A variational approach to chaotic dynamics in periodically forced nonlinear oscillators[*]

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Abstract

We prove that a class of problems containing the classical periodically forced pendulum equation displays the main features of chaotic dynamics. The approach is based on the construction of multibump type heteroclinic solutions to periodic orbits by the use of global variational methods.

1 Introduction

The purpose of this paper is to produce evidence of chaotic dynamics in the equation

$$\ddot{u}(t) + W'(t, u(t)) = h(t) \tag{1}$$

through the existence of heteroclinic type solutions. In (1), W' stands for $\frac{\partial W}{\partial u}$, and the functions W and h satisfy the following assumptions:

(H1) $W \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}; \mathbf{R})$ is 1–periodic in t and S–periodic in u

(H2) $h \in \mathcal{C}(\mathbf{R}; \mathbf{R})$ is 1–periodic and satisfies $\int_0^1 h(t)\, dt = 0$.

Assumptions (H1) and (H2), which we will use throughout the paper without further repetition, are satisfied by the periodically forced pendulum equation $\ddot{u} + \sin u = h(t)$, where h has mean value zero. We believe that the applicability of our results to this classical mechanical model is one of the points of interest of the present work.

In order to describe our results, let $V(t, u) = W(t, u) - h(t)u$ and let $L(u) = \frac{1}{2}\dot{u}^2 - V(t, u)$ be the Lagrangian associated to (1). It is well known that equation (1)

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admits, under assumptions (H1) and (H2), an ordered family of 1–periodic solutions which can be obtained as global minimizers of the action functional

$$f(u) = \int_0^1 L(u)\, dt$$

over the space E_1 of 1–periodic, H^1_{loc} functions. Let u_0 and u_1 be two *consecutive* minimizers (see Definition 2.1). We are interested in solutions q of (1) which are asymptotic to prescribed states in $\{u_0, u_1\}$ when t tends to $\pm\infty$. To simplify notation, we will write $q(-\infty) = u_0$ instead of $\lim_{t\to-\infty}(q(t) - u_0(t)) = 0$ for example, and similar expressions at $+\infty$. A solution is called homoclinic if it is asymptotic to the same state at $-\infty$ and $+\infty$, and heteroclinic if the asymptotic states are different.

The search for homoclinic or heteroclinic solutions is a classical subject, which has been deeply studied by the use of geometrical methods (see e.g. [20]); in this context, the results are generally of a perturbative nature. Starting from an integrable system, one analyzes the dynamics of a new system obtained as a *small* perturbation of the former. More recently, a global approach based on variational arguments has been shown to be useful in proving existence and multiplicity of homoclinic solutions for certain classes of problems. The use of the variational approach is often convenient because it does not require the system to be a small perturbation of a simpler one, needs in general only mild nondegeneracy conditions, and is powerful enough to detect the principal features of chaotic dynamics. This approach has been very extensively applied, starting from the late eighties, to problems of Duffing kind with various types of time dependence (see [4], [7], [8], [11], [16], [17], [19], and references therein for an introduction to the subject). The object of these papers is to produce evidence of chaotic dynamics by the construction of *multibump* type homoclinic orbits. In this type of problem the solutions are homoclinic to a rest point of the system.

In contrast, only a few papers are devoted to the study of homoclinic or heteroclinic solutions to *periodic* orbits via a global variational approach, a reference model in this case being the forced pendulum equation. This is possibly due to the lack of a simple functional formulation for such problems, which, by their very nature, force one to work with nonintegrable functions. The major contribution to the removal of this obstacle is the work [13] by P. H. Rabinowitz, where the author managed to construct a functional J (see (2)) whose minimizers are the desired solutions, opening in this way the road to a global approach. In the papers [14], [15], [10], [6], [1], which followed [13], existence and multiplicity of heteroclinics, homoclinics and more general multibump and chaotic solutions were obtained both for scalar problems and for systems. The common feature of all these papers, and the idea that led Rabinowitz to construct the functional J, is the use of a *reversibility* assumption: it is required that the potential V be an even function of time. No paper that we know of deals with the general, nonreversible case from a nonperturbative point of view.

The purpose of our work is to show that for scalar problems one can drop the

reversibility assumption and obtain the same type of results as in the nonreversible case.

The main difficulty one has to face when working without reversibility assumptions is to show that the functional J still makes sense and is fit to obtain the desired results. After the construction of the functional, the typical way to prove existence of multibump and chaotic trajectories can be roughly synthesized in the following scheme.

First, one usually needs some nondegeneracy assumption on the asymptotic states u_0 and u_1. This means for example that u_0 and u_1 are required to be nondegenerate as minimizers, or at least isolated; these assumptions have been used in [12] and [13], [14], [15] respectively. Then, the first basic types of heteroclinic solutions can be found; in our problem these are an orbit q_0 that connects u_0 to u_1 (namely $q_0(-\infty) = u_0$ and $q_0(+\infty) = u_1$) and an orbit q_1 that connects u_1 to u_0. The basic nature of q_0 and q_1 is their appearance as minimizers of the functional J over suitable classes of functions. These orbits are then used as building blocks for more complicated types of solutions, in the spirit of the shadowing lemma; they give rise to multibump type orbits, which are solutions that oscillate between the states u_0 and u_1 a prescribed number of times and are asymptotic to the required states at $\pm\infty$. In order to do this, one needs a further nondegeneracy condition on the structure of the basic heteroclinic solutions. This assumption takes the place of the transversality condition in the perturbative approach and prevents, for example, the problem from being autonomous. Finally, a rather careful analysis is needed in order to obtain estimates independent of the number of bumps, so that one can pass to the limit and obtain solutions with infinitely many bumps, which display the chaotic nature of the dynamics.

With respect to this general scheme our work is also characterized, in addition to the removal of the reversibility assumption, by the following features.

The nondegeneracy conditions on u_0 and u_1 are replaced with the requirement that u_0 and u_1 be consecutive. We work throughout the paper in the order interval $[u_0, u_1]$; this weakens the classical assumptions and shows that our results are in general valid for variational problems possessing two global minimizers, without assumptions on the space periodicity of the potential. On the other hand, the existence of consecutive minimizers is *necessary* in order to obtain multibump solutions (see [6]).

Next, the nondegeneracy condition on the structure of basic heteroclinics is a suitable modification of the one introduced in [6] for the scalar reversible case (and later adapted in [15] to reversible systems). Roughly it says that the sets of orbits connecting u_0 to u_1 and u_1 to u_0 must be disconnected (see $(*)$ at the beginning of Section 4). This assumption is then compared to the standard nondegeneracy hypotheses in the perturbative approach. We prove that when u_0 and u_1 are hyperbolic (which we never require), our condition is equivalent to the fact that the stable manifold of u_0 (resp. u_1) and the unstable manifold of u_1 (resp. u_0) do not coincide.

Below we state our main results; in their statements, when we label u with an

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index, that index must be read mod 2.

Theorem 1.1 (Multibump solutions). *Assume u_0 and u_1 are consecutive minimizers and let ($*$) hold. Then for every $\delta > 0$ small enough there exists $m = m(\delta) \in \mathbf{N}$ such that for every sequence $(p_i)_{i \in \mathbf{Z}} \subset \mathbf{Z}$ such that $p_{i+1} - p_i \geq 4m$ and for every $j, k \in \mathbf{Z}$ with $j < k$, there exists a classical solution q of (1) satisfying*

$$u_0(t) < q(t) < u_1(t) \qquad \text{for all } t \in \mathbf{R},$$

$$q(-\infty) = u_j, \qquad q(+\infty) = u_{k+1}$$

and, for all $i = j, \ldots, k$,

$$|q(p_i - m) - u_i(p_i - m)| \leq \delta \qquad and \qquad |q(p_i + m) - u_{i+1}(p_i + m)| \leq \delta.$$

Theorem 1.2 (∞–bump solutions). *Under the same assumptions as in Theorem 1.1, for every $\delta > 0$ small enough there exists $m = m(\delta) \in \mathbf{N}$ such that for every sequence $(p_i)_{i \in \mathbf{Z}} \subset \mathbf{Z}$ such that $p_{i+1} - p_i \geq 4m$, there exists a classical solution q of (1) satisfying*

$$u_0(t) < q(t) < u_1(t) \qquad \text{for all } t \in \mathbf{R},$$

and

$$|q(p_i - m) - u_i(p_i - m)| \leq \delta \qquad and \qquad |q(p_i + m) - u_{i+1}(p_i + m)| \leq \delta$$

for all $i \in \mathbf{Z}$.

In addition to these results, one could also work in a space of periodic functions to obtain the existence of infinitely many periodic orbits with large periods. This requires only minor modifications of our arguments, and we will not deepen this point; we refer the reader to [9] for a similar construction in a different setting.

The above results show that some of the principal features of chaotic dynamics are embodied in equation (1); that is, one has sensitive dependence from the initial conditions, existence of infinitely many periodic orbits with diverging periods, and existence of an uncountable number of bounded, nonperiodic trajectories.

The nature of our methods also shows that the set of forcing terms h for which (1) displays the described features is open in the space of continuous periodic functions with mean value zero.

As final remark, we point out that through the multibump structure obtained by Theorem 1.1 one can easily show the Poincaré map associated to (1) has positive topological entropy.

The main drawback of our method is that it relies deeply on the order structure of $\mathbf{R}$, so that for the time being, it does not seem to extend to systems. The problem of the existence of heteroclinics to periodic motions for systems in the nonreversible (nonperturbative) case is, as far as we know, completely open.

This paper is structured as follows: Section 2 contains the estimates needed for the definition and the use of the functional J. In Section 3, we prove the existence of

the basic heteroclinic solutions. The multibump construction and the main results are given in Section 4. Finally, the comparison with the classical results in the hyperbolic case is the object of Section 5.

Notation.   We use the symbols L^p and H^1 to denote the spaces $L^p(0,1)$ and $H^1(0,1)$, endowed with their usual norms. The symbols L_{loc}^∞ and H_{loc}^1 stand for $L_{loc}^\infty(\mathbf{R};\mathbf{R})$ and $H_{loc}^1(\mathbf{R};\mathbf{R})$.

If u_0 and u_1 are 1–periodic (continuous) functions such that $u_0(t) \le u_1(t)$ for all t, we say that $u \in [u_0, u_1]$ if

$$u_0(t) \le u(t) \le u_1(t) \ \ \forall t \in \mathrm{dom}(u).$$

Whenever E is a set of continuous functions we will write $u \in E \cap [u_0, u_1]$ to mean that $u \in E$ and $u_0(t) \le u(t) \le u_1(t) \ \ \forall t \in \mathrm{dom}(u),$.

2   Functional setting and estimates

This section is devoted to the study of the the main properties of the functional that we will use to construct homoclinic solutions. These properties require a number of estimates on the action of both periodic and nonperiodic functions.

To begin with we define, for $n \in \mathbf{N}$, the space of n–periodic functions

$$E_n = \{u \in H_{loc}^1(\mathbf{R};\mathbf{R}) \ / \ u(t+n) = u(t) \ \text{a.e.}\},$$

the Lagrangian $L(u) = \frac{1}{2}\dot{u}^2 - V(t,u)$, the action functional over $H^1(0,1)$

$$f(u) = \int_0^1 L(u)\, dt,$$

and we set

$$c_p = \min_{E_1} f(u).$$

By classical arguments, the value c_p is attained at an ordered family of functions. It is a family because of the spatial periodicity of the potential W, and it is ordered since two absolute minimizers cannot cross (see [18]).

Definition 2.1 *We say that $u_0, u_1 \in E_1$ are consecutive minimizers of the periodic problem associated to equation (1) if*

1. $f(u_0) = f(u_1) = c_p$,

2. $u_0(t) < u_1(t)$ for all $t \in [0,1]$,

3. $u \in E_1 \cap [u_0, u_1]$ and $f(u) = c_p$ imply $u \in \{u_0, u_1\}$.

Note that u_0 and u_1 need not be isolated in E_1: there might be other minimizers accumulating to u_0 from below or to u_1 from above. The existence of consecutive minimizers is thus a weaker condition than the isolatedness of minimizers and actually in [6] it is shown that this condition is necessary for the existence of multibump solutions. From now on we will always assume that two consecutive minimizers u_0 and u_1 are given.

We now recall the definition of the functional J. For $k \in \mathbf{Z}$ and $q \in H^1_{loc}(\mathbf{R}; \mathbf{R}) \cap [u_0, u_1]$ we set

$$a_k(q) = \int_k^{k+1} L(q) \, dt - c_p$$

and we define (formally, for the moment) a functional J as

$$J(q) = \sum_{k \in \mathbf{Z}} a_k(q). \tag{2}$$

This functional has been introduced by P. H. Rabinowitz in [13] in the context of time–reversible equations, and has been later used in [14], [15], [6], [1] to deal again with reversible problems. The reversibility assumption implies that each $a_k(q)$ is nonnegative, so that the series is well defined (convergent or divergent to $+\infty$), see [13]. The functional J has been used in the quoted papers to prove existence of multibump type solutions, both for scalar problems and for systems. The argument used is minimization over suitable classes of functions.

In the present work there is no reversibility assumption, and this means that the terms $a_k(q)$ can be negative. It is therefore not clear *a priori* if J is bounded from below or even well defined (the series might in principle be undetermined). One of the purposes of this paper is to show that for *scalar* problems the functional J is still the right tool to use to prove existence of multibump and chaotic trajectories.

In order to carry out our program we need a series of estimates, which we now describe. We first recall the following equality.

Lemma 2.2 *For every $n \in \mathbf{N}$,*

$$\min_{u \in E_n} \int_0^n L(u) \, dt = nc_p \tag{3}$$

and the minimum is attained at 1–periodic functions (the minimizers of f over E_1).

Proof. Equality (3) is perhaps well–known. Since we don't know a precise reference we outline here a simple proof. Let u be a minimizer over E_n (the existence of such function is straightforward); then, since the problem is 1–periodic, the function $v(t) = u(t + 1)$ is a minimizer as well. But since u and v have the same mean value and they cannot cross (by regularity arguments and uniqueness in the Cauchy problem), it must be $u(t + 1) = u(t)$ for all t, which proves the statement. ∎

This result shows in particular that if $u \in E_n \cap [u_0, u_1]$ satisfies $\int_0^n L(u) \, dt = nc_p$, then $u \in \{u_0, u_1\}$, since u_0 and u_1 are consecutive.

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In many parts of this paper we will need a way to measure the distance of a function from a set of periodic minimizers. For computational reasons we will use the notion of distance described in the following definition.

Definition 2.3 *We denote by $\mathcal{M}$ the (closed) set of 1–periodic minimizers of f and by $\mathcal{U}$ the set $\{u_0, u_1\}$. We also set*

$$\mathcal{M}(t) = \{u(t) \,/\, u \in \mathcal{M}\} \qquad and \qquad \mathcal{U}(t) = \{u(t) \,/\, u \in \mathcal{U}\}.$$

If v is a continuous function defined on a compact set I we define as its distance from $\mathcal{M}$ the number

$$\hat{d}(v, \mathcal{M}) = \max_{t \in I} \min_{u \in \mathcal{M}} |v(t) - u(t)|, \tag{4}$$

and likewise for its distance from $\mathcal{U}$.

Remark 2.4 i) Notice that it may result $\hat{d}(v, \mathcal{M}) = 0$ and $v \notin \mathcal{M}$; this happens if v lies in a continuum of minimizers. ii) One of the reasons for measuring in this way the distance is the following immediate consequence of the definition: if $\hat{d}(v, \mathcal{U}) \geq \delta$, there is a point t^* where *both* $|v(t^*) - u_0(t^*)| \geq \delta$ and $|v(t^*) - u_1(t^*)| \geq \delta$.

The next proposition defines a constant that we will use repeatedly.

Proposition 2.5 *For every $\delta > 0$ there exists $\rho(\delta) > 0$ such that for all $n \in \mathbf{N}$*

$$\inf\{\int_0^n L(v)\,dt - nc_p \,/\, v \in E_n, \ \hat{d}(v, \mathcal{M}) \geq \delta\} \geq \rho(\delta). \tag{5}$$

Moreover, if $v \in E_n$ satisfies

$$\operatorname{dist}(v(k), \mathcal{M}(k)) \geq \delta \qquad for\ all \qquad k = 1, \ldots, n-1, \tag{6}$$

then

$$\int_0^n L(v)\,dt - nc_p \geq n\rho(\delta). \tag{7}$$

Proof. Call $\rho(\delta, n)$ the left–hand–side of (5). Then $\rho(\delta, n) > 0$ for all $n \in \mathbf{N}$ as one easily checks by standard compactness properties and the application of Lemma 2.2 (see [13] for a similar argument). We now show that $\rho(\delta, n) \geq \rho(\delta, 1)$, so that (5) holds with $\rho(\delta) := \rho(\delta, 1)$ uniformly in n.

We proceed by induction on n. If $n = 1$ the inequality is trivial. We assume therefore that $\rho(\delta, n-1) \geq \rho(\delta, 1)$ and we show that $\rho(\delta, n) \geq \rho(\delta, 1)$. To see this, let $v \in E_n$ verify $\hat{d}(v, \mathcal{M}) \geq \delta$ and define $g : [0, n-1] \to \mathbf{R}$ as $g(t) = v(t+1) - v(t)$. The function g is continuous and vanishes at some $\tau \in [0, n-1]$, for otherwise we would have $v(k+1) > v(k)$ (or the reversed inequality) for all $k = 0, \ldots, n-1$, which contradicts the periodicity of v. We have therefore a point τ where $v(\tau) = v(\tau+1)$. It is not restrictive (by shifting time, if necessary) to assume that $\tau \in (0, n-1)$. Let v_1 be the 1–periodic extension of $v_{|[\tau, \tau+1]}$ and let

$$v_{n-1}(t) = \begin{cases} v(t) & \text{if } t \in [0, \tau] \\ v(t+1) & \text{if } t \in [\tau, n-1]; \end{cases}$$

then $v_1 \in E_1$, $v_{n-1} \in E_{n-1}$ and at least one between $\hat{d}(v_1, \mathcal{M}) \geq \delta$ and $\hat{d}(v_{n-1}, \mathcal{M}) \geq \delta$ is true. Noticing that

$$\int_0^n L(v)\, dt = \int_0^{n-1} L(v_{n-1})\, dt + \int_{n-1}^n L(v_1)\, dt,$$

we see that by the inductive assumption,

$$\int_0^n L(v)\, dt \geq (n-1)c_p + \rho(\delta, 1) + c_p = nc_p + \rho(\delta, 1).$$

Since this inequality holds for all $v \in E_n$ such that $\hat{d}(v, \mathcal{M}) \geq \delta$, we have that $\rho(\delta, n) \geq \rho(\delta, 1)$.

To prove the second part we again proceed by induction. If $n = 1$ there is nothing to prove; we therefore assume that (7) is true for $n - 1$ and we prove it for n. To do this take v in E_n satisfying (6) and repeat the above argument to construct v_1 and v_{n-1}. Notice now that when $\text{dist}(v(k), \mathcal{M}(k)) \geq \delta$ for all $k = 1, \dots, n - 1$, then *both* v_1 and v_{n-1} satisfy (6) in E_1 and E_{n-1} respectively. Therefore, by the inductive assumption,

$$\int_0^n L(v)\, dt = \int_0^{n-1} L(v_{n-1})\, dt + \int_{n-1}^n L(v_1)\, dt \geq (n-1)c_p + (n-1)\rho(\delta) + c_p + \rho(\delta),$$

and the proof is complete. ∎

We now turn to estimates on nonperiodic functions. The one provided by the following proposition will play a central role in the rest of the paper. Before stating it we observe that Proposition 2.5 holds also for $\delta = 0$. In this case, of course, we obtain $\rho(0) = 0$.

Proposition 2.6 *There exists a positive constant $\hat{C}$ such that for every $\delta \geq 0$ and for every $n \in \mathbf{N}$, if $q \in H^1(0, n) \cap [u_0, u_1]$ satisfies $\hat{d}(q, \mathcal{U}) \geq \delta$, then*

$$\int_0^n L(q)\, dt - nc_p \geq \rho(\delta) - \hat{C}|q(n) - q(0)|. \tag{8}$$

Proof. If $\hat{d}(q, \mathcal{U}) \geq \delta$, there is a point $t^* \in [0, n]$ where $u_1(t^*) - q(t^*) \geq \delta$ and $q(t^*) - u_0(t^*) \geq \delta$. We assume that $t^* \in [\frac{n}{2}, n]$, the other case being symmetric.

Define $v : [0, \frac{1}{2}] \to \mathbf{R}$ as $v(t) = (1 - 2t)(q(n) - q(0))$ and set

$$\varphi(t) = \begin{cases} q(t) + v(t) & \text{if } t \in [0, \frac{1}{2}] \\ q(t) & \text{if } t \in [\frac{1}{2}, n]; \end{cases}$$

then $\varphi \in E_n$, because $\varphi(0) = \varphi(n)$. Notice that

$$\hat{d}(\varphi, \mathcal{M}) \geq \min_{u \in \mathcal{M}} |\varphi(t^*) - u(t^*)| \geq \delta$$

since $u_0(t^*) < \varphi(t^*) < u_1(t^*)$. Applying Proposition 2.5 to φ yields

$$\int_0^n L(\varphi)\, dt - nc_p \geq \rho(\delta). \tag{9}$$

We now evaluate

$$\int_0^n L(\varphi)\, dt = \int_0^{\frac{1}{2}} L(q+v)\, dt + \int_{\frac{1}{2}}^n L(q)\, dt = \frac{1}{2}\int_0^{\frac{1}{2}} [\dot{q}^2 + \dot{v}^2 + 2\dot{q}\dot{v}]\, dt$$

$$- \int_0^{\frac{1}{2}} [V(t,q+v) - V(t,q)]\, dt - \int_0^{\frac{1}{2}} V(t,q)\, dt + \int_{\frac{1}{2}}^n L(q)\, dt$$

$$\leq \int_0^n L(q)\, dt + M\int_0^{\frac{1}{2}} |v|\, dt + \frac{1}{2}\int_0^{\frac{1}{2}} \dot{v}^2\, dt + \int_0^{\frac{1}{2}} \dot{q}\dot{v}\, dt,$$

where we have denoted by M the Lipschitz constant of V.

A direct evaluation of the last three integrals shows that there is a constant $\hat{C}$ such that

$$\int_0^n L(\varphi)\, dt \leq \int_0^n L(q)\, dt + \hat{C}|q(n) - q(0)|.$$

Combining this with the above estimate shows that

$$\int_0^n L(q)\, dt - nc_p \geq \rho(\delta) - \hat{C}|q(n) - q(0)|,$$

which is the desired bound. ∎

The previous estimate has a very important consequence on the behavior of the functional J.

Proposition 2.7 *There exists $b \in \mathbf{R}$ such that for all $q \in H^1_{loc}(\mathbf{R}; \mathbf{R}) \cap [u_0, u_1]$ and all $n \in \mathbf{N}$,*

$$\sum_{k=0}^{n-1} a_k(q) = \int_0^n L(q)\, dt - nc_p \geq b. \tag{10}$$

Proof. Notice that for all q as in the statement there results $|q(n) - q(0)| \leq \max u_1 - \min u_0$. Applying Proposition 2.6 with $\delta = 0$ yields

$$\sum_{k=0}^{n-1} a_k(q) = \int_0^n L(q)\, dt - nc_p \geq -\hat{C}(\max u_1 - \min u_0) =: b,$$

which is the desired estimate. ∎

It is convenient to retain from the preceding proposition that

$$\liminf_{n\to+\infty} \sum_{k=-n}^{n-1} a_k(q) \geq b \tag{11}$$

for all $q \in H^1_{loc} \cap [u_0, u_1]$.

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Remark 2.8 We cannot hope, in general, to obtain a better estimate with $b \geq 0$. This is due to the fact that in the nonreversible case there generally exist functions $v \in H^1(0,1)$ such that $f(v) - c_p < 0$. Actually, if b were positive or zero, the whole problem we are dealing with could be tackled with the easier arguments from the reversible case.

We now begin the study of the behavior of the functional J.

Lemma 2.9 *For all $q \in H^1_{loc} \cap [u_0, u_1]$,*

$$if \quad \limsup_{n \to +\infty} \sum_{k=-n}^{n-1} a_k(q) = +\infty, \qquad then \qquad J(q) = +\infty,$$

in the sense that the series $J(q)$ is well defined and diverges to $+\infty$.

Proof. Suppose on the contrary that

$$\liminf_{n \to +\infty} \sum_{k=-n}^{n-1} a_k(q) = \lambda < +\infty$$

(λ is finite by (11)) and choose a number $M > \lambda + 1 - 2b$, where b is the (negative) constant defined in the previous proposition. Let $h < H$ be two positive integers such that

$$\sum_{k=-h}^{h-1} a_k(q) \geq M \qquad \text{and} \qquad \sum_{k=-H}^{H-1} a_k(q) \leq \lambda + 1;$$

then we have

$$\sum_{k=-H}^{-h-1} a_k(q) + \sum_{k=h}^{H-1} a_k(q) = \sum_{k=-H}^{H-1} a_k(q) - \sum_{k=-h}^{h-1} a_k(q) \leq \lambda + 1 - M < 2b,$$

which contradicts Proposition 2.7. ∎

We can now prove that the functional J is well defined.

Proposition 2.10 *The functional J is well defined from $H^1_{loc} \cap [u_0, u_1]$ to $\mathbf{R} \cup \{+\infty\}$ (either the series converges or it diverges to $+\infty$). Moreover, if $q \in H^1_{loc} \cap [u_0, u_1]$ and $J(q) < +\infty$, then $q(\pm\infty) \in \mathcal{U}$.*

Proof. We first prove the second part, namely that if q does not converge to u_0 or u_1 as $t \to \pm\infty$, then $J(q) = +\infty$. We work with $t \to +\infty$. This assumption implies that there exist a number $\delta > 0$ and two divergent sequences $k_j \in \mathbf{N}$ and $t_j \in [k_j, k_{j+1})$ such that

$$|q(k_{j+1}) - q(k_j)| = o(1) \quad \text{as } j \to +\infty$$

and
$$\operatorname{dist}(q(t_j), \mathcal{U}(t_j)) \geq \delta \quad \text{for all} \ \ j \in \mathbf{N}.$$

Relabeling, if necessary, we can assume that $|q(k_{j+1}) - q(k_j)| < \frac{\rho(\delta)}{2\hat{C}}$ for all $j \in \mathbf{N}$, where $\hat{C}$ is the constant provided by Proposition 2.6. Applying this proposition in every $[k_j, k_{j+1}]$ we obtain

$$\sum_{k=k_j}^{k_{j+1}-1} a_k(q) \geq \rho(\delta) - \hat{C}|q(k_{j+1}) - q(k_j)| > \frac{1}{2}\rho(\delta).$$

Now, recalling Proposition 2.7, we obtain, for all $n \in \mathbf{N}$,

$$\sum_{k=-k_n}^{k_n-1} a_k(q) \geq b + \sum_{k=k_0}^{k_n-1} a_k(q) = b + \sum_{j=0}^{n-1}\sum_{k=k_j}^{k_{j+1}-1} a_k(q) \geq b + \frac{n}{2}\rho(\delta),$$

which shows that

$$\limsup_{n} \sum_{k=-n}^{n-1} a_k(q) = +\infty,$$

and by Lemma 2.9 we conclude that $J(q) = +\infty$. This proves the second part of the Proposition.

To complete the proof it is necessary to show that J is well defined (convergent or divergent to $+\infty$) also when q tends to u_0 or u_1 as $t \to \pm\infty$. To see this, assume for definiteness that q tends to u_0 and to u_1 when t tends to $-\infty$ and $+\infty$ respectively (the other cases being analogous).

Working indirectly, suppose that

$$\lambda = \liminf_{n} \sum_{k=-n}^{n-1} a_k(q) < \limsup_{n} \sum_{k=-n}^{n-1} a_k(q) = \Lambda,$$

and notice that we can assume $\Lambda < +\infty$, otherwise, by Lemma 2.9, $J(q) = +\infty$, and there is nothing left to prove; also, $\lambda \geq b$ by (11).

Let n_j, m_j be two divergent sequences of positive integers such that for all j there results $n_j + 1 < m_j < n_{j+1} - 1$ and

$$\lim_{j \to +\infty} \sum_{k=-m_j}^{m_j-1} a_k(q) = \lambda \quad \text{and} \quad \lim_{j \to +\infty} \sum_{k=-n_j}^{n_j-1} a_k(q) = \Lambda.$$

Now as $j \to \infty$ we have

$$\sum_{k=-m_j}^{-n_j-1} a_k(q) + \sum_{k=n_j}^{m_j-1} a_k(q) = \sum_{k=-m_j}^{m_j-1} a_k(q) - \sum_{k=-n_j}^{n_j-1} a_k(q) \leq \lambda - \Lambda + o(1) < \frac{\lambda - \Lambda}{2}. \quad (12)$$

On the other hand, as $j \to \infty$, $q(-m_j)$ and $q(-n_j)$ tend to $u_0(0)$, while $q(m_j)$ and $q(n_j)$ tend to $u_1(0)$. Therefore, by Proposition 2.6, as $j \to \infty$ we have

$$\sum_{k=-m_j}^{-n_j-1} a_k(q) \geq -\hat{C}|q(-n_j) - q(-m_j)| = o(1) \quad \text{and likewise for} \quad \sum_{k=n_j}^{m_j-1} a_k(q)$$

which contradict (12). This shows that $\lambda = \Lambda$, and the proof is complete. ∎

In the next sections we will need the following estimate of the level of functions having the same behavior at $\pm\infty$.

Proposition 2.11 *Assume* $q \in H^1_{loc} \cap [u_0, u_1]$, $q \notin \mathcal{U}$, *verifies* $q(\pm\infty) = u_0$ *or* $q(\pm\infty) = u_1$. *Then* $J(q) > 0$.

Proof. Assume for definiteness that $q(\pm\infty) = u_0$. Since $q \not\equiv u_0$, there are $n_0 \in \mathbf{N}$ and $\delta > 0$ such that $\hat{d}(q_{|[-n,n]}, \mathcal{U}) \geq \delta$ and $|q(n) - q(-n)| \leq \frac{\rho(\delta)}{2\hat{C}}$ for all $n \geq n_0$. Then, by Proposition 2.6,

$$\sum_{k=-n}^{n-1} a_k(q) \geq \rho(\delta) - \hat{C}|q(n) - q(-n)| \geq \frac{1}{2}\rho(\delta)$$

for all $n \geq n_0$. This shows that $J(q) > 0$. ∎

We now establish a property that we will use in a while.

Proposition 2.12 *The sublevels of* J *in* $H^1_{loc} \cap [u_0, u_1]$ *are bounded in* H^1_{loc}.

Proof. Let $K \in \mathbf{R}$ and let $q \in H^1_{loc} \cap [u_0, u_1]$ satisfy $J(q) \leq K$; clearly $||q||_{L^\infty(\mathbf{R})}$ is bounded independently of q. Applying Proposition 2.7 we see that for every $n \in \mathbf{Z}$,

$$K \geq J(q) = a_n(q) + \sum_{k<n} a_k(q) + \sum_{k>n} a_k(q) \geq a_n(q) + 2b;$$

therefore, recalling the definition of $a_n(q)$,

$$\frac{1}{2}||\dot{q}||^2_{L^2(n,n+1)} \leq K - 2b + c_p + \max_{\substack{t \in [0,1] \\ x \in [\min u_0, \max u_1]}} |V(t,x)| =: \gamma.$$

This means that $||\dot{q}||^2_{L^2(-n,n)} \leq 4n\gamma$, which is the desired bound. ∎

We close the section with the definition of some "cut–off" operators. To do this we first recall a "gluing lemma" from [6], to which we refer for its proof.

Lemma 2.13 *There exist positive constants* $\bar{\varepsilon}$ *and* C_1 *such that for all* $\delta \in [0, \bar{\varepsilon}]$, *there exists* $w \in H^1(0,1) \cap [u_0, u_1]$ *such that*

$$\begin{cases} w(0) = u_0(0) \\ w(1) = u_0(1) + \delta \\ |a_0(w)| \leq C_1\delta. \end{cases}$$

Remark 2.14 One can of course also show the existence of functions $w \in H^1 \cap [u_0, u_1]$ connecting $u_0(0) + \delta$ to $u_0(1)$ or $u_1(0)$ to $u_1(1) - \delta$, or $u_1(0) - \delta$ to $u_1(1)$ with the property that $|a_0(w)| \leq C_1 \delta$. Clearly this estimate is still true (as long as $\delta \leq \bar{\varepsilon}$) working in any $[p, p+1]$, with $p \in \mathbf{Z}$, instead of $[0, 1]$, and replacing a_0 by a_p. The name "w" will be reserved for this kind of functions; it is agreed that $|a_p(w)| \leq C_1 \delta$ for all such w and all p.

We now define (two families of) cut–off operators from $H^1_{loc} \cap [u_0, u_1]$ to itself which will be used to glue a function q to u_0 or u_1.

Definition 2.15 *For $i = 0, 1$, $p \in \mathbf{Z}$ and $q \in H^1_{loc} \cap [u_0, u_1]$, we set*

$$\chi_i^+(p)q(t) = \begin{cases} q(t) & \text{if } t \leq p \\ w(t) & \text{if } t \in [p, p+1] \\ u_i(t) & \text{if } t \geq p+1 \end{cases} \quad \text{and} \quad \chi_i^-(p)q(t) = \begin{cases} u_i(t) & \text{if } t \leq p-1 \\ w(t) & \text{if } t \in [p-1, p] \\ q(t) & \text{if } t \geq p, \end{cases}$$

where w is, in the first case, a function that connects $q(p)$ to $u_i(p+1)$ in the spirit of Lemma 2.13, while w connects $u_i(p-1)$ to $q(p)$ in the second case.

Lemma 2.16 *Assume $q \in H^1_{loc} \cap [u_0, u_1]$ is such that $J(q)$ is finite. Let $p \in \mathbf{Z}$ be a point where $|q(p) - u_i(p)| \leq \delta$, for some $i = 0, 1$ and some $\delta < \bar{\varepsilon}$ (this number is defined in the preceding lemma). Then*

$$|J(\chi_i^+(p)q) - \sum_{k \leq p-1} a_k(q)| \leq C_1 \delta \quad \text{and} \quad |J(\chi_i^-(p)q) - \sum_{k \geq p} a_k(q)| \leq C_1 \delta.$$

Proof. It follows directly from the definition of the operators and Lemma 2.13. ∎

3 Heteroclinic solutions

In this section we prove the existence of the simplest type of connecting orbits, namely the heteroclinic (or one–bump) solutions between u_0 and u_1. These will be used in Section 4 to construct multibump type and chaotic trajectories.

The scheme of the proof does not differ too much from the one adopted in [13] or [6], since the technical estimates relative to the nonreversible case have been proved in Section 2.

To begin with we let

$$\Gamma_0 = \{q \in H^1_{loc} \cap [u_0, u_1] \,/\, q(-\infty) = u_0, \, q(+\infty) = u_1\}$$

be the set of functions connecting u_0 to u_1 and we denote by Γ_1 the analogous set of functions connecting u_1 to u_0. We also define

$$c_0 = \inf_{\Gamma_0} J \quad \text{and} \quad c_1 = \inf_{\Gamma_1} J,$$

and we notice that these numbers are both finite (see the comment following Proposition 2.7).

The next result establishes the existence of heteroclinic solutions.

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Theorem 3.1 *There exist $q_0 \in \Gamma_0$ and $q_1 \in \Gamma_1$ such that $J(q_0) = c_0$ and $J(q_1) = c_1$. The functions q_0 and q_1 solve the equation of motion (1).*

Proof. Assuming for a moment the existence of q_0 and q_1, we remark that the second statement of the Theorem is not trivial, since the orbits are found by *constrained* minimization (the constraint being $q \in [u_0, u_1]$ in the definition of the classes Γ_i). For this reason we should show that q_0 and q_1 do not touch u_0 and u_1. However this has been shown to be true in the reversible case in [6] with an argument that does not depend on reversibility and that can be repeated here without changes. We omit therefore further details on this and we concentrate on the existence of minimizing orbits.

We prove the existence of q_0. Let $q_n \in \Gamma_0$ be a minimizing sequence and let $\delta > 0$ be small. Because of the boundary conditions, for every q_n there is a unique point t_n such that $q_n(t_n) = u_0(t_n) + \delta$ and $q_n(t) < u_0(t) + \delta$ for all $t < t_n$. By an integer shift of the time we can achieve that $t_n \in [0, 1)$ for every n; recall also that the value of $J(q_n)$ is not altered by integer time shifts.

The sequence q_n is bounded in H^1_{loc} (Proposition 2.12) and therefore it contains a subsequence (still denoted q_n) such that

$$q_n \to q_0 \quad \text{weakly in } H^1_{loc} \quad \text{and} \quad \text{strongly in } L^\infty_{loc}.$$

We now show that $J(q_0)$ is convergent. To see this use Proposition 2.7 to obtain that for every $q \in H^1_{loc} \cap [u_0, u_1]$ for which $J(q)$ is finite and for every $p \in \mathbf{N}$,

$$\sum_{k=-p}^{p-1} a_k(q) = J(q) - \sum_{k<-p} a_k(q) - \sum_{k\geq p} a_k(q) \leq J(q) - 2b.$$

Then by weak lower semicontinuity on bounded intervals we obtain that for every $p \in \mathbf{N}$,

$$\sum_{k=-p}^{p-1} a_k(q_0) \leq \liminf_n \sum_{k=-p}^{p-1} a_k(q_n) \leq \liminf_n (J(q_n) - 2b) = c_0 - 2b,$$

namely that $J(q_0)$ is convergent (apply Lemma 2.9 and Proposition 2.10). The preceding inequality also implies, via Proposition 2.10, that the limits $q_0(\pm\infty)$ exist and are in $\mathcal{U}$. By construction we see that $q_0(-\infty) = u_0$, so that we must prove that q_0 has the right behavior at $+\infty$. Notice that $q_0 \not\equiv u_0$ because $q_0(t^*) = u_0(t^*) + \delta$ for some $t^* \in [0, 1]$, by our choice of t_n.

We assume that $q_0(+\infty) = u_0$ and we show that this leads to a contradiction. Let

$$\varepsilon = \min\left(\bar{\varepsilon}, \frac{1}{2}\frac{\rho(\delta)}{\hat{C} + C_1}\right),$$

where $\hat{C}$, C_1 and $\bar{\varepsilon}$ are the constants provided by Proposition 2.6 and Lemma 2.13. For this ε choose an integer $N \geq 1$ where $q_0(N) \leq u_0(N) + \frac{\varepsilon}{2}$. By the uniform convergence of q_n, for all n large enough we have $q_n(N) \leq u_0(N) + \varepsilon$.

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Now, since $q_n(-\infty) = u_0$ for all n, we see that for all integers p less than a convenient p_n, there results

$$|q_n(p) - q_n(N)| \leq \varepsilon.$$

Moreover, since $q_n(t_n) = u_0(t_n) + \delta$ for some $t_n \in [0,1)$ and all n, we also have that

$$\hat{d}(q_{n|[p,N]}, \mathcal{U}) \geq \delta$$

for all $p \leq p_n$ and all large n. This means, by Proposition 2.6, that

$$\sum_{k=p}^{N-1} a_k(q_n) \geq \rho(\delta) - \hat{C}\varepsilon,$$

and, this being true for all $p \leq p_n$, we finally obtain

$$\sum_{k<N} a_k(q_n) \geq \rho(\delta) - \hat{C}\varepsilon$$

for every large $n \in \mathbf{N}$. Now, recalling Lemma 2.16, we evaluate

$$J(\chi_0^-(N)q_n) \leq \sum_{k \geq N} a_k(q_n) + C_1\varepsilon = J(q_n) - \sum_{k<N} a_k(q_n) + C_1\varepsilon$$

$$\leq J(q_n) - \rho(\delta) + \hat{C}\varepsilon + C_1\varepsilon = J(q_n) - \frac{1}{2}\rho(\delta) < c_0$$

for n large. Since $\chi_0^-(N)q_n \in \Gamma_0$, this is a contradiction, and our claim is proved.

Finally we prove that $J(q_0) = c_0$. Suppose this is false; then $J(q_0) = c_0 + \sigma$ for some $\sigma > 0$. Let $\varepsilon < \frac{\sigma}{16\hat{C}}$ and take $p_\varepsilon \in \mathbf{N}$ such that

$$q_0(-p) \leq u_0(-p) + \varepsilon \qquad \text{and} \qquad q_0(p) \geq u_1(p) - \varepsilon \qquad \text{for all integers } p \geq p_\varepsilon.$$

It is not restrictive to assume that also

$$\sum_{k=-p}^{p-1} a_k(q_0) \geq c_0 + \frac{3}{4}\sigma \qquad \text{for all } p \geq p_\varepsilon. \tag{13}$$

We now choose a $p \geq p_\varepsilon$ and we notice that since

$$\sum_{k=-p}^{p-1} a_k(q_0) \leq \liminf_n \sum_{k=-p}^{p-1} a_k(q_n),$$

we can fix n_0 such that

$$\sum_{k=-p}^{p-1} a_k(q_n) \geq \sum_{k=-p}^{p-1} a_k(q_0) - \frac{\sigma}{4} \qquad \text{for all } n \geq n_0.$$

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At the same time we can also think that n_0 is so large that

$$q_n(-p) \le u_0(-p) + 2\varepsilon \qquad \text{and} \qquad q_n(p) \ge u_1(p) - 2\varepsilon \qquad \text{for all } n \ge n_0.$$

Now, by Proposition 2.6,

$$\sum_{k<-p} a_k(q_n) \ge -2\hat{C}\varepsilon \qquad \text{and} \qquad \sum_{k\ge p} a_k(q_n) \ge -2\hat{C}\varepsilon \qquad \text{for all } n \ge n_0.$$

Combining this and the above estimate we see that for all $n \ge n_0$

$$J(q_n) \ge -2\hat{C}\varepsilon + \sum_{k=-p}^{p-1} a_k(q_n) - 2\hat{C}\varepsilon \ge \sum_{k=-p}^{p-1} a_k(q_0) - \frac{\sigma}{4} - 4\hat{C}\varepsilon \ge c_0 + \frac{\sigma}{4}$$

by (13) and our choice of ε. This contradicts the fact that $J(q_n)$ tends to c_0, and the proof is complete. ∎

The orbits q_0 and q_1 satisfy the following monotonicity property.

Corollary 3.2 *Let $q_0 \in \Gamma_0$ and $q_1 \in \Gamma_1$ be the solutions found in Theorem 3.1. Then*

$$q_0(t) < q_0(t+1) \qquad and \qquad q_1(t) > q_1(t+1) \qquad for \ all \ t \in \mathbf{R}.$$

Proof. We prove the inequality concerning q_0. To this aim, assume that at some $t_0 \in \mathbf{R}$ there results $q_0(t_0) \ge q_0(t_0 + 1)$. Then pick $t_1 > t_0$ where $q_0(t_1) < q_0(t_1 + 1)$, which exists since $q_0(+\infty) = u_1$. Defining $g(t) = q_0(t+1) - q_0(t)$ on $[t_0, t_1]$, we see that g must vanish at some point $t^* \in [t_0, t_1]$. Setting

$$\bar{q}(t) = \begin{cases} q_0(t) & \text{if } t \le t^* \\ q_0(t+1) & \text{if } t \ge t^*, \end{cases}$$

it is easy to check that $J(\bar{q}) < J(q_0)$, since we have taken out of q_0 a 1-periodic function away from $\mathcal{U}$. This is impossible, and the inequality is proved. ∎

Remark 3.3 We observe that the solutions found by means of the preceding Theorem also verify natural boundary conditions on the derivatives. Stating it only for q_0, we have

$$\lim_{t\to-\infty} (\dot{q}_0(t) - \dot{u}_0(t)) = \lim_{t\to+\infty} (\dot{q}_0(t) - \dot{u}_1(t)) = 0.$$

This can be easily proved exactly like in [6], Corollary 3.10.

Remark 3.4 The sign of c_0 and c_1 is unknown; however we can say that $c_0 + c_1 > 0$. To see this let t^* be a point where $q_0(t^*) = q_1(t^*)$ (which exists because of the boundary conditions) and let

$$q^-(t) = \begin{cases} q_0(t) & \text{if } t \le t^* \\ q_1(t) & \text{if } t \ge t^* \end{cases} \qquad \text{and} \qquad q^+(t) = \begin{cases} q_1(t) & \text{if } t \le t^* \\ q_0(t) & \text{if } t \ge t^*. \end{cases}$$

Then, as it is readily seen, $J(q_0) + J(q_1) = J(q^-) + J(q^+)$, so that $c_0 + c_1 = J(q^-) + J(q^+) > 0$ because of Proposition 2.11.

Remark 3.5 The fact that one between c_0 and c_1 may be negative affects the semicontinuity properties of J. Indeed, assume for instance that $c_1 < 0$, let t_n be the largest time where $q_0(t_n) = q_1(t_n - n)$ and set

$$\bar{q}_n(t) = \begin{cases} q_0(t) & \text{if } t \le t_n \\ q_1(t - n) & \text{if } t \ge t_n. \end{cases}$$

Then it is easy to see that $J(\bar{q}_n) = c_0 + c_1 + o(1)$ as $n \to +\infty$ and that $\bar{q}_n$ tends weakly to q_0 in H^1_{loc}. Therefore

$$J(q_0) = c_0 > c_0 + c_1 = \liminf_n J(\bar{q}_n),$$

and J is not (sequentially) weakly lower semicontinuous.

4 Multibump–type and chaotic orbits

The construction of multibump and chaotic solutions for equation (1) is based on a shadowing–like argument in the spirit of [15], [6] and [1], where it was carried out in the reversible case. It requires a suitable adaptation of a nondegeneracy condition which has been introduced in [6] and that we are now going to state. First of all we describe roughly what we mean by multibump solutions (a precise definition can be found in the statement of the theorems). We use this term to indicate solutions that, given a number $\delta > 0$, are alternately "closer than δ" to the periodic states u_0 and u_1 on prescribed time intervals, and are asymptotic to u_0 or u_1 as $t \to \pm\infty$. We consider as a "bump" here the change of state of an orbit (from close to u_0 to close to u_1, for example). The asymptotic behavior of a multibump solution q is obtained by imposing the desired values of $q(\pm\infty)$ in $\mathcal{U}$.

We now turn to the description of the nondegeneracy condition. That such a condition is necessary to obtain a rich structure of solutions is evident from the fact that no multibump solution is present in autonomous problems. This condition is therefore strictly related to the explicit time dependence in the equation and replaces the classical assumptions on the intersection of the stable and unstable manifolds of u_0 and u_1 that are used in the perturbative approach (see for example [20], [3]). Roughly it says that there must not be too many heteroclinics between u_0 and u_1. To state it precisely, let

$$\mathcal{S}_0 = \{q(0) \, / \, q \in \Gamma_0, \ J(q) = c_0 \} \qquad \text{and} \qquad \mathcal{S}_1 = \{q(0) \, / \, q \in \Gamma_1, \ J(q) = c_1 \}.$$

These are infinite subsets of the interval $(u_0(0), u_1(0))$ which accumulate at its boundary points (to see this examine the sequences $q_0(k)$ and $q_1(k)$ with $k \in \mathbf{Z}$); if the problem is autonomous, clearly $\mathcal{S}_0 = \mathcal{S}_1 = (u_0(0), u_1(0))$.

The condition we require from now on is the following.

$$(*) \qquad\qquad \mathcal{S}_0 \ne (u_0(0), u_1(0)) \qquad \text{and} \qquad \mathcal{S}_1 \ne (u_0(0), u_1(0)).$$

In Section 5 we analyze the relationship between $(*)$ and the classical conditions; for our present purposes we only make use of the following important consequence of $(*)$.

Proposition 4.1 *If* (∗) *holds, then for every* $\hat{\delta} > 0$ *there exist* $\delta_i \in (0, \hat{\delta})$, $i = 0, \ldots, 3$ *and two positive numbers* Λ_0, Λ_1 *such that*

$$\inf\{J(q) \ / \ q \in \Gamma_0, \ q(0) = u_0(0) + \delta_0, \ or \ q(0) = u_1(0) - \delta_2 \} \geq c_0 + \Lambda_0 \qquad (14)$$

and

$$\inf\{J(q) \ / \ q \in \Gamma_1, \ q(0) = u_1(0) - \delta_1, \ or \ q(0) = u_0(0) + \delta_3 \} \geq c_1 + \Lambda_1. \qquad (15)$$

We will use condition (∗) in the sense given by the previous proposition, namely, there are points as close as we wish to $u_0(0)$ and to $u_1(0)$ through which no minimizing heteroclinic can pass. The proof of proposition 4.1 can be found in [6] or [15] in the reversible case. Since the argument used in those papers does not depend on reversibility it can be repeated here without changes, and therefore we omit it.

We now proceed to the first step in the construction of multibump solutions. To this end, we begin by fixing some constants.

Definition 4.2 *We set*

$$\bar{d} = \frac{1}{2} \min_t (u_1(t) - u_0(t))$$

and we fix a positive number $\hat{\delta}$ *such that*

$$\hat{\delta} < \min\left(\bar{\varepsilon}, \bar{d}, \frac{\rho(\bar{d})}{\hat{C} + 2C_1}\right), \qquad (16)$$

where ρ, $\hat{C}$, C_1 *and* $\bar{\varepsilon}$ *are defined in Propositions 2.5, 2.6 and Lemma 2.13.*

Let now $\hat{\delta}$ be as in the previous definition, take $\delta_0, \ldots, \delta_3$ as in Proposition 4.1, and let $m \in \mathbf{N}$. We define the sets

$$X_0 = \{q \in H^1_{loc} \cap [u_0, u_1] \ / \ q(-m) \leq u_0(-m) + \delta_0 \ and \ q(m) \geq u_1(m) - \delta_2 \}$$

and

$$X_1 = \{q \in H^1_{loc} \cap [u_0, u_1] \ / \ q(-m) \geq u_1(-m) - \delta_1 \ and \ q(m) \leq u_0(m) + \delta_3 \}$$

These sets depend on m, a constant which will be appropriately fixed later. For the moment we will agree that m is large enough so that X_0 and X_1 contain two heteroclinic solutions $q_0 \in \Gamma_0$ and $q_1 \in \Gamma_1$ respectively.

The multibump solutions we are interested in will be found among orbits which "look like" gluing of translates of functions in X_0 and X_1. The argument will once again be minimization of J over suitable sets of functions. The main problem is to show that the minimizers are free from the constraint imposed as in X_0 and X_1.

Remark 4.3 Some notation conventions are in order. From now on we will agree that when we label c, Λ, u, q, Γ, X and $\chi^{\pm}$ with an index, that index is considered mod 2, so that for example q_i is q_0 if i is even, and q_1 otherwise. This will simplify some formulas.

Keeping this in mind we now define some subsets of $H^1_{loc} \cap [u_0, u_1]$ which we will need to construct multibump solutions.

Definition 4.4 *Let $P = (p_i)_{i \in \mathbf{Z}} \subset \mathbf{Z}$ be a bi–infinite set of integers such that $p_{i+1} - p_i \geq 4m$ for all i. We set, for every pair of integers $j \leq k$,*

$$\Gamma(j, k) = \{q \in H^1_{loc} \cap [u_0, u_1] \,/\, q(-\infty) = u_j, \;\; q(+\infty) = u_{k+1},$$

$$q(\cdot + p_i) \in X_i \quad \forall\, i = j, \dots, k\}$$

and

$$c(j, k) = \inf_{\Gamma(j,k)} J. \tag{17}$$

Remark 4.5 From now on we consider the set P as fixed. Notice that functions in $\Gamma(j, k)$ satisfy "natural" boundary conditions with respect to the constraints. For instance, if j is even, then any $q \in \Gamma(j, k)$ is close to u_0 at $t = p_j - m$ and it is required to be asymptotic to u_0 at $-\infty$. Notice also the particular case $c(j, j) = c_j$ (recall Remark 4.3).

The first step in the construction of multibump solution is to show that the levels $c(j, k)$ are attained.

Proposition 4.6 *Let $P \subset \mathbf{Z}$ be as above and let $j \leq k$ be integers. Then there exists $q_{jk} \in \Gamma(j, k)$ such that $J(q_{jk}) = c(j, k)$.*

Proof. It is a variant of the proof of Theorem 3.1. If $j = k$, the statement follows trivially from the fact that in this case the minimizers are q_0 and q_1. Let therefore $j < k$ and consider a minimizing sequence $q_n \in \Gamma(j, k)$ for J. Then, by Proposition 2.12, q_n is bounded in H^1_{loc}, and therefore it possesses a subsequence, still denoted q_n such that

$$q_n \to q_{jk} \quad \text{weakly in } H^1_{loc} \text{ and strongly in } L^\infty_{loc}.$$

Clearly q_{jk} verifies the constraints at $p_i \pm m$ for all $i = j, \dots, k$. To prove that $q_{jk} \in \Gamma(j, k)$ we must show that q_{jk} satisfies the appropriate conditions at $\pm\infty$. Before doing this we notice that as in the proof of Theorem 3.1, $J(q)$ is finite, so that by Proposition 2.10, the limits $q(\pm\infty)$ exist and are in $\mathcal{U}$. We carry out the details only at $+\infty$, and assuming that k is odd (the other cases being similar). If k is odd, then $q_{jk}(p_k + m) \leq u_0(p_k + m) + \delta_3$ and we must check that $q_{jk}(+\infty) = u_0$. Arguing indirectly, we assume that $q_{jk}(+\infty) = u_1$ and we show that this leads to a contradiction.

Choose an integer $N > p_k + m$ so that

$$\hat{d}(q_{jk | [p_k + m, N]}, \mathcal{U}) > \bar{d},$$

which is possible since $q_{jk}(+\infty) = u_1$. This inequality also holds for q_n, as soon as n is large enough, because of L^∞_{loc} convergence. Moreover, since $q_n \in \Gamma(j, k)$, there

results $q_n(p) \leq u_0(p) + \delta_3$ for all integers p larger than some p_n. This means that $|q_n(p) - q_n(p_k + m)| \leq \delta_3$ for all large p, so that the application of Proposition 2.6 to q_n on all intervals $[p_k + m, p]$ yields

$$\sum_{i \geq p_k + m} a_i(q_n) \geq \rho(\bar{d}) - \hat{C}\delta_3.$$

Recalling Lemma 2.16 we evaluate, as $n \to +\infty$,

$$J(\chi_0^+(p_k + m)q_n) \leq \sum_{i < p_k + m} a_i(q_n) + C_1\delta_3 = J(q_n) - \sum_{i \geq p_k + m} a_i(q_n) + C_1\delta_3$$

$$\leq J(q_n) - \rho(\bar{d}) + (\hat{C} + C_1)\delta_3 = c(j, k) - \rho(\bar{d}) + (\hat{C} + C_1)\delta_3 + o(1).$$

Since $\chi_0^+(p_k + m)q_n \in \Gamma(j, k)$, this contradicts the definition of $c(j, k)$ (recall Definition 4.2); therefore $q_{jk}(+\infty) = u_0$.

To complete the proof we should also check that $J(q_{jk}) = c(j, k)$; this can be done exactly as in the proof of Theorem 3.1, and we do not repeat the details. ∎

The functions q_{jk} found in the previous proposition do not, in general, solve equation (1). This is due to the constrains imposed at the points $p_i \pm m$. In the remaining part of this section we prove that by appropriately choosing m, the minimizers q_{jk} are free from constraints, so that they solve the problem in a classical sense. In order to do this, and also in order to obtain estimates independent of the number of points p_i, some care must be taken.

Definition 4.7 *We choose a positive $\varepsilon < \bar{\varepsilon}$ ($\bar{\varepsilon}$ is defined in Lemma 2.13) such that*

$$\varepsilon < \frac{\min(\Lambda_0, \Lambda_1)}{4C_1}, \tag{18}$$

and we require from now on that the number m used to define P is large enough so that the following inequalities hold:

$$q_0(2m) > u_1(2m) - \delta_1 \quad and \quad q_0(-2m) < u_0(-2m) + \delta_3, \tag{19}$$

$$q_1(2m) < u_0(2m) + \delta_0 \quad and \quad q_1(-2m) > u_1(-2m) - \delta_2. \tag{20}$$

and

$$m \geq \frac{2C_1\hat{\delta}}{\rho(\varepsilon)}. \tag{21}$$

The following proposition establishes a key inequality for the rest of the argument. Here we only need that m is chosen so that (19) and (20) hold.

Proposition 4.8 *Let $j \leq l < k$ be integers. Then*

$$c(j, k) < c(j, l) + c(l + 1, k).$$

Proof. We only prove the Proposition for l even, the other case being analogous. Let q_{jl} and $q_{l+1,k}$ be minimizers of J over $\Gamma(j,l)$ and $\Gamma(l+1,k)$ respectively. We claim that it is not restrictive to assume that

$$q_{jl}(p_{l+1} - m) > u_1(p_{l+1} - m) - \delta_1, \tag{22}$$

$$q_{l+1,k}(p_l + m) > u_1(p_l + m) - \delta_2, \tag{23}$$

in the sense that we can choose minimizers q_{jl} and $q_{l+1,k}$ satisfying these requirements. Indeed, if $l = j$, this is certainly true by taking as q_{jl} the function $q_0(\cdot - p_l)$: recalling the monotonicity property of Corollary 3.2, we obtain

$$q_{jl}(p_{l+1} - m) = q_0(p_{l+1} - m - p_l) > q_0(2m) > u_1(2m) - \delta_1 = u_1(p_{l+1} - m) - \delta_1$$

by our choice of the points in P and (19). In the same way one can check (23) when $l = k - 1$. Proving (22) when $l > j$ is more delicate.

Let $\bar{q} = q_0(\cdot - p_l)$. Without being too precise, we note that $\bar{q}$ is close to u_0 at $p_{l-1} - m$ and verifies $\bar{q}(p_{l+1} - m) > u_1(p_{l+1} - m) - \delta_1$. On the other hand, q_{jl} is close to u_1 at $p_{l-1} - m$; if (22) is violated, then we see that $\bar{q}$ and q_{jl} must cross at some $t^* \in [p_{l-1} - m, p_{l+1} - m]$. Set

$$q^+(t) = \begin{cases} q_{jl}(t) & \text{if } t \leq t^* \\ \bar{q}(t) & \text{if } t \geq t^* \end{cases} \qquad \text{and} \qquad q^-(t) = \begin{cases} \bar{q}(t) & \text{if } t \leq t^* \\ q_{jl}(t) & \text{if } t \geq t^* \end{cases}$$

and observe that $q^+ \in \Gamma(j,l)$ and satisfies (22), and $q^- \in \Gamma_0$. Now, as it is easily seen,

$$J(q^+) \leq J(q^+) + J(q^-) - c_0 = J(q_{jl}) + J(\bar{q}) - c_0 = c(j,l),$$

so that we can replace q_{jl} by q^+ and get (22). In a similar way one can prove (23) when $l < k - 1$.

To complete the proof we use q_{jl} and $q_{l+1,k}$ to estimate $c(j,k)$. Using arguments similar to those in the first part of the proof, it is not difficult to show that q_{jl} and $q_{l+1,k}$ must cross at some $t^* \in [p_l - m, p_{l+1} + m]$. Setting this time

$$q^-(t) = \begin{cases} q_{jl}(t) & \text{if } t \leq t^* \\ q_{l+1,k}(t) & \text{if } t \geq t^* \end{cases} \qquad \text{and} \qquad q^+(t) = \begin{cases} q_{l+1,k}(t) & \text{if } t \leq t^* \\ q_{jl}(t) & \text{if } t \geq t^*, \end{cases}$$

we see that $q^- \in \Gamma(j,k)$ (because it satisfies the right constraints at $p_l + m$ and $p_{l+1} - m$ by (22), (23)) and that $q^+(\pm\infty) = u_1$, so that $J(q^+) > 0$ by Proposition 2.11. ¿From this we obtain

$$J(q^-) < J(q^-) + J(q^+) = J(q_{jl}) + J(q_{l+1,k}) = c(j,l) + c(l+1,k),$$

from which the desired inequality follows at once.   ∎

We finally show that the minimizers q_{jk} found by Proposition 4.6 provide the solutions we seek. The last step consists in showing that the functions q_{jk} are "free from constraints".

The next proposition completes therefore the proof of the existence of multibump solutions.

Proposition 4.9 *Assume* (∗) *holds and let m be fixed as in Definition 4.7. For all integers $j < k$, let $q_{jk} \in \Gamma(j,k)$ be the minimizer found in Proposition 4.6. Then q_{jk} is free from constraints, in the sense that*

$$u_0(t) < q_{jk}(t) < u_1(t) \qquad \text{for all } t \in \mathbf{R}, \tag{24}$$

$$q_{jk}(p_i - m) < u_0(p_i - m) + \delta_0 \quad and \quad q_{jk}(p_i + m) > u_1(p_i + m) - \delta_2, \quad i \ even,$$

$$q_{jk}(p_i - m) > u_1(p_i - m) - \delta_1 \quad and \quad q_{jk}(p_i + m) < u_0(p_i + m) + \delta_3, \quad i \ odd$$

for all $j \leq i \leq k$. In particular, therefore, q_{jk} solves equation (1).

Proof. The strict inequality (24) is a consequence of the minimizing properties of u_0, u_1 and q_{jk}; we omit its proof, as we did in Theorem 3.1, since the details can be found in [6]. To prove the other inequalities we set $q = q_{jk}$ for notational convenience. We first show that

$$\sum_{h=p_i+m}^{p_{i+1}-m-1} a_h(q) \leq 2C_1\hat{\delta} \qquad \text{for } i = j, \ldots, k-1. \tag{25}$$

To see this, let w_1 be a function that connects $q(p_i+m)$ to $u_{i+1}(p_i+m+1)$ in the spirit of Lemma 2.13, and likewise, let w_2 be a function that connects $u_{i+1}(p_{i+1} - m - 1)$ to $q(p_{i+1} - m)$. Setting

$$\bar{u} = \begin{cases} w_1 & \text{in } [p_i + m, p_i + m + 1] \\ u_{i+1} & \text{in } [p_i + m + 1, p_{i+1} - m - 1] \\ w_2 & \text{in } [p_{i+1} - m - 1, p_{i+1} - m], \end{cases}$$

we see that

$$\sum_{h=p_i+m}^{p_{i+1}-m-1} a_h(q) \leq \sum_{h=p_i+m}^{p_{i+1}-m-1} a_h(\bar{u}) \leq 2C_1\hat{\delta},$$

because q is a minimizer; this proves (25).

Let now $I_i = [p_i + m, p_{i+1} - m]$. We prove that there exists $n_i \in I_i \cap \mathbf{Z}$ such that

$$|q(n_i) - u_{i+1}(n_i)| < \varepsilon \qquad \text{for all } i = j, \ldots, k-1, \tag{26}$$

where ε is the constant introduced in Definition 4.7. First of all, we observe that $q(t) \neq \frac{1}{2}(u_1(t) + u_0(t))$ for all $t \in I_i$. Indeed, if this is not the case, then $\hat{d}(q_{|I_i}, \mathcal{U}) \geq \bar{d}$, and, by Proposition 2.6,

$$\sum_{h=p_i+m}^{p_{i+1}-m-1} a_h(q) \geq \rho(\bar{d}) - \hat{C}|q(p_{i+1} - m) - q(p_i + m)| \geq \rho(\bar{d}) - \hat{C}\hat{\delta},$$

contradicting, via (25), the choice of $\hat{\delta}$ made in (16).

Keeping in mind the inequalities that q satisfies at the endpoints of I_i, we therefore see that

$$|q(t) - u_i(t)| > \frac{1}{2}(u_1(t) - u_0(t)) \qquad \text{for all } t \in I_i,$$

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so that $\hat{d}(q_{|I_i},\mathcal{U}) = ||q - u_{i+1}||_{L^\infty(I_i)}$ and

$$\text{dist}(q(n),\mathcal{U}(n)) = |q(n) - u_{i+1}(n)| \quad \text{for all } n \in I_i \cap \mathbf{Z}.$$

Assume now that (26) is violated, so that for some $j \leq i \leq k - 1$ there results

$$\text{dist}(q(n),\mathcal{U}(n)) = |q(n) - u_{i+1}(n)| \geq \varepsilon \qquad \text{for all } n \in I_i \cap \mathbf{Z}. \tag{27}$$

Let $\bar{q} = \chi_{i+1}^-(p_i+m) \circ \chi_{i+1}^+(p_{i+1}-m)q$ restricted to the interval $[p_i+m-1, p_{i+1}-m+1]$; then $\bar{q}(p_i + m - 1) = \bar{q}(p_{i+1} - m + 1)$, and, by (27),

$$\text{dist}(\bar{q}(n),\mathcal{U}(n)) \geq \varepsilon \qquad \text{for all } n \in I_i \cap \mathbf{Z}.$$

Applying Proposition 2.5 to $\bar{q}$ yields

$$\sum_{h=p_i+m}^{p_{i+1}-m-1} a_h(q) \geq \sum_{h=p_i+m-1}^{p_{i+1}-m} a_h(\bar{q}) - 2C_1\hat{\delta} \geq (p_{i+1} - p_i - 2m + 2)\rho(\varepsilon) - 2C_1\hat{\delta},$$

which, taken (25) into account, violates the choice of m made in (21) (recall also that $p_{i+1} - p_i \geq 4m$). We have thus proved the existence of the points n_i satisfying (26).

To complete the proof we first argue in the case $i \neq j, k$. We let $q_l = \chi_i^+(n_{i-1})q$, $q_r = \chi_{i+1}^-(n_i)q$ and $q_c = \chi_i^-(n_{i-1}) \circ \chi_{i+1}^+(n_i)q$. Notice that

$$q_l \in \Gamma(j, i-1), \qquad q_r \in \Gamma(i+1, k) \quad \text{and} \qquad q_c \in \Gamma_i.$$

Assume now that q, and therefore q_c violates one of the inequalities in the statement of the proposition. Then, by Proposition 4.1, $J(q_c) \geq c_i + \Lambda_i$. Hence, by Proposition 4.8,

$$c_i + \Lambda_i \leq J(q_c) \leq J(q) - J(q_l) - J(q_r) + 4C_1\varepsilon$$
$$\leq c(j, k) - c(j, i-1) - c(i+1, k) + 4C_1\varepsilon < c(i, i) + 4C_1\varepsilon = c_i + 4C_1\varepsilon,$$

which contradicts the choice of ε in (18).

Finally, if $i = j$, one simply chooses a point n_{j-1} so close to $-\infty$ that $|q(n_{j-1}) - u_j(n_{j-1})| < \varepsilon$ (this point exists because $q(-\infty) = u_j$) and repeats the above argument with the only change that now $J(q_l) > 0$ because $q_l(\pm\infty) = u_j$. Obvious modifications when $i = k$. The proof is complete. ∎

We summarize the results obtained so far in a single theorem. In its statement recall the "mod 2" convention (Remark 4.3).

Theorem 4.10 *Assume u_0 and u_1 are consecutive minimizers and let $(*)$ hold. Then for every $\delta > 0$ small enough there exists $m = m(\delta) \in \mathbf{N}$ such that for every sequence $(p_i)_{i \in \mathbf{Z}} \subset \mathbf{Z}$ such that $p_{i+1} - p_i \geq 4m$ and for every $j, k \in \mathbf{Z}$ with $j < k$, there exists a classical solution q of (1) satisfying*

$$u_0(t) < q(t) < u_1(t) \qquad \text{for all } t \in \mathbf{R},$$

$$q(-\infty) = u_j, \qquad q(+\infty) = u_{k+1}$$

and, for all $i = j, \ldots, k$,

$$|q(p_i - m) - u_i(p_i - m)| \le \delta \qquad and \qquad |q(p_i + m) - u_{i+1}(p_i + m)| \le \delta. \qquad (28)$$

Remark 4.11 The theorem establishes the existence of multibump solutions. Its statement could be made more precise, in the spirit of the shadowing lemma, in the sense that solutions q could be found satisfying the inequality $|q(t) - u_{i+1}(t)| \le \delta$ in the whole intervals $[p_i + m, p_{i+1} - m]$; to achieve this one must only choose a bit more carefully the various constants used in the proof. We won't detail this point.

A great deal of technicalities in the proof of Theorem 4.10 have been introduced in order to obtain estimates independent of the number of bumps. This allows us, with a classical argument, to establish the existence of solutions having infinitely many bumps, which give further evidence of the chaotic nature of the dynamics associated to equation (1).

Theorem 4.12 *Assume u_0 and u_1 are consecutive minimizers and let $(*)$ hold. Then for every $\delta > 0$ small enough there exists $m = m(\delta) \in \mathbf{N}$ such that for every sequence $(p_i)_{i \in \mathbf{Z}} \subset \mathbf{Z}$ such that $p_{i+1} - p_i \ge 4m$ there exists a classical solution q of (1) satisfying*

$$u_0(t) < q(t) < u_1(t) \qquad for \ all \ t \in \mathbf{R},$$

and

$$|q(p_i - m) - u_i(p_i - m)| \le \delta \qquad and \qquad |q(p_i + m) - u_{i+1}(p_i + m)| \le \delta \qquad (29)$$

for all $i \in \mathbf{Z}$.

Proof. The argument is classical. We repeat it here for completeness. For every $N \in \mathbf{N}$, let q_N be a multibump solution in $\Gamma(-N, N)$; this is found by means of Theorem 4.10. Since q_N is bounded in $L^\infty(\mathbf{R})$ independently of N, and since each q_N solves (1), we see that q_N is uniformly bounded in $\mathcal{C}^2(\mathbf{R}; \mathbf{R})$. Therefore, by the Ascoli–Arzelà Theorem, there is a subsequence of q_N convergent in $\mathcal{C}^1_{loc}(\mathbf{R}; \mathbf{R})$ to some q. The fact that every q_N solves (1) shows that q solves (1) as well. Properties (29) are clearly preserved by local uniform convergence, so that q is the required solution. $\blacksquare$

The previous theorem yields the existence of an uncountable number of bounded solutions to (1). If the points p_i are chosen in a nonperiodic fashion, then these solutions will not be periodic. Therefore we obtain as a by–product the existence of an uncountable number of bounded nonperiodic orbits. On the other hand, choosing the points p_i in a periodic way produces, with a slight modification of our arguments, infinitely many periodic solutions of (1) with arbitrarily large periods. These solutions "shadow" multibump homoclinics in each period (see [9] for a related result).

5   Comparison with the classical conditions

All the results obtained in the previous part of the paper have been proved for consecutive minimizers and under assumption $(*)$. We now wish to relate these assumptions with some classical conditions used in the geometric and perturbative approach to homoclinics.

In order to do this we place ourselves in a setting where the classical conditions can be stated, namely we assume that the periodic minimizers u_0 and u_1 are hyperbolic. In this context (as it has been known since the time of Poincaré) the standard assumption that guarantees the existence of chaotic features in the dynamics associated to (1) is the transversality of the intersection of the stable and unstable manifolds relative to u_0 and u_1. Condition $(*)$ is strictly related to the intersection properties of these manifolds. However, as it can be understood from the autonomous case, it is only a "branch" of these manifolds that one needs to take into account; recall also that we are not assuming that u_1 is a space translate of u_0, so that we are not in a position to pass to the quotient in the phase plane. Moreover, we have always worked in the region $[u_0, u_1]$, and therefore we are only interested in what takes place in that region. For these purposes we now define stable and unstable manifolds of u_0 and u_1 relative to $[u_0, u_1]$.

Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be the time–one map associated to equation (1), namely

$$\phi(x, y) = (u(1), \dot{u}(1)) \qquad \text{if and only if} \qquad \begin{cases} \ddot{u} + W'(t, u) = h(t) \\ u(0) = x \\ \dot{u}(0) = y. \end{cases}$$

The periodic minimizers u_0 and u_1 give rise to fixed points of ϕ.

Recall that in presence of hyperbolicity the orbits u_i possess global stable and unstable manifolds defined, for $i = 0, 1$, as

$$W^s(u_i) = \bigcup_{n \leq 0} \phi^n(W^s_{loc}(u_i)) \qquad \text{and} \qquad W^u(u_i) = \bigcup_{n \geq 0} \phi^n(W^u_{loc}(u_i)).$$

Roughly speaking, we now want to select from these manifolds the branches such that (denoting P_1 the projection onto the space variable), $P_1 W^s(u_i) \subset [u_0(0), u_1(0)]$, and likewise for W^u. This corresponds, in the unforced pendulum equation, to selecting the separatrices lying in the strip $[-\pi, \pi] \times \mathbf{R}$.

To achieve this goal we first need a simple preliminary lemma. It is perhaps well known, but we prove it anyway due to the lack of a precise reference. We carry out the computations only for u_0, everything being symmetric when dealing with u_1.

Recall that E_n is the space of (H^1_{loc}) n–periodic functions, and set

$$\lambda_1(n) = \inf_{v \in E_n \setminus \{0\}} \frac{\int_0^n \dot{v}^2 \, dt - \int_0^n W''(t, u_0) v^2 \, dt}{\int_0^n v^2 \, dt}.$$

Since u_0 minimizes $\int_0^n L(u) \, dt$ over E_n (Lemma 2.2), we know that $\lambda_1(n) \geq 0$ for all n; moreover, since u_0 is now supposed to be hyperbolic, $\lambda_1 := \lambda_1(1) > 0$ (a proof of this can be found in [2]).

Lemma 5.1 *For all $n \in \mathbf{N}$ there results $\lambda_1(n) = \lambda_1$.*

Proof. Let $\varphi \in E_n$ be a positive eigenfunction corresponding to $\lambda_1(n)$. Then φ solves

$$- \ddot{\varphi} - W''(t, u_0)\varphi = \lambda_1(n)\varphi; \tag{30}$$

notice that, since $W''(\cdot, u_0(\cdot))$ is 1–periodic, the function $\varphi(\cdot + 1)$ solves (30) as well, and therefore also $z(t) := \varphi(t) - \varphi(t+1)$ is a solution. Now z has zero mean value, so that it cannot solve (30) unless it is identically zero. This means that φ is 1-periodic, and therefore $\lambda_1(n)$ must be an eigenvalue of the 1–periodic problem. Moreover, since φ is positive, $\lambda_1(n)$ must be the first eigenvalue, namely $\lambda_1(n) = \lambda_1$.
∎

The previous lemma provides a minimum of convexity necessary to prove a uniqueness result for solutions close to u_0.

Proposition 5.2 *There exists $\sigma > 0$ such that for every $\delta \in (0, \sigma)$ there exists $\varepsilon > 0$ with the property that for all $x \in [u_0(0), u_0(0) + \varepsilon]$, the problem*

$$\begin{cases} \ddot{q} + W'(t, q) = h(t) & in \ [0, +\infty) \\ q(0) = x, \quad q(+\infty) = u_0 \\ u_0(t) \leq q(t) \leq u_0(t) + \delta & in \ [0, +\infty) \end{cases} \tag{31}$$

has a unique classical solution.

Proof. The existence of q follows with slight modifications of arguments used in the first part of the paper. Indeed it is enough to minimize the functional $J^+(q) = \sum_{k \geq 0} a_k(q)$ over the set

$$\Gamma_x = \{q \in H^1_{loc}(\mathbf{R}^+; \mathbf{R}) \cap [u_0, u_1] \ / \ q(0) = x, q(+\infty) = u_0\}.$$

This yields a function q such that $J^+(q) < +\infty$, $q(0) = x$ and $q(+\infty) \in \mathcal{U}$. In order to prove that $q(+\infty) = u_0$ and the remaining requirements, it is enough to show that $u_0(t) < q(t) < u_0(t) + \delta$ for all $t \geq 0$. But this also follows easily from arguments already used in the preceding sections, and we do not give the details of the computations.

The proof of uniqueness is very similar to the proof of Theorem 5.4 in [6]; we will therefore be rather sketchy at some points. Let $\lambda_1 > 0$ be the eigenvalue defined above, and let $\eta > 0$ be such that

$$|W''(t, a) - W''(t, b)| \leq \frac{\lambda_1}{2} \qquad \text{for all } t \geq 0 \text{ and all } a, b \text{ such that } |a - b| < \eta;$$

this choice is possible due to the uniform continuity of W''. Now if uniqueness is violated, for $\sigma = \frac{\eta}{2}$ there exist $\delta < \sigma$, a small x and two solutions of (31), q_1 and q_2. Letting $\psi = q_2 - q_1$, we see that ψ solves

$$\ddot{\psi} + W'(t, q_1 + \psi) - W'(t, q_1) = 0 \qquad \text{for } t \geq 0 \tag{32}$$

and satisfies $\psi(0) = 0$, $\psi(+\infty) = 0$ and $||\psi||_{L^\infty(\mathbf{R}^+)} \leq \delta$.

A standard application of the mean value Theorem (see [6]) shows that (32) can be written

$$\ddot{\psi} + W''(t, u_0)\psi + a(t)\psi = 0 \tag{33}$$

where $|a(t)| = |W''(t, q_1 + \theta_t \psi) - W''(t, u_0)| \leq \frac{\lambda_1}{2}$ for all $t \geq 0$, since $|q_1 + \theta_t \psi - u_0| \leq 2\delta < \eta$.

For every $n \in \mathbf{N}$ define now $\psi_n : [0, n+1] \to \mathbf{R}$ as

$$\psi_n(t) = \begin{cases} \psi(t) & \text{if } t \in [0, n] \\ (n + 1 - t)\psi(n) & \text{if } t \in [n, n+1]; \end{cases}$$

then $\psi_n(n + 1) = \psi_n(0) = 0$ and therefore, by Lemma 5.1,

$$\int_0^{n+1} \dot{\psi}_n^2 \, dt - \int_0^{n+1} W''(t, u_0)\psi_n^2 \, dt \geq \lambda_1 \int_0^{n+1} \psi_n^2 \, dt.$$

Recalling the definition of ψ_n, we see that the preceding inequality implies that

$$\int_0^n \dot{\psi}^2 \, dt - \int_0^n W''(t, u_0)\psi^2 \, dt \geq \lambda_1 \int_0^n \psi^2 \, dt + o(1) \qquad \text{as } n \to \infty. \tag{34}$$

On the other hand, multiplying (33) by ψ and integrating over $[0, n]$ yields, as $n \to \infty$,

$$\int_0^n \dot{\psi}^2 \, dt - \int_0^n W''(t, u_0)\psi^2 \, dt = \int_0^n a(t)\psi^2 + \psi(n)\dot{\psi}(n) \leq \frac{\lambda_1}{2} \int_0^n \psi^2 \, dt + o(1)$$

Comparing this and (34) shows that as $n \to \infty$,

$$\frac{\lambda_1}{2} \int_0^n \psi^2 \, dt \leq o(1),$$

which proves that $\psi \equiv 0$, namely that $q_1 = q_2$. ∎

We are now in a position to define stable and unstable manifolds relative to $[u_0, u_1]$.

Let $\varepsilon > 0$ be given by Proposition 5.2 and, for all $x \in [u_0(0), u_0(0) + \varepsilon]$, denote by q_x the unique solution to (31) also given by Proposition 5.2. We define the local stable manifold of u_0 relative to $[u_0, u_1]$ as

$$\mathcal{W}_{loc}^s(u_0) = \{(x, \dot{q}_x(0)) \in \mathbf{R}^2 \ / \ x \in [u_0(0), u_0(0) + \varepsilon] \}$$

and the stable manifold of u_0 relative to $[u_0, u_1]$ as

$$\mathcal{W}^s(u_0) = \bigcup_{n \leq 0} \phi^n(\mathcal{W}_{loc}^s(u_0)).$$

In a similar way (using obvious variants of Proposition 5.2) one can define the unstable manifold of u_0 relative to $[u_0, u_1]$ and the analogous manifolds $\mathcal{W}^s(u_1)$ and $\mathcal{W}^u(u_1)$ corresponding to u_1.

We now relate our assumptions with some classical conditions. In presence of hyperbolic periodic orbits it is well known that multibump solutions and some features of chaotic dynamics are present provided the stable and unstable manifolds of the periodic orbits intersect transversally. We show that condition $(*)$ is weaker than the transversality assumption.

Theorem 5.3 *Let u_0 and u_1 be consecutive hyperbolic minimizers. Then*

$(*)$ $\qquad\qquad \mathcal{S}_0 \neq (u_0(0), u_1(0)) \qquad and \qquad \mathcal{S}_1 \neq (u_0(0), u_1(0))$

if and only if

$$\mathcal{W}^s(u_0) \neq \mathcal{W}^u(u_1) \qquad and \qquad \mathcal{W}^u(u_0) \neq \mathcal{W}^s(u_1).$$

Proof. If $(*)$ holds, then by Theorem 4.10 we can find a solution q asymptotic to u_0 both at $-\infty$ and at $+\infty$. Then it is readily seen that $(q(0), \dot{q}(0)) \in \mathcal{W}^s(u_0)$; however, since $q(-\infty) = u_0$, there also results that $(q(0), \dot{q}(0)) \notin \mathcal{W}^u(u_1)$. In the same way one proves that $(q(0), \dot{q}(0))$ belongs to $\mathcal{W}^u(u_0)$ but not to $\mathcal{W}^s(u_1)$. Working with a solution doubly asymptotic to u_1, the remaining relations can be proved similarly. Therefore $\mathcal{W}^s(u_0) \neq \mathcal{W}^u(u_1)$ and $\mathcal{W}^u(u_0) \neq \mathcal{W}^s(u_1)$ both hold.

Conversely, assume for example that there is a point $(x, y) \in \mathcal{W}^s(u_0) \setminus \mathcal{W}^u(u_1)$; this means that, for a convenient $m \geq 0$, the point $(\bar{x}, \bar{y}) = \phi^m(x, y)$ belongs to $\mathcal{W}^s_{loc}(u_0)$. We now prove that $\bar{x} \notin \mathcal{S}_1$. Suppose this is not the case. Then there exists a heteroclinic minimizer $q \in \Gamma_1$ such that $q(0) = \bar{x}$; by choosing a larger m, if necessary, we can make sure that q also verifies, for all $t \geq 0$, $u_0(t) \leq q(t) \leq u_0(t) + \delta$ (this is the quantity provided by Proposition 5.2). Then by uniqueness it must be $\dot{q}(0) = \bar{y}$; this means that $(\bar{x}, \bar{y})$, and therefore (x, y) belongs to $\mathcal{W}^u(u_1)$, since $q(-\infty) = u_1$, which contradicts the choice of (x, y). Therefore we must have $\bar{x} \notin \mathcal{S}_1$. The other case is handled in a similar way. ∎

Remark 5.4 Theorem 5.3 shows that working with consecutive minimizers and assuming $(*)$ is in some sense more natural than using the classical conditions. First of all in presence of hyperbolicity $(*)$ is weaker than the usual transversality assumption. Moreover $(*)$ can be used, like we did, also without hyperbolicity, when the classical condition cannot be stated. Finally we observe that the variational characterization of the numbers c_0 and c_1 which is used to introduce $(*)$ shows that the validity of that condition is stable under small perturbations. This allows one to say, for example, that the set of h's for which $(*)$ holds is open in the space of continuous, zero mean valued periodic functions.

Remark 5.5 As a final remark we observe that a family of multibump solutions like those provided by Theorem 4.10 ensures the positivity of the topological entropy for the map ϕ. We do not prove this statement because its proof is essentially the same as those in [17] and [5].

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