ON THE UNIFIED FIELD THEORY. V

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The general existence theorem established in Note III ceases to apply when the data of the problem is ascribed over a characteristic surface S_3. In treating this important exceptional case we arrive at an existence theorem which is, roughly speaking, analogous to the existence theorem for the ordinary wave-equation when the data is given over a characteristic surface and which also bears certain resemblances to a theorem of Hadamard.[1] This theorem is capable of dynamical application. It follows from it that there exists an infinite number of sets of integrals h^i_α of the field equations such that each set of integrals and their first derivatives assume the same values over a characteristic surface S_3; in fact, the conditions under which these integrals exist are such as to preclude the occurrence of more than a single set of integrals in case the surface S_3 is not a characteristic. This result leads to the interpretation of the characteristic surfaces as gravitational and electro-magnetic wave surfaces in the four-dimensional continuum. A brief discussion of these wave surfaces will be given in Note VI.

1. Let us denote for the moment by $X^1, \ldots, X^4$ the coördinates of the continuum; let us also denote by H^α_i the contravariant components of the fundamental vectors and by $G^{\alpha\beta}$ the contravariant components of the fundamental metric tensor. A characteristic surface S_3 is then a three-dimensional surface $\Phi(X) = 0$ such that over it the equation

$$G^{\alpha\beta}\frac{\partial\Phi}{\partial X^\alpha}\frac{\partial\Phi}{\partial X^\beta} = 0 \tag{1.1}$$

is satisfied. Suppose that P is not a *singular point* on the hypersurface $\Phi = 0$, i.e., all first derivatives of the function Φ do not vanish at P, and consider another surface S^*_3 defined by $\Psi(X) = 0$ which passes through the point P. This latter surface will be subjected to the following condition, the reason for which will later appear.

C_1. *The inequality*

$$G^{\alpha\beta}\frac{\partial\Phi}{\partial X^\alpha}\frac{\partial\Psi}{\partial X^\beta} \gtrless 0 \tag{1.2}$$

holds at the point P.

The above condition is independent of the orientation of the fundamental vectors. It is therefore possible, and we shall later find it expedient, to orientate the vector configurations throughout the continuum in the following manner.

C₂. *The vector configurations are so oriented that the inequalities*

$$H_2^\alpha \frac{\partial \Phi}{\partial X^\alpha} \gtrless 0 \tag{1.3}$$

$$\sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \begin{vmatrix} H_1^\alpha & H_1^\beta \\ H_2^\alpha & H_2^\beta \end{vmatrix} \frac{\partial \Phi}{\partial X^\alpha} \frac{\partial \Psi}{\partial X^\beta} \gtrless 0 \tag{1.4}$$

$$\sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \begin{vmatrix} H_3^\alpha & H_3^\beta \\ H_4^\alpha & H_4^\beta \end{vmatrix} \frac{\partial \Phi}{\partial X^\alpha} \frac{\partial \Psi}{\partial X^\beta} \gtrless 0 \tag{1.5}$$

are satisfied at the point P[2].

We shall refer to the above conditions C_1 and C_2 as *normal conditions*. It is evident that C_2 might also be regarded as giving conditions on the surfaces S_3 and S_3^*. However, the above point of view has been preferred as it does not involve an apparent restriction on the characteristic surface S_3.

In consequence of condition C_1 it follows that the rank of the matrix

$$\begin{Vmatrix} \frac{\partial \Phi}{\partial X^1} & \frac{\partial \Phi}{\partial X^2} & \frac{\partial \Phi}{\partial X^3} & \frac{\partial \Phi}{\partial X^4} \\ \frac{\partial \Psi}{\partial X^1} & \frac{\partial \Psi}{\partial X^2} & \frac{\partial \Psi}{\partial X^3} & \frac{\partial \Psi}{\partial X^4} \end{Vmatrix} \tag{1.6}$$

must be two at the point P. In fact, if the rank of this matrix were less than two, condition C_1 would fail to be satisfied. It is therefore possible to choose functions $\Omega(X)$ and $\Xi(X)$ such that the jacobian of the coördinate transformation

$$x^1 = \Phi(X),\ x^2 = \Psi(X),\ x^3 = \Omega(X),\ x^4 = \Xi(X)$$

will not vanish at P; this means that the above transformation possesses an inverse throughout the neighborhood of the point P. Under this transformation the equation of the characteristic surface S_3 assumes the form $x^1 = 0$, while the equation of the auxiliary surface S_3^* becomes $x^2 = 0$. Denoting the components of the fundamental vectors and the fundamental metric tensor in the (x) coördinate system by the usual designation of h and g, respectively, we have from (1.1) that the contravariant component $g^{11} = 0$ over S_3; also by C_1 we have that $g^{12} \gtrless 0$ at P. Moreover, the condition C_2 results in the fact that the inequalities

$$h_2^1 \gtrless 0,\ h_1^1 h_2^2 - h_2^1 h_1^2 \gtrless 0,\ h_3^1 h_4^2 - h_4^1 h_3^2 \gtrless 0 \tag{1.7}$$

are satisfied by the contravariant components h_i^α at the point P. In treating the problem of the determination of integrals of the field equations by the specification of data over a characteristic surface S_3 we shall assume in the following work the normalization of this problem afforded by the selection of the (x) coördinate system and the assertion of the conditions C_1 and C_2.

2. We shall find it expedient in making certain calculations to introduce a slight change in notation. Let us put

$$\alpha = h^1_1,\ \beta = h^1_2,\ \gamma = h^1_3,\ \delta = h^1_4$$

$$a = h^2_1,\ b = h^2_2,\ c = h^2_3,\ d = h^2_4$$

for the contravariant components h^α_i; also

$$W = \alpha^2 - \beta^2 - \gamma^2 - \delta^2, \quad W^* = \alpha a - \beta b - \gamma c - \delta d.$$

The two-rowed determinants formed from the matrix

$$\begin{Vmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{Vmatrix}$$

will also enter into our calculations so that we shall use the abbreviations

$$\begin{array}{lll} A = \alpha b - \beta a, & B = \alpha c - \gamma a, & C = \alpha d - \delta a \\ D = \beta c - \gamma b & E = \beta d - \delta b, & F = \gamma d - \delta c. \end{array}$$

The above quantity W is nothing more than the contravariant component g^{11} and hence vanishes over the characteristic surface S_3; also W^* stands for the contravariant component g^{12} so that $W^* \gtrless 0$ at P. Finally the inequalities $\beta \gtrless 0$, $A \gtrless 0$ and $F \gtrless 0$ are equivalent to (1.7).

3. Consider the matrix represented by Table 1. Each row in table 1 corresponds to an equation III (2.2) or III (3.1) determined by the indicated values of j, k, l or j, respectively, and the elements in any column are the coefficients of the derivatives at the top of this column; more precisely each row in Table 1 corresponds to a set of four equations but the above terminology is convenient and has been used throughout this note. It is necessary to deduce a number of lemmas regarding certain matrices and determinants formed from the elements in Table 1.

Let us denote by M_1 the matrix formed from the elements in the first six columns in Table 1. The fourth order determinant in the upper left-hand corner of M_1 will be denoted by J. More generally a determinant constructed from the elements in rows $l, m, \ldots, n$ and columns $p, q, \ldots, r$ in Table 1 will be designated by the symbol

$$\begin{vmatrix} l & m & \ldots & n \\ p & q & \ldots & r \end{vmatrix}.$$

Let us consider the eight determinants of order five in M_1 which are formed by bordering J in all possible manners. Expansions of these determinants show that

$$\begin{vmatrix}1\,2\,3\,4\,5\\1\,2\,3\,4\,5\end{vmatrix} = \begin{vmatrix}1\,2\,3\,4\,6\\1\,2\,3\,4\,6\end{vmatrix} = -\ \alpha\beta\gamma W, \quad \begin{vmatrix}1\,2\,3\,4\,5\\1\,2\,3\,4\,6\end{vmatrix} = -\ \begin{vmatrix}1\,2\,3\,4\,6\\1\,2\,3\,4\,5\end{vmatrix} = \alpha^2\delta W.$$

$$\begin{vmatrix}1\,2\,3\,4\,7\\1\,2\,3\,4\,5\end{vmatrix} = \begin{vmatrix}1\,2\,3\,4\,8\\1\,2\,3\,4\,6\end{vmatrix} = -\ \beta^2\gamma W, \quad \begin{vmatrix}1\,2\,3\,4\,7\\1\,2\,3\,4\,6\end{vmatrix} = -\ \begin{vmatrix}1\,2\,3\,4\,8\\1\,2\,3\,4\,5\end{vmatrix} = \alpha\beta\delta W.$$

Since $W = 0$ at a point P on a characteristic surface S_3 it follows by a theorem in algebra[3] that if $J \neq 0$ at P the rank of the matrix M_1 is four at P. We can, in fact, easily prove a more specific result, namely, that every determinant of order five in M_1 contains W as a factor.[4] As we shall later have use of this result we state it as the following

jkl / j	$\frac{\partial h^i_{2,1}}{\partial x^1}$	$\frac{\partial h^i_{4,3}}{\partial x^1}$	$\frac{\partial h^i_{4,1}}{\partial x^1}$	$\frac{\partial h^i_{3,2}}{\partial x^1}$	$\frac{\partial h^i_{4,2}}{\partial x^1}$	$\frac{\partial h^i_{3,1}}{\partial x^1}$	$\frac{\partial h^i_{4,1}}{\partial x^2}$	$\frac{\partial h^i_{3,2}}{\partial x^2}$	$\frac{\partial h^i_{4,2}}{\partial x^2}$	$\frac{\partial h^i_{3,1}}{\partial x^2}$	$\frac{\partial h^i_{2,1}}{\partial x^2}$	$\frac{\partial h^i_{4,3}}{\partial x^2}$
123	$-\gamma$	0	0	$-\alpha$	0	β	0	$-a$	0	b	$-c$	0
$j = 4$	0	$-\gamma$	α	0	$-\beta$	0	a	0	$-b$	0	0	$-c$
$j = 3$	0	δ	0	$-\beta$	0	α	0	$-b$	0	a	0	d
124	$-\delta$	0	β	0	$-\alpha$	0	b	0	$-a$	0	$-d$	0
134	0	$-\alpha$	γ	0	0	δ	c	0	0	$-d$	0	$-a$
$j = 2$	α	0	0	γ	$-\delta$	0	0	c	d	0	a	0
234	0	$-\beta$	0	$-\delta$	γ	0	0	$-d$	c	0	0	$-b$
$j = 1$	β	0	δ	0	0	γ	d	0	0	c	b	0

TABLE 1

LEMMA I. *Every determinant of order five of the matrix M_1 has the form $WR(\alpha, \beta, \gamma, \delta)$, where $R(\alpha, \beta, \gamma, \delta)$ denotes a polygnomial in the variables indicated.*

Let us denote by M_2 the matrix determined by the elements in the first four rows and first six columns in Table 1. The following lemma can then be proved.

LEMMA II. *The matrix M_2 is of rank four at a point P of a characteristic surface.*

This lemma shows in particular that the rank of M_1 is four at P. To prove Lemma II we consider the determinant

$$\begin{vmatrix}1 & 2 & 3 & 4\\3 & 4 & 5 & 6\end{vmatrix} = (\alpha - \beta)^2$$

contained in M_2. The assumption that the above determinant vanishes gives $\alpha^2 = \beta^2$ and since $W = 0$ at P it follows that $\gamma = \delta = 0$ at P. Hence $F = 0$ at P contrary to condition C_2 in Sect. 1. This proves Lemma II.

By calculation we have that[5]

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{vmatrix} = 4A^2F^2 + (B^2 + C^2 - D^2 - E^2)^2.$$

Since $A \gtrless 0$ and $F \gtrless 0$ we obtain the result stated in the following

LEMMA III. *The inequality*

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{vmatrix} > 0$$

is satisfied at the point P.

4. We shall apply the symbols U and V to represent components h^i_{jk} as indicated by the scheme

$$U \sim h^i_{2,1},\ h^i_{4,3};\quad V \sim h^i_{4,1},\ h^i_{3,2},\ h^i_{4,2},\ h^i_{3,1}.$$

Equations III (2.2) and III (3.1) can now be solved for first derivatives of the V with respect to x^1 and x^2 in consequence of the fact that the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{vmatrix}$$

does not vanish at P by Lemma III. This solution gives

$$\frac{\partial V}{\partial x^1} = \sum R(h) \frac{\partial U}{\partial x^\alpha} + \sum R(h) \frac{\partial V}{\partial x^\beta} + \bigstar \tag{4.1}$$

$$(\alpha = 1, 2, 3, 4;\ \beta = 3, 4)$$

$$\frac{\partial V}{\partial x^2} = W \sum R \frac{\partial U}{\partial x^1} + \sum R \frac{\partial U}{\partial x^\alpha} + \sum R \frac{\partial U}{\partial x^\beta} + \bigstar \tag{4.2}$$

$$(\alpha = 2, 3, 4;\ \beta = 3, 4)$$

where R denotes a rational function of the h^i_α and the $\bigstar$ as usual denotes terms of lower order than those which have been written down explicitly. The occurrence of the quantity W as a factor in the first term of the right member of (4.2) follows as a result of Lemma III; this fact is essential in our later work.

By multiplying the four equations corresponding to the first four rows in Table 1 by suitably chosen quantities p, q, r, s and adding to the equation corresponding to the seventh row in Table 1 it is possible to obtain an equation in which the coefficients of the derivatives of the components U and V with respect to x^1 will vanish at P. This follows from Lemmas

I and II. Similarly it is possible to multiply the four equations corresponding to the first four rows in Table 1 by quantities $\bar{p}, \bar{q}, \bar{r}, \bar{s}$, and add to the equation corresponding to the last row in Table 1 so as to obtain an equation in which the coefficients of the above derivatives will likewise vanish at P. We can, in fact, construct in an obvious manner a system of linear equations in the unknown quantities p, q, r, s or $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ with the non-vanishing determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{vmatrix}$$

and solve by Cramer's rule. This gives

$$p = \frac{-\alpha\delta}{\alpha^2 - \beta^2}, \quad q = \frac{-\beta\gamma}{\alpha^2 - \beta^2}, \quad r = \frac{\beta\delta}{\alpha^2 - \beta^2}, \quad s = \frac{\alpha\gamma}{\alpha^2 - \beta^2},$$

$$\bar{p} = \frac{\beta\gamma}{\alpha^2 - \beta^2}, \quad \bar{q} = \frac{-\alpha\delta}{\alpha^2 - \beta^2}, \quad \bar{r} = \frac{-\alpha\gamma}{\alpha^2 - \beta^2}, \quad \bar{s} = \frac{\beta\delta}{\alpha^2 - \beta^2}.$$

In the two equations so obtained the coefficients of the derivatives in in the last six columns of Table 1, i.e., the derivatives of the quantities U and V with respect to x^2, will be given by the matrix

$$\begin{Vmatrix} u & v & w & x & y & z \\ -v & u & -x & w & -z & y \end{Vmatrix}$$

in which $$u = \frac{\gamma A}{\alpha^2 - \beta^2}, \quad v = \frac{\beta E - \alpha C}{\alpha^2 - \beta^2}, \quad w = \frac{\alpha B - \beta D}{\alpha^2 - \beta^2},$$

$$x = \frac{-\delta A}{\alpha^2 - \beta^2}, \quad y = \frac{-\alpha F}{\alpha^2 - \beta^2}, \quad z = \frac{\gamma D + \delta E - bW}{\alpha^2 - \beta^2}.$$

Let us call the four equations corresponding to the first four rows in Table 1 the system L_1 and the two equations which we have constructed by the above process, the system L_2. Now differentiate L_1 with respect to x^2 and L_2 with respect to x^1. On combining the resulting equations we have a system L_3 of six equations which can be solved for the second derivatives with respect to x^1, x^2 of the quantities U and V. Table 2 indicates the determinant of the coefficients of these derivatives in L_3. Expanding[5] this determinant and neglecting additive terms containing W we obtain the simple quantity $(2\beta W^*)^2$ which does not vanish at P in consequence of the normal conditions imposed in Sect. 1; this fact permits the unique determination of the

0	$-\alpha$	0	β	$-\gamma$	0
α	0	$-\beta$	0	0	$-\gamma$
0	$-\beta$	0	α	0	δ
β	0	$-\alpha$	0	$-\delta$	0
u	v	w	x	y	z
$-v$	u	$-x$	w	$-z$	y

TABLE 2

above second derivatives and leads to a system of equations L_3 of the form

$$\left.\begin{array}{c}\dfrac{\partial^2 U}{\partial x^1 \partial x^2}\\[2ex] \dfrac{\partial^2 V}{\partial x^1 \partial x^2}\end{array}\right\} = W\sum R\frac{\partial^2 U}{\partial x^1\partial x^1} + \sum R\frac{\partial^2 U}{\partial x^\alpha \partial x^\beta} + \sum R\frac{\partial^2 V}{\partial x^\alpha\partial x^\beta} + \bigstar. \quad (4.3)$$

$$(\alpha \neq 1 \text{ if } \beta = 1, 2)$$

It is to be observed that the first term in the right member of these equations contains W as a factor and that terms involving second derivatives of V with respect to x^1 do not appear.

5. Let us write I (4.5) or III (2.1) in the form

$$\frac{\partial h_2^i}{\partial x^1} = \frac{\partial h_1^i}{\partial x^2} + \sum RU + \sum RV \quad (5.1a)$$

$$\left.\begin{array}{l}\dfrac{\partial h_3^i}{\partial x^1} = \dfrac{\partial h_1^i}{\partial x^3} + \sum RU + \sum RV\\[2ex] \dfrac{\partial h_3^i}{\partial x^2} = \dfrac{\partial h_2^i}{\partial x^3} + \sum RU + \sum RV\end{array}\right\} \quad (5.1b)$$

$$\left.\begin{array}{l}\dfrac{\partial h_4^i}{\partial x^1} = \dfrac{\partial h_1^i}{\partial x^4} + \sum RU + \sum RV\\[2ex] \dfrac{\partial h_4^i}{\partial x^2} = \dfrac{\partial h_2^i}{\partial x^4} + \sum RU + \sum RV\\[2ex] \dfrac{\partial h_4^i}{\partial x^3} = \dfrac{\partial h_3^i}{\partial x^4} + \sum RU + \sum RV\end{array}\right\} \quad (5.1c)$$

Other equations of this type likewise result from the condition $W = 0$ over the surface S_3; in fact we have

$$\sum_{\sigma=2}^{4} g^{1\sigma} h_\tau^1 \frac{\partial h_\sigma^\tau}{\partial x^\alpha} = 0 \ (\alpha = 2, 3, 4) \quad (5.2)$$

over S_3. Now $\alpha \neq 0$ at P since $\alpha = 0$ would give $\beta = \gamma = \delta = 0$ at P and we would have a contradiction with the normal conditions imposed in Sect. 1. Making use of this fact and also the fact that $W^* \neq 0$ at P equations (5.2) can be solved for the derivatives of the covariant component h_2^1 with respect to x^2, x^3 and x^4. Hence (5.2) can be given the form

$$\frac{\partial h_2^1}{\partial x^2} = \ldots; \frac{\partial h_2^1}{\partial x^3} = \ldots; \frac{\partial h_2^1}{\partial x^4} = \ldots, \quad (5.3)$$

where the dots denote terms of the sort occurring in the right members of (5.1) with the exception of the derivatives in the left members of (5.3). It is to be borne in mind that equations (5.3) hold only over the surface S_3 and must not, therefore, be differentiated with respect to the coördinate x^1 in the process of finding derivatives of the component h^1_2 which we shall later employ. This circumstance, however, causes no difficulty since such differentiations can be made on the equation (5.1a) which contains the derivative of h^1_2 with respect to x^1 in its left member.

6. Suppose that each component U is defined over the surfaces S_3 and $S_3{}^*$ as an analytic function of the surface coördinates. We represent this by writing

$$\left.\begin{aligned} U &= J(x^2, x^3, x^4) \quad \text{for } x^1 = 0 \\ U &= K(x^1, x^3, x^4) \quad \text{for } x^2 = 0 \end{aligned}\right\} \tag{6.1}$$

where it is to be understood that over the two-dimensional surface S defined by $x^1 = x^2 = 0$ the functions J and K are identical, i.e.,

$$J(0, x^3, x^4) = K(0, x^3, x^4).$$

Over the surface S_2 we shall ascribe the components V as indicated by the equation

$$V = L(x^3, x^4) \quad \text{for } x^1 = x^2 = 0. \tag{6.2}$$

Similarly the components h^i_α will be defined throughout the four-dimensional continuum, over the above surfaces S_3, S_2 and along the curve S_1 given by $x^1 = x^2 = x^3 = 0$ as shown by the following equations

$$\begin{array}{c|c} h^i_1 = P^i(x^1, x^2, x^3, x^4) & h^i_2 = Q^i(x^2, x^3, x^4) \\ {[i = 1, 2, 3, 4]} & \begin{bmatrix} i = 2, 3, 4 \\ x^1 = 0 \end{bmatrix} \\ \hline h^i_3 = R^i(x^3, x^4) & h^i_4 = S^i(x^4) \\ \begin{bmatrix} i = 1, 2, 3, 4 \\ x^1 = x^2 = 0 \end{bmatrix} & \begin{bmatrix} i = 1, 2, 3, 4 \\ x^1 = x^2 = x^3 = 0 \end{bmatrix} \end{array}. \tag{6.3}$$

A value $(h^1_2)_0$ will be assigned the component h^1_2 at the point P, which is now taken as the origin of the (x) coördinate system, such that the conditions $W = 0$ and $\beta \lessgtr 0$ are satisfied at P. It will, moreover, be assumed that the values of the h^i_α in (6.3) are such that the remainder of the normal conditions imposed in Sect. 1, namely, $W^* \gtrless 0$, $A \gtrless 0$ and $F \gtrless 0$ hold likewise at the point P; there is also the condition, which is a consequence of the underlying postulates of the space of distant parallelism, that the determinant $| h^\alpha_i | \gtrless 0$ at P.

The specification of the above data is sufficient to determine the power series expansions of the components h^i_α in the neighborhood of the point P. We observe, for example, that the quantities in the right members of (4.1) and (4.2) are known at P so that the left members of these equations are determined. Likewise the left members of (5.1) and (5.3) are determined at P since all quantities occurring in the right members of these equations are known from the assignment of the above data. In other words, all first derivatives of the quantities U, V and h^i_α are determined at the point P. To assist in the calculation of the higher derivatives of these quantities we lay down certain *rules of anteriority* which indicate the order in which these derivatives are to be determined. If $\mathfrak{A}$ and $\mathfrak{B}$ denote derivatives of U or V we shall say that $\mathfrak{A}$ is anterior to $\mathfrak{B}$ when one of the following conditions is satisfied:

(1) $\mathfrak{A}$ is of lower order than $\mathfrak{B}$;

(2) $\mathfrak{A}$ is of the same order as $\mathfrak{B}$ but involves fewer differentiations with respect to x^1;

(3) $\mathfrak{A}$ is of the same order as $\mathfrak{B}$, involves the same number of differentiations with respect to x^1, but fewer differentiations with respect to x^2;

(4) $\mathfrak{A}$ is of the same order as $\mathfrak{B}$, involves the same number of differentiations with respect to x^1 and the same number of differentiations with respect to x^2, but $\mathfrak{A}$ is a derivative of a quantity U whereas $\mathfrak{B}$ is a derivative of a quantity V.

The rules of anteriority for the derivatives of the components h^i_α can be made on the basis of the idea of the "cote" which we have previously used (footnote 4 to Note III). Let us assign "cotes" to the h^i_α and the coördinates x^α as follows:

$$\begin{array}{lll} x^\alpha & \text{has "cote"} & -\alpha \\ h^1_2 & \text{has "cote"} & 5 \\ h^i_\alpha & \text{has "cote"} & \alpha \\ \multicolumn{3}{l}{\quad (i \neq 1 \text{ if } \alpha = 2)} \end{array}$$

Then if $\mathfrak{C}$ and $\mathfrak{D}$ denote derivatives of components h^i_α we shall say that $\mathfrak{C}$ is anterior to $\mathfrak{D}$ if:

(1) $\mathfrak{C}$ is of lower order than $\mathfrak{D}$;

(2) $\mathfrak{C}$ is of the same order as $\mathfrak{D}$ but the "cote" of $\mathfrak{C}$ is algebraically less than that of $\mathfrak{D}$.[6]

Finally we shall say that $\mathfrak{A}$ is anterior to $\mathfrak{C}$ if $\mathfrak{A}$ is of lower order than $\mathfrak{C}$ and conversely that $\mathfrak{C}$ is anterior to $\mathfrak{A}$ if $\mathfrak{C}$ is of lower order than $\mathfrak{A}$.

Now differentiate one of the equations (4.1), (4.2) or (4.3) any number of times with respect to the coördinates x^α and evaluate at the point P. The resulting equation will then express the derivative in its left member as a sum of polynomials in anterior derivatives. In case we are dealing with an equation (4.2) the vanishing of the factor W at P prevents the occurrence in the right member of (4.2) of derivatives which are not anterior to the derivative in the left member of this equation; a similar remark applies to an equation (4.3). If we differentiate one of the equations (5.3) any number of times with respect to coördinates x^2, x^3, x^4 and evaluate at P the derivatives in the right member of the resulting equation will be anterior to the derivative in the left member. This will likewise be the case if we differentiate one of the equations (5.1) any number of times with respect to the coördinates x^α except in the case of an equation (5.1b) which involves in its right member a derivative of the component h^1_2; in this case, however, the derivative of the component h^1_2 resulting from differentiation of the equation (5.1b) can be eliminated by an obvious substitution so as to obtain an equation in which the right member contains only derivatives anterior to the derivative in its left member. From the above considerations it is obvious that any derivative of U, V or h^i_α can be determined at P when the values are known at P of all derivatives anterior to the derivative in question. It follows, therefore, that the power series expansions of the components h^i_α about the point P will be fully determined by the value $(h^1_2)_0$ of the component h^1_2 at P and the specification of data given by (6.1), (6.2) and (6.3) in accordance with the condition $|h^\alpha_i| \gtrless 0$ at P and the normal conditions in Sect. 1.

7. It was shown in Note II that the expression

$$16K(3, r + 1) + 8K(2, r + 1) \tag{7.1}$$

constituted a lower bound to the number of arbitrary derivatives of the quantities U and V of the $(r + 1)$st order; also the method explained in Sect. 4 of Note III enables us to say that there cannot be less than $4K(4, r + 2)$ derivatives of the h^i_α of order $(r + 1)$ to which arbitrary values can be assigned at the point P when the condition $W = 0$ over the surface S_3 is disregarded. When this latter condition is taken into consideration it follows, therefore, that the difference between $4K(4, r + 2)$ and $K(3, r + 1)$, or

$$4K(4, r + 1) + 3K(3, r + 1) + 4K(2, r + 1) + 4 \tag{7.2}$$

is a lower bound to the number of derivatives of the h^i_α of the $(r + 1)$st order to which arbitrary values can be assigned at the point P.

If we differentiate r times the equations (4.1), (4.2), (5.1), (5.3) and $r - 1$ times the equations (4.3), and then form the conditions of integrability of the resulting equations we obtain a system M involving the U, V and h^i_α and the derivatives of these quantities to the order $(r + 1)$ at most. Now the data specified by (6.3) shows that the number of arbitrary derivatives of h^i_α of the $(r + 1)$st order at P is at most equal to (7.2); the number of arbitrary derivatives of U and V of the $(r + 1)$st order is easily seen to be equal to (7.1) in consequence of (6.1) and (6.2). If, therefore, one of the equations of M is not satisfied identically and if, moreover, this equation involves at least one of the derivatives of the components U, V or h^i_α of order $(r + 1)$ we shall be led to a contradiction with the fact that (7.1) or (7.2) constitutes a lower bound to the number of these derivatives which are arbitrary at the point P; in case an equation of M does not involve a derivative of U, V, or h^i_α of order $(r + 1)$ and yet is not satisfied identically we shall have a similar contradiction with regard to the lower bound for derivatives of order less than $(r + 1)$. It follows, therefore, that all conditions of integrability involved in the determination of the power series expansions of the h^i_α must be satisfied and, hence, that these expansions are unique. As the power series expansions of the h^i_α can be shown to converge within a sufficiently small neighborhood at the point P we arrive at the following[7]

EXISTENCE THEOREM. *Let us specify the functions U over the surfaces S_3 and S^*_3 and also the functions V over the surface S_2 as arbitrary analytic functions of the surface coördinates represented by equations* (6.1) *and* (6.2), *respectively. Moreover, let us specify the value of the component h^1_2 at P, i.e., $x^\alpha = 0$, and also the values of the remaining components h^i_α as arbitrary analytic functions in accordance with* (6.3), *the above values being taken subject to the following conditions:*

$$W = 0, \; |h^\alpha_i| \gtrless 0, \; \beta \gtrless 0$$
$$W^* \gtrless 0, \; A \gtrless 0, \; F \gtrless 0$$

at the point P. Then there exists one, and only one, set of functions h^i_α given by convergent power series expansions about the point P, which constitutes a set of integrals of the field equations III (3.1) *and for which the surface S_3 is a characteristic.*

It should be observed that the normal conditions incorporated in the statement of the above existence theorem have been imposed merely to avoid the ambiguity which would otherwise occur in the selection of certain non-vanishing determinants formed from the coefficients of certain of the previous equations; these conditions in no way restrict the characteristic surface S_3.

8. Suppose that we confine our differentiations of the equations (4.1), (4.2), (4.3), (5.1) and (5.3) in the process of determining the power series expansions of the functions h^i_α about the point P to differentiations involving the indices x^2, x^3, x^4 alone; this is evidently possible from the rules of anteriority in Sect. 6. We then arrive at a determination of the quantities h^i_α and their first derivatives over the characteristic surface

S_3. Now if we consider the functions K in (6.1) which we can suppose to be of the form

$$\varphi_3(x^1)^3 + \varphi_4(x^1)^4 + \ldots$$

we see that this determination of the h^i_α and their first derivatives over S_3 is independent of the φ_k. Different selections of the functions $\varphi_k(x^3, x^4)$ will give different sets of integrals h^i_α of the field equations and we arrive at the fact that *there exists an infinite number of sets of integrals h^i_α of the field equations such that each set of integrals and their first derivatives assume the same values over a characteristic surface S_3.*

The above result can be extended in the following manner: Let us supplement the above process by allowing a single differentiation with respect to the coördinate x^1. Then the quantities h^i_α as well as their first and second derivatives will be determined over S_3 independently of the functions φ_4, φ_5, ... and there is an analogous result when we restrict ourselves to $p(>1)$ differentiations with respect to the coördinate x^1. Let us say that two sets of integrals h^i_α of the field equations have contact of order C over a characteristic surface S_3 if the two sets of functions h^i_α and all their derivatives to those of order C inclusive, but not derivatives of the $(C + 1)$st order, assume the same values over S_3. We can then say that there exists an infinite number of sets of integrals h^i_α of the field equations having contact of order $C(\geqq 1)$ with one another over a characteristic surface S_3. It is important to notice that if the surface S_3 were not a characteristic the functional data common to each of these sets of integrals would be sufficient to give a unique determination of a set of integrals h^i_α of the field equations.

[1] J. Hadamard, *Lecons sur la Propagation des Ondes*, Hermann (1903), pp. 296–310.

[2] See footnote 6 to Note III.

[3] Bôcher, *Introduction to Higher Algebra*, MacMillan, p. 54 (1929).

[4] Let Q denote any determinant of order five in M_1. Assuming $J \neq 0$, we have $Q = 0$ whenever $W = 0$ by the result in the text. Hence $JQ = 0$ for $W = 0$ without restricting J to non-vanishing values. Hence $JQ \equiv WT$. Since the polynomial W is irreducible it follows from the last equation that either J or Q must involve W as a factor. But J is equal to $\alpha^2\delta^2 + \beta^2\gamma^2$ so that W is not a factor of J; hence W is a factor of Q. While the above discussion is sufficiently general to admit the possibility of J having zero values, the vanishing of J at the point P would, as a matter of fact, be in contradiction to the normal conditions in Sect. 1.

[5] See footnote 3 to Note III.

[6] A "cote" of a derivative is the integer obtained by adding to the "cote" of the function which is differentiated, the "cotes" of all the variables of differentiation, distinct or not.

[7] The proof of convergence involved here will be given later in a comprehensive exposition of the present theory.