}}(u-s)^{3/2}\psi^{1/3}\,dx\right\}^2 &\leq \left\{\int_{\{u>s\}} u\psi^{1/3}u^{1/2}\,dx\right\}^2 \\ &\leq \left\{\int_\Omega u^3\psi\,dx\right\}^{2/3}\left\|u\right\|_{L^1(B(x_0,R)\cap\Omega)}|\Omega|^{1/3} \\ &\leq \varepsilon\int_\Omega u^3\psi\,dx + \frac{1}{3}\Big(\frac{2}{3\varepsilon}\Big)^2|\Omega|\left\|u\right\|^3_{L^1(B(x_0,R)\cap\Omega)} \end{aligned} \tag{11.10}$$

for $\varepsilon > 0$. Therefore, writing $C_\varepsilon = \frac{4}{27}\varepsilon^{-2}$, we have

$$\begin{aligned}\|\nabla w\|_1^2 &\leq \frac{9}{2\log s}\int_{B(x_0,R)\cap\Omega}\left(u\log u + e^{-1}\right)dx\int_\Omega |\nabla u|^2\,\psi\,dx \\ &\quad + \frac{A^2}{2}\varepsilon\int_\Omega u^3\psi\,dx + \frac{A^2}{2}C_\varepsilon\,|\Omega|\,\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}.\end{aligned}$$

Finally, from (11.10) and $\psi^{1/2} \leq \psi^{1/3}$ it follows that

$$\left\|w\right\|_1^2 \leq \varepsilon\int_\Omega u^3\psi\,dx + C_\varepsilon|\Omega|\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}.$$

These relations, combined with (11.7), imply

$$\begin{aligned}&\left(\frac{1}{4} - K^2\Big(\frac{A^2}{2}+1\Big)\varepsilon\right)\int_\Omega u^3\psi\,dx \\ &\quad\leq \frac{9K^2}{\log s}\int_{B(x_0,R)\cap\Omega}\left(u\log u + e^{-1}\right)dx\cdot\int_\Omega |\nabla u|^2\,\psi\,dx \\ &\qquad + K^2C_\varepsilon\,|\Omega|\left(\frac{A^2}{2}+1\right)\left\|u\right\|^3_{L^1(B(x_0,R)\cap\Omega)} + \frac{5}{4}s^3\,|\Omega|\,.\end{aligned}$$

Taking $\varepsilon > 0$ as

$$\frac{1}{4} - K^2\Big(\frac{A^2}{2}+1\Big)\varepsilon = \frac{1}{8},$$

we obtain (11.9) and the proof is complete. □

We are discretizing in space the argument of Chapter 4. We assume $T_{\max} < +\infty$ and take the blowup set $\mathcal{S}$ of $u = u(\cdot, t)$ as $t \uparrow T_{\max}$. Generic positive constants are denoted as $C_1, C_2, \ldots$, successively. In case that their dependence on the parameter, say $\alpha, \beta, \ldots$, must be referred to explicitly, we write them as $C_\alpha, C_{\alpha,\beta}, \ldots$, and so forth.

In the previous argument, we showed that the first equation of (11.2), provided with the boundary condition, implies

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = 0,$$

and hence (1.11) follows from $u > 0$ in $\overline{\Omega} \times (0, T_{\max})$:

$$\|u(t)\|_1 = \|u_0\|_1 \equiv \lambda. \tag{11.11}$$

Then, from the L^1 estimate to the second equation of (11.2), we obtain

$$\sup_{0 \le t < T_{\max}} \left\{ \|v(t)\|_{W^{1,q}(\Omega)} + \|v(t)\|_p \right\} < +\infty \tag{11.12}$$

for $q \in [1, 2)$ and $p \in [1, \infty)$. Here, we remember that $W^{m,q}(\Omega)$ denotes the usual Sobolev space; the set of q-integrable functions up to the m-th order of differentiation.

The global version of the following lemma is shown in the proof of Theorem 4.1 by the maximal regularity theorem (Chapter 4). Here, we apply Moser's iteration scheme for the proof.

Lemma 11.2 *A point $x_0 \in \overline{\Omega}$ is a blowup point of u if and only if*

$$\limsup_{t \uparrow T_{\max}} \int_{B(x_0,R)\cap\Omega} (u \log u)(x,t)\,dx = +\infty$$

holds for any small $R > 0$.

Proof: The "if" part is obvious, because $x_0 \notin \mathcal{S}$ implies

$$\sup_{0 \le t < T_{\max}} \|u(t)\|_{L^\infty(B(x_0,R)\cap\Omega)} < +\infty$$

for $0 < R \ll 1$ by the definition. To prove the "only if" part, we suppose

$$\sup_{0 \le t < T_{\max}} \int_{B(x_0,R)\cap\Omega} (u \log u)(x,t)\,dx < +\infty \tag{11.13}$$

for some $0 < R \ll 1$. Then, localizing the estimates [14, 50, 110], we shall show $x_0 \notin \mathcal{S}$.

For this purpose, first, we take $R' \in (0, R)$ and $\psi = (\varphi_{x_0,R',R})^6$. Multiplying $u\psi$ to the first equation of (11.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi\,dx + \int_\Omega |\nabla u|^2\,\psi\,dx + \int_\Omega u\nabla u \cdot \nabla\psi\,dx$$
$$= \int_\Omega u\,(\nabla v \cdot \nabla u)\,\psi\,dx + \int_\Omega u^2 \nabla v \cdot \nabla\psi\,dx. \tag{11.14}$$

Here, the first term of the right-hand side of (11.14) is treated by the second equation of (11.2) as

$$
\begin{aligned}
\int_\Omega u(\nabla v \cdot \nabla u)\psi \, dx &= \frac{1}{2}\int_\Omega \left(\nabla u^2 \cdot \nabla v\right)\psi \, dx \\
&= -\frac{1}{2}\int_\Omega (u^2 \Delta v)\psi \, dx - \frac{1}{2}\int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx \\
&= \frac{1}{2}\int_\Omega u^3 \psi \, dx - \frac{1}{2}\int_\Omega u^2 v \psi \, dx - \frac{1}{2}\int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx \\
&\le \frac{1}{2}\int_\Omega u^3 \psi \, dx - \frac{1}{2}\int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx.
\end{aligned}
$$

Therefore, the right-hand side of (11.14) is estimated from above by

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega u^3 \psi \, dx + \frac{1}{2}\int_\Omega u^2 \nabla v \dot{\nabla} \psi \, dx \\
&\qquad = \frac{1}{2}\int_\Omega u^3 \psi \, dx - \frac{1}{2}\int_\Omega v \nabla u^2 \cdot \nabla \psi \, dx - \frac{1}{2}\int_\Omega u^2 v \Delta \psi \, dx,
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \psi \, dx + \int_\Omega |\nabla u|^2 \psi \, dx \\
&\qquad \le \frac{1}{2}\int_\Omega u^3 \psi \, dx - \int_\Omega u \nabla u \cdot \nabla \psi \, dx \\
&\qquad\quad - \frac{1}{2}\int_\Omega v \nabla u^2 \cdot \nabla \psi \, dx - \frac{1}{2}\int_\Omega u^2 v \Delta \psi \, dx. \qquad (11.15)
\end{aligned}
$$

Now, Young's inequality is applied to each term of the right-hand side except for the first one:

$$
\begin{aligned}
\left| \int_\Omega u \nabla u \cdot \nabla \psi \, dx \right| &\le A \int_\Omega u \psi^{1/3} \cdot |\nabla u| \, \psi^{1/2} \, dx \\
&\le A |\Omega|^{1/6} \left\{ \int_\Omega u^3 \psi \, dx \right\}^{1/3} \left\{ \int_\Omega |\nabla u|^2 \psi \, dx \right\}^{1/2} \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^2 \psi \, dx + \frac{1}{3}\int_\Omega u^3 \psi \, dx + \frac{4A^6 |\Omega|}{3},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\Big| \int_\Omega v \nabla u^2 \cdot \nabla \psi \, dx \Big| &\leq A \int_\Omega v \cdot u \psi^{1/3} \cdot |\nabla u| \, \psi^{1/2} \, dx \\
&\leq A \|v\|_6 \Big\{ \int_\Omega u^3 \psi \, dx \Big\}^{1/3} \Big\{ \int_\Omega |\nabla u|^2 \psi \, dx \Big\}^{1/2} \\
&\leq \frac{1}{4} \int_\Omega |\nabla u|^2 \, \psi \, dx + \frac{1}{3} \int_\Omega u^3 \psi \, dx + \frac{4A^6 \|v\|_6^6}{3},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\Big| \int_\Omega u^2 v \Delta \psi \, dx \Big| &\leq \frac{B}{2} \int_\Omega v \cdot u^2 \psi^{2/3} \, dx \\
&\leq \frac{B}{2} \|v\|_3 \Big\{ \int_\Omega u^3 \psi \, dx \Big\}^{2/3} \\
&\leq \frac{1}{3} \int_\Omega u^3 \psi \, dx + \frac{B^3 \|v\|_3^3}{6}.
\end{aligned}
$$

Therefore, from (11.12) we obtain

$$
\frac{d}{dt} \int_\Omega u^2 \psi \, dx + \int_\Omega |\nabla u|^2 \, \psi \, dx \leq \int_\Omega u^3 \psi \, dx + C_1.
$$

Here, we apply (11.9) and (11.13) to the right-hand side of this inequality. Making $s \gg 1$, we have

$$
\frac{d}{dt} \int_\Omega u^2 \psi \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, \psi \, dx \leq C_2.
$$

This implies

$$
\sup_{0 \leq t < T_{\max}} \int_\Omega u(\cdot, t)^2 \psi \, dx < +\infty. \tag{11.16}
$$

We proceed to the second step, multiplying $u^2 \psi$ to the first equation of (11.2). In this case, we obtain

$$
\begin{aligned}
\frac{1}{3} \frac{d}{dt} \int_\Omega u^3 \psi \, dx + 2 \int_\Omega u \, |\nabla u|^2 \, \psi \, dx + \int_\Omega u^2 \nabla u \cdot \nabla \psi \, dx \\
= 2 \int_\Omega u^2 \, (\nabla v \cdot \nabla u) \, \psi \, dx + \int_\Omega u^3 \nabla v \cdot \nabla \psi \, dx.
\end{aligned}
$$

For $w = u^{3/2}$, this means

$$\frac{1}{3}\frac{d}{dt}\int_\Omega w^2\psi\,dx + \frac{8}{9}\int_\Omega |\nabla w|^2\,\psi\,dx + \frac{2}{3}\int_\Omega w\nabla w\cdot\nabla\psi\,dx$$
$$= \frac{4}{3}\int_\Omega w\,(\nabla v\cdot\nabla w)\,\psi\,dx + \int_\Omega w^2\nabla v\cdot\nabla\psi\,dx. \quad (11.17)$$

Here, using the second equation of (11.2), we have

$$\begin{aligned}\int_\Omega w\big(\nabla v\cdot\nabla w\big)\psi\,dx &= \frac{1}{2}\int_\Omega \big(\nabla v\cdot\nabla w^2\big)\psi\,dx\\ &\le \frac{1}{2}\int_\Omega uw^2\psi\,dx - \frac{1}{2}\int_\Omega w^2\nabla v\cdot\nabla\psi\,dx\\ &= \frac{1}{2}\int_\Omega w^{8/3}\psi\,dx - \frac{1}{2}\int_\Omega w^2\nabla v\cdot\nabla\psi\,dx.\end{aligned}$$

Therefore, the right-hand side of (11.17) is estimated from above by

$$\begin{aligned}&\frac{2}{3}\int_\Omega w^{8/3}\psi\,dx + \frac{1}{3}\int_\Omega w^2\nabla v\cdot\nabla\psi\,dx\\ &\quad\le \frac{2}{3}\Big\{\int_\Omega w^3\psi\,dx\Big\}^{8/9}|\Omega|^{1/9} - \frac{1}{3}\int_\Omega v\nabla w^2\cdot\nabla\psi\,dx\\ &\qquad - \frac{1}{3}\int_\Omega w^2 v\Delta\psi\,dx\\ &\quad\le \frac{2}{3}\int_\Omega w^3\psi\,dx - \frac{1}{3}\int_\Omega v\nabla w^2\cdot\nabla\psi\,dx\\ &\qquad - \frac{1}{3}\int_\Omega w^2 v\Delta\psi\,dx + \frac{2}{3}\cdot\Big(\frac{8}{9}\Big)^8\cdot\frac{|\Omega|}{9}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega w^2\psi\,dx + \frac{8}{9}\int_\Omega |\nabla w|^2\,\psi\,dx + \frac{2}{3}\int_\Omega w\nabla w\cdot\nabla\psi\,dx\\ &\quad\le \frac{2}{3}\int_\Omega w^3\psi\,dx - \frac{1}{3}\int_\Omega v\nabla w^2\cdot\nabla\psi\,dx - \frac{1}{3}\int_\Omega w^2 v\Delta\psi\,dx\\ &\qquad + \frac{2}{3}\cdot\Big(\frac{8}{9}\Big)^8\cdot\frac{|\Omega|}{9},\end{aligned}$$

which has a similar form of (11.15).

Here, inequality (11.16) means

$$\sup_{0\le t<T_{\max}}\int_\Omega w^{4/3}(x,t)\psi\,dx < +\infty$$

and hence we have

$$\sup_{0 \le t < T_{\max}} \int_{B(x_0, R') \cap \Omega} (w \log w)(x,t)\, dx < +\infty$$

and

$$\sup_{0 \le t < T_{\max}} \left\| w(t) \right\|_{L^1(B(x_0, R') \cap \Omega)} < +\infty.$$

Therefore, taking $R'' \in (0, R')$, we can repeat the above argument, with u, R, and $\psi = \left(\varphi_{x_0, R', R}\right)^6$, replaced by w, R', and $\psi_1 = \left(\varphi_{x_0, R'', R'}\right)^6$, respectively. Then, similarly to (11.16) it follows that

$$\sup_{0 \le t < T_{\max}} \left\| w(t) \right\|^{2/3}_{L^2(B(x_0, r) \cap \Omega)} = \sup_{0 \le t < T_{\max}} \left\| u(t) \right\|_{L^3(B(x_0, r) \cap \Omega)} < +\infty$$

for any $r \in (0, R)$, because $R' \in (0, R)$ and $R'' \in (0, R')$ are arbitrary.

From the second equation of (11.2) this implies

$$\sup_{0 \le t < T_{\max}} \left\| v(t) \right\|_{W^{2,3}(B(x_0, r') \cap \Omega)} < +\infty$$

for $r' \in (0, r)$ by the local L^3 estimate, and hence it holds that

$$\sup_{0 \le t < T_{\max}} \left\| v(t) \right\|_{C^1(B(x_0, r) \cap \Omega)} < +\infty \tag{11.18}$$

for any $r \in (0, R)$. On the other hand, repeating the above argument once more, we obtain

$$\sup_{0 \le t < T_{\max}} \left\| u(t) \right\|_{L^4(B(x_0, r) \cap \Omega)} < +\infty. \tag{11.19}$$

These estimates, (11.18) and (11.19), are sufficient to set up the iteration scheme by putting $\psi = (\varphi_{x_0, r', r})^6$ for $r' \in (0, r)$. This $\psi(x)$ is different from the previous one, as the support is reduced, but there will arise no confusion.

Thus, for $p \ge 1$, we multiply $u^p \psi^{p+1}$ by the first equation of (11.2) and obtain

$$\begin{aligned} &\frac{d}{dt} \frac{1}{p+1} \int_\Omega (u\psi)^{p+1}\, dx \\ &\quad = -\int_\Omega \nabla(u^p \psi^{p+1}) \cdot \nabla u\, dx + \int_\Omega u \nabla(u^p \psi^{p+1}) \cdot \nabla v\, dx \\ &\quad = -\mathrm{I} + \mathrm{II}. \end{aligned}$$

Here, we have

$$
\begin{aligned}
\mathrm{I} &= \int_\Omega \left(p u^{p-1} \psi^{p+1} \nabla u + u^p \nabla \psi^{p+1} \right) \cdot \nabla u \, dx \\
&= \frac{4p}{(p+1)^2} \int_\Omega \left| \nabla u^{\frac{p+1}{2}} \right|^2 \psi^{p+1} \, dx + \frac{1}{p+1} \int_\Omega \nabla \psi^{p+1} \cdot \nabla u^{p+1} \, dx \\
&= \frac{4p}{(p+1)^2} \int_\Omega \left| \nabla u^{\frac{p+1}{2}} \right|^2 \psi^{p+1} \, dx + \frac{4}{p+1} \int_\Omega \psi^{\frac{p+1}{2}} \nabla u^{\frac{p+1}{2}} \\
&\quad \cdot u^{\frac{p+1}{2}} \nabla \psi^{\frac{p+1}{2}} \, dx = \left\{ \frac{4p}{(p+1)^2} - \frac{2}{p+1} \right\} \int_\Omega \left| \nabla u^{\frac{p+1}{2}} \right|^2 \psi^{p+1} \, dx \\
&\quad + \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi)^{\frac{p+1}{2}} \right|^2 dx - \frac{2}{p+1} \int_\Omega u^{p+1} \left| \nabla \psi^{\frac{p+1}{2}} \right|^2 dx
\end{aligned}
$$

and the right-hand side is estimated from below by

$$
\begin{aligned}
&\frac{2}{p+1} \int_\Omega \left| \nabla (u\psi)^{\frac{p+1}{2}} \right|^2 dx - \frac{p+1}{2} \int_\Omega u^{p+1} \psi^{p-1} \left| \nabla \psi \right|^2 dx \\
&\quad \geq \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi)^{\frac{p+1}{2}} \right|^2 dx - \frac{A^2(p+1)}{2} \int_\Omega (u\psi)^{p+\frac{2}{3}} u^{\frac{1}{3}} \, dx \\
&\quad \geq \frac{2}{p+1} \int_\Omega \left| \nabla (u\psi)^{\frac{p+1}{2}} \right|^2 dx - \frac{A^2(p+1)}{2} \left\| u_0 \right\|_1^{\frac{1}{3}} \\
&\quad \cdot \left\{ \int_\Omega (u\psi)^{1+\frac{3}{2}p} \, dx \right\}^{\frac{2}{3}}.
\end{aligned}
$$

On the other hand, estimate (11.18) means

$$
L \equiv \sup_{0 \leq t < T_{\max}} \left\| \nabla v(t) \right\|_{L^\infty(B(x_0,r)\cap\Omega)} < +\infty
$$

and hence we obtain

$$
\begin{aligned}
\mathrm{II} &\le L \int_\Omega \left| u \nabla \left(u^p \psi^{p+1} \right) \right| dx \\
&= L \int_\Omega \left| \frac{p}{p+1} \nabla (u\psi)^{p+1} + u^{p+1} \psi^p \nabla \psi \right| dx \\
&\le \frac{Lp}{p+1} \int_\Omega \left| \nabla (u\psi)^{p+1} \right| dx + L(p+1) \int_\Omega u^{p+1} \psi^p \left| \nabla \psi \right| dx \\
&\le \frac{2pL}{p+1} \int_\Omega (u\psi)^{\frac{p+1}{2}} \left| \nabla (u\psi)^{\frac{p+1}{2}} \right| dx \\
&\quad + LA(p+1) \int_\Omega (u\psi)^{p+\frac{5}{6}} u^{\frac{1}{6}} \, dx \\
&\le \frac{1}{p+1} \int_\Omega \left| \nabla (u\psi)^{\frac{p+1}{2}} \right|^2 dx + 4L^2(p+1) \int_\Omega (u\psi)^{p+1} \, dx \\
&\quad + LA(p+1) \left\| u_0 \right\|_1^{\frac{1}{6}} \left\{ \int_\Omega (u\psi)^{1+\frac{6}{5}p} \, dx \right\}^{\frac{5}{5}}.
\end{aligned}
$$

These relations are summarized, for $u_1 = u\psi$, as

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega u_1^{p+1} \, dx \le & - \int_\Omega \left| \nabla u_1^{\frac{p+1}{2}} \right|^2 dx + C_3 (p+1)^2 \int_\Omega u_1^{p+1} \, dx \\
& + C_3 (p+1)^2 \left\{ \int_\Omega u_1^{\frac{3p+2}{2}} \, dx \right\}^{\frac{2}{3}} \\
& + C_3 (p+1)^2 \left\{ \int_\Omega u_1^{\frac{6p+5}{5}} \, dx \right\}^{\frac{5}{6}}
\end{aligned}
\tag{11.20}
$$

with a constant $C_3 > 0$ independent of $p \ge 1$. Therefore, now we can argue as in Alikakos [2].

For this purpose, we make use of Gagliardo–Nirenberg's inequality in the form of

$$
\|w\|_{L^q(\Omega)} \le K \left(\|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right)^{\frac{1-(1/q)}{2}} \|w\|_{L^1(\Omega)}^{1/q}, \tag{11.21}
$$

where $K \ge 1$ is a constant independent of $q \in [1, q_0]$ if $q_0 > 1$ is prescribed in advance. This K is again different from the previous one, but there will arise no confusion.

First, we apply (11.21) for $w = u_1^{(p+1)/2}$ and $q = \frac{3p+2}{p+1} \in [\frac{5}{2}, 3)$:

$$\begin{aligned} C_3(p+1)^2\Big\{\int_\Omega u_1^{\frac{3p+2}{2}}\,dx\Big\}^{\frac{2}{3}} &= C_3(p+1)^2\|w\|_q^{\frac{2q}{3}} \\ &\le C_3(p+1)^2K^{\frac{2q}{3}}\left(\|\nabla w\|_2^2 + \|w\|_2^2\right)^{(q-1)/3} \cdot \|w\|_1^{\frac{2}{3}} \\ &= C_3(p+1)^2K^{\frac{6p+4}{3p+3}}\Big\{\int_\Omega\Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx\Big\}^{\frac{2p+1}{3p+3}} \\ &\quad \cdot \Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx\Big\}^{\frac{2}{3}}. \end{aligned}$$

By means of $\frac{6p+4}{3p+3} < 2$ and $\frac{2p+1}{3p+3} < \frac{2}{3}$, the right-hand side is estimated from above by

$$\begin{aligned} &C_3K^2(p+1)^2\Big\{\frac{1}{4}\Big(\int_\Omega\Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx + 1\Big)\Big\}^{\frac{2}{3}} \\ &\cdot 4^{\frac{2}{3}}\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx\Big\}^{\frac{2}{3}} \le \frac{1}{6}\Big\{\int_\Omega\Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx + 1\Big\} \\ &\qquad + \frac{16}{3}\cdot C_3^3K^6\cdot(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\}^2. \end{aligned}$$

Next, we apply (11.21) for $w = u_1^{(p+1)/2}$ and $q = \frac{12p+10}{5p+5} \in \big[\frac{22}{10}, \frac{12}{5}\big)$. This time, we have

$$\begin{aligned} C_3(p+1)^2\Big\{\int_\Omega u_1^{\frac{6p+5}{5}}\,dx\Big\}^{5/6} &= C_3(p+1)^2\|w\|_q^{5q/6} \\ &\le C_3(p+1)^2K^{5q/6}\left(\|\nabla w\|_2^2 + \|w\|_2^2\right)^{5(q-1)/12} \cdot \|w\|_1^{5/6} \\ &= C_3K^{5q/6}(p+1)^2\Big\{\int_\Omega\Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx\Big\}^{\frac{7p+5}{12p+12}} \\ &\quad \cdot \Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx\Big\}^{5/6} \end{aligned}$$

and the right-hand side is estimated from above by

$$
\begin{aligned}
& C_3K^2(p+1)^2\left\{\frac{2}{7}\Big(\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx + 1\Big)\right\}^{\frac{7}{12}} \\
&\quad \cdot \Big\{\Big(\frac{7}{2}\Big)^{7/10}\int_\Omega u_1^{\frac{p+1}{2}}\,dx\Big\}^{5/6} \le \frac{1}{6}\int_\Omega \Big(\Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 + u_1^{p+1}\Big)\,dx \\
&\quad + \frac{1}{6} + \frac{5}{12}\cdot\Big(\frac{7}{2}\Big)^{7/5} C_3^{12/5} K^{24/5}(p+1)^{24/5}\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\}^2 \\
&\quad \le \frac{1}{6}\Big\{\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + u_1^{p+1}\,dx + 1\Big\} \\
&\quad + \frac{5}{12}\cdot\Big(\frac{7}{2}\Big)^{7/5} C_3^{12/5} K^{24/5}(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\}^2.
\end{aligned}
$$

Finally, we apply (11.21) for $w = u_1^{(p+1)/2}$ and $q = 2$ and obtain

$$
\begin{aligned}
& C_3(p+1)^2\int_\Omega u_1^{p+1}\,dx \le C_3K^2(p+1)^2 \\
&\quad \cdot \left\{\frac{1}{3}\Big(\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx\Big)\right\}^{\frac{1}{2}} \cdot 3^{1/2}\int_\Omega u_1^{\frac{p+1}{2}}\,dx \\
&\quad \le \frac{1}{6}\Big\{\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx + 1\Big\} \\
&\quad + \frac{3^{1/2}}{2}\cdot C_3^2K^4(p+1)^4\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\} \\
&\quad \le \frac{1}{6}\Big\{\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx + \int_\Omega u_1^{p+1}\,dx + 1\Big\}^2 \\
&\quad + \frac{3^{1/2}}{2}\cdot C_3^2(p+1)^6K^4\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\}^2.
\end{aligned}
$$

In this way, from (11.20) we can deduce

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega u_1^{p+1}\,dx + \frac{1}{2}\int_\Omega \Big|\nabla u_1^{\frac{p+1}{2}}\Big|^2 dx \\
&\qquad \le \frac{1}{2}\int_\Omega u_1^{p+1}\,dx + C_4(p+1)^6\Big\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\Big\}^2.
\end{aligned}
$$

However, again by (11.21) for $q = 2$, we have

$$
\begin{aligned}
\|w\|_2 &\le K^2\Big(\|\nabla w\|_2^2 + \|w\|_2^2\Big)^{\frac{1}{2}}\|w\|_1 \\
&\le \frac{1}{4}\Big(\|\nabla w\|_2^2 + \|w\|_2^2\Big) + K^4\|w\|_1^2
\end{aligned}
$$

and hence we obtain

$$\left\|u_1^{\frac{p+1}{2}}\right\|_2^2 \le \frac{1}{3}\left\|\nabla u_1^{\frac{p+1}{2}}\right\|_2^2 + \frac{4K^4}{3}\left\|u_1^{\frac{p+1}{2}}\right\|_1^2.$$

Thus, we have

$$\frac{d}{dt}\int_\Omega u_1^{p+1}\,dx + \frac{1}{3}\int_\Omega \left|\nabla u_1^{\frac{p+1}{2}}\right|^2 dx \\ \le C_4(p+1)^6\left\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\right\}^2 + \frac{2}{3}K^2\left\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx\right\}^2,$$

and therefore it holds that

$$\frac{d}{dt}\int_\Omega u_1^{p+1}\,dx + \int_\Omega u_1^{p+1}\,dx \le C_5(p+1)^6\left\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\right\}^2$$

for $t \in [0, T_{\max})$.

From this differential inequality, we can conclude

$$\sup_{0\le t<T_{\max}}\left\{\int_\Omega u_1^{p+1}\,dx + 1\right\} \\ \le C_5 \max\left\{(p+1)^6 \sup_{0\le t<T_{\max}}\left\{\int_\Omega u_1^{\frac{p+1}{2}}\,dx + 1\right\}^2,\ \|u_0\|_\infty^{p+1}|\Omega| + 1\right\}$$

for any $p \ge 1$. Therefore,

$$\Phi_k = \sup_{0\le t<T_{\max}}\int_\Omega u_1^{2^k}\,dx + 1$$

satisfies

$$\Phi_{k+1} \le C_5 \max\left\{2^{6(k+1)}\Phi_k^2,\ (|\Omega|+1)\left(\|u_0\|_\infty + 1\right)^{2^{k+1}}\right\} \\ \le C_5 2^{6(k+1)} \max\left\{\Phi_k^2,\ \left(\|u_0\|_\infty + 1\right)^{2^{k+1}}\right\}$$

for $k = 1, 2, \ldots$. This implies

$$\Phi_{k+1} \le C_5^{2^{k-1}-1}\cdot 2^{\sum_{\ell=2}^{k} 6(\ell+1)2^{k-\ell}}\cdot \max\left\{\Phi_2^{2^{k-1}},\ d^{2^{k+1}}\right\}$$

for $k = 2, 3, \ldots$, where $d = \|u_0\|_\infty + 1$, and hence we obtain

$$\sup_{0\le t<T_{\max}}\left\{\int_\Omega u_1^{2^{k+1}}\,dx\right\}^{\frac{1}{2^{k+1}}} \le \Phi_{k+1}^{\frac{1}{2^{k+1}}} \\ \le C_5^{\frac{2^{k-1}-1}{2^{k+1}}}\cdot 2^{6\sum_{j=1}^{\infty} j2^{-j}}\cdot \max\left\{\Phi_2^{1/4},\ d\right\}.$$

Making $k \to +\infty$, we have

$$\sup_{0 \le t < T_{\max}} \left\| u_1(t) \right\|_\infty \le C_5 \max \left\{ \left(\sup_{0 \le t < T_{\max}} \left\| u_1(t) \right\|_4^4 + 1 \right)^{1/4},\ d \right\}.$$

Using (11.19), we obtain

$$\sup_{0 \le t < T_{\max}} \left\| u_1(t) \right\|_\infty = \sup_{0 \le t < T_{max}} \left\| u(t)\psi \right\|_\infty < +\infty,$$

and hence it holds that

$$\limsup_{t \uparrow T_{\max}} \left\| u(t) \right\|_{L^\infty(B(x_0, r') \cap \Omega)} < +\infty.$$

This means $x_0 \notin \mathcal{S}$ and the proof is complete. □

Localizing the argument of [73], now we show the following lemma.

Lemma 11.3 *It holds that*

$$\frac{d}{dt} \int_\Omega (u \log u) \psi \, dx + \frac{1}{4} \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx \le 2 \int_\Omega u^2 \psi \, dx + C_6 \quad (11.22)$$

for $t \in (0, T_{\max})$.

Proof: From the first equation of (11.2), we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega (u \log u)\, \psi \, dx &= \int_\Omega u_t (\log u + 1) \psi \, dx \\ &= - \int_\Omega \nabla u \cdot \nabla \left((\log u + 1)\, \psi \right) \, dx \\ &\quad + \int_\Omega u \nabla v \cdot \nabla \left((\log u + 1)\, \psi \right) \, dx \\ &= -\mathrm{I} + \mathrm{II}. \end{aligned}$$

Here, the second equation of (11.2) applies and it holds that

$$\begin{aligned} \mathrm{II} &= \int_\Omega \psi \nabla v \cdot \nabla u \, dx + \int_\Omega u \left(\log u + 1 \right) \nabla v \cdot \nabla \psi \, dx \\ &= - \int_\Omega u \nabla \cdot \left(\psi \nabla v \right) \, dx + \int_\Omega u \left(\log u + 1 \right) \nabla v \cdot \nabla \psi \, dx \\ &= \int_\Omega u \psi (u - v) \, dx + \int_\Omega u \log u \nabla v \cdot \nabla \psi \, dx. \end{aligned}$$

We also have

$$I = \int_\Omega u^{-1}|\nabla u|^2 \psi \, dx + \int_\Omega (\log u + 1) \, \nabla u \cdot \nabla \psi \, dx$$

and hence

$$\begin{aligned} \frac{d}{dt} & \int_\Omega (u \log u)\psi \, dx + \int_\Omega u^{-1} \, |\nabla u|^2 \, \psi \, dx + \int_\Omega uv\psi \, dx \\ & = \int_\Omega u^2 \psi \, dx - \int_\Omega (\log u + 1) \, \nabla u \cdot \nabla \psi \, dx \\ & \quad + \int_\Omega (u \log u) \nabla v \cdot \nabla \psi \, dx. \end{aligned} \tag{11.23}$$

Now, we use an elementary inequality valid for $\alpha > 0$ and $0 < \beta < 2$:

$$\big(|\log u| + 1\big)^\alpha u^\beta \le u^2 + C_{\alpha,\beta} \qquad (u > 0).$$

Then, the second term of the right-hand side of (11.23) is estimated from above by

$$\begin{aligned} & \left| \int_\Omega (\log u + 1) \nabla u \cdot \nabla \psi \, dx \right| \le A \int_\Omega \big(|\log u| + 1\big) u^{1/2} \psi^{1/3} \\ & \quad \cdot u^{-1/2} \, |\nabla u| \, \psi^{1/2} \, dx \le A \, |\Omega|^{1/6} \left\{ \int_\Omega \big(|\log u| + 1\big)^3 u^{3/2} \psi \, dx \right\}^{1/3} \\ & \quad \cdot \left\{ \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx \right\}^{1/2} \le A \, |\Omega|^{1/6} \, 2^{1/2} \\ & \quad \cdot \left\{ \int_\Omega u^2 \psi \, dx + C_{3,3/2} \, |\Omega| \right\}^{1/3} \cdot \left\{ \frac{1}{2} \int_\Omega u^{-1} \, |\nabla u|^2 \, \psi \, dx \right\}^{1/2} \\ & \quad \le \frac{1}{4} \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx + \frac{1}{3} \int_\Omega u^2 \psi \, dx + \frac{4A^6 \, |\Omega|}{3} + \frac{C_{3,3/2} \, |\Omega|}{3}. \end{aligned}$$

The third term of the right-hand side of (11.23) is equal to

$$\begin{aligned} - \int_\Omega v \nabla \cdot (u \log u \nabla \psi) \, dx = & - \int_\Omega v (\log u + 1) \nabla u \cdot \nabla \psi \, dx \\ & - \int_\Omega (vu \log u) \, \Delta \psi \, dx, \end{aligned}$$

where each term of the right-hand side is estimated from above as follows:

$$\begin{aligned}
&\left| \int_\Omega v(\log u + 1)\nabla u \cdot \nabla\psi \, dx \right| \\
&\quad \leq A \int_\Omega v \cdot u^{1/2} \left(|\log u| + 1\right) \psi^{1/3} \cdot u^{-1/2} \, |\nabla u| \, \psi^{1/2} \, dx \leq A \|v\|_6 2^{1/2} \\
&\qquad \cdot \left\{ \int_\Omega u^{3/2} \left(|\log u| + 1\right)^3 \psi \, dx \right\}^{1/3} \cdot \left\{ \frac{1}{2} \int_\Omega u^{-1} \, |\nabla u|^2 \, \psi \, dx \right\}^{1/2} \\
&\qquad \leq \frac{1}{4} \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx + \frac{1}{3} \int_\Omega u^2 \psi \, dx + \frac{4A^6 \|v\|_6}{3} + \frac{C_{3,3/2}|\Omega|}{3},
\end{aligned}$$

$$\begin{aligned}
\left| \int_\Omega (vu \log u) \Delta\psi \, dx \right| &\leq A \int_\Omega v \, |u \log u| \, \psi^{2/3} \, dx \\
&\leq A \|v\|_3 2^{\frac{2}{3}} \cdot \left\{ \frac{1}{2} \int_\Omega |u \log u|^{3/2} \, \psi \, dx \right\}^{2/3} \\
&\leq \frac{1}{3} \int_\Omega u^2 \psi \, dx + \frac{4A^3 \|v\|_3^3}{3} + \frac{C_{3/2,3/2} \, |\Omega|}{3}.
\end{aligned}$$

Inequality (11.22) follows from (11.12) and the proof is complete. □

We are ready to prove the key lemma.

Lemma 11.4 *The blowup set $\mathcal{S}$ of u is finite.*

Proof: First, we show that there is $\varepsilon_0 > 0$ such that any $x_0 \in \mathcal{S}$ and $0 < R \ll 1$ admit the estimate

$$\limsup_{t \uparrow T_{\max}} \int_{B(x_0, R) \cap \Omega} u(\cdot, t) \, dx \geq \varepsilon_0. \tag{11.24}$$

For this purpose, we take $R' \in (0, R)$ and $\psi = \left(\varphi_{x_0, R', R}\right)^6$. Then, from (11.22) and (11.8), it follows that

$$\frac{d}{dt} \int_\Omega (u \log u) \psi \, dx + \frac{1}{4} \left(1 - 16K^2 \int_{B(x_0, R) \cap \Omega} u \, dx \right) \cdot \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx \leq C_7.$$

Therefore, if

$$\limsup_{t \uparrow T_{\max}} \int_{B(x_0, R) \cap \Omega} u(\cdot, t) \, dx < \varepsilon_0 \equiv \frac{1}{16K^2},$$

then it holds that

$$\limsup_{t\uparrow T_{\max}} \int_{B(x_0,R')\cap\Omega} (u\log u)(\cdot,t)\,dx \le \limsup_{t\uparrow T_{\max}} \int_{\Omega} (u\log u)(\cdot,t)\psi\,dx < +\infty.$$

This implies $x_0 \notin \mathcal{S}$ by Lemma 11.2, a contradiction. Thus, we have proven (11.24) for each $x_0 \in \mathcal{S}$ and $0 < R \ll 1$.

Now, we apply (5.11), or equivalently,

$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\psi\,dx\right| \le B\left\|u_0\right\|_1 + \frac{1}{2}\left\|\rho_\psi\right\|_{L^\infty(\Omega\times\Omega)}\left\|u_0\right\|_1^2 \tag{11.25}$$

with

$$\rho_\psi(x,y) = \nabla\psi(x)\cdot\nabla_x G(x,y) + \nabla\psi(y)\cdot\nabla_y G(x,y).$$

Actually, it is a consequence of the method of symmetrization, because the first equation of (11.2) implies

$$\frac{d}{dt}\int_\Omega u\psi\,dx = \int_\Omega u_t\psi\,dx = \int_\Omega u\Delta\psi\,dx + \int_\Omega u\nabla v\cdot\nabla\psi\,dx$$

by $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega} = 0$.

Here, it is obvious that

$$\left|\int_\Omega u(\cdot,t)\Delta\psi\,dx\right| \le B\left\|u_0\right\|_1,$$

while we have

$$\int_\Omega u\nabla v\cdot\nabla\psi = \frac{1}{2}\int_\Omega\int_\Omega \rho_\psi(x,y)u(x,t)u(y,t)\,dxdy$$

with

$$\rho_\psi \in L^\infty(\Omega\times\Omega).$$

Thus, it holds that

$$\left|\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)\,dxdy\right| \le \left\|\rho\right\|_{L^\infty(\Omega\times\Omega)}\left\|u_0\right\|_1^2$$

and inequality (11.25) follows.

This fact confirms that the value

$$\lim_{t\uparrow T_{\max}}\int_\Omega u(\cdot,t)\psi dt = \int_\Omega u_0(x)\psi\,dx + \int_0^{T_{\max}}\Big(\frac{d}{dt}\int_\Omega u(\cdot,t)\psi\,dx\Big)dt$$

exists for $\psi = (\varphi_{x_0,R',R})^6$. Since $0 < R \ll 1$ is arbitrary in (11.24), it is improved as

$$\liminf_{t\uparrow T_{\max}} \int_{B(x_0,R)\cap\Omega} u(\cdot,t)\,dx \geq \lim_{t\uparrow T_{\max}} \int_\Omega u(\cdot,t)\psi\,dx$$
$$\geq \limsup_{t\uparrow T_{\max}} \int_{B(x_0,R')\cap\Omega} u(\cdot,t)\,dx \geq \varepsilon_0.$$

This inequality holds for any $x_0 \in \mathcal{S}$ and $0 < R \ll 1$, and therefore from (11.11) it follows that

$$\sharp\mathcal{S} \leq \|u_0\|_1/\varepsilon_0 < +\infty.$$

This indicates the finiteness of blowup points and the proof is complete. □

After the finiteness of blowup points is proven, formation of the chemotactic collapse (6.5) is a consequence of the localization of the argument of Chapter 4. Here, we apply Brezis–Merle's type of inequality instead of the Trudinger–Moser inequality.

In fact, putting

$$\mathcal{S}_\varepsilon = \cup_{x_0\in\mathcal{S}} B(x_0,\varepsilon)$$

for $0 < \varepsilon \ll 1$, we obtain

$$\sup_{0\leq t<t_{\max}} \|u(t)\|_{L^\infty(\Omega\setminus\mathcal{S}_\varepsilon)} < +\infty$$

and therefore the relation

$$\sup_{0\leq t<T_{\max}} \|\nabla v(t)\|_{L^\infty(\Omega\setminus\mathcal{S}_{2\varepsilon})} < +\infty$$

follows from the second equation of (11.2) and the elliptic regularity. Then, the first and the second equations of (11.2) assure

$$\|u\|_{C^{2+\theta,1+\theta/2}(\Omega\setminus\mathcal{S}_{3\varepsilon}\times[0,T_{\max}))} < +\infty \tag{11.26}$$

and

$$\|v\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega}\setminus\mathcal{S}_{4\varepsilon}\times[0,T_{\max})\right)} < +\infty \tag{11.27}$$

for $\theta \in (0,1)$ by the parabolic and elliptic regularities [58, 85]. In particular, it holds that

$$\sup_{0\leq t<T_{\max}} \|u_t(t)\|_{C\left(\overline{\Omega}\setminus\mathcal{S}_{3\varepsilon}\right)} < +\infty$$

and

$$f(x) = u(x,0) + \int_0^{T_{\max}} u_t(x,t)dt$$
$$= \lim_{t\uparrow T_{\max}} u(x,t) \geq 0 \tag{11.28}$$

exists for any $x \in \overline{\Omega} \setminus \mathcal{S}$. Convergence (11.28) is locally uniform on $\overline{\Omega} \setminus \mathcal{S}$, and relation (11.6) follows from (11.11) and Fatou's lemma.

The family

$$\left\{u(x,t)\,dx \;\middle|\; 0 \leq t < T_{\max}\right\} \subset \mathcal{M}(\overline{\Omega})$$

is bounded, and therefore is sequentially precompact $*$-weakly as $t \uparrow T_{\max}$. Now, we shall show the following lemma.

Lemma 11.5 *It holds that*

$$\liminf_{t\uparrow T_{\max}} \int_{B(x_0,R)\cap\Omega} u(x,t)\,dx \geq m_*(x_0) \tag{11.29}$$

for each $x_0 \in \mathcal{S}$ *and* $0 < R \ll 1$*, where*

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega)\,, \\ 4\pi & (x_0 \in \partial\Omega)\,. \end{cases}$$

This lemma implies Theorem 11.1 as follows. First, any sequence $t_k \uparrow T_{\max}$ admits a subsequence $\left\{t_k'\right\}$ and a measure $\mu(dx) \in \mathcal{M}(\overline{\Omega})$ such that

$$w^* - \lim_{k\to\infty} u(x,t_k')\,dx = \mu(dx).$$

Since $\mu(dx) - f(x)\,dx \in \mathcal{M}(\overline{\Omega})$ has the support on the finite set $\mathcal{S}$ and (11.29) holds, we have $m' : \mathcal{S} \to [4\pi, +\infty)$ satisfying $m\Big|_{\mathcal{S}\cap\Omega} \geq 8\pi$ and

$$\mu(dx) = \sum_{x_0\in\mathcal{S}} m'(x_0)\delta_{x_0}(dx) + f(x)\,dx.$$

However, from the proof of Lemma 11.4 we have the existence of

$$\lim_{t\uparrow T_{\max}} \int_\Omega u(x,t)\varphi(x)\,dx$$

for any $\varphi \in C^2(\overline{\Omega})$ in $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$, and the value $m'(x_0)$ is independent of the choice of $\{t_k\}$ or $\left\{t'_k\right\}$. This implies (6.5).

To prove Lemma 11.5 we take $x_0 \in \mathcal{S}$ and $0 < R' < R \ll 1$. Then, letting $\varphi = \varphi_{x_0,R',R}$, we introduce the localized Lyapunov function

$$W_\varphi(t) = \int_\Omega \left\{ u \log u - uv + \frac{1}{2}\left(|\nabla v|^2 + v^2\right) \right\} \varphi \, dx$$

and show the following lemma.

Lemma 11.6 *It holds that*

$$\frac{d}{dt} W_\varphi(t) + \int_\Omega u \left| \nabla(\log u - v) \right|^2 \varphi \, dx = \frac{d}{dt} \int_\Omega u\varphi \, dx + R_1(u, v, \varphi) \tag{11.30}$$

with

$$\begin{aligned} R_1(u, v, \varphi) = & \int_\Omega \left[(1 - v)\nabla u - (u \log u - uv + v_t)\nabla v \right] \cdot \nabla\varphi \, dx \\ & + \int_\Omega (u \log u)\Delta\varphi \, dx. \end{aligned}$$

Proof: Multiplying $(\log u - v)\varphi$ by the first equation of (11.2), we have

$$\begin{aligned} \int_\Omega u_t (\log u - v)\varphi \, dx &= \int_\Omega \left[\nabla \cdot (\nabla u - u\nabla v) \right] (\log u - v)\varphi \, dx \\ &= -\int_\Omega u |\nabla(\log u - v)|^2 \varphi \, dx \\ &\quad - \int_\Omega (\log u - v)(\nabla u - u\nabla v) \cdot \nabla\varphi \, dx. \end{aligned} \tag{11.31}$$

Here, it holds that

$$\begin{aligned} \int_\Omega u_t (\log u - v)\varphi \, dx = & \frac{d}{dt} \int_\Omega (u \log u - uv)\varphi \, dx \\ & - \frac{d}{dt} \int_\Omega u\varphi \, dx + \int_\Omega u v_t \varphi \, dx \end{aligned} \tag{11.32}$$

and

$$\begin{aligned} \int_\Omega (\log u)\nabla u \cdot \nabla\varphi \, dx &= -\int_\Omega u \nabla \cdot (\log u \nabla\varphi) \, dx \\ &= -\int_\Omega \left\{ (u \log u)\Delta\varphi + \nabla u \cdot \nabla\varphi \right\} \, dx \end{aligned} \tag{11.33}$$

by $\frac{\partial \varphi}{\partial \nu}\big|_{\partial\Omega} = 0$.

Using the second equation of (11.2), we have

$$\int_\Omega u v_t \varphi\, dx = \int_\Omega (-\Delta v + v) v_t \varphi\, dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right)\varphi\, dx + \int_\Omega v_t \nabla v \cdot \nabla\varphi\, dx.$$

This, together with (11.31)–(11.33), implies

$$\frac{d}{dt}W_\varphi + \int_\Omega u\,|\nabla(\log u - v)|^2\,\varphi\, dx$$
$$= \frac{d}{dt}\int_\Omega u\varphi\, dx + \int_\Omega (u\log u)\Delta\varphi\, dx$$
$$+ \int_\Omega \left[(1-v)\nabla u - (u\log u - uv + v_t)\nabla v\right]\cdot\nabla\varphi\, dx$$

and the proof is complete. □

Now, we show the following lemma.

Lemma 11.7 *If $x_0 \in \mathcal{S}$ and $\varphi = \varphi_{x_0,R',R}$ for $0 < R' < R \ll 1$, then*

$$W^* \equiv \sup_{0\le t<T_{\max}} W_\varphi(t) < +\infty \tag{11.34}$$

and

$$\limsup_{t\uparrow T_{\max}} \int_\Omega |\nabla v|^2\,\varphi\, dx = +\infty. \tag{11.35}$$

Proof: We recall (11.30) and put

$$F(t) = W_\varphi(t) - \int_0^t R_1(u, v, \varphi)ds - \int_\Omega u\varphi\, dx.$$

Then, from (11.11), (11.26), and (11.27) we obtain

$$\left|\int_\Omega u\varphi\, dx\right| \le \|u_0\|_1$$

and

$$\sup_{0\le t<T_{max}} |R_1(u, v, \varphi)| < +\infty.$$

Also, by Lemma 11.6, F is monotone decreasing in $t \in [0, T_{\max})$ and therefore (11.34) follows.

Now, we have

$$\int_\Omega (u \log u)\varphi \, dx \le W^* + \int_\Omega uv\varphi \, dx,$$

and therefore Lemma 11.2 guarantees

$$\limsup_{t \uparrow T_{\max}} \int_\Omega uv\varphi \, dx = +\infty.$$

Then, using Young's inequality, we have

$$\begin{aligned} a \int_\Omega uv\varphi \, dx &\le \int_\Omega (u \log u)\, \varphi \, dx + \frac{1}{e} \int_\Omega e^{av} \varphi \, dx \\ &\le W_\varphi + \int_\Omega uv\varphi \, dx + \frac{1}{e} \int_\Omega e^{av} \varphi \, dx \\ &\le W^* + \int_\Omega uv\varphi \, dx + \frac{1}{e} \int_\Omega e^{av} \varphi \, dx, \end{aligned}$$

and hence it follows that

$$(a-1) \int_\Omega uv\varphi \, dx \le \frac{1}{e} \int_\Omega e^{av} \varphi \, dx + W^*,$$

where a is a constant. Therefore, we have

$$\limsup_{t \uparrow T_{\max}} \int_\Omega e^{av} \varphi \, dx = +\infty$$

for $a > 1$, and (11.35) follows from the following lemma. □

Lemma 11.8 *If $a > 0$ and $\varphi = \varphi_{x_0, R', R}$ for $x_0 \in \mathcal{S}$ and $0 < R' < R \ll 1$, then the inequality*

$$\int_\Omega e^{av} \varphi \, dx \le C_8 \exp\left(\frac{a^2}{8\pi} \int_\Omega |\nabla v|^2 \, \varphi \, dx \right) \tag{11.36}$$

holds for $t \in [0, T_{\max})$. If $x_0 \in \Omega$, then this inequality is improved as

$$\int_\Omega e^{av} \varphi \, dx \le C_9 \exp\left(\frac{a^2}{16\pi} \int_\Omega |\nabla v|^2 \, \varphi \, dx \right). \tag{11.37}$$

Proof: We apply the inequalities [23, 102] in Chapter 4 valid for $w \in X$:

$$\log\left(\int_\Omega e^w\,dx\right) \leq \frac{1}{2\pi^*}\left\|\nabla w\right\|_2^2 + \frac{1}{|\Omega|}\int_\Omega w\,dx + K,$$

where

$$\pi^* = \begin{cases} 4\pi & \left(X = H^1(\Omega)\right), \\ 8\pi & \left(X = H_0^1(\Omega)\right). \end{cases}$$

In fact, it holds that

$$\sup_{0 \leq t < T_{max}} \left\|v(t)\right\|_{C^1\left(\overline{B(x_0,R)\cap\Omega}\setminus B(x_0,R')\right)} < +\infty$$

by (11.27), so that for the case of $x_0 \in \mathcal{S} \cap \partial\Omega$ we have

$$\begin{aligned} \int_\Omega e^{av}\varphi\,dx &\leq \int_{B(x_0,R')\cap\Omega} e^{av}\,dx + \int_{B(x_0,R)\cap\Omega\setminus B(x_0,R')} e^{av}\,dx \\ &\leq \int_\Omega e^{av\varphi}\,dx + C_{10} \leq e^K \exp\left(\frac{a^2}{8\pi}\left\|\nabla(v\varphi)\right\|_2^2 + \frac{a\|v\|_1}{|\Omega|}\right) \\ &\quad + C_{10} \leq \left(e^{K+aC_{11}} + C_{10}\right)\exp\left(\frac{a^2}{8\pi}\int_\Omega |\nabla v|^2\varphi\,dx\right) \end{aligned}$$

by (11.12). This shows (11.36). Inequality (11.37) for $x_0 \in \mathcal{S} \cap \Omega$ is shown similarly, and the proof is complete. □

Now, we localize an inequality used in Chapter 4.

Lemma 11.9 *We have*

$$\int_\Omega uv\varphi\,dx \leq \int_\Omega (u\log u)\varphi\,dx + M_\varphi \log\left(\int_\Omega e^v\varphi\,dx\right) - M_\varphi \log M_\varphi \qquad (11.38)$$

for $M_\varphi = \int_\Omega u\varphi\,dx$.

Proof: Since $-\log s$ is convex, Jensen's inequality applies as

$$\begin{aligned} -\log\left(\frac{1}{M_\varphi}\int_\Omega e^v\varphi\,dx\right) &= -\log\left(\int_\Omega \frac{e^v}{u}\frac{u}{M_\varphi}\varphi\,dx\right) \\ &\leq \int_\Omega \left\{-\log\left(\frac{1}{u}e^v\right)\frac{u}{M_\varphi}\varphi\right\}dx \\ &= -\frac{1}{M_\varphi}\int_\Omega \left\{u\log\left(\frac{e^v}{u}\right)\varphi\right\}dx. \end{aligned}$$

This means (11.38). □

We are ready to complete the proof of Theorem 11.1.

Proof of Lemma 11.5: Having proven that $\lim_{t\uparrow T_{\max}} \|u\varphi\|_1$ exists, we suppose

$$\lim_{t\uparrow T_{\max}} M_\varphi(t) = \lim_{t\uparrow T_{\max}} \|u\varphi\|_1 < m_* \tag{11.39}$$

and derive a contradiction.

In fact, in the case that $x_0 \in \Omega$ we have (11.37), and therefore

$$\begin{aligned} \frac{1}{2}\int_\Omega \big(|\nabla v|^2 + v^2\big)\varphi &= W_\varphi - \int_\Omega (u\log u - uv)\varphi\, dx \\ &\le W_\varphi + M_\varphi \log\Big(\int_\Omega e^v \varphi\, dx\Big) - M_\varphi \log M_\varphi \\ &\le W^* + \frac{M_\varphi}{16\pi}\int_\Omega |\nabla v|^2\, \varphi\, dx + M_\varphi \log \frac{C_9}{M_\varphi} \end{aligned}$$

by (11.34) and (11.38). This means

$$\frac{1}{2}\Big(1 - \frac{M_\varphi}{8\pi}\Big)\int_\Omega |\nabla v|^2 \varphi\, dx \le W^* + M_\varphi \log\frac{C_9}{M_\varphi} \le C_{12},$$

and therefore

$$\limsup_{t\uparrow T_{\max}} \int_\Omega |\nabla v|^2\, \varphi\, dx < +\infty$$

by (11.39) and $m_* = 8\pi$, a contradiction. The case $x_0 \in \partial\Omega$ is treated similarly, and the proof is complete. □

12
Concentration Lemma

In this chapter we show that the mass quantization of collapse occurs if the solution *blows up in infinite time* [148]. It is uncertain whether such a solution exists or not. Actually, it is suspected that the blowup in infinite time occurs only when the solution converges to a singular stationary solution, and therefore, in that case, the total mass $\lambda = \|u_0\|_1$ must be quantized as $\lambda \in 4\pi\mathbf{N}$. This question is open, but an important tool is exploited, which we call the *concentration lemma*. (This is different from the lemma given by [23].)

Similarly to the case dealt with in the previous chapter, the mass quantization of collapse of the solution blowing-up in infinite time is valid for other systems, provided they also have the properties of positivity and total mass preserving of the solution, decrease of the free energy, and the physical field associated with a kernel uniformly bounded under the symmetrization process. But we concentrate on (11.2) for simplicity:

$$
\begin{aligned}
u_t &= \nabla \cdot (\nabla u - u \nabla v) && \text{in} \quad \Omega \times (0, T), \\
0 &= \Delta v - av + u && \text{in} \quad \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 && \text{on} \quad \partial\Omega \times (0, T), \\
u\big|_{t=0} &= u_0 && \text{on} \quad \Omega,
\end{aligned}
\tag{12.1}
$$

Here, $u_0 = u_0(x) \geq 0$ is a smooth function, $a > 0$ is a constant, $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and ν denotes the outer unit normal vector. Furthermore, we put $a = 1$.

Under these assumptions, we can show that if $u = u(x,t)$ is a solution global in time, then any $t_n \uparrow +\infty$ admits $\left\{t_n'\right\} \subset \{t_n\}$ and $0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B}\{t_n'\})$ such that

$$u(x,t_n')\,dx \rightharpoonup \sum_{x_0 \in \mathcal{B}(\{t_n'\})} m_*(x_0)\delta_{x_0}(dx) + f(x)\,dx \tag{12.2}$$

in $\mathcal{M}(\overline{\Omega})$, where $\mathcal{B}(\{t_n'\})$ denotes the set of exhausted blowup points so that $x_0 \in \overline{\Omega}$ belongs to $\mathcal{B}(\{t_n'\})$ if and only if there is $\{x_n'\} \subset \overline{\Omega}$ such that $u(x_n', t_n') \to +\infty$, and

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

It should be noted that the number of exhausted blowup points is not prescribed in (12.2), and the case $\mathcal{B}(\{t_n'\}) = \emptyset$ is admitted.

Here, we confirm the notation used in this chapter. First,

$$\mathcal{B}(\{t_n'\})$$

denotes the *blowup set* of $\{u(t_n')\}$, so that $x_0 \in \mathcal{B}(\{t_n'\})$ if and only if there exist $\{t_n''\} \subset \{t_n'\}$ and $\{x_n\} \subset \overline{\Omega}$ satisfying $t_n'' \uparrow +\infty$, $x_n \to x_0$, and $u(x_n, t_n'') \to +\infty$. We say that x_0 is *exhausted* as a blowup point of $\left\{u(t_n')\right\}$, if $t_n'' = t_n'$ holds in the above relation. By definition, the blowup set $\mathcal{B}(\{t_n'\})$ is exhausted, if its any element is so. Next, we say that the solution blows up in infinite time if

$$T_{\max} = +\infty \quad \text{and} \quad \limsup_{t \to +\infty} \|u(t)\|_\infty = +\infty$$

hold. In this case, we have $\mathcal{B}\left(\{t_n'\}\right) \neq \emptyset$ if $\lim_{n\to\infty} \|u(t_n)\|_\infty = +\infty$, but the possibility of the oscillating formation of collapses is still kept in (12.2).

We proceed to the *concentration lemma*. This is a criterion for a sequence of solutions to have a converging subsequence using the *concentration function*. This type of result is obtained when Ω is a flat torus [152], and the proof is done along a similar line.

Theorem 12.1 *If $\left\{u_0^n\right\}_{n\geq 1}$ is a family of smooth nonnegative functions defined on $\overline{\Omega}$ satisfying*

$$\sup_{n\geq 1} \left\|u_0^n\right\|_1 = \Lambda < +\infty,$$

$u^n = u^n(x,t)$ denotes the solution to (12.1) whose initial value is u_0^n, the supremum of the existence time of u^n is denoted by $T^n = T^n_{\max} > 0$, and

$$\limsup_{n\to\infty} \left\|u_0^n\right\|_{L^1(B(x_0,R)\cap\Omega)} < m_*(x_0) \tag{12.3}$$

for $x_0 \in \overline{\Omega}$ and $R > 0$, then there exist $T > 0$, $r > 0$, and $\left\{u^{n'}\right\} \subset \{u^n\}$ satisfying

$$\sup_{n'} \left\|u^{n'}\right\|_{C^{2+\theta,1+\theta/2}\left(B(x_0,r)\cap\Omega\times[\tau,\min(T,T^{n'}_{\max}))\right)} < +\infty \tag{12.4}$$

for any $\tau \in (0,T)$, where $\theta \in (0,1)$. If $R > 0$ in (12.3) is independent of $x_0 \in \overline{\Omega}$, then we have $T_* \equiv \liminf_n T^n_{\max} > 0$. Furthermore, in this case, $\{u^n\}$ has a subsequence converging in $C^{2,1}(\overline{\Omega}\times(0,T])$ for some $T \in (0,T_*)$.

Here and henceforth, we take the agreement that the term

$$\left\|u^{n'}\right\|_{C^{2+\theta,1+\theta/2}\left(B(x_0,r)\cap\Omega\times[\tau,\min(T,T^{n'}_{\max}))\right)}$$

in (12.4) has the meaning only when $\tau < T^{n'}_{\max}$.

We use several facts obtained in the previous chapters for the proof. First, we use (11.7), a form of the Gagliardo–Nirenberg inequality, specifying the constant $K_1 > 0$ determined by $\Omega \subset \mathbf{R}^2$:

$$\left\|w\right\|_2^2 \leq K_1^2\left(\left\|\nabla w\right\|_1^2 + \left\|w\right\|_1^2\right),$$

which is valid for all $w \in W^{1,1}(\Omega)$. We use also (5.11) valid for $\psi = \psi(x) \in C^2(\overline{\Omega})$ with $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$, that is,

$$\left|\frac{d}{dt}\int_\Omega \psi(x)u(x,t)\,dx\right| \leq \frac{K_2\lambda^2}{2}\left\|\nabla\psi\right\|_{W^{1,\infty}(\Omega)} + \lambda\left\|\Delta\psi\right\|_\infty \tag{12.5}$$

for $\lambda = \|u_0\|_1$, where $K_2 > 0$ is a constant determined again by Ω. Given $x_0 \in \overline{\Omega}$, we take the cut-off function $\varphi = \varphi_{x_0,R',R}$ for $0 < R' < R \ll 1$ defined in Chapter 5, and put

$$\psi = (\varphi_{x_0,R',R})^6.$$

This smooth $\psi = \psi_{x_0,R',R}$ satisfies $0 \leq \psi \leq 1$ in Ω,

$$\psi(x) = \begin{cases} 1 & (x \in B(x_0,R')\cap\Omega), \\ 0 & (x \in \Omega\setminus B(x_0,R)), \end{cases}$$

$\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$, and

$$\left|D^{\alpha}\psi\right| \le A(R-R')^{-1}\psi^{5/6} \quad (|\alpha|=1),$$
$$\left|D^{\alpha}\psi\right| \le B(R-R')^{-2}\psi^{2/3} \quad (|\alpha|=2),$$

with the absolute constants $A > 0$ and $B > 0$. In this chapter, we use the order on $(R - R')$ in these inequalities. Finally, $|D|$ denotes the area of measurable set $D \subset \mathbf{R}^2$.

In the following lemma, $u = u(x,t)$ denotes the solution to (12.1) with the smooth initial value $u_0 = u_0(x) \ge 0$, and the supremum of its existence time is denoted by $T = T_{\max} > 0$. This lemma, in contrast with Theorem 12.1, describes a uniform estimate depending only on $\Lambda \ge \|u_0\|_1$. More precisely, the constants T_1 and C_1 stated here are explicitly determined by K_1, K_2, Λ, R_1 and K_1, K_2, Λ, R_1, τ, respectively. The proof is done by examining the proof of Theorem 11.1 carefully.

Lemma 12.1 *There exists $T_1 > 0$ determined by $R_1 > 0$ such that if*

$$\left\|u_0\right\|_{L^1(B(x_0,3R_1)\cap\Omega)} < 1/(64K_1^2), \tag{12.6}$$

then each $\tau \in (0, T_1)$ admits $C_1 > 0$ satisfying

$$\left\|u\right\|_{C^{2+\theta,1+\theta/2}\left(B(x_0,R_1)\cap\overline{\Omega}\times[\tau,\min(T_1,T_{\max}))\right)} \le C_1$$

for $x_0 \in \overline{\Omega}$.

Proof: To make the description simple, we write K, R, T, and λ for K_1, R_1, T_1, and Λ, respectively. Given $R > 0$ and $x_0 \in \overline{\Omega}$, we take $\psi = \psi_{x_0,5R/2,3R}$ and apply (12.5):

$$\left|\frac{d}{dt}\int_{\Omega} u\psi\,dx\right| \le \frac{K_2\lambda^2}{2}\left\|\nabla\psi\right\|_{W^{1,\infty}} + \lambda\left\|\Delta\psi\right\|_{\infty}.$$

Thus, if $T_1 \in (0, T_{\max})$ is taken as

$$\left(\frac{K_2\lambda^2}{2}\left\|\nabla\psi\right\|_{W^{1,\infty}} + \lambda\left\|\Delta\psi\right\|_{\infty}\right)T_1 \le 1/(64K^2),$$

then, from (12.6) it follows that

$$\sup_{t\in[0,T_1]}\left\|u(t)\right\|_{L^1(\Omega\cap B(x_0,5R/2))} \le 1/(32K^2). \tag{12.7}$$

Next, we take $\psi_1 = \psi_{x_0,2R,5R/2}$. Then, for $R \in (0, 1)$, inequalities (11.8) and (11.22) of the previous chapter are written as

$$\int_\Omega u^2\psi_1\,dx \le 2K^2 \int_{\Omega\cap B(x_0,5R/2)} u\,dx \cdot \int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx + K^2\Big(\frac{A^2}{8R^2}+1\Big)\lambda^2 \tag{12.8}$$

and

$$\frac{d}{dt}\int_\Omega (u\log u)\,\psi_1\,dx + \frac{1}{4}\int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx \le 2\int_\Omega u^2\psi_1\,dx + LR^{-6}, \tag{12.9}$$

respectively, where $L > 0$ is a constant determined by $|\Omega|$ and λ. Henceforth, C_i $(i = 2, 3, \dots)$ denote positive constants determined by Ω, λ, and R. Then, (12.8) and (12.9) imply

$$\frac{d}{dt}\int_\Omega (u\log u)\,\psi_1\,dx + \frac{1}{4}\Big(1 - 16K^2\int_{B(x_0,5R/2)} u\,dx\Big) \cdot \int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx \le C_2$$

for $t \in [0, T_{\max})$. Therefore, from (12.7) we have

$$\frac{d}{dt}\int_\Omega (u\log u)\,\psi_1\,dx + \frac{1}{8}\int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx \le C_2 \tag{12.10}$$

for $t \in [0, T_1]$.

In what follows, the time variable t is restricted in $t \in [0, T_1]$. Then, from (12.8) it follows that

$$\int_\Omega u^2\psi_1\,dx \le \frac{1}{16}\int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx + K^2\Big(\frac{A^2}{8R^2}+1\Big)\lambda^2, \tag{12.11}$$

and hence

$$\frac{d}{dt}\int_\Omega (u\log u)\,\psi_1\,dx + 2\int_\Omega u^2\psi_1\,dx \le C_3$$

holds true. In terms of

$$J(t) \equiv \int_\Omega \big(u\log u + e^{-1}\big)\psi_1\,dx \ge 0,$$

this implies

$$\frac{dJ}{dt} + 3J^{3/2} \leq C_3 - 2\int_\Omega u^2\psi_1\,dx + 3\int_\Omega \left(u\log u + e^{-1}\right)^{3/2}\psi_1\,dx\cdot|\Omega|^{1/2}. \quad (12.12)$$

Since the elementary inequality

$$u\log u \leq 4u^{5/4}$$

is valid in $u > 0$, the right-hand side of (12.12) is estimated from above by

$$C_4\Big(1 + \int_\Omega u^{15/8}\psi_1\,dx\Big) - 2\int_\Omega u^2\psi_1\,dx \leq C_5.$$

Therefore, it holds that

$$\frac{d}{dt}J(t) + 3J(t)^{3/2} \leq C_5$$

for $t \in [0, T_1]$.

We also have

$$\frac{d}{dt}t^{-2} + 3\big(t^{-2}\big)^{3/2} = t^{-3},$$

and the function $F(J) \equiv J_+^{3/2}$ is continuously differentiable in $J \in \mathbf{R}$. The standard comparison theorem is applicable, and if $T \in (0, T_1)$ is so taken as in $T^3C_5 \leq 1$, then, for $t \in (0, T)$ it holds that $t^{-3} \geq C_5$. This implies that the inequality

$$J(t) = \int_\Omega \big(u\log u + e^{-1}\big)\psi_1\,dx \leq t^{-2} \quad (12.13)$$

holds in $t \in (0, T]$. Then, by (12.10), inequality (12.13) gives

$$\frac{1}{8}\int_t^{t_1} K(s)ds \leq t^{-2} + C_2T \quad (12.14)$$

for $0 < t \leq t_1 \leq T$, where

$$K(t) = \int_\Omega u^{-1}\,|\nabla u|^2\,\psi_1\,dx.$$

Now we repeat the proof of Lemma 11.2 based on (12.13), taking $t = \tau/2$ as the initial time for given $\tau \in (0, T)$. Henceforth, C_i ($i = 6, 7, 8, \dots$) denote positive constants determined by τ besides Ω, λ, and R. Its first step assures

$$\frac{d}{dt}\int_\Omega u^2\psi_2\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, \psi_2\, dx \le C_6 \tag{12.15}$$

for $t \in [\tau/2, T]$, where $\psi_2 = \psi_{x_0,3R/2,2R}$. Here, we have

$$\frac{4}{\tau}\int_{\tau/2}^{3\tau/4} K(s)ds \le C_7$$

by (12.14), and therefore there exists $\tau_1 \in (\tau/2, 3\tau/4)$ satisfying $K(\tau_1) \le C_7$. This implies

$$\int_\Omega u^2\psi_1\, dx \le C_8$$

at $t = \tau_1$ by (12.11), and inequality (12.15) now guarantees

$$\int_\Omega u^2\psi_2\, dx + \frac{1}{2}\int_{\tau_1}^{t} K_1(s)ds \le C_9 \tag{12.16}$$

for $t \in [\tau_1, T]$, where $K_1(t) = \displaystyle\int_\Omega |\nabla u|^2\, \psi_2\, dx$. In particular,

$$\int_{3\tau/4}^{7\tau/8} \left\|\nabla u(s)\right\|^2_{L^2(\Omega\cap B(x_0,3R/2))} ds \le 2C_9$$

follows, and we have $\tau_2 \in (3\tau/4, 7\tau/8)$ satisfying

$$\left\|\nabla u(\tau_2)\right\|_{L^2(\Omega\cap B(x_0,3R/2))} \le C_{10} \tag{12.17}$$

similarly. This implies

$$\|u(\tau_2)\|_{L^p(\Omega\cap B(x_0,3R/2))} = O(1)$$

for any $p \in [1, \infty)$ fixed, by Sobolev's inequality.

Based on this fact, we can perform the second step of Lemma 11.2, taking $t = \tau_2$ as the initial value. We set $R_1 = (3R)/2$ for the moment, and take $R_2 \in (R, R_1)$. Then, given $p \in [1, \infty)$, we can prescribe $C > 0$ satisfying

$$\sup_{t\in[\tau_2,T]} \|u(t)\|_{L^p(\Omega\cap B(x_0,R_2))} \le C.$$

The second equation of (12.1) now gives

$$\sup_{t\in[7\tau/8,T]} \left\| v(t) \right\|_{W^{2,p}(\Omega\cap B(x_0,R_3))} \leq C'$$

with the prescribed $C' > 0$, where $R_3 \in (R, R_2)$ is fixed arbitrarily. Thus, we obtain

$$\sup_{t\in[7\tau/8,T]} \left\| v(t) \right\|_{C^{1+\theta}(\Omega\cap B(x_0,R_3))} \leq C_{11} \tag{12.18}$$

by taking $p > 2$, where $\theta \in (0, 1)$.

Now, we take $R_4 \in (R, R_3)$ and set $\psi_3 = \psi_{x_0,R_4,R_3}$. Multiplying $-\nabla \cdot (\psi_3 \nabla u)$ to the first equation of (12.1), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \psi_3 \, dx + \int_\Omega |\Delta u|^2 \, \psi_3 \, dx \\
&= \int_\Omega [\nabla \cdot (u\nabla v)] \, \nabla \cdot (\psi_3 \nabla u) \; dx - \int_\Omega (\Delta u) \, \nabla\psi_3 \cdot \nabla u \, dx \\
&= \int_\Omega [\nabla \cdot (u\nabla v)] \, (\Delta u) \, \psi_3 \, dx \\
&\quad + \int_\Omega [\nabla \cdot (u\nabla v) - \Delta u] \, \nabla u \cdot \nabla \psi_3 \, dx \\
&= \int_\Omega \Big(\nabla u \cdot \nabla v - u^2 + uv \Big) \, (\Delta u) \, \psi_3 \, dx \\
&\quad - \int_\Omega (\Delta u) \, \nabla u \cdot \nabla \psi_3 \, dx \\
&\quad + \int_\Omega \Big(\nabla u \cdot \nabla v - u^2 + uv \Big) \, (\nabla u \cdot \nabla \psi_3) \; dx,
\end{aligned}$$

where we make use of the second equation of (12.1). By means of (12.18), the right-hand side is estimated from above by

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\Delta u|^2 \, \psi_3 \, dx + \int_\Omega u^4 \psi_3 \, dx + \int_\Omega u^2 v^2 \psi_3 \, dx \\
&+ C_{12} \int_\Omega |\nabla u|^2 \left(|\nabla v|^2 + 1 \right) \psi_3^{3/2} \, dx \\
&\leq \frac{1}{2} \int_\Omega |\Delta u|^2 \, \psi_3 \, dx + C_{13} \Big(\int_\Omega |\nabla u|^2 \, \psi_2 \, dx + 1 \Big),
\end{aligned}$$

and hence it follows that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 \, \psi_3 \, dx + \int_\Omega |\Delta u|^2 \, \psi_3 \leq 2C_{13} \Big(\int_\Omega |\nabla u|^2 \, \psi_2 \, dx + 1 \Big)$$

for $t \in [7\tau/8, T]$. This implies

$$\frac{16}{\tau}\int_{7\tau/8}^{15\tau/16} \left\| \Delta u(s) \right\|^2_{L^2(\Omega \cap B(x_0,R_4))} ds \le C_{14}$$

by (12.16) and (12.17), and therefore there is

$$\tau_3 \in (7\tau/8, 15\tau/16)$$

such that

$$\left\| \Delta u(\tau_3) \right\|^2_{L^2(\Omega \cap B(x_0,R_4))} \le C_{14}.$$

Then, we have

$$\left\| u(\tau_3) \right\|_{L^\infty(\Omega \cap B(x_0,R_4))} \le C_{15} \tag{12.19}$$

by Morrey's inequality and $\sup_{t \in [0,T_{\max})} \left\| u(t) \right\|_1 \le \lambda$.

Based on (12.19), we can perform the final part of the proof of Lemma 11.2. First, the iteration scheme gives

$$\sup_{t \in [\tau_3, T]} \left\| u(t) \right\|_{L^\infty(\Omega \cap B(x_0,R_5))} \le C_{16}$$

for $R_5 \in (R, R_4)$. Then the standard elliptic and parabolic estimates guarantee

$$\left\| u \right\|_{C^{2+\theta,1+\theta/2}(B(x_0,R)\times[\tau,T])} \le C$$

with the prescribed constant $C > 0$, and the proof is complete. □

Given $\{u_0^n\}_{n\ge1}$ of Theorem 12.1, we choose $\{u_0^{n'}\} \subset \{u_0^n\}$ and $\mu_0(dx) \in \mathcal{M}(\overline{\Omega})$ satisfying

$$u_0^{n'}(x)\,dx \rightharpoonup \mu_0(dx)$$

in $\mathcal{M}(\overline{\Omega})$ as $n' \to \infty$. We write n for n' for simplicity. The constants $T_2, T_3, \ldots, R_2, R_3, \ldots$, and $C_2, C_3, \ldots$ prescribed below are different from those in the proof of Lemma 12.1.

By means of $\mu_0(\overline{\Omega}) \le \Lambda$, we have $\sharp\mathcal{S} \le 64K_1^2\Lambda < +\infty$, where

$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid \mu_0(\{x_0\}) \ge 1/(64K_1^2) \right\}. \tag{12.20}$$

Each $x_0 \in \overline{\Omega} \setminus \mathcal{S}$ admits $R_1 > 0$ and $N_1 \ge 1$ such that

$$\left\| u_0^n \right\|_{L^1(B(x_0,3R_1)\cap\Omega)} < 1/(64K_1^2)$$

for $n \geq N_1$. Then, Lemma 12.1 guarantees the existence of $T_1 > 0$ such that each $\tau \in (0, T_1)$ admits $C_1 > 0$ satisfying

$$\left\|u^n\right\|_{C^{2+\theta,1+\theta/2}\left(B(x_0,R_1)\cap\overline{\Omega}\times[\tau,\min(T_1,T^n_{\max}))\right)} \leq C_1$$

for $n \geq N_1$. These R_1, N_1, T_1, and C_1 depend on x_0. But if we set

$$\mathcal{S}_s = \{x \in \overline{\Omega} \mid \operatorname{dist}(x, \mathcal{S}) < s\}$$

for $s > 0$, then the standard argument of covering guarantees the existence of $T_2 = T_2(s) > 0$ such that each $\tau \in (0, T_2)$ admits $C_2 = C_2(s, \tau) > 0$, satisfying

$$\left\|u^n\right\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega}\setminus\mathcal{S}_s\times[\tau,\min(T_2,T^n_{\max}))\right)} \leq C_2 \tag{12.21}$$

for $n \geq 1$. Then, the following lemma is obtained.

Lemma 12.2 *Under the assumptions of Theorem 12.1, given $x_0 \in \mathcal{S}\cap\Omega$, we define $R_2 > 0$ and $\delta_1 > 0$, satisfying $B(x_0, 4R_2) \subset \Omega$,*

$$\overline{B(x_0, 4R_2)} \cap \mathcal{S} = \{x_0\},$$

and

$$\left\|u_0^n\right\|_{L^1(B(x_0,3R_2))} \leq 8\pi - 2\delta_1$$

for n sufficiently large, where $\mathcal{S}$ denotes the set defined by (12.20). Then, there is $T_3 > 0$ such that each $\tau \in (0, T_3)$ admits $C_3 > 0$ satisfying

$$\|u^n\|_{C^{2+\theta,1+\theta/2}\left(B(x_0,R_2)\times[\tau,\min(T_3,T^n_{\max}))\right)} \leq C_3.$$

It is obvious that this lemma is a consequence of the following lemma.

Lemma 12.3 *If $u = u(x,t)$ is the classical solution to (12.1) with the smooth initial value $u_0 = u_0(x) \geq 0$ and $T = T_{\max} \in (0, +\infty]$ denotes the supremum of its existence time, then there are $0 < T \leq \hat{T} < T_{\max}$ determined by $\delta > 0$ and $R > 0$ such that if $B(x_0, 4R) \subset \Omega$,*

$$\left\|u_0\right\|_{L^1(B(x_0,3R))} \leq 8\pi - 2\delta$$

and $\tau \in (0, T)$, then we have $\hat{C} > 0$ determined by

$$\ell = \left\|u\right\|_{C^{2+\theta,1+\theta/2}(B(x_0,4R)\setminus B(x_0,R/4)\times[\tau/4,\hat{T}])}, \tag{12.22}$$

τ, and R, such that

$$\left\|u\right\|_{C^{2+\theta,1+\theta/2}(B(x_0,R)\times[\tau,T])} \leq \hat{C}, \tag{12.23}$$

where $\theta \in (0, 1)$.

In the radially symmetric case, this lemma can be proven by the method of Biler [14], and therefore we apply the rearrangement technique to treat the general case.

Thus, given a nonnegative measurable function f, we put

$$\mu(\sigma) = \left|\left\{x \in \Omega \;\middle|\; f(x) > \sigma\right\}\right|$$

and

$$f^*(s) = \inf\left\{\sigma > 0 \;\middle|\; \mu(\sigma) \leq s\right\}$$

for $s \in [0, |\Omega|]$, so that $f^*(s)$ denotes the *monotone decreasing rearrangement* of $f(x)$. Let us confirm

$$\rho = f^*(\mu(\rho)).$$

The following are fundamental propositions of this type of rearrangement, and are obtained by Mossino and Rakotoson [104], where $1 \leq p \leq \infty$.

Proposition 12.2 *If $f \in W_0^{1,p}(\Omega)$ is a nonnegative function and $\delta \in (0, |\Omega|)$, then $f^* \in W^{1,p}((\delta, |\Omega|))$ holds.*

Proposition 12.3 *If $f = f(t) \in H^1(0, T; L^p(\Omega))$ is a nonnegative function, then it holds that*

$$f^* = f(t)^* \in H^1\left(0, T; L^p\left((0, |\Omega|)\right)\right)$$

and

$$\left\|\partial_t f^*(t)\right\|_{L^p((0,|\Omega|))} \leq \left\|\partial_t f(t)\right\|_{L^p(\Omega)}$$

for a.e. $t \in [0, T]$. Furthermore, the function

$$F(s,t) = \int_0^s f^*(\sigma, t) d\sigma$$

is in

$$F \in L^\infty\big((0, |\Omega|) \times (0, T)\big) \cap H^1\big(0, T; W^{1,p}((0, |\Omega|))\big) \bigcap_{\delta > 0} L^2\big(0, T; W^{2,p}((\delta, |\Omega|))\big)$$

and satisfies

$$\int_{\{u>s\}} (\partial_t f)\, dx = \partial_t F\left(\left|\{u > s\}\right|, t\right)$$

for a.e. $(s, t) \in (0, |\Omega|) \times (0, T)$.

Beginning the proof of Lemma 12.3, we take the agreement that a positive constant determined by ℓ besides τ, possibly changing from line to line, is henceforth denoted by C_ℓ.

First, the second equation of (12.1) implies

$$\left\| v \right\|_{C^{2+\theta,1+\theta/2}(B(x_0,3R)\setminus B(x_0,R/3)\times[\tau/2,T]))} \le C_\ell.$$

Now, we take $\varphi = \varphi_{x_0,R,2R}$. Given $s \in [0, |B(x_0, 3R)|)$ and $t \in [0, T_{\max})$, we define

$$\mu(s,t) = \left| \left\{ x \in B(x_0, 3R) \;\middle|\; (u\varphi)(x,t) > s \right\} \right|$$

and

$$(u\varphi)^*(s,t) = \inf \left\{ \sigma > 0 \;\middle|\; \mu(\sigma, t) \le s \right\}.$$

Then the function

$$k(s,t) = \int_0^s (u\varphi)^*(s',t)ds'$$

is Lipschitz continuous in (s,t) with k_s, and it holds that

$$\int_{\{x\in B(x_0,3R)|(u\varphi)(x,t)>s\}} \frac{\partial}{\partial t}(u\varphi)\, dx = \frac{\partial k}{\partial t}\left(\mu(s,t),t\right) \tag{12.24}$$

for a.e. $(s,t) \in (0, |B(x_0, 3R)|) \times (0, T_{\max})$ by Proposition 12.3. We have also

$$k\left(\mu(s,t),t\right) = \int_{\{x\in B(x_0,3R)|(u\varphi)(x,t)>s\}} u\varphi\, dx.$$

Now, we use the following lemma [38].

Lemma 12.4 *Under the assumptions of the previous lemma, there is a constant $L > 0$ determined by ℓ and R such that*

$$\frac{\partial k}{\partial t} - 4\pi s \frac{\partial^2 k}{\partial s^2} - (k + Ls)\frac{\partial k}{\partial s} - Ls \le 0 \tag{12.25}$$

for a.e. $(s,t) \in (0, |B(x_0, 2R)|) \times (\tau/2, T)$. Furthermore, we have

$$k\Big|_{s=0} = 0 \quad \text{and} \quad \frac{\partial k}{\partial s}\Big|_{s=|B(x_0,2R)|} = 0 \tag{12.26}$$

for $t \in [\tau/2, T]$.

Proof: Equalities of (12.26) are obtained immediately. To show (12.25), we take $t \in (\frac{\tau}{2}, T)$. Given $\rho \in (0, (u\varphi)^*(0, t))$ and $h > 0$, we set

$$T_{\rho h}(s) = \begin{cases} 0 & (s \le \rho), \\ s - \rho & (\rho < s \le \rho + h), \\ h & (s > \rho + h). \end{cases}$$

Then it holds that $T_{\rho h}((u\varphi)(\cdot, t)) \in W_0^{1,\infty}(B(x_0, 2R))$.

Using the first equation of (12.1), we have

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial t}(u\varphi) \cdot T_{\rho h}(u\varphi)\,dx &= \int_\Omega u_t \varphi \cdot T_{\rho h}(u\varphi)\,dx \\ &= -\int_\Omega \nabla u \cdot \big(\varphi \nabla T_{\rho h}(u\varphi) + T_{\rho h}(u\varphi)\nabla\varphi\big)\,dx \\ &\quad + \int_\Omega u\nabla v \cdot \big(\varphi \nabla T_{\rho h}(u\varphi) + T_{\rho h}(u\varphi)\nabla\varphi\big)\,dx \\ &= -\int_\Omega \nabla(u\varphi) \cdot \nabla T_{\rho h}(u\varphi)\,dx + \int_\Omega (u\varphi)\nabla v \cdot \nabla T_{\rho h}(u\varphi)\,dx \\ &\quad + \int_\Omega T_{\rho h}(u\varphi)\,(-\nabla\cdot(u\nabla\varphi) - \nabla u \cdot \nabla\varphi + u\nabla v \cdot \nabla\varphi)\,dx. \end{aligned} \tag{12.27}$$

Since the supports of $|\nabla\varphi|$ and $|\Delta\varphi|$ are contained in

$$\overline{B(x_0, 2R) \setminus B(x_0, R)},$$

the function $g = -\nabla \cdot (u\nabla\varphi) - \nabla u \cdot \nabla\varphi + u\nabla v \cdot \nabla\varphi$ satisfies

$$\|g\|_{L^\infty(\Omega\times[\tau/2,T])} \le C_\ell.$$

Therefore, the last term of the right-hand side of (12.27) is treated by

$$\begin{aligned} &\left|\frac{1}{h}\int_\Omega T_{\rho h}(u\phi) g\,dx\right| \\ &\qquad = \left|\frac{1}{h}\left\{\int_{\{(u\varphi)>\rho+h\}} hg\,dx + \int_{\{\rho<(u\varphi)\le\rho+h\}} (u\varphi - \rho) g\,dx\right\}\right| \\ &\qquad \le \|g\|_\infty\,|\{(u\varphi) > \rho + h\}| + \|g\|_\infty\,|\{\rho < (u\varphi) \le \rho + h\}|. \end{aligned}$$

This implies

$$\limsup_{h\downarrow 0}\left|\frac{1}{h}\int_\Omega T_{\rho h}(u\varphi) g\,dx\right| \le \|g\|_\infty \mu(\rho, t).$$

Next, the first term of the right-hand side of (12.27) is treated by

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\int_\Omega \nabla(u\varphi)\cdot\nabla T_{\rho h}(u\varphi)\,dx &= \lim_{h\downarrow 0}\frac{1}{h}\left\{\int_{\{(u\varphi)>\rho\}}|\nabla(u\varphi)|^2\,dx-\int_{\{(u\varphi)>\rho+h\}}|\nabla(u\varphi)|^2\,dx\right\}\\ &= -\frac{\partial}{\partial\rho}\int_{\{(u\varphi)>\rho\}}|\nabla(u\varphi)|^2\;dx.\end{aligned}$$

Finally, in terms of

$$S_{\rho h}(s)=\int_0^s s\frac{d}{ds}T_{\rho h}(s)ds=\begin{cases}0 & (s\le\rho),\\ \frac{1}{2}\left(s^2-\rho^2\right) & (\rho<s\le\rho+h),\\ h\left(\rho+\frac{h}{2}\right) & (s>\rho+h),\end{cases}$$

we have

$$\begin{aligned}\int_\Omega (u\varphi)\nabla v\cdot\nabla T_{\rho h}(u\varphi)\,dx &= \int_\Omega \nabla v\cdot\nabla S_{\rho h}(u\varphi)\,dx\\ &= \int_\Omega (u-v)S_{\rho h}(u\varphi)\,dx\end{aligned}$$

from the second equation of (12.1).

The second term of the right-hand side of (12.27) is treated by

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\int_\Omega (u\varphi)\nabla v\cdot\nabla T_{\rho h}(u\varphi)\,dx &= \lim_{h\downarrow 0}\frac{1}{2h}\int_{\{\rho<(u\varphi)\le\rho+h\}}(u-v)\left\{(u\varphi)^2-\rho^2\right\}dx\\ &\quad+\lim_{h\downarrow 0}\int_{\{(u\varphi)>\rho+h\}}(u-v)\left(\rho+\frac{h}{2}\right)dx\\ &= \rho\int_{\{(u\varphi)>\rho\}}(u-v)\,dx\le\rho\int_{\{(u\varphi)>\rho\}}u\,dx=(u\varphi)^*\left(\mu(\rho,t),t\right)\\ &\quad\cdot\left\{\int_{\{(u\varphi)>\rho\}}u\varphi\,dx+\int_{\{(u\varphi)>\rho\}\cap(B(x_0,2R)\setminus B(x_0,R))}u(1-\varphi)\,dx\right\}\\ &\le\frac{\partial k}{\partial s}\left(\mu(\rho,t),t\right)\left\{k\left(\mu(\rho,t),t\right)+\ell\mu(\rho,t)\right\}.\end{aligned}$$

On the other hand, relation (12.24) is applicable to the left-hand side of (12.27). We have

$$\lim_{h\downarrow 0}\frac{1}{h}\int_\Omega\frac{\partial}{\partial t}(u\varphi)\cdot T_{\rho h}(u\varphi)\,dx=\int_{\{(u\varphi)>\rho\}}\frac{\partial}{\partial t}(u\varphi)\,dx=\frac{\partial k}{\partial t}\left(\mu(\rho,t),t\right).$$

Those relations are summarized as

$$\frac{\partial k}{\partial t}(\mu(\rho,t),t) - \frac{\partial}{\partial \rho}\int_{\{(u\varphi)>\rho\}} |\nabla(u\varphi)|^2\, dx \\ \le \frac{\partial k}{\partial s}(\mu(\rho,t),t)\,(\ell\mu(\rho,t) + k(\mu(\rho,t),t)) + L'\mu(\rho,t)$$

for a.e. $(\rho, t) \in (0, |B(x_0, 2R)|) \times (\tau/2, T)$ with a constant $L' > 0$ determined by ℓ.

From this stage we can argue similarly to the proof of Lemma 4 of [38]. In fact, the co-area formula and isoperimetric inequality imply

$$4\pi\mu\Big(-\frac{\partial\mu}{\partial\rho}\Big)^{-1} \le -\frac{\partial}{\partial\rho}\int_{\{(u\varphi)>\rho\}} |\nabla(u\varphi)|^2\, dx$$

and hence

$$\Big(-\frac{\partial\mu}{\partial\rho}\Big)^{-1} \le \frac{1}{4\pi\mu}\Big(-\frac{\partial k}{\partial t} + k\frac{\partial k}{\partial s}\Big) + \frac{1}{4\pi}\Big(\ell\frac{\partial k}{\partial s} + L'\Big)$$

follows for a.e. (ρ, t). This implies

$$-\frac{\partial(u\varphi)^*}{\partial s}(s,t) = -\frac{\partial^2 k}{\partial s^2}(s,t) \\ \le \frac{1}{4\pi s}\Big(-\frac{\partial k}{\partial t}(s,t) + k(s,t)\frac{\partial k}{\partial s}(s,t)\Big) + \frac{1}{4\pi}\big(\ell k(s,t) + L'\big)$$

for a.e. (s, t). Now, setting $L = \max\{\ell, L'\}$, we get the conclusion. □

We are ready to give the following proof.

Proof of Lemma 12.3: We take $\hat{T} \in (0, T_{\max})$ as small as

$$\hat{T}^{-2}\,|B(x_0, 2R)| \ge 1$$

and

$$\sup_{t\in[0,\hat{T}]} \big\|u(t)\big\|_{L^1(B(x_0,2R))} \le 8\pi - \delta.$$

Actually, the latter relation is obtained by (12.5). Then, we take ℓ by (12.22) and L according to Lemma 12.4. Furthermore, letting

$$m = 8\pi - \frac{\delta}{2} \quad \text{and} \quad \sigma_0 = \frac{2m}{\delta} - 1, \tag{12.28}$$

we take $T \in (0, \hat{T})$ as small as

$$\frac{\delta m\sigma_0^2}{2} - T(1+\sigma_0)\left\{LT + 2 + T^3(1+\sigma_0)^2\right\} \geq 0. \tag{12.29}$$

Given $\tau \in (0, T)$, we take $\tilde{\tau} = \tau/2$. Then, we rescale (s,t) to (y,t) by $y = s(t-\tilde{\tau})^{-2}$. We see that the function $j(y,t) = k(s,t)$ solves

$$\begin{aligned} j_t - 4\pi y(t-\tilde{\tau})^{-2} j_{yy} - \left\{j + L(t-\tilde{\tau})^2 y + 2(t-\tilde{\tau})y\right\}(t-\tilde{\tau})^{-2} j_y \\ \leq L(t-\tilde{\tau})^2 y \end{aligned}$$

for a.e. $(y,t) \in \big(0, (t-\tilde{\tau})^{-2}\,|B(x_0, 2R)|\big) \times (\tilde{\tau}, T)$. It holds also that

$$j\big(\,|B(x_0,2R)|\,(t-\tilde{\tau})^{-2}, t\big) = k\big(\,|B(x_0,2R)|\,, t\big) \leq 8\pi - \delta$$

for $t \in [0, T]$.

Here, we take the function

$$J(y) = \frac{m\sigma_0 y}{1+\sigma_0 y}$$

and apply the comparison theorem. First, we have $J(1) = 8\pi - \delta$. Then, putting

$$\begin{aligned} \mathcal{L}(J) \equiv{} & J_t - 4\pi y\,(t-\tilde{\tau})^{-2}\, J_{yy} \\ & - \big(J + L(t-\tilde{\tau})^2 y + 2(t-\tilde{\tau})y\big)(t-\tilde{\tau})^{-2} J_y - L(t-\tilde{\tau})^2 y, \end{aligned}$$

we obtain

$$\begin{aligned} & \mathcal{L}(J) = (t-\tilde{\tau})^{-2} \cdot (1+\sigma_0 y)^{-3} \cdot y \\ & \cdot \left\{\delta m\sigma_0^2/2 - (t-\tilde{\tau})(1+\sigma_0 y) \cdot \big(L(t-\tilde{\tau}) + 2 + (t-\tilde{\tau})^3(1+\sigma_0 y)^2\big)\right\}. \end{aligned}$$

by (12.28). Therefore, we have

$$\mathcal{L}J \geq 0 \geq \mathcal{L}j$$

for a.e. (s,t) by (12.29) and

$$j(y,\tilde{\tau}) = \lim_{t\downarrow\tau} k\big((t-\tilde{\tau})^2 y, t\big) = k(0,t) = 0 \leq J(y)$$

for $y \in (0, 1]$. We have also

$$j(1,t) = k\big((t-\tilde{\tau})^2, t\big) \leq k\big(\,|B(x_0,2R)|\,, t\big) < 8\pi - \delta = J(1)$$

for $t \in [0, T]$ by $T^{-2} |B(x_0, 2R)| \geq 1$ and

$$J(0) = 0 = j(0, t).$$

Then, the inequality

$$j(y, t) \leq J(y)$$

follows for $(y, t) \in (0, 1] \times [\tilde{\tau}, T]$ similarly to Proposition A.1 of [38].

This implies

$$k_s(0, t)(t - \tilde{\tau})^2 = j_y(0, t) = \lim_{h \downarrow 0} \frac{j(h, t)}{h} \leq \lim_{h \downarrow 0} \frac{J(h)}{h} = J_y(0) = m\sigma_0$$

for $t \in [\tilde{\tau}, T]$, and hence

$$\left\| u(t) \right\|_{L^\infty(B(x_0, 2R) \times [\tau, T])} \leq 4\tau^{-2} m\sigma_0$$

holds true. We have obtained (12.23), and the proof is complete. □

The case $x_0 \in \partial\Omega \cap \mathcal{S}$ is treated by the same idea. First, we take $R_3 \in (0, R)$ as

$$\overline{B(x_0, 4R_3)} \cap \mathcal{S} = \{x_0\}.$$

Without loss of generality, we assume $x_0 = 0$ and $\nu = (0, -1)$ at x_0. We have a conformal mapping

$$X = (X_1, X_2) : \Omega \cap B(0, 4R_3) \to \mathbf{R}^2_+$$

satisfying

$$X(\partial\Omega \cap B(0, 4R_3)) \subset \partial\mathbf{R}^2_+ \quad \text{and} \quad \frac{\partial X}{\partial x}(0) = \text{id},$$

where $\mathbf{R}^2_+ = \{(X_1, X_2) \mid X_2 > 0\}$. We assume also

$$B(0, 2R_3) \subset X(B(0, 3R_3))$$

and

$$X(B(0, R_3/2)) \subset B(0, R_3).$$

As in Chapter 5, we take a smooth function φ_1 satisfying $0 \leq \varphi_1 \leq 1$,

$$\varphi_1(x) = \begin{cases} 1 & (x \in B(0, R_3/2)), \\ 0 & (x \notin B(0, 3R_3)), \end{cases}$$

and $(\partial\varphi_1)/(\partial\nu) = 0$ on $\partial\Omega$. Making $R_3 > 0$ smaller, we assume

$$\varepsilon \equiv \sup_{x \in B(0,3R_3)\cap\Omega} \left| \frac{\partial X}{\partial x}(x) - \mathrm{id} \right| < \frac{1}{5},$$

furthermore. We put $T_4 = T_3(R_3/2)$ and $\ell_1 = C_3(R_3/2, \tau)$ in (12.21) for $\tau \in (0, T_4)$. This means

$$\left\| u^n \right\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega}\setminus\mathcal{S}_{R_3/2}\times[\tau,\min(T_4,T_{\max}))\right)} \leq \ell_1. \tag{12.30}$$

The following lemma is a modification of Lemma 12.4, where

$$P^n(s,t) = \int_0^s (u^n\varphi_1)^*(\theta, t)d\theta.$$

Lemma 12.5 *There is* $L \geq 0$ *satisfying*

$$\partial_t P^n - 2\pi(1-5\varepsilon)s\partial_s^2 P^n - \left(P^n + Ls\right)\partial_s P^n - Ls \leq 0 \tag{12.31}$$

for a.e. $(s,t) \in (0, |\Omega|) \times \left(\tau, \min\{T_4, T_{\max}^n\}\right)$ *and* $n \geq 1$. *It also holds that*

$$P^n\Big|_{s=0} = 0 \quad \textit{and} \quad \partial_s P^n\Big|_{s=|\Omega|} = 0$$

for $t \in \left(\tau, \min\{T_4, T_{\max}^n\}\right)$.

Proof: We drop the index n and put

$$T_5 = \min(T_4, T_{\max}).$$

Let

$$\widetilde{u\varphi_1}(\xi,t) = u\left(X^{-1}(\xi), t\right)\cdot\varphi_1\left(X^{-1}(\xi)\right)$$

and

$$\widetilde{v\varphi_1}(\xi,t) = v\left(X^{-1}(\xi), t\right)\cdot\varphi_1\left(X^{-1}(\xi)\right).$$

The even extensions of $\widetilde{u\varphi_1}$ and $\widetilde{v\varphi_1}$ with respect to $\partial\mathbf{R}_+^2$ are denoted by $\overline{u\varphi_1}$ and $\overline{v\varphi_1}$, respectively. Then, we see that $(u\varphi_1)^*$ is locally absolutely continuous on $(0, |\Omega|]$ by Lemma 12.2.

Taking $\tau \in (0, T_5)$. For $t \in (\tau, T_5)$, $\rho \in (0, \|u(t)\varphi_1\|_\infty)$, and $h > 0$, we put that

$$T_{\rho,h}(s) = \begin{cases} 0 & (s \leq \rho)\,, \\ s-\rho & (\rho < s \leq \rho + h)\,, \\ h & (s > \rho + h)\,. \end{cases}$$

Then it holds that

$$T_{\rho,h}(\overline{u\varphi_1}(\cdot,t)) \in W^{1,\infty}(B(0,2R_3))$$

and

$$T_{\rho,h}(u(\cdot,t)\varphi_1) \in W^{1,\infty}(\Omega).$$

Using the first equation of (12.1), we obtain

$$\begin{aligned}\int_\Omega (u\varphi_1)_t\, T_{\rho,h}(u\varphi_1)\,dx &= \int_\Omega u_t\varphi_1 T_{\rho,h}(u\varphi_1)\,dx \\ &= \int_\Omega \nabla\cdot(\nabla u - u\nabla v)\,\varphi_1\cdot T_{\rho,h}(u\varphi_1)\,dx \\ &= -\int_\Omega \nabla(u\varphi_1)\cdot\nabla T_{\rho,h}(u\varphi_1)\,dx \\ + \int_\Omega (u\varphi_1)\nabla v\cdot\nabla T_{\rho,h}(u\varphi_1)\,dx &- \int_\Omega T_{\rho,h}(u\varphi_1)\cdot g\,dx \\ &= -\mathrm{I} + \mathrm{II} - \mathrm{III} \end{aligned} \tag{12.32}$$

with

$$g = \nabla\cdot(u\nabla\varphi_1) + \nabla u\cdot\nabla\varphi_1 - u\nabla v\cdot\nabla\varphi_1.$$

Then, we estimate each term of the right-hand side as follows.

First, for the third term we have

$$\operatorname{supp}\ |\nabla\varphi_1| \cap B(0,R_3/2) = \emptyset,$$

and hence from (12.30) it follows that

$$\|g\|_{L^\infty(\Omega\times[\tau,T_5])} \le L_1,$$

where $L_1 > 0$ is a constant. This implies

$$\begin{aligned}\limsup_{h\downarrow 0}\left|\frac{\mathrm{III}}{h}\right| &= \limsup_{h\downarrow 0}\left|\frac{1}{h}\int_\Omega T_{\rho,h}\left(u(x,t)\varphi_1(x)\right) g(x,t)\,dx\right| \\ &\le L_1 \limsup_{h\downarrow 0}\frac{1}{h}\int_{\{u(\cdot,t)\varphi_1>\rho\}} h \le L_1\mu(\rho,t),\end{aligned}$$

where

$$\mu(\rho,t) = \left|\left\{x\in\Omega \;\middle|\; u(x,t)\varphi_1(x) > \rho\right\}\right|.$$

To estimate the first term of the right-hand side of (12.32), we use the fact that X is conformal. This implies

$$\int_E |\nabla f|^2 \, dx = \int_{X(E)} \left|\nabla \tilde{f}\right|^2 d\xi = \frac{1}{2} \int_{\overline{X}(E)} \left|\nabla \overline{f}\right|^2 d\xi$$

for any $f \in C^1\left(\overline{B(0, 3R_3) \cap \Omega}\right)$ and any measurable set $E \subset \overline{B(0, 3R_3) \cap \Omega}$, where $\overline{X}(E)$ denotes the even extension of $X(E)$. Therefore, it holds that

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{I}{h} &= \lim_{h\downarrow 0} \frac{1}{h} \int_\Omega \nabla(u\varphi_1) \cdot \nabla T_{\rho,h}(u\varphi_1) \, dx \\
&= \lim_{h\downarrow 0} \frac{1}{h} \left\{ \int_{\{u\varphi_1 > \rho\}} |\nabla(u\varphi_1)|^2 \, dx - \int_{\{u\varphi_1 > \rho + h\}} |\nabla(u\varphi_1)|^2 \, dx \right\} \\
&= \frac{1}{2} \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho + h \geq \overline{u\varphi_1} > \rho\}} \left|\nabla_\xi \left(\overline{u\varphi_1}\right)\right|^2 d\xi \\
&= -\frac{1}{2} \frac{\partial}{\partial \rho} \int_{\{\overline{u\varphi_1} > \rho\}} \left|\nabla_\xi \left(\overline{u\varphi_1}\right)\right|^2 d\xi.
\end{aligned}$$

Here, applying the co-area formula and the isoperimetric inequality, we obtain

$$4\pi m \left(-\frac{\partial m}{\partial \rho} \right)^{-1} \leq -\frac{\partial}{\partial \rho} \int_{\{\overline{u\varphi_1} > \rho\}} \left|\nabla_\xi \left(\overline{u\varphi_1}\right)\right|^2 d\xi$$

for a.e. ρ, where

$$m = m(s, t) \equiv \left| \left\{ \xi \in \operatorname{supp} \overline{\varphi_1} \;\middle|\; \overline{u\varphi_1}(\xi, t) > s \right\} \right|.$$

We have also

$$\begin{aligned}
\det \left(\frac{\partial X}{\partial x} \right) &= \frac{\partial X_1}{\partial x_1} \cdot \frac{\partial X_2}{\partial x_2} - \frac{\partial X_1}{\partial x_2} \cdot \frac{\partial X_2}{\partial x_1} \\
&\geq (1 - \varepsilon)^2 - \varepsilon^2 = 1 - 2\varepsilon
\end{aligned}$$

and

$$\det \left(\frac{\partial X}{\partial x} \right) \leq (1 + \varepsilon)^2 + \varepsilon^2 \leq 1 + 3\varepsilon$$

by $\varepsilon \in (0, 1/2)$. Therefore, it holds that

$$m(\rho, t) = \int_{\{\overline{u\varphi_1} > \rho\}} d\xi = \frac{1}{2} \int_{\{u\varphi_1 > \rho\}} \det \left(\frac{\partial X}{\partial x} \right) dx \geq \frac{1 - 2\varepsilon}{2} \mu(\rho, t)$$

and

$$
\begin{aligned}
-\frac{\partial m}{\partial \rho}(\rho, t) &= \lim_{h\downarrow 0} \frac{1}{h} \left(m(\rho, t) - m(\rho + h, t) \right) \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho+h \geq \overline{u\varphi_1} > \rho\}} d\xi \leq \frac{1+3\varepsilon}{2} \lim_{h\downarrow 0} \frac{1}{h} \int_{\{\rho+h \geq u\varphi_1 > \rho\}} dx \\
&= -\frac{1+3\varepsilon}{2} \frac{\partial \mu}{\partial \rho}(\rho, t).
\end{aligned}
$$

These relations are summarized as

$$
\lim_{h\downarrow 0} \frac{I}{h} \geq \frac{1}{2} \cdot 4\pi \cdot \frac{\mu}{1+3\varepsilon} \cdot (1-2\varepsilon) \cdot \Big(-\frac{\partial \mu}{\partial \rho} \Big)^{-1}
$$

or equivalently,

$$
\mu^{-1} \Big(-\frac{\partial \mu}{\partial \rho} \Big) \lim_{h\downarrow 0} \frac{I}{h} \geq \frac{1}{2} \cdot 4\pi \cdot \frac{1-2\varepsilon}{1+3\varepsilon} \geq 2\pi \cdot (1-5\varepsilon)
$$

for a.e. ρ.

To handle with the second term of the right-hand side of (12.32), we put

$$
\begin{aligned}
S_{\rho,h}(s) &= \int_0^s \tau \frac{d}{d\tau} T_{\rho,h}(\tau) d\tau \\
&= \begin{cases} 0 & (s \leq \rho), \\ \frac{1}{2}(s^2 - \rho^2) & (\rho < s \leq \rho + h), \\ h\left(\rho + \frac{h}{2}\right) & (s > \rho + h). \end{cases}
\end{aligned}
$$

Then, we have

$$
\mathrm{II} = \int_\Omega \nabla v \cdot \nabla S_{\rho,h}(u\varphi_1)\, dx = \int_\Omega (u - v) S_{\rho,h}(u\varphi_1)\, dx
$$

from the second equation of (12.1). This implies

$$
\begin{aligned}
\lim_{h\downarrow 0} \frac{\mathrm{II}}{h} &= \lim_{h\downarrow 0} \frac{1}{h} \int_\Omega (u\varphi_1) \nabla v \cdot \nabla T_{\rho,h}(u\varphi_1)\, dx \\
&= \lim_{h\downarrow 0} \frac{1}{2h} \int_{\{\rho+h \geq u\varphi_1 > \rho\}} (u - v) \left\{ (u\varphi_1)^2 - \rho^2 \right\} dx \\
&\quad + \lim_{h\downarrow 0} \int_{\{u\varphi_1 > \rho + h\}} (u - v) \Big(\rho + \frac{h}{2} \Big) dx = \rho \int_{\{u\varphi_1 > \rho\}} (u - v)\, dx \\
&\leq \rho \int_{\{u\varphi_1 > \rho\}} u\varphi_1\, dx + \rho \int_{\{u\varphi_1 > \rho\}} u(1 - \varphi_1)\, dx.
\end{aligned}
$$

Here, we have

$$\int_{\{u\varphi_1>\rho\}} u(1-\varphi_1)\,dx \le \ell_1\mu(\rho,t)$$

by (12.30) and supp $(1-\varphi_1)\cap B(0,R_3/2)=\emptyset$. Also, we have

$$\int_{\{u\varphi_1>\rho\}} u\varphi_1\,dx = \int_0^{\mu(\rho,t)} (u\varphi_1)^* d\theta = P(\mu(\rho,t),t)$$

and

$$\rho = (u\varphi)^*(\mu(\rho,t),t) = P_s(\mu(\rho,t),t).$$

Thus, we obtain

$$\lim_{h\downarrow 0}\frac{\mathrm{II}}{h} \le \left(P(\mu(\rho,t),t)+\ell_1\mu(\rho,t)\right) P_s\left(\mu(\rho,t),t\right).$$

On the other hand, the left-hand side of (12.32) is treated by Lemma 12.3. We have

$$\left|\{u\varphi_1=\rho\}\right| = 0 \quad \text{for} \quad \text{a.e.} \quad \rho$$

and hence it follows that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_\Omega (u\varphi_1)_t T_{\rho,h}(u\varphi)\,dx = \int_{\{u\varphi>\rho\}} (u\varphi_1)_t\,dx = P_t(\mu(\rho,t),t)$$

for any $t\in(0,T_5)$. Therefore, equality (12.32) implies

$$\begin{aligned}2\pi(1-5\varepsilon) \le \mu^{-1}\Big(-\frac{\partial\mu}{\partial\rho}\Big)\Big\{ &- P_t(\mu(\rho,t)) \\ &+ \big(P(\mu(\rho,t),t)+\ell_1\mu(\rho,t)\big)P_s(\mu(\rho,t),t)+L_1\mu(\rho,t)\Big\}.\end{aligned}$$

Then, integrating this inequality in $\rho\in(\rho_1,\rho_2)\subset[0,|\Omega|)$, we get

$$\begin{aligned}&2\pi(1-5\varepsilon)\,(\rho_2-\rho_1) \\ &\qquad\le \int_{\mu(\rho_2,t)}^{\mu(\rho_1,t)} s^{-1}\big\{-P_t(s,t)+(P(s,t)+\ell_1 s)P_s(s,t)+L_1 s\big\}ds.\end{aligned}$$

Here, we have

$$\rho_2-\rho_1 = (u\varphi_1)^*\left(\mu(\rho_2,t),t\right) - (u\varphi_1)^*\left(\mu(\rho_1,t),t\right),$$

and it holds that

$$\begin{aligned} 0 &\le -2\pi(1-5\varepsilon)\partial_s^2 P(s,t) \\ &= -2\pi(1-2\varepsilon)\left[\partial_s(u\varphi_1)^*\right](s,t) \\ &\le s^{-1}\left(-P_t + (P+\ell_1 s)P_s + L_1 s\right). \end{aligned}$$

Finally, we have $P_s \ge 0$; hence inequality (12.31) holds with $L = \max(\ell_1, L_1)$. The latter part of the lemma is immediate and the proof is complete. □

Here, we use the following lemma, where

$$\mathcal{L}(h) = \partial_t h - Bs\partial_s^2 h - (h + Ds)\partial_s h - Ds$$

is a second-order parabolic differential operator with the inhomogeneous term $-Ds$, and $B > 0$ and $D \ge 0$ are constants.

Lemma 12.6 *Let $A = A(t) \in C^1([0,T])$ be a positive function, and $f = f(s,t)$ and $g = g(s,t)$ be measurable functions defined on*

$$Q_T = \{(s,t) \mid 0 < t < T,\ 0 < s < A(t)\},$$

satisfying the following conditions for any $\delta > 0$:

(i) $f,\ g,\ f_t,\ g_t,\ f_s,\ g_s \in L^\infty(Q_T)$.

(ii) $\sup_{0\le t\le T}\left\{\|f(t)\|_{W^{2,1}(\delta,A(t))} + \|g(t)\|_{W^{2,1}(\delta,A(t))}\right\} < +\infty$.

(iii) $\mathcal{L}(f) \le \mathcal{L}(g)$ *a.e. in* Q_T.

(iv) $0 = f(0,t) \le g(0,t)$ *and* $f(A(t),t) \le g(A(t),t)$ *for* $t \in [0,T]$.

(v) $f(s,0) \le g(s,0)$ *for* $s \in [0, A(0)]$ *and* $g \ge 0$ *in* Q_T.

Then, the inequality $f \le g$ holds on Q_T.

Proof: Let

$$w = f - g \quad \text{and} \quad w_+ = \max\{w, 0\}.$$

By means of $g \ge 0$ and $f(0,t) = 0$, we have

$$w_+(s,t) \le f(s,t) \le c_1 s \tag{12.33}$$

in Q_T, where $c_1 = \|f_s\|_{L^\infty(Q_T)}$. In particular,

$$w_+/s \in L^\infty(Q_T)$$

holds true.

We have also

$$w_t - Bsw_{ss} - Dsw_s - (f \cdot f_s - g \cdot g_s) \leq 0$$

a.e. in Q_T. Multiplying w_+/s, we integrate it in $s \in [\delta, A(t)]$ for $\delta \in \big(0, \min_{t\in[0,T]} A(t)\big)$. Then, we get

$$\frac{1}{2}\frac{d}{dt}\int_\delta^{A(t)} s^{-1}w_+^2 ds \leq \int_\delta^{A(t)} Bw_{ss}w_+ ds + \frac{D}{2}\big[w_+^2\big]_{s=\delta}^{s=A(t)} + \int_\delta^{A(t)} (f \cdot f_s - g \cdot g_s)\, s^{-1}w_+ ds = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where $t \in [0, T]$. Here, we have

$$I = -Bw_s(\delta, t)w_+(\delta, t) - B\int_\delta^{A(t)} (w_{+s})^2\, ds$$

and

$$\mathrm{II} = -\frac{\mathrm{D}}{2}\,(\mathrm{w}_+(\delta, \mathrm{t}))^2 \leq 0$$

by $w_+(A(t), t) = 0$. Furthermore,

$$\mathrm{III} = \int_\delta^{A(t)} s^{-1}w_+^2 g_s ds + \int_\delta^{A(t)} s^{-1}w_+w_s f ds \leq c_2\int_\delta^{A(t)} \frac{w_+^2}{s} ds + c_1\int_\delta^{A(t)} w_+\,|w_{+s}|\, ds$$

holds by $w = f - g$ and (12.33), where $c_2 = \|g_s\|_{L^\infty(Q_T)}$.

Therefore, using

$$c_1\int_\delta^{A(t)} w_+\,|w_{+s}|\, ds \leq \frac{B}{2}\int_\delta^{A(t)} (w_{+s})^2\, ds + \frac{c_1^2\|A\|_{C([0,T])}}{2B}\int_\delta^{A(t)} s^{-1}w_+^2 ds,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\delta^{A(t)} s^{-1}w_+^2 ds + \frac{B}{2}\int_\delta^{A(t)} (w_{+s})^2\, ds \leq \left(c_2 + \frac{c_1^2\|A\|_{C([0,T])}}{2B}\right)\int_\delta^{A(t)} s^{-1}w_+^2 ds + B\,|w_s(\delta, t)w_+(\delta, t)|\,.$$

This implies

$$\frac{1}{2}\Big[\int_\delta^{A(t)} s^{-1} w_+^2 ds\Big]_{t=t_1}^{t=t_2} \le \left(c_2 + \frac{c_1^2 \|A\|_{C([0,T])}}{2B}\right) \int_{t_1}^{t_2} dt \cdot \int_\delta^{A(t)} s^{-1} w_+^2 ds + B \int_{t_1}^{t_2} dt \cdot \big| w_s(\delta, t) w_+(\delta, t) \big|$$

for $0 < t_1 < t_2 < T$. Here, $w_+(s, 0) = 0$ holds, and we have

$$\lim_{\delta \downarrow 0} \int_{t_1}^{t_2} |w_s(\delta, t) w_+(\delta, t)| \, dt = 0.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \int_0^{A(t)} s^{-1} w_+^2 ds \le \left(c_2 + \frac{c_1^2 \|A\|_{C([0,T])}}{2B}\right) \int_0^{A(t)} s^{-1} w_+^2 ds$$

follows for $t \in [0, T]$. This implies $\int_0^{A(t)} s^{-1} w_+(s, t)^2 ds = 0$ and the proof is complete. □

Now, we show the following lemma.

Lemma 12.7 *Suppose* $x_0 \in \mathcal{S} \cap \partial\Omega$ *in Theorem 12.1, and let us take* $R_3 \in (0, R)$ *and* $\delta_2 > 0$ *satisfying* $\overline{B(x_0, 4R_3)} \cap \mathcal{S} = \{x_0\}$ *and*

$$\|u_0^n\|_{L^1(B(x_0, 4R_3) \cap \Omega)} \le 4\pi - 2\delta_2$$

for $n \ge 1$. *Then, there is* $T_6 > 0$ *such that each* $\tau \in (0, T_6)$ *admits* $C_4 > 0$ *satisfying*

$$\|u^n\|_{C^{2+\theta, 1+\theta/2}\left(B(x_0, R_3) \cap \overline{\Omega} \times [\tau, \min(T_6, T_{\max}^n))\right)} \le C_4.$$

Proof: We omit the index n, and put $x_0 = 0$ without loss of generality. Making $R_3 > 0$ smaller, we assume that ε in (12.31) satisfies $\delta_2/(16\pi) > 2\varepsilon$.

Next, we take $\psi = \varphi_2$ in (5.11), where $0 \le \varphi_2 \le 1$,

$$\varphi_2(x) = \begin{cases} 1 & (x \in B(0, 3R_3)), \\ 0 & (x \notin B(0, 4R_3)), \end{cases}$$

and $(\partial\varphi_2)/(\partial\nu) = 0$ on $\partial\Omega$:

$$\left|\frac{d}{dt}\int_\Omega \psi(x)u(x,t)\,dx\right| \le \frac{K\lambda^2}{2}\left\|\nabla\psi\right\|_{C^1(\overline{\Omega})} + \lambda\left\|\Delta\psi\right\|_\infty.$$

This provides $T_7 > 0$ determined by Λ, Ω, and R_3 such that

$$2T_7^{-2}\left|\Omega \cap \operatorname{supp}\varphi_1\right| \ge 1$$

and

$$\sup_{t\in[0,T_7]}\left\|u(t)\right\|_{L^1(B(0,3R_3)\cap\Omega)} \le 4\pi - \frac{2\delta_2}{3}. \tag{12.34}$$

We take $\tau_1 \in (0, T_{\max})$ and set

$$j(\zeta,t) = P(s,t)$$

for $\zeta = s(t-\tau_1)^{-2}$. Putting

$$\begin{aligned}\mathcal{M}(j) \equiv \partial_t j - 2\pi\left(1 - \frac{\delta_1}{16\pi}\right)\zeta(t-\tau_1)^{-2} j_{\zeta\zeta} \\ - \left\{ j + L(t-\tau_1)^2\zeta + (t-\tau_1)\zeta \right\}(t-\tau_1)^{-2} j_\zeta - L(t-\tau_1)^2\zeta,\end{aligned}$$

we obtain

$$\mathcal{M}(j) \le 0$$

for a.e. $t \in (\tau_1, \min(T_7, T_{\max}))$ and $\zeta \in \left(0, (t-\tau_1)^{-2}\left|\Omega\right|\right)$ by

$$\delta_2/(16\pi) > 5\varepsilon, \qquad j_{\zeta\zeta} \le 0,$$

and (12.31).

Putting

$$J(\zeta) = \frac{m\sigma_0\zeta}{1+\sigma_0\zeta}$$

with $m = 4\pi - (\delta_2/2)$ and $\sigma_0 = (2m/\delta_2) - 1$, we have

$$J(1) = \frac{m\sigma_0}{1+\sigma_0} = \frac{\sigma_0\delta_2}{2} = m - \frac{\delta_2}{2} = 4\pi - \delta_2$$

and

$$P(\left|\Omega\right|, t) \le 4\pi - 2\delta_2$$

by (12.34). Hence it holds that

$$j(1,\tau_1) = P\big((t-\tau_1)^2, t\big) \le P\left(|\Omega|, t\right) \le 4\pi - \frac{3\delta_2}{2} = J(1) - \frac{\delta_2}{2} \tag{12.35}$$

for $t \in [0, T_7]$.

Next, we have

$$\begin{aligned} \zeta^{-1}&(t-\tau_1)^2(1+\sigma_0\zeta)^3\mathcal{M}(J) \\ &= \left\{4\pi\Big(1-\frac{\delta_2}{16\pi}\Big)m\sigma_0^2 - m^2\sigma_0^2\right\} - (t-\tau_1)(1+\sigma_0\zeta) \\ &\quad \cdot \left\{Lm\sigma_0(t-\tau_1) + m\sigma_0 + L(t-\tau_1)^3(1+\sigma_0\zeta)^2\right\} \\ &\ge \frac{\delta_2 m\sigma_0^2}{4} - (t-\tau_1)(1+\sigma_0\zeta) \\ &\quad \cdot \left\{Lm\sigma_0(t-\tau_1) + m\sigma_0 + L(t-\tau_1)^3(1+\sigma_0\zeta)^2\right\}. \end{aligned}$$

Therefore, taking $T_8 \in (\tau_1, T_7]$ in

$$\frac{\delta_2 m\sigma_0^2}{4} - T_8(1+\sigma_0)\left\{Lm\sigma_0 T_8 + m\sigma_0 + LT_8^3(1+\sigma_0)^2\right\} \ge 0,$$

we obtain

$$\mathcal{M}(J) \ge 0 \ge \mathcal{M}(j) \tag{12.36}$$

for $(\zeta, t) \in (0,1] \times [\tau_1, T_8]$. Finally, we have

$$\begin{aligned} j(\zeta,\tau_1) &= \lim_{t\downarrow\tau_1} P((t-\tau_1)^2\zeta, t) \\ &= P(0,\tau_1) = 0 < J(\zeta) \end{aligned} \tag{12.37}$$

for $\zeta \in (0,1]$. Using

$$\begin{aligned} j_\zeta(\zeta,t) &= (t-\tau_1)^2(u\varphi_1)^*\big((t-\tau_1)^2\zeta, t\big) \\ &\le (t-\tau_1)^2\left\|u(t)\right\|_\infty \end{aligned}$$

and

$$J_\zeta(\zeta) \ge (m\sigma_0)(1+\sigma_0)^{-2},$$

we can find τ_2 in $0 < \tau_2 - \tau_1 \ll 1$ satisfying

$$j(\zeta, \tau_2) \leq J(\zeta) \tag{12.38}$$

for $\zeta \in [0, 1]$.

Now, we apply Lemma 12.6 for

$$f(s,t) = P(s,t), \quad g(s,t) = J\left(s(t-\tau_1)^{-2})\right), \quad A(t) = (t-\tau_1)^2,$$

and

$$Q_T = \{(s,t) \mid \tau_2 < t < T_9,\ s \in (0, A(t))\},$$

where $T_9 \in (\tau_2, \min\{T_8, T_{\max}\})$ is arbitrary. In fact, conditions (i) and (ii) are obvious, while conditions (iii), (iv), and (v) follow from (12.36), (12.37), (12.35), $P(0,t) = J(0) = 0$, and (12.38). Thus, we obtain

$$P(s,t) \leq J\big(s(t-\tau_1)^{-2}\big)$$

for any $t \in [\tau_2, T_9]$ and $s \in [0, (t-\tau_1)^2]$.

Since T_9 is arbitrary, this means

$$P(s,t) \leq \frac{m\sigma_0 s}{(t-\tau_1)^2 + \sigma_0 s}$$

for $t \in [\tau_2, \min(T_8, T_{\max}))$ and $s \in [0, (t-\tau_1)^2]$. Finally, τ_2 and τ_1 are also arbitrary, and hence

$$P(s,t) \leq \frac{m\sigma_0 s}{t^2 + \sigma_0 s}$$

follows for $t \in (0, \min(T_8, T_{\max}))$ and $s \in (0, t^2]$.

Combining this with $P(0,t) = 0$, we obtain

$$\begin{aligned}(u\varphi)^*(0,t) = P_s(0,t) &\leq \partial_s\left(\frac{m\sigma_0 s}{t^2 + \sigma_0 s}\right)\Big|_{s=0} \\ &= (m\sigma_0)t^{-2},\end{aligned}$$

or $\left\|u(t)\varphi_1\right\|_{L^\infty(\Omega)} \leq m\sigma_0 t^{-2}$. Then, the standard bootstrap argument guarantees the conclusion. The proof is complete. □

We conclude this section with the following proof.

Proof of Theorem 12.1: Inequality (12.4) is a consequence of (12.21) and Lemmas 12.2 and 12.7.

To prove

$$T_* \equiv \liminf_{n\to\infty} T^n_{\max} > 0$$

in the latter case, we take $\{n'\} \subset \{n\}$ such that

$$T_* = \lim_{n'\to\infty} T^{n'}_{\max}.$$

Then, we have $\{u^{n''}\} \subset \{u^{n'}\}$ and $\mu_0(dx) \in \mathcal{M}(\overline{\Omega})$, satisfying

$$u^{n''} \rightharpoonup \mu_0(dx)$$

in $\mathcal{M}(\overline{\Omega})$ and the finite set $\mathcal{S}$ as above. The constants T_3, T_6, C_3, and C_4 are taken to be uniform in $x_0 \in \mathcal{S}$ in Lemmas 12.2 and 12.7.

Combining this with (12.21), we have

$$T_* = \lim_{n\to\infty} T^{n''}_{\max} > 0.$$

Furthermore, there exists $T \in (0, T_*)$ such that any $\tau \in (0, T)$ admits $C > 0$ satisfying

$$\left\| u^{n''} \right\|_{C^{2+\theta,1+\theta/2}\left(\overline{\Omega}\times[\tau,T]\right)} \leq C.$$

Therefore, $\{u^{n''}\} \subset C^{2,1}\left(\overline{\Omega} \times (0, T]\right)$ has a converging subsequence. Then, repeating the above argument, we obtain $T_* > 0$, and the proof is complete. □

13
Weak Solution

Our concern is focused on the problem of mass quantization,

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega), \end{cases}$$

in (11.3):

$$u(x,t)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx, \tag{13.1}$$

which arises as $t \uparrow T_{\max} < +\infty$ in $\mathcal{M}(\overline{\Omega})$. Here, $u = u(x,t)$ denotes the classical solution to (11.2):

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u\nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0,T),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T),$$
$$u\big|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \tag{13.2}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $a > 0$ is a constant.

To solve this problem we define the weak solution and study its blowup criterion as in Theorem 5.1. More precisely, formulation (5.8) to (13.2) makes it possible to introduce a measure-valued solution, and then the argument developed for the proof of Theorem 5.1 is applicable. If the initial measure contains a collapse

$$m(x_0)\delta_{x_0}(dx)$$

with $m(x_0) > m_*(x_0)$, then the solution does not exist even locally in time. This means that if a weak solution is constructed after the blowup time, then the collapses formed at the blowup time must have the quantized mass. In other words, mass quantization of collapses of the classical blowup solution is reduced to its post-blowup continuation. This principle was noticed by [147], and the present chapter is devoted to its proof. It is extended also to the case of a blowup in infinite time as described in the previous chapter.

In 1991, Victory [182] showed that a weak solution for the Fokker–Planck equation exists globally in time, provided that the initial distribution is regular. However, this is not the case in (13.2). In fact, although this system is the adiabatic limit of the Fokker–Planck equation, the status of particles treated in these systems is different. Thus, the mean field associated with (13.2) is thicker than the one for the Fokker–Planck equation, and consequently the strategy of constructing a weak solution globally in time to show the mass quantization of a collapse in the Keller–Segel system has not been successful. However, if we combine the weak solution with the *backward self-similar transformation*, then we can achieve our purpose; that is, we obtain the mass quantization of a collapse by the *parabolic envelope* and the blowup criterion of the rescaled system in Chapter 15.

The *weak formulation* of the classical solution to (13.1) has already been introduced as (5.8), in terms of the Green's function $G = G(x, x')$ of $-\Delta + a$ in Ω under the Neumann boundary condition, that is,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)\psi(x)\,dx - \int_\Omega u(x,t)\Delta\psi(x)\,dx \\ = \frac{1}{2}\iint_{\Omega\times\Omega} \rho_\psi(x,y)u(x,t)u(x',t)\,dx\,dx' \qquad (13.3)\end{aligned}$$

with $\rho_\psi = \rho_\psi(x, x')$ defined by

$$\rho_\psi(x,x') = \nabla\psi(x)\cdot\nabla_x G(x,x') + \nabla\psi(x')\cdot\nabla_{x'}G(x,x'), \qquad (13.4)$$

where $\psi = \psi(x)$ is a C^2 function defined on $\overline{\Omega}$ satisfying $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$. Introducing the notion of a *weak solution*, we take into account that the classical solution satisfies $\|u(t)\|_1 = \|u_0\|_1$ for $t \in [0, T_{\max})$. Therefore it is natural to put the weak solution into the space of measures. We recall also that convergence (13.1) holds in $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$, the set of measures on $\overline{\Omega}$, where $\rightharpoonup$ denotes the $*$-weak convergence. Henceforth, we shall write

$$\int_{\overline{\Omega}} \eta(x)\mu(dx) = \langle \eta, \mu \rangle_{C(\overline{\Omega}),\mathcal{M}(\overline{\Omega})} \tag{13.5}$$

for $\eta \in C(\overline{\Omega})$ and $\mu \in \mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$.

In Chapter 5, we have shown that $\rho_\psi \in L^\infty(\Omega \times \Omega)$ holds, but the argument provides more precise profiles. To describe this, we put

$$X = \left\{ \psi \in C^2(\overline{\Omega}) \mid \frac{\partial \psi}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}, \tag{13.6}$$

and $\mathcal{E} = \mathcal{E}_0 + C\left(\overline{\Omega} \times \overline{\Omega}\right) \subset L^\infty(\Omega \times \Omega)$ for

$$\mathcal{E}_0 = \left\{ \rho_\psi \mid \psi \in X \right\},$$

and take $\Omega_0 \subset\subset \Omega$, the covering

$$\Omega \setminus \Omega_0 \subset \bigcup_{k=1}^{m} B(x_k, r_k)$$

with $x_k \in \partial\Omega$ and $r_k > 0$, and the conformal mapping $X_k : B(x_k, r_k) \to \mathbf{R}^2$ satisfying $X_k(x_k) = 0$, $X_k\left(B(x_k, r_k) \cap \Omega\right) \subset \mathbf{R}^2_+$, and

$$X_k\left(B(x_k, r_k) \cap \partial\Omega\right) \subset \partial\mathbf{R}^2_+.$$

We define

$$G_k(x, x') = \frac{1}{2\pi} \log \frac{1}{|X_k(x) - X_k(x')|} + \frac{1}{2\pi} \log \frac{1}{|X_k(x) - X_k(x')_*|}$$

and

$$G_0(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|},$$

where $X_* = (X_1, -X_2)$ for $X = (X_1, X_2)$. Also, putting $U_0 = \Omega_0$ and $U_k = B(x_k, r_k)$ for $k = 1, 2, \ldots, m$, we take the partition of unity associated with the covering $\{U_k\}_{k=0}^m$ of $\overline{\Omega}$, denoted by $\{\varphi_k\}_{k=0}^m$. Then, given $\psi \in X$, we define

$$\rho^0_\psi(x, x') = \sum_{k=0}^{m} \left(\nabla\psi_k(x) \cdot \nabla_x G_k(x, x') + \nabla\psi_k(x') \cdot \nabla_{x'} G_k(x, x') \right)$$

for $\psi_k = \psi \cdot \varphi_k$.

Setting

$$\mathcal{E}_{00} = \left\{ \rho_\psi^0 \mid \psi \in X \right\},$$

we have

$$\mathcal{E} = \mathcal{E}_{00} + C\left(\overline{\Omega} \times \overline{\Omega}\right).$$

If $\psi \in X$ satisfies $\rho_\psi \in C(\overline{\Omega} \times \overline{\Omega})$, then

$$\lim_{h \to 0} \frac{(\nabla\psi(x+h) - \nabla\psi(x)) \cdot h}{|h|^2} = \lim_{h \to 0} \frac{D^2\psi(x)[h,h]}{|h|^2}$$

exists for any $x \in \overline{\Omega}$, where $D^2\psi$ denotes the Hess matrix of ψ. Diagonalizing $D^2\psi(x)$, we see that this is the case that $D^2\psi(x) = a(x)E$ holds, where E denotes the unit matrix and $a(x)$ is a scalar continuous function. Then, it follows that

$$\psi(x) = a\,|x|^2 + \mathbf{b} \cdot x + c$$

with some $a, c \in \mathbf{R}$ and $\mathbf{b} \in \mathbf{R}^2$, where $a = 0$ and $\mathbf{b} = 0$ follow from

$$\frac{\partial\psi}{\partial\nu} = (2ax + b) \cdot \nu = 0$$

on $\partial\Omega$. This means $\mathcal{E}_{00} \cap C(\overline{\Omega} \times \overline{\Omega}) = \{0\}$, and hence we obtain

$$\mathcal{E} = \mathcal{E}_{00} \oplus C\left(\overline{\Omega} \times \overline{\Omega}\right).$$

If $\rho \in \mathcal{E}_{00}$ satisfies $\rho = \rho_{\psi_1} = \rho_{\psi_2}$ for $\psi_1, \psi_2 \in X$, on the other hand, then it follows that $\rho_\psi = 0 \in C(\overline{\Omega} \times \overline{\Omega})$ for $\psi = \psi_1 - \psi_2$. This implies that $\psi(x)$ is a constant as we have seen, and therefore the mapping

$$[\psi] \in X/\mathbf{R} \quad \mapsto \quad \rho_\psi^0 \in \mathcal{E}_{00}$$

is an isomorphism. Thus, $\mathcal{E}_{00} \cong X/\mathbf{R}$ holds true, provided with the norm

$$\left\| [\psi] \right\|_X = \sum_{|\alpha|=1,2} \left\| D^\alpha \psi \right\|_\infty$$

for given $[\psi] \in X/\mathbf{R}$.

From this isomorphism, $\mathcal{E}$ is a separable Banach space, with the norm provided from $L^\infty(\Omega \times \Omega)$, and hence

$$L^1(0,T;\mathcal{E})' \cong L^\infty_*(0,T;\mathcal{E}')$$

follows from *Strassen's theorem* [70, 97]. Here, $T > 0$ is arbitrary and $L^\infty_*(0, T; \mathcal{E}')$ denotes the set of $\mathcal{E}'$-valued, $*$-weakly measurable, and essentially bounded functions on $(0, T)$. We note that $\mathcal{E}'$ is not separable, and the theorem of Pettis concerning the strong measurability does not work [191].

With these preparations, now we can introduce the notion of a weak solution to (13.2). We say that $\mu = \mu(dx, t)$ is a *weak solution* to (13.2) if the following conditions are satisfied, using the notations (13.6) and (13.5):

1. It belongs to $C_*\left([0, T), \mathcal{M}(\overline{\Omega})\right)$, that is,

$$\mu(dx, t) \in \mathcal{M}(\overline{\Omega}) = C'(\overline{\Omega})$$

   holds for $t \in [0, T)$ and the mapping

$$t \in [0, T) \quad \mapsto \quad \int_{\overline{\Omega}} \eta(x)\mu(dx, t) \tag{13.7}$$

   is continuous for each $\eta \in C(\overline{\Omega})$.

2. It is nonnegative and satisfies $\mu(dx, 0) = \mu_0(dx)$.

3. There exists $\nu = \nu(t) \geq 0$ belonging to $L^\infty_*\left(0, T'; \mathcal{E}'\right)$ for any $T' < T$ such that

$$\nu(t)\big|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu(dx\, dx', t) \tag{13.8}$$

   for a.e. $t \in (0, T)$.

4. The mapping defined by (13.7) is absolutely continuous if $\eta = \psi \in X$, and then the relation

$$\frac{d}{dt}\int_{\overline{\Omega}} \psi(x)\mu(dx, t) = \int_{\overline{\Omega}} \Delta\psi(x)\mu(dx, t) + \frac{1}{2}\left\langle \rho_\psi, \nu(t)\right\rangle_{\mathcal{E},\mathcal{E}'} \tag{13.9}$$

   holds for a.e. $t \in (0, T)$ for $\rho_\psi \in \mathcal{E}$ defined by (13.4).

Any classical solution $u = u(x, t)$ is regarded as a weak solution by $\mu(dx, t) = u(x, t)\, dx$. If $\mu(dx, t)$ is a weak solution, then we have

$$\left\|\mu(t)\right\|_{\mathcal{M}(\overline{\Omega})} = \int_{\overline{\Omega}} \mu(dx, t) = \int_{\overline{\Omega}} \mu_0(dx) = \left\|\mu_0\right\|_{\mathcal{M}(\overline{\Omega})} \tag{13.10}$$

for $t \in [0, T)$. This is obtained by putting $\psi = 1$ in (13.9) because $\mu = \mu(dx, t)$ is nonnegative. Furthermore, we have

$$\begin{aligned}\|\nu(t)\|_{\mathcal{E}'} &= \sup\left\{\langle\rho, \nu(t)\rangle_{\mathcal{E},\mathcal{E}'} \;\middle|\; \|\rho\|_{L^\infty(\Omega\times\Omega)} = 1,\ \rho \in \mathcal{E}\right\} \\ &\geq \sup\left\{\langle\eta, \nu(t)\rangle_{\mathcal{E}',\mathcal{E}} \;\middle|\; \|\eta\|_{L^\infty(\Omega\times\Omega)} = 1,\ \eta \in C(\overline{\Omega} \times \overline{\Omega})\right\} \\ &= \left\|\mu(t)\right\|^2_{\mathcal{M}(\overline{\Omega})} = \left\{\int_{\overline{\Omega}} \mu_0(dx)\right\}^2\end{aligned}$$

for a.e. $t \in (0, T)$ by (13.8) and $1 \in C(\overline{\Omega} \times \overline{\Omega})$. On the other hand, we have $\|\rho\|_\infty - \rho \geq 0$ in $\mathcal{E}$ for any $\rho \in \mathcal{E}$, and hence it follows that $\langle\|\rho\|_\infty - \rho, \nu(t)\rangle \geq 0$. This implies

$$\langle\rho, \nu(t)\rangle \leq \|\rho\|_\infty\left\{\int_{\overline{\Omega}} \mu_0(dx)\right\}^2$$

for a.e. $t \in (0, T)$. Similarly, we have

$$-\langle\rho, \nu(t)\rangle \leq \|\rho\|_\infty \left\{\int_{\overline{\Omega}} \mu_0(dx)\right\}^2,$$

and hence the relation

$$\left\|\nu(t)\right\|_{\mathcal{E}'} = \left\|\mu_0\right\|^2_{\mathcal{M}(\overline{\Omega})} \tag{13.11}$$

holds for a.e. $t \in (0, T)$. In particular, we have $\nu \in L^\infty_*(0, T; \mathcal{E}')$. For $\psi \in X$, on the other hand, equality (13.11) implies

$$\left|\frac{d}{dt}\int_{\overline{\Omega}} \psi(x)\mu(dx, t)\right| \leq \left\|\Delta\psi\right\|_\infty \left\|\mu_0\right\|_{\mathcal{M}(\overline{\Omega})} + \frac{1}{2}\left\|\rho_\psi\right\|_\infty \left\|\mu_0\right\|^2_{\mathcal{M}(\overline{\Omega})}$$

for a.e. $t \in (0, T)$. Combined with (13.10), this gives the existence of

$$\lim_{t\uparrow T} \mu(dx, t) = \mu(dx, T)$$

$*$-weakly in $\mathcal{M}(\overline{\Omega})$, and therefore the continuation after $t = T$ is examined by the existence of the weak solution locally in time with this $\mu(dx, T)$ as the initial measure. Thus, we study the existence and nonexistence of the weak solution to (13.2) locally in time, given the initial measure $\mu_0 = \mu_0(dx)$.

If

$$\mu_0(dx) = \mu^0_s(dx) + f(x)\, dx \tag{13.12}$$

denotes the Lebesgue – Radon – Nikodym decomposition of $\mu_0(dx)$, then $\mu^0_s(dx)$ ($\perp$ dx) is singular and $f(x)\, dx = \mu^0_{a.c.}(dx)$ is absolutely continuous with the nonnegative density function $f = f(x) \in L^1(\Omega)$. In this case, we have the following theorem.

Theorem 13.1 *If $\mu_s^0(\{x\}) < m_*(x)$ holds for any $x \in \overline{\Omega}$, then there is a weak solution to* (13.2) *locally in time, denoted by $\mu = \mu(dx,t) \in C_*\big([0,T), \mathcal{M}(\overline{\Omega})\big)$. This $\mu = \mu(dx,t)$ satisfies $\mu(dx,t) = u(x,t)\,dx$ for $0 < t \ll 1$ with a smooth $u = u(x,t) \geq 0$, and therefore we have the smoothing effect to the solution in this case.*

Theorem 13.2 *If there is $x_0 \in \overline{\Omega}$ satisfying*

$$\mu_s^0(\{x_0\}) > m_*(x_0),$$

then system (13.2) admits no weak solution.

As a consequence of Theorem 13.2, if

$$\mu = \mu(dx,t) \in C_*\left([0,T), \mathcal{M}(\overline{\Omega})\right)$$

is a weak solution of (13.2), $\mu(dx,t) = \mu_s(dx,t) + f(x,t)\,dx$ is its Lebesgue–Radon–Nikodym decomposition, and $S_i(t)$ denotes the set of isolated points of supp $\mu_s(dx,t)$, then it holds that

$$\mu_s(dx,t)\big|_{S_i(t)} = \sum_{k=1}^{N(t)} m_k(t)\delta_{x_k(t)}(dx) \tag{13.13}$$

with $N(t) \leq \lambda = \|\mu_0\|_{\mathcal{M}(\overline{\Omega})}$, $x_k(t) \in \overline{\Omega}$, and $m_k(t) \leq m_*(x_k(t))$. Coming back to the classical solution, the case $T^*_{\max} > T_{\max}$, referred to as the post-blowup continuation, can occur only when $m(x_0) = m_*(x_0)$ is satisfied for any $x_0 \in \mathcal{S}$ in (13.1), where $T^*_{\max}$ denotes the supremum of existence time of $\mu = \mu(dx,t)$ as a weak solution.

On the other hand, the first part of Theorem 13.1 is a consequence of the following principle concerning the construction of the weak solution. To prove the second part, we make use of the concentration lemma of the previous chapter. First, we show the following.

Theorem 13.3 *Let $\{u^n(x,t)\}$ be a family of classical solutions of* (13.2)*, where each $u^n = u^n(x,t)$ possesses the smooth initial value $u^n(\cdot,0) = u_0^n \geq 0$ satisfying*

$$u_0^n(x)\,dx \rightharpoonup \mu_0(dx)$$

in $\mathcal{M}(\overline{\Omega})$ with some measure $\mu_0 = \mu_0(dx)$ defined on $\overline{\Omega}$. Suppose that

$$T = \liminf_{n\to\infty} T^n_{\max} > 0$$

holds, where $T^n_{\max} \in (0, +\infty]$ denotes the supremum of the existence time of $u^n = u^n(x,t)$ as the classical solution. Then, there is $\{u^{n'}\} \subset \{u^n\}$ satisfying

$$u^{n'}(x,t)\,dx \rightharpoonup \mu(dx,t)$$

in $C_*\left([0,T), \mathcal{M}(\overline{\Omega})\right)$, where $\mu = \mu(dx,t)$ is a weak solution of (13.2) *with the initial measure $\mu_0 = \mu_0(dx) \geq 0$.*

Proof: From the assumption, we have $\Lambda = \sup_n \|u_0^n\|_1 < +\infty$. Putting

$$\mu^n(dx,t) = u^n(x,t)\,dx \geq 0, \qquad \mu_0^n(dx) = \mu^n(dx,0),$$

and

$$\nu^n(dx\,dx',t) = u^n(x,t)u^n(x',t)\,dx\,dx' \geq 0,$$

we have

$$\begin{aligned}\left\|\mu^n(t)\right\|_{\mathcal{M}(\overline{\Omega})} &= \int_{\overline{\Omega}} \mu^n(dx,t) = \left\|u_0^n\right\|_1 \leq \Lambda \\ \nu^n(t)\big|_{C(\overline{\Omega}\times\overline{\Omega})} &= \mu^n \otimes \mu^n(dx\,dx',t) \\ \left\|\nu^n(t)\right\|_{\mathcal{E}'} &= \left\|u^n(t)\right\|_1^2 \leq \Lambda^2,\end{aligned}$$

and

$$\begin{aligned}&\int_{\overline{\Omega}} \psi(x)\mu^n(dx,t) - \int_{\overline{\Omega}} \psi(x)\mu_0^n(dx) \\ &\qquad = \int_0^t ds \int_{\overline{\Omega}} \Delta\psi(x)\mu^n(dx,s) + \frac{1}{2}\int_0^t \left\langle \rho_\psi, \nu^n(s)\right\rangle_{\mathcal{E},\mathcal{E}'} ds\end{aligned}$$

for large n, where $\psi \in X$ and $t \in [0,T)$. Then, we can take $\{\mu^{n'}\} \subset \{\mu^n\}$ satisfying

$$\mu^{n'}(dx,t) \rightharpoonup \hat{\mu}(dx,t) \geq 0$$

in $L^\infty_*\left(0,T;\mathcal{M}(\overline{\Omega})\right) \cong L^1\left(0,T;C(\overline{\Omega})\right)'$ and

$$\nu^{n'}(dx\,dx',t) \rightharpoonup \nu(t) \geq 0$$

and in $L^\infty_*\left(0,T;\mathcal{E}'\right) \cong L^1\left(0,T;\mathcal{E}\right)'$, respectively. This implies

$$\left\|\hat{\mu}(\cdot,t)\right\|_{\mathcal{M}(\overline{\Omega})} \leq \Lambda, \qquad \left\|\nu(t)\right\|_{\mathcal{E}'} \leq \Lambda^2,$$

and

$$\nu(t)\big|_{C(\overline{\Omega}\times\overline{\Omega})} = \hat{\mu}\otimes\hat{\mu}(dx\,dx',t)$$

for a.e. $t \in (0,T)$. Furthermore, it holds that

$$\int_{\overline{\Omega}} \psi(x)\hat{\mu}(dx,t) - \int_{\overline{\Omega}} \psi(x)\mu_0(dx) \\ = \int_0^t ds \int_{\overline{\Omega}} \Delta\psi(x)\hat{\mu}(dx,s) + \frac{1}{2}\int_0^t \left\langle \rho_\psi, \nu(s)\right\rangle_{\mathcal{E},\mathcal{E}'} ds \tag{13.14}$$

for a.e. $t \in (0,T)$, where $\psi \in X$.

Since X is separable, we have dense $X_0 \subset X$ and measure zero $I_0 \subset (0,T)$ such that (13.14) holds for any $\psi \in X_0$ and $t \in (0,T)\setminus I_0$. Here, the right-hand side is continuous in $t \in [0,T)$, and therefore we have $h = h_\psi(t) \in C([0,T))$ such that

$$\int_{\overline{\Omega}} \psi(x)\hat{\mu}(dx,t) = h_\psi(t)$$

for $t \in [0,T) \setminus I_0$. This implies

$$\sup_{t\in[0,T)} \left|h_\psi(t)\right| \leq \Lambda \left\|\psi\right\|_\infty,$$

and there is $\mu(dx,t) \in \mathcal{M}(\overline{\Omega})$ satisfying

$$h_\psi(t) = \int_{\overline{\Omega}} \psi(x)\mu(dx,t)$$

for any $t \in [0,T)$ and $\psi \in X_0$. Then, it holds that

$$\hat{\mu}(dx,t) = \mu(dx,t)$$

in $\mathcal{M}(\overline{\Omega})$ for $t \in [0,T) \setminus I_0$. Furthermore, this $\mu = \mu(dx,t)$ satisfies

$$\nu(t)\big|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu\otimes\mu(dx\,dx',t) \tag{13.15}$$

for a.e. $t \in [0,T)$, and (13.14) for any $\psi \in X_0$ and $t \in [0,T)$ with $\hat{\mu}(dx,t)$ replaced by $\mu(dx,t)$:

$$\int_{\overline{\Omega}} \psi(x)\mu(dx,t) - \int_{\overline{\Omega}} \psi(x)\mu_0(dx) \\ = \int_0^t ds \int_{\overline{\Omega}} \Delta\psi(x)\mu(dx,s) + \frac{1}{2}\int_0^t \left\langle \rho_\psi, \nu(s)\right\rangle_{\mathcal{E},\mathcal{E}'} ds. \tag{13.16}$$

We can extend this equality of $t \in [0, T)$ to any $\psi \in X$ with the (Lipschitz) continuity of

$$t \in [0, T) \quad \mapsto \quad \int_{\overline{\Omega}} \psi(x)\mu(dx, t)$$

in terms of the right-hand side of (13.16). We have also

$$\sup_{t \in [0,T)} \left\| \mu(\cdot, t) \right\|_{\mathcal{M}(\overline{\Omega})} \leq \Lambda$$

and therefore $\mu = \mu(dx, t) \in C_* \left([0, T), \mathcal{M}(\overline{\Omega}) \right)$ follows from the left-hand side, because $X \subset C(\overline{\Omega})$ is dense and the continuity is preserved under the uniform convergence. We have also (13.15) for a.e. $t \in (0, T)$. If $\psi \in X$, the mapping

$$t \in [0, T) \quad \mapsto \quad \int_{\overline{\Omega}} \psi(x)\mu(dx, t)$$

is absolutely continuous by (13.16), and relation (13.9) holds for a.e. $t \in (0, T)$ with $\mu(dx, 0) = \mu_0(dx)$. It holds also (13.8) for a.e. $t \in (0, T)$, and therefore $\mu = \mu(dx, t) \in C_* \left([0, T), \mathcal{M}(\overline{\Omega}) \right)$ is a weak solution to (13.2). □

Now we give the following.

Proof of Theorem 13.1: We take $\Phi \in C_0^\infty(\mathbf{R}^2)$ with $0 \leq \Phi \leq 1$, supp $\Phi \subset \overline{B(0, 1)}$, and $\int_{\mathbf{R}^2} \Phi(x)\, dx = 1$. Then, we put $H(x, s) = s^{-2}\Phi(x/s)$ for $s > 0$ and

$$u_0^n(x) = \int_{\overline{\Omega}} H(x - y, n^{-1})\mu_0(dy) \;\text{ for }\; n = 1, 2, \ldots.$$

This $u_0^n = u_0^n(x)$ is a nonnegative smooth function satisfying

$$u_0^n(x)\, dx \;\rightharpoonup\; \mu_0(dx)$$

in $\mathcal{M}(\overline{\Omega})$.

Let $u^n = u^n(x, t)$ be the classical solution to (13.2) with the initial value $u_0 = u_0^n(x)$, and $T_{\max}^n \in (0, +\infty]$ be the supremum of its existence time. If $T_{\max}^n < +\infty$, we have a blowup point, denoted by $x_n \in \overline{\Omega}$. Then, we have (13.1) with $m(x_0) \geq m_*(x_0)$, and hence it holds that

$$\liminf_{t \uparrow T_{\max}^n} \left\| u^n(t) \right\|_{L^1(\Omega \cap B(x_n, R))} \geq m_*(x_n) \tag{13.17}$$

for any $R > 0$. Let a subsequence $\{x_{n'}\}$ of $\{x_n\}$ converge to $x_0 \in \overline{\Omega}$. In the case of $x_0 \in \Omega$, we may assume $m_*(x_{n'}) = 8\pi$ for any n':

$$\liminf_{t \uparrow T_{\max}^{n'}} \left\| u^{n'}(t) \right\|_{L^1(\Omega \cap B(x_{n'}, R))} \geq 8\pi.$$

Otherwise, we have $m_*(x_0) = 4\pi$. In any case we can replace (13.17) by

$$\liminf_{t \uparrow T_{\max}^{n'}} \left\| u^{n'}(t) \right\|_{L^1(\Omega \cap B(x_{n'}, R))} \geq m_*(x_0). \tag{13.18}$$

From the assumption to $\mu_0(dx)$ and its outer regularity, we have $R > 0$ and $\delta > 0$ satisfying

$$\mu_0 \left(\overline{\Omega} \cap B(x_0, 3R) \right) < m_*(x_0) - \delta.$$

Now, we recall (5.11) valid for $\psi \in X$:

$$\left| \frac{d}{dt} \int_\Omega \psi(x) u^{n'}(x, t) \, dx \right| \leq \frac{K\Lambda^2}{2} \left\| \nabla \psi \right\|_{W^{1,\infty}(\Omega)} + \Lambda \left\| \Delta \psi \right\|_\infty. \tag{13.19}$$

Here, $K > 0$ is a constant determined by Ω. Taking $\psi = \varphi_{x_{n'}, R, 2R}$, we get

$$\left| \frac{d}{dt} \int_\Omega \psi(x) u^{n'}(x, t) \, dx \right| \leq C(\Lambda + \Lambda^2)(R^{-1} + R^{-2})$$

with a constant $C > 0$, which implies that

$$\begin{aligned} \left\| u^{n'}(t) \right\|_{L^1(\Omega \cap B(x_{n'}, R))} &\leq \left\| u_0^{n'} \right\|_{L^1(\Omega \cap B(x_{n'}, 2R))} \\ &\quad + \int_0^t \left| \frac{d}{ds} \int_\Omega \psi(x) u^{n'}(x, s) \, dx \right| ds \\ &\leq \left\| u_0^{n'} \right\|_{L^1(\Omega \cap B(x_{n'}, 2R))} + tC(\Lambda + \Lambda^2)(R^{-1} + R^{-2}) \end{aligned}$$

for $t \in [0, T_{\max}^{n'})$.

Here, we have

$$\begin{aligned} \limsup_{n' \to \infty} \left\| u_0^{n'} \right\|_{L^1(\Omega \cap B(x_{n'}, 2R))} &\leq \limsup_{n' \to \infty} \left\| u_0^{n'} \right\|_{L^1(\Omega \cap B(x_0, 5R/2))} \\ &\leq \mu_0 \left(\overline{\Omega} \cap B(x_0, 3R) \right) < m_*(x_0) - \delta, \end{aligned}$$

and therefore for n' sufficiently large it holds that

$$\left\| u^{n'}(t) \right\|_{L^1(\Omega \cap B(x_{n'}, R))} < m_*(x_0) - \delta + tC(\lambda + \lambda^2)(R^{-2} + R^{-1}).$$

Making $t \uparrow T_{\max}^{n'}$, we obtain

$$T_{\max}^{n'} \geq \frac{\delta}{C(\lambda + \lambda^2)(R^{-2} + R^{-1})}$$

by (13.18). This implies

$$\hat{T} \equiv \lim_{n' \to \infty} T_{\max}^{n'} > 0$$

and Theorem 13.3 is applicable. Thus, we get a weak solution

$$\mu = \mu(dx, t) \in C_*\big([0, \hat{T}), \mathcal{M}(\overline{\Omega})\big)$$

to (13.2) with the given initial measure $\mu_0 = \mu_0(dx)$.

To prove the latter part, we use the concentration lemma, Theorem 12.1. In fact, from the assumption and the outer regularity, each $x \in \overline{\Omega}$ admits $r > 0$ satisfying

$$\mu_0\,(B(x, r)) < m_*(x).$$

Then, we have a constant $\delta > 0$ and a covering $\{B(x_k, r_k)\}_{k=1}^n$ of $\overline{\Omega}$ such that

$$\mu_0\,(B(x_k, r_k)) < m_*(x_k) - \delta$$

for $k = 1, 2, \ldots, n$, and therefore there exists $R > 0$ such that any $x \in \overline{\Omega}$ admits k satisfying $B(x, R) \subset B(x_k, r_k)$. This implies

$$\mu_0\,(B(x, R)) \leq m_*(x) - \delta$$

for any $x \in \overline{\Omega}$, which guarantees assumption (12.3) of Theorem 12.1:

$$\limsup_{n \to \infty} \left\| u_0^n \right\|_{L^1(B(x_0, R) \cap \Omega)} < m_*(x_0)$$

with $u_0^n = u_0^n(x) \geq 0$, denoting the regularization of $\mu_0 = \mu_0(dx)$ defined above.

Therefore, we have $T \in (0, \hat{T}]$, a subsequence $\left\{u^{n''}\right\} \subset \left\{u^{n'}\right\}$, and $u \in C^{2,1}\left(\overline{\Omega} \times (0, T]\right)$ with the classical solution $u^{n''}(x, t)$ with the initial value $u_0^{n''} = u_0^{n''}(x)$ converging to $u(x, t)$ locally uniformly in $(x, t) \in \overline{\Omega} \times (0, T]$. This gives

$$\mu(dx, t) = u(x, t)\, dx$$

for $t \in (0, T]$, and the proof is complete. □

Now we turn to the proof of Theorem 13.2. Let $\mu = \mu(dx,t)$ be a weak solution of (13.2) with the supremum of its existence time, denoted by $T^*_{\max}$. Given an open set $\omega \subset \mathbf{R}^3$ with $\omega \cap \overline{\Omega} \neq \emptyset$, we say that $\mu = \mu(dx,t)$ is extended in $\omega \cap \overline{\Omega}$ after $T^*_{\max}$, if the following conditions are satisfied for $T > T^*_{\max}$.

1. It has an extension to $C_*\left([0,T), \mathcal{M}(\overline{\Omega})\right)$, denoted by the same symbol, satisfying $\mu(dx,t) \geq 0$ and
$$\sup_{t\in[0,T)} \left\|\mu(t)\right\|_{\mathcal{M}(\overline{\Omega})} \leq \left\|\mu_0\right\|_{\mathcal{M}(\overline{\Omega})}.$$
2. It admits an extension of $\nu(t)$, denoted by the same symbol, as in $0 \leq \nu = \nu(t) \in L^\infty_*\left(0, T'; \mathcal{E}'\right)$ for any $T' < T$, satisfying (13.8) and (13.9) for a.e. $t \in (0,T)$, where ψ is an arbitrary function in $C^2_0(\omega)$ satisfying $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial\Omega$.

The supremum of such T is denoted by
$$T^*_{\max}(\omega).$$
Similarly to (13.11), the inequality
$$\left\|\nu(t)\right\|_{\mathcal{E}'} \leq \left\|\mu_0\right\|^2_{\mathcal{M}(\overline{\Omega})} \tag{13.20}$$
is proven for a.e. $t \in (0, T^*_{\max}(\omega))$. However, equality (13.9) is involved by the Green's function $G = G(x,y)$ on the whole domain and the above notion does not mean the time extension of $\mu = \mu(dx,t)$ on $\omega \cap \overline{\Omega}$ as a solution to (13.2).

Using the cut-off function $\varphi = \varphi_{x_0,R',R}(x)$ introduced in Chapter 5, we now shall show the following.

Theorem 13.4 *If there exists $x_0 \in \overline{\Omega}$ satisfying*
$$\mu_0\left(\{x_0\}\right) > m_*(x_0)$$
and
$$\int_{\overline{\Omega}} |x - x_0|^2\, \varphi_{x_0,R,2R}(x)^4 \mu_0(dx) = o(R^2)$$
as $R \downarrow 0$, then
$$T^*_{\max}\left(B(x_0,R)\right) = o(R^2)$$
follows.

Proof: The proof is similar to that of the blowup criterion using the second moment. First, the derivation of inequality (5.35) is valid for the weak solution, and therefore we have

$$\frac{dI_R}{dt}(t) \leq J_R(0) + a\big(R^{-1}t^{1/2}\big) + BR^{-1}I_R(t)^{1/2},$$

where

$$I_R(t) = \int_{\overline{\Omega}} m(x)\psi_R(x)\mu(dx, t)$$
$$M_R(t) = \int_{\overline{\Omega}} \psi_R(x)\mu(dx, t)$$
$$J_R(t) = 4M_R(t) - \frac{4M_R(t)^2}{m_*(x_0)} + 8BR^{-1}I_{4R}(t)^{1/2}$$

for $0 < R \ll 1$, $\psi_R = \varphi^4_{x_0,R,2R}$,

$$m(x) = \begin{cases} |x - x_0|^2 & (x_0 \in \Omega), \\ |X(x)|^2 / \left|\frac{\partial X}{\partial x}(x_0)\right| & (x_0 \in \partial\Omega), \end{cases}$$

and

$$B = C_*\big(\lambda^{3/2} + \lambda^{1/2}\big),$$
$$a(s) = C_*\big(\lambda^{1/2} + \lambda^3\big)(s^2 + s)$$
$$\lambda = \mu_0(\overline{\Omega})$$

with a constant $C_* > 0$ determined by Ω.

From this inequality, if $J_R(0) \leq -A < 0$,

$$\frac{1}{R^2}I_R(0) < \left(\frac{A}{24B}\right)^2 \quad \text{and} \quad I_R(0) < \frac{R^2}{6}a^{-1}\left(\frac{A}{4}\right)^2,$$

then it holds that

$$T \equiv a^{-1}(A/4)^2 \cdot R^2 \geq T^*_{\max}\,(B(x_0, R))$$

from the proof of Theorem 14.1. Then, letting $R \downarrow 0$, we can make $A \downarrow 0$ from the assumption, and the proof is complete. □

Now we give the following.

Proof of Theorem 13.2: The measure

$$\sigma_0(dx) = \mu_0(dx) - \mu_0(\{x_0\})\,\delta_{x_0}(dx)$$

is nonnegative and satisfies $\sigma_0(\{x_0\}) = 0$. Hence it holds that

$$\begin{aligned}\frac{1}{R^2}\int_{\overline{\Omega}} &|x - x_0|^2\,\varphi_{x_0,R,2R}(x)^4\mu_0(dx)\\ &= \frac{1}{R^2}\int_{\overline{\Omega}} |x - x_0|^2\,\varphi_{x_0,R,2R}(x)^4\sigma_0(dx)\\ &\le 4\sigma_0\left(\overline{\Omega}\cap B(x_0, 2R)\right) = o(1)\end{aligned}$$

as $R \downarrow 0$. From this relation, combined with the assumption

$$\mu_s^0(\{x_0\}) > m_*(x_0),$$

we see that Theorem 13.4 is applicable. Then we obtain $T^*_{\max} = 0$ by

$$T^*_{\max} \le T^*_{\max}(B(x_0, R)).$$

The proof is complete. □

14
Hyperparabolicity

We are concerned with the classical solution $u = u(x, t)$ to (11.2):

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0, T),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$
$$u\big|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \tag{14.1}$$

and study the problem of mass quantization,

$$m(x_0) = m_*(x_0) \tag{14.2}$$

in (11.3):

$$u(x, t)\,dx \ \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx \tag{14.3}$$

as $t \uparrow T_{\max}$. Here, $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, and

$$m_*(x_0) = \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

In this chapter, we show that the mass quantization (14.2) occurs to (14.3) if the blowup point $x_0 \in \mathcal{S}$ is *hyperparabolic*. This means

$$\lim_{t \uparrow T_{\max}} M_{bR(t),x_0}(t) = m(x_0) \tag{14.4}$$

for any $b > 0$, where $R(t) = (T - t)^{1/2}$,

$$M_{R,x_0}(t) = \int_\Omega \psi_{R,x_0}(x) u(x,t)\, dx,$$

$T = T_{\max}$, and $\psi_{R,x_0} = \varphi_{x_0,R,2R}$, and the existence of the limit in the left-hand side of (14.4) is also assumed. This suggests the necessity of the backward self-similar transformation for the proof of Theorem 1.2.

First, we note that relation (14.3) implies

$$\lim_{R \downarrow 0} \lim_{t \uparrow T_{\max}} M_{R,x_0}(t) = m(x_0).$$

Next, $y = (x - x_0)/R(t)$ is the standard backward self-similar variable, and it always holds that

$$\limsup_{t \uparrow T_{\max}} M_{bR(t),x_0}(t) \le m(x_0),$$

and the hyperparabolicity of $x_0 \in \mathcal{S}$ means

$$\lim_{b \downarrow 0} \liminf_{t \uparrow T_{\max}} M_{bR(t),x_0}(t) \ge m(x_0).$$

Thus, at the hyperparabolic blowup point, the process of the formation of collapses, (14.3), is reduced to the infinitely small parabolic region, which we call the *hyperparabola*, which is associated with the backward self-similar transformation. This is not the case with subcritical nonlinearity (Chapter 1), and motivated by this we next study the rescaled solution. We show that if it develops the singularity, then it is a sum of delta functions, which we call *subcollapses*. Their masses are quantized similarly to the case formed by the blowup in infinite time of the solution to the prescaled system. In the next chapter we show that the total mass of the collapse in consideration of the prescaled solution, denoted by $m(x_0)$, is preserved under the transformation $y_b = (x - x_0)/b(T - t)^{1/2}$ if we make $b \to +\infty$. Therefore, if the residual term of the limit measure of that rescaled solution vanishes, then the collapse mass $m(x_0)$ satisfies $m(x_0)/m_*(x_0) \in \mathbf{N}$. This virtual infinitely wide parabolic region is called the *parabolic envelope*.

If the rescaled solution develops the singularity, then the blowup point is called *type (II)*. Its blowup mechanism is simple, and in the next chapter we will show that the limit measure described above is composed of one subcollapse located on the origin. This means $\lim_{k\to\infty} M_{bR(t_k),x_0}(t_k) = m(x_0)$ for any $b > 0$ if $t_k \to T_{\max}$ satisfies

$$\lim_{k\to\infty} \sup_{x\in\Omega,\ |x-x_0|\le CR(t_k)} R(t_k)^2 u(x,t_k) = +\infty$$

for some $C > 0$, namely, the type (II) blowup point is regarded as hyperparabolic along the above time sequence $t_k \to T_{\max}$.

The following theorem motivates us to introduce the method of rescaling in the study of mass quantization, where it is shown that the hyperparabolicity implies the mass quantization. It is a weak version of Theorem 1.2, and the proof is not hard.

Theorem 14.1 *If $x_0 \in \mathcal{S}$ is hyperparabolic, then $m(x_0) = m_*(x_0)$.*

Proof: Similarly to the proof of Theorem 13.4, we start with inequality (5.35):

$$\frac{d}{dt} I_R(t) \le J_R(0) + a\big(R^{-1}t^{1/2}\big) + BR^{-1}I_R(t)^{1/2},$$

where

$$\begin{aligned} I_R(t) &= \int_\Omega m(x)u(x,t)\psi_R(x)\,dx \\ M_R(t) &= \int_\Omega u(x,t)\psi_R(x)\,dx \\ J_R(t) &= 4M_R(t) - \frac{4M_R(t)^2}{m_*(x_0)} + 8BR^{-1}I_{4R}(t)^{1/2} \end{aligned}$$

with $0 < R \ll 1$, $\psi_R = \varphi^4_{x_0,R,2R}$,

$$m(x) = \begin{cases} |x-x_0|^2 & (x_0 \in \Omega), \\ |X(x)|^2 / \left|\frac{\partial X}{\partial x}(x_0)\right| & (x_0 \in \partial\Omega), \end{cases}$$

and

$$\begin{aligned} Bz &= C_*\big(\lambda^{3/2} + \lambda^{1/2}\big), \\ a(s) &= C_*\big(\lambda^{1/2} + \lambda^3\big)(s^2 + s), \end{aligned}$$

with constant $C_* > 0$ determined by Ω.

First, we take the case that

$$J_R(0) = -A < 0 \tag{14.5}$$

and

$$T \equiv a^{-1}(A/4)^2 \cdot R^2 < T_{\max}. \tag{14.6}$$

Then, for $t \in [0, T]$ we have

$$a(R^{-1}t^{1/2}) \leq a(R^{-1}T^{1/2}) = A/4,$$

and hence

$$\frac{dI_R}{dt} \leq -\frac{A}{4} + BR^{-1}I_R^{1/2}$$

holds true. If

$$\frac{I_R(0)}{R^2} < \left(\frac{A}{24B}\right)^2$$

and

$$I_R(0) < \frac{A}{6} \cdot T = \frac{R^2}{6}a^{-1}\left(\frac{A}{4}\right)^2,$$

then we have

$$\left.\frac{dI_R}{dt}\right|_{t=0} \leq -\frac{A}{6},$$

and therefore the inequalities

$$\frac{I_R(t)}{R^2} < \left(\frac{A}{24B}\right)^2$$

and

$$\frac{dI_R}{dt} \leq -\frac{A}{6}$$

are preserved in $t \in [0, T]$. This implies

$$I_R(T) \leq I_R(0) - \frac{A}{6} \cdot T < 0,$$

a contradiction. In other words,

$$\frac{I_R(0)}{R^2} \geq \min\left\{\frac{a^{-1}(A/4)^2}{6}, \left(\frac{A}{24B}\right)^2\right\}$$

holds in this case of (14.5) and (14.6).

Otherwise, we have either $J_R(0) \geq 0$ or $J_R(0) < 0$ and $T \geq T_{\max}$, and the latter case is written simply as

$$-J_R(0) \geq 4 \cdot a\left(T_{\max}^{1/2}/R\right). \tag{14.7}$$

Thus, all possibilities are classified into either (14.7) or

$$\frac{I_R(0)}{R^2} \geq \min\left\{\frac{1}{6}a^{-1}\Big(\min\Big(0, -\frac{J_R(0)}{4}\Big)\Big), \min\Big(0, \frac{-J_R(0)}{24B}\Big)^2\right\}.$$

Therefore, since system (14.1) is autonomous in t, the following alternatives hold for each $R \in (0, 1]$ and $t \in [0, T_{\max})$:

1. $-J_R(t) \geq 4 \cdot a\left((T_{\max} - t)^{1/2}/R\right)$
2. $\frac{I_R(t)}{R^2} \geq \min\left\{\frac{1}{6}a^{-1}\big(\min\big(0, -\frac{J_R(t)}{4}\big)\big), \min\big(0, -\frac{J_R(t)}{24B}\big)^2\right\}$.

Now we show the following lemma using

$$R(t) = (T_{\max} - t)^{1/2}.$$

Lemma 14.1 *If $x_0 \in \mathcal{S}$ is hyperparabolic, then we have*

$$\lim_{t \uparrow T_{\max}} \frac{I_{bR(t)}(t)}{R(t)^2} = 0 \tag{14.8}$$

for each $b > 0$.

Proof: By the assumption it holds that

$$\lim_{t \uparrow T_{\max}} \left\{M_{bR(t)}(t) - M_{\varepsilon R(t)}(t)\right\} = 0$$

for each $\varepsilon \in (0, b)$. Here, we have

$$\begin{aligned}
\frac{I_{bR(t)}(t)}{R(t)^2} &= \frac{1}{R(t)^2}\int_\Omega m(x)\psi_{bR(t)}(x)u(x,t)\,dx \\
&= \frac{1}{R(t)^2}\int_\Omega m(x)\left(\psi_{bR(t)}(x)-\psi_{\varepsilon R(t)}(x)\right)u(x,t)\,dx \\
&\quad + \frac{1}{R(t)^2}\int_\Omega m(x)\psi_{\varepsilon R(t)}(x)u(x,t)\,dx \\
&= \frac{1}{R(t)^2}\int_{\left\{x\in\Omega \,\middle|\, |x-x_0|\le 2bR(t)\right\}} m(x)\left(\psi_{bR(t)}(x)-\psi_{\varepsilon R(t)}(x)\right) \\
&\quad \cdot u(x,t)\,dx + \frac{1}{R(t)^2}\int_\Omega m(x)\psi_{\varepsilon R(t)}(x)u(x,t)\,dx \\
&\le Cb^2\int_\Omega \left(\psi_{R(t)}(x)-\psi_{\varepsilon R(t)}(x)\right)u(x,t)\,dx + C\varepsilon^2\lambda \\
&= Cb^2\left\{M_{R(t)}(t)-M_{\varepsilon R(t)}(t)\right\} + C\varepsilon^2\lambda
\end{aligned}$$

with a constant $C > 0$ determined by Ω. Therefore, making $t \uparrow T_{\max}$ and then $\varepsilon \downarrow 0$, we obtain (14.8) and the proof is complete. □

Returning to the proof of Theorem 14.1, first we use

$$\begin{aligned}
J_{bR(t)}(t) &= 4M_{bR(t)}(t) - \frac{4M_{bR(t)}(t)^2}{m_*(x_0)} + 8Bb^{-1}R(t)^{-1}I_{4bR(t)}(t)^{1/2} \\
&\to \quad 4m(x_0) - \frac{4m(x_0)^2}{m_*(x_0)}
\end{aligned}$$

as $t \to T_{\max}$ by (14.4) and Lemma 14.1. Applying the alternatives (i) and (ii) with $R = bR(t)$ for each $t \in [0, T_{\max})$, we get

$$4m(x_0) - \frac{4m(x_0)^2}{m_*(x_0)} \begin{cases} \le -4a(b^{-1}) \\ \text{or} \\ \ge 0 \end{cases} \tag{14.9}$$

again by Lemma 14.1.

Here, the first alternative is impossible if $b > 0$ is small because $\lim_{s\to+\infty} a(s) = +\infty$ holds true. Then the second alternative implies

$$m(x_0) \le m_*(x_0)$$

by $m(x_0) > 0$, and the proof is complete. □

Two alternatives in (14.9) correspond almost to the types (I) and (II) of $x_0 \in \mathcal{S}$, respectively, and here, we can emphasize the importance of the rescaled system in accordance with the standard backward self-similar transformation. In the latter part of this chapter, we show the formation of subcollapses in the rescaled solution, supposing $T = T_{\max} < +\infty$ in (14.1) and $x_0 \in \mathcal{S}$. Thus, we define the transformation

$$z(y,s) = (T-t)u(x,t) \quad \text{and} \quad w(y,s) = v(x,t) \tag{14.10}$$

with

$$y = (x - x_0)/(T-t)^{1/2} \quad \text{and} \quad s = -\log(T-t).$$

Then it follows that

$$\begin{aligned} \left. \begin{aligned} z_s &= \nabla \cdot (\nabla z - z\nabla w - yz/2)t) \\ 0 &= \Delta w + z - ae^{-s}w \end{aligned} \right\} \quad &\text{in} \quad \mathcal{O}, \\ \partial z/\partial \nu = \partial w/\partial \nu = 0 \quad &\text{on} \quad \Gamma, \\ z\big|_{s=-\log T} = z_0 \quad &\text{in} \quad \mathcal{O}_0, \end{aligned} \tag{14.11}$$

where $\mathcal{O} = \ cup_{s>-\log T}\mathcal{O}_s \times \{s\}$, $\Gamma = \cup_{s>-\log T}\Gamma_s \times \{s\}$, and $z_0(y) = Tu_0(x_0 + T^{1/2}y)$ with $\mathcal{O}_s = e^{s/2}(\Omega - \{x_0\})$ and $\Gamma_s = \partial\mathcal{O}_s = e^{s/2}(\partial\Omega - \{x_0\})$.

Since this (z,w) is regarded as a solution to (14.11) globally in time, a relation similar to (12.2) is proven. Henceforth, if $x_0 \in \partial\Omega$, then H denotes the half-space in $\mathbf{R}^2$ containing $0 \in \partial H$, with ∂H parallel to the tangential line of $\partial\Omega$ at $x = x_0$. Furthermore, we put

$$L = \begin{cases} H & (x_0 \in \partial\Omega), \\ \mathbf{R}^2 & (x_0 \in \Omega). \end{cases}$$

The following theorem describes the mass quantization of subcollapses [149, 151].

Theorem 14.2 *Any $s_n \to +\infty$ admits a subsequence $\{s_n'\} \subset \{s_n\}$ such that*

$$z(y, s_n')dy \rightharpoonup \sum_{y_0 \in \mathcal{B}} m_*(y_0)\delta_{y_0}(dy) + g(y)dy \tag{14.12}$$

in $\mathcal{M}(\mathbf{R^2})$ as $n \to \infty$, where $g = g(y) \in L^1(L)$ is a nonnegative function and $\mathcal{B} \subset \overline{L}$ denotes the set of exhausted blowup points of $\left\{z(\cdot, s_n')\right\}$, namely, $y_0 \in \mathcal{B}$ if and only if there exists $\{y_n\} \subset \mathbf{R}^2$ such that $z(y_n, s_n') \to +\infty$ as $n \to \infty$, where zero extension of $z = z(y,s)$ is taken where it is not defined. The case $\mathcal{B} = \emptyset$ is also admitted.

Before starting the proof, we confirm several fundamental properties of (z, w) regarded as a solution to (14.11). Henceforth, C_i $(i = 1, 2, \dots)$ denote generic positive constants. First, the L^1 estimate of the prescaled system guarantees

$$\sup_{t\in[0,T_{\max})} \left\{ \left\| \nabla v(t) \right\|_q + \left\| v(t) \right\|_p \right\} \leq C_1$$

if $q \in [1, 2)$ and $p \in [1, \infty)$. In what follows, we shall argue the case $x_0 \in \partial\Omega$ mostly, because the other is easier to treat. Thus, we have $L = H$, and take $R_1 > 0$ and the conformal mapping

$$X : \overline{B(x_0, 4R_1) \cap \Omega} \to \mathbf{R^2}$$

such that

$$\begin{aligned} X(x_0) &= 0, \\ X\left(B(x_0, 4R_1) \cap \Omega\right) &\subset \mathbf{R}^2_+, \\ X\left(B(x_0, 4R_1) \cap \partial\Omega\right) &\subset \partial\mathbf{R}^2_+, \\ \frac{\partial X}{\partial x}(x_0) &= \mathrm{id}. \end{aligned}$$

Then, for $x, x' \in \overline{B(x_0, R_1) \cap \Omega}$ we obtain

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')|} + \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')_*|} + K(x, x'),$$

where

$$K \in C^{\theta,1+\theta} \cap C^{1+\theta,\theta} \left(\overline{B(x_0, R_1) \cap \Omega} \times \overline{B(x_0, R_1) \cap \Omega} \right),$$

$G = G(x, x')$ denotes the Green's function for $-\Delta + a$ in Ω under the Neumann boundary condition, $\theta \in (0, 1)$, and $X_* = (x_1, -x_2)$ for $X = (x_1, x_2)$. This implies

$$\left| \nabla_x G(x, x') \right| \leq C_2 \left(\frac{1}{|x - x'|} + 1 \right) \tag{14.13}$$

for x and $x' \in \overline{\Omega}$ with $x \neq x'$. Defining

$$\mathcal{G}(y, y', s) = G(e^{-s/2} y + x_0, e^{-s/2} y' + x_0)$$

for $y, y' \in \overline{B(x_0, e^{s/2}R_1) \cap \mathcal{O}(s)}$, we have also

$$\mathcal{G}(y, y', s) = \frac{1}{2\pi} \log \frac{1}{|Y(y,s) - Y(y',s)|} + \frac{1}{2\pi} \log \frac{1}{|Y(y,s) - Y(y',s)_*|} + \mathcal{K}(y, y', s) + s/(2\pi),$$

where $Y(y,s) = e^{s/2} X(e^{-s/2}y + x_0)$ and

$$\mathcal{K}(\cdot,\cdot,s) = \mathcal{K}(y, y', s) = K(e^{-s/2}y + x_0, e^{-s/2}y' + x_0) \in C^{\theta,1+\theta} \cap C^{1+\theta,\theta}\big(\overline{B(x_0, e^{s/2}R_1) \cap \Omega} \times \overline{B(x_0, e^{s/2}R_1) \cap \Omega}\big).$$

This $\mathcal{G} = \mathcal{G}(\cdot,\cdot,s)$ is nothing but the Green's function $-\Delta + ae^{-s}$ in $\mathcal{O}(s)$ under the Neumann boundary condition on $\partial\mathcal{O}(s)$.

Given $y_0 \in \overline{H}$ and $0 < R' < R \ll 1$, we introduce the cut-off function $\Phi = \Phi^s_{y_0,R',R}$ defined for $s \gg 1$, modifying $\varphi = \varphi_{x_0,R',R}$ of Chapter 5. Thus, we take $\Phi = \varphi$ in the case of $y_0 \in H$. If $y_0 \in \partial H$, then we take $\Phi = \zeta_{r',r} \circ Y(\cdot, s)$ instead of $\varphi = \zeta_{r',r} \circ X$. Then this smooth $\Phi = \Phi^s_{y_0,R',R}$ satisfies $0 \le \Phi \le 1$, $\Phi = 1$ in $B(y_0, R') \cap \mathcal{O}_s$, $\Phi = 0$ in $\mathcal{O}_s \setminus B(y_0, R)$, and $\frac{\partial \Phi}{\partial \nu} = 0$ on $\partial\mathcal{O}(s)$.

First, we use

$$\frac{\partial Y}{\partial s} = \frac{1}{2} e^{s/2}\big(X(e^{-s/2}y + x_0) - X(x_0)\big) - \frac{1}{2}\frac{\partial X}{\partial x}(e^{-s/2}y + x_0)y = \frac{1}{2}\Big[\frac{\partial X}{\partial x}(\theta e^{-s/2}y + x_0) - \frac{\partial X}{\partial x}(e^{-s/2} + x_0)\Big]y$$

for $\theta \in (0,1)$, which implies

$$\Big|\frac{\partial Y}{\partial s}(y,s)\Big| = O\big(e^{-s/2}\,|y|^2\big).$$

On the other hand, we have $\frac{\partial \Psi}{\partial s} = \nabla \zeta_{r',r} \frac{\partial Y}{\partial s}$ and therefore $r' \le |Y(y,s)| \le r$ holds if this term does not vanish. Writing

$$e^{-s/2}Y = X\big(e^{-s/2}y + x_0\big),$$

we see that this implies $|y| = O(1)$. In this way, the s-dependence of Φ is mild, and we have

$$\Big\|\frac{\partial}{\partial s}\Phi^s_{y_0,R',R}\Big\|_\infty \le C_3 e^{-s/2}.$$

Defining $\Psi(y,s)=\Phi^s_{y_0,R',R}(y)^6$, we have

$$|\nabla\Psi| \leq A(R-R')^{-1}\Psi^{5/6}$$

and $|D^\alpha\Psi| \leq B(R-R')^{-2}\psi^{2/3}$ for $|\alpha|=2$, where A, B are positive constants. In the estimates exposed below, the dependence in $R>0$ or $R'>0$ is not essential, and is not indicated explicitly. The following lemma is proven by the previous arguments, and we omit the proof.

Lemma 14.2 *It holds that* $\|\Xi\|_{L^\infty(\mathcal{O}(s)\times\mathcal{O}(s))} \leq C_4$ *for* $s \gg 1$, *where*

$$\begin{aligned}\Xi &= \Xi_\Psi(y,y',s)\\ &= \nabla_y\Psi(y,s)\cdot\nabla_y\mathcal{G}(y,y',s)+\nabla_{y'}\Psi(y',s)\cdot\nabla_{y'}\mathcal{G}(y,y',s).\end{aligned}$$

We can also show the following lemma.

Lemma 14.3 *It holds that*

$$\left|\frac{d}{ds}\int_{\mathcal{O}(s)} z\Psi dy\right| \leq C_5$$

for $s \gg 1$.

Proof: Writing $\varphi(x,t)=\Psi(y,s)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}(u\varphi) &= \frac{\partial}{\partial s}(z\Psi)+\nabla_y\cdot(yz\Psi/2)\\ &= \big(z_s+\nabla_y\cdot(yz/2)\big)\Psi + z\left(\Psi_s+y\cdot\nabla\Psi/2\right)\\ &= z\left(\Psi_s+y\cdot\nabla\Psi/2\right)+\left(\nabla\cdot(\nabla z - z\nabla w)\right)\Psi\end{aligned}$$

and therefore it follows that

$$\begin{aligned}\frac{d}{ds}\int_{\mathcal{O}(s)} z\Psi dy &= (T-t)\frac{d}{dt}\int_\Omega u\varphi\, dx = \int_\Omega (T-t)\frac{\partial}{\partial t}(u\varphi)\, dx\\ &= \int_{\mathcal{O}(s)}\left\{z\left(\Psi_s+y\cdot\nabla\Psi/2\right)+\left(\nabla\cdot(\nabla z-z\nabla w)\right)\Psi\right\}dy\\ &= \int_{\mathcal{O}(s)} z\left(\Psi_s+y\cdot\nabla\Psi/2\right)dy+\int_{\mathcal{O}(s)} z\Delta\Psi dy\\ &\quad +\int_{\mathcal{O}(s)} z\nabla w\cdot\nabla\Psi dy = \mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad (14.14)\end{aligned}$$

Here, we have

$$|\mathrm{I}|+|\mathrm{II}| \leq C_6\lambda,$$

where $\lambda = \|u_0\|_1$. On the other hand, from the method of symmetrization we have

$$\begin{aligned} \mathrm{III} &= \iint_{\mathcal{O}(s)\times\mathcal{O}(s)} z(y,s)\nabla_y \Psi(y,s)\cdot \nabla_y \mathcal{G}(y,y',s) z(y',s)dy\,dy' \\ &= \frac{1}{2}\iint_{\mathcal{O}(s)\times\mathcal{O}(s)} \Xi_\Psi(y,y',s) z(y,s) z(y',s) dy\,dy'. \end{aligned}$$

Then Lemma 14.2 guarantees

$$\left|\mathrm{III}\right| \le \frac{1}{2} C_4 \lambda^2$$

and the proof is complete. □

We also use the following lemma.

Lemma 14.4 *Given* $y_0 \in \overline{H}$, $0 < R \ll 1$, *and* $q \in [1,2)$, *we have*

$$\left\|\nabla w\right\|_{L^q(B(y_0,R)\cap\mathcal{O}(s))} \le C_7 \tag{14.15}$$

for $s \gg 1$.

Proof: In fact, by (14.13) we have

$$\begin{aligned} \left|\nabla_y \mathcal{G}(y,y',s)\right| &= \left|e^{-s/2}\nabla_x G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0)\right| \\ &\le C_2\Big(\frac{1}{|y-y'|} + 1\Big) \end{aligned}$$

for $s \gg 1$. Then, it follows that

$$\begin{aligned} &\Big\{ \int_{B(y_0,R)\cap\mathcal{O}(s)} \left|\nabla_y w(y,s)\right|^q dy \Big\}^{1/q} \\ &\quad = \Big(\int_{B(y_0,R)\cap\mathcal{O}(s)} \Big| \int_{\mathcal{O}(s)} \nabla_y \mathcal{G}(y,y',s) z(y',s) dy' \Big|^q \Big)^{1/q} \\ &\quad \le C_8 \Big\{ \int_{\mathcal{O}(s)} z(y',s)dy' \Big\}^{(q-1)/q} \\ &\qquad \cdot \Big\{ \int_{B(y_0,R)\cap\mathcal{O}(s)} \int_{\mathcal{O}(s)} z(y',s) \Big(\frac{1}{|y-y'|} + 1\Big)^q dy' dy \Big\}^{1/q}. \end{aligned}$$

This implies (14.15) by $\|z(s)\|_{L^1(\mathcal{O}_s)} \le \lambda$ and the proof is complete. □

Fundamental structures of the concentration lemma in Chapter 12 are space localization and time discretization. In spite of the linear term $yz/2$, estimates given above are enough to guarantee them for (14.11). Thus, we have the following lemma [149].

Lemma 14.5 *If*

$$\limsup_{n\to\infty} \left\| z(s_n) \right\|_{L^1(B(y_0,R)\cap\mathcal{O}(s_n))} < m_*(y_0)$$

holds for $y_0 \in \overline{L}$, $R > 0$, *and* $s_n \to +\infty$, *then there exist* $\tau > 0$, $R' \in (0, R)$, *and* $\{s_n'\} \subset \{s_n\}$ *such that*

$$\sup_n \left\| z \right\|_{C^{2+\theta,1+\theta/2}(\overline{B(y_0,R')\cap\mathcal{O}(s)})\times[s_n'-\tau,s_n'+\tau]} < +\infty \tag{14.16}$$

for $\theta \in (0, 1)$.

From the elliptic regularity, we have

$$\sup_n \left\| \nabla w \right\|_{C^{1+\theta,\theta/2}\left(\overline{B(y_0,R')\cap\mathcal{O}(s)}\right)\times[s_n'-\tau/2,s_n'+\tau/2]} < +\infty$$

by (14.16). The counterpart of the above lemma is the rescaled version of the local blowup criterion discussed in Chapter 5. We continue to take the case $x_0 \in \partial\Omega$, and also $y_0 \in \partial H$. Then, we have $m_*(y_0) = 4\pi$ and put that $\Psi_i(\cdot, s) = \Psi^s_{y_0,4^i R,2\cdot 4^i R}$ and $m(y, s) = |Y(y, s) - Y(y_0, s)|^2$ for $0 < R \ll 1$, $s \gg 1$, and $i = 1, 2$. Recalling the requirement $\frac{\partial X}{\partial x}(x_0) = \mathrm{id}$ in Lemma 5.3, we obtain the following lemma similarly to the prescaled system [149].

Lemma 14.6 *It holds that*

$$\begin{aligned}
&\left| \tilde{\Xi}(y, y', s) + \frac{2}{\pi}\Psi_1(y, s)\Psi_2(y', s) \right| \\
&\qquad \le C_9 R^{-1} \left(|y - y_0| + \left|y' - y_0\right| \right) \Psi_1(y, s)^{1/2}\Psi_2(y', s) \\
&\qquad\qquad + C_9 R^{-1} \left|y' - y_0\right| \Psi_2(y', s)^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Xi}(y, y', s) = &\left[\nabla(m\Psi_1)(y, s) \cdot \nabla_y \mathcal{G}(y, y', s) \right] \Psi_2(y', s) \\
&+ \left[\nabla(m\Psi_1)(y') \cdot \nabla_{y'} \mathcal{G}(y, y', s) \right] \Psi_2(y, s).
\end{aligned}$$

Now we can argue similarly to the prescaled system. First, as in (14.14) we have

$$\frac{d}{ds}\int_{\mathcal{O}(s)} zm\Psi_1 dy = \int_{\mathcal{O}(s)} \nabla\cdot(\nabla z - z\nabla w)(zm\Psi_1)dy$$
$$+ \int_{\mathcal{O}(s)} \Big((m\Psi_1)_s + \frac{1}{2}\nabla(ym\Psi_1)\Big) z dy = \mathrm{IV} + \mathrm{V}.$$

Here, we have

$$|m_s| + |\nabla m| \le C_{10} m^{1/2}$$

and

$$|\Psi_{1s}| + |\nabla \Psi_1| \le C_{10}\Psi_1^{5/6},$$

and hence it follows that

$$|V| \le C_{11}\lambda^{1/2}\Big\{\int_{\mathcal{O}(s)} mz\Psi_1 dy\Big\}^{1/2}.$$

On the other hand, Lemma 14.6 and the method of symmetrization guarantees

$$\mathrm{IV} \le 4M_1 - \frac{1}{\pi}M_1^2 + C_{12}R^{-1}\lambda^{3/2}I_1^{1/2} + C_{12}\lambda\left(M_2 - M_1\right),$$

where

$$M_i = \int_{\mathcal{O}(s)} z\Psi_i dy \quad \text{and} \quad I_i = \int_{\mathcal{O}(s)} mz\Psi_i dy$$

for $i = 1, 2$. Hence it follows that

$$\frac{d}{ds}I_1 \le 4M_1 - \frac{1}{\pi}M_1^2 + C_{13}\big(\lambda^{1/2} + \lambda^{3/2}\big)R^{-1}I_1^{1/2}$$
$$+ C_{13}\lambda\left(M_2 - M_1\right),$$

and the following lemma is obtained including the case $y_0 \in H$ [149].

Lemma 14.7 *If*

$$\int_{\mathcal{O}(s_0)\cap B(y_0,R)} z(y,s_0)dy > m_*(y_0)$$

holds for $y_0 \in \overline{H}$*,* $s_0 \gg 1$*, and* $0 < R \ll 1$*, then there exists* $\eta > 0$ *determined by* $\lambda = \|u_0\|_1$ *and* $\int_{\mathcal{O}(s)\cap B(y_0,R)} z(y,s_0)dy > m_*(y_0)$ *such that*

$$\frac{1}{R^2}\int_{\mathcal{O}(s_0)\cap B(y_0,4R)} |y-y_0|^2 z(y,s_0)dy < \eta$$

then $z = z(\cdot,s)$ *must blow up in finite time, which is not the case.*

Lemmas 14.5 and 14.7 are enough to provide the following proof.

Proof of Theorem 14.2: Taking the zero extension of $z(\cdot, s)$ to $\mathbf{R}^2 \setminus \overline{\mathcal{O}_s}$, we obtain $\|z(s)\|_{L^1(\mathbf{R}^2)} = \lambda$, and therefore Lemma 14.5 guarantees the existence of $\{s_n'\} \subset \{s_n\}$ such that

$$z(y, s_n')dy \rightharpoonup \sum_{y_0 \in \mathcal{B}} m(y_0)\delta_{y_0}(dy) + g(y)dy$$

in $\mathcal{M}(\mathbf{R}^2)$, where $\mathcal{B} \subset \overline{L}$ denotes the set of exhausted blowup points of $\left\{z(\cdot, s_n')\right\}$, $m(y_0) \geq m_*(y_0)$, and $g = g(y) \in L^1(\mathbf{R}^2)$ is a nonnegative function with the support contained in $\overline{L}$.

If $m(y_0) > m_*(y_0)$ holds for some $y_0 \in \mathcal{B}$, then there are $\delta > 0$, n_0, and $R_0 > 0$ such that

$$\int_{\mathcal{O}(s_n') \cap B(y_0, R)} z(y, s_n')dy > m_*(y_0) + \delta$$

holds for any $n \geq n_0$ and $R \in (0, R_0]$. Then, we can take $\eta > 0$ of Lemma 14.7 subject to λ and δ with the relation that

$$\frac{1}{R^2} \int_{\mathcal{O}(s_n') \cap B(y_0, R)} |y - y_0|^2 \, z(y, s_n')dy < \eta$$

for some $n \geq n_0$ and $R \in (0, R_0]$. This is impossible and we obtain $m(y_0) = m_*(y_0)$. The proof is complete. □

15
Quantized Blowup Mechanism

Motivated by Theorem 14.1, we introduced the standard backward self-similar transformation in the previous chapter. Supposing $T = T_{\max} < +\infty$ and $x_0 \in \mathcal{S}$, we define $R(t) = (T-t)^{1/2}$, $y = (x-x_0)/R(t)$, and $s = -\log(T-t)$. Then $z(y,s) = (T-t)u(x,t)$ satisfies (14.11), which is a similar system to (14.1), and this $\{z(\cdot,s)\}$ is regarded as its global semiorbit. Similarly to the collapse formed in infinite time in the prescaled system, quantized *subcollapses* are formed in infinite time in this rescaled system, stated as Theorem 14.2. Thus, any $s_n \to +\infty$ admits a subsequence $\{s_n'\} \subset \{s_n\}$ satisfying (14.12) in $\mathcal{M}(\mathbf{R}^2)$:

$$z(y, s_n')dy \rightharpoonup \mu_0(dy), \tag{15.1}$$

where supp $\mu_0(dy) \subset \overline{L}$,

$$\mu_0(dy) = \sum_{y_0 \in \mathcal{B}} m_*(y_0)\delta_{y_0}(dy) + g(y)dy, \tag{15.2}$$

$0 \leq g \in L^1(L) \cap C(\overline{L} \setminus \mathcal{B})$, and

$$L = \begin{cases} \mathbf{R}^2 & (x_0 \in \Omega), \\ H & (x_0 \in \partial\Omega). \end{cases}$$

Here, H denotes the half-space in $\mathbf{R}^2$ with ∂H containing the origin and parallel to the tangent line of $\partial\Omega$ at x_0, $\mathcal{B}$ is the set of exhausted blowup points of

$\{z(\cdot, s_n')\}$, which may be empty, and the zero extension of $z = z(y, s)$ is taken where it is not defined. We noticed also that if the residual term g vanishes in (15.2), then $m(x_0)/m_*(x_0) \in \mathbf{N}$ follows from the *parabolic envelope* described below.

We can show, more strongly, that $m(x_0) = m_*(x_0)$ always holds, using the forward self-similar transformation applied to the backward rescaled system limit. If x_0 is a type (II) blowup point, furthermore, then the limit measure $\mu_0(dy)$ of (15.1) is always equal to $m_*(x_0)\delta_0(dy)$. If it is type (I), then we have a profile of emergence, and the local free energy of the parabolic region diverges to $+\infty$. Thus, this chapter completes the proof of Theorem 1.2. New devices are the parabolic envelope, reverse second moment, and forward self-similar transformation.

We begin with the introduction of the parabolic envelope indicating the infinitely wide parabolic region. It is a virtual notion, and what we actually show is the following lemma.

Lemma 15.1 *We have*

$$m(x_0) = \mu_0(\overline{L}) \equiv \sum_{y_0 \in \mathcal{B}} m_*(y_0) + \int_L g(y)dy \tag{15.3}$$

in (15.2)*, where $m(x_0)$ denotes the collapse mass at $x_0 \in \mathcal{S}$ defined by (14.3), which arises in the classical solution $u = u(x, t)$ to* (14.1) *as $t \uparrow T = T_{\max} < +\infty$:*

$$u(x, t)\, dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\, dx. \tag{15.4}$$

Proof: Given $x_0 \in \mathcal{S}$, we take $\psi = \varphi^4_{x_0, R, 2R}$ for $0 < R' < R \ll 1$ and set

$$M_R(t) = \int_\Omega \psi(x)u(x, t)\, dx$$

Relation (15.4) implies

$$\lim_{R \downarrow 0} \lim_{t \to T} M_R(t) = m(x_0),$$

while in Chapter 5 it is proven that

$$\left| \frac{d}{dt} M_R(t) \right| \leq C(\lambda^2 + \lambda)(R^{-2} + R^{-1}) \tag{15.5}$$

for $t \in (0, T)$ with a constant $C > 0$ determined by Ω. Hence we obtain

$$|M_R(T) - M_R(t)| \leq C(\lambda^2 + \lambda)(R^{-2} + R^{-1})(T - t)$$

for

$$M_R(T) = \lim_{t \to T} M_R(t) = \sum_{x_0 \in \mathcal{S}} m(x_0)\psi(x_0) + \int_\Omega \psi(x) f(x)\, dx.$$

Taking $b > 0$ arbitrarily, we put

$$R = bR(t) = b(T - t)^{1/2}$$

in (15.5). Then, we obtain

$$\left| M_{bR(t)}(T) - M_{bR(t)}(t) \right| \leq C(\lambda^2 + \lambda)\big(b^{-2} + b^{-1}(T - t)^{1/2}\big), \tag{15.6}$$

and therefore for

$$\overline{m}_b(x_0) = \limsup_{t \to T} M_{bR(t)}(t)$$

and

$$\underline{m}_b(x_0) = \liminf_{t \to T} M_{bR(t)}(t),$$

it holds that

$$m(x_0) - C(\lambda^2 + \lambda)b^{-2} \leq \underline{m}_b(x_0) \leq \overline{m}_b(x_0) \leq m(x_0) + C(\lambda^2 + \lambda)b^{-2}$$

by $m(x_0) = \lim_{t \to T} M_{bR(t)}(T)$. This implies

$$\overline{m}_b(x_0) - C(\lambda^2 + \lambda)b^{-2} \leq m(x_0) \leq \underline{m}_b(x_0) + C(\lambda^2 + \lambda)b^{-2}. \tag{15.7}$$

Here, we have

$$\int_{B(x_0, R) \cap \Omega} u(x, t)\, dx \leq M_R(t) \leq \int_{B(x_0, 2R) \cap \Omega} u(x, t)\, dx$$

and hence it follows that

$$\int_{B(0,b)} z(y, s) dy \leq M_{bR(t)}(t) \leq \int_{B(0,2b)} z(y, s) dy.$$

This implies

$$\mu_0\left(B(0, b - 1)\right) \leq \underline{m}_b(x_0) \leq \overline{m}_b(x_0) \leq \mu_0\left(B(0, 2b + 1)\right),$$

and we obtain

$$\lim_{b \to +\infty} \underline{m}_b(x_0) = \lim_{b \to +\infty} \overline{m}_b(x_0) = \mu_0\big(\mathbf{R}^2\big) = \mu_0\left(\overline{L}\right).$$

Then, (15.3) holds by (15.7). □

Each $x_0 \in \partial\Omega$ admits the conformal mapping $X : B(x_0, R) \cap \Omega \to \mathbf{R}^2_+$ for $0 < R \ll 1$, with the properties described in the previous chapter. If $x_0 \in \Omega$, on the other hand, we put simply $X = \mathrm{id}$, the identity operator. In any case, we take $\varphi \in C^2_0(\overline{L})$ with $\varphi(x_0) \neq 0$ satisfying $\frac{\partial\varphi}{\partial\nu} = 0$ on ∂L, and set $\Phi = \varphi \circ Y(\cdot, s)$ for $s \gg 1$, where $Y(y, s) = e^{s/2}X(e^{-s/2}y + x_0)$. Then, similarly to (14.14) we obtain

$$\frac{d}{ds}\int_{\mathcal{O}(s)} z\Phi dy = \int_{\mathcal{O}(s)} z\left(\Phi_s + y\cdot\nabla\Phi/2\right)dy + \int_{\mathcal{O}(s)} z\Delta\Phi dy \\ +\frac{1}{2}\iint_{\mathcal{O}(s)\times\mathcal{O}(s)} \Xi_\Phi(y, y', s)z(y, s)z(y', s)dy\,dy', \tag{15.8}$$

where

$$\Xi_\Phi(y, y', s) = \nabla_y\Phi(y, s)\cdot\nabla_y\mathcal{G}(y, y'.s) + \nabla_{y'}\Phi(y', s)\cdot\nabla_{y'}\mathcal{G}(y, y', s)$$

and

$$\mathcal{G}(y, y', s) = G\big(e^{-s/2}y + x_0, e^{-s/2}y' + x_0\big).$$

Here, we can regard $Y = Y(y, s)$ as the transformation $y \in \overline{\mathcal{O}(s)} \mapsto Y \in \overline{H}$, and in this case it follows that

$$\int_{\mathcal{O}(s)} z\Phi dy = \int_H \tilde{\Phi}\tilde{z}dY$$

for $\tilde{\Phi}(Y) = \Phi(y)$ and

$$\tilde{z}(Y, s) = \left|\det\Big(\frac{\partial y}{\partial Y}\Big)\right| z(y, s).$$

We have $y = y(Y, s) \to Y$ as $s \to +\infty$ uniformly in Y in the neighborhood of $0 \in \overline{H}$ up to its second and first derivatives with respect to Y and s, respectively. If $x_0 \in \Omega$, first, we have

$$\mathcal{G}(y, y', s) = G_0(y, y') + \frac{s}{4\pi} + K\Big(e^{-s/2}y + x_0, e^{-s/2}y' + x_0\Big)$$

for $s \gg 1$, where $K \in C^{\theta,1+\theta} \cap C^{1+\theta,\theta}\left(B(x_0, R) \times B(x_0, R)\right)$, $0 < R \ll 1$, $\theta \in (0, 1)$, and

$$G_0(y, y') = \frac{1}{2\pi}\log\frac{1}{|y - y'|},$$

and therefore it follows that

$$\nabla_y \left(\mathcal{G}(y, y', s) - G_0(y, y') \right) \rightarrow 0 \tag{15.9}$$

locally uniformly in $(y, y') \in \mathbf{R}^2 \times \mathbf{R}^2$ as $s \rightarrow +\infty$. If $x_0 \in \partial\Omega$, on the contrary, we have

$$\begin{aligned}\mathcal{G}(y, y', s) = G_0(y, y') & \\ & + \frac{1}{2\pi} \log \frac{\left| y - y' \right|}{|Y(y, s) - Y(y', s)|} + \frac{1}{2\pi} \log \frac{\left| y - y'_* \right|}{|Y(y, s) - Y(y', s)_*|} \\ & + K\left(e^{-s/2} y + x_0, e^{-s/2} y' + x_0 \right) + \frac{s}{2\pi}\end{aligned}$$

for $s \gg 1$, where Y_* denotes the reflection of Y with respect to ∂H,

$$G_0(y, y') = \frac{1}{2\pi} \log \frac{1}{|y - y'|} + \frac{1}{2\pi} \log \frac{1}{\left| y - y'_* \right|},$$

$K \in C^{\theta, 1+\theta} \cap C^{1+\theta, \theta}\left(B(x_0, R) \cap \overline{\Omega} \times B(x_0, R) \cap \overline{\Omega} \right)$, $0 < R \ll 1$, and $\theta \in (0, 1)$. Therefore, relation (15.9) holds as $s \rightarrow +\infty$ even in this case, uniformly in (Y, Y') in a neighborhood of $0 \in \overline{H} \times \overline{H}$ through the transformation $y \in \overline{\mathcal{O}(s)} \mapsto Y \in \overline{H}$.

These relations are sufficient to guarantee the generation of the weak solution from the family $\{z(\cdot + s_n)\}$, when the zero extension is taken to $z = z(y, s)$ where it is not defined. Thus, any $s_n \rightarrow +\infty$ admits $\left\{ s'_n \right\} \subset \{s_n\}$ and $\mu = \mu(dy, s) \in \mathcal{M}(\mathbf{R}^2)$ defined for all $s \in \mathbf{R}$ such that

$$\operatorname{supp} \mu(dy, s) \subset \overline{L}$$

and

$$z(y, \cdot + s'_n) dy \rightharpoonup \mu(dy, \cdot)$$

in $C_*\left((-\infty, +\infty), \mathcal{M}(\mathbf{R}^2) \right)$, and this $\mu(dy, s)$ becomes a weak solution for

$$\begin{aligned} z_s &= \nabla \cdot (\nabla z - z \nabla p), && \text{in} \quad L \times (-\infty, \infty) \\ \frac{\partial z}{\partial \nu} &= 0 && \text{on} \quad \partial L \times (-\infty, \infty), \end{aligned} \tag{15.10}$$

where $p = w + \frac{|y|^2}{4}$,

$$\nabla_y w(y, s) = \int_L \nabla_y G_0(y, y') z(y', s) dy,$$

and

$$G_0(y, y') = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|y-y'|} & (x_0 \in \Omega), \\ \frac{1}{2\pi} \log \frac{1}{|y-y'|} + \frac{1}{2\pi} \log \frac{1}{|y-y'_*|} & (x_0 \in \partial\Omega). \end{cases}$$

More precisely, if $C^m_\infty(\overline{L})$ denotes the closure of $C^m_0(\overline{L})$ in $W^{m,\infty}(L)$, and $\overline{L}_\infty$ indicates the one-point compactification of $\mathbf{R}^2$, usually denoted by $\mathbf{R}^2 \cup \{\infty\}$ in the case of $L = \mathbf{R}^2$, and its portion cut by ∂H in the other case of $L = H$, then there is $0 \leq \nu = \nu(s) \in L^\infty_*\left(-\infty, +\infty; \mathcal{E}'\right)$ such that

$$\nu(s)\big|_{C_0(\overline{L})\oplus\mathbf{R}\times C_0(\overline{L})\oplus\mathbf{R}} = \mu \otimes \mu(dy\,dy', s)$$

for almost every $s \in (-\infty, \infty)$,

$$s \in (-\infty, \infty) \quad \mapsto \quad \int_{\overline{L}} \varphi(y)\mu(dy, s)$$

is locally absolutely continuous for each $\varphi \in X^0$, and

$$\frac{d}{ds}\int_{\overline{L}} \varphi(y)\mu(dy, s) = \int_{\overline{L}} (\Delta\varphi + y \cdot \nabla\varphi/2)\mu(dy, s) + \frac{1}{2}\left\langle \rho^0_\varphi, \nu(s)\right\rangle_{\mathcal{E},\mathcal{E}'}$$

for almost every $s \in (-\infty, \infty)$, where

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_0 \oplus C(\overline{L}_\infty \times \overline{L}_\infty) \subset L^\infty(L \times L) \\ \mathcal{E}_0 &= \left\{\rho^0_\varphi \mid \varphi \in X^0\right\} \\ \rho^0_\varphi(y, y') &= \nabla\varphi(y) \cdot \nabla_y G_0(y, y') + \nabla\varphi(y') \cdot \nabla_{y'} G_0(y, y') \\ X^0 &= \left\{\varphi \in C^2_\infty(\overline{L}) \mid \frac{\partial\varphi}{\partial\nu}\Big|_{\partial L} = 0\right\}. \end{aligned}$$

If $\mu(dy, s) = \mu_s(dy, s) + \mu_{a.c.}(dy, s)$ indicates the Lebesgue - Radon - Nikodym decomposition, then it holds that

$$\mu_{a.c.}(dy, s) = F(y, s)dy$$

with $0 \leq F(\cdot, s) = F(y, s) \in L^1(L)$,

$$\mu_s(dy, s) = \sum_{y_0 \in \mathcal{B}_s} m_*(y_0)\delta_{y_0}(dy) \tag{15.11}$$

for a finite set $\mathcal{B}_s \subset \overline{L}$, and $\mu(\overline{L}, s) = m(x_0)$ by Theorem 14.2, and therefore we obtain

$$8\pi \cdot \sharp\,(L \cap \mathcal{B}_s) + 4\pi \cdot \sharp\,(\partial L \cap \mathcal{B}_s) + \mu_{a.c.}(L, s) = m(x_0) = \mu(\overline{L}, s)$$

by Lemma 15.1.

The set

$$\mathcal{D} = \bigcup_{s \in \mathbf{R}} \left(\overline{L} \setminus \mathcal{B}_s \right) \times \{s\}$$

is relatively open in $\overline{L} \times \mathbf{R}$, because

$$s \in (-\infty, +\infty) \mapsto \mu(dy, s) \in \mathcal{M}(\overline{L})$$

is continuous and we have the classification (15.11) of the singular part of $\mu(dy, s)$. Then Lemma 14.5 guarantees the smoothness of $F = F(y, s) \geq 0$ in $\mathcal{D}$, and there we have

$$F_s = \nabla \cdot (\nabla F - F \nabla p) \tag{15.12}$$

with a smooth function $p = p(y, s)$.

Thanks to the parabolic envelope, the proof of mass quantization, $m(x_0) = m_*(x_0)$, has been thus reduced to the study of the weak solution $\mu = \mu(dy, s)$ to the limiting backward rescaled system (15.10).We next show that $\mu(\{0\}, 0) > m_*(x_0)$ implies its blowup, and hence is a contradiction. This fact is proven by the study of (15.10), and therefore we can assume $L = \mathbf{R}^2$, using an even extension of the solution in the other case of $L = H$.

Here, we apply the method of the (reverse) second moment. In more detail, if $\varphi(y) = A(|y|^2) \in C_0^2(\mathbf{R}^2)$, then we have

$$\begin{aligned}
&\left(\nabla \varphi(y) - \nabla \varphi(y')\right) \cdot (y - y') \\
&\quad = 2\left(A'(|y|^2)y - A'(|y'|^2)y'\right) \cdot (y - y') \\
&\quad = 2A'(|y|^2)|y - y'|^2 + 2\left(A'(|y|^2) - A'(|y'|^2)\right)y' \cdot (y - y') \\
&\quad = \left[A'(|y|^2) + A'(|y'|^2)\right]|y - y'|^2 \\
&\qquad + \left(A'(|y|^2) - A'(|y'|^2)\right) \cdot \left(|y|^2 - |y'|^2\right)
\end{aligned}$$

because the left-hand side is symmetric with respect to (y, y'), and therefore it holds that

$$\begin{aligned}
\left\langle \rho_\varphi^0, \nu(s) \right\rangle_{\mathcal{E},\mathcal{E}'} &= -\frac{1}{2\pi} \langle I, \nu(s) \rangle_{\mathcal{E},\mathcal{E}'} \\
&\quad - \frac{1}{2\pi} \iint_{\mathbf{R}^2 \times \mathbf{R}^2} \left[A'(|y|^2) + A'(|y'|^2)\right] \mu \otimes \mu(dy\, dy', s)
\end{aligned}$$

for

$$I = I(y, y') = \frac{\left(A'(|y|^2) - A'(|y'|^2)\right) \cdot \left(|y|^2 - |y'|^2\right)}{|y - y'|^2}.$$

This implies

$$\frac{d}{ds}\int_{\mathbf{R}^2} A(|y|^2)\mu(dy,s) = \int_{\mathbf{R}^2}\Big\{4A''(|y|^2)\,|y|^2 + 4A'(|y|^2) + |y|^2\,A'(|y|^2) - \frac{m(x_0)}{2\pi}A'(|y|^2)\Big\}\mu(dy,s) - \frac{1}{4\pi}\,\langle I, \nu(s)\rangle_{\mathcal{E},\mathcal{E}'}\,.$$

Taking $R \geq 1$ here, we specify $A = A(s)$ as follows:

$$\begin{aligned} 0 \leq A'(s) &\leq 1 \quad (s \geq 0),\\ -R^2 \leq A(s) &\leq 0 \quad (s \geq 0),\\ A(s) &= \begin{cases} s - R^2 & (0 \leq s \leq R^2/4),\\ 0 & (s \geq 4R^2). \end{cases}\end{aligned}$$

In this case, we have $C_0 > 0$ such that

$$4A''(|y|^2)\,|y|^2 + |y|^2\,A'(|y|^2) \leq C_0\big(A(|y|^2) + R^2\big)$$

for all $y \in \mathbf{R}^2$. We have also

$$\big|I(y,y')\big| = \big\{\chi_{B_{2R}\times\mathbf{R}^2}(y,y') + \chi_{\mathbf{R}^2\times B_{2R}}(y,y')\big\} \cdot \frac{\big|A'(|y|^2) - A'(|y'|^2)\big|\cdot\big||y|^2 - |y'|^2\big|}{|y-y'|^2},$$

and divide the first term of the right-hand side by

$$\chi_{B_{2R}\times\mathbf{R}^2}(y,y') = \chi_{B_{2R}\times B_{4R}}(y,y') + \chi_{B_{2R}\times B_{4R}^c}(y,y').$$

First, we apply the mean value theorem and obtain

$$\begin{aligned}\chi_{B_{2R}\times B_{4R}}\,(y,y')&\frac{\big|A'(|y|^2) - A'(|y'|^2)\big|\cdot\big||y|^2 - |y'|^2\big|}{|y-y'|^2}\\ &\leq \big\|A''\big\|_\infty |y+y'|^2\chi_{B_{4R}\times B_{4R}}(y,y')\\ &\leq 2\big\|A''\big\|_\infty\big(|y|^2\chi_{B_{4R}\times\mathbf{R}^2}(y,y') + |y'|^2\chi_{\mathbf{R}^2\times B_{4R}}(y,y')\big). \quad (15.13)\end{aligned}$$

Here, we have $C_1 > 0$ such that

$$|y|^2 \leq C_1\big(A(|y|^2) + R^2\big)$$

for all $y \in B_{4R}$, and therefore the right-hand side of (15.13) is estimated from above by $2C_1\big\|A''\big\|_\infty$ times

$$\big\{A(|y|^2) + R^2\big\} + \big\{A(|y'|^2) + R^2\big\}.$$

Next, we have

$$
\begin{aligned}
&\chi_{B_{2R}\times B_{4R}^c}(y,y')\frac{\left|A'(|y|^2)-A'(|y'|^2)\right|\cdot\left||y|^2-|y'|^2\right|}{|y-y'|^2} \\
&\qquad = \chi_{B_{2R}\times B_{4R}^c}(y,y')A'(|y|^2)\cdot\frac{|y'|^2-|y|^2}{|y-y'|^2} \\
&\qquad \leq \chi_{B_{2R}\times B_{4R}^c}(y,y')\left|\frac{y'+y}{y'-y}\right| \leq C_2\chi_{\mathbf{R}^2\times B_{4R}^c}(y,y')
\end{aligned}
$$

by $0 \leq A' \leq 1$, where

$$
C_2 = \sup_{(y,y')\in B_{2R}\times B_{4R}^c}\left|\frac{y'+y}{y'-y}\right| < +\infty.
$$

We have $A(|y'|^2)+R^2 = R^2 \geq 1$ for $y' \in B_{4R}^c$, and therefore this term is estimated from above by

$$
C_2\bigl(A(|y'|^2)+R^2)\bigr).
$$

Since $I(y,y')$ is symmetric, the other terms are treated similarly. Putting

$$
C_3 = C_0 + \frac{2C_1}{\pi}m(x_0)\left\|A''\right\|_\infty + \frac{C_2}{2\pi}m(x_0),
$$

we obtain

$$
\begin{aligned}
&\frac{d}{ds}\int_{\mathbf{R}^2}\bigl(A(|y|^2)+R^2\bigr)\mu(dy,s) = \frac{d}{ds}\int_{\mathbf{R}^2}A(|y|^2)\mu(dy,s) \\
&\leq C_3\int_{\mathbf{R}^2}\bigl(A(|y|^2)+R^2\bigr)\mu(dy,s) + \left\{4-\frac{m(x_0)}{2\pi}\right\}\int_{\mathbf{R}^2}A'(|y|^2)\mu(dy,s)
\end{aligned}
$$

by $\nu(s) \geq 0$ and $\nu(s)|_{C(\mathbf{R}^2\cup\{\infty\}\times\mathbf{R}^2\cup\{\infty\})} = \mu\otimes\mu(dy\,dy',s)$

Here, we use $\delta > 0$ satisfying

$$
A(s) + R^2 + A'(s) \geq \delta
$$

for all $s \geq 0$. Then we obtain

$$
\int_{\mathbf{R}^2}A'(|y|^2)\mu(dy,s) \geq \delta m(x_0) - \int_{\mathbf{R}^2}\bigl(A(|y|^2)+R^2\bigr)\mu(dy,s),
$$

and therefore

$$
\begin{aligned}
&\frac{d}{ds}\int_{\mathbf{R}^2}\bigl(A(|y|^2)+R^2\bigr)\mu(dy,s) \\
&\qquad \leq C_3\cdot\int_{\mathbf{R}^2}\bigl(A(|y|^2)+R^2\bigr)\mu(dy,s) + \delta m(x_0)\left\{4-\frac{m(x_0)}{2\pi}\right\}
\end{aligned}
$$

for almost all $s \in \mathbf{R}$ in case $m(x_0) > 8\pi$.

From this inequality, we see that if $m(x_0) > 8\pi + \varepsilon$ holds with $\varepsilon > 0$, then there is $\eta > 0$ such that if

$$\int_{\mathbf{R}^2} \big(A(|y|^2) + R^2\big)\mu(dy, 0) < \eta,$$

then

$$\int_{\mathbf{R}^2} \big(A(|y|^2) + R^2\big)\mu(dy, s) < 0$$

for $s \gg 1$, a contradiction by $A(s) + R^2 > 0$ for $s > 0$. In other words, we have

$$\int_{\mathbf{R}^2} \big(A(|y|^2) + R^2\big)z(y, 0)dy \geq \eta \tag{15.14}$$

in case $m(x_0) > 8\pi + \varepsilon$; that is, the concentration, indicated by

$$\int_{\mathbf{R}^2} \big(A(|y|^2) + R^2\big)z(y, 0)dy < \eta$$

implies the nearly mass quantization, $m(x_0) \leq 8\pi + \varepsilon$.

However, we can remove this concentration condition using the forward self-similar transformation. This argument, due to Kurokiba and Ogawa [84] concerning the prescaled system on the whole domain $\mathbf{R}^2$, established that $\|u_0\|_1 > 8\pi$ implies the blowup in finite time of the solution. In more detail, in the problem on the whole (or a half) plane, such a concentration condition in terms of the second moment can be hidden behind the forward self-similar transformation, and $\|u_0\|_1 \leq 8\pi$ must be always satisfied for the classical solution to exist globally in time.

To apply this argument, we note that the first equation of (15.10) is written as

$$z'_{s'} - \nabla' \cdot (y'z'/s') = \nabla' \cdot \big(\nabla' z' - z'\nabla w'\big) \tag{15.15}$$

by $y' = e^{-s/2}y$ and $s' = -e^{-s}$, where $z'(y', s') = z(y, s)$ and $w'(y', s') = w(y, s)$. This form is easier to find the *forward self-similar transformation* to this rescaled system. In fact, (15.15) is invariant under the transformation of $z'_\mu(y'z') = \mu^2 z'(\mu y', \mu^2 s')$ and $w'_\mu(y', s') = w'(\mu y', \mu^2 s')$. Using this structure, we can show the mass quantization, $m(x_0) = m_*(x_0)$, or $m(x_0) = 8\pi$ under the above reduction to $L = \mathbf{R}^2$ by the even extension, as follows.

First, if $m(x_0) > 8\pi + \varepsilon$, then we have (15.14). Next, system (15.10) is invariant under the s-translation, and therefore it must hold that

$$\int_{\mathbf{R}^2} \big(A(|y|^2) + R^2\big)\mu(dy, s) \geq \eta$$

for all $s \in \mathbf{R}$. However, system (15.10) is invariant also under the forward self-similar transformation,

$$z^\lambda(y, s) = \lambda^2 e^{-s} z(\lambda e^{-s/2} y, -\lambda^2 e^{-s})$$
$$w^\lambda(y, s) = w(\lambda e^{-s/2} y, -\lambda^2 e^{-s}),$$

and therefore we have

$$\int_{\mathbf{R}^2} \big(A(\left|y\right|^2) + R^2\big) d\mu^\lambda(dy, s) \geq \eta$$

for any $s \in \mathbf{R}$ and $\lambda > 0$, where

$$\mu^\lambda(dy, s) = \lambda^2 e^{-s} \mu(\lambda e^{-s/2} dy, -\lambda^2 e^{-s}).$$

Using $\tilde{y} = \lambda e^{-s/2} y$, $\tilde{s} = \lambda^2 e^{-s}$, this means

$$\int_{\mathbf{R}^2} \big(A(\tilde{s}\,|\tilde{y}|^2) + R^2\big)\mu(d\tilde{y}, -\tilde{s}) \geq \eta,$$

that is,

$$\int_{\mathbf{R}^2} \big(A(\tilde{s}\,|y|^2) + R^2\big)\mu(dy, -\tilde{s}) \geq \eta$$

for all $\tilde{s} > 0$.

Again, we use the translation invariance of this system. Applying the above inequality for $\mu_{\tilde{s}}(dy, s) = \mu(dy, s + \tilde{s})$, we obtain

$$\int_{\mathbf{R}^2} \big(A(\tilde{s}\,|y|^2) + R^2\big)\mu_{\tilde{s}}(dy, -\tilde{s})\, dy = \int_{\mathbf{R}^2} \big(A(\tilde{s}\,|y|^2) + R^2\big)\mu(dy, 0)\, dy \geq \eta,$$

but this implies a contradiction by making $\tilde{s} \downarrow 0$, because $0 \leq A(\tilde{s}\,|y|^2) + R^2 \leq R^2$ and $A(\tilde{s}\,|y|^2) + R^2 \to 0$ for any $y \in \mathbf{R}^2$ and therefore the left-hand side converges 0 by the dominated convergence theorem. This means $m(x_0) \leq 8\pi + \varepsilon$, and therefore $m(x_0) \leq 8\pi$ because $\varepsilon > 0$ is arbitrary. Thus, we have proven the following theorem.

Theorem 15.1 *Mass quantization $m(x_0) = m_*(x_0)$ holds in* (15.4).

If $x_0 \in \mathcal{S}$ is of type (II), then it holds that $F(y, 0) = 0$, where $\mu_{a.c.}(dy, s) = F(y, s)dy$. Then, the strong maximum principle or the unique continuation theorem applied to (15.12) assures $\mu_{a.c.}(dy, s) = 0$ for any $s \in \mathbf{R}^2$, and therefore we have

$$\mu(dy, s) = m_*(x_0)\delta_{y(s)}(dy)$$

with $s \in \mathbf{R} \mapsto y(s) \in \overline{L}$ locally absolutely continuous by Theorem 15.1.

To study this case in more detail, for the moment, we assume $x_0 \in \Omega$ and examine

$$\varphi(y) = (R^2 - |y|^2)_+.$$

In fact, the other case of $x_0 \in \partial\Omega$ is reduced to the study of this case by the even extension of $\mu(dy, s)$.

First, this function $\varphi = \varphi(y)$ satisfies

$$\Delta\varphi(y) + y \cdot \nabla\varphi(y)/2 = \varphi(y) - (R^2 + 4)$$

and

$$\rho^0_\varphi(y, y') = \frac{1}{\pi}$$

in the regions where $\varphi(y) > 0$ and $\varphi(y), \varphi(y') > 0$, respectively, that is, $B_R = B(0, R)$ and $B_R \times B_R$. Therefore, as far as $y(s) \in B_R$, we have

$$\begin{aligned}\frac{d}{ds}\int_{\mathbf{R}^2}&(R^2 - |y|^2)_+\mu(dy, s)\\ &= \int_{\mathbf{R}^2}(R^2 - |y|^2)_+\mu(dy, s) - (R^2 + 4)\mu(B_R, s) + \frac{1}{2\pi}\mu(B_R, s)^2\end{aligned}$$

or, equivalently,

$$\frac{d}{ds}\,|y(s)|^2 = |y(s)|^2\,.$$

Therefore, we see that

$$|y(s)| = |y(0)|\,e^{s/2}$$

holds for any $s \in \mathbf{R}$, by making R large.

Now, we show that only $y(0)=0$ is admitted by the parabolic envelope, and we have always

$$\mu_0(dy) = m_*(x_0)\delta_0(dy).$$

For this purpose, we fix $s\in\mathbf{R}$, and define $t_n\in(0,T)$ by $s_n'+s=-\log(T-t_n)$. Then, putting $t=t_n$ in (15.6), we obtain

$$\left|M_{bR(t_n)}(t_n)-M_{bR(t_n)}(T)\right|\leq C(\lambda^2+\lambda)\big(b^{-2}+R(t_n)b^{-1}\big),$$

where $R(t)=(T-t)^{1/2}$. Here, we have

$$M_{bR(t_n)}(t_n)=\int_{\mathcal{O}(s)}\psi_n(y)z(y,s+s_n')dy$$

for

$$\psi_n(y)=\varphi^4_{x_0,bR(t_n),2bR(t_n)}(x_0+R(t_n)y)$$

and this function converges to some $\xi_b(y)\in C_0(\mathbf{R}^2)$ uniformly in $y\in\mathbf{R}^2$ as $n\to\infty$, where it holds that

$$\xi_b(y)=\begin{cases}1 & (|y|<b)\,,\\ 0 & (|y|>2b)\,.\end{cases}$$

Thus, we obtain

$$\lim_{n\to\infty}M_{bR(t_n)}(t_n)=\int_{\overline{L}}\xi_b(y)\mu(dy,s)=m_*(x_0)\xi_b(y(s))$$

and therefore

$$\left|m_*(x_0)\xi_b(y(s))-m_*(x_0)\right|\leq C(\lambda^2+\lambda)b^{-2}.$$

Here, $s\in\mathbf{R}$ and $b>0$ are arbitrary, and if $y(0)\neq 0$, then we obtain $m_*(x_0)=0$ by making $s\to+\infty$ and then $b\to+\infty$. This is a contradiction, and we have $y(0)=0$.

If $x_0\in\mathcal{S}$ satisfies, more strongly,

$$\lim_{t\to T}\sup_{x\in\Omega,\ |x-x_0|\leq CR(t)}R(t)^2u(x,t)=+\infty, \tag{15.16}$$

then we have

$$z(y,s)dy \rightharpoonup m_*(x_0)\delta_0(dy)$$

as $s \to +\infty$. In this it holds that

$$\lim_{t \to T} M_{bR(t),x_0} = m_*(x_0) = m(x_0) \tag{15.17}$$

for any $b > 0$, and therefore this x_0 is hyperparabolic. Thus, we have proven the following theorem.

Theorem 15.2 *If $x_0 \in \mathcal{S}$ is of type (II) and*

$$\lim_{n \to +\infty} \sup_{x \in \Omega,\ |x - x_0| \le CR(t_n)} R(t_n)^2 u(x, t_n) = +\infty \tag{15.18}$$

for some $t_n \to T = T_{\max}$ and $C > 0$, then we have

$$z(y, s_n + \cdot)dy \ \rightharpoonup\ m_*(x_0)\delta_0(dy) \tag{15.19}$$

in $C_*\left((-\infty, +\infty), \mathcal{M}(\mathbf{R}^2)\right)$ for $s_n \to +\infty$ defined by

$$s_n = -\log(T - t_n).$$

Here, the zero extension it taken to $z = z(y, s)$ in the region where it is not defined and $z(y, s) = (T - t)u(x, t)$ for $y = (x - x_0)/(T - t)^{1/2}$ and $T = T_{\max} < +\infty$. If (15.16) *holds for some $C > 0$, more strongly, then $x_0 \in \mathcal{S}$ is hyperparabolic and we have* (15.17) *for any $b > 0$.*

Actual existence of the type (I) blowup point, on the other hand, is open. If $x_0 \in \mathcal{S}$ is such a point, then we obtain a classical solution to (15.10) as we have seen. Formally, this solution is subject to the Lyapunov function

$$\mathcal{H}(z) = \int_L z\left(\log z - 1 - \frac{|y|^2}{4}\right)dy - \frac{1}{2}\iint_{L \times L} G_0(y, y')z(y, s)z(y', s)dy\,dy',$$

and it holds that

$$\frac{d}{ds}\mathcal{H}(z) + \int_L z\,|\nabla\,(\log z - p)|^2\,dy = 0.$$

This formula is actually justified if $z = z(y, s)$ decays sufficiently fast as $|y| \to +\infty$, and then there should be a stationary solution contained in the ω- or α-limit set of $\{\mu(d\xi, s)\}$. Here, the stationary problem is given by $\log z - p =$ constant, or

$$-\Delta w = \frac{\lambda e^p}{\int_L e^p dy} \quad \text{in} \quad L,$$

$$\frac{\partial w}{\partial \nu} = 0 \qquad \text{on} \quad \partial L \tag{15.20}$$

with $p = w + \frac{|y|^2}{4}$, but no bounded radially symmetric solution to (15.20) exists for any λ [63]. This suggests that if there is a type (I) blowup point, then the collapse is formed from the *wedge of the parabolic envelope* without asymptotical symmetry or boundedness. Such a mechanism may be interesting from the biological point of view, but may not exist mathematically.

We can confirm also that the local free energy diverges to $+\infty$ around this type of blowup point. More precisely,

$$\lim_{t \to T} \mathcal{F}_{x_0, bR(t)}(u(t)) = +\infty \tag{15.21}$$

holds for any $b > 0$, where $T = T_{\max} < +\infty$, $R(t) = (T - t)^{1/2}$, and

$$\begin{aligned}\mathcal{F}_{x_0,R}(u) = &\int_\Omega u(\log u - 1)\psi_{x_0,R}(x)\, dx \\ &- \frac{1}{2}\iint_{\Omega\times\Omega} G(x, x')u(x,t)u(x',t)\psi_{x_0,R}(x)\psi_{x_0,R}(x')\, dx\, dx'\end{aligned}$$

for $\psi_{x_0,R} = \varphi^4_{x_0,R,2R}$. In fact, if $x_0 \in \mathcal{S}$ is of type (I), then any $s_n \to +\infty$ admits $\{s'_n\} \subset \{s_n\}$ and a smooth $z = z(y) > 0$ satisfying $\int_L z(y)dy = m_*(x_0)$ and

$$z(y, s'_n) - z(y) \ \to\ 0$$

locally uniformly in $Y \in \overline{L}$, and therefore

$$\begin{aligned}\mathcal{F}_{bR(t)}(u(t)) = &\int_L z(\log z - 1)\xi_b dy \\ &- \frac{1}{2}\iint_{L\times L}\big(G_0(y, y') + K(x_0, x_0)\big) z(y)z(y')\xi_b(y)\xi_b(y')dy\, dy' \\ &+ o(1) - \log R(t)\cdot\left\{2\int_L z\xi_b dy - \frac{2}{m_*(x_0)}\Big(\int_L z\xi_b dy\Big)^2 + o(1)\right\}\end{aligned}$$

as $t \to T$ by

$$G_0(x, x') = G_0(y, y') - \frac{4}{m_*(x_0)}\log R(t).$$

The right-hand side diverges to $+\infty$ by

$$\int_L z\xi_b dy < m_*(x_0),$$

and hence (15.21) follows.

We can summarize that if the type (I) blowup point exists, then the local entropy is swept away to the wedge of parabolic envelope, while the quantized concentration mass comes from this wedge, and the proof of Theorem 1.2 is complete.

16
Theory of Dual Variation

The expanding cosmos is the origin of nonequilibrium.
It creates order, induces complexity, and brings life.
— H. Tanaka

This chapter is the epilogue. We summarize the argument and give a new formulation applicable to other theories.

So far, we have discovered the quantized blowup mechanism of the mean field of many self-interacting particles, subject to the total mass conservation, decrease of the free energy, compensated compactness via the symmetrization of potential kernels, and certain scaling invariance. We studied

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u\nabla v) \\ 0 &= \Delta v - av + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0,T),$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T),$$
$$u\big|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \tag{16.1}$$

as a typical example, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, and ν is the unit outer normal vector on $\partial\Omega$. It is the simplified system of chemotaxis in mathematical biology, describing the chemotactic feature of cellular slime molds, but it is also the description of the nonequilibrium mean field of self-attractive particles subject to the second law of thermodynamics in the theory of statistical mechanics.

The unique existence of the classical solution locally in time is proven, and the solution becomes positive and regular for $t > 0$ if $u_0 \not\equiv 0$. The supremum of its existence time is denoted by $T_{\max} \in (0, +\infty]$, and $T_{\max} < +\infty$ is referred to as the blowup of the solution in finite time. In this case, it holds that

$$u(x,t)\,dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)\,dx \tag{16.2}$$

as $t \uparrow T_{\max}$ in $\mathcal{M}(\overline{\Omega})$ with

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega), \\ 4\pi & (x_0 \in \partial\Omega), \end{cases} \tag{16.3}$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightharpoonup$ the $*$-weak convergence there. Actually, $\mathcal{S}$ in (16.2) is the blowup set of $u(\cdot, t)$, and $x_0 \in \mathcal{S}$ if and only if there are $x_k \to x_0$ and $t_k \uparrow T_{\max}$ in $x_k \in \overline{\Omega}$ satisfying $u(x_k, t_k) \to +\infty$. We have

$$\lim_{t \uparrow T_{\max}} \left\| u(t) \right\|_\infty = +\infty$$

and hence $\mathcal{S} \neq \emptyset$ holds in the case of $T_{\max} < +\infty$. Therefore, (16.2) with (16.3) implies the sharp estimate of the number of blowup points,

$$\sharp\,(\partial\Omega \cap \mathcal{S}) + 2 \cdot \sharp\,(\Omega \cap \mathcal{S}) \leq \left\| u_0 \right\|_1 / (4\pi).$$

The equality $m(x_0) = m_*(x_0)$ in (16.2) is referred to as the mass quantization of collapses. It has been suspected from the hierarchy of systems in statistical mechanics, that is, the global existence of the weak solution of the Fokker–Planck equation and mass and location quantization of the blowup family of solutions to the Liouville–Gel'fand equation, which describe the kinetic and the equilibrium states of the mean field, respectively. The actual proof is associated with the backward self-similar transformation, and the blowup point $x_0 \in \mathcal{S}$ is classified into two types, namely, it is of type (I) if

$$\limsup_{t \uparrow T_{\max}} \sup_{\substack{x \in \Omega, \\ |x - x_0| \leq CR(t)}} R(t)^2 u(x,t) < +\infty$$

for any $C > 0$ and of type (II) for the other case that

$$\limsup_{t \uparrow T_{\max}} \sup_{\substack{x \in \Omega, \\ |x - x_0| \leq CR(t)}} R(t)^2 u(x,t) = +\infty$$

for some $C > 0$, where $R(t) = (T_{\max} - t)^{1/2}$. The important notion introduced here is the *parabolic envelope*, the infinitely wide parabolic region as $b \uparrow +\infty$ of

$$\left\{ (x,t) \in \overline{\Omega} \times [0, T_{\max} \;\middle|\; |x - x_0| \leq bR(t) \right\},$$

that is, the whole blowup mechanism is enveloped there and it holds that

$$\lim_{b \to +\infty} \limsup_{t \uparrow T_{\max}} \left| \int_\Omega \psi_{x_0, bR(t)}(x) u(x,t)\, dx - m(x_0) \right| = 0,$$

where $\psi = \psi_{x_0,R}(x)$ is the cut-off function around x_0 with the support radius $2R > 0$ and $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$.

If $x_0 \in \mathcal{S}$ is of type (II) and $t_k \uparrow T_{\max}$ satisfies

$$\lim_{k \to \infty} \sup_{\substack{x \in \Omega, \\ |x - x_0| \leq CR(t)}} R(t_k)^2 u(x, t_k) = +\infty$$

for some $C > 0$, then it holds that

$$z(y, s_k) dy \rightharpoonup m_*(x_0) \delta_0(dy)$$

in $\mathcal{M}(\mathbf{R}^2)$ as $k \to \infty$. Here, $z(y,s) = R(t)^2 u(x,t)$ with the zero extension taken where it is not defined, $y = (x - x_0)/R(t)$, and $s = s_k$ is defined from $t = t_k$ by $s = -\log(T_{\max} - t)$. Thus, the type (II) blowup point is fixed at first. Then, it attracts particles asymptotically radially symmetric and creates a mass quantization collapse. Concentration, compared with aggregation, is strong, and the rescaled $z = z(y,s)$ develops delta singularity $m_*(x_0)\delta_0(dz)$, called the subcollapse, at the origin. While actual existence of the type (I) blowup point is open to question, if it exists then it takes a profile of emergence in the sense of Kauffman as

$$\lim_{t \uparrow T_{\max}} \mathcal{F}_{x_0, bR(t)}\left(u(t)\right) = +\infty$$

holds for any $b > 0$, where $\mathcal{F}_{x_0,R}(u)$ denotes the local free energy defined by

$$\begin{aligned} \mathcal{F}_{x_0,R}(u) = & \int_\Omega \psi_{x_0,R}(x) u \left(\log u - 1\right)(x)\, dx \\ & - \frac{1}{2} \iint_{\Omega \times \Omega} \psi_{x_0,R}(x) \psi_{x_0,R}(x') G(x,x') u \otimes u(x,x')\, dx\, dx'. \end{aligned}$$

Here, $G = G(x, x')$ denotes the Green's function for $-\Delta + a$ under the Neumann boundary condition. In this connection, we note that the global free energy $\mathcal{F}(u(t))$ always decreases as the requirement of the second law of thermodynamics, where

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)(x)\, dx - \frac{1}{2} \iint_{\Omega\times\Omega} G(x, x') u \otimes u(x, x')\, dx\, dx'.$$

On the other hand, formation of collapses around this type of blowup point may be nonradially symmetric or may not decay rapidly at the infinite point in the rescaled variable, because the rescaled system is formally provided with the Lyapunov function. We conclude that around the type (I) blowup point, if it exists, mass and free energy are exchanged at the wedge of the parabolic envelope, with a clean "self" of mass quantized collapse being created.

The quantized blowup mechanism of the nonstationary state described above comes from that of the stationary state, and this story is called the nonlinear quantum mechanics. In fact, the stationary state of (16.1) is realized as the nonlinear eigenvalue problem

$$-\Delta v + av = \frac{\lambda e^v}{\int_\Omega e^v\, dx} \quad \text{in} \quad \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \tag{16.4}$$

with $\lambda = \|u_0\|_1$, and the quantized blowup mechanism at this level arises in the blowup family of solutions, namely, if

$$\left\{(\lambda_k, v_k)\right\}_{k=1}^{\infty}$$

is a family of solutions for (16.4) for $\lambda = \lambda_k$ and $v = v_k$, satisfying $\lambda_k \to \lambda_0 \in [0, \infty)$ and $\|v_k\|_\infty \to +\infty$, then the blowup set of $\{v_k\}$, denoted by $\mathcal{S} \subset \overline{\Omega}$ is finite, and passing through a subsequence, it holds that

$$u_k(x)\, dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_*(x_0) \delta_{x_0}(dx)$$

in $\mathcal{M}(\overline{\Omega})$ as $k \to \infty$, where

$$u_k = \frac{\lambda_k e^{v_k}}{\int_\Omega e^{v_k}\, dx}.$$

In particular, $\lambda_0 \in 4\pi\mathcal{N}$, and furthermore, we have

$$\nabla_x \Big(m_*(x_0) K(x, x_0) + \sum_{x_0' \in \mathcal{S}\setminus\{x_0\}} m_*(x_0') G(x, x_0') \Big)\Big|_{x = x_0} = 0 \tag{16.5}$$

for each $x_0 \in \mathcal{S}$, where only tangential derivative is taken in (16.5) if $x_0 \in \partial\Omega$, and

$$K(x, x') = G(x, x') + \begin{cases} \frac{1}{2\pi} \log |x - x'| & (x \in \Omega), \\ \frac{1}{\pi} \log |x - x'| & (x \in \partial\Omega) \end{cases}$$

represents the regular part of the Green's function $G = G(x, x')$.

This kind of quantization was first observed by [114, 115] for the Gel'fand problem

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v \, dx} \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega, \tag{16.6}$$

and [7] proved the converse, that is, the singular perturbation. Calculation of the topological degree was done based on these facts [30, 31, 87]. On the other hand, [113, 162] established the uniqueness of the solution to (16.6) for simply connected Ω and $\lambda \in (0, 8\pi)$. There, the Morse index of the stationary solution $v = v(x)$ is shown to be equal to the number of eigenvalues in $\mu < 1$ minus one of the eigenvalue problem

$$\begin{aligned} -\Delta\phi = \mu u \phi \quad & \text{in} \quad \Omega, \\ \phi = \text{constant} \quad & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} ds = 0, & \end{aligned} \tag{16.7}$$

where

$$u = \frac{\lambda e^v}{\int_\Omega e^v \, dx}. \tag{16.8}$$

This Morse index is induced from the variational structure of (16.6), associated with the functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v \, dx\right) + c \tag{16.9}$$

defined for $v \in H_0^1(\Omega)$. Using c prescribed later, we can treat this system by the general theory.

Independently, [185] showed the same fact for $u = u(x) > 0$ satisfying

$$(-\Delta_D)^{-1} u = \log u + \text{constant} \quad \text{in} \quad \Omega, \qquad \|u\|_1 = 1. \tag{16.10}$$

In more precision, this problem was introduced by the stationary state of the similar system to (16.1),

$$\left.\begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v) \\ 0 &= \Delta v + u \end{aligned}\right\} \quad \text{in} \quad \Omega\times(0,T),$$
$$\left.\begin{aligned} \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\times(0,T). \tag{16.11}$$

It is subject to the decrease of the free energy defined by

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)\,dx - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\,dx\,dx',$$

where $G = G(x, x')$ denotes the Green's function for $-\Delta$ under the Dirichlet boundary condition. Problem (16.10) is nothing but the Euler equation for the variational problem $\delta\mathcal{F}(u) = 0$ under the constraint $\|u\|_1 = \lambda$, and the Morse index of the solution is defined by the maximal dimension of the linear subspaces in which the associated quadratic form is negative. Thus, what [185] showed is that this index is equal to the number of eigenvalues in $\mu < 1$ minus 1 for (16.7). On the other hand, these problems (16.7) are equivalent through (16.8) and

$$v = (-\Delta_D)^{-1}u, \tag{16.12}$$

and in this way, these two variational structures concerning v and u are equivalent up to the Morse indices. This is important to us, because the structure of elliptic problem (16.6) is known in detail and the Morse index is easier to calculate, while the dynamics of (16.11) are subject to the decrease of the free energy and the local dynamics around the stationary solution is controlled by its Morse index. Actually, this observation, combined with the global bifurcation diagram of the equilibrium state, led us to the mass quantization of collapses of the nonstationary solution (Chapter 15) and the relation between the dynamical and linearized stabilities (Chapter 9).

However, the dynamical equivalence of these variations is proven also by the general theory, called *dual variation* in this book. Since the Lyapunov function for the full system takes a role in developing this theory, we describe the situation for

$$\left.\begin{aligned} u_t &= \nabla\cdot(\nabla u - u\nabla v) \\ \tau v_t &= \Delta v + u \end{aligned}\right\} \quad \text{in} \quad \Omega\times(0,T),$$
$$\left.\begin{aligned} \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\times(0,T). \tag{16.13}$$

In fact, defining $\mathcal{W}$ by

$$\mathcal{W}(u, v) = \int_\Omega u(\log u - 1)\, dx + \frac{1}{2}\left\|\nabla v\right\|_2^2 - \langle v, u\rangle\,,$$

we have

$$\frac{d}{dt}\mathcal{W}\big(u(t), v(t)\big) + \tau\left\|v_t(t)\right\|_2^2 + \int_\Omega u\left|\nabla\,(\log u - v)\,\right|^2(\cdot, t)\, dx = 0, \tag{16.14}$$

where $u = u(\cdot, t),\ v = v(\cdot, t)$ is the classical solution to (16.13) and $\langle\ ,\ \rangle$ denotes the duality:

$$\langle v, u\rangle = \int_\Omega uv\, dx.$$

In the simplified system we have (16.12), and this $\mathcal{W}$ is reduced to the free energy:

$$\mathcal{W}\big|_{v=(-\Delta_D)^{-1}u} = \mathcal{F}. \tag{16.15}$$

We have, furthermore,

$$u = \frac{\lambda e^v}{\int_\Omega e^v\, dx}$$

in the stationary state, because in this case

$$\log u - v = \text{constant} \quad \text{and} \quad \left\|u\right\|_1 = \lambda$$

follow from (16.14). If we take $c = \lambda\log\lambda - \lambda$ in (16.9), that is,

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\left\|\nabla v\right\|_2^2 - \lambda\log\left(\int_\Omega e^v\, dx\right) + \lambda\log\lambda - \lambda$$

for $v \in H_0^1(\Omega)$, then it holds that

$$\mathcal{W}\big|_{u=\lambda e^v/\int_\Omega e^v\, dx} = \mathcal{J}_\lambda. \tag{16.16}$$

We call these relations, (16.15) and (16.16), the *unfolding Legendre transformation*, because the stationary states given by u and v, (16.10) and (16.6), respectively, are realized by $\delta\mathcal{F}(u) = 0$ on $\|u\|_1 = \lambda$ and $\delta\mathcal{J}_\lambda(v) = 0$ on $H_0^1(\Omega)$, respectively. On the other hand, we have the *minimality*,

$$\mathcal{W}(u, v) \geq \max\left\{\mathcal{F}(u), \mathcal{J}_\lambda(v)\right\}, \tag{16.17}$$

where $\|u\|_1 = \lambda$. In fact, the first inequality is a direct consequence of Schwarz's inequality, while the second inequality is proven by Jensen's inequality. These inequalities are applicable to derive the global existence of the solution to (16.11) or (16.13) in the case of $\lambda = \|u_0\|_1 < 8\pi$. However, we can show them by the general theory.

The theory of *dual variation* guarantees the splitting of the stationary state into each component, indicating the particle density and field distribution, together with their variational and dynamical equivalence. The above unfolding and minimality, on the other hand, are enough to establish its stability. Some systems describing the mean field are provided with only semi-unfolding and semi-minimality, from which we can derive the stability of one component.

We are ready to begin the abstract theory. Let X be a Banach space over $\mathbf{R}$. Its dual space and the paring are denoted by X^* and $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{X,X^*}$, respectively. Given $F : X \to [-\infty, +\infty]$, we define its *Legendre transformation* by

$$F^*(p) = \sup_{x\in X} \{\langle x, p\rangle - F(x)\} \qquad (p \in X^*).$$

Then, *Fenchel–Moreau's theorem* says that if

$$F : X \to (-\infty, +\infty]$$

is proper, convex, lower semicontinuous, then so is $F^* : X^* \to (-\infty, +\infty]$, and the *second Legendre transformation* defined by

$$F^{**}(x) = \sup_{p\in X^*} \left\{\langle x, p\rangle - F^*(p)\right\} \qquad (x \in X)$$

is equal to $F(x)$ [43].

Given proper, convex, lower semicontinuous $F, G : X \to (-\infty, +\infty]$, we put

$$\Phi(x, y) = F(x + y) - G(x).$$

and define the effective domains of F, G by

$$D(F) = \left\{x \in X \;\middle|\; F(x) < +\infty\right\},$$
$$D(G) = \left\{x \in X \;\middle|\; G(x) < +\infty\right\}.$$

Each $x \in D(G)$ induces a proper, convex, lower semicontinuous mapping

$$y \in X \mapsto \Phi(x, y) \in (-\infty, +\infty]$$

and its Legendre transformation is given by

$$W(x, p) = \sup_{y \in X} \left\{ \langle y, p \rangle - \Phi(x, y) \right\} \qquad (p \in X^*)$$

and thus

$$W(x, \cdot) : X^* \to (-\infty, +\infty]$$

is proper, convex, lower semicontinuous. Sometimes

$$L(x, p) = -W(x, p)$$

is referred to as the *Lagrange function*. Then, for $(x, p) \in D(G) \times X^*$ we have

$$\begin{aligned} W(x, p) &= \sup_{y \in X} \left\{ \langle y + x, p \rangle - F(x + y) + G(x) - \langle x, p \rangle \right\} \\ &= F^*(p) + G(x) - \langle x, p \rangle . \end{aligned} \tag{16.18}$$

Putting $W(x, p) = +\infty$ for $x \notin D(G)$, we obtain (16.18) for any $(x, p) \in X \times X^*$.

Next, given $p \in X^*$, we put

$$J^*(p) = \begin{cases} F^*(p) - G^*(p) & (p \in D(F^*)) , \\ +\infty & \text{(otherwise)} \end{cases} \tag{16.19}$$

and obtain

$$\begin{aligned} \inf_{x \in X} W(x, p) &= F^*(p) - \sup_{x \in X} \left\{ \langle x, p \rangle - G(x) \right\} \\ &= F^*(p) - G^*(p) = J^*(p) \end{aligned}$$

for $p \in D(F^*)$. It is valid even for $p \notin D(F^*)$ by (16.18) and (16.19). Similarly, for $x \in X$ we set

$$J(x) = \begin{cases} G(x) - F(x) & (x \in D(G)) , \\ +\infty & \text{(otherwise)} \end{cases} \tag{16.20}$$

and obtain

$$\begin{aligned} \inf_{p \in X^*} W(x, p) &= G(x) - \sup_{p \in X^*} \left\{ \langle x, p \rangle - F^*(p) \right\} \\ &= G(x) - F^{**}(x) = J(x) \end{aligned}$$

for $x \in D(G)$, which is valid even for $x \notin D(G)$ by (16.18) and (16.20). Thus, we have

$$\begin{aligned} D(J) \quad &= \left\{ x \in X \mid J(x) \neq \pm\infty \right\} = D(G) \cap D(F) \\ D(J^*) &= \left\{ p \in X^* \mid J^*(p) \neq \pm\infty \right\} = D(G^*) \cap D(F^*) \end{aligned}$$

and

$$\begin{aligned} \inf_{x \in X} W(x, p) &= J^*(p) \left(p \in X^* \right) \\ \inf_{p \in X^*} W(x, p) &= J(x) \, (x \in X) \, . \end{aligned} \tag{16.21}$$

Relation (16.21) implies

$$\inf_{(x,p) \in X \times X^*} W(x, p) = \inf_{p \in X^*} J^*(p) = \inf_{x \in X} J(x), \tag{16.22}$$

which is called the *Toland duality* [173], [174].

The above global theory can be localized by the *subdifferential*. In fact, given $F : X \to [-\infty, +\infty]$, $x \in X$, and $p \in X^*$, we say $p \in \partial F(x)$, $x \in \partial F^*(p)$ if

$$\begin{aligned} F(y) &\geq F(x) + \langle y - x, p \rangle \qquad \text{(for any } y \in X), \\ F^*(q) &\geq F^*(p) + \langle x, q - p \rangle \quad \text{(for any } q \in X^*), \end{aligned}$$

respectively. It is obvious that $\partial F(x) \neq \emptyset$ implies $x \in D(F)$, but if $F : X \to (-\infty, +\infty]$ is proper, convex, and lower semicontinuous, then

$$x \in \partial F^*(p) \quad \Leftrightarrow \quad p \in \partial F(x), \tag{16.23}$$

and *Fenchel–Moreau's identity*

$$F(x) + F^*(p) = \langle x, p \rangle \tag{16.24}$$

holds [43].

With these preparations, we can show the first part of the theory of dual variation, the variational equivalence.

Theorem 16.1 *Let $F, G : X \to (-\infty, +\infty]$ be proper, convex, lower semicontinuous, and $W = W(x, p)$ be defined by* (16.18). *Given $\hat{x} \in X$, $\hat{p} \in X^*$, we take the set of minimizers of $p \in X^*$, $x \in X$ in*

$$J(\hat{x}) = \inf_{p \in X^*} W(\hat{x}, p), \qquad J^*(\hat{p}) = \inf_{x \in X} W(x, \hat{p}),$$

denoted by $A^*(\hat{x})$ *and* $A(\hat{p})$*, respectively. We say that* $\hat{x} \in X$ *and* $\hat{p} \in X^*$ *are critical points of* J *and* J^* *if* $\partial G(\hat{x}) \cap \partial F(\hat{x}) \neq \emptyset$*,* $\partial G^*(\hat{p}) \cap \partial F^*(\hat{p}) \neq \emptyset$*, respectively, and that* $(\hat{x}, \hat{p})$ *is a critical point of* W *if* $0 \in \partial_x W(\hat{x}, \hat{p})$*,* $0 \in \partial_p W(\hat{x}, \hat{p})$ *holds true. Then, we have*

$$A^*(x) = \partial F(x), \qquad A(p) = \partial G^*(p) \tag{16.25}$$

for each $(x, p) \in X \times X^*$ *and furthermore, the following items are equivalent:*

1. $(\hat{x}, \hat{p}) \in X \times X^*$ *is a critical point of* W.
2. $\hat{x} \in X$ *is a critical point of* J *and it holds that* $\hat{p} \in \partial G(\hat{x}) \cap \partial F(\hat{x})$.
3. $\hat{p} \in X^*$ *is a critical point of* J^* *and it holds that* $\hat{x} \in \partial F^*(\hat{p}) \cap \partial G^*(\hat{p})$.

Finally, we have

$$W(\hat{x}, \hat{p}) = J(\hat{x}) = J^*(\hat{p}) \tag{16.26}$$

in this case.

Proof: In fact, from (16.18) and (16.23) we have

$$\begin{aligned} &0 \in \partial_x W(x, p) = 0 \quad \Leftrightarrow \quad p \in \partial G(x) \quad \Leftrightarrow \quad x \in \partial G^*(p), \\ &0 \in \partial_p W(x, p) = 0 \quad \Leftrightarrow \quad x \in \partial F^*(p) \quad \Leftrightarrow \quad p \in \partial F(x), \end{aligned} \tag{16.27}$$

for each $(x, p) \in X \times X^*$. Given $x \in X$, we take $p \in A^*(x)$, which attains

$$J(x) = \inf_{p \in X^*} W(x, p),$$

or equivalently, $0 \in \partial_p W(x, p)$. Thus, $A^*(x) = \partial F(x)$ holds by (16.27). Relation $A(p) = \partial G^*(p)$ follows similarly, and the first part, (16.25), is proven. The second part, the equivalence of those three items is obtained also by (16.27), because $(\hat{x}, \hat{p}) \in X \times X^*$ is a critical point of

$$W = W(x, p) = F^*(p) + G(x) - \langle x, p \rangle$$

if and only if $\hat{p} \in \partial G(\hat{x})$ and $\hat{x} \in \partial F^*(\hat{p})$. Finally, (16.26) follows from (16.25), $\hat{p} \in \partial G(\hat{x})$, and $\hat{x} \in \partial F^*(\hat{p})$:

$$\begin{aligned} W(\hat{x}, \hat{p}) &= F^*(\hat{p}) + G(\hat{x}) - \left\langle \hat{x}, \hat{p} \right\rangle \\ &= F^*(\hat{p}) - G^*(\hat{p}) \\ &= G(\hat{x}) - F(\hat{x}). \end{aligned}$$

The proof is complete. □

We have the equivalence of

$$\hat{p} \in \partial G(\hat{x}) \cap \partial F(\hat{x}) \quad \Leftrightarrow \quad \hat{x} \in \partial F^*(\hat{p}) \cap \partial G^*(\hat{p}),$$

and therefore each critical point of J, J^* produces that of J^*, J, respectively. We call This correspondence the Legendre transformation of critical points, or their duality. The *principle of dual variation* means the production of these critical points of J and J^* from this duality, or equivalently, that the critical point $(\hat{x}, \hat{p}) \in X \times X^*$ of $W = W(x, p)$ is characterized as for each element $\hat{x}$, $\hat{p}$ to be a critical point of J, J^*, respectively. We can prove the equivalence of these critical points up to their Morse indices under reasonable assumptions, as in the special case of (16.1). If a (local) dynamical system

$$t \in [0, T) \mapsto (x(t), p(t))$$

is given and $W = W(x, p)$ acts as a Lyapunov function, then we call critical points of W the *stationary state*.

The structures (16.15), (16.16), and (16.17) of the unfolding and minimality are the second part of the abstract theory.

Theorem 16.2 *Given proper, convex, lower semicontinuous functionals F, G : $X \to (-\infty, +\infty]$, we take $W = W(x, p)$, $J = J(x)$, and $J^* = J^*(p)$ by (16.18), (16.20), and (16.19), respectively. Then, it holds that*

$$W\big|_{p \in \partial F(x)} = J, \qquad W\big|_{x \in \partial G^*(p)} = J^*, \tag{16.28}$$

and

$$W(x, p) \geq \max \left\{ J(x), J^*(p) \right\}, \tag{16.29}$$

for each $(x, p) \in X \times X^*$.

Proof: To prove the unfolding (16.28), we note that $p \in \partial F(x)$ implies

$$F^*(p) - \langle x, p \rangle = -F(x)$$

from Fenchel–Moreau's identity (16.24). This implies the first equality of (16.28), and the second equality is proven similarly. On the other hand, the minimality (16.29) is a direct consequence of (16.21). □

Unfolding and minimality imply the stability of the stationary state as follows.

Theorem 16.3 *Let a proper, convex, and lower semicontinuous functional* $F : X \to (-\infty, +\infty]$ *be given with* $J : X \to [-\infty, +\infty]$ *and* $W : X \times X^* \to [-\infty, +\infty]$ *satisfying*

$$W\big|_{p\in\partial F(x)} = J \quad \text{and} \quad W(x, p) \geq J(x)$$

for any $(x, p) \in X \times X^*$. *Let* $(\hat{x}, \hat{p}) \in D(W) \subset X \times X^*$ *be in*

$$\hat{p} \in \partial F(\hat{x}) \cap Y_* \quad \text{and} \quad \hat{x} \in Y_0,$$

where Y_0 *is a closed subset of a Banach space* Y *continuously imbedded in* X, *and* Y_* *is a Banach space continuously embedded in* X^*. *Suppose that* $\hat{x}$ *is a linearized stable local minimizer of* $J|_{Y_0}$ *in the sense that for some* $\varepsilon_0 > 0$, *any* $\varepsilon \in (0, \varepsilon_0/4]$ *admits* $\delta > 0$ *such that*

$$x \in Y_0, \quad \left\|x - \hat{x}\right\|_Y < \varepsilon_0, \quad J(x) - J(\hat{x}) < \delta \quad \Rightarrow \quad \left\|x - \hat{x}\right\|_Y < \varepsilon. \tag{16.30}$$

Suppose, finally, that $W|_{Y_0\times Y_*}$ *is continuous at* $(\hat{x}, \hat{p})$. *Then, if*

$$\left\{(x(t), p(t))\right\}_{0\leq t<T} \subset Y_0 \times Y_*$$

is given with $t \in [0, T) \mapsto x(t) \in Y_0$ *continuous and*

$$t \in [0, T) \quad \mapsto \quad W\,(x(t), p(t)) \tag{16.31}$$

nonincreasing, then any $\varepsilon \in (0, \varepsilon_0/4]$ *admits* $\delta > 0$ *such that*

$$\left\|x(0) - \hat{x}\right\|_Y < \delta \quad \text{and} \quad \left\|p(0) - \hat{p}\right\|_{Y_*} < \delta \tag{16.32}$$

imply

$$\left\|x(t) - \hat{x}\right\|_Y < \varepsilon \qquad (0 \leq t < T). \tag{16.33}$$

Similarly, if $G : X \to (-\infty, +\infty]$ *is proper, convex, lower semicontinuous,* $J^* : X^* \to [-\infty, +\infty]$ *satisfies that*

$$W\big|_{x\in\partial G^*(p)} = J^* \quad \text{and} \quad W(x, p) \geq J^*(x)$$

for any $(x, p) \in X \times \hat{X}$, $(\hat{x}, \hat{p}) \in D(W)$ *is in*

$$\hat{x} \in \partial G^*(\hat{p}) \quad \text{and} \quad \hat{p} \in Y_{0*},$$

respectively, where Y_{0*} is a closed set in Y_*, $\hat{p}$ is a linearized stable local minimizer of $J^*\big|_{Y_{0*}}$ in the sense that any $\varepsilon \in (0, \varepsilon_0]$ admits $\delta > 0$ such that

$$p \in Y_{0*}, \quad \left\|p - \hat{p}\right\|_{Y_*} < \varepsilon_0, \quad J^*(p) - J^*(\hat{p}) < \delta \\ \Rightarrow \quad \left\|p - \hat{p}\right\|_{Y_*} < \varepsilon,$$

$t \in [0, T) \mapsto p(t) \in Y_{0*}$ is continuous with (16.31) decreasing, $\hat{x} \in \partial G^*(\hat{p})$, and $W\big|_{Y \times Y_{0*}}$ is continuous at $(\hat{x}, \hat{p})$, then any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that (16.32) implies

$$\left\|p(t) - \hat{p}\right\|_{Y_*} < \varepsilon \qquad (0 \leq t < T).$$

Proof: We show the former part. In fact, given $\varepsilon \in (0, \varepsilon_0/4]$, we take $\delta = \delta_1 > 0$ in (16.30). Since $W\big|_{Y_0 \times Y_*}$ is continuous at $(\hat{x}, \hat{p})$, there exists $\delta \in (0, \varepsilon_0/2]$ such that

$$\left\|x(0) - \hat{x}\right\|_Y < \delta \quad \text{and} \quad \left\|p(0) - \hat{p}\right\|_{Y_*} < \delta \tag{16.34}$$

imply

$$W\left(x(0), p(0)\right) - W(\hat{x}, \hat{p}) < \delta_1. \tag{16.35}$$

On the other hand, we have

$$W(x, p) \geq J(x) \geq J(\hat{x}) = W(\hat{x}, \hat{p})$$

for any $(x, p) \in Y_0 \times X^*$ with $\|x - \hat{x}\|_Y < \varepsilon_0$ from the assumption. Therefore, as far as

$$\left\|x(t) - \hat{x}\right\|_Y < \varepsilon_0 \tag{16.36}$$

we have

$$\begin{aligned} 0 &\leq J\left(x(t)\right) - J(\hat{x}) \leq W\left(x(t), p(t)\right) - J(\hat{x}) \\ &\leq W\left(x(0), p(0)\right) - W(\hat{x}, \hat{p}) < \delta_1. \end{aligned} \tag{16.37}$$

Now, we have

$$\left\|x(0) - \hat{x}\right\|_Y < \delta \leq \varepsilon_0/2.$$

Then, if there is $t_0 \in (0, T)$ satisfying $\|x(t_0) - \hat{x}\|_Y = \varepsilon_0/2$, then we have (16.36) and hence (16.37) for $t = t_0$. This implies from (16.30) (with $\delta = \delta_1$) that

$$\|x(t) - \hat{x}\|_Y < \varepsilon \leq \varepsilon_0/4, \tag{16.38}$$

a contradiction. Therefore, since $t \in [0, T) \mapsto x(t) \in Y_0 \subset Y$ is continuous, the relation

$$\left\| x(t) - \hat{x} \right\|_Y < \varepsilon_0/2$$

keeps to hold for $t \in [0, T)$, and hence (16.36) in particular. Again this implies (16.37) and (16.38) for any $t \in [0, T)$, and the proof is complete. □

To infer (16.32), the continuity of W at $(x, p = (\hat{x}, \hat{p})$ can be replaced by the first case of (16.34) and (16.35) for the initial value $(x(0), p(0))$. By Damlamian [35], Toland duality was observed in the free boundary problem for plasma confinement, between the formulations of Berestycki and Brezis [10] and Temam [172]. In the Penrose–Fife system [129], on the other hand, exact duality cannot be observed, while semi-unfolding and semi-minimality are valid, which provide stability for the field component. We have several examples of a dual variation or semi-dual variation in mean field theories. Here, we show how the abstract theory is realized in the system of chemotaxis, particularly in (16.11), where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$.

For this problem, we take $X = H_0^1(\Omega)$ with the Gel'fand triple $X \hookrightarrow L^2(\Omega) \hookrightarrow X^*$. Then, the *dual entropy functional* $F : X \to (-\infty, +\infty]$ is defined by

$$F(v) = \lambda \log \left(\int_\Omega e^v \, dx \right) - \lambda \log \lambda + \lambda,$$

which is proper, convex, and lower semicontinuous. We have

$$D(F) = \left\{ v \in X \;\Big|\; \int_\Omega e^v \, dx < +\infty \right\},$$

$\partial F(v) \neq \emptyset$ for any $v \in D(F)$, and

$$u \in \partial F(v) \quad \Leftrightarrow \quad u = \frac{\lambda e^v}{\int_\Omega e^v \, dx}.$$

The *entropy functional* is defined by its Legendre transformation,

$$F^*(u) = \begin{cases} \displaystyle\int_\Omega u(\log u - 1)\, dx & \left(u \in X^* \cap L^1(\Omega),\ u \geq 0,\ \left\|u\right\|_1 = \lambda\right), \\ +\infty & (\text{otherwise}). \end{cases}$$

It holds that

$$D(F^*) = \left\{ u \in X^* \;\middle|\; u \geq 0,\ u \in L \log L(\Omega),\ \|u\|_1 = \lambda \right\},$$

$v \in \partial F^*(u)$ if and only if $u \in D(F^*)$ and

$$v = \log u + \text{constant} \in X,$$

where $L \log L(\Omega)$ denotes the Zygmund space. On the other hand, putting

$$G(v) = \frac{1}{2}\|\nabla v\|_2^2,$$

we obtain a proper, convex, and lower semicontinuous mapping

$$G : X \to (-\infty, +\infty).$$

The operator $-\Delta_D$ induces the isomorphism $\hat{A} : X \to X^*$, and we have

$$G^*(u) = \frac{1}{2}\left\langle \hat{A}^{-1}u, u \right\rangle$$

for $u \in X^*$. Then, the Lyapunov function of this system is realized as

$$W(v, u) = F^*(u) + G(v) - \langle v, u \rangle$$

and the equilibrium is described by

$$0 \in \partial_v W(\overline{v}, \overline{u}), \qquad 0 \in \partial_u W(\overline{v}, \overline{u})$$

or equivalently,

$$\overline{u} = \hat{A}^{-1}\overline{v}, \qquad \overline{v} \in \partial F^*(\overline{u}).$$

From Theorem 9.1, this relation is transformed into the conditions on $\overline{u}$ and $\overline{v}$, separately, that is, to be critical points of

$$\begin{aligned} J(v) &= G(v) - F(v) \\ &= \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log\left(\int_\Omega e^v \, dx\right) + \lambda \log \lambda - \lambda \end{aligned}$$

defined for $v \in X$ and

$$\begin{aligned} J^*(u) &= F^*(u) - G^*(u) \\ &= \int_\Omega u(\log u - 1)\, dx - \frac{1}{2}\left\langle \hat{A}^{-1}u, u \right\rangle \end{aligned}$$

defined for $u \in X^* \cap L^1(\Omega)$, $u \geq 0$, $\|u\|_1 = \lambda$, respectively. These conditions are equivalent to

$$\overline{v} \in X, \quad \int_\Omega e^{\overline{v}}\, dx < +\infty,$$
$$\hat{A}\overline{v} = \frac{\lambda e^{\overline{v}}}{\int_\Omega e^{\overline{v}}\, dx} \in X^* \tag{16.39}$$

and

$$\overline{u} \in X \cap L \log L(\Omega), \quad \overline{u} \geq 0, \quad \left\|\overline{u}\right\|_1 = \lambda$$
$$\hat{A}^{-1}\overline{u} = \log \overline{u} + \text{constant} \in X, \tag{16.40}$$

respectively. The exact correspondence of the Morse indices is known, but here we only make use of the equivalence of the linearized stability of those $\overline{u}$ and $\overline{v}$, applying Theorem 16.3 for

$$Y = D(J) = D(G) \cap D(F) = \left\{ v \in X \;\middle|\; \int_\Omega e^v\, dx < +\infty \right\}$$

and

$$Y_{0*} = D(J^*) = \left\{ u \in X^* \;\middle|\; u \in L \log L(\Omega),\ u \geq 0,\ \left\|u\right\|_1 = \lambda \right\}$$
$$\subset Y_* = X^* \cap L \log L(\Omega).$$

Thus, there is $\varepsilon_0 > 0$ such that if $u = u(\cdot, t)$, $v = v(\cdot, t)$ is a solution to (16.11) for $t \in [0, T)$, then any $\varepsilon \in (0, \varepsilon_0/4]$ admits $\delta > 0$ such that

$$\left\|v(\cdot, 0) - \overline{v}\right\|_X < \delta, \quad \left\|u(\cdot, 0) - \overline{u}\right\|_{X^* \cap L \log L} < \delta,$$
$$\left\|u(\cdot, 0)\right\|_1 = \lambda = \left\|\overline{u}\right\|_1$$

implies

$$\left\|v(t) - \overline{v}\right\|_X < \varepsilon, \qquad \left\|u(\cdot, t) - \overline{u}\right\|_{X^* \cap L \log L} < \varepsilon$$

for any $t \in [0, T)$. This result is valid for any space dimension and also to the full system

$$\left.\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ \tau v_t &= \Delta v + u \end{aligned}\right\} \quad \text{in} \quad \Omega \times (0, T),$$
$$\left.\begin{aligned} \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega \times (0, T).$$

with $\tau > 0$.

Next, we apply the theory of unfolding of minimality to study the behavior of the solution globally in time for (16.1). In fact, first, from the Trudinger–Moser inequality, we have

$$\inf_{v\in X} J(v) > -\infty \tag{16.41}$$

in the case of $n = 2$ and $\lambda = 8\pi$. Next, Theorem 9.1 guarantees the equivalence of the boundedness from below of J on X and that J^* on X^*, and hence it follows that

$$\inf_{u\in X^*\cap L\log L,\ u\geq 0,\ \|u\|_1=\lambda} J^*(u) > -\infty$$

in this case. Furthermore, the Trudinger–Moser inequality again guarantees the imbedding $L\log L(\Omega) \hookrightarrow X^*$ for $n = 2$, and hence it holds that

$$\inf\left\{ \int_\Omega u(\log u - 1)\, dx - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\, dx\, dx' \,\Big|\, u \geq 0,\ \|u\|_1 = 8\pi \right\} > -\infty, \tag{16.42}$$

where $G = G(x, x')$ denotes the Green's function. Inequality (16.42), valid for $n = 2$, is regarded as the dual form of the Trudinger–Moser–Onofri inequality.

From (16.41), each $\lambda < 8\pi$ admits a constant C_1 such that

$$\begin{aligned} J(v) &= \frac{1}{2}\left\|\nabla v\right\|_2^2 - \lambda\log\left(\int_\Omega e^v\, dx\right) + \lambda\log\lambda - \lambda \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda}{8\pi}\right)\left\|\nabla v\right\|_2^2 - C_1 \end{aligned}$$

for any $v \in X$. Therefore, if $\left\|u_0\right\|_1 = \lambda < 8\pi$ in (16.11), then we have

$$\sup_{t\in[0,T)} \left\|\nabla v(\cdot,t)\right\|_2 \leq C_2$$

with a constant $C_2 > 0$ determined by λ, because of (16.29), and

$$\sup_{t\in[0,T)} W\left(v(t), u(t)\right) \leq W\left(u(0), v(0)\right).$$

Similarly, from (16.42) we have

$$J^*(u) \geq \left(1 - \frac{\lambda}{8\pi}\right)\int_\Omega u\log u\, dx - C_3$$

for any $u \in L \log L(\Omega)$ in $u \geq 0$ and $\|u\| = \lambda < 8\pi$, and this implies that

$$\sup_{t\in[0,T)} \int_\Omega (u \log u)(x,t)\, dx \leq C_4. \tag{16.43}$$

Then, from Moser's iteration scheme or the maximal regularity we can derive $T_{\max} = +\infty$ and the uniform boundedness of $u(\cdot, t)$:

$$\sup_{t\in[0,T)} \left\| u(\cdot,t) \right\|_\infty \leq C_5.$$

Inequality (16.43) is also derived from the Trudinger–Moser inequality and

$$\int_\Omega (u \log u - uv)\ dx + \lambda \log \left(\int_\Omega e^v\, dx \right) - \lambda \log \lambda \geq 0 \tag{16.44}$$

which is valid for $u \in L \log L(\Omega)$, $u \geq 0$, $\|u\|_1 = \lambda$ [14, 50, 110]. Inequality (16.44) follows from Jensen's inequality, but it is also a consequence of the minimality

$$W(v,u) \geq J(v).$$

In the simplified system, we have

$$W(v,u) = \mathcal{F}(u)$$

and hence from the minimality it follows that

$$\mathcal{F}\left(u(t)\right) \geq J\left(v(t)\right) \qquad (0 \leq t < T_{\max}).$$

On the other hand, from the quantization of the blowup mechanism of the stationary state we have

$$j_\lambda = \inf\{J(v) \mid v \in E_\lambda\} > -\infty$$

for $\lambda \in [0,\infty) \setminus \mathcal{N}$, where E_λ denotes the set of critical points of J on X. Furthermore, we have

$$\frac{d}{dt}\int_\Omega u \log u\, dx \leq 2K^2\lambda + 4\,|\Omega| \exp\left(4K^2 \int_\Omega u \log u\, dx + 4K^2 e^{-1}\,|\Omega|\right)$$

with a constant $K > 0$ determined by Ω, and if $T_{\max} = +\infty$ and

$$\lim_{t\to+\infty} \int_\Omega u \log u(x,t)\, dx < +\infty,$$

then we have $t_k \to +\infty$, $\delta > 0$, and $C > 0$ such that

$$\int_\Omega (u \log u)(x,t)\, dx \leq C \qquad (t \in [t_k, t_k + \delta]) .$$

This implies the nonempty form of the ω-limit set of $(u(t), v(t))$, and therefore

$$W(v_0, u_0) \geq \lim_{t \to +\infty} W\left(v(t), u(t)\right) \geq j_\lambda$$

holds true. Since

$$\lim_{t \uparrow T_{\max}} \int_\Omega (u \log u)(x,t)\, dx = +\infty$$

follows in the case of $T_{\max} < +\infty$, we obtain the criterion of [69], that is,

$$W(v_0, u_0) < j_\lambda \quad \Rightarrow \quad \lim_{t \to T_{\max}} \int_\Omega (u \log u)(x,t)\, dx = +\infty. \qquad (16.45)$$

Nonlinear quantum mechanics is just an episode of the mathematical theory of statistical mechanics. It asserts the control of the total set of stationary states on the global dynamics of nonstationary states. This story, we are convinced, is valid for each theory of mean fields, where the self-interaction is caused in terms of the field created by particles. The principle of dual variation arises in this context, that is, the study on the stationary states of nonlinear systems, where interaction is described in terms of the field created by particles. It assures that the stationary state in these hierarchies splits into the problems on fields and particles, each of which is provided with the variational structure, dynamically equivalent to each other. We have unified such a structure in the Toland duality for the system of chemotaxis and also for the free boundary problem in plasma confinement. In both cases, concentration of the particle density is observed. On the other hand, the Penrose–Fife system has the property of unfolding-minimality in the field component, and the same is true for the Euler-Poisson equation describing the evolution of gaseous stars. Consequently, we can discuss the stability of the equilibrium field in these systems by introducing variational structure for the field component. On the other hand, we have the other systems controlled by a different form of the dual variation, where the stable stationary state is realized as a saddle point of the Lagrange function. Actually, it is associated with the Kuhn–Tucker duality, and especially the dynamics surrounding the degenerate stable equilibrium are interesting. The study of these theories of dual variation is still in progress and will be clarified in the future.

In the context of the theory of self-organization, the type (I) blowup point is interesting. It assures emergence coming from the wedge of the parabolic envelope, where entropy and mass are exchanged to create a clean self with the quantized mass, which reminds us of the principle asserted in system biology that the expanding cosmos is the origin of life. This remarkable fact is suggested in the story of nonlinear quantum mechanics, where the theory of dual variation takes on a role to control the set of stationary states as well as the local dynamics around them. Finally, even the type (II) blowup point can take on the profile of emergence in the other scaling of space-time, that is, the emergence is a corollary of the collapse mass quantization and the blowup envelope.

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