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Progress in Nonlinear Differential Equations and Their Applications

Søren Fournais
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Spectral Methods in Surface Superconductivity

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Spectral Methods in Surface Superconductivity

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Contents

Preface

General Presentation

The analysis of mathematical problems connected with the theory of superconductivity has been intensively developed in the last decade. For concreteness, in this introduction we will only discuss the two-dimensional case. Also, let us stress from the beginning that in this book we will not discuss at all the microscopic BCS-theory of superconductivity. An accepted basic model of superconductivity is the Ginzburg–Landau functional involving a pair $(\psi, \mathbf{A})$, where ψ is a wave function and $\mathbf{A}$ is a magnetic potential on an open set $\Omega \subset \mathbb{R}^2$, which is defined by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx \\ + \kappa^2 \int_\Omega \Big|\sigma \operatorname{curl} \mathbf{A} - \sigma\beta\Big|^2 \, dx \, . \tag{1}$$

Here ψ is called the order parameter, $\mathbf{A}$ is a magnetic potential, and β—or rather $\kappa\sigma\beta$—is the external[1] magnetic field. The field $\operatorname{curl} \mathbf{A}$ is called the induced magnetic field. The parameter $\kappa > 0$ is characteristic of the sample. In the physics literature, one usually makes the distinction between type I materials, corresponding to small κ, and type II materials, corresponding to large values of κ. For some model problems in the entire space the transition between the two types takes place at the value $\kappa = 1/\sqrt{2}$. Mathematically, this leads to the analysis of various asymptotic regimes like $\kappa \to 0$ or $\kappa \to +\infty$. It is this last case that will be the subject of the present book. In order to measure the dependence on the magnitude of the external magnetic field, we have written the external magnetic field in terms of the parameter σ. Thus, we think of β as being some fixed function and σ as measuring the strength of the field.

[1] Sometimes also called the applied magnetic field.

We will generally assume that Ω is a bounded, simply connected subset of $\mathbb{R}^2$ (or $\mathbb{R}^3$). As Ω is bounded, proving the existence of a minimizer of $\mathcal{G}$ is a rather standard problem. The minimizer should satisfy the Euler–Lagrange equation, which in this context is called the Ginzburg–Landau system (see [S-JdG]).

The minimizers will describe the properties of the material when submitted to the external magnetic field; i.e., $|\psi(x)|^2$ measures the level of superconductivity (density of Cooper pairs in the language of physicists) of the material near the point x. One traditionally distinguishes among three types of possible minimizers. We say that a minimizer $(\psi, \mathbf{A})$ is superconducting if ψ never vanishes, normal if ψ is identically 0, and mixed if ψ has zeroes but does not vanish identically.

This naturally leads physicists to "define" two critical fields. The lower one (or first critical field) corresponds to the transition from superconducting to mixed states and is denoted by $H_{C_1}(\kappa)$. In other words, this corresponds to the appearance of zeroes for ψ (usually called vortices) when increasing the external field, and many authors have worked on this phenomenon.

It is believed that when κ is large, there exists a zone where the minimizers correspond to mixed states. So the highest critical field, or third critical field, which is denoted by $H_{C_3}(\kappa)$, corresponds to the transition from mixed states to normal states. When this process is viewed in the opposite direction—magnetic field strength decreasing across H_{C_3}—this is the phenomenon called the onset of superconductivity. It can indeed be observed that for large external magnetic fields the minimizer is "normal"—that is, $\psi \equiv 0$. So we are interested in describing what happens when we decrease this external magnetic field. What can be shown is that for large κ, superconductivity first appears at the boundary. This is called surface superconductivity, and the precise description of this phenomenon will occupy a large part of the text.

The reader may wonder why we have until now only introduced $H_{C_1}(\kappa)$ and $H_{C_3}(\kappa)$. The last critical field (called the second critical field) of interest is $H_{C_2}(\kappa)$, and denotes the field corresponding, inside the "mixed zone", to the transition from having minimizers, which are localized very near the boundary, to having minimizers that are significatively nonzero in regions far from the boundary.

Because we are mainly interested in understanding what happens around $H_{C_3}(\kappa)$, it is natural to analyze when the normal solution $(0, \mathbf{F})$ (with $\operatorname{curl} \mathbf{F} = \beta$) is a local minimum of the functional. This leads naturally to the question of positivity of the Hessian of $\mathcal{G}$. Due to the particular form of the functional and to the choice of the point where we compute the Hessian, this positivity is immediately related to the positivity of the operator $-\Delta_{\kappa\sigma\mathbf{F}} - \kappa^2$, where $-\Delta_{\kappa\sigma\mathbf{F}}$ is a Schrödinger operator with magnetic field

$$-\Delta_{\kappa\sigma\mathbf{F}} := (-i\nabla + \kappa\sigma\mathbf{F})^2 \ , \tag{2}$$

corresponding to the self-adjoint realization with a magnetic Neumann boundary condition:

$$\nu \cdot (-i\nabla + \kappa\sigma\mathbf{F})\psi = 0 \quad \text{on } \partial\Omega\,, \tag{3}$$

where ν is the interior normal vector at the boundary. This is a linear problem, which is, of course, related to the determination of the lowest eigenvalue—or ground state energy—of the operator in question. When κ is large, this becomes a semiclassical problem where the role of the Planck constant h is played by $1/(\kappa\sigma)$. So the first part of the book will mainly be devoted to the techniques leading to a very accurate spectral analysis of $-\Delta_{\kappa\sigma\mathbf{F}}$.

Not surprisingly, there are strong links between the questions of determining whether the normal solution is a local minimum or whether it is a global minimum—mathematicians may, in fact, wonder why physicists have not distinguished in their terminology between these local and global critical fields—but the second problem is effectively nonlinear. It is the aim of the second part of this book to develop the necessary nonlinear tools to solve this question rather completely and pursue the analysis to the zone $]H_{C_2}(\kappa), H_{C_3}(\kappa)[$. Note that here we are complementary with the scope of the excellent recent book by Sandier and Serfaty [SaS3], which treats mainly the zone $]H_{C_1}(\kappa), H_{C_2}(\kappa)[$. Furthermore, we will not discuss, except for illustrating some phenomena, what has been done in numerical analysis (see [DuGP] and references therein).

Organization of the Book

As the presentation suggests, this book is divided into two main parts:

- The first part is devoted to the spectral analysis of the Schrödinger operator with magnetic field.
- The second part concentrates on the analysis of the Ginzburg–Landau functional and is mainly nonlinear.

For a first reading, or for a graduate course on the subject, we suggest restricting to the 2D situation and reading Chapters 1–4, 7, and 8 from the linear part and Chapters 10, 12, and 13 from the nonlinear part. The remaining chapters in the linear part, Chapters 5, 6, and 9, are somewhat more technical and can be skipped at a first reading. The same applies to Chapters 11 and 14 in the nonlinear part. The final chapters, 15 and 16, contain more specialized topics.

Linear analysis: Spectral analysis of Schrödinger operators with magnetic fields

Chapter 1 is a short introduction to the spectral theory for the Schrödinger operator with magnetic field. We analyze successively the following basic questions:

- How do we define the self-adjoint extensions?
- What are the basic properties of the spectrum?

We finish the chapter by presenting rough bounds for the ground state energy.

Chapter 2 is devoted to the analysis of diamagnetism, which will play an important role in the analysis of the critical fields in the nonlinear part. Diamagnetism means that the (ground state) energy is an increasing function of the magnetic field.

The elementary Chapter 3 contains a precise analysis of various one-dimensional problems that are fundamental for understanding the spectral analysis in the higher-dimensional situations. In particular, we start with a complete analysis of a family of one-dimensional models that are the basis of the explanation of surface superconductivity.

Chapter 4 concentrates on the spectral analysis of two-dimensional models. We apply the results of the analysis in the previous chapter to various model geometries. The magnetic field is assumed to be constant and the domains are successively $\mathbb{R}^2$, $\mathbb{R}^{2,+}$, and the infinite sector. In the analysis of general domains in later chapters these model geometries will be used in comparison arguments to obtain precise spectral information.

Chapter 5 gives a detailed presentation of the case of the disc or of its exterior. This will play a basic role in the analysis of curvature effects.

Chapter 6 is concerned with the same questions in the three-dimensional case. We concentrate mainly on $\mathbb{R}^3$ and $\mathbb{R}^{3,+}$. The magnetic field is still constant, but the interesting fact is that the analysis depends strongly on the angle between the vector normal to the boundary and the magnetic vector field. As a side product, the results obtained justify the assumption done by de Gennes and Saint-James [S-JdG] that taking the magnetic vector field tangent to the boundary gives the lowest energy.

We recall in Chapter 7 the main techniques in semiclassical analysis: harmonic approximation, decay estimates. Although this material is already present in various books, we feel it was necessary to give a brief account of the standard techniques before extending them to the problems arising in the case with boundary.

In Chapter 8, we present the methods allowing one to arrive at the two-term asymptotics of the ground state energy in the two-dimensional case and to the localization property of the ground state within the boundary and close to the points of maximal curvature. We mention under what assumptions one can get a complete asymptotics, and we conclude with an analysis of diamagnetism.

Chapter 9 is analogous to Chapter 8 but for three-dimensional domains. This chapter is more descriptive because it would be too technical to prove all the results presented. We refer to Helffer–Morame [HeM6] for proofs. As an application, we show in detail how the main results on the localization of the ground state can be applied to the question of diamagnetism.

Nonlinear analysis: The Ginzburg–Landau functional with magnetic field

This part is devoted to the analysis of the onset of superconductivity. We mainly concentrate our analysis on the high-κ limit, which is the case where the large field—or semiclassical—analysis is relevant. Also, we mainly restrict our attention to the regime where the strength of the external magnetic field is "above $H_{C_2}(\kappa)$", i.e., we do not study vortices.

Chapter 10 comes back in detail to the general presentation of the Ginzburg–Landau functional. We recall the rather standard proof of the existence of a minimizer and prove that the normal solution is a global minimum for large external magnetic fields. We marginally treat some questions about type I superconductors.

Chapter 11 explores a priori estimates that are needed to relate the nonlinear question to the linear one. These are elliptic estimates, but compared to the "classical" case, we need to control the uniformity with respect to different parameters. In the second part of this chapter, we analyze what can be achieved through the technique of "blowing up" initiated in this context by Lu–Pan [LuP3].

In Chapter 12, we discuss decay estimates in the direction normal to the boundary. As explained above, when σ is much larger than κ, one expects superconductivity to be localized near the boundary. For sufficiently large fields, the techniques of Agmon estimates can be used to prove this (see [HeP1]). However, we also consider weaker estimates due to Almog (see [Al2, Al4]), which are valid for all $\sigma > \kappa$.

Chapter 13 is devoted to a complete analysis of the critical field $H_{C_3}(\kappa)$, corresponding to the transition, when decreasing the strength of the external magnetic field, between normal minimizers and non-normal minimizers. Here we follow Lu–Pan [LuP3], Helffer–Pan [HeP1], [Pa6] with some recent improvements in the approach due to Fournais–Helffer [FoH3, FoH4, FoH6].

Chapter 14 describes what happens when continuing to decrease the strength of the external magnetic field. One would like to understand how the onset of superconductivity, which has been shown to start from the points of maximal curvature at the boundary, will extend to the whole boundary by a nonlinear mechanism of uniformization inside the boundary. We follow here the works by Lu–Pan, Fournais–Helffer [FoH2], Almog–Helffer [AlH], and Pan [Pa2] (for the region close to $H_{C_2}(\kappa)$ see also Sandier–Serfaty [SaS2] and Aftalion–Serfaty [AfS]).

Chapter 15 gives a short presentation of the case with corners. This is a case where the literature in physics is quite developed [BeR] and which leads to interesting conjectures that are confirmed both experimentally and numerically. This also gives a good opportunity to show the tunneling effects occurring inside the boundary between the different corners in the case of a regular polygon. We refer here to the works of Pan [Pa1], Pan–Kwek

[PaK], Jadallah [Ja], Bonnaillie, Bonnaillie-Noël–Dauge, and Bonnaillie-Noël–Fournais [Bon1, Bon2, BonD, BonF].

Chapter 16 presents various extensions. The Ginzburg–Landau functional is the simplest model corresponding to a superconducting sample surrounded by a vacuum. Other models are proposed to better take into account the exterior of the superconducting sample. This can, for example, lead to other boundary conditions (like the de Gennes boundary conditions considered by Lu–Pan [LuP3], [Pa4] and Kachmar [Kac1, Kac2]).

Additionally, we will discuss problems related to the existence of holes or of periodically perforated structures.

Finally, we will discuss how the techniques used in this book can also be useful in the analysis of liquid crystals. In particular, we present the analogy due to de Gennes [dGe4] between the problems analyzed here in superconductivity and the transition smectic–nematic occurring in liquid crystals. Many recent papers have been devoted to this subject [BaCLP, Pa5, Pa7, Pa8, JoP].

We conclude this chapter with a short presentation of open problems in the field.

Each chapter (except this Preface and Chapter 16) ends with a Notes section containing comments and references.

We have added at the end of the book various appendices containing somewhat standard material in order to make the book self-contained.

We conclude the book by giving a fairly complete bibliography on the subject of the book.

About the History of the Subject

We limit ourselves in this presentation to the phenomenon of the onset of superconductivity and refer to [SaS3] for the discussion of other aspects like the appearance of vortices.

One can surely find the original problem in the first papers by Ginzburg or Landau [Gin] or [GiL] starting from the 1950s, but one usually refers to the paper by D. Saint-James and P-G. de Gennes [S-JdG] of 1963 as the initial reference for a theoretical explanation for the onset of superconductivity. These authors were mainly interested in the analysis of a sample Ω in $\mathbb{R}^3$ delimited by two hyperplanes. Assuming that the external magnetic field is parallel to the boundary, the authors reduced the problem to a family of one-dimensional problems, which will play an important role throughout the analysis.

So the first mathematical results in this direction are based on a fine analysis of one-dimensional problems (see Bolley [Bol], Bolley–Helffer [BolH1]–[BolH4], and also Aftalion [Af1] and the survey by Aftalion and Troy [AfT]). All these works appeared in the 1990s.

As explained, for example, in the lectures of Rubinstein [Ru], the analysis of effectively two-dimensional problems arises later, at first through rather formal papers (Chapman [Ch1, Ch2], Chapman–Howison–Ockendon [ChHO]) proposing formal constructions of minimizers. A completely rigorous approach starts with the papers [LuP1, LuP2] by Lu and Pan. Then three mathematical papers appeared that played an important role for the further development of the subject: The formal expansion by Bernoff–Sternberg suggesting the role of the boundary curvature, the fine analysis of the case of the disc by Bauman–Phillips–Tang (1998) [BaPT], and the paper by Giorgi–Phillips (1999) [GioP]. Then began a period of intense activity by Lu–Pan [LuP3]–[LuP7] on one side and del Pino–Felmer–Sternberg [dPiFS] on the other side. The semiclassical character of the questions allowed Helffer and Morame to bring all the semiclassical technology around the WKB constructions and Agmon estimates into the subject. This led to the solution of a conjecture (initially due to [BeS]) about the two-term asymptotics of the third critical field (Helffer–Morame [HeM3], Helffer–Pan [HeP1]) and gave new possibilities for the analysis of the problem in dimension 3 (Lu–Pan [LuP7], Pan [Pa6], Helffer–Morame [HeM4, HeM6]).

More recent works were developed in three directions:

- case of corners (Jadallah–Rubinstein–Sternberg (1999) [JaRS], Jadallah (2001) [Ja], Pan (2002) [Pa1], Bonnaillie (2003-2005) [Bon1, Bon2], Bonnaillie-Noël–Dauge (2006) [BonD], and Bonnaillie-Noël–Fournais (2007) [BonF]),
- fine analysis of all the definitions of the third critical field (Fournais–Helffer),
- analysis of the region between $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$ (Pan (2002) [Pa2], Fournais–Helffer (2005) [FoH1], Almog–Helffer (2007) [AlH]).

Comparison with the Existing Literature and Prerequisites

We make an effort to keep the text reasonably self-contained, having graduate students and researchers in mind. The reader is supposed to have a good knowledge of elementary spectral analysis, Hilbertian analysis, and the elliptic theory in PDE. For the spectral theory, the books by Reed and Simon [ReS] is more than enough, and the reader can also look at [LeB] (in French) or to the notes of an unpublished course [He8].

When Schrödinger operators with magnetic fields are concerned, one should also mention the surveys by Helffer [He4, Hel5, He9], Mohamed–Raikov [MoR], [He6] for the relations with superconductivity, and the book by Thaller [Th], which is mainly devoted to the Dirac operator but contains interesting information on magnetic problems. Other aspects in semiclassical analysis are presented in the books by Helffer [He2], Dimassi–Sjöstrand [DiS], Robert

[Ro], Kolokoltsov [Ko] (in connection with results of Maslov's school), and A. Martinez (in the spirit of microlocal analysis) [Mart]. Concerning superconductivity, one should mention in mathematics recent books by Bethuel–Brezis–Helein [BeBH], Sandier–Serfaty [SaS3], Hoffmann–Tang [HoT], and surveys like the lectures by Rubinstein [Ru] or Sternberg [St]. The collective book edited by Berger and Rubinstein [BeR] also contains a lot of information on the problems with holes. The techniques appearing in this book might have applications for related problems in the theory of Bose–Einstein condensates, see [Af2] and [LSSY].

We should also mention in the physics literature the course by de Gennes [dGe2] and the books by Saint-James, Sarma, and Thomas [S-JST], Tilley–Tilley [TiT], and Tinkham [Ti].

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Notation

We will work with domains $\Omega \subset \mathbb{R}^n$, which will generally be assumed simply connected (for convenience) and regular or piecewise regular (polygons). For a point $x \in \partial\Omega$, we denote by $\nu(x)$ the unit interior normal vector to the boundary. Also, for a general point $x \in \Omega$, we define

$$t(x) := \mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y| \,. \tag{4}$$

Some model operators will appear repeatedly in the text. We therefore fix the following definitions [see (3.1) and (3.9)]:

$$\mathfrak{h}_0 := -\frac{d^2}{d\tau^2} + \tau^2 \quad \text{on} \quad L^2(\mathbb{R})\,, \tag{5}$$

$$\mathfrak{h}^{N,\xi} := -\frac{d^2}{d\tau^2} + (\tau + \xi)^2 \quad \text{on} \quad L^2(\mathbb{R}^+)\,, \tag{6}$$

with Neumann boundary condition at the origin.

The scalar product in $L^2(\Omega)$ is denoted by

$$\langle f \mid g \rangle = \int_\Omega \overline{f(x)}\, g(x)\; dx \,. \tag{7}$$

We will use the standard Sobolev spaces $W^{s,p}$. For integer values of s, these are given by

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) \,:\, D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq s\}\,. \tag{8}$$

When s is a positive integer, the norm on $W^{s,p}(\Omega)$ is,

$$\|u\|_{W^{s,p}(\Omega)} := \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^p(\Omega)}\,.$$

For $s \in \mathbb{R}$, the space $W^{s,p}(\Omega)$ is defined by duality (negative values of s) and interpolation (noninteger values of s). See, for instance, [Ad] for details. In the case $p = 2$, we will also use the standard symbol H^s for $W^{s,p}$, i.e.,

$$H^s(\Omega) := W^{s,2}(\Omega)\,.$$

These spaces will sometimes be combined with the suffixes "comp" or "loc" to denote "compact support" or "locally". For example, a distribution f belongs to $H^2_{\mathrm{loc}}(\Omega)$ if

$$\phi f \in H^2(\Omega)\,, \qquad \text{for all } \phi \in C_0^\infty(\Omega)\,.$$

Also, the magnetic generalizations of these spaces will be use; for example, for a given vector function (magnetic vector potential) $\mathbf{A}$, the space $H^1_{\mathbf{A}}(\Omega)$ is given by the functions f with $f \in L^2(\Omega)$ and $(-i\nabla + \mathbf{A})f \in L^2(\Omega)$. This space is given its natural norm.

Furthermore, we will use Hölder spaces. Let us fix the definition of the norm in the Hölder spaces $C^{n,\alpha}$. For a smooth bounded domain Ω, $n \in \mathbb{N}$, $\alpha \in]0,1[$, the space $C^{n,\alpha}(\overline{\Omega})$ is the set of functions u with the nth-order derivatives being Hölder continuous of degree α in $\overline{\Omega}$ and such that the norm

$$\|u\|_{C^{n,\alpha}(\overline{\Omega})} := \sum_{|\beta|\leq n} \|\partial^\beta u\|_{L^\infty(\Omega)} + \sum_{|\beta|=n} \sup_{x,y\in\overline{\Omega}} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha} \tag{9}$$

is finite. In the case when $\alpha = 0$, the last sum is omitted.

Part I

Linear Analysis

1

Spectral Analysis of Schrödinger Operators

1.1 The Magnetic Schrödinger Operator

Let Ω be an open set in $\mathbb{R}^n$, $\mathbf{A} = (A_1, A_2, \dots, A_n)$ be a C^∞ vector field on $\overline{\Omega}$, corresponding to the so-called magnetic potential, V (which may depend on B) be a $C^\infty(\overline{\Omega})$ real-valued function, corresponding to the electric potential, and $B > 0$ be a (large) parameter, playing the role of the strength of the magnetic field. The vector field $\mathbf{A}$ corresponds more intrinsically to a 1-form

$$\omega_{\mathbf{A}} = \sum_{j=1}^{n} A_j \, dx_j \, . \tag{1.1}$$

One can then associate to $\omega_{\mathbf{A}}$ a 2-form called the magnetic field σ_β:

$$\sigma_\beta := d\omega_{\mathbf{A}} = \sum_{j<k} \beta_{jk} \, dx_j \wedge dx_k \, . \tag{1.2}$$

When $n = 2$, the unique coefficient β_{12} defines (in a fixed system of coordinates) a function, more simply denoted by

$$x \mapsto \beta(x) = \operatorname{curl} \mathbf{A} = \partial_{x_1} A_2 - \partial_{x_2} A_1 \, ,$$

also called the magnetic field.

When $n = 3$, the magnetic field is identified with a magnetic vector β, by the Hodge map:

$$\beta = (\beta_1, \beta_2, \beta_3) = (\beta_{23}, -\beta_{13}, \beta_{12}) = \operatorname{curl} \mathbf{A} \, , \tag{1.3}$$

with the usual definition of curl. All these objects can be defined more generally on a Riemannian manifold (with notions like connections, curvature, etc.), but that is outside the scope of this book.

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We would like to discuss the spectrum of self-adjoint realizations of the Schrödinger operator in an open set Ω in $\mathbb{R}^n$:

$$P_{B\mathbf{A},V,\Omega} = \sum_{j=1}^{n}(-i\,\partial_{x_j} + BA_j)^2 + V(x;B)\,.$$

For this abstract question we will generally absorb the parameter B in the vector potential and thus write

$$P_{\mathbf{A},V,\Omega} = p_{\mathbf{A}}^2 + V = -\nabla_{\mathbf{A}}^2 + V = -\Delta_{\mathbf{A}} + V\,,$$

with

$$p_{\mathbf{A}} := -i\nabla + \mathbf{A} = -i\nabla_{\mathbf{A}}\,,$$

and

$$\nabla_{\mathbf{A}} := \nabla + i\mathbf{A}, \qquad \Delta_{\mathbf{A}} := (\nabla + i\mathbf{A})^2.$$

Notice that one can perform a *gauge transformation*, i.e., a conjugation by $e^{i\phi}$. Then, since $e^{-i\phi}p_{\mathbf{A}}e^{i\phi} = p_{\mathbf{A}+\nabla\phi}$, we get the unitary equivalence[1] of $P_{\mathbf{A},V,\Omega}$ and $P_{\mathbf{A}+\nabla\phi,V,\Omega}$. Notice that $\operatorname{curl}\nabla\phi = 0$, so the magnetic field is unchanged by the change of gauge.

1.2 Self-Adjointness

Our main interest is the analysis of the bottom of the spectrum of $P_{\mathbf{A},V,\Omega}$. The open set Ω can be bounded or the whole space $\mathbb{R}^n$. Many physically interesting situations correspond to $n = 2, 3$. In the case of a bounded open set Ω, we will consider the Dirichlet realization or the Neumann realization.

The Dirichlet realization

The Dirichlet realization corresponds to taking the so-called Friedrichs extension associated with the quadratic form:

$$C_0^\infty(\Omega;\mathbb{C}) \ni u \mapsto Q^D_{\mathbf{A},V,\Omega}(u) := \int_\Omega |\nabla_{\mathbf{A}}u(x)|^2 + V(x)|u(x)|^2\,dx\,. \tag{1.4}$$

The existence of the Friedrichs extension follows immediately if one can prove that the quadratic form is semibounded from below, i.e., the existence of a constant C such that:

$$\int_\Omega |\nabla_{\mathbf{A}}u(x)|^2 + V(x)|u(x)|^2\,dx \geq -C\|u\|^2\,, \qquad \forall u \in C_0^\infty(\Omega)\,. \tag{1.5}$$

[1] Of course, that will not become a rigorous statement before the domains of the operators in question are defined.

When Ω is regular and bounded (and $V, \mathbf{A}$ are smooth), the form domain of the operator is

$$\mathcal{V}^D(\Omega) = H_0^1(\Omega)\,. \tag{1.6}$$

Using the Lax–Milgram lemma, we can then associate a self-adjoint operator $P^D_{\mathbf{A},V}$ (which is denoted by $P^D_{\mathbf{A},V,\Omega}$ when we need to stress the domain Ω in question) in the following way. We consider the sesquilinear form $q^D_{\mathbf{A},V,\Omega}$ defined on $\mathcal{V}^D(\Omega) \times \mathcal{V}^D(\Omega)$ by

$$(u,v) \mapsto q^D_{\mathbf{A},V,\Omega}(u,v) = \int_\Omega \Big(\overline{\nabla_{\mathbf{A}} u(x)} \cdot \nabla_{\mathbf{A}} v(x) \, + \, V(x)\overline{u(x)}\, v(x) \Big)\, dx\,.$$

The space $\mathcal{V}^D(\Omega) \times \mathcal{V}^D(\Omega)$ is called the form domain of the sesquilinear form $q^D_{\mathbf{A},V,\Omega}$. The domain of the operator is defined as the subspace of the u's in $\mathcal{V}^D(\Omega)$ such that $v \mapsto q^D_{\mathbf{A},V,\Omega}(u,v)$ extends as a continuous linear form on L^2, and we denote by $P^D_{\mathbf{A},V} u$ this element identified by Riesz's theorem to an element in L^2. So we have

$$\langle P^D_{\mathbf{A},V} u \,|\, v \rangle_{L^2(\Omega)} = q^D_{\mathbf{A},V,\Omega}(u,v)\,, \quad \forall v \in \mathcal{V}^D(\Omega)\,. \tag{1.7}$$

More concretely, observing that (1.7) is equivalent to

$$\langle P^D_{\mathbf{A},V} u \,|\, v \rangle_{L^2(\Omega)} = q^D_{\mathbf{A},V,\Omega}(u,v)\,, \quad \forall v \in C_0^\infty(\Omega)\,, \tag{1.8}$$

this leads to

$$D(P^D_{\mathbf{A},V,\Omega}) := \{ u \in \mathcal{V}^D(\Omega) \,|\, P_{\mathbf{A},V,\Omega} u \in L^2(\Omega) \}\,, \tag{1.9}$$

where $D(H)$ denotes the domain of the operator H. The operator $P^D_{\mathbf{A},V,\Omega}$ is simply defined, for $u \in D(P^D_{\mathbf{A},V,\Omega})$, by

$$P^D_{\mathbf{A},V,\Omega} u = P_{\mathbf{A},V,\Omega} u\,.$$

Using a regularity theorem, this domain can be characterized, if Ω is assumed to be regular, as

$$D(P^D_{\mathbf{A},V,\Omega}) = H_0^1(\Omega) \cap H^2(\Omega)\,. \tag{1.10}$$

In most cases under consideration, the operator $P^D_{\mathbf{A},V,\Omega}$ will have a compact resolvent[2] and the spectrum will consist of a nondecreasing sequence of eigenvalues denoted by $\{\lambda_j^D(\mathbf{A}, V, \Omega)\}$. We will sometimes omit some or all of the variables and write, for example, λ_1^D or $\lambda_1^D(\Omega)$ if it is clear from the context what the other variables are. Also, in the case when the shape of $\mathbf{A}$ is fixed and B is a parameter measuring the strength of the field, we will write $\lambda_j^D(B) := \lambda_j^D(B\mathbf{A})$.

[2] For bounded regular Ω, compactness follows from (1.6) and the compactness of the inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

The Neumann realization

The Neumann realization corresponds to first taking the Friedrichs extension of the quadratic form:

$$C^\infty(\overline{\Omega};\mathbb{C}) \ni u \mapsto Q^N_{\mathbf{A},V,\Omega}(u) := \int_\Omega |\nabla_{\mathbf{A}} u(x)|^2 + V(x)|u(x)|^2\,dx\,. \tag{1.11}$$

Again, the existence of the Friedrichs extension follows from semiboundedness, i.e., if there exists a constant C such that

$$\int_\Omega |\nabla_{\mathbf{A}} u(x)|^2 + V(x)|u(x)|^2\,dx \geq -C\|u\|^2\,,\quad \forall u \in C^\infty(\overline{\Omega})\,. \tag{1.12}$$

When Ω is regular and bounded (and V and $\mathbf{A}$ are smooth), the form domain of the operator is

$$\mathcal{V}^N(\Omega) = H^1(\Omega)\,. \tag{1.13}$$

We then associate to this quadratic form a self-adjoint operator, which is denoted by $P^N_{\mathbf{A},V}$ or $P^N_{\mathbf{A},V,\Omega}$, in the classical way. Denoting by $q = q^N_{\mathbf{A},V,\Omega}$ the sesquilinear form associated to $Q^N_{\mathbf{A},V,\Omega}$, the domain of the operator is defined as the subspace in $\mathcal{V}^N(\Omega)$ of the u's such that $\mathcal{V}^N(\Omega) \ni v \mapsto q(u,v)$ admits a continuous extension to $L^2(\Omega)$. When Ω is regular, the domain of the operator can be characterized as

$$D(P^N_{\mathbf{A},V,\Omega}) = \{u \in H^2(\Omega)\mid \nu\cdot(-i\nabla + \mathbf{A})u = 0 \;\text{ on } \partial\Omega\,\}\,. \tag{1.14}$$

Here, for $x \in \partial\Omega$, $\nu(x)$ denotes the unit interior normal vector to $\partial\Omega$ at x and the condition

$$\nu\cdot(-i\nabla + \mathbf{A})u = 0 \;\text{ on } \partial\Omega \tag{1.15}$$

is called the magnetic Neumann boundary condition. This characterization involves the Green–Riemann formula and a regularity result for the magnetic Laplacian. We can then define $P^N_{\mathbf{A},V,\Omega}u \in L^2(\Omega)$ by

$$\langle P^N_{\mathbf{A},V,\Omega}u \mid v\rangle_{L^2} = q(u,v)\,,\qquad \forall v \in \mathcal{V}^N(\Omega)\,.$$

The eigenvalues of the Neumann Schrödinger operator will be denoted by $\{\lambda^N_j(\mathbf{A},V,\Omega)\}$. The conventions about notation discussed for Dirichlet eigenvalues also apply to the λ^N_j.

Remark 1.2.1.
Let us for a moment reintroduce the dependence on the parameter B. Clearly, $D(P^D_{B\mathbf{A},V,\Omega})$ does not depend on B. This is obvious from (1.10). *However, for the Neumann operator, we see from* (1.14) *that if we want a domain independent of B, we need to impose the condition*

$$\nu(x)\cdot\mathbf{A}(x) = 0 \quad \textit{for all } x \in \partial\Omega\,.$$

Under this condition the magnetic Neumann condition becomes the usual Neumann condition. We discuss in Appendix D (Proposition D.1.1) how to arrive at this situation via a gauge transformation.

The case of $\mathbb{R}^n$

In the case of $\mathbb{R}^n$, it is in general more difficult to characterize the domain of the operator. When $V \geq -C$, it is easy to characterize the form domain, which is

$$\mathcal{V}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \nabla_{\mathbf{A}} u \in L^2(\mathbb{R}^n)\,,\ (V+C)^{\frac{1}{2}} u \in L^2(\mathbb{R}^n) \right\}. \quad (1.16)$$

The domain of the associated operator is then given by

$$D(P_{\mathbf{A},V}) := \{u \in \mathcal{V}(\mathbb{R}^n)\,,\ P_{\mathbf{A},V} u \in L^2(\mathbb{R}^n)\}. \quad (1.17)$$

In the general case, if the operator is semibounded on $C_0^\infty(\mathbb{R}^n)$ in the sense of (1.5), it has been proved by Simader [Sima] that the operator is essentially self-adjoint. The essential self-adjointness means that the Friedrichs extension is the unique self-adjoint extension in $L^2(\mathbb{R}^n)$ starting from $C_0^\infty(\mathbb{R}^n)$ and that $D(P_{\mathbf{A},V})$ satisfies in this case

$$D(P_{\mathbf{A},V}) = \{u \in L^2(\mathbb{R}^n)\,,\ P_{\mathbf{A},V} u \in L^2(\mathbb{R}^n)\}. \quad (1.18)$$

We include here the proof of essential self-adjointness.

Theorem 1.2.2.
Suppose that $P = (-i\nabla + \mathbf{A})^2 + V$ *is semibounded on* $C_0^\infty(\mathbb{R}^n)$ *and that* $V \in C^0(\mathbb{R}^n)$, $\mathbf{A} \in C^1(\mathbb{R}^n)$. *Then* P *is essentially self-adjoint.*

Proof.
Since P is semibounded, we may assume, possibly replacing P by $P + C$ for some constant $C \geq 0$, that

$$\langle u \mid Pu \rangle \geq \|u\|^2\,, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \quad (1.19)$$

Here we recall that the scalar product of two functions f and g in $L^2(\mathbb{R}^n)$ is introduced in (7). Inequality (1.19) extends by density to distributions $u \in H^1_{\mathrm{comp}}(\mathbb{R}^n)$ (the H^1 distributions with compact support),

$$\|\nabla_{\mathbf{A}} u\|^2 + \int_{\mathbb{R}^n} V(x)|u(x)|^2\,dx \geq \|u\|^2\,, \quad \forall u \in H^1_{\mathrm{comp}}(\mathbb{R}^n). \quad (1.20)$$

According to the general criterion of essential self-adjointness, it suffices to verify that the range $R(P)$ is dense. Suppose that $f \in L^2(\mathbb{R}^n)$ is such that

$$\langle f \mid Pu \rangle = 0\,, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \quad (1.21)$$

We have to show that $f = 0$.

We first observe that (1.21) implies that

$$((-i\nabla + \mathbf{A})^2 + V)f = 0\,,$$

in the sense of distributions. Standard elliptic regularity theory for the Laplacian (with our assumptions on V and $\mathbf{A}$ in mind) implies then that $f \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$.

We now introduce a family of cutoff functions, ζ_k, by

$$\zeta_k(x) := \zeta(x/k)\,, \quad \forall k \in \mathbb{N}\,, \tag{1.22}$$

where $\zeta \in C_0^\infty(\mathbb{R}^n)$ satisfies $0 \le \zeta \le 1$, $\zeta = 1$ on the unit ball $\mathrm{D}(0,1)$, and $\operatorname{supp} \zeta \subset \mathrm{D}(0,2)$.

For any $u \in C_0^\infty(\mathbb{R}^n)$, we have the identity

$$\begin{aligned}
&\int \overline{\nabla_{\mathbf{A}}(\zeta_k f)} \cdot \nabla_{\mathbf{A}}(\zeta_k u)\, dx + \int \zeta_k^2 V(x) \overline{f(x)} u(x)\, dx \\
&\quad = \langle f \mid P(\zeta_k^2 u)\rangle + \int |\nabla \zeta_k(x)|^2 \overline{f(x)} u(x)\, dx \\
&\qquad + \int \zeta_k(x) \nabla \zeta_k(x) \cdot \left[\overline{f(x)} \nabla_{\mathbf{A}} u(x) - u(x) \overline{\nabla_{\mathbf{A}} f(x)}\right] dx\,.
\end{aligned} \tag{1.23}$$

When f satisfies (1.21), we get

$$\begin{aligned}
&\int \overline{\nabla_{\mathbf{A}}(\zeta_k f)} \cdot \nabla_{\mathbf{A}}(\zeta_k u)\, dx + \int \zeta_k(x)^2 V(x) \overline{f(x)} u(x)\, dx \\
&\quad = \int |\nabla \zeta_k(x)|^2 \overline{f(x)} u(x)\, dx \\
&\qquad + \int \zeta_k(x) \nabla \zeta_k(x) \cdot \left[\overline{f(x)} \nabla_{\mathbf{A}} u(x) - u(x) \overline{\nabla_{\mathbf{A}} f(x)}\right] dx\,.
\end{aligned} \tag{1.24}$$

This formula can be extended by continuity to functions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, we can take $u = f$ and obtain

$$\begin{aligned}
&\|\nabla_{\mathbf{A}}(\zeta_k f)\|^2 + \int \zeta_k(x)^2 V(x) |f(x)|^2\, dx \\
&\quad = \Re \left\{ \|\nabla_{\mathbf{A}}(\zeta_k f)\|^2 + \int \zeta_k(x)^2 V(x) |f(x)|^2\, dx \right\} \\
&\quad = \int |\nabla \zeta_k(x)|^2\, |f(x)|^2\, dx\,.
\end{aligned} \tag{1.25}$$

Using (1.20), (1.25), the definition of ζ_k, and taking the limit $k \to \infty$, we get

$$\begin{aligned}
\|f\|^2 &= \lim_{k\to\infty} \|\zeta_k f\|^2 \\
&\le \limsup_{k\to\infty} \left(\|\nabla_{\mathbf{A}}(\zeta_k f)\|^2 + \int_{\mathbb{R}^n} V(x) |\zeta_k(x) f(x)|^2\, dx \right) \\
&= \limsup_{k\to\infty} \int |\nabla \zeta_k(x)|^2\, |f(x)|^2\, dx = 0\,.
\end{aligned} \tag{1.26}$$

This finishes the proof of Theorem 1.2.2. □

1.3 Spectral Theory

All the operators introduced earlier are self-adjoint. If one denotes by $\mathcal{H}$ one of these unbounded operators, one can analyze its spectrum, defined as the complement in $\mathbb{C}$ of the resolvent set $\rho(\mathcal{H})$ corresponding to the points $z \in \mathbb{C}$ such that $(\mathcal{H} - z)^{-1}$ exists. The spectrum $\sigma(\mathcal{H})$ is a closed set contained in $\mathbb{R}$. The spectrum contains in particular the set of the eigenvalues of $\mathcal{H}$. We recall that λ is an eigenvalue if there exists a nonzero vector $u \in D(\mathcal{H})$ such that $\mathcal{H}u = \lambda u$. The multiplicity of λ is the dimension of $\mathrm{Ker}(\mathcal{H} - \lambda)$. We call discrete spectrum $\sigma_d(\mathcal{H})$ the subset of the spectrum consisting of eigenvalues of finite multiplicity that are isolated points of $\sigma(\mathcal{H})$. The following standard theorem plays an important role in the theory.

Theorem 1.3.1.
For all $\lambda \in \mathbb{C}$ and all $u \in D(\mathcal{H})$, we have

$$\mathrm{dist}(\lambda, \sigma(\mathcal{H}))\, \|u\| \leq \|(\mathcal{H} - \lambda)u\|\,. \tag{1.27}$$

An elementary consequence that will be used quite often in the book is that if we find a normalized u in $D(\mathcal{H})$ such that, for some $\epsilon > 0$,

$$\|(\mathcal{H} - \lambda)u\| \leq \epsilon\,, \tag{1.28}$$

then

$$d(\lambda, \sigma(\mathcal{H})) \leq \epsilon\,.$$

Therefore, approximate eigenfunctions—also called quasimodes—i.e., functions u satisfying (1.28) for some (small) ϵ, are very useful for locating the spectrum.

Finally, the essential spectrum of $\mathcal{H}$—denoted by $\sigma_{\mathrm{ess}}(\mathcal{H})$—is defined to be the closed set:

$$\sigma_{\mathrm{ess}}(\mathcal{H}) = \sigma(\mathcal{H}) \setminus \sigma_d(\mathcal{H})\,. \tag{1.29}$$

In this book, we will mainly be interested in the analysis of the infimum of the spectrum of $\mathcal{H}$ as a function of the various parameters (mainly B). Depending on the assumptions, this infimum could correspond to an eigenvalue or to the bottom of the essential spectrum.

Using the min-max characterization (see Appendix A), the infimum of the spectrum of $\mathcal{H} = P_{B\mathbf{A},V}$ is determined by

$$\inf(\sigma(P_{B\mathbf{A},V})) = \inf_{u \in \mathcal{V}\setminus\{0\}} Q_{B\mathbf{A},V}(u)/\|u\|^2\,, \tag{1.30}$$

where $\mathcal{V}$ denotes the form domain of the quadratic form $Q_{B\mathbf{A},V}$. In order to determine if the infimum corresponds to an eigenvalue, it is consequently enough to find a nontrivial u in the form domain $\mathcal{V}$ such that

$$Q_{B\mathbf{A},V}(u) < \inf(\sigma_{\mathrm{ess}}(P_{B\mathbf{A},V}))\|u\|^2\,. \tag{1.31}$$

An easy case where the infimum of the spectrum is an eigenvalue is when $\sigma_{\rm ess}(P_{B\mathbf{A},V}) = \emptyset$, corresponding to the case when $\mathcal{H}$ has compact resolvent. To verify this last property, it is enough to show that the injection of $\mathcal{V}$ in L^2 is compact. This is in particular the case (for Dirichlet and Neumann boundary conditions) when Ω is regular and bounded. When Ω is unbounded, it is possible to determine the bottom of the essential spectrum using Persson's theorem (see Appendix B).

Example 1.3.2. .
Let us consider $P_{B^2V} := -\Delta + B^2 V$ on $\mathbb{R}^n$, where V is a C^∞ potential tending to 0 at ∞ and such that $\inf_{x\in\mathbb{R}^n} V(x) < 0$.

Then, if B is large enough, there exists at least one eigenvalue for P_{B^2V}. We note that the essential spectrum is $[0,+\infty[$. The proof of the existence of this eigenvalue is elementary. If $x_{\rm min}$ is a point such that $V(x_{\rm min}) = \inf_x V(x)$, it is enough to show that, with $\phi_B(x) = \exp(-B|x-x_{\rm min}|^2)$, the quotient $\frac{\langle P_{B^2V}\phi_B \mid \phi_B\rangle}{B^2\,\|\phi_B\|^2}$ tends to $V(x_{\rm min}) < 0$ as $B \to +\infty$.

Actually, we can produce an arbitrary number N of eigenvalues below the essential spectrum, under the condition that $B \in [B_N, +\infty[$, for B_N large enough.

1.4 Preliminary Estimates for the Dirichlet Realization

1.4.1 Lower bounds

We start by giving the following very basic result, Lemma 1.4.1. The lemma immediately yields a lower bound to the spectrum of the Dirichlet realization; see (1.34). But this lemma will also be very useful for the Neumann realization, since it is the fundamental reason behind the boundary localization, which will be proven in Chapters 8 and 9.

Lemma 1.4.1.
Let $n \in \{2,3\}$ and let $\operatorname{curl}\mathbf{A} = \beta$. *We use the conventions that if $n=3$, $\beta = (\beta_1,\beta_2,\beta_3)$ is a vector and that if $n=2$, $\beta=\beta_1$ is a function. Then, for all $u \in C_0^\infty(\Omega)$ and all j, we have the inequality*

$$\|\nabla_{\mathbf{A}} u\|^2 = \langle P_{\mathbf{A},\Omega} u \mid u\rangle \geq \int_\Omega \beta_j(x)|u(x)|^2\,dx\,. \tag{1.32}$$

Of course, (1.32) is only interesting if $\beta_j(x)$ is positive in Ω. Notice (see the proof) that it is important that the function u has compact support.

Proof.
The basic point is to observe that

$$\beta_j(x) = -i[\partial_{x_k} + iA_k, \partial_{x_\ell} + iA_\ell]\,, \tag{1.33}$$

for suitable k, ℓ. We then write

$$\beta_j(x)u(x)\,\overline{u(x)} = -i\,\overline{u(x)}\,(X_k X_\ell u)(x) \;+ i\,\overline{u(x)}\,(X_\ell X_k u)(x)\,,$$

with $X_\ell = \partial_{x_\ell} + iA_\ell$.

Integrating over Ω and performing an integration by parts, we get

$$\int_\Omega \beta_j(x)|u(x)|^2\,dx = i\,\langle X_k u \mid X_\ell u\rangle - i\langle X_\ell u \mid X_k u\rangle = 2\Im\langle X_\ell u \mid X_k u\rangle\,.$$

Applying the Cauchy–Schwarz inequality, we then find

$$\int_\Omega \beta_j(x)|u(x)|^2\,dx \le \|X_k u\|^2 + \|X_\ell u\|^2\,,$$

which yields (1.32). □

1.4.2 Two-dimensional case

In the two-dimensional situation, Lemma 1.4.1 leads, for the Dirichlet realization and when $\beta(x) \ge 0$, to the easy but useful estimate,

$$\inf \sigma(P^D_{\mathbf{A}}) \ge \inf_{x\in\overline{\Omega}} \beta(x) =: b\,. \tag{1.34}$$

Note that the converse inequality is asymptotically (as $B \to \infty$) true. The proof is rather easy. This will later—in Chapter 8—be carried out in a more systematic way after the analysis of model operators, but let us simply look here for Gaussian quasimodes. In a system of coordinates where $x = 0$ denotes a minimum of β—which is assumed to be inside Ω—and in a gauge where

$$\mathbf{A}(x_1, x_2) = \frac{1}{2} b(-x_2, x_1) + \mathcal{O}(|x|^2)\,,$$

we consider the quasimode

$$u(x;B) := \rho^{\frac{1}{2}} b^{\frac{1}{4}} B^{\frac{1}{2}} \exp\big(-\rho\sqrt{b}B|x|^2\big)\chi(x)\,,$$

where χ is a cutoff function equal to 1 in a neighborhood of 0 and $\rho > 0$ has to be determined. The optimal ρ is computed by minimizing (with respect to ρ) the energy corresponding to the constant magnetic field $b = 1$ and $B = 1$:

$$\left(\int \left|\left(\partial_{y_1} - \frac{i}{2}y_2\right) u_\rho(y)\right|^2 + \left|\left(\partial_{y_2} + \frac{i}{2}y_1\right) u_\rho(y)\right|^2\,dy\right) / \|u_\rho\|^2\,,$$

with

$$u_\rho(y) = \pi^{-\frac{1}{4}} \rho^{\frac{1}{2}} \exp\left(-\frac{\rho}{2} y^2\right)\,. \tag{1.35}$$

One easily gets that this quantity is minimized for $\rho = 1/2$ and that the corresponding energy is 1. The control of the remainders is easy, and we get, using (1.30),

$$\inf \sigma(P^D_{B\mathbf{A}}) \leq Bb + \mathcal{O}(B^{\frac{1}{2}})\,. \tag{1.36}$$

So we have proven[3] (in the two-dimensional case):

Theorem 1.4.2.
Suppose $\beta \geq 0$. The smallest eigenvalue $\lambda_1^D(B)$ of the Dirichlet realization $P^D_{B\mathbf{A},\Omega}$ of $P_{B\mathbf{A},\Omega}$ satisfies

$$\frac{\lambda_1^D(B)}{B} = b + o(1)\,, \tag{1.37}$$

as $B \to +\infty$, with b defined in (1.34).

Remark 1.4.3.
For the Dirichlet realization and when the magnetic field is constant, one can show, by taking a Gaussian centered as far as possible from the boundary, the existence of $\alpha > 0$ such that

$$\frac{\lambda_1^D(B)}{B} = b + \mathcal{O}(\exp -\alpha B)\,, \tag{1.38}$$

as $B \to +\infty$.

Except for the case of the disc [see (5.1)*] the optimal α is unknown, but the construction of quasimodes suggests that it should be the square of the inner radius of Ω.*

1.4.3 The case of three or more dimensions

Let us state Theorem 1.4.2 in a more general case. Let us extend at each point β_{jk} as an antisymmetric matrix (more intrinsically, this is the matrix of the 2-form σ_β). Then the eigenvalues of the matrix $i\beta$ are real and one can see that if λ is an eigenvalue of $i\beta$, with corresponding eigenvector u, then $\overline{u}$ is an eigenvector relative to the eigenvalue $-\lambda$, since β has real coefficients. If the $\widehat{\lambda}_j$ denote the eigenvalues of $i\beta$ counted with multiplicity, then one can define

$$\mathrm{tr}^+\,\beta(x) = \sum_{j:\lambda_j(x)>0} \widehat{\lambda}_j(x)\,. \tag{1.39}$$

[3] We leave the proof of (1.36) in the case where the minimum of $\beta(x)$ is attained at the boundary to the reader. This affects only the remainder term.

The extension of Theorem 1.4.2 is then

Theorem 1.4.4.
The smallest eigenvalue $\lambda_1^D(B)$ of the Dirichlet realization $P^D_{B\mathbf{A},\Omega}$ of $P_{B\mathbf{A},\Omega}$ satisfies

$$\frac{\lambda_1^D(B)}{B} = \inf_{x\in\Omega} \operatorname{tr}^+(\beta(x)) + o(1)\,, \tag{1.40}$$

as $B \to +\infty$.

The idea for the proof is to first treat the constant field case, and then to make a partition of unity. For the constant field case, after a change of variable, we will get, for $n = 2m$, the model

$$\sum_{j=1}^{m} [-(\partial_{x_j})^2 - (\partial_{x_{j+m}} + i\widehat{\lambda}_j x_j)^2]\,,$$

and for $n = 2m+1$, the model

$$-\partial^2_{2m+1} + \sum_{j=1}^{m} [-(\partial_{x_j})^2 - (\partial_{x_{j+m}} + i\widehat{\lambda}_j x_j)^2]\,,$$

with

$$\sum_{j=1}^{m} |\widehat{\lambda}_j| = \operatorname{tr}^+ \beta\,.$$

1.5 Perturbation Theory for Small B

Although our main interest in this book is the case of large B, it is also interesting to discuss the opposite case, which also appears in the physics literature.

If $\mathbf{A}$ satisfies

$$\mathbf{A}\cdot\nu \equiv 0 \text{ on } \partial\Omega \qquad \text{and} \qquad \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega\,, \tag{1.41}$$

then the domain of the Neumann realization $P^N_{B\mathbf{A},\Omega}$ is fixed (see Remark 1.2.1) and the dependence on B is analytic. Assume that Ω is bounded and connected and has smooth boundary. Then the resolvent of $P^N_{B\mathbf{A},\Omega}$ is compact and the spectrum is discrete. So the family of operators $\{P^N_{B\mathbf{A},\Omega}\}_B$ is a holomorphic family of type (A) (see Appendix C). We can then apply analytic perturbation theory to the analysis of the ground state energy.

For $B = 0$, $P^N_{B\mathbf{A},\Omega}$ is simply the Neumann realization of $-\Delta$ in Ω. The ground state energy is 0 and this is a simple eigenvalue (Ω is assumed to be connected). The associated L^2-normalized eigenfunction φ_{10} can be chosen as the constant function

$$\varphi_{10} = \frac{1}{|\Omega|^{\frac{1}{2}}}\,, \tag{1.42}$$

where $|\Omega|$ denotes the volume (area) of Ω: $|\Omega| = \int_\Omega dx$.

We consequently know that for B small ($B \in]-B_0, +B_0[$) enough, the ground state energy remains simple and admits the convergent expansion

$$\lambda_1^N(B\mathbf{A}) = \sum_{j\geq 1} B^j \lambda_{1j}\,. \tag{1.43}$$

We will proceed to compute λ_{11} and λ_{12}.

This can be done by using formal expansions in the following way. We look for an eigenvalue admitting the expansion (1.43) and an associated eigenfunction

$$\varphi_1(B\mathbf{A}) \sim \sum_{j\geq 0} B^j \varphi_{1j}\,. \tag{1.44}$$

Moreover, without loss of generality, we may assume that φ_{1j} is orthogonal to φ_{10} for $j \geq 1$. This can be rewritten in the form

$$\int_\Omega \varphi_{1j}\, dx = 0\,, \tag{1.45}$$

for $j = 1, \dots, n$.

We now write that

$$P^N_{B\mathbf{A},\Omega}\,\varphi_1(B\mathbf{A}) - \lambda_1^N(B\mathbf{A})\,\varphi_1(B\mathbf{A}) \sim 0\,, \tag{1.46}$$

in the sense of formal series in powers of B. This means more precisely that, when expanding the left-hand side of (1.46) in powers of B, each of the coefficients in the expansion should vanish.

We note that with our choice of gauge,

$$-\Delta_{B\mathbf{A}} = -\Delta + 2iB\mathbf{A}\cdot\nabla + B^2|\mathbf{A}|^2\,.$$

We denote by $\mathcal{R}_0$ the operator defined by

$$\mathcal{R}_0 := (I - \Pi_0)\Delta^{-1}(I - \Pi_0)\,,$$

where Π_0 is the projector on the first eigenfunction φ_{10}.

Computing the coefficients of each of the powers of B in (1.46), we get equations determining the λ_{1j}, φ_{1j}. Due to our choice of φ_{10}, it is clear that the coefficient of B^0 is 0. Let us look at the coefficient of B. We get

$$-\Delta\varphi_{11} - \lambda_{11}\varphi_{10} = -2i\mathbf{A}\cdot\nabla\varphi_{10} = 0\,. \tag{1.47}$$

A necessary condition (take the scalar product with φ_{10}, i.e., simply integrate the equation over Ω) is that

$$\lambda_{11} = 0\,, \tag{1.48}$$

and we can consequently choose $\varphi_{11} = 0$.

Remark 1.5.1.
The fact that λ_{11} vanishes is not a surprise. We should indeed have

$$\lambda_1^N(B\mathbf{A}) \geq 0\,,$$

by the positivity of the quadratic form.

Let us now look at the coefficient of B^2. Taking account of the previous equation, we obtain

$$-\Delta\varphi_{12} - \lambda_{12}\varphi_{10} + |\mathbf{A}|^2\varphi_{10} = 0\,. \tag{1.49}$$

This equation can be solved if and only if

$$\lambda_{12} = \frac{1}{|\Omega|}\int_\Omega |\mathbf{A}(x)|^2 dx\,. \tag{1.50}$$

This gives the value of λ_{12} , which is nonzero if and only if the magnetic field $\operatorname{curl}\mathbf{A}$ is not identically 0 (See Appendix D). We are also very happy to verify that λ_{12} is positive, which is natural from the positivity of $\lambda_1(B\mathbf{A})$. For this value of λ_{12} , one can then define φ_{12} by

$$\varphi_{12} = \mathcal{R}_0\left(|\mathbf{A}|^2\,\varphi_{10}\right) = \frac{1}{|\Omega|^{\frac{1}{2}}}\mathcal{R}_0|\mathbf{A}|^2\,. \tag{1.51}$$

It is easy to see that one can continue to solve the equations by recursion. The necessary solvability condition indeed determines λ_{1j} at each step and the solution is unique due to (1.45).

We have therefore proven the following result.

Proposition 1.5.2.
Let $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$, be smooth, connected, and bounded. Suppose that $\mathbf{A}$ satisfies $\operatorname{curl}\mathbf{A} \neq 0$ and (1.41). Then λ_{12} defined by (1.50) satisfies $\lambda_{12} > 0$, and

$$\lambda_1^N(B\mathbf{A}) - \lambda_{12}\,B^2 = \mathcal{O}(B^3)\,, \quad \textit{as } B \to 0\,. \tag{1.52}$$

Remark 1.5.3.
Observing that the complex conjugation $u \mapsto \Gamma u = \overline{u}$ intertwines $-\Delta_{B\mathbf{A}}$ and $-\Delta_{(-B\mathbf{A})}$, i.e., satisfies

$$\Gamma \circ \Delta_{B\mathbf{A}} = \Delta_{(-B\mathbf{A})} \circ \Gamma\,, \tag{1.53}$$

one can actually show that all the λ_{1j} 's with j odd vanish in the expansion (1.43). So we have

$$\lambda_1^N(B\mathbf{A}) \sim \sum_{\ell \geq 1, j = 2\ell} \lambda_{1j}B^{2\ell}\,. \tag{1.54}$$

One can also observe that the functions φ_{1j} are real for j even and purely imaginary for j odd.

Remark 1.5.4.
Without Assumption (1.41), λ_{12} *can be more intrinsically written as*

$$\frac{1}{|\Omega|} \inf_{\phi \in H^2(\Omega)} \int_\Omega |\mathbf{A} + \nabla\phi|^2 dx \,. \tag{1.55}$$

Observe that (1.41) *is satisfied if and only if the infimum of* (1.55) *is realized for* $\phi = 0$ *.*

1.6 Notes

1. The main references for this chapter are the book by Kato [Kat1], the series of articles by Avron–Herbst–Simon [AvHS1]–[AvHS3], the paper by Combes–Schrader–Seiler [CoSS], and the contribution of Leinfelder–Simader [LeS] on self-adjointness.
2. For an introduction to spectral theory and most of the material presented in the three first sections, students can consult the books [ReS, LiL, LeB].
3. The abstract criterion of self-adjointness can be found in [ReS, Theorem X. 26, Vol. II].
4. Theorem 1.2.2 is proven under weaker conditions on the electric potential V in [Sima]. We follow here [He8], where the case without magnetic field was considered. We show here that one can modify the proof in order to accommodate the magnetic case also. One can find in [CyFKS, Chapter 1] a statement of Leinfelder–Simader [LeS] giving a criterion of self-adjointness under the weaker condition that $\mathbf{A} \in L^4_{\rm loc}$ and $\operatorname{div} \mathbf{A} \in L^2_{\rm loc}$. This condition is necessary in order to have $\Delta_{\mathbf{A}}\phi \in L^2$ for $\phi \in C_0^\infty$.
5. We have mainly discussed the Dirichlet case (which is the most standard one) and the Neumann case, which is the basic case in view of our applications. Note that in the physics literature on superconductivity, one finds other boundary conditions—the so-called de Gennes boundary conditions (more commonly called Robin's boundary condition in the mathematics literature)—which take the form
$$\nu \cdot \nabla_{\mathbf{A}} u = \gamma u \text{ on } \partial\Omega \,, \tag{1.56}$$
where γ is a real parameter with a physical interpretation. The case $\gamma = 0$ corresponds to the Neumann case. The de Gennes boundary condition appears when studying the L^2-normalized minimizers of the quadratic form
$$H^1(\Omega) \ni u \mapsto Q_\gamma(u) = Q^N(u) + \gamma \int_{\partial\Omega} |u|^2 d\sigma \,, \tag{1.57}$$
where Q^N is defined in (1.11) and $d\sigma$ is the induced measure on $\partial\Omega$. In the context of superconductivity, we refer to [dGe1, Kac1, Kac2, LuP3, Pa4].
6. The lower bound obtained in Lemma 1.4.1 appears already in the work of Avron–Herbst–Simon [AvHS1]. It is actually closely related to the standard proof of the uncertainty principle.

7. An extension of Theorem 1.4.2 appears (actually in a still more general situation) in [Me], [Ho, Vol. III, Chapter 22.3], and [HeM2]. The question arises in some of these references for rather different problems occurring in the analysis of partial differential equations like Gårding's inequality or hypoellipticity of operators with double characteristics.
8. The problem of the perturbation theory for small B analyzed in Section 1.5 appears in the one-dimensional case in the work of Bolley–Helffer [BolH2, BolH3] for the analysis of a one-dimensional reduced model corresponding to the functional
$$(f, A) \mapsto \int_{-d}^{+d} \kappa^{-2} f'(x)^2 + (1 - f(x)^2)^2 + A(x)^2 f(x)^2 + (A'(x)^2 - h)^2 \, dx \,, \tag{1.58}$$
which occurs already in the work of Ginzburg–Landau [GiL] when modeling a 2D problem for $\Omega =]-d, +d[\times \mathbb{R}$. The small parameter there is the Ginzburg–Landau parameter κ and for a minimizer of the functional (f, A), the wave function f is nearly constant.
In the three-dimensional case, the question appears in the work of Pan [Pa3] (for a problem coming from the analysis of Type I superconductors) and [Pa7] (in the context of problems for liquid crystals) and more recently in [HeP2, HeP3].

2

Diamagnetism

2.1 Preliminaries

For Schrödinger operators, the inclusion of a magnetic field raises the energy. This is the consequence of a basic inequality due to Kato. A variation on this question—namely, monotonicity of the ground state energy as a function of the parameter B—is of central importance in the theory of superconductivity and is discussed in Section 2.3 here and repeatedly in the book.

Without loss of generality, set $B = 1$ in this first section.

Theorem 2.1.1 (Diamagnetic inequality).
Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ *be in* $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ *and suppose that* $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ *is such that* $(-i\nabla + \mathbf{A})f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ *. Then* $|f| \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\big|\nabla|f|\big| \leq \big|(-i\nabla + \mathbf{A})f\big| \tag{2.1}$$

almost everywhere.

In the proof of this theorem we will clearly need to differentiate the absolute value. We state this result as a proposition.

Proposition 2.1.2.
Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\nabla f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *. Then also* $\nabla|f| \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and with the notation*

$$\operatorname{sign} z = \begin{cases} \frac{\overline{z}}{|z|}\,, & \text{for } z \neq 0\,, \\ 0\,, & \text{for } z = 0\,, \end{cases} \tag{2.2}$$

we have

$$\nabla|f|(x) = \Re\{\operatorname{sign}(f(x))\nabla f(x)\} \ \text{ for almost every } x \in \mathbb{R}^n\,. \tag{2.3}$$

In particular,

$$\big|\nabla|f|\big| \leq |\nabla f|\,,$$

almost everywhere in $\mathbb{R}^n$ *.*

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Proof of Proposition 2.1.2.
Suppose first that $u \in C^\infty(\mathbb{R}^n)$ and define $|z|_\epsilon = \sqrt{|z|^2+\epsilon^2} - \epsilon$, for $z \in \mathbb{C}$ and $\epsilon > 0$. We observe that

$$0 \leq |z|_\epsilon \leq |z| \qquad \text{and} \qquad \lim_{\epsilon \to 0} |z|_\epsilon = |z|\,.$$

Then the function $|u|_\epsilon$, defined, for $x \in \mathbb{R}^n$, by

$$|u|_\epsilon(x) = |u(x)|_\epsilon\,,$$

belongs to $C^\infty(\mathbb{R}^n)$ and

$$\nabla |u|_\epsilon = \frac{\Re(\overline{u}\,\nabla u)}{\sqrt{|u|^2+\epsilon^2}}\,. \tag{2.4}$$

Now let f be as in the proposition and define f_δ as the convolution

$$f_\delta = f * \rho_\delta\,,$$

with ρ_δ being a standard approximation of unity for convolution. Explicitly, we take a $\rho \in C_0^\infty(\mathbb{R}^n)$ with

$$\rho \geq 0\,, \qquad \int_{\mathbb{R}^n} \rho(x)\,dx = 1\,,$$

and define $\rho_\delta(x) := \delta^{-n}\rho(x/\delta)$, for $x \in \mathbb{R}^n$ and $\delta > 0$. Then $f_\delta \to f$, $|f_\delta| \to |f|$, and $\nabla f_\delta \to \nabla f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\delta \to 0$.

Take a test function $\phi \in C_0^\infty(\mathbb{R}^n)$. We may extract a subsequence $\{\delta_k\}_{k\in\mathbb{N}}$ (with $\delta_k \to 0$ for $k \to \infty$) such that $f_{\delta_k}(x) \to f(x)$ for almost every $x \in \operatorname{supp}\phi$. We restrict our attention to this subsequence. For simplicity of notation, we omit the k from the notation and write $\lim_{\delta\to 0}$ instead of $\lim_{k\to\infty}$.

We now calculate, using dominated convergence and (2.4),

$$\begin{aligned}
\int (\nabla\phi)|f|\,dx &= \lim_{\epsilon\to 0}\int (\nabla\phi)|f|_\epsilon\,dx \\
&= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int (\nabla\phi)|f_\delta|_\epsilon\,dx \\
&= -\lim_{\epsilon\to 0}\lim_{\delta\to 0}\int \phi\frac{\Re(\overline{f_\delta}\nabla f_\delta)}{\sqrt{|f_\delta|^2+\epsilon^2}}\,dx\,.
\end{aligned}$$

Using the pointwise convergence of $f_\delta(x)$ and $\|\nabla f_\delta - \nabla f\|_{L^1(\operatorname{supp}\phi)} \to 0$, we can take the limit $\delta \to 0$ and get

$$\int (\nabla\phi)|f|\,dx = -\lim_{\epsilon\to 0}\int \phi\frac{\Re(\overline{f}\nabla f)}{\sqrt{|f|^2+\epsilon^2}}\,dx\,. \tag{2.5}$$

Now, $\phi\nabla f \in L^1(\mathbb{R}^n)$ and $\overline{f(x)}\,(|f(x)|^2+\epsilon^2)^{-1/2} \to \operatorname{sign} f(x)$ as $\epsilon \to 0$. So we get (2.3) from (2.5) by dominated convergence. □

Proof of Theorem 2.1.1.
Since $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $f \in L^2_{\text{loc}}(\mathbb{R}^n)$, the assumption $(\nabla + i\mathbf{A})f \in L^2_{\text{loc}}(\mathbb{R}^n)$ implies that $\nabla f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore, we can use Proposition 2.1.2 to conclude that (2.3) holds for f. Since $\Re\{\text{sign}(f) i\mathbf{A} f\} = 0$, we can rewrite (2.3) as

$$\nabla|f| = \Re\{\text{sign}(f)(\nabla + i\mathbf{A})f\}\,, \tag{2.6}$$

and therefore, since $|z| \geq |\Re(z)|$ for all $z \in \mathbb{C}$, we get (2.1). □

Using Theorem 2.1.1 we now get, by the variational characterization of the ground state energy, the comparison for Dirichlet eigenvalues,

$$\inf\ \sigma\left(P^D_{\mathbf{A},\Omega} + V\right) \geq \inf\ \sigma\left(-\Delta^D_\Omega + V\right)\,, \tag{2.7}$$

where $-\Delta^D_\Omega$ denotes the Dirichlet Laplacian in Ω.
Also, a similar result is true in the case of Neumann boundary conditions:

$$\inf\ \sigma\left(P^N_{\mathbf{A},\Omega} + V\right) \geq \inf\ \sigma\left(-\Delta^N_\Omega + V\right)\,, \tag{2.8}$$

where $-\Delta^N_\Omega$ denotes the Neumann Laplacian in Ω.

Inequality (2.8) admits a kind of converse inequality showing its optimality.

Proposition 2.1.3.
Suppose that $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is bounded, smooth, and connected, that $\mathbf{A} \in C^1(\overline{\Omega})$, and that $V \in L^\infty(\Omega)$. Let $\lambda_1(\mathbf{A})$ be the ground state of $P^N_{\mathbf{A}}$. Then the following three properties are equivalent:

1. *$P^N_{\mathbf{A}}$ and $P^N_{\mathbf{A}=\mathbf{0}}$ are unitarily equivalent.*
2. *$\lambda_1(\mathbf{A}) = \lambda_1(0)$.*
3. *$\mathbf{A}$ satisfies the following two conditions:*

$$\beta := \text{curl}\,\mathbf{A} = 0 \tag{2.9}$$

and

$$\frac{1}{2\pi}\int_\gamma \omega_{\mathbf{A}} = \frac{1}{2\pi}\int_\gamma \mathbf{A}\cdot dx \in \mathbb{Z} \tag{2.10}$$

on any closed path γ in Ω.

Proof.
Clearly, 1 implies 2.

Let us now prove the statement that 3 implies 1. First, let us observe that, when Ω is simply connected, condition (2.10) is a consequence of (2.9), by Green's formula. Now, even for nontrivial topology, conditions (2.9) and (2.10) permit the construction of a multivalued function, $\phi_{\mathbf{A}}$, such that $\nabla\phi_{\mathbf{A}} = -\mathbf{A}$ and $\exp(i\phi_{\mathbf{A}})$ is welldefined. This function $\phi_{\mathbf{A}}$ is defined by taking some $x_0 \in \Omega$ and then writing

$$\phi_{\mathbf{A}}(x) = \int_{\gamma(x_0,x)} \omega_{\mathbf{A}}\,,$$

where $\gamma(x_0, x)$ is a path in Ω joining x_0 and x and the integral is independent of the choice of path (modulo $2\pi\mathbb{Z}$). This permits us to define the C^1 function on $\overline{\Omega}$:

$$\Omega \ni x \mapsto U_{\mathbf{A}}(x) = \exp(-i\phi_{\mathbf{A}}(x))\,. \tag{2.11}$$

The associated multiplication operator $U_{\mathbf{A}}$ gives the unitary equivalence with the problem corresponding to $\mathbf{A} = 0$. Thus, we have established that 3 implies 1.

Let us finish by giving the proof that 2 gives 3. This requires the use of more advanced techniques, including some positivity results that are beyond the scope of this book. Let $u_{\mathbf{A}}$ be a normalized ground state of $P^N_{\mathbf{A}}$. By elliptic regularity theory we conclude that $u_{\mathbf{A}} \in C^1(\overline{\Omega})$. Define $\rho_{\mathbf{A}} := |u_{\mathbf{A}}|$. The diamagnetic inequality and assumption 2 imply that

$$\lambda_1(\mathbf{A} = 0) = \int_\Omega |\nabla \rho_{\mathbf{A}}|^2 + V\rho_{\mathbf{A}}^2\, dx\,,$$

and so we conclude that $\rho_{\mathbf{A}}$ is a ground state for $P^N_{\mathbf{A}=0}$. By elliptic regularity theory we therefore conclude that $\rho_{\mathbf{A}} \in C^1(\overline{\Omega})$, and the Harnack inequality [GiT, Corollary 9.25] implies that $\rho_{\mathbf{A}} > 0$ in $\overline{\Omega}$.

Thus, we can write

$$e^{i\phi} = \frac{u_{\mathbf{A}}}{\rho_{\mathbf{A}}}\,,$$

for some multivalued function ϕ such that $\nabla\phi$ is welldefined and continuous and $e^{i\phi}$ is of class C^1. Now a calculation gives

$$\begin{aligned}\lambda_1(\mathbf{A}) &= \int_\Omega |(-i\nabla + \mathbf{A})u_{\mathbf{A}}|^2 + V|u_{\mathbf{A}}|^2\, dx \\ &= \int_\Omega |\nabla\rho_{\mathbf{A}}|^2 + V|\rho_{\mathbf{A}}|^2\, dx + \int_\Omega \rho_{\mathbf{A}}^2 |\nabla\phi + \mathbf{A}|^2\, dx\,.\end{aligned} \tag{2.12}$$

So we can conclude that $\mathbf{A} = -\nabla\phi$, from which (2.9) and (2.10) clearly follow. □

Remark 2.1.4.

It is instructive to look at the model on the circle with the magnetic Laplacian $(-i\frac{d}{d\theta} + a)^2$*, where* a *is a real constant corresponding to the magnetic potential. So the magnetic field is zero and the spectrum can easily be found to be described by the sequence* $(n+a)^2$ $(n \in \mathbb{Z})$ *with corresponding eigenfunctions* $\theta \mapsto \exp(in\theta)$.

We immediately see that, confirming the general statement, the ground state energy, which is equal to $\operatorname{dist}(-a, \mathbb{Z})^2$*, increases when a magnetic potential is introduced. We also observe that the multiplicity of the ground state is* 1 *except when* $d(a, \mathbb{Z}) = 1/2$*. We note finally that the number of eigenvalues which are strictly less than* 1*, is* 1 *for* $a = 0$*, and* 2 *for* $a \in]0, 1[$*. This shows that although the ground state energy becomes higher when we introduce a magnetic potential, this is not the case for the second one.*

2.2 Diamagnetic Estimates

One can actually give a more quantitative version of the previous proof in the two-dimensional case. We only consider the situation when Ω is regular, bounded, and simply connected in $\mathbb{R}^2$.

Let us start with the upper bound. Let u_0 be the normalized, positive ground state of $-\Delta + V$. For any ϕ in $C^\infty(\overline{\Omega})$,[1] we can use $u_\phi = u_0 \exp i\phi$ as a test function. Using formula (2.12) and the min-max principle, we have

$$\lambda_1(\mathbf{A}) \leq \lambda_1(\mathbf{A} = \mathbf{0}) + \int_\Omega |\nabla \phi + \mathbf{A}|^2 u_0^2 \, dx \, .$$

This implies

$$\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}) \leq \left(\sup_{x\in\Omega} u_0(x) \right)^2 \left(\inf_{\phi \in C^\infty(\overline{\Omega})} \int_\Omega |\nabla \phi + \mathbf{A}|^2 \, dx \right) .$$

So there exists by Proposition D.2.2 a constant $C_{\Omega,V}$ such that

$$\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}) \leq C_{\Omega,V} \, \| \operatorname{curl} \mathbf{A} \|^2_{H^{-1}(\Omega)} \, , \tag{2.13}$$

where $H^{-1}(\Omega)$ denotes the dual of $H^1_0(\Omega)$.

Let us now look for a lower bound. Again using (2.12), we first write

$$\int_\Omega (|\nabla \rho_{\mathbf{A}}|^2 + V \rho_{\mathbf{A}}^2) \, dx \leq \lambda_1(\mathbf{A}) \, . \tag{2.14}$$

We would like to estimate $\rho_{\mathbf{A}} - u_0$ as $\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})$ tends to 0 .

For this, we first write

$$\rho_{\mathbf{A}} = \alpha u_0 + u^\perp \, , \tag{2.15}$$

where $u^\perp$ is orthogonal to u_0 in $L^2(\Omega)$.

Denoting by $\lambda_2(\mathbf{0})$ the second eigenvalue of the Neumann realization of $-\Delta + V$ in Ω , we immediately deduce from (2.14) the inequality

$$(1 - \alpha^2) \leq \frac{\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})}{\lambda_2(\mathbf{0}) - \lambda_1(\mathbf{0})} \, ,$$

which leads to

$$\| \rho_{\mathbf{A}} - u_0 \|^2 \leq 2 \frac{\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})}{\lambda_2(\mathbf{0}) - \lambda_1(\mathbf{0})} \, . \tag{2.16}$$

This control is not sufficient. We need a control in L^∞ . One can then find, using the Sobolev embedding theorem combined with interpolation, for any $\theta \in \,]0, 1/2[$, a constant C_θ and $p_\theta > 2(1-\theta)/(1-2\theta)$ such that

$$\| \rho_{\mathbf{A}} - u_0 \|_{L^\infty} \leq C \| \rho_{\mathbf{A}} - u_0 \|^\theta_{L^2} \| \rho_{\mathbf{A}} - u_0 \|^{1-\theta}_{W^{1,p_\theta}} \, .$$

[1] In the non simply connected case, one should replace this condition by $\exp i\phi \in C^\infty(\Omega)$.

So, for some constant $C > 0$ depending only on Ω and V, we get

$$\|\rho_{\mathbf{A}} - u_0\|_{L^\infty} \leq C\big(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})\big)^{\frac{\theta}{2}} \Big((\|\nabla\rho_{\mathbf{A}}\|_{L^{p_\theta}} + \|\rho_{\mathbf{A}}\|_{L^{p_\theta}})^{1-\theta} + C\Big)\,.$$

We now use the magnetic Sobolev spaces $W^{2,2}_{\mathbf{A}}(\Omega)$ and get, using the diamagnetic inequality,

$$\|\,|u_{\mathbf{A}}|\,\|_{W^{1,p_\theta}(\Omega)} \leq C\,\|u_{\mathbf{A}}\|_{W^{2,2}_{\mathbf{A}}(\Omega)}\,, \tag{2.17}$$

and

$$\begin{aligned}\|\rho_{\mathbf{A}} - u_0\|_{L^\infty} &\leq C(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}))^{\frac{\theta}{2}} \Big((\|\nabla\rho_{\mathbf{A}}\|_{L^{p_\theta}} + \|\rho_{\mathbf{A}}\|_{L^{p_\theta}})^{1-\theta} + C\Big)\\ &\leq C(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}))^{\frac{\theta}{2}}\Big\{\Big(\|u_{\mathbf{A}}\|_{W^{2,2}_{\mathbf{A}}(\Omega)}\Big)^{1-\theta} + C\Big\}\,.\end{aligned}$$

We next use the estimate

$$\sum_{j,k} \|D_j D_k \psi\|^2_{L^2(\Omega)} \leq 3\,\|\operatorname{curl}\mathbf{A}\|^2_\infty \|\psi\|^2_2 + 2\,\|\mathcal{H}\psi\|^2_2\,, \tag{2.18}$$

which is a particular case of what will be proved in (11.5) and where the D_j's correspond to magnetic differentiation: $D_j = \partial_{x_j} + iA_j$ and $\mathcal{H} = \sum_j D_j^2$.

This leads, for given V and Ω and for any $\theta \in\,]0,1/2[$, to the existence of a constant C_θ such that for any $\mathbf{A}$,

$$\begin{aligned}\|\rho_{\mathbf{A}} - u_0\|_{L^\infty} &\leq C_\theta(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}))^{\frac{\theta}{2}}\\ &\qquad \times \Big(\|\operatorname{curl}\mathbf{A}\|_\infty + (\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})) + C_\theta\Big)\,.\end{aligned} \tag{2.19}$$

We now come back to the end of our control of a lower bound. Starting from

$$\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}) \geq \int_\Omega \rho^2_{\mathbf{A}} |\nabla\phi + \mathbf{A}|^2\,dx \geq \inf_{x\in\Omega} |\rho_{\mathbf{A}}(x)|^2 \|\operatorname{curl}\mathbf{A}\|^2_{H^{-1}(\Omega)} \tag{2.20}$$

(curl is continuous from L^2 into H^{-1}), we obtain the following converse statement of (2.13). For any $\theta \in\,]0,1/2[$, there exists a constant $C_\theta > 0$ such that

$$\begin{aligned}\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}) &\geq \frac{1}{C_\theta}\Big\{1 - C_\theta(\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0}))^{\frac{\theta}{2}}\\ &\qquad \times \Big(\|\operatorname{curl}\mathbf{A}\|_\infty + (\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{0})) + C_\theta\Big)\Big\}\|\operatorname{curl}\mathbf{A}\|^2_{H^{-1}(\Omega)}\,.\end{aligned} \tag{2.21}$$

This is rather good except for the fact that the norm $\|\operatorname{curl}\mathbf{A}\|_{L^\infty}$ (instead of $\|\operatorname{curl}\mathbf{A}\|_{H^{-1}(\Omega)}$) appears in the right-hand side.

Remark 2.2.1.
Replacing $\mathbf{A}$ by $B\mathbf{A}$ and letting $B \to 0$, it is interesting to compare the result of this section with the result we have obtained in Section 1.5.

2.3 Monotonicity of the Ground State Energy for Large Field

As discussed earlier, the diamagnetic inequality (2.1) implies that the ground state energy increases when a magnetic field is applied. Consider a fixed magnetic potential $\mathbf{A}$ and let $\lambda_1(B)$ be the ground state energy of the operator $(-i\nabla + B\mathbf{A})^2 + V$ (either in a domain and with boundary conditions, or on $\mathbb{R}^n$). One can now ask whether the function $B \mapsto \lambda_1(B)$ is monotone non-decreasing for all $B > 0$. This is generally not true. However, for large B, positive results can be obtained.

We consider the Neumann operator $P^N_{B\mathbf{A},V}$ in a domain Ω and assume that Ω, V are such that $P^N_{B\mathbf{A},V}$ has compact resolvent for all (sufficiently large) $B > 0$. So the spectrum of $P^N_{B\mathbf{A},V}$ consists of a sequence of eigenvalues (of finite multiplicity) tending to infinity; in particular, the degeneracy of the ground state energy is finite. Let $B \in \mathbb{R}$ and let n be the degeneracy of the ground state $\lambda^N_1(B)$. For simplicity, from now on we write more briefly

$$\lambda_1(B) = \lambda^N_1(B)\,.$$

Furthermore, we will assume that we have chosen a gauge such that

$$\mathbf{A}(x) \cdot \nu(x) = 0 \quad \text{for all } x \in \partial\Omega\,. \tag{2.22}$$

This implies that the domain $\mathcal{D}(P^N_{B\mathbf{A},V})$ is independent of B; see Remark 1.2.1. We are now in a situation where we can apply analytic perturbation theory to $P^N_{B\mathbf{A},V}$ (see Theorem C.2.2 in Appendix C). Thus, there exist $\epsilon > 0$, n analytic functions

$$(B-\epsilon, B+\epsilon) \ni \theta \mapsto \phi_j(\theta) \in H^2(\Omega) \setminus \{0\}\,,$$

for $j = 1, \dots, n$, and n analytic functions

$$(B-\epsilon, B+\epsilon) \ni \theta \mapsto E_j(\theta) \in \mathbb{R}\,,$$

such that

$$P^N_{\theta\mathbf{A},V}\phi_j(\theta) = E_j(\theta)\phi_j(\theta)\,,$$
$$E_j(B) = \lambda_1(B)\,.$$

We may choose ϵ sufficiently small in order to have the existence of $j_+, j_- \in \{1, \dots, n\}$ such that

$$\begin{aligned} &\text{for } \theta > B: \qquad E_{j_+}(\theta) = \min_{j\in\{1,\dots,n\}} E_j(\theta)\,,\\ &\text{for } \theta < B: \qquad E_{j_-}(\theta) = \min_{j\in\{1,\dots,n\}} E_j(\theta)\,. \end{aligned} \tag{2.23}$$

Define the left and right derivatives of $\lambda_1(B)$:

$$\lambda'_{1,\pm}(B) := \lim_{\epsilon\to 0_\pm} \frac{\lambda_1(B+\epsilon)-\lambda_1(B)}{\epsilon}\,. \tag{2.24}$$

Notice that

$$\lambda'_{1,+}(B) = E'_{j_+}(B)\,, \qquad \lambda'_{1,-}(B) = E'_{j_-}(B)\,; \tag{2.25}$$

in particular, $\lambda'_{1,+}(B) = \lambda'_{1,-}(B)$ if $j_+ = j_-$. Notice also that since λ_1 is the minimum of the E_j, we must have

$$\lambda'_{1,+}(B) \le \lambda'_{1,-}(B)\,. \tag{2.26}$$

Proposition 2.3.1.
For all $B\in\mathbb{R}$, the one-sided derivatives $\lambda'_{1,\pm}(B)$ exist and satisfy

$$\lambda'_{1,\pm}(B) = -2\Re\langle \phi_{j_\pm}\,|\,\mathbf{A}\cdot(-i\nabla + B\mathbf{A})\phi_{j_\pm}\rangle\,.$$

Proof.
By (2.25) we need to prove that

$$E'_{j_\pm}(B) = -2\Re\langle \phi_{j_\pm}\,|\,\mathbf{A}\cdot(-i\nabla + B\mathbf{A})\phi_{j_\pm}\rangle\,.$$

But this result is just first-order perturbation theory (and is called the Feynman–Hellmann formula). □

Using the specific algebraic structure of $P^N_{B\mathbf{A},V}$ and the variational principle, one can prove the following result.

Proposition 2.3.2.
Suppose that Ω is bounded with smooth boundary. Then

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \ge \limsup_{\epsilon\to 0_+} \epsilon^{-1} \liminf_{B\to\infty} \big(\lambda_1(B+\epsilon)-\lambda_1(B)\big)\,, \tag{2.27}$$

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \le \liminf_{\epsilon\to 0_-} \epsilon^{-1} \limsup_{B\to\infty} \big(\lambda_1(B+\epsilon)-\lambda_1(B)\big)\,. \tag{2.28}$$

Proof.
Let $\epsilon > 0$. Then

$$\begin{aligned}\lambda'_{1,+}(B) &= -2\Re\langle \phi_{j_+}(B)\,|\,\mathbf{A}\cdot(-i\nabla + B\mathbf{A})\phi_{j_+}(B)\rangle\\ &= \frac{1}{\epsilon}\langle \phi_{j_+}(B)\,|\,\big(P^N_{(B+\epsilon)\mathbf{A},V} - P^N_{B\mathbf{A},V} - \epsilon^2|\mathbf{A}|^2\big)\phi_{j_+}(B)\rangle\,.\end{aligned}$$

Therefore, the variational principle implies

$$\lambda'_{1,+}(B) \ge \frac{\lambda_1(B+\epsilon)-\lambda_1(B)}{\epsilon} - \epsilon\|\mathbf{A}\|^2_{L^\infty(\Omega)}\,.$$

Upon letting $B\to+\infty$ and then $\epsilon\to 0$, we get (2.27).

The proof of (2.28) is similar (taking $\epsilon < 0$ reverses the inequalities) and is omitted. □

Proposition 2.3.2 implies that the derivative of $\lambda_1(B)$ (for large values of B) can be estimated from knowledge of the asymptotics of $\lambda_1(B)$.

Corollary 2.3.3.
Suppose that Ω is bounded with smooth boundary. Suppose that $\mathbf{A}, V$ are smooth functions and that $\Omega, \mathbf{A}, V$ are such that there exist $\alpha \in \mathbb{R}$ and a function g, satisfying $g \in o(B)$, and

$$\lambda_1(B) = \alpha B + g(B) + o(1)\,, \ \text{as } B \to +\infty\,. \tag{2.29}$$

Then

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) \geq \alpha + \limsup_{\epsilon\to 0_+} \epsilon^{-1} \liminf_{B\to\infty} \big(g(B+\epsilon) - g(B)\big)\,, \tag{2.30}$$

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \alpha + \liminf_{\epsilon\to 0_-} \epsilon^{-1} \limsup_{B\to\infty} \big(g(B+\epsilon) - g(B)\big)\,. \tag{2.31}$$

In particular, if for all $\epsilon \in \,]0,1[\,$,

$$\lim_{B\to\infty} g(B+\epsilon) - g(B) = 0\,, \tag{2.32}$$

then the limits $\lim_{B\to\infty} \lambda'_{1,+}(B)$ and $\lim_{B\to\infty} \lambda'_{1,-}(B)$ exist and

$$\lim_{B\to\infty} \lambda'_{1,+}(B) = \lim_{B\to\infty} \lambda'_{1,-}(B) = \alpha\,. \tag{2.33}$$

Remark 2.3.4.
In the case of the disc (see Chapter 4), the assumption (2.32) *is not satisfied, and we need the more general result of* (2.30) *and* (2.31)*. We will actually in specific cases (see, for example, the proof of Theorem 8.5.1) combine the general idea with other techniques in order to obtain the same conclusion but without knowing as precise an asymptotic result as* (2.29)*. In those cases, one will actually not take ϵ to be small, but rather to grow as a function of B.*

Remark 2.3.5.
Let $\gamma \in [0,1[\,$; then $g(\theta) = \theta^\gamma$ satisfies (2.32)*. Thus, if there exist powers $\gamma_1, \ldots, \gamma_m \in [0,1[$ and $\alpha, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$, such that (as $B \to \infty$)*

$$\lambda_1(B) = \alpha B + \sum_{j=1}^m \alpha_j B^{\gamma_j} + o(1)\,,$$

then Corollary 2.3.3 implies that

$$\lim_{B\to+\infty} \lambda'_{1,\pm}(B) = \alpha\,.$$

Proof of Corollary 2.3.3.
The estimates (2.30) and (2.31) are immediate from Proposition 2.3.2. Furthermore, suppose (2.32) is satisfied. Then the last terms in (2.30) and (2.31) vanish. Therefore, we get (2.33) by recalling (2.26). □

2.4 Kato's Inequality

In order to obtain stronger results on essential self-adjointness than Theorem 1.2.2, a useful tool is the so-called Kato inequality. We present it in the magnetic version in Theorem 2.4.2.

Let us start with the case without magnetic field.

Theorem 2.4.1 (Kato's inequality).
Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ *such that* $\Delta f \in L^1_{\text{loc}}(\mathbb{R}^n)$. *Then we have the inequality*

$$\Delta|f| \geq \Re\{\operatorname{sign}(f)\Delta f\}\,, \tag{2.34}$$

almost everywhere, where $\operatorname{sign} f$ *was defined in* (2.2).

The proof of Theorem 2.4.1 follows the same steps as the proof of Proposition 2.1.2. That is, one first considers smooth functions f and the regularized absolute value

$$|z|_\epsilon = \sqrt{|z|^2 + \epsilon^2} - \epsilon\,,$$

and calculates directly. One then considers a sequence f_δ of smooth approximations to f. Taking first δ and then ϵ to zero, one obtains the desired inequality. We leave the details to the reader (see [ReS, Vol. 2, Section X.4]).

Theorem 2.4.2 (Kato's magnetic inequality).
Let $\mathbf{A} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. *Then, for all* $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ *with* $(-i\nabla + \mathbf{A})^2 f \in L^2_{\text{loc}}(\mathbb{R}^n)$, *we have the inequality*

$$\Delta|f| \geq -\Re\{\operatorname{sign}(f)(-i\nabla + \mathbf{A})^2 f\}\,, \tag{2.35}$$

where $\operatorname{sign} f$ *was defined in* (2.2).

Proof of Theorem 2.4.2.
We only give the proof under the extra regularity assumption, $\mathbf{A} \in C^2(\mathbb{R}^n)$. In that case the assumption $(-i\nabla + \mathbf{A})^2 f \in L^2_{\text{loc}}(\mathbb{R}^n)$ and standard elliptic regularity imply that $f \in H^2_{\text{loc}}(\mathbb{R}^n)$, in particular that

$$\Delta f, \nabla f \in L^1_{\text{loc}}(\mathbb{R}^n)\,. \tag{2.36}$$

Suppose now that u is smooth. Then we can calculate as follows:

$$\nabla|u|_\epsilon = \frac{\Re\{\overline{u}\,\nabla u\}}{\sqrt{|u|^2+\epsilon^2}} = \frac{\Re\{\overline{u}\,(\nabla + i\mathbf{A})u\}}{\sqrt{|u|^2+\epsilon^2}}\,. \tag{2.37}$$

We therefore find

$$\begin{aligned}
\sqrt{|u|^2+\epsilon^2}\,\Delta|u|_\epsilon &= \operatorname{div}(\sqrt{|u|^2+\epsilon^2}\,\nabla|u|_\epsilon) - \big|\nabla|u|_\epsilon\big|^2 \\
&= \Re\big\{\overline{\nabla u}\cdot(\nabla + i\mathbf{A})u + \overline{u}\,\operatorname{div}((\nabla + i\mathbf{A})u)\big\} - \big|\nabla|u|_\epsilon\big|^2 \\
&= \big|(\nabla + i\mathbf{A})u\big|^2 - \big|\nabla|u|_\epsilon\big|^2 \\
&\quad + \Re\big\{i\mathbf{A}\overline{u}\cdot(\nabla + i\mathbf{A})u + \overline{u}\,\operatorname{div}((\nabla + i\mathbf{A})u)\big\}\,.
\end{aligned} \tag{2.38}$$

By (2.37) and using the Cauchy–Schwarz inequality, we get

$$\left|(\nabla + i\mathbf{A})u\right|^2 \geq \left|\nabla |u|_\epsilon\right|^2 .$$

So (2.38) implies that, for a smooth u,

$$\Delta |u|_\epsilon \geq \Re \frac{\overline{u}\,(\nabla + i\mathbf{A})^2 u}{\sqrt{|u|^2 + \epsilon^2}} . \tag{2.39}$$

The end of the proof now follows the same lines as the proof of Proposition 2.1.2, i.e., (2.39) holds for suitably smoothened versions f_δ of f. By taking the limit $\delta \to 0$ followed by the limit $\epsilon \to 0$, in the sense of distributions, one arrives at (2.35).

If one only knows that $\mathbf{A} \in C^1$, it is not immediate to conclude (2.36). The details in this case can be found in [ReS, Vol. 2, Section X.4]. □

2.5 Notes

1. The diamagnetic inequality first appeared in [Kat2]. We refer also to Sections 7.20–7.21 in [LiL] for additional comments on our Section 2.1.
2. The Aharonov–Bohm effect appears as a basic *Gedanken Experiment* in the interpretation of quantum mechanics [AhB]. It says that although the magnetic field is identically zero, the magnetic potential has an effect through the circulation of its magnetic potential along paths that are not homotopic to a point. This can typically occur in two-dimensional domains with holes. Although the first effect considered by Aharonov and Bohm was related to scattering theory, the effect analyzed in the present chapter (Proposition 2.1.3) corresponds to an analogous interpretation. This effect was first mentioned in a paper by Lavine–O'Caroll [LaOC] that gives a heuristic proof of the phenomenon later justified in [He3]. The proof given here is a little simpler than the original one, profiting from the fact that we consider the Neumann condition.
3. Diamagnetism appears also in various other questions. Let us mention its connection with Hardy inequalities and the applications to complex analysis [ChF].
   An interesting model case is the Dirichlet realization of the magnetic Laplacian in $\Omega = \mathbb{R}^2 \setminus \{0\}$ or in $\Omega = \mathrm{D}(0,1) \setminus \{0\}$, when the magnetic potential is
   $$\mathbf{A} = \frac{\alpha}{|x|^2 + |y|^2}(-y, x) .$$
   The corresponding magnetic field vanishes identically (in polar coordinates, we have $\omega_{\mathbf{A}} = \alpha\, d\theta$), but the flux around a positively oriented simple curve around the origin is equal to $2\pi\alpha$. In this example, one can,

for instance, explicitly measure the diamagnetic effect. Using polar coordinates and a unitary transformation, one is indeed led to the analysis of the family of Dirichlet operators (indexed by $m \in \mathbb{Z}$):

$$-\frac{d^2}{dr^2} + \frac{(\alpha - m)^2 - \frac{1}{4}}{r^2}\,,$$

on $L^2(\,]0,+\infty[\,)$ or $L^2(\,]0,1[\,)$.
When $\alpha = 1/2$, the ground state has multiplicity 2 as for the toy model on the circle (see Remark 2.1.4). This has interesting consequences in superconductivity (see in the book [BeR] the contributions of Rubinstein (after Berger–Rubinstein) and Helffer, M. and T. Hoffmann-Ostenhof, and Owen).
There is a long list of references related to this operator—see [LaW] or [Ba] and references therein. These authors also consider cases with holes.

4. In Section 2.2, we can also write, starting from (2.20), the estimate

$$\lambda^N_{\mathbf{A}} - \lambda^N_{\mathbf{0}} \geq \int_\Omega \rho^2_{\mathbf{0}} |\nabla\phi + \mathbf{A}|^2 dx - 2\int_\Omega \rho_{\mathbf{0}} |\rho_A - \rho_{\mathbf{0}}| |\nabla\phi + \mathbf{A}|^2\, dx\,. \quad (2.40)$$

This yields

$$\lambda^N_{\mathbf{A}} - \lambda^N_{\mathbf{0}} \geq \frac{1}{C}\int_\Omega |\nabla\phi + \mathbf{A}|^2 dx - C\|\rho_{\mathbf{A}} - \rho_{\mathbf{0}}\|_2 \|\nabla\phi + \mathbf{A}\|^2_4\,. \quad (2.41)$$

From (2.41) we find, for some constant C (independent of $\mathbf{A}$),

$$\sqrt{\lambda^N_{\mathbf{A}} - \lambda^N_{\mathbf{0}}}\left(\sqrt{\lambda^N_{\mathbf{A}} - \lambda^N_{\mathbf{0}}} + \inf_\phi \|\nabla\phi + \mathbf{A}\|^2_4\right) \geq \frac{1}{C}\inf_\phi \|\nabla\phi + \mathbf{A}\|^2_2\,. \quad (2.42)$$

5. On the subject of diamagnetism, we would also like to mention the contribution of Erdös [Er2, Er3] and his survey [Er4], which has one section devoted to this question and contains many references.
6. For the applications of Kato's inequality to self-adjointness questions, we refer also to [ReS, Vol. 2, Section X.4].

3

Models in One Dimension

Many of the spectral problems that we encounter later in the text can be completely described in terms of one-dimensional problems. These 1D problems have to be analyzed in detail in order to understand the original higher-dimensional questions. That is the objective of the present chapter.

3.1 The Harmonic Oscillator on $\mathbb{R}$

The most important operator in this book is the harmonic oscillator. Actually, the main role will be played by this operator in its realization on a half-axis, but before analyzing that, we recall some results on the case without boundary. Let

$$\mathfrak{h}_0 := -\frac{d^2}{dt^2} + t^2\,, \tag{3.1}$$

whose domain is

$$D(\mathfrak{h}_0) = B^2(\mathbb{R})\,, \tag{3.2}$$

where, for $k \in \mathbb{N}$, $B^k(\mathbb{R})$ is defined as

$$B^k(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : t^p u^{(q)}(t) \in L^2(\mathbb{R})\,,\ \forall p, q \in \mathbb{N} \text{ s.t. } p + q \leq k\}\,. \tag{3.3}$$

The space B^k is equipped with the natural norm,

$$\|u\|_{B^k} = \sum_{p+q\leq k} \|t^p u^{(q)}\|_{L^2}\,. \tag{3.4}$$

Of course, $\mathfrak{h}_0$ is the unique self-adjoint operator associated with the quadratic form on $B^1(\mathbb{R})$:

$$B^1(\mathbb{R}) \ni u \mapsto \int_{-\infty}^{+\infty} \left(|u'(t)|^2 + t^2|u(t)|^2\right) dt\,. \tag{3.5}$$

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This operator has compact resolvent and its spectrum is known explicitly as

$$\sigma(\mathfrak{h}_0) = \{(2j-1)\,,\ j \in \mathbb{N}^*\}\,, \tag{3.6}$$

each eigenvalue having finite multiplicity. Moreover, the corresponding orthonormal basis of eigenfunctions $\{\varphi_j\}$ [associated with the eigenvalue $(2j-1)\,,$ $j \in \mathbb{N}^*$] are generated from the Gaussian

$$\varphi_1(t) = \pi^{-\frac{1}{4}} \exp(-t^2/2)\,,$$

through the action of the creation operator

$$L = -\frac{d}{dt} + t\,,$$

by the formula:

$$\varphi_j = c_j L^{j-1} \varphi_1\,, \tag{3.7}$$

where $c_j \in \mathbb{R}$ is a normalization constant.

One can recognize the Schwartz space $\mathcal{S}(\mathbb{R})$ as

$$\mathcal{S}(\mathbb{R}) = \cap_k B^k(\mathbb{R})\,. \tag{3.8}$$

One can show the following proposition by the difference-quotients method

Proposition 3.1.1.
For all $k \in \mathbb{N}$, the restriction of $\mathfrak{h}_0$ to $B^k(\mathbb{R})$ defines an isomorphism of $B^{k+2}(\mathbb{R})$ onto $B^k(\mathbb{R})$ and hence of $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$.

One way to see this property is to observe that $\mathcal{S}(\mathbb{R})$ can be described as the subspace of the functions in $L^2(\mathbb{R})$, whose coefficients in the orthonormal basis φ_j are in the space of the rapidly decreasing sequences $\mathfrak{s}(\mathbb{N})$.

Remark 3.1.2.
The same proof gives that, for any $\lambda \notin \sigma(\mathfrak{h}_0)$, $\mathfrak{h}_0 - \lambda$ maps $B^{k+2}(\mathbb{R})$ onto $B^k(\mathbb{R})$ for all $k \in \mathbb{N}$ and $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$.

3.2 Harmonic Oscillator on a Half-Axis

Let us begin with the analysis of a family of ordinary differential operators, whose study will play an important role in the analysis of various examples. For $\xi \in \mathbb{R}$, we consider the Neumann realization $\mathfrak{h}^{N,\xi}$ in $L^2(\mathbb{R}^+)$ associated with the operator $-\frac{d^2}{dt^2} + (t+\xi)^2$, i.e.,

$$\mathfrak{h}^{N,\xi} := -\frac{d^2}{dt^2} + (t+\xi)^2, \qquad \mathcal{D}(\mathfrak{h}^{N,\xi}) = \{u \in B^2(\mathbb{R}^+) \mid u'(0) = 0\}\,. \tag{3.9}$$

Here the $B^k(\mathbb{R}^+)$ are defined similarly to the $B_k(\mathbb{R})$, and let us observe that

$$\cap_{k\in\mathbb{N}} B^k(\mathbb{R}^+) = \mathcal{S}(\overline{\mathbb{R}^+})\,.$$

3.2.1 Elementary properties of $\mathfrak{h}^{N,\xi}$

It is easy to see that the operator $\mathfrak{h}^{N,\xi}$ has compact resolvent. This operator is indeed associated with the quadratic form

$$B^1(\mathbb{R}^+) \ni u \mapsto q^{(N,\xi)}(u) := \int_0^{+\infty} \left(|u'(t)|^2 + (t+\xi)^2 |u(t)|^2\right)\, dt\,, \qquad (3.10)$$

where [see (3.3)]

$$B^1(\mathbb{R}^+) := \{u \in L^2(\mathbb{R}^+)\,,\, tu \in L^2(\mathbb{R}^+)\,,\, \text{and}\, u'(t) \in L^2(\mathbb{R}^+)\}\,. \qquad (3.11)$$

So the form domain is $B^1(\mathbb{R}^+)$ and it is a standard exercise to show that the injection of $B^1(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ is compact.

The domain of $\mathfrak{h}^{N,\xi}$ can be determined as

$$D(\mathfrak{h}^{N,\xi}) = \{u \in B^2(\mathbb{R}^+) \,|\, u'(0) = 0\}\,. \qquad (3.12)$$

Moreover, the lowest eigenvalue $\mu(\xi)$ of $\mathfrak{h}^{N,\xi}$ is simple. For this point, the following simple argument can be used. Suppose by contradiction that the eigenspace is (at least) of dimension 2. Then we can find in this eigenspace an eigenstate u such that $u(0) = u'(0) = 0$. But then it should be identically 0 by Cauchy uniqueness. This argument actually gives that all eigenvalues are simple.

Suppose that f is a ground state, normalized in L^2. Using Proposition 2.1.2, we see that $|f|$ has a lower energy. Therefore, we may assume that $f \geq 0$. Assume that $f(t_0) = 0$. By positivity, we must have $f'(t_0) = 0$, and we therefore get by Cauchy uniqueness that $f \equiv 0$. This is in contradiction to the normalization condition. So we see that the ground state will be strictly positive.

We can therefore introduce

Definition 3.2.1.
The function φ_ξ is the unique, strictly positive, L^2-normalized ground state of $\mathfrak{h}^{N,\xi}$ associated to $\mu(\xi)$.

Proposition 3.2.2.
The function $\mathbb{R} \ni \xi \mapsto \mu(\xi)$ is continuous and satisfies

1. *$\mu(\xi) > 0$, for all $\xi \in \mathbb{R}$.*
2. *At $+\infty$ we have the limit,*

$$\lim_{\xi \to +\infty} \mu(\xi) = +\infty\,. \qquad (3.13)$$

3. *At the origin the value is*

$$\mu(0) = 1\,. \qquad (3.14)$$

4. *At $-\infty$ we have*

$$\lim_{\xi \to -\infty} \mu(\xi) = 1\,. \qquad (3.15)$$

5. There exists $\xi_0 \in \mathbb{R}^-$ *such that*

$$\mu(\xi_0) = \inf_{\xi \in \mathbb{R}} \mu(\xi) < 1\,. \tag{3.16}$$

Furthermore, the second eigenvalue $\mu_2(\xi)$ *satisfies*

$$\mu_2(\xi) \geq 1\,, \qquad \forall \xi \in \mathbb{R}\,. \tag{3.17}$$

Proof.
The min-max characterization shows that $\xi \mapsto \mu(\xi)$ is a continuous function. Also, the operator $\mathfrak{h}^{N,\xi}$ is clearly positive, so $\mu(\xi) > 0$.

To prove (3.13), we estimate for $\xi \geq 0$,

$$\mathfrak{h}^{N,\xi} \geq -\frac{d^2}{dt^2} + t^2 + \xi^2 \geq \xi^2,$$

and this gives

$$\mu(\xi) > \xi^2\,, \qquad \forall \xi \geq 0\,. \tag{3.18}$$

Clearly, (3.13) follows from this.

To prove (3.14), we use the fact that the lowest eigenvalue of the Neumann realization of $-\frac{d^2}{dt^2} + t^2$ in $\mathbb{R}^+$ is the same as the lowest eigenvalue of $-\frac{d^2}{dt^2} + t^2$ in $\mathbb{R}$, but restricted to the even functions, which is also the same as the lowest eigenvalue of $-\frac{d^2}{dt^2} + t^2$ in $\mathbb{R}$ [see (3.6)].

Moreover, the derivative of μ at 0 is strictly positive [see (3.21) or (3.29) below]. Hence, using also (3.14), we get the inequality in (3.16). This weak result can also be obtained, without proving that μ is C^1, by minimizing the quadratic form associated with $\mathfrak{h}^{(N)}(\xi)$ over a family of Gaussians.

It is a little more difficult to prove (3.15). For the upper bound, we observe that $\mu(\xi) \leq \lambda(\xi)$, where $\lambda(\xi)$ is the eigenvalue of the Dirichlet realization $\mathfrak{h}^{D,\xi}$. By the monotonicity of $\lambda(\xi)$, it is easy to see that $\lambda(\xi) \geq 1$ and that $\lambda(\xi) \to 1$ as $\xi \to -\infty$. Another way is to use the function $t \mapsto \exp -\frac{1}{2}(t+\xi)^2$ as a test function.

For the converse statement, we start from the eigenfunction $t \mapsto \varphi_\xi(t)$ and show some uniform decay of $\varphi_\xi(t)$ near 0 as $\xi \to -\infty$. It is actually enough to write that for any $\xi < 0$ we have

$$\int_0^{+\infty} |\varphi_\xi'(t)|^2\, dt + \int_0^{+\infty} (t+\xi)^2 |\varphi_\xi(t)|^2\, dt \leq \mu(\xi) \leq \lambda(\xi) \leq \lambda(0) = 3\,.$$

This implies that, for any $R > 0$,

$$\int_0^R |\varphi_\xi(t)|^2\, dt \leq \frac{3}{(R+\xi)^2}\,, \qquad \forall \xi < -R\,. \tag{3.19}$$

We can now use the function $x \mapsto \chi(x-\xi)\varphi_\xi(x-\xi)$, with χ with support in $]0,+\infty[$ such that $\chi = 1$ on $[1,+\infty[$, as a test function for the harmonic oscillator in $\mathbb{R}$—$\mathfrak{h}_0$—and, applying the min-max principle, we obtain the existence of $C > 0$ such that

$$1 \leq \mu(\xi) + \frac{C}{|\xi|}\,, \qquad \forall \xi \leq -C\,. \tag{3.20}$$

This completes the proof of (3.15).

Combining the previous results, we obtain that the infimum in (3.16) is actually a minimum (and strictly less than 1). This finishes the proof of (3.16).

The assertion about the second eigenvalue was proven in [Fr1] and is based on the following idea. If u_2 is indeed an eigenfunction associated with $\mu_2(\xi)$, u_2, which is orthogonal to the strictly positive first eigenfunction, admits at least[1] one zero $x_2(\xi)$ in $\mathbb{R}^+$. So the restriction of u_2 to $]x_2(\xi), +\infty[$ is an eigenfunction of the Dirichlet realization of the harmonic oscillator

$$-\frac{d^2}{dt^2} + (t+\xi)^2\,, \qquad \text{in} \quad]x_2(\xi), +\infty[\,.$$

So $\mu_2(\xi)$ is larger than the lowest eigenvalue of this problem. By monotonicity of the Dirichlet problem (see Example A.2.2), we get that $\mu_2(\xi)$ is higher than the lowest eigenvalue of the harmonic oscillator on $\mathbb{R}$, which is equal to 1. □

3.2.2 Variation of μ and Feynman–Hellmann formula.

It is a little more work (see Appendix C) to show that the eigenfunction depends analytically on ξ. Actually, observing that $\mathfrak{h}^N(\xi)$ is a holomorphic family of type (A) with domain given in (3.12), we have the proposition.

Proposition 3.2.3.
The eigenvalue $\mu(\xi)$ and the corresponding eigenfunction φ_ξ are analytic with respect to ξ.

Properties (3.13), (3.14), and (3.15) are completed by the following proposition.

Proposition 3.2.4.
The eigenvalue μ admits a minimum Θ_0, which is attained at a unique point $\xi_0 < 0$, and satisfies $\Theta_0 \in]0,1[$. Moreover, this minimum is nondegenerate and μ is strictly decreasing on $]-\infty, \xi_0]$ from 1 to Θ_0 and strictly increasing on $[\xi_0, +\infty[$ from Θ_0 to $+\infty$.

Remark 3.2.5.
In Proposition 3.2.8 ahead we will improve the lower bound to $\Theta_0 > 1/2$.

Proof.
Let us first establish the following identity:

$$\mu'(\xi) = [\mu(\xi) - \xi^2]\, \varphi_\xi(0)^2\,. \tag{3.21}$$

[1] Actually, exactly one by Sturm–Liouville theory.

To get (3.21), we use the Feynman–Hellmann formula (3.29) ahead.

$$\mu'(\xi) = 2\int_0^{\infty} (t+\xi)|\varphi_\xi(t)|^2 dt. \tag{3.22}$$

One proceeds by writing $2(t+\xi) = \frac{d}{dt}(t+\xi)^2$ and integrating by parts. This yields

$$\mu'(\xi) = -2\int_0^{\infty} (t+\xi)^2 \varphi_\xi(t)\varphi_\xi'(t)dt - \xi^2\varphi_\xi(0)^2.$$

Using the eigenvalue equation for φ_ξ inside the integral sign yields (3.21).

From (3.21), it follows that, for any critical point ξ_c of μ in $\mathbb{R}^-$,

$$\mu''(\xi_c) = -2\xi_c\varphi_{\xi_c}^2(0) > 0\,. \tag{3.23}$$

So any negative critical point will be a local minimum. Thus, there can be at most one negative critical point. The existence of a negative critical point follows from Proposition 3.2.2. It also follows from (3.23) that any positive critical point will be a local maximum. However, $\lim_{\xi\to+\infty}\mu(\xi) = +\infty$. Combining these two pieces of information, we find that no positive critical point exists. Finally, (3.21) implies that $\mu'(0) > 0$.

In conclusion, there exists a unique minimum $\xi_0 < 0$ such that

$$\Theta_0 = \inf_\xi \mu(\xi) = \mu(\xi_0) < 1\,. \tag{3.24}$$

Moreover, by (3.21),

$$\Theta_0 = \xi_0^2\,. \tag{3.25}$$

□

Remark 3.2.6.
In the case of the Dirichlet realization, we have a similar formula:

$$\lambda'(\xi) = (\varphi_\xi^D)'(0)^2\,, \tag{3.26}$$

where φ_ξ^D is the ground state of $\mathfrak{h}^{D,\xi}$ and this immediately shows the monotonicity. Note that $(\varphi_\xi^D)'(0) \neq 0$ (by the Cauchy uniqueness theorem), so λ' is strictly positive.

This formula is actually a particular case of a general formula (called Rellich's formula) for the Dirichlet realization of a Schrödinger operator.

We now further consider the properties of $\xi \mapsto \mu(\xi)$ and $\varphi_\xi(\cdot)$, which are related to the Feynman–Hellmann formula. We differentiate the identity[2] with respect to ξ:

$$\mathfrak{h}^N(\xi)\varphi(\,\cdot\,;\xi) = \mu(\xi)\varphi(\,\cdot\,;\xi)\,. \tag{3.27}$$

[2] We change a little the notations for $\mathfrak{h}^{N,\xi}$ [this becomes $\mathfrak{h}^N(\xi)$] and φ_ξ [this becomes $\varphi(\,\cdot\,;\xi)$)] in order to have an easier way of writing the differentiation.

We obtain

$$(\partial_\xi \mathfrak{h}^N(\xi) - \mu'(\xi))\varphi(\,\cdot\,;\xi) + (\mathfrak{h}^N(\xi) - \mu(\xi))(\partial_\xi\varphi)(\,\cdot\,;\xi) = 0\,. \tag{3.28}$$

Taking the scalar product with φ_ξ in $L^2(\mathbb{R}^+)$, we obtain the so-called Feynman–Hellmann formula:

$$\mu'(\xi) = \langle\, \partial_\xi \mathfrak{h}^N(\xi)\varphi_\xi \mid \varphi_\xi\rangle = 2\int_0^{+\infty} (t+\xi)|\varphi_\xi(t)|^2 dt\,. \tag{3.29}$$

Taking the scalar product with $(\partial_\xi\varphi)(\cdot;\xi)$, we obtain the identity:

$$\begin{aligned}&\langle(\partial_\xi \mathfrak{h}^N(\xi) - \mu'(\xi))\varphi(\,\cdot\,;\xi) \mid (\partial_\xi\varphi)(\cdot;\xi)\rangle\\ &\quad + \langle(\mathfrak{h}^N(\xi) - \mu(\xi))(\partial_\xi\varphi)(\,\cdot\,;\xi) \mid (\partial_\xi\varphi)(\,\cdot\,;\xi)\rangle = 0\,.\end{aligned} \tag{3.30}$$

In particular, for $\xi = \xi_0$, we obtain

$$\begin{aligned}&\big\langle(\partial_\xi \mathfrak{h}^N(\xi_0)\varphi(\,\cdot\,;\xi_0) \mid (\partial_\xi\varphi)(\cdot;\xi_0)\big\rangle\\ &\quad + \big\langle(\mathfrak{h}^N(\xi_0) - \mu(\xi_0))(\partial_\xi\varphi)(\,\cdot\,;\xi_0) \mid (\partial_\xi\varphi)(\,\cdot\,;\xi_0)\big\rangle = 0\,.\end{aligned} \tag{3.31}$$

We observe that the second term is positive [and with some extra work coming back to (3.28) strictly positive]:

$$\langle(\partial_\xi \mathfrak{h}^N(\xi_0))\varphi)(\,\cdot\,;\xi_0) \mid (\partial_\xi\varphi)(\,\cdot\,;\xi_0)\rangle < 0\,. \tag{3.32}$$

Let us define for later use

$$I_2 := -\frac{1}{4}\langle(\partial_\xi \mathfrak{h}^N(\xi_0))\varphi)(\,\cdot\,;\xi_0) \mid (\partial_\xi\varphi)(\,\cdot\,;\xi_0)\rangle\,. \tag{3.33}$$

Let us differentiate (3.28) once more with respect to ξ:

$$\begin{aligned}&2(\partial_\xi \mathfrak{h}^N(\xi) - \mu'(\xi))\partial_\xi\varphi(\,\cdot\,;\xi) + (\mathfrak{h}^N(\xi) - \mu(\xi))(\partial_\xi^2\varphi)(\,\cdot\,;\xi)\\ &\quad + (\partial_\xi^2 \mathfrak{h}^N(\xi) - \mu''(\xi))\varphi(\,\cdot\,;\xi) = 0\,.\end{aligned} \tag{3.34}$$

Taking the scalar product with φ_ξ and $\xi = \xi_0$, we obtain from (3.32) that

$$\mu''(\xi_0) = 2 + 2\langle\, \partial_\xi \mathfrak{h}^N(\xi_0)\varphi(\,\cdot\,;\xi_0) \mid \partial_\xi\varphi(\,\cdot\,;\xi_0)\rangle < 2\,. \tag{3.35}$$

We also notice that

$$\frac{\mu''(\xi_0)}{2} = 1 - 4I_2\,. \tag{3.36}$$

3.2.3 Formulas for the moments

We set

$$u_0 = \varphi_{\xi_0} \tag{3.37}$$

and introduce the constant

$$\mathcal{C}_1 = \frac{u_0^2(0)}{3}\,. \tag{3.38}$$

The next formula (3.39) is simply a rephrasing of (3.23):

$$\frac{\mu''(\xi_0)}{2} = 3\mathcal{C}_1\sqrt{\Theta_0}\,. \tag{3.39}$$

Define M_k to be the kth moment, centered at $-\xi_0$, of the measure $u_0^2(t)\,dt$:

$$M_k = \int_{\mathbb{R}^+} (t+\xi_0)^k u_0^2(t)\;dt\,. \tag{3.40}$$

These moments can be calculated as follows.

Lemma 3.2.7.
The first moments can be expressed by the following formulas:

$$M_0 = 1\,,\qquad M_1 = 0\,,\qquad M_2 = \frac{\Theta_0}{2}\,,\qquad M_3 = \frac{u_0^2(0)}{6} > 0\,. \tag{3.41}$$

Proof.
Define for $\alpha > 0$ the following L^2-normalized functions:

$$u_{0,\alpha}(t) := \sqrt{\alpha}u_0(\alpha t)\,. \tag{3.42}$$

After a change of variable, we see that

$$\Theta_0 = \int_0^\infty \alpha^{-2}|u_{0,\alpha}'(t)|^2 + (\alpha t+\xi_0)^2 u_{0,\alpha}^2(t)\,dt$$

for all $\alpha > 0$. Differentiating this function at $\alpha = 1$ and using (3.29), we get

$$\int_0^{+\infty} u_0'(t)^2\,dt = \int_0^{+\infty} (t+\xi_0)^2 u_0(t)^2\,dt = \frac{\Theta_0}{2}\,.$$

To have formulas for higher-order moments, we observe the following identities:

$$\begin{aligned}(\mathfrak{h}^N(\xi_0) - \Theta_0)&(2pu_0' - p'u_0)\\ &= u_0\left(p^{(3)} - 4\left((t+\xi_0)^2 - \Theta_0\right))p' - 4(t+\xi_0)p\right)\,,\end{aligned} \tag{3.43}$$

for $p \in C^\infty(\mathbb{R}^+)$, and

$$\langle(\mathfrak{h}^N(\xi_0) - \Theta_0)v\;|u_0\rangle = u_0(0)v'(0)\,, \tag{3.44}$$

for $v \in \mathcal{S}(\overline{\mathbb{R}^+})$. So for any polynomial p, we get

$$\begin{aligned}\langle(\mathfrak{h}^N&(\xi_0) - \Theta_0)(2pu_0' - p'u_0)\;|u_0\rangle\\ &= u_0^2(0)\left(2p(0)(\xi_0^2 - \Theta_0) - p^{(2)}(0)\right) = -u_0^2(0)p^{(2)}(0)\,,\end{aligned}$$

where we have used (3.25). Taking $p(t) = (t+\xi_0)^2$ gives the formula for M_3. □

As an application, we give a lower bound of Θ_0 :

Proposition 3.2.8.
We have $\Theta_0 > 1/2$.

Proof.
By the lower bound to the harmonic oscillator on the half-axis, we can estimate

$$1 < \int_0^\infty |u_0'(t)|^2 + t^2 u_0^2(t)\, dt\,.$$

Inserting

$$t^2 = (t-\xi_0)^2 + 2\xi_0(t-\xi_0) + \xi_0^2\,,$$

applying the formula for M_1, as well as the definition of u_0 , we find

$$1 < \Theta_0 + \xi_0^2 = 2\Theta_0\,.$$

This clearly finishes the proof. □

3.2.4 On the regularized resolvent

Finally, in some more specialized applications, we will need the following mapping properties of the regularized resolvent.

Lemma 3.2.9.
Let $P_0 = \mathfrak{h}^N(\xi_0) - \Theta_0$. *For* $\phi \perp u_0$, *we can define* $P_0^{-1}\phi$ *as the unique solution* f *to*

$$P_0 f = \phi\,, \qquad f \perp u_0\,. \tag{3.45}$$

Let $R_0 \in \mathcal{L}(L^2(\mathbb{R}^+))$ *be the regularized resolvent:*

$$R_0\phi = \begin{cases} 0\,, & \phi \parallel u_0\,,\\ P_0^{-1}\phi, & \phi \perp u_0 \end{cases} \tag{3.46}$$

(and extended by linearity). Then R_0 *is continuous from* $\mathcal{S}(\overline{\mathbb{R}^+})$ *into* $\mathcal{S}(\overline{\mathbb{R}^+})$. *Moreover, for any* $\alpha \geq 0$, R_0 *is continuous in* $L^2(\mathbb{R}^+\,;\, \exp(-\alpha t)\, dt)$.

Sketch of proof.
Using the local regularity up to the boundary of P_0 (as a differential operator), we first get that R_0 sends $\mathcal{S}(\overline{\mathbb{R}^+})$ into $C^\infty(\overline{\mathbb{R}^+})$. For the control at $+\infty$, we then observe, after cutting away from 0 , that the problem is reduced to the question of inverting the harmonic oscillator $\mathfrak{h}_0 - \Theta_0$ on $\mathcal{S}(\mathbb{R})$, which is a standard result (see Remark 3.1.2).

For the last statement, we can also observe that, for $\epsilon_0 > 0$ small enough, the operator

$$\exp(-\epsilon t^2) \cdot \left(-\frac{d^2}{dt^2} + (t+\xi_0)^2 - \Theta_0\right)^{-1} \cdot \exp(\epsilon t^2)\,,$$

which can also be seen as

$$\left(-\left(\frac{d}{dt}+2\epsilon t\right)^2+(t+\xi_0)^2-\Theta_0\right)^{-1},$$

is welldefined for any $\epsilon \in [-\epsilon_0, \epsilon_0]$ and extends continuously to $L^2(\mathbb{R})$ and by using, for example, a global pseudodifferential calculus to $\mathcal{S}(\mathbb{R})$. □

Remark 3.2.10.
By similar techniques, it is easy to prove that the eigenfunctions of $\mathfrak{h}^N(\xi)$ *are in* $\mathcal{S}(\overline{\mathbb{R}^+})$.

3.3 Montgomery's Model

We briefly discuss a one-dimensional model that is useful for the description of the results in dimension 3. Although not directly used later, it can be interesting to see how it appears first in the analysis of "magnetic bottles".

We consider in $\mathbb{R}^2_{x,y}$, and for some parameter $\kappa > 0$ the operator

$$P := -\partial_x^2 + \left(-i\partial_y + \frac{\kappa}{2}x^2\right)^2 . \tag{3.47}$$

The magnetic potential is $\mathbf{A} = (0, \kappa x^2/2)$, and we have

$$\operatorname{curl} \mathbf{A} = \kappa x .$$

So the magnetic field vanishes along the line $\{x = 0\}$. Let us describe the spectral analysis of this model. After a Fourier transform in the y-variable, we first get

$$\hat{P} = -\partial_x^2 + \left(\eta + \frac{\kappa}{2}x^2\right)^2 .$$

This leads to the analysis of the family, parametrized by $\eta \in \mathbb{R}$, of self-adjoint operators on $L^2(\mathbb{R})$:

$$\hat{P}(\eta) = -\frac{d^2}{dx^2} + \left(\eta + \frac{\kappa}{2}x^2\right)^2 .$$

Using a simple dilation, we get

$$\inf \sigma(\hat{P}) = \inf_\eta \inf \sigma\big(\hat{P}(\eta)\big) = \left|\frac{\kappa}{2}\right|^{\frac{2}{3}} \inf_\rho \inf \sigma\left(-\frac{d^2}{dr^2} + (r^2+\rho)^2\right) . \tag{3.48}$$

Let us summarize some properties of the family of operators

$$S(\rho) = -\frac{d^2}{dr^2} + (r^2+\rho)^2 , \tag{3.49}$$

and of the corresponding ground state ψ^ρ.

Theorem 3.3.1.

1. *There exists a unique* $\rho = \rho_{\min}$ *such that*

$$\hat{\nu}_0 := \inf_\rho \inf \sigma\left(-\frac{d^2}{dr^2} + (r^2+\rho)^2\right) = \inf \sigma\left(-\frac{d^2}{dr^2} + (r^2+\rho_{\min})^2\right) . \tag{3.50}$$

2. ψ^ρ *belongs to* $\mathcal{S}(\mathbb{R})$ *and is even.*

Sketch of proof.
Except for the uniqueness of the minimum, the proof of these statements is not too difficult.

It is immediate to see that the lowest eigenvalue $\hat{\nu}(\rho)$ of $S(\rho)$ tends to $+\infty$ as $\rho \to +\infty$. Also, first-order perturbation theory gives that $\hat{\nu}'(\rho) > 0$ for $\rho \geq 0$.

To analyze the behavior as $\rho \to -\infty$, it is suitable to do a dilation $r = \sqrt{-\rho}\, s$, which leads to the analysis of

$$(-\rho)^2 \left(-h^2 \frac{d^2}{ds^2} + (s^2-1)^2\right) ,$$

with $h = (-\rho)^{-3/2}$ small. Semiclassical analysis is therefore relevant and it is easy to show, using harmonic approximation, as it will be explained in Chapter 7, that

$$\hat{\nu}(\rho) \sim 2|\rho|^{\frac{1}{2}} \quad \text{as } \rho \to -\infty .$$

It is then clear that the continuous function $\hat{\nu}(\rho)$ admits a minimum for a strictly negative $\rho_{\min}$, which is given, using the Feynman–Hellmann formula, by

$$\rho_{\min} = -\int r^2 |\psi^{\rho_{\min}}(r)|^2 \, dr . \tag{3.51}$$

□

3.4 A Model Occurring in the Analysis of Infinite Sectors

When looking at problems in an infinite sector of opening angle α, we get—after various changes of variables and a gauge transformation—the question of determining the lower bound for the quadratic form

$$Q_\alpha(u) = \alpha \int_{]0,+\infty[\times]-\frac{1}{2},+\frac{1}{2}[} \left(2t|(\partial_t - i\eta)u|^2 + \frac{1}{2\alpha^2 t}|\partial_\eta u|^2\right) dt d\eta ,$$

where we minimize over the L^2-normalized functions in the variational space

$$\mathcal{V} = \left\{ u \in L^2 \,|\, t^{-1/2}\partial_\eta u \in L^2, \sqrt{t}(\partial_t - i\eta)u \in L^2 \right\} ,$$

with $L^2 = L^2(]0,+\infty[\times]-1/2,+1/2[)$.

When analyzing the asymptotics of the model in a sector with a small opening angle, it is natural to think that, as $\alpha \to 0$, the ground state will be essentially constant in the η variable. This leads us to restrict attention to the subspace of the η invariant functions. On this space the quadratic form Q_α takes the reduced form

$$Q_\alpha^{\rm red}(u) = 2\alpha \int_0^{+\infty} \left(t|\partial_t u|^2 + \frac{t}{12}|u|^2 \right) dt\,.$$

Dividing by α, the differential operator associated with the quadratic form $Q_\alpha^{\rm red}$ is

$$L^{\rm mean} = -2\frac{d}{dt}t\frac{d}{dt} + \frac{t}{6}\,, \qquad (3.52)$$

which consequently plays the role of a "mean-value" operator (after integrating over the angle variable). This operator is easily seen to have compact resolvent. Furthermore, its ground state energy and corresponding ground state are given by

$$\lambda^{\rm mean} = \frac{1}{\sqrt{3}}\,, \qquad u(t) = \frac{1}{\sqrt{3}} \exp\left(-\frac{t}{2\sqrt{3}}\right)\,. \qquad (3.53)$$

3.5 Notes

1. The method of difference-quotients, which can be used in the proof of Proposition 3.1.1, is explained in [LiM] or [GiT].
2. The family $\mathfrak{h}^{N,\xi}$ of model problems on $\mathbb{R}^+$, first appeared in the work of de Gennes. In physics books, one usually gets a partial estimate by considering a problem on the line with the potential $(|t| + \xi)^2$ for $\xi < 0$. The complete mathematical analysis first appeared in the works of Bolley [Bol] and Bolley–Helffer [BolH1], who discovered the role of (3.21). The proof of (3.21) in full generality was obtained by Dauge–Helffer [DaH].
3. The half-axis models, $\mathfrak{h}^{N,\xi}$, reappeared a few years later in the analysis of 2D problems. The first one is the case of the disc [BaPT], which will be discussed in detail later. We should also mention the papers of Sternberg and collaborators (cf. [BeS, dPiFS]), who discovered new relations between the moments, and at about the same time the work of Lu–Pan (see [LuP3] and references therein). The general formula for the moments for $k > 3$ given in [BeS] is

$$4kM_k = (k-1)\big\{4\xi_0^2 M_{k-2} + (k-2)[\xi_0^{k-3}\phi_0^2(0) + (k-3)M_{k-4}]\big\}. \qquad (3.54)$$

4. The estimate of the value of Θ_0 is often carried out in physics texts by using appropriate Gaussians as test functions. That yields quite good upper bounds. The lower bound of Proposition 3.2.8 is given in [LuP3].

We also state the following numerical values from [Bon1, Bon3] (containing also a rigorous analysis of the error):

$$\mathcal{C}_1 \sim 0.254\,, \qquad |\xi_0| \sim 0.768\,. \tag{3.55}$$

Another approach is using (Weber) special functions. If $u(t;\lambda)$ is the unique solution in $\mathbb{R}$ of

$$\left(-\frac{d^2}{dt^2} + t^2 - \lambda\right) u = 0\,,$$

such that $\lim_{t\to+\infty} t^{(1-\lambda)/2} e^{t^2/2} u(t;\lambda) = 1$, then $\Theta_0 = \xi_0^2$ can be recovered as the solution of $(\partial_t u)(\xi, \xi^2) = 0$. This has been implemented in Mathematica by M. Persson and yields the same numerical values as above (3.55). This approach can, in principle, give arbitrarily good precision.

5. The global pseudodifferential calculus is, for example, presented in [He1] and in [Ho] in the usual context of the Weyl calculus.
6. Montgomery's model appears in a problem in sub-Riemannian geometry: *Can we hear the shape of a zero locus* [Mon] (see also [HeM2]). It also appears in the theory of analytic hypoellipticity. Its role in three-dimensional superconductivity problems was first discovered by Pan [Pa6] and then exploited by Helffer–Morame [HeM5, HeM6], see also [HeK1, HeK3]. The fact that there is a unique minimum was numerically observed by Bolley and discussed in Pan–Kwek [PaK]. A proof is given in [He10], who obtains in addition the nondegeneracy of the minimum.
7. Section 3.4 follows [Bon1], who gives much more complete information (complete expansion in powers of α as $\alpha \to 0$). The reader interested in the sector problem can continue his or her reading of this question in Section 4.4, where a universal upper bound of the ground state energy is given.

4

Constant Field Models in Dimension 2: Noncompact Case

Before we analyze the general situation and the possible differences between the Dirichlet problem and the Neumann problem, it is useful—and actually a part of the proof for the general case—to analyze particular model geometries.

4.1 Preliminaries in Dimension 2

Let us consider, in a regular domain Ω in $\mathbb{R}^2$, the Neumann realization (or the Dirichlet realization) of the operator $P_{B\mathbf{F}}$ with

$$\mathbf{F}(x_1, x_2) = \frac{1}{2}(-x_2, x_1)\,. \tag{4.1}$$

The magnetic Neumann boundary condition [as introduced in (3)] is the natural condition considered in the theory of superconductivity; see Chapter 10. We will assume $B > 0$. If the domain is invariant by dilation, one can reduce the analysis to $B = 1$. Let us denote by $\lambda_1^N(B,\Omega)$ and $\lambda_1^D(B,\Omega)$ the infimum of the spectrum of the Neumann and Dirichlet realizations, respectively, of $P_{B\mathbf{F}}$ in Ω. Depending on Ω, this infimum can correspond to an eigenvalue (if Ω is bounded) or to a point in the essential spectrum (for example, if $\Omega = \mathbb{R}^2$ or if $\Omega = \mathbb{R}^{2,+}$). The analysis of basic examples will be crucial for the general study of the problem.

4.2 The Case of $\mathbb{R}^2$

We would like to analyze the spectrum of $P_{B\mathbf{F}}$ more compactly denoted by:

$$S_B := \left(-i\partial_{x_1} - \frac{B}{2}x_2\right)^2 + \left(-i\partial_{x_2} + \frac{B}{2}x_1\right)^2\,. \tag{4.2}$$

We first look at the self-adjoint realization in $\mathbb{R}^2$. Let us briefly show how one can analyze its spectrum. We leave as an exercise to show that the

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spectrum (or the discrete spectrum) of two self-adjoint operators S and T is the same if there exists a unitary operator U such that

$$U(S \pm i)^{-1} U^{-1} = (T \pm i)^{-1} .$$

We note that this implies that U sends the domain of S onto the domain of T.

In order to determine the spectrum of the operator S_B, we perform a succession of unitary conjugations.

Step 1:
The first one, U_1, is defined, for $f \in L^2(\mathbb{R}^2)$, by

$$U_1 f = \exp\left(iB\frac{x_1 x_2}{2}\right) f . \tag{4.3}$$

It satisfies

$$S_B U_1 f = U_1 S_B^1 f , \qquad \forall f \in \mathcal{S}(\mathbb{R}^2) , \tag{4.4}$$

with

$$S_B^1 := (-i\partial_{x_1})^2 + (-i\partial_{x_2} + Bx_1)^2 . \tag{4.5}$$

Remark 4.2.1. .
U_1 is a very special case of what is called a gauge transformation. More generally, as was done in the proof of Proposition 2.1.3 [see (2.11)], we can consider $U = \exp(i\phi)$, where $\exp(i\phi)$ is C^∞.

If $\Delta_{\mathbf{A}} := -\sum_j (-i\partial_{x_j} + A_j)^2$ is a general Schrödinger operator associated with the magnetic potential A, then $U^{-1}\Delta_{\mathbf{A}} U = \Delta_{\tilde{\mathbf{A}}}$, where $\tilde{\mathbf{A}} = \mathbf{A} + \operatorname{grad}\phi$. Here we observe that $\beta := \operatorname{curl}\mathbf{A} = \operatorname{curl}\tilde{\mathbf{A}}$. The associated magnetic field is unchanged in a gauge transformation. We are discussing in this section the very special (but important!) case when the magnetic potential is constant.

Step 2:
We now have to analyze the spectrum of S_B^1. Observing that the operator has constant coefficients with respect to the x_2-variable, we perform a partial Fourier transform with respect to the x_2-variable:

$$U_2 = \mathcal{F}_{x_2 \mapsto \xi_2} , \tag{4.6}$$

and get by conjugation, on $L^2(\mathbb{R}^2_{x_1,\xi_2})$,

$$S_B^2 := (-i\partial_{x_1})^2 + (\xi_2 + Bx_1)^2 . \tag{4.7}$$

Step 3:
We now introduce a third unitary transform U_3:

$$(U_3 f)(y_1, \xi_2) = f(x_1, \xi_2) , \qquad \text{with } y_1 = x_1 + \frac{\xi_2}{B} , \tag{4.8}$$

and we obtain the operator

$$S_B^3 := -\partial_y^2 + B^2 y^2 , \tag{4.9}$$

operating on $L^2(\mathbb{R}^2_{y,\xi_2})$.

The operator S_B^3 depends only on the y-variable. It is easy to find an orthonormal basis of eigenfunctions for this operator. We observe indeed that if $f \in L^2(\mathbb{R}_{\xi_2})$ (with $\|f\| = 1$), and if ϕ_n is the nth eigenfunction of the harmonic oscillator, as defined in (3.7), then

$$(x, \xi_2) \mapsto |B|^{\frac{1}{4}} f(\xi_2) \cdot \phi_n(|B|^{\frac{1}{2}} y)$$

is an eigenfunction corresponding to the eigenvalue $(2n-1)|B|$. So each eigenspace has an infinite dimension. An orthonormal basis of this eigenspace can be given by vectors $e_j(\xi_2)|B|^{1/4}\phi_n(|B|^{1/2}y)$ where $\{e_j\}$ $(j \in \mathbb{N})$ is a basis of $L^2(\mathbb{R})$. We consequently have an empty discrete spectrum, and the infimum of the spectrum (which is also the infimum of the essential spectrum) is B. The eigenvalues (which are of infinite multiplicity!) are usually called Landau levels.

4.3 The Case of $\mathbb{R}^{2,+}$

We now consider the case of the half-space:

$$\mathbb{R}^{2,+} = \{(x_1, x_2) \mid x_1 > 0\} .$$

For the analysis of the spectrum of the Neumann realization of the Schrödinger operator with constant magnetic field S_B in $\mathbb{R}^{2,+}$, we start as in the case of $\mathbb{R}^2$ until we arrive at (4.7). We can take $B = 1$ because a dilation will permit us to get the general case. We can no longer use a translation to arrive to the harmonic oscillator, because $\mathbb{R}^+$ is not invariant by translation. So we arrive at the analysis of the operator

$$S_{B=1}^{2,N} := -\partial_{x_1}^2 + (\xi_2 + x_1)^2 ,$$

on $L^2(\mathbb{R}_{x_1}^+ \times \mathbb{R}_{\xi_2})$.

Rewriting $L^2(\mathbb{R}_{x_1}^+ \times \mathbb{R}_{\xi_2})$ as a Hilbertian integral,

$$L^2(\mathbb{R}_{x_1}^+ \times \mathbb{R}_{\xi_2}) = \int_{\mathbb{R}}^{\oplus} \{L^2(\mathbb{R}_{x_1}^+)\}\, d\xi_2 ,$$

we can rewrite, with the notations of Section 3.2,

$$S_{B=1}^2 = \int_{\mathbb{R}}^{\oplus} \mathfrak{h}^{N,\xi_2}\, d\xi_2 .$$

Then we can use the preliminary study in dimension 1 developed in Section 3.2. Using a standard theorem on the Hilbertian integral of operators, we get, with $\mu_j(\xi)$ denoting the jth eigenvalue of $\mathfrak{h}^N(\xi)$, that

$$\sigma(S_1^{N,\mathbb{R}^{2,+}}) = \overline{\bigcup_j \mu_j(\mathbb{R})} . \tag{4.10}$$

Using the dilation, we get

$$\sigma(S_B^{N,\mathbb{R}^{2,+}}) = |B|\,\sigma(S_1^{N,\mathbb{R}^{2,+}})\,. \tag{4.11}$$

So the bottom of the spectrum is given by

$$\inf \sigma(S_B^{N,\mathbb{R}^{2,+}}) = \inf_{\xi} \mu(\xi)\,|B| = \Theta_0\,|B|\,. \tag{4.12}$$

Similarly, for the Dirichlet realization, we find

$$\inf \sigma(S_B^{D,\mathbb{R}^{2,+}}) = \inf_{\xi\in\mathbb{R}} \lambda(\xi)\,|B| = |B|\,. \tag{4.13}$$

4.4 The Case of an Infinite Sector

We consider the Neumann realization of the Schrödinger operator with $B = 1$ in a sector:

$$\Omega_\alpha := \left\{(x_1,x_2)\in\mathbb{R}^2 \mid |x_2| \leq \tan\left(\frac{\alpha}{2}\right) x_1\right\}\,,$$

i.e., $P^N_{\mathbf{F},\Omega_\alpha}$, where $\mathbf{F}(x_1,x_2) := (-x_2/2, x_1/2)$ generates a constant magnetic field. We define

$$\mu^{\mathrm{sect}}(\alpha) := \inf \sigma(P^N_{\mathbf{F},\Omega_\alpha})\,. \tag{4.14}$$

One can first show, using Persson's theorem (see Appendix B), that the infimum of the essential spectrum is equal to Θ_0 :

$$\inf \sigma_{\mathrm{ess}}(P^N_{\mathbf{F},\Omega_\alpha}) = \Theta_0\,. \tag{4.15}$$

So the question is to know if there exists an eigenvalue below the essential spectrum. A simple result is:

$$\lim_{\alpha\to 0} \frac{\mu^{\mathrm{sect}}(\alpha)}{\alpha} = \frac{1}{\sqrt{3}}\,. \tag{4.16}$$

Computing the energy of the quasimode u_α ,

$$\Omega_\alpha \ni (x,y) = (\rho\cos\phi, \rho\sin\phi) \mapsto u_\alpha(x,y) := c\exp\left(i\frac{\rho^2\beta^2\phi}{2}\right)\exp\left(-\frac{\beta\rho^2}{4}\right)\,,$$

with $\beta = \alpha/\sqrt{3+\alpha^2}$ and $c = \beta^{1/4}\alpha^{-1/2}$ chosen such that the L^2-norm in the sector is 1 , one has the universal estimate

$$\mu^{\mathrm{sect}}(\alpha) \leq \frac{\alpha}{\sqrt{3+\alpha^2}}\,, \tag{4.17}$$

which gives (4.16). This also answers the question of the existence of an eigenvalue below Θ_0 under the condition that

$$\frac{\alpha}{\sqrt{3+\alpha^2}} < \Theta_0\,.$$

Let us mention three interesting conjectures.

Conjecture 4.4.1.
For any $\alpha \in]0,\pi[$ *, there exists at least one eigenvalue* $\mu^{\mathrm{sect}}(\alpha)$ *below* Θ_0 *.*

At present it is only known that $\mu^{\mathrm{sect}}(\alpha) < \Theta_0$ for all $\alpha \in]0,\pi/2]$.

Conjecture 4.4.2.
The map $]0,\pi[\ni \alpha \mapsto \mu^{\mathrm{sect}}(\alpha)$ *is monotonically increasing.*

Conjecture 4.4.3.
For $\alpha \in [\pi, 2\pi[$ *, the infimum of the spectrum is* Θ_0 *.*

Very persuasive numerical evidence has been obtained[1] by Bonnaillie-Noël (see Fig. 4.1).

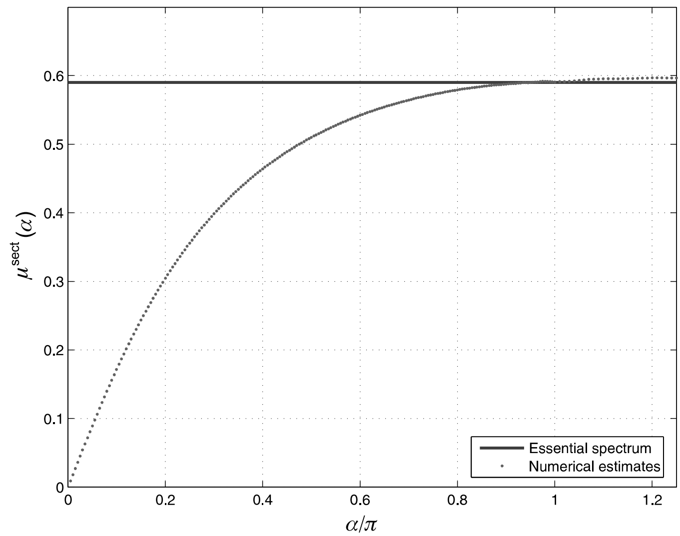

Figure 4.1. $\mu^{\mathrm{sect}}(\alpha)$ vs. α/π for $\alpha \in [0, 1.25\pi]$.

We end by giving the following theorem, whose proof is left to the reader.

Theorem 4.4.4.
Suppose that $\mu^{\mathrm{sect}}(\alpha) < \Theta_0$ *and that* ψ *is an eigenfunction of* $P^N_{\mathbf{F},\Omega_\alpha}$ *with eigenvalue* $\lambda < \Theta_0$ *. Then there exist positive constants* ϵ *and* C *such that*

$$\int_{\Omega_\alpha} e^{\epsilon|x|}\big\{|\psi(x)|^2 + |p_{\mathbf{F}}\psi(x)|^2\big\}\,dx \leq C\|\psi\|_2^2 \,. \tag{4.18}$$

[1] The authors thank Virginie Bonnaillie-Noël for allowing them to reproduce her graph here.

4.5 Notes

1. The case of $\mathbb{R}^2$ is completely standard and appears at the very beginning of quantum mechanics [Fo, Lan]. The eigenvalues are called the Landau levels.
2. The theory of Hilbertian integrals can be found in [ReS, Vol. IV, Chapter XIII], and we mainly use their Theorem XIII.85.
3. The case of $\mathbb{R}^{2,+}$ appears in the analysis of surface superconductivity by de Gennes. It is treated mathematically in [Bol] and later in [LuP5], which contains in addition an analysis of the L^∞-spectrum (both for $\mathbb{R}^2$ and $\mathbb{R}^{2,+}$), which is useful in the blow-up arguments (see Chapter 11).
4. Section 4.4 can be seen as the natural follow-up to Section 3.4.
5. After preliminary results devoted to the case $\Omega = \mathbb{R}^+ \times \mathbb{R}^+$ and obtained by [Ja] (using a result by [Al1]) and [Pa1], a more systematic analysis in angular sectors was carried out by Bonnaillie in [Bon1, Bon2].
6. The limiting behavior for small angles (4.16) is proved in [Bon1, Bon2]. We do not show the lower bound, which is more difficult. For earlier, nonrigorous work, see [SP].
7. The construction of quasimodes follows an idea of Bonnaillie–Fournais published in [Bon1], which is reminiscent of constructions of [BrDFM].

5

Constant Field Models in Dimension 2: Discs and Their Complements

5.1 Introduction

In this section, we consider the disc and the complement of the disc in $\mathbb{R}^2$. We will study the operator

$$P_{B\mathbf{F}} = (-i\nabla + B\mathbf{F})^2,$$

with Neumann boundary conditions, and where $\operatorname{curl}\mathbf{F} = 1$.

The constant curvature models—the disc and its complement—are important for the precise understanding of general domains Ω. Consider a small neighborhood of a boundary point x_0. As a first approximation, we may consider approximating by a straight boundary. This becomes the half-space model considered in the previous chapter. A better approximation is obtained by considering the disc (or complement of a disc) with curvature equal to the curvature of $\partial\Omega$ at x_0. Thus, even if one is only interested in bounded domains Ω, one needs to consider both bounded and unbounded constant curvature models.

First, we state a result for the case of Dirichlet boundary conditions. As $R\sqrt{B}$ becomes large, the following asymptotics holds:

$$\lambda_1^D(B, \mathrm{D}(0,R)) - B \sim 2^{\frac{3}{2}}\pi^{-\frac{1}{2}}B^{\frac{3}{2}}R\exp\left(-\frac{BR^2}{2}\right). \tag{5.1}$$

We will actually not use this result in what follows. Our main concern is indeed the case of Neumann boundary conditions. We will be interested in obtaining a fine asymptotic formula for the ground state energy. In the literature, one can find the following:

Proposition 5.1.1.

$$\lambda_1^N(B, \mathrm{D}(0,1)) = \Theta_0 B - \mathcal{C}_1\sqrt{B} + \mathcal{O}(1). \tag{5.2}$$

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However, in order to apply Corollary 2.3.3 to prove the monotonicity of the ground state energy, we need better control of the remainder. That will be the result of Theorem 5.3.1.

We will start with an accurate discussion of an approximating model. The analysis of this approximating model in Section 5.2 is much more technical than most of this book. The reader not particularly interested in this calculation may safely skip Section 5.2.

5.2 A Perturbed Model

In order to understand the effect of the boundary curvature for two-dimensional models, we need to consider a perturbed version of $\mathfrak{h}^{N,\xi}$. Let η be small enough, and we will choose for definiteness

$$\eta \in]0, 1/100[\,. \tag{5.3}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a standard cutoff function

$$\chi(t) = 1, \text{ for } |t| \leq 1\,, \qquad 0 \leq \chi \leq 1\,, \quad \text{and} \quad \operatorname{supp} \chi \subset [-2, 2]\,,$$

and define the function ℓ on $\mathbb{R}^+$ by

$$\ell(\tau) := \tau \chi(2B^{-\eta}\tau). \tag{5.4}$$

Notice that ℓ depends on B and η, though we will not include this dependence in the notation.

We observe that

$$\ell(\tau) = \tau\,, \qquad \text{if } \tau \leq B^\eta/2\,, \tag{5.5}$$

$$\ell(\tau) = 0\,, \qquad \text{if } \tau \geq B^\eta\,, \tag{5.6}$$

and that

$$0 \leq \ell \leq B^\eta\,. \tag{5.7}$$

Consider, for $\delta, B \geq B_0$ (with B_0 large enough[1]), the quadratic form $q_{\eta,\delta,B}$,

$$\begin{aligned} q_{\eta,\delta,B}[\phi] = \int_0^\infty & \left(1 - \frac{\ell(\tau)}{\sqrt{B}}\right)^{-1} \left((\tau + \xi_0) + B^{-\frac{1}{2}} \left(\delta - \frac{\ell(\tau)^2}{2}\right)\right)^2 |\phi(\tau)|^2 \\ & + \left(1 - \frac{\ell(\tau)}{\sqrt{B}}\right) |\phi'(\tau)|^2 \, d\tau\,, \end{aligned} \tag{5.8}$$

defined on the space $B^1(\mathbb{R}^+)$.

[1] The condition $B_0^{\frac{1}{2}-\eta} \geq 2$ is enough at this stage. It allows us to have the lower bound $(1 - \ell/\sqrt{B}) \geq 1/2$.

This closed quadratic form defines an unbounded operator on

$$L^2\left(\mathbb{R}^+;\left(1-\frac{\ell(\tau)}{\sqrt{B}}\right)d\tau\right)$$

(with Neumann boundary condition at 0). Denote by $\{\lambda_j(q_{\eta,\delta,B})\}_{j\in\mathbb{N}}$ the increasing sequence of eigenvalues of the self-adjoint operator associated with $q_{\eta,\delta,B}$. Then we have the following result.

Proposition 5.2.1.
Let $\eta\in]0,1/100[$ *. There exist positive constants* C*,* c_0*,* M*, and* B_0 *such that if* $B\geq B_0$ *, then*

- *If* $|\delta|\geq M\,B^{\frac{1}{4}+\eta}$ *, then*

$$\lambda_1(q_{\eta,\delta,B})\geq\Theta_0+c_0\min\left(1,\frac{\delta^2}{B}\right). \tag{5.9}$$

- *If* $|\delta|\leq M\,B^{\frac{1}{4}+\eta}$ *, then*

$$\lambda_2(q_{\eta,\delta,B})\geq 1\,, \tag{5.10}$$

and

$$\left|\lambda_1(q_{\eta,\delta,B})-\left(\Theta_0-\mathcal{C}_1B^{-\frac{1}{2}}+\lambda_2(\delta)B^{-1}\right)\right|\leq C\left(\frac{1+|\delta|^3}{B^{\frac{3}{2}}}\right), \tag{5.11}$$

where λ_2 *is given by the expression*

$$\lambda_2(\delta)=3\mathcal{C}_1\sqrt{\Theta_0}\big((\delta-\check{\delta}_0)^2+\mathcal{C}_0\big)\,, \tag{5.12}$$

for universal constants $\check{\delta}_0$ *and* $\mathcal{C}_0$ *and with* $\mathcal{C}_1$ *from* (3.38).

Remark 5.2.2.
In particular, it follows from Proposition 5.2.1 that

$$\inf_{\delta\in\mathbb{R}}\lambda_1(q_{\eta,\delta,B})=\Theta_0-\mathcal{C}_1B^{-1/2}+3\mathcal{C}_1\sqrt{\Theta_0}\mathcal{C}_0B^{-1}+\mathcal{O}(B^{-\frac{3}{2}})\,. \tag{5.13}$$

Proof.
Recall the unperturbed quadratic form $q^{(N,\xi)}$ from (3.10). Using (5.7), we get, for all $\phi\in B^1(\mathbb{R}^+)$,

$$\begin{aligned}q_{\eta,\delta,B}[\phi]&\geq(1-B^{\eta-1/2})\\&\quad\times\int_0^{+\infty}|\phi'(\tau)|^2+\left|\tau+\xi_0+B^{-\frac{1}{2}}\delta-B^{-\frac{1}{2}}\frac{\ell(\tau)^2}{2}\right|^2|\phi(\tau)|^2\,d\tau\,.\end{aligned} \tag{5.14}$$

We then use the inequality

$$\left|\tau+\xi_0+B^{-\frac{1}{2}}\delta-B^{-\frac{1}{2}}\frac{\ell(\tau)^2}{2}\right|^2\geq(1-\epsilon)|\tau+\xi_0+B^{-\frac{1}{2}}\delta|^2-\frac{1}{\epsilon}B^{-1}\ell(\tau)^4\,,$$

with $\epsilon = B^{2\eta-\frac{1}{2}}$ and get the existence of a constant $C > 0$, such that for all $\delta \in \mathbb{R}$,

$$q_{\eta,\delta,B}[\phi] \geq (1 - CB^{2\eta-1/2}) q^{(N,\xi_0+\delta/\sqrt{B})}[\phi] - CB^{2\eta-1/2} \int |\phi(\tau)|^2 \, d\tau \,. \quad (5.15)$$

This implies by the variational characterization of eigenvalues (see Appendix A) that, for any $j \geq 1$,

$$\lambda_j(q_{\eta,\delta,B}) \geq (1 - CB^{2\eta-1/2}) \, \mu_j(\xi_0 + \delta/\sqrt{B}) - CB^{2\eta-1/2} \,. \quad (5.16)$$

In particular, using the nondegeneracy result (3.23), which implies the existence of $c_0 > 0$ such that

$$\mu(\xi) \geq \Theta_0 + c_0 \min\left(|\xi - \xi_0|^2, 1\right) \,,$$

we obtain

$$\lambda_1(q_{\eta,\delta,B}) \geq (1 - CB^{2\eta-1/2}) \left[\Theta_0 + c_0 \min(1, \delta^2/B)\right] - CB^{2\eta-1/2} \,.$$

This again implies that (for M sufficiently large)

$$\lambda_1(q_{\eta,\delta,B}) \geq \Theta_0 + c_0' \min(1, \delta^2/B), \quad \text{for all } |\delta| \geq MB^{\frac{1}{4}+\eta} \,, \quad (5.17)$$

where $c_0' > 0$, and thus proves (5.9).

From now on, we consider only values of δ such that $|\delta| \leq MB^{\frac{1}{4}+\eta}$; in particular, $\delta/\sqrt{B}$ is bounded.

To get the reverse inequality to (5.16), let us consider an eigenfunction f_j of $\mathfrak{h}^{N,\xi}$ with eigenvalue $\mu_j(\xi)$. As mentioned in Remark 3.2.10, f_j decays exponentially at $+\infty$. Applying Proposition A.1.3 to the subspace $V := \text{Span}\{f_1, \ldots, f_n\}$, where the f_j's are taken with $\xi = \xi_0 + \delta/\sqrt{B}$, we get

$$\lambda_j(q_{\eta,\delta,B}) \leq \mu_j(\xi_0 + \delta/\sqrt{B}) + CB^{2\eta-1/2} \,, \quad (5.18)$$

where the constant C is uniform for $|\delta| \leq MB^{\frac{1}{4}+\eta}$. Combining (5.16) and (5.18), we find

$$\left|\lambda_j(q_{\eta,\delta,B}) - \mu_j(\xi_0 + \delta/\sqrt{B})\right| \leq CB^{2\eta-1/2} \,. \quad (5.19)$$

In particular, we can conclude, using (3.17), that $q_{\eta,\delta,B}$ admits exactly one eigenvalue below 1 for B sufficiently large. This proves (5.10).

The self-adjoint operator $\mathfrak{h}(\delta, B)$ associated with $q_{\eta,\delta,B}$ [on the Hilbert space $L^2(\mathbb{R}^+; (1 - \ell/\sqrt{B})d\tau)$] is the following differential operator (with Neumann boundary condition):

$$\begin{aligned} \mathfrak{h}(\delta, B) = &- \left(1 - \frac{\ell(\tau)}{\sqrt{B}}\right)^{-1} \frac{d}{d\tau} \left(1 - \frac{\ell(\tau)}{\sqrt{B}}\right) \frac{d}{d\tau} \\ &+ \left(1 - \frac{\ell(\tau)}{\sqrt{B}}\right)^{-2} \left((\tau + \xi_0) + B^{-\frac{1}{2}} \left(\delta - \frac{\ell(\tau)^2}{2}\right)\right)^2 . \end{aligned} \quad (5.20)$$

We will write an explicit test function for $\mathfrak{h}(\delta, B)$ in (5.25), giving $\lambda_1(q_{\eta,\delta,B})$ up to an error of order $o(B^{-1})$.

For $f \in \mathcal{S}(\overline{\mathbb{R}^+})$ we have

$$\left\|\mathfrak{h}(\delta, B)f - \left(\mathfrak{k}_0 + B^{-\frac{1}{2}}\mathfrak{k}_1 + B^{-1}\mathfrak{k}_2\right)f\right\|_{L^2(\mathbb{R}^+)} = \mathcal{O}\left(\frac{1+|\delta|^2}{B^{3/2}}\right), \tag{5.21}$$

with

$$\begin{aligned}
\mathfrak{k}_0 &:= -\frac{d^2}{d\tau^2} + (\tau+\xi_0)^2 \quad (= \mathfrak{h}^{N,\xi_0}),\\
\mathfrak{k}_1 &:= \frac{d}{d\tau} + 2(\tau+\xi_0)\left(\delta - \frac{\tau^2}{2}\right) + 2\tau(\tau+\xi_0)^2,\\
\mathfrak{k}_2 &:= \tau\frac{d}{d\tau} + \left(\delta - \frac{\tau^2}{2}\right)^2 + 4\tau(\tau+\xi_0)\left(\delta - \frac{\tau^2}{2}\right) + 3\tau^2(\tau+\xi_0)^2.
\end{aligned} \tag{5.22}$$

Notice, as part of the argument leading to (5.21), that ℓ can be replaced by τ up to errors in $\mathcal{O}(B^{-\infty})$, since

$$\int_0^\infty (\ell(\tau)-\tau)^2|f^{(k)}(\tau)|^2\,d\tau = \mathcal{O}(B^{-\infty}),$$

(and similarly for expressions like $\int_0^\infty \ell'(\tau)^2|f'(\tau)|^2\,d\tau$).

We will only consider $\mathfrak{k}_1$ and $\mathfrak{k}_2$ as differential operators acting on $\mathcal{S}(\overline{\mathbb{R}^+})$—we do not consider their possible self-adjoint extensions in a given Hilbert space.

Let u_0 be the known normalized ground state of $\mathfrak{h}^{N,\xi_0}$ with eigenvalue Θ_0. Let R_0 be the regularized resolvent considered in Lemma 3.2.9. Let $\lambda_1(\delta)$ and $\lambda_2(\delta)$ be given by

$$\lambda_1 := \langle u_0 \,|\, \mathfrak{k}_1 u_0\rangle \tag{5.23}$$

and

$$\lambda_2 := \lambda_{2,1} + \lambda_{2,2}, \text{ with } \lambda_{2,1} := \langle u_0 \,|\, \mathfrak{k}_2 u_0\rangle,\ \lambda_{2,2} := \langle u_0 \,|\, (\mathfrak{k}_1 - \lambda_1)u_1\rangle.$$

Here the inner products are the usual inner products in $L^2(\mathbb{R}^+, d\tau)$. The functions u_1 and u_2 are given by

$$u_1 := -R_0(\mathfrak{k}_1 - \lambda_1)u_0, \quad u_2 := -R_0\big\{(\mathfrak{k}_1-\lambda_1)u_1 + (\mathfrak{k}_2 - \lambda_2)u_0\big\}. \tag{5.24}$$

Notice that $u_0 \in \mathcal{S}(\overline{\mathbb{R}^+})$ and that, by Lemma 3.2.9, R_0 maps $\mathcal{S}(\overline{\mathbb{R}^+})$ (continuously) to itself. Therefore, u_0, u_1, u_2 (and their derivatives) are rapidly decreasing functions on $\mathbb{R}^+$. Furthermore, each function satisfies the Neumann boundary condition at 0.

Our trial state is defined by

$$\psi := u_0 + B^{-\frac{1}{2}}u_1 + B^{-1}u_2. \tag{5.25}$$

We will need to make explicit how the objects above depend on δ. We can rewrite $\mathfrak{k}_1$ as

$$\mathfrak{k}_1 := \frac{d}{d\tau} + (2\delta - \xi_0^2)(\tau + \xi_0) + (\tau + \xi_0)^3. \tag{5.26}$$

From this, (3.38), and (3.41), it is immediate that

$$\lambda_1 = -\mathcal{C}_1\,. \tag{5.27}$$

In particular, λ_1 is independent of δ. Also, for some $u_{1,0}, u_{1,1} \in \mathcal{S}(\overline{\mathbb{R}^+})$,

$$u_1 = \delta u_{1,1} + u_{1,0}\,. \tag{5.28}$$

Notice that

$$\lambda_{2,2} = -\langle u_0 \,|\, (\mathfrak{k}_1 - \lambda_1) R_0 (\mathfrak{k}_1 - \lambda_1) u_0 \rangle\,.$$

Hence, we get that $\lambda_2(\delta)$ is a quadratic polynomial as a function of δ. We find that the coefficient of δ^2 is

$$1 - 4\langle u_0 \,|\, (\tau + \xi_0) R_0 (\tau + \xi_0) u_0 \rangle\,. \tag{5.29}$$

Therefore, also u_2 is quadratic in δ,

$$u_2 = \delta^2 u_{2,2} + \delta u_{2,1} + u_{2,0}\,, \tag{5.30}$$

with $u_{2,2}, u_{2,1}, u_{2,0} \in \mathcal{S}(\overline{\mathbb{R}^+})$.

A calculation (using that $|\delta|/\sqrt{B} \le 1$, (5.21) and the decay of the involved functions) gives

$$\begin{aligned}
&\big\|\big\{\mathfrak{h}(\delta, B) - \big(\Theta_0 + \lambda_1 B^{-\frac{1}{2}} + \lambda_2 B^{-1}\big)\big\}\psi\big\| \\
&= \big\|\big\{\mathfrak{k}_0 - \Theta_0 + (\mathfrak{k}_1 - \lambda_1) B^{-\frac{1}{2}} + (\mathfrak{k}_2 - \lambda_2) B^{-1}\big\}\psi\big\| + \mathcal{O}\left(\frac{1 + |\delta|^2}{B^{3/2}}\right) \\
&= \big\|B^{-\frac{3}{2}}[(\mathfrak{k}_1 - \lambda_1) u_2 + (\mathfrak{k}_2 - \lambda_2) u_1] + B^{-2}(\mathfrak{k}_2 - \lambda_2) u_2\big\| + \mathcal{O}\left(\frac{1 + |\delta|^2}{B^{3/2}}\right) \\
&= \mathcal{O}\left(\frac{1 + |\delta|^3}{B^{3/2}}\right)\,.
\end{aligned} \tag{5.31}$$

Here $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R}^+; (1 - \frac{\ell}{\sqrt{B}}) d\tau)$. Furthermore,

$$\|\psi\|_{L^2(\mathbb{R}^+; (1 - \frac{\ell}{\sqrt{B}}) d\tau)} = 1 + \mathcal{O}\left(\frac{1 + |\delta|}{B^{\frac{1}{2}}}\right)\,. \tag{5.32}$$

By the spectral Theorem 1.3.1, we get, combining (5.31) and (5.32),

$$\operatorname{dist}\big(\Theta_0 + \lambda_1 B^{-\frac{1}{2}} + \lambda_2 B^{-1}, \sigma(\mathfrak{h}(\delta, B))\big) = \mathcal{O}\left(\frac{1 + |\delta|^3}{B^{3/2}}\right)\,.$$

Since $\mathfrak{h}(\delta, B)$ has only one eigenvalue below 1, we have therefore proved that

$$\lambda_1(q_{\eta,\delta,B}) = \Theta_0 + \lambda_1 B^{-\frac{1}{2}} + \lambda_2 B^{-1} + \mathcal{O}\left(\frac{1+|\delta|^3}{B^{3/2}}\right). \tag{5.33}$$

In view of (3.28), (3.35), and (3.36), we recognize that the coefficient of δ^2 in $\lambda_2(\delta)$ given in (5.29), can be written as

$$\frac{\mu''(\xi_0)}{2} = 3\mathcal{C}_1|\xi_0| = 3\mathcal{C}_1\sqrt{\Theta_0} = 1 - 4I_2\,.$$

Notice that $3\mathcal{C}_1\sqrt{\Theta_0} > 0$, which expresses the nondegeneracy of the minimum of μ at ξ_0. Since λ_2 is quadratic in δ, there exist therefore $\check{\delta}_0$ and $\mathcal{C}_0 \in \mathbb{R}$ such that

$$\lambda_2(\delta) = 3\mathcal{C}_1\sqrt{\Theta_0}\big((\delta - \check{\delta}_0)^2 + \mathcal{C}_0\big)\,.$$

The constants $\check{\delta}_0$ and $\mathcal{C}_0$ are indeed defined by

$$3\mathcal{C}_1\sqrt{\Theta_0}\,\mathcal{C}_0 = \min_{\delta} \lambda_2(\delta) = \lambda_2(\check{\delta}_0)\,.$$

Inserting this information in (5.33) finishes the proof of Proposition 5.2.1. □

5.3 Asymptotics of the Ground State Energy for the Disc

In this subsection, we will state a precise asymptotic estimate that will help us settle the question of diamagnetism for the disc. We remind the reader that the spectral parameters $\mathcal{C}_1$, Θ_0, and ξ_0 were introduced in (3.24), (3.25), and (3.38).

Theorem 5.3.1 (Eigenvalue asymptotics for the disc)**.**
Suppose that $\Omega = \mathrm{D}(0,1)$ is the unit disc. Define $\delta(m,B)$, for $m \in \mathbb{Z}$, $B > 0$, by

$$\delta(m,B) := m - \frac{B}{2} - \xi_0\sqrt{B}. \tag{5.34}$$

Then there exist (computable) constants $\mathcal{C}_0$ and $\check{\delta}_0 \in \mathbb{R}$ such that if

$$\widehat{\Delta}_B := \inf_{m\in\mathbb{Z}} |\delta(m,B) - \check{\delta}_0|\,, \tag{5.35}$$

then, for all $\eta > 0$,

$$\lambda_1(B) = \Theta_0 B - \mathcal{C}_1\sqrt{B} + 3\mathcal{C}_1\sqrt{\Theta_0}\big(\widehat{\Delta}_B^2 + \mathcal{C}_0\big) + \mathcal{O}(B^{\eta-\frac{1}{2}})\,. \tag{5.36}$$

When $\Omega = \mathbb{R}^2 \setminus \overline{\mathrm{D}(0,1)}$ is the exterior of the unit disc, a similar statement holds. Let us define

$$\delta^{\mathrm{ext}}(m,B) := -m - \frac{B}{2} + \xi_0\sqrt{B}, \qquad \widehat{\Delta}_B^{\mathrm{ext}} := \inf_{m\in\mathbb{Z}} |\delta^{\mathrm{ext}}(m,B) - \check{\delta}_0|\,.$$

Then, with the same constants as above,

$$\lambda_1(B) = \Theta_0 B + \mathcal{C}_1\sqrt{B} + 3\mathcal{C}_1\sqrt{\Theta_0}\big((\Delta_B^{\mathrm{ext}})^2 + \mathcal{C}_0\big) + \mathcal{O}(B^{\eta-\frac{1}{2}})\,. \tag{5.37}$$

Remark 5.3.2.
As the proof will show, the constants $\mathcal{C}_0, \check{\delta}_0$ are the constants introduced in (5.12)*. We recall that they can be expressed in terms of spectral data for the basic operator $\mathfrak{h}^{N,\xi_0}$ discussed in Section 3.2.*

Remark 5.3.3.
For the exterior of the disc, one first observes, using Persson's characterization in Appendix B, that the bottom of the essential spectrum is B and one can show that when B is large, there exists at least one eigenvalue below B. Therefore, it follows in particular that the ground state is a discrete eigenvalue for large B.

Proof of Theorem 5.3.1.
We start by giving the proof in the case of the disc. Afterwards we briefly indicate the changes in the case of the exterior domain.

For brevity, we write $\lambda_1(B)$ instead of $\lambda_1(B, \mathrm{D}(0,1))$. A simple argument with a trial state, which will be detailed later in Section 8.2.1 for all domains with smooth boundary, gives the preliminary upper bound:

$$\lambda_1(B) \le \Theta_0 B + o(B)\,. \tag{5.38}$$

Let $\mathrm{D}(t) = \{x \in \mathbb{R}^2 \;:\; |x| \le t\}$ be the disc with radius t. Let $\widetilde{Q}_B$ be the quadratic form

$$\widetilde{Q}_B[u] = \int_{\mathrm{D}(1)\setminus \mathrm{D}(\frac{1}{2})} \big|(-i\nabla + B\mathbf{F})u\big|^2\,dx\,,$$

with domain $\{u \in H^1(\mathrm{D}(1)\setminus \mathrm{D}(\frac{1}{2})) \;:\; u(x) = 0 \text{ on } |x| = \frac{1}{2}\}$. Let $\tilde{\lambda}_1(B)$ be the lowest eigenvalue of the corresponding self-adjoint operator. Using the Agmon estimates in the normal direction, which will be proven in a more general situation in Theorem 8.2.4, we obtain

$$\lambda_1(B) \le \tilde{\lambda}_1(B) = \lambda_1(B) + \mathcal{O}(B^{-\infty})\,. \tag{5.39}$$

The first inequality in (5.39) is immediate by the variational principle. The second estimate follows by using a cutoff version of the ground state ψ of $P^N_{B\mathbf{F},\mathrm{D}(1)}$ as a trial state in $\widetilde{Q}_B$, since ψ decays exponentially away from $\{|x| = 1\}$ by the normal Agmon estimates.

Let $\widetilde{\mathcal{H}}(B)$ be the self-adjoint operator associated with the quadratic form $\widetilde{Q}_B$. In the remainder of the proof, we will use the fact that a similar result on exponential decay holds for the first eigenfunctions of $\widetilde{\mathcal{H}}(B)$.

Proposition 5.3.4.
For any $\rho \in]0,1[$, *there exist positive constants* α, B_0, *and* C *such that if* $\widetilde{\psi} \in \mathcal{D}\big(\widetilde{\mathcal{H}}(B)\big)$ *satisfies*

$$\widetilde{\mathcal{H}}(B)\widetilde{\psi} = \widetilde{\lambda}(B)\widetilde{\psi}\,,$$

with

$$B \geq B_0, \qquad \widetilde{\lambda}(B) \leq (1-\rho)B\,,$$

then

$$\int_{\mathrm{D}(1)\setminus\mathrm{D}(\frac{1}{2})} e^{\alpha\sqrt{B}(1-|x|)}\big\{|\widetilde{\psi}(x)|^2 + B^{-1}|p_{B\mathbf{F}}\widetilde{\psi}(x)|^2\big\}\,dx \leq C\,\|\widetilde{\psi}\|_2^2\,. \tag{5.40}$$

Proposition 5.3.4 is essentially a particular case of Theorem 8.2.4 and hence the proof is omitted here.

By changing to boundary coordinates [see Section F.1—if (r,θ) are usual polar coordinates, then $t = 1-r$, $s=\theta$], the quadratic form $\widetilde{Q}_B[u]$ and the L^2-norm become

$$\widetilde{Q}_B[u] = \int_0^{2\pi}\int_0^{1/2} (1-t)^{-1}|(-i\partial_s + B\tilde{A}_1)u|^2 + (1-t)|\partial_t u|^2\,dtds\,, \tag{5.41}$$

$$\|u\|_{L^2}^2 = \int_0^{2\pi}\int_0^{1/2} (1-t)|u|^2\,dtds\,, \tag{5.42}$$

with $\tilde{A}_1(s,t) = \frac{1}{2} - t + \frac{t^2}{2}$. Here we have used Lemma F.1.1, and [see (F.8)]

$$\gamma_0 = \frac{\int_\Omega \operatorname{curl}\mathbf{F}\,dx}{|\partial\Omega|} = \frac{1}{2}\,,$$

for the disc.

Performing the scaling $\tau = \sqrt{B}\,t$ and decomposing u in Fourier modes, we find

$$\tilde{\lambda}_1(B) = B\inf_{m\in\mathbb{Z}} e_{\delta(m,B),B}\,. \tag{5.43}$$

Here the function $\delta(m,B)$ was defined in (5.34) and $e_{\delta,B}$ is the lowest eigenvalue of the self-adjoint operator $\widetilde{\mathfrak{h}}_{\delta,B}$ associated with the quadratic form $\tilde{q}_{\delta,B}$:

$$\tilde{q}_{\delta,B}[\phi] = \int_0^{\sqrt{B}/2}\Big(1-\frac{\tau}{\sqrt{B}}\Big)^{-1}\Big((\tau+\xi_0) + B^{-\frac{1}{2}}\Big(\delta - \frac{\tau^2}{2}\Big)\Big)^2|\phi(\tau)|^2 + \Big(1-\frac{\tau}{\sqrt{B}}\Big)|\phi'(\tau)|^2\,d\tau\,. \tag{5.44}$$

This quadratic form is considered to be a form defined on the space

$$L^2\big((0,\sqrt{B}/2);\Big(1-\frac{\tau}{\sqrt{B}}\Big)\,d\tau\big)\,.$$

Let $\rho, \eta \in]0, 1/100[$ and let $\widetilde{\phi}$ be a normalized ground state for $\widetilde{\mathfrak{h}}_{\delta,B}$. So

$$\int_0^{\sqrt{B}/2} (1 - \tfrac{\tau}{\sqrt{B}})|\widetilde{\phi}(\tau)|^2 \, d\tau = 1 \,. \tag{5.45}$$

Then

- either
$$e_{\delta(m,B),B} \geq (1 - \rho) \,, \tag{5.46}$$
- or we can apply Proposition 5.3.4 to $e^{-ims}\widetilde{\phi}(\sqrt{B}t)$ to see that $\widetilde{\phi}$ decays exponentially in τ .

Using the rough bound (5.38), it suffices to consider the cases where $e_{\delta(m,B),B} < (1 - \rho)$. Let $\chi \in C_0^\infty(\mathbb{R})$, with $\chi \equiv 1$ on a neighborhood of 0 , and define

$$\overline{\phi}(\tau) := \chi(B^{-\eta}\tau)\widetilde{\phi}(\tau)$$

(extended by 0 to a function on $\mathbb{R}^+$). Then the decay estimate in Proposition 5.3.4 implies that [with the localized form $q_{\eta,\delta(m,B),B}$ defined in (5.8)],

$$\begin{aligned} e_{\delta(m,B),B} &= \tilde{q}_{\delta(m,B),B}[\widetilde{\phi}] = \tilde{q}_{\delta(m,B),B}[\overline{\phi}] + \mathcal{O}(B^{-\infty}) \\ &= q_{\eta,\delta(m,B),B}[\overline{\phi}] + \mathcal{O}(B^{-\infty}) \\ &\geq \lambda_1(q_{\eta,\delta(m,B),B}) \int |\overline{\phi}(\tau)|^2 \left(1 - \frac{\ell}{\sqrt{B}}\right) d\tau + \mathcal{O}(B^{-\infty}) \,. \end{aligned} \tag{5.47}$$

Proposition 5.3.4 also implies that [with ℓ from (5.4)]

$$\int |\overline{\phi}(\tau)|^2 \left(1 - \frac{\ell}{\sqrt{B}}\right) d\tau = 1 + \mathcal{O}(B^{-\infty}) \,.$$

Thus,

$$e_{\delta(m,B),B} \geq \lambda_1(q_{\eta,\delta(m,B),B}) + \mathcal{O}(B^{-\infty}) \,. \tag{5.48}$$

Recall that $\lambda_1(q_{\eta,\delta(m,B),B})$ was estimated in Proposition 5.2.1. In particular, with the constant M from Proposition 5.2.1, we have

$$\lambda_1(q_{\eta,\delta(m,B),B}) \geq \Theta_0 + c_0 M^2 B^{2\eta - \frac{1}{2}} \,,$$

for $|\delta| \geq M B^{\frac{1}{4}+\eta}$. Therefore, such values of δ correspond to a ground state energy that is too large compared to the desired one and can be neglected.

For $|\delta| \leq M B^{\frac{1}{4}+\eta}$, we can use the explicit test function ψ from (5.25) (in the proof of Proposition 5.2.1) as a trial state for $\tilde{q}_{\delta,B}$. This will complete (5.48) with a corresponding upper bound; i.e., we find

$$e_{\delta(m,B),B} = \lambda_1(q_{\eta,\delta(m,B),B}) + \mathcal{O}\left(\frac{1 + |\delta|^3}{B^{\frac{3}{2}}}\right) \,. \tag{5.49}$$

Remembering (5.39), (5.43), and Proposition 5.2.1, this finishes the proof of Theorem 5.3.1 in the case of the disc.

When $\Omega = \mathbb{R}^2 \setminus \overline{D(0,1)}$, we still have Agmon estimates. In this case the curvature $k = -1$, and since the curvature is negative, the boundary coordinates can be used on the entire domain. We find

$$\gamma_0 = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{F}\cdot\gamma'(s)\,ds = -\frac{1}{2}\,;$$

therefore,

$$\tilde{A}_1^{\mathrm{ext}}(s,t) = -\frac{1}{2} - t - \frac{t^2}{2}\,,$$

and the quadratic form becomes

$$Q_B^{\mathrm{ext}}[u] = \int_0^{2\pi}\int_0^{\infty} (1+t)^{-1}|(-i\partial_s + B\tilde{A}_1^{\mathrm{ext}})u|^2 + (1+t)|\partial_t u|^2\,dtds\,, \tag{5.50}$$

with

$$\|u\|_{L^2}^2 = \int_0^{2\pi}\int_0^{\infty} t\,(1+t)|u|^2\,dtds\,. \tag{5.51}$$

After scaling and decomposition in Fourier modes, we find

$$\tilde{\lambda}_1^{\mathrm{ext}}(B) = B\inf_{m\in\mathbb{Z}} e^{\mathrm{ext}}_{\delta(m,B),B}\,, \tag{5.52}$$

with $e^{\mathrm{ext}}_{\delta,B}$ being the lowest eigenvalue of the quadratic form $\tilde{q}^{\mathrm{ext}}_{\delta,B}$,

$$\begin{aligned}\tilde{q}^{\mathrm{ext}}_{\delta,B}[\phi] = \int_0^{\infty} &\Big(1+\frac{\tau}{\sqrt{B}}\Big)^{-1}\Big((\tau+\xi_0) - B^{-\frac{1}{2}}\Big(\delta - \frac{\tau^2}{2}\Big)\Big)^2|\phi(\tau)|^2 \\ &+ \Big(1+\frac{\tau}{\sqrt{B}}\Big)|\phi'(\tau)|^2\,d\tau\end{aligned} \tag{5.53}$$

on the space $L^2\big(\mathbb{R}^+;(1+\frac{\tau}{\sqrt{B}})d\tau\big)$.

In the notation and sense from (5.21) and (5.22), we therefore find

$$\mathfrak{h}^{\mathrm{ext}}(\delta,B) = \mathfrak{k}_0^{\mathrm{ext}} + B^{-\frac{1}{2}}\mathfrak{k}_1^{\mathrm{ext}} + B^{-1}\mathfrak{k}_2^{\mathrm{ext}} + \mathcal{O}\left(\frac{1+|\delta|^3}{B^{\frac{3}{2}}}\right)\,, \tag{5.54}$$

with

$$\mathfrak{k}_0^{\mathrm{ext}} = \mathfrak{k}_0\,,\qquad \mathfrak{k}_1^{\mathrm{ext}} = -\mathfrak{k}_1\,,\qquad \mathfrak{k}_2^{\mathrm{ext}} = \mathfrak{k}_2\,. \tag{5.55}$$

Therefore,

$$\lambda_1^{\mathrm{ext}} = -\lambda_1\,,\qquad \lambda_2^{\mathrm{ext}} = \lambda_2\,. \tag{5.56}$$

This finishes the proof of the exterior case. □

Remark 5.3.5.
In the case of the disc, let $\lambda_1(B,m)$ *denote the ground state energy of* $P^N_{B,\Omega}$ *restricted to the space of angular momentum* m *, i.e., functions that have the form* $e^{-im\theta}f(r)$ *in polar coordinates* (r,θ)*. Then, using* (5.39), (5.43), *and* (5.46), *we obtain that*

- *either* $\lambda_1(B,m) \geq \frac{99}{100}B$ *,*
- *or* $\lambda_1(B,m) = Be_{\delta(m,B),B} + \mathcal{O}(B^{-\infty})$ *.*

It is rather easy to deduce from the calculations above that if m *is fixed then* $\liminf_{B\to\infty} \lambda_1(B,m)/B > \Theta_0$ *. Thus, the integer minimal* $m(B)$ *such that* $\lambda_1(B,m(B)) = \inf_j \lambda_1(B,j)$ *will have to change with* B *, and this implies crossings of eigenvalues by continuity. This means that we have the existence of a sequence* $\{B_n\}_{n\in\mathbb{N}}$ *with* $B_n \to \infty$ *and such that* $\lambda_1(B_n)$ *has multiplicity at least* 2 *.*

If, on the other hand, for $n \in \mathbb{N}$ *, we define* B_n *as the positive solution to*

$$\delta(n,B_n) := n - \frac{B_n}{2} - \xi_0\sqrt{B_n} = \delta_0 \,,$$

then we have $\widehat{\Delta}_{B_n} = 0$ *. Now using the asymptotics of* $\lambda_1(B_n,n)$ *and those of* $\lambda_1(B_n,m)$ *, we observe that for* n *large,*

$$\lambda_1(B_n,n) = \inf_m \lambda_1(B_n,m) \,.$$

Now (5.36) *implies that*

$$\lambda_1(B_n,n) = \Theta_0 B_n - \mathcal{C}_1\sqrt{B_n} + 3\mathcal{C}_1\sqrt{\Theta_0}\mathcal{C}_0 + o(1) \,,$$

and for all $m \neq n$ *,*

$$\lambda_1(B_n,m) \geq \Theta_0 B_n - \mathcal{C}_1\sqrt{B_n} + 3\mathcal{C}_1\sqrt{\Theta_0}(1+\mathcal{C}_0) + o(1) \,.$$

This proves the existence of a spectral gap of asymptotic size larger than $3\mathcal{C}_1\sqrt{\Theta_0}$ *along the sequence* $\{B_n\}$ *.*

Similar arguments give the following two statements:

- *There exists* $B_0 > 0$ *such that for* $B \geq B_0$ *, the maximal degeneracy of the ground state energy* $\lambda_1(B)$ *is equal to* 2 *.*
- *The spectral gap* $\lambda_2(B) - \lambda_1(B)$ *satisfies*

$$\limsup_{B\to\infty} \big(\lambda_2(B) - \lambda_1(B)\big) = 3\mathcal{C}_1\sqrt{\Theta_0} \,.$$

5.4 Application to the Monotonicity

Proposition 5.4.1.
Let Ω *be the unit disc,* $\mathrm{D}(0,1)$ *, or the exterior of the unit disc,* $\mathbb{R}^2 \setminus \overline{\mathrm{D}(0,1)}$ *. Then the left- and right-hand derivatives* $\lambda'_{1,\pm}(B)$ *exist and satisfy*

$$\begin{aligned}
&\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B)\,, \\
&\liminf_{B\to+\infty} \lambda'_{1,+}(B) \geq \Theta_0 - \frac{3}{2}\mathcal{C}_1|\xi_0| > 0\,, \qquad (5.57)\\
&\limsup_{B\to+\infty} \lambda'_{1,-}(B) \leq \Theta_0 + \frac{3}{2}\mathcal{C}_1|\xi_0|\,.
\end{aligned}$$

In particular, $B \mapsto \lambda_1(B)$ is strictly increasing for large B.

Proof of Proposition 5.4.1.
We only consider the case of the unit disc, since the calculation is similar for $\Omega = \mathbb{R}^2 \setminus \overline{\mathrm{D}(0,1)}$.

The numerical fact that $\Theta_0 > \frac{3}{2}\mathcal{C}_1|\xi_0|$ follows from known identities. We give the following short argument. From (3.39), we get that

$$3\mathcal{C}_1|\xi_0| = 1 - 4I_2\,, \qquad (5.58)$$

where I_2 [introduced in (3.33)] satisfies $I_2 > 0$. In particular, $3\mathcal{C}_1|\xi_0| < 1$. Since it is known from Proposition 3.2.8 that $\Theta_0 > 1/2$, this proves the desired statement (see also Note 4 to Chapter 3 for numerical values).

We now prove (5.57). Let $g(B) = -\mathcal{C}_1\sqrt{B} + 3\mathcal{C}_1\sqrt{\Theta_0}\big(\widehat{\Delta}_B^2 + \mathcal{C}_0\big)$, $\alpha = \Theta_0$. By Corollary 2.3.3, we have to estimate $\limsup_{B\to\infty} g(B+\epsilon) - g(B)$ and $\liminf_{B\to\infty} g(B+\epsilon) - g(B)$. As discussed in Remark 2.3.4, we only need to consider the oscillating part of the function g.

Notice that $0 \leq \widehat{\Delta}_B \leq 1/2$, for all $B > 0$. Furthermore, consider $B > 1$, $\epsilon > 0$. Let $m_0 \in \mathbb{Z}$ be such that

$$\widehat{\Delta}_{B+\epsilon} = \left| m_0 - \frac{B+\epsilon}{2} - \xi_0\sqrt{B+\epsilon} - \check{\delta}_0 \right|.$$

Then, since $-1 < \xi_0 < 0$,

$$\begin{aligned}
\widehat{\Delta}_{B+\epsilon} - \widehat{\Delta}_B &\geq \left| m_0 - \frac{B+\epsilon}{2} - \xi_0\sqrt{B+\epsilon} - \check{\delta}_0 \right| - \left| m_0 - \frac{B}{2} - \xi_0\sqrt{B} - \check{\delta}_0 \right| \\
&\geq -\left| -\frac{\epsilon}{2} - \xi_0 \frac{\epsilon}{\sqrt{B+\epsilon}+\sqrt{B}} \right| \geq -\frac{\epsilon}{2}. \qquad (5.59)
\end{aligned}$$

Therefore,

$$\widehat{\Delta}_{B+\epsilon}^2 - \widehat{\Delta}_B^2 = (\widehat{\Delta}_{B+\epsilon} + \widehat{\Delta}_B)(\widehat{\Delta}_{B+\epsilon} - \widehat{\Delta}_B) \geq -\frac{\epsilon}{2}\,,$$

and we get

$$\liminf_{B\to+\infty} \frac{g(B+\epsilon) - g(B)}{\epsilon} \geq -\frac{3}{2}\mathcal{C}_1\sqrt{\Theta_0}\,.$$

The estimate of (5.57) follows by taking the limit $\epsilon \to 0$. The proof of the upper bound on the left-side derivative is similar and is omitted.

The attentive reader may have realized a small problem in the argument in the proof of Proposition 2.3.2—on which Corollary 2.3.3 depends—in the case when $\Omega = \mathbb{R}^2 \setminus \overline{D(0,1)}$. Since the vector potential $\mathbf{F}(x_1, x_2) := \frac{1}{2}(-x_2, x_1)$ is unbounded, one cannot estimate the term

$$\int_\Omega |\mathbf{F}(x)|^2 |\psi(x)|^2 \, dx$$

by the square of the L^∞-norm of $\mathbf{F}$, which is infinite. However, by the Agmon estimates (see Theorem B.5.1), the eigenfunction ψ decays exponentially in the radial variable, so we find that

$$\int_\Omega |\mathbf{F}(x)|^2 |\psi(x)|^2 \, dx \leq C \,,$$

with a constant C independent of B. From this point the proof is identical to the case of the disc. □

We can easily obtain results for other constant curvature domains by applying scaling to the disc models. Let $k \in \mathbb{R}$. Then the model with constant boundary curvature k is

$$\Omega_k := \begin{cases} D(0, k^{-1}) \,, & \text{if } k > 0 \,, \\ \mathbb{R}^2 \setminus \overline{D(0, |k^{-1}|)} \,, & \text{if } k < 0 \,, \\ \mathbb{R}^{2,+} , & \text{if } k = 0 \,. \end{cases}$$

By scaling we see that, for $k \neq 0$,

$$\lambda_1(B, \Omega_k) = k^2 \lambda_1 \left(\frac{B}{k^2}, \Omega_{\mathrm{sign}(k)} \right) . \tag{5.60}$$

Therefore, we find the following corollary of Theorem 5.3.1.

Corollary 5.4.2.
There exist positive constants C and B_0 such that if Ω_k is the domain with constant boundary curvature k, then for all $k \in \mathbb{R}$, $B \geq B_0 k^2$,

$$\left| \lambda_1(B, \Omega_k) - \left(\Theta_0 B - \mathcal{C}_1 k \sqrt{B} \right) \right| \leq C \, k^2 \,. \tag{5.61}$$

5.5 Notes

1. The case of Dirichlet boundary conditions was considered by Erdös in connection with an isoperimetric inequality [Er2]. Estimate (5.1), which is a small improvement of his result, was proved in [HeM3] using the techniques of [BolH1].

2. The case of the disc with Neumann conditions was first considered by Bauman–Phillips–Tang [BaPT] (see also [HoT]) who obtain (5.2). Another proof with a worse remainder term was given in [HeM3]. More formal computations were given in [S-J] for the disc and [BeBZ] for its complement. These contributions indicate the role of the sign of the curvature.
3. The case of the complement to the disc was first given in [HeM3] with a less precise remainder term.
4. For the purpose of the analysis of the monotonicity of the ground state energy or for the analysis of the multiplicity, new improvements were needed. They first appeared in [FoH4] and are explained in detail in the present chapter.

6

Models in Dimension 3: $\mathbb{R}^3$ or $\mathbb{R}^{3,+}$

In the analysis of the magnetic Schrödinger operator with Neumann boundary condition in an open set $\Omega \subset \mathbb{R}^3$, the first two models to analyze are the constant field case in $\mathbb{R}^3$ and the constant field case in $\mathbb{R}^{3,+}$. The latter model will permit us to understand the effect of the boundary.

6.1 The Case of $\mathbb{R}^3$

We start with the Schrödinger operator with constant magnetic field $\beta = \operatorname{curl} \mathbf{F}$ in dimension 3. After possibly performing a rotation in $\mathbb{R}^3$ and a gauge transformation, we arrive at the model:

$$P_{B\mathbf{F}} = -\partial_{x_1}^2 + (-i\partial_{x_2} + Bx_1)^2 - \partial_{x_3}^2 \,, \tag{6.1}$$

with

$$B = \|\beta\| \,,$$

being the strength of the magnetic field. Here we can take the partial Fourier transform with respect to x_2 and x_3 in order to get the operator

$$-\partial_{x_1}^2 + (\xi_2 + Bx_1)^2 + \xi_3^2 \,.$$

When $B \neq 0$, we can translate in the x_1-variable and get the operator on $L^2(\mathbb{R}^3)$:

$$-\partial_{y_1}^2 + (By_1)^2 + \xi_3^2 \,.$$

It is then easy to see that the spectrum is $[B, +\infty[$.

Remark 6.1.1.
Unitarily implementing the scaling $x \mapsto B^{-1/2}x$, one finds that $P_{B\mathbf{F}}$ is unitarily equivalent to $B^{-1}P_{\mathbf{F}}$. Therefore, one only needs to go through the discussion above in the case $B = 1$.

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Remark 6.1.2.
One can also give explicit quasimodes to show that $\inf \sigma(P_{\mathbf{F}}) \leq 1$. *For a general constant* $\beta \in \mathbb{R}^3$ *with* $|\beta| = 1$, *we consider the gauge where* $\mathbf{A} = \frac{1}{2}\beta \times x$. *We consider coordinates* $x = (x_\perp, x_\parallel)$ *corresponding to the perpendicular and parallel components of the vector* x *with respect to* β. *Then ground state quasimodes can be written as*

$$\epsilon^{-1/4} e^{-\epsilon x_\parallel^2} e^{-x_\perp^2/4}.$$

More generally, one can make a "magnetic translation" of these functions to have them localized near an arbitrary point $y = (y_\perp, y_\parallel) \in \mathbb{R}^3$*:*

$$u_\epsilon(x) = \epsilon^{-1/4} e^{-\epsilon(x_\parallel - y_\parallel)^2} e^{iy_\perp \times x_\perp} e^{-(x_\perp - y_\perp)^2/4}, \tag{6.2}$$

where $y_\perp \times x_\perp$ *is the two-dimensional cross-product.*

6.2 The Case of $\mathbb{R}^{3,+}$

We now investigate the case of $\mathbb{R}^{3,+} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 > 0\}$. In this case, there is one geometric parameter, namely, the angle between the magnetic field and the boundary $\{x_1 = 0\}$. We will see that the infimum of the spectrum is a monotone continuous function of this angle with the lowest value Θ_0 being attained for the magnetic field parallel to the boundary and the highest value 1 being attained when the magnetic field is perpendicular to the boundary.

6.2.1 An easy upper bound

We would proceed as in the case of $\mathbb{R}^3$, but our rotations have to conserve $\mathbb{R}^{3,+}$ and its boundary. Let $\beta = (\beta_{23}, \beta_{13}, \beta_{12})$, and let us start from the particular gauge choice

$$P(\beta) := -\partial_{x_1}^2 + (-i\partial_{x_2} + \beta_{12}x_1)^2 + (-i\partial_{x_3} - \beta_{13}x_1 + \beta_{23}x_2)^2$$

in $\mathbb{R}^{3,+}$. After scaling, we can assume that $\beta_{12}^2 + \beta_{13}^2 + \beta_{23}^2 = 1$.

More precisely, the operator $P(\beta)$ is defined as the positive operator associated with the closed quadratic form

$$u \mapsto \int_{\mathbb{R}^{3,+}} |\partial_{x_1} u|^2 + |(-i\partial_{x_2} + \beta_{12}x_1)u|^2 + |(-i\partial_{x_3} - \beta_{13}x_1 + \beta_{23}x_2)u|^2 \, dx\,,$$

with domain

$$\Big\{u \in L^2(\mathbb{R}^{3,+}) \,\Big|\, \partial_{x_1} u \in L^2(\mathbb{R}^{3,+})\,, \quad (-i\partial_{x_2} + \beta_{12}x_1)u \in L^2(\mathbb{R}^{3,+})\,,$$
$$(-i\partial_{x_3} - \beta_{13}x_1 + \beta_{23}x_2)u \in L^2(\mathbb{R}^{3,+})\Big\}.$$

Let us start by noticing the following inequality:

$$\inf \sigma(P(\beta)) \leq 1\,. \tag{6.3}$$

Proof.
We can implement the quasimodes of Remark 6.1.2 in order to prove (6.3). We only sketch the geometric idea. For a given $\epsilon > 0$, we choose the localization point y far away from the boundary. We can now insert the u from (6.2) in the quadratic form restricted to $\mathbb{R}^{3,+}$, and realize that the difference with the form on the entire space is exponentially small in the distance from y to the boundary.

We have thereby established (6.3). □

6.2.2 Preliminary reductions

We return to the differential operator $P(\beta)$. After performing a rotation in the (x_2, x_3)-variables, we can assume that the new magnetic field $\tilde{B}$ satisfies $\tilde{\beta}_{12} = 0$, the new $\tilde{\beta}_{13}$ satisfying

$$\tilde{\beta}_{13}^2 = \beta_{12}^2 + \beta_{13}^2\,.$$

So we have now reduced to the problem of analyzing

$$P(\beta_1, \beta_2) := -\partial_{x_1}^2 - \partial_{x_2}^2 + (-i\partial_{x_3} + \beta_1 x_1 + \beta_2 x_2)^2\,,$$

in $\{x_1 > 0\}$, where

$$\beta_1^2 + \beta_2^2 = 1\,.$$

Here we have

$$\beta_1^2 = \beta_{23}^2\,, \qquad \beta_2^2 = \beta_{12}^2 + \beta_{13}^2\,.$$

So we arrive at the following model:

$$\mathfrak{L}(\vartheta, -i\partial_t) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (-i\partial_t + x_1 \cos\vartheta + x_2 \sin\vartheta)^2\,,$$

which only depends on the parameter ϑ. Geometrically, $\sin\vartheta = \beta \cdot \nu$ (where ν is the interior normal vector), and so ϑ is the angle between the magnetic field and the (tangent plane to the) boundary.

By a partial Fourier transform in the t-variable, we arrive at

$$\mathfrak{L}(\vartheta, \tau) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (\tau + x_1 \cos\vartheta + x_2 \sin\vartheta)^2\,,$$

in $\{x_1 > 0\}$ and with Neumann boundary condition on $\{x_1 = 0\}$. It is enough to consider the variation with respect to $\vartheta \in [0, \pi/2]$.

The bottom of the spectrum is given by

$$\varsigma(\vartheta) := \inf \sigma\left(\mathfrak{L}(\vartheta, -i\partial_t)\right) = \inf_{\tau} (\inf \sigma(\mathfrak{L}(\vartheta, \tau)))\,. \tag{6.4}$$

Lemma 6.2.1.

1. *If* $\vartheta \in]0, \pi/2]$ *, then* $\sigma(\mathfrak{L}(\vartheta,\tau))$ *is independent of* τ *.*
2. *The function* $\vartheta \mapsto \varsigma(\vartheta)$ *is continuous on* $]0, \pi/2[$ *.*
3. *At the endpoints* ς *takes the values*

$$\varsigma(0) = \Theta_0 < 1\,, \qquad \textit{and} \qquad \varsigma\left(\frac{\pi}{2}\right) = 1\,. \tag{6.5}$$

Proof.
The proof of the first assertion is easy since the translation $x_2 \mapsto x_2 + \tau/\sin\vartheta$, exchanges $\mathfrak{L}(\vartheta,\tau)$ and $\mathfrak{L}(\vartheta,0)$.

We now prove the second assertion. After the change of variables $y_1 = \cos\vartheta\, x_1$, $y_2 = \sin\vartheta\, x_2$,[1] we get a continuous family of operators with a fixed domain. Using the min-max principle (see Appendix A), the result follows easily.

To prove the first inequality in (6.5), we first observe that

$$\mathfrak{L}(0, -i\partial_t) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (-i\partial_t + x_1)^2\,.$$

After a partial Fourier transform in x_2 and t, we thus have to analyze the bottom of the spectrum of the family:

$$\mathfrak{L}(0,\tau,\xi_2) := -\partial_{x_1}^2 + \xi_2^2 + (x_1+\tau)^2\,.$$

This infimum is obtained as the infimum over $\tau \in \mathbb{R}$ of the spectrum of the family:

$$\mathfrak{L}(0,\tau,0) = -\partial_{x_1}^2 + (x_1+\tau)^2\,.$$

This is the model $\mathfrak{h}^N(\tau)$ that was analyzed in Chapter 3.

To prove the second inequality in (6.5), we start from

$$\mathfrak{L}\left(\frac{\pi}{2},\tau\right) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (\tau + x_2)^2\,.$$

The infimum of the spectrum is the same as the bottom of the Neumann realization of

$$-\partial_{x_1}^2 - \partial_{x_2}^2 + x_2^2\,,$$

in $\{x_1 > 0\}$. This is easily computed (by separation of variables) as being equal to 1 (the infimum of the spectrum of the harmonic oscillator). □

6.2.3 Spectral bounds

Lemma 6.2.2.
For $\vartheta \in [0, \pi/2]$ *, we have the bounds*

$$\varsigma(\vartheta) \le \Theta_0 \cos\vartheta + \sin\vartheta\,, \tag{6.6}$$

[1] Here we use the restrictions on ϑ.

and

$$\varsigma(\vartheta) \geq \Theta_0(\cos\vartheta)^2 + (\sin\vartheta)^2 \,. \tag{6.7}$$

Furthermore, the function $[0,\pi/2] \ni \vartheta \mapsto \varsigma(\vartheta)$ *is monotone increasing.*

Notice that by Lemma 6.2.1 the bounds (6.6) and (6.7) are valid—actually with strict inequalities—for $\vartheta \in \{0,\pi/2\}$. Thus, it suffices to consider $\vartheta \in]0,\pi/2[$. Also, by combining Lemma 6.2.1 with (6.6) and (6.7), one finds that ς is continuous at 0 and $\pi/2$. Notice though that the upper bound (6.6) is weaker than (6.3) for ϑ near but below $\pi/2$.

Proof.
We will prove (6.6) for $\vartheta \in]0,\pi/2[$. Let us write

$$\begin{aligned}\mathfrak{L}(\vartheta,0) = &-\partial_{x_1}^2 + (x_1\cos\vartheta + z)^2 - \partial_{x_2}^2 + (x_2\sin\vartheta - z)^2 \\ &+ 2(x_1\cos\vartheta + z)(x_2\sin\vartheta - z)\,.\end{aligned}$$

Use as quasimode the product of the eigenfunction attached to the lowest eigenvalue of $-\partial_{x_1}^2 + (x_1\cos\vartheta + z)^2$ and of the eigenfunction attached to the lowest eigenvalue of $-\partial_{x_2}^2 + (x_2\sin\vartheta - z)^2$. This gives, by a good choice of z ($z = \xi_0\sqrt{\cos\vartheta}$), the desired upper bound.

We will now prove the lower bound (6.7). We only need to consider the case when $\vartheta \in]0,\pi/2]$. According to Lemma 6.2.1, we can take $\tau = 0$ and we have to analyze

$$\mathfrak{L}(\vartheta) := \mathfrak{L}(\vartheta,0) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (x_1\cos\vartheta + x_2\sin\vartheta)^2. \tag{6.8}$$

Let us introduce a parameter $\rho \in [0,1]$ and associate the following decomposition

$$\mathfrak{L}(\vartheta) := P_1(\rho,\vartheta) + P_2(\rho,\vartheta), \tag{6.9}$$

with

$$\begin{aligned}P_1 &:= -\,\partial_{x_1}^2 + \rho^2(x_1\cos\vartheta + x_2\sin\vartheta)^2\,,\\ P_2 &:= -\,\partial_{x_2}^2 + (1-\rho^2)(x_1\cos\vartheta + x_2\sin\vartheta)^2\,.\end{aligned} \tag{6.10}$$

We will find a lower bound of the spectrum of $\mathfrak{L}(\vartheta)$ by considering the sum of the lower bounds of the spectra of the two operators P_1 and P_2. Easy computations lead to

$$\inf\sigma(P_1(\rho,\vartheta)) = \rho\Theta_0\cos\vartheta \tag{6.11}$$

and

$$\inf\sigma(P_2(\rho,\vartheta)) = \sqrt{1-\rho^2}\sin\vartheta\,. \tag{6.12}$$

Choosing $\rho = \cos\vartheta$, we obtain (6.7).

We finally prove the monotonicity. The case $\vartheta = 0$ is a consequence of (6.7) and (6.5). The case $\vartheta = \pi/2$ follows from (6.3) and (6.5). We therefore assume that $\vartheta \in]0, \pi/2[$ and consider $\mathfrak{L}(\vartheta)$ in $\mathbb{R}^{2,+}$.

We first use the change of variables

$$u_1 = x_1 \cos\vartheta + x_2 \sin\vartheta\,, \qquad u_2 = -x_1 \sin\vartheta + x_2 \cos\vartheta\,,$$

whose inverse is given by

$$x_1 = u_1 \cos\vartheta - u_2 \sin\vartheta\,, \qquad x_2 = u_1 \sin\vartheta + u_2 \cos\vartheta\,.$$

This shows that $\mathfrak{L}(\vartheta, 0)$ is unitarily equivalent to

$$L' = -\partial_{u_1}^2 - \partial_{u_2}^2 + u_1^2\,, \tag{6.13}$$

in $\{u_1 > \tan\vartheta\, u_2\}$.

A new change of variables,

$$y_1 = -u_1\,, \qquad y_2 = -\tan\vartheta\, u_2\,,$$

shows that this problem is unitarily equivalent to the Neumann realization of

$$L^{\text{new}} = -\partial_{y_1}^2 - \tan^2(\vartheta)\partial_{y_2^2}^2 + y_1^2\,, \tag{6.14}$$

in $\{y_2 > y_1\}$.

By unitary equivalence, $\varsigma(\vartheta)$ is the bottom of the spectrum of this new operator. The monotonicity is immediate from (6.14), via the min-max principle. □

We have $\varsigma(\vartheta) \le 1$. If the inequality was strict, i.e., if we were able to prove strict monotonicity directly, we would get, by combining with our result on the essential spectrum, that $\varsigma(\vartheta)$ is an eigenvalue.

Unfortunately, the strict monotonicity is clear only when we know that $\varsigma(\vartheta)$ is an eigenvalue. Note that our "rough" upper bound does not give the result, and so we are obliged to prove an upper bound of $\varsigma(\vartheta)$ in another way, thereby showing that it is less than 1 as soon as $\vartheta < \pi/2$. We will give such a result in Section 6.2.5.

For the behavior near $\vartheta = 0$, we will later need

$$\varsigma(\vartheta) = \Theta_0 + \delta_0|\vartheta| + \mathcal{O}(\vartheta^2)\,, \tag{6.15}$$

as $\vartheta \to 0$, with

$$\delta_0 = \sqrt{\frac{\mu''(\xi_0)}{2}}\,. \tag{6.16}$$

We recall from (3.23) that

$$\delta_0 > 0\,. \tag{6.17}$$

The expansion (6.15) is a consequence of the more general property

Proposition 6.2.3.
The function ς admits at 0 the following asymptotic expansion:

$$\varsigma(\vartheta) \sim \Theta_0 + \sum_{n\geq 1} \alpha_n |\vartheta|^n \,. \tag{6.18}$$

Sketch of the proof.
After a scaling $(y_1, y_2) = (x_1(\cos\vartheta)^{\frac{1}{2}}, x_2 \sin\vartheta/(\cos\vartheta)^{\frac{1}{2}})$, the operator $\mathfrak{L}(\vartheta)$ introduced in (6.8) becomes $\cos\vartheta\; P(\epsilon)$, where $P(\epsilon)$ is defined on $\mathbb{R}^{2,+}$ by

$$P(\epsilon) := -\epsilon^2 \partial_{y_2}^2 - \partial_{y_1}^2 + (y_1 + y_2)^2 \,,$$

with $\epsilon = \tan\vartheta$.

Then a Born–Oppenheimer technique is relevant for getting the complete expansion, which will not be detailed in this book. For the weaker result [the lower bound in (6.15), which is the only important point], one can simply observe that

$$\inf \sigma(P(\epsilon)) \geq \inf \sigma(-\epsilon^2 \partial_t^2 + \mu(t)) \,, \tag{6.19}$$

where $\mu(\xi)$ is the lowest eigenvalue of $\mathfrak{h}^{N,\xi}$ introduced in (3.9). Here $-\epsilon^2\partial_t^2 + \mu(t)$ is considered as an operator on $L^2(\mathbb{R})$. Observing that, by (6.16) and (6.17), we have a nondegenerate well at ξ_0, we can then use the semiclassical analysis (see Section 7.1 in the next chapter) for a one-well problem to get, via the harmonic approximation, the suitable lower bound. □

6.2.4 Analysis of the essential spectrum

Proposition 6.2.4.
If $\vartheta \in]0, \pi/2]$, then the essential spectrum of $\mathfrak{L}(\vartheta)$ is contained in $[1, +\infty[$.

Proof.
Using Persson's criterion (see Appendix B), we have to show that if the support of u is in $\{x_1 > R\} \cup \{|x_2| > R\}$, then we have

$$\langle \mathfrak{L}(\vartheta) u \,|\, u\rangle \geq (1 - \epsilon(R)) \|u\|^2 \,,$$

with $\epsilon(R) \to 0$ as $R \to +\infty$.

We start by reducing to the two cases $\operatorname{supp} u \subset \{x_1 > R\}$ and $\operatorname{supp} u \subset \{|x_2| > R\}$. Let $f_1^2 + f_2^2 = 1$ be a partition of unity on $\mathbb{R}$ with $f_1 = 1$ on $[-1/2, 1/2]$, $\operatorname{supp} f_1 \subset [-1, 1]$. Define $f_{j,R}(x) = f_j(x_2/R)$ and $u_j = f_{j,R} u$. Then, by an integration by parts,

$$\langle \mathfrak{L}(\vartheta) u \,|\, u\rangle \geq \langle \mathfrak{L}(\vartheta) u_1 \,|\, u_1\rangle + \langle \mathfrak{L}(\vartheta) u_2 \,|\, u_2\rangle - \frac{C}{R^2}\|u\|^2 \,. \tag{6.20}$$

Furthermore, $\operatorname{supp} u_2 \subset \{|x_2| > R\}$ and—using that $\operatorname{supp} u \subset \{x_1 > R\} \cup \{|x_2| > R\}$—$\operatorname{supp} u_1 \subset \{x_1 > R\}$.

We treat the first case, $\{x_1 > R\}$, by observing that one can use the Dirichlet result or, better, the lower bound of the operator in $\mathbb{R}^2$. After a rotation, the operator is isospectral to $-\partial_{s_1}^2 - \partial_{s_2}^2 + s_1^2$, whose spectrum starts at the value 1.

For the second case, $\{|x_2| > R\}$, one uses the decomposition (6.9), (6.10) with $\rho = \cos\vartheta$. Under the assumption that the support of u is contained in $\{|x_2| \geq R\}$, we have

$$\langle P_1(\rho,\vartheta)u \mid u\rangle \geq \cos\vartheta\,\rho \inf_{|x_2|\geq R} \mu(x_2 \cot(\vartheta))\,\|u\|^2\,,$$

where $\mu(\tau)$ is the first eigenvalue of the Neumann realization $\mathfrak{h}^{N,\tau}$ of the operator $-\frac{d^2}{dt^2} + (t+\tau)^2$ in $\mathbb{R}^+$.

This gives, for R large enough,

$$\langle P_1(\rho,\vartheta)u \mid u\rangle \geq \cos\vartheta\,\rho\,\mu(-R\cot(\vartheta))\,\|u\|^2\,,$$

when $\operatorname{supp} u \subset \{|x_2| > R\}$. Combining this estimate with (6.12), we get the result by recalling [see (3.15) and (3.13)] the behavior of μ at ∞:

$$\lim_{\tau\to-\infty} \mu(\tau) = 1 \qquad \text{and} \qquad \lim_{\tau\to+\infty} \mu(\tau) = +\infty\,.$$

□

6.2.5 A refined upper bound: $\varsigma(\vartheta) < 1$

In this section, we will prove the strict inequality

$$\varsigma(\vartheta) < 1\,, \quad \text{for all} \quad \vartheta \in]0, \tfrac{\pi}{2}[\,. \tag{6.21}$$

For this analysis, we come back to L^{new}, which was introduced in (6.14). For simplicity, we define α by

$$\frac{1}{\alpha} = \tan\vartheta^{\,2}\,,$$

and we note that $\alpha > 0$ if $\vartheta < \pi/2$. We now introduce:

$$f(t) = \exp -\frac{t^2}{2}\,, \quad \text{and} \quad F(t) = \int_{-\infty}^{t} \exp -s^2\, ds\,.$$

We observe that F is strictly positive, that

$$\lim_{t\to+\infty} F(t) = \sqrt{\pi}\,, \tag{6.22}$$

and that

$$F(t) \sim \frac{1}{2t} \exp -t^2\,,\ \text{as } t \to -\infty\,. \tag{6.23}$$

We shall apply the min-max principle (see Appendix A) with the test function:

$$\Psi(y_1, y_2) = f(y_1)g(y_2)\,,$$

with g to be determined in $L^2(\mathbb{R})$.

Integrating Ψ^2 in the domain $\{y_2 > y_1\}$, we first have

$$\|\Psi\|^2 = \int_{-\infty}^{+\infty} g(y_2)^2 F(y_2) dy_2\,.$$

Let us now compute the energy associated with Ψ. We first get

$$\begin{aligned}\langle L^{\mathrm{new}}\Psi \mid \Psi\rangle &= \int_{-\infty}^{+\infty} g(y_2)^2 \left(\int_{-\infty}^{y_2} \left(f'(y_1)^2 + y_1^2 f(y_1)^2\right) dy_1\right) dy_2 \\ &\quad + \frac{1}{\alpha}\int_{-\infty}^{+\infty} g'(y_2)^2 F(y_2) dy_2\,.\end{aligned}$$

After a first integration by parts, we get

$$\begin{aligned}\langle L^{\mathrm{new}}\Psi \mid \Psi\rangle = \|\Psi\|^2 &+ \int_{-\infty}^{+\infty} g(y_2)^2 f(y_2) f'(y_2)\, dy_2 \\ &+ \frac{1}{\alpha}\int_{-\infty}^{+\infty} g'(y_2)^2 F(y_2) dy_2\,,\end{aligned}$$

and then, after a second integration by parts,

$$\begin{aligned}\langle L^{\mathrm{new}}\Psi \mid \Psi\rangle &= \|\Psi\|^2 - \int_{-\infty}^{+\infty} g(y_2) g'(y_2) f(y_2)^2\, dy_2 + \frac{1}{\alpha}\int_{-\infty}^{+\infty} g'(y_2)^2 F(y_2) dy_2 \\ &= \|\Psi\|^2 + \Sigma_\alpha(g)\,, \qquad (6.24)\end{aligned}$$

where

$$\Sigma_\alpha(g) := \int_{-\infty}^{+\infty} g'(y_2)\left(\frac{1}{\alpha} g'(y_2) F(y_2) - g(y_2) F'(y_2)\right) dy_2\,.$$

We observe that we will have finished the proof of (6.21) if we find a $g \in L^2$ such that $\Sigma_\alpha(g)$ is strictly negative. Let us first see what happens if we try to have the sum inside the integral vanish. A natural try is then to solve the equation

$$\frac{1}{\alpha} g'(y_2) F(y_2) - g(y_2) F'(y_2) = 0\,,$$

which leads to $g = c\, g_\alpha$, with $g_\alpha = F^\alpha$. Notice that F^α is not in L^2 at $+\infty$.

We can compute $\Sigma_\alpha(g_{\hat\alpha})$ for more general $\hat\alpha$. We get

$$\Sigma_\alpha(g_{\hat\alpha}) = \hat\alpha(\hat\alpha\alpha^{-1} - 1)\int_{-\infty}^{+\infty} f^4(y_2) F^{2\hat\alpha - 1}(y_2)\, dy_2\,.$$

Let us first confirm that this integral is welldefined. There is no problem at $+\infty$, because F tends to a constant and f is exponentially decreasing. Near $-\infty$, F decreases like f^2 (see above), and so the integral converges for all $\hat{\alpha} > 0$. Now the expression is clearly negative if $0 < \hat{\alpha} < \alpha$.

We now choose such an $\hat{\alpha}$. **But $g_{\hat{\alpha}}$ is not in L^2 at $+\infty$.** So we are obliged to introduce a cutoff function χ_n defined by

$$\chi_n(t) = \chi\left(\frac{t}{n}\right),$$

where χ is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$. We now take $g = g_{\hat{\alpha},n} = \chi_n(t) g_{\hat{\alpha}}(t)$. We observe that, due to (6.22), the corresponding $\|\Psi_{\hat{\alpha},n}\|^2$ increases like n as $n \to +\infty$. More precisely, we have

$$-C + n\pi^{\hat{\alpha}+1} \leq \|\Psi_{\hat{\alpha},n}\|^2 \leq (2n)\pi^{\hat{\alpha}+1} + C.$$

Let us compare $\Sigma_\alpha(g_{\hat{\alpha},n})$ and $\Sigma_\alpha(g_{\hat{\alpha}})$ as $n \to +\infty$. We have

$$g'_{\hat{\alpha},n}(t) = \frac{1}{n}\chi'\left(\frac{t}{n}\right) g_{\hat{\alpha}}(t) + \chi\left(\frac{t}{n}\right) g'_{\hat{\alpha}}(t).$$

The more problematic term is:

$$\frac{1}{n^2}\int_{-\infty}^{+\infty} \chi'\left(\frac{t}{n}\right)^2 g_{\hat{\alpha}}^2(t) F(t)\, dt.$$

But this term is less than $\frac{C}{n^2}\|\Psi_{\hat{\alpha},n}\|^2$, that is of order $n \times \mathcal{O}(1/n^2) = \mathcal{O}(1/n)$. The other terms appearing in the computation of $\Sigma_\alpha(g_{\hat{\alpha},n}) - \Sigma_\alpha(g_{\hat{\alpha}})$ are $\mathcal{O}(1/n)$. Now, observing that $\Sigma_\alpha(g_{\hat{\alpha}}) < 0$, we get, for n large enough, that

$$\langle L^{\mathrm{new}}\Psi \mid \Psi\rangle \leq \Sigma(g_{\hat{\alpha}}) + \frac{C}{n} + \|\Psi_{\hat{\alpha},n}\|^2 < \|\Psi_{\hat{\alpha},n}\|^2.$$

This shows property (6.21).

6.2.6 Application

Coming back to the initial problem, we have shown that

$$\inf \sigma(P(\beta)) \geq (\Theta_0(\beta_{13}^2 + \beta_{12}^2) + \beta_{23}^2)(\beta_{13}^2 + \beta_{12}^2 + \beta_{23}^2)^{-\frac{1}{2}}.$$

Moreover, one verifies that we have equality when $\beta_{23} = 0$:

$$\inf \sigma(P(\beta)) = (\Theta_0(\beta_{13}^2 + \beta_{12}^2))^{\frac{1}{2}}.$$

This clearly shows that when $|\beta| = 1$ is fixed, the energy is minimal when the magnetic field is parallel to the hyperplane $x_1 = 0$.

6.3 Notes

1. The case of $\mathbb{R}^3$ was also studied by Fock and Landau [Fo, Lan].
2. The first results can be found in Kato [Kat1] and Avron–Herbst–Simon [AvHS1]–[AvHS3].
3. The results presented in Section 6.2 were first obtained by Lu–Pan in [LuP7], Pan in [Pa6], and then rewritten in papers by Helffer–Morame [HeM3]–[HeM6], see also [Ar2].
4. Further estimates and identities for the half-space model are obtained in the recent work [Ra3].
5. Proposition 6.2.3 has been established in [HeM4].

7

Introduction to Semiclassical Methods for the Schrödinger Operator with a Large Electric Potential

In this chapter, we present one of the basic techniques for analyzing the ground state energy (also called lowest eigenvalue or principal eigenvalue) of a Schrödinger operator in the case when the electric potential V has non-degenerate minima, in the limit of large coupling constant B. This problem turns out to be a semiclassical problem.

7.1 Harmonic Approximation

7.1.1 Upper bounds

The case of the one-dimensional Schrödinger operator

We start with the simplest one-well problem:

$$P_{B^2 v} := -\frac{d^2}{dx^2} + B^2 v(x)\,, \tag{7.1}$$

where v is a C^∞-function tending to $+\infty$ at $|x| = \infty$ and having a unique minimum at 0 with

$$v(0) = 0\,.$$

Let us assume that

$$v''(0) > 0\,. \tag{7.2}$$

In this very simple case, the harmonic approximation is an elementary exercise. We first consider the harmonic oscillator associated with 0:

$$-\frac{d^2}{dx^2} + \frac{1}{2} v''(0) B^2 x^2\,. \tag{7.3}$$

This means that we replace the potential v by its quadratic approximation at 0, namely, $\frac{1}{2}v''(0)x^2$, and consider the associated Schrödinger operator.

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Using the dilation $x = B^{-\frac{1}{2}} y$, we observe that this operator is unitarily equivalent to

$$B\left[-\frac{d^2}{dy^2} + \frac{1}{2} v''(0) y^2\right] . \tag{7.4}$$

Consequently, as we have seen in Section 3.1, the eigenvalues are given by

$$\lambda_n(B) = B\,\lambda_n(1) = (2n+1) B \sqrt{\frac{v''(0)}{2}} , \tag{7.5}$$

and the corresponding eigenfunctions are

$$u_n^B(x) = B^{\frac{1}{4}} u_n(B^{\frac{1}{2}} x) . \tag{7.6}$$

Here[1]

$$u_1(y) = \left(\frac{v''(0)}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\sqrt{\frac{v''(0)}{2}}\, y^2} ,$$

and, by recursion,

$$u_n = c_n \left(\frac{d}{dy} - \sqrt{\frac{v''(0)}{2}} y\right) u_{n-1} , \tag{7.7}$$

where c_n is a normalization constant.

It is easy to see that

$$u_n(y) = P_n(y) e^{-\frac{1}{2}\sqrt{\frac{v''(0)}{2}}\, y^2} ,$$

where P_n is a polynomial of degree $(n-1)$ that can also be obtained recursively.

We now return to the full operator $P_{B^2 v}$. For simplicity, we will only consider the first eigenvalue. We consider the function $u_1^{B,\mathrm{app}}$,

$$x \mapsto \chi(x) u_1^B(x) = \chi(x) \left(\frac{B v''(0)}{2\pi}\right)^{\frac{1}{4}} \exp\left(-\sqrt{\frac{v''(0)}{2}}\,\frac{B}{2}\, x^2\right) ,$$

where χ is compactly supported in a small neighborhood of 0 and equal to 1 in a smaller neighborhood of 0 . Note here that the H^1-norm of this function over the complement to a neighborhood of 0 is exponentially small as $B \to +\infty$. We now get

$$\left(P_{B^2 v} - B \sqrt{\frac{v''(0)}{2}}\right) u_1^{B,\mathrm{app}} = \mathcal{O}(B^{\frac{1}{2}}) . \tag{7.8}$$

[1] We normalize by assuming that the L^2-norm of u_n^B is one. For the first eigenvalue, we have seen that, by assuming in addition that the function is strictly positive, we determine $u_1^B(x)$ completely.

The coefficients corresponding to the commutation of $P_{B^2 v}$ and χ give exponentially small terms, and the main contribution is

$$B^2 \left\| (v(x) - \frac{1}{2} v''(0) x^2) \chi(x) u_1^B(x) \right\|_{L^2} ,$$

which is easily estimated as $\mathcal{O}(B^{\frac{1}{2}})$, observing that

$$\left| v(x) - \frac{1}{2} v''(0) x^2 \right| \leq C |x|^3 \quad \text{for } |x| \leq 1 .$$

The spectral theorem (Theorem 1.3.1) applied to (7.8) gives the existence of $C > 0$ and B_0 such that, for any $B \geq B_0$, there exists an eigenvalue $\lambda(B)$ of $P_{B^2 v}$ such that

$$\left| \lambda(B) - B \sqrt{\frac{v''(0)}{2}} \right| \leq C\, B^{\frac{1}{2}} .$$

In particular, we get the inequality

$$\lambda_1(B) \leq B \sqrt{\frac{v''(0)}{2}} + C\, B^{\frac{1}{2}} . \tag{7.9}$$

Combining with a lower bound (which will be obtained in the next subsection), one can actually prove that

$$\left| \lambda_1(B) - B \sqrt{\frac{v''(0)}{2}} \right| \leq C\, B^{\frac{1}{2}} . \tag{7.10}$$

The harmonic approximation in general: Upper bounds

In the multidimensional case, we can proceed essentially in the same way. The analysis of the quadratic case

$$H(-i\partial_x, Bx) := -\Delta + \frac{1}{2} B^2 \langle Ax \mid x \rangle$$

can be done explicitly by diagonalizing A via an orthogonal matrix U. There is a corresponding unitary transformation on $L^2(\mathbb{R}^n)$ defined by

$$(\mathcal{U} f)(x) = f(U^{-1} x)$$

such that

$$\mathcal{U}^{-1} H \mathcal{U} = \sum_{j=1}^{n} \left(-\partial_{y_j}^2 + \frac{\mu_j}{2} B^2 y_j^2 \right) .$$

Using the Hermite functions as quasimodes, we get the upper bounds by $B \sum_{j=1}^n \sqrt{\mu_j/2} + \mathcal{O}(B^{\frac{1}{2}})$ as in the one-dimensional case.

Case of multiple minima

When there is more than one minimum, one can apply the above construction near each of the minima. The upper bound for the ground state is obtained by taking the infimum over all the minima of the upper bound attached to each minimum.

7.1.2 Harmonic approximation in general: Lower bounds

It is rather standard[2] to show the existence of a constant C and, for any $R > 0$, of a covering of $\mathbb{R}^n$, by balls of radius R, $\mathrm{D}(x^j, R)$ $(j \in \mathcal{J})$, and of a corresponding partition of unity with $\operatorname{supp} \phi_j^R \subset \mathrm{D}(x^j, R)$, such that,

$$\sum_{j\in\mathcal{J}} (\phi_j^R)^2 = 1\,, \qquad \sum_{\ell=1}^{n}\sum_{j\in\mathcal{J}} |\partial_{x_\ell}\phi_j^R|^2 \leq \frac{C}{R^2}\,. \tag{7.11}$$

Using this partition of unity, we can then write, for all $u \in C_0^\infty$,

$$\begin{aligned}\langle P_{B^2V} u \mid u\rangle &= \sum_j \langle P_{B^2V}\phi_j^R u \mid \phi_j^R u\rangle - \sum_{j,\ell} \| |\partial_{x_\ell}\phi_j^R| u\|^2 \\ &\geq \sum_j \langle P_{B^2V}\phi_j^R u \mid \phi_j^R u\rangle - \frac{C}{R^2}\|u\|^2\,. \end{aligned} \tag{7.12}$$

We now suppose

$$R \in]0,1]\,.$$

Possibly taking a larger C, we can in addition assume either that the balls are centered at the minima of V (denoted by x^{j_k}, $k \in \mathcal{K}$) or that the balls are at a distance at least R/C from these minima.

In the first case, we observe that

$$\left|\langle P_{B^2V}\phi_j^R u \mid \phi_j^R u\rangle - \langle P^k_{B^2V}\phi_j^R u \mid \phi_j^R u\rangle\right| \leq C\, B^2 R^3 \|\phi_j^R u\|^2\,, \tag{7.13}$$

where $P^k_{B^2V}$ is the quadratic approximation model at the minimum x^{j_k}, i.e., the operator obtained by replacing V by its quadratic approximation

$$V^k(x) = \inf V + \frac{1}{2}\langle \operatorname{Hess} V(x^{j_k})(x - x^{j_k}) \,|\, (x - x^{j_k})\rangle\,,$$

if the ball is centered at the minimum.

In the second case, we use the fact that the minima of V are nondegenerate, and find

$$\langle P_{B^2V}\phi_j^R u \mid \phi_j^R u\rangle \geq \left(B^2 \inf V + \frac{B^2R^2}{C}\right) \|\phi_j^R u\|^2\,. \tag{7.14}$$

---

[2] One can first construct a covering by balls of radius 1 and an associate partition of unity. We then get the general family by a dilation.

The optimization between the two errors appearing in (7.12) and (7.13) leads to the choice $R^{-2} = B^2 R^3$, that is to the choice

$$R = B^{-\frac{2}{5}}\,, \tag{7.15}$$

and we then observe that $R^2/C = B^{-\frac{4}{5}}/C$, which is dominant in comparison with B^{-1} as $B \to +\infty$. We then get the lower bound

$$\frac{\lambda_1(B)}{B^2} \geq \inf V + B^{-2}(\inf_k \lambda_1(B, x^{j_k})) - CB^{-\frac{6}{5}}\,, \tag{7.16}$$

where the infimum is over the various minima x^{j_k} (assumed to be nondegenerate) and $\lambda_1(B, x^{j_k})$ denotes the lowest eigenvalue of the harmonic approximation at x^{j_k} $P^k_{B^2V}$. Recall, by the explicit calculation on the harmonic oscillator, that $\lambda_1(B, x^{j_k})$ has order of magnitude B [see (7.5)], and so the error term in (7.16) is indeed small.

7.1.3 The case with magnetic field

Let us consider two situations.

V has a nondegenerate minimum.

The first case is the case when V has a nondegenerate minimum at 0, with $V(0) = 0$. In this case, the model that gives the approximation is

$$\sum_{j=1}^{n}(-i\partial_{x_j} + BA^0_j)^2 + \frac{B^2}{2}\langle V''(0)x \mid x\rangle\,,$$

where $\mathbf{A}^0$ is a linear magnetic potential generating the constant magnetic field $\beta_{jk} = \beta_{jk}(0)$:

$$A^0_j(x) = \frac{1}{2}\Big(\sum_k \beta_{jk}x_k\Big)\,.$$

Therefore, in a suitable gauge [note that by a linear gauge, one can first reduce to the case when $\mathbf{A}(0) = 0$],

$$\mathbf{A}(x) - \mathbf{A}^0(x) = \mathcal{O}(|x|^2)\,.$$

After the dilation $x = B^{-\frac{1}{2}}y$, we get the operator

$$B\left(\sum_{j=1}^{n}(-i\partial_{y_j} + A^0_j)^2 + \frac{1}{2}\langle V''(0)y \mid y\rangle\right)\,,$$

whose spectrum can be determined explicitly.

Let us treat the two-dimensional case as an exercise. We start from

$$-\partial_{x_1}^2 + (-i\partial_{x_2} + Bx_1)^2 + \frac{h_1}{2}x_1^2 + \frac{h_2}{2}x_2^2\,,$$

with $h_1 > 0, h_2 > 0$.

A partial Fourier transform with respect to the x_2-variable leads to

$$-\partial_{x_1}^2 + (\xi_2 + Bx_1)^2 + \frac{h_1}{2}x_1^2 - \frac{h_2}{2}\partial_{\xi_2}^2\,.$$

A dilation leads to the standard Schrödinger operator

$$-\partial_t^2 - \partial_s^2 + \left(\sqrt{\frac{h_2}{2}}s + Bt\right)^2 + \frac{h_1}{2}t^2\,.$$

So we have proved the isospectrality of the initial operator with a standard Schrödinger operator, with quadratic electric potential

$$V^{\text{new}}(s,t) = \left(\sqrt{\frac{h_2}{2}}s + Bt\right)^2 + \frac{h_1}{2}t^2\,.$$

Its ground state energy is immediately computed as

$$\lambda(B) = \sqrt{\lambda(0)^2 + B^2}\,, \quad \text{with} \quad \lambda(0) = \left(\sqrt{h_1} + \sqrt{h_2}\right)/\sqrt{2}\,.$$

In this formula, one explicitly sees the diamagnetic effect announced in Section 2.1 and also that

$$\lambda(B) - |B| \leq \lambda(0)\,, \tag{7.17}$$

which is more specific to the quadratic case (paramagnetic inequality).

Lower bounds.

The lower bound is obtained similarly to the case without a magnetic field once we have observed that

$$\Re\langle P_{B\mathbf{A},B^2V}u \mid u\rangle = \sum_j \langle P_{B\mathbf{A},B^2V}\phi_j^R u \mid \phi_j^R u\rangle - \sum_{j,\ell} \||\partial_{x_\ell}\phi_j^R|\, u\|^2\,. \tag{7.18}$$

Then, for the balls containing the minima, we must replace the magnetic potential by its affine approximation at the minimum and control the remainder. Note that there is a "small" additional difficulty (of the same type as for the manifold case) of controlling the term corresponding to the approximation of the magnetic potential.

Let us more precisely describe what is going on. A new control is only necessary for the balls centered at one of the minima. The idea is that the harmonic approximation at the minimum [we choose one of the minima, taking coordinates such that 0 is the minimum of V, so $V(0)=0$ and $\nabla V(0)=0$] has to be replaced by

$$P_B^{\mathrm{app},0} := \sum_\ell (-i\partial_{x_\ell} + BA_\ell^{\mathrm{lin}}(x))^2 + \frac{B^2}{2}\,\mathrm{Hess}\,V(0)x\cdot x\,.$$

We recall from the previous section that this spectrum is known and equal to B times the spectrum computed for $B=1$, as immediately seen by the dilation $x = B^{-\frac{1}{2}}y$.

After a gauge transformation, we can assume that

$$\mathbf{A}(x) - \mathbf{A}^{\mathrm{lin}}(x) = \mathcal{O}(|x|^2)$$

and note that the magnetic field generated by $\mathbf{A}^{\mathrm{lin}}(x)$ is the value of the magnetic field generated by $\mathbf{A}$ evaluated at 0.

We now take $R = B^{-\frac{2}{5}}$ and write

$$\begin{aligned}\langle P_{B\mathbf{A},B^2V}\phi_j^R u \mid \phi_j^R u\rangle &\geq \langle P_B^{\mathrm{app},0}\phi_j^R u \mid \phi_j^R u\rangle - CB^{\frac{4}{5}}\|\phi_j^R u\|^2\\ &\quad - B\int |(\mathbf{A}(x)-\mathbf{A}^{\mathrm{lin}}(x))\phi_j^R u|\cdot|(-i\nabla + B\mathbf{A}^{\mathrm{lin}}(x))\phi_j^R u|\,dx\,.\end{aligned}$$

This leads first (omitting the reference to R, which is now fixed) to

$$\begin{aligned}\langle P_{B\mathbf{A},B^2V}\phi_j u \mid \phi_j u\rangle \geq \langle P_B^{\mathrm{app},0}\phi_j u \mid \phi_j u\rangle - CB^{\frac{4}{5}}\|\phi_j u\|^2&\\ - CB^{\frac{1}{5}}\|\phi_j u\|\cdot\|(-i\nabla + B\mathbf{A}^{\mathrm{lin}}(x))\phi_j u\|&\,.\end{aligned}$$

Using the Cauchy–Schwarz inequality with some (to be determined) weight $\rho(B)$, we obtain

$$\begin{aligned}&\langle P_{B\mathbf{A},B^2V}\phi_j u \mid \phi_j u\rangle\\ &\quad\geq \langle P_B^{\mathrm{app},0}\phi_j u \mid \phi_j u\rangle - CB^{\frac{4}{5}}\|\phi_j u\|^2\\ &\qquad - CB^{\frac{1}{5}}\left(\frac{1}{\rho(B)^2}\|\phi_j u\|^2 + \rho(B)^2\|(\nabla + iB\mathbf{A}^{\mathrm{lin}}(x))\phi_j u\|^2\right)\\ &\quad\geq (1 - C\,B^{\frac{1}{5}}\rho(B)^2)\,\langle P_B^{\mathrm{app},0}\phi_j u \mid \phi_j u\rangle\\ &\qquad - CB^{\frac{4}{5}}\|\phi_j u\|^2 - C\,B^{\frac{1}{5}}\,\rho(B)^{-2}\|\phi_j u\|^2\,.\end{aligned}$$

The choice of $\rho(B) = B^{-\frac{3}{10}}$ leads to

$$\langle P_{B\mathbf{A},B^2V}\phi_j^R u \mid \phi_j^R u\rangle \geq (1 - C\,B^{-\frac{2}{5}})\,\langle P_B^{\mathrm{app},0}\,\phi_j^R u \mid \phi_j^R u\rangle - C\,B^{\frac{4}{5}}\|\phi_j^R u\|^2\,.$$

We are now essentially in the same situation as in the case without magnetic field.

Magnetic wells

We would like to describe a case where no electric potential is present. We consider the rather generic case when $\beta \in C^\infty(\overline{\Omega})$ satisfies, for some $(x_0, y_0) \in \Omega$,

$$\beta(x,y) > b := \beta(z_0) > 0, \quad \forall (x,y) \in \overline{\Omega} \setminus \{(x_0,y_0)\}, \tag{7.19}$$

and we assume that the minimum is nondegenerate:

$$\operatorname{Hess} \beta(x_0, y_0) > 0\,. \tag{7.20}$$

We introduce in this case the notation

$$a = \operatorname{Tr}\left(\frac{1}{2} \operatorname{Hess} \beta(x_0, y_0)\right)^{1/2}\,. \tag{7.21}$$

Theorem 7.1.1.
If $\mathbf{A} \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$, *and if the hypotheses* (7.19) *and* (7.20) *are satisfied, then*

$$\lambda_1^D(B\mathbf{A}) = \left(b + \frac{a^2}{2bB}\right) B + o(1)\,, \tag{7.22}$$

where $\lambda_1^D(B\mathbf{A})$ *denotes the ground state energy of the Dirichlet realization.*

The proof is based on the analysis of the simpler model where, near 0,

$$\beta(x,y) = b + \alpha x^2 + \beta y^2. \tag{7.23}$$

In this case, we can also choose (after a gauge transformation) a magnetic potential $\mathbf{A}(x,y)$ such that

$$A_1(x,y) = 0 \qquad \text{and} \qquad A_2(x,y) = bx + \frac{\alpha}{3}x^3 + \beta x y^2\,. \tag{7.24}$$

When β vanishes, other models should be considered. An interesting case is the case when β vanishes along a line. This is related to Montgomery's model described in Section 3.3.

7.2 Decay of Eigenfunctions and Applications

7.2.1 Introduction

As we have already seen when comparing the spectrum of the harmonic oscillator and that of the Schrödinger operator, it could be quite important to know a priori how the eigenfunction associated with an eigenvalue $\lambda(B)$ decays in the "classically forbidden region"—that is, the set of the x's such that $B^2V(x) > \lambda(B)$. The Agmon estimates give a very efficient way to control such a decay.

Let us start with very weak notions of localization. For a family $B \mapsto \psi_B$ of L^2-normalized functions defined in Ω, we will say that the family ψ_B lives (resp. fully lives) in a closed set U of $\overline{\Omega}$ if, for any neighborhood $\mathcal{V}(U)$ of U,

$$\liminf_{B\to+\infty} \int_{\mathcal{V}(U)\cap\Omega} |\psi_B(x)|^2\, dx > 0\,,$$

respectively,

$$\lim_{B\to+\infty} \int_{\mathcal{V}(U)\cap\Omega} |\psi_B(x)|^2\, dx = 1\,.$$

For example, we expect that the ground state of the Schrödinger operator $-\Delta + B^2V(x)$ fully lives in $V^{-1}(\inf V)$. Similarly, we expect that if[3]

$$\limsup_{B\to+\infty} \frac{\lambda(B)}{B^2} \leq E < \inf \sigma_{\rm ess}(B^{-2}P_{B^2V}) - \epsilon_0$$

(for $\epsilon_0 > 0$) and if ψ_B is an eigenfunction associated with $\lambda(B)$, then ψ_B will fully live in $V^{-1}(]-\infty, E])$. This is the way we can understand that in the semiclassical limit (remembering that the semiclassical parameter h is $1/B$) the quantum mechanics should recover the classical mechanics.

Of course, the above is very heuristic, but there are more accurate mathematical notions like the frequency set (see [Ro]) permitting us to give a mathematical formulation to the above vague statements.

Once we have determined a closed set U, where ψ_B fully lives (and hopefully the smallest), it is interesting to discuss the behavior of ψ_B outside U, and to measure how ψ_B decays in this region.

To illustrate the discussion, we can start with the very explicit example of the harmonic oscillator. The ground state $x \mapsto \pi^{-\frac{1}{4}}\, B^{\frac{1}{4}} \exp(-\frac{B}{2}x^2)$ of $-\frac{d^2}{dx^2} + B^2x^2$ lives at 0 and is exponentially decaying in any interval $[a,b]$ such that $0 \notin [a,b]$. This is this type of result that we will recover but without having an explicit expression for ψ_B.

7.2.2 Energy inequalities

The main but basic tool is a very simple identity for the Schrödinger operator $P_{B\mathbf{A},B^2V}$.

Proposition 7.2.1.
Let Ω be a bounded open domain in $\mathbb{R}^n$ with C^2 boundary. Let $V \in C^0(\overline{\Omega};\mathbb{R})$, $\mathbf{A} \in C^0(\overline{\Omega};\mathbb{R}^n)$ and ϕ be a realvalued Lipschitzian function on $\overline{\Omega}$. Then, for any $u \in C^2(\overline{\Omega};\mathbb{C})$ satisfying

- *either the Dirichlet condition $u\big|_{\partial\Omega} = 0$,*

[3] This is the case, in particular, when $\liminf_{|x|\to+\infty} V(x) > \inf V$.

- *or the magnetic Neumann condition* $\nu \cdot (\nabla u + iB\mathbf{A}u)\big|_{\partial\Omega} = 0\,,$

we have

$$\int_\Omega |\nabla_{B\mathbf{A}}\,(e^{B\phi}u)|^2\,dx + B^2\int_\Omega (V - |\nabla\phi|^2)e^{2B\phi}|u|^2\,dx$$
$$= \Re\left(\int_\Omega e^{2B\phi}\,\overline{(P_{B\mathbf{A},B^2V}u)(x)}\cdot u(x)\,dx\right)\,. \tag{7.25}$$

Proof.
In the case when ϕ is a $C^2(\overline{\Omega})$-function and $\mathbf{A} = 0$, this is an immediate consequence of the Green–Riemann formula:

$$\int_\Omega \overline{\nabla v}\cdot\nabla w\,dx = -\int_\Omega \overline{\Delta v}\cdot w\,dx - \int_{\partial\Omega}\overline{(\partial v/\partial\nu)}\cdot w\,d\sigma_{\partial\Omega}\,, \tag{7.26}$$

where $\partial v/\partial\nu$ is the normal derivative at the boundary

$$(\partial v/\partial\nu) = (\nu\cdot\nabla v)\big|_{\partial\Omega}\,.$$

This gives in particular

$$\int_\Omega \overline{\nabla v}\cdot\nabla w\,dx = -\int_\Omega \overline{\Delta v}\cdot w\,dx\,, \tag{7.27}$$

for all $v, w \in C^2(\overline{\Omega})$ such that $w\big|_{\partial\Omega} = 0$ or $\partial v/\partial\nu\big|_{\partial\Omega} = 0$. This can actually be extended to $v, w \in H^1_0(\Omega)$.

We then observe (we still treat the case when $\mathbf{A} = 0$)

$$\begin{aligned}\Re\int e^{2B\phi}\overline{(-\Delta u)}\cdot u\,dx &= \Re\int \overline{(\nabla u)}\cdot(\nabla e^{2B\phi}u)\,dx\\ &= \Re\int \overline{((\nabla - B\nabla\phi)\,e^{B\phi}u)}\cdot(\nabla + B\nabla\phi)e^{B\phi}u)\,dx\\ &= \int |(\nabla e^{B\phi}u)|^2\,dx\ - B^2\int |\nabla\phi|^2|e^{B\phi}u|^2\,dx\,.\end{aligned}$$

The case when $\mathbf{A}$ is nonzero is treated similarly. Using the gauge invariance, one can first treat the case when $\mathbf{A}\cdot\nu$ vanishes at the boundary.

To treat more general ϕ's, we just write ϕ as a limit as $\epsilon \to 0$ of $\phi_\epsilon = \chi_\epsilon * \phi$, where $\chi_\epsilon(x) = \chi(x/\epsilon)\,\epsilon^{-n}$ is the standard mollifier, and we remark that $\nabla\phi$ is almost everywhere the limit of $\nabla\phi_\epsilon = \nabla\chi_\epsilon * \phi$. When $\mathbf{A}$ is nonzero, we must additionally use

$$\begin{aligned}\int_\Omega \overline{\nabla_{B\mathbf{A}}v}\cdot\nabla_{B\mathbf{A}}w\,dx = &-\int_\Omega \overline{\Delta_{B\mathbf{A}}v}\cdot w\,dx\\ &-\int_{\partial\Omega}\overline{(\partial v/\partial\nu + iB\mathbf{A}\cdot\nu\,v)}\cdot w\,d\sigma_{\partial\Omega}\,.\end{aligned} \tag{7.28}$$

This identity can be used in the two cases considered in the proposition. □

Note that the proposition is also true for $u \in H^2(\Omega)$.

7.2.3 The Agmon distance

The Agmon metric associated with an energy E and a potential V is defined as $(V-E)_+dx^2$, where dx^2 is the standard metric on $\mathbb{R}^n$. This metric is degenerate and vanishes identically in the "classical" region: $\{x \mid V(x) \leq E\}$. Associated with the Agmon metric, we define a natural distance

$$(x,y) \mapsto d_{(V-E)_+}(x,y)$$

by taking the infimum:

$$d_{(V-E)_+}(x,y) = \inf_{\gamma \in C^{1,pw}([0,1];x,y)} \int_0^1 [(V(\gamma(t))-E)_+]^{\frac{1}{2}} |\gamma'(t)| dt\,, \tag{7.29}$$

where $C^{1,pw}([0,1];x,y)$ is the set of the piecewise C^1 paths in $\mathbb{R}^n$ connecting x and y:

$$C^{1,pw}([0,1];x,y) = \{\gamma \in C^{1,pw}([0,1];\mathbb{R}^n)\,,\ \gamma(0)=x\,,\ \gamma(1)=y\}\,. \tag{7.30}$$

When there is no ambiguity, we shall write more simply

$$d_{(V-E)_+} = d\,. \tag{7.31}$$

Similarly to the Euclidean case [which corresponds to $(V-E)_+$ replaced by 1], we obtain the following properties:

- Triangle inequality

$$|d(x',y)-d(x,y)| \leq d(x',x)\,, \qquad \forall x,x',y \in \mathbb{R}^m\,. \tag{7.32}$$

- $$|\nabla_x d(x,y)|^2 \leq (V-E)_+(x)\,, \tag{7.33}$$

  almost everywhere.

We observe that the second inequality is satisfied for other distances like

$$d(x,U) = \inf_{y \in U} d(x,y)\,.$$

The most useful case will be the case when U is the set $\{x \mid V(x) \leq E\}$. In this case, $d(x,U)$ is the distance to the classical region associated with the energy E. In this case, we will write

$$d_E(x) = d(x, \{x \mid V(x) \leq E\})\,. \tag{7.34}$$

As these notions are expressed in terms of metrics, they can easily be extended to manifolds.

7.2.4 Decay of eigenfunctions for the Schrödinger operator

When u_B is a normalized eigenfunction of the Dirichlet realization in Ω satisfying

$$P_{B\mathbf{A},B^2V} u_B = \lambda_B u_B \,,$$

then the identity (7.25) roughly states that $\exp(B\phi)\ u_B$ is well controlled (in L^2) in a region

$$\Omega_1(\epsilon_1, B) = \left\{ x \mid V(x) - |\nabla\phi(x)|^2 - \frac{\lambda_B}{B^2} > \epsilon_1 > 0 \right\} ,$$

by $\exp(\sup_{\Omega\setminus\Omega_1} B\phi(x))$. The choice of a suitable ϕ (possibly depending on B) is related to the Agmon metric $(V-E)_+\, dx^2$ when $\lambda_B/B^2 \to E$ as $B \to +\infty$. The typical choice is $\phi(x) = (1-\epsilon) d_E(x)$, where $d_E(x)$ is introduced in (7.34). In this case, we get that the eigenfunction is localized inside a small neighborhood of the classical region and we can measure the decay of the eigenfunction outside the classical region by

$$\exp[(1-\epsilon)Bd_E(x)]\ u_B = \mathcal{O}(\exp \epsilon B)\,, \tag{7.35}$$

for any $\epsilon > 0$.

More precisely, we get the following theorem:

Theorem 7.2.2. *:*
Let us assume that V is C^∞, semibounded and satisfies

$$\liminf_{|x|\to\infty} V > \inf V\,. \tag{7.36}$$

Let E be such that

$$\inf V \leq E < \liminf_{|x|\to\infty} V\,. \tag{7.37}$$

Let u_B be a (family of L^2-) normalized eigenfunctions such that

$$P_{B\mathbf{A},B^2V} u_B = \lambda_B u_B \,, \tag{7.38}$$

with

$$\limsup_{B\to+\infty} \frac{\lambda_B}{B^2} \leq E\,. \tag{7.39}$$

Then, for all $\epsilon > 0$ and all compact $K \subset \mathbb{R}^m$, there exists a constant $C_{\epsilon,K}$ such that for B large enough,

$$\|\nabla_{B\mathbf{A}}(e^{Bd_E} u_B)\|_{L^2(K)} + \|e^{Bd_E} u_B\|_{L^2(K)} \leq C_{\epsilon,K} \exp \epsilon B\,. \tag{7.40}$$

Remark 7.2.3.
Useful improvements in the case when $E = \min V$ and when the minima are nondegenerate can be obtained by controlling what is going on near the minima of V more carefully with respect to B. It is also possible to control the eigenfunction at ∞. This was actually the initial goal of Agmon (see also Theorem B.5.1).

Proof.
Let us choose some $\delta \in]0,1[$. We shall use the identity (7.25) with

- V replaced by $V - \frac{\lambda_B}{B^2}$,
- $\phi = (1-\delta) d_E(x)$,
- $P_{B\mathbf{A},B^2V}$ replaced by $-\Delta_{B\mathbf{A}} + B^2 V - \lambda_B$.

Let

$$\Omega_\delta^+ = \{x \in \Omega\,,\ V(x) > E + \delta\}\,, \qquad \Omega_\delta^- = \{x \in \Omega\,,\ V(x) \leq E + \delta\}\,.$$

We deduce from (7.25)

$$\int_\Omega |\nabla_{B\mathbf{A}}(\exp B\phi\, u_B)|^2 dx + B^2 \int_{\Omega_\delta^+} \left(V - \frac{\lambda_B}{B^2} - |\nabla\phi|^2\right) \exp 2B\phi\, |u_B|^2\, dx$$
$$\leq B^2 \sup_{x\in\Omega_\delta^-} \left| V(x) - \frac{\lambda_B}{B^2} - |\nabla\phi|^2 \right| \left(\int_{\Omega_\delta^-} \exp 2B\phi\, |u_B|^2\, dx \right)\,.$$

Then, for some constant C independent of $B \in [B_0, +\infty[$ and $\delta \in]0,1[$, we get

$$\int_\Omega |\nabla_{B\mathbf{A}}(\exp B\phi\, u_B)|^2 dx + B^2 \int_{\Omega_\delta^+} \left(V - \frac{\lambda_B}{B^2} - |\nabla\phi|^2\right) \exp 2B\phi\, |u_B|^2\, dx$$
$$\leq C B^2 \int_{\Omega_\delta^-} \exp 2B\phi\, |u_B|^2\, dx\,.$$

Let us observe now that on Ω_δ^+ we have [with $\phi = (1-\delta) d(\cdot, U)$]

$$V - \frac{\lambda_B}{B^2} - |\nabla\phi|^2 \geq (2-\delta)\delta^2 + o(1)\,,$$

as $B \to \infty$. Choosing $B(\delta)$ large enough, we then get, for any $B \in [B(\delta), +\infty[$,

$$V - \frac{\lambda_B}{B^2} - |\nabla\phi|^2 \geq \delta^2\,.$$

This permits us to get the estimate

$$\int_\Omega |\nabla_{B\mathbf{A}}(\exp B\phi\, u_B)|^2 dx + \delta^2 B^2 \int_{\Omega_\delta^+} \exp 2B\phi\, |u_B|^2\, dx$$
$$\leq C\ B^2 \int_{\Omega_\delta^-} \exp 2B\phi\, |u_B|^2\, dx\,,$$

and finally,

$$\int_\Omega |\nabla_{B\mathbf{A}}(\exp B\phi\, u_B)|^2 dx + \delta^2 B^2 \int_\Omega \exp 2B\phi\, |u_B|^2\, dx \leq \tilde{C} B^2 \exp a(\delta) B\,,$$

where $a(\delta) = 2 \sup_{x\in\Omega_\delta^-} \phi(x)$. We now observe that $\lim_{\delta\to 0} a(\delta) = 0$ and the end of the proof is then easy. □

Remark 7.2.4.
When V has a unique nondegenerate minimum, the estimate can be improved when, for some $C_0 > 0$, λ_B belongs to the interval $[EB^2, EB^2 + C_0 B]$. We take in the previous proof

$$\delta = CB^{-1}, \quad \textit{and} \quad \phi = d_E - CB^{-1} \inf(\log(Bd_E), \log C)\,,$$

for some $C \geq 1$. We observe indeed that $V - E$, d_E, and $|\nabla d_E|^2$ are equivalent in the neighborhood of the well. We can then replace $C_{\epsilon,K} \exp(\epsilon B)$ in (7.40) by $C_K\, B^{N_0}$ for some $N_0 \in \mathbb{R}$.

7.2.5 Applications

As an example of an application, we can compare different Dirichlet problems corresponding to different open sets Ω_1 and Ω_2 containing a unique well U associated with an energy E. If, for example, $\Omega_1 \subset \Omega_2$, we can prove the existence of a bijection b between the spectrum of $B^{-2} P_{B\mathbf{A}, B^2 V, \Omega_1}$ in an interval $I(B)$ tending (as $B \to +\infty$) to E and the corresponding spectrum of $B^{-2} P_{B\mathbf{A}, B^2 V, \Omega_2}$ such that $|b(\lambda) - \lambda| = \mathcal{O}(\exp -BS)$ [under a weak assumption on the spectrum at $\partial I(B)$]. Here S is any constant such that

$$0 < S < d_{(V-E)_+}(\partial \Omega_1, U)\,.$$

This can actually be improved (using more sophisticated perturbation theory) as $\mathcal{O}_\epsilon(\exp -2B(S - \epsilon))$, for any $\epsilon > 0$.

Let us just give a hint about the proof. If $(u_B^{(2)}, \lambda_B^{(2)})$ is a family of spectral pairs of the Dirichlet realization of the Schrödinger operator in Ω_2, then, if χ is a cutoff function with compact support in Ω_1, which is equal to 1 on a neighborhood of U, we can use $\chi u_B^{(2)}$ as a quasimode for the realization in Ω_1. We observe indeed that

$$(-\Delta_{B\mathbf{A}} + B^2 V - \lambda_B^{(2)})(\chi u_B^{(2)}) = -2(\nabla \chi) \cdot (\nabla_{B\mathbf{A}} u_B^{(2)}) - (\Delta \chi) u_B^{(2)}\,.$$

Hence, the choice of χ and the Agmon decay estimates on $u_B^{(2)}$ permit us to show that the right-hand side is exponentially small as stated.

7.2.6 The case with magnetic fields but without electric potential

In this case, there is no hope to use the result for V, which does not create any localization. The idea is that the role previously played by $V(x)$ is replaced by $|\beta(x)|/B$ for the two-dimensional case [or more generally by $x \mapsto \mathrm{tr}^+(\beta(x)]$. This is due to (1.32) in the case $n = 2$ [$\beta(x)$ of constant sign] and to their extensions. The Agmon distance will be associated with $\frac{1}{B}\, [\mathrm{tr}^+(\beta(x)) - \inf_x \mathrm{tr}^+(\beta(x))]\, dx^2$.

The proof is in two steps: treatment of the case with constant magnetic field and then partition of unity to control the comparison with this case.

This explains, because of the presence of B^{-1} before $|\beta|$, that the decay is measured through a weight of the type $\exp -\alpha\sqrt{B}\phi$, where $\alpha \in]0,1[$ and ϕ should satisfy

$$|\nabla\phi|^2 \leq \mathrm{tr}^+(\beta(x)) - \inf_x \mathrm{tr}^+(\beta(x)),$$

outside a neighborhood of the magnetic well, which is the set of points where $\mathrm{tr}^+(\beta(x)) = \inf_x \mathrm{tr}^+(\beta(x)$. We will come back to this in Chapter 8.

7.3 Notes

1. The aim of this chapter was to explain the semiclassical techniques mainly due to Helffer–Sjöstrand and Simon. The presented material appears in many books or courses (Cycon–Froese–Kirsch–Simon [CyFKS], Helffer [He2, He7, He9], Dimassi–Sjöstrand [DiS], Hislop–Sigal [HiS], Martinez [Mart]) that emphasize various aspects of the theory. We have rewritten the results in the version of a large coupling constant in order to immediately have the needed applications. So the large parameter B corresponds to the small parameter $h = B^{-1}$ which plays the role of the Planck constant in semiclassical analysis.
2. For the harmonic approximation, we follow Simon's approach (see [Simo]). Another approach is described in [He2] and another variant in [DiS]. The reader can also look at another presentation in Chapter 11 of [CyFKS].
3. The specific semiclassical properties of Schrödinger operators were mainly developed to answer questions from Solid-state physics (see [HeS5] and references therein).
4. Note that in the case of the harmonic approximation on a manifold, there is another term that leads to a small change in the argument (see [Simo]). The Laplacian indeed has the form $\sum_{ij} g^{-1/2}\partial_{x_i} g g^{ij} \partial_{x_j} g^{-1/2}$ after a change of function in order to come back to the self-adjoint case.
5. The explicit computations of $\lambda(B)$ are particular cases of theorems due to Matsumoto [Mat] or Matsumoto–Ueki [MatU], but they are actually much older and appear in the analysis of the Gårding–Melin inequality [Me]. We can, for example, refer to Hörmander [Ho, Vol. 3, Lemma 22.3.1 (p. 360)].
6. The fact that inequality (7.17) (which says that the ground state energy of the Pauli operator $P_{B^2V} - |B|$ is lower than in the case without magnetic field) cannot be extended for more general situations has been shown by Avron–Simon [AvS] and Helffer [He3] using the Aharonov–Bohm effect.
7. The detailed proof of Theorem 7.1.1 can be found in [HeM3].
8. Models with a vanishing magnetic field along lines were proposed by Montgomery [Mon]. See also Section 3.3 and the discussion around Theorem 8.6.2. More examples can be treated (see Helffer–Morame [HeM2, HeM3] and more recently Helffer–Kordyukov [HeK1]).

9. The Agmon estimates were developed first in [Ag]. Agmon was actually more interested in the behavior of eigenfunctions at spatial infinity. In the semiclassical context, we refer to [He2] or to the original papers of Helffer–Sjöstrand [HeS1] or Simon [Simo] for details and complements. Microlocal versions of these Agmon estimates are discussed in [Mart].
10. That the derivative of a Lipschitz function can be defined almost everywhere is a standard result due to Rademacher. See, for example, [DiS, p. 50].
11. The decay properties of eigenfunctions were also the object of many contributions. Let us mention Helffer–Sjöstrand [HeS4], Brummelhuis [Bru], Helffer–Nourrigat [HeN2] for typically magnetic effects, Erdös, Martinez, Nakamura, and Sordoni for Gaussian decay properties (see [Er1, So, MartS] and references therein).
12. It can be useful to extend the decay properties of eigenfunctions to the decay properties of the kernel of the resolvent of the operator. The reader is invited to look in [DiS, Proposition 6.6].

8

Large Field Asymptotics of the Magnetic Schrödinger Operator: The Case of Dimension 2

In this chapter, we study the asymptotics of the ground state energy of the magnetic Neumann operator $P^N_{B\mathbf{A},\Omega}$ as the field strength B tends to infinity. We also obtain the localization properties of the ground state. These results are combined to analyze the question of the monotonicity of the ground state energy.

8.1 Main Results

We recall that we have given a rough asymptotic estimate for the ground state energy of the Dirichlet realization, $P^D_{B\mathbf{A},\Omega}$, in dimension 2 (see Theorem 1.4.2) and that by the min-max principle this also gives an upper bound in the case of Neumann boundary conditions. Of course, the case of the Dirichlet realization does not lead to really new phenomena in comparison with the case $\Omega = \mathbb{R}^n$, at least if the condition

$$b < b \tag{8.1}$$

is satisfied, where we introduced the notations

$$\inf_{x\in\overline{\Omega}} |\beta(x)| = b\,, \qquad \inf_{x\in\partial\Omega} |\beta(x)| = b'\,. \tag{8.2}$$

For the Neumann Laplacian, however, the introduction of a boundary can lead to interesting new phenomena. The first "rough" theorem for the Neumann realization is the following:

Theorem 8.1.1.
Suppose that $\Omega \subset \mathbb{R}^2$ *is bounded and smooth. Then*

$$\lim_{B\to\infty} \frac{1}{B} \inf \sigma(P^N_{B\mathbf{A},\Omega}) = \min(b, \Theta_0 b')\,, \tag{8.3}$$

where Θ_0 *is the constant from* (3.24).

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Recall that the constant $\Theta_0 < 1$. Therefore, in the special case where β is constant, we have $\Theta_0 b' < b$. This spectral fact is responsible for the phenomenon of surface superconductivity, as we will see later.

One important theorem that we would like to present is

Theorem 8.1.2.
Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded. If the magnetic field is constant and nonzero, then any ground state corresponding to the Neumann realization is localized as $B \to \infty$ near the boundary of Ω.

These two theorems are not necessarily optimal. In many cases one can give more precise asymptotic expressions for the ground state energy. Also, in many cases the ground state will be localized in a smaller region, i.e., only near a part of the boundary. We give a more precise result for the case of a constant magnetic field.

Theorem 8.1.3.
Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded. If the magnetic field is constant and nonzero, then any ground state corresponding to the Neumann realization is localized as $B \to \infty$ near the points on the boundary where the boundary curvature is maximal.

8.2 Proof of Theorem 8.1.1

8.2.1 Upper bounds

The case when $b = 0$ can be treated independently. The upper bounds are based on the construction of suitable quasimodes. Gaussians can be used in the case when $b \leq \Theta_0 b'$—just as in the proof of Theorem 1.4.2. In the case when $\Theta_0 b' < b$, one should use trial functions obtained by multiplying a boundary tangential Gaussian by a "normal" solution constructed with the help of the first eigenfunction of the model on $\mathbb{R}^+$ (see Section 3.2). More precisely, let x_0 be a point on the boundary where $|\beta(x_0)| = b'$. We can take a system of coordinates $x \mapsto (s,t)$ such that $t(x)$ denotes the distance to the boundary and $s(x)$ is a parametrization of the boundary with $s(x_0) = 0$ (see Section F.1 for details). In these coordinates, the leading-order term of the operator as $B \to \infty$ will look like

$$-\partial_t^2 + (-i\partial_s + Bb't)^2$$

on the half-plane $t > 0$. (More correctly, we should consider $\mathbb{S}^1 \times]0, t_0]$ with Neumann boundary conditions at $t = 0$ and Dirichlet conditions at $t = t_0$.)

The first guess in order to have the lowest possible energy is to consider the function

$$(t,s) \mapsto B^{\frac{1}{4}} e^{i\rho_0 s\sqrt{B}} u_0(B^{\frac{1}{2}} \tau_0 t)\,,$$

where $\mathbb{R}^+ \ni v \mapsto u_0(v)$ is the eigenfunction for the half-line model with $\xi = \xi_0$ and magnetic field equal to 1 (τ_0 and ρ_0 being suitably chosen) in order to get the minimal energy (for the moment, it is an L^∞-eigenfunction).

This leads to the equation

$$-B\tilde{\beta}^2 u_0''(B^{\frac{1}{2}}\tau_0 t) + (B^{\frac{1}{2}}\rho_0 - Bb't)^2 u_0(B^{\frac{1}{2}}\tau_0 t) = \Theta_0 Bb' u_0(B^{\frac{1}{2}}\tau_0 t)\,.$$

So we should take the pair $(\tau_0, \rho_0) = (\sqrt{b'}, \xi_0\tau_0)$.

It then remains to localize with a cutoff function $t \mapsto \chi(t)$ with compact support in $[0, t_0[$ and to localize in the s-direction with a function $s \mapsto \chi_0(s)$ with support in a neighborhood of 0. So the trial function that we choose (for a B-independent constant C and for $\alpha > 0$ arbitrary) is

$$\begin{aligned}\phi_0(t, s; B) = C\ B^{\frac{5}{16}}\ \chi(t)\,\chi_0(s)\\ \times \exp(-\alpha B^{\frac{1}{4}} s^2)\exp(i\xi_0\sqrt{Bb'}s)\ u_0(\sqrt{Bb'}t)\,.\end{aligned} \tag{8.4}$$

Computing the energy of this trial function gives

$$\lambda_1^N(B\mathbf{A}) \leq B\min(b, \Theta_0 b') + o(B)\,, \tag{8.5}$$

which is enough for the analysis of the decay and proves the upper bound part in Theorem 8.1.1.

8.2.2 Lower bounds

Let $0 \leq \rho \leq 1$. We first claim that there exists C such that, for any $R_0 > 0$, we can, by scaling a standard partition of unity of $\mathbb{R}^2$, and by restricting it to $\overline{\Omega}$, find a partition of unity χ_j^B satisfying in Ω

$$\sum_j |\chi_j^B|^2 = 1\,, \tag{8.6}$$

$$\sum_j |\nabla\chi_j^B|^2 \leq C\,R_0^{-2}\,B^{2\rho}\,, \tag{8.7}$$

and

$$\operatorname{supp}(\chi_j^B) \subset Q_j = \mathrm{D}(z_j, R_0\,B^{-\rho})\,, \tag{8.8}$$

where $\mathrm{D}(c, r)$ denotes the open disc in $\mathbb{R}^2$ of center c and radius r. Moreover, we can add the property that:

$$\text{either } \operatorname{supp}\chi_j \cap \partial\Omega = \emptyset\,, \quad \text{or } z_j \in \partial\Omega\,. \tag{8.9}$$

According to the two alternatives in (8.9), we can decompose the sum in (8.6) in the form

$$\sum = \sum_{\text{int}} + \sum_{\text{bnd}}\,,$$

where "int" refers to the j's such that $z_j \in \Omega$ and "bnd" refers to the j's such that $z_j \in \partial\Omega$.

We now implement this partition of unity in the following way

$$Q(u) = \sum_j Q(\chi_j^B u) - \sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \,, \qquad \forall u \in H^1_{B\mathbf{A}}(\Omega)\,. \tag{8.10}$$

Here $Q = Q^N_{B\mathbf{A},\Omega}$ denotes the magnetic quadratic form as defined in (1.11). We can rewrite the right-hand side of (8.10) as the sum of three (types of) terms:

$$Q(u) = \sum_{\text{int}} Q(\chi_j^B u) + \sum_{\text{bnd}} Q(\chi_j^B u) - \sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \,, \ \forall u \in H^1_{B\mathbf{A}}(\Omega)\,. \tag{8.11}$$

For the last term on the right side of (8.11), we get, using (8.7),

$$\sum_j \| \, |\nabla \chi_j^B| \, u \, \|^2 \leq C \, B^{2\rho} \, R_0^{-2} \, \|u\|^2 \,. \tag{8.12}$$

This measures the price to pay when using a fine partition of unity: **If ρ is large, the error due to this localization will be in $\mathcal{O}(B^{2\rho})$.**

We shall later optimize the choice of ρ or of R_0 for our various problems (note that taking R_0 large will only be interesting when $\rho = 1/2$).

The first term on the right-hand side (8.11) can be estimated from below using (1.32). The support of $\chi_j^B u$ is indeed contained in Ω. So we have

$$\sum_{\text{int}} Q(\chi_j^B u) \geq B \sum_{\text{int}} \int \beta(x) |\chi_j^B u|^2 \, dx \,. \tag{8.13}$$

The second term on the right-hand side of (8.11) is the more delicate and corresponds to the specificity of the Neumann problem. We have to find a lower bound for $Q(\chi_j^B u)$ for some j such that $z_j \in \partial\Omega$. We emphasize that z_j depends on B, so we have to be careful in the control of the uniformity.

We will use the boundary coordinates (s,t) defined in Section F.1. Let $z \in \partial\Omega$ and consider functions u supported in the small disc $\mathrm{D}(z, B^{-\rho})$ (where the magnetic parameter B is sufficiently large). We have (F.5) after a change of coordinates. We now choose a convenient gauge. Define

$$\tilde{A}_1(s,t) := -\int_0^t \big(1 - t' k(s)\big) \tilde{\beta}(s,t') \, dt', \qquad \tilde{A}_2(s,t) := 0.$$

With a suitable gauge change, i.e., with the substitution $\tilde{v} := e^{iB\phi} u$ for some function ϕ, we have for $\operatorname{supp} u \subset \mathrm{D}(z, R_0 B^{-\rho})$,

$$\begin{aligned} &\int |(-i\nabla + B\mathbf{A})u|^2 \, dx \\ &\quad = \int (1 - tk(s))^{-1} \big|(-i\partial_s + B\tilde{A}_1)\tilde{v}\big|^2 + (1 - tk(s)) |\partial_t \tilde{v}|^2 \, ds dt \,. \end{aligned} \tag{8.14}$$

Define

$$k_0 := k(0)\,, \qquad \overline{A}(s,t) := -\tilde\beta(0,0)\left(t - \frac{1}{2}t^2 k(0)\right),$$
$$\Delta k(s) := k(s) - k(0)\,, \qquad \tilde b(s,t) := \big(1 - tk(s)\big)\tilde\beta(s,t) - \big(1 - tk(0)\big)\tilde\beta(0,0)\,,$$
$$\tilde a_1(s,t) := -\int_0^t \tilde b(s,t')\,dt'\,.$$

Then we have the estimates in the support of $\tilde v$:

$$|\Delta k| \le CR_0B^{-\rho}\,, \qquad |\tilde b(s,t)| \le CR_0B^{-\rho}\,, \qquad |\tilde a_1(s,t)| \le CR_0B^{-\rho}t\,.$$

Of course, since $t = \mathcal{O}(B^{-\rho})$, one can obtain a t-independent estimate, but we keep the t-dependence for later use.

Let B be so large that $2^{-1} \le (1 - tk(s)) \le 2$ on $\operatorname{supp}\tilde v$. Then we can make the following comparison between (8.14) and the similar constant field, constant curvature formula:

$$\begin{aligned}
&\int |(-i\nabla + B\mathbf{A})u|^2\,dx \\
&\quad \ge (1-\eta)\int (1-tk_0)^{-1}|(-i\partial_s + B\overline{A})\tilde v|^2 + (1-tk_0)|\partial_t \tilde v|^2\,dsdt \\
&\qquad - C\int t\Delta k\big\{\big|(-i\partial_s + B\tilde A_1)\tilde v\big|^2 + |\partial_t\tilde v|^2\big\}\,dsdt \\
&\qquad - \eta^{-1}\int (1-tk_0)^{-1}B^2\tilde a_1^2|\tilde v|^2\,dsdt,
\end{aligned} \tag{8.15}$$

for any $0 < \eta < 2^{-1}$ and any u with $\operatorname{supp} u \subset \mathrm{D}(z, R_0B^{-\rho})$. The first term on the right-hand side is the quadratic form corresponding to constant curvature and constant magnetic field, so we can estimate

$$\begin{aligned}
&\int (1-tk_0)^{-1}|(-i\partial_s + B\overline{A})\tilde v|^2 + (1-tk_0)|\partial_t\tilde v|^2\,dsdt \\
&\qquad \ge \big(\Theta_0 B\beta(z) - \mathcal{C}_1 k\sqrt{B\beta(z)} - C\big)\|\tilde v\|_2^2\,,
\end{aligned} \tag{8.16}$$

using Corollary 5.4.2. Notice that this estimate is uniform, since the boundary curvature is uniformly bounded.

The second term on the right-hand side is estimated by

$$\begin{aligned}
&C\int t\Delta k\big\{\big|(-i\partial_s + B\tilde A_1)\tilde v\big|^2 + |\partial_t\tilde v|^2\big\}\,dsdt \\
&\qquad \le C\widehat{C}B^{-2\rho}\int |(-i\nabla + B\mathbf{A})u|^2\,dx\,,
\end{aligned} \tag{8.17}$$

and consequently involves the left-hand side. Here we use the property that $0 \le t \le CB^{-\rho}$ on $\operatorname{supp}\tilde v$.

The third term is estimated by

$$\eta^{-1} \int (1 - tk_0)^{-1} B^2 \tilde{a}_1^2 |\tilde{v}|^2 \, dsdt \leq \widetilde{C} \eta^{-1} B^{2-4\rho} \, \|\widetilde{v}\|^2 . \tag{8.18}$$

To get a first but nonoptimal estimate, we choose $R_0 = 1$, $\eta = B^{\frac{1}{2}-2\rho}$, $\rho = 3/8$, and conclude from (8.15) and (8.16)–(8.18) that

$$\int |(-i\nabla + B\mathbf{A})u|^2 \, dx \geq \big(\Theta_0 B\beta(z) - CB^{\frac{3}{4}}\big) \|u\|^2 , \tag{8.19}$$

for all u such that $\operatorname{supp} u \subset \mathrm{D}(z, B^{-\rho})$.

Combining this with (8.10), (8.12), and (8.13), we find the lower bound inherent in Theorem 8.1.1. More precisely, we find constants C and B_0 such that, $\forall u \in H^1_{B\mathbf{A}}(\Omega)$ and $\forall B \geq B_0$,

$$\begin{aligned} Q(u) \geq{} & B \sum_{\text{int}} \int \beta(x) |\chi_j^B u|^2 \, dx \\ & + \Theta_0 B \sum_{\text{bnd}} \int \beta(z_j) |\chi_j^B u|^2 \, dx - CB^{\frac{3}{4}} \sum_j \int |\chi_j^B u|^2 \, dx . \end{aligned} \tag{8.20}$$

Upon replacing $\beta(z_j)$ by $\beta(x)$ in each of the terms in the boundary sum, we have actually proved the following.

Proposition 8.2.1.
There exist positive constants C and B_0 such that, with

$$U_\beta(x) := \begin{cases} B\beta(x) , & d(x, \partial\Omega) \geq B^{-\frac{3}{8}} , \\ \Theta_0 B\beta(x) , & d(x, \partial\Omega) < B^{-\frac{3}{8}} , \end{cases} \tag{8.21}$$

we have

$$\int_\Omega |(-i\nabla + B\mathbf{A})u|^2 \, dx \geq \int_\Omega (U_\beta(x) - CB^{\frac{3}{4}}) |u(x)|^2 \, dx , \tag{8.22}$$

for all $u \in H^1_{B\mathbf{A}}(\Omega)$ and all $B \geq B_0$.

In particular, we get the following version of the lower bound corresponding to Theorem 8.1.1.

Proposition 8.2.2.
There exist positive constants C and B_0 such that, for all $B \geq B_0$, we have the estimate

$$\lambda_1(B\mathbf{A}) \geq B \min(b, \Theta_0 b') - C\, B^{\frac{3}{4}} . \tag{8.23}$$

We can also make the choice $\rho = 1/2$, $\eta = B^{-1/8}$ and R_0 large in (8.15). This gives an estimate that may look weaker than Proposition 8.2.1, but that will be more efficient in the study of decay. The reason is that the boundary zone now has the right length scale, namely $B^{-1/2}$. The result analogous to Proposition 8.2.1 is

Proposition 8.2.3. .
There exist $C, B_0 > 0$ *and, for all* $R_0 > 0$, *there exists* $C(R_0)$ *such that with*

$$U_\beta^{(2)}(x) := \begin{cases} B\beta(x)\,, & d(x,\partial\Omega) \geq R_0 B^{-\frac{1}{2}}\,, \\ BC(R_0)\beta(x)\,, & d(x,\partial\Omega) \leq R_0 B^{-\frac{1}{2}}\,, \end{cases} \tag{8.24}$$

we have

$$\int_\Omega |(-i\nabla + B\mathbf{A})u|^2\,dx \geq \int_\Omega \left(U_\beta^{(2)}(x) - C\frac{B}{R_0^2}\right)|u(x)|^2\,dx\,, \tag{8.25}$$

for all $u \in H^1_{B\mathbf{A}}(\Omega)$ *and all* $B \geq B_0$.

8.2.3 Agmon's estimates

We will prove that ground states localize near the boundary in the case where

$$\Theta_0 b' < b \tag{8.26}$$

is satisfied.

We first observe that if Φ is a real and uniformly Lipschitzian function and if u is in the domain of the Neumann realization of $P^N_{B\mathbf{A},\Omega}$, then we have by a simple integration by parts

$$\begin{aligned} &\Re\langle P_{B\mathbf{A},\Omega} u \mid e^{2\sqrt{B}\Phi}u\rangle \\ &\quad = \Re\langle(-i\nabla + B\mathbf{A})u \mid (-i\nabla + B\mathbf{A})e^{2\sqrt{B}\Phi}u\rangle \\ &\quad = \langle(-i\nabla + B\mathbf{A})e^{\sqrt{B}\Phi}u \mid (-i\nabla + B\mathbf{A})e^{\sqrt{B}\Phi}u\rangle - B\big\||\nabla\Phi|\,e^{\sqrt{B}\Phi}u\big\|^2 \\ &\quad = Q_{B\mathbf{A}}(e^{\sqrt{B}\Phi}u) - B\big\||\nabla\Phi|\,e^{\sqrt{B}\Phi}u\big\|^2\,. \end{aligned} \tag{8.27}$$

We now take u to be an eigenfunction associated with the lowest eigenvalue $\lambda_1(B\mathbf{A})$. This gives

$$\lambda_1(B\mathbf{A})\|e^{\sqrt{B}\Phi}u\|^2 = Q_{B\mathbf{A}}(e^{\sqrt{B}\Phi}u) - B\big\||\nabla\Phi|\,e^{\sqrt{B}\Phi}u\big\|^2\,. \tag{8.28}$$

We will obtain strong decay estimates by implementing the upper bound (8.5) and the lower bound of Proposition 8.2.3. Let us take

$$\Phi(x) = \alpha\max(d(x,\partial\Omega), R_0 B^{-\frac{1}{2}}),$$

where $\alpha > 0$ has to be determined, and let us apply Proposition 8.2.3. We first write

$$Q_{B\mathbf{A}}(e^{\sqrt{B}\Phi}u) \geq \int \left(U_\beta^{(2)}(x) - C\frac{B}{R_0^2}\right)\left|e^{\sqrt{B}\Phi(x)}u(x)\right|^2 dx\,. \tag{8.29}$$

Implementing (8.26), (8.5) becomes

$$\lambda_1^N(B\mathbf{A}) \leq \Theta_0\, b'\, B + o(B)\,. \tag{8.30}$$

Using (8.28), we now obtain

$$\begin{aligned}\int_{\{t(x)\geq R_0 B^{-\frac{1}{2}}\}} & \left[\beta(x) - \Theta_0 b' - o(1) - \frac{C}{R_0^2} - \alpha^2\right] e^{2\sqrt{B}\Phi(x)} |u(x)|^2\, dx \\ & \leq \int_{\{t(x)\leq R_0 B^{-\frac{1}{2}}\}} \left[\Theta_0 b' - C(R_0)\beta(x) + o(1) + \frac{C}{R_0^2}\right] e^{2\sqrt{B}\Phi(x)} |u(x)|^2\, dx \\ & \leq C'(R_0) \int_{\{t(x)\leq R_0 B^{-\frac{1}{2}}\}} |u(x)|^2\, dx. \end{aligned} \tag{8.31}$$

This gives the main ingredient of the proof for the following theorem:

Theorem 8.2.4. .
Under condition (8.26)*, there exist* $C > 0$ *,* $\alpha > 0$*, and* $B_0 > 0$ *, such that if* $B \geq B_0$ *and* u_B *is the ground state of* $P^N_{B\mathbf{A},\Omega}$ *, then*

$$\int_\Omega e^{2\alpha\sqrt{B}\, d(x,\partial\Omega)} \left\{|u_B(x)|^2 + B^{-1}|p_{B\mathbf{A}} u_B(x)|^2\right\} dx \leq C\, \|u_B\|^2\,. \tag{8.32}$$

More generally, for $\delta > 0$ *, there exist* $C, \alpha, B_0 > 0$ *such that if* u_B *is an eigenfunction of* $P^N_{B\mathbf{A},\Omega}$ *with eigenvalue smaller than* $(1-\delta)Bb$ *, and* $B \geq B_0$ *, then* (8.32) *holds.*

Remark 8.2.5.
Notice that Theorem 8.2.4 is a precise version of Theorem 8.1.2.

Note that condition (8.26) is always satisfied when β is constant, because $\Theta_0 < 1$ and in that case $b = b'$.

Proof.
We only consider the case of the ground state. From (8.31), we see that for all $\alpha < \sqrt{b - \Theta_0 b'}$, we can choose R_0 sufficiently large and get (for large B) the inequality

$$\int e^{2\sqrt{B}\Phi} |u(x)|^2\, dx \leq C(\alpha, R_0) \int_{\{t(x)\leq R_0 B^{-\frac{1}{2}}\}} |u(x)|^2\, dx\,. \tag{8.33}$$

From this we deduce the estimate on $\|e^{\alpha\sqrt{B} d(x,\partial\Omega} u\|_2$ in (8.32). The other part of (8.32) is a consequence of (8.33) and (8.28). □

Remark 8.2.6.
On the contrary, when $b < \Theta_0 b'$ *, the ground state decays exponentially outside any fixed neighborhood of* $\beta^{-1}(b)$ *in* $\overline{\Omega}$*. Note that in this case the boundary condition does not affect the localization of the ground state or the asymptotics of the ground state energy (exponentially small effect). The decay is then estimated by the weight* $\exp -[\alpha_0\sqrt{B} d_{\beta-b}(x)]$ *, where, for a given* $x \in \Omega$ *,* $d_{\beta-b}(x)$ *denotes the Agmon distance of* x *to the minima of the strength of the magnetic field* β *attached to the potential* $y \mapsto \beta(y) - b$ *.*

In applications it is often not exponential but polynomial weights that occur. Theorem 8.2.4 has the following useful corollary.

Corollary 8.2.7.
Suppose that (8.26) *is satisfied. Then there exists* $C_n > 0$ *such that*

$$\int t(x)^n \{|u_B(x)|^2 + B^{-1}|p_{B\mathbf{A}} u_B(x)|^2\} \, dx \leq C_n B^{-\frac{n}{2}} \|u_B\|^2 . \tag{8.34}$$

Remark 8.2.8.
Upon inserting Corollary 8.2.7 in (8.15)*, we get (choosing* $\eta = B^{-\rho}$, $\rho = 1/3$*), the following improvement of* (8.23)*. There exist* C *and* B_0 *such that, for* $B \geq B_0$ *,*

$$\lambda_1(B\mathbf{A}) \geq \min(b, \Theta_0 b') \; B - C \, B^{\frac{2}{3}} . \tag{8.35}$$

We recall that the optimal result is in $\mathcal{O}(B^{1/2})$ *.*

The next result, which is useful in the analysis of the monotonicity of λ_1^N , is a rather weak localization result inside the boundary. The proof is analogous to the proof of Theorem 8.2.4 but instead uses Proposition 8.2.1.

Proposition 8.2.9.
Suppose that (8.26) *is satisfied and that in addition the restriction of* $\beta(x)$ *to the boundary is not constant. Then, for any neighborhood* $\mathcal{V}(\partial\Omega)$ *of*

$$\widetilde{n}(\partial\Omega) := \{x \in \partial\Omega \,|\, |\beta(x)| = b'\} , \tag{8.36}$$

there exist $\eta > 0$, $B_0 > 0$, *and* $C > 0$ *such that, for* $B \geq B_0$, *any normalized ground state* u_B *satisfies*

$$\int_{\Omega \setminus \mathcal{V}(\partial\Omega)} |u_B(x)|^2 \, dx \leq C \, \exp(-\eta B^{\frac{1}{2}}) .$$

8.3 Constant Magnetic Field

In this section, we will obtain more precise asymptotics in the important special case of a constant magnetic field. We therefore assume that

$$\beta(x) = 1 , \qquad \forall \, x \in \Omega. \tag{8.37}$$

In this case, Theorem 8.1.1 becomes

Proposition 8.3.1.
In the case of a constant magnetic field, we have

$$\lambda_1(B) = \Theta_0 B + o(B). \tag{8.38}$$

Also, the exponential localization in Theorem 8.2.4 holds. We will prove that in the case of a constant magnetic field the next term in the expansion is determined by the maximum of the boundary curvature.

Theorem 8.3.2.
Suppose that Ω *is bounded and smooth and that* $\beta(x) \equiv 1$. *Then*

$$\lambda_1(B) = \Theta_0 B - \mathcal{C}_1 k_{\max}\sqrt{B} + \mathcal{O}(B^{\frac{1}{3}}), \tag{8.39}$$

where

$$k_{\max} := \max\{k(s) \,\big|\, s \in \partial\Omega\}\,,$$

$k(s)$ *denotes the curvature of the boundary at the point* s , *and* $\mathcal{C}_1$ *is the constant defined in* (3.38).

Proof.
Lower bound:
The argument is the same as for the proof of Theorem 8.1.1. In particular, we have (8.10) and (8.12). We will choose $R_0 = 1$. For the boundary terms, we notice that since the magnetic field is constant, the function $\tilde{b}$ satisfies

$$\tilde{b}(s,t) = -t(\Delta k)(s)\,,$$

and thus

$$\tilde{a}_1(s,t) = \frac{1}{2}t^2(\Delta k)(s)\,.$$

Therefore, (8.15) is improved to

$$\begin{aligned}
&\int |(-i\nabla + B\mathbf{A})u|^2\,dx \\
&\quad \geq (1-\eta)\int (1-tk_0)^{-1}|(-i\partial_s + B\overline{A})\tilde{v}|^2 + (1-tk_0)|\partial_t \tilde{v}|^2\,dsdt \\
&\qquad - C\int t\Delta k\big\{\big|(-i\partial_s + \tilde{A}_1)\tilde{v}\big|^2 + |\partial_t\tilde{v}|^2\big\}\,dsdt \\
&\qquad - C\eta^{-1}B^2\int t^4(\Delta k)^2|\tilde{v}|^2\,dsdt\,,
\end{aligned} \tag{8.40}$$

for all u such that $\operatorname{supp} u \subset \mathrm{D}(z, B^{-\rho})$.

Estimating $\Delta k = \mathcal{O}(B^{-\rho})$ and putting together the different pieces, we get the inequality

$$\begin{aligned}
&\int |(-i\nabla + B\mathbf{A})u|^2\,dx \\
&\quad \geq B\sum_{\text{int}}\int |\chi_j^B u|^2\,dx + \sum_{\text{bnd}}(1-\eta)\int (\Theta_0 B - \mathcal{C}_1 k(z_j)\sqrt{B} - C)|\chi_j^B u|^2\,dx \\
&\qquad - CB^{-\rho}\sum_{\text{bnd}}\int t(x)|p_{B\mathbf{A}}(\chi_j u)|^2\,dx - C\eta^{-1}B^{2-2\rho}\sum_{\text{bnd}}\int t(x)^4|\chi_j u|^2\,dx \\
&\qquad - CB^{2\rho}\|u\|^2\,,
\end{aligned} \tag{8.41}$$

for all $u \in H^1_{B\mathbf{A}}(\Omega)$. In the case where $u = u_B$ is a ground state of $P^N_{B\mathbf{A},\Omega}$, we can apply Theorem 8.2.4 and find

$$\sum_{\text{bnd}} \int t(x)|p_{B\mathbf{A}}(\chi_j u_B)|^2\,dx \leq CB^{\frac{1}{2}}\,, \tag{8.42}$$

$$\sum_{\text{bnd}} \int t(x)^4|\chi_j u_B|^2\,dx \leq CB^{-2}\,. \tag{8.43}$$

Using that $\Theta_0 < 1$ and choosing $\eta = B^{-\frac{1}{2}-\rho}, \rho = 1/6$, we therefore find (8.39).

Upper bound:

For the upper bound we choose u to be localized near $z \in \partial\Omega$, where the boundary curvature is maximal. This is done in such a way that only one term—the corresponding boundary term—in the sum for the lower bound is nonzero. Then by (8.14) we get, similarly to (8.40),

$$\begin{aligned}
&\int |(-i\nabla + B\mathbf{A})u|^2\,dx \\
&\quad\leq (1+\eta)\int (1-tk_0)^{-1}|(-i\partial_s + B\overline{A})\tilde{v}|^2 + (1-tk_0)|\partial_t\tilde{v}|^2\,dsdt \\
&\qquad + C\int t\Delta k\big\{\big|(-i\partial_s + B\tilde{A}_1)\tilde{v}\big|^2 + |\partial_t\tilde{v}|^2\big\}\,dsdt \\
&\qquad + C\eta^{-1}B^2\int t^4(\Delta k)^2|\tilde{v}|^2\,dsdt\,.
\end{aligned} \tag{8.44}$$

Choose

$$\tilde{v} = \chi(B^\rho s)\chi(B^\rho t)u_{1,B,k_0}(s,t)\,,$$

where u_{1,B,k_0} is a normalized ground state for the constant curvature model of curvature k_0 as analyzed in Chapter 5. The corresponding eigenvalue $\lambda_1(B,\Omega_{k_0})$ satisfies, according to (5.61), $\lambda_1(B,\Omega_{k_0}) = \Theta_0 B - \mathcal{C}_1 k_0\sqrt{B} + \mathcal{O}(1)$. Choosing $\rho = 1/6$, $\eta = B^{-\frac{1}{2}-\rho}$ as for the lower bound, we find (using the decay in t of u_{1,B,k_0}; see Corollary 8.2.7)

$$\int |(-i\nabla + B\mathbf{A})u|^2\,dx \leq \Big[(1+\eta)\lambda_1(B,\Omega_{k_0}) + C'B^{\frac{1}{3}}\Big]\|u\|^2\,. \tag{8.45}$$

This finishes the proof of the upper bound. □

Looking more carefully at the proof of the lower bound in Theorem 8.3.2, one can actually see that we have also proved the following proposition.

Proposition 8.3.3.

Under the assumptions of Theorem 8.3.2, there exists $C_0 > 0$ such that we have

$$Q^N_{B\mathbf{A},\Omega}(u) \geq \int_\Omega W^1_B(x)|u(x)|^2\,dx\,, \qquad \forall u \in H^1_{B\mathbf{A}}(\Omega)\,, \tag{8.46}$$

for any $B \geq 1$ *. Here* W_B^1 *is defined by*

$$W_B^1(x) := \begin{cases} B\,, & \text{if } \operatorname{dist}(x;\partial\Omega) > 2B^{-\frac{1}{6}}\,, \\ \Theta_0 B - \mathcal{C}_1\, B^{\frac{1}{2}} k(s) - C_0\, B^{\frac{1}{3}}\,, & \text{if } \operatorname{dist}(x;\partial\Omega) \leq 2B^{-\frac{1}{6}}\,. \end{cases} \tag{8.47}$$

In particular, we have, also using the upper bound of $\lambda_1(B\mathbf{A})$ and with u_B being a normalized ground state,

$$\begin{aligned} &Q^N_{B\mathbf{A},\Omega}(u_B) - \lambda_1(B\mathbf{A}) \\ &\quad \geq \int_\Omega (W_B^1(x) - \lambda_1(B\mathbf{A}))|u_B(x)|^2\, dx \\ &\quad \geq \mathcal{C}_1 B^{\frac{1}{2}} \int_{\{t(x)\leq 2B^{-1/6}\}} (k_{\max} - k(s) - C_0 B^{-\frac{1}{6}})|u_B(x)|^2\, dx\,. \end{aligned} \tag{8.48}$$

As for the proof of the previous Agmon estimates in this chapter, we get

Theorem 8.3.4.
Under the assumptions of Theorem 8.3.2, we have the following localization. There exist $\delta > 0$ *and for any* $\epsilon > 0$ *,* $C_\epsilon > 0$ *and* $B_\epsilon > 0$ *such that, for all* $B \geq B_\epsilon$ *,*

$$\| e^{\delta B^{\frac{1}{4}} \hat{d}(x,n(\partial\Omega),B)}\; u_B \| \leq C_\epsilon \exp(\epsilon B^{\frac{1}{4}})\,. \tag{8.49}$$

Here $n(\partial\Omega)$ *is the set*

$$n(\partial\Omega) := \{ z \in \partial\Omega \mid k(z) = k_{\max} \} \tag{8.50}$$

of the points of maximal curvature,

$$\hat{d}(x, n(\partial\Omega), B) = \hat{d}_{\partial\Omega}(s(x), n(\partial\Omega))\chi(d(x,\partial\Omega)) + B^{\frac{1}{4}} d(x,\partial\Omega)\,, \tag{8.51}$$

and $\hat{d}_{\partial\Omega}(s, n(\partial\Omega))$ *is the Agmon distance to* $n(\partial\Omega)$ *associated with the metric* $(\kappa_{max} - \kappa(s))\, ds^2$ *.*

As an immediate corollary, we have

Corollary 8.3.5.
Under the assumptions of Theorem 8.3.2, then, for any neighborhood $\mathcal{V}(\partial\Omega)$ *of* $n(\partial\Omega)$ *in* $\overline{\Omega}$ *, there exist* $\eta > 0$ *and* $C > 0$ *such that, for large* B *,*

$$\int_{\Omega\setminus\mathcal{V}(\partial\Omega)} |u_B(x)|^2\; dx \leq C\; \exp(-\eta B^{\frac{1}{4}})\,.$$

Remark 8.3.6.
We can use the localization estimate to improve the error bound in the eigenvalue asymptotics. Thus, when the maxima of the boundary curvature are nondegenerate, we can improve the error bound from $\mathcal{O}(B^{1/3})$ *to* $\mathcal{O}(B^{1/4})$ *in Theorem 8.3.2. We refer to the next section for precise statements.*

8.4 Refined Expansions and Spectral Gap

We now present finer results that give a complete asymptotic expansion of $\lambda_1(B\mathbf{A})$ under a (generically satisfied) nondegeneracy assumption on Ω. The argument used in the proof is somewhat more technically involved than in the rest of the book, so we limit ourselves to stating the result without proof and refer to the original article for details. We define $\lambda_n(B\mathbf{A})$ to be the nth eigenvalue of $P^N_{B\mathbf{A},\Omega}$, in particular,

$$\lambda_1(B\mathbf{A}) = \inf \sigma(P^N_{B\mathbf{A},\Omega}) .$$

Then the result is as follows.

Theorem 8.4.1.
Suppose that Ω is a smooth, bounded domain, that its curvature $\partial\Omega \ni s \mapsto k(s)$ at the boundary has a unique maximum,

$$k(s) < k(s_0) =: k_{\max} , \qquad \textit{for all } s \neq s_0 , \tag{8.52}$$

and that the maximum is nondegenerate, i.e.,

$$k_2 := -k''(s_0) \neq 0 . \tag{8.53}$$

Then, for all $n \in \mathbb{N} \setminus \{0\}$, there exists a sequence $\{\zeta_j^{(n)}\}_{j=1}^{\infty} \subset \mathbb{R}$ (which can be calculated recursively to any order) such that $\lambda_n(B)$ admits the following asymptotic expansion (for large B):

$$\begin{aligned}\lambda_n(B\mathbf{A}) \sim \Theta_0 B - k_{\max}\mathcal{C}_1 B^{\frac{1}{2}} + \mathcal{C}_1 \Theta_0^{\frac{1}{4}} \sqrt{\frac{3k_2}{2}}(2n-1)B^{\frac{1}{4}} \\ + B^{\frac{1}{8}} \sum_{j=0}^{\infty} \zeta_j^{(n)} B^{-\frac{j}{8}} .\end{aligned} \tag{8.54}$$

For possible applications to bifurcations from the normal state in superconductivity (see Section 13.5), it is important to calculate the splitting between the ground state energy and the first excited eigenvalues of $P^N_{B\mathbf{A},\Omega}$. Let us define

$$\Delta(B) = \lambda_2(B) - \lambda_1(B) . \tag{8.55}$$

Corollary 8.4.2.
Under the hypothesis from Theorem 8.4.1, $\Delta(B)$ admits the following asymptotics:

$$\Delta(B) \sim \mathcal{C}_1 \Theta_0^{\frac{1}{4}} \sqrt{6k_2} B^{\frac{1}{4}} + B^{\frac{1}{8}} \sum_{j=0}^{\infty} B^{-\frac{j}{8}} \widehat{\xi}_j , \tag{8.56}$$

where $\widehat{\xi}_j = \zeta_j^{(2)} - \zeta_j^{(1)}$.

The case when Ω is a disc has been presented in Chapter 5. In this case, the splitting $\Delta(B)$ turns out to vanish for a sequence of values of B tending to ∞. This is a complication in the analysis of bifurcation. Thus, in some sense, the more "generic" situation considered in Theorem 8.4.1 has a nicer property. We recall that for the disc we get from Remark 5.3.5 that

$$0 = \liminf_{B\to\infty} \Delta(B) < \limsup_{B\to\infty} \Delta(B) \ < +\infty\,.$$

We recall also that in the case of a domain with a unique corner (with a sufficiently small angle), we have ([Bon2], [BonD], and [BonF]—see also Chapter 15)

$$\liminf_{B\to\infty} \frac{\Delta(B)}{B} > 0\,.$$

In our case, (8.56) implies

$$\lim_{B\to\infty} \frac{\Delta(B)}{B^{\frac{1}{4}}} > 0\,.$$

Of course, if there are multiple minima and symmetries, one expects by tunneling analysis an exponentially small gap between the lowest eigenvalues.

We will come back to this point in Chapter 15, which is devoted to domains with corners.

8.5 Monotonicity

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with a regular boundary. Let $\mathbf{F}(x) = (F_1(x), F_2(x)) = (-x_2/2, x_1/2)$ such that $\operatorname{curl} \mathbf{F} = 1$. We consider $\mathcal{H}(B) = P^N_{B\mathbf{F},\Omega}$ and will show that in the present situation, we can obtain the monotonicity of $\lambda_1(B)$ with much less information on the asymptotics of $\lambda_1(B)$ than required by the general Corollary 2.3.3.

We use the definitions concerning the geometry of the boundary defined in Section F.1; in particular, the boundary is parametrized (by arc-length) by $\gamma(s)$, $s \in \frac{|\partial\Omega|}{2\pi}\mathbb{S}^1$, and $k(s)$ denotes the curvature at the point $\gamma(s)$. We recall that $k_{\max}$ denotes the maximum of the boundary curvature.

Theorem 8.5.1.
The one-sided derivatives,

$$\lambda'_{1,+}(B) = \lim_{\epsilon\to 0_+} \frac{\lambda_1(B+\epsilon) - \lambda_1(B)}{\epsilon}\,, \quad \lambda'_{1,-}(B) = \lim_{\epsilon\to 0_+} \frac{\lambda_1(B) - \lambda_1(B-\epsilon)}{\epsilon}\,,$$

exist for all $B > 0$ and $\lambda'_{1,+}(B)$ satisfies

$$\liminf_{B\to\infty} \lambda'_{1,+}(B) > 0\,. \tag{8.57}$$

Furthermore, if Ω *is not a disc, then the limit actually exists and satisfies*

$$\lim_{B\to\infty} \lambda'_{1,-}(B) = \lim_{B\to\infty} \lambda'_{1,+}(B) = \Theta_0 \,. \tag{8.58}$$

If Ω *is a disc, then*

$$\limsup_{B\to\infty} \lambda'_{1,+}(B) > \Theta_0 \,, \qquad 0 < \liminf_{B\to\infty} \lambda'_{1,+}(B) < \Theta_0 \,.$$

In particular, in any case, there exists $B_0 > 0$ *such that* $B \mapsto \lambda_1(B)$ *is strictly increasing on* $[B_0, \infty[$ *.*

If we could obtain sufficiently precise asymptotic expansions of $\lambda_1(B)$, a result like Theorem 8.5.1 would follow from Corollary 2.3.3. However, as the argument below illustrates, we can obtain the same conclusion from a weaker asymptotic expansion combined with information on the localization of the magnetic ground state. This strategy does not work in the case of the disc. However, due to the symmetry of the question for that domain, special techniques have already been applied in Chapter 5 to settle that special case. Thus, the structure of the proof of Theorem 8.5.1 is as follows. If Ω is not a disc, then there exists a part of the boundary where the ground state ψ will be very small. Thus, we can choose a gauge such that $|\widehat{\mathbf{A}}\,\psi| \ll 1$ (for large B and in the L^2-sense), where $\widehat{\mathbf{A}}$ is the vector field $\mathbf{F}$ in the new gauge. This new input allows us to differentiate the leading-order asymptotics for $\lambda_1(B)$.

Notice that if Ω is not a disc, then it satisfies the following assumption:

Assumption 8.5.2.
Let $n(\partial\Omega)$ *denote the set of boundary points of maximal curvature as defined in* (8.50)*. Then*

$$n(\partial\Omega) \neq \partial\Omega \,.$$

We recall the Agmon estimates, Theorem 8.2.4, which we state in the following form:

Lemma 8.5.3 (Normal Agmon estimates)**.**
There exist positive constants α*,* M*, and* C *such that if* $B \geq 1$ *and* $\psi_1(\,\cdot\,;B)$ *is a ground state of* $\mathcal{H}(B)$ *, then*

$$\begin{aligned}\int_\Omega e^{2\alpha\sqrt{B}\,\mathrm{dist}(x,\partial\Omega)} &\left\{ |\psi_1(x;B)|^2 + \frac{1}{B} |p_{B\mathbf{F}}\psi_1(x;B)|^2 \right\} dx \\ &\leq C \int_{\{\sqrt{B}\,\mathrm{dist}(x,\partial\Omega) \leq M\}} |\psi_1(x;B)|^2 \, dx \,. \end{aligned} \tag{8.59}$$

In particular, for all $N > 0$ *,*

$$\int \mathrm{dist}(x,\partial\Omega)^N |\psi_1(x;B)|^2 \, dx = \mathcal{O}(B^{-N/2}) \,. \tag{8.60}$$

Also, Theorem 8.3.4 implies that ground states are localized near $n(\partial\Omega)$. We actually only need the following very weak version of this localization.

Lemma 8.5.4.
Let $\epsilon_0 > 0$. Then, for all $N > 0$, there exist $C_N > 0$ and $B_N > 0$ such that if $\psi_1(\,\cdot\,; B)$ is a normalized ground state for $\mathcal{H}(B)$, then

$$\int_{\{\mathrm{dist}(x,n(\partial\Omega))\geq\epsilon_0\}} |\psi_1(x;B)|^2\, dx \leq C_N\, B^{-N}\,, \qquad \forall B \geq B_N\,.$$

We will also need to use the boundary coordinates (s,t) defined in Section F.1.

Lemma 8.5.5.
Let us define for $\epsilon \leq \min(t_0/2, |\partial\Omega|/2)$ and $s_0 \in \partial\Omega$

$$\Omega(\epsilon, s_0) := \{x = \Phi(s,t) \,\big|\, t \leq \epsilon, |s - s_0| \geq \epsilon\}\,.$$

Then there exists $\phi \in C^\infty(\Omega)$ such that $\widehat{\mathbf{A}} = \mathbf{F} + \nabla\phi$ satisfies

$$|\widehat{\mathbf{A}}(x)| \leq C\ \mathrm{dist}(x, \partial\Omega)\,,$$

for $x \in \Omega(\epsilon, s_0)$.

Proof.
Let $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$ be the magnetic 1-form written in the (s,t)-coordinates,

$$F_1\, dx + F_2\, dy = \widetilde{A}_1\, ds + \widetilde{A}_2\, dt\,.$$

Taking the exterior derivative, and using $dx \wedge dy = |D\Phi| ds \wedge dt$, we find

$$\mathrm{curl}_{s,t}\, \widetilde{\mathbf{A}}(s,t) = \partial_s \widetilde{A}_2 - \partial_t \widetilde{A}_1 = (1 - tk(s))\,.$$

Since $\{(s,t)\,|\,t \leq \epsilon\,, |s - s_0| \geq \epsilon\}$ is simply connected, there exists a function $\widetilde{\phi} \in C^\infty(\Phi^{-1}(\Omega(\epsilon, s_0)))$ such that

$$\widetilde{\mathbf{A}}(s,t) + \nabla_{s,t}\widetilde{\phi}(s,t) = (t - t^2 k(s)/2, 0)\,.$$

Let $\chi \in C^\infty(\overline{\Omega})$,

$$\chi = 1 \quad \text{on} \quad \{x\,|\,t \leq \epsilon, |s - s_0| \geq \epsilon\}\,,$$
$$\chi = 0 \quad \text{on} \quad \{x\,|\,\mathrm{dist}(x, \partial\Omega) \geq 2\epsilon \text{ or } |s - s_0| \leq \epsilon/2\}\,,$$

and define $\phi(x) = \widetilde{\phi}(\Phi^{-1}(x))\chi(x)$. Then ϕ solves the problem. □

Proof of Theorem 8.5.1.
The existence of $\lambda'_{1,+}(B), \lambda'_{1,-}(B)$ follows from analytic perturbation theory. We recall that the case of the disc was already considered in Proposition 5.4.1,

and so it remains to consider the case where Ω is not the disc. Thus, Ω satisfies Assumption 8.5.2. Therefore, there exist $s_0 \in [-|\partial\Omega|/2, |\partial\Omega|/2]$ and $0 < \epsilon_0 < \min(t_0/2, |\partial\Omega|/4)$ such that

$$[s_0 - 2\epsilon_0, s_0 + 2\epsilon_0] \cap n(\partial\Omega) = \emptyset\,.$$

Let $\widehat{\mathbf{A}}$ be the vector potential defined in Lemma 8.5.5, $\widehat{Q}_B$ the quadratic form $u \mapsto \widehat{Q}_B(u) = \int_\Omega |-i\nabla u + B\widehat{\mathbf{A}}u|^2 dx$, and $\widehat{\mathcal{H}}(B)$ the associated operator. Then $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$ are unitarily equivalent and thus have the same spectrum. Let $\psi_1^+(\,\cdot\,;\beta)$ be a normalized ground state of $\widehat{\mathcal{H}}(B)$ for β in a right neighborhood of B and such that $\beta \mapsto \psi_1^+(\,\cdot\,;\beta) \in L^2$ is smooth—the existence of such a ground state was discussed in Section 2.3. Then we get the following expression for the right-derivative:

$$\begin{aligned}\lambda_{1,+}'(B) = \langle \widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\,|\,p_{B\mathbf{A}}\psi_1^+(\,\cdot\,;B)\rangle& \\ + \langle p_{B\mathbf{A}}\psi_1^+(\,\cdot\,;B)\,|\,\widehat{\mathbf{A}}\psi_1^+(\,\cdot\,;B)\rangle\,.& \qquad (8.61)\end{aligned}$$

We now obtain, for any $b > 0$,

$$\begin{aligned}\lambda_{1,+}'(B) &= \frac{\widehat{Q}_{B+b}(\psi_1^+(\,\cdot\,;B)) - \widehat{Q}_B(\psi_1^+(\,\cdot\,;B))}{b} - b\int_\Omega |\widehat{\mathbf{A}}(x)|^2\,|\psi_1^+(x;B)|^2\,dx \\ &\geq \frac{\lambda_1(B+b) - \lambda_1(B)}{b} - b\int_\Omega |\widehat{\mathbf{A}}(x)|^2\,|\psi_1^+(x;B)|^2\,dx\,. \qquad (8.62)\end{aligned}$$

By Lemma 8.5.5, we can estimate

$$\begin{aligned}\int_\Omega |\widehat{\mathbf{A}}(x)|^2\,|\psi_1^+(x;B)|^2\,dx \leq C\int_\Omega \operatorname{dist}(x,\partial\Omega)^2|\psi_1^+(x;B)|^2\,dx& \\ + \|\widehat{\mathbf{A}}\|_\infty^2 \int_{\Omega\setminus\Omega(\epsilon_0,s_0)} |\psi_1^+(x;B)|^2\,dx\,.& \qquad (8.63)\end{aligned}$$

Combining Lemmas 8.5.3 and 8.5.4, we therefore find the existence of a constant $C > 0$ such that

$$\int_\Omega |\widehat{\mathbf{A}}(x)|^2\,|\psi_1^+(x;B)|^2\,dx \leq C\,B^{-1}\,. \qquad (8.64)$$

We now choose $b = \eta\,B$, where $\eta > 0$ is arbitrary. By the asymptotics (8.38) for $\lambda_1(B)$, we therefore find

$$\liminf_{B\to\infty} \lambda_{1,+}'(B) \geq \Theta_0 - \eta\,C\,. \qquad (8.65)$$

Since η was arbitrary, this implies

$$\liminf_{B\to\infty} \lambda_{1,+}'(B) \geq \Theta_0\,. \qquad (8.66)$$

Applying the same argument to the derivative from the left, $\lambda'_{1,-}(B)$, we get (the inequality gets turned since $\beta < 0$)

$$\limsup_{B\to\infty} \lambda'_{1,-}(B) \leq \Theta_0 \,. \tag{8.67}$$

Since, by perturbation theory,

$$\lambda'_{1,+}(B) \leq \lambda'_{1,-}(B) \text{ for all } B\,,$$

we get (8.58). □

8.6 Extensions

8.6.1 Nonconstant magnetic fields with boundary localization

If $\Theta_0 b' < b$ and if the restriction of $\beta(x)$ to the boundary is not constant, i.e., with the notation introduced in (8.36) if

$$\widetilde{n}(\partial\Omega) \neq \partial\Omega\,, \tag{8.68}$$

we can get the monotonicity of the ground state energy in the large field limit.

We can find $\phi \in C^\infty(\overline{\Omega})$ and $\widehat{\mathbf{A}}$ such that $\widehat{\mathbf{A}} = \mathbf{A} + \nabla\phi$ and $\widehat{A}$ vanishes on $\partial\Omega$ except in a neighborhood of a point where $|\beta(x)|$ is maximum. It is indeed enough to apply Lemma 8.5.5, whose proof can be modified in order to extend it to the case of a general magnetic field.

We can then follow the proof of Theorem 8.5.1 given in the case of non-constant curvature. We then use Proposition 8.2.1 (and its application in Corollary 8.2.7 to the decay), Proposition 8.2.9, and Theorem 8.1.1 (and its proof) for the asymptotic expansion of $\lambda_1(B\mathbf{A})$.

8.6.2 Interior localization

If $\Theta_0 b' > b > 0$, i.e., the ground state should be localized in the interior of Ω, we can also use the techniques described in this book.

Theorem 8.6.1.
Suppose $\Theta_0 b' > b > 0$ and that $b = \beta(z_0)$ is attained at a unique point $z_0 \in \Omega$. Suppose furthermore that

$$\operatorname{Hess}\beta(z_0) > 0.$$

Then the following asymptotics holds:

$$\lambda_1(B\mathbf{A}) = B + \frac{1}{4}\operatorname{tr}(\operatorname{Hess}\beta(z_0)) + o(1)\,, \tag{8.69}$$

as $B \to +\infty$.

This asymptotic estimate is sufficiently precise to apply Corollary 2.3.3 and thereby get the monotonicity of the ground state energy.

8.6.3 Montgomery's model revisited

The Montgomery model appears more generally in the following context [PaK]. If, in the two-dimensional case, the magnetic field $\beta = \operatorname{curl} \mathbf{A}$ has non-degenerate zeroes in Ω and if we denote by $\mathcal{Z}(\beta)$ the subset of the zeroes of β in $\overline{\Omega}$, then the ground state energy $\lambda_1(B\mathbf{A})$ of the magnetic Laplacian associated to the magnetic field $B\beta$ can be estimated asymptotically as $B \to +\infty$. More precisely, we have the theorem

Theorem 8.6.2 (Pan–Kwek)**.**

$$\lim_{B \to +\infty} \frac{\lambda_1(B\mathbf{A})}{B^{\frac{2}{3}}} = [\widehat{\alpha}_1(\beta)]^{\frac{2}{3}}\,, \tag{8.70}$$

where

$$\widehat{\alpha}_1(\beta) = \min\left\{ \frac{1}{2}\hat{\nu}_0^{\frac{3}{2}} \inf_{x \in \Omega \cap \mathcal{Z}(\beta)} |\nabla \beta(x)|\,,\ \inf_{x \in \partial\Omega \cap \mathcal{Z}(\beta)} \widehat{\zeta}(\vartheta(x))^{\frac{3}{2}} |\nabla \beta(x)| \right\}\,, \tag{8.71}$$

where $\vartheta(x)$ denotes the angle between $\operatorname{curl} \beta$ *and the tangent vector of $\partial\Omega$ at x and $\widehat{\zeta}$ denotes the lowest eigenvalue of $-\Delta_{\mathbf{A}_\vartheta}$ in $\mathbb{R}^{2,+}$ with*

$$\mathbf{A}_\vartheta = -\frac{|x|^2}{2}(\cos\vartheta, \sin\vartheta)\ .$$

8.7 Notes

1. Theorem 8.1.1 and 8.1.2 first appeared in [LuP4]. Theorem 8.1.3 was proved in [HeM3].
2. The large magnetic field limit and the semiclassical limit $h \searrow 0$ are clearly equivalent, since
$$\int_\Omega |(-i\nabla_x + B\mathbf{A}(x))u(x)|^2\,dx = B^2 \int_\Omega |(-\tfrac{i}{B}\nabla_x + \mathbf{A}(x))u(x)|^2\,dx\,.$$
The implementation of semiclassical techniques for the analysis of the magnetic ground state first appeared in [HeS4] and then in [HeM2]. Very roughly, it is shown in [HeM2] that if $\Omega = \mathbb{R}^n$, then $B|\operatorname{curl} A(x)|$ plays the role of an effective electric potential. By this we mean that the analysis of the operator, $-\Delta + B|\operatorname{curl} \mathbf{A}(x)|$, gives good information about the localization of the ground state. The case of domains with boundary was less analyzed.
3. More precise results concerning the case when $b = 0$ are obtained under additional conditions in [HeM2].
4. Note also that the upper bound involving $b = \inf \beta$ can also be obtained by using [HeM3]. Following the same paper or [dPiFS], one can improve the $o(B)$ into $\mathcal{O}(B^{1/2})$, without additional assumptions.

5. Formula (8.10) is sometimes called the IMS formula (see [CyFKS], in the context of N-body problems) but is actually much older (see [Me, Ho]).
6. The points where the minimum of $|\beta|$ is attained are sometimes called magnetic wells for the energy b. The decay of the ground state outside the wells can be estimated (cf. [Bru, HeN2]) as a function of the Agmon distance associated with the Agmon metric $(|\beta|-b)dx^2$, where dx^2 denotes the Euclidean metric. We recall that this estimate is very easy to get from (1.32) in the special case when $n=2$ and when the magnetic field has a constant sign.
7. The decay estimates established in Theorem 8.2.4 can be found in various forms (weighted L^2- or L^∞-estimates) in [LuP4, HeM3] and [dPiFS].
8. As in the case where $\mathbf{A}=0$ but where an electric potential V is added, it is possible to discuss the various possible asymptotics depending on the properties of β near the minima (cf. [HeM2, HeM3, Mon, Sh, Ue1, Ue2] or more recently [PaK]). But these results are mainly devoted to the case of $\mathbb{R}^n$ or of a compact, boundaryless manifold and admit relatively simple extensions for the Dirichlet problem, but as we see in the whole book, this property is no more true in the case of the Neumann realization. The infimum b of $|\beta(x)|$ on $\overline{\Omega}$ is not necessarily the right quantity for analyzing the bottom of the spectrum as (8.1) is satisfied. Of course, by direct comparison of the variational spaces corresponding to Dirichlet and Neumann, one knows that the smallest eigenvalue $\lambda_1^N(B\mathbf{A})$ of the Neumann realization $P^N_{B\mathbf{A},\Omega}$ of $P_{B\mathbf{A},\Omega}$ is bounded from above by $\lambda_1^D(B\mathbf{A})$ [but the lower bound (1.37) is not correct in general].
9. In the case of a constant magnetic field, the two-term expansion given in Theorem 8.3.2 was conjectured by Bernoff–Sternberg [BeS], but the complete proof was achieved by Helffer–Morame in [HeM6] (see [HeM6, Theorems 10.3 and 11.1]). The localization at the points of maximal curvature has been verified numerically (see Fig. 5.9, p. 61 in [HoS]). The extension of the two-term asymptotics to the case of nonconstant magnetic fields is done by Raymond in [Ra2] and [Ar1]–[Ar5].
10. The results of Section 8.4 were obtained [FoH2], but stated in terms of the semiclassical limit. Formal expansions previously appeared in [BeS]. If the uniqueness condition in (8.52) is replaced by the assumption that there are a finite number of maxima [for which (8.53) is assumed to hold], there will exist sequences of eigenvalues $z^{(n)}(B)$ corresponding to each maximum. This also follows by the same techniques with a little extra work.
11. The analysis of the tunneling effect was done in the case of the Schrödinger operators by Helffer–Sjöstrand [HeS1, HeS2] and Simon in the 1980s (see also the books [He2] or [DiS]). The rigorous analysis of the tunneling inside the boundary is open (see Bonnaillie [Bon1] for a discussion inspired by the analysis of the tunnel effect and Chapter 15 in this book devoted to the case of domains with corners).

12. The assumption that Ω is bounded is included for convenience only. An adaptation of the techniques presented in this book would permit the omission of this assumption (see also Chapter 5, where we treat the exterior of the disc).
13. It follows easily from the proof that Theorem 8.4.1 holds without change in the case of a nonconstant field $\beta(z) = \operatorname{curl} \mathbf{A}(z)$ provided $\beta(z)$ satisfies β is constant on a neighborhood of the boundary $\partial\Omega$, or even more generally, that $\beta \equiv \beta_0 + \beta'$, where β_0 is a constant and β' vanishes to infinite order on $\partial\Omega$. Extensions are considered in [Ra2].
14. Results similar to (8.57) were first proved in [FoH3] under extra assumptions. This was due to the fact that the complete asymptotics of Theorem 8.4.1 was used as an input. The most prominent domain excluded in this approach is the disc—where the curvature is constant. However, [FoH3] includes a special analysis of the disc—essentially repeated here in Chapter 5—proving that Theorem 8.5.1 remains true in that case.
15. What remained was the study of all the other nongeneric cases. Also, it seemed desirable to be able to establish Theorem 8.5.1 without using the existence of a complete asymptotic expansion, since such expansions are difficult to obtain and their structure depends heavily on the different kinds of maxima of the boundary curvature.
16. Theorem 8.6.1 is a simplified version of [HeM3, Theorem 7.2].

9

Main Results for Large Magnetic Fields in Dimension 3

In dimension 3, the general strategy is very similar to the two-dimensional situation, but the geometry is somewhat more complicated. Therefore, for some of the proofs, we will give only the main ideas and refer to the original papers for details.

9.1 Main Results for Variable Magnetic Fields

We are concerned with the behavior of the ground state energy of magnetic Schrödinger operators—with nonzero magnetic field $\beta = \operatorname{curl} \mathbf{F}$ and Neumann boundary conditions—as the strength of the magnetic field becomes large. In this chapter, $\Omega \subset \mathbb{R}^3$ denotes a three-dimensional domain.

For simplicity of notation, we write $\mathcal{H}(B) = P^N_{B\mathbf{F},\Omega}$ for the magnetic Laplacian defined in Section 1.2. Similarly, we write $Q_B = Q^N_{B\mathbf{F},\Omega}$ for the associated quadratic form. As in the two-dimensional situation we define

$$\lambda_1(B) := \inf \sigma(\mathcal{H}(B)) \,, \tag{9.1}$$

i.e., the lowest eigenvalue of $\mathcal{H}(B)$. The main result is the following extension of Proposition 8.2.2 in the two-dimensional case. In order to state the result, let us define the function $\vartheta(x)$, which gives the angle between $\beta(x)$ and the tangent plane at x. More precisely,

$$\partial\Omega \ni x \mapsto \vartheta(x) \in [0, \pi/2]$$

is given by

$$\vartheta(x) = \arcsin\left(\frac{|\beta(x)\cdot\nu(x)|}{|\beta(x)|}\right) . \tag{9.2}$$

So, $\pi/2 - \vartheta(x)$ denotes the angle between $\beta(x)$ and $\nu(x)$ at $x \in \partial\Omega$.

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Theorem 9.1.1.
We assume that Ω *is a bounded, smooth domain in* $\mathbb{R}^3$ *and that a smooth magnetic field* β *is given such that* $\inf_{x\in\Omega}|\beta(x)|\neq 0$. *There exist constants* C *and* B_0 *such that, for all* $B\geq B_0$,

$$\left|\lambda_1(B)-B\min\Big(\inf_{x\in\Omega}|\beta(x)|,\inf_{x\in\partial\Omega}\varsigma(\vartheta(x))|\beta(x)|\Big)\right|\leq C\,|B|^{\frac{3}{4}}\,. \tag{9.3}$$

Here we recall that the map $\vartheta\mapsto\varsigma(\vartheta)$ *was introduced in* (6.4).

We will prove only the lower bound. This proof is quite analogous to the proof of Proposition 8.2.2, given our analysis of the three-dimensional models carried out in Chapter 6.

Proof of Theorem 9.1.1.
As in the two-dimensional case, we introduce a partition of unity attached to a covering of Ω by small balls of radius $R(B)=B^{-\rho}$. The choice of $\rho>0$ will be done later, but, in any case, we observe that

$$R(B)\in\,]0,1]\,,$$

for $B\geq 1$—which is a condition that we always impose (we are interested in the large B asymptotics). We start from the same partition of unity [see (8.6), (8.7)],

$$\sum_{j\in\mathcal{J}}(\chi_j^B)^2=1\,,\qquad\qquad \sum_{\ell=1}^{3}\sum_{j\in\mathcal{J}}|\partial_{x_\ell}\chi_j^B|^2\leq\frac{C}{R(B)^2}\,. \tag{9.4}$$

We can additionally assume that each ball either does not intersect the boundary ("interior balls") or is centered at a point of the boundary ("boundary balls"). Using this partition of unity, we can then write that, for $u\in H^1(\Omega)$,

$$\|\nabla_{B\mathbf{F}}u\|^2\geq\sum_j\|\nabla_{B\mathbf{F}}(\chi_j^B u)\|^2-CR(B)^{-2}\sum_j\|\chi_j^B u\|^2\,.$$

We now distinguish between the case when j corresponds to a ball inside Ω and the case when the ball is centered at a point of $\partial\Omega$ and write

$$\sum_j=\sum_j^{\mathrm{int}}+\sum_j^{\mathrm{bnd}}\,. \tag{9.5}$$

Estimates for the interior balls
We do not repeat the estimates for the interior balls, which are quite analogous to the case of dimension 2. They lead to the choice of $\rho=3/8$ and so

$$R(B)=B^{-\frac{3}{8}}\,, \tag{9.6}$$

and to the estimate

$$\|\nabla_{B\mathbf{F}}(\chi_j^B u)\|^2 \geq B|\beta(x^j)|\,\|\chi_j^B u\|^2 - \hat{C}B^{\frac{3}{4}}\|\chi_j^B u\|^2 . \tag{9.7}$$

Estimates for the boundary balls
We proceed essentially in the same way as in the 2D case with two additional difficulties. We first take a system (see Section F.2.1) of tubular coordinates $y = \Theta(x)$ locally sending Ω on a half-space $\{y_3 > 0\}$, $\partial\Omega$ on $\{y_3 = 0\}$, and the center of the balls at $(0,0,0)$. Moreover, the metric is the identity at $(0,0,0)$ and hence the Jacobian is 1 at $(0,0,0)$. After a gauge transformation, we can assume[1] that

$$\mathbf{F}(x^j) = 0 .$$

The estimate that will be established at the end [see (9.8)] is gauge invariant, and so there is no loss of generality.

In the new coordinates $y = (y_1, y_2, y_3) = (r, s, t)$, the magnetic potential is defined by

$$\sum_j \mathbf{F}_j dx_j = \sum_j \widetilde{\mathbf{F}}_j dy_j .$$

The approximation of the quadratic form in the new coordinates is done by replacing $\widetilde{\mathbf{F}}_j(y)$ by its linear part at $(0,0,0)$ denoted by $\widetilde{\mathbf{F}}_j^{\mathrm{lin}}(y)$ so that

$$|\widetilde{\mathbf{F}}(y) - \widetilde{\mathbf{F}}^{\mathrm{lin}}(y)| \leq C|y|^2 ,$$

and by replacing in the Jacobian and the gradient the metric $g_{ij}(y)$ by the flat metric δ_{ij}.

The associated approximating operator is a Schrödinger operator with constant magnetic field of strength $|\beta(x^j)|$ (in the new variable y), but we now have to consider the Neumann realization in the half-space. This time a new parameter appears, which is the angle $\vartheta(x^j)$ introduced in (9.2). The bottom of the spectrum is given by $|\beta(x^j)|\varsigma(\vartheta(x^j))$ by (6.4).

We now follow the proof of the 2D case and note that, on the support of the ball $\mathrm{D}(x^j, R)$, we have, for v with $\operatorname{supp} v \subset \mathrm{D}(x^j, R)$,

$$\|v\|^2 = (1 + \mathcal{O}(R)) \int |\widetilde{v}(y)|^2 dy .$$

Here we have defined $\widetilde{v}$ by

$$\widetilde{v}(y) = v(\Theta^{-1}(y)) .$$

[1] We keep the notation $\mathbf{F}$ for this new magnetic potential, which may depend on the index j. Notice that the involved local gauge transformation possibly depends on j as well.

We now follow the proof of the "interior" case using the change of variables and we get, with $\widetilde{\chi}_j(y) := \chi_j^B(\Theta^{-1}(y))$,

$$\|\nabla_{B\mathbf{F}}(\chi_j^B u)\|^2 \geq (1 - CR)\|\widetilde{\nabla}_{B\mathbf{F}^{\mathrm{lin}}}(\widetilde{\chi}_j\widetilde{u})\|^2 - CB^2R^4\|\widetilde{\chi}_j\widetilde{u}\|^2 - CBR^2\|\widetilde{\chi}_j\widetilde{u}\|\|\widetilde{\nabla}_{B\mathbf{F}^{\mathrm{lin}}}(\widetilde{\chi}_j\widetilde{u})\| .$$

Proceeding in the same way, this leads, for any r , to

$$\begin{aligned}\|\nabla_{B\mathbf{F}}\chi_j u\|^2 &\geq (1 - CR)(1 - BR^2r^2)\, B\varsigma(\vartheta(x^j))\|\widetilde{\chi}_j\widetilde{u}\|^2 \\ &\quad - CB^2R^4\|\widetilde{\chi}_j\widetilde{u}\|^2 - CB\,R^2\,r^{-2}\|\widetilde{\chi}_j\widetilde{u}\|^2 \\ &\geq (1 - CR)^2(1 - BR^2r^2)\, B\,\varsigma(\vartheta(x^j))\|\chi_j u\|^2 \\ &\quad - C(1 + CR)B^2R^4\|\chi_j u\|^2 - C(1 + CR)B\,R^2\,r^{-2}\|\chi_j u\|^2 .\end{aligned}$$

The choice of the parameter $r = B^{-1/4}$ leads, together with the choice of R done in (9.6), to the lower bound

$$\|\nabla_{B\mathbf{F}}\chi_j u\|^2 \geq B\,\varsigma(\vartheta(x^j))\|\chi_j u\|^2 - \widehat{C}B^{\frac{3}{4}}\|\chi_j u\|^2 .$$

It remains to observe that the variation of $x \mapsto \varsigma(\vartheta(x))$ is uniformly controlled on the ball $\mathrm{D}(x^j, R)$, and we obtain

$$\|\nabla_{B\mathbf{F}}\chi_j u\|^2 \geq B\int \varsigma(\vartheta(x))|\chi_j(x)u(x)|^2\,dx - \widetilde{C}B^{\frac{3}{4}}\|\chi_j u\|^2 . \tag{9.8}$$

Summing up the contributions of the "interior" terms and the "boundary" terms, we have obtained the proof of the following proposition, which is the three-dimensional analog of Proposition 8.2.1.

Proposition 9.1.2.
We assume that Ω is a bounded, smooth domain in $\mathbb{R}^3$ and that a smooth magnetic field β is given such that $\inf_{x\in\overline{\Omega}}|\beta(x)| \neq 0$. Then, there exist constants C and B_0 such that, if

$$U_\beta(x) = \begin{cases} B|\beta(x)| & \text{if } d(x,\partial\Omega) \geq B^{-\frac{3}{8}} ,\\ \varsigma(\vartheta(x))|\beta(x)| & \text{if } d(x,\partial\Omega) < B^{-\frac{3}{8}} ,\end{cases} \tag{9.9}$$

then

$$Q_B(u) \geq \int_\Omega (U_\beta(x) - CB^{\frac{3}{4}})|u(x)|^2\,dx , \tag{9.10}$$

for all $u \in H^1(\Omega)$ and all $B \geq B_0$.

Here $x \mapsto \varsigma(\vartheta(x))$ is the extension of the function initially defined on $\partial\Omega$ to a tubular neighborhood of $\partial\Omega$, which is independent of $t = t(x)$ in the boundary coordinates.

As in the previous chapter, this proposition plays an important role for the control of the decay of the eigenfunctions and immediately implies Theorem 9.1.1. □

9.2 Refined Results for Constant Magnetic Fields

We now concentrate on the case when the magnetic field β is constant and describe the more accurate results that can be obtained in this case. This will be a step in the proof (under a generic assumption on the domain Ω) that the mapping $B \mapsto \lambda_1(B)$ is monotonically increasing for sufficiently large values of B.

Consider the case where the magnetic field is constant; after normalization we assume

$$|\beta| = 1\,. \tag{9.11}$$

Then the minimum in (9.3) is given by the term $\inf_{x\in\partial\Omega} \varsigma(\vartheta(x))$ and the infimum is obtained at the points of $\partial\Omega$ where $\varsigma(\vartheta(x))$ is minimal and hence $|\vartheta(x)|$ is minimal.

By Stokes' formula, we have

$$\int_{\partial\Omega} \sin\vartheta(x)\, d\sigma_{\partial\Omega} = \int_{\partial\Omega} \operatorname{curl}\mathbf{A}\cdot\nu\, d\sigma_{\partial\Omega} = \int_{\Omega} \operatorname{div}\operatorname{curl}\mathbf{A}\, dx = 0\,. \tag{9.12}$$

So there exists at least one point $x_0 \in \partial\Omega$ such that $\vartheta(x_0) = 0$, and it is then natural to introduce the (nonempty) set of boundary points where $\vartheta(x) = 0$, i.e., where β is tangent to $\partial\Omega$,

$$\Sigma := \{x \in \partial\Omega \,\big|\, \beta\cdot\nu(x) = 0\}\,. \tag{9.13}$$

We will work under the following geometric assumption.

Assumption 9.2.1.

1. *The domain Ω is a bounded, open set of $\mathbb{R}^3$ with smooth boundary.*
2. *On Σ, the differential d^T of the function $\partial\Omega \ni x \mapsto \beta\cdot\nu(x)$ is nonzero:*

$$d^T(\beta\cdot\nu(x)) \neq 0\,. \tag{9.14}$$

3. *The set of points where β is tangent to Σ is finite.*

Under these assumptions, Σ is a regular submanifold of $\partial\Omega$. Therefore, Σ is a disjoint union of regular curves. We choose an orientation on each such curve, and define the normal curvature at the point $x \in \Sigma$ by

$$k_n(x) := K_x(T(x)\wedge\nu(x), \beta)\,. \tag{9.15}$$

Here K denotes the second fundamental form on $\partial\Omega$, and $T(x)$ is the oriented, unit tangent vector to Σ at the point x. A computation gives

$$|k_n(x)| = |d^T(\beta\cdot\nu(x))|\,. \tag{9.16}$$

Hence, we have

$$k_n(x) \neq 0\,, \quad \forall x \in \Sigma\,. \tag{9.17}$$

Assumption 9.2.1 seems generic. It is, for instance, satisfied for ellipsoids, whereas a domain containing a cylindrical boundary piece with axis parallel to β will violate this assumption.

We will need a two-term asymptotics of the ground state energy of $\mathcal{H}(B)$.

Theorem 9.2.2.
With Θ_0 *from* (3.24) *and with* Ω *satisfying Assumption 9.2.1, we have, when* $\beta(x)$ *is constant of module* 1,

$$\lambda_1(B) = \Theta_0 B + \widehat{\gamma}_0 B^{\frac{2}{3}} + \mathcal{O}(B^{\frac{2}{3}-\eta}), \tag{9.18}$$

for some $\eta > 0$.

Here $\widehat{\gamma}_0$ *is defined by*

$$\widehat{\gamma}_0 := \inf_{x\in\Sigma} \widetilde{\gamma}_0(x), \tag{9.19}$$

$$\widetilde{\gamma}_0(x) := 2^{-\frac{2}{3}} \widehat{\nu}_0\, \delta_0^{\frac{1}{3}}\, |k_n(x)|^{\frac{2}{3}} \Big(1 - (1-\delta_0)\Big|T(x)\cdot\beta\Big|^2\Big)^{\frac{1}{3}}, \tag{9.20}$$

where $\widehat{\nu}_0$ *and* δ_0 *are defined in* (3.50) *and* (6.16).

Remark 9.2.3.
The two last items in Assumption 9.2.1 are not used for the proof of the upper bound.

9.3 Some Heuristics Around the Proof of Theorem 9.2.2

Giving in this book a complete proof of Theorem 9.2.2 would lead us too far. Therefore, we would like to discuss a simpler model that plays the role of approximating model at a point of Σ and that will permit us to understand the appearance of the first two coefficients in (9.18).

We concentrate our analysis on the model

$$P_0 := (hD_r - \sin\theta\, t)^2 + \left(hD_s + \cos\theta\, t + \gamma\frac{r^2}{2}\right)^2 + h^2 D_t^2, \tag{9.21}$$

on $\mathbb{R}^2_{r,s}\times\mathbb{R}^+_t$.

Here we have divided the operator by B^2 and introduced the semiclassical parameter

$$h = B^{-1}.$$

The three coordinates (r,s,t) (see Section F.2.2) should be interpreted in the following way. The hyperplane $t=0$ corresponds to the boundary and $r=0$ determines the curve (parametrized by s) on which the magnetic field vanishes. The parameter γ corresponds to a curvature that can be defined intrinsically at each point of Σ. Finally, $\theta + \pi/2$ denotes the angle inside $t=0$ of the magnetic field with the tangent to the curve.

Our model corresponds to the following magnetic vector field:

$$\beta = (-\cos\theta, -\sin\theta, \gamma r)\ . \tag{9.22}$$

Having the construction of quasimodes in mind or to find a lower bound for the bottom of the spectrum, we perform the dilation

$$r = h^{\frac{1}{3}}\hat{r}\ ,\ t = h^{\frac{1}{2}}\hat{t}\ ,\ s = h^{\frac{1}{2}}\hat{s}\,.$$

We first get

$$\widetilde{P}_0 := h\left((h^{\frac{1}{6}}D_{\hat{r}} - \sin\theta\,\hat{t})^2 + \left(D_{\hat{s}} + \cos\theta\,t + h^{\frac{1}{6}}\gamma\frac{\hat{r}^2}{2}\right)^2 + D_{\hat{t}}^2\right) \tag{9.23}$$

on $\mathbb{R}^2_{r,s}\times\mathbb{R}^+_t$. We then take a partial Fourier transform in the $\hat{s}$-variable (with the dual variable denoted by $\hat{\sigma}$) and obtain

$$\widehat{P}_0 := h\left((h^{\frac{1}{6}}D_{\hat{r}} - \sin\theta\,\hat{t})^2 + \left(\hat{\sigma} + \cos\theta\,t + h^{\frac{1}{6}}\gamma\frac{\hat{r}^2}{2}\right)^2 + D_{\hat{t}}^2\right)\,. \tag{9.24}$$

Remark 9.3.1.
From the point of view of the construction of suitable test functions, it may be better to think that we are looking for formal eigenfunctions for $\widetilde{P}_0$ *in the form*

$$\hat{u}(\hat{r},\hat{t};h)\exp(i\hat{\sigma}\hat{s}) \tag{9.25}$$

corresponding in the initial variables to formal solutions of P_0 *in the form*

$$u(r,t,s;h) = h^{-\frac{1}{6}-\frac{1}{4}}\hat{u}(h^{-\frac{1}{3}}r, h^{-\frac{1}{2}}t)\exp(ih^{-\frac{1}{2}}\hat{\sigma}s)\,. \tag{9.26}$$

Part of the task is to find the optimal $\hat{\sigma} = \hat{\sigma}(h)$ *in this trial state. These formal trial states actually have to be localized by suitable cutoff functions in order to get elements of the Hilbert space. See the end of the construction.*

On the contrary, when thinking of lower bounds, it is better to keep the point of view of partial Fourier transform and to think that we will try to minimize over $\hat{\sigma}$ *.*

After division by h, we find the operator (omitting the hats)

$$P_1 := D_t^2 + (t - h^{\frac{1}{6}}L_1(\widetilde{\sigma}))^2 + h^{\frac{1}{3}}L_2(\widetilde{\sigma})^2\,, \tag{9.27}$$

with

$$L_1(\widetilde{\sigma}) = \sin\theta D_r - \cos\theta\left(\widetilde{\sigma} + \frac{\gamma}{2}r^2\right)\,, \tag{9.28}$$

$$L_2(\widetilde{\sigma}) = \cos\theta D_r + \sin\theta\left(\widetilde{\sigma} + \frac{\gamma}{2}r^2\right)\,, \tag{9.29}$$

and

$$\widetilde{\sigma} = h^{-\frac{1}{6}} \hat{\sigma}\,.$$

We assume

$$\theta \neq \frac{\pi}{2}$$

(the situation $\theta = \pi/2$ corresponds to the particular case when the magnetic vector field is tangent to Σ) and will rewrite $\widetilde{\sigma}$ as

$$\widetilde{\sigma} = \widehat{\sigma}_0 h^{-\frac{1}{6}} + \widehat{\sigma}\,.$$

Here $\widehat{\sigma}_0$ is a constant to be determined, and $\widehat{\sigma}$ will be written as a formal power series in h; see (9.34).

After a gauge transformation,

$$u \mapsto \exp -i \left(\tan\theta \hat{\sigma}_0 h^{-\frac{1}{6}} r \right) \, u\,, \tag{9.30}$$

which leads to

$$P_2(\sigma_0, \widehat{\sigma}) := D_t^2 + (t - \sigma_0 - h^{\frac{1}{6}} L_1(\widehat{\sigma}))^2 + h^{\frac{1}{3}} L_2(\widehat{\sigma})^2\,, \tag{9.31}$$

with

$$\sigma_0 = \frac{\widehat{\sigma}_0}{\cos\theta}\,.$$

We now look for a solution having the form

$$u(t,r,h) \sim \sum_{j\geq 0} u_j(t,r) h^{\frac{j}{6}}\,, \tag{9.32}$$

$$\lambda(h) \sim \sum_{j\geq 0} \lambda_j h^{\frac{j}{6}}\,, \tag{9.33}$$

$$\widehat{\sigma}(h) = \sum_{j\geq 1} \sigma_j h^{\frac{j-1}{6}}\,, \tag{9.34}$$

satisfying

$$(P_2(\sigma_0, \widehat{\sigma}) - \lambda(h)) u(t,r,h) \sim 0\,, \tag{9.35}$$

with $\widehat{\sigma} = \widehat{\sigma}(h)$.

The goal is to determine for which pair $(\sigma_0, \widehat{\sigma}(h))$ one can find a minimal asymptotic $\lambda(h)$ in the limit $h \to 0$.

We will limit our analysis to the first three terms, which permit us to understand the main points and are enough for our constructions.

Expanding in powers of $h^{1/6}$, **the first equation** reads

$$[D_t^2 + (t - \sigma_0)^2 - \lambda_0] u_0(t,r) = 0\,, \tag{9.36}$$

and we look for

$$u_0(t,r) = \varphi_0(t)w_0(r)\,, \tag{9.37}$$

with $\|w_0\|_{L^2(\mathbb{R})} = 1$.

Using the notation of Section 3.2, (9.36) is satisfied if

$$(\mathfrak{h}^N(-\sigma_0) - \lambda_0)\varphi_0 = 0\,. \tag{9.38}$$

We choose σ_0 such that λ_0 is minimal. We know that this minimum is unique and corresponds to

$$\sigma_0 = -\xi_0 \quad \text{and} \quad \lambda_0 = \Theta_0\,, \tag{9.39}$$

and we choose the corresponding positive normalized eigenfunction [see (3.2.1)]

$$\varphi_0 = \varphi_{\xi_0}\,, \tag{9.40}$$

(with the Neumann condition at 0) and ψ_0 remains free for the moment. We recall from (3.29) at $\xi = \xi_0$ that φ_0 satisfies

$$\int_0^{+\infty} (t+\xi_0)\varphi_0(t)^2\,dt = 0\,. \tag{9.41}$$

We now look at the coefficient of $h^{1/6}$. The second equation reads

$$[\mathfrak{h}^N(\xi_0) - \Theta_0]u_1(t,r) - 2(t+\xi_0)L_1(\sigma_1)u_0 - \lambda_1 u_0 = 0\,. \tag{9.42}$$

We rewrite this in the form

$$[\mathfrak{h}^N(\xi_0) - \Theta_0]u_1(t,r) - 2(t+\xi_0)\varphi_0(t)L_1(\sigma_1)w_0 - \lambda_1\varphi_0 w_0 = 0\,. \tag{9.43}$$

Multiplying this last equation by $\varphi_0(t)$ and integrating over $t \in \mathbb{R}^+$, we get, using (9.41), the necessary condition

$$\lambda_1 = 0\,. \tag{9.44}$$

Then, with φ_1 solution of

$$[\mathfrak{h}^N(\xi_0) - \Theta_0]\varphi_1 = 2(t+\xi_0)\varphi_0\,, \qquad \int_0^{+\infty} \varphi_0\varphi_1\,dt = 0\,, \tag{9.45}$$

i.e.,

$$\varphi_1 = R^0\big((t+\xi_0)\varphi_0\big)\,,$$

with the notation from Lemma 3.2.9, we can take

$$u_1(t,r) = w_1(r)\varphi_0(t)\,, \tag{9.46}$$

with

$$w_1(r) = L_1(\sigma_1) w_0(r) \,. \tag{9.47}$$

Let us now look at the crucial equation expressing the $\mathcal{O}(h^{1/3})$ balance.

$$\begin{aligned} &[\mathfrak{h}^N(\xi_0) - \Theta_0] u_2(t,r) - 2(t+\xi_0) L_1(\sigma_1) u_1 \\ &\quad + \big(L_1(\sigma_1)^2 + L_2(\sigma_1)^2 - \lambda_2\big) u_0 = 0 \,. \end{aligned} \tag{9.48}$$

Using our previous choices, we get

$$\begin{aligned} &[\mathfrak{h}^N(\xi_0) - \Theta_0] u_2(t,r) - 2(t+\xi_0) \varphi_1 L_1(\sigma_1)^2 w_0 \\ &\quad + \varphi_0 \big(L_1(\sigma_1)^2 + L_2(\sigma_1)^2 - \lambda_2\big) w_0 = 0 \,. \end{aligned} \tag{9.49}$$

A necessary solution for solving is obtained as before by multiplying by φ_0 and integrating over $t \in]0, +\infty[$. We get

$$\delta_0 L_1(\sigma_1)^2 w_0 + L_2(\sigma_1)^2 w_0 - \lambda_2 w_0 = 0 \,, \tag{9.50}$$

with

$$\delta_0 = 1 - 2 \int_0^{+\infty} (t+\xi_0) \varphi_1(t) \varphi_0(t) \, dt \,. \tag{9.51}$$

It is then natural to introduce

$$P_3(\sigma_1) := \delta_0 L_1(\sigma_1)^2 + L_2(\sigma_1)^2 \,. \tag{9.52}$$

Our aim is now to minimize the bottom of the spectrum of $P_3(\sigma_1)$ over σ_1 .

Let us now show that by a gauge transform, we can rewrite $P_3(\sigma_1)$ in the form

$$P_4(\sigma_1) = c D_r^2 + d(r^2 - \rho)^2 \,. \tag{9.53}$$

We look for a gauge function on the form

$$t(\theta, r) = \alpha(\theta) \left(\frac{\gamma}{6} r^3 + \sigma_1 r \right) \,, \tag{9.54}$$

such that

$$P_4(\sigma_1) := e^{-it(\theta,r)} P_3(\sigma_1) e^{it(\theta,r)} \,. \tag{9.55}$$

The function $\alpha(\theta)$ in (9.54) is chosen such that the coefficient of the operator $(\frac{\gamma}{2} r^2 + \sigma_1) D_r + D_r (\frac{\gamma}{2} r^2 + \sigma_1)$ vanishes. This leads to

$$\alpha(\theta) = \frac{\sin\theta \, \cos\theta \, (1 - \delta_0)}{\delta_0 \sin^2\theta + \cos^2\theta} \,. \tag{9.56}$$

Of course, we have

$$c = \cos^2\theta + \delta_0 \sin^2\theta \,, \tag{9.57}$$

and

$$\rho = 2\sigma_1/\gamma\,. \tag{9.58}$$

But for this value of $\alpha(\theta)$, we get:

$$d = \left(\frac{\gamma}{2}\right)^2 \left(\delta_0(\cos\theta + \alpha(\theta)\sin\theta)^2 + (-\sin\theta + \alpha(\theta)\cos\theta)^2\right)\,.$$

After computation, this gives

$$d = \delta_0 \left(\frac{\gamma}{2}\right)^2 /(\delta_0 \sin^2\theta + \cos^2\theta)\,. \tag{9.59}$$

We now rescale the operator $cD_r^2 + d(r^2-\rho)^2$. This means that we perform a new scaling:

$$r = \left(\frac{c}{d}\right)^{\frac{1}{6}} r'\,,$$

such that $P_5(\sigma_1)$ becomes in the new coordinates

$$P_5(\sigma_1) = d^{\frac{1}{3}} c^{\frac{2}{3}} \left[D_{r'}^2 + ((r')^2 - \rho')^2\right]\,, \tag{9.60}$$

with

$$\rho' = \left(\frac{c}{d}\right)^{-\frac{1}{3}} \rho\,.$$

We observe that c and d are independent of σ_1. So in order to minimize the bottom of the spectrum of the initial operator over σ_1, we will have to minimize the bottom of the spectrum of the operator $\left(D_{r'}^2 + ((r')^2 - \rho')^2\right)$ over ρ', which is obtained for $\rho' = \rho_{\min}$, and take the value $\hat{\nu}_0$ introduced in (3.50). This corresponds to

$$\sigma_{1,\min} = \frac{\gamma}{2} \left(\frac{c}{d}\right)^{\frac{1}{3}} \rho_{\min}\,, \tag{9.61}$$

with

$$\frac{c}{d} = (\cos^2\theta + \delta_0 \sin^2\theta)^2/[\delta_0(\gamma/2)^2]\,. \tag{9.62}$$

So the infimum over σ_1 of the bottom of the spectrum of $P_5(\sigma_1)$ is given for this value of $\sigma_1 = \sigma_{1,\min}$ by

$$d^{\frac{1}{3}} c^{\frac{2}{3}} \hat{\nu}_0 = \left(\frac{1}{2}\right)^{\frac{2}{3}} \delta_0^{\frac{1}{3}} |\gamma|^{\frac{2}{3}} (\delta_0 \sin^2\theta + \cos^2\theta)^{\frac{1}{3}} \hat{\nu}_0\,. \tag{9.63}$$

We have consequently found $w_0 \neq 0$, σ_1 such that λ_2 is as small as possible. One can then find u_2 as a solution in the form

$$u_2 = \varphi_2 w_2\,,$$

where φ_2 is the solution orthogonal to φ_0 of

$$[\mathfrak{h}^N(\xi_0) - \Theta_0]\varphi_2 = 2(t+\xi_0)\varphi_1 - (1-\delta_0)\varphi_0\,,$$

and

$$w_2 = L_1(\sigma_1)^2 w_0\,.$$

Reintroducing the hats, we have found σ_0 and σ_1 and an L^2-normalized $u^{(2)} \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^+)$ such that the energy $\langle P_2(\sigma_0)u^{(2)}, u^{(2)}\rangle$ of

$$u^{(2)}(\hat r; h) := c_h \left(\sum_{j=0}^{2} u_j(\hat r, \hat t) h^{\frac{j}{6}}\right)$$

satisfies

$$\langle P_2(\sigma_0)u^{(2)}, u^{(2)}\rangle = \Theta_0 + h^{\frac{1}{3}}\hat\nu_0(\tfrac{1}{2})^{\frac{2}{3}}\delta_0^{\frac{1}{3}}|\gamma|^{\frac{2}{3}}(\delta_0\sin^2\theta + \cos^2\theta)^{\frac{1}{3}} + \mathcal{O}(h^{\frac{1}{2}})\,. \tag{9.64}$$

In particular, we have proved the following:

Lemma 9.3.2.
The ground state energy of the family of model operators $P_2(\sigma_0, \hat\sigma)$ satisfies

$$\inf_{\sigma_0,\sigma_1} (\inf\sigma(P_2(\sigma_0,\sigma_1))) \leq \Theta_0 + h^{\frac{1}{3}}\hat\nu_0(\tfrac{1}{2})^{\frac{2}{3}}\delta_0^{\frac{1}{3}}|\gamma|^{\frac{2}{3}}(\delta_0\sin^2\theta + \cos^2\theta)^{\frac{1}{3}} + \mathcal{O}(h^{\frac{1}{2}})\,. \tag{9.65}$$

Moreover, the upper bound is obtained for $\sigma_0 = -\xi_0$ and $\sigma_1 = \sigma_{1,\min}$ defined in (9.61).

Sketch for the lower bound.

We can use a vector-valued version of the analysis of $\mathfrak{h}^N(\xi)$. This leads, using an abstract functional calculus, to

$$P_1(\widetilde\sigma) \geq \mu(h^{\frac{1}{3}}L_1(\widetilde\sigma)) + h^{\frac{1}{3}}L_2(\widetilde\sigma)^2\,.$$

Modulo a localization argument for which we refer to [HeM6], we can then use the quadratic approximation of μ to replace $\mu(h^{\frac{1}{3}}L_1(\widetilde\sigma)) + h^{\frac{1}{3}}L_2(\widetilde\sigma)^2$ by $\Theta_0 + h^{\frac{1}{3}}P_3(\widetilde\sigma)$ and we can then use the previous analysis.

Quasimodes for P_0

Having in mind formulas (9.25) and (9.26), we define

$$\begin{aligned} u^h(r,s,t) &= c_h h^{-\frac{1}{6}-\frac{1}{4}-\frac{\delta}{2}} u^{(2)}(\hat r, \hat t)\exp(ih^{-\frac{1}{6}}\widetilde\sigma s)\ \exp(i\hat\sigma_0\tan\theta h^{-\frac{1}{2}} r) \\ &\quad \times \chi(h^{-\delta}s)\chi(h^{-\delta+\frac{1}{3}}r/C)\chi(h^{-\delta+\frac{1}{2}}t/C)\,, \end{aligned} \tag{9.66}$$

with $\delta \in]\frac{5}{18}, \frac{1}{3}[$ and χ a cutoff function around the origin.

Remark 9.3.3.
In our application, γ and θ are not independent but should satisfy

$$\gamma = k_n(x)\,, \qquad \theta = \theta(x)\,,$$

for some x in Σ. Here $k_n(x)$ was defined in (9.15) *and $\theta(x)$ is defined by*

$$\theta(x) = \arcsin\left(T(x)\cdot\frac{\beta(x)}{|\beta(x)|}\right), \tag{9.67}$$

where $T(x)$ is a unit-vector tangent to Σ at x.

This suggests that we have to look for a minimum over Σ of the expression

$$\gamma^2\left(\delta_0\sin^2\theta(x) + \cos^2\theta(x)\right),$$

that is,

$$\ell^2 := \inf_{x\in\Sigma}\left\{(k_n(x))^2(\delta_0\sin^2\theta(x)+\cos^2\theta(x))\right\}. \tag{9.68}$$

This explains formulas (9.19) *and* (9.20)*.*

9.4 Localization Estimates

We still consider the case of constant magnetic field β under Assumption 9.2.1. We start by stating the decay in the direction normal to the boundary. We will often use the notation

$$t(x) := \operatorname{dist}(x, \partial\Omega)\,. \tag{9.69}$$

Now, if $\phi \in C_0^\infty(\Omega)$, i.e., has support away from the boundary, Lemma 1.4.1 implies that

$$Q_B(\phi) \geq B\|\phi\|_2^2\,. \tag{9.70}$$

As in dimension 2, this inequality (and the fact that $\Theta_0 < 1$) implies that ground states are exponentially localized near the boundary (in the sense of Theorem 9.4.1). We give the result without repeating the proof.

Theorem 9.4.1.
Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set with smooth boundary. Then there exist positive constants C, a_1, and B_0 such that

$$\begin{aligned}\int_\Omega e^{2a_1 B^{\frac{1}{2}} t(x)}\Big(|\psi_B(x)|^2 + B^{-1}|(-i\nabla + B\mathbf{F})\psi_B(x)|^2\Big)\,dx \\ \leq C\,\|\psi_B\|_2^2\,,\end{aligned} \tag{9.71}$$

for all $B \geq B_0$ and all ground states ψ_B of the operator $\mathcal{H}(B)$.

We will mainly use this localization result in the following form.

Corollary 9.4.2.
Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set with smooth boundary. Then for all $n \in \mathbb{N}$, there exist $C_n > 0$ and $B_n > 0$ such that

$$\int t(x)^n |\psi_B(x)|^2 \, dx \leq C_n B^{-\frac{n}{2}} \|\psi_B\|_2^2 \,,$$

for all $B \geq B_n$ and all ground states ψ_B of the operator $\mathcal{H}(B)$.

We now define tubular neighborhoods of the boundary as follows. For $\epsilon > 0$, define

$$B(\partial\Omega, \epsilon) = \{x \in \Omega \,:\, t(x) \leq \epsilon\}\,. \tag{9.72}$$

For sufficiently small ϵ_0, we have that for all $x \in B(\partial\Omega, 2\epsilon_0)$ exists a unique point $y = y(x) \in \partial\Omega$ such that $t(x) = \operatorname{dist}(x, y)$. We fix such an ϵ_0 in the rest of the chapter.

We extend the definition of ϑ introduced in (9.2) to the tubular neighborhood $B(\partial\Omega, 2\epsilon_0)$ by $\vartheta(x) := \vartheta(y(x))$. In order to obtain localization estimates in the variable normal to Σ, we use the following operator inequality, which is a particular case of Proposition 9.1.2.

Proposition 9.4.3.
Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set with smooth boundary. Let B_0 be chosen such that $B_0^{-3/8} = \frac{1}{2}\epsilon_0$ and define, for $B \geq B_0, C > 0$, and $x \in \Omega$,

$$W_B(x) := \begin{cases} B - CB^{\frac{3}{4}}\,, & t(x) \geq B^{-\frac{3}{8}}\,, \\ B\varsigma(\vartheta(x)) - CB^{\frac{3}{4}}\,, & t(x) < B^{-\frac{3}{8}}\,, \end{cases} \tag{9.73}$$

where ς is the function defined in (6.4).

Then

$$\mathcal{H}(B) \geq W_B\,, \tag{9.74}$$

for all $B \geq B_0$, if C is sufficiently large.

We use this energy estimate to prove Agmon-type estimates on the boundary.

Theorem 9.4.4.
Suppose that $\Omega \subset \mathbb{R}^3$ satisfies Assumption 9.2.1. Define, for $x \in \partial\Omega$,

$$d_\Sigma(x) := \operatorname{dist}(x, \Sigma)\,,$$

and extend d_Σ to a tubular neighborhood of the boundary by $d_\Sigma(x) := d_\Sigma(y(x))$ [where $y(x)$ is the unique boundary point closest to x].

Then there exist positive constants C, a_2, and B_0 such that

$$\int_{B(\partial\Omega,\epsilon_0)} e^{2a_2 B^{\frac{1}{4}} d_\Sigma(x)} |\psi_B(x)|^2\, dx \leq C\|\psi_B\|_2^2\,, \tag{9.75}$$

for all $B \geq B_0$ and all ground states ψ_B of $\mathcal{H}(B)$.

It is useful to collect the following easy consequence.

Corollary 9.4.5.
Suppose that $\Omega \subset \mathbb{R}^3$ satisfies Assumption 9.2.1. Then, for all $n \in \mathbb{N}$, there exist $C_n > 0$ and $B_n > 0$ such that

$$\int_{B(\partial\Omega,\epsilon_0)} d_\Sigma(x)^n |\psi_B(x)|^2\, dx \leq C_n B^{-\frac{n}{4}} \|\psi_B\|_2^2\,, \tag{9.76}$$

for all $B \geq B_n$ and all ground states ψ_B of $\mathcal{H}(B)$.

Proof of Theorem 9.4.4.
We may clearly assume that $\|\psi_B\|_2 = 1$. Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ satisfy that χ_1 is decreasing, $\chi_1 \equiv 0$ on $[2, +\infty[$, $\chi_1 \equiv 1$ on $]-\infty, 1]$, χ_2 is increasing, $\chi_2 \equiv 1$ on $[2, +\infty[$, and $\chi_2 \equiv 0$ on $]-\infty, 1]$.

By the standard localization formula, we find, since $\mathcal{H}(B)\psi_B = \lambda_1(B)\psi_B$,

$$\begin{aligned}
&\lambda_1(B) \left\| \chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma} \psi_B \right\|_2^2 \\
&\quad = Q_B\left[\chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma} \psi_B\right] \\
&\qquad - \int \left| \nabla\left(\chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma}\right)\right|^2 |\psi_B|^2\, dx\,.
\end{aligned} \tag{9.77}$$

Using Theorem 9.4.3, we estimate

$$\begin{aligned}
&Q_B\left[\chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma} \psi_B\right] \\
&\qquad \geq \int W_B(x) \left|\chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma} \psi_B\right|^2 dx\,.
\end{aligned} \tag{9.78}$$

Also,

$$\begin{aligned}
&\left|\nabla\left(\chi_1\left(\frac{t}{\epsilon_0}\right) \chi_2(B^{\frac{1}{4}} d_\Sigma) e^{aB^{\frac{1}{4}} d_\Sigma}\right)\right|^2 \\
&\quad \leq 2\left|\nabla\chi_1\left(\frac{t}{\epsilon_0}\right)\right|^2 \chi_2^2(B^{\frac{1}{4}} d_\Sigma) e^{2aB^{\frac{1}{4}} d_\Sigma} \\
&\qquad + 2\left|\nabla\chi_2(B^{\frac{1}{4}} d_\Sigma)\right|^2 \chi_1^2\left(\frac{t}{\epsilon_0}\right) e^{2aB^{\frac{1}{4}} d_\Sigma} \\
&\qquad + 2a^2 B^{\frac{1}{2}} \chi_1^2\left(\frac{t}{\epsilon_0}\right) \chi_2^2(B^{\frac{1}{4}} d_\Sigma) e^{2aB^{\frac{1}{4}} d_\Sigma}.
\end{aligned} \tag{9.79}$$

Combining (9.77), (9.78), and (9.79), we find

$$\int \big(W_B(x) - \lambda_1(B) - 2a^2 B^{\frac12}\big)\chi_1^2\left(\frac{t}{\epsilon_0}\right)\chi_2^2(B^{\frac14} d_\Sigma)e^{2aB^{\frac14} d_\Sigma}|\psi_B|^2\,dx$$
$$\leq C\int_{B(\partial\Omega,2\epsilon_0)\setminus B(\partial\Omega,\epsilon_0)} e^{2aB^{\frac14} d_\Sigma}|\psi_B|^2\,dx$$
$$+ CB^{\frac12}e^{4a}\int_{\{x\in B(\partial\Omega,2\epsilon_0)\,:\,B^{\frac14} d_\Sigma(x)\leq 2\}} |\psi_B(x)|^2\,dx\,. \tag{9.80}$$

Since Ω is bounded there exists $D > 0$ such that $d_\Sigma(x) \leq D$ for all x. Thus we can estimate, with a_1 being the constant from Theorem 9.4.1,

$$\int_{B(\partial\Omega,2\epsilon_0)\setminus B(\partial\Omega,\epsilon_0)} e^{2aB^{\frac14} d_\Sigma}|\psi_B|^2\,dx$$
$$\leq e^{2aB^{\frac14}D^{3/2}}e^{-2a_1B^{\frac14}\epsilon_0}\int_\Omega e^{2a_1B^{\frac14}t(x)}|\psi_B|^2\,dx$$
$$\leq Ce^{2B^{\frac14}(aD^{\frac32}-a_1\epsilon_0)}\|\psi_B\|_2^2 = \mathcal{O}(B^{-\infty})\,, \tag{9.81}$$

where the last estimate holds if a is sufficiently small.

Notice now that Assumption (9.15) implies that $\beta\cdot N$ vanishes exactly at order 1 on Σ. Therefore, using the boundedness of Ω, there exists a constant $C > 0$ such that

$$C^{-1}d_\Sigma(x) \leq \vartheta(x) \leq Cd_\Sigma(x)\,, \tag{9.82}$$

for all $x \in B(\partial\Omega, 2\epsilon_0)$. Therefore, using that ς is monotone, that $\delta_0 > 0$ [from (6.17)], and the upper bound on $\lambda_1(B)$ [from (9.18)], we find that

$$W_B(x) - \lambda_1(B) - 2a^2B^{\frac12} \geq c_0B^{\frac34}\,, \tag{9.83}$$

for some $c_0 > 0$ and for all $x \in B(\partial\Omega, 2\epsilon_0)\cap\{B^{1/4}d_\Sigma(x)\geq 1\}$, if a is sufficiently large and B is sufficiently large.

Inserting (9.81) and (9.83) in (9.80) yields

$$\int \chi_1^2\left(\frac{t}{\epsilon_0}\right)\chi_2^2(B^{\frac14}d_\Sigma)e^{2aB^{\frac14}d_\Sigma}|\psi_B|^2\,dx \leq C\,. \tag{9.84}$$

Since $e^{2aB^{1/4}d_\Sigma}$ is bounded when $B^{1/4}d_\Sigma \leq 2$, (9.75) follows from (9.84). □

Consider now the set $\mathcal{M}_\Sigma \subset \Sigma$ where the function $\widetilde{\gamma}_0$ is minimized:

$$\mathcal{M}_\Sigma := \{x\in\Sigma\,:\,\widetilde{\gamma}_0 = \widehat{\gamma}_0\}\,. \tag{9.85}$$

Theorem 9.4.6.
Suppose that $\Omega \subset \mathbb{R}^3$ satisfies Assumption 9.2.1 and let $\delta > 0$. Then, for all $N > 0$, there exist C_N and $B_N > 0$ such that if ψ_B is a ground state of $\mathcal{H}(B)$, then

$$\int_{\{x\in\Omega\,:\,\mathrm{dist}(x,\mathcal{M}_\Sigma)\geq\delta\}} |\psi_B(x)|^2\, dx \leq C_N B^{-N}\,, \tag{9.86}$$

for all $B \geq B_N$.

Sketch of the proof.
We refer to the analysis of the model done in Section 9.3 for an idea of this proof, which is actually very long and technical. One can then use Agmon estimates using the weight $\exp[\alpha B^{1/3}(\widetilde{\gamma}_0(x) - \widehat{\gamma}_0)]$ for some $\alpha > 0$. This actually gives a decay in $\exp[-\alpha(\delta)B^{1/3}]$ for some $\alpha(\delta) > 0$. □

9.5 The Derivative of $\lambda_1(B)$

In this section, we prove how one can derive the monotonicity result from the known asymptotics of the ground state energy and localization estimates for the ground state itself. This section also is restricted to the case of constant magnetic field β and Ω satisfying Assumption 9.2.1. Of course, monotonicity would follow from the general Corollary 2.3.3 if a sufficiently precise asymptotics [up to order $o(1)$] of $\lambda_1(B)$ were available. However, we do not know any genuinely three-dimensional example where such an asymptotics is known. Therefore, we will combine the idea of the proof of Corollary 2.3.3 with various localization estimates from the previous section.

Theorem 9.5.1.
Let $\beta \in \mathbb{S}^2$ be a constant magnetic field and let $\Omega \subset \mathbb{R}^3$ satisfy Assumption 9.2.1. Let $\{\Sigma_1, \dots, \Sigma_n\}$ be the collection of disjoint smooth curves making up Σ. We assume that for all j there exists $x \in \Sigma_j$ such that $\widetilde{\gamma}_0(x) > \widehat{\gamma}_0$. Then the directional derivatives $\lambda'_{1,\pm} := \lim_{\beta\to 0\pm}[\lambda_1(B+\beta) - \mu(B)]/\beta$, exist and satisfy

$$\lim_{B\to\infty} \lambda'_{1,+}(B) = \lim_{B\to\infty} \lambda'_{1,-}(B) = \Theta_0\,. \tag{9.87}$$

In particular, there exists $B_0 \geq 0$ such that $B \mapsto \lambda_1(B)$ is strictly increasing on $[B_0, +\infty[$.

Based on these estimates, the proof of Theorem 9.5.1 is very similar to the two-dimensional case. But we first need an adapted gauge transformation.

Proposition 9.5.2.
Let d_Σ be the function defined in Theorem 9.4.4 and let Σ_j be one of the curves making up Σ. Let $s_0 \in \Sigma_j$ and define, for $\epsilon > 0$,

$$\Omega(\epsilon, s_0) = \{x \in \Omega\ :\ d_\Sigma(x) < \epsilon \text{ and } \mathrm{dist}(x, s_0) > \epsilon\}\,. \tag{9.88}$$

Then if ϵ is sufficiently small, there exists a function $\phi \in C^\infty(\overline{\Omega})$ such that $\widehat{\mathbf{A}} := \mathbf{F} + \nabla\phi$ satisfies

$$|\widehat{\mathbf{A}}(x)| \leq C\Big(t(x) + d_\Sigma(x)^2\Big)\,,$$

for all $x \in \Omega(\epsilon, s_0)$.

An easy localization argument shows that we can carry out the above gauge change simultaneously at each Σ_j.

Corollary 9.5.3.
Let $(s_1, \dots, s_N) \in \Sigma_1 \times \cdots \times \Sigma_N$ and define, for $\epsilon > 0$,

$$\begin{aligned}&\Omega\big(\epsilon, (s_1, \dots, s_N)\big)\\ &\qquad = \{x \in \Omega \,:\, \operatorname{dist}(x, \Sigma) < \epsilon \ \textit{and}\ \min_j \operatorname{dist}(x, s_j) > \epsilon\}\,.\end{aligned} \tag{9.89}$$

Then if ϵ is sufficiently small, there exists a function $\phi \in C^\infty(\overline{\Omega})$ such that $\widehat{\mathbf{A}} := \mathbf{F} + \nabla\phi$ satisfies

$$|\widehat{\mathbf{A}}(x)| \leq C\Big(t(x) + d_\Sigma(x)^2\Big)\,,$$

for all $x \in \Omega\big(\epsilon, (s_1, \dots, s_N)\big)$.

Proof of Proposition 9.5.2.
We use the adapted coordinates (r, s, t) near Σ_j defined in Section F.2. Let Σ_j be parametrized by arc-length as

$$\frac{|\Sigma_j|}{2\pi}\mathbb{S}^1 \ni s \mapsto \Sigma_j(s) \in \partial\Omega\,.$$

Given a point $x \in \Omega$ sufficiently close to Σ_j, there exist a unique point $y(x) \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, y(x))$ and a unique point $\Sigma_j(s(x)) \in \Sigma_j$ such that $\operatorname{dist}_{\partial\Omega}(y(x), \Sigma_j) = \operatorname{dist}_{\partial\Omega}(y(x), \Sigma_j(s(x)))$, where $\operatorname{dist}_{\partial\Omega}$ denotes the geodesic distance on the boundary. The coordinates (r, s, t) associated with the point x now satisfy

$$|r| = \operatorname{dist}_{\partial\Omega}(y(x), \Sigma_j)\,, \qquad s = s(x)\,, \qquad t = \operatorname{dist}(x, \partial\Omega)\,.$$

Notice that there exists a constant $C > 0$ such that

$$C^{-1} d_\Sigma(x) \leq |r(x)| \leq C d_\Sigma(x)\,,$$

and so we may replace d_Σ by r in the proposition.

Let $\widetilde{A}_1 dr + \widetilde{A}_2 ds + \widetilde{A}_3 dt$ be the magnetic 1-form $\omega_{\mathbf{A}} = \mathbf{A} \cdot dx$ written in the new coordinates (r, s, t). Also, write the corresponding magnetic 2-form, $d\omega_{\mathbf{A}}$, as

$$\widetilde{B}_{12} dr \wedge ds + \widetilde{B}_{13} dr \wedge dt + \widetilde{B}_{23} ds \wedge dt\,.$$

Clearly,

$$\widetilde{B}_{ij} = \partial_i \widetilde{A}_j - \partial_j \widetilde{A}_i \,, \tag{9.90}$$

for $i < j$ and where we identify $(1,2,3)$ with (r,s,t) for the derivatives.

The magnetic field β corresponds to the magnetic 2-form via the Hodge-map. In particular, since β is tangent to $\partial\Omega$ at Σ, we get

$$\widetilde{B}_{12}(0,s,0) = 0\,. \tag{9.91}$$

We now find a particular solution $\widetilde{\mathbf{A}}$ to (9.90). We make the *Ansatz*

$$\widetilde{A}_1 = -\int_0^t \widetilde{B}_{13}(r,s,\tau)\,d\tau\,, \tag{9.92}$$

$$\widetilde{A}_2 = -\int_0^t \widetilde{B}_{23}(r,s,\tau)\,d\tau + \psi_2(r,s)\,, \tag{9.93}$$

$$\widetilde{A}_3 = 0\,. \tag{9.94}$$

Using the relation $d(d\omega_{\mathbf{A}}) = 0$, we see that the above *Ansatz* gives a solution if ψ_2 is chosen as

$$\psi_2(r,s) = \int_0^r \widetilde{B}_{12}(\rho,s,0)\,d\rho\,. \tag{9.95}$$

We can verify by inspection that with these choices

$$|\widetilde{\mathbf{A}}| \leq C(r^2 + t)\,. \tag{9.96}$$

By transporting this $\widetilde{\mathbf{A}}$ back to the original coordinates, we get the existence of an $\widehat{\mathbf{A}}$ with

$$\operatorname{curl}\widehat{\mathbf{A}} = 1, \qquad |\widehat{\mathbf{A}}(x)| \leq C(t(x) + d_\Sigma(x)^2)\,.$$

Since $\Omega(\epsilon, s_0)$ is simply connected (for sufficiently small ϵ), $\widehat{\mathbf{A}}$ is gauge equivalent to $\mathbf{F}$ and the proposition is proved. □

Proof of Theorem 9.5.1.
The strategy is the same as the one applied in the proof of Theorem 8.5.1. We will use a convenient gauge together with localization estimates in order to prove monotonicity using a less precise asymptotics than the one required by the general statement of Corollary 2.3.3.

Applying analytic perturbation theory to $\mathcal{H}(B)$, we get the existence of $\lambda'_{1,\pm}(B)$.

Let $\Sigma = \cup_{j=1}^N \Sigma_j$ be the decomposition of Σ in disjoint closed curves and let $s_j \in \Sigma_j$ be a point with $\widetilde{\gamma}(s_j) > \widehat{\gamma}_0$. Let $\Omega\big(\epsilon, (s_1, \ldots, s_N)\big)$ be as defined in (9.89) with ϵ so small that

$$\widetilde{\gamma}(x) > \widehat{\gamma}_0\,,$$

for all x in the set

$$\{x \in \Omega \,:\, \operatorname{dist}(x,\Sigma) < \epsilon \text{ and } \min \operatorname{dist}(x, s_j) \leq \epsilon\}\,.$$

Define $\widehat{\mathbf{A}}$ to be the vector potential defined in Corollary 9.5.3. Let $\widehat{Q}_B$ be the quadratic form

$$H^1(\Omega) \ni u \mapsto \widehat{Q}_B(u) = \int_\Omega |-i\nabla u + B\widehat{\mathbf{A}}u|^2\,dx\,,$$

and let $\widehat{\mathcal{H}}(B)$ be the associated operator. Then $\widehat{\mathcal{H}}(B)$ and $\mathcal{H}(B)$ are unitarily equivalent: $\widehat{\mathcal{H}}(B) = e^{iB\phi}\mathcal{H}(B)e^{-iB\phi}$, for some ϕ independent of B.

It follows from the strategy of the proof of Theorem 8.5.1 that it suffices to prove

$$B^{\frac{2}{3}} \int_\Omega |\widehat{\mathbf{A}}(x)|^2\, |\psi_1^+(x;B)|^2\,dx \leq C\,, \tag{9.97}$$

for some constant C independent of B. [Then we can take $b := MB^{\frac{2}{3}-\eta}$, in (8.62) with η from (9.18) and $M > 0$ arbitrarily large and proceed as in the proof of Theorem 8.5.1].

Thus, it remains to prove (9.97).

By Corollary 9.5.3, we can estimate

$$\begin{aligned}\int_\Omega &|\widehat{\mathbf{A}}(x)|^2\, |\psi_1^+(x;B)|^2\,dx \\ &\leq C\int_{\Omega(\epsilon,(s_1,\ldots,s_N))} (t^2+r^4)|\psi_1^+(x;B)|^2\,dx \\ &\quad + \|\widehat{\mathbf{A}}\|_\infty^2 \int_{\Omega\setminus\Omega(\epsilon,(s_1,\ldots,s_N))} |\psi_1^+(x;B)|^2\,dx\,. \end{aligned} \tag{9.98}$$

Combining Corollaries 9.4.2 and 9.4.5 and Theorem 9.4.6, we therefore find the existence of a constant $C > 0$ such that

$$\int_\Omega |\widehat{\mathbf{A}}(x)|^2\, |\psi_1^+(x;B)|^2\,dx \leq C\,B^{-1}\,, \tag{9.99}$$

which is stronger than the estimate (9.97) needed. □

9.6 Notes

1. The localization of the ground state around Σ is mentioned in the physics literature, at least for the case of the sphere. One can find in Chapter 4 of [S-JST] a physical presentation of the problem we are considering. We place particular emphasis on their Section 4.3, where they analyze

(with partially heuristic arguments) the angular dependence of the nucleation field (which is the third critical field). For type II superconductors, they write

Superconductivity is not entirely destroyed for $H_{C_2} < H < H_{C_3}$ *. A superconducting sheath remains close to the surface parallel to the applied field. Conversely, when the field is decreased below* H_{C_3} *, a superconducting sheath appears at the surface before superconductivity is restored in the bulk at* $H = H_{C_2}(\kappa)$ *. If the sample is a long cylinder with the applied field parallel to the axis, the sheath will cover all the surface of the cylinder. If it is a sphere, the sheath will be restricted to a small zone near the equatorial plane when* H *is close to* H_{C_3} *. When the field is decreased toward* $H_{C_2}(\kappa)$ *, the sheath will progressively extend up to the poles.*

Note, however, that in this chapter we have only analyzed the linear problem. But this text could give the main motivation for the second part of the book.

Notice also that a precise spectral analysis in the case of the sphere is contained in [FoP].

2. The fact that the ground state energy is minimal when the magnetic field is tangent to the boundary is at the origin of the choice of some one-dimensional models occurring often in the literature. Let us consider $\mathbb{R}^{3,+}$ (or $\mathbb{R}^2 \times]-d,+d[$) and assume that the external magnetic field is tangent to the boundary $x_3 = 0$. Then it is natural to minimize the Ginzburg–Landau functional over $\mathbf{A}$'s that have the same property. One can restrict the functional to vector potentials in the form $\mathbf{A}(x_1,x_2,x_3) = (a(x_3),0,0)$. This leads by minimization to the reduced model mentioned in Note 8 to Chapter 1 [see (1.58)].
3. Theorem 9.1.1 was first obtained in [LuP7] with some additional information concerning the decay appearing in [HeM4]. We do not repeat the proof of the upper bound, which is completely analogous to the one given in Section 8.2.1. Note that the optimal result without additional assumption is with a remainder in $\mathcal{O}(|B|^{2/3})$ which was obtained in [HeM6] and in [Pa6] (in the constant magnetic field case, but this assumption is not used).
4. The proof of Theorem 9.2.2 was achieved in [HeM6]. The corresponding upper bound was also given in [Pa6], and a less general geometric situation was studied in [HeM4].
5. Pan [Pa6] obtained for the upper bound in (9.18) the probably optimal remainder in $\mathcal{O}(B^{7/12})$.
6. The results presented in this chapter are due to Lu–Pan [LuP7], Almog [Al3], Helffer–Morame [HeM4]–[HeM6], Pan [Pa6], and the recent paper of Fournais–Helffer [FoH6].

Part II

Nonlinear Analysis

10

The Ginzburg–Landau Functional

10.1 The Problem in Superconductivity

Let us describe the mathematical problem. It is naturally posed for domains in $\mathbb{R}^3$, but for cylindrical domains in $\mathbb{R}^3$, it is natural (though not completely justified mathematically) to consider a functional defined in a domain $\Omega \subset \mathbb{R}^2$, where Ω is the cross-section of the cylinder. This explains why we also consider models in $\mathbb{R}^2$. The behavior of a sample of material can be read off from the properties of the minimizers $(\psi, \mathbf{A})$ of the Ginzburg–Landau functional (free energy) $\mathcal{G}$ to be defined below.

10.1.1 The functional

Let $d = 2$ or 3 and consider a domain $\Omega \subset \mathbb{R}^d$ and another domain $\widetilde{\Omega}$ such that $\Omega \subseteq \widetilde{\Omega}$. The most important cases are $\widetilde{\Omega} = \Omega$ and $\widetilde{\Omega} = \mathbb{R}^d$. In this book, we will always consider the cases where Ω is connected and simply connected. The Ginzburg–Landau functional is defined by

$$\mathcal{G}_{\Omega,\kappa,\sigma}(\psi, \mathbf{A}) = \int_{\Omega} |p_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx + (\kappa\sigma)^2 \int_{\Omega} |\operatorname{curl} \mathbf{A} - \beta|^2 \, dx \,. \tag{10.1}$$

Here the function ψ is called the order parameter (or sometimes the wave function) and $\mathbf{A}$ is a magnetic potential. The symbol β denotes a magnetic vector field and is called the external magnetic field or the applied magnetic field. In the case $d = 2$, β is just a function in, say, L^2_{loc} , whereas when $d = 3$, β is the curl of some vector field with components in H^1_{loc} , hence satisfying $\operatorname{div} \beta = 0$. The parameter $\kappa > 0$ (the Ginzburg–Landau parameter) depends on the material, and $\sigma > 0$ (or rather the product $\kappa\sigma$) is a measure of the strength of the external magnetic field. In the present book, we are concerned with the analysis of the asymptotic regime $\kappa \to +\infty$.

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We will mainly use two different choices of $\widetilde{\Omega}$ depending on whether $d = 2$ or 3. This is partly for physical reasons that will be explained below, but also partly in order to follow what we consider the custom in the subject. Therefore, we define

$$\mathcal{G}(\psi, \mathbf{A}) = \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) := \begin{cases} \mathcal{G}_{\Omega,\kappa,\sigma}(\psi, \mathbf{A}) & \text{when } d = 2\,, \\ \mathcal{G}_{\mathbb{R}^3,\kappa,\sigma}(\psi, \mathbf{A}) & \text{when } d = 3\,. \end{cases} \tag{10.2}$$

In the three-dimensional situation, we will sometimes write $\mathcal{G} = \mathcal{G}^{3D}$ in order to stress the dimension of the space.

In order to avoid certain technicalities for the Neumann problem in non-smooth domains, in Chapter 15 we will deviate from convention (10.2). In Chapter 15, we will consider (polygonal) $\Omega \subset \mathbb{R}^2$, but the functional will be associated with the pair $(\Omega, \widetilde{\Omega} = \mathbb{R}^2)$.

Next, we will discuss the two cases in (10.2) separately below.

10.1.2 The two-dimensional functional

In the two-dimensional situation, the original functional $\widehat{\mathcal{G}}$ is defined on functions $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\widehat{\mathcal{G}}(\psi, \mathbf{A}) = \int_\Omega |p_{\kappa\sigma\mathbf{A}}\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \, dx + (\kappa\sigma)^2 \int_{\mathbb{R}^2} |\operatorname{curl} \mathbf{A} - \beta|^2 \, dx\,. \tag{10.3}$$

It will be convenient to subtract the constant $\frac{\kappa^2}{2}|\Omega|$ from the functional, and this leads to the new functional $\mathcal{G}_{\mathbb{R}^2,\kappa,\sigma}$. This just changes the zero-point of the "energy" and has no physical consequence.

The function β (magnetic field) is initially defined in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$, but since Ω is simply connected, one can replace the domain of integration in the field integral from $\mathbb{R}^2$ to Ω (see Section 10.5). Thus, we end up with $\beta \in L^2(\Omega)$ and $\mathcal{G}$ as defined above, i.e.,

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |p_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx + (\kappa\sigma)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - \beta|^2 \, dx\,. \tag{10.4}$$

We will sometime write $\mathcal{G} = \mathcal{G}_{\kappa,\sigma}$ or even $\mathcal{G} = \mathcal{G}_{\kappa,\sigma,\beta}$, if we want to emphasize the choice of parameters involved in the definition of the functional. Note that if $\psi \equiv 0$ and $\mathbf{A}$ is such that $\operatorname{curl} \mathbf{A} = \beta$, then $\mathcal{G}(\psi, \mathbf{A}) = 0$. The above change of zero for the energy is motivated by the fact that we will, in particular, study the behavior of minimizers of $\mathcal{G}$ near such a state (called the normal state in physics).

The natural domain of the functional is $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$. However, due to the gauge invariance of $\mathcal{G}$ (see Section D.1), it is better to restrict the functional to the smaller set $H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, where

$$H^1_{\rm div}(\Omega) = \Big\{\mathcal{V} = (V_1, V_2) \in H^1(\Omega, \mathbb{R})^2 \;\Big|\; \operatorname{div} \mathcal{V} = 0 \text{ in } \Omega\,,\ \mathcal{V}\cdot\nu = 0 \text{ on } \partial\Omega\Big\}\,. \tag{10.5}$$

The space $H^1_{\rm div}(\Omega)$ inherits the topology (norm) from $H^1(\Omega, \mathbb{R}^2)$. We will generally consider the functional on this space if not specified otherwise.

We define the Ginzburg–Landau ground state energy to be the infimum of the functional, i.e.

$$E(\kappa, \sigma) := \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\rm div}(\Omega)} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A})\,, \tag{10.6}$$

and we observe, using the previously mentioned gauge invariance, that

$$E(\kappa, \sigma) := \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1(\Omega)} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A})\,. \tag{10.7}$$

As Ω is bounded, the existence of a minimizer is rather standard, so the infimum is actually a minimum. We will prove this existence in the next section. However, in general, one does not expect uniqueness of minimizers. A minimizer should satisfy the Euler–Lagrange equation, which is called in this context the Ginzburg–Landau system.

Using (10.7), this equation reads

$$\left.\begin{aligned} p^2_{\kappa\sigma\mathbf{A}}\psi &= \kappa^2(1-|\psi|^2)\psi\,,\\ \operatorname{curl}\big(\operatorname{curl}\mathbf{A} - \beta\big) &= -\tfrac{1}{\kappa\sigma}\Re\Big(\overline{\psi}\, p_{\kappa\sigma\mathbf{A}}\psi\Big) \end{aligned}\right\} \quad \text{in} \quad \Omega\,, \tag{10.8a}$$

$$\left.\begin{aligned} \nu\cdot p_{\kappa\sigma\mathbf{A}}\psi &= 0\,,\\ \operatorname{curl}\mathbf{A} \ - \ \beta &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{10.8b}$$

Here, for $\mathbf{A} = (A_1, A_2)$, $\operatorname{curl}\mathbf{A} = \partial_{x_1}A_2 - \partial_{x_2}A_1$, and

$$\operatorname{curl}^2\mathbf{A} = (\partial_{x_2}(\operatorname{curl}\mathbf{A}), -\partial_{x_1}(\operatorname{curl}\mathbf{A}))\,.$$

Notice that the weak formulation of (10.8) is

$$\Re\int_\Omega \left(\overline{p_{\kappa\sigma\mathbf{A}}\phi}\cdot p_{\kappa\sigma\mathbf{A}}\psi - \kappa^2(1-|\psi|^2)\overline{\phi}\psi\right)\,dx = 0\,, \tag{10.9a}$$

$$\int_\Omega (\operatorname{curl}\alpha)(\operatorname{curl}\mathbf{A} - \beta)\,dx = -\frac{1}{\kappa\sigma}\int_\Omega \Re\big(\overline{\psi}\,p_{\kappa\sigma\mathbf{A}}\psi\big)\,\alpha\,dx\,, \tag{10.9b}$$

for all $(\phi, \alpha) \in H^1(\Omega)\times H^1(\Omega, \mathbb{R}^2)$.

The analysis of the system (10.8) can be performed by PDE techniques. We note that this system is nonlinear, that $H^1(\Omega)$ is, when Ω is bounded and regular in $\mathbb{R}^2$, compactly embedded in $L^p(\Omega)$ for all $p \in [1, +\infty[$, and that, if $\operatorname{div}\mathbf{A} = 0$, $\operatorname{curl}^2\mathbf{A} = (-\Delta A_1, -\Delta A_2)$.

Actually, the nonlinearity is weak in the sense that the principal part is a linear elliptic system. One can show in particular that the solution in $H^1(\Omega, \mathbb{C})\times H^1_{\rm div}(\Omega)$ of the elliptic system (10.8) is actually, when Ω is regular, in $C^\infty\big(\overline{\Omega}\big)$ (see Theorem E.2.1 in Appendix E).

10.1.3 The three-dimensional functional

Most of the discussion and notation from the two-dimensional case carries over to the three-dimensional case. However, notice that, whereas in two dimensions a magnetic field is just (in a given system of coordinates[1]) any *function* [the image of the operator curl on say $H^1_{\mathrm{loc}}(\mathbb{R}^2\,,\mathbb{R}^2)$ is all of L^2_{loc} by Proposition D.2.1], in three dimensions a magnetic field $\mathbf{b}$ has to satisfy that $\operatorname{div}\mathbf{b} = 0$.

This is the reason why in 3D it is not immediate to replace the field integral $\int_{\mathbb{R}^3} |\operatorname{curl}\mathbf{A} - \beta|^2$ by the same integral over Ω. Hence, what in the future we call a magnetic field is always a vector field β that is associated with some $\mathbf{F} \in H^1_{\mathrm{loc}}(\mathbb{R}^3\,,\mathbb{R}^3)$ such that

$$\beta = \operatorname{curl}\mathbf{F} \;\text{ and }\; \operatorname{div}\mathbf{F} = 0\,. \tag{10.10}$$

We now define the Ginzburg–Landau functional in three dimensions by

$$\begin{aligned}\mathcal{G}_{\kappa,\sigma}(\psi,\mathbf{A}) = \mathcal{G}^{3D}_{\kappa,\sigma}(\psi,\mathbf{A}) = \int_\Omega |p_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\,dx \\ + (\kappa\sigma)^2\int_{\mathbb{R}^3} |\operatorname{curl}\mathbf{A} - \beta|^2\,dx\,.\end{aligned} \tag{10.11}$$

Here $\psi \in H^1(\Omega,\mathbb{C})$ and $\mathbf{A}$ lies in the space $\dot{H}^1_{\mathrm{div},\mathbf{F}}$ that we are presently defining. We consider the case of smooth, bounded Ω. Let $\dot{H}^1(\mathbb{R}^3)$ denote the homogeneous Sobolev space, i.e., the closure of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$u \mapsto \|u\|_{\dot{H}^1(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)}\,.$$

Then the natural variational space for the functional $\mathcal{G}^{3D}$ is $H^1(\Omega) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}$, where

$$\dot{H}^1_{\mathrm{div},\mathbf{F}} := \{\mathbf{A}\;:\;\operatorname{div}\mathbf{A} = 0\,,\text{ and }\mathbf{A} - \mathbf{F} \in \dot{H}^1(\mathbb{R}^3)\}\,. \tag{10.12}$$

We note that minimizing over $\dot{H}^1_{\mathrm{div},\mathbf{F}}$ is the same as minimizing over $\dot{H}^1_{\mathbf{F}}$, with

$$\dot{H}^1_{\mathbf{F}} := \{\mathbf{A}\;:\;\mathbf{A} - \mathbf{F} \in \dot{H}^1(\mathbb{R}^3)\}\,. \tag{10.13}$$

Given some $\mathbf{A}$ in $\dot{H}^1_{\mathbf{F}}$, one can indeed always find $\widetilde{\mathbf{A}} \in \dot{H}^1_{\mathrm{div},\mathbf{F}}$ and $\phi \in H^2_{\mathrm{loc}}$ such that $\nabla\phi \in \dot{H}^1(\mathbb{R}^3)$ and $\widetilde{\mathbf{A}} - \mathbf{A} = \nabla\phi$. This is a consequence of the properties of the operator $\nabla\Delta^{-1}$ on $\mathbb{R}^3$. Using this remark, minimizers of $\mathcal{G}$ are weak solutions of the Euler–Lagrange equations

[1] The identification is through the map $f \mapsto f(x,y)dx \wedge dy$.

$$p_{\kappa\sigma\mathbf{A}}^2\psi = \kappa^2(1-|\psi|^2)\psi \qquad \text{in} \quad \Omega\,, \tag{10.14a}$$

$$\operatorname{curl}\big(\operatorname{curl}\mathbf{A}-\beta\big) = -\frac{1}{\kappa\sigma}\Re(\overline{\psi}\,p_{\kappa\sigma\mathbf{A}}\psi)\,1_\Omega \qquad \text{in} \quad \mathbb{R}^3\,, \tag{10.14b}$$

$$(p_{\kappa\sigma\mathbf{A}}\psi)\cdot\nu = 0 \qquad \text{on} \quad \partial\Omega\,. \tag{10.14c}$$

The analysis of the regularity of this system is more delicate. This is due to the fact that in the right-hand side of (10.14b) we have introduced a cutoff function 1_Ω. This will be discussed further in Appendix E.

10.2 The Existence of a Minimizer

Using the discussion in the previous section, we can impose without loss of generality the condition that $\mathbf{A}\in H^1_{\mathrm{div}}(\Omega)$ (resp. $\mathbf{A}\in \dot{H}^1_{\mathrm{div},\mathbf{F}}$).

Theorem 10.2.1.
Suppose that $\Omega\subset\mathbb{R}^2$ is bounded, smooth, and simply connected. For all $\kappa,\sigma\in\mathbb{R}^+$ and $\beta\in L^2(\Omega)$, the functional $\mathcal{G}$ on $H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)$ has a minimizer.

Similarly, if $\Omega\subset\mathbb{R}^3$ is bounded and has Lipschitz continuous boundary, then for all $\kappa,\sigma\in\mathbb{R}^+$ and $\beta\in L^2(\mathbb{R}^3)$, the functional $\mathcal{G}$ has a minimizer in $H^1(\Omega)\times\dot{H}^1_{\mathrm{div},\mathbf{F}}$.

Furthermore, minimizers satisfy the Ginzburg–Landau systems in two and three dimensions [(10.8) and (10.14)] respectively.

Proof.
We start by giving the proof in dimension 2.
Let $(\psi_n,\mathbf{A}_n)\in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty}\mathcal{G}(\psi_n,\mathbf{A}_n) = \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)}\mathcal{G}(\psi,\mathbf{A})\,. \tag{10.15}$$

Step 1. $\{(\psi_n,\mathbf{A}_n)\}$ is bounded in $H^1(\Omega)\times H^1(\Omega)$.
By using that $\widetilde{\mathcal{G}}$ is the sum of three positive terms, we get the existence of a constant $E_0>0$ such that

$$T_n \le E_0\,, \tag{10.16}$$

where T_n is any of the three terms

$$\int_\Omega |(\nabla+i\kappa\sigma\mathbf{A}_n)\psi_n|^2\,dx,\quad \int_\Omega(|\psi_n|^2-1)^2\,dx\,,\quad \int_\Omega|\operatorname{curl}\mathbf{A}_n-\beta|^2\,dx\,.$$

Since β is a fixed function in $L^2(\Omega)$ and $\operatorname{div}\mathbf{A}_n=0$, we get from Proposition D.2.1 that $\{\mathbf{A}_n\}$ is uniformly bounded in $H^1(\Omega)$.

Using the Cauchy–Schwarz inequality and the inequality

$$2ab\le \epsilon a^2+\epsilon^{-1}b^2 \text{ for any } \epsilon>0\,,$$

notice that

$$\int_\Omega (|\psi_n|^2 - 1)^2 \, dx = \int_\Omega \big(|\psi_n|^4 - 2|\psi_n|^2 + 1\big) \, dx$$
$$\geq \|\psi_n\|_4^4 - 2\|\psi_n\|_4^2 \sqrt{|\Omega|} \geq \frac{1}{2}\|\psi_n\|_4^4 - 2|\Omega| \,.$$

Therefore, $\{\psi_n\}$ is uniformly bounded in $L^4(\Omega)$, and therefore—again using the Cauchy–Schwarz inequality—in $L^2(\Omega)$.

The boundedness of $\{\mathbf{A}_n\}$ in $H^1(\Omega)$ implies, by the Sobolev embedding theorem, that $\{\mathbf{A}_n\}$ is uniformly bounded in $L^4(\Omega)$. Combined with the L^4-bound on ψ_n , this gives the uniform boundedness of $\{\mathbf{A}_n\psi_n\}$ in $L^2(\Omega)$. So, considering the uniform bound,

$$\int_\Omega |\nabla\psi_n + i\kappa\sigma\mathbf{A}_n\psi_n|^2 \, dx \leq E_0 \,,$$

this implies that $\{\psi_n\}_n$ is uniformly bounded in $H^1(\Omega)$.

Step 2. A weak limit is a minimizer.

We now extract a subsequence, again denoted by $\{(\psi_n, \mathbf{A}_n)\}$, converging weakly in $H^1(\Omega) \times H^1(\Omega)$ to some $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1(\Omega)$.

Of course, by taking the limit, we obtain

$$\operatorname{div} \mathbf{A} = 0 \quad \text{ in } \Omega \,, \tag{10.17}$$

in the sense of distributions.

Furthermore, since the inclusion of $H^1(\Omega)$ in $H^s(\Omega)$ is compact for all $s < 1$ and the restriction $H^s(\Omega) \hookrightarrow L^2(\partial\Omega)$ is continuous for all $s > 1/2$, we also get

$$\mathbf{A} \cdot \nu = 0 \quad \text{ on } \partial\Omega \,.$$

Thus, $\mathbf{A} \in H^1_{\mathrm{div}}(\Omega)$. We can estimate:

$$\int_\Omega |\operatorname{curl} \mathbf{A} - \beta|^2 \, dx = \lim_{n\to+\infty} \langle \operatorname{curl} \mathbf{A} - \beta \mid \operatorname{curl} \mathbf{A}_n - \beta\rangle_{L^2\times L^2}$$
$$\leq \|\operatorname{curl} \mathbf{A} - \beta\|_2 \liminf_{n\to+\infty} \|\operatorname{curl} \mathbf{A}_n - \beta\|_2 \,.$$

Therefore,

$$\int_\Omega |\operatorname{curl} \mathbf{A} - \beta|^2 \, dx \leq \liminf_{n\to+\infty} \int_\Omega |\operatorname{curl} \mathbf{A}_n - \beta|^2 \, dx \,. \tag{10.18}$$

The same type of calculation gives

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{A})\psi|^2 \, dx \leq \liminf_{n\to+\infty} \int_\Omega |(\nabla + i\kappa\sigma\mathbf{A}_n)\psi_n|^2 \, dx \,. \tag{10.19}$$

The compactness of the Sobolev embedding

$$H^1(\Omega) \hookrightarrow L^p(\Omega) \ \text{ for } \frac{1}{p} > \frac{1}{2} - \frac{1}{d}$$

(if d is the dimension, here $d = 2$), hence for $p = 2, 4$, implies that

$$\int_\Omega (|\psi|^2 - 1)^2 \, dx \ = \lim_{n\to+\infty} \int_\Omega (|\psi_n|^2 - 1)^2 \, dx \, . \tag{10.20}$$

Combining (10.15) with (10.17)–(10.20) shows that $(\psi, \mathbf{A})$ is a minimizer. This finishes the proof in the two-dimensional case.

The proof in the three-dimensional case is similar. By (D.17), Hölder's inequality, and the boundedness of Ω, we find that

$$\|\mathbf{A} - \mathbf{F}\|_{L^p(\Omega)} \leq C \, \|\operatorname{curl} \mathbf{A} - \beta\|_{L^2(\mathbb{R}^3)} \, ,$$

for all $p \leq 6$. From here the proof is identical to the 2D case. □

10.3 Basic Properties for Solutions of the Ginzburg–Landau Equations

As we have seen, minimizers are solutions of the Ginzburg–Landau equations, but many properties are true for general solutions of these equations. The first important property is

Proposition 10.3.1.
If $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2)$ *is a (weak) solution to* (10.8), *then*

$$\|\psi\|_{L^\infty(\Omega)} \leq 1 \, . \tag{10.21}$$

The same is true for solutions to the three-dimensional GL system (10.14).

We only give the explicit proof in the 2D case. We first indicate the idea of an alternative proof using the maximum principle.
Sketch of a proof via the maximum principle
Assuming the C^2-regularity of the boundary and of the stationary point (up to the boundary), we can apply the maximum principle to the function

$$u(x) = |\psi(x)|^2 \, . \tag{10.22}$$

We observe that u satisfies

$$\frac{1}{2}\Delta u + \kappa^2 u(1-u) = |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 \, . \tag{10.23}$$

This equation is a direct consequence of the first Ginzburg–Landau equation (10.8a): We multiply the equation for ψ by $\bar{\psi}$ and take the real part.

Formula (10.23) is then a consequence of the identity

$$\Re\left(\Delta_{\kappa\sigma\mathbf{A}}\psi\cdot\bar{\psi}\right) = \frac{1}{2}\Delta(|\psi|^2) \; - \; |(\nabla + i\kappa\sigma\mathbf{A})\psi|^2 \,,$$

with

$$\Delta_{\kappa\sigma\mathbf{A}} = (\nabla + i\kappa\sigma\mathbf{A})^2 \,.$$

From (10.23), we get

$$\frac{1}{2}\Delta u + \kappa^2 u(1-u) \geq 0 \,. \qquad (10.24)$$

Now if u admits a maximum that is greater than 1, then we get a contradiction as follows. If this maximum is attained at a point $x_0 \in \Omega$, we have indeed $\Delta u(x_0) \leq 0$ and $\kappa^2 u(x_0)(1-u(x_0)) < 0$ in contradiction with (10.24). If the maximum is attained at the boundary, we should additionally use the fact that u satisfies the usual Neumann boundary condition.

Instead of giving the necessary justifications for the above proof, we prefer to give a proof that does not imply showing regularity properties.

Proof of Proposition 10.3.1.
With the notation $[t]_+ = \max(t,0)$, we introduce

$$\Omega_+ := \{x \in \Omega \; : \; |\psi(x)| > 1\} \,,$$

and the following functions on Ω_+:

$$f := \frac{\psi}{|\psi|} \,, \qquad \tilde{\psi} := [|\psi| - 1]_+ f \,.$$

Notice that $[t]_+ = (t + |t|)/2$, and so applying Proposition 2.1.2 twice, we see that

$$[|\psi| - 1]_+ \in H^1(\Omega) \quad \text{ and } \quad \nabla[|\psi| - 1]_+ = 1_{\Omega_+}\nabla[|\psi| - 1]_+ = 1_{\Omega_+}\nabla|\psi| \,.$$

Let $\chi \in C^\infty(\mathbb{R})$ be increasing and satisfy

$$\chi(t) = 0 \text{ on } t \leq \frac{1}{4} \,, \qquad \chi(t) = 1 \text{ on } t \geq \frac{3}{4} \,,$$

and define

$$G(z) = \chi(|z|)\frac{z}{|z|} \,, \qquad \tilde{f} := G(\psi) \,.$$

Then, since G is smooth with bounded derivatives and $\psi \in H^1(\Omega)$, the chain rule gives that $\tilde{f} = G(\psi) \in H^1(\Omega)$ (see, for instance, [LiL, Theorem 6.16]). Furthermore,

$$\tilde{\psi} = [|\psi| - 1]_+ \tilde{f} \,,$$

and so

$$(\nabla + i\kappa\sigma\mathbf{A})\tilde{\psi} = 1_{\Omega_+}\tilde{f}\nabla|\psi| + [|\psi| - 1]_+(\nabla + i\kappa\sigma\mathbf{A})\tilde{f}\,.$$

Now, clearly,

$$1_{\Omega_+}(\nabla + i\kappa\sigma\mathbf{A})\psi = 1_{\Omega_+}(\nabla + i\kappa\sigma\mathbf{A})(|\psi|\tilde{f}) = 1_{\Omega_+}\left\{\tilde{f}\nabla|\psi| + |\psi|(\nabla + i\kappa\sigma\mathbf{A})\tilde{f}\right\}.$$

Therefore,

$$\begin{aligned}&\Re\Big\{\overline{(\nabla + i\kappa\sigma\mathbf{A})\tilde{\psi}}\,\cdot\,(\nabla + i\kappa\sigma\mathbf{A})\psi\Big\}\\&\quad= 1_{\Omega_+}\Big(\big|\nabla|\psi|\big|^2 + (|\psi| - 1)|\psi|\,\big|(\nabla + i\kappa\sigma\mathbf{A})\tilde{f}\big|^2\Big).\end{aligned}$$

Here we used the fact that, on Ω_+, we have $|f| = |\tilde{f}| = 1$. Therefore,

$$f\,\nabla\overline{f} + \overline{f}\,\nabla f = \nabla|f|^2 = 0\,,$$

and so $1_{\Omega_+} f\,\nabla\overline{f}$ takes values in $i\mathbb{R}^2$.

Thus, we have, by (10.9) and the support of $\tilde{\psi}$,

$$\begin{aligned}0 &= \Re\Big\{\int_\Omega \overline{(\nabla + i\kappa\sigma\mathbf{A})\tilde{\psi}}\,(\nabla + i\kappa\sigma\mathbf{A})\psi + \overline{\tilde{\psi}}(|\psi|^2 - 1)\psi\,dx\Big\}\\&= \int_{\Omega_+}\big|\nabla|\psi|\big|^2 + (|\psi| - 1)|\psi|\,\big|(\nabla + i\kappa\sigma\mathbf{A})\tilde{f}\big|^2 + (1 + |\psi|)(|\psi| - 1)^2|\psi|\,dx\,.\end{aligned}$$

Since the integrand is nonnegative, we easily conclude that Ω_+ has measure zero. ☐

Using Proposition 10.3.1, we can get various a priori estimates on solutions to the Ginzburg–Landau equations (10.8).

Lemma 10.3.2.
Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth, and let $\beta \in L^2(\Omega)$ be given. Then for all $p \geq 2$, there exists a constant $C = C(p) > 0$ such that for all solutions $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\rm div}(\Omega)$ to (10.8), we have

$$\|p^2_{\kappa\sigma\mathbf{A}}\psi\|_p \leq \kappa^2\|\psi\|_p\,, \tag{10.25}$$

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa\|\psi\|_2\,, \tag{10.26}$$

$$\|\operatorname{curl}\mathbf{A} - \beta\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa\sigma}\|\psi\|_\infty\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_p\,. \tag{10.27}$$

Furthermore, there exists a constant $C_2 > 0$ such that

$$\|\operatorname{curl}\mathbf{A} - \beta\|_2 \leq \frac{C_2}{\sigma}\|\psi\|_2\|\,\psi\|_4\,. \tag{10.28}$$

Proof.
Since, by Proposition 10.3.1,

$$0 \leq 1 - |\psi|^2 \leq 1\,, \tag{10.29}$$

the inequality (10.25) is immediate from (10.8a). Multiplying the equation for ψ in (10.8a) by $\overline{\psi}$ and integrating over Ω, one obtains (10.26), again using (10.29).

Since, by definition,

$$\operatorname{curl}(\operatorname{curl}\mathbf{A} - \beta) = \left(\partial_{x_2}(\operatorname{curl}\mathbf{A} - \beta), -\partial_{x_1}(\operatorname{curl}\mathbf{A} - \beta)\right)\,,$$

it follows immediately from the equation for $\mathbf{A}$ in (10.8a) that

$$\|\nabla(\operatorname{curl}\mathbf{A} - \beta)\|_p \leq \frac{1}{\kappa\sigma}\,\|\psi\|_\infty \|p_{\kappa\sigma\mathbf{A}}\psi\|_p\,. \tag{10.30}$$

Since $\operatorname{curl}\mathbf{A} - \beta$ vanishes on $\partial\Omega$, (10.27) follows from (10.30) by the Poincaré inequality, Theorem E.5.1.

Finally, we prove (10.28). For this we use (10.9b) with $\alpha := \mathbf{A} - \mathbf{F}$. Here $\mathbf{F}$ is the unique vector field (see Section D.1) in $H^1_{\mathrm{div}}(\Omega)$ such that

$$\operatorname{curl}\mathbf{F} = \beta \quad \text{and} \quad \operatorname{div}\mathbf{F} = 0 \quad \text{in } \Omega\,, \tag{10.31}$$

$$\mathbf{F}\cdot\nu = 0 \quad \text{on } \partial\Omega\,. \tag{10.32}$$

Applying Hölder's inequality yields

$$\|\operatorname{curl}\mathbf{A} - \beta\|_2^2 \leq \frac{1}{\kappa\sigma}\,\|\psi\|_4\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_2\,\|\mathbf{A} - \mathbf{F}\|_4\,.$$

Thus, using a Sobolev inequality and (D.7), we get

$$\|\operatorname{curl}\mathbf{A} - \beta\|_2 \leq \frac{C}{\kappa\sigma}\,\|\psi\|_4\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_2\,. \tag{10.33}$$

The estimate (10.28) follows upon inserting (10.26) in (10.33). □

The 3D version of Lemma 10.3.2 is slightly harder to prove.

Lemma 10.3.3.
Let $\Omega \subset \mathbb{R}^3$ be bounded and smooth, and let β be given as in (10.10) *and satisfying $\beta \in L^\infty(\mathbb{R}^3)$. Then for all $p \leq 6$, there exists a constant $C = C(p) > 0$ such that for all solutions $(\psi, \mathbf{A}) \in H^1(\Omega) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}$ to* (10.14), *we have*

$$\|p^2_{\kappa\sigma\mathbf{A}}\psi\|_p \leq \kappa^2\,\|\psi\|_p\,, \tag{10.34}$$

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa\,\|\psi\|_2\,, \tag{10.35}$$

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \leq C\left\{\frac{1}{\sigma}\|\psi\|_\infty\,\|\psi\|_2 + \frac{1}{\kappa\sigma}\,\|\psi\|_\infty\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_p\right\}. \tag{10.36}$$

Finally, there exists a constant $C_2 > 0$ *such that*

$$\| \operatorname{curl} \mathbf{A} - \beta \|_2 \leq \frac{C_2}{\sigma} \| \psi \|_2 \, \| \psi \|_4 \,. \tag{10.37}$$

Proof.
The proofs of (10.34) and (10.35) are identical to the corresponding estimates for the 2D case [see (10.25) and (10.26)]. The same applies to the proof of (10.37) except that one has to apply Proposition D.3.2.

So we only have to prove (10.36). From (D.17) we find that there exists $C > 0$ such that

$$\| \mathbf{A} - \mathbf{F} \|_{L^6(\mathbb{R}^3)} \leq C \, \| \operatorname{curl} \mathbf{A} - \beta \|_{L^2(\mathbb{R}^3)} \,. \tag{10.38}$$

Since $\operatorname{div}(\mathbf{A} - \mathbf{F}) = 0$, (10.14b) can be reformulated as

$$\Delta(\mathbf{A} - \mathbf{F}) = -\frac{1}{\kappa\sigma} \Re(\overline{\psi} \, p_{\kappa\sigma\mathbf{A}} \psi) \, 1_\Omega \quad \text{in} \quad \mathbb{R}^3 \,. \tag{10.39}$$

Let $\mathrm{D}(0, R)$ be the open ball of radius R around the origin. Using the elliptic regularity for the Laplacian (see Theorem E.4.1), we obtain, for all $p' \in [1, \infty[$, $R > 0$, the existence of a constant $C_{p'}(R)$ such that

$$\begin{aligned} &\| \mathbf{A} - \mathbf{F} \|_{W^{2,p'}(\mathrm{D}(0,R))} \\ &\quad \leq C_{p'}(R) \left(\| \mathbf{A} - \mathbf{F} \|_{L^{p'}(\mathrm{D}(0,2R)} + \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^{p'}(\Omega)} \right) \,. \end{aligned}$$

In particular, for $p' \leq 6$, we can apply the estimate (10.38) and the compactness of $\overline{\mathrm{D}(0, 2R)}$ to get, for $p' \leq 6$,

$$\begin{aligned} &\| \mathbf{A} - \mathbf{F} \|_{W^{2,p'}(\mathrm{D}(0,R))} \\ &\quad \leq C'_{p'}(R) \left(\| \operatorname{curl} \mathbf{A} - \beta \|_{L^2(\mathbb{R}^3)} + \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^{p'}(\Omega)} \right) \,. \end{aligned} \tag{10.40}$$

Let R be chosen so big that $\Omega \subset \mathrm{D}(0, R - 1)$. Using once again Theorem E.4.1 and the Sobolev embedding theorem, we find for any $p \in [1, \infty[$,

$$\begin{aligned} \| \mathbf{A} - \mathbf{F} \|_{W^{2,p}(\Omega)} &\leq C \left(\| \mathbf{A} - \mathbf{F} \|_{L^p(\mathrm{D}(0,R))} + \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^p(\Omega)} \right) \\ &\leq C \left(\| \mathbf{A} - \mathbf{F} \|_{W^{2,2}(\mathrm{D}(0,R))} + \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^p(\Omega)} \right) \,. \end{aligned}$$

Reimplementing (10.40) with $p' = 2$, we obtain

$$\begin{aligned} \| \mathbf{A} - \mathbf{F} \|_{W^{2,p}(\Omega)} \leq C \Big(&\| \operatorname{curl} \mathbf{A} - \beta \|_{L^2(\mathbb{R}^3)} + \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^2(\Omega)} \\ &+ \frac{1}{\kappa\sigma} \| \psi \|_\infty \, \| p_{\kappa\sigma\mathbf{A}} \psi \|_{L^p(\Omega)} \Big) \,. \end{aligned} \tag{10.41}$$

Multiplying (10.14b) by $\mathbf{A}-\mathbf{F}$ and integrating by parts yields

$$\begin{aligned}\|\operatorname{curl}\mathbf{A}-\beta\|^2_{L^2(\mathbb{R}^3)} &= -\frac{1}{\kappa\sigma}\int_\Omega (\mathbf{A}-\mathbf{F})\Re(\overline{\psi}\, p_{\kappa\sigma\mathbf{A}}\psi)\,dx \\ &\leq \frac{1}{\kappa\sigma}\|\mathbf{A}-\mathbf{F}\|_{L^2(\Omega)}\|\psi\|_\infty\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_{L^2(\Omega)} \\ &\leq \frac{C}{\kappa\sigma}\|\mathbf{A}-\mathbf{F}\|_{L^6(\Omega)}\|\psi\|_\infty\,\|p_{\kappa\sigma\mathbf{A}}\psi\|_{L^2(\Omega)}\,.\end{aligned}$$

Implementing the estimates (10.38) and (10.35), we obtain

$$\|\operatorname{curl}\mathbf{A}-\beta\|_{L^2(\mathbb{R}^3)} \leq C\sigma^{-1}\|\psi\|_\infty\,\|\psi\|_{L^2(\Omega)}\,. \tag{10.42}$$

Thus, (10.41) combined with (10.35) and (10.42) yields (10.36). □

10.4 The Result of Giorgi–Phillips

We observe that $(0,\mathbf{F})$ is a trivial critical point of the functional $\mathcal{G}$, i.e., a trivial solution of the Ginzburg–Landau system (10.8). The pair $(0,\mathbf{F})$ is often called the *normal state* or *normal solution.* The situation is similar in the 3D case but $\mathbf{F}$ appears already in our discussion of the domain.

It is natural to discuss—as a function of σ—whether this pair is a local or global minimizer. When σ is large, one will show that this solution is effectively the unique global minimizer. One says that in this case the superconductivity is destroyed. In other words, the order parameter is identically zero in Ω.

Let us give a rather simple proof of this result that roughly says (see Theorem 10.4.1 for the precise statement) that $(0,\mathbf{F})$ is the unique minimizer of the functional when the strength of the exterior magnetic field is sufficiently large. We will actually show this result for the solutions of the associated Ginzburg–Landau system.

So we assume that we have a **nonnormal** stationary point $(\psi,\mathbf{A})$ for $\mathcal{G}$. This means that $(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\rm div}(\Omega)$ [resp. $(\psi,\mathbf{A})\in H^1(\Omega)\times \dot{H}^1_{{\rm div},\mathbf{F}}$ in the 3D case] is a solution of (10.8) [resp. (10.14)] and

$$\int_\Omega |\psi(x)|^2\,dx > 0\,. \tag{10.43}$$

By (10.26), (10.27), and (10.21), and using (D.7) for controlling $\|\mathbf{A}-\mathbf{F}\|^2$ in Ω by $\|\operatorname{curl}\mathbf{A}-\beta\|^2$, we get

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2 + (\kappa\sigma)^2\|\mathbf{A}-\mathbf{F}\|_2^2 \leq C_\Omega\kappa^2\|\psi\|_2^2\,. \tag{10.44}$$

We now compare $\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2\,dx$ and $\int_\Omega |(\nabla + i\kappa\sigma\mathbf{A})\psi|^2\,dx$. A trivial estimate is

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2\,dx \leq 2\,\|(\nabla + i\kappa\sigma\mathbf{A})\psi\|^2 + 2(\kappa\sigma)^2\|(\mathbf{A}-\mathbf{F})|\psi|\,\|^2\,. \quad (10.45)$$

Implementing (10.21) and (10.44) gives

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2\,dx \leq 2C_\Omega\kappa^2 \int_\Omega |\psi(x)|^2\,dx\,. \quad (10.46)$$

Since ψ satisfies (10.43), we obtain

$$\lambda_1^N(\sigma\kappa\mathbf{F}) \leq 2C_\Omega\kappa^2\,. \quad (10.47)$$

We observe that, by Proposition 2.1.3, $\lambda_1^N(\sigma\kappa\mathbf{F}) > 0$. So by combining Propositions 1.5.2 and 8.2.2 [and the continuity of $B \mapsto \lambda_1^N(B\mathbf{F})$], we get the existence of a constant $C_0 > 0$ such that

$$\lambda_1^N(\sigma\kappa\mathbf{F}) \geq \frac{1}{C_0}\min(\sigma\kappa, (\sigma\kappa)^2)\,. \quad (10.48)$$

Thus, we find that if a nontrivial stationary point $(\psi, \mathbf{A})$ exists, then

$$\sigma \leq C(1+\kappa)\,.$$

This can be reformulated as the following theorem. Before we formally state the result let us note that the same argument can be applied in 3D. In that case, the lower bound to the eigenvalue is given in Theorem 9.1.1.

Theorem 10.4.1 (Giorgi–Phillips)**.**
2D. *Let $\Omega \subset \mathbb{R}^2$ be smooth, bounded, and simply connected, and let the function β in* (10.8) *be continuous and satisfy*

$$\beta(x) \geq c > 0\,, \qquad \forall x \in \Omega\,.$$

Then there exists a constant $C = C(\Omega, c)$ such that if

$$\sigma \geq C\max\{\kappa, 1\}\,,$$

then the pair $(0, \mathbf{F})$ is the unique solution to (10.8) *in $H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$.*
3D
Let $\Omega \subset \mathbb{R}^3$ be smooth, bounded, and simply connected, and let β be a continuous vector field in (10.14) *satisfying* (10.10) *and*

$$|\beta(x)| \geq c > 0\,, \qquad \forall x \in \Omega\,.$$

Then there exists a constant $C = C(\Omega, c)$ such that if

$$\sigma \geq C\max\{\kappa, 1\}\,,$$

then $(0, \mathbf{F})$ is the unique solution to (10.14) *in $H^1(\Omega) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}$.*

We emphasize that the result is true for any $\kappa > 0$.

10.5 $\mathcal{G}_\Omega$ vs. $\mathcal{G}_{\mathbb{R}^d}$

Here we briefly discuss the question of how the magnetic energy should enter the functional. Shall we integrate over Ω or over $\mathbb{R}^2$ (or $\mathbb{R}^3$) for the quantity $\int |\beta - \operatorname{curl} \mathbf{A}|^2$ appearing in the definition of the GL functional?

The 2D case

In two dimensions it does not matter (for simply connected Ω) whether the field integral is taken over Ω or over $\mathbb{R}^2$. We recall the outline of the proof, in order to be able to see how it breaks down in the three-dimensional case. For simplicity, we only consider an open set Ω, which is star-shaped with respect to the origin. One can then split a magnetic field b defined on $\mathbb{R}^2$ as $b = b^1 + b^2$ with b^1 supported in $\overline{\Omega}$ and b^2 supported in Ω^c. One can now choose a vector potential $\mathbf{a}^2$, defined on $\mathbb{R}^2$ and supported in Ω^c, such that $\operatorname{curl} \mathbf{a}^2 = b^2$ as a distributional equation in $\mathbb{R}^2$. In the starshaped case, the explicit formula (Poincaré gauge)

$$\mathbf{a}^2(x) = \int_0^1 b^2(tx) \times tx \, dt$$

shows this immediately. Here $b^2 \times x = (-x_2 b^2, x_1 b^2)$. In the non-starshaped case, the same result is true, but the proof involves a bit of algebraic topology—the topological condition on Ω being that Ω is homeomorphic to the unit disc. Define (we omit κ, σ for simplicity)

$$E_\Omega = \inf_{(\psi,\mathbf{A})} \mathcal{G}_\Omega(\psi, \mathbf{A}), \quad E_{\mathbb{R}^2} = \inf_{(\psi,\mathbf{A})} \mathcal{G}_{\mathbb{R}^2}(\psi, \mathbf{A}). \tag{10.49}$$

We aim to prove that

$$E_\Omega = E_{\mathbb{R}^2}. \tag{10.50}$$

The inequality $E_\Omega \leq E_{\mathbb{R}^2}$ is easy—it suffices to consider a minimizing sequence for $\mathcal{G}_{\mathbb{R}^2}$ and restrict the $\mathbf{A}$'s to Ω.

To get the other inequality, let $(\psi, \mathbf{A}) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2)$ be given. Let $\tilde{A} \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ be an extension of $\mathbf{A}$ to the entire plane, with $\operatorname{curl} \tilde{A} = \beta$ outside a (large) compact set, and define

$$b^1 + b^2 = \operatorname{curl} \tilde{A} - 1_{\Omega^c}\beta$$

as above. Let

$$\operatorname{curl} \mathbf{a}^2 = b^2 = (\operatorname{curl} \tilde{A} - \beta) 1_{\Omega^c}$$

in $\mathbb{R}^2$, as before, with $\operatorname{supp} \mathbf{a}^2 \subset \mathbb{R}^2 \setminus \Omega$, and define

$$\bar{A} = \tilde{A} - \mathbf{a}^2.$$

Then

$$\operatorname{curl} \bar{A} = 1_\Omega \cdot \operatorname{curl} \mathbf{A} + \beta 1_{\Omega^c}.$$

In particular, in the domain Ω, we have

$$\operatorname{curl} \bar{A} = \operatorname{curl} \mathbf{A} ,$$

and, therefore, there exists a gauge function ϕ such that

$$\int_\Omega |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 \, dx = \int_\Omega |(-i\nabla + \kappa\sigma\bar{A})e^{i\phi}\psi|^2 \, dx .$$

So $\mathcal{G}_\Omega[e^{i\phi}\psi, \bar{A}] = \mathcal{G}_{\mathbb{R}^2}[\psi, \mathbf{A}]$. This gives the desired converse inequality.

The 3D case

The situation is not as simple in 3D. The "algebraic topology" remains valid; i.e., magnetic fields supported in Ω^c can be generated by vector potentials supported in the same set.[2] The difference lies in the meaning of the words "magnetic field". In 3D a magnetic field $\mathbf{b}$ has to satisfy $\operatorname{div} \mathbf{b} = 0$, and upon splitting $\mathbf{b} = \mathbf{b}1_\Omega + \mathbf{b}1_{\Omega^c}$, this condition on the divergence is generally not satisfied by either term on the right-hand side. In two dimensions, a "magnetic field" on $\mathbb{R}^2$ is any function, and therefore the splitting does not cause any problems.

10.6 Critical Fields

It follows from Theorem 10.4.1 that for fixed κ and for sufficiently large σ, the only minimizer (or—more generally—stationary point) of $\mathcal{G}_{\kappa,\sigma}$ is the normal state $(0, \mathbf{F})$. It is then natural to try to follow the property of the minimizers when decreasing σ starting from $+\infty$ and to determine when the trivial solution is no longer a global minimum or a local minimum. This suggests defining the third critical field $H_{C_3}(\kappa)$ mentioned in the Preface as follows:

$$H_{C_3}(\kappa) := \inf\{\sigma > 0 \, : \, (0, \mathbf{F}) \text{ is the unique minimizer of } \mathcal{G}_{\kappa,\sigma}\} . \tag{10.51}$$

Of course, one can make a similar definition in the 3D case.

Another critical field mentioned in the Preface is $H_{C_2}(\kappa)$ corresponding to the transition from the surface superconducting state to the bulk superconducting state. That second critical field is much harder to define; in fact, no rigorous definition exists in the literature. However, the term is widely used and has an intuitive meaning. Many results exist to indicate that, in the limit where κ is large, minimizers show "bulk" behavior for $b\sigma < \kappa$ and "surface concentration" for $b\sigma > \kappa$, where

$$b := \inf_{x\in\Omega} \beta(x) .$$

[2] This, of course, depends strongly on the topology. As before, the precise condition is that Ω and the unit ball are homeomorphic.

In this book, we will allow ourselves to use the term H_{C_2} in discussions. For the purpose of mathematical clarity, we *define*

$$H_{C_2}(\kappa) := \frac{\kappa}{b}\,. \tag{10.52}$$

10.7 Notes

1. The Ginzburg–Landau model is described in all textbooks on superconductivity (see, for example, Saint-James–Sarma–Thomas [S-JST], Tilley–Tilley [TiT], or Tinkham [Ti]). In the mathematics literature, many articles have appeared devoted to the analysis of this model. In dimension one, one can refer to the work of Bolley [Bol] in the late 1980s. For the first mathematical discussions in the two- or three-dimensional cases, we mention Du–Gunzburger–Peterson [DuGP] and Chapman–Howison–Ockendon [ChHO] and the basic material also appears in the books by Bethuel–Brézis–Hélein [BeBH], Sandier–Serfaty [SaS3] or in the lectures of Rubinstein [Ru].
2. Our proof of Proposition 10.3.1 is inspired by the proof of [DuGP].
3. The systems considered here are a particular (simple) case of Agmon–Douglis–Nirenberg systems [AgDN1, AgDN2]. In appendix E we give a short presentation of the needed material and the involved bootstrap argument.
4. Our presentation of the Giorgi–Phillips theorem is close to the original proof [GioP] with some simplifications. One can also find a presentation of this result in the book of Hoffmann and Tang [HoT]. The new point is that our statement concerns not only minimizers but any critical point of the GL functional.
5. The argument for the discussion between the two possible definitions of the GL functional is essentially taken from [DuH]. We do not treat in this book the case with holes (see, however, the last chapter for a short discussion).
6. The first definition of $H_{C_3}(\kappa)$ was proposed by Lu–Pan in [LuP3]. We will discuss this and related definitions in Chapter 13 (two and three dimensions, smooth boundary) and Chapter 15 (2D case with corners). We will complete the list of references in the specific chapter devoted to the analysis of this critical field.
7. There is an extensive literature on boundary value problems in Lipschitz domains. We mention here [Ne], [GeK], [BuCS], [BuC], [BuG], and the book by Maz'ja [Maz]. See also [GeM].

11

Optimal Elliptic Estimates

11.1 Introduction

In this chapter, we establish estimates on "how far" the induced magnetic vector potential $\mathbf{A}$ is allowed to be from the external field $\mathbf{F}$, when $(\psi, \mathbf{A})$ is a solution to the Ginzburg–Landau equations. It is in particular proved (for more precise statements; see, for instance, Theorem 11.3.1) that in the entire region between $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$, i.e., for $\sigma \geq \kappa$, we have that $\mathbf{A} - \mathbf{F} = o(1)$, in suitable norms.

These estimates come in two types:
The first set of estimates is deduced from the ellipticity of the Ginzburg–Landau system. In this way one obtains the desired inequalities in (Sobolev) norms, $W^{s,p}$, for $p < +\infty$ (by embedding theorems, estimates in Hölder norms, $C^{s',\alpha}$, $\alpha < 1$, are also obtained). The challenge here is to get inequalities with the right dependence on the magnetic field strength. This part of the analysis is valid in a large parameter regime and is essentially functional analytical.

The second set of estimates corresponds to the case $p = \infty$ above. They are obtained using a very different technique, called a "blow-up argument". In this technique, one rescales balls of size κ^{-1} to unity, thereby obtaining (after suitable extractions of subsequences) a limit function that is a solution of a natural limit equation. In many interesting cases this limit equation has only trivial solutions, and this information can then be transformed back to the original solutions of the Ginzburg–Landau system. The estimates obtained in this way are often much stronger than the first set of estimates described above.

11.2 Hölder Estimates

In both dimensions 2 and 3, the starting point is a formula expressing the L^2-norm of mixed second derivatives of a function in terms of the L^2-norm of

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the magnetic Laplacian on the function and lower-order terms involving the magnetic field itself.

11.2.1 The case of two dimensions

For convenience, we will use the following notation for the magnetic derivatives:

$$\mathbf{D} = (D_1, D_2) = (-i\nabla + B\mathbf{A})\,. \tag{11.1}$$

The magnetic Laplacian is now the operator

$$\mathcal{H} := \mathbf{D}^2 = D_1^2 + D_2^2\,.$$

Proposition 11.2.1.
Let $\Omega \subset \mathbb{R}^2$ be a regular bounded domain. Suppose that $\psi \in W^{2,2}(\Omega)$ satisfies magnetic Neumann boundary conditions

$$\nu \cdot D\psi\big|_{\partial\Omega} = 0\,. \tag{11.2}$$

Then

$$\sum_{j,k} \|D_jD_k\psi\|_2^2 = B^2 \int_\Omega (\operatorname{curl}\mathbf{A})^2|\psi|^2\,dx + \int_\Omega |\mathcal{H}\psi|^2\,dx + 2B\int_\Omega (\operatorname{curl}\mathbf{A})\Im(D_1\psi\,\overline{D_2\psi})\,dx\,. \tag{11.3}$$

Proof of Proposition 11.2.1.
Notice the following magnetic commutator:

$$[D_j, D_k] = -iB(\partial_jA_k - \partial_kA_j)\,.$$

Therefore, a calculation, also using the divergence theorem, yields

$$\begin{aligned}\sum_{j,k} \|D_jD_k\psi\|_2^2 = B^2 \int_\Omega (\operatorname{curl}\mathbf{A})^2|\psi|^2\,dx + \int_\Omega |\mathcal{H}\psi|^2\,dx \\ + 2B\int_\Omega (\operatorname{curl}\mathbf{A})\;\Im(D_1\psi\,\overline{D_2\psi})\,dx \\ + \Re\int_{\partial\Omega}\Big\{(\nu\cdot\overline{\mathbf{D}\psi})\;\mathcal{H}\psi + \sum_{j,k}\overline{D_k\psi}\;\nu_j\;D_kD_j\psi\Big\}\,d\sigma\,.\end{aligned}$$

This formula holds for all functions ψ in $C^\infty(\overline{\Omega})$. Proposition 11.2.1 will follow from this calculation once we prove that the boundary term vanishes.

We now assume the Neumann boundary condition. Thus, $(\nu\cdot\mathbf{D}\psi) = 0$, and so the first boundary term vanishes. We can rewrite the second boundary term as follows:

$$\Re\int_{\partial\Omega}\sum_{j,k}\overline{D_k\psi}\;\nu_j\;D_kD_j\psi\,d\sigma = \Re(a+b)\,,$$

with

$$a := \int_{\partial\Omega} \sum_{j,k} \overline{D_k\psi}\, D_k\nu_j D_j\psi\, d\sigma\,, \qquad b := i\int_{\partial\Omega} \sum_{j,k} \overline{D_k\psi}\,(\partial_k\nu_j) D_j\psi\, d\sigma\,.$$

To analyze a and b, we introduce a unit vector τ parallel to the boundary and we define $D_\tau := \tau\cdot\mathbf{D}$, $D_\nu := \nu\cdot\mathbf{D}$.

Let us start by proving that $\Re(b)$ vanishes. Taking the real part, we find

$$\Re(b) = \frac{i}{2}\int_{\partial\Omega} \langle \mathbf{D}\psi \,|\, M\mathbf{D}\psi\rangle_{\mathbb{C}^2}\, d\sigma\,, \tag{11.4}$$

where M is the matrix with entries $M_{j,k} = \partial_j\nu_k - \partial_k\nu_j$. It clearly suffices to prove that the integrand is real in order to conclude that $\Re(b) = 0$. Writing $\mathbf{D}\psi = (D_\tau\psi)\tau + (D_\nu\psi)\nu$ and using the boundary condition, we find that the integrand satisfies

$$\langle \mathbf{D}\psi \,|\, M\mathbf{D}\psi\rangle_{\mathbb{C}^2} = |D_\tau\psi|^2 \langle \tau \,|\, M\tau\rangle_{\mathbb{C}^2}\,,$$

which is manifestly real since M, τ are real. Thus, $\Re(b) = 0$.

Using the Neumann boundary condition and the fact that (τ,ν) is an orthogonal basis for $\mathbb{C}^2$, we can rewrite a as

$$a = \int_{\partial\Omega} \overline{D_\tau\psi}\, D_\tau D_\nu\psi\, d\sigma\,.$$

Since (the vector-field part in) D_τ is a derivative along the boundary, and since $D_\nu\psi\big|_{\partial\Omega} = 0$, we find $D_\tau D_\nu\psi\big|_{\partial\Omega} = 0$. Thus, a clearly vanishes.

This finishes the proof of Proposition 11.2.1. □

Applying Hölder's inequality to the result of Proposition 11.2.1 yields an interesting elliptic inequality for 2D magnetic problems with Neumann boundary conditions.

Lemma 11.2.2.
Let $\Omega \subset \mathbb{R}^2$ be a regular domain and let $\beta \in L^\infty(\Omega)$ be an external magnetic field. Suppose that $\psi \in C^\infty(\overline{\Omega})$ satisfies magnetic Neumann boundary conditions. Then, for all $p_1, p_2 \in [1,+\infty]$, we have

$$\begin{aligned}\sum_{j,k} \|D_jD_k\psi\|_2^2 \le 3B^2\|\beta\|_\infty^2\|\psi\|_2^2 + 2\|\mathcal{H}\psi\|_2^2 + 2B^2\|\operatorname{curl}\mathbf{A} - \beta\|_{2p_1}^2\|\psi\|_{2q_1}^2 \\ + 2B\|\operatorname{curl}\mathbf{A} - \beta\|_{p_2}\|\mathbf{D}\psi\|_{2q_2}^2\,,\end{aligned} \tag{11.5}$$

where q_j is the conjugate exponent to p_j; i.e., $p_j^{-1} + q_j^{-1} = 1$.

Proof.
The proof is direct using the identity in Proposition 11.2.1—replacing $\operatorname{curl}\mathbf{A}$ by $(\operatorname{curl}\mathbf{A} - \beta) + \beta$—and Hölder's inequality. The term $B\|\beta\|_\infty\|\mathbf{D}\psi\|_2^2$ is estimated as

$$B\|\beta\|_\infty\|\mathbf{D}\psi\|_2^2 = B\|\beta\|_\infty\langle\psi, \mathcal{H}\psi\rangle \le B^2\|\beta\|_\infty^2\|\psi\|_2^2 + \|\mathcal{H}\psi\|_2^2\,,$$

where the Neumann boundary condition is used to get the identity. □

11.2.2 The case of three dimensions

The same calculation as in the 2D case yields (using of course that ψ satisfies the Neumann condition)

$$\sum_{j,k} \|D_j D_k \psi\|_2^2 = B^2 \int_\Omega (\operatorname{curl} \mathbf{A})^2 |\psi|^2 \, dx + \int_\Omega |\mathcal{H}\psi|^2 \, dx \\ + 2B \int_\Omega (\operatorname{curl} \mathbf{A}) \cdot \Im \begin{pmatrix} D_2\psi \, \overline{D_3\psi} \\ D_3\psi \, \overline{D_1\psi} \\ D_1\psi \, \overline{D_2\psi} \end{pmatrix} dx + \Re b \,, \tag{11.6}$$

with

$$b := i \int_{\partial\Omega} \sum_{j,k} \overline{D_k\psi} \, (\partial_k \nu_j) \, D_j\psi \, d\sigma \,, \qquad \operatorname{curl} \mathbf{A} = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix} . \tag{11.7}$$

In the three-dimensional case, we are not able to prove that b vanishes, but this boundary term can be controlled as follows by trace theorems.

Since the derivatives of ν are bounded, we can estimate

$$|b| \leq C \|\mathbf{D}\psi\|^2_{L^2(\partial\Omega)} = C \sum_{j=1}^{3} \|D_j\psi\|^2_{L^2(\partial\Omega)} \,.$$

Notice that the elementary identity

$$|u(0)|^2 = -\int_0^\infty \frac{d}{dt} |u(t)|^2 \, dt = -2 \int_0^\infty |u(t)| \, |u(t)|' \, dt \,,$$

for $u \in H^1(\mathbb{R}^+)$, implies the inequality

$$|u(0)| \leq \sqrt{2} \, \|u\|_{L^2(\mathbb{R}^+)} \|u'\|_{L^2(\mathbb{R}^+)} \,.$$

Implementing this inequality (with u replaced by χu, where χ localizes closely to 0) in the boundary coordinates introduced in Appendix F, we see that there exists a constant $C' > 0$ such that, for all $\epsilon < 1$ and all $f \in H^1(\Omega)$, we have

$$\|f\|^2_{L^2(\partial\Omega)} \leq C' \epsilon^{-1} \|f\|_2^2 + \epsilon \|f\|^2_{H^1(\Omega)} \,.$$

We will choose $\epsilon = 1/2C$ and apply the resulting inequality to each of the functions $f = |D_j\psi|$. This gives, with a new constant C,

$$|b| \leq C \sum_j \|D_j\psi\|_2^2 + \frac{1}{2} \sum_{j,k} \|\partial_k |D_j\psi|\|_2^2 \leq C \sum_j \|D_j\psi\|_2^2 + \frac{1}{2} \sum_{j,k} \|D_k D_j \psi\|_2^2 \,,$$

where we used the diamagnetic inequality, Theorem 2.1.1, to get the last estimate.

Combining with (11.6), we thereby get

$$\sum_{j,k} \|D_j D_k \psi\|_2^2 \leq C\|\mathbf{D}\psi\|_2^2 + 2B^2 \int_\Omega (\operatorname{curl} \mathbf{A})^2 |\psi|^2 \, dx + 2 \int_\Omega |\mathcal{H}\psi|^2 \, dx$$
$$+ 4B \left| \int_\Omega (\operatorname{curl} \mathbf{A}) \cdot \Im \begin{pmatrix} D_2\psi \, \overline{D_3\psi} \\ D_3\psi \, \overline{D_1\psi} \\ D_1\psi \, \overline{D_2\psi} \end{pmatrix} dx \right| . \tag{11.8}$$

The 3D result analogous to Lemma 11.2.2 is the following.

Lemma 11.2.3.
Let $\Omega \subset \mathbb{R}^3$ be a domain with compact smooth boundary and let $\beta \in L^\infty(\Omega)$ be an exterior magnetic field [see (10.10)*]. Then there exists a constant $C > 0$ such that, for all $p_1, p_2 \in [1, +\infty]$ and all $\psi \in C^\infty(\overline{\Omega})$ satisfying the magnetic Neumann condition, we have*

$$\sum_{j,k} \|D_j D_k \psi\|_2^2 \leq C\Big\{ B^2 \|\beta\|_\infty^2 \|\psi\|_2^2 + \|\mathbf{D}\psi\|_2^2 + \|\mathcal{H}\psi\|_2^2 \tag{11.9}$$
$$+ B^2 \| \operatorname{curl} \mathbf{A} - \beta\|_{2p_1}^2 \|\psi\|_{2q_1}^2 + B \| \operatorname{curl} \mathbf{A} - \beta\|_{p_2} \|\mathbf{D}\psi\|_{2q_2}^2 \Big\} ,$$

where q_j is the conjugate exponent to p_j ; i.e., $p_j^{-1} + q_j^{-1} = 1$.

11.3 Regularity of Solutions of the Ginzburg–Landau System

11.3.1 The case of two dimensions

We recall that the Ginzburg–Landau system was introduced in (10.8), the vector field $\mathbf{F}$ in (10.31), and the basic inequality

$$\|\psi\|_\infty \leq 1 \,, \tag{11.10}$$

was given in Proposition 10.3.1. Also, recall that, without loss of generality, by a gauge transformation we can assume that the vector potential $\mathbf{A}$ belongs to the space $H^1_{\mathrm{div}}(\Omega)$, i.e., satisfies

$$\mathbf{A} \in H^1(\Omega, \mathbb{R}^2) \,, \qquad \operatorname{div} \mathbf{A} = 0 \quad \text{in } \Omega \,, \qquad \mathbf{A} \cdot \nu = 0 \quad \text{on } \partial\Omega \,. \tag{11.11}$$

Theorem 11.3.1.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let $\beta \in C^\infty(\overline{\Omega})$. Then there exist a constant C and, for any $\alpha \in]0,1[$ and $p \in]1, +\infty[$, constants $\widehat{C}_\alpha$ and $\widetilde{C}_p$ such that if $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ is any solution of the Ginzburg–Landau system (10.8) *with parameters κ , $\sigma > 0$, then*

$$\sum_{j,k} \|D_j D_k \psi\|_2 \leq C(1 + \kappa\sigma + \kappa^2)\|\psi\|_2, \tag{11.12}$$

$$\| \operatorname{curl} \mathbf{A} - \beta\|_{C^{0,\alpha}(\overline{\Omega})} \leq \widehat{C}_\alpha \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty , \tag{11.13}$$

and

$$\| \operatorname{curl} \mathbf{A} - \beta\|_{W^{1,p}(\Omega)} \leq \widetilde{C}_p \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty . \tag{11.14}$$

Remark 11.3.2.

- *Using the $W^{k,p}$-regularity of the curl-div system (Proposition D.2.5), we obtain from* (11.14) *the estimate*

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \leq \widetilde{D}_p \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty . \tag{11.15}$$

  Hence, using the Sobolev embedding theorem,

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leq \widehat{D}_\alpha \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty, \tag{11.16}$$

  for all $\alpha \in [0, 1[$.
- *In the applications, σ is of the same order as κ , and so* (11.16) *gives that* $(\mathbf{A} - \mathbf{F})$ *is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ in this regime, for any $\alpha < 1$.*

Proof of Theorem 11.3.1.
We use Lemma 11.2.2 with $p_1 = 1$, $p_2 = \infty$, and $B = \kappa\sigma$. After inserting (10.8a) and the results of Lemma 10.3.2, we find

$$\sum_{j,k} \|D_j D_k \psi\|_2^2 \leq C \left\{ \big(1 + \kappa^4 + (\kappa\sigma)^2\big) \|\psi\|_2^2 + \kappa^3 \sigma \|\psi\|_2^2 \, \| \operatorname{curl} \mathbf{A} - \beta\|_\infty \right\} . \tag{11.17}$$

Using a Sobolev inequality and the fact that

$$\|\psi\|_2 \leq |\Omega|^{\frac{1}{2}} ,$$

which is an immediate consequence of (11.10), this becomes, with a new constant C and, for any $\epsilon \in]0, 1]$,

$$\begin{aligned} &\sum_{j,k} \|D_j D_k \psi\|_2^2 \\ &\quad \leq C\Big\{\epsilon^{-1}\big(1 + \kappa^4 + (\kappa\sigma)^2\big) \|\psi\|_2^2 + \epsilon(\kappa\sigma)^2 \| \operatorname{curl} \mathbf{A} - \beta\|_{W^{1,p}}^2 \Big\} . \end{aligned} \tag{11.18}$$

We now apply a Sobolev inequality and the (pointwise) diamagnetic inequality (2.1)—with $f = (-i\partial_{x_k} + \kappa\sigma A_k)\psi$—to (10.27), in order to get

$$\begin{aligned} \|\operatorname{curl}\mathbf{A} - \beta\|^2_{W^{1,p}(\Omega)} &\leq \frac{C'}{(\kappa\sigma)^2}\|\psi\|^2_\infty \left(\sum_{j,k}\|D_jD_k\psi\|^2_2 + \|p_{\kappa\sigma A}\psi\|^2_2\right) \\ &\leq \frac{C'}{(\kappa\sigma)^2}\|\psi\|^2_\infty \sum_{j,k}\|D_jD_k\psi\|^2_2 + \frac{C'}{\sigma^2}\|\psi\|^2_\infty\,\|\psi\|^2_2\,, \end{aligned} \tag{11.19}$$

where the last inequality follows from (10.26).

Inserting (11.19) in (11.18) and choosing ϵ sufficiently small yields the desired (11.12).

Once (11.12) is established, we get (11.14) from (11.19). Finally, (11.13) follows from (11.14) and a Sobolev inequality. This finishes the proof of Theorem 11.3.1. □

11.3.2 The case of three dimensions

In three dimensions, the Ginzburg–Landau functional $\mathcal{G}^{3\mathrm{D}}$ is given in (10.11) and the Ginzburg–Landau system is stated in (10.14). Recall that in this case $\mathbf{A}$ belongs to $\dot{H}^1_{\mathrm{div},\mathbf{F}}$ as defined in (10.12). Also, the inequality (10.21) remains true in the 3D case.

As we will see ahead, the fact that we do not have a boundary condition for $\mathbf{A}$ will be both a simplification and a complication.

Theorem 11.3.3.
Let $\Omega \subset \mathbb{R}^3$ be a smooth, bounded domain and let $\beta \in L^\infty(\mathbb{R}^3)$ be given. Then, for all $\alpha < 1/2$ and all $1 \leq p \leq 6$, there exist constants C_α, C_p such that for all $\kappa, \sigma > 0$ and all solutions $(\psi, \mathbf{A}) \in W^{1,2}(\Omega) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}$ to (10.14),

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \leq C_p \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma}\|\psi\|_\infty\,\|\psi\|_2\,, \tag{11.20}$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_\alpha \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma}\|\psi\|_\infty\,\|\psi\|_2\,. \tag{11.21}$$

Proof.
By (10.36), Sobolev embeddings, the diamagnetic inequality, and (10.35), we find for $p \leq 6$ the estimate

$$\begin{aligned} &\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \\ &\quad\leq C\left(\frac{1}{\sigma}\|\psi\|_\infty\,\|\psi\|_2 + \frac{1}{\kappa\sigma}\|\psi\|_\infty\,\big\||p_{\kappa\sigma\mathbf{A}}\psi|\big\|_{W^{1,2}(\Omega)}\right) \\ &\quad\leq C'\left(\frac{1}{\sigma}\|\psi\|_\infty\|\psi\|_2 + \frac{1}{\kappa\sigma}\,\|\psi\|_\infty\sum_{j,k}\|D_jD_k\psi\|_2\right). \end{aligned} \tag{11.22}$$

We will use (11.22) for $p = 6$ and the Sobolev inequality

$$\| \operatorname{curl} \mathbf{A} - \beta \|_\infty \leq C \, \|\mathbf{A} - \mathbf{F}\|_{W^{2,6}(\Omega)} \,. \tag{11.23}$$

We use Lemma 11.2.3 with $p_1 = 1$ and $p_2 = +\infty$, and find, by implementing (10.14a), (10.21), (10.35), and (10.42),

$$\sum_{j,k} \|D_j D_k \psi\|_2^2 \quad \leq C' \left\{ (1 + (\kappa\sigma)^2 + \kappa^4) \|\psi\|_2^2 + \kappa^3 \sigma \| \operatorname{curl} \mathbf{A} - \beta \|_\infty \, \|\psi\|_2^2 \right\} .$$

So, using (11.10), we get, for all $\epsilon > 0$,

$$\sum_{j,k} \|D_j D_k \psi\|_2 \leq C \Big\{ (1 + \kappa\sigma + \kappa^2 + \epsilon^{-1}\kappa^2) \|\psi\|_2 + \epsilon\kappa\sigma \| \operatorname{curl} \mathbf{A} - \beta \|_\infty \Big\} . \tag{11.24}$$

Choosing ϵ sufficiently small and inserting (11.10), (11.23), and (11.24) in (11.22) with $p = 6$, we get

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,6}(\Omega)} \leq C \, \frac{1 + \kappa^2 + \kappa\sigma}{\kappa\sigma} \|\psi\|_\infty \|\psi\|_2 \,. \tag{11.25}$$

Now using Sobolev embeddings, we have proved Theorem 11.3.3. □

11.4 Asymptotic Estimates in Two Dimensions

11.4.1 Nonexistence of solutions to certain partial differential equations

We will use the notation $\tilde{\mathbf{F}}$ for any vector potential on $\mathbb{R}^2$ or on the half-space $\mathbb{R}^{2,+} := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0\}$ satisfying $\operatorname{curl} \tilde{\mathbf{F}} = 1$.

From Section 4.2 we know that the natural self-adjoint extension of the differential operator $(-i\nabla + \tilde{\mathbf{F}})^2$ on $L^2(\mathbb{R}^2)$ has spectrum

$$\sigma(-i\nabla + \tilde{\mathbf{F}})^2_{L^2(\mathbb{R}^2)} = \big\{ 2j + 1, j \in \mathbb{N} \big\} .$$

We also consider the Neumann realization $\mathcal{H}$ of the same operator but restricted to the half-space $\mathbb{R}^{2,+}$. That is the operator studied in Section 4.3, where we found

$$\inf \sigma(\mathcal{H}) = \Theta_0 \,. \tag{11.26}$$

In this section, we will consider the following partial differential equations, for $\beta_0 > 0$:

$$(-i\nabla + \beta_0 \tilde{\mathbf{F}})^2 \psi = \lambda \beta_0 \psi \qquad \text{on } \mathbb{R}^2 , \text{ with } \lambda < 1 , \tag{11.27a}$$

$$(-i\nabla + \beta_0 \tilde{\mathbf{F}})^2 \psi = \lambda \beta_0 (1 - S^2 |\psi|^2) \psi \qquad \text{on } \mathbb{R}^2 , \text{ with } 0 \leq \lambda \leq 1 , \tag{11.27b}$$

$$(-i\nabla + \beta_0\tilde{\mathbf{F}})^2\psi = \lambda\beta_0\psi \qquad \text{on } \mathbb{R}^2_+\,,\ \text{with } \lambda < \Theta_0\,, \tag{11.27c}$$

$$(-i\nabla + \beta_0\tilde{\mathbf{F}})^2\psi = \lambda\beta_0(1 - S^2|\psi|^2)\psi \qquad \text{on } \mathbb{R}^2_+\,,\ \text{with } 0 \leq \lambda \leq \Theta_0\,. \tag{11.27d}$$

The last two equations, (11.27c), (11.27d), are considered with Neumann boundary condition, i.e.,

$$\nu\cdot(-i\nabla + \beta_0\tilde{\mathbf{F}})\psi\big|_{\partial\mathbb{R}^2_+} = 0\,.$$

In order for this boundary condition to be well defined, we assume that

$$\psi \in H^2_{\mathrm{loc}}(\mathbb{R}^2_+)\,.$$

Also, we assume that

- the parameter $S \geq 0$ in (11.27b) verifies $S \neq 0$ when $\lambda = 1$,
- the parameter $S \geq 0$ in (11.27d) satisfies $S \neq 0$ when $\lambda = \Theta_0$.

The linear problems (11.27a), (11.27c) have no nontrivial solutions in L^2. That follows directly from the definition of the spectrum. We will prove that they do not have any nontrivial bounded solutions either.

Proposition 11.4.1.
Let (ψ, λ) be a solution to one of the equations (11.27a)–(11.27d) *with λ in the indicated interval and ψ being globally bounded. Then $\psi = 0$.*

Proof.
We only consider the cases on $\mathbb{R}^2_+$ since the other statements follow by the same arguments. Also, we can reduce to the case $\beta_0 = 1$ by scaling.

Let $\mathcal{H}$ be the operator $(-i\nabla + \tilde{\mathbf{F}})^2$ with the Neumann boundary condition. We will prove that a nonzero bounded solution to (11.27c) or (11.27d) will provide a contradiction to (11.26) through the variational principle.

Let $\psi \in L^\infty(\mathbb{R}^2_+) \setminus \{0\}$ be a solution to (11.27c). Define

$$\langle x\rangle = \sqrt{x^2+1}\,, \tag{11.28}$$

and notice that $|\nabla\langle x\rangle| \leq 1$. Define, furthermore,

$$\chi_R(x) = \exp(-\langle x\rangle/R)\,. \tag{11.29}$$

Since ψ is bounded, we get that $\chi_R\psi \in L^2(\mathbb{R}^{2,+})$. We see that the Neumann condition $\nu\cdot(-i\nabla + \tilde{\mathbf{F}})(\chi_R\psi)\big|_{\partial\mathbb{R}^2_+} = 0$ is satisfied. So, using (11.27c), (11.26), and an integration by parts, we get

$$\begin{aligned}\Theta_0\|\chi_R\psi\|_2^2 &\leq \langle \chi_R\psi \,|\, \mathcal{H}(\chi_R\psi)\rangle \\ &\leq \lambda\|\chi_R\psi\|_2^2 + \frac{1}{R^2}\int_{\mathbb{R}^{2,+}} \big|\chi_R(x)\big|^2\,|\psi(x)|^2\,dx\,. \end{aligned}\tag{11.30}$$

Therefore,

$$(\Theta_0 - \lambda - R^{-2})\|\chi_R\psi\|_2^2 \leq 0\,, \tag{11.31}$$

which for sufficiently large values of R implies that $\psi = 0$. This finishes the proof for (11.27c).

We now prove the nonexistence of nontrivial bounded solutions to (11.27d). Let $\psi \in L^\infty(\mathbb{R}^2_+)\setminus\{0\}$ be a solution to (11.27d) and let the rest of the notation be as in the previous case. If $\lambda = 0$ or $S = 0$, (11.27d) is the same as (11.27c), and so we may assume that $0 < \lambda \leq \Theta_0$ and $S > 0$. Furthermore, after replacing ψ by $S\psi$, we may assume that $S = 1$.

Integrating by parts, we obtain in analogy to (11.30)

$$\Theta_0\|\chi_R\psi\|_2^2 \leq \langle \chi_R\psi \,|\, \mathcal{H}(\chi_R\psi)\rangle \leq \lambda\|\chi_R\psi\|_2^2 - \lambda \int_{\mathbb{R}^{2,+}} \chi_R(x)^2|\psi(x)|^4\,dx + \frac{1}{R^2}\int_{\mathbb{R}^{2,+}} |\chi_R(x)|^2\,|\psi(x)|^2\,dx\,. \tag{11.32}$$

Since $\lambda \leq \Theta_0$, this implies, using the Cauchy–Schwarz inequality, that

$$\lambda \int_{\mathbb{R}^{2,+}} \chi_R(x)^2|\psi(x)|^4\,dx \leq R^{-2}\Big\{\int_{\mathbb{R}^{2,+}} \chi_R(x)^2|\psi(x)|^4\,dx \int \chi_R^2(x)\,dx\Big\}^{1/2}.$$

So

$$\lambda \int_{\mathbb{R}^{2,+}} \chi_R(x)^2|\psi(x)|^4\,dx \leq R^{-4}\int \chi_R^2(x)\,dx \leq CR^{-2}\,. \tag{11.33}$$

By taking the limit $R \to +\infty$, this implies that $\psi = 0$. □

11.4.2 Extraction of convergent subsequences

The nonexistence result of Proposition 11.4.1 will be combined with a compactness result that states that under certain circumstances we can construct bounded solutions to (11.27).

In this section, we will consider sequences of solutions to the Ginzburg–Landau equations. In particular, we will consider sequences $\{\mathbf{A}_n\}_n$ of vector potentials. We stress that the vector potential $\mathbf{A}$ has components (A_1, A_2) and similarly $\mathbf{A}_n = (A_1^n, A_2^n)$.

Lemma 11.4.2.
Suppose that the external magnetic field β belongs to $C^\infty(\overline{\Omega})$. Suppose we are given a sequence $\{(P_n, \kappa_n, \sigma_n)\}_{n\in\mathbb{N}} \subset \overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+$, and let $(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n} \in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)$ be an associated sequence of solutions to (10.8) [with $(\kappa,\sigma) = (\kappa_n,\sigma_n)$ in the equation] with $\psi_n \neq 0$. Define $S_n := \|\psi_n\|_\infty$. Assume that $\kappa_n \to \infty$ and that $\kappa_n/\sigma_n \to \Lambda \in \mathbb{R}^+$.

Then there exist $P \in \overline{\Omega}$, $S \in [0,1]$, $f \in \mathbb{C}$, and $\beta_0 \in \mathbb{R}$ such that—after possibly extracting a subsequence—we have

$$P_n \to P\,, \qquad S_n \to S\,, \qquad \psi_n(P_n) \to f\,, \qquad \operatorname{curl}\mathbf{A}_n(P_n) \to \beta_0\,, \tag{11.34}$$

as $n \to \infty$.

Furthermore:

Case 1

If

$$\sqrt{\kappa_n \sigma_n}\, \mathrm{dist}(P_n, \partial\Omega) \to \infty\,, \tag{11.35}$$

then there exist

- *a function* $\varphi \in C^\infty(\mathbb{R}^2)$*, satisfying* $|\varphi(0)| = |f|/S$ *and* $\|\varphi\|_\infty \leq 1$*,*
- *a linear vector potential* $\widetilde{\mathbf{F}}$*, with* $\mathrm{curl}\, \widetilde{\mathbf{F}} = 1$*,*

such that

$$(-i\nabla + \beta_0 \widetilde{\mathbf{F}})^2 \varphi = \Lambda(1 - S^2|\varphi|^2)\varphi \qquad \text{in } \mathbb{R}^2\,. \tag{11.36}$$

Case 2

If there exists $C > 0$ *such that*

$$\sqrt{\kappa_n \sigma_n}\, \mathrm{dist}(P_n, \partial\Omega) \leq C\,, \tag{11.37}$$

then there exist

- *a function* $\varphi \in C^\infty(\overline{\mathbb{R}^{2,+}})$*, satisfying* $\|\varphi\|_\infty \leq 1$ *and* $|\varphi(0)| = |f|/S$*,*
- *a linear vector potential* $\widetilde{\mathbf{F}}$ *with* $\mathrm{curl}\, \widetilde{\mathbf{F}} = 1$*,*

such that

$$\begin{aligned} (-i\nabla + \beta_0 \widetilde{\mathbf{F}})^2 \varphi &= \Lambda(1 - S^2|\varphi|^2)\varphi \qquad && \text{in } \mathbb{R}^{2,+} \\ e_2 \cdot (-i\nabla + \beta_0 \widetilde{\mathbf{F}})\varphi &= 0 && \text{on } \partial\mathbb{R}^{2,+}. \end{aligned} \tag{11.38}$$

Note that, up to extraction of a subsequence, we can always ensure that case 1 or case 2 occurs.

Proof.
The proof of (11.34) is elementary since $\overline{\Omega}$ is compact, $|\psi_n(x)| \leq 1$ for all n and x, and we use (11.16).

Since β is regular, we get from Corollary D.2.6 that also $\mathbf{F} \in C^\infty(\overline{\Omega})$. Therefore, we can use (11.15), with $\mathbf{A} = \mathbf{A}_n$, to conclude that the sequence $\{\mathbf{A}_n\}_n$ is bounded in $W^{2,p}(\Omega)$, for all $p < \infty$. By compactness of the inclusion $W^{2,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ for $s < 2$, we may further extract a convergent subsequence (still denoted by $\mathbf{A}_n$) in $W^{s,p}(\Omega)$. Furthermore, for a given $\alpha < 1$, we may choose p sufficiently big and s sufficiently close to 2 in order to have the inclusion $W^{s,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$. Thus, we get the existence of some $\overline{\mathbf{A}} \in (\cap_{\alpha<1} C^{1,\alpha}(\overline{\Omega})) \cap (\cap_{s<2,p} W^{s,p}(\Omega))$ such that, for all $\alpha < 1$, $s < 2$, and $p > 1$,

$$\mathbf{A}_n \to \overline{\mathbf{A}} \quad \text{in} \quad C^{1,\alpha}(\overline{\Omega}) \cap W^{s,p}(\Omega)\,.$$

We now identify the field generated by $\overline{\mathbf{A}}$. The inequality (10.28) holds for $\mathbf{A}_n$ with a constant C independent of n (only depending on Ω). By passing to the limit (using Proposition 10.3.1), we find that

$$\mathrm{curl}\, \overline{\mathbf{A}} = \beta\,. \tag{11.39}$$

From here we split the proof into two parts, depending on whether we are in case 1 or case 2.

Limiting equation for case 1.
Let us define, for any $R>0$ the following functions on the disc $\mathrm{D}(0,R)$:

$$\mathbf{a}_n(y) := \sqrt{\kappa_n\sigma_n}\left(\mathbf{A}_n\left(P_n+\frac{y}{\sqrt{\kappa_n\sigma_n}}\right)-\mathbf{A}_n(P_n)\right),$$

$$\varphi_n(y) := S_n^{-1}e^{-i\sqrt{\kappa_n\sigma_n}\mathbf{A}_n(P_n)\cdot y}\psi_n\left(P_n+\frac{y}{\sqrt{\kappa_n\sigma_n}}\right).$$

Since we are in case 1, $\mathbf{a}_n$, φ_n are defined on $\mathrm{D}(0,R)$ for all n sufficiently large.

Let us introduce the linear function

$$\bar{\mathbf{F}}(y) := \big(\operatorname{Jac}\overline{\mathbf{A}}(P)\big)y,$$

with $(\operatorname{Jac}\overline{\mathbf{A}})_{jk} = (\partial_{x_j}\overline{A}_k)$.

Since $P_n \to P$ and $\mathbf{A}_n \to \overline{\mathbf{A}}$ in $C^{1,\alpha}(\overline{\Omega})$, we find that

$$\mathbf{a}_n \to \bar{\mathbf{F}},$$

in $C^\alpha\big(\overline{\mathrm{D}(0,R)}\big)$ for all R.

By (11.39), we obtain

$$\operatorname{curl}\bar{\mathbf{F}} = \beta_0 \quad \text{in} \quad \mathbb{R}^2.$$

Thus, $\bar{\mathbf{F}} = \beta_0\tilde{\mathbf{F}}$, where

$$\operatorname{curl}\tilde{\mathbf{F}} = 1.$$

The equation for ψ in (10.8a) implies, since $\operatorname{div}\mathbf{a}_n = 0$, that

$$-\Delta\varphi_n - 2i\mathbf{a}_n\cdot\nabla\varphi_n + |\mathbf{a}_n|^2\varphi_n = \frac{\kappa_n}{\sigma_n}(1-S_n^2|\varphi_n|^2)\varphi_n. \tag{11.40}$$

Notice that (11.16) implies that for all $\alpha<1$, $\|\mathbf{a}_n\|_{C^\alpha(\mathrm{D}(0,R))} \le C_\alpha(R)$ for some $C_\alpha(R)>0$. Also, we have $\|\varphi_n\|_\infty \le 1$. Elliptic regularity (see Theorem E.3.2) now implies, since $\{\kappa_n/\sigma_n\}$, $\{S_n\}$ are bounded uniformly in n, the existence of a constant $C'_\alpha(R)>0$ such that

$$\|\varphi_n\|_{C^{1,\alpha}(\overline{\mathrm{D}(0,\frac{2R}{3})})} \le C'_\alpha(R).$$

Now applying Theorem E.3.1, we obtain

$$\|\varphi_n\|_{C^{2,\alpha}(\overline{\mathrm{D}(0,\frac{R}{2})})} \le C'_\alpha(R).$$

Since the inclusion $C^{2,\alpha}(\overline{\mathrm{D}(0,R/2)}) \hookrightarrow C^{2,\alpha'}(\overline{\mathrm{D}(0,R/2)})$ is compact for any $\alpha'<\alpha$, we may, for any $\alpha<1$, $R\ge 1$, extract a subsequence—denoted by $\{\varphi_n^R\}$—having a limit in the $C^{2,\alpha}(\overline{\mathrm{D}(0,R/2)})$ topology. A "diagonal sequence" argument now gives the existence of a subsequence $\{\tilde{\varphi}_n\}$ of the original sequence $\{\varphi_n\}$ and a $\varphi \in C^{2,\alpha}(\mathbb{R}^2)$ such that

$$\lim_{n\to\infty} \|\tilde{\varphi}_n - \varphi\|_{C^{2,\alpha}(\overline{\mathrm{D}(0,R)})} = 0\,,$$

for all $R > 0$. In particular,

$$|\varphi(0)| = \lim_{n\to\infty} |\tilde{\varphi}_n(0)| = \lim_{n\to\infty} \frac{|\psi_n(P_n)|}{S_n} = \frac{|f|}{S}\,.$$

Passing to the limit in (11.40) we obtain that φ satisfies (11.36). We can now use elliptic regularity to obtain the additional $(C^\infty(\mathbb{R}^2))$ regularity of φ.

Limiting equation for case 2.
The idea in the second case is the same as before, but things are complicated slightly by the presence of the boundary. We transform the equation into boundary coordinates (s,t) in order to find a model on the half-plane.

The boundary coordinates are defined in Section F.1. We repeat some of the definitions, since we will need to consider a sequence of centered coordinate changes.

Since we are in case 2, $P \in \partial\Omega$. Let $Q_n \in \partial\Omega$ be the unique (for n sufficiently large) boundary point such that $|P_n - Q_n| = \operatorname{dist}(P_n, \partial\Omega)$. Let $\mathcal{O}$ be a (sufficiently small) neighborhood of P, let $\gamma : [-s_0, s_0] \to \partial\Omega$ be a smooth parametrization of the boundary with $\gamma(0) = P$, $|\gamma'(s)| = 1$, and let $\nu(s)$ be the inward normal vector to $\partial\Omega$ at the point $\gamma(s)$. We may assume that $\{\gamma'(s), \nu(s)\}$ is a positively oriented basis. Define the coordinate change

$$\Phi \; :]-s_0, s_0[\times]0, t_0[\to \Omega \cap \mathcal{O}\,,$$

by $\Phi(s,t) = \gamma(s) + t\nu(s)$. For $s_0, t_0, \mathcal{O}$ sufficiently small, the map Φ is a diffeomorphism.

Let γ_n be as γ above, but with $\gamma_n(0) = Q_n$. We now define Φ_n to be the same construction but with γ replaced by γ_n and s_0 replaced by $s_0/2$. Since $Q_n \to P$ as $n \to \infty$, the image of Φ_n will contain $\Phi\big(]-s_0/4, s_0/4[\times]0, t_0[\big)$ when n is large.

Define

$$\begin{aligned} \tilde{\psi}_n &:= \psi_n \circ \Phi_n\,, & \tilde{\mathbf{A}}_n &:= (\operatorname{Jac}\Phi_n)^t(\mathbf{A}_n \circ \Phi_n)\,,\\ J_n &:= |\det \operatorname{Jac}\Phi_n|\,, & M_n &= \{M^n_{j,k}\} := \big[(\operatorname{Jac}\Phi_n)^t(\operatorname{Jac}\Phi_n)\big]^{-1}\,. \end{aligned}$$

Notice that $M_n\big|_{t=0} = Id$ and that the boundary condition $\nu \cdot \mathbf{A}_n\big|_{\partial\Omega} = 0$ implies that

$$e_2 \cdot \tilde{\mathbf{A}}_n\big|_{t=0} = 0\,,$$

where $e_2 = (0,1)$.

Implementing this change of variables in (10.8a) for ψ_n yields

$$J_n^{-1}(-i\nabla+\kappa_n\sigma_n\tilde{\mathbf{A}}_n)\cdot\big[J_nM_n(-i\nabla+\kappa_n\sigma_n\tilde{\mathbf{A}}_n)\tilde{\psi}_n\big]=\kappa_n^2(1-|\tilde{\psi}_n|^2)\tilde{\psi}_n\,,$$
$$e_2\cdot(-i\nabla+\kappa_n\sigma_n\tilde{\mathbf{A}}_n)\tilde{\psi}_n\Big|_{t=0}=0\,.$$

Let us calculate $\operatorname{curl}\tilde{\mathbf{A}}_n$. We use the geometric fact that $\nu_n'(s)=-k_n(s)\gamma_n'(s)$, where $k_n(s)$ is the curvature of the boundary at the point $\gamma_n(s)$. Then

$$\tilde{\mathbf{A}}_n=(\tilde{A}_1^n,\tilde{A}_2^n)=\big((1-tk_n(s))\gamma_n'(s)\cdot\mathbf{A}_n(\Phi_n(s,t))\,,\ \nu_n(s)\cdot\mathbf{A}_n(\Phi_n(s,t))\big)\,.$$

A direct calculation now yields

$$\operatorname{curl}\tilde{\mathbf{A}}_n:=\partial_s\tilde{A}_2^n-\partial_t\tilde{A}_1^n=(1-tk(s))(\operatorname{curl}\mathbf{A}_n)\big|_{\Phi_n(s,t)}\,. \tag{11.41}$$

Define $y_n:=\Phi_n^{-1}(P_n)$ and $z_n:=\sqrt{\kappa_n\sigma_n}y_n$. Since we are in case 2, $\{z_n\}$ is bounded and we may assume that $z_n\to z\in\mathbb{R}^{2,+}$.

We proceed to rescale as before. Define, with $\zeta=(\sigma,\tau)$,

$$\mathbf{a}_n(\zeta):=\frac{\tilde{\mathbf{A}}_n(\zeta/\sqrt{\kappa_n\sigma_n})-\tilde{\mathbf{A}}_n(0)}{1/\sqrt{\kappa_n\sigma_n}}\,,\qquad j_n(\zeta):=J_n(\zeta/\sqrt{\kappa_n\sigma_n})\,,$$
$$\varphi_n(\zeta):=S_n^{-1}e^{-i\sqrt{\kappa_n\sigma_n}\tilde{\mathbf{A}}_n(0)\cdot\zeta}\psi_n(\zeta/\sqrt{\kappa_n\sigma_n})\,,\quad m_n(\zeta):=M_n(\zeta/\sqrt{\kappa_n\sigma_n})\,.$$

We denote the components of $\mathbf{a}_n, m_n$ in the natural way, i.e., $\mathbf{a}_n=(a_1^n,a_2^n)$, $m_n=\{m_{j,k}^n\}_{j,k=1}^2$. Remember also the relations

$$m_n\big|_{\tau=0}=Id\,,\qquad e_2\cdot\mathbf{a}_n\big|_{\tau=0}=0\,.$$

We get the resulting equation for the scaled function φ_n:

$$j_n^{-1}(-i\nabla+\mathbf{a}_n)\cdot\big[j_nm_n(-i\nabla+\mathbf{a}_n)\varphi_n\big]=\frac{\kappa_n}{\sigma_n}(1-S_n^2|\varphi_n|^2)\varphi_n\,, \tag{11.42}$$
$$e_2\cdot(-i\nabla+\mathbf{a}_n)\varphi_n\big|_{\tau=0}=0\,.$$

By (11.15), $\{\mathbf{A}_n\}$ is a bounded sequence in $W^{2,p}(\Omega)$ for all $p<\infty$. Therefore, $\{\tilde{\mathbf{A}}_n\}$ is bounded in $W^{2,p}(]-s_0/4,s_0/4[\times]0,t_0[)$, and $\{\mathbf{a}_n\}$ is bounded in $W^{1,p}\big(\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}\big)$ for all $R>0$. Also, it is immediate that the matrix m_n is uniformly (in n) bounded in $C^1(\overline{D(0,R)\cap\mathbb{R}^{2,+}})$ for all $R>0$. We note for later use, that the bounds on $\{\mathbf{A}_n\}$ imply compactness in $W^{s,p}_{\rm loc}(]-s_0/4,s_0/4[\times]0,t_0[)$ for any $s<2$. So we may extract a convergent subsequence with limit $\overline{\mathbf{A}}$. It is now immediate from the definition that

$$\mathbf{a}_n\to D\overline{\mathbf{A}}(0)\qquad\text{in } W^{s,p}_{\rm loc}(\mathbb{R}^2), \tag{11.43}$$

(for any $s<1$) along the same subsequence. Notice that the limit is a linear vector potential.

Ahead we will use the standard results on elliptic regularity recalled in Chapter E on (11.42) to conclude that for all $\alpha<1$,

$$\{\varphi_n\}_n \text{ is bounded in } C^{1,\alpha}\big(\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}\big) \text{ for all } R>0\,. \tag{11.44}$$

To prove (11.44), we rewrite the equation for φ_n as follows:

$$-\operatorname{div}(m_n\nabla\varphi_n)+\mathbf{b}_n\cdot\nabla\varphi_n+c_n\varphi_n=f_n\,, \tag{11.45}$$

with

$$f_n:=\frac{\kappa_n}{\sigma_n}(1-S_n^2|\varphi_n|^2)\varphi_n+i\Big(\sum_{j,k}m_{j,k}^n\,\partial_j a_k^n\Big)\varphi_n\,,$$

and with the standard Neumann boundary condition $e_2\cdot\nabla\varphi_n\big|_{\tau=0}=0$. Here $\{f_n\}$ is uniformly bounded in $L^\infty\big(\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}\big)$ (for all $R>0$) since $\|\varphi_n\|_\infty\leq 1$, and the coefficients $\mathbf{b}_n,c_n$ are uniformly bounded in $W^{1,p}\big(\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}\big)$ for any $p<\infty$ and hence in $L^\infty\big(\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}\big)$.

In order to remove the boundary condition we extend by reflection. We denote extended functions by a superscript tilde. These functions will be defined by the fact that they are extensions of the original functions and that they are even or odd under the symmetry $(\sigma,\tau)\mapsto(\sigma,-\tau)$. These symmetry properties are as follows

$$\begin{aligned}&\tilde\varphi_n\,,\ \tilde m_{1,1}^n\,,\ \tilde m_{2,2}^n\,,\ \tilde b_1^n\,,\ \tilde c_n\,,\ \tilde f \quad \text{are even,}\\ &\tilde m_{1,2}^n\,,\ \tilde m_{2,1}^n\,,\ \tilde b_2^n \quad \text{are odd.}\end{aligned}$$

Since $m_n\big|_{\tau=0}=Id$, the matrix $\tilde m_n$ thus defined is uniformly Lipschitz continuous and $\tilde\varphi_n$ satisfies the extended version of (11.45) (with symbols having a superscript tilde). Clearly, the boundedness properties of $\mathbf{b}_n$, c_n imply that $\tilde{\mathbf{b}}_n,\tilde c_n$ are bounded in $L^\infty(\mathrm{D}(0,R))$ for all R. We can now apply the "interior" estimate Theorem E.3.2, to this extended equation and conclude that, for all $\alpha<1$,

$$\begin{aligned}\|\varphi_n\|_{C^{1,\alpha}(\overline{\mathrm{D}(0,R)\cap\mathbb{R}^{2,+}})}&\leq\|\tilde\varphi_n\|_{C^{1,\alpha}(\overline{\mathrm{D}(0,R)})}\\&\leq C\big(\|\tilde\varphi_n\|_{L^\infty(\mathrm{D}(0,2R))}+\|\tilde f_n\|_{L^\infty(\mathrm{D}(0,2R))}\big)\\&\leq C'\big(\|\varphi_n\|_{L^\infty(\mathrm{D}(0,2R)\cap\mathbb{R}^{2,+})}+\|f\|_{L^\infty(\mathrm{D}(0,2R)\cap\mathbb{R}^{2,+})}\big)\,.\end{aligned} \tag{11.46}$$

Using that $\|\varphi_n\|_\infty\leq 1$, we can control $\|f_n\|_{L^\infty}$ by $\|\varphi_n\|_{L^\infty}$. We have therefore proven (11.44).

With (11.44) established, we can proceed essentially as in case 1. Let $\alpha<1$. A diagonal sequence argument, as for case 1, gives the existence of $\varphi\in C^{1,\alpha}(\mathbb{R}^{2,+})$ with

$$\varphi(0)=|f|/S,\qquad \|\varphi\|_\infty\leq 1 \tag{11.47}$$

such that (eventually after extraction of a subsequence)

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{C^{1,\alpha}\left(\overline{D(0,R)\cap\mathbb{R}^{2,+}}\right)} = 0\,,$$

for all $R > 0$.

We now pass to the limit in (11.42) in the weak sense, i.e., we let $u \in C_0^1(\mathbb{R}^2)$ be arbitrary and consider the equation obtained by integrating by parts once in (11.42), namely

$$\int [-i\nabla(j_n^{-1}u) + \mathbf{a}_n(j_n^{-1})] \cdot j_n m_n [-i\nabla(\varphi_n) + \mathbf{a}_n(\varphi_n)]\,dx$$
$$= \frac{\kappa_n}{\sigma_n}\int (1 - S_n^2|\varphi_n|^2)\varphi_n u\,dx. \tag{11.48}$$

Upon passing to the limit, we obtain that

$$\int (-i\nabla u + \beta_0\tilde{\mathbf{F}}u)\cdot(-i\nabla\varphi + \beta_0\tilde{\mathbf{F}}\varphi)\,dx = \int \Lambda(1 - S^2|\varphi|^2)\varphi u\,dx. \tag{11.49}$$

Also the Neumann boundary condition is obtained by taking the limit in (11.42). In conclusion φ satisfies (11.38) in the weak sense. Here we used (11.41) to conclude that the limiting linear vector field obtained in (11.43)—which we denote by $\beta_0\tilde{\mathbf{F}}$—satisfies $\operatorname{curl}\tilde{\mathbf{F}} = 1$. We can now use elliptic regularity to obtain additional regularity ($C^\infty(\overline{\mathbb{R}^{2,+}})$) of φ. □

11.4.3 Asymptotic estimates

We will now combine the nonexistence result Proposition 11.4.1 with the "compactness" result in Lemma 11.4.2 to obtain strong estimates on solutions to the Ginzburg–Landau equations.

Actually, our first result, Proposition 11.4.4, only uses the extraction of convergent subsequences from (the proof of) Lemma 11.4.2.

Remark 11.4.3.
Our results will be stated under the assumption that κ/σ has a positive lower bound. Actually, when the magnetic field β is nonvanishing, it follows from Theorem 10.4.1 that this assumption is automatically satisfied for nontrivial solutions.

Proposition 11.4.4.
Suppose that $\beta \in C^\infty(\overline{\Omega})$. Let $0 < \lambda_{\min} \leq \lambda_{\max}$. There exist constants D_0, D_1 such that if

$$\kappa \geq D_0, \qquad \lambda_{\min} \leq \kappa/\sigma \leq \lambda_{\max},$$

then any solution $(\psi, \mathbf{A})$ of (10.8) satisfies

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_{C(\overline{\Omega})} \leq D_1\sqrt{\kappa\sigma}\|\psi\|_\infty\,, \tag{11.50}$$

$$\| \operatorname{curl} \mathbf{A} - \beta \|_{C^1(\overline{\Omega})} \leq \frac{D_1}{\sqrt{\kappa\sigma}} \, \|\psi\|_\infty^2 \,, \tag{11.51}$$

$$\| \operatorname{curl} \mathbf{A} - \beta \|_{C^2(\overline{\Omega})} \leq D_1 \, \|\psi\|_\infty^2 \,. \tag{11.52}$$

Proof of Proposition 11.4.4.
Proof of (11.50). Suppose (11.50) is wrong. Then there exist a sequence $(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n}$ of solutions to (10.8) and a corresponding sequence of points $\{P_n\} \subset \Omega$ such that

$$\frac{|p_{\kappa_n \sigma_n \mathbf{A}_n} \psi_n(P_n)|}{\sqrt{\kappa_n \sigma_n} \|\psi_n\|_\infty} \to \infty \,.$$

After extracting subsequences as in the proof of Lemma 11.4.2, we find (along the converging subsequence)

$$\lim_{n\to\infty} \frac{|p_{\kappa_n \sigma_n \mathbf{A}_n} \psi_n(P_n)|}{\sqrt{\kappa_n \sigma_n} \|\psi_n\|_\infty} = |(-i\nabla + \beta_0 \tilde{\mathbf{F}}) \varphi(z)| < \infty \,,$$

where $z = 0$ in case 1 and $z = \lim_{n\to\infty} \sqrt{\kappa_n \sigma_n} \Phi_n^{-1}(P_n)$ in case 2. This yields a contradiction, and so we conclude that (11.50) is correct.
Proof of (11.51). This inequality is a consequence of (11.50). Remember that

$$\operatorname{curl}^2 \mathbf{A} := (\partial_{x_2} \operatorname{curl} \mathbf{A} \,, \, -\partial_{x_1} \operatorname{curl} \mathbf{A}) \,.$$

Thus, by the Ginzburg–Landau equation (10.8a) and (11.50),

$$\begin{aligned} \|\nabla(\operatorname{curl} \mathbf{A} - \beta)\|_\infty &= \| \operatorname{curl}(\operatorname{curl} \mathbf{A} - \beta) \|_\infty \\ &= \frac{1}{\kappa\sigma} \| \Re\{\overline{\psi} \, p_{\kappa\sigma\mathbf{A}} \psi\} \|_\infty \leq \frac{C}{\sqrt{\kappa\sigma}} \|\psi\|_\infty^2 \,. \end{aligned} \tag{11.53}$$

This is (11.51) for the derivatives.

Furthermore, since $\operatorname{curl} \mathbf{A} - \beta = 0$ on $\partial\Omega$ and Ω is bounded, we can integrate (11.53) "from the boundary" and find

$$\| \operatorname{curl} \mathbf{A} - \beta \|_\infty \leq \frac{C}{\sqrt{\kappa\sigma}} \|\psi\|_\infty^2 \,.$$

This finishes the proof of (11.51).
Proof of (11.52). The proof of this inequality follows the same idea as the proof of the pair of inequalities (11.50)–(11.51). One needs to take one extra derivative and consequently prove the existence of $C > 0$ and $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$,

$$\frac{1}{\kappa\sigma} \|\nabla(\overline{\psi} \, p_{\kappa\sigma\mathbf{A}} \psi)\|_\infty \leq C \|\psi\|_\infty^2 \,. \tag{11.54}$$

As before, if (11.54) was wrong, there would exist a sequence $(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n}$ of solutions and a corresponding sequence of points $\{P_n\} \subset \overline{\Omega}$ such that

$$\lim_{n\to+\infty} \frac{|\nabla(\overline{\psi_n}\, p_{\kappa_n\sigma_n \mathbf{A}_n}\psi_n)|(P_n)}{\kappa_n\sigma_n\|\psi_n\|_\infty^2} = +\infty\,.$$

After extracting subsequences, we find, following the proof of Lemma 11.4.2, that

$$\lim_{n\to+\infty} \frac{|\nabla(\overline{\psi_n} p_{\kappa_n\sigma_n \mathbf{A}_n}\psi_n)|(P_n)}{\kappa_n\sigma_n\|\psi_n\|_\infty^2} = \left|\nabla\left(\overline{\varphi}(-i\nabla + \beta_0\tilde{\mathbf{F}})\varphi\right)(z)\right| < +\infty\,.$$

This yields a contradiction. □

Our next result shows that ψ must be "small" in the region that is not "classically allowed". We will discuss such results further in Chapter 12 using other methods.

Proposition 11.4.5.
Let $\beta \in C^\infty(\overline{\Omega})$ be strictly positive and let $0 < \Lambda_{\max} < 1$ be given. Define

$$\mathcal{A}_B := \{x \in \partial\Omega \,|\, \Lambda_{\max} \geq \Theta_0\beta(x)\}, \qquad \mathcal{A}_I := \{x \in \Omega \,|\, \Lambda_{\max} \geq \beta(x)\}\,.$$

Then there exist positive constants κ_0 and C such that if $(\psi, \mathbf{A})_{\kappa,\sigma}$ is a solution to (10.8) *with $\psi \neq 0$, and*

$$\kappa > \kappa_0\,, \qquad \kappa/\sigma \leq \Lambda_{\max}\,,$$

and $P \in \overline{\Omega}$ is such that $|\psi(P)| = \|\psi\|_\infty$, then

$$\operatorname{dist}(P, \mathcal{A}_B \cup \mathcal{A}_I) \leq \frac{C}{\sqrt{\kappa\sigma}}\,.$$

Proof.
Suppose Proposition 11.4.5 is false. Then (with standard notation) there exists a sequence $(P_n, \kappa_n, \sigma_n, \psi_n, \mathbf{A}_n)$ such that

$$\begin{aligned} &\kappa_n \to \infty\,, && \kappa_n/\sigma_n \leq \Lambda_{\max}, \\ &|\psi_n(P_n)| = \|\psi_n\|_\infty\,, && \sqrt{\kappa_n\sigma_n}\operatorname{dist}(P_n, \mathcal{A}_B \cup \mathcal{A}_I) \to \infty\,. \end{aligned} \tag{11.55}$$

We will also suppose that $\sqrt{\kappa_n\sigma_n}\operatorname{dist}(P_n, \partial\Omega) \to \infty$, the contrary case being treated analogously. By case 1 in Lemma 11.4.2, we find a continuous solution $\varphi \in L^\infty(\mathbb{R}^2)$ to (11.36) with $|\varphi(0)| = 1$, $\Lambda \leq \Lambda_{\max}$, and $S \leq 1$. By Proposition 11.4.1 for the case (11.27b), applied with $\lambda = \Lambda_{\max}/\beta_0$ and $S \leq 1$, we therefore have $\varphi \equiv 0$, in contradiction to $|\varphi(0)| = 1$. Thus, no such sequence can exist and Proposition 11.4.5 is true. □

Remark 11.4.6.
In the special case when $\beta \equiv 1$ and

$$\Theta_0 < \Lambda_{\max} < 1\,,$$

the result of Proposition 11.4.5 is that maxima of $|\psi|$ are located within a distance $1/\sqrt{\kappa\sigma}$ of $\partial\Omega$. In other words ψ is in some sense concentrated near the boundary.

The next result is of the same spirit.

Proposition 11.4.7.
Suppose that $\beta \equiv 1$ *. Let* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy* $g(\kappa) \to 0$ *as* $\kappa \to \infty$ *. Then there exists a function* $\tilde{g}$ *with* $\tilde{g}(\kappa) \to 0$ *as* $\kappa \to \infty$ *such that if*

$$\kappa(\Theta_0^{-1} - g(\kappa)) \le \sigma \le \kappa(\Theta_0^{-1} + g(\kappa)), \tag{11.56}$$

then any solution $(\psi, \mathbf{A})_{\kappa,\sigma}$ *of* (10.8) *satisfies*

$$\|\psi\|_\infty \le \tilde{g}(\kappa). \tag{11.57}$$

In Section 13.4, we will give a more quantitative estimate valid in a smaller parameter region.

Proof of Proposition 11.4.7.
The proof goes by contradiction. If Proposition 11.4.7 is false, then there exist $\epsilon_0 > 0$ and a sequence $\{(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n}\}$ of solutions to (10.8) such that

$$(\Theta_0^{-1} - g(\kappa_n)) \le \sigma_n/\kappa_n \le (\Theta_0^{-1} + g(\kappa_n)),$$
$$\kappa_n \to \infty,$$

and

$$\|\psi_n\|_\infty \ge \epsilon_0.$$

Choose $P_n \in \overline{\Omega}$ such that $\|\psi_n\|_\infty = |\psi_n(P_n)|$. We now proceed to extract subsequences as described above. We may assume that either case 1 or case 2 is satisfied. In case 1, we find the limiting equation (11.36) with $\Lambda = \Theta_0$ and $S \ge \epsilon_0$. Proposition 11.4.1 implies, since $\Theta_0 \le 1$, that $\varphi \equiv 0$. However, by assumption,

$$|\varphi(0)| = \lim_{n\to\infty} |\tilde{\varphi}_n(0)| = \lim_{n\to\infty} \frac{|\psi_n(P_n)|}{\|\psi_n\|_\infty} = 1. \tag{11.58}$$

This is a contradiction, so we conclude that case 1 cannot occur.

Since case 1 cannot occur, we necessarily find that case 2 occurs. Thus, the limiting equation becomes (11.38) with $\Lambda = \Theta_0$, $S \ge \epsilon_0$. By Proposition 11.4.1, $\varphi \equiv 0$, but

$$|\varphi(z)| = \lim_{n\to\infty} |\varphi_n(z_n)| = \lim_{n\to\infty} \frac{|\psi_n(P_n)|}{\|\psi_n\|_\infty} = 1.$$

Thus, case 2 is also impossible, and we conclude that Proposition 11.4.7 is satisfied. □

11.5 Asymptotic Estimates in Three Dimensions

In this section, we will carry out a similar analysis in the three-dimensional situation. We will generally be rather brief when arguments are too similar to the 2D case to warrant a repetition.

11.5.1 Nonexistence of solutions to certain partial differential equations

Let $\beta \in \mathbb{S}^2$. We will use the notation $\tilde{\mathbf{F}}$ for any vector potential on $\mathbb{R}^3$ or on the half-space $\mathbb{R}^3_+$ such that $\operatorname{curl} \tilde{\mathbf{F}} = \beta$.

The spectrum of the (Neumann) realizations of $(-i\nabla + \tilde{\mathbf{F}})^2$ on $L^2(\mathbb{R}^3)$ or $L^2(\mathbb{R}^{3,+})$ was determined in Chapter 6. In particular—in the half-space case—it depends on the angle ϑ between β and the boundary.

Just like the case of dimension 2 we will consider the linear and nonlinear equations

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda\psi \qquad \text{on } \mathbb{R}^3, \text{ with } \lambda < 1\,, \tag{11.59}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda(1 - S^2|\psi|^2)\psi \qquad \text{on } \mathbb{R}^3, \text{ with } 0 \leq \lambda \leq 1\,, \tag{11.60}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda\psi \qquad \text{on } \mathbb{R}^{3,+}\,, \text{ with } \lambda < \varsigma(\vartheta)\,, \tag{11.61}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda(1 - S^2|\psi|^2)\psi \qquad \text{on } \mathbb{R}^{3,+}, \text{ with } 0 \leq \lambda \leq \varsigma(\vartheta)\,. \tag{11.62}$$

Equations (11.61), (11.62) are considered with the Neumann boundary condition, i.e., $\nu\cdot(-i\nabla+\tilde{\mathbf{F}})\psi\big|_{\partial\mathbb{R}^{3,+}} = 0$, and we assume that $\psi \in H^2_{\text{loc}}(\mathbb{R}^{3,+})$. Also, we assume that the parameter $S \geq 0$ in (11.60) verifies $S \neq 0$ when $\lambda = 1$, and similarly, the parameter $S \geq 0$ in (11.62) satisfies $S \neq 0$ when $\lambda = \varsigma(\vartheta)$.

Proposition 11.5.1.
Let (ψ, λ) be a solution to one of (11.59), (11.60), (11.61) *or* (11.62) *with λ in the indicated interval and ψ globally bounded. Then $\psi = 0$.*

The proof of Proposition 11.5.1 is identical to the 2D result (Proposition 11.4.1), and will be omitted.

11.5.2 Three-dimensional asymptotic estimates

The extraction of subsequences is essentially identical to the two-dimensional case.

Lemma 11.5.2.
Suppose that the external magnetic field β belongs to $C^\infty(\mathbb{R}^3)$. Suppose we are given a sequence $\{(P_n, \kappa_n, \sigma_n)\}_{n\in\mathbb{N}}$ in $\Omega\times\mathbb{R}^+\times\mathbb{R}^+$, and let $\{(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n}\}$ be an associated sequence of solutions in $H^1(\Omega)\times\dot{H}^1_{\text{div},\mathbf{F}}$ to (10.14) *with $\psi_n \neq 0$. Define $S_n := \|\psi_n\|_\infty$. Assume that $\kappa_n \to \infty$ and that $\kappa_n/\sigma_n \to \Lambda \in \mathbb{R}^+$.*

Then there exist $P \in \overline{\Omega}$, $S \in [0,1]$, $f \in \mathbb{C}$, *and* $\beta_0 \in \mathbb{R}^3$ *such that—after possibly extracting a subsequence—we have*

$$P_n \to P\,, \qquad S_n \to S\,, \qquad \psi_n(P_n) \to f\,, \qquad \operatorname{curl} \mathbf{A}_n(P_n) \to \beta_0\,, \quad (11.63)$$

as $n \to \infty$.

Furthermore:

Case 1

If

$$\sqrt{\kappa_n \sigma_n}\, \operatorname{dist}(P_n, \partial\Omega) \to \infty\,, \tag{11.64}$$

then there exist a function $\varphi \in C^\infty(\mathbb{R}^3)$, *with* $\|\varphi\|_\infty \leq 1$ *and* $|\varphi(0)| = |f|/S$ *and a (linear) vector potential* $\widetilde{\mathbf{F}}$ *with* $\operatorname{curl} \widetilde{\mathbf{F}} = \beta_0$, *and such that*

$$(-i\nabla + \widetilde{\mathbf{F}})^2 \varphi = \Lambda(1 - S^2|\varphi|^2)\varphi \qquad \text{in } \mathbb{R}^3\,. \tag{11.65}$$

Case 2

If there exists $C > 0$ *such that*

$$\operatorname{dist}(P_n, \partial\Omega) \leq \frac{C}{\sqrt{\kappa_n \sigma_n}}\,, \tag{11.66}$$

then there exist a function $\varphi \in C^\infty(\overline{\mathbb{R}^{3,+}})$, *with* $\|\varphi\|_\infty \leq 1$ *and* $|\varphi(0)| = |f|/S$ *and a linear vector potential* $\widetilde{\mathbf{F}}$ *with* $\operatorname{curl} \widetilde{\mathbf{F}} = \beta_0$, *and such that*

$$\begin{aligned} (-i\nabla + \widetilde{\mathbf{F}})^2 \varphi &= \Lambda(1 - S^2|\varphi|^2)\varphi \qquad && \text{in } \mathbb{R}^{3,+}\,, \\ e_3 \cdot (-i\nabla + \widetilde{\mathbf{F}})\varphi &= 0 && \text{on } \partial\mathbb{R}^{3,+}. \end{aligned} \tag{11.67}$$

The analysis in the three-dimensional case is similar to the two-dimensional one, and so we state only a representative result without proof.

Proposition 11.5.3.
Let $\beta \in C^\infty(\mathbb{R}^3)$ *satisfy*

$$|\beta(x)| \geq c > 0\,,$$

for all $x \in \Omega$, *and let* $\Lambda_{\max} \in]0,1[$ *be given. Define*

$$\mathcal{A}_B := \{x \in \partial\Omega \,|\, \Lambda_{\max} \geq \zeta(\vartheta(x))|\beta(x)|\}, \qquad \mathcal{A}_I := \{x \in \Omega \,|\, \Lambda_{\max} \geq |\beta(x)|\}\,.$$

Then there exist positive constants κ_0 *and* $C > 0$ *such that if* $(\psi, \mathbf{A})_{\kappa,\sigma}$ *is a solution to* (10.14) *with* $\psi \neq 0$, *and*

$$\kappa > \kappa_0\,, \qquad \kappa/\sigma \leq \Lambda_{\max}\,,$$

and $P \in \overline{\Omega}$ *is such that* $|\psi(P)| = \|\psi\|_\infty$, *then*

$$\operatorname{dist}(P, \mathcal{A}_B \cup \mathcal{A}_I) \leq \frac{C}{\sqrt{\kappa\sigma}}\,.$$

11.6 Notes

1. The elliptic estimates are very useful and can be found in various publications (cf. [LuP3]–[LuP7], [HeP1], [Al2]–[Al4]) in the context of superconductivity.
2. In this chapter we have followed rather closely the review paper [FoH5] with a few improvements and extensions.
3. Theorem 11.3.3 is inspired by [Pa3, Lemma 3.3].
4. Proposition 11.4.1 is reminiscent of [LuP3, Proposition 2.5]. Our proof is based on an idea we learned from Almog.
5. Proposition 11.4.4 is a slightly improved version of [HeP1, Prop. 4.2] and [Pa2, Lemma 7.1].
6. In the case of Schrödinger operators without magnetic fields, the strong relation between the spectrum and the existence of generalized eigenfunctions is well known (Sch'nol's theorem, see [Sch'n] or in [CyFKS, Chapter 2]). Proposition 11.4.1 can be seen as a generalization of that result.
7. Formula (11.3) appears in [LuP4] with an additional boundary term. Here we show that this boundary term vanishes in the case of the magnetic Neumann condition.
8. Note 1 to Chapter 9 also relates to the present chapter. In particular, Proposition 11.5.3 is a mathematical proof of some of the phenomena discussed in [S-JST] and quoted in the note.
9. Clearly, one can obtain many more results in 3 dimensions—both the direct generalization of the 2D case and real 3D cases where the angle of the magnetic field with the boundary comes into play. Some results in this direction are given in [Pa3, Pa6].

12

Decay Estimates

12.1 Introduction

In this chapter, which will mainly concentrate on the 2D case, we will discuss the decay of ψ away from the boundary. Suppose for the purpose of this discussion that $\beta = 1$, i.e., that the magnetic field is constant. We have seen that—for the linear problem—eigenfunctions corresponding to low eigenvalues are concentrated near the boundary. As will be discussed below, this result carries over to solutions of the nonlinear Ginzburg–Landau equations. Proposition 11.4.5 can also be seen as such a result. There are various techniques to establish rigorous and precise versions of this statement, and the objective of this chapter is to discuss some of these approaches.

For purposes of understanding, the reader may consider the case of constant magnetic field only. However, we will allow general magnetic fields (with regularity C^α for some $\alpha > 0$), satisfying the assumption introduced in (8.26). For the facility of the reader, we recall here the assumption

$$b > \Theta_0 b', \tag{12.1}$$

with

$$b = \inf_{x\in\Omega} \beta(x)\,, \qquad b' = \inf_{x\in\partial\Omega} \beta(x)\,. \tag{12.2}$$

We study solutions to the two-dimensional Ginzburg–Landau equations (10.8) or sometimes their three-dimensional version (10.14).

In Section 12.2, we will use the magnetic Agmon technique from Chapter 8 combined with the elliptic estimates from Chapter 11 to obtain exponential decay estimates for the 2D problem and for magnetic field strengths satisfying, for some $\delta > 0$,

$$\frac{b\sigma}{\kappa} > 1 - \delta\,.$$

Since the proof of the elliptic estimates in Chapter 11 is somewhat involved, we also give another self-contained proof of the Agmon estimates in

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Section 12.5. This approach has the advantage of being directly applicable to domains with limited regularity. In this section, we also treat the 3D problem.

The rest of the chapter is mainly concerned with pushing the lower limit of validity of the decay estimates downward toward $H_{C_2}(\kappa)$, i.e., to

$$b\sigma \approx \kappa\,.$$

We concentrate for simplicity on the 2D situation. In Section 12.3, we give a new proof of (an improved version of) an estimate by Almog:

$$\int_\Omega |\psi(x)|^4\,dx \leq C/\kappa\,. \tag{12.3}$$

For large fields, this follows from the Agmon estimates, but the estimate is valid for all σ above κ/b (and slightly below—see Theorem 12.3.1 for the precise statement). We also recall the original proof in Section 12.6. That original proof is very interesting but very different from the techniques generally used in this book.

The advantage of (12.3) is the large range of validity. The disadvantage is the lack of decay rate. However, it can be used as input to an Agmon estimate with κ, σ in an extended parameter regime. That is Theorem 12.4.1, which is our best decay estimate.

We end this introduction by recalling some inequalities for solutions to (10.8) that will be used repeatedly all through the chapter. By Lemma 10.3.2 combined with Proposition 10.3.1, we find

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa\,\|\psi\|_2\,, \tag{12.4}$$

$$\|\psi\|_4^2 \leq \|\psi\|_2\,, \tag{12.5}$$

and

$$\sigma\|\,\mathrm{curl}\,\mathbf{A} - \beta\|_2 \leq C_0\,\|\psi\|_\infty\|\psi\|_2 \leq C_0\,\|\psi\|_2\,. \tag{12.6}$$

We also recall the notation used throughout the text

$$t(x) := \mathrm{dist}(x, \partial\Omega)\,.$$

12.2 Nonlinear Agmon Estimates

The important technique of Agmon estimates can also give exponential localization to the boundary for nonlinear problems. This is an adaptation of the technique for linear problems—as introduced in previous chapters and more specifically in Section 8.2.3—to the nonlinear Ginzburg–Landau equations. Notice that in the equation for ψ in (10.8), the nonlinearity has a specific sign. We can express that equation as

$$-\Delta^N_{\kappa\sigma\mathbf{A}}\psi + V(\psi)\psi = \kappa^2\psi\,, \tag{12.7}$$

where $V(\psi)$ is the potential

$$x \mapsto V(\psi)(x) := \kappa^2 |\psi(x)|^2 \geq 0\,. \tag{12.8}$$

The positivity of the potential allows us to discard this nonlinear term in the analysis and thus to argue exactly as for the linear case. Of course, when $|\psi|$ is not small—which is expected for $b\sigma$ near κ—it will be rather expensive to omit the nonlinear term, which explains why the Agmon estimates are not optimal in that region.

We consider first the case of general magnetic fields β and smooth domains Ω. Here we use the elliptic estimates from Chapter 11. Notice that in that chapter we used in particular the regularity of the curl-div system in Sobolev spaces $W^{k,p}$ (see Appendix D). That is, though standard, a rather heavy theorem. It therefore seems appropriate to also derive the Agmon estimates more directly using simpler arguments. We give such an argument in Section 12.3.

Theorem 12.2.1 (Agmon estimates)**.**
Suppose that $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with smooth boundary. Suppose that $\beta \in C^\infty(\overline{\Omega})$. Then, for all $c_0 \in]0, b[$ and all $\alpha < \sqrt{c_0}$, there exist $\kappa_0 > 0$ and $C > 0$ such that if

$$\sigma(b - c_0) > \kappa\,, \qquad \kappa \geq \kappa_0\,,$$

then

$$\int_\Omega e^{2\alpha\sqrt{\kappa\sigma}t(x)} \left\{ |\psi(x)|^2 + \frac{1}{\kappa\sigma} |p_{\kappa\sigma\mathbf{A}}\psi(x)|^2 \right\} dx \leq C \int_{\{\sqrt{\kappa\sigma}t(x) \leq 1\}} |\psi(x)|^2\, dx\,, \tag{12.9}$$

for all solutions $(\psi, \mathbf{A})_{\kappa,\sigma}$ to the Ginzburg–Landau equations (10.8).

Notice that Corollary 12.5.5 is an interesting consequence of these decay estimates.

Remark 12.2.2.
The reader may have noticed that we do not impose an assumption like (12.1) *in the above theorem. However, if such a condition is not imposed, one risks that the theorem is empty in the sense that the only solution to* (10.8) *with $\kappa < b\sigma$ is the trivial solution $(0, \mathbf{F})$. This will essentially follow from the analysis of the critical field $H_{C_3}(\kappa)$ in Chapter 13.*

Remark 12.2.3.
Coming back to the second equation of the Ginzburg–Landau system (10.8a), *we can deduce the following estimate in the two-dimensional case from* (12.9)*:*

$$\int_\Omega e^{2\alpha\sqrt{\kappa\sigma}t(x)} \, |\nabla(\operatorname{curl}\mathbf{A} - \beta)|\, dx \leq \frac{C}{\sqrt{\kappa\sigma}} \int_\Omega |\psi(x)|^2\, dx\,. \tag{12.10}$$

By integration from the boundary, we also get an estimate for $\operatorname{curl}\mathbf{A} - \beta$.

Proof of Theorem 12.2.1.
Let $\chi \in C^\infty(\Omega)$ have compact support in Ω. First using (1.32) and (10.9a), we obtain

$$\kappa\sigma \int_\Omega \operatorname{curl} \mathbf{A}\, |\chi\psi|^2\, dx \le \int_\Omega |p_{\kappa\sigma\mathbf{A}}(\chi\psi)|^2\, dx$$
$$= \int_\Omega |\nabla\chi|^2|\psi|^2 + \kappa^2|\chi\psi|^2(1-|\psi|^2)\, dx\,. \qquad (12.11)$$

We will now deduce from (12.11) the existence of $C > 0$ such that

$$\kappa\sigma\Big(b - \frac{C}{\sqrt{\kappa\sigma}}\Big) \int_\Omega |\chi\psi|^2\, dx \le \int_\Omega |\nabla\chi|^2|\psi|^2 + \kappa^2|\chi\psi|^2(1-|\psi|^2)\, dx\,. \quad (12.12)$$

This will result from the control of $\int_\Omega |\operatorname{curl}\mathbf{A} - \beta||\chi\psi|^2\, dx$. Here we simply use (11.51) in order to get

$$\int_\Omega |\operatorname{curl}\mathbf{A} - \beta||\chi\psi|^2\, dx \le \frac{C}{\sqrt{\kappa\sigma}}\|\chi\psi\|^2\,.$$

We will discard the negative term $-\kappa^2\int|\chi\psi|^2|\psi|^2$ in (12.12) and finally obtain

$$\kappa\sigma\Big(b - \frac{C}{\sqrt{\kappa\sigma}}\Big) \int_\Omega |\chi\psi|^2\, dx \le \int_\Omega \left(|\nabla\chi|^2|\psi|^2 + \kappa^2|\chi\psi|^2\right)\, dx\,. \qquad (12.13)$$

We make the following choice of function χ. Let $f \in C^\infty(\mathbb{R})$, with

$$f \equiv 1 \text{ on } [1,\infty[\,, \qquad f \equiv 0 \text{ on }]-\infty, \frac{1}{2}]\,, \qquad (12.14)$$

and define

$$\chi(x) := e^{\alpha\sqrt{\kappa\sigma}t(x)} f(\sqrt{\kappa\sigma}t(x))\,.$$

Note that, for any $\epsilon > 0$, we can find $C(\epsilon)$ s.t.

$$|\nabla\chi|^2 \le (1+\epsilon)\alpha^2\kappa\sigma\chi^2 + C(\epsilon)\kappa\sigma e^{2\alpha\sqrt{\kappa\sigma}t(x)}|f'(\sqrt{\kappa\sigma}t(x))|^2\,.$$

Therefore, (12.13) becomes, using the support properties of f and f',

$$\Big(b - \frac{\kappa}{\sigma} - C_1\frac{1}{\sqrt{\kappa\sigma}} - \alpha^2(1+\epsilon)\Big)\int_{\{\sqrt{\kappa\sigma}t(x)\ge\frac{1}{2}\}} e^{2\alpha\sqrt{\kappa\sigma}t(x)}|\psi(x)|^2\, dx$$
$$\le C'(\epsilon)\int_{\{\sqrt{\kappa\sigma}t(x)\le 1\}} |\psi(x)|^2\, dx\,. \qquad (12.15)$$

Clearly, (12.15) implies the bound on $\|e^{\alpha\sqrt{\kappa\sigma}t}\psi\|_2^2$ in (12.9), upon taking ϵ sufficiently small that

$$b - \frac{\kappa}{\sigma} - \alpha^2(1+\epsilon) > 0\,.$$

The estimate on $\|e^{\alpha\sqrt{\kappa\sigma}t}p_{\kappa\sigma}\psi\|_2^2$ follows by inserting the bound on $\|e^{\alpha\sqrt{\kappa\sigma}t}\psi\|_2^2$ in (12.13). □

12.3 Weak Decay Estimates

In this section, we will consider both the two- and three-dimensional cases. If $d = 3$, we will suppose that the external magnetic field β is constant of unit length. After possibly performing a fixed rotation of the coordinates, we may assume that $\beta = (0,0,1)$. In the two-dimensional case, we work with general external fields as before.

In this section, we establish "weak" decay estimates. In the proof of these estimates, we do not use Proposition 11.4.4. Thus, the estimates are valid under very general assumptions. The estimates are weak in the sense that no decay rate is given. However, they have the strong advantages of being valid

1. for domains with limited regularity,
2. even when $\sigma \approx \kappa$, i.e., all the way down to the critical field called $H_{C_2}(\kappa)$.

The results in this section will later be used to give a second proof of Agmon-type estimates.

We start with a calculation similar to (12.13) but without using (11.51). Let $f \in C^\infty(\mathbb{R})$ be a standard nondecreasing cutoff function, as in (12.14).

Let $\lambda > 0$ and define $\chi_\lambda : \Omega \to \mathbb{R}$ by

$$\chi_\lambda(x) := f(t(x)/\lambda)\,.$$

Then χ_λ is a Lipschitz function and $\operatorname{supp} \chi_\lambda \subset \Omega$.

Arguing as in (12.11)–(12.13) but using a simple Cauchy–Schwarz inequality instead of invoking (11.51), we find

$$\begin{aligned}\kappa\sigma b\|\chi_\lambda\psi\|_2^2 &- \kappa\sigma\|\operatorname{curl}\mathbf{A} - \beta\|_2\|\chi_\lambda\psi\|_4^2 \\ &\leq \int_\Omega |\nabla\chi_\lambda|^2|\psi|^2 + \kappa^2|\chi_\lambda\psi|^2(1-|\psi|^2)\,dx\,. \end{aligned} \tag{12.16}$$

Here we used for the 3D situation that the magnetic field is constant (actually constant direction would suffice), in order to avoid an additional error term in the first inequality in (12.11).

Using (12.6)—notice that (12.6) remains valid in 3D by (10.37)—we get from (12.16)

$$\begin{aligned}&\kappa(b\sigma - \kappa)\|\chi_\lambda\psi\|_2^2 \\ &\quad\leq C_0\kappa\|\psi\|_2\|\chi_\lambda\psi\|_4^2 - \kappa^2\int_\Omega \chi_\lambda^2|\psi|^4\,dx + \|f'\|_\infty^2\lambda^{-2}\int_{\{t(x)\leq\lambda\}}|\psi(x)|^2\,dx \\ &\quad\leq \frac{C_0^2}{2}\|\psi\|_2^2 - \frac{\kappa^2}{2}\int_\Omega \chi_\lambda^2|\psi|^4\,dx \\ &\qquad\quad + \|f'\|_\infty^2\lambda^{-2}\int_{\{t(x)\leq\lambda\}}|\psi(x)|^2\,dx + \frac{\kappa^2}{2}\int_\Omega(\chi_\lambda^4 - \chi_\lambda^2)|\psi|^4\,dx\,.\end{aligned}$$

Notice that since $\chi_\lambda \leq 1$, the last integral is negative and we thus find by dividing the integral $\|\psi\|_2^2$ in two:

$$\left\{\left(\frac{b\sigma}{\kappa}-1\right)-\frac{C_0^2}{2\kappa^2}\right\}\|\chi_\lambda\psi\|_2^2+\frac{1}{2}\int_\Omega|\psi|^4\,dx \leq \frac{C_0^2}{2\kappa^2}\int_\Omega(1-\chi_\lambda^2)|\psi|^2\,dx$$
$$+\frac{\|f'\|_\infty^2}{\kappa^2\lambda^2}\int_{\{t(x)\leq\lambda\}}|\psi(x)|^2\,dx+\frac{1}{2}\int_{\{t(x)\leq\lambda\}}|\psi(x)|^4\,dx\,. \tag{12.17}$$

Thus,

$$\left\{\left(\frac{b\sigma}{\kappa}-1\right)-\frac{C_0^2}{2\kappa^2}\right\}\|\chi_\lambda\psi\|_2^2+\frac{1}{2}\int_\Omega|\psi|^4\,dx$$
$$\leq\left(\frac{\|f'\|_\infty^2}{\kappa^2\lambda^2}+\frac{C_0^2}{2\kappa^2}+\frac{1}{2}\right)\int_{\{t(x)\leq\lambda\}}|\psi(x)|^2\,dx\,. \tag{12.18}$$

Here C_0 remains the constant from (12.6). This is the basic inequality.

Theorem 12.3.1 (L^4-estimate)**.**
Let $\Omega\subset\mathbb{R}^d$, with $d=2$ or 3, be a bounded, smooth, or polygonal domain. If $d=2$, let β be a continuous magnetic field with $b>0$. If $d=3$, we suppose that $\beta\in\mathbb{S}^2$ is constant. Then there exists $\kappa_0>0$ and for all $C>0$, there exists a $C'>0$ such that if $(\psi,\mathbf{A})_{\kappa,\sigma}$ is a solution to (10.8) with $\kappa>\kappa_0$ and

$$\frac{b\sigma}{\kappa}\geq 1-C\kappa^{-1/2}\,, \tag{12.19}$$

then

$$\|\psi\|_4^4\leq C'\begin{cases}\kappa^{-1} & \text{for } b\sigma<\kappa\,,\\ \|\psi\|_\infty^2\,\kappa^{-1} & \text{for } b\sigma\geq\kappa\,.\end{cases} \tag{12.20}$$

Using the Hölder inequality, we obtain an L^2 bound:

Corollary 12.3.2.
Under the assumptions of Theorem 12.3.1, we have

$$\|\psi\|_2\leq c\kappa^{-1/4}\,. \tag{12.21}$$

Proof of Theorem 12.3.1.
We take $\lambda=\kappa^{-1}$ in (12.18). Since there exists a constant $\widehat{C}_1>0$ (depending only on Ω) such that

$$\text{meas}\{x\ :\ t(x)\leq\lambda\}\leq\widehat{C}_1\lambda\,,$$

for all $\lambda\in]0,2]$, the right side of (12.18) clearly satisfies the correct bound. Assumption (12.19) implies that

$$\left\{\left(\frac{b\sigma}{\kappa}-1\right)-\frac{\delta}{2\kappa^2}\right\}\|\chi_\lambda\psi\|_2^2\geq-\frac{C''}{\sqrt{\kappa}}\|\psi\|_2^2\geq-\frac{C''\sqrt{|\Omega|}}{\sqrt{\kappa}}\|\psi\|_4^2\,.$$

If $b\sigma\geq\kappa$, we can discard the positive term $(\frac{b\sigma}{\kappa}-1)\|\chi_\lambda\psi\|_2^2$ on the left side in (12.18). □

12.4 Nonlinear Agmon Estimates II

Using Theorem 12.3.1, we can improve the Agmon estimates. In the proof we use (11.50), and so Theorem 12.4.1 is dependent on Chapter 11. Hence, the domain Ω has to satisfy the strong regularity assumptions of that chapter; in particular, Theorem 12.4.1 does not apply to domains with corners.

Theorem 12.4.1.
Suppose that $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with smooth boundary. Suppose that $\beta \in C^\infty\left(\overline{\Omega}\right)$ and let $\delta > 0$. Then there exist $\alpha, C > 0$ and $\kappa_0 > 0$ such that if

$$\sigma b > \kappa + \kappa^{-\frac{1}{4}+\delta}, \qquad \kappa \geq \kappa_0,$$

then

$$\int_\Omega e^{2\alpha\sqrt{\kappa(b\sigma-\kappa)}t(x)} \left\{|\psi(x)|^2 + \frac{1}{\kappa\sigma}|p_{\kappa\sigma\mathbf{A}}\psi(x)|^2\right\} dx \leq C \int_{\{\sqrt{\kappa(b\sigma-\kappa)}t(x)\leq 1\}} |\psi(x)|^2\, dx, \tag{12.22}$$

for all solutions $(\psi, \mathbf{A})_{\kappa,\sigma}$ to the Ginzburg–Landau equations (10.8).

Proof.
We follow the proof of Theorem 12.2.1, but we will estimate $|\operatorname{curl}\mathbf{A} - \beta|$ in a different manner. Choose $\eta > 0$ sufficiently small that

$$\frac{5+2\eta}{4+2\eta} > \frac{5}{4} - \frac{\delta}{2}. \tag{12.23}$$

By the Sobolev inequality and (10.27), we have

$$\begin{aligned}\|\operatorname{curl}\mathbf{A} - \beta\|_\infty &\leq C\|\operatorname{curl}\mathbf{A} - \beta\|_{W^{1,2+\eta}} \leq \frac{C'}{\kappa\sigma}\|\psi\|_\infty\, \|p_{\kappa\sigma\mathbf{A}}\psi\|_{2+\eta} \\ &\leq \frac{C'}{\kappa\sigma}\|\psi\|_\infty\, \|p_{\kappa\sigma\mathbf{A}}\psi\|_2^{2/(2+\eta)} \|p_{\kappa\sigma\mathbf{A}}\psi\|_\infty^{\eta/(2+\eta)}. \end{aligned}\tag{12.24}$$

The norm $\|p_{\kappa\sigma\mathbf{A}}\psi\|_\infty$ is controlled by (11.50) and the norm $\|p_{\kappa\sigma\mathbf{A}}\psi\|_2$ by (12.4), so we get (with new constants C and C')

$$\begin{aligned}\|\operatorname{curl}\mathbf{A} - \beta\|_\infty &\leq \frac{C}{\kappa\sigma}(\kappa\|\psi\|_2)^{2/(2+\eta)}(\kappa\sigma)^{\eta/(4+2\eta)} \\ &\leq C'\kappa^{-(5+2\eta)/(4+2\eta)}, \end{aligned}\tag{12.25}$$

where we used (12.21) to get the last estimate. Therefore, by the choice of η, (12.13) becomes

$$\kappa\sigma(b - C'\kappa^{-\frac{5}{4}+\frac{\delta}{2}}) \int_\Omega |\chi\psi|^2\, dx \leq \int_\Omega \left(|\nabla\chi|^2|\psi|^2 + \kappa^2|\chi\psi|^2\right) dx. \tag{12.26}$$

We now modify the definition of the function χ to be

$$\chi(x) = e^{\alpha\sqrt{\kappa(b\sigma-\kappa)}t(x)} f(\sqrt{\kappa(b\sigma-\kappa)}t(x)). \tag{12.27}$$

The rest of the proof is identical to that of Theorem 12.2.1 and will be omitted. □

12.5 Nonlinear Agmon Estimates III

In this section, we will use the weak decay estimates to obtain Agmon-type exponential decay estimates for solutions to the Ginzburg–Landau equations. We first need the following energy estimate for functions located away from the boundary.

Lemma 12.5.1.
Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be a bounded, smooth, or polygonal domain and let β be a continuous magnetic field with $b > 0$. If $d = 3$, we assume that $\beta \in \mathbb{S}^2$ is constant. There exists a universal constant $C(\Omega)$ such that if $(\psi, \mathbf{A})$ is a solution to (10.8) *or* (10.14)*, then, for all $\phi \in C_0^\infty(\Omega)$, we have*

$$\|(-i\nabla + \kappa\sigma\mathbf{A})\phi\|_2^2 \geq \kappa b\sigma\Big(1 - C(\Omega)\sqrt{\frac{\kappa}{b\sigma}}\,\|\psi\|_2\Big)\|\phi\|_2^2\,. \tag{12.28}$$

Proof.
In the 2D case, we proceed as follows. We estimate, for $\phi \in C_0^\infty(\Omega)$, using Lemma 1.4.1,

$$\begin{aligned}\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 &\geq \kappa\sigma \int_\Omega (\operatorname{curl}\mathbf{A})\,|\phi|^2\,dx \\ &\geq \kappa\sigma b\|\phi\|_2^2 - \kappa\sigma\|\operatorname{curl}\mathbf{A} - \beta\|_2\|\phi\|_4^2\,. \end{aligned}\tag{12.29}$$

By the Sobolev inequality, for $\phi \in C_0^\infty(\mathbb{R}^2)$, and scaling, we get the existence of a a universal constant C_{Sob} such that, for all $\eta > 0$,

$$\|\phi\|_4^2 \leq C_{\text{Sob}}\Big(\eta\big\|\nabla|\phi|\big\|_2^2 + \eta^{-1}\|\phi\|_2^2\Big) \leq C_{\text{Sob}}\Big(\eta\big\|p_{\kappa\sigma\mathbf{A}}\phi\big\|_2^2 + \eta^{-1}\|\phi\|_2^2\Big)\,, \tag{12.30}$$

where we used the diamagnetic inequality to get the second estimate. Combining (12.29), (12.6), and (12.30) and choosing $\eta = \sqrt{1/\kappa\sigma b}$, we find:

$$\Big(1 + C_{\text{Sob}}\sqrt{\frac{\kappa}{b\sigma}}\,\|\psi\|_2\Big)\,\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \geq \kappa\sigma b\,\Big(1 - C_{\text{Sob}}\sqrt{\frac{\kappa}{b\sigma}}\,\|\psi\|_2\Big)\,\|\phi\|_2^2\,. \tag{12.31}$$

Thus (12.28) follows from (12.31).

In the 3D case, the scaling behavior is different so we choose different exponents. First of all, in (12.29) we use the Hölder inequality instead of the Cauchy–Schwarz inequality to get

$$\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \geq \kappa\sigma b\|\phi\|_2^2 - \kappa\sigma\|\operatorname{curl}\mathbf{A} - \beta\|_3\|\phi\|_3^2\,. \tag{12.32}$$

Now the 3-norm scales correctly in 3D, and we get

$$\|\phi\|_3^2 \leq C_{\mathrm{Sob}}\Big(\eta\big\|p_{\kappa\sigma\mathbf{A}}\phi\big\|_2^2 + \eta^{-1}\|\phi\|_2^2\Big). \tag{12.33}$$

We can combine (10.35) and (10.36) of Lemma 10.3.3 to get

$$\|\operatorname{curl}\mathbf{A} - \beta\|_{L^3(\Omega)} \leq C\|\operatorname{curl}\mathbf{A} - \beta\|_{W^{1,2}(\Omega)} \leq \widehat{C}\sigma^{-1}\|\psi\|_2\,. \tag{12.34}$$

Combining these estimates, we get (12.31) as in the 2D case.

This finishes the proof of Lemma 12.5.1. □

Remark 12.5.2.
It is clear from the proof that the estimate of Lemma 12.5.1 is not optimal. In particular, we have made a simple and convenient choice of exponents in the Hölder inequalities instead of striving for optimality.

In particular, using the estimate on $\|\psi\|_2$ from Corollary 12.3.2, we find

Lemma 12.5.3.
Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be a bounded, smooth, or polygonal domain and let β be a continuous magnetic field with $b > 0$. If $d = 3$, we assume that $\beta \in \mathbb{S}^2$ is constant. There exists a constant C' such that if $b\sigma > \kappa$ and $(\psi, \mathbf{A})$ is a solution to (10.8) *or* (10.14)*, then for all $\phi \in C_0^\infty(\Omega)$, we have*

$$\|(-i\nabla + \kappa\sigma\mathbf{A})\phi\|_2^2 \geq \kappa b\sigma\left(1 - \frac{C'}{\sqrt[4]{\kappa}}\right)\|\phi\|_2^2\,. \tag{12.35}$$

By standard arguments, Lemma 12.5.3 implies Agmon estimates in the interior, i.e., the conclusion of Theorem 12.2.1. We restate the theorem including also the 3D case.

Theorem 12.5.4.
Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be a bounded, smooth or polygonal domain and let β be a continuous magnetic field with $b > 0$. If $d = 3$, we assume that $\beta \in \mathbb{S}^2$ is constant. Then, for all $c_0 \in \,]0, b[$ and all $\alpha < \sqrt{c_0}$, there exist $\kappa_0 > 0$ and $C > 0$ such that if

$$\sigma(b - c_0) > \kappa\,, \qquad \kappa \geq \kappa_0\,,$$

then the Agmon estimate (12.9) *holds for all solutions $(\psi, \mathbf{A})_{\kappa,\sigma}$ to the Ginzburg–Landau equations* (10.8) *or* (10.14).

Proof.
We proceed as in the proof of Theorem 12.2.1 but using Lemma 12.5.3. Applying that lemma instead of (11.51), the left-hand side of (12.13) is replaced by

$$\kappa\sigma b\left(1 - \frac{C_1'}{\sqrt[4]{\kappa}}\right)\|\chi\psi\|_2^2\,.$$

Following the proof, we see that the only difference is that the lower-order factor $C_1/\sqrt{\kappa\sigma}$ in (12.15) is replaced by $C_1'/\sqrt[4]{\kappa}$, which is of still lower order. □

Corollary 12.5.5.
Let the assumptions of Theorem 12.5.4 be satisfied. Then, for any $p \geq 2$, *there exists a constant* $C_p > 0$ *such that*

$$\|\psi\|_2 \leq C_p(\kappa\sigma)^{-\frac{p-2}{4p}}\|\psi\|_p\,. \tag{12.36}$$

In particular, we have

$$\|\psi\|_2 \leq C_\infty(\kappa\sigma)^{-\frac{1}{4}}\|\psi\|_\infty \leq C_\infty(\kappa\sigma)^{-\frac{1}{4}}\,. \tag{12.37}$$

Proof.
Applying the Hölder inequality to (12.9), we get

$$\|\psi\|_2 \leq C\left\{\int_{\{\sqrt{\kappa\sigma}t(x)\leq 1\}} dx\right\}^{\frac{1}{q}}\|\psi\|_p\,,$$

with $\frac{1}{q}+\frac{1}{p}=\frac{1}{2}$. Estimating

$$\int_{\{\sqrt{\kappa\sigma}t(x)\leq 1\}} dx \leq \frac{C'}{\sqrt{\kappa\sigma}}$$

yields the result. □

12.6 Almog's L^4 Bound

We will now prove a generalization of Almog's estimate (12.3), which in the original version only considered the case of constant magnetic field β . Of course, this estimate is essentially the same as Theorem 12.3.1, but the method of proof is very different. An important ingredient is the implementation of the elliptic estimates from Proposition 11.4.4.

Theorem 12.6.1.
Suppose that $\Omega \subset \mathbb{R}^2$ *is smooth and bounded, let* $\beta \in C^\infty(\overline{\Omega})$ *with* $b > 0$, *and let* $\Lambda > b$ *be given. Then there exist positive constants* κ_0 *and* C *such that if* $(\psi, \mathbf{A})$ *is a solution to* (10.8) *and*

$$\kappa \geq \kappa_0\,, \qquad \kappa \leq b\sigma \leq \Lambda\kappa\,,$$

then

$$\int_\Omega |\psi(x)|^4\,dx \leq \frac{C}{\kappa}\|\psi\|_\infty^2 \leq \frac{C}{\kappa}\,. \tag{12.38}$$

Remark 12.6.2.
We can actually extend the argument to $b\sigma \geq \kappa - \mathcal{O}(\sqrt{\kappa})$, thereby obtaining the same range of validity as for Theorem 12.3.1.

Proof.
We will, as usual, work in the gauge where $\mathbf{A} \in H^1_{\rm div}(\Omega)$ is defined in (10.5). Define

$$h := -\operatorname{curl}\mathbf{A} + \beta \qquad \text{and} \qquad u := \kappa\sigma h + \frac{1}{2}|\psi|^2\,.$$

An explicit, though tedious, calculation using (10.8) yields an equation for u. One can verify that [see (10.23), with a different "u"]

$$\kappa\sigma\Delta h = \kappa\sigma|\psi|^2 \operatorname{curl}\mathbf{A} - 2\Re\big[\overline{\partial_{x_1}\psi}(i\partial_{x_2}\psi)\big] - 2\kappa\sigma\Re\big[\overline{\psi}(A_1\partial_{x_2}\psi - A_2\partial_{x_1}\psi)\big]\,, \tag{12.39}$$

and

$$\frac{1}{2}\Delta|\psi|^2 = -\kappa^2(1-|\psi|^2)|\psi|^2 + |\nabla\psi|^2 + \kappa^2\sigma^2\mathbf{A}^2|\psi|^2 + 2\kappa\sigma\mathbf{A}\cdot\Im(\overline{\psi}\nabla\psi)\,. \tag{12.40}$$

Therefore, with

$$\hat{J} := (\partial_{x_1}\psi - i\partial_{x_2}\psi) + i\kappa\sigma(A_1 - iA_2)\psi\,,$$

we find

$$\Delta u = |\psi|^2(\kappa\sigma\operatorname{curl}\mathbf{A} - \kappa^2) + \kappa^2|\psi|^4 + |\hat{J}|^2\,. \tag{12.41}$$

Integrating (12.41) over Ω yields

$$\begin{aligned}
\kappa^2\int_\Omega |\psi(x)|^4\,dx + \int_\Omega |\psi(x)|^2(\kappa\sigma\operatorname{curl}\mathbf{A} - \kappa^2)\,dx &\leq \int_\Omega \Delta u(x)\,dx \\
= \int_{\partial\Omega} \nu\cdot\nabla u\,d\sigma(x) = -\kappa\sigma\int_{\partial\Omega} \nu\cdot\nabla(\operatorname{curl}\mathbf{A} - \beta)\,d\sigma(x) & \\
\leq C\sqrt{\kappa\sigma}\|\psi\|^2_\infty\,, &
\end{aligned} \tag{12.42}$$

where we have used that ψ satisfies the Neumann condition (remember that $\mathbf{A}\cdot\nu = 0$ on the boundary) and the last inequality follows from the elliptic estimate (11.51).

Now using the property that the parameters satisfy $b\sigma \geq \kappa$ combined with (11.51) gives

$$|\psi|^2\,(\sigma\ \operatorname{curl}\mathbf{A} - \kappa) \geq -C|\psi|^2\,.$$

Therefore, (12.42) combined with Proposition 10.3.1 and the boundedness of Ω implies (12.38). □

12.7 Power Law Decay Above $H_{C_2}(\kappa)$

Theorem 12.7.1.
Let $\Omega \subset \mathbb{R}^2$ be bounded and have smooth boundary. Let $\beta \in C^\infty(\overline{\Omega})$ satisfy $b > 0$. Then there exist positive constants C_0 and C_1 such that if

$$b\sigma \geq \kappa + C_0\,,$$

and $(\psi, \mathbf{A})_{\kappa,\sigma}$ is a solution of (10.8), *then*

$$\int_\Omega (\kappa t(x))^4 |\psi(x)|^4 + (\kappa t(x))^2 \Big\{ |\psi(x)|^2 + \kappa^{-2} |p_{\kappa\sigma\mathbf{A}}\psi|^2 \Big\}\, dx \leq C_1\,. \tag{12.43}$$

Proof of Theorem 12.7.1.
We start by proving the L^4 bound inherent in (12.43). Upon multiplying (10.8a) by $(\kappa t)^4 \overline{\psi}$ and integrating over Ω, we get, after an integration by parts,

$$\begin{aligned}\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\kappa^2 t^2 \psi)|^2\, dx - \int_\Omega \left|\nabla(\kappa^2 t^2)\right|^2 |\psi|^2\, dx \\ = \kappa^2 \int_\Omega (\kappa t)^4 |\psi|^2\, dx - \kappa^2 \int_\Omega (\kappa t)^4 |\psi|^4\, dx\,.\end{aligned} \tag{12.44}$$

Now, since $t^2\psi$ vanishes on $\partial\Omega$, we can estimate, using (11.51),

$$\begin{aligned}\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\kappa^2 t^2 \psi)|^2\, dx &\geq \kappa\sigma \int_\Omega (\operatorname{curl}\mathbf{A})\, |\kappa^2 t^2 \psi|^2\, dx \\ &\geq \kappa\sigma(b - C/\sqrt{\kappa\sigma}) \|\kappa^2 t^2 \psi\|_2^2\,.\end{aligned} \tag{12.45}$$

If the constant C_0 in Theorem 12.7.1 is sufficiently large, we have

$$\kappa\sigma(b - C/\sqrt{\kappa\sigma}) \geq \kappa^2$$

and therefore obtain the estimate

$$\kappa^2 \int_\Omega (\kappa t)^4 |\psi|^4\, dx \leq \int_\Omega \left|\nabla(\kappa^2 t^2)\right|^2 |\psi|^2\, dx \leq 4\kappa^2 \int_\Omega (\kappa t)^2 |\psi|^2\, dx\,. \tag{12.46}$$

We proceed by applying the Hölder inequality and find

$$\|\kappa t \psi\|_4^2 \leq 4\sqrt{|\Omega|}\,. \tag{12.47}$$

This is the desired L^4-estimate.

The L^2-estimate in (12.43) follows from the L^4-estimate and Hölder's inequality.

Upon multiplying (10.8a) by $(\kappa t)^2 \overline{\psi}$ instead of $(\kappa t)^4 \overline{\psi}$, we get the following modification of (12.44):

$$\begin{aligned}\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\kappa t \psi)|^2\, dx + \kappa^2 \int_\Omega (\kappa t)^2 |\psi|^4\, dx \\ = \int_\Omega \left|\nabla(\kappa t)\right|^2 |\psi|^2\, dx + \kappa^2 \int_\Omega (\kappa t)^2 |\psi|^2\, dx\,.\end{aligned} \tag{12.48}$$

Using the L^2 bound deduced from the L^4 bound from (12.47) and the L^∞ bound of ψ, we can estimate the two terms on the r.h.s. of (12.48) and arrive (after division by κ^2) at

$$\int_\Omega |p_{\kappa\sigma\mathbf{A}}(t\psi)|^2\,dx \leq C\,. \tag{12.49}$$

The desired estimate,

$$\int_\Omega t^2|p_{\kappa\sigma\mathbf{A}}\psi|^2\,dx \leq C\,,$$

follows from (12.49) and the L^∞ bound of ψ. □

As a corollary to the proof above, we get the following estimate.

Corollary 12.7.2.
Let $\Omega \subset \mathbb{R}^2$ be bounded, simply connected and have smooth boundary, and let $\beta \in C^\infty(\overline{\Omega})$ satisfy $b > 0$. Then there exist positive constants C_0 and C_1 such that if

$$b\sigma \geq \kappa + C_0\,,$$

then, for all $p \in [2, \infty[$,

$$\int_\Omega |\psi(x)|^p\,dx \leq \frac{C_1}{\kappa}\,. \tag{12.50}$$

Proof of Corollary 12.7.2.
By Proposition 10.3.1, it suffices to consider the case $p = 2$. Using (12.48) and a spectral estimate similar to (12.45), we find

$$\int_\Omega (\kappa t(x))^2|\psi(x)|^4\,dx \leq C\|\psi\|_2^2\,. \tag{12.51}$$

We now calculate/estimate as follows:

$$\begin{aligned}\|\psi\|_2^2 &= \int_{\{t\leq\kappa^{-1}\}} |\psi(x)|^2\,dx + \int_{\{t\geq\kappa^{-1}\}} (\kappa t(x)|\psi(x)|^2)(\kappa t(x))^{-1}\,dx \\ &\leq C\kappa^{-1} + \sqrt{\int_\Omega \kappa^2 t(x)^2|\psi(x)|^4\,dx} \times \sqrt{\int_{\{t\geq\kappa^{-1}\}} (\kappa t(x))^{-2}\,dx}\,.\end{aligned} \tag{12.52}$$

Combining (12.51) with the inequality $\int_{\{t\geq\kappa^{-1}\}}(\kappa t(x))^{-2}\,dx \leq C\,\kappa^{-1}$ (12.52) implies that

$$\|\psi\|_2^2 \leq C'\kappa^{-1} + C'\|\psi\|_2\kappa^{-1/2}\,,$$

or equivalently,

$$\|\psi\|_2^2 \leq C''\kappa^{-1}\,.$$

□

Clearly, Almog's L^4-estimate (12.3), which can be extended to $p \geq 4$, is contained in Corollary 12.7.2 but for a slightly reduced parameter regime. We also see from Corollary 12.7.2 that (12.21) is not optimal.

12.8 Notes

1. We refer to (and follow) [HeM3, HeP1, FoH3] for implementations of the Agmon estimates in the present context.
2. The "weak decay estimates" obtained in Theorem 12.3.1 first appeared in [BonF] in a somewhat weaker L^2 version. More precisely, it is shown that there exists a $C > 0$ such that if $(\psi, \mathbf{A})_{\kappa,\sigma}$ is a solution to (10.8) with

$$\kappa(b\sigma - \kappa) \geq 1\,,$$

then

$$\|\psi\|_2^2 \leq C \int_{\{\sqrt{\kappa(b\sigma-\kappa)}\, t(x) \leq 1\}} |\psi(x)|^2\, dx \leq \frac{C'}{\sqrt{\kappa(b\sigma - \kappa)}}\,. \tag{12.53}$$

3. In this chapter, we restrict in the 3D case to constant magnetic field β. That restriction permits us to directly use the lower bound $P_{\mathbf{F}}^D \geq |\beta|$ [by (1.32)]. Similar results can, however, be obtained for more general magnetic fields but with a bit more work, since for a variable magnetic field one can obtain a generalization of the above lower bound but with an error term. For details and applications, see [HeM5].
4. The Agmon estimates give L^2-exponential decay of ψ and its derivative when $b\sigma/\kappa \geq 1 + \epsilon$. By Sobolev inequalities and using the Ginzburg–Landau equation, this implies that ψ is exponentially small in the L^∞ sense away from a narrow boundary region.
   When $b\sigma \leq \kappa$, the L^∞ norm is no longer small in the interior of Ω. This follows from L^4-estimates by [SaS2]. These have been strengthened to L^∞-estimates in a recent work (see [FoH7], in answer to a conjecture of Aftalion–Serfaty [AfS]—appearing in a less formalized way in [SaS2]). So we have, for any $\delta > 0$, a constant C_δ such that

$$|\psi(x)| \leq C_\delta \left(1 - \frac{\sigma b}{\kappa}\right)$$

   when $\operatorname{dist}(x, \partial\Omega) \geq \delta$.
   Actually, one can even get estimates away from a κ-dependent neighborhood of the boundary; see [SaS2, FoH7] for the precise statement (see also [FoK2] for more precise results in the region where $b\sigma \approx \kappa$).
5. The Agmon estimate of Theorem 12.4.1 was inspired by a suggestion of the referee of the present book.
6. The bound in Theorem 12.7.1 is inspired by the somewhat overlooked result [Pa2, Theorem 4.1].

13

On the Third Critical Field H_{C_3}

Using the spectral asymptotics of the Neumann Laplacian with magnetic field, we give precise estimates on the critical field H_{C_3}, describing the onset of superconductivity in type II superconductors. Furthermore, we prove that the definitions of this field corresponding to local minimizers and global minimizers coincide.

13.1 Critical Fields and Spectral Theory

13.1.1 Critical fields

It follows from Theorem 10.4.1 that, for fixed κ and sufficiently large σ, the only minimizer (or—more generally—stationary point) of $\mathcal{G}_{\kappa,\sigma}$ is the normal state $(0, \mathbf{F})$. This corresponds to the observation in physics that a large magnetic field breaks the superconductivity of a given material. An important question in the literature has been to define and calculate the critical field where this transition takes place.

The first definition, which was proposed by Lu–Pan is

$$H_{C_3}(\kappa) := \inf\{\sigma > 0 \,:\, (0, \mathbf{F}) \text{ is the unique minimizer of } \mathcal{G}_{\kappa,\sigma}\}\,.$$

At this point one should be careful because a priori one cannot be sure that the transition from ψ nonvanishing to $\psi \equiv 0$ takes place at a unique value of σ—there could be a region of transitions back and forth. Thus, one should define upper and lower critical fields by

$$\underline{H}_{C_3}(\kappa) = \inf\{\sigma > 0 : (0, \mathbf{F}) \text{ is a minimizer of } \mathcal{G}_{\kappa,\sigma}\}\,, \tag{13.1}$$

$$\overline{H}_{C_3}(\kappa) = \inf\{\sigma > 0 \,:\, (0, \mathbf{F}) \text{ is the unique minimizer of } \mathcal{G}_{\kappa,\sigma'} \text{ for all } \sigma' > \sigma\}\,. \tag{13.2}$$

Recall that we have fixed the choice of gauge by our choice of variational space. If one releases this constraint, we would have to replace “unique” by “unique

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up to change of gauge" in the preceding definition. Note that in (13.1) the word "unique" is not present but appears in the definition of $\overline{H}_{C_3}(\kappa)$. This is because we want $\underline{H}_{C_3}(\kappa)$ to be as small as possible and $\overline{H}_{C_3}(\kappa)$ to be as large as possible. By Theorem 10.4.1, we have

$$\overline{H}_{C_3}(\kappa) \leq C \max(\kappa, 1)\,,$$

and clearly

$$\underline{H}_{C_3}(\kappa) \leq H_{C_3}(\kappa) \leq \overline{H}_{C_3}(\kappa)\,. \tag{13.3}$$

The aim of this chapter is to prove that these different definitions coincide for large κ and that actually they also coincide with similarly defined local fields. The local fields are easier to calculate, and we will therefore be able to give good asymptotic expansions of $H_{C_3}(\kappa)$ once we have established that the local and global definitions coincide.

The main point is to investigate the strong connections between the critical field $H_{C_3}(\kappa)$ and the smallest magnetic Neumann eigenvalue $\lambda_1^N(\kappa\sigma\mathbf{A})$. One first observes the following elementary lemma:

Lemma 13.1.1.

- *If* $\lambda_1^N(\kappa\sigma\mathbf{F}) < \kappa^2$, *then* $\mathcal{G}_{\kappa,\sigma}$ *has a nontrivial minimizer, with energy strictly less than the energy of the normal solution.*
- *If* $\mathcal{G}_{\kappa,\sigma}$ *has a nontrivial minimizer* $(\psi, \mathbf{A})$, *then* $\lambda_1^N(\kappa\sigma\mathbf{A}) < \kappa^2$.

Proof.
Notice that the normal state $(0, \mathbf{F})$ has energy

$$\mathcal{G}_{\kappa,\sigma}(0, \mathbf{F}) = 0\,. \tag{13.4}$$

For the first statement, it is easy to see that if u_1 is a normalized eigenfunction associated with $\lambda_1^N(\kappa\sigma\mathbf{F})$ and if we consider the couple $(\mu u_1, \mathbf{F})$, then we get a negative energy for $|\mu| \neq 0$, small enough. We indeed have

$$\mathcal{G}_{\kappa,\sigma}(\mu u_1, \mathbf{F}) = |\mu|^2(\lambda_1^N(\kappa\sigma\mathbf{F}) - \kappa^2) + \frac{\kappa^2|\mu|^4}{2}\|u_1\|_4^4\,.$$

For the second statement, we observe that if the minimizer satisfies $\psi \neq 0$, then

$$0 \geq \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) > \|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2\,,$$

which implies, using the variational characterization of the ground state energy, that $\lambda_1^N(\kappa\sigma\mathbf{A}) < \kappa^2$. □

The previous proof also gives an upper bound to the infimum of the Ginzburg–Landau functional $(\psi, \mathbf{A}) \mapsto \mathcal{G}(\psi, \mathbf{A})$. Optimizing with respect to μ in the proof of the previous lemma indeed gives

$$\inf_{\psi,\mathbf{A}} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A}) \leq -\frac{(\lambda_1^N(\kappa\sigma\mathbf{F}) - \kappa^2)^2}{2\kappa^2 \int |u_1(x)|^4\, dx}\,. \tag{13.5}$$

We will see in Section 13.5 that this upper bound is rather optimal when $\lambda_1^N(\kappa\sigma\mathbf{F})$ is a simple eigenvalue, sufficiently far from the second one.

Remark 13.1.2.
Using the first Ginzburg–Landau equation, a minimizer $(\psi, \mathbf{A})$ *satisfies* (12.7) *and* (12.8)*. If one shows by a priori estimates that* $\mathbf{A}$ *is near* $\mathbf{F}$ *and that* ψ *is small in* L^∞ *in the asymptotic regime considered here (such properties were reviewed in Chapter 11), it is not too surprising to think that the analysis presented in the first part of the book of the behavior of the ground state energy of* $p^2_{B\mathbf{F}}$ *as* $B \to \infty$ *will still be valid for the order parameter* ψ *corresponding to the minimizer* $(\psi, \mathbf{A})$ *.*

The discussion around Lemma 13.1.1 naturally leads to the following questions:

- Does the equation in σ ,
$$\lambda_1(\kappa\sigma) = \kappa^2 ,$$
have a unique solution (for κ large enough)?
Here and in the rest of this section, we will use the notation
$$\lambda_1(B) = \lambda_1^N(B\mathbf{F}, \Omega)$$
for the lowest eigenvalue of the Neumann realization of the magnetic Laplacian in Ω , $\mathcal{H}(B) = P^N_{B\mathbf{F},\Omega}$ $(\Omega \subset \mathbb{R}^d$, $d = 2, 3)$.
- Is this unique solution the critical field $H_{C_3}(\kappa)$?

Theorem 8.5.1 gives an affirmative answer to the first question in the case when $\beta = \operatorname{curl}\mathbf{F}$ is constant—see also some other cases treated in Section 8.6 and see Theorem 9.5.1 for a three-dimensional result.

In order to analyze the second question in some generality, let us define the following subsets of the positive real axis:

$$\mathcal{N}(\kappa) := \{\sigma > 0 \,|\, \mathcal{G}_{\kappa,\sigma} \text{ has a nontrivial minimizer}\} , \tag{13.6}$$
$$\mathcal{N}^{\mathrm{loc}}(\kappa) := \{\sigma > 0 \,|\, \lambda_1(\kappa\sigma) < \kappa^2\} , \tag{13.7}$$
$$\mathcal{N}^{\mathrm{sc}}(\kappa) := \{\sigma > 0 \,|\, \text{The Ginzburg–Landau equations have nontrivial solutions}\} . \tag{13.8}$$

Recall that the functional $\mathcal{G}$ is defined by (10.1), (10.2) and that the Ginzburg–Landau equations are (10.8) in 2D and (10.14) in 3D.

We define local fields and generalized fields by

$$\begin{aligned} \overline{H}^{\mathrm{loc}}_{C_3}(\kappa) &:= \sup \mathcal{N}^{\mathrm{loc}}(\kappa) , \quad \underline{H}^{\mathrm{loc}}_{C_3}(\kappa) := \inf \mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{loc}}(\kappa) , \\ \overline{H}^{\mathrm{sc}}_{C_3}(\kappa) &:= \sup \mathcal{N}^{\mathrm{sc}}(\kappa) , \quad \underline{H}^{\mathrm{sc}}_{C_3}(\kappa) := \inf \mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{sc}}(\kappa) . \end{aligned} \tag{13.9}$$

The global fields—defined in (13.1) and (13.2)—obviously have similar relations to the set $\mathcal{N}(\kappa)$. Also, one easily verifies by calculating the Hessian of the functional that the local fields are determined by the values where the normal solution $(0, \mathbf{F})$ is a *not unstable* local minimum of $\mathcal{G}_{\kappa,\sigma}$.

13.1.2 Main results

Our main result (combining Theorems 13.1.3 and 13.1.4) below is that all the critical fields above are contained in the interval $[\underline{H}_{C_3}^{\rm loc}(\kappa), \overline{H}_{C_3}^{\rm loc}(\kappa)]$ when κ is large. More precisely (see Proposition 13.1.7), the sets $\mathcal{N}(\kappa)$, $\mathcal{N}^{\rm loc}(\kappa)$, and $\mathcal{N}^{\rm sc}(\kappa)$ coincide for large values of κ. The proof we give is identical for the two- and three-dimensional situations. For simplicity, as elsewhere in this book, we restrict in the 3D case to a constant magnetic field.

We first observe the following general inequalities.

Theorem 13.1.3.
Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be a bounded, simply connected domain with smooth boundary. The following general relations hold between the different definitions of H_{C_3} :

$$\underline{H}_{C_3}^{\rm loc}(\kappa) \leq \underline{H}_{C_3}(\kappa)\,, \tag{13.10}$$

$$\overline{H}_{C_3}^{\rm loc}(\kappa) \leq \overline{H}_{C_3}(\kappa)\,. \tag{13.11}$$

For large values of κ, we have a converse statement to (13.11).

Theorem 13.1.4.
Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be a bounded, simply connected domain with smooth boundary. If $d = 2$, suppose that the external magnetic field β satisfies

$$0 < \Theta_0 b' < b\,. \tag{13.12}$$

If $d = 3$, we suppose that $\beta \in \mathbb{S}^2$ is constant.

Then there exists $\kappa_0 > 0$ such that for $\kappa \geq \kappa_0$,

$$\overline{H}_{C_3}^{\rm loc}(\kappa) = \overline{H}_{C_3}(\kappa)\,. \tag{13.13}$$

Furthermore, if the function $B \mapsto \lambda_1(B\mathbf{F})$ is strictly increasing for large B, then all the critical fields coincide for large κ and are given by the unique solution H to the equation

$$\lambda_1(\kappa H) = \kappa^2\,. \tag{13.14}$$

Remark 13.1.5.
One may ask what happens when $\sigma = H_{C_3}(\kappa)$. Are there nontrivial minimizers of $\mathcal{G}_{\kappa,\sigma}$ for that value of σ? Consider for simplicity the case where $B \mapsto \lambda_1(B)$ is strictly increasing for large B. At $\sigma = H_{C_3}(\kappa)$ we have $\lambda_1(\kappa\sigma) = \kappa^2$, and we therefore conclude from (13.26) that Δ vanishes, which implies that $\psi = 0$. Thus, no nontrivial minimizer (or more generally no nontrivial stationary point) can exist at the critical value. A more quantitative version of this result is given in Proposition 13.4.1, in which $\|\psi\|_\infty$ is estimated in terms of the quantity $\kappa^2 - \lambda_1(\kappa\sigma)$.

13.1.3 Proofs

Theorem 13.1.3 follows from the following easy lemma whose proof is left to the reader.

Lemma 13.1.6.
We have the following inclusions for all values of κ *:*

$$\mathcal{N}^{\mathrm{loc}}(\kappa) \subseteq \mathcal{N}(\kappa) \subseteq \mathcal{N}^{\mathrm{sc}}(\kappa)\,.$$

The remainder of this section will be devoted to the proof of the converse inclusion for large values of κ (see Proposition 13.1.7). Clearly, this implies the proof of Theorem 13.1.4.

Proposition 13.1.7.
Let $\Omega \subset \mathbb{R}^d$ *,* $d = 2$ *or* 3 *, be smooth, bounded, and simply connected. If* $d = 2$ *, suppose that the external magnetic field* β *satisfies* (13.12)*. If* $d = 3$ *, we suppose that the magnetic field* $\beta \in \mathbb{S}^2$ *is constant. Then there exists* $\kappa_0 > 0$ *such that*

$$\mathcal{N}^{sc}(\kappa) = \mathcal{N}(\kappa) = \mathcal{N}^{\mathrm{loc}}(\kappa),$$

for all $\kappa \geq \kappa_0$ *.*

Proof.
Using Lemma 13.1.6, it only remains to prove the inclusion

$$\mathcal{N}^{sc}(\kappa) \subseteq \mathcal{N}^{\mathrm{loc}}(\kappa)\,.$$

Suppose that $(\psi, \mathbf{A})$ is a solution of (10.8) or (10.14) with $\psi \neq 0$. From Theorem 10.4.1, we get that

$$\sigma \leq C\kappa\,. \tag{13.15}$$

Using Theorem 8.1.1 in the 2D case and Theorem 9.1.1 in the 3D case (notice that in this case $b' = b = 1$), we see that, for any $\epsilon > 0$, there exists $\kappa_\epsilon > 0$ such that, if $\kappa \geq \kappa_\epsilon$,

$$\left[0, \left(\frac{1}{\Theta_0 b'} - \epsilon\right)\kappa\right] \subset \mathcal{N}^{\mathrm{loc}}(\kappa)\,. \tag{13.16}$$

So it suffices to consider σ satisfying, for some $c > 0$ and some arbitrary $\epsilon > 0$,

$$c \leq \frac{\kappa}{\sigma} \leq \Theta_0 b' + \epsilon\,. \tag{13.17}$$

In particular, we may take ϵ such that $\Theta_0 b' + \epsilon < b$. Let us define

$$\Delta := \kappa^2 \|\psi\|_2^2 - \|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2\,. \tag{13.18}$$

Multiplying the first equation in (10.8), respectively (10.14), by $\overline{\psi}$ and integrating over Ω, we first obtain

$$\kappa^2\|\psi\|_4^4 = \Delta\,, \tag{13.19}$$

and the nontriviality of $(\psi, \mathbf{A})$ implies the inequality

$$0 < \kappa^2\|\psi\|_4^4 = \Delta\,. \tag{13.20}$$

Combining (13.27), (13.28) ahead with (13.20) yields

$$(\kappa\sigma)^2\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)}^2 \leq C\frac{\Delta}{(\kappa\sigma)^{\frac{1}{4}}}\,. \tag{13.21}$$

We now estimate, for arbitrary $\eta > 0$,

$$\begin{aligned} -\Delta &= \|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2 \\ &\geq (1-\eta)\|p_{\kappa\sigma\mathbf{F}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2 - \eta^{-1}(\kappa\sigma)^2\,\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)}^2\,\|\psi\|_4^2\,. \end{aligned} \tag{13.22}$$

Thus, using (13.28), (13.20), and (13.21), we obtain

$$-\Delta \geq (\lambda_1(\kappa\sigma\mathbf{F})-\kappa^2)\|\psi\|_2^2 - \eta C\lambda_1(\kappa\sigma\mathbf{F})\frac{\Delta^{\frac{1}{2}}}{\kappa(\kappa\sigma)^{\frac{1}{4}}} - \eta^{-1}C\frac{\Delta^{\frac{3}{2}}}{\kappa(\kappa\sigma)^{\frac{1}{4}}}\,. \tag{13.23}$$

It clearly follows from Theorem 8.1.1, respectively Theorem 9.1.1 in three dimensions, that there exists a $C > 0$ such that

$$\lambda_1(\kappa\sigma\mathbf{F}) \leq C\sigma\kappa \tag{13.24}$$

if $\sigma\kappa \geq 1$.

Thus, we can choose $\eta = \sqrt{\Delta/\kappa\sigma}$ and find that (13.23) gives the existence of κ_0 and C such that, for $\kappa \geq \kappa_0$,

$$0 \geq -\Delta \geq (\lambda_1(\kappa\sigma\mathbf{F}) - \kappa^2)\|\psi\|_2^2 - C\frac{(\kappa\sigma)^{\frac{1}{4}}}{\kappa}\Delta\,. \tag{13.25}$$

Using (13.15), we find that, for sufficiently large κ,

$$0 < (1 - C\kappa^{-\frac{1}{2}})\Delta \leq [\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F})]\|\psi\|_2^2\,. \tag{13.26}$$

Thus, since $\psi \neq 0$, we conclude that $\lambda_1(\kappa\sigma\mathbf{F}) < \kappa^2$, which is what we needed to prove. □

Proposition 13.1.8.
Suppose that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is smooth, simply connected, and bounded. Suppose that β satisfies (13.12) *if $d = 2$ and that $\beta \in \mathbb{S}^2$ is constant if $d = 3$. Then there exists a $C_1 > 0$ such that*

$$\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)} \leq \frac{C_1}{\sigma}\|\psi\|_4\|\psi\|_2\,, \tag{13.27}$$

for all solutions $(\psi, \mathbf{A})$ to the Ginzburg–Landau equations.

Furthermore, if ϵ is sufficiently small, then there exist $\kappa_0 > 0$ and $C_2 > 0$ such that if

$$\kappa \leq (\Theta_0 b' + \epsilon)\sigma\,,$$

then

$$\|\psi\|_2^2 \leq \frac{C_2}{(\kappa\sigma)^{\frac{1}{4}}}\,\|\psi\|_4^2\,, \tag{13.28}$$

when $\kappa \geq \kappa_0$ and for all solutions $(\psi, \mathbf{A})$ to the Ginzburg–Landau equations.

Proof.
Corollary 12.5.5 with $p = 4$ yields (13.28) for $d = 2$ or 3. For $d = 2$, we use (10.28), (D.7), and a Sobolev inequality to get (13.27). The estimate (13.27) in 3D is a consequence of (10.37) and Proposition D.3.2. □

13.2 Asymptotics of the Critical Field in 2D

A central question in the mathematical treatment of type II superconductors is to establish the asymptotic behavior of $H_{C_3}(\kappa)$ for large values of κ. We see from Theorem 13.1.4 that this asymptotic behavior can be read directly from the asymptotics of $\lambda_1(B)$. In particular, we get, using Theorem 8.1.1.

Theorem 13.2.1.
Suppose that $\Omega \subset \mathbb{R}^2$, β satisfy the assumptions of Theorem 13.1.4. Then

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0 b'} + o(\kappa)\,.$$

The case of a constant magnetic field has been the focus of much attention in the literature. In that case, we see from Theorem 8.5.1 combined with Theorem 13.1.4 that the critical fields coincide and we get the following asymptotics, using Theorem 8.3.2.

Theorem 13.2.2.
Suppose that $\beta = 1$ and that $\Omega \subset \mathbb{R}^2$ is bounded, smooth, and simply connected. Then

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{\mathcal{C}_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{3}})\,. \tag{13.29}$$

Remark 13.2.3.
As mentioned in Remark 8.3.6, the error bound in Theorem 8.3.2 is not optimal. If one uses the optimal error bound $\mathcal{O}(B^{1/4})$, (13.29) improves to

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{\mathcal{C}_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}})\,. \tag{13.30}$$

Of course, one can also use the more precise eigenvalue asymptotics for special geometries as in Theorem 8.4.1 or in Corollary 5.4.2 to give better estimates on $H_{C_3}(\kappa)$ in the corresponding cases. We leave the details to the reader.

13.3 Asymptotics of the Critical Field in 3D

It should be noted from Chapter 9 that obtaining precise spectral asymptotics is technically much more involved in 3D than in 2D. In particular, we have at present no three-dimensional analog of Theorem 8.5.1. Thus, we do not in general know that the interval $[\underline{H}^{\mathrm{loc}}_{C_3}(\kappa), \overline{H}^{\mathrm{loc}}_{C_3}(\kappa)]$ collapses to a point for large values of κ, even when β is constant.

As in 2D, Theorem 13.1.4 reduces the *nonlinear* question of the asymptotics of H_{C_3} (whatever the definition) to the simpler *linear* question of asymptotics of $\lambda_1(B)$. For example, we can use the asymptotics (9.18) under certain geometric assumptions on Ω. Combining this result with Theorem 13.1.4, one gets a two-term asymptotics for $H_{C_3}(\kappa)$:

Theorem 13.3.1.
Suppose Ω is a smooth, bounded, simply connected domain in $\mathbb{R}^3$ satisfying Assumption 9.2.1. Then

$$H_{C_3}(\kappa) - \left(\frac{\kappa}{\Theta_0} - \widehat{\gamma}_0 \Theta_0^{-\frac{5}{3}} \kappa^{\frac{1}{3}}\right) = o(\kappa^{\frac{1}{3}}) , \tag{13.31}$$

where $H_{C_3}(\kappa)$ denotes any of the six different (upper or lower) critical fields defined above and the geometric constant $\widehat{\gamma}_0$ was defined in (9.19).

The proof of Theorem 13.3.1 is immediate. By Theorems 13.1.3 and 13.1.4, it suffices to prove that $\underline{H}^{\mathrm{loc}}_{C_3}(\kappa)$ and $\overline{H}^{\mathrm{loc}}_{C_3}(\kappa)$ have the asymptotics given by (13.31). But this follows easily from the asymptotics of $\lambda_1(B)$ given in Theorem 9.2.2.

Theorem 13.1.4 together with the monotonicity result established in Theorem 9.5.1 gives

Corollary 13.3.2.
Suppose that Ω satisfies Assumption 9.2.1. Then there exists a $\kappa_0 > 0$ such that for $\kappa \in [\kappa_0 , \infty[$, one has

$$\underline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \underline{H}_{C_3}(\kappa) = \underline{H}^{\mathrm{sc}}_{C_3}(\kappa) = \overline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \overline{H}_{C_3}(\kappa) = \overline{H}^{\mathrm{sc}}_{C_3}(\kappa) . \tag{13.32}$$

13.4 Amplitude Near the Onset

In this section, we restrict for simplicity to two-dimensional domains. From Proposition 10.3.1, we know that minimizers $(\psi, \mathbf{A})$ of the Ginzburg–Landau functional satisfy the estimate $\|\psi\|_{L^\infty} \leq 1$ independently of the values of κ, σ. Furthermore, the elliptic estimate of Proposition 11.4.7 tells us that, in the case of a constant magnetic field,

$$\|\psi\|_\infty = o(1)\,,$$

for large κ when σ is near $H_{C_3}(\kappa)$. Here we will give more precise estimates on the magnitude of $\|\psi\|_\infty$.

Proposition 13.4.1.
Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with smooth boundary. Suppose that the magnetic field β satisfies

$$0 < \Theta_0 b' < b\,.$$

Then, for all $\epsilon > 0$ and $\epsilon' > 0$, there exist constants C and κ_0 such that if $(\psi, \mathbf{A})$ is a nontrivial solution to (10.8) with

$$\frac{\kappa}{\sigma} \leq b - \epsilon'\,, \qquad \kappa \geq \kappa_0\,,$$

then

$$\|\psi\|_{L^\infty(\Omega)} \leq C\,\kappa^{-\frac{1}{2}+\epsilon}\,\sqrt{\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F})}\,. \tag{13.33}$$

Remark 13.4.2.
The estimate (13.33) can also be expressed in terms of the distance to the critical field, i.e., $|\overline{H}_{C_3}(\kappa) - \sigma|$. Suppose that Ω and β are such that[1]

$$C_0 := \limsup_{B\to\infty} |\lambda_1'(B\mathbf{F})| < \infty\,.$$

Then, using Theorem 13.1.4 (and a continuity argument), we can write (for large values of κ)

$$\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F}) = \int_{\kappa\sigma}^{\kappa\overline{H}_{C_3}(\kappa)} \lambda_1'(B\mathbf{F})\,dB \leq (C_0+1)\,\kappa\,|\overline{H}_{C_3}(\kappa) - \sigma|\,.$$

Therefore, the estimate (13.33) becomes,

$$\|\psi\|_{L^\infty(\Omega)} \leq C_\epsilon\,\kappa^\epsilon\,\sqrt{|\overline{H}_{C_3}(\kappa) - \sigma|}\,, \qquad \forall \epsilon > 0\,. \tag{13.34}$$

Let $(\psi, \mathbf{A})$ be a solution to (10.8). We will define a "nonlinear spectral distance" δ_ψ by

$$\delta_\psi := \frac{\kappa^2\|\psi\|_2^2 - \|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2}{\|\psi\|_2^2}\,. \tag{13.35}$$

[1] In general, the derivative $\lambda_1'(B\mathbf{F})$ will not exist for all B. So one should rather replace $\lambda_1'(B\mathbf{F})$ by the maximum of the left- and right-hand derivatives $\max(|\lambda_{1,-}'(B\mathbf{F})|, |\lambda_{1,+}'(B\mathbf{F})|)$. For simplicity, we omit this point in the discussion.

With the parameter Δ from the proof of Proposition 13.1.7, we have

$$\delta_\psi = \frac{\Delta}{\|\psi\|_2^2}\,.$$

Hence, from (13.19), we obtain that $\delta_\psi > 0$ if $\psi \neq 0$.

The proposition will be a consequence of the following lemma.

Lemma 13.4.3.
Under the assumptions of Proposition 13.4.1, for all $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $\epsilon' > 0$ such that $\epsilon_1 \leq 1 + \epsilon_2$, there exist constants C and $\kappa_0 > 0$ such that if $(\psi, \mathbf{A})$ is a nontrivial solution to (10.8), with

$$\frac{\kappa}{\sigma} \leq b - \epsilon', \qquad \kappa \geq \kappa_0\,,$$

then

$$\lambda \leq C \delta_\psi^{\frac{1}{2}} \kappa^{\epsilon_1+\epsilon_2} \mu^{1+\epsilon_1}\,, \tag{13.36}$$

where $\lambda = \|\psi\|_\infty$ and μ is defined by $\lambda\mu = \|\psi\|_2$.

Proof of Lemma 13.4.3.
We recall from (10.26) and (13.19) the estimates

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2 \leq \kappa^2 \|\psi\|_2^2\,, \qquad\qquad \|\psi\|_4^4 = \frac{\delta_\psi}{\kappa^2}\|\psi\|_2^2\,. \tag{13.37}$$

Furthermore, from Proposition 11.4.4 we get the inequality

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_\infty \leq C\sqrt{\kappa\sigma}\|\psi\|_\infty\,. \tag{13.38}$$

By the Sobolev inequality and using interpolation theory, we get that, for all (p, s) satisfying $ps > 2$ and $0 < s \leq 1$, there exist constants $\widehat{C}$ and C such that

$$\lambda \leq \widehat{C}\, \||\psi|\|_{W^{s,p}} \leq C\, \|\psi\|_p^{1-s} \|\nabla|\psi|\|_p^s + C\, \|\psi\|_p\,.$$

We then use the diamagnetic and Hölder inequalities on the right-hand side:

$$\lambda \leq C\|\psi\|_p^{1-s}\, \|p_{\kappa\sigma\mathbf{A}}\psi\|_p^s + C\lambda\mu^{\frac{2}{p}}\,.$$

Using Corollary 12.5.5, with $p = \infty$, we get

$$\mu \leq C\kappa^{-\frac{1}{2}}\,. \tag{13.39}$$

So for κ large enough, we obtain

$$\lambda \leq C\|\psi\|_p^{1-s}\, \|p_{\kappa\sigma\mathbf{A}}\psi\|_p^s\,.$$

We now apply the Hölder inequality for each term on the right-hand side:

$$\lambda \leq C\big\{\lambda^{p-4}\|\psi\|_4^4\big\}^{\frac{1-s}{p}} \Big\{\|p_{\kappa\sigma\mathbf{A}}\psi\|_\infty^{p-2}\|p_{\kappa\sigma\mathbf{A}}\psi\|_2^2\Big\}^{\frac{s}{p}}\,.$$

We use (13.37) and (13.38) to get

$$\lambda \leq C\left(\lambda^{p-2}\mu^2\frac{\delta_\psi}{\kappa^2}\right)^{\frac{1-s}{p}}(\lambda^p\kappa^p\mu^2)^{\frac{s}{p}} = C\lambda^{1-2\frac{1-s}{p}}\delta_\psi^{\frac{1-s}{p}}\mu^{\frac{2}{p}}\kappa^{s-2\frac{1-s}{p}}\,. \tag{13.40}$$

This implies that

$$\lambda \leq C\delta_\psi^{\frac{1}{2}}\mu^{\frac{1}{1-s}}\kappa^{\frac{ps}{2(1-s)}-1}\,. \tag{13.41}$$

Write $1/(1-s) = 1+\epsilon_1$ and $ps = 2+2\epsilon_2/(1+\epsilon_1)$. Then we find (13.33). □

Proof of Proposition 13.4.1.
The estimate (13.26) implies that

$$\delta_\psi \leq 2(\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F}))\,, \tag{13.42}$$

for large κ. Thus, (13.33) follows from (13.36) with $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon/2$, (13.42), and (13.39). □

Remark 13.4.4.
If $(\psi, \mathbf{A})$ *is a global nontrivial minimizer, we have*

$$0 > \mathcal{G}(\psi,\mathbf{A}) \geq -\frac{\Delta}{2} = -\frac{\kappa^2}{2}\|\psi\|_4^4 = -\frac{1}{2}\frac{\Delta^2}{\kappa^2\|\psi\|_4^4}\,.$$

Using (13.26)*, this finally gives*

$$\mathcal{G}(\psi,\mathbf{A}) \geq -\frac{1}{2}(1+C\kappa^{-\frac{1}{4}})\frac{[\kappa^2-\lambda_1(\kappa\sigma\mathbf{F})]^2\|\psi\|_2^4}{\kappa^2\|\psi\|_4^4}\,. \tag{13.43}$$

It is interesting to compare this with (13.5).

If we indeed are able to prove in some case that $\psi/\|\psi\|_2$ *is close to* u_1 *for* σ *sufficiently close to* $H_{C_3}(\kappa)$*, we will get an estimate of the energy of the minimizer.*

For this last point, we could try to use ψ *as a quasimode for* $\Delta_{\kappa\sigma\mathbf{A}}$ *or* $\Delta_{\kappa\sigma\mathbf{F}}$ *together with a lower bound of* $\lambda_2(\kappa\sigma\mathbf{F}) - \lambda_1(\kappa\sigma\mathbf{F})$*, which we have, for example, in the case when* β *is constant and when the curvature has a unique nondegenerate maximum at the boundary. This will be developed in the next section.*

13.5 Energy Near the Onset

Near H_{C_3} we have $\sigma \approx \kappa/\Theta_0$, in particular,

$$C^{-1}\kappa \leq \sigma \leq C\kappa\,,$$

for some constant $C > 0$. We take the assumptions of Theorem 8.4.1, and so we have

$$\lambda_2(\kappa\sigma) - \lambda_1(\kappa\sigma) \approx \kappa^{\frac{1}{2}} . \tag{13.44}$$

Recall the notation $u_1 = u_{1,B}$ for a normalized ground state of the Neumann realization $P^N_{B\mathbf{F},\Omega}$. Our aim is to prove

Proposition 13.5.1.
Under the assumptions of Theorem 8.4.1, there exists a function $g(\alpha)$ on $[0, \alpha_0]$ such that

$$\lim_{\alpha \to 0} g(\alpha) = 0$$

and such that if

$$\kappa^2 - \alpha\kappa^{\frac{1}{2}} \le \lambda_1(\kappa\sigma F) \le \kappa^2 , \tag{13.45}$$

then we have

$$\inf \mathcal{G}(\psi, \mathbf{A}) = -(1 + g(\alpha))(1 + C\kappa^{-\frac{1}{4}}) \frac{[\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F})]^2}{2\kappa^2 \|u_{1,\kappa\sigma}\|_4^4} . \tag{13.46}$$

Remark 13.5.2.
Under these assumptions (see Theorem 8.4.1), the strength σ satisfies for some $\widehat{\alpha}_0$ and for α small enough

$$\lambda_1(\kappa\sigma\mathbf{F}) \le \kappa^2 \le \lambda_2(\kappa\sigma\mathbf{F}) - \widehat{\alpha}_0\kappa^{\frac{1}{2}} . \tag{13.47}$$

Remark 13.5.3.
Condition (13.45) *can also be written in the form*

$$0 \le H_{C_3}(\kappa) - \sigma \le \frac{\alpha}{C}\kappa^{-\frac{1}{2}} . \tag{13.48}$$

Proof.
What remains to be proven is the lower bound. The upper bound was indeed obtained in (13.5). According to (13.43), it is enough to show that the L^4 norm of $\psi/\|\psi\|_2$ tends to the L^4 norm of u_1 as $\alpha \to 0$ (we actually only need an asymptotic lower bound by the L^4 norm of u_1).

Let us assume that we have proven the two following lemmas under the assumptions of the proposition.

Lemma 13.5.4.
There exists $c(\alpha, \sigma, \kappa) \in \mathbb{C}$ such that $|c(\alpha, \sigma, \kappa)| = 1$ and

$$\left\| \frac{\psi}{\|\psi\|_2} - c(\alpha, \sigma, \kappa) u_1 \right\|_2 = o(1) , \tag{13.49}$$

uniformly with respect to the other parameters as $\alpha \to 0$.

Lemma 13.5.5.

$$\|u_1\|_4^4 \approx \kappa^{\frac{5}{4}} \quad \textit{and} \quad \|u_1\|_6^6 \approx \kappa^{\frac{5}{2}} . \tag{13.50}$$

Using the elementary inequality

$$|b|^4 \geq |a|^4 - 4|b-a||a|^3\,,$$

for real numbers a, b, we get for any pair of functions v, w in $L^6(\Omega)$,

$$\int_\Omega |v|^4 dx \geq \int_\Omega |w|^4 dx - 4\|v-w\|_2 \|w\|_6^3\,. \tag{13.51}$$

Applying this inequality with $v = \psi/\|\psi\|_2$ and $w = cu_1$, and Lemmas 13.5.4 and 13.5.5, yields

$$\int_\Omega (\psi/\|\psi\|_2)^4\, dx \geq \int_\Omega |u_1|^4\, dx - o(1)\kappa^{\frac{5}{4}} = (1-o(1)) \int_\Omega |u_1|^4\, dx\,, \tag{13.52}$$

as $\alpha \to 0$.

So the proof of the proposition is achieved modulo the proofs of the two lemmas. □

Proof of Lemma 13.5.4.
We come back to (13.22) but replace (13.23) with

$$-\Delta \geq (1-\eta)\|p_{\kappa\sigma\mathbf{F}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2 - \eta^{-1} C \frac{\Delta^{\frac{3}{2}}}{\kappa(\kappa\sigma)^{\frac{1}{4}}}\,. \tag{13.53}$$

We now choose $\eta = \beta\kappa^{-3/2}$. This leads to

$$-\Delta \geq (1-\eta)\left(\|p_{\kappa\sigma\mathbf{F}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2\right) - \beta\kappa^{\frac{1}{2}}\|\psi\|_2^2 - \Delta\left(C\beta^{-1}\Delta^{\frac{1}{2}}\right)\,, \tag{13.54}$$

with β to be chosen suitably.

We need at this stage an upper bound for Δ. Using (13.26) and the upper bound of $\|\psi\|_2^2$ obtained in Corollary 12.7.2, we find, using our assumption,

$$\Delta \leq C\left(\kappa^2 - \lambda_1(\kappa\sigma F)\right)^2 \kappa^{-1} \leq C\alpha^2\,. \tag{13.55}$$

So we can take $\beta = \alpha^{1/2}$ and finally obtain

$$0 \geq -\Delta(1-\mathcal{O}(\alpha^{\frac{1}{2}})) \geq \left(\|p_{\kappa\sigma\mathbf{F}}\psi\|_2^2 - \kappa^2\|\psi\|_2^2\right) - C\sqrt{\alpha}\kappa^{\frac{1}{2}}\|\psi\|_2^2\,, \tag{13.56}$$

which implies

$$\|p_{\kappa\sigma\mathbf{F}}\psi\|_2^2 \leq (\kappa^2 + C\sqrt{\alpha}\kappa^{\frac{1}{2}}))\|\psi\|^2\,. \tag{13.57}$$

This should be interpreted as the fact that $\psi/\|\psi\|$ is a quasimode for $-\Delta_{\kappa\sigma\mathbf{F}}$. This is indeed the case if the quantity

$$(\kappa^2 + C\sqrt{\alpha}\kappa^{\frac{1}{2}} - \lambda_1(\kappa\sigma\mathbf{F}))\kappa^{-\frac{1}{2}}$$

is small. The smallness of that quantity is assured by our assumption and by (13.45). Hence, by abstract analysis, we obtain the existence of c such that $|c| = 1$ and

$$\left\|\frac{\psi}{\|\psi\|_2} - cu_1\right\|^2 \leq 2\frac{\left(\kappa^2 + C\sqrt{\alpha}\kappa^{\frac{1}{2}} - \lambda_1(\kappa\sigma\mathbf{F})\right)}{\lambda_2(\kappa\sigma\mathbf{F}) - \lambda_1(\kappa\sigma\mathbf{F})} \leq \widehat{C}\sqrt{\alpha}\,. \tag{13.58}$$

□

Proof of Lemma 13.5.5.
This is a consequence of the WKB approximation of the ground state of the linear model and as such part of the proof of Theorem 8.4.1 given in [FoH2]. In boundary coordinates and near the maximum of the curvature, this approximation takes the following form (see (2.2) in [FoH2]),

$$u_1 \sim (\kappa\sigma)^{\frac{5}{16}} \chi(s,t) \exp i\xi_0 (\kappa\sigma)^{\frac{1}{4}} s \ \times G((\kappa\sigma)^{\frac{1}{16}} s)\, u_0((\kappa\sigma)^{\frac{1}{4}} t)\,, \tag{13.59}$$

where G is a Gaussian, χ is a cutoff function at the point of maximal curvature $(s,t) = 0$, and $u_0 = \varphi_{\xi_0}$ was introduced in (3.37). This leads to

$$\|u_1\|_4^4 \approx (\kappa\sigma)^{\frac{5}{8}} \qquad \text{and} \qquad \|u_1\|_6^6 \approx (\kappa\sigma)^{\frac{5}{4}}\,,$$

and observing that $\sigma \approx \kappa$, we have proven the lemma. □

Remark 13.5.6.
This proof also gives the following approximation of $\|\psi\|_2$*:*

$$\|\psi\|_2 = (1 + \epsilon(\alpha) + \mathcal{O}(\kappa^{-\frac{1}{4}}))\frac{1}{\kappa}\frac{1}{\|u_1\|_4^2}\left(\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F})\right)^{\frac{1}{2}}\,, \tag{13.60}$$

with $\lim_{\alpha\to 0} \epsilon(\alpha) = 0$.

Using the asymptotic behavior of the right-hand side together with (13.50)*, we get*

$$\|\psi\|_2 \approx \kappa^{-\frac{13}{8}}\left(\kappa^2 - \lambda_1(\kappa\sigma\mathbf{F})\right)^{\frac{1}{2}}\,. \tag{13.61}$$

13.6 Notes

1. The results of this part were initiated in the series of papers by Lu–Pan [LuP3, LuP4, LuP5, LuP7]. In particular, they were the first to propose a clear mathematical definition for H_{C_3}.
   Then the first point was to observe that many other definitions of this critical field were possible.
   The second point was to have a good asymptotics of these various critical fields.
   The third point was to try to get better asymptotics. Any improvement in this direction was leading to the conclusion that all these possible critical fields have the same asymptotics.
   This was a good motivation for showing that all these critical fields coincide in the large κ regime. This result was proved in [FoH3] (see [FoH4] for later improvements).
2. Lemma 13.1.1 was stated in [LuP3]. Also Theorem 13.2.1 was obtained in [LuP5].

3. The analysis of the 3D case is very close to the presentation in [FoH6], where a slightly modified functional is also considered. Theorem 13.3.1 (which appeared in [FoH6]) is an affirmative answer to a conjecture in [Pa6], however; the conjecture is stated without the geometric Assumption 9.2.1, which originated in the work [HeM6]. Notice however, the typo in [FoH6] which has been corrected in (13.31).
   The main point here is that—for generic domains—the 3D case does not present new phenomena in the nonlinear part. This is only true for external field strengths that are very close to the third critical field. For the linear part, we have used mainly the results presented in Chapter 9.
4. The use of (13.51) in the proof of Proposition 13.5.1 is inspired by an argument appearing in [AfH] in a similar context.

14

Between H_{C_2} and H_{C_3} in Two Dimensions

Between $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$, superconductivity is for large κ confined to the boundary. This follows from the decay estimates in Chapter 12. In the present chapter, we will give leading-order energy estimates in this parameter region, which are valid when we are not too close to either of the critical fields. These energy estimates indicate that superconductivity is essentially uniformly distributed over the entire boundary region.

14.1 Introduction

In this chapter, we will assume that the dimension is 2 and that the external magnetic field is constant, i.e., we take

$$\beta \equiv 1\,, \tag{14.1}$$

in (10.4). In that case, we have, by (13.30),

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{\mathcal{C}_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}})\,. \tag{14.2}$$

We will consider field strengths $\sigma = \sigma(\kappa)$ below $H_{C_3}(\kappa)$. The results in this chapter will concern σ's such that $\sigma \to \infty$ and $H_{C_3}(\kappa) - \sigma \to \infty$ as $\kappa \to \infty$. Thus, we define the positive quantity

$$\rho = \rho(\kappa) := H_{C_3}(\kappa) - \sigma\,. \tag{14.3}$$

We recall that a complementary analysis was carried out in Section 13.5, which was devoted to the case when we were much closer to H_{C_3}:

$$\rho(\kappa) = o(\kappa^{-\frac{1}{2}})\,.$$

Recall that the Ginzburg–Landau ground state energy $E(\kappa, \sigma)$ was defined in (10.6). The main result of this chapter is an asymptotic formula for the ground state energy.

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Theorem 14.1.1.
1. Suppose that

$$\sigma = (\mathfrak{b} + o(1))\kappa\,, \quad \textit{for some } \mathfrak{b} \in [1, \Theta_0^{-1}[\,.$$

Then there exists a constant $E_{\mathfrak{b}} > 0$ such that

$$E(\kappa, \sigma) = -\sqrt{\kappa\sigma} E_{\mathfrak{b}} |\partial\Omega| + o(\kappa)\,. \tag{14.4}$$

2. For $\mathfrak{b}$ sufficiently close to Θ_0^{-1},

$$E_{\mathfrak{b}} = \frac{1}{2\mathfrak{b}} \|f_{\zeta(\mathfrak{b}^{-1}),\mathfrak{b}^{-1}}\|_{L^4(\mathbb{R}^+)}^4\,, \tag{14.5}$$

where $\zeta(\lambda)$ will be introduced in Definition 14.2.4 and the function $f_{\zeta,\lambda}$ will be defined in Proposition 14.2.1.
3. If $\mathfrak{b} = \Theta_0^{-1}$ and $\rho \to +\infty$, then

$$E(\kappa, \sigma) = -\frac{\Theta_0^{\frac{5}{2}} |\partial\Omega| + o(1)}{2\|u_0\|_4^4} \frac{\rho^2}{\kappa}\,. \tag{14.6}$$

Remark 14.1.2.
The heuristics behind Theorem 14.1.1 is as follows.

- *In the case corresponding to (14.6), σ is very close to H_{C_3}. This forces the minimizer of the Ginzburg–Landau functional to be approximately equal to the function u_0 in the normal variable. It roughly has the structure*

$$\psi \approx \lambda e^{i\xi_0\kappa\sigma s} u_0(\sqrt{\kappa\sigma} t)\,, \tag{14.7}$$

  where (s,t) are boundary coordinates, ξ_0, u_0 are the parameters/functions from the model problem in 1 dimension, and λ is a normalization parameter. In particular, $|\psi|$ is (up to approximation errors) constant on the boundary.
- *When σ is slightly farther away from H_{C_3}, corresponding to (14.5), the structure of ψ as $e^{i\kappa\sigma\zeta s} \times$ (function of t) remains, but the parameter ζ and the function of t are now determined by a nonlinear one-dimensional problem (14.16).*
- *Finally, in the case corresponding to (14.4), we are not able to prove that the minimizer essentially has a product structure as in the previous cases. However, one gets the uniform distribution of energy along the boundary by a weaker argument.*

The present chapter is devoted to the proof of Theorem 14.1.1. By the decay estimates of Chapter 12, superconductivity is localized to a region near the boundary, which, after a change of coordinates, can be identified with a cylinder. Section 14.3 is devoted to the study of the effective model on this cylinder. However, the most important ingredient of the proof is the analysis of the effective one-dimensional model obtained from the cylinder model after a Fourier transformation. That analysis is carried out in Section 14.2. Finally, in Sections 14.4 and 14.5, we collect the estimates to finish the proof of Theorem 14.1.1.

14.2 A Nonlinear One-Dimensional Problem

14.2.1 Presentation

Let $\mu(\xi)$ be the lowest eigenvalue of $\mathfrak{h}^{N,\xi}$ defined in Section 3.2. Suppose that λ is a parameter such that

$$\lambda \in]\Theta_0, 1[\,. \tag{14.8}$$

Then, using the properties of μ established in Proposition 3.2.4, there exist two real points $z_1(\lambda)$ and $z_2(\lambda)$ such that

$$z_1(\lambda) < \xi_0 < z_2(\lambda) < 0 \qquad \text{and} \qquad \mu^{-1}([\Theta_0, \lambda[) = \,]\, z_1(\lambda)\,, z_2(\lambda)\,[\,.$$

Clearly, $z_1(\lambda)$ is decreasing, $z_2(\lambda)$ is increasing, and

$$\lim_{\lambda\to+1} z_1(\lambda) = -\infty\,, \qquad \lim_{\lambda\to+1} z_2(\lambda) = 0\,. \tag{14.9}$$

Let us consider, for $z \in \mathbb{R}$ and for λ_1 and λ_2 in $]0, +\infty[$, the functional

$$\begin{aligned} B^1(\mathbb{R}^+) \ni \phi \mapsto & \,\widetilde{\mathcal{E}}_{z,\lambda_1,\lambda_2}(\phi) \\ := & \int_0^\infty |\phi'(\tau)|^2 + (\tau + z)^2|\phi(\tau)|^2 + \frac{\lambda_2}{2}|\phi(\tau)|^4 - \lambda_1|\phi(\tau)|^2\, d\tau\,, \end{aligned} \tag{14.10}$$

where $B^1(\mathbb{R}^+)$ is introduced in (3.3).

Let us also introduce

$$\widetilde{b}(z, \lambda_1, \lambda_2) := \inf_{\phi\in B^1(\mathbb{R}^+)} \widetilde{\mathcal{E}}_{z,\lambda_1,\lambda_2}(\phi)\,. \tag{14.11}$$

Notice that the scaling relation $\widetilde{\mathcal{E}}_{z,\lambda_1,\lambda_2}(t\phi) = t^2\widetilde{\mathcal{E}}_{z,\lambda_1,t^2\lambda_2}(\phi)$, which is satisfied for all $\phi \in B^1(\mathbb{R}^+)$ and $t > 0$, implies that

$$\widetilde{b}(z, \lambda_1, \lambda_2) = t^2\widetilde{b}(z, \lambda_1, t^2\lambda_2)\,. \tag{14.12}$$

In particular, if f minimizes $\widetilde{\mathcal{E}}_{z,\lambda_1,\lambda_1}$, then $\sqrt{\lambda_1/\lambda_2}\, f$ minimizes $\widetilde{\mathcal{E}}_{z,\lambda_1,\lambda_2}$ and

$$\widetilde{b}(z, \lambda_1, \lambda_2) = \frac{\lambda_1}{\lambda_2} b(z, \lambda_1, \lambda_1)\,. \tag{14.13}$$

Thus, we can reduce our attention to the case $\lambda_1 = \lambda_2$. We define corresponding quantities without tildes for this case, i.e.,

$$\mathcal{E}_{z,\lambda}(\phi) := \int_0^\infty |\phi'(\tau)|^2 + (\tau + z)^2|\phi(\tau)|^2 + \frac{\lambda}{2}|\phi(\tau)|^4 - \lambda|\phi(\tau)|^2\, d\tau\,, \tag{14.14}$$

$$b(z, \lambda) := \inf_{\phi\in B^1(\mathbb{R}^+)} \mathcal{E}_{z,\lambda}(\phi)\,. \tag{14.15}$$

Proposition 14.2.1.
For all $z \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, the functional $\mathcal{E}_{z,\lambda}$ admits a nonnegative minimizer $f_{z,\lambda} \in B^1(\mathbb{R}^+)$. The minimizer satisfies the Euler–Lagrange equation

$$\begin{cases} f'_{z,\lambda}(0) = 0\,, \\ -f''_{z,\lambda} + (\tau + z)^2 f_{z,\lambda} = \lambda f_{z,\lambda}(1 - |f_{z,\lambda}|^2)\,. \end{cases} \tag{14.16}$$

Moreover, we have

$$b(z,\lambda) = \inf_{\phi \in B^1(\mathbb{R}^+)} \mathcal{E}_{z,\lambda}(\phi) = -\frac{\lambda}{2}\|f_{z,\lambda}\|_4^4\,, \tag{14.17}$$

and the inequality

$$\|f_{z,\lambda}\|_\infty \leq 1\,. \tag{14.18}$$

Proof.
The existence of a minimizer is left to the reader. It follows from Proposition 2.1.2 that

$$\mathcal{E}_{z,\lambda}(f) \geq \mathcal{E}_{z,\lambda}(|f|)\,.$$

Hence, we have equality for a minimizer and minimizers can be chosen to be nonnegative.

The energy identity (14.17) is obtained by multiplying by $\overline{f_{z,\lambda}}$ in (14.16) and integrating over $]0,+\infty[$. Finally, the inequality (14.18) is the analog of Proposition 10.3.1 and can be proved similarly (or using the maximum principle). □

The preceding proposition does not tell us whether $f_{z,\lambda}$ is trivial (i.e., identically 0) or not. This question is analyzed in the next proposition.

Proposition 14.2.2.
Let $\lambda \in]0,+\infty[$ and $z \in \mathbb{R}$ be given. Then the following properties hold.

1. *The equation*

$$\begin{cases} u'(0) = 0\,, \\ -u'' + (\tau + z)^2 u = \lambda u(1 - |u|^2) \end{cases} \tag{14.19}$$

   admits nontrivial bounded solutions if and only if $\mu(z) < \lambda$.
2. *If $u \in L^\infty(\mathbb{R}^+)$ satisfies* (14.19), *then $\|u\|_\infty \leq 1$. Furthermore, u is in $L^2(\mathbb{R}^+)$ and*

$$\int_0^\infty e^{\alpha\tau}\left(|u(\tau)|^2 + |u'(\tau)|^2\right) d\tau < \infty\,, \tag{14.20}$$

   for all $\alpha > 0$.

3. *If* $\mu(z) < \lambda < 1$, *the only nontrivial bounded solution (up to multiplication by a unit scalar) of* (14.19) *is a minimizer of* $\mathcal{E}_{z,\lambda}$. *In particular, any bounded solution of* (14.19) *has the form* $u = c|u|$ *with* $|c| = 1$ *and any minimizer has this form.*
4. *If* $\mu(z) < \lambda < 1$, *the nonnegative minimizer* $f_{z,\lambda}$ *of* $\mathcal{E}_{z,\lambda}$ *is unique, is strictly positive, and is the (nonnormalized) ground state of the Neumann realization on* $L^2(\mathbb{R}^+)$ *of*

$$-\frac{d^2}{d\tau^2} + (\tau + z)^2 - \lambda(1 - |f_{z,\lambda}(\tau)|^2)\,. \tag{14.21}$$

5. *If* $\lambda < 1$, *the map* $]z_1(\lambda), z_2(\lambda)[\ni z \mapsto f_{z,\lambda} \in B^2(\mathbb{R}^+)$ *is* C^∞.

Proof.

We first prove 1.

Suppose that $\mu(z) \geq \lambda$ and that $u \in L^\infty(\mathbb{R}^+)$ solves (14.19). Let $\chi \in C_0^\infty(\mathbb{R})$ be a standard cutoff function: $\chi(t) = 1$ on $[-1, 1]$, $\operatorname{supp}\chi \subset [-2, 2]$ and define $\chi_N(\tau) := \chi(\tau/N)$. Using the min-max characterization of $\mu(z)$ together with (14.19), we obtain the existence of $C > 0$ such that, for all $N \geq 1$,

$$\begin{aligned}
\mu(z)\|\chi_N u\|_2^2 &\leq \|(\chi_N u)'\|_2^2 + \|(\tau + z)(\chi_N u)\|_2^2 \\
&= \int_0^\infty \chi_N^2(\tau)\overline{u(\tau)}\,[-u''(\tau) + (\tau + z)^2 u(\tau)]\,d\tau + \|u\chi_N'\|_2^2 \\
&\leq \lambda \int_0^{+\infty} (1 - |u(\tau)|^2)\,|\chi_N(\tau)u(\tau)|^2\,d\tau + C\frac{\|u\|_\infty^2}{N}\,.
\end{aligned}$$

Using the fact that $\mu(z) \geq \lambda$, we therefore get

$$\lambda \lim_{N\to+\infty} \int_0^{+\infty} |u(\tau)|^2 |\chi_N(\tau)u(\tau)|^2\,d\tau = 0\,,$$

and conclude that $u = 0$.

If, on the other hand, $\mu(z) < \lambda$, we can consider, for any $s > 0$, the function $s\phi_z$, where ϕ_z is the ground state of the operator $\mathfrak{h}^{N,z}$ studied in Section 3.2. It is easy to see that $\mathcal{E}_{z,\lambda}(s\phi_z) < 0$ for sufficiently small s. Therefore, the minimizer of $\mathcal{E}_{z,\lambda}$ is a nontrivial solution of (14.19), which is bounded by (14.18).

We next prove 2.

The uniform estimate $\|u\|_\infty \leq 1$ is a standard consequence of the maximum principle.

To prove that $u \in L^2$, define $\mathfrak{h}_u$ to be the Neumann realization of

$$-\frac{d^2}{d\tau^2} + (\tau + z)^2 - \lambda(1 - |u(\tau)|^2)\,.$$

Then (as a formal differential operator)

$$\mathfrak{h}_u u = 0\,. \tag{14.22}$$

Also, $(\chi_N u)$ (with χ_N from the previous argument) is a Weyl sequence for $\mathfrak{h}_u$; i.e.,

$$\frac{\langle \chi_N u \,|\, \mathfrak{h}_u(\chi_N u)\rangle}{\|\chi_N u\|_2^2} \to 0\,,$$

as $N \to \infty$. This implies that $0 \in \sigma(\mathfrak{h}_u)$. Since the potential

$$(\tau + z)^2 - \lambda(1 - |u(\tau)|^2)$$

tends to $+\infty$ for large τ, $\sigma(\mathfrak{h}_u)$ is discrete and consists of eigenvalues. Thus, 0 is an eigenvalue of $\mathfrak{h}_u$ and there exists a nontrivial function $v \in L^2(\mathbb{R}^+)$ in the domain of $\mathfrak{h}_u$ such that $\mathfrak{h}_u v = 0$. We may assume that $v(0) = u(0)$. Observing that $v'(0) = u'(0) = 0$, the Cauchy uniqueness theorem implies that $v = u$; i.e., $u \in L^2(\mathbb{R}^+)$.

The decay estimate (14.20) follows from the Agmon technique described in Chapter 12 combined with the easy inequality

$$\begin{aligned}\int_0^{+\infty} |\phi'(\tau)|^2 + (\tau+z)^2|\phi(\tau)|^2\,d\tau &\geq \int_0^{+\infty} (\tau+z)^2|\phi(\tau)|^2\,d\tau \\ &\geq M^2 \int_{|z|+M}^{\infty} |\phi(\tau)|^2\,d\tau\,,\end{aligned}$$

for all $\phi \in B^1(\mathbb{R}^+)$ and all $M > 0$.

We now prove 3 and 4.

We first prove that nontrivial bounded solutions have a sign. Let $\mathfrak{h}_u$ be the operator from the proof of 2. Suppose that $u \in L^\infty(\mathbb{R}^+) \setminus \{0\}$ solves (14.19). Clearly, $\mathfrak{h}_u$ satisfies the operator inequality

$$\mathfrak{h}_u \geq \mathfrak{h}^{N,z} - \lambda\,,$$

with $\mathfrak{h}^{N,z}$ from (3.9). Thus, (3.17), the min-max principle, and the simplicity of eigenvalues in dimension 1 imply that the second eigenvalue $\mu_2(\mathfrak{h}_u)$ of $\mathfrak{h}_u$ satisfies

$$\mu_2(\mathfrak{h}_u) > 0\,.$$

Therefore, 2 and (14.22) imply that u is the ground state of $\mathfrak{h}_u$. Since ground states can be chosen to be positive this proves that u has a sign. In the process, we have proven 4. Moreover, u is necessarily strictly positive by the Cauchy uniqueness theorem. It remains to prove the uniqueness part of 3. Let u_1 and u_2 be two nonnegative and nontrivial solutions of (14.19). Integrating by parts yields

$$\begin{aligned}\int_0^{+\infty} u_1'(\tau)u_2'(\tau)d\tau + \int_0^{+\infty} (\tau+z)^2 u_1(\tau)u_2(\tau)\,d\tau \\ - \lambda \int_0^{+\infty} u_1(\tau)u_2(\tau)\,d\tau = -\lambda \int_0^{+\infty} u_1(\tau)^3 u_2(\tau)\,d\tau\,.\end{aligned}$$

Exchanging the roles of u_1 and u_2 , we get

$$\int_0^{+\infty} u_1(\tau)^3 u_2(\tau)\, d\tau = \int_0^{+\infty} u_2(\tau)^3 u_1(\tau)\, d\tau\,.$$

From this we deduce that if $u_1 \geq u_2$ or $u_2 \geq u_1$, then $u_1 = u_2$.

It remains to exclude the case when one of these two last conditions is not satisfied. Again by Cauchy uniqueness, we have only to exclude the case when

$$u_1(0) > u_2(0)$$

and when there is a point $\tau_{\min} > 0$ such that

$$u_1(\tau) > u_2(\tau) \quad \text{for} \quad \tau \in]0, \tau_{\min}[\quad \text{and} \quad u_1(\tau_{\min}) = u_2(\tau_{\min})\,.$$

We start from the Wronskian identity

$$-\left(u_2^2 \left(\frac{u_1}{u_2} - 1\right)'\right)' + \lambda u_1 u_2 (u_1^2 - u_2^2) = 0\,.$$

We multiply by $\frac{u_1}{u_2} - 1$ and integrate over $]0, \tau_{\min}[$. This gives

$$\int_0^{\tau_{\min}} u_2^2 \left[\left(\frac{u_1}{u_2} - 1\right)'\right]^2 d\tau + \lambda \int_0^{\tau_{\min}} u_1(u_1 + u_2)(u_1 - u_2)^2\, d\tau = 0\,.$$

This is the sum of two positive terms, the second one being strictly positive, and this sum should vanish, hence a contradiction.

We finally prove 5.

This is a consequence of the implicit function theorem applied to the map

$$\begin{aligned} &]z_1(\lambda), z_2(\lambda)[\, \times \{f \in B^2(\mathbb{R}^+)\ :\ f'(0) = 0\} \to L^2(\mathbb{R}^+), \\ &(z, f) \mapsto -f'' + (t+z)^2 f - \lambda f + \lambda f^3\,. \end{aligned}$$

We observe indeed that the operator $-\frac{d^2}{dt^2} + (t+z)^2 - \lambda + 3\lambda f_{z,\lambda}^2$ (the derivative with respect to f) satisfies

$$-\frac{d^2}{dt^2} + (t+z)^2 - \lambda + 3\lambda f_{z,\lambda}^2 = \left\{-\frac{d^2}{dt^2} + (t+z)^2 - \lambda + \lambda f_{z,\lambda}^2\right\} + 2\lambda f_{z,\lambda}^2$$

and is consequently invertible using what we have established in 4 (in particular, the strict positivity of $f_{z,\lambda}$).

This finishes the proof of Proposition 14.2.2. □

Proposition 14.2.3.
Suppose that $\lambda \in]\Theta_0, 1[$. *Then there exists* $z_0 \in \big]z_1(\lambda), z_2(\lambda)\big[$ *such that*

$$b(z_0,\lambda) = \inf_{z\in\mathbb{R}} b(z,\lambda) = \inf_{z\in\mathbb{R},\phi\in B^1(\mathbb{R}^+)} \mathcal{E}_{z,\lambda}(\phi)$$

and

$$\int_0^\infty (\tau + z_0)\big|f_{z_0,\lambda}(\tau)\big|^2\, d\tau = 0\,. \tag{14.23}$$

Proof of Proposition 14.2.3.
One can observe that $b(z,\lambda) < 0$ for $z \in]z_1(\lambda), z_2(\lambda)[$ and that $b(z,\lambda) = 0$ for $z \notin]z_1(\lambda), z_2(\lambda)[$. Moreover, b is a C^∞-function on $]z_1(\lambda), z_2(\lambda)[$. Thus, the existence of a minimum is an exercise that can be left to the reader. Notice that (14.23) is the Euler–Lagrange equation with respect to z . □

It is then natural to introduce the smallest z_0 realizing the infimum.

Definition 14.2.4.
For $\lambda \in [\Theta_0, 1)$, *we define* $\zeta = \zeta(\lambda)$ *by*

$$\zeta(\lambda) := \begin{cases} \min\{z_0 \in \mathbb{R} \,:\, b(z_0,\lambda) = \inf_z b(z,\lambda)\}, & \lambda > \Theta_0\,, \\ \xi_0, & \lambda = \Theta_0\,. \end{cases} \tag{14.24}$$

Clearly, the value at $\lambda = \Theta_0$ is fixed by continuity at that point [since $z_1(\lambda), z_2(\lambda) \to \xi_0$ as $\lambda \searrow \Theta_0$].

Remark 14.2.5.
A natural question is to ask whether the minimum is attained in a single point, i.e., if

$$\{z_0 \in \mathbb{R} \,:\, b(z_0,\lambda) = \inf_z b(z,\lambda)\} = \{\zeta(\lambda)\}\,.$$

We will show that this is indeed the case when λ *is sufficiently close to* Θ_0 .

14.2.2 Bifurcation analysis

We will study here the bifurcation for the nonlinear equation (14.19) near the solution $z = \xi_0$, $\lambda = \Theta_0$, $u = 0$. We use a variant of the standard method due to Lyapunov–Schmidt.

We consider $\lambda \in]\Theta_0, 1[$ and z with $\mu(z) < \lambda$. Let us define φ_z as in (3.2.1) to be the positive ground state of $\mathfrak{h}^{N,z}$ and let Π_z be the L^2-projection on $\operatorname{Span}\varphi_z$. Finally, let us introduce the regularized resolvent of $\mathfrak{h}^{N,z}$ by

$$R_{z,\lambda} := \big(\mathfrak{h}^{N,z} - \lambda\big)^{-1}(1 - \Pi_z)\,.$$

Let us recall the main properties.

Lemma 14.2.6.
Suppose that $\lambda < 1$, $z < 0$. *Then the regularized resolvent* $R_{z,\lambda}$ *maps* $L^2(\mathbb{R}^+)$ *into* $\{u \in B^2(\mathbb{R}^+) \ : \ u'(0) = 0\}$. *Furthermore, if* $K \subset]-\infty, 0[\times[\Theta_0, 1[$ *is compact, then there exists a constant* $C_1 = C_1(K)$ *such that*

$$\|R_{z,\lambda}u\|_{B^1(\mathbb{R}^+)} \leq C_1\|u\|_{L^2(\mathbb{R}^+)} \,, \tag{14.25}$$

for all $(z, \lambda) \in K$.

Proof.
The regularized resolvent maps $L^2(\mathbb{R}^+)$ continuously to the operator domain $\mathcal{D}(\mathfrak{H}^{N,z})$ given by (3.9), in particular to the quadratic form domain. It is immediate that the natural norm on the quadratic form domain is equivalent to the $B^1(\mathbb{R}^+)$ norm. This gives the continuous maps

$$L^2(\mathbb{R}^+) \stackrel{R_{z,\lambda}}{\to} \mathcal{D}(\mathfrak{H}^{N,z}) \stackrel{id}{\to} B^1(\mathbb{R}^+) \,, \tag{14.26}$$

where the first map is uniformly bounded for λ uniformly below 1 [by (3.17)] and the second is uniformly bounded for z varying in compact sets. Note that $(\lambda, z) \mapsto R_{\lambda,z}$ is smooth with values in $\mathcal{L}(L^2(\mathbb{R}^+), B^2(\mathbb{R}^+))$. □

Let $f_{z,\lambda}$ be the unique positive solution to (14.19). If we introduce

$$\epsilon = \langle f_{z,\lambda} \,|\, \varphi_z \rangle \,, \tag{14.27}$$

we get from (14.19) the system of equations

$$(\lambda - \mu(z))\epsilon = \langle f_{z,\lambda}^3 \,|\, \varphi_z \rangle, \tag{14.28}$$

$$f_{z,\lambda} + \lambda R_{z,\lambda} f_{z,\lambda}^3 = \epsilon\varphi_z \,. \tag{14.29}$$

Let us also introduce the map $G = G_{z,\lambda}$ by

$$B^1(\mathbb{R}^+) \ni u \mapsto G(u) = -\lambda R_{z,\lambda} u^3 \,. \tag{14.30}$$

Note that (14.29) reads

$$f_{z,\lambda} - G(f_{z,\lambda}) = \epsilon\varphi_z \,. \tag{14.31}$$

In order to invert this last equation, we have to analyze the properties of G . First, we have.

Lemma 14.2.7.
The map G *maps the space* $B^1(\mathbb{R}^+)$ *to itself. Furthermore, for all compact sets* $K \subset]-\infty, 0[\times[\Theta_0, 1[$, *there exists a constant* C_3 *such that*

$$\|G(u)\|_{B^1(\mathbb{R}^+)} \leq C_3\|u\|^3_{B^1(\mathbb{R}^+)} \,, \tag{14.32}$$

for all $u \in B^1(\mathbb{R}^+)$.

Moreover, $G_{z,\lambda}$ *depends smoothly on the parameters* (z, λ) .

Proof.
This is an immediate consequence of Lemma 14.2.6. We just have to observe in addition that $B^1(\mathbb{R}^+)$ is an algebra and that, by the standard Sobolev embedding theorem, we have

$$B^1(\mathbb{R}^+) \hookrightarrow W^{1,2}(\mathbb{R}^+) \hookrightarrow L^\infty(\mathbb{R}^+)\,,$$

with continuous injection. □

This immediately implies the following contraction property.

Lemma 14.2.8.
Let $K \subset]-\infty, 0[\times[\Theta_0, 1[$ be compact. Then there exists an $R > 0$ such that if $u \in B^1(\mathbb{R}^+)$ with $\|u\|_{B^1(\mathbb{R}^+)} \le R$ and if $(z, \lambda) \in K$, then

$$\|G_{z,\lambda}(u)\|_{B^1(\mathbb{R}^+)} \le \frac{1}{2}\|u\|_{B^1(\mathbb{R}^+)}\,.$$

Therefore, we can define the inverse of $I - G$ by

$$u \mapsto \sum_{j=0}^{\infty} G^j(u)\,,$$

which is defined in the ball centered at 0 and of radius R in $B^1(\mathbb{R}^+)$.

In order to have an a priori control of $f_{z,\lambda}$ in $B^1(\mathbb{R}^+)$, we will use the following lemma.

Lemma 14.2.9.
Let $K \subset]-\infty, 0[\times[\Theta_0, 1[$ be compact. Then there exists a constant $C > 0$ such that if $(z, \lambda) \in K$, $u \in B^1(\mathbb{R}^+)$, and $\mathcal{E}_{z,\lambda}(u) \le 0$, then

$$\|u\|_{B^1(\mathbb{R}^+)} \le C\sqrt{\lambda - \Theta_0}\,. \tag{14.33}$$

Proof.
Using the compactness of K , there exists a constant C_1 , independent of z , such that

$$\|u\|^2_{B^1(\mathbb{R}^+)} \le C_1 \int_0^\infty |u'(\tau)|^2 + (\tau + z)^2|u(\tau)|^2\, d\tau\,. \tag{14.34}$$

Using the negativity of the energy, we find

$$\|u\|^2_{B^1(\mathbb{R}^+)} \le C_1\lambda\|u\|_2^2\,. \tag{14.35}$$

Using again the negativity of the energy, we also find

$$\begin{aligned} \frac{\lambda}{2}\|u\|_4^4 &\le \lambda\|u\|_2^2 - \int_0^\infty |u'(\tau)|^2 + (\tau + z)^2|u(\tau)|^2\, d\tau \\ &\le (\lambda - \mu(z))\|u\|_2^2 \le (\lambda - \Theta_0)\|u\|^2_{B^1(\mathbb{R}^+)}\,. \end{aligned} \tag{14.36}$$

We finally need to use the decay inherent in the space $B^1(\mathbb{R}^+)$ to control the L^2 norm by the L^4 norm. That is achieved using the Hölder inequality:

$$\begin{aligned} \|u\|_2^2 &= \int_0^{+\infty} |u(\tau)||(1+\tau)u(\tau)|(1+\tau)^{-1}\, d\tau \\ &\le \|u\|_4 \, \|(1+\tau)u\|_2 \, \|(1+\tau)^{-1}\|_4 \le C_2 \|u\|_4 \, \|u\|_{B^1(\mathbb{R}^+)} \,. \end{aligned} \tag{14.37}$$

Combining (14.35)–(14.37) yields

$$\|u\|^2_{B^1(\mathbb{R}^+)} \le C \sqrt[4]{\lambda - \Theta_0} \, \|u\|^{\frac{3}{2}}_{B^1(\mathbb{R}^+)} \,, \tag{14.38}$$

(for some new constant C, which may be chosen depending only on K), from which the lemma follows. □

Observing from (14.17) that $\mathcal{E}_{z,\lambda}(f_{z,\lambda}) < 0$ and applying Lemma 14.2.9 with $u = f_{z,\lambda}$, we get

$$\|f_{z,\lambda}\|_{B^1(\mathbb{R}^+)} \le C\sqrt{\lambda - \Theta_0} \,. \tag{14.39}$$

So we get

$$f_{z,\lambda} = \sum_{j=0}^{\infty} G^j(\epsilon\varphi_z) = \sum_{j=0}^{+\infty} \epsilon^{3^j} G^j(\varphi_z) \,, \tag{14.40}$$

with ϵ introduced in (14.27), which satisfies for some constant $C(K)$

$$0 \le \epsilon \le C(K)\sqrt{\lambda - \Theta_0} \,. \tag{14.41}$$

Note that the series (in ϵ) has a positive radius of convergence and that there exists $\epsilon_0(K)$ such that the right-hand side in (14.40) is normally convergent in the $B^1(\mathbb{R}^+)$ norm for $\epsilon \in \mathrm{D}(0, \epsilon_0(K))$.

We denote by $L(\epsilon, z, \lambda)$ the sum of the series, which is a C^∞-function (actually analytic) in all the arguments in $\cup_K \left(\mathrm{D}(0, \epsilon_0(K)) \times K\right)$ with values in $B^1(\mathbb{R}^+)$. It is useful to write L in the form

$$L(\epsilon, z, \lambda) = \epsilon M(\epsilon^2, z, \lambda) \,, \tag{14.42}$$

where $M(\eta, z, \lambda)$ is defined by

$$M(\eta, z, \lambda) = \sum_{j=0}^{+\infty} \eta^{\frac{3^j - 1}{2}} G^j(\varphi_z) \,, \tag{14.43}$$

for $(\eta, \lambda, z) \in \cup_K \left(\mathrm{D}(0, \eta_0(K)) \times K\right)$, with $\eta_0(K) = \epsilon_0(K)^2$.

Hence, we can write

$$f_{z,\lambda} = \epsilon M(\epsilon^2, z, \lambda) \,.$$

Note that

$$M(0, z, \lambda) = \varphi_z \,. \tag{14.44}$$

Inserting this equality into (14.28), we get

$$(\lambda - \mu(z))\epsilon = \epsilon^3 \langle M(\epsilon^2, z, \lambda)^3 \,|\, \varphi_z \rangle \,.$$

We look for positive, nontrivial solutions; therefore, we have to impose $\epsilon > 0$. So we can rewrite the equation in the form

$$(\lambda - \mu(z)) = \epsilon^2 m(\epsilon^2, z, \lambda)\,, \tag{14.45}$$

where $(\eta, z, \lambda) \mapsto m(\eta, z, \lambda)$ is a C^∞-function such that

$$m(0, z, \lambda) = \int_0^{+\infty} \varphi_z^4(\tau)\, d\tau \neq 0\,.$$

We first easily solve the equation for ζ and η in a small complex neighborhood of 0

$$\nu = \eta\, m(\eta, z, \lambda)$$

and get

$$\eta = \nu\, n(\nu, z, \lambda)\,,$$

with

$$n(0, z, \lambda) = \frac{1}{\|\varphi_z\|_4^4} > 0\,.$$

When $\lambda - \mu(z) > 0$, we finally recover ϵ by

$$\epsilon(z, \lambda) = \sqrt{\lambda - \mu(z)}\sqrt{n(\lambda - \mu(z), z, \lambda)}\,, \tag{14.46}$$

and the desired (unique) positive solution $f_{z,\lambda}$ is given by

$$f_{z,\lambda} = \epsilon(z, \lambda) M(\eta(z, \lambda), z, \lambda)\,, \tag{14.47}$$

with

$$\eta(z, \lambda) = (\lambda - \mu(z)) n(\lambda - \mu(z), z, \lambda)\,. \tag{14.48}$$

This shows that $(z, \lambda) \mapsto \eta(z, \lambda)$ and hence $(z, \lambda) \mapsto M(\eta(z, \lambda), z, \lambda)$ are C^∞-functions for z near ξ_0 and λ near (and above) Θ_0.

Let us finish by proving that the minimum $\zeta(\lambda)$ is attained for a unique value of z_0 when $\lambda - \Theta_0$ is small. We use (14.23) (which is satisfied for any minimum of b) linking ϵ, λ, and z_0 and implement (14.47). This gives

$$\begin{aligned} 0 &= \int_0^{+\infty} (\tau + z_0)|f_{z_0,\lambda}(\tau)|^2 d\tau \\ &= \epsilon^2 \int_0^{+\infty} (\tau + z_0)|M(\eta(z_0, \lambda), z_0, \lambda)|^2 d\tau\,. \end{aligned} \tag{14.49}$$

Using the Feynman–Hellmann formula (3.29) and (14.44), we find

$$\mu'(z) = \int_0^{+\infty} (\tau + z)\varphi_z(\tau)^2\, d\tau = \int_0^{+\infty} (\tau + z)|M(0, z_0, \lambda)|^2\, d\tau$$
$$= -\int_0^{+\infty} (\tau + z)\big(|M(\eta(z_0, \lambda), z_0, \lambda)|^2 - |M(0, z_0, \lambda)|^2\big)\, d\tau\,,$$

where we used (14.49) to get the last equality. By (14.43), we find

$$\mu'(z_0) = \eta(z_0, \lambda)^2 a(z_0, \lambda)\,,$$

with $a(z, \lambda)$ being a smooth function.

Inserting (14.48) yields that a minimum should be a solution of the equation

$$\mu'(z) = (\lambda - \mu(z))\tilde{a}(z, \lambda)\,,$$

for a new smooth function $\tilde{a}$. Consider the smooth function

$$(z, \lambda) \mapsto g(z, \lambda) := \mu'(z) - (\lambda - \mu(z))\tilde{a}(z, \lambda)\,.$$

Using the fact that $\mu'(\xi_0) = 0$, $\mu(\xi_0) = \Theta_0$, we get

$$\frac{\partial g}{\partial z}(\xi_0, \Theta_0) = \mu''(\xi_0) > 0\,,$$

by (3.23). Therefore, the implicit function theorem implies that there exists a unique (smooth) solution $\zeta(\lambda)$ to the equation $g(z, \lambda) = 0$ for λ sufficiently close to Θ_0.

Hence, we have proven the following refinement of Proposition 14.2.3.

Proposition 14.2.10.
There exists $\eta_0 > 0$ such that for any $\lambda \in]\Theta_0, \Theta_0 + \eta_0]$, there exists a unique $\zeta(\lambda) \in]z_1(\lambda), z_2(\lambda[$ such that

$$b(\zeta(\lambda), \lambda) = \inf_z b(z, \lambda)\,.$$

Moreover,

- *$\lambda \mapsto \zeta(\lambda)$ is a C^∞-function on $[\Theta_0, \Theta_0 + \eta_0]$ with $\zeta(\Theta_0) = \xi_0$.*
- *$\lambda \mapsto f^2_{\zeta(\lambda),\lambda}$ is a B^1-valued C^∞-function on $[\Theta_0, \Theta_0 + \eta_0]$.*

For the calculation of the ground state energy, we will need the following result.

Lemma 14.2.11.
Let $u_0 = \varphi_{\xi_0}$ be the normalized ground state of $\mathfrak{h}^{N,\xi_0}$ as in Section 3.2. We have

$$\|f_{\zeta(\lambda),\lambda}\|_4^4 = \frac{(\lambda - \Theta_0)^2}{\|u_0\|_4^4}(1 + o(1))\,, \tag{14.50}$$

as $\lambda \to \Theta_0$.

Proof.
By the convergence of the series (14.40) in the B^1 norm and therefore in the L^4 norm we get

$$\|f_{\zeta(\lambda),\lambda}\|_4^4 = |\epsilon|^4 \|\varphi_{\zeta(\lambda)}\|_4^4 + \mathcal{O}(|\epsilon|^6)\,.$$

By continuity in ζ , we find $\|\varphi_{\zeta(\lambda)}\|_4 = \|u_0\|_4(1+o(1))$, which, combined with (14.46), yields

$$\|f_{\zeta(\lambda),\lambda}\|_4^4 = \frac{\big(\lambda - \mu(\zeta(\lambda))\big)}{\|u_0\|_4^4}(1+o(1))\,.$$

Since

$$\mu(\zeta(\lambda)) = \Theta_0 + \mathcal{O}\big((\zeta(\lambda)-\xi_0)^2\big) = \Theta_0 + \mathcal{O}\big((\lambda-\Theta_0)^2\big)\,,$$

this finishes the proof of (14.50). □

The lemma permits us to immediately get

Proposition 14.2.12.
There exists $\eta_0 > 0$, such that $\lambda \mapsto \inf_z b(z,\lambda)$ is a C^∞-function on the interval $[\Theta_0, \Theta_0+\eta_0]$ and satisfies

$$\inf_z b(z,\lambda) = -\frac{\lambda}{2}\frac{\big(\lambda-\Theta_0\big)^2}{\|u_0\|_4^4}(1+o(1))\,, \tag{14.51}$$

as $\lambda \to \Theta_0$.

14.2.3 The spectral estimate

The following proposition will be very important in the analysis of the 2D problem in half-cylinders. Let us introduce the following closed and symmetric quadratic form on $B^1(\mathbb{R}^+)$:

$$q_{\alpha,\lambda}(\phi) := \int_0^{+\infty} |\phi'(\tau)|^2 + (\tau+\zeta+\alpha)^2|\phi(\tau)|^2 - \lambda(1-|f_{\zeta,\lambda}(\tau)|^2)|\phi(\tau)|^2\,d\tau\,, \tag{14.52}$$

with $\zeta = \zeta(\lambda)$.

Furthermore, let

$$\gamma(\alpha,\lambda) := \inf \sigma\left(q_{\alpha,\lambda}\right)$$

be the infimum of the spectrum of the unique self-adjoint operator $\mathfrak{h}_{\alpha,\lambda}$ associated with $q_{\alpha,\lambda}$. Then we have the following property.

Proposition 14.2.13.
There exists $\epsilon > 0$ such that for $\lambda \in [\Theta_0, \Theta_0+\epsilon[$, we have

$$\inf_{\alpha\in\mathbb{R}} \gamma(\alpha,\lambda) = 0\,.$$

Proof.
Clearly, $q_{\alpha,\lambda}$ is the quadratic form of the Neumann realization $\mathfrak{h}_{\alpha,\lambda}$ on $L^2(\mathbb{R}^+)$ of the differential operator

$$-\frac{d^2}{d\tau^2} + (\tau+\zeta+\alpha)^2 - \lambda(1-|f_{\zeta,\lambda}(\tau)|^2)\,.$$

By (14.16), $f_{\zeta,\lambda}$ is an eigenfunction of $\mathfrak{h}_{0,\lambda}$ with eigenvalue 0. Since $f_{\zeta,\lambda}$ is positive, it is necessarily the ground state. Therefore,

$$\gamma(0,\lambda) = 0\,. \tag{14.53}$$

Furthermore, we will see that

$$\frac{\partial\gamma}{\partial\alpha}(0,\lambda) = 0\,. \tag{14.54}$$

For the proof of this, define $u = u(\cdot;\alpha,\lambda)$ to be the positive, normalized ground state of $\mathfrak{h}_{\alpha,\lambda}$. The family $\alpha \mapsto \mathfrak{h}_{\alpha,\lambda}$ is a holomorphic family of type (A). By perturbation theory (see Appendix C), the map $\alpha \mapsto \gamma(\alpha,\lambda)$ is analytic in α and

$$\frac{\partial\gamma}{\partial\alpha}(\alpha,\lambda) = 2\int_0^{+\infty} (\tau+\alpha+\zeta)|u|^2\,d\tau\,. \tag{14.55}$$

Thus,

$$\frac{\partial\gamma}{\partial\alpha}(0,\lambda) = \frac{2}{\|f_{\zeta,\lambda}\|_2^2}\int_0^{+\infty} (\tau+\zeta)\big|f_{\zeta,\lambda}(\tau)\big|^2\,d\tau = 0\,,$$

by (14.23).

We will prove in Lemma 14.2.17 that

$$\gamma_{\alpha\alpha}(0,\lambda) > 0\,. \tag{14.56}$$

Now, because $\lambda \mapsto |f_{\zeta(\lambda),\lambda}|^2$ is a C^∞-function on $[\Theta_0, \Theta_0+\eta_0]$, one gets by a rather standard perturbation argument that $(\alpha,\lambda)\mapsto\gamma(\alpha,\lambda)$ is a C^∞-function.

From (14.53), (14.54), and (14.56), it follows that there exist α_1 and $\epsilon_1 > 0$ such that if $\lambda < \Theta_0 + \epsilon_1$, then

$$\gamma(\alpha,\lambda) \geq 0 \qquad \text{for all } |\alpha| \leq \alpha_1\,.$$

Furthermore, it is clear that $\gamma(\alpha,\lambda) \geq \mu(\zeta+\alpha) - \lambda$, and we know that we have the convergence $\zeta(\lambda) \to \xi_0$ as $\lambda \to \Theta_0$. Therefore, using our knowledge of the function $\mu(\xi)$, we see that there exists $\epsilon_2 < \epsilon_1$ such that if $\lambda < \Theta_0 + \epsilon_2$ and $|\alpha| \geq \alpha_1$, then

$$\gamma(\alpha,\lambda) \geq 0\,.$$

This finishes the proof modulo the proof of (14.56). □

Proof of (14.56)

There are two possible proofs. The first one is done by direct computation, which will be established in the next lemma. A second one could be obtained by extending to complex α. We present the first one.

Lemma 14.2.14.
For any $\lambda \in]\Theta_0, 1[$, *we have, with* $\zeta = \zeta(\lambda)$, $f = f_{\zeta(\lambda),\lambda}$,

$$\frac{1}{2}\gamma_{\alpha\alpha}(0,\lambda)\,\|f\|_2^2 = -\zeta f(0)^2 - 2\lambda \int_0^{\infty} f(\tau)^2 f'(\tau)^2\,d\tau - \frac{\lambda^2}{4}\|f\|_4^4 . \tag{14.57}$$

Proof.
This is a tricky computation. We write for brevity $f = f_{\zeta,\lambda}$, and as before $u = u(\cdot;\alpha,\lambda)$ is the normalized ground state of $\mathfrak{h}_{\alpha,\lambda}$. As in [Pa2], we introduce the function

$$H(t) := f'(t)^2 - (t+\zeta)^2 f(t)^2 + \lambda f(t)^2 - \frac{\lambda}{2} f(t)^4 , \tag{14.58}$$

and recall that

$$f = \|f\|_2 u \quad \text{if} \quad \alpha = 0 . \tag{14.59}$$

First, we immediately see that

$$H'(t) = -2(t+\zeta) f^2(t) , \tag{14.60}$$

and hence, in view of the behavior of f at $+\infty$ and of the definition of ζ, we obtain by integration

$$H(0) = 0 . \tag{14.61}$$

Differentiating the equation satisfied by u,

$$\mathfrak{h}_{\alpha,\lambda} u = \gamma(\alpha,\lambda) u , \tag{14.62}$$

with respect to α, with $u_\alpha(t) := \frac{\partial u}{\partial \alpha}(t;\alpha,\lambda)$, we obtain

$$\begin{cases} -u_\alpha'' + (t+\alpha+\zeta)^2 u_\alpha - \lambda(1-f^2) u_\alpha = \gamma u_\alpha + \gamma_\alpha u - 2(t+\alpha+\zeta) u , \\ u_\alpha'(0) = 0 . \end{cases} \tag{14.63}$$

Let us introduce v by

$$u_\alpha(t;\alpha,\lambda) = f(t)\,v(t) . \tag{14.64}$$

We observe that

$$v'(0) = 0 . \tag{14.65}$$

We now write

$$u_\alpha' = f'v + fv' , \qquad u_\alpha'' = f''v + 2f'v' + fv'' .$$

Substituting in (14.63), we obtain

$$\begin{aligned} &- f''v - 2f'v' - fv'' + (t+\alpha+\zeta)^2 fv - \lambda(1-f^2)fv \\ &\quad = \gamma fv + \gamma_\alpha u - 2(t+\alpha+\zeta)u\,. \end{aligned}$$

We will now evaluate at $\alpha = 0$. By the definition of ζ, we have

$$\gamma = 0, \qquad \gamma_\alpha = 0, \qquad\qquad \text{at } \alpha = 0.$$

Using the equation satisfied by f, we obtain, for $\alpha = 0$,

$$-2f'v' - fv'' = -2(t+\zeta)u\,.$$

We now multiply this equation by f and get from (14.59) that, at $\alpha = 0$,

$$-2ff'v' - f^2v'' = -2(t+\zeta)uf = -2(t+\zeta)\frac{f^2}{\|f\|_2}\,,$$

which can be written, using (14.60), in the form

$$\left(f^2v'\right)' = -\frac{H'}{\|f\|_2}\,, \tag{14.66}$$

for $\alpha = 0$.

Consequently, by integration over $]0,+\infty[$, and using the conditions at 0 (14.55) and (14.65), we get

$$H = -f^2v'\|f\|_2 \qquad \text{for } \alpha = 0\,. \tag{14.67}$$

Let us start an independent computation. We differentiate (14.55) with respect to α and obtain for $\alpha = 0$

$$\gamma_{\alpha\alpha}(0,\lambda) = 2\left(1 + 2\int_0^{+\infty}(\tau+\zeta)u(\tau)u_\alpha(\tau)\,d\tau\right)\,. \tag{14.68}$$

This leads after an integration by parts and using (14.61) to

$$\frac{\gamma_{\alpha\alpha}(0,\lambda)}{2} = 1 - \frac{1}{\|f\|_2}\int_0^{+\infty} H'v\,d\tau = 1 + \frac{1}{\|f\|_2}\int_0^{+\infty} Hv'\,d\tau\,.$$

Hence, having (14.67) in mind, we get

$$\frac{\gamma_{\alpha\alpha}(0,\lambda)}{2} = 1 - \frac{1}{\|f\|_2^2}\int_0^{+\infty}\frac{H^2}{f^2}\,d\tau\,. \tag{14.69}$$

Recall the definition of H. We see that (14.69) expresses the second partial derivative $\gamma_{\alpha\alpha}(0,\lambda)$ purely in terms of the solution f. We will now use the equation satisfied by f in order to control the sign of $\gamma_{\alpha\alpha}(0,\lambda)$ (for small λ).

Notice, using the equation satisfied by f, that

$$H = f'^2 - ff'' + \frac{\lambda}{2} f^4 \,. \tag{14.70}$$

Thus,

$$\int_0^\infty \frac{H^2}{f^2}\, d\tau = \int_0^\infty \left(\frac{H[(f')^2 - ff'']}{f^2} + \frac{\lambda}{2} H f^2 \right) d\tau \,. \tag{14.71}$$

We split the integral into two parts. Consider first

$$\int_0^\infty \frac{H[(f')^2 - ff'']}{f^2} d\tau = \int_0^\infty H \left(\frac{-f'}{f} \right)' d\tau = \int_0^\infty \frac{f'}{f} H'\, d\tau \,.$$

Note, for controlling the boundary terms in the integration by parts, that H decays exponentially rapidly at $+\infty$ and that $f'(\tau)/f(\tau)$ behaves like τ at $+\infty$ by the theory of ordinary differential equations (as presented in [Sib]). Using (14.60) and a further integration by parts yields

$$\int_0^\infty \frac{H[(f')^2 - ff'']}{f^2} d\tau = -\int_0^\infty 2(\tau + \zeta) f f'\, d\tau = \zeta f^2(0) + \|f\|_2^2 \,. \tag{14.72}$$

We also evaluate the other term in (14.71), using (14.70) and an integration by parts:

$$\begin{aligned} \int_0^\infty H f^2\, d\tau &= \int f^2 \{(f')^2 - ff'' + \frac{\lambda}{2} f^4\}\, d\tau \\ &= 4 \int_0^\infty f^2 (f')^2\, d\tau + \frac{\lambda}{2} \|f\|_4^4 \,. \end{aligned} \tag{14.73}$$

Combining (14.69), and (14.71)–(14.73), we get (14.57). This achieves the proof of the lemma. □

Remark 14.2.15.
In Lemma 14.2.14, $\zeta(\lambda)$ can be any critical value $\zeta^c(\lambda) \in]z_1(\lambda), z_2(\lambda)[$ such that

$$\int_0^{+\infty} (\tau + \zeta^c(\lambda)) |f_{\zeta^c(\lambda),\lambda}(\tau)|^2\, d\tau = 0 \,.$$

Remark 14.2.16.
Using (14.61)*, one can see that*

$$\lambda - \zeta^c(\lambda)^2 = \frac{\lambda}{2} f^2_{\zeta^c(\lambda),\lambda}(0) \,. \tag{14.74}$$

So $\zeta^c(\lambda)$ is a solution of the implicit equation

$$\lambda - z^2 = \frac{\lambda}{2} f^2_{z,\lambda}(0) \,.$$

This gives another way to show the uniqueness of $\zeta^c(\lambda)$ for λ close to Θ_0. We have indeed shown that the map $(z,\lambda) \mapsto f^2_{z,\lambda}$ is C^∞ with value in B^1. This implies that $(z,\lambda) \mapsto f^2_{z,\lambda}(0)$ is C^∞.

So, taking the limit $\lambda \to \Theta_0$ in (14.57) and the properties of $f_{\zeta(\lambda),\lambda}$ as $\lambda \to \Theta_0$, we obtain the equivalent of (3.24).

Lemma 14.2.17.

$$\lim_{\lambda \to \Theta_0} \frac{1}{2}\gamma_{\alpha\alpha}(0,\lambda) = -\xi_0 \varphi_{\xi_0}^2(0) > 0\,. \tag{14.75}$$

Remark 14.2.18.
A variant of the Feynman–Hellmann formula gives

$$b''(\zeta(\lambda),\lambda)) = \|f_{\zeta^c(\lambda),\lambda}\|_2^2\, \frac{\partial^2 \gamma}{\partial \alpha^2}(0,\lambda)\,.$$

So there exists $\delta_0 > 0$ *such that, for any* $\lambda \in]\Theta_0, \Theta_0 + \delta_0]$, $b(\cdot,\lambda)$ *has in* $]z_1(\lambda), z_2(\lambda)[$ *a nondegenerate minimum at* $\zeta(\lambda)$.

14.3 Models on Half-Cylinders

Theorem 14.3.1.
For $\omega \in]0,+\infty[$, $\lambda_1 \in [\Theta_0, 1]$, *and* $\lambda_2 \in \mathbb{R}^+$, *let us consider the functional*

$$\begin{aligned} \mathcal{H}_\omega \ni \psi \mapsto E_\omega^{\mathrm{cyl}}(\psi,\lambda_1,\lambda_2) \\ := \int_{-\pi/\omega}^{\pi/\omega} \left(\int_0^\infty |(i\nabla + \xi_1 \hat{i}_2)\psi|^2 - \lambda_1 |\psi|^2 + \frac{\lambda_2}{2}|\psi|^4\, d\xi_1 \right) d\xi_2\,, \end{aligned} \tag{14.76}$$

where

$$|(i\nabla + \xi_1 \hat{i}_2)\psi|^2 = |i\partial_{\xi_1}\psi|^2 + |(i\partial_{\xi_2} + \xi_1)\psi|^2$$

and

$$\begin{aligned} \mathcal{H}_\omega = \Big\{ \psi \in H^1_{\mathrm{mag}}\big(\mathbb{R}^+ \times]-L,L[\big)\,,\ \forall L > 0 \ \Big| \\ \exists z \in \mathbb{R} :\ \psi(\xi_1, \xi_2 + 2\pi/\omega) = e^{-iz\frac{2\pi}{\omega}}\psi(\xi_1,\xi_2) \Big\}\,. \end{aligned}$$

Let $\psi_{\lambda_1,\lambda_2}$ *be the function*

$$\mathbb{R}^+ \times \mathbb{R} \ni (\xi_1,\xi_2) \mapsto \psi_{\lambda_1,\lambda_2}(\xi_1,\xi_2) := \sqrt{\frac{\lambda_1}{\lambda_2}}\, e^{-i\zeta(\lambda_1)\xi_2} f_{\zeta(\lambda_1),\lambda_1}(\xi_1)\,. \tag{14.77}$$

Then, for all $M > 1$, *there exists* $\epsilon > 0$ *such that*

$$E_\omega^{\mathrm{cyl}}(\psi,\lambda_1,\lambda_2) \geq E_\omega^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2},\lambda_1,\lambda_2) \tag{14.78}$$

for all $\lambda_1 \in]\Theta_0, \Theta_0 + \epsilon[$, $\omega > 0$, $\lambda_2 \in [M^{-1}, M]$, *and* $\psi \in \mathcal{H}_\omega$.

Here

$$H^1_{\mathrm{mag}}\big(\mathbb{R}^+\times]-L,L[\big) := \Big\{\psi\in L^2\big(\mathbb{R}^+\times]-L,L[\big)\ :\ |(i\nabla+\xi_1\hat{i}_2)\psi|\in L^2\big(\mathbb{R}^+\times]-L,L[\big)\Big\}\,.$$

Remark 14.3.2.
Clearly, $\psi_{\lambda_1,\lambda_2}$ is in $\mathcal{H}_\omega$ [take $z=\zeta(\lambda_1)$]. Hence, the theorem states that it is the global minimizer of E^{cyl}_ω in $\mathcal{H}_\omega$. Inserting (14.17) *and* (14.13), *we can express the minimal energy as*

$$E^{\mathrm{cyl}}_\omega(\psi_{\lambda_1,\lambda_2},\lambda_1,\lambda_2)=\frac{2\pi\lambda_1}{\omega\lambda_2}b\big(\zeta(\lambda_1)\big)=-\frac{\pi\lambda_1^2}{\omega\lambda_2}\|f_{\zeta(\lambda_1),\lambda_1}\|_4^4\,. \tag{14.79}$$

Proof.
Consider first functions in $\mathcal{H}_\omega$ that are given in the form

$$(\xi_1,\xi_2)\mapsto\psi(\xi_1,\xi_2):=g_\zeta(\xi_1)e^{-i\zeta\xi_2}v\,, \tag{14.80}$$

with

$$g_\zeta:=\sqrt{\frac{\lambda_1}{\lambda_2}}f_{\zeta,\lambda_1}\,,\qquad \zeta=\zeta(\lambda_1)\,,$$

and v periodic:

$$v(\xi_1,\xi_2)=v(\xi_1,\xi_2+\tfrac{2\pi}{\omega})\,.$$

Then

$$E^{\mathrm{cyl}}_\omega(\psi,\lambda_1,\lambda_2) = \int_{-\pi/\omega}^{\pi/\omega}\int_0^\infty |(i\nabla+(\xi_1+\zeta)\hat{i}_2)g_\zeta v|^2-\lambda_1|g_\zeta v|^2+\frac{\lambda_2}{2}|g_\zeta v|^4\,d\xi_1d\xi_2\,.$$

By periodicity, we can expand v in an L^2-convergent Fourier series as

$$v(\xi_1,\xi_2)=\sum_{n=-\infty}^{\infty}v_n(\xi_1)e^{-in\omega\xi_2}\,.$$

Then, using Parseval's theorem, we get

$$\begin{aligned}E^{\mathrm{cyl}}_\omega(\psi,\lambda_1,\lambda_2)=\frac{2\pi}{\omega}\sum_n\int_0^{+\infty}&\Big\{\big|(g_\zeta v_n)'\big|^2+(n\omega+\xi_1+\zeta)^2|g_\zeta v_n|^2\\&-\lambda_1|g_\zeta v_n|^2+\lambda_2 g_\zeta^2|g_\zeta v_n|^2\Big\}\,d\xi_1\\&-\lambda_2\frac{2\pi}{\omega}\sum_n\int_0^{+\infty}g_\zeta^4|v_n|^2\,d\xi_1+\frac{\lambda_2}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^{+\infty}|g_\zeta v|^4\,d\xi_1d\xi_2\,.\end{aligned}$$

Again using Parseval's theorem, the last two terms can be combined and we see that

$$E_\omega^{\mathrm{cyl}}(\psi,\lambda_1,\lambda_2) = \sum_n \frac{2\pi}{\omega}\int_0^{+\infty} \Big\{\big|(g_\zeta v_n)'\big|^2 + (n\omega+\xi_1+\zeta)^2|g_\zeta v_n|^2 - \lambda_1|g_\zeta v_n|^2 + \lambda_2 g_\zeta^2|g_\zeta v_n|^2\Big\}\,d\xi_1 + \frac{\lambda_2}{2}\int_{-\pi/\omega}^{\pi/\omega}\int_0^{+\infty} g_\zeta^4\Big((|v|^2-1)^2-1\Big)\,d\xi_1 d\xi_2\,.$$

Using (14.13) and (14.17), we see that

$$E_\omega^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2},\lambda_1,\lambda_2) = -\frac{\lambda_2}{2}\frac{2\pi}{\omega}\int_0^{+\infty} g_\zeta^4\,d\xi_1\,, \tag{14.81}$$

and therefore,

$$E_\omega^{\mathrm{cyl}}(\psi,\lambda_1,\lambda_2) - E_\omega^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2},\lambda_1,\lambda_2) \geq \sum_n \frac{2\pi}{\omega}\gamma(n\omega,\lambda_1,\lambda_2)\int_0^{+\infty}|g_\zeta v_n|^2\,d\xi_1\,. \tag{14.82}$$

This finishes the proof under Assumption (14.80) if ϵ is so small that Proposition 14.2.13 can be applied.

To prove (14.78) for all $\psi \in \mathcal{H}_\omega$, we now consider functions of the form

$$(\xi_1,\xi_2) \mapsto \psi_0(\xi_1,\xi_2) = g_\zeta(\xi_1)e^{-iz\xi_2}v\,, \quad \text{with } v(\xi_1,\xi_2) = v(\xi_1,\xi_2+\tfrac{2\pi}{\omega})\,. \tag{14.83}$$

Consider first the case when $\omega \in \mathbb{R}^+$ satisfies

$$\frac{\zeta - z}{\omega} = \frac{p}{q} \text{ for some pair } (p,q) \in \mathbb{Z}\times\mathbb{N}\,. \tag{14.84}$$

Clearly, if ψ_0 satisfies (14.83) for some $\omega \in \mathbb{R}^+$, then it also satisfies (14.83) for $\omega/\hat{q}$, for every $\hat{q} \in (\mathbb{N}\setminus\{0\})$. Moreover, it is easy to show that

$$E_{\omega/\hat{q}}^{\mathrm{cyl}}(\psi_0) = \hat{q}E_\omega^{\mathrm{cyl}}(\psi_0)\,, \qquad E_{\omega/\hat{q}}^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2}) = \hat{q}E_\omega^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2})\,. \tag{14.85}$$

We now choose $\hat{q} = q$, and observe that, according to (14.84), $\hat{\omega} = \omega/q$ satisfies

$$\frac{\zeta - z}{\hat{\omega}} \in \mathbb{Z}\,. \tag{14.86}$$

But in this case, ψ_0 admits the representation (14.80), and hence

$$E_{\hat{\omega}}^{\mathrm{cyl}}(\psi_0) \geq E_{\hat{\omega}}^{\mathrm{cyl}}(\psi_{\lambda_1,\lambda_2})\,.$$

Coming back to ω and using (14.85), we have the proof of (14.78) when ω satisfies (14.84) (with the additional condition that z is fixed).

The proof of (14.78) in the general case now follows immediately from the density of the rational numbers in $\mathbb{R}$. □

14.4 Proof of (14.5) and (14.6)

14.4.1 Lower bounds

The main part of this argument will also be valid in the case of (14.4), at least for $b > 1$, where the exponential decay estimates of Theorem 12.2.1 are valid.

We will use the parameter

$$\varepsilon := \frac{1}{\sqrt{\kappa\sigma}}\,.$$

Let $(\psi, \mathbf{A})$ be a minimizer of $\mathcal{G}$. First, we need to make a localization to the boundary region. Let $1 = f_1^2(t) + f_2^2(t)$ be a standard partition of unity on $[0, \infty[$. We choose f_1 to be nonincreasing and satisfying

$$f_1(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 0 & \text{if } t \geq 2\,. \end{cases} \tag{14.87}$$

Consider $\psi_j(x) = f_j(t(x)/\varepsilon M)\psi(x)$ [in (14.95), we will choose $M = C|\log \varepsilon|$ for some large constant C]. For ε small enough, one can change to boundary coordinates on the support of $f_1(t(x)/\varepsilon M)$. Then, by the localization formula [see (8.10)],

$$\begin{aligned}\mathcal{G}(\psi, \mathbf{A}) =& \mathcal{G}(\psi_1, \mathbf{A}) + \mathcal{G}(\psi_2, \mathbf{A}) + \frac{\kappa^2}{2}\int \left(1 - f_1^4\left(\tfrac{t(x)}{\varepsilon M}\right) - f_2^4\left(\tfrac{t(x)}{\varepsilon M}\right)\right)|\psi(x)|^4\,dx \\ &- (\varepsilon M)^{-2}\int \left(\left|f_1'\left(\tfrac{t(x)}{\varepsilon M}\right)\right|^2 + \left|f_2'\left(\tfrac{t(x)}{\varepsilon M}\right)\right|^2\right)|\psi(x)|^2\,dx\,. \end{aligned} \tag{14.88}$$

Consider first the third term in (14.88). Since

$$1 = (f_1^2 + f_2^2)^2 = f_1^4 + f_2^4 + 2f_1^2 f_2^2\,,$$

this term is positive. We will therefore discard it for the lower bound.

Proposition 11.4.4 tells us that $\operatorname{curl}\mathbf{A} \approx 1$, and therefore, since ψ_2 has compact support in Ω, and using Lemma 1.4.1,

$$\mathcal{G}(\psi_2, \mathbf{A}) \geq \varepsilon^{-2}(1 - \mathcal{O}(\kappa^{-1}))\|\psi_2\|_2^2 \geq 0\,.$$

So we can ignore this positive term for the lower bound.

The Agmon estimates, Theorem 12.2.1, combined with the properties of the support of $(|f_1'|^2 + |f_2'|^2)(t(x)/(\varepsilon M))$ can be used to bound the localization errors as follows:

$$\begin{aligned}
(\varepsilon M)^{-2} &\int (|f_1'|^2 + |f_2'|^2)\left(\frac{t(x)}{\varepsilon M}\right)|\psi(x)|^2\,dx \\
&\leq C(\varepsilon M)^{-2}\int_{\left\{1\leq \frac{t(x)}{\varepsilon M}\leq 2\right\}} e^{-\frac{\alpha t(x)}{\varepsilon}}\left(e^{\frac{\alpha t(x)}{\varepsilon}}|\psi(x)|^2\right)dx \\
&\leq C(\varepsilon M)^{-2}e^{-\alpha M}\int e^{\frac{\alpha t(x)}{\varepsilon}}|\psi(x)|^2\,dx \\
&\leq C'(\varepsilon M)^{-2}\,e^{-\alpha M}\int_{\{t(x)<c_0\varepsilon\}}|\psi(x)|^2\,dx \\
&\leq C'(\varepsilon M)^{-2}\,e^{-\alpha M}\|\psi_1\|_2^2\,. \qquad (14.89)
\end{aligned}$$

Here we used, in the last line, the fact that $M\to\infty$ as $\varepsilon\to 0$; therefore (for ε sufficiently small),

$$\int_{\{t(x)<c_0\varepsilon\}}|\psi(x)|^2\,dx \leq \|\psi_1\|_2^2\,.$$

For sufficiently large M, we find

$$(\varepsilon M)^{-2}\int(|f_1'|^2+|f_2'|^2)\left(\frac{t(x)}{\varepsilon M}\right)|\psi(x)|^2\,dx \leq C\varepsilon^{-2}\,e^{-\frac{\alpha M}{2}}\,\|\psi_1\|_2^2\,.$$

From these estimates and (14.88), we find

$$\mathcal{G}(\psi,\mathbf{A}) \geq \int |(-i\nabla+\kappa\sigma\mathbf{A})\psi_1|^2 - \kappa^2(1+Ce^{-\frac{\alpha M}{2}})|\psi_1|^2 + \frac{\kappa^2}{2}|\psi_1|^4\,dx\,.$$

Upon changing to boundary coordinates (see Section F.1), this integral becomes:

$$\begin{aligned}
\int_0^{|\partial\Omega|}\int_{\{t\leq 2M\varepsilon\}} &\Big\{|\partial_t\phi|^2 + (1-tk(s))^{-2}|(-i\partial_s+\kappa\sigma\tilde{A}_1)\phi|^2 \\
&-\kappa^2(1+Ce^{-\frac{\alpha M}{2}})|\phi|^2 + \frac{\kappa^2}{2}|\phi|^4\Big\}(1-tk(s))\,dt\,ds\,, \qquad (14.90)
\end{aligned}$$

where

$$\phi = e^{-i\kappa\sigma\varphi}\psi_1(\Phi(s,t))\,, \qquad (14.91)$$

and $\tilde{A}_1$ is defined in Lemma F.1.1. Here the gauge transformation $e^{-i\kappa\sigma\varphi}$ is chosen in order to have $\tilde{A}_2=0$. It also follows from Lemma F.1.1 combined with Proposition 11.4.4 that

$$\tilde{A}_1(s,t) = \gamma_\varepsilon - t + \frac{k(s)t^2}{2} + \mathcal{O}(\varepsilon t^2) \qquad (14.92)$$

(uniformly in ε) in a fixed neighborhood of the boundary. Again using Proposition 11.4.4,

$$\gamma_\varepsilon := \frac{1}{|\partial\Omega|}\int_\Omega \operatorname{curl}\mathbf{A}\,dx = \frac{|\Omega|}{|\partial\Omega|} + \mathcal{O}(\varepsilon)\,. \tag{14.93}$$

In order to have a simple model operator, we want to replace $\tilde{A}_1(s,t)$ by $\gamma_\varepsilon - t$. Therefore, we estimate

$$|(-i\partial_s + \kappa\sigma\tilde{A}_1)\phi|^2 \geq (1-\varepsilon)|(-i\partial_s + \kappa\sigma(\gamma_\varepsilon - t))\phi|^2$$
$$+ (\kappa\sigma)^2(1-\varepsilon^{-1})\big|\big(\tilde{A}_1 - (\gamma_\varepsilon - t)\big)\phi\big|^2\,. \tag{14.94}$$

Using the Agmon estimates, Theorem 12.2.1, and the simple inequality

$$|\tilde{A}_1(s,t) - (\gamma_\varepsilon - t)| \leq Ct^2\,,$$

which is valid on $\operatorname{supp}\phi$ and deduced from (14.92), we find

$$(\kappa\sigma)^2\int \big|\big(\tilde{A}_1(s,t) - (\gamma_\varepsilon - t)\big)\phi\big|^2\,ds\,dt$$
$$\leq C(\kappa\sigma)^2\|t^4 e^{-\frac{\alpha t}{\varepsilon}}\|_\infty \int e^{\frac{\alpha t}{\varepsilon}}|\phi|^2\,ds\,dt \leq C'\|\phi\|_2^2\,.$$

We use the Agmon estimates and Theorem 12.2.1 [and the boundedness of the curvature $k(s)$] to replace all factors of $(1-tk(s))$ by $1+\mathcal{O}(\varepsilon)$. Upon choosing

$$M = C_M|\log\varepsilon| \tag{14.95}$$

(for a large constant C_M), we get

$$\mathcal{G}[\psi,\mathbf{A}] \geq (1-C\varepsilon)\tilde{Q}[\phi] - \kappa^2(1+C\varepsilon)\int_0^{|\partial\Omega|}\int_0^\infty |\phi(s,t)|^2\,ds\,dt$$
$$+ \frac{\kappa^2}{2}(1-C\varepsilon)\int_0^{|\partial\Omega|}\int_0^\infty |\phi(s,t)|^4\,ds\,dt\,,$$

where

$$\tilde{Q}[\phi] = \int_0^{|\partial\Omega|}\int_0^\infty |\partial_t\phi|^2 + |(-i\partial_s - \kappa\sigma(\gamma_\varepsilon - t))\phi|^2\,dt\,ds\,.$$

We finally change coordinates $(s,t) = \varepsilon(\xi_1,\xi_2)$. We introduce

$$\tilde{\phi}(\xi_1,\xi_2) := \phi(\varepsilon\xi_1,\varepsilon\xi_2)\,. \tag{14.96}$$

The inequality thereby becomes

$$\mathcal{G}[\psi,\mathbf{A}] \geq (1-C\varepsilon)Q_\varepsilon[\tilde{\phi}] - \frac{\kappa}{\sigma}(1+C\varepsilon)\|\tilde{\phi}\|_2^2 + \frac{\kappa}{2\sigma}(1-C\varepsilon)\|\tilde{\phi}\|_4^4\,. \tag{14.97}$$

Here, with $\Gamma_\varepsilon = -\gamma_\varepsilon/\varepsilon$,

$$Q_\varepsilon[\tilde{\phi}] = \int_0^{|\partial\Omega|/\varepsilon} d\xi_1\int_0^{+\infty} |\partial_{\xi_2}\tilde{\phi}|^2 + |(-i\partial_{\xi_1} + \xi_2 + \Gamma_\varepsilon)\tilde{\phi}|^2\,d\xi_2\,.$$

Using the functional E^{cyl} defined in (14.76), we recognize (14.97) as stating that

$$\mathcal{G}[\psi, \mathbf{A}] \geq (1 - C\varepsilon) E^{\text{cyl}}_{\frac{|\partial\Omega|}{2\varepsilon}} \left(e^{i\Gamma_\varepsilon \xi_1} \tilde{\phi}, \frac{\kappa}{\sigma} \frac{1 + C\varepsilon}{1 - C\varepsilon}, \frac{\kappa}{\sigma} \right). \tag{14.98}$$

Proof of (14.5) *and* (14.6).
In the case of these two results, we are in a situation where the assumptions of Theorem 14.3.1 are satisfied. We therefore continue the estimate in (14.98) as follows, using (14.79):

$$\begin{aligned} \mathcal{G}[\psi, \mathbf{A}] &\geq (1 - C\varepsilon) E^{\text{cyl}}_{\frac{2\pi\varepsilon}{|\partial\Omega|}} \left(e^{i\Gamma_\varepsilon \xi_1} \tilde{\phi}, \frac{\kappa}{\sigma} \frac{1 + C\varepsilon}{1 - C\varepsilon}, \frac{\kappa}{\sigma} \right) \\ &\geq -(1 + C'\varepsilon) \frac{|\partial\Omega|}{2\varepsilon} \frac{\kappa}{\sigma} \| f_{\zeta(\frac{\kappa}{\sigma} \frac{1+C\varepsilon}{1-C\varepsilon}, \frac{\kappa}{\sigma}), \frac{\kappa}{\sigma} \frac{1+C\varepsilon}{1-C\varepsilon}} \|_4^4 . \end{aligned} \tag{14.99}$$

In the case of (14.5), we have $\kappa/\sigma \to b^{-1} > \Theta_0$, and we find

$$\mathcal{G}[\psi, \mathbf{A}] \geq -(1 + o(1)) \frac{|\partial\Omega|}{2b} \sqrt{\kappa\sigma} \| f_{\zeta(b^{-1}), b^{-1}} \|_4^4 , \tag{14.100}$$

which finishes the proof of (14.5).

In the case of (14.6), we have to use the bifurcation analysis for λ_1 near Θ_0. Notice that in the case under consideration, we have

$$\frac{1}{\Theta_0} - \frac{\sigma}{\kappa} \approx \frac{\rho}{\kappa}, \qquad \text{as } \kappa \to \infty .$$

The bifurcation analysis Lemma 14.2.11 yields

$$\| f_{\zeta(\frac{\kappa}{\sigma} \frac{1+C\varepsilon}{1-C\varepsilon}), \frac{\kappa}{\sigma} \frac{1+C\varepsilon}{1-C\varepsilon}} \|_4^4 = \frac{\rho^2}{\kappa^2} \Theta_0^2 \| u_0 \|_4^{-4} (1 + o(1)) .$$

Thus, we find, using $\sigma \approx \kappa/\Theta_0$, that

$$\begin{aligned} E(\kappa, \sigma) &\geq -\sqrt{\kappa\sigma} \frac{|\partial\Omega|}{2b} \frac{\rho^2}{\kappa^2} \Theta_0^2 \| u_0 \|_4^{-4} (1 + o(1)) \\ &\geq -\frac{|\partial\Omega|}{2\| u_0 \|_4^4} \Theta_0^{5/2} \frac{\rho^2}{\kappa} (1 + o(1)) . \end{aligned} \tag{14.101}$$

This finishes the proof of the lower bound in (14.6). □

14.4.2 Upper bounds

We will give the upper bounds corresponding to (14.5) and (14.6). To get a good upper bound, we can use an explicit test configuration. We choose $\mathbf{A} = \mathbf{F}$ (the external field). For ψ, we write (in the boundary coordinates defined in Section F.1)

$$\psi(s,t) = e^{i\kappa H\varphi + i[\zeta]_\varepsilon s/\varepsilon}\lambda g(t/\varepsilon)\chi(t)\,. \tag{14.102}$$

We will proceed to define the different parts of ψ.

The function χ is smooth and localizes to the boundary region. If t_0 is the constant from Section F.1 defining the boundary region, the function χ is chosen to be nonincreasing and satisfying

$$\chi \in C^\infty(\mathbb{R}), \qquad \chi(t) = \begin{cases} 0 & \text{if } t \geq \frac{3t_0}{4}\,, \\ 1 & \text{if } t \leq \frac{t_0}{2}\,. \end{cases} \tag{14.103}$$

This localization near the boundary allows us to use the boundary coordinates (s,t).

We will write ζ for $\zeta(b^{-1})$ defined by (14.24). The symbol $[\zeta]_\varepsilon$ denotes an "integer part" of ζ:

$$[\zeta]_\varepsilon = \max\left\{ z \in \tfrac{2\pi\varepsilon}{|\partial\Omega|}\mathbb{Z} \,\middle|\, z \leq \zeta - \varepsilon^{-1}\frac{|\Omega|}{|\partial\Omega|} \right\}. \tag{14.104}$$

The term $\varepsilon^{-1}\frac{|\Omega|}{|\partial\Omega|}$ counterbalances the (topological) constant γ_0 in (14.107) below. Ideally, we would use $\zeta - \varepsilon^{-1}\frac{|\Omega|}{|\partial\Omega|}$ instead of $[\zeta]_\varepsilon$, but in order for ψ to be welldefined on Ω, we need it to be periodic in s. This is assured by using $[\zeta]_\varepsilon$.

The constant λ is defined by

$$\lambda = \begin{cases} 1 & \text{if } b < \Theta_0^{-1}\,, \\ \sqrt{\frac{\Theta_0}{\|u_0\|_4^4}\frac{\varrho}{\kappa}} & \text{if } b = \Theta_0^{-1}\,. \end{cases} \tag{14.105}$$

The function g is, of course, either u_0 or $f_{\zeta,b^{-1}}$:

$$g = \begin{cases} f_{\zeta,b^{-1}}, & \text{if } b < \Theta_0^{-1}\,, \\ u_0, & \text{if } b = \Theta_0^{-1}\,. \end{cases} \tag{14.106}$$

Let $\tilde{\mathbf{A}} = (\tilde{A}_1, \tilde{A}_2)$ be the vector potential $\mathbf{F}$ transformed to boundary coordinates. Using Lemma F.1.1, we choose φ such that, with $\gamma_0 = |\Omega|/|\partial\Omega|$,

$$\begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \end{pmatrix} := \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} + \nabla_{(s,t)}\varphi = \begin{pmatrix} \gamma_0 - t + k(s)\frac{t^2}{2} \\ 0 \end{pmatrix}. \tag{14.107}$$

With all these choices, ψ from (14.102) is defined and we can proceed to calculate $\mathcal{G}[\psi,\mathbf{F}]$. Using Section F.1, we will calculate in boundary coordinates

$$\begin{aligned}
\mathcal{G}[\psi,\mathbf{F}] = &\int (1-tk(s))^{-1}|(-i\partial_s + \kappa H\tilde{A}_1)\psi|^2\,ds\,dt \\
&+ \int \left\{ |(-i\partial_t + \kappa H\tilde{A}_2)\psi|^2 - \kappa^2|\psi|^2 \right\}(1-tk(s))\,ds\,dt \\
&+ \tfrac{1}{2}\kappa^2 \int |\psi|^4(1-tk(s))\,ds\,dt\,.
\end{aligned}$$

Upon calculating $\mathcal{G}[\psi, \mathbf{F}]$, we therefore find

$$\begin{aligned}\mathcal{G}[\psi, \mathbf{F}] &= \lambda^2 \int (1 - tk(s))^{-1} \Big|(\kappa H \hat{A}_1 + \varepsilon^{-1}[\zeta]_\varepsilon)(g(t/\varepsilon)\chi(t))\Big|^2 ds\, dt \\ &\quad + \lambda^2 \int (1 - tk(s)) \big\{ \big|\partial_t\, (g(t/\varepsilon)\chi(t))\,\big|^2 + \kappa^2 \big|g(t/\varepsilon)\chi(t))\big|^2 \big\}\, dsdt \\ &\quad + \frac{\kappa^2}{2}\lambda^4 \int \big|g(t/\varepsilon)\chi(t))\big|^4 (1 - tk(s))\, ds\, dt\,. \end{aligned} \tag{14.108}$$

We write this, after a change of variable in t, as

$$\mathcal{G}[\psi, \mathbf{F}] = I + II\,, \tag{14.109}$$

with

$$\begin{aligned} I := \lambda^2 \varepsilon \int_0^\infty \int_0^{|\partial\Omega|} &\Big|(\kappa H \hat{A}_1(s, \varepsilon\tau) + \varepsilon^{-1}[\zeta]_\varepsilon)(g(\tau)\chi(\varepsilon\tau))\Big|^2 \\ &\times (1 - \varepsilon\tau k(s))^{-1}\, ds\, d\tau \end{aligned} \tag{14.110}$$

and

$$\begin{aligned} II := \lambda^2 \varepsilon \iint (1 - \varepsilon\tau k(s)) &\Big\{ \big|\varepsilon^{-1}\partial_\tau\, (g(\tau)\chi(\varepsilon\tau))\,\big|^2 + \kappa^2 \big|g(\tau)\chi(\varepsilon\tau))\big|^2 \Big\}\, dsd\tau \\ + \frac{\kappa^2}{2}\lambda^4\varepsilon \iint (1 - \varepsilon\tau k(s)) &\big|g(\tau)\chi(\varepsilon\tau))\big|^4\, ds\, d\tau\,, \end{aligned} \tag{14.111}$$

with the same limits on the integrals as for I.

We start by estimating I. We keep in mind the fast decay of g; cf. Remark 3.2.10 and (14.20). This will allow us to replace χ by the constant 1 and $(1 - \varepsilon\tau k(s))$ by $1 + \mathcal{O}(\varepsilon)$. We calculate, inserting the definitions of $\hat{A}_1$ and γ_0,

$$\begin{aligned} &\kappa H \hat{A}_1(s, \varepsilon\tau) + \varepsilon^{-1}[\zeta]_\varepsilon \\ &\quad = \varepsilon^{-1} \left\{ (\zeta - \tau) + \frac{1}{2}\varepsilon k(s)\tau^2 + \left([\zeta]_\varepsilon - \left(\zeta - \varepsilon^{-1}\frac{|\Omega|}{|\partial\Omega|} \right) \right) \right\}. \end{aligned} \tag{14.112}$$

The last term, $[\zeta]_\varepsilon - (\zeta - \varepsilon^{-1}\frac{|\Omega|}{|\partial\Omega|})$ is uniformly bounded by ε. Using the fast decay of g, we can therefore (uniformly) estimate all terms depending on s and do the s integral in order to get

$$I = \lambda^2 \varepsilon^{-1} |\partial\Omega| \int_0^\infty (\zeta - \tau)^2 |g(\tau)|^2\, d\tau + \mathcal{O}(\lambda^2)\,. \tag{14.113}$$

Similarly, we can use the rapid decay of g on II to get

$$\begin{aligned} II = \lambda^2 \varepsilon^{-1} |\partial\Omega| \int_0^\infty |g'(\tau)|^2\, d\tau - \lambda^2\kappa^2\varepsilon |\partial\Omega| \int_0^\infty |g(\tau)|^2\, d\tau \\ + \frac{\kappa^2}{2}\lambda^4 \varepsilon \int_0^\infty |g(\tau)|^4\, d\tau + \mathcal{O}(\lambda^2 + \lambda^4). \end{aligned} \tag{14.114}$$

Collecting the terms I and II, using $\lambda \leq 1$ and the definition of ε, we find

$$\mathcal{G}[\psi, \mathbf{F}] \leq \varepsilon^{-1}|\partial\Omega|\lambda^2 \int \Big\{|g'(\tau)|^2 + (\tau - \zeta)^2|g(\tau)|^2 - \frac{\kappa}{\sigma}|g(\tau)|^2 + \lambda^2\frac{\kappa}{2\sigma}|g(\tau)|^4\Big\}\, d\tau + C\lambda^2\,, \tag{14.115}$$

for some constant $C > 0$.

In the case where $b < \Theta_0^{-1}$, we have $\lambda = 1$, $\kappa/\sigma = b^{-1} + o(1)$, $g = f_{\zeta, b^{-1}}$, and we get, using (14.17),

$$\mathcal{G}[\psi, \mathbf{F}] \leq -\varepsilon^{-1}|\partial\Omega|\frac{1}{2b}\|f_{\zeta,b^{-1}}\|_4^4 + o(\varepsilon^{-1})\,. \tag{14.116}$$

We recognize (14.116) as being the upper bound corresponding to (14.5).

In the case where $b = \Theta_0^{-1}$, we have $g = u_0$, $\zeta = \xi_0$, and therefore, using the L^2-normalization of u_0,

$$\int |g'(\tau)|^2 + (\tau - \zeta)^2|g(\tau)|^2\, d\tau = \Theta_0\|u_0\|_2^2 = \Theta_0\,. \tag{14.117}$$

Also, since $\sigma = \Theta_0^{-1}\kappa - \rho + \mathcal{O}(1)$,

$$\frac{\kappa}{\sigma} = \Theta_0 + \Theta_0^2\frac{\rho}{\kappa} + \mathcal{O}\left(\frac{\rho^2}{\kappa^2} + \kappa^{-1}\right),$$

$$\int \frac{\kappa}{\sigma}|g(\tau)|^2\, d\tau = \Theta_0 + \Theta_0^2\frac{\rho}{\kappa} + \mathcal{O}\left(\frac{\rho^2}{\kappa^2} + 1\right),$$

$$\int \lambda^2\frac{\kappa}{2\sigma}|g(\tau)|^4\, d\tau = \frac{1}{2}\lambda^2\Theta_0\|u_0\|_4^4\left(1 + \mathcal{O}\left(\frac{\rho}{\kappa}\right)\right).$$

Therefore, (14.115) becomes

$$\mathcal{G}[\psi, \mathbf{F}] \leq \varepsilon^{-1}|\partial\Omega|\lambda^2\Big\{-\Theta_0^2\frac{\rho}{\kappa} + \frac{1}{2}\lambda^2\Theta_0\|u\|_4^4 + \mathcal{O}\left(\frac{\rho^2}{\kappa^2} + \kappa^{-1} + \lambda^2\frac{\rho}{\kappa}\right)\Big\} + \mathcal{O}(\lambda^2)\,. \tag{14.118}$$

Inserting the (optimal) value of λ from (14.105) yields the upper bound corresponding to (14.6).

14.5 Idea of the Proof of (14.4)

The proof of (14.4) is somewhat more complicated. This is mainly due to the fact that Theorem 14.3.1 is only valid for λ_1 near Θ_0, i.e., for $\mathfrak{b}$ near Θ_0^{-1}. When this is not the case, we do not know that minimizers have a product structure in boundary coordinates. Therefore, the argument is more indirect and we cannot give the constant $E_{\mathfrak{b}}$ as explicitly as when $\mathfrak{b}$ is near Θ_0^{-1}. We only explain the main ideas of the proof and refer for details to the original work of Pan [Pa2].

By the decay estimates of Chapter 12, superconductivity is a boundary phenomenon in the parameter domain in question. Therefore, one can restrict the functional to the boundary region where the boundary coordinates (s,t) are defined. Also, by Proposition 11.4.4, one can replace the induced magnetic vector potential $\mathbf{A}$ by the vector potential $\mathbf{F}$ generating the constant exterior magnetic field. After a scaling of the coordinates, one finds the following functional on cylinders [with notation as in (14.76)]:

$$\psi \mapsto \int_{-L}^{L}\int_0^{\infty} |(i\nabla + \xi_1 \hat{i_2})\psi|^2 - \lambda|\psi|^2 + \frac{\lambda}{2}|\psi|^2 \, d\xi_1 \, d\xi_2 \,, \tag{14.119}$$

where ψ is restricted to periodic H^1-functions $\psi(-L,\xi_2) = \psi(L,\xi_2)$. The parameter λ equals (up to small errors) the quotient κ/σ, i.e., $\mathfrak{b}^{-1}$.

Define $t(L,\lambda)$ to be the ground state energy of this functional. After the scaling, the perimeter $2L$ of the cylinder is of the order of magnitude $\sqrt{\kappa\sigma}$, i.e., very large. One therefore needs to prove the existence of a constant $C_\lambda > 0$ such that

$$\frac{t(L,\lambda)}{2L} \to -C_\lambda, \qquad \text{as} \qquad L \to \infty \,. \tag{14.120}$$

In order to prove (14.120), one needs the ideas from Chapter 12 to prove that the energy density is concentrated near the boundary of the cylinder. The other input is a subadditivity inequality stating the existence of some constant C_0 such that

$$t(L_1 + L_2, \lambda) \geq t(L_1,\lambda) + t(L_2,\lambda) - C_0 \,, \qquad \forall L_1, L_2 \,. \tag{14.121}$$

Backtracking the scalings and changes of coordinates, (14.4) follows from (14.120). □

14.6 Notes

1. The first mathematical contributions to the subject considered in this chapter are again to be found in the work of Lu–Pan [LuP3]. Then came various efforts to improve and clarify the initial contribution. Near $H_{C_3}(\kappa)$, one should mention the work of Fournais–Helffer [FoH1] devoted to the phenomenon of uniformization along the boundary when leaving the third critical zone with lower exterior fields. The other important contribution is then the paper by Pan [Pa2], which was later complemented by the paper of Almog–Helffer [AlH]. We have chosen here to give a new and improved presentation of what happens near H_{C_3}.

2. Theorem 14.1.1 is a combination of [Pa2, FoH1, AlH]. The first result (14.4) was proven in [Pa2]. The identification of the constant for $\mathfrak{b}$ close to Θ_0^{-1} was carried out in [AlH]. The last energy asymptotics (14.6) was proven in [FoH1] (see also [LuP3]) for a reduced parameter domain.
3. One may ask if (14.7) can be justified in the uniform $\| \cdot \|_\infty$-topology and not just in an energy norm. That question has been pursued in [AlH]. Under a reasonable—but unproven—continuity hypothesis, they conclude that there is an obstruction of topological nature to such a uniform convergence.
4. The reference [Pa2] contains a number of interesting conjectures on the minimizer.
5. We refer, for example, to [AbMR] for the implicit function theorem in Banach spaces, which is used at the end of the proof of Proposition 14.2.2.
6. Standard references for bifurcation theory include [CrR] (bifurcation from simple eigenvalues). The method is also called the Lyapunov–Schmidt method.
7. Lemma 14.2.14 is a variant of a computation mentioned in [AlH] and inspired by a calculation by Almog (personal communication).
8. One should also mention that coming from $H_{C_1}(\kappa)$, other authors, including Sandier, Serfaty, Aftalion, and Almog ([AfS], [Al6], [SaS2]), have started to explain what is going on near the second critical field. As known by physicists starting with A. A. Abrikosov, this is related to the appearance of vortex lattices called Abrikosov lattices (see [BeR, p. 100, Section 4.2] and also Section 16.5).

15

On the Problems with Corners

15.1 Introduction

It has been observed in the physics literature that the transition field $H_{C_3}(\kappa)$ is significantly larger when the domain Ω has corners than for samples of the same material but with a smooth cross section. In this chapter, we analyze this phenomenon. It still results in the value of $H_{C_3}(\kappa)$ being completely determined by the corresponding linear spectral problem. The experimentally observed change in $H_{C_3}(\kappa)$ is due to a decrease of the first eigenvalue for the magnetic Neumann problem when a domain has a corner. Eigenfunctions corresponding to the lowest eigenvalues will be localized near the corners and their leading-order asymptotics for a large field controlled by the model of an infinite sector considered in Section 4.4.

In this chapter, for simplicity we will only consider polygons. The first part of the chapter will develop the necessary (linear) spectral theory for the magnetic Neumann operator with magnetic field on polygons. The second part will be devoted to the analysis of the Ginzburg–Landau functional on polygonal domains.

In the case of regular domains (without corners), one has the asymptotics (cf. Theorem 13.2.2)

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \mathcal{O}(1)\,, \qquad \text{(regular domains)}$$

where the leading correction depends on the maximal curvature of the boundary. The corresponding result for polygons, Theorem 15.3.7, gives the asymptotics

$$H_{C_3}(\kappa) = \frac{\kappa}{\Lambda_1} + \mathcal{O}(1)\,, \qquad \text{(corners)}$$

for some spectrally defined $\Lambda_1 < \Theta_0$.

We observe that the corners—which can be seen as points where the curvature is infinite—change the leading-order term of $H_{C_3}(\kappa)$. Thus, there is a

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large parameter regime of magnetic field strengths between κ/Θ_0 and $H_{C_3}(\kappa)$, where superconductivity in the sample must be dominated by the corners.

Since the conjectures in Section 4.4 have not yet been proven, we need to add the technical Assumption 15.1.1. If Conjectures 4.4.1 and 4.4.3 were established, this assumption would hold for any convex polygon.

We consider a bounded open subset $\Omega \subset \mathbb{R}^2$ whose boundary is a polygon. We denote by

$$\Sigma = \{\mathsf{s}_1, \dots, \mathsf{s}_N\} \tag{15.1}$$

the set of vertices of $\partial\Omega$. Notice that for a polygon we always have $N = \#\Sigma \geq 3$. Recall that the ground state energies, $\mu^{\rm sect}$, for magnetic operators in angular sectors were defined in (4.14). We will work under the following assumption on the domain.

Assumption 15.1.1.
We assume that Ω is a convex polygon Ω such that if we denote by α_{s} the angle at the vertex s (measured toward the interior), then

$$\mu^{\rm sect}(\alpha_{\mathsf{s}}) < \Theta_0 \qquad \textit{for all } \mathsf{s} \in \Sigma\,. \tag{15.2}$$

We define

$$\Lambda_1 := \min_{\mathsf{s}\in\Sigma} \mu^{\rm sect}(\alpha_{\mathsf{s}})\,. \tag{15.3}$$

When having a fixed numbering $\{\mathsf{s}_1, \dots, \mathsf{s}_N\}$ of the vertices, we will also write α_j instead of α_{s_j}.

15.2 Large Field Analysis in Domains with Corners

15.2.1 Agmon estimates near corners for the linear problem

The proof of Theorem 8.2.4 goes through unchanged and yields that eigenstates corresponding to low eigenvalues are exponentially localized near the boundary:

Theorem 15.2.1.
Let $\delta > 0$.

Then there exist positive constants ϵ, C, and B_0 such that

$$\int e^{\epsilon\sqrt{B}\,{\rm dist}(x,\partial\Omega)}\big\{|\psi_B(x)|^2 + B^{-1}|p_{B\mathbf{F}}\psi_B(x)|^2\big\}\,dx \leq C\|\psi_B\|_2^2\,,$$

for all $B \geq B_0$ and all eigenfunctions ψ_N of $P^N_{B\mathbf{F},\Omega}$ with eigenvalue $\lambda(B)$ satisfying

$$\lambda(B) \leq (1-\delta)B\,.$$

In order to prove exponential localization near the corners for minimizers of $\mathcal{G}_{\kappa,\sigma}$, we will need the operator inequality (15.4) (compare to Proposition 8.3.3).

Theorem 15.2.2.
Let $\delta > 0$. Then there exist constants $M_0 > 0$ and $B_0 > 0$ such that if $B \geq B_0$, then $P^N_{B\mathbf{F},\Omega}$ satisfies the operator inequality

$$P^N_{B\mathbf{F},\Omega} \geq U_B\,, \tag{15.4}$$

where U_B is the potential given by

$$U_B(x) := \begin{cases} (\mu^{\text{sect}}(\alpha_{\mathsf{s}}) - \delta)B & \text{if } \operatorname{dist}(x,\mathsf{s}) \leq \frac{M_0}{\sqrt{B}}\,, \\ (\Theta_0 - \delta)B & \text{if } \operatorname{dist}(x,\Sigma) > \frac{M_0}{\sqrt{B}}\,, \\ (1-\delta)B & \text{if } \operatorname{dist}(x,\partial\Omega) > \frac{M_0}{\sqrt{B}}\,. \end{cases}$$

Proof of Theorem 15.2.2.
Suppose for simplicity that the corners $\{\mathsf{s}_j\}_{j=1}^N$ are numbered such that s_j and s_{j+1} are connected by a smooth curve Γ_j. We will use a cyclic numbering such that $\mathsf{s}_{N+1} = \mathsf{s}_1$.

For $j = 1, \ldots, N$, choose a smooth, simply connected domain $\widetilde{\Omega}_j$ such that $\Gamma_j \subset \partial\widetilde{\Omega}_j$ and such that Ω and $\widetilde{\Omega}_j$ lie (locally) on the same side of Γ_j.

Define $\tilde{\lambda}^{(j)}(B) := \lambda_1^N(B\mathbf{F},\widetilde{\Omega}_j)$. By Theorem 8.3.2, we have

$$\tilde{\lambda}^{(j)}(B) = \Theta_0 B + \mathcal{O}(\sqrt{B})\,. \tag{15.5}$$

Let $\chi_1 \in C^\infty(\mathbb{R})$ be nonincreasing and satisfy $\chi_1(t) = 1$ for $t \leq 1$, $\chi_1(t) = 0$ for $t \geq 2$.

Define, for $M > 0$ and $j = 1, \ldots, N$,

$$\chi_{\text{cor}}^{(j)}(x) := \chi_1(\sqrt{B}\operatorname{dist}(x,\mathsf{s}_j)/M)\,.$$

Define also, for $L, M > 0$ and $j = 1, \ldots, N$, the Lipschitz functions

$$\chi_{\text{side}}^{(j)}(x) := \sqrt{1 - \chi_1^2(\sqrt{B}\operatorname{dist}(x,\mathsf{s}_j)/M)}\,\chi_1(\sqrt{B}L\operatorname{dist}(x,\Gamma_j)/M)\,.$$

We choose and fix L (depending only on the smallest opening angle α_{s} of the corners) such that the supports of the $\chi_{\text{side}}^{(j)}$ are disjoint. Finally, we define the Lipschitz function χ_{int} by

$$\chi_{\text{int}}^2 = 1 - \sum_j (\chi_{\text{cor}}^{(j)})^2 - \sum_j (\chi_{\text{side}}^{(j)})^2\,.$$

Let $\phi \in H^2(\Omega)$. The standard localization formula yields for some constant $C > 0$,

$$\langle \phi \,|\, P^N_{B\mathbf{F},\Omega}\phi\rangle \geq \sum_{j=1}^N \langle \chi^{(j)}_{\rm cor}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi^{(j)}_{\rm cor}\phi)\rangle + \sum_{j=1}^N \langle \chi^{(j)}_{\rm side}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi^{(j)}_{\rm side}\phi)\rangle$$

$$+ \langle \chi_{\rm int}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi_{\rm int}\phi)\rangle - C\frac{B}{M^2}\|\phi\|^2 \,. \tag{15.6}$$

We can compare the corner contributions with the infinite sector with opening angle α_j and get

$$\langle \chi^{(j)}_{\rm cor}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi^{(j)}_{\rm cor}\phi)\rangle \geq B\mu^{\rm sect}(\alpha_j)\|\chi^{(j)}_{\rm cor}\phi\|_2^2 \,. \tag{15.7}$$

We can compare the contributions from the sides with $\widetilde{\Omega}_j$ and get, using (15.5),

$$\langle \chi^{(j)}_{\rm side}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi^{(j)}_{\rm side}\phi)\rangle \geq \tilde{\lambda}^{(j)}\|\chi^{(j)}_{\rm side}\phi\|_2^2 \geq (\Theta_0 B - C\sqrt{B})\|\chi^{(j)}_{\rm side}\phi\|_2^2 \,. \tag{15.8}$$

Finally, the interior piece is estimated using Lemma 1.4.1:

$$\langle \chi_{\rm int}\phi \,|\, P^N_{B\mathbf{F},\Omega}(\chi_{\rm int}\phi)\rangle \geq B\|\chi_{\rm int}\phi\|_2^2 \,. \tag{15.9}$$

Combining (15.6)–(15.9) gives Theorem 15.2.2 upon choosing M_0 sufficiently large. □

One can use Theorem 15.2.2 to prove localization estimates near the corners for the linear problem. We do not give the details, since the proof is a repetition of ideas from Chapter 8 and since we will give the proof of the corresponding statement for the nonlinear problem. So we only state the main result.

Theorem 15.2.3.
Let $\delta > 0$. Then there exist positive constants ϵ , C , and B_0 such that

$$\int e^{\epsilon\sqrt{B}\,{\rm dist}(x,\Sigma)}\big\{|\psi_B(x)|^2 + B^{-1}|p_{B\mathbf{F}}\psi_B(x)|^2\big\}\,dx \leq C\|\psi_B\|_2^2 \,,$$

for all $B \geq B_0$ and all eigenfunctions ψ_B of $P^N_{B\mathbf{F},\Omega}$ with eigenvalue $\lambda(B)$ satisfying

$$\lambda(B) \leq (\Theta_0 - \delta)B \,.$$

15.2.2 Eigenvalue asymptotics

Definition 15.2.4.
Let Ω be a bounded polygon. We denote by

- *Λ_n the nth eigenvalue of the model operator $\oplus_{\mathsf{s}\in\Sigma}P^N_{\mathbf{F},\Omega_{\alpha_\mathsf{s}}}$,*
- *K_Ω the largest integer K such that $\Lambda_K < \Theta_0$,*
- *$\lambda_n(B)$ the nth eigenvalue (counted with multiplicity) of the magnetic Neumann Laplacian $(-i\nabla + B\mathbf{F})^2$ on Ω .*

In particular,

$$\Lambda_1 = \min_{\mathsf{s}\in\Sigma} \mu_1(\alpha_{\mathsf{s}})\,.$$

Theorem 15.2.5.
Let $n \leq K_\Omega$. Then

$$\lambda_n(B) = \Lambda_n B + \mathcal{O}(B^{-\infty})\,. \tag{15.10}$$

Proof.
We only give the main ideas and leave the details to the reader. Define $\delta = \min_{j<k} |\mathsf{s}_j - \mathsf{s}_k|$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a standard localization function: $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq 2$.

Let ψ_n be the eigenfunction corresponding to Λ_n and let $\Omega_{\alpha_{\mathsf{s}}}$ be the corresponding angular sector. We apply the scaling $\psi_n(x) \mapsto B\psi_n(\sqrt{B}x)$ and a (magnetic) rigid motion in order to move the vertex of $\Omega_{\alpha_{\mathsf{s}}}$ to the corresponding corner s of Ω . Furthermore, we localize near the corner s by multiplication by $\chi(4\,\mathrm{dist}(x,\mathsf{s})/\delta)$. We denote the corresponding function $\phi_n \in L^2(\Omega)$.

Using the exponential decay result, Theorem 4.4.4, $\|\phi_n\|_2 = 1 + \mathcal{O}(B^{-\infty})$, and $\{\phi_1, \ldots, \phi_n\}$ span a linear space of dimension n . Furthermore, a calculation yields

$$\langle \phi_j \,|\, (-i\nabla + B\mathbf{F})^2 \phi_k\rangle = \Lambda_k \delta_{j,k} B + \mathcal{O}(B^{-\infty}). \tag{15.11}$$

By Proposition A.1.3, this yields the upper bound in (15.10).

The proof of the corresponding lower bound follows from the inverse procedure: Let $\{\psi_1, \ldots, \psi_n\}$ be the first n eigenfunctions of $P_{B\mathbf{F},\Omega}$. By Theorem 15.2.3, these functions are exponentially localized near the corners. By localization with $\chi(4\,\mathrm{dist}(x,\Sigma)/\delta)$ and applying the inverse scalings/translations we obtain a family of n functions $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\}$ in $\oplus_{\mathsf{s}\in\Sigma} L^2(\Omega_{\alpha_{\mathsf{s}}})$. Again using the min-max principle, Proposition A.1.3, we get the lower bound. □

15.2.3 Monotonicity of $\lambda_1(B)$

By Theorem 15.2.5, we have a complete asymptotics of $\lambda_1(B)$. The proof of Corollary 2.3.3 easily carries over to the case with corners. Therefore, we have

Proposition 15.2.6.
Let Ω be a bounded polygon. The limits of $\lambda'_{1,+}(B)$ and $\lambda'_{1,-}(B)$ as $B \to +\infty$ exist and are equal, and we have

$$\lim_{B\to+\infty} \lambda'_{1,+}(B) = \lim_{B\to+\infty} \lambda'_{1,-}(B) = \Lambda_1 > 0\,.$$

Therefore, $B \mapsto \lambda_1(B)$ is strictly increasing for large B .

Corollary 15.2.7.
Let Ω be a bounded polygon. The equation $\lambda_1(\kappa\sigma) = \kappa^2$ in σ has a unique solution $\sigma(\kappa)$ for κ sufficiently large. Furthermore,

$$\sigma(\kappa) = \frac{\kappa}{\Lambda_1} + \mathcal{O}(\kappa^{-\infty})\,.$$

15.2.4 The tunneling effect between corners

We describe heuristically the case of the n-regular polygon. We then show how we arrive at an $n \times n$ interaction matrix with symmetry properties. We then reproduce some of the pictures obtained by Bonnaillie-Noël–Dauge–Martin–Vial.

The "proof" is based on various steps and is inspired by the analysis of the spectrum of the Schrödinger operator with multiple wells in the presence of symmetries.

Invariance by rotation

We consider the polygon to be placed with its center at the origin. The first point is that we have a finite group G_n of symmetry generated by the rotation by $2\pi/n$ around the origin in $\mathbb{R}^2$, which is denoted by g_n. This group has n elements g_n^j ($j = 0, \dots, n-1$) and we have $g_n^n = 1$. The group acts on $L^2(\mathbb{R}^n)$ as follows:

$$(M(g_n)u)(x) = u(g_n^{-1}x)\,,$$

where we verify that $M(g_n)$ commutes with $(-i\nabla + B\mathbf{F})^2$.

Selected orthonormal basis of $\mathcal{E}^n$

The second point is that according to the previous sections, if we consider $\mu^{\text{sect}}(2\pi/n)$ the ground state energy of the reference problem in a sector of opening angle $2\pi/n$, then we can show that there exist n eigenvalues exponentially close to $\mu^{\text{sect}}(2\pi/n)\,B$, the rest of the spectrum being at a distance of order B/C_n for some constant $C_n > 0$.

Moreover, all the corresponding eigenfunctions are exponentially localized in the union of the n corners.

Lemma 15.2.8.
There exists an orthonormal basis (e_j) $(j = 1, \dots, n)$ of the eigenspace $\mathcal{E}^n(B)$ corresponding to the n lowest eigenvalues such that

- *e_j is exponentially localized in a corner and exponentially close (in L^2) to the ground state of the corresponding corner,*
- *$M(g_n)e_j = e_{j+1}$.*

The proof consists of starting with the eigenfunction ψ_1^B relative to a corner s_1 defined by

$$\psi_1^B(x) = B^{\frac{1}{4}} \exp i\frac{B}{2}(x \wedge s_1)\, \psi_1(B^{\frac{1}{2}}(\mathcal{R}_1^{-1}(x - s_1)))\,.$$

Here ψ_1 is the eigenfunction in the sector $\Omega_{2\pi/n}$ and $\mathcal{R}_1$ is the rotation sending this sector on the corner s_1 of our polygon.

We now introduce $f_1 = \chi_1\psi_1^B$, where χ_1 is cutoff function equal to one in a neighborhood of the corner s_1 and depending only on the distance to s_1.

Due to the fact that ψ_1 decays exponentially at ∞ in the infinite sector by Theorem 4.4.4, this gives a good quasimode for $(-i\nabla + B\mathbf{F})^2$. We indeed have

$$(-\nabla + iB\mathbf{F})^2 f_1 = \mu_1 \left(\frac{2\pi}{n}\right) f_1 + \mathcal{O}\left(\exp -\frac{1}{C} B^{\frac{1}{2}} |x - s_1|\right) . \tag{15.12}$$

We then define f_j by

$$f_j = M(g)^{j-1} f_1 ,$$

and project these f_j on the spectral space relative to the n-first eigenfunctions to get a basis g_j :

$$g_j = \Pi_0 f_j ,$$

Its Gram matrix $\langle g_j \,|\, g_k \rangle$ is exponentially close to the identity and we can orthonormalize by the Gram–Schmidt procedure to get the orthonormal basis e_j . Throughout this construction we have respected the symmetries. In particular, we have

$$M(g_n) e_j = e_{j+1} , \qquad \forall j \in \mathbb{Z}/n\mathbb{Z} .$$

□

The interaction matrix

Once we have constructed the basis $\{e_1, \ldots, e_n\}$, we can introduce the interaction matrix, which is simply the matrix $\mathcal{M}$ of $(-\nabla + iB\mathbf{F})^2$ restricted to $\mathcal{E}^n(B)$ in the basis (e_j) . This permits us to identify $\mathcal{E}^n(B)$ with $\ell^2(\mathbb{Z}/n\mathbb{Z})$, and the action of $M(g_n)$ simply becomes the shift operator τ whose corresponding matrix is given by $\tau_{j,k} = \delta_{j+1,k}$.

The restricted Hamiltonian $\mathcal{M}$ is a self-adjoint matrix that commutes with τ . It can therefore be written as

$$\mathcal{M} = \lambda \tau^0 + I_1 \tau + \sum_{k=2}^{n-1} I_k \tau^k ,$$

for some coefficients $I_k \in \mathbb{C}$. Observing that

$$\tau^* = \tau^{-1} = \tau^{n-1} ,$$

the self-adjointness implies that

$$\lambda = \bar{\lambda} , \qquad \text{and} \qquad I_k = \overline{I_{n-k}} \qquad \text{for } k = 1, \ldots, n-1 .$$

All these matrices share the property of being diagonalizable in the same orthonormal basis of eigenfunctions u_k $(k = 1, \ldots, n)$ whose coordinates in our selected basis are given by

$$(u_k)_\ell = \omega_n^{(k-1)\ell} ,$$

with

$$\omega_n := \exp(2i\pi/n) .$$

Braid structure of the eigenvalues

Let us, for example, look at the case $n = 3$. Then $\mathcal{M}$ takes the form

$$\mathcal{M} = \begin{pmatrix} \lambda & I_1 & \overline{I_1} \\ \overline{I_1} & \lambda & I_1 \\ I_1 & \overline{I_1} & \lambda \end{pmatrix} .$$

Writing $I_1 = \rho \exp i\theta$, the eigenvalues are given by

$$\lambda_1 = \lambda + 2\rho \cos\theta\,,\ \lambda_2 = \lambda + 2\rho\cos\left(\theta + \frac{2\pi}{3}\right)\,,\ \lambda_3 = \lambda + 2\rho\cos\left(\theta + \frac{4\pi}{3}\right)\,.$$

It is easy to show that ρ decreases exponentially rapidly as a function of B. This is a consequence of the exponential decay of e_j away from the corner s_j, permitting us to show the existence of $C > 0$ and $B_0 > 0$ such that

$$\rho \leq C \exp -\frac{1}{C} B^{\frac{1}{2}}\,, \qquad \forall B \geq B_0\,.$$

But the asymptotics of ρ (or a lower bound) is not determined. The function $B\theta(B)$, if chosen continuously as a function of B for avoiding jumps of 2π, is expected to depend asymptotically on B in a linear way. So it is natural to look at the map $\theta \mapsto \big(\cos\theta, \cos(\theta + \frac{2\pi}{3}), \cos(\theta + \frac{4\pi}{3})\big)$. The graph of this function immediately gives the right prediction for the braid structure of the first three eigenvalues of $(-\nabla + iB\mathbf{F})^2$. This leads us to predict a crossing of eigenvalues, hence a change of multiplicity, when $\cos\theta = \cos(\theta + \frac{2\pi}{3})$, which leads to $\theta = 2\pi/3$ and $\theta = 5\pi/3$. Other crossings occur for $\cos(\theta + \frac{2\pi}{3}) = \cos(\theta + \frac{4\pi}{3})$, i.e., for $\theta = 0$ and $\theta = \pi$ and for $\cos(\theta + \frac{4\pi}{3}) = \cos\theta$, which corresponds to $\theta = 4\pi/3$ and $\theta = \pi/3$. All together, we can predict crossings for $\theta = k\pi/3$ ($k \in \mathbb{Z}$).

Remark 15.2.9.
We suspect that for B large, we have

$$\theta \sim \frac{1}{n}|\Omega| B\,.$$

This leads us to predict that the values B_k where a crossing occurs satisfy

$$B_{k+1} - B_k \sim \frac{\pi}{|\Omega|}\,. \tag{15.13}$$

For $n = 4$, $\mathcal{M}$ assumes the form

$$\mathcal{M} = \begin{pmatrix} \lambda & I_1 & I_2 & \overline{I_1} \\ \overline{I_1} & \lambda & I_1 & I_2 \\ I_2 & \overline{I_1} & \lambda & I_1 \\ I_1 & I_2 & \overline{I_1} & \lambda \end{pmatrix} .$$

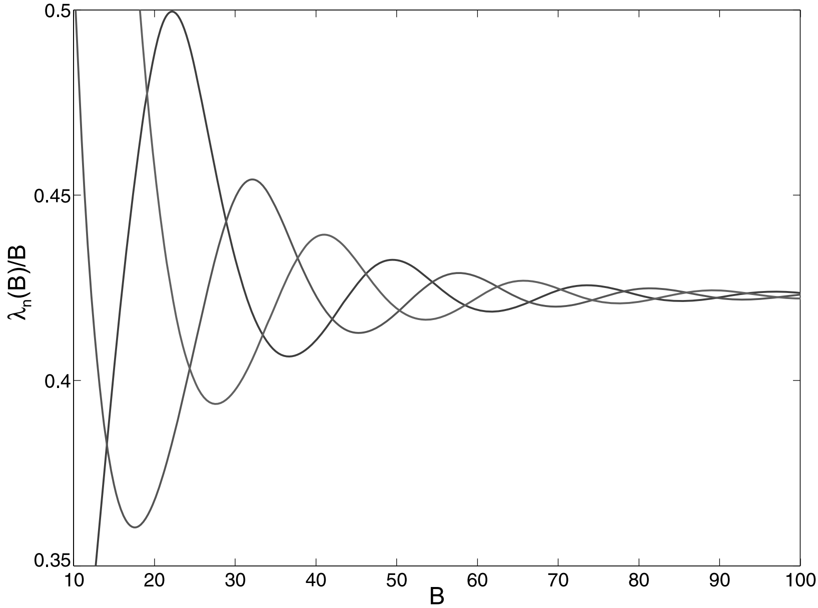

Figure 15.1. $\lambda_n(B)/B$ as a function of B for the equilateral triangle and for $n = 1, 2, 3$. Notice that $\Theta_0 \approx 0.59$ is, as expected, larger than $\lim_{B\to+\infty} \lambda_n(B)/B$.

We note that I_2 is real. Probably I_2 (which corresponds to a tunneling effect between opposite corners) is exponentially small in comparison with $|I_1| = \rho$. One can indeed suspect that $I_2 \approx \rho^2$. In any case, we have four eigenvalues given by

$$\lambda_1 = \lambda + I_2 + 2\rho\cos\theta\,, \qquad \lambda_2 = \lambda - I_2 + 2\rho\cos\left(\theta + \frac{\pi}{2}\right)\,,$$

$$\lambda_3 = \lambda + I_2 - 2\rho\cos\theta\,, \qquad \lambda_4 = \lambda - I_2 - 2\rho\cos\left(\theta + \frac{\pi}{2}\right)\,.$$

Independently of I_2, crossings of λ_1 and λ_3 occur when $\theta = (2k+1)\frac{\pi}{2}$ $(k \in \mathbb{Z})$ and crossings of λ_2 and λ_4 occur when $\theta = k\pi$ $(k \in \mathbb{Z})$.

When $I_2 = 0$, λ_1 and λ_2 cross for $\theta = 3\pi/4$ and $\theta = 7\pi/4$. This crossing is transverse but remains when I_2/ρ is small at a close value of θ. Similarly, λ_3 and λ_4 cross for the same value for $I_2 = 0$ (so we have two double eigenvalues). The appearance of an $I_2 \neq 0$ keeps the two crossings but destroys the phenomenon of appearance for the same value of θ of two distinct double eigenvalues. Finally, one observes similar properties for $\theta = \pi/4$ and $5\pi/4$ with crossings between λ_2 and λ_3 and λ_4 and λ_1.

15.3 Nonlinear Analysis

15.3.1 Basic estimates

In the case of domains with corners, we deviate from the convention (10.2) applied in the rest of the book and introduce

$$\mathcal{G}_{\kappa,\sigma}(\psi,\mathbf{A}) = \int_\Omega |p_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\,dx$$
$$+ (\kappa\sigma)^2 \int_{\mathbb{R}^2} |\operatorname{curl}\mathbf{A} - 1|^2\,dx\,. \tag{15.14}$$

Here we have immediately restricted to the case of a constant exterior magnetic field. This functional is studied much in the spirit of our analysis of the 3D functional.

Let us define

$$\mathbf{F}(x) := \Big(-\frac{x_2}{2}, \frac{x_1}{2}\Big)\,. \tag{15.15}$$

Contrary to the case of three dimensions, it is not easy to prove that minimizers of $\mathcal{G}$ exist. This has to do with the lack of existence of a homogeneous Sobolev inequality [like (D.16)] in 2D. We therefore have to restrict attention to $\mathbf{A}$ such that $\mathbf{A}-\mathbf{F}$ belongs to the space $W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)$—a somewhat smaller space than $\dot{H}^1_{\mathbf{F}}$. See (D.12) in Appendix D for the definition of $W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)$.

Theorem 15.3.1.
For all $\kappa,\sigma>0$, there exists a (possibly nonunique) minimizer of $\mathcal{G}_{\kappa,\sigma}$ defined on $(\psi,\mathbf{A})$ such that $(\psi,\mathbf{A}-\mathbf{F}) \in W^{1,2}(\Omega)\times W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)$.

Furthermore, minimizers are weak solutions of the Ginzburg–Landau equations:

$$p^2_{\kappa\sigma\mathbf{A}}\psi = \kappa^2(1-|\psi|^2)\psi \qquad \textit{in} \quad \Omega\,, \tag{15.16a}$$

$$\operatorname{curl}^2\mathbf{A} = -\tfrac{1}{\kappa\sigma}\Re(\overline{\psi}\,p_{\kappa\sigma\mathbf{A}}\psi)1_\Omega \qquad \textit{in} \quad \mathbb{R}^2\,, \tag{15.16b}$$

$$(p_{\kappa\sigma\mathbf{A}}\psi)\cdot\nu = 0 \qquad \textit{on} \quad \partial\Omega\,. \tag{15.16c}$$

We can use the gauge invariance (see Lemma D.2.7) to impose the condition

$$\operatorname{div}\mathbf{A} = 0\,. \tag{15.17}$$

This determines $\mathbf{A}$ up to an additive constant. We choose (again using the gauge freedom) this constant such that (D.14) holds, i.e.,

$$\|\mathbf{A}-\mathbf{F}\|_{W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)} = \inf_{a\in\mathbb{R}^2}\|(\mathbf{A}-\mathbf{F})-a\|_{W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)}$$
$$\leq C\|\operatorname{curl}\mathbf{A}-1\|_2\,, \tag{15.18}$$

for some universal constant C.

In this chapter, we will always study solutions of (15.16) under the additional gauge choice implied by (15.17) and (15.18). With this choice, $\operatorname{curl}\mathbf{A}-1$ controls the local $W^{1,2}$ norm by (D.15); in particular, there exists a $C>0$ such that

$$\|\mathbf{A}-\mathbf{F}\|_{W^{1,2}(\Omega)} \leq C\|\operatorname{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)}\,. \tag{15.19}$$

By a Sobolev inequality, this again implies that $\|\operatorname{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)}$ controls $\|\mathbf{A}-\mathbf{F}\|_{L^p(\Omega)}$ for all $p<\infty$.

Lemma 15.3.2.
Let Ω be a bounded polygonal domain and let $(\psi,\mathbf{A})$ be a (weak) solution to (15.16)*. Then* $\operatorname{curl}(\mathbf{A}-\mathbf{F})=0$ *on* $\mathbb{R}^2\setminus\overline{\Omega}$.

Proof.
The second equation, (15.16b), reads in the exterior of Ω (in the sense of distributions), using the fact that $\operatorname{curl}\mathbf{F}=1$,

$$\big(\partial_2\operatorname{curl}(\mathbf{A}-\mathbf{F}),-\partial_1\operatorname{curl}(\mathbf{A}-\mathbf{F})\big)=0\,.$$

Thus, we see that $\operatorname{curl}(\mathbf{A}-\mathbf{F})$ is constant on $\mathbb{R}^2\setminus\overline{\Omega}$. Since $\operatorname{curl}(\mathbf{A}-\mathbf{F})\in L^2(\mathbb{R}^2)$, we get the conclusion. □

Rereading the proofs from Chapter 10 with the new definition of the functional [and using Lemma 15.3.2 in the proof of (15.22)], we now get the following results for polygonal domains:

- Solutions to (15.16) satisfy the inequality
$$\|\psi\|_\infty \leq 1\,. \tag{15.20}$$
- Solutions to (15.16) satisfy the inequalities (cf. Lemma 10.3.2)
$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa\|\psi\|_2\,, \tag{15.21}$$
$$\|\operatorname{curl}\mathbf{A}-1\|_{L^2(\mathbb{R}^2)} \leq \frac{C_1}{\sigma}\|\psi\|_2\|\psi\|_{L^4(\Omega)}\,, \tag{15.22}$$
for some constant $C_1>0$.
- There exists a constant $C>0$ such that if
$$\sigma \geq C\max(1,\kappa)\,, \tag{15.23}$$
then the normal state, $(0,\mathbf{F})$, is the unique solution to (15.16).

The statement of (15.23) in Chapter 10, i.e., Theorem 10.4.1, contains the assumption that Ω is smooth. However, upon inspecting the proof, one realizes that this assumption was only needed in order to know that $B\mapsto\lambda_1(B)$ increases like constant $\times\, B$ for large values of B. This was established for polygons in the previous section; therefore, Theorem 10.4.1 is also valid for polygons.

We will mainly use (15.22) combined with (15.19) and a Sobolev inequality to give a bound on the $L^4(\Omega)$ norm:

Lemma 15.3.3.
There exists a constant $C > 0$ such that, for all solutions $(\psi, \mathbf{A})$ to (15.16), we have

$$(\kappa\sigma)^2 \|\mathbf{A} - \mathbf{F}\|^2_{L^4(\Omega)} \leq C\,\kappa^2\,\|\psi\|_4^2\,\|\psi\|_2^2\,. \tag{15.24}$$

15.3.2 Nonlinear Agmon estimates

Normal estimates

In the variable normal to the boundary we have the exponential estimate of Theorem 12.5.4. The input to these estimates is the inequalities (12.4)–(12.6), which by (15.20)–(15.22) are also valid for polygons. Therefore, Theorem 12.5.4 also holds for solutions to (15.16). In particular, the "rough bound" established in Theorem 12.3.1 holds for solutions to (15.16).

Rough bounds on $\|\psi\|_2^2$

For stronger fields, superconductivity is essentially localized to the corners.

Theorem 15.3.4 (Decay estimate on the boundary)**.**
Suppose that Ω satisfies Assumption 15.1.1. For $\mu \in (\Lambda_1, \Theta_0)$, define

$$\Sigma' = \Sigma'(\mu) := \{\mathsf{s} \in \Sigma \,|\, \mu_1(\alpha_{\mathsf{s}}) \leq \mu\} \qquad \textit{and}$$
$$b := \inf_{\mathsf{s}\in\Sigma\setminus\Sigma'} \{\mu_1(\alpha_{\mathsf{s}}) - \mu\} \tag{15.25}$$

(in the case $\Sigma = \Sigma'$, we set $b := \Theta_0 - \mu$).

There exist $\kappa_0 > 0$, $C > 0$, $C' > 0$, and $M > 0$ such that if $(\psi, \mathbf{A})_{\kappa,\sigma}$ is a solution of (15.16) with

$$\frac{\sigma}{\kappa} \geq \mu^{-1}, \qquad \kappa \geq \kappa_0\,, \tag{15.26}$$

then

$$\|\psi\|_2^2 \leq C \int_{\{\kappa\ \mathrm{dist}(x,\Sigma')\leq M\}} |\psi(x)|^2\,dx \leq \frac{C'}{\kappa^2}\,. \tag{15.27}$$

Proof.
To prove this result, we follow the same procedure as in the proof of Theorem 12.3.1.

Let $\delta = b/2$, and let $M_0 = M_0(\delta)$ be the constant from Theorem 15.2.2. Let $\chi \in C^\infty(\mathbb{R})$ be a standard nondecreasing cutoff function,

$$\chi = 1 \quad \text{on } [1, \infty[\,, \qquad \chi = 0 \quad \text{on }]-\infty, 1/2[\,,$$

and let $\lambda = 2M_0/\sqrt{\kappa\sigma}$. Define $\chi_\lambda : \Omega \to \mathbb{R}$, by

$$\chi_\lambda(x) = \chi\left(\frac{\operatorname{dist}(x, \Sigma')}{\lambda}\right).$$

Then χ_λ is a Lipschitz function and $\operatorname{supp}\chi_\lambda \cap \Sigma' = \emptyset$. Combining the standard localization formula and (15.16), we find as previously

$$\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\chi_\lambda\psi)|^2\,dx - \int_\Omega |\nabla\chi_\lambda|^2|\psi|^2\,dx$$
$$= \Re\langle\chi_\lambda^2\psi, P^N_{\kappa\sigma\mathbf{A},\Omega}\psi\rangle \leq \kappa^2\|\chi_\lambda\psi\|_2^2\,. \tag{15.28}$$

We need a lower bound to $\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\chi_\lambda\psi)|^2\,dx$. Notice that

$$\operatorname{supp}\chi_\lambda \cap \partial\Omega \neq \emptyset\,;$$

so we cannot use the basic lower bound from (1.32). Therefore, we will introduce the constant magnetic field $\mathbf{F}$ for which we do have such an estimate, namely Theorem 15.2.2. We can write

$$\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\chi_\lambda\psi)|^2\,dx \geq (1-\varepsilon)\int_\Omega |p_{\kappa\sigma\mathbf{F}}(\chi_\lambda\psi)|^2\,dx$$
$$-\,\varepsilon^{-1}\int_\Omega (\kappa\sigma)^2|\mathbf{F}-\mathbf{A}|^2(\chi_\lambda\psi)|^2\,dx\,. \tag{15.29}$$

Theorem 15.2.2 and the choice of λ imply that

$$\int_\Omega |p_{\kappa\sigma\mathbf{F}}(\chi_\lambda\psi)|^2\,dx \geq \left(\inf_{\mathsf{s}\in\Sigma\setminus\Sigma'}\mu_1(\alpha_{\mathsf{s}}) - \delta\right)\kappa\sigma\|\chi_\lambda\psi\|_2^2$$
$$= \left(\mu + \frac{b}{2}\right)\kappa\sigma\|\chi_\lambda\psi\|_2^2\,. \tag{15.30}$$

We now have to give a lower bound to the second part of the right side of (15.29). We can estimate

$$\int_\Omega (\kappa\sigma)^2|\mathbf{F}-\mathbf{A}|^2|\chi_\lambda\psi|^2\,dx \leq (\kappa\sigma)^2\|\mathbf{A}-\mathbf{F}\|^2_{L^4(\Omega)}\,\|\chi_\lambda\psi\|_4^2\,. \tag{15.31}$$

By Lemma 15.3.3 and (15.20),

$$(\kappa\sigma)^2\|\mathbf{F}-\mathbf{A}\|^2_{L^4(\Omega)} \leq C\kappa^2\sigma^2\|\mathbf{F}-\mathbf{A}\|^2_{W^{1,2}(\Omega)} \leq \tilde{C}\kappa^2\|\psi\|_2^2\,. \tag{15.32}$$

Let us now estimate $\|\chi_\lambda\psi\|_4^2$. According to (12.20)—which is valid for solutions to (15.16) by the discussion above—and the property of the cutoff function $0 \leq \chi_\lambda \leq 1$, we can deduce that

$$\|\chi_\lambda \psi\|_4^2 \leq \frac{C}{\sqrt{\kappa}} . \tag{15.33}$$

Inserting (15.30)–(15.33) in (15.29), we obtain

$$\int_\Omega |p_{\kappa\sigma\mathbf{A}}(\chi_\lambda\psi)|^2\,dx \geq (1-\varepsilon)\left(\mu + \frac{b}{2}\right)\kappa\sigma\|\chi_\lambda\psi\|_2^2 - C\varepsilon^{-1}\kappa^{3/2}\|\psi\|_2^2 . \tag{15.34}$$

We insert (15.34) in (15.28). Then

$$\begin{aligned}\left[(1-\varepsilon)\left(\mu+\frac{b}{2}\right)\kappa\sigma - \kappa^2 - C\varepsilon^{-1}\kappa^{3/2}\right] &\int_{\{\mathrm{dist}(x,\Sigma')\geq\lambda\}} |\psi|^2\,dx \\ \leq (C\varepsilon^{-1}\kappa^{3/2} + \|\chi'\|_\infty^2\lambda^{-2}) &\int_{\{\mathrm{dist}(x,\Sigma')\leq\lambda\}} |\psi|^2\,dx . \end{aligned}\tag{15.35}$$

Assumption (15.26) leads to the lower bound

$$(1-\varepsilon)\left(\mu+\frac{b}{2}\right)\kappa\sigma - \kappa^2 - C\varepsilon^{-1}\kappa^{3/2} \geq \frac{b}{4}\kappa\sigma , \tag{15.36}$$

as soon ε is small enough and κ large enough.

Once ε is fixed and with $\lambda = 2M_0/\sqrt{\kappa\sigma}$, we find

$$C\varepsilon^{-1}\kappa^{3/2} + \|\chi'\|_\infty\lambda^{-2} \leq c\,\kappa\sigma . \tag{15.37}$$

Combining (15.35)–(15.37), we deduce

$$\int_{\{\mathrm{dist}(x,\Sigma')\geq\lambda\}} |\psi|^2\,dx \leq C\int_{\{\mathrm{dist}(x,\Sigma')\leq\lambda\}} |\psi|^2\,dx . \tag{15.38}$$

It follows easily that

$$\|\psi\|_2^2 \leq (C+1)\int_{\{\mathrm{dist}(x,\Sigma')\leq\lambda\}} |\psi|^2\,dx .$$

Inserting the choice $\lambda = 2M_0/\sqrt{\kappa\sigma}$ and the condition (15.26) on σ, this clearly implies (15.27). □

Exponential localization

We will use the rough bound on ψ to obtain exponential (Agmon) estimates near the corners. The strategy is the same as in Section 12.5.

Theorem 15.3.5.
Suppose that Ω satisfies Assumption 15.1.1, let $\mu > 0$ satisfy

$$\min_{\mathsf{s}\in\Sigma} \mu_1(\alpha_{\mathsf{s}}) < \mu < \Theta_0 ,$$

and define

$$\Sigma' := \{\mathsf{s}\in\Sigma \,|\, \mu_1(\alpha_{\mathsf{s}}) \leq \mu\} .$$

There exist constants $\kappa_0 > 0$, $M > 0$, $C > 0$, *and* $\epsilon > 0$ *such that if*

$$\kappa \geq \kappa_0\,, \qquad \frac{\sigma}{\kappa} \geq \mu^{-1}\,,$$

and $(\psi, \mathbf{A})$ *is a solution of* (15.16), *then*

$$\int_\Omega e^{\epsilon\sqrt{\kappa\sigma}\,\mathrm{dist}(x,\Sigma')}\left(|\psi(x)|^2 + \frac{1}{\kappa\sigma}|p_{\kappa\sigma\mathbf{A}}\psi(x)|^2\right)dx$$

$$\leq C \int_{\{x:\sqrt{\kappa\sigma}\,\mathrm{dist}(x,\Sigma')\leq M\}} |\psi(x)|^2\,dx\,.$$

We will need the following lemma, which is similar to Lemma 12.5.1.

Lemma 15.3.6.
Suppose that $\Omega \subset \mathbb{R}^2$ *satisfies Assumption 15.1.1. For* $\mu \in (\Lambda_1, \Theta_0)$, *define*

$$\Sigma' := \{\mathsf{s} \in \Sigma \,\big|\, \mu_1(\alpha_\mathsf{s}) \leq \mu\} \quad \textit{and} \quad b := \inf_{\mathsf{s}\in\Sigma\setminus\Sigma'}\{\mu_1(\alpha_\mathsf{s}) - \mu\} \tag{15.39}$$

(in the case $\Sigma = \Sigma'$, *we set* $b := \Theta_0 - \mu$*).*

There exists $M_0 > 0$ *such that if* $(\psi, \mathbf{A})$ *is a solution of* (15.16), *then for all* $\phi \in C^\infty(\overline{\Omega})$ *such that* $\mathrm{dist}(\mathrm{supp}\,\phi, \Sigma') \geq M_0/\sqrt{\kappa\sigma}$, *we have*

$$\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \geq \mu\kappa\sigma\left(1 + \frac{b}{4}\right)\|\phi\|_2^2\,, \tag{15.40}$$

for $\kappa\sigma$ *sufficiently large.*

Proof.
Let $\delta = b/2$ and let $M_0 = M_0(\delta)$ be the constant from Theorem 15.2.2. We estimate, for $\phi \in C^\infty(\overline{\Omega})$ such that $\mathrm{dist}(\mathrm{supp}\,\phi, \Sigma') \geq M_0/\sqrt{\kappa\sigma}$,

$$\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \geq (1-\varepsilon)\int_\Omega |p_{\kappa\sigma\mathbf{F}}\phi|^2\,dx - \varepsilon^{-1}\int_\Omega (\kappa\sigma)^2|\mathbf{F}-\mathbf{A}|^2\,|\phi|^2\,dx\,. \tag{15.41}$$

Using Theorem 15.2.2 and the support properties of ϕ, we have

$$\int_\Omega |p_{\kappa\sigma\mathbf{F}}\phi|^2\,dx \geq \left(\inf_{\mathsf{s}\in\Sigma\setminus\Sigma'}\mu_1(\alpha_\mathsf{s}) - \delta\right)\kappa\sigma\|\phi\|_2^2 = \left(\mu + \frac{b}{2}\right)\kappa\sigma\|\phi\|_2^2\,. \tag{15.42}$$

Using the Cauchy–Schwarz inequality, Lemma 15.3.3, and Theorem 15.3.4, we can bound the last term of (15.41) as follows:

$$\int_\Omega (\kappa\sigma)^2|\mathbf{F}-\mathbf{A}|^2\,|\phi|^2\,dx \leq (\kappa\sigma)^2\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)}^2\,\|\phi\|_4^2$$

$$\leq C\kappa^2\|\psi\|_2^2\|\phi\|_4^2 \leq \tilde{C}\big\||\phi|\big\|_4^2\,. \tag{15.43}$$

We use the Sobolev inequality (12.30) in (15.43) and estimate $\|\nabla|\phi|\|_2^2$, using the diamagnetic inequality, by $\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2$ to obtain

$$\int_\Omega (\kappa\sigma)^2 |\mathbf{F}-\mathbf{A}|^2\,|\phi|^2\,dx \le C_{\mathrm{Sob}}\big(\eta\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 + \eta^{-1}\|\phi\|_2^2\big)\,. \tag{15.44}$$

Combining (15.41), (15.42) and (15.44), we deduce that

$$\left(1+\frac{C_{\mathrm{Sob}}\eta}{\varepsilon}\right)\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \ge \left\{(1-\varepsilon)\left(\mu+\frac{b}{2}\right)\kappa\sigma - \frac{C_{\mathrm{Sob}}}{\varepsilon\eta}\right\}\|\phi\|_2^2\,. \tag{15.45}$$

We choose $\eta = C_{\mathrm{Sob}}/(\varepsilon^2\kappa\sigma)$; then (15.45) becomes

$$\left(1+\frac{C_{\mathrm{Sob}}^2}{\varepsilon^3\kappa\sigma}\right)\|p_{\kappa\sigma\mathbf{A}}\phi\|_2^2 \ge \kappa\sigma\left\{(1-\varepsilon)\left(\mu+\frac{b}{2}\right)-\varepsilon\right\}\|\phi\|_2^2\,. \tag{15.46}$$

If we choose ε sufficiently small and independent of κ,σ (actually, since $\mu + b/2 \le 1$, $\varepsilon = b/8$ will do), then (15.40) follows. □

By standard arguments, Lemma 15.3.6 implies the Agmon estimates given in Theorem 15.3.5.

Proof of Theorem 15.3.5.
The function $t'(x) := \mathrm{dist}(x,\Sigma')$ defines a Lipschitz continuous function on Ω. In particular, $|\nabla t'| \le 1$. Let $\chi \in C^\infty(\mathbb{R})$ be a nondecreasing function satisfying

$$\chi = 1 \quad \text{on } [1,\infty[\,, \qquad \chi = 0 \quad \text{on } [-\infty, 1/2[\,.$$

Define the function χ_M on Ω by $\chi_M(x) = \chi(t'(x)\sqrt{\kappa\sigma}/M)$. By Lemma 15.3.6, there exists $\beta > 0$ such that if $M, \kappa\sigma$ are sufficiently large, then

$$\int_\Omega \left|p_{\kappa\sigma\mathbf{A}}\big(e^{\epsilon\sqrt{\kappa\sigma}t'}\chi_M\psi\big)\right|^2 dx \ge \mu\kappa\sigma(1+\beta)\|e^{\epsilon\sqrt{\kappa\sigma}t'}\chi_M\psi\|_2^2.$$

Using the localization formula and the assumption $\sigma/\kappa \ge \mu^{-1}$, there exists some constant C independent of κ, σ, ϵ, and M such that

$$\begin{aligned}\beta\mu\|e^{\epsilon\sqrt{\kappa\sigma}t'}\chi_M\psi\|_2^2 &\le C\epsilon^2\|\nabla t'\|_\infty^2\|e^{\epsilon\sqrt{\kappa\sigma}t'}\chi_M\psi\|_2^2 \\ &\quad + \frac{C\|\nabla t'\|_\infty^2}{M^2}\int_\Omega e^{2\epsilon\sqrt{\kappa\sigma}t'(x)}\left|\chi'\left(\frac{t'(x)\sqrt{\kappa\sigma}}{M}\right)\psi(x)\right|^2 dx\,.\end{aligned} \tag{15.47}$$

We achieve the proof of Theorem 15.3.5 with arguments similar to the ones of the proof of Theorem 9.4.1. □

15.3.3 Equality of critical fields

In order to define the critical field $H_{C_3}(\kappa)$, one runs into the same problems of a priori non-uniqueness as in the smooth case. Thus, we can define upper and lower fields as in (13.1), (13.2) and also local versions as in (13.9). We will prove that in the case with corners these definitions also coincide for large values of κ.

Theorem 15.3.7.
Suppose that Ω satisfies Assumption 15.1.1. Then there exists $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$, then the equation

$$\lambda_{1,\Omega}(\kappa\sigma) = \kappa^2$$

has a unique solution $H = H_{C_3}^{\mathrm{loc}}(\kappa)$. Furthermore, if κ_0 is chosen sufficiently large, then for $\kappa \geq \kappa_0$, the critical fields defined in (13.1), (13.2), *and* (13.9) *coincide and satisfy*

$$\underline{H}_{C_3}(\kappa) = \overline{H}_{C_3}(\kappa) = H_{C_3}^{\mathrm{loc}}(\kappa)\,. \tag{15.48}$$

Finally, the critical field satisfies

$$H_{C_3}(\kappa) = \frac{\kappa}{\Lambda_1} + \mathcal{O}(\kappa^{-\infty}), \qquad \textit{for} \quad \kappa \to \infty\,. \tag{15.49}$$

Proof of Theorem 15.3.7.
By Corollary 15.2.7 it only remains to prove (15.48). Actually, we will prove that Proposition 13.1.7 remains true for domains with corners with essentially unchanged proof. As in the proof of that proposition, the only nontrivial point is the inclusion (for large κ)

$$\mathcal{N}^{sc}(\kappa) \subseteq \mathcal{N}^{\mathrm{loc}}(\kappa)\,.$$

We now let $(\psi, \mathbf{A})$ be a nontrivial solution to (15.16). Since we have (15.23) and the leading term of the asymptotics of $\lambda_1(B)$, it clearly suffices to consider the case where

$$C^{-1} \leq \frac{\kappa}{\sigma} \leq \Lambda_1 + \epsilon\,,$$

for some $C > 0$ and $\epsilon > 0$ (small). Also, by Lemma 15.3.3,

$$\|\mathbf{A} - \mathbf{F}\|_{L^4(\Omega)} \leq \frac{C}{\sigma} \|\psi\|_4 \, \|\psi\|_2\,. \tag{15.50}$$

By Corollary 12.5.5, we find, for some $C > 0$,

$$\|\psi\|_2^2 \leq C \, \frac{\|\psi\|_4^2}{(\kappa\sigma)^{\frac{1}{4}}}\,. \tag{15.51}$$

With these estimates established, the proof of Proposition 13.1.7 goes through unchanged. ☐

15.3.4 Energy asymptotics in corners

Finally, we discuss leading-order energy asymptotics in the parameter regime dominated by the corners, i.e., $1 \ll \sigma - \kappa/\Theta_0 \leq H_{C_3}(\kappa) - \kappa/\Theta_0$. The result below, Theorem 15.3.8, can be seen as a partial converse statement to Theorem 15.3.5 in that all corners that are spectrally permitted will contribute to the leading-order of the ground state energy.

We recall that the angular sectors Ω_α were defined in Section 4.4. Let $\alpha \in]0,\pi[$ be such that $\mu^{\rm sect}(\alpha) < \Theta_0$. Define, for $\mu_1, \mu_2 > 0$, the following functional $J^\alpha_{\mu_1,\mu_2}$:

$$J^\alpha_{\mu_1,\mu_2}[\psi] = \int_{\Omega_\alpha} \left\{ |(-i\nabla + \mathbf{F})\psi|^2 - \mu_1|\psi|^2 + \frac{\mu_2}{2}|\psi|^4 \right\} dx\,, \tag{15.52}$$

with domain $\{\psi \in L^2(\Omega_\alpha)\,|\,(-i\nabla + \mathbf{F})\psi \in L^2(\Omega_\alpha)\}$. Define also the corresponding ground state energy

$$E^\alpha_{\mu_1,\mu_2} := \inf_\psi J^\alpha_{\mu_1,\mu_2}[\psi]\,.$$

The main result on the ground state energy of the Ginzburg–Landau functional in the parameter regime dominated by the corners is the following.

Theorem 15.3.8.
Suppose $\kappa/\sigma(\kappa) \to \mu \in \mathbb{R}_+$ as $\kappa \to \infty$, where $\mu < \Theta_0$. Let $(\psi, \mathbf{A}) = (\psi,\mathbf{A})_{\kappa,\sigma(\kappa)}$ be a minimizer of $\mathcal{G}_{\kappa,\sigma(\kappa)}$.

Then

$$\mathcal{G}_{\kappa,\sigma(\kappa)}[\psi,\mathbf{A}] \to \sum_{\mathsf{s}\in\Sigma} E^{\alpha_{\mathsf{s}}}_{\mu,\mu}\,, \tag{15.53}$$

as $\kappa \to \infty$.

Remark 15.3.9.
Proposition 15.3.10 states that $E^{\alpha_{\mathsf{s}}}_{\mu,\mu} = 0$ unless $\mu_1(\alpha_{\mathsf{s}}) < \mu$. So only corners satisfying this spectral condition contribute to the ground state energy, in agreement with the localization estimate from Theorem 15.3.5.

Basic properties

We give the following proposition without proof, since it is completely analogous to the similar statements for $\mathcal{G}_{\kappa,\sigma}$.

Proposition 15.3.10.
The map $]0,\Theta_0[\times\mathbb{R}_+ \ni (\mu_1,\mu_2) \mapsto E^\alpha_{\mu_1,\mu_2}$ is continuous.

Suppose that $\mu_1 < \Theta_0$. If $\mu_1 \leq \mu^{\rm sect}(\alpha)$, then $E^\alpha_{\mu_1,\mu_2} = 0$ and $\psi = 0$ is a minimizer.

If $\mu_1 > \mu^{\rm sect}(\alpha)$, there exists a nontrivial minimizer ψ_0 of $J^\alpha_{\mu_1,\mu_2}$. Furthermore, there exist constants a and $C > 0$ such that

$$\int_{\Omega_\alpha} e^{2a|x|}\big(|\psi_0(x)|^2 + |(-i\nabla + \mathbf{F})\psi_0|^2\big)\,dx \leq C\,. \tag{15.54}$$

Finally, ψ_0 satisfies the uniform bound,

$$\|\psi_0\|_\infty \leq \frac{\mu_1}{\mu_2}\,.$$

One easily verifies the following scaling property.

Proposition 15.3.11.
Let $T > 0$. Then the functional

$$\psi \mapsto \int_{\Omega_\alpha} |(-i\nabla + T^{-2}\mathbf{F})\psi|^2 - \mu_1 T^{-2}|\psi|^2 + \frac{\mu_2}{2}T^{-2}|\psi|^4\,dx\,,$$

defined on $\{\psi \in L^2(\Omega_\alpha)\,\big|\,(-i\nabla + T^{-2}\mathbf{F})\psi \in L^2(\Omega_\alpha)\}$, is minimized by

$$\tilde{\psi}_0(y) = \psi_0\big(\frac{y}{T}\big)\,,$$

where ψ_0 is the minimizer of $J^\alpha_{\mu_1,\mu_2}$.
In particular,

$$\inf_\psi \int_{\Omega_\alpha} |(-i\nabla + T^{-2}\mathbf{F})\psi|^2 - \mu_1 T^{-2}|\psi|^2 + \frac{\mu_2}{2}T^{-2}|\psi|^4\,dx = E^\alpha_{\mu_1,\mu_2}\,.$$

By the continuity of $E^\alpha_{\mu_1,\mu_2}$, we get the following consequence.

Proposition 15.3.12.
Suppose that $\lim_{\kappa\to+\infty}\kappa/\sigma(\kappa) := \mu < \Theta_0$ and that $d_1(\kappa)$ and $d_2(\kappa) \to 1$ as $\kappa \to \infty$. Then the ground state energy of the functional

$$\psi \mapsto \int_{\Omega_\alpha} |p_{\kappa\sigma\mathbf{F}}\psi|^2 - d_1(\kappa)\kappa^2|\psi|^2 + d_2(\kappa)\frac{\kappa^2}{2}|\psi|^4\,dx \tag{15.55}$$

tends to $E^\alpha_{\mu,\mu}$ as $\kappa \to \infty$.

Proof of Theorem 15.3.8.
Upper bounds
We indicate here how to obtain the inequality

$$\inf_{(\psi,\mathbf{A})} \mathcal{G}_{\kappa,\sigma(\kappa)}[\psi,\mathbf{A}] \leq \sum_{\mathsf{s}\in\Sigma} E^{\alpha_\mathsf{s}}_{\mu,\mu} + o(1)\,, \tag{15.56}$$

which is the "easy" part of (15.53).

The inequality (15.56) follows from a calculation with an explicit trial state. The test functions will be of the form $\mathbf{A} = \mathbf{F}$ and

$$\psi(x) = \sum_{\mathsf{s}\in\Sigma}\psi_\mathsf{s}(\Phi_\mathsf{s}(x)), \qquad \text{with} \qquad \psi_\mathsf{s}(y) = e^{i\kappa\sigma\eta_\mathsf{s}}\psi^{\alpha_\mathsf{s}}_{\mu,\mu}(\sqrt{\kappa\sigma}y)\chi(|y|)\,.$$

Here $\eta_\mathsf{s} \in C^\infty(\mathbb{R}^2,\mathbb{R})$ is a gauge function, χ is a standard cutoff function, $\chi = 1$ on a, neighborhood of 0 , $\operatorname{supp}\chi \subset \mathrm{D}(0,r)$, with $r = \frac{1}{2}\min_{j<k}|\mathsf{s}_j - \mathsf{s}_k|$, and

$\psi^{\alpha_{\mathsf{s}}}_{\mu,\mu}$ is the minimizer of $J^{\alpha_{\mathsf{s}}}_{\mu,\mu}$. The coordinate change Φ_{s} is the rigid motion mapping $\Omega \cap \mathrm{D}(\mathsf{s}, r)$ to $\Omega_{\alpha_{\mathsf{s}}} \cap \mathrm{D}(0, r)$.

The proof of (15.56) is a straightforward calculation similar to the lower bound (given below) and will be omitted. Notice though that the decay estimates (15.54) for the minimizers $\psi^{\alpha_{\mathsf{s}}}_{\mu,\mu}$ imply that the function defined by $\psi^{\alpha_{\mathsf{s}}}_{\mu,\mu}(\sqrt{\kappa\sigma}y)[1-\chi(|y|)]$ is exponentially small.

Lower bounds

Let $(\psi, \mathbf{A})$ be a minimizer of $\mathcal{G}_{\kappa,\sigma}$. Define $\chi_1 \in C^\infty(\mathbb{R})$ to be a standard localization function, χ_1 is nonincreasing, $\chi_1(t) = 1$ for $t \leq 1$, $\chi_1(t) = 0$ for $t \geq 2$.

For $\mathsf{s} \in \Sigma$, let

$$\phi_{\mathsf{s}}(x) = \chi_1\left(\frac{4\,\mathrm{dist}(x, \mathsf{s})}{\delta}\right),$$

where $\delta := \min j < k|\mathsf{s}_j - \mathsf{s}_k|$.

Notice that $\phi_{\mathsf{s}}\phi_{\mathsf{s}'} = 0$, when $\mathsf{s} \neq \mathsf{s}'$. Therefore, using Theorem 15.3.5, the localization formula, and the estimate $\|\psi\|_\infty \leq 1$, we can write

$$\begin{aligned}\mathcal{G}_{\kappa,\sigma}[\psi, \mathbf{A}] &\geq \sum_{\mathsf{s}\in\Sigma} \mathcal{G}_{\kappa,\sigma}[\phi_{\mathsf{s}}\psi, \mathbf{A}] + \mathcal{O}(\kappa^{-\infty}) \qquad (15.57)\\ &= \sum_{\mathsf{s}\in\Sigma} \int_\Omega |p_{\kappa\sigma\mathbf{A}}(\phi_{\mathsf{s}}\psi)|^2 - \kappa^2|\phi_{\mathsf{s}}\psi|^2 + \frac{\kappa^2}{2}|\phi_{\mathsf{s}}\psi|^4\,dx + \mathcal{O}(\kappa^{-\infty}).\end{aligned}$$

By Lemma 15.3.3, (15.20), and Theorem 15.3.5, we get

$$(\kappa\sigma)^2\|\mathbf{A} - \mathbf{F}\|_4^2 \leq C''\kappa^2\|\psi\|_2^2 \leq C'''. \qquad (15.58)$$

Thus, we can estimate

$$\begin{aligned}&\int_\Omega \left|p_{\kappa\sigma\mathbf{A}}(\phi_{\mathsf{s}}\psi)\right|^2 dx\\ &\qquad\geq (1-\kappa^{-\frac{1}{2}})\int_\Omega \left|p_{\kappa\sigma\mathbf{F}}(\phi_{\mathsf{s}}\psi)\right|^2 dx - \kappa^{\frac{1}{2}}(\kappa\sigma)^2\|\mathbf{A}-\mathbf{F}\|_4^2\|\phi_{\mathsf{s}}\psi\|_4^2\\ &\qquad\geq (1-\kappa^{-\frac{1}{2}})\int_\Omega \left|p_{\kappa\sigma\mathbf{F}}(\phi_{\mathsf{s}}\psi)\right|^2 dx - C\kappa^{-\frac{1}{2}}, \qquad (15.59)\end{aligned}$$

where we used the inequality (consequence of Theorem 15.3.5)

$$\|\phi_{\mathsf{s}}\psi\|_4^2 \leq \sqrt{\int_\Omega |\psi|^2\,dx} \leq \sqrt{C\int_{\{\mathrm{dist}(x,\Sigma)\leq M\kappa^{-1}\}} dx} \leq C'\kappa^{-1}.$$

Combining (15.59) and (15.57), we find

$$\mathcal{G}_{\kappa,\sigma}[\psi,\mathbf{A}] \geq (1-\kappa^{-\frac{1}{2}})\sum_{\mathsf{s}\in\Sigma}\int_\Omega \Big\{|p_{\kappa\sigma\mathbf{F}}(\phi_{\mathsf{s}}\psi)|^2 - \frac{\kappa^2}{1-\kappa^{-\frac{1}{2}}}|\phi_{\mathsf{s}}\psi|^2 + \frac{\kappa^2}{2(1-\kappa^{-\frac{1}{2}})}|\phi_{\mathsf{s}}\psi|^4\Big\}\,dx + \mathcal{O}(\kappa^{-\frac{1}{2}})\,. \tag{15.60}$$

For fixed $\mathsf{s}\in\Sigma$ (and using the fact that Ω is a polygon), the integral is—up to a rigid motion and a gauge transformation—of the type considered in (15.55). We therefore get by Proposition 15.3.12:

$$\mathcal{G}_{\kappa,\sigma(\kappa)}[\psi,\mathbf{A}] \geq \sum_{\mathsf{s}\in\Sigma} E^{\alpha_{\mathsf{s}}}_{\mu,\mu} + o(1)\,.$$

This finishes the proof of Theorem 15.3.8. □

15.4 Notes

1. We have chosen for simplicity to present the case of polygons. This permits us to directly implement the results on infinite sectors. For results on more general domains with corners, see below.
2. The observation that the superconductivity first appears at the corner for type II superconductors was mentioned in the physics literature. Semi-rigorous results were obtained by Molshalkov et al. We refer, for example, to [BeR, Section 4.3.6].
3. At the mathematical level, Jadallah, Rubinstein, and Sternberg first exposed the corner effect for $H_{C_3}(\kappa)$ in [JaRS]. Further results were obtained by Jadallah [Ja] and Pan [Pa1]. Here we mainly refer to contributions by Bonnaillie [Bon1, Bon2], Bonnaillie-Noël–Dauge [BonD], Bonnaillie-Noël–Dauge–Vial [BonDMV], and Bonnaillie-Noël–Fournais [BonF]. In particular, an analysis similar to the present chapter, but for general curvilinear domains instead of polygons, is done in [BonD], [BonF].
4. The linear spectral problem has been studied in depth in the case of corners in [Bon1, Bon2, BonD].
5. The tunneling effect is discussed in the thesis of Bonnaillie [Bon1]. She implements ideas that were developed for the Schrödinger operator in [HeS1, HeS2]. This was further discussed in Bonnaillie-Noël–Dauge [BonD] and numerically in the work by Bonnaillie-Noël–Dauge–Martin–Vial [BonDMV]. It is rather surprising that the numerical system MÉLINA permits one to follow exponentially close eigenvalues so accurately.
   The computations for the triangle were done for us more recently by Bonnaillie-Noël.
6. One can do the same kind of discussion about the tunneling in a case of regular domains with a G_n symmetry. Then we can play with the points of

maximal curvature assuming that they are nondegenerate and exchanged by the rotation.

7. For more general domains, the nonlinear part has been carried out in [BonF].
8. The question about the existence of minimizers has been studied by several authors; see [Gio], [GioS], and references therein. The solution is to use the correct variational space $W^{1,2}_{0,0}$. Some of these technical points are discussed briefly in Section D.2.3.

16

On Other Models in Superconductivity and Open Problems

In this chapter, we will briefly describe some questions for which the techniques developed in this book have been (or could be) useful for understanding some asymptotic regimes for other problems occurring in superconductivity. We will also discuss some open questions.

16.1 On Josephson's Junctions

In [Kac3] (extending previous results of [ChDG, Gio, GioJ]), Kachmar analyzes the situation of two open sets Ω_1 and Ω_2—Ω_1 representing the superconducting material and Ω_2 surrounding Ω_1 and playing the role of a metallic material. Mathematically, this corresponds to adding to the previously introduced functional $\mathcal{G}$ (used with $\Omega = \Omega_1$) a second functional associated with Ω_2 taking the form

$$\mathcal{G}^J_{\Omega_2}(\psi, \mathbf{A}) := \int_{\Omega_2} \left\{ \frac{1}{m} |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 + a\kappa^2|\psi|^2 + \mu\sigma^2 |\frac{1}{\mu}\operatorname{curl}\mathbf{A} - 1|^2 \right\} dx\,,$$

with $a > 0$, $m > 0$, and $\mu > 0$.

In physics, the existence of a surrounding material is often modeled through the de Gennes boundary condition—instead of studying the Ginzburg–Landau equations with Neumann boundary condition, one replaces the first equation in (10.8b) by an equation of the type (1.56). A rather complete analysis of the modified functional has been performed. This new situation requires

- a spectral analysis of new models in dimension 1,
- a semiclassical analysis of problems with a transmission condition at the boundary between Ω_1 and Ω_2.

Through this approach, Kachmar could also obtain an alternative explanation of the de Gennes condition. In certain parameter regimes, the results coincide

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with those obtained for the much studied "usual" Ginzburg–Landau model, including in particular the two-term expansion for the upper critical field obtained by Helffer–Pan [HeP1] and the identification of the critical field H_{C_3} [FoK1].

16.2 Analogy with Liquid Crystals

Based on de Gennes' theory of analogies between liquid crystals and superconductivity, Pan introduced in [Pa5] a critical wave number Q_{c_3} (which is an analog of the upper critical field H_{C_3} for superconductors) and predicted the existence of a surface smectic state, which is supposed to be an analog of the surface superconducting state. It is then interesting to analyze the existence of the surface smectic state of liquid crystals. We refer here to recent contributions, including [BaCLP], [Al7], and [HeP2, HeP3], and references therein.

Let us recall the Landau–de Gennes functional of liquid crystals [dGeP, dGe4]. After some simplifying assumptions, this energy functional takes the form

$$\begin{aligned}\mathcal{E}[\psi,\mathbf{n}] = \int_\Omega \Big\{ & |\nabla_{q\mathbf{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \\ & + K_1|\operatorname{div}\mathbf{n}|^2 + K_2|\mathbf{n}\cdot\operatorname{curl}\mathbf{n}+\tau|^2 + K_3|\mathbf{n}\times\operatorname{curl}\mathbf{n}|^2 \Big\}\, dx\,, \end{aligned} \tag{16.1}$$

where Ω is the region occupied by the liquid crystal, ψ is a complex-valued function called the *order parameter*, $\mathbf{n}$ is a real vector field of unit length called the *director field*, q is a real number called the *wave number*, τ is a real number referring to the *chiral pitch* in some liquid crystal materials, K_1 , K_2 , and K_3 are positive constants called *elastic coefficients*, and κ is a positive constant that depends on the material and temperature. As in [Pa5], we call κ the *Ginzburg–Landau parameter* of the liquid crystal.

We are interested in the properties of the global minimizers of $\mathcal{E}$ without prescribing boundary data for the director fields. As explained in [Pa5], the natural space for the variational problems of (16.1) is

$$\mathbb{V}(\Omega) = H^1(\Omega,\mathbb{C}) \times V(\Omega,\mathbb{S}^2)\,,$$

where[1]

$$\begin{aligned} V(\Omega,\mathbb{R}^3) &= \{\mathbf{u}\in L^2(\Omega,\mathbb{R}^3): \quad \operatorname{div}\mathbf{u}\in L^2(\Omega), \quad \operatorname{curl}\mathbf{u}\in (L^2(\Omega,\mathbb{R}^3))^3\},\\ V(\Omega,\mathbb{S}^2) &= \{\mathbf{n}\in V(\Omega,\mathbb{R}^3): \quad |\mathbf{n}(x)| = 1 \quad \text{a.e. in } \Omega\}. \end{aligned} \tag{16.2}$$

[1] The space $V(\Omega,\mathbb{R}^3)$ was denoted by $H(\operatorname{curl},\operatorname{div},\Omega)$ in Dautray–Lions [DaL].

$V(\Omega, \mathbb{R}^3)$ is a Hilbert space with the inner product and norm defined by

$$\begin{aligned}(\mathbf{u}, \mathbf{v})_V &= \int_\Omega \operatorname{div} \mathbf{u} \ \operatorname{div} \mathbf{v} + \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} + \mathbf{u} \cdot \mathbf{v} \, dx \,, \\ \|\mathbf{u}\|_V &= \left\{ \| \operatorname{div} \mathbf{u} \|_2^2 + \| \operatorname{curl} \mathbf{u} \|_2^2 + \|\mathbf{u}\|_2^2 \right\}^{\frac{1}{2}} \,.\end{aligned} \tag{16.3}$$

Define

$$C(K_1, K_2, K_3, \kappa, q, \tau) = \inf_{(\psi, \mathbf{n}) \in \mathbb{V}(\Omega)} \mathcal{E}[\psi, \mathbf{n}] \,.$$

We assume that Ω is a bounded, simply connected domain in $\mathbb{R}^3$ with smooth boundary.

According to de Gennes' theory [dGe3, dGeP], $\psi = 0$ for a nematic phase, and $\psi \neq 0$ for a smectic phase. Hence, a nontrivial minimizer $(\psi, \mathbf{n})$ of (16.1) where $\psi \neq 0$ describes a smectic state, and a trivial critical point $(0, \mathbf{n})$ corresponds to the nematic state.

The set of trivial critical points of $\mathcal{E}$ is given by $(0, \mathbf{n})$ with $\mathbf{n} \in \mathcal{C}(\tau)$, where $\mathbf{n} \in \mathcal{C}(\tau)$ if and only if $\mathbf{n} \in V(\Omega, \mathbb{S}^2)$ and $\mathbf{n}$ satisfies

$$\operatorname{div} \mathbf{n} = 0 \,, \qquad \mathbf{n} \cdot \operatorname{curl} \mathbf{n} + \tau = 0 \,, \qquad \mathbf{n} \times \operatorname{curl} \mathbf{n} = 0 \,. \tag{16.4}$$

For a unit-length vector field, (16.4) is equivalent to

$$\operatorname{div} \mathbf{n} = 0 \,, \qquad \operatorname{curl} \mathbf{n} + \tau \mathbf{n} = 0 \,. \tag{16.5}$$

The set $\mathcal{C}(\tau)$ of all solutions of (16.5) in $V(\Omega, \mathbb{S}^2)$ consists of the vector fields

$$\mathcal{N}_\tau^Q \equiv Q \mathcal{N}_\tau (Q^t x) \,, \quad Q \in SO(3) \,, \tag{16.6}$$

where

$$\mathcal{N}_\tau (x_1, x_2, x_3) = (\cos \tau x_3, \sin \tau x_3, 0) \,. \tag{16.7}$$

One can consequently first consider the reduced Ginzburg–Landau functional $\mathcal{G}_\mathbf{A}$, associated with a magnetic potential $\mathbf{A}$, which is defined on $H^1(\Omega)$ by

$$\mathcal{G}_\mathbf{A}[\psi] = \int_\Omega |\nabla_\mathbf{A} \psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \, dx \,. \tag{16.8}$$

For convenience, we also write $\mathcal{G}_\mathbf{A}[\psi]$ as $\mathcal{G}[\psi, \mathbf{A}]$. Define

$$c_g = \inf_{\mathbf{n} \in \mathcal{C}(\tau)} \inf_{\psi \in H^1(\Omega, \mathbb{C})} \mathcal{G}_{q\mathbf{n}}[\psi] = \inf_{\mathbf{n} \in \mathcal{C}(\tau), \psi \in H^1(\Omega, \mathbb{C})} \mathcal{G}[\psi, q\mathbf{n}] \,. \tag{16.9}$$

It can be shown that c_g is a good approximation of the minimal value of $\mathcal{E}$ for large K_j's. Then one can analyze the behavior of the minimizers of $\mathcal{G}$. There are mainly two cases.

- When $\tau = 0$, we meet essentially the case that was analyzed in this book: analysis of the spectrum of a magnetic Neumann Laplacian with constant magnetic field, with as new point the question of minimizing over the direction of the external field. So this supposes there is a uniform control with respect of this external vector field. This is not a problem in the strictly convex case.

- When $\tau \neq 0$, we already have a problem in order to obtain the optimal asymptotics. The magnetic field is no longer constant but only of constant length. A partial analysis of this has been carried out in [Pa9] and continued in [HeP2].

16.3 Perforated Structures

In this book, we have assumed for simplification that our domains Ω were simply connected. There is an extensive literature about the phenomena arising when holes are present. In the linear part, we have mentioned some specific phenomena due to the Aharonov–Bohm effect. We do not have room here to describe all these phenomena (Little–Parks effects, antidot lattices, superconducting micronetworks, etc.), and we prefer to refer to the book [BeR] edited by Berger and Rubinstein, which is devoted to an overview of what is known, mathematically and physically, on this subject (see also [JiZ],[AlB]). There are still many open questions related to the discussions in that book. We refer to the paper by Frank [Fr1] for preliminary results on antidot lattices.

16.4 Pinning

Here we just present the kind of problems one can find (following a paper of [CaR]). Similar considerations also appear in [KiZ].

When considering the presence of impurities in the superconducting sample, one usually introduces the slightly modified energy functional

$$\mathcal{G}_a(\psi, \mathbf{A}) = \int_\Omega |(\nabla + i\kappa\mathbf{A})\psi|^2 + \kappa^2|\operatorname{curl}\mathbf{A} - \beta|^2 + \frac{\kappa^2}{2}(a(x) - |\psi|^2)^2 dx\,, \tag{16.10}$$

where a is a positive smooth (to simplify) function on $\overline{\Omega}$ with values close to 1. This modified Ginzburg–Landau functional has been studied by many authors in the context of vortex pinning. The idea is that if a mixed state ψ presents a vortex, i.e., a point where ψ vanishes, then the vortex has to be situated in the region where $a < 1$ (see [Ru, Section 3]). This "pinning" of the vortices is important in applications, for their implication in the loss of energy.

In the spirit of the present book, we are interested in a different regime. Our concern is to consider the case where a is slightly less than 1 in Ω, but that there exists a small region where it reaches the value 1. Then we expect that the superconductivity appears in that region, and we would like to have some knowledge on the corresponding critical field. This model can be understood as a study of the nucleation phenomenon (the onset of superconductivity) for a superconducting sample of bad quality. As in the case $a = 1$, we have to consider the linearization of the Euler–Lagrange equation for the functional $\mathcal{G}_a$ at $\psi = 0$, and we obtain (with $\operatorname{curl} \mathbf{F} = \beta$)

$$\begin{cases} -(\nabla + i\sigma\kappa\mathbf{F})^2\psi = \kappa^2 a(x)\psi\,, & \text{in } \Omega\,, \\ (\nabla + i\sigma\kappa\mathbf{F})\psi \cdot \nu = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{16.11}$$

We define $h = 1/\kappa\sigma$, and write (16.11) as

$$\begin{cases} (h\nabla + i\mathbf{F})^2\psi + \frac{\lambda}{\sigma^2}V(x)\psi = \frac{1-\lambda}{\sigma^2}\psi\,, & \text{in } \Omega\,, \\ (h\nabla + i\mathbf{F})\psi \cdot \nu = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{16.12}$$

where we have used the notation

$$a = (1-\lambda) - \lambda V = 1 - \lambda(1+V)\,. \tag{16.13}$$

Here V is a smooth function on $\overline{\Omega}$ with values in $[-1, 0]$, close to 0 in almost all of Ω, with a unique minimum in $\bar{\Omega}$, and $\lambda > 0$ is a real constant that measures the amplitude of the fluctuations of a below the value 1. At least when this minimum is attained in Ω, one meets a problem similar to the one solved in [HeS4] where the main effect is due to the electric potential. One can also play with the size of λ by permitting it to be h-dependent. Hence, we get localization near minima of the electric potential, and surface superconductivity is destroyed by the pinning. There is an extensive literature on the subject of pinning, but it is mainly devoted to the analysis of vortices, which is outside the scope of the present book. We refer to [AfSS, Af2] and references therein.

16.5 Abrikosov Lattices

Abrikosov's theory [Ab] about lattices was predictive in the sense that at the time there was no experiment to motivate the theory. The main point was that one can consider the Schrödinger operator with constant magnetic field B associated to a lattice Γ in $\mathbb{R}^2$ and discuss the ground state energy of the operator restricted to a space $L^2(\mathbb{R}^2/\Gamma, F)$, where F is a one-dimensional fiber bundle over $\mathbb{R}^2/\Gamma$. A quantization condition of the magnetic field appears automatically, and the basic question—which should explain some phenomena in superconductivity—becomes to minimize the quantity

$$I := \int_{\mathbb{R}^2/\Gamma} |\phi_1(x,y)|^4 \, dxdy\,,$$

over the various lattices of fixed area, where ϕ_1 is the normalized ground state.

Abrikosov first wrongly predicted that the square lattice was minimizing before realizing that the hexagonal lattice (in another terminology, the triangular lattice) gives a lower energy. As in (13.5) and (13.43), the quantity appears in the analysis of a bifurcation (see [Od, Du1, Du2, Ay1, Ay2, BGT]).

One of the basic tools is to look for solutions $(\psi, \mathbf{A})$ of the Ginzburg–Landau system that are gauge periodic in the sense that given a lattice Γ in $\mathbb{R}^2$, for any $\gamma \in \Gamma$, there exists ϕ_γ such that

$$\psi(x+\gamma) = \exp i\phi_\gamma(x)\psi\,, \qquad \mathbf{A}(x+\gamma) = \mathbf{A}(x) + \nabla\phi_\gamma(x)\,.$$

Important references about the minimization of I over the lattices Γ with fixed area are [KRA, Mon1, Al8] and in a different context [NoV, AfBN].

The Abrikosov lattices play an important role in the understanding of the vortex structure near $H_{C_2}(\kappa)$ (see [Al6, AfS]).

16.6 Open Problems

16.6.1 Spectral theory

1. For 1D problems, one open problem (see Section 3.3) is to study the infimum over the parameter ρ of the lowest eigenvalue of the generalized Montgomery operator $D_t^2 + (t^{k+1} - \rho)^2$. It can be shown that this infimum is attained in at least one point, but it is expected to be unique and nondegenerate. This problem was recently settled for $k = 1$ (the original Montgomery model) but remains open for $k > 1$ (see [He10] and references therein).
2. For 2D problems, some of the most attractive open problems are the conjectures 4.4.1, 4.4.2 and 4.4.3 concerning the ground state energy for infinite sectors.
3. It would be interesting to have fine results in the case of the ball in $\mathbb{R}^3$. Part of this is contained in [FoP].
4. The corresponding three-dimensional problems, i.e., the analysis of magnetic models in 3D wedges, are widely open (see [Pa1] for first results). A motivation for this analysis could come from the theory of liquid crystals.
5. The precise quantitative analysis of the tunneling effect is open in the two main cases: case of corners and case of regular domains in dimension 2 with isolated points of nondegenerate maximal curvature. In particular, one would like to explain the effect of tunneling on the location of the eigenfunctions and on the multiplicity. This question is presented in [Pa8] (Problem 2.2.6). It could be useful to reduce the question to a nonlinear spectral problem for a pseudodifferential operator on the boundary.
6. Another problem is to extend all the fine theory in dimension 3 proposed for constant magnetic fields to magnetic fields of constant norm. This problem appears naturally in the theory of liquid crystals, where the magnetic field satisfies the additional condition $\operatorname{curl} B = -\tau B$. In the constant magnetic field case, the determination of a third term in the expansion (9.18) (or of a complete expansion) should be done in the spirit

of (8.54) (for the 2D case) under the assumption that the function $\widetilde{\gamma}_0(x)$ introduced in (9.20) admits a unique minimum on Σ. In particular, this will permit us to estimate the splitting between the first eigenvalue and the second one. Some analysis of this question has been carried out in [Pa5, Pa7, Pa9] and [HeP2].

7. Even in the case of constant magnetic field there remains quite some work in three dimensions. For example, one may ask if monotonicity of $B \mapsto \lambda_1(B)$ holds for sufficiently large B (case of general smooth Ω) without any additional conditions. A notable special case where this monotonicity has not yet been established is for Ω equal to a ball; however, that will be contained in the work [FoP].
8. In the case with nonconstant magnetic field, other natural questions occur in dimension 3 around the second term in formula (9.3) in Theorem 9.1.1. Here there are two natural cases depending on the comparison between $\inf_{x\in\Omega} |\beta(x)|$ and $\inf_{x\in\partial\Omega} \varsigma(\vartheta(x))|\beta(x)|$. In each case, the most generic assumption is to assume that the infimum is attained at a unique point and that the corresponding function $|\beta(x)|$ or $\varsigma(\vartheta(x))|\beta(x))$ has nondegenerate minima. This problem is considered in [Ra3].
9. Finally, as mentioned in Problem 2.2.9 in [Pa8], the question of estimating the ground state energy in the case of weakly smooth magnetic fields [for example, with $\beta \in W^{1,2}(\Omega)$ or $\beta \in L^2(\Omega)$] is open.

16.6.2 Nonlinear theory

In the nonlinear part, some of the most attractive open problems are related to the analysis described in Chapter 14.

1. Can one extend the validity of Proposition 14.2.13 to all $\lambda \in [\Theta_0, 1[$? This is important since it would immediately give an answer to the next question.
2. Does the identity (14.5) hold for all $\mathfrak{b} \in]1, \Theta_0^{-1}[$?
3. A related question is to describe the solutions to the nonlinear model problem

$$-\Delta_{\mathbf{F}}\psi = \lambda(1 - |\psi|^2) \quad \text{in } \mathbb{R}^2_+ , \qquad \nu \cdot \nabla\psi = 0 \quad \text{on } \partial\mathbb{R}^2_+ , \qquad (16.14)$$

   where we choose $\mathbf{F} = (-x_2, 0)$.
   The problem is to determine if all the bounded solutions to (16.14) have the form $e^{i\zeta x_2} f(x_1)$, where f is a solution to

$$-f''(t) + (t+\zeta)^2 f = \lambda(1 - |f^2|)f \quad \text{on } \mathbb{R}^+, \qquad f'(0) = 0 . \qquad (16.15)$$

   This is [Pa2, Conjecture 2].
4. One may ask if (14.7) can be justified in the uniform $\|\cdot\|_\infty$ topology and not just in an energy norm. Negative indications are given in [AlH]. However, the question remains unclarified (see also Note 3 to Chapter 14).

5. It would be useful to carry through the analysis similar to Chapter 14 in the three-dimensional case.
6. The next problem is to improve the understanding of the emergence of the Abrikosov lattices. This is on the borderline between the scope of the present text and [SaS3], in which Abrikosov lattices are shown to appear for magnetic field strengths near (and below) H_{C_2}. More precisely, let us suppose that $\sigma = \mathfrak{b}\kappa$ for some $\mathfrak{b} \in]1-\epsilon, 1[$ (ϵ small). Is it true that, as $\kappa \to \infty$, the zeroes of a Ginzburg–Landau minimizer ψ will (approximately) be organized in a regular lattice. One could even take this problem to the next level by asking whether the lattice is the hexagonal one.
7. A very different open problem is to follow the bifurcations from the normal state when σ is decreased below H_{C_3}. The work [Sa] is in this direction but considers a different asymptotic regime.
8. In the case of the disc, the analysis of the bifurcation for values of $\kappa\sigma$ giving double eigenvalues is also open (see [BaPT]).

A

Min-Max Principle

A.1 Main Result

We now give a very flexible criterion for the determination of the bottom of the spectrum and the bottom of the essential spectrum. This flexibility comes from the fact that we do not need explicit knowledge of the various eigenspaces.

Theorem A.1.1.
Let A be a self-adjoint semibounded operator of domain $D(A) \subset H$. Let us introduce

$$\lambda_n(A) = \sup_{\psi_1,\psi_2,\ldots,\psi_{n-1}} \inf_{\substack{\phi \in [\mathrm{span}\,(\psi_1,\ldots,\psi_{n-1})]^\perp; \\ \phi \in D(A) \text{ and } \|\phi\|=1}} \langle A\phi \mid \phi\rangle\,. \tag{A.1}$$

Then either
(a) $\lambda_n(A)$ is the nth eigenvalue when ordering the eigenvalues in increasing order (and counting the multiplicity) and A has discrete spectrum in $]-\infty, \lambda_n(A)]$
or
(b) $\lambda_n(A)$ corresponds to the bottom of the essential spectrum.
In the second case, we have $\lambda_j(A) = \lambda_n(A)$ for all $j \geq n$.

Remark A.1.2.
In the case when the operator has compact resolvent, case (b) does not occur and the supremum in (A.1) *is a maximum. Similarly, the infimum is a minimum. This explains the traditional terminology "min-max principle" for this theorem (though one may argue that "max-min" would be more correct).*

Note that the proof also gives the following proposition.

Proposition A.1.3.
Suppose that there exist $a \in \mathbb{R}$ and an n-dimensional subspace $V \subset D(A)$ such that

$$\langle A\phi \mid \phi\rangle \leq a\|\phi\|^2\,, \qquad \forall \phi \in V\,, \tag{A.2}$$

is satisfied. Then we have the inequality

$$\lambda_n(A) \leq a\,. \tag{A.3}$$

Corollary A.1.4.
Under the same assumption as in Proposition A.1.3, if in addition a is below the bottom of the essential spectrum of A, then A has at least n eigenvalues (counted with multiplicity).

Remark A.1.5.
In continuation of Example 1.3.2, one can show that, for any $\epsilon > 0$ and any N, there exists $h_0 > 0$ such that for $h \in]0, h_0]$, $P_{h,V}$ has at least N eigenvalues in $[\inf V, \inf V + \epsilon]$.

A first natural extension of Theorem A.1.1 is obtained by

Theorem A.1.6.
Let A be a self-adjoint, semibounded operator and $\mathcal{V}(A)$ its form domain.[1] *Then*

$$\lambda_n(A) = \sup_{\psi_1,\psi_2,\dots,\psi_{n-1}} \inf_{\substack{\phi \in [\mathrm{span}\,(\psi_1,\dots,\psi_{n-1})]^\perp; \\ \phi \in \mathcal{V}(A) \text{ and } \|\phi\| = 1}} \langle A\phi \mid \phi\rangle\,. \tag{A.4}$$

A.2 Applications

- It is very often useful to apply the min-max principle by taking the minimum over a dense set in $\mathcal{V}(A)$.
- The min-max principle permits one to control the continuity of the eigenvalues with respect to parameters. For example, the lowest eigenvalue $\lambda_1(\epsilon)$ of $-\frac{d^2}{dx^2}+x^2+\epsilon x^4$ increases with respect to ϵ. One can show that $\epsilon \mapsto \lambda_1(\epsilon)$ is right continuous on $[0, +\infty[$. [The reader may assume (see [He1]) that the corresponding eigenfunction is in $\mathcal{S}(\mathbb{R})$ for $\epsilon \geq 0$.]
- The min-max principle permits one to give an upper bound on the bottom of the spectrum and the comparison between the spectra of two operators. If $A \leq B$ in the sense that $\mathcal{V}(B) \subset \mathcal{V}(A)$ and[2]

  $$\langle Au \mid u\rangle \leq \langle Bu \mid u\rangle\,, \qquad \forall u \in \mathcal{V}(B)\,,$$

  then

  $$\lambda_n(A) \leq \lambda_n(B)\,.$$

  We get similar conclusions if the inclusion holds for the operator domains, i.e., $D(B) \subset D(A)$.

[1] Associated by completion to the form $u \mapsto \langle u \mid Au\rangle$ initially defined on $D(A)$.
[2] It is enough to verify the inequality on a dense set in $\mathcal{V}(B)$.

Example A.2.1 (Comparison between Dirichlet and Neumann).
Let Ω be a bounded, regular, connected, open set in $\mathbb{R}^d$. Then the nth eigenvalue of the Neumann realization of $P_{\mathbf{A},V} = -\Delta_{\mathbf{A}} + V$ is less than or equal to the nth eigenvalue of the Dirichlet realization. The proof is immediate if we observe the inclusion of the form domains.

Example A.2.2 (Monotonicity with respect to the domain).
Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ be two bounded, regular, open sets. Then the nth eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_2 is less than or equal to the nth eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_1. We observe that we can indeed identify $H_0^1(\Omega_1)$ with a subspace of $H_0^1(\Omega_2)$ by extending with 0 in $\Omega_2 \setminus \Omega_1$.

Note that this monotonicity result is wrong for the Neumann problem.

B

Essential Spectrum and Persson's Theorem

In this appendix, we will describe an easy method for determining the bottom of the essential spectrum of a Schrödinger operator by using a criterion that is quite analogous to the variational characterization of the bottom of the spectrum. We recall that the essential spectrum is by definition the complement—within the spectrum—of the discrete spectrum, which corresponds to the isolated eigenvalues of finite multiplicity. This analysis is inspired by Agmon's book [Ag] (see also [HiS, Chapter 14]).

B.1 The Statement

Theorem B.1.1.
Let V be a real-valued, semibounded potential and $\mathbf{A}$ a magnetic potential in $C^1(\mathbb{R}^n)$. Let $\mathcal{H} = -\Delta_{\mathbf{A}} + V$ be the corresponding self-adjoint, semibounded Schrödinger operator. Then the bottom of the essential spectrum is given by

$$\inf \sigma_{\text{ess}}(\mathcal{H}) = \Sigma(\mathcal{H}) \,, \tag{B.1}$$

where

$$\Sigma(\mathcal{H}) := \sup_{K \subset \mathbb{R}^n} \left[\inf_{\|\phi\|=1} \{ \langle \phi \mid \mathcal{H}\phi \rangle \mid \phi \in C_0^\infty(\mathbb{R}^n \setminus K) \} \right] , \tag{B.2}$$

where the supremum is over all compact subsets $K \subset \mathbb{R}^n$.

Essentially, this is a corollary of Weyl's theorem. We will indeed play with the fact that

Lemma B.1.2.

$$\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H} + W) \,,$$

for any regular potential W with compact support.

Let us now give the detailed proof. Extensions exist for the case with boundary (see [Bon1]).

B.2 Preliminary Lemmas

It is useful to find weights ρ such that the form domain of the operator $\mathcal{H}$ has a continuous injection in L^2_ρ. This idea also has applications in the analysis of the compactness of the resolvent. We introduce

Definition B.2.1.
For any $y \in \mathbb{R}^n$ and $R > 0$, we define $\Lambda_R(y, \mathcal{H})$ by

$$\Lambda_R(y, \mathcal{H}) = \inf \left\{ \frac{\langle \mathcal{H}\phi \mid \phi \rangle}{\|\phi\|^2} \mid \phi \in C_0^\infty(\mathrm{D}(y, R)) \right\} . \tag{B.3}$$

In other words, using the characterization of the bottom of the spectrum, $\Lambda_R(y, \mathcal{H})$ is the lowest eigenvalue of the Dirichlet realization of $-\Delta_{\mathbf{A}} + V$ in the ball $\mathrm{D}(y, R)$.

The function $x \mapsto \Lambda_R(x, \mathcal{H})$ will play the role of the weight ρ alluded to above as shown by the following:

Lemma B.2.2.
For all $\epsilon > 0$, there exists R_ϵ such that

$$\langle \mathcal{H}\phi \mid \phi \rangle \geq \int_{\mathbb{R}^n} (\Lambda_R(x, \mathcal{H}) - \epsilon) |\phi(x)|^2 \, dx \, , \tag{B.4}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$ and all $R \geq R_\epsilon$.

Proof of Lemma B.2.2.
Let ζ be a real-valued function in $C_0^\infty(\mathbb{R}^n)$ such that

$$\zeta(x) = 0 \text{ if } |x| \geq \frac{1}{2} \qquad \text{and} \qquad \int_{\mathbb{R}^n} |\zeta(x)|^2 \, dx = 1 \, . \tag{B.5}$$

For $R > 0$ and $y \in \mathbb{R}^n$, let

$$\zeta_{R,y}(x) = \zeta\Big(\frac{x-y}{R}\Big) , \qquad \zeta_R(x) = \zeta_{R,0}(x) \, . \tag{B.6}$$

We immediately get the existence of a constant C such that, for all $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$, and all $R > 0$,

$$|\nabla_x \zeta_{R,y}(x)|^2 \leq \frac{C}{R^2} \, . \tag{B.7}$$

We now use the standard localization identity

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|(\nabla_A(\zeta_{R,y}\phi))(x)|^2 + V(x) |\zeta_{R,y}(x)\phi(x)|^2 \right) dx \\ &\quad - \int_{\mathbb{R}^n} |(\nabla \zeta_{R,y})(x)|^2 |\phi(x)|^2 \, dx = \Re \int_{\mathbb{R}^n} \overline{(\mathcal{H}\,\phi)(x)} \, \zeta_{R,y}(x)^2 \phi(x) \, dx \, . \end{aligned} \tag{B.8}$$

The first term on the left-hand side is estimated from below by

$$\int_{\mathbb{R}^n} \left(|(\nabla_{\mathbf{A}}(\zeta_{R,y}\phi))(x)|^2 + V(x)|\zeta_{R,y}(x)\phi(x)|^2\right) dx$$
$$= \langle \mathcal{H}(\zeta_{R,y}\phi), \zeta_{R,y}\phi\rangle \geq \Lambda_{\frac{R}{2}}(y,\mathcal{H}) \int_{\mathbb{R}^n} |\phi(x)\zeta_{R,y}(x)|^2 dx\,. \tag{B.9}$$

Now we observe that when $|x-y| \leq R/2$, we have $\mathrm{D}(y,R/2) \subset \mathrm{D}(x,R)$ and therefore

$$\Lambda_{\frac{R}{2}}(y,\mathcal{H}) \geq \Lambda_R(x,\mathcal{H})\,. \tag{B.10}$$

This leads to the following lower bound

$$\int_{\mathbb{R}^n} \left(|(\nabla_{\mathbf{A}}(\zeta_{R,y}\phi))(x)|^2 + V(x)|\zeta_{R,y}(x)\phi(x)|^2\right) dx$$
$$\geq \int_{\mathbb{R}^n} \Lambda_R(x,\mathcal{H})|\phi(x)\zeta_{R,y}(x)|^2 dx\,. \tag{B.11}$$

So the left-hand side of (B.8) is estimated from below as

$$\int_{\mathbb{R}^n} \left(|(\nabla_{\mathbf{A}}(\zeta_{R,y}\phi))|^2 + V(x)|\zeta_{R,y}\phi|^2\right) dx - \int_{\mathbb{R}^n} |(\nabla\zeta_{R,y})|^2|\phi|^2\, dx$$
$$\geq \int_{\mathbb{R}^n} \Lambda_R(x,\mathcal{H})|\phi(x)\zeta_{R,y}(x)|^2 dx \; - CR^{-2}\int_{\mathrm{D}(y,R)} |\phi(x)|^2\, dx\,. \tag{B.12}$$

We have consequently obtained

$$\Re\int_{\mathbb{R}^n} \overline{(\mathcal{H}\phi)(x)}\; \zeta_{R,y}(x)^2\phi(x)\, dx$$
$$\geq \int_{\mathbb{R}^n} \Lambda_R(x,\mathcal{H})|\phi(x)\zeta_{R,y}(x)|^2\, dx \; - CR^{-2}\cdot\int_{\mathrm{D}(y,R)} |\phi(x)|^2\, dx\,. \tag{B.13}$$

We now integrate this inequality with respect to y:

$$R^n\cdot\Re\int_{\mathbb{R}^n} \overline{(\mathcal{H}\phi)(x)}\; \phi(x)\, dx$$
$$\geq R^n\int_{\mathbb{R}^n} \Lambda_R(x,\mathcal{H})|\phi(x)|^2 dx \; - \tilde{C}R^{n-2}\int_{\mathbb{R}^n} |\phi(x)|^2\, dx\,. \tag{B.14}$$

Dividing by R^n gives

$$\Re\int_{\mathbb{R}^n} \overline{(\mathcal{H}\phi)(x)}\; \phi(x)\, dx \geq \int_{\mathbb{R}^n} \Lambda_R(x,\mathcal{H})|\phi(x)|^2 dx \; - \tilde{C}R^{-2}\int_{\mathbb{R}^n} |\phi(x)|^2\, dx\,. \tag{B.15}$$

The lemma is obtained by taking $R_\epsilon = (\tilde{C}/\epsilon)^{1/2}$. This finishes the proof of Lemma B.2.2. □

The relationship between the family of $\Lambda_R(x,\mathcal{H})$ and $\Sigma(\mathcal{H})$ is described by the following:

Lemma B.2.3.
With $\Sigma(\mathcal{H})$ *from* (B.2) *and* $\Lambda_R(x,\mathcal{H})$ *from* (B.3), *we have*

$$\Sigma(\mathcal{H}) = \lim_{R\to+\infty} \liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H})\,. \tag{B.16}$$

Proof of Lemma B.2.3.
Step 1.
Let K be a compact subset of $\mathbb{R}^n$ and $R>0$. Then if $|x|$ is large enough, the ball $\mathrm{D}(x,R)$ is contained in $\mathbb{R}^n\setminus K$. Therefore, for such an x, we have

$$\inf\left\{\frac{\langle \mathcal{H}\phi\,|\,\phi\rangle}{\|\phi\|^2}\;|\;\phi\in C_0^\infty(\mathbb{R}^n\setminus K)\,,\ \phi\neq 0\right\}\leq \Lambda_R(x,\mathcal{H})\,,$$

and consequently,

$$\inf\left\{\frac{\langle \mathcal{H}\phi\,|\,\phi\rangle}{\|\phi\|^2}\;|\;\phi\in C_0^\infty(\mathbb{R}^n\setminus K)\,,\ \phi\neq 0\right\}\leq \liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})\,. \tag{B.17}$$

The left-hand side is independent of R; so we can take the limit $R\to+\infty$ in (B.17) and get

$$\inf\left\{\frac{\langle \mathcal{H}\phi\,|\,\phi\rangle}{\|\phi\|^2}\;|\;\phi\in C_0^\infty(\mathbb{R}^n\setminus K)\,,\ \phi\neq 0\right\}\leq \lim_{R\to+\infty}\liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})\,. \tag{B.18}$$

Here we note that the limit on the right-hand side of (B.18) exists because the map $R\mapsto \liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})$ is a monotonically decreasing function.

Now the right-hand side of (B.18) is independent of K. Taking the infimum over K, we get

$$\Sigma(\mathcal{H}) \leq \lim_{R\to+\infty} \liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H})\,. \tag{B.19}$$

This is the first part of the statement in the lemma.
Step 2.
Let us now show the reverse inequality. Coming back to the definition of $\liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})$, we get, for any $\epsilon>0$ and any R, that there exists R_0 such that, for all $\phi\in C_0^\infty(\mathbb{R}^n\setminus\overline{\mathrm{D}(0,R_0)})$, we have

$$\int_{\mathbb{R}^n}\Lambda_R(x,\mathcal{H})|\phi(x)|^2\,dx \geq \left(\liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})-\epsilon\right)\|\phi\|^2\,. \tag{B.20}$$

Therefore, (B.4) and (B.20) imply, for any $R\geq R_\epsilon$, the existence of R_0 such that, for any $\phi\in C_0^\infty(\mathbb{R}^n\setminus\overline{\mathrm{D}(0,R_0)})$ $(\phi\neq 0)$,

$$\frac{\langle \mathcal{H}\phi\,|\,\phi\rangle}{\|\phi\|^2}\geq\left(\liminf_{|x|\to+\infty}\Lambda_R(x,\mathcal{H})-2\epsilon\right)\,. \tag{B.21}$$

Therefore, coming back to the definition of $\Sigma(\mathcal{H})$, we get that, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that, for all $R \geq R_\epsilon$,

$$\Sigma(\mathcal{H}) \geq \left(\liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H}) - 2\epsilon \right). \tag{B.22}$$

So the only restriction on R is that $R \geq R_\epsilon$. Therefore, we can take the limit $R \to +\infty$:

$$\Sigma(\mathcal{H}) \geq \lim_{R\to+\infty} \left(\liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H}) - 2\epsilon \right). \tag{B.23}$$

But we can take the limit $\epsilon \to 0$ in (B.23) and get

$$\Sigma(\mathcal{H}) \geq \lim_{R\to+\infty} \left(\liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H}) \right). \tag{B.24}$$

This is the second part of the statement in the lemma. This finishes the proof of Lemma B.2.3. □

B.3 Proof of the Inequality $\inf \boldsymbol{\sigma}_{\rm ess}(\boldsymbol{\mathcal{H}}) \geq \boldsymbol{\Sigma}(\boldsymbol{\mathcal{H}})$

To prove this inequality, we first use Lemma B.2.2 and get, for any $\epsilon > 0$, the existence of $R > 0$ such that

$$\langle \mathcal{H}\phi \,|\, \phi \rangle \geq \int_{\mathbb{R}^n} \left(\Lambda_R(x,\mathcal{H}) - \frac{\epsilon}{2} \right) |\phi(x)|^2 \, dx \,, \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n) \,. \tag{B.25}$$

Since, by Lemma B.2.3,

$$\liminf_{|x|\to+\infty} \Lambda_R(x,\mathcal{H}) \geq \Sigma(\mathcal{H}) \,,$$

it follows that, for any $\epsilon > 0$, there exists a_ϵ such that

$$\Lambda_R(x,\mathcal{H}) \geq \Sigma(\mathcal{H}) - \frac{\epsilon}{2} \,,$$

for $|x| \geq a_\epsilon$.

On the other hand, we have

$$\Lambda_R(x,\mathcal{H}) \geq \inf \sigma(\mathcal{H}) \,,$$

and there consequently exists a constant C such that

$$\Lambda_R(x,\mathcal{H}) \geq \Sigma(\mathcal{H}) - C \,.$$

We now choose a function W with compact support such that

$$W(x) \geq C \,, \qquad \forall x \in \mathrm{D}(0, a_\epsilon) \,.$$

We consider $\mathcal{H}+W$ and obtain, from (B.25), that, for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\langle (\mathcal{H}+W)\phi \mid \phi\rangle \geq \int \big(W(x) + \Lambda_R(x,\mathcal{H}) - \frac{\epsilon}{2}\big)|\phi(x)|^2 dx$$
$$\geq (\Sigma(\mathcal{H}) - \epsilon)\int |\phi|^2 dx\,. \tag{B.26}$$

This can be interpreted as

$$\inf \sigma(\mathcal{H}+W) \geq \Sigma(\mathcal{H}) - \epsilon\,. \tag{B.27}$$

We now observe that

$$\inf \sigma_{\mathrm{ess}}(\mathcal{H}) = \inf \sigma_{\mathrm{ess}}(\mathcal{H}+W) \tag{B.28}$$

and

$$\inf \sigma_{\mathrm{ess}}(\mathcal{H}+W) \geq \inf \sigma(\mathcal{H}+W)\,. \tag{B.29}$$

This leads to

$$\inf \sigma_{\mathrm{ess}}(\mathcal{H}) \geq \Sigma(\mathcal{H}) - \epsilon\,, \qquad \forall \epsilon > 0\,,$$

and finally to

$$\inf \sigma_{\mathrm{ess}}(\mathcal{H}) \geq \Sigma(\mathcal{H})\,, \tag{B.30}$$

which corresponds to the first statement in Persson's theorem.

B.4 Proof of the Inequality $\inf \sigma_{\mathrm{ess}}(\mathcal{H}) \leq \Sigma(\mathcal{H})$

Let us show the reverse inequality. Let $\mu < \inf \sigma_{\mathrm{ess}}(\mathcal{H})$ and let $E_{]-\infty,\mu]}$ be the spectral projection that has finite rank (we are below the essential spectrum). We first observe that there exists a finite orthonormal system of eigenfunctions such that

$$E_{]-\infty,\mu]} = \sum_i \langle \cdot \mid \phi_i\rangle \phi_i\,. \tag{B.31}$$

From this we get that, for any ϵ, there exists R_ϵ such that

$$\int_{|x|\geq R_\epsilon} |E_{]-\infty,\mu]}\phi|^2 \leq \epsilon \|\phi\|^2\,, \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n)\,. \tag{B.32}$$

We now get

$$\big\langle \mathcal{H}\phi \mid \phi\big\rangle = \langle \mathcal{H}(I-E(\mu))\phi \mid (I-E(\mu))\phi\big\rangle + \big\langle \mathcal{H}E(\mu)\phi \mid E(\mu)\phi\big\rangle$$
$$\geq \mu\big\langle (I-E(\mu))\phi \mid (I-E(\mu))\phi\rangle - C\langle E(\mu)\phi \mid E(\mu)\phi\big\rangle\,.$$

But we can write

$$\Sigma(\mathcal{H}) \geq \inf\left\{\frac{\langle \mathcal{H}\phi \mid \phi\rangle}{\|\phi\|^2} \;\Big|\; \phi \in C_0^\infty(\mathbb{R}^n \setminus (\mathrm{D}(0,R_\epsilon))\right\} \geq \mu\|\phi\|^2 - \epsilon(C+\mu)\|\phi\|^2\,.$$

As $\epsilon \to 0$, we get

$$\Sigma(\mathcal{H}) \geq \mu$$

and, letting μ tend to $\sigma_{\mathrm{ess}}(\mathcal{H})$, we finally obtain

$$\Sigma(\mathcal{H}) \geq \sigma_{\mathrm{ess}}(\mathcal{H}) . \tag{B.33}$$

This finishes the proof of Persson's theorem.

B.5 Agmon Estimates and Essential Spectrum

Theorem B.5.1.
Let $\mathbf{A}$ be a C^∞ vector field and V be a C^∞ semibounded potential and $\mathcal{H} = P_{\mathbf{A},V}$. If u is an eigenfunction associated to an eigenvalue λ of a self-adjoint Schrödinger operator $\mathcal{H}$ on $\mathbb{R}^n$ with

$$\lambda < \inf \sigma_{\mathrm{ess}}(\mathcal{H}) , \tag{B.34}$$

then, for all $\alpha < 1$, there exists C_α such that

$$\int |u(x)|^2 \exp(2\alpha \sqrt{\inf \sigma_{\mathrm{ess}}(\mathcal{H}) - \lambda}|x|) \, dx < +\infty . \tag{B.35}$$

This theorem was first given in [Ag, Theorem 4.1 and Corollary 4.2, Chapter 4] in the case without magnetic field and is the origin of the name "Agmon estimates".

Remark B.5.2.
The theorem also holds for the exterior of an open set. In the case of the Schrödinger operator with constant magnetic field B in $\mathbb{R}^n$, we obtain that, for $\lambda < \mathrm{tr}_+ B$, the eigenfunction u_λ satisfies, for any $\alpha < 1$,

$$\int |u(x)|^2 \exp(2\alpha \sqrt{\mathrm{tr}_+ B - \lambda}|x|) \, dx < +\infty . \tag{B.36}$$

Other results on the decay at ∞ can be found in [Bru] and [HeN2].

Remark B.5.3.
The theorem also holds in the case of a sector S. Here the bottom of the essential spectrum is Θ_0. So for $\lambda < \Theta_0$, the eigenfunction u_λ attached to an eigenvalue $\lambda < \Theta_0$ of $(-i\nabla + \mathbf{F})^2$ in a sector satisfies, for any $\alpha < 1$,

$$\int_S |u(x)|^2 \exp(2\alpha \sqrt{\Theta_0 - \lambda}|x|) \, dx < +\infty . \tag{B.37}$$

B.6 Essential Spectrum for the Schrödinger Operator with Magnetic Field

Theorem B.6.1.
Let $\mathbf{A}$ *be a magnetic potential in* $C^2(\mathbb{R}^n)$ *such that* $\operatorname{curl}\mathbf{A} = \beta$ *, and assume that*

$$\sum_{\gamma,j,k\,,\;1\leq|\gamma|\leq 3} |D_x^\gamma \beta_{jk}(x)| \leq C\rho(x)^{-1} \tag{B.38}$$

for some slowly varying $\rho(x)$ *such that* $\lim_{|x|\to+\infty}\rho(x) = +\infty$ *.*

Let $\mathcal{H} = -\Delta_{\mathbf{A}}$ *be the corresponding self-adjoint, semibounded Schrödinger operator. Then the essential spectrum satisfies*

$$\inf \sigma_{\text{ess}}(\mathcal{H}) \geq \liminf_{|x|\to+\infty} \operatorname{tr}^+ \beta(x)\,. \tag{B.39}$$

The same result is also true in the exterior of a bounded domain. We note that there is a specific difficulty. The lower bound depends only on the behavior of the magnetic field at ∞. Condition (B.38) can be forgotten in the case of dimension 2, if the magnetic field has constant sign.

In the general case, this result is a consequence of [HeM1]. There are essentially two cases. Either $\operatorname{tr}_+\beta(x)$ tends to $+\infty$ and the operator is with compact resolvent or $\operatorname{tr}_+\beta(x)$ does not tend to $+\infty$. In this case, the essential spectrum is obtained by considering the union of the spectra of limiting Schrödinger operators with constant magnetic field obtained by taking all the possible limits

$$\beta_{jk}^\infty = \lim_{n\to+\infty} \beta_{jk}(x_n)\,,$$

for which $\lim_{n\to+\infty}|x_n| = +\infty$. The result is then easy and follows from the analysis of the case with constant magnetic field.

Remark B.6.2.
This result is a consequence of Theorem 1.5 in [HeM1] (see also [Hel5, MoR, Mo]). A semiclassical version is also given in [HeM2, p. 44, formulas (1.14)–(1.16)] .

C

Analytic Perturbation Theory

C.1 Main Goals

In this appendix, we will recall the main definitions and main results concerning type (A) and type (B) self-adjoint holomorphic families of operators. Although some of the results are quite old (see Rellich [Re]), we mainly refer to Chapter 7 in [Kat2].

C.2 Main Results

Definition C.2.1.
A family $T(\xi)$ of unbounded operators on a Hilbert space $\mathcal{H}$ and defined for ξ in a domain D_0 of $\mathbb{C}$ is said to be a holomorphic family of type (A) if

1. *The domain $\mathcal{D}(T(\xi))$ is independent of $\xi \in \mathcal{D}_0$. We denote it by $\mathbf{D}$.*
2. *For every u in $\mathbf{D}$, $\xi \mapsto T(\xi)u$ is holomorphic for $\xi \in D_0$.*

The main theorem used in this book is a theorem due to Rellich (see Theorem 3.9 in [Kat2] or in a less precise form Theorem XII.3 in [ReS, Volume IV]):

Theorem C.2.2.
Let $T(\xi)$ be a self-adjoint family of type (A) defined for ξ in a neighborhood $\mathcal{V}(I_0)$ of an interval I_0 of the real axis. Furthermore, let $T(\xi)$ have compact resolvent. Then all eigenvalues of $T(\xi)$ can be represented by functions that are holomorphic in some neighborhood of I_0 . More precisely, there are a sequence of scalar-valued functions $\mu_n(\xi)$ and a sequence of vector-valued functions $\varphi_n(\xi)$, all holomorphic in an n-dependent complex neighborhood $\mathcal{V}_n(I_0)$ of I_0 , such that for $\xi \in I_0$, the $\mu_n(\xi)$ represent all the repeated eigenvalues of $T(\xi)$ and the $\varphi_n(\xi)$ form a complete orthonormal family of the associated eigenfunctions of $T(\xi)$.

The basic idea in the proof is to reduce the problem to a similar problem for a family of finite-dimensional self-adjoint matrices $\xi \mapsto M(\xi)$. This can be done by using a spectral projector.

There is a weaker notion introduced by Kato corresponding to families of type (B). The starting point is then a family of sesquilinear forms that satisfies the following property.

Definition C.2.3.
We will say that a family $q(\xi)$ of unbounded sesquilinear forms on $\mathcal{H} \times \mathcal{H}$ (where $\mathcal{H}$ is an Hilbert space) that is defined for ξ in a domain D_0 of $\mathbb{C}$ is a holomorphic family of type (a) if

1. *Each $q(\xi)$ is sectorial and the form domain $\mathcal{D}(q(\xi))$ is independent of $\xi \in \mathcal{D}_0$ and dense in $\mathcal{H}$. We denote it by $\mathbf{D_q}$.*
2. *For every $u, v \in \mathbf{D_q}$, $\xi \mapsto q(\xi)(u,v)$ is holomorphic for $\xi \in D_0$.*

This leads to the following definition.

Definition C.2.4.
A family $T(\xi)$ of unbounded operators on a Hilbert space $\mathcal{H}$ and defined for ξ in a domain D_0 of $\mathbb{C}$ is said to be a holomorphic family of type (B) if $T(\xi)$ is the maximal operator associated with the sesquilinear form $q(\xi)$, where $q(\xi)$ is a holomorphic family of type (a).

It is clear that a family of type (A) is of type (B), but this new notion is weaker as will be shown below.

As mentioned in [Kat1], Theorem C.2.2 is also valid for the self-adjoint holomorphic families of type (B).

C.3 Basic Examples

- The family $\xi \mapsto \mathfrak{h}^{N,\xi}$ introduced in (3.9) is a self-adjoint family of type (A) in $D_0 = \mathbb{C}$. We have indeed in this case $\mathcal{H} = L^2(\mathbb{R}^+)$ and a fixed domain
$$D(\mathfrak{h}^{N,\xi}) = \{u \in B^2(\mathbb{R}^+)\,,\, u'(0) = 0\}\,.$$
- The family $\mathbb{R} \ni B \mapsto Q^N_{B\mathbf{A},V,\Omega}$ introduced in (1.11) can be seen, as Ω is bounded as the restriction to the real of a type (a) family whose form domain is $H^1(\Omega)$.
So, in general, the associated family of operators $P^N_{B\mathbf{A},V,\Omega}$ is a type (B) self-adjoint holomorphic family. The magnetic Neumann condition
$$\nu \cdot (-i\nabla + B\mathbf{A})u = 0 \;\text{ on } \partial\Omega$$
is in general B-dependent. So this family, whose domain is given in (1.14), is not a type (A) family.

- But we can get, using Proposition D.1.1, a type (A) family after a gauge transform.
  We get indeed a type (A) family under the condition that

$$\mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\,.$$

D

About the Curl-Div System

We will review in this appendix the main results needed about the curl-div system in our analysis of the Ginzburg–Landau functional. Most of the material was established in the context of problems in mechanics (see Temam [Te], Duvaut–Lions [DuL] and Girault–Raviart [GirR]).

D.1 Discussion About Reduced Spaces and Gauge Invariance

We first show the proposition

Proposition D.1.1.
Given $\widehat{\mathbf{A}} \in H^1(\Omega)$ on a regular, connected, open set Ω, one can always find a gauge transform, i.e., a function $\varphi \in H^2(\Omega)$, such that $\mathbf{A} := \widehat{\mathbf{A}} - \nabla\varphi$ satisfies

$$\operatorname{div} \mathbf{A} = 0 \text{ in } \Omega, \qquad \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{D.1}$$

Proof.
The proof is standard and very simple. Given a general $\widehat{\mathbf{A}}$, we look for $\varphi \in H^2(\Omega)$ such that

$$\operatorname{div}\left(\widehat{\mathbf{A}} - \nabla\varphi\right) = 0 \text{ in } \Omega, \qquad \left(\widehat{\mathbf{A}} - \nabla\varphi\right) \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{D.2}$$

With the definitions $f := \operatorname{div} \widehat{\mathbf{A}} \in L^2(\Omega)$, $g := \widehat{\mathbf{A}} \cdot \nu|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, this reads

$$\Delta\varphi = f \text{ in } \Omega, \qquad \nu \cdot \nabla\varphi = g \text{ on } \partial\Omega\,. \tag{D.3}$$

This is an inhomogeneous Neumann problem, whose solution is unique if we add the condition that

$$\int_\Omega \varphi\, dx = 0\,. \tag{D.4}$$

The proof can be done in two steps. We first reduce to the homogeneous case by choosing $\psi \in H^2(\Omega)$ such that

$$\partial_\nu \psi = \widehat{\mathbf{A}} \cdot \nu \text{ on } \partial\Omega\,. \tag{D.5}$$

Then $\chi = \phi + \psi$, should be a solution of

$$\Delta\chi = \operatorname{div} \widehat{\mathbf{A}} + \Delta\psi\,, \qquad \partial_\nu \chi = 0 \text{ on } \partial\Omega\,.$$

This last equation can be solved if the right-hand side is orthogonal to constants, that is, if

$$\int_\Omega \left(\operatorname{div} \widehat{\mathbf{A}} + \Delta\psi\right)\, dx = 0\,.$$

But this is an immediate consequence of (D.5) and the Green–Riemann formula. We then find the unique solution χ by adding the condition

$$\int_\Omega \chi\, dx = \int_\Omega \psi\, dx\,.$$

□

Remark D.1.2.
We note that in this proof we do not need to assume that Ω is simply connected. The case when Ω is not connected can also be treated by considering each connected component.

D.2 About the Curl-Div System in Two Dimensions

D.2.1 H^1-regularity

The basic Hilbert space is $H^1_{\rm div}(\Omega)$, which is defined by

$$H^1_{\rm div}(\Omega) = \left\{\mathcal{V} = (V_1, V_2) \in H^1(\Omega)^2 \;\middle|\; \operatorname{div} \mathcal{V} = 0 \text{ in } \Omega\,,\, \mathcal{V}\cdot\nu = 0 \text{ on } \partial\Omega\right\}. \tag{D.6}$$

We will need the following standard result (see, for example, [Te, Appendix 1]) on the curl-div system.

Proposition D.2.1.
If Ω is bounded, simply connected, and has regular boundary, then curl *defines an isomorphism from $H^1_{\rm div}(\Omega)$ onto $L^2(\Omega)$.*

In particular, there exists a constant $C > 0$ such that for all $\mathbf{a} \in H^1_{\rm div}(\Omega)$, we have

$$\|\mathbf{a}\|_{H^1(\Omega)} \le C \|\operatorname{curl} \mathbf{a}\|_2\,. \tag{D.7}$$

Actually, we also need the corresponding version for the larger space $H^0_{\mathrm{div}}(\Omega)$, which is defined by

$$H^0_{\mathrm{div}}(\Omega) = \{\mathbf{A} \in L^2(\Omega, \mathbb{R}^2)\,,\ \mathrm{div}\,\mathbf{A} = 0 \ \text{ and } \mathbf{A} \cdot \nu = 0 \ \text{ on } \partial\Omega\}. \tag{D.8}$$

Proposition D.2.2.
If Ω is a bounded, regular, simply connected open set in $\mathbb{R}^2$, then the map $\mathbf{A} \mapsto \mathrm{curl}\,\mathbf{A}$ defines an isomorphism from $H^0_{\mathrm{div}}(\Omega)$ onto $H^{-1}(\Omega)$.

The reader could think that there is a problem in the above definition which involves a trace on the boundary. We can use here the following result (see, for example, [GirR, Theorem 2.6]).

Lemma D.2.3.
The map defined on $C^\infty(\overline{\Omega}, \mathbb{R}^2)$ by $\mathbf{A} \mapsto \mathbf{A} \cdot \nu|_{\partial\Omega}$ admits a unique continuous extension from $\{\mathbf{A} \in L^2(\Omega, \mathbb{R}^2)\,,\ \mathrm{div}\,\mathbf{A} \in L^2(\Omega)\}$ into $H^{-1/2}(\partial\Omega)$.

Remark D.2.4.
When $\mathrm{div}\,\mathbf{A} = 0$, one can also write

$$\mathbf{A} \cdot \nu = 0 \ \textit{on}\ \partial\Omega\,, \tag{D.9}$$

in the form

$$\langle \mathbf{A}\,,\ \nabla\phi\rangle = 0\,, \qquad \forall \phi \in C^\infty(\overline{\Omega})\,. \tag{D.10}$$

Proof of Proposition D.2.2.
For the surjectivity, one can use the property that if $\beta \in H^{-1}(\Omega)$ [resp. in $L^2(\Omega)$], there exists a unique $\psi \in H^1_0(\Omega)$ [resp. in $H^2(\Omega) \cap H^1_0(\Omega)$] solving $\Delta\psi = \beta$. Then $\mathbf{A} = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$ gives a solution in $H^0_{\mathrm{div}}(\Omega)$ [resp. in $H^1_{\mathrm{div}}(\Omega)$]. For the injectivity, we use the property that Ω is simply connected. □

D.2.2 L^p-regularity for the curl-div system

We denote, for $k \in \mathbb{N}$, by $W^{k,p}_{\mathrm{div}}(\Omega)$ the space

$$W^{k,p}_{\mathrm{div}}(\Omega) = \{\mathbf{A} \in W^{k,p}(\Omega)\,,\ \mathrm{div}\,\mathbf{A} = 0 \text{ and } \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\}\,. \tag{D.11}$$

Then we have the following L^p-regularity for the curl-div system.

Proposition D.2.5.
Let $1 \leq p < \infty$. If $\mathbf{A} \in W^{1,p}_{\mathrm{div}}(\Omega)$ satisfies $\mathrm{curl}\,\mathbf{A} \in W^{k,p}(\Omega)$, for some $k \geq 0$, then $\mathbf{A} \in W^{k+1,p}_{\mathrm{div}}(\Omega)$.

Proof.
If $\mathbf{A}$ belongs to $W^{1,p}_{\mathrm{div}}(\Omega)$ and $\mathrm{curl}\,\mathbf{A} \in L^p(\Omega)$, then there exists $\psi \in W^{2,p}(\Omega)$ such that

$$\mathbf{A} = (-\partial_{x_2}\psi, \partial_{x_1}\psi)\,, \qquad -\Delta\psi = \mathrm{curl}\,\mathbf{A}\,, \qquad \text{with } \psi = 0 \text{ on } \partial\Omega\,.$$

This is simply the Dirichlet L^p problem for the Laplacian (see Section D.1). The result we need for proving the proposition is then that if $-\Delta\psi$ is in addition in $W^{k,p}(\Omega)$, then $\psi \in W^{k+2,p}(\Omega)$. This is simply an L^p-regularity result for the Dirichlet problem for the Laplacian, which is described in Section E.4. Coming back to the definition of ψ, we get $\mathbf{A} \in W^{k+1,p}(\Omega)$. □

Corollary D.2.6.
If $\mathbf{A} \in W^{1,p}_{\mathrm{div}}(\Omega)$ *for some* $p \in [1,+\infty[$ *and satisfies* $\operatorname{curl}\mathbf{A} \in C^\infty(\overline{\Omega})$, *then*

$$\mathbf{A} \in C^\infty(\overline{\Omega};\mathbb{R}^2)\,.$$

D.2.3 The curl-div system in the corner case

We discuss here the necessary justifications in the case related to Chapter 15. The domain Ω is a polygon and the Ginzburg–Landau functional is defined by (15.14).

As usual, we assume $\psi \in W^{1,2}(\Omega)$, but for $\mathbf{A}$ the correct choice is that $\mathbf{A} - \mathbf{F} \in W^{1,2}_{0,0}(\mathbb{R}^2)$, where $W^{1,2}_{0,0}(\mathbb{R}^2)$ is the weighted Sobolev space:

$$\begin{aligned} W^{1,2}_{0,0}(\mathbb{R}^2) \in \Big\{ u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \,:\, & \frac{u}{\sqrt{1+x^2}\,\log(2+x^2)} \in L^2(\mathbb{R}^2)\,, \\ & \nabla u \in L^2(\mathbb{R}^2) \Big\}\,. \end{aligned} \tag{D.12}$$

Notice that the constant functions belong to $W^{1,2}_{0,0}(\mathbb{R}^2)$ (but higher-degree polynomials do not). Clearly, $W^{1,2}_{0,0}(\mathbb{R}^2)$ equipped with the natural inner product is a Hilbert space.

This space is sufficiently large to lift all magnetic fields:

Lemma D.2.7.
For all $u \in L^2(\mathbb{R}^2)$, *there exists* $\mathbf{A} \in W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)$ *with*

$$\operatorname{curl}\mathbf{A} = u\,, \qquad \operatorname{div}\mathbf{A} = 0\,.$$

Furthermore, this uniquely determines $\mathbf{A}$ *up to a constant.*

Lemma D.2.8.
We have the following elementary identity for all $\mathbf{A} \in C^\infty_0(\mathbb{R}^2,\mathbb{R}^2)$:

$$\|D\mathbf{A}\|_2^2 = \int_{\mathbb{R}^2} |\operatorname{div}\mathbf{A}|^2 + |\operatorname{curl}\mathbf{A}|^2\,dx\,. \tag{D.13}$$

Furthermore, there exists $C>0$ *such that for all* $\mathbf{A} \in W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)$,

$$\frac{1}{C}\inf_{a\in\mathbb{R}^2} \|\mathbf{A}-a\|_{W^{1,2}_{0,0}(\mathbb{R}^2,\mathbb{R}^2)} = \|D\mathbf{A}\|_2^2 = \int_{\mathbb{R}^2} |\operatorname{div}\mathbf{A}|^2 + |\operatorname{curl}\mathbf{A}|^2\,dx\,. \tag{D.14}$$

Clearly, the inf in (D.14) is actually a minimum.

Notice that—since the weight $[\sqrt{1+x^2}\log(2+x^2)]^{-1}$ is bounded away from zero on any bounded set—for all bounded $\omega \subset \mathbb{R}^2$, there exists $C > 0$ such that

$$\|u\|_{W^{1,2}(\omega)} \leq C\|u\|_{W^{1,2}_{0,0}(\mathbb{R}^2)}\,. \tag{D.15}$$

Combined with (D.14), (D.15) will give that (after gauge transformations) an energy-minimizing sequence will have $\mathbf{A}$ components converging strongly in $L^p(\Omega)$. This is what is needed for the proof of the existence of a minimizer. For details, see references in the notes to Chapter 15.

D.3 About the Curl-Div System in Three Dimensions

When we study the curl-div system on the entire space $\mathbb{R}^3$, things are rather simple. Notice that in 3D we have the following homogeneous Sobolev inequality:

$$\|u\|_{L^6(\mathbb{R}^3)} \leq S_3\|\nabla u\|_{L^2(\mathbb{R}^3)}\,, \tag{D.16}$$

for some constant $S_3 > 0$ and all $u \in C_0^\infty(\mathbb{R}^3)$. Also, as is easily seen by taking the Fourier transform, the norms

$$\|\nabla \mathbf{a}\|_{L^2(\mathbb{R}^3)} \qquad \text{and} \qquad \|\operatorname{curl}\mathbf{a}\|_{L^2(\mathbb{R}^3)} + \|\operatorname{div}\mathbf{a}\|_{L^2(\mathbb{R}^3)}$$

are equivalent. In particular, we get the existence of a constant $C > 0$ such that

$$\|\mathbf{a}\|_{L^6(\mathbb{R}^3)} \leq C\|\operatorname{curl}\mathbf{a}\|_{L^2(\mathbb{R}^3)}\,, \tag{D.17}$$

for all $\mathbf{a}$ with compact support and satisfying $\operatorname{div}\mathbf{a} = 0$.

In the next theorem, we will use the homogeneous Sobolev space $\dot{H}(\mathbb{R}^3)$. This is defined as the closure of of $C_0^\infty(\mathbb{R}^3)$ under the norm $f \mapsto \|\nabla f\|_2$. The norm on $\dot{H}(\mathbb{R}^3)$ is $\|f\|_{\dot{H}(\mathbb{R}^3)} := \|\nabla f\|_2$.

Theorem D.3.1 (Ellipticity of the curl-div system)**.**
There exists a constant $C > 0$ *such that for all (magnetic fields)* $\mathbf{b} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ *with* $\operatorname{div}\mathbf{b} = 0$, *there exists a unique* $\mathbf{a} \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ *such that*

$$\operatorname{curl}\mathbf{a} = \mathbf{b}\,, \qquad \operatorname{div}\mathbf{a} = 0\,.$$

This solution satisfies the estimate

$$\|\mathbf{a}\|_{\dot{H}^1} \leq C\|\mathbf{b}\|_{L^2}\,. \tag{D.18}$$

Proof.
An argument for this standard result is given in [GioP]. It is based on the elementary fact that for $f \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$, one has

$$\|f\|_{\dot{H}^1} = \int_{\mathbb{R}^3} |\operatorname{div} f|^2 + |\operatorname{curl} f|^2 \, dx \,. \tag{D.19}$$

With $\Gamma(x) = 1/(4\pi|x|)$ being the fundamental solution of the Laplacian, the desired solution is (formally) $\mathbf{a} = -\operatorname{curl}(\Gamma * \mathbf{b})$. □

Proposition D.3.2.
Let $2 \leq p \leq 6$ and let $\Omega \subset \mathbb{R}^3$ have bounded measure. Then there exists a constant $C_p > 0$ such that for all $\mathbf{b} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} \mathbf{b} = 0$, the solution $\mathbf{a}$ given in Theorem D.3.1 satisfies the estimate

$$\|\mathbf{a}\|_{L^p(\Omega)} \leq C_p \|\mathbf{b}\|_{L^2(\mathbb{R}^3)} \,. \tag{D.20}$$

Proof.
By (D.18) and the standard three-dimensional Sobolev estimate,

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C_{\mathrm{Sob}} \|f\|_{\dot{H}^1} \,, \quad \forall f \in \dot{H}^1(\mathbb{R}^3) \,, \tag{D.21}$$

the desired estimate holds for $p = 6$. Since Ω has finite measure, Hölder's inequality implies that $\|\mathbf{a}\|_{L^p(\Omega)} \leq C \|\mathbf{a}\|_{L^6(\Omega)}$, for $p \leq 6$. □

E

Regularity Theorems and Precise Estimates in Elliptic PDE

E.1 Introduction

Here we recall some standard regularity theorems for elliptic partial differential equations and refer for proofs to [GiT]. Some aspects of the L^p theory are not in the book of Gilbarg and Trudinger and the extension to systems in full generality can be found in Agmon–Douglis–Nirenberg [AgDN1, AgDN2]. We emphasize that our systems are usually very particular and that a direct approach following the scalar case is always possible. The proofs of these regularity theorems involve estimates that play an important role in the attack of the nonlinear problems. In the problems we meet, the lower-order terms of our nonlinear equations or systems depend on the solution itself. A direct use of the regularity theorems for linear PDE is not possible and one needs to go through bootstrap arguments.

E.2 Bootstrap Arguments for Nonlinear Problems

E.2.1 The case of dimension 2

Typically, the first operator in the Ginzburg–Landau system is the operator

$$\psi \mapsto -\Delta_{\kappa\sigma\mathbf{A}}\psi - \kappa^2\psi(1-|\psi|^2)\,, \tag{E.1}$$

which we would like to consider as the operator $\psi \mapsto L\psi$ with

$$L = -\Delta + \sum_{j=1}^{2} b_j \partial_{x_j} + c\,, \tag{E.2}$$

where

$$b_j = -2i\kappa\sigma\, A_j\,, \qquad c = -2i\kappa\sigma\, \operatorname{div}\mathbf{A} + \kappa^2\sigma^2|\mathbf{A}|^2 - \kappa^2(1-|\psi|^2)\,. \tag{E.3}$$

So the regularity of the coefficients will depend on the regularity of ψ and $\mathbf{A}$ starting with the initial regularity

- $\psi \in L^2(\Omega)$, $(-\nabla + i\mathbf{A})\psi \in L^2(\Omega)$,
- $\mathbf{A} \in H^1(\Omega)$, $\operatorname{div} \mathbf{A} = 0$, and $\mathbf{A} \cdot \nu|_{\partial\Omega} = 0$.

Actually, it could be better to start with

$$-\Delta\psi = 2i\kappa\sigma A_j \partial_{x_j}\psi - \kappa^2\sigma^2|\mathbf{A}|^2\psi + \kappa^2(1-|\psi|^2)\psi\,, \tag{E.4}$$

with Neumann condition (here we use our choice of gauge for $\mathbf{A}$)

$$\partial_\nu \psi\big|_{\partial\Omega} = 0\,. \tag{E.5}$$

Let us look at the right-hand side. In dimension 2, using also the fact that $\psi \in L^\infty(\Omega)$ (see Proposition 10.3.1), we observe, using the initial regularity and the Sobolev embedding theorem, that the right-hand side in (E.4) is in L^p for any $p \in [1, +\infty[$, and that $\psi \in W^{1,2}(\Omega)$. The regularity of the Neumann problem in L^p , which will be recalled in Theorem E.4.7, gives $\psi \in W^{2,p}$, for any $p \in]1, 2]$; hence, by Sobolev's embedding theorem, $\psi \in C^{0,\alpha}$ with $\alpha < 1$ and in $W^{1,p}(\Omega)$ for any $p \in]1, +\infty[$. Using again this last information, we get $\psi \in W^{2,p}$ for any $p \in [1, +\infty[$.

There is a need to improve the regularity on $\mathbf{A}$. We look at the second line of (10.8a):

$$\operatorname{curl}(\operatorname{curl} \mathbf{A} - \beta) = -\frac{1}{\kappa\sigma}\Re\left(\overline{\psi}\, p_{\kappa\sigma\mathbf{A}}\psi\right)\,, \tag{E.6}$$

which implies

$$\operatorname{curl} \mathbf{A} - \beta \in H^1(\Omega)\,. \tag{E.7}$$

Using $\beta \in C^\infty(\overline{\Omega})$, $\operatorname{div} \mathbf{A} = 0$, and the boundary condition, we obtain, by the regularity of the curl-div system [Proposition D.2.5 ($p = 2$)], that $\mathbf{A} \in H^2(\Omega)$ and hence $\mathbf{A} \in C^{0,\alpha}$ for any $\alpha < 1$ and $\mathbf{A} \in W^{1,p}$ for any $p \in [1, +\infty[$.

The recursion is then easy, where we can alternatively play with (E.4) for ψ , (E.6) for $\operatorname{curl} \mathbf{A}$, and the curl-div equation for $\mathbf{A}$ and the associated regularity theorems. Hence, we have proved

Theorem E.2.1.
If $d = 2$, if $\Omega \subset \mathbb{R}^2$ has a C^∞ and bounded boundary, and if $(\psi, \mathbf{A})$ belongs to $H^1(\Omega) \times H^1_{\operatorname{div}}(\Omega)$ and is a solution of (10.8)*, then ψ and $\mathbf{A}$ are in $C^\infty(\overline{\Omega})$.*

E.2.2 The case of three dimensions

The 3D case is more delicate. We now look at the three-dimensional case but consider for simplicity the Ginzburg–Landau functional [defined in (10.1)] with $\widetilde{\Omega} = \Omega$. The main difference with two dimensions is that—by the Sobolev embedding theorem—we have worse regularity. Our solution $(\psi, \mathbf{A})$ satisfies a variant of (10.14) this time. We will limit ourselves to the regularity in (the interior of) Ω .

We first obtain that $\psi \in W^{1,2}(\Omega)$ should satisfy

$$-\Delta\psi = 2i\kappa\sigma \sum_{j=1}^{3} A_j \partial_{x_j}\psi - \kappa^2\sigma^2|\mathbf{A}|^2\psi + \kappa^2(1-|\psi|^2)\psi\,, \tag{E.8}$$

and, using the fact that $\mathbf{A} \in L^6(\Omega)$, we get $-\Delta\psi \in L^{3/2}(\Omega)$, and by elliptic regularity $\psi \in W^{2,3/2}(\Omega)$; hence, by the Sobolev embedding theorem, $\psi \in W^{1,3}(\Omega)$. So there exists $\alpha > 0$ such that $\psi \in C^{0,\alpha}(\overline{\Omega})$.

We take the gauge for which

$$\operatorname{div}\mathbf{A} = 0\,, \qquad \mathbf{A}\cdot\nu\big|_{\partial\Omega} = 0\,.$$

The vector potential $\mathbf{A}$ is in $H^1(\Omega)$ and satisfies in Ω

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}) = -\frac{1}{\kappa\sigma}\Re(\overline{\psi}\, p_{\kappa\sigma\mathbf{A}}\psi)\,, \qquad (\operatorname{curl}\mathbf{A}-\beta)\times\nu\big|_{\partial\Omega} = 0\,.$$

Using that $\mathbf{A}$ is divergence-free, we obtain the system

$$\begin{gathered} -\Delta(\mathbf{A}-\mathbf{F}) = -\frac{1}{\kappa\sigma}\Re(\overline{\psi}\, p_{\kappa\sigma\mathbf{A}}\psi)\,, \\ \operatorname{curl}\mathbf{A}\times\nu\big|_{\partial\Omega} = 0\,, \qquad \mathbf{A}\cdot\nu\big|_{\partial\Omega} = 0\,. \end{gathered} \tag{E.9}$$

This is an elliptic system in the sense of Agmon–Douglis–Nirenberg. In order to verify this property at the boundary, we should freeze the problem at one point of the boundary and see if the obtained system in the half-space is well-posed.

Choosing a point x_0 on the boundary and using the invariance by rotation of the system, we can assume that $\nu(x_0) = (0,0,1)$ and we obtain the following problem in $\mathbb{R}^3_+$:

$$\begin{gathered} -\Delta\widetilde{\mathbf{A}} = \mathbf{G}\,, \\ \widetilde{\mathbf{A}}_3 = 0 \text{ for } x_3 = 0\,, \quad \partial_{x_3}\widetilde{\mathbf{A}}_1 = \partial_{x_3}\widetilde{\mathbf{A}}_2 = 0 \text{ for } x_3 = 0\,. \end{gathered} \tag{E.10}$$

So the frozen half-space problem is completely decoupled in three independent problems for each component of $\widetilde{\mathbf{A}}$, the problem being the standard Dirichlet or Neumann problem.

This system has consequently the same regularity properties as the Laplacian. In particular, we first get that $(\mathbf{A}-\mathbf{F}) \in H^2(\Omega)$, so $\mathbf{A} \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha > 0$.

Coming back to (E.8), we can now show that $\psi \in H^2(\Omega)$ and $\psi \in C^{1,\alpha}(\Omega)$ for any $\alpha \in]0,1[$ and the recursion is then easy.

Remark E.2.2.
For the functional considered in Section 10.1.3, (10.14b) *permits us to get* $\mathbf{A} \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ *and by recursion we get* $\psi \in C^\infty(\Omega)$, $\mathbf{A} \in C^\infty(\Omega)$, *and* $\psi \in C^\infty(\mathbb{R}^3 \setminus \Omega)$, $\mathbf{A} \in C^\infty(\mathbb{R}^3 \setminus \Omega)$.

E.3 Schauder Hölder Estimates

E.3.1 Interior estimates

Theorem E.3.1.
Let Ω be an open set in $\mathbb{R}^n$ and L be the differential operator

$$L := -\sum_{i,j} a_{ij}\partial_{x_i}\partial_{x_j} + \sum_i b_i \partial_{x_i} + c\,, \tag{E.11}$$

where a_{ij} , b_i , and c are in $C^{0,\alpha}(\overline{\Omega})$ and

$$\Big|\sum_{ij} a_{ij}(x)\xi_i\xi_j\Big| \geq \Lambda|\xi|^2\,, \qquad \forall x\in\Omega, \xi\in\mathbb{R}^n\,, \tag{E.12}$$

for some $\Lambda > 0$. Then, for any $\Omega' \subset\subset \Omega$, there exists a constant C depending only[1] *on Λ , $d(\Omega', \Omega^c)$, and the norms in $C^{0,\alpha}(\overline{\Omega})$ of the coefficients, such that for any $u \in C^{2,\alpha}(\overline{\Omega})$, we have*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega'})} \leq C\left(\|u\|_{C^{0,\alpha}(\overline{\Omega})} + \|Lu\|_{C^{0,\alpha}(\overline{\Omega})}\right). \tag{E.13}$$

We also need the following variant (when beginning a bootstrap argument). Notice the slight change in the definition of the operator:

Theorem E.3.2.
Let Ω be an open set in $\mathbb{R}^n$ and L be the differential operator

$$\mathcal{L} := -\sum_{i,j} \partial_{x_i} a_{ij}\partial_{x_j} + \sum_i b_i \partial_{x_i} + c, \tag{E.14}$$

where a_{ij} is in $C^{0,\alpha}(\overline{\Omega})$, b_i and c are in $L^\infty(\Omega)$ and satisfying (E.12). *Then, for any $\Omega' \subset\subset \Omega$, there exists a constant C depending only on Λ , $d(\Omega', \Omega^c)$, and the corresponding norms of the coefficients in $\overline{\Omega}$, such that for any $u \in C^{2,\alpha}(\overline{\Omega})$, we have*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega'})} \leq C\left(\|u\|_{C^{0}(\overline{\Omega})} + \|\mathcal{L}u\|_{C^{0}(\overline{\Omega})}\right). \tag{E.15}$$

Remark E.3.3.
Note that it is quite important in the applications to have $\|u\|_{C^0(\overline{\Omega})}$ and not only $\|u\|_{C^{0,\alpha}(\overline{\Omega})}$ on the right-hand side of (E.15).

The last statement is a regularity statement.

[1] Of course, it also depends on n , the diameter of Ω , and α , but they are supposed to be fixed.

Theorem E.3.4.
Let Ω be an open set in $\mathbb{R}^n$ and L be the differential operator (E.11)*, where a_{ij}, b_i, c are in $C^{0,\alpha}(\Omega)$, and satisfying* (E.12)*. Then if $u \in C^2(\Omega)$ satisfies $Lu = f$ and if $f \in C^{0,\alpha}(\Omega)$, then $u \in C^{2,\alpha}(\Omega)$. Moreover, if above a_{ij}, b_i and c are in $C^{k,\alpha}(\Omega)$ for some $k \geq 0$, then if $f \in C^{k,\alpha}(\Omega)$, $u \in C^{k+2,\alpha}(\Omega)$.*

Corollary E.3.5.
Let u be a $C^2(\Omega)$ solution of the equation $Lu = f$ in an open set Ω, where f and the coefficients of the elliptic operator L are in $C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$.

E.3.2 Boundary estimates

We now present the analogous results corresponding to the case with boundary. We meet in our problems the Dirichlet case and the Neumann case.

Dirichlet problem

Theorem E.3.6 (A priori estimates)**.**
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary and let L be the differential operator (E.11)*, where a_{ij}, b_i, and c are in $C^{0,\alpha}(\overline{\Omega})$ and satisfying* (E.12)*. Then there exists a constant C depending only on Λ, and on the corresponding norms of the coefficients in $\overline{\Omega}$, such that, for any $u \in C^{2,\alpha}(\overline{\Omega})$, we have*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\left(\|u\|_{C^{0,\alpha}(\overline{\Omega})} + \|\gamma_o u\|_{C^{1,\alpha}(\partial\Omega)} + \|Lu\|_{C^{0,\alpha}(\overline{\Omega})}\right), \tag{E.16}$$

where $u \mapsto \gamma_0 u$ is the trace operator on $\partial\Omega$, and

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|u\|_{C^{0,\alpha}(\overline{\Omega})} + \|\gamma_o u\|_{C^{2,\alpha}(\partial\Omega)} + \|Lu\|_{C^{0,\alpha}(\overline{\Omega})}\right). \tag{E.17}$$

Theorem E.3.7 (Hölder regularity)**.**
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, and L be the differential operator (E.11)*, where a_{ij}, b_i, c are in $C^{0,\alpha}(\overline{\Omega})$, and satisfying* (E.12)*. Then if $u \in C^2(\overline{\Omega})$ satisfies $Lu = f$ in Ω and if $f \in C^{0,\alpha}(\overline{\Omega})$ and $\gamma_0 u \in C^{2,\alpha}(\partial\Omega)$, then $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover, if above $\partial\Omega$ has regularity $C^{k+2,\alpha}$ and a_{ij}, b_i and c are in $C^{k,\alpha}(\overline{\Omega})$, then if $f \in C^{k,\alpha}(\overline{\Omega})$ and $\gamma_0 u \in C^{k+2,\alpha}(\partial\Omega)$, $u \in C^{k+2,\alpha}(\overline{\Omega})$.*

Corollary E.3.8 (C^∞-regularity)**.**
Let u be a $C^2(\overline{\Omega})$ solution of the equation $Lu = f$ in an open set Ω with smooth (C^∞) boundary, where f and the coefficients of the elliptic operator L are in $C^\infty(\overline{\Omega})$. Then if $\gamma_0 u \in C^\infty(\partial\Omega)$, $u \in C^\infty(\overline{\Omega})$.

Remark E.3.9.
There is also a similar result in the analytic category [LiM].

Neumann problem

Theorem E.3.10 (A priori estimate)**.**
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary and let L be the differential operator (E.11)*, where a_{ij} , b_i , and c are in $C^{0,\alpha}(\overline{\Omega})$ and satisfying* (E.12)*. Then there exists a constant C depending only on Λ and the corresponding norms of the coefficients in $\overline{\Omega}$ such that for any $u \in C^{2,\alpha}(\overline{\Omega})$, we have*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \left(\|u\|_{C^0(\overline{\Omega})} + \|\gamma_1 u\|_{C^{1,\alpha}(\partial\Omega)} + \|Lu\|_{C^{0,\alpha}(\overline{\Omega})} \right) , \qquad \text{(E.18)}$$

where $u \mapsto \gamma_1 u$ is the trace operator of the normal derivative of u on $\partial\Omega$.

Theorem E.3.11 (Hölder regularity)**.**
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary and L be the differential operator (E.11)*, where a_{ij}, b_i, c are in $C^{0,\alpha}(\overline{\Omega})$, and satisfying* (E.12)*. Then if $u \in C^2(\overline{\Omega})$ satisfies $Lu = f$ in Ω and if $f \in C^{0,\alpha}(\overline{\Omega})$ and $\gamma_1 u \in C^{1,\alpha}(\partial\Omega)$, then $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover, if in addition above Ω has $C^{k+2,\alpha}$ boundary, if a_{ij}, b_i, c are in $C^{k,\alpha}(\overline{\Omega})$, then if $f \in C^{k+1,\alpha}(\overline{\Omega})$ and $\gamma_1 u \in C^{k+1,\alpha}(\partial\Omega)$, $u \in C^{k+2,\alpha}(\overline{\Omega})$.*

Corollary E.3.12 (C^∞-regularity)**.**
Let u be a $C^2(\overline{\Omega})$ solution of the equation $Lu = f$ in an open set Ω with smooth boundary, where f and the coefficients of the elliptic operator L are in $C^\infty(\overline{\Omega})$. Then if $\gamma_1 u \in C^\infty(\partial\Omega)$, $u \in C^\infty(\overline{\Omega})$.

This will mainly be applied when u satisfies the Neumann condition $\gamma_1 u = 0$. The theorem with the magnetic Neumann condition is also true under the condition that the magnetic potential satisfies the regularity condition $\mathbf{A} \cdot \nu \in C^{1,\alpha}$ on the boundary. In any case, when establishing these theorems, one can usually work in a gauge where $\mathbf{A} \cdot \nu = 0$ on $\partial\Omega$.

The proof of all these theorems can either be direct or involve the reflection method.

E.4 Schauder L^p-Estimates

We refer to [GeG] for a good presentation of the L^p theory for boundary elliptic problems. However, the presentation is for operators with smooth coefficients. Some of the results below can also be found in [GiT], but—in particular for the boundary estimates—we refer to the original paper [AgDN2].

E.4.1 Interior estimates

Theorem E.4.1 (A priori estimates)**.**
Let Ω be an open set in $\mathbb{R}^n$ and L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\overline{\Omega})$, b_i and c are in $L^\infty(\Omega)$ and satisfying* (E.12)*. Then for any*

$\Omega' \subset\subset \Omega$, there exists a constant C depending only on Λ, $d(\Omega', \Omega^c)$, and the corresponding norms of the coefficients in $\overline{\Omega}$ such that for any $u \in W^{2,p}(\overline{\Omega})$, we have

$$\|u\|_{W^{2,p}(\Omega')} \leq C \left(\|u\|_{L^p(\Omega)} + \|Lu\|_{L^p(\Omega)}\right) . \tag{E.19}$$

Theorem E.4.2 (Local regularity).
Let Ω be an open set in $\mathbb{R}^n$ and L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\Omega)$, b_i and c are in $L^\infty_{\mathrm{loc}}(\Omega)$ and satisfying* (E.12)*. Then if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies $Lu \in L^p_{\mathrm{loc}}(\Omega)$ in the weak sense, then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. Moreover, if Ω has $C^{k+2,\alpha}$ boundary, a_{ij} is in $C^{k,1}(\Omega)$, b_i and c are in $C^{k,\alpha}(\Omega)$, then $Lu \in W^{k,p}_{\mathrm{loc}}(\Omega)$ implies $u \in W^{k+2,p}_{\mathrm{loc}}(\Omega)$.*

Corollary E.4.3.
If Ω has C^∞ boundary, and the coefficients a_{ij}, b_i, and c are in $C^\infty(\Omega)$, then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $Lu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

E.4.2 Boundary estimates

Dirichlet problem

Theorem E.4.4 (A priori estimate).
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary (for some $\alpha > 0$) and let L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\overline{\Omega})$ and b_i and c are in $C^0(\overline{\Omega})$ and satisfying* (E.12)*.*

Then there exists a constant C depending only on Λ and the corresponding norms of the coefficients in $\overline{\Omega}$ such that for any $u \in W^{2,p}(\Omega)$, we have

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|u\|_{L^p(\Omega)} + \|\gamma_0 u\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|Lu\|_{L^p(\Omega)}\right) . \tag{E.20}$$

Theorem E.4.5 (Regularity).
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary (for some $\alpha > 0$) and let L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\overline{\Omega})$ and b_i and c are in $C^0(\overline{\Omega})$ and satisfying* (E.12)*. Then if $u \in W^{1,p}(\Omega)$ satisfies $Lu \in L^p(\Omega)$ and $\gamma_0 u \in W^{2-\frac{1}{p},p}$, then $u \in W^{2,p}(\Omega)$.*

More generally, if, for some $k \geq 0$, Ω has $C^{k+2,\alpha}$ boundary, a_{ij} is in $C^{k,1}(\overline{\Omega})$, b_i and c are in $C^k(\overline{\Omega})$, $Lu \in W^{k,p}(\Omega)$, $u \in W^{1,p}(\Omega)$ and $\gamma_0 u \in W^{k+2-\frac{1}{p},p}(\partial\Omega)$, then $u \in W^{k+2,p}(\Omega)$.

Neumann problem

Theorem E.4.6 (A priori estimates).
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary (for some $\alpha > 0$) and let L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\overline{\Omega})$ and b_i and c are in $C^0(\overline{\Omega})$ and satisfying* (E.12)*.*

Then there exists a constant C depending only on Λ and the corresponding norms of the coefficients in $\overline{\Omega}$ such that for any $u \in W^{2,p}(\Omega)$, we have

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left(\|u\|_{L^p(\Omega)} + \|\gamma_1 u\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \|Lu\|_{L^p(\Omega)} \right) . \qquad \text{(E.21)}$$

Theorem E.4.7 (Regularity)**.**
Let Ω be an open set in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary (for some $\alpha > 0$) and let L be the differential operator (E.11)*, where a_{ij} is in $C^{0,1}(\overline{\Omega})$ and b_i and c are in $C^0(\overline{\Omega})$ and satisfying* (E.12)*. Then, if $u \in W^{1,p}(\Omega)$ satisfies $Lu \in L^p(\Omega)$ and $\gamma_1 u = 0$, then $u \in W^{2,p}(\Omega)$.*

If in addition, for some $k \geq 0$, Ω has $C^{k+2,\alpha}$ boundary, a_{ij} is in $C^{k,1}(\overline{\Omega})$, b_i and c are in $C^k(\overline{\Omega})$, $Lu \in W^{k,p}(\Omega)$, and $\gamma_1 u = 0$, then $u \in W^{k+2,p}(\Omega)$.

E.5 Poincaré Inequality

We recall the following Poincaré-type inequality [Bre, Corollaire IX.19]. We stress that no regularity of $\partial\Omega$ is needed.

Theorem E.5.1.
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then, for all $p \in [1, \infty[$, there exists a constant $C = C(p, \Omega)$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$.

F

Boundary Coordinates

F.1 The Two-Dimensional Case

Let Ω be a smooth, simply connected[1] domain in $\mathbb{R}^2$. Let

$$\gamma : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \to \partial\Omega$$

be a parametrization of the boundary with $|\gamma'(s)| = 1$ for all s. Let $\nu(s)$ be the unit vector, normal to the boundary, pointing inward at the point $\gamma(s)$. We choose the orientation of the parametrization γ to be counterclockwise, so

$$\det\big(\gamma'(s), \nu(s)\big) = 1\,.$$

The curvature $k(s)$ of $\partial\Omega$ at the point $\gamma(s)$ is now given in this parametrization by

$$\gamma''(s) = k(s)\nu(s)\,.$$

The map Φ defined by

$$\begin{aligned} &\Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[\to \Omega\,,\\ &(s,t) \mapsto \gamma(s) + t\nu(s) \end{aligned} \tag{F.1}$$

is clearly a diffeomorphism, when t_0 is sufficiently small, with image

$$\Phi\big(\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[\big) = \{x \in \Omega \big| \operatorname{dist}(x,\partial\Omega) < t_0\} =: \Omega_{t_0}\,.$$

Furthermore, with the function $t(x)$ defined in (4), $t(\Phi(s,t)) = t$.

The inverse Φ^{-1} defines a system of coordinates for a tubular neighborhood of $\partial\Omega$ in $\overline{\Omega}$ that we can use locally or semiglobally.

[1] In the non-simply connected case, the construction below will give coordinates in a neighborhood of any connected component of the boundary.

If $\mathbf{A}$ is a vector field on Ω_{t_0} with $\beta = \operatorname{curl}\mathbf{A}$, we define the associated fields in (s,t)-coordinates by

$$\tilde{A}_1(s,t) = (1-tk(s))\mathbf{A}(\Phi(s,t))\cdot\gamma'(s)\,, \quad \tilde{A}_2(s,t) = \mathbf{A}(\Phi(s,t))\cdot\nu(s)\,, \tag{F.2}$$

$$\tilde{\beta}(s,t) = \beta(\Phi(s,t))\,. \tag{F.3}$$

Then

$$\partial_s\tilde{A}_2 - \partial_t\tilde{A}_1 = (1-tk(s))\tilde{\beta}\,. \tag{F.4}$$

Furthermore, for all $u \in H^1(\Omega_{t_0})$, we have, with $v = u\circ\Phi$,

$$\begin{aligned}\int_{\Omega_{t_0}} |(-i\nabla+\mathbf{A})u|^2\,dx = \int\Big\{&(1-tk(s))^{-2}\Big|(-i\partial_s+\tilde{A}_1)v\Big|^2\\ &+\Big|(-i\partial_t+\tilde{A}_2)v\Big|^2\Big\}(1-tk(s))\,ds\,dt\,,\end{aligned}$$

$$\int_{\Omega_{t_0}} |u(x)|^2\,dx = \int |v(s,t)|^2(1-tk(s))\,ds\,dt\,. \tag{F.5}$$

The next lemma is quite useful for the fine analysis in a tubular neighborhood of the boundary and gives a kind of normal form.

Lemma F.1.1.
Suppose Ω is a bounded, simply connected domain with smooth boundary and let t_0 be the constant from (F.1). *Let θ be a given function on Ω_{t_0} such that the corresponding $\tilde{\theta}$ is t-independent. Then there exists a constant $C>0$ such that, if $\mathbf{A}$ is a magnetic vector potential in Ω with*

$$\operatorname{curl}\mathbf{A} = \theta \quad \text{on } \partial\Omega\,, \tag{F.6}$$

and with $\tilde{\mathbf{A}}$ defined as in (F.2), *then there exists a gauge function $\varphi(s,t)$ on $\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[$ such that $\bar{\mathbf{A}} = \tilde{\mathbf{A}} - \nabla_{(s,t)}\varphi$ satisfies*

$$\bar{\mathbf{A}}(s,t) = \begin{pmatrix}\bar{A}_1(s,t)\\ \bar{A}_2(s,t)\end{pmatrix} = \begin{pmatrix}\gamma_0 - \tilde{\theta}(s,0)t + \frac{t^2k(s)}{2} + t^2b(s,t)\\ 0\end{pmatrix}\,, \tag{F.7}$$

where

$$\gamma_0 = \frac{1}{|\partial\Omega|}\int_\Omega \operatorname{curl}\mathbf{A}\,dx\,, \tag{F.8}$$

and b satisfies the estimate

$$\|b\|_{L^\infty(\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,\frac{t_0}{2}[)} \le C\|\nabla\operatorname{curl}\mathbf{A}-\theta\|_{C^0(\overline{\Omega_{t_0}})}\,. \tag{F.9}$$

Furthermore, if $[s_0,s_1]$ is a subset of $\mathbb{R}/(|\partial\Omega|\mathbb{Z})$ with $s_1-s_0<|\partial\Omega|$, then we may choose φ on $]s_0,s_1[\times]0,t_0[$ such that

$$\bar{\mathbf{A}}(s,t) = \begin{pmatrix}\bar{A}_1(s,t)\\ \bar{A}_2(s,t)\end{pmatrix} = \begin{pmatrix}-\tilde{\theta}(s,0)t + \frac{t^2k(s)}{2} + t^2b(s,t)\\ 0\end{pmatrix}\,, \tag{F.10}$$

with b still satisfying the estimate (F.9).

Proof.
Notice first that

$$\int_0^{|\partial\Omega|} \bar{A}_1(s,0)\,ds = \int_0^{|\partial\Omega|} \bar{\mathbf{A}}\cdot\gamma'(s)\,ds = \int_\Omega \operatorname{curl}\mathbf{A}\,dx\,.$$

This determines γ_0 which is a global quantity associated with $\operatorname{curl}\mathbf{A}$.

Let us write

$$f = \operatorname{curl}\mathbf{A} - \theta\,, \qquad \tilde{f}(s,t) = f(\Phi(s,t))\,, \qquad \tilde{f}' = \frac{\tilde{f}}{t}\,.$$

Then $\|\tilde{f}'\|_{L^\infty} \leq C\|\nabla f\|_{C^0(\overline{\Omega_{t_0}})}$ and, using (F.4), we obtain

$$\partial_s\tilde{A}_2 - \partial_t\tilde{A}_1 = (1-tk(s))(\tilde{\theta}(s,t) + t\tilde{f}') = (1-tk(s))(\tilde{\theta}(s,0) + t\tilde{f}')\,.$$

Define

$$\varphi(s,t) = \int_0^t \tilde{A}_2(s,t')\,dt' + \Big(\int_0^s \tilde{A}_1(s',0)\,ds' - s\gamma_0\Big)\,. \tag{F.11}$$

Then φ is a welldefined continuous function on $\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[$. We pose $\bar{A} = \tilde{A} - \nabla\varphi$ and find

$$\bar{A}(s,t) = \begin{pmatrix}\bar{A}_1(s,t)\\ \bar{A}_2(s,t)\end{pmatrix} = \begin{pmatrix}\bar{A}_1(s,t)\\ 0\end{pmatrix}\,,$$

with

$$\begin{aligned}\partial_t\bar{A}_1(s,t) &= -(\partial_s\tilde{A}_2 - \partial_t\tilde{A}_1) = -(1-tk(s))(\tilde{\theta}(s,0) + t\tilde{f}')\,,\\ \bar{A}_1(s,0) &= \gamma_0\,.\end{aligned}$$

Therefore,

$$\bar{A}_1(s,t) = \gamma_0 - \tilde{\theta}(s,0)t + \frac{t^2k(s)}{2} - \int_0^t t'(1-t'k(s))\tilde{f}'(s,t')\,dt'\,,$$

and we get (F.7) by applying l'Hôpital's rule to the integral.

When we only consider a (simply connected) part $]s_0,s_1[\times]0,t_0[$ of the ring $\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times]0,t_0[$, we can omit the term $s\gamma_0$ in (F.11) since we do not need φ to be periodic. □

F.2 Adapted Coordinates in the Three-Dimensional Case

F.2.1 Tubular coordinates

Let $\partial\Omega \ni x \mapsto \phi(x) = (y_1,y_2)$ be local coordinates on the boundary and G the metric induced by the 3D Euclidean metric g_0 on $\partial\Omega$ in these coordinates.

Then for $t_0 > 0$ small enough (and considering an open set ω on which ϕ^{-1} is welldefined), we can consider the map

$$\omega \times]0, t_0[\ni (y_1, y_2, y_3) \mapsto \Phi(y) = \phi^{-1}(y_1, y_2) + \nu(\phi^{-1}(y_1, y_2))y_3 \,, \qquad \text{(F.12)}$$

where $\nu(x)$ is the interior normal unit vector at the point $x \in \partial\Omega$. This defines a diffeomorphism of $\omega \times]0, t_0[$ onto $\mathcal{V}$ in Ω_{t_0} and its inverse defines local coordinates on $\mathcal{V}$, $\mathcal{V} \ni x \mapsto y(x)$ such that

$$y_3(x) = \text{dist}(x, \partial\Omega) \,. \qquad \text{(F.13)}$$

Then we get by direct computation the form of the standard flat metric g_0 in these new coordinates:

$$\begin{aligned} g_0 &= \sum_{1 \le i,j \le 3} g_{ij} \, dy_i \otimes dy_j \\ &= dy_3 \otimes dy_3 + G + 2y_3 \sum_{1 \le i,j \le 2} \left\langle \frac{\partial \nu}{\partial y_i} \Big| \frac{\partial x}{\partial y_j} \right\rangle dy_i \otimes dy_j \\ &\quad + y_3^2 \sum_{1 \le i,j \le 2} \left\langle \frac{\partial \nu}{\partial y_i} \Big| \frac{\partial \nu}{\partial y_j} \right\rangle dy_i \otimes dy_j \,. \end{aligned} \qquad \text{(F.14)}$$

Remark F.2.1.
We frequently denote the map $x \mapsto y_3(x) = \text{dist}(x, \partial\Omega)$ by $x \mapsto t(x)$. Let us also observe that there is some freedom in the choice of the boundary coordinates. We will explain in the next subsection how to construct coordinates adapted to a given curve in the boundary.

F.2.2 Local coordinates near a curve inside the boundary

Let Σ be a curve in $\partial\Omega$ parametrized by arc length on some interval I ($I = [-a, +a]$): $\Sigma = \{\gamma(s);\ s \in I\}$. So we have $|\gamma'(s)| = 1$. Then, there exists a neighborhood $\mathcal{W}_{x_0}$ of $x_0 = \gamma(0)$ in $\partial\Omega$, such that, for any $z \in \mathcal{W}_{x_0} \cap \Sigma$, there exists a unique geodesic Λ_z through z and normal to Σ. The neighborhood $\mathcal{W}_{x_0}$ of x_0 can also be chosen such that

$$\forall x \in \mathcal{W}_{x_0} \,,\ \exists!\ z = z(x) \in \Sigma \cap \mathcal{W}_{x_0} \ \text{ s.t. } d_{\partial\Omega}(x, z) = d_{\partial\Omega}(x, \Sigma) \,, \qquad \text{(F.15)}$$

where $d_{\partial\Omega}(\,.\,,\,.\,)$ denotes the distance on $\partial\Omega$.

Then, there exist an open set S of $\mathbb{R}^2$ and a regular diffeomorphism

$$\phi : \mathcal{W}_{x_0} \to S, \ \phi(x) = (r, s) \text{ with } \pm r = d_{\partial\Omega}(x, \Sigma) = d_{\partial\Omega}(x, \gamma(s)) \,. \qquad \text{(F.16)}$$

We observe that

$$x(0, s) = \gamma(s) \,.$$

We choose a positive orientation (and this determines the choice of the sign of r) by imposing

$$\frac{\partial x}{\partial r}(0,s) \wedge \frac{\partial x}{\partial s}(0,s) = \nu(\gamma(s))\,, \tag{F.17}$$

where $\nu(x)$ is the interior normal of $\partial\Omega$ at the point $x \in \partial\Omega$. Then (r,s) are local coordinates in $\mathcal{W}_{x_0}$ and observing that, for any fixed s, $r \mapsto x(r,s)$ is a parametrization by arc lengths of the geodesic $\Lambda_{\gamma(s)}$, we have

$$\Big|\frac{\partial x}{\partial r}(r,s)\Big| = 1\,, \tag{F.18}$$

and

$$\Big\langle \frac{\partial x}{\partial r}(0,s)\,\Big|\,\frac{\partial x}{\partial s}(0,s)\Big\rangle = 0\,. \tag{F.19}$$

More precisely we have the following Lemma.

Lemma F.2.2.
In the above local coordinates, the metric G on $\partial\Omega$ is diagonal:

$$G = dr \otimes dr + \alpha(r,s)\, ds \otimes ds\,. \tag{F.20}$$

On the curve Σ, we have

$$\alpha(0,s) = 1\,, \quad \frac{\partial \alpha}{\partial r}(0,s) = -2\kappa_g(s)\,, \quad \textit{and} \quad \frac{\partial \alpha}{\partial s}(0,s) = 0\,, \tag{F.21}$$

where $\kappa_g(s)$ denotes the geodesic curvature of the curve Σ at $\gamma(s)$.

Remark F.2.3.
In the coordinates (r,s) the second fundamental form

$$K = K_{11}\, dr \otimes dr \;+\; K_{12} dr \otimes ds \;+\; K_{21}\, ds \otimes dr \;+\; K_{22}\, ds \otimes ds\,,$$

satisfies

$$\begin{aligned}
K_{11}(r,s) &= \Big\langle \frac{\partial^2 x}{\partial r^2}(r,s)\,\Big|\,\nu(x(r,s))\Big\rangle\,,\\
K_{22}(r,s) &= \Big\langle \frac{\partial^2 x}{\partial s^2}(r,s)\,\Big|\,\nu(x(r,s))\Big\rangle\,,\\
K_{12}(r,s) &= \Big\langle \frac{\partial^2 x}{\partial r \partial s}(r,s)\,\Big|\,\nu(x(r,s))\Big\rangle\,,\\
K_{21}(r,s) &= K_{12}(r,s)\,.
\end{aligned}$$

The function $K_{11}(r,s)$ is the normal curvature of the geodesic $\Lambda_{\gamma(s)}$ at $x(r,s)$ and the function $K_{22}(0,s) = \kappa_n(\gamma(s))$ is the normal curvature of the curve Σ at $x(0,s) = \gamma(s)$.

F.2.3 Local coordinates near a curve on the boundary

We come back to previous computations and relate them to the curvatures. Let $\phi(x) = (y_1, y_2)$ be local coordinates of the boundary as defined in the previous subsection. We have observed in (F.14) that

$$g_0 = dy_3 \otimes dy_3 + \sum_{1\leq i,j\leq 2} [G_{ij}(y_1,y_2) - 2y_3 K_{ij}(y_1,y_2) + y_3^2 L_{ij}] dy_i \otimes dy_j \,, \quad \text{(F.22)}$$

where

$$G = \sum_{1\leq i,j\leq 2} G_{ij}\, dy_i \otimes dy_j \,,$$

$$K = \sum_{1\leq i,j\leq 2} K_{ij}\, dy_i \otimes dy_j \,,$$

$$L = \sum_{1\leq i,j\leq 2} L_{ij}\, dy_i \otimes dy_j = \sum_{1\leq i,j\leq 2} \Big\langle \frac{\partial \nu}{\partial y_i} \,\Big|\, \frac{\partial \nu}{\partial y_j} \Big\rangle dy_i \otimes dy_j \,.$$

The forms G, K, and L are respectively called the first, second, and third fundamental forms on $\partial\Omega$. If we take local coordinates $(y_1, y_2) = (r, s)$ on the boundary given by Lemma F.2.2, the sesquilinear form becomes

$$\begin{aligned} q^h_{\mathbf{A}}(u) = \int_{\mathcal{V}_{x_0}} |g|^{\frac{1}{2}} \Big[&|-ih\partial_{y_3} + \widetilde{A}_3 u|^2 \qquad \text{(F.23)} \\ &+ \sum_{1\leq i,j\leq 2} g^{ij} (-ih\partial_{y_i} u + \widetilde{A}_i u) \cdot \overline{(-ih\partial_{y_j} u + \widetilde{A}_j u)} \,\Big]\, dy_1 dy_2 dy_3 \,, \end{aligned}$$

for u supported in $\mathcal{V}_{x_0}$, and the associated differential operator is

$$\begin{aligned} P^h_{\mathbf{A}} = (-ih\partial_{y_3} + \widetilde{A}_3)^2 &+ \frac{h}{2i} |g|^{-1} (\partial_{y_3}|g|)(-ih\partial_{y_3} + \widetilde{A}_3) \\ &+ |g|^{-\frac{1}{2}} \sum_{1\leq i,j\leq 2} (-ih\partial_{y_j} + \widetilde{A}_j)(|g|^{\frac{1}{2}} g^{ij} (-ih\partial_{y_i} + \widetilde{A}_i)) \,. \qquad \text{(F.24)} \end{aligned}$$

If we now consider the coordinates $(y_1, y_2) = (r, s)$ and complete by $t = y_3$ introduced in Remark F.2.1, then

$$|g| = \alpha(r,s) - 2t\big[\alpha(r,s) K_{11}(r,s) + K_{22}(r,s)\big] + t^2 \epsilon_3(r,s,t) \,, \qquad \text{(F.25)}$$

and, for $1 \leq i,j \leq 2$,

$$(g^{ij})_{1\leq i,j\leq 2} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} + 2t \begin{pmatrix} K_{11} & \alpha^{-1} K_{12} \\ \alpha^{-1} K_{21} & \alpha^{-2} K_{22} \end{pmatrix} + t^2 R \,, \qquad \text{(F.26)}$$

where ϵ_3 and R_{ij} are smooth functions.

F.2.4 More magnetic geometry

We will apply the previous considerations in the case where the curve Σ is defined by (9.13). We assume for simplicity that the magnetic field $\beta = \operatorname{curl} \mathbf{A}$ is constant and we can assume, without loss of generality, that

$$\mathbf{A}(x) = \frac{b}{2}(0, -x_3, x_2)\,, \tag{F.27}$$

for some fixed $b > 0$.

Let Ω be a bounded open set of $\mathbb{R}^3$ with regular boundary $\partial\Omega$. We now assume that Σ defined in (9.13) is regular.

Remark F.2.4.
We observe that this assumption is satisfied when Ω is strictly convex.

In this situation, we can introduce the following definition:

Definition F.2.5.
At each point x of Σ, we introduce the normal curvature along the magnetic field β by

$$\kappa_{n,B}(x) := K_x \left(\gamma' \wedge \nu\,,\, \frac{\beta}{|\beta|} \right)\,, \tag{F.28}$$

where K denotes the second fundamental form on the surface $\partial\Omega$.

A calculation gives the following identity:

Lemma F.2.6.

$$\kappa_{n,B} = k_n\,, \tag{F.29}$$

with $\kappa_{n,B}$ from (F.28) *and k_n from* (9.15).

Similarly to $\kappa_{n,B}$, we can define

$$\kappa_{t,B}(s) = K\Big(\gamma'(s), \frac{\beta}{|\beta|}\Big)\,. \tag{F.30}$$

We observe that we have

$$\begin{aligned} \kappa_{n,B}(s) &= K\Big(\frac{\partial}{\partial r}, \frac{\beta}{|\beta|}\Big) = \cos\theta(s) K_{11}(0,s) + \sin\theta(s) K_{12}(0,s)\,, \\ \kappa_{t,B}(s) &= K\Big(\frac{\partial}{\partial s}, \frac{\beta}{|\beta|}\Big) = \cos\theta(s) K_{12}(0,s) + \sin\theta(s) K_{22}(0,s)\,. \end{aligned} \tag{F.31}$$

Let us observe that the angle $\theta(s)$ is not "free" in our picture. In fact, we have the geometrical constraint:

Proposition F.2.7.
The assumption that β is constant (of norm equal to one) and tangent to the surface $\partial\Omega$ along the curve Σ implies that

$$\kappa_{t,B}(x) = 0, \qquad \forall\, x \in \Sigma\,. \tag{F.32}$$

Furthermore, if $s \mapsto \gamma(s)$ *is a parametrization of* Σ *by arc length, with*

$$\theta(s) = \arcsin\langle \gamma'(s) \mid \beta \rangle\,,$$

then

$$\theta'(s) = \kappa_g(\gamma(s))\,, \qquad \forall\, s\,. \tag{F.33}$$

Lemma F.2.8.
In the case when $\partial\Omega$ *is strictly convex* $(K > 0)$, *then* (F.32) *implies that*

$$\kappa_{n,B} \neq 0\,, \qquad \forall\, x \in \Sigma\,. \tag{F.34}$$

When $\theta(s) = 0$, we deduce from (F.31) and (F.32) that $K_{12}(x(s)) = 0$. So the curvature matrix K becomes diagonal.

Proof of Lemma F.2.8.
We observe that (F.31) can be rewritten in the form

$$\begin{pmatrix} \kappa_{n,B} \\ \kappa_{t,B} \end{pmatrix} = K \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{F.35}$$

Observing that K is invertible when Ω is strictly convex (K is actually strictly positive), we immediately see that $|\kappa_{n,B}| + |\kappa_{t,B}| \neq 0$. □

Example F.2.9.
In the case of the ellipsoid $\{a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \leq 1\}$, *it is interesting to compute our invariants. Take for simplification the case when* $\beta = (0, 0, 1)$. *Then* Σ *is the intersection of the ellipsoid with* $x_3 = 0$. *So we get an ellipse in this plane. We can now observe that the vector field* β *is orthogonal to* Σ. *We observe that, at the point* (x_1, x_2, x_3) *on the boundary of the ellipsoid, we have*

$$\langle \beta \mid \nu \rangle = -\frac{a_3 x_3}{\sqrt{a_1^2 x_1^2 + a_2^2 x_2^2}}\,.$$

This leads to

$$|\kappa_{n,B}(x_1, x_2, 0)| = \frac{a_3}{\sqrt{a_1^2 x_1^2 + a_2^2 x_2^2}}\,.$$

The minimum of $\kappa_{n,B}$ *is then obtained at the point where* $\sqrt{a_1^2 x_1^2 + a_2^2 x_2^2}$ *is maximal. If we assume, for example, that* $a_1 > a_2$, *we get that this maximum is obtained at* $x_2 = x_3 = 0$ *and equal to* a_1.

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Index