CONFORMAL TENSORS

(*First Note*)

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1. In his treatment of the problem of the equivalence of two quadratic differential forms, E. B. Christoffel[1] was led in 1869 to a sequence of invariants which we now designate as the curvature tensor and its successive covariant derivatives. The process of forming any invariant of this sequence after the first, or curvature tensor, is known as covariant differentiation and can be applied to any arbitrarily given tensor to deduce a tensor of higher rank; however, the significance and indeed the justification of this process of covariant differentiation lies in the fact that it leads to the construction of a sequence of invariants suitable for the complete algebraic characterization of the quadratic differential form, or of the Riemann space, to adopt a more geometrical terminology. We mention this fact in particular because the point of view hereby expressed is fundamental in the following work.

The analogous problem of constructing a sequence of tensor invariants capable of characterizing the *conformal Riemann space* has been recognized for a number of years among geometers. Various aspects of this problem have been treated by Cartan, Schouten, J. M. Thomas, Veblen, Weyl and others. The following discussion will, however, be related primarily to two of my previous notes in These PROCEEDINGS and on this account we shall give a brief statement of the contents of these notes.[2] In the first of these it was recognized that the underlying analytical theory of the conformal Riemann space could be described as the invariant theory of a quadratic differential form

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} G_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

of weight $-\frac{2}{n}$, where n denotes the dimensionality number of the space; the second note led to the definition of a connection with components ${}^0\Gamma^i_{j\alpha}$ obeying a law of transformation

$$ {}^0\overline{\Gamma}^q_{j\alpha}\, u^i_q = \frac{\partial u^i_j}{\partial x^\alpha} + {}^0\Gamma^i_{p\mu}\, u^p_j\, u^\mu_\alpha \tag{1.1}$$

under an arbitrary analytic transformation of the x coördinates. In the above equations (1.1), as in all following equations, we adopt the summation convention with the understanding that Greek indices assume the values 0, 1, ..., n and Latin indices the values 0, 1, ..., n, ∞ unless the contrary is indicated; here ∞ is introduced as a convenient symbol in place of $n + 1$ as used by some writers. The quantities u^i_k in (1.1) are defined as the elements of the matrix

$$
\begin{array}{c|c|ccc|c|}
 & 0 & 1 & \cdots & n & \infty \\
\hline
0 & 1 & 0 & \cdots & 0 & 0 \\
\hline
1 & \bar{\psi}_1 & \dfrac{\partial x^1}{\partial \bar{x}^1} & \cdots & \dfrac{\partial x^n}{\partial \bar{x}^1} & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n & \bar{\psi}_n & \dfrac{\partial x^1}{\partial \bar{x}^n} & \cdots & \dfrac{\partial x^n}{\partial \bar{x}^n} & 0 \\
\hline
\infty & \frac{1}{2}\bar{\psi}_\sigma\bar{\psi}^\sigma & \dfrac{\partial x^1}{\partial \bar{x}^\sigma}\bar{\psi}^\sigma & \cdots & \dfrac{\partial x^n}{\partial \bar{x}^\sigma}\bar{\psi}^\sigma & (x\bar{x})^{\frac{2}{n}} \\
\hline
\end{array}
$$

in such a way that u^i_k stands for the element in the i$^{\text{th}}$ column and k$^{\text{th}}$ row of this matrix; also

$$\bar{\psi}_\sigma = \frac{\partial \log (x\bar{x})}{\partial \bar{x}^\sigma}, \quad \bar{\psi}^\sigma = \bar{G}^{\sigma\tau}\bar{\psi}_\tau$$

$$(\sigma,\tau = 1, \ldots, n)$$

where $(x\bar{x})$ denotes the jacobian determinant of the coördinate transformation. As a knowledge of certain of the components ${}^0\Gamma^i_{j\alpha}$ is necessary in our later work we shall observe that these are defined by

$${}^0\Gamma^i_{k0} = -\frac{1}{n}\,\delta^i_k; \quad {}^0\Gamma^i_{0\gamma} = -\frac{1}{n}\,\delta^i_\gamma,$$

$${}^0\Gamma^\alpha_{\beta\gamma} = K^\alpha_{\beta\gamma}; \quad {}^0\Gamma^0_{\beta\gamma} = \left(\frac{n}{n-2}\right)Q_{\beta\gamma}; \quad {}^0\Gamma^\infty_{\beta\gamma} = -\frac{1}{n}\,G_{\beta\gamma}, \quad (\alpha,\beta,\gamma = 1,\ldots,n)$$

$${}^0\Gamma^\alpha_{\infty\beta} = \left(\frac{n}{n-2}\right)Q^\alpha_\beta; \quad {}^0\Gamma^0_{\infty\gamma} = {}^0\Gamma^\infty_{\infty\gamma} = 0. \quad (\alpha,\beta,\gamma = 1, \ldots, n).$$

Reference may be had to the second of the above-mentioned papers for the explicit definition of the quantities Q and K in the above expressions. The quantities $G_{\alpha\beta}(\alpha,\beta = 1, \ldots, n)$ which have the law of transformation

$$\bar{G}_{\alpha\beta} = (x\bar{x})^{-\frac{2}{n}}\,G_{\mu\nu}\,u^\mu_\alpha\,u^\nu_\beta \quad (\text{indices} = 1, \ldots, n) \qquad (1.2)$$

constitute the components of the *fundamental conformal tensor.* In addition to the above equations we shall also need the equations

$$\bar{G}^{\alpha\beta} = (x\bar{x})^{\frac{2}{n}} G^{\mu\nu} u^{\alpha}_{\mu} u^{\beta}_{\nu} \quad (\text{indices} = 1, \ldots, n) \tag{1.3}$$

which give the law of transformation of the contravariant components $G^{\mu\nu}$ of the fundamental conformal tensor.

It is evident that equations (1.1) can be used in the usual manner to construct from an arbitrarily given tensor T, a covariant derivative for which the values 0, 1, ..., n only will be assumed by the added index of the differentiation; on account of the restricted range of this index, this process of covariant differentiation cannot, however, be used to form a second covariant derivative of tensor character. This suggests that we "complete" the above covariant derivative by defining the values of its components for the value ∞ of the index of differentiation.[3] Using the idea of completing this covariant derivative we introduce in this note a method which succeeds for all cases for which a certain constant K does not vanish, the constant K depending on the dimensionality number n of the space, the number of contravariant indices, the number of covariant indices and the weight of the given tensor. An extension of this method which enables us to complete a tensor whose components involve two Greek indices of restricted range 0, 1, ..., n is given in § 3. Corresponding to the above case, this latter method fails to apply whenever another constant L, depending on the same quantities as K, is equal to zero. This method can in general be applied to the conformal curvature tensor as defined in the second of my above-mentioned notes in These PROCEEDINGS, since for this tensor $L = 0$ only if $n = 4$; hence we arrive at covariant derivatives of the conformal curvature tensor having the significance that the equations of transformation of their components express integrability conditions of (1.1). It turns out that if the dimensionality number n is odd our method leads to the formation of an infinite sequence of covariant derivatives of the conformal curvature tensor. For n even but different from 4, the constant K will always vanish at some stage of the process of forming this sequence so that our method of constructing successive covariant derivatives cannot be continued.

A slight formal modification of a well-known lemma on partial differential equations[4] suffices to show that, for n odd, the conformal curvature tensor and its successive covariant derivatives constitute, when combined with the fundamental conformal tensor G, a complete set of tensor invariants of the conformal Riemann space.

In a second note on this same subject we shall treat the exceptional cases $K = 0$ and $L = 0$ on the basis of the fundamental idea of covariant differentiation, namely, the construction of conditions of integrability in tensorial form of the equations of transformation of the components of a tensor.

2. *Covariant derivative of an arbitrary tensor T.* DEFINITION. *A complete relative conformal tensor T of weight W or simply a relative conformal tensor of weight W, is an entity with components $T^{p\ldots q}_{r\ldots s}$ which depend on the coördinates $x^1, \ldots, x^n$ alone and which have the transformation*

$$\bar{T}^{i\ldots j}_{k\ldots l}\, u^p_i \ldots u^q_j = |u^a_b|^W\, T^{p\ldots q}_{r\ldots s}\, u^r_k \ldots u^s_l \tag{2.1}$$

when the coördinates undergo an arbitrary analytic transformation whose jacobian determinant does not vanish identically. If, however, one or more of the indices in these equations are limited to the restricted range 0, 1, ..., *n the entity T so defined is called an incomplete conformal tensor of weight W.* The quantity $|u^a_b|$ in (2.1) denotes the determinant of order $n+2$ formed from the quantities u^a_b and in fact is such that

$$|u^a_b| = (x\bar{x})^{\frac{n+2}{n}}.$$

Observing first that

$$\frac{\partial |u^a_b|}{\partial \bar{x}^\alpha} = W|u^a_b|^W \{{}^0\Gamma^h_{h\alpha} - {}^0\Gamma^h_{h\sigma}\, u^\sigma_\alpha\},$$

we find by differentiating (2.1) with respect to $\bar{x}^\alpha$ and eliminating derivatives of the u^i_k which occur, by means of (1.1), that

$$\bar{T}^{i\ldots j}_{k\ldots l,\alpha}\, u^p_i \ldots u^q_j = |u^a_b|^W\, T^{p\ldots q}_{r\ldots s,\mu}\, u^r_k \ldots u^s_l\, u^\mu_\alpha, \tag{2.2}$$

where

$$\begin{aligned} T^{p\ldots q}_{r\ldots s,\mu} = \frac{\partial T^{p\ldots q}_{r\ldots s}}{\partial x^\mu} &+ T^{h\ldots q}_{r\ldots s}\, {}^0\Gamma^p_{h\mu} + \ldots + T^{p\ldots h}_{r\ldots s}\, {}^0\Gamma^q_{h\mu} \\ &- T^{p\ldots q}_{h\ldots s}\, {}^0\Gamma^h_{r\mu} - \ldots - T^{p\ldots q}_{r\ldots h}\, {}^0\Gamma^h_{s\mu} - WT^{p\ldots q}_{r\ldots s}\, {}^0\Gamma^h_{h\mu}, \end{aligned} \tag{2.3}$$

and there is an analogous expression for the components appearing in the left members of (2.2); in the following it will be assumed without special mention that whenever a set of quantities such as $T^{p\ldots q}_{r\ldots s,\mu}$ are defined by equations of the type (2.3) the corresponding barred quantities are defined in an analogous manner.

Let N denote the number of indices $p \ldots q$ and M the number of indices $r \ldots s$ appearing in the components of the above tensor T. Then (2.3) show that

$$T^{p\ldots q}_{r\ldots s,0} = \left[\frac{M - N + (n+2)W}{n}\right] T^{p\ldots q}_{r\ldots s}. \tag{2.4}$$

Now we can form a conformal tensor from the incomplete conformal tensor whose components occur in (2.2) if we can define a set of quantities

$T^{p\dots q}_{r\dots s,\infty}$ depending on the coördinates $x^1, \dots, x^n$ alone and having equations of transformation

$$\overline{T}^{i\dots j}_{k\dots l,\infty}\, u^p_i \dots u^q_j = |u^a_b|^W \{ T^{p\dots q}_{r\dots s,\infty}\, u^\infty_\infty + T^{p\dots q}_{r\dots s,0}\, u^0_\infty + \sum_{\sigma=1}^{n} T^{p\dots q}_{r\dots s,\sigma}\, u^\sigma_\infty \} \quad u^r_k \dots u^s_l. \tag{2.5}$$

In fact (2.2) and (2.5) combine to give

$$\overline{T}^{i\dots j}_{k\dots l,f}\, u^p_i \dots u^q_j = |u^a_b|^W\, T^{p\dots q}_{r\dots s,g}\, u^r_k \dots u^s_l\, u^g_f. \tag{2.6}$$

The conformal tensor whose components appear in (2.2) will be called the incomplete covariant derivative and the tensor whose components appear in (2.6), the complete covariant derivative, or simply the covariant derivative of the given tensor T.

To construct the desired quantities $T^{p\dots q}_{r\dots s,\infty}$ transforming by (2.5) we first differentiate (2.2) with respect to $\bar{x}^\beta$, obtaining

$$\overline{T}^{i\dots j}_{k\dots l,\alpha\beta}\, u^p_i \dots u^q_j = |u^a_b|^W \{ T^{p\dots q}_{r\dots s,\mu\nu}\, u^\mu_\alpha u^\nu_\beta + T^{p\dots q}_{r\dots s,\sigma}\, {}^0\overline{\Gamma}^\infty_{\alpha\beta}\, u^\sigma_\infty \}\, u^r_k \dots u^s_l, \tag{2.7}$$

where

$$\begin{aligned} T^{p\dots q}_{r\dots s,\mu\nu} = \frac{\partial T^{p\dots q}_{r\dots s,\mu}}{\partial x^\nu} &+ T^{h\dots q}_{r\dots s,\mu}\, {}^0\Gamma^p_{h\nu} + \dots + T^{p\dots h}_{r\dots s,\mu}\, {}^0\Gamma^q_{h\nu} \\ &- T^{p\dots q}_{h\dots s,\mu}\, {}^0\Gamma^h_{r\nu} - \dots - T^{p\dots q}_{r\dots h,\mu}\, {}^0\Gamma^h_{s\nu} \\ &- T^{p\dots q}_{r\dots s,\sigma}\, {}^0\Gamma^\sigma_{\mu\nu} - W T^{p\dots q}_{r\dots s,\mu}\, {}^0\Gamma^h_{h\nu}. \end{aligned}$$

In particular we can deduce from these latter equations and (2.4) that

$$T^{p\dots q}_{r\dots s,0\nu} = \left[\frac{M - N + (n+2)W + 1}{n}\right] T^{p\dots q}_{r\dots s,\nu}, \tag{2.8}$$

$$T^{p\dots q}_{r\dots s,\mu 0} = \left[\frac{M - N + (n+2)W + 1}{n}\right] T^{p\dots q}_{r\dots s,\mu}, \tag{2.9}$$

$$T^{p\dots q}_{r\dots s,00} = \left[\frac{M - N + (n+2)W + 1}{n}\right]\left[\frac{M - N + (n+2)W}{n}\right] T^{p\dots q}_{r\dots s}. \tag{2.10}$$

Now restrict the values of α and β in (2.7) to the range $1, \dots, n$ and rewrite the right members of these equations so as to obtain a corresponding restriction of range for the indices μ and ν, i.e., we form the equations

$$\begin{aligned} \overline{T}^{i\dots j}_{k\dots l,\alpha\beta}\, u^p_i \dots u^q_j = |u^a_b|^W \{ & T^{p\dots q}_{r\dots s,\mu\nu}\, u^\mu_\alpha u^\nu_\beta + T^{p\dots q}_{r\dots s,0\nu}\, u^0_\alpha u^\nu_\beta \\ &+ T^{p\dots q}_{r\dots s,\mu 0}\, u^\mu_\alpha u^0_\beta + T^{p\dots q}_{r\dots s,00}\, u^0_\alpha u^0_\beta \\ &+ T^{p\dots q}_{r\dots s,\nu}\, {}^0\overline{\Gamma}^\infty_{\alpha\beta}\, u^\nu_\infty + T^{p\dots q}_{r\dots s,0}\, {}^0\overline{\Gamma}^\infty_{\alpha\beta}\, u^0_\infty \}\, u^r_k \dots u^s_l \end{aligned} \tag{2.11}$$

with the understanding that the indices α,β,μ,ν assume values $1, \ldots, n$. If we multiply (2.11) by $\overline{G}^{\alpha\beta}$ and sum on the indices α and β, taking account of (1.3), (2.4), (2.8), (2.9) and (2.10) we obtain a system of equations which can be written

$$\overline{T}^{i\ldots j}_{k\ldots l,\alpha\beta}\,\overline{G}^{\alpha\beta}\,u^p_i \ldots u^q_j = \left|u^a_b\right|^W \Big\{ T^{p\ldots q}_{r\ldots s,\mu\nu}\,G^{\mu\nu}\,u^\infty_\infty + \left[\frac{2M-2N+2(n+2)W+2-n}{n}\right](T^{p\ldots q}_{r\ldots s,0}\,u^0_\infty + T^{p\ldots q}_{r\ldots s,\nu}\,u^\nu_\infty)\Big\}\,u^r_k \ldots u^s_l. \tag{2.12}$$

Hence, if the constant K defined by

$$K = 2M - 2N + 2(n+2)W + 2 - n$$

does not vanish, we can put

$$T^{p\ldots q}_{r\ldots s,\infty} = \left(\frac{n}{K}\right)\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} T^{p\ldots q}_{r\ldots s,\mu\nu}\,G^{\mu\nu}, \tag{2.13}$$

and the components $T^{p\ldots q}_{r\ldots s,\infty}$ so defined will transform by (2.5). We thus obtain the complete covariant derivative with components $T^{p\ldots q}_{r\ldots s,g}$ transforming in accordance with (2.6).

If $K = 0$, equations (2.12) show the existence of a conformal tensor $\mathfrak{T}$ of weight $W + \dfrac{2}{n+2}$ having the components

$$\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} T^{p\ldots q}_{r\ldots s,\mu\nu}\,G^{\mu\nu}.$$

The constant K will not vanish for the tensor $\mathfrak{T}$ so that it is possible to construct the complete conformal covariant derivative of this tensor. It is, in fact, evident that we can form the infinite sequence of successive covariant derivatives of the tensor $\mathfrak{T}$.

3. *An Extension of the Preceding Method.*—Before applying the above theory to the problem of constructing conformal tensor invariants suitable for the characterization of the conformal space we must consider the problem of completing a tensor D having components $D^{p\ldots q}_{r\ldots s\mu\nu}$ which are skew-symmetric in the indices μ and ν and such that

$$D^{p\ldots q}_{r\ldots s0\nu} \equiv D^{p\ldots q}_{r\ldots s\mu 0} \equiv 0.$$

Differentiation of the equations of transformation of the components of the tensor D, namely,

$$\overline{D}^{i\ldots j}_{k\ldots l\alpha\beta}\,u^p_i \ldots u^q_j = \left|u^a_b\right|^W D^{p\ldots q}_{r\ldots s\mu\nu}\,u^r_k \ldots u^s_l\,u^\mu_\alpha\,u^\nu_\beta$$

and elimination of the derivatives of the u^i_k which occur, by (1.1), gives

$$\bar{D}^{i\ldots j}_{k\ldots l\alpha\beta\gamma}\, u^p_i \ldots u^q_j = \left| u^a_b \right|^W \left\{ D^{p\ldots q}_{r\ldots s\mu\nu\pi}\, u^\mu_\alpha u^\nu_\beta u^\pi_\gamma + D^{p\ldots q}_{r\ldots s\mu\nu} \left({}^0\bar{\Gamma}^\infty_{\alpha\gamma}\, u^\mu_\infty\, u^\nu_\beta + {}^0\bar{\Gamma}^\infty_{\beta\gamma}\, u^\mu_\alpha u^\nu_\infty \right) \right\} u^r_k \ldots u^s_l, \qquad (3.2)$$

where

$$D^{p\ldots q}_{r\ldots s\mu\nu\pi} = \frac{\partial D^{p\ldots q}_{r\ldots s\mu\nu}}{\partial x^\pi} + D^{h\ldots q}_{r\ldots s\mu\nu}\, {}^0\Gamma^p_{h\pi} + \ldots + D^{p\ldots h}_{r\ldots s\mu\nu}\, {}^0\Gamma^q_{h\pi} - D^{p\ldots q}_{h\ldots s\mu\nu}\, {}^0\Gamma^h_{r\pi} - \ldots - D^{p\ldots q}_{r\ldots h\mu\nu}\, {}^0\Gamma^h_{s\pi} - D^{p\ldots q}_{r\ldots s\sigma\nu}\, {}^0\Gamma^\sigma_{\mu\pi} - D^{p\ldots q}_{r\ldots s\mu\sigma}\, {}^0\Gamma^\sigma_{\nu\pi} - WD^{p\ldots q}_{r\ldots s\mu\nu}\, {}^0\Gamma^h_{h\pi}.$$

If as before N and M denote the number of indices $p\ldots q$ and $r\ldots s$, respectively, then it follows from the last formula that

$$D^{p\ldots q}_{r\ldots s0\nu\pi} = \frac{1}{n} D^{p\ldots q}_{r\ldots s\pi\nu},$$

$$D^{p\ldots q}_{r\ldots s\mu 0\pi} = \frac{1}{n} D^{p\ldots q}_{r\ldots s\mu\pi},$$

$$D^{p\ldots q}_{r\ldots s\mu\nu 0} = \left[\frac{M - N + (n+2)W + 2}{n}\right] D^{p\ldots q}_{r\ldots s\mu\nu}.$$

Equations (3.2) can accordingly be written

$$\bar{D}^{i\ldots j}_{k\ldots l\alpha\beta\gamma}\, u^p_i \ldots u^q_j = \left| u^a_b \right|^W \left\{ D^{p\ldots q}_{r\ldots s\mu\nu\pi}\, u^\mu_\alpha u^\nu_\beta u^\pi_\gamma + \frac{1}{n} D^{p\ldots q}_{r\ldots s\pi\nu}\, u^0_\alpha u^\nu_\beta u^\pi_\gamma + \frac{1}{n} D^{p\ldots q}_{r\ldots s\mu\pi}\, u^\mu_\alpha u^0_\beta u^\pi_\gamma + \left[\frac{M - N + (n+2)W + 2}{n}\right] D^{p\ldots q}_{r\ldots s\mu\nu}\, u^\mu_\alpha u^\nu_\beta u^0_\gamma + D^{p\ldots q}_{r\ldots s\mu\nu} \left({}^0\Gamma^\infty_{\alpha\gamma}\, u^\mu_\infty\, u^\nu_\beta + {}^0\Gamma^\infty_{\beta\gamma}\, u^\mu_\alpha\, u^\nu_\infty \right) \right\} u^r_k \ldots u^s_l \qquad (3.3)$$

in which it is to be understood that the indices $\alpha,\beta,\gamma,\mu,\nu,\pi$ admit only the restricted range $1, \ldots, n$. Multiplying (3.3) by $\bar{G}^{\beta\gamma}$ and summing on the indices β and γ, we obtain after some reductions

$$\bar{D}^{i\ldots j}_{k\ldots l\alpha\beta\gamma}\, \bar{G}^{\beta\gamma}\, u^p_i \ldots u^q_j = \left| u^a_b \right|^w \left\{ D^{p\ldots q}_{r\ldots s\mu\nu\pi}\, G^{\nu\pi}\, u^\mu_\alpha\, u^\infty_\infty + \left(\frac{L}{n}\right) D^{p\ldots q}_{r\ldots s\mu\nu}\, u^\mu_\alpha\, u^\nu_\infty \right\} u^r_k \ldots u^s_l \qquad (3.4)$$

where the indices $\alpha,\beta,\gamma,\mu,\nu,\pi$ admit the range $1, \ldots, n$ only and the integer L is defined by

$$L \equiv M - N + (n+2)W + 4 - n.$$

Since

$$D^{p\ldots q}_{r\ldots s0\nu\pi}\, G^{\nu\pi} \equiv 0$$

owing to the skew-symmetric property of the components of the incomplete conformal tensor D, it follows that (3.4) holds also when regarded as equations for the transformation of the quantities

$$D^{p\dots q}_{r\dots s\mu\nu\pi}\, G^{\nu\pi}, \tag{3.5}$$

where the index μ takes on the full range of values $0, 1, \dots, n$ in conformity with the general convention; in the following we shall regard (3.4) as holding in this sense.

If $L \neq 0$, we put

$$D^{p\dots q}_{r\dots s\mu\infty} \equiv -D^{p\dots q}_{r\dots s\infty\mu} \equiv \left(\frac{n}{L}\right) D^{p\dots q}_{r\dots s\mu\nu\pi}\, G^{\nu\pi},$$

$$D^{p\dots q}_{r\dots s0\infty} \equiv D^{p\dots q}_{r\dots s\infty 0} \equiv D^{p\dots q}_{r\dots s\infty\infty} \equiv 0.$$

Then (3.1) and (3.4) combine to give

$$D^{i\dots j}_{k\dots lcd}\, u^p_i \dots u^q_j = \left| u^a_b \right|^W D^{p\dots q}_{r\dots sfg}\, u^r_k \dots u^s_l u^f_c u^g_d,$$

and we have succeeded in completing the incomplete conformal tensor D; from their definition the components of the complete conformal tensor D are skew-symmetric in their last two indices, i.e.,

$$D^{p\dots q}_{r\dots sfg} \equiv -D^{p\dots q}_{r\dots sgf}.$$

When $L = 0$, equations (3.4) lead to the definition of an incomplete conformal tensor of weight $W + \dfrac{2}{n+2}$ with components (3.5); it is obvious that a method analogous to the above can in general be used to complete this tensor after which its successive covariant derivatives can be constructed.

4. *An Application of the Method of* § 3.—As a first application of the method of the preceding paragraph we can consider the components of an incomplete conformal tensor D to be defined by

$$D^{p\dots q}_{r\dots s\mu\nu} \equiv T^{p\dots q}_{r\dots s,\mu\nu} - T^{p\dots q}_{r\dots s,\nu\mu}, \tag{4.1}$$

Cf. equation (2.7). The tensor D is then the direct analogue of the skew-symmetric part of the second covariant derivative of a given conformal tensor T. It is therefore always possible to complete the tensor D defined by (4.1) even when our method for constructing the complete covariant derivative of the tensor T fails to apply, since the integers K and L cannot vanish simultaneously.

5. *The Complete Conformal Curvature Tensor* ($n \neq 4$).—The important application of the method of § 3 and the one in fact for which this method was designed, is to complete the curvature tensor whose components

${}^0B^i_{k\alpha\beta}$ were deduced in the second of my above mentioned notes in These PROCEEDINGS (see § 1). That is, we put

$$D^i_{k\alpha\beta} \equiv {}^0B^i_{k\alpha\beta};$$

the requisite conditions on the components $D_{k\alpha\beta}^{\ i}$ are then satisfied since the components ${}^0B_{k\alpha\beta}^{\ i}$ in addition to being skew-symmetric in the indices α and β, are equal to zero whenever the values $\alpha = 0$ or $\beta = 0$ are assumed. For this case $L = 4 - n$ and hence we can use the method of § 3 to complete the curvature tensor whenever n is different from 4. We thus obtain the components ${}^0B^i_{klm}$ of the complete conformal curvature tensor.

6. *Tensors Derived from the Conformal Curvature Tensor.*—Starting with the complete conformal curvature tensor, $n \neq 4$, let us consider the problem of forming its successive covariant derivatives. We have $N = 1$ and $W = 0$ so that the integer K is equal to $2\,M - n$ for this case. Hence if n is an odd integer, greater than or equal to 3, we can construct the infinite sequence of complete covariant derivatives of the complete conformal curvature tensor. Let us indicate this by writing

$$\boxed{n \text{ odd}} \qquad {}^0B^i_{klm};\ {}^0B^i_{klmp};\ {}^0B^i_{klmpq};\ \ldots.$$

as the components of the curvature tensor and its successive covariant derivatives.

If n is even and greater than 4 the constant K will always vanish at some stage of the process of forming the infinite sequence of complete covariant derivatives of the curvature tensor. For example, if $n = 6$, $K = 0$ for the conformal curvature tensor; if $n = 8$, we can construct the first complete conformal covariant derivative of the conformal curvature tensor but then find that $K = 0$ for this covariant derivative, etc. The following indications for $n = 6$, 8 and 10 enable us to see at a glance the behavior of these sequences for the above and higher values of the dimensionality number.

$$\boxed{n = 6} \qquad {}^0B^i_{klm}.$$

$$\boxed{n = 8} \qquad {}^0B^i_{klm};\ {}^0B^i_{klmp}.$$

$$\boxed{n = 10} \qquad {}^0B^i_{klm};\ {}^0B^i_{klmp};\ {}^0B^i_{klmpq}.$$

As shown these sequences end after a finite number of terms after which our method for completing the covariant derivative fails to apply. Use of the quantities P_{ik} defined in the second of my above-mentioned notes in These PROCEEDINGS will evidently lead to an entirely covariant form of the components of the conformal curvature tensor and its covariant derivatives.

[1] E. B. Christoffel, *J. reine angew. Math.* (*Crelle*), **70**, 46 and 241 (1869).

[2] T. Y. Thomas, "Invariants of Relative Quadratic Differential Forms," These PROCEEDINGS, **11**, 722–725 (1925); also "On Conformal Geometry," *Ibid.*, **12**, 352–359 (1926).

[3] O. Veblen completed the covariant derivative by imposing a set of invariant conditions involving components of conformal tensors. These conditions correspond to our equations (2.13) but differ from these latter equations in the important respect that they involve only derivatives of the first order. See "Differential Invariants and Geometry," *Atti del congresso internazionale dei matematici*, **1**, [6] 181–189 (1928); also, "Conformal Tensors and Connections," These PROCEEDINGS, **14**, 735–745 (1928). Under certain conditions this method enables one to complete the covariant derivative of a tensor T in a very satisfactory manner. However, Veblen's method does not lead to a sequence of conformal tensor invariants, analogous to the Riemann curvature tensor and its successive covariant derivatives, by means of which the equivalence or non-equivalence of two conformal spaces can be determined. On this account the method of the present note is preferable in my opinion.

[4] See, for example, L. P. Eisenhart, *Non-Riemannian Geometry* (New York) 1927, pp. 14–18, where references to the literature are to be found.

ON THE CHANGES OF SIGN OF THE DERIVATIVES OF A FUNCTION DEFINED BY A LAPLACE INTEGRAL

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Let a real function $f(x)$ be defined by a Laplace integral

$$f(x) = \int_0^\infty e^{-xt}\,\varphi(t)dt,$$

where $\varphi(t)$ is a real continuous function in the interval $0 \leqq t < \infty$, the integral converging for $x > 0$. The purpose of this note is to announce certain results concerning the changes of sign of $f(x)$ and its derivatives as affected by the changes of sign of $\varphi(t)$.

It was known to Laguerre[1] that $f(x)$ cannot have more changes of sign than $\varphi(t)$. Since

$$f^{(k)}(x) = (-1)^k \int_0^\infty e^{-xt}\, t^k \varphi(t)\, dt,$$

it follows that $f^{(k)}(x)$ cannot have more changes of sign than $\varphi(t)$. We are able to show that it has exactly as many as $\varphi(t)$ for all k sufficiently large. Moreover, we show that if a change of sign of $\varphi(t)$ is at $t = a$, then one of the changes of sign of $f^{(k)}(x)$ will be at a point x_k such that