APPROACHES TO

Quantum Gravity

Toward a New Understanding of Space, Time and Matter

Edited by Daniele Oriti

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Toward a New Understanding of Space, Time and Matter

The theory of quantum gravity promises a revolutionary new understanding of gravity and spacetime, valid from microscopic to cosmological distances. Research in this field involves an exciting blend of rigorous mathematics and bold speculations, foundational questions and technical issues.

Containing contributions from leading researchers in this field, this book presents the fundamental issues involved in the construction of a quantum theory of gravity and building up a quantum picture of space and time. It introduces the most current approaches to this problem, and reviews their main achievements. Each part ends in questions and answers, in which the contributors explore the merits and problems of the various approaches. This book provides a complete overview of this field from the frontiers of theoretical physics research for graduate students and researchers.

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK
Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521860451

First published in print format 2009

ISBN-13 978-0-511-51640-5 eBook (EBL)

ISBN-13 978-0-521-86045-1 hardback

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Preface

Quantum Gravity is a dream, a theoretical need and a scientific goal. It is a theory which still does not exist in complete form, but that many people claim to have had glimpses of, and it is an area of research which, at present, comprises the collective efforts of hundreds of theoretical and mathematical physicists.

This yet-to-be-found theory promises to be a more comprehensive and complete description of the gravitational interaction, a description that goes beyond Einstein's General Relativity in being possibly valid at all scales of distances and energy; at the same time it promises to provide a new and deeper understanding of the nature of space, time and matter.

As such, research in Quantum Gravity is a curious and exciting blend of rigorous mathematics and bold speculations, concrete models and general schemata, foundational questions and technical issues, together with, since recently, tentative phenomenological scenarios.

In the past three decades we have witnessed an amazing growth of the field of Quantum Gravity, of the number of people actively working in it, and consequently of the results achieved. This is due to the fact that some approaches to the problem started succeeding in solving outstanding technical challenges, in suggesting ways around conceptual issues, and in providing new physical insights and scenarios. A clear example is the explosion of research in string theory, one of the main candidates to a quantum theory of gravity, and much more. Another is the development of Loop Quantum Gravity, an approach that attracted much attention recently, due to its successes in dealing with many long standing problems of the canonical approach to Quantum Gravity. New techniques have been then imported to the field from other areas of theoretical physics, e.g. Lattice Gauge Theory, and influenced in several ways the birth or growth of even more directions in Quantum Gravity research, including for example discrete approaches. At the same time, Quantum Gravity has been a very fertile ground and a powerful motivation for developing new mathematics as well as alternative ways of thinking about spacetime and matter, which in turn have triggered the exploration of other promising avenues toward a Quantum Gravity theory.

I think it is fair to say that we are still far from having constructed a satisfactory theory of Quantum Gravity, and that any single approach currently being considered is too incomplete or poorly understood, whatever its strengths and successes may be, to claim to have achieved its goal, or to have proven to be the only reasonable way to proceed.

On the other hand every single one of the various approaches being pursued has achieved important results and insights regarding the Quantum Gravity problem. Moreover, technical or conceptual issues that are unsolved in one approach have been successfully tackled in another, and often the successes of one approach have clearly come from looking at how similar difficulties had been solved in another.

It is even possible that, in order to achieve our common goal, formulate a complete theory of Quantum Gravity and unravel the fundamental nature of space and time, we will have to regard (at least some of) these approaches as different aspects of the same theory, or to develop a more complete and more general approach that combines the virtues of several of them. However strong faith one may have in any of these approaches, and however justified this may be in light of recent results, it should be expected, purely on historical grounds, that none of the approaches currently pursued will be understood in the future in the same way as we do now, even if it proves to be the right way to proceed. Therefore, it is useful to look for new ideas and a different perspective on each of them, aided by the the insights provided by the others. In no area of research a "dogmatic approach" is less productive, I feel, than in Quantum Gravity, where the fundamental and complex nature of the problem, its many facets and long history, combined with a dramatically (but hopefully temporarily) limited guidance from Nature, suggest a very open-minded attitude and a very critical and constant re-evaluation of one's own strategies.

I believe, therefore, that a broad and well-informed perspective on the various present approaches to Quantum Gravity is a necessary tool for advancing successfully in this area.

This collective volume, benefiting from the contributions of some of the best Quantum Gravity practitioners, all working at the frontiers of current research, is meant to represent a good starting point and an up-to-date support reference, for both students and active researchers in this fascinating field, for developing such a broader perspective. It presents an overview of some of the many ideas on the table, an introduction to several current approaches to the construction of a Quantum Theory of Gravity, and brief reviews of their main achievements, as well as of the many outstanding issues. It does so also with the aim of offering a comparative perspective on the subject, and on the different roads that Quantum Gravity researchers are following in their searches. The focus is on non-perturbative aspects of Quantum Gravity and on the fundamental structure of space and time. The variety of approaches presented is intended to ensure that a variety of ideas and mathematical techniques will be introduced to the reader.

More specifically, the first part of the book (Part I) introduces the problem of Quantum Gravity, and raises some of the fundamental questions that research in Quantum Gravity is trying to address. These concern for example the role of locality and of causality at the most fundamental level, the possibility of the notion of spacetime itself being emergent, the possible need to question and revise our way of understanding both General Relativity and Quantum Mechanics, before the two can be combined and made compatible in a future theory of Quantum Gravity. It provides as well suggestions for new directions (using the newly available tools of category theory, or quantum information theory, etc.) to explore both the construction of a quantum theory of gravity, as well as our very thinking about space and time and matter.

The core of the book (Parts II–IV) is devoted to a presentation of several approaches that are currently being pursued, have recently achieved important results, and represent promising directions. Among these the most developed and most practiced are string/M-theory, by far the one which involves at present the largest amount of scholars, and loop quantum gravity (including its covariant version, i.e. spin foam models). Alongside them, we have various (and rather different in both spirit and techniques used) discrete approaches, represented here by simplicial quantum gravity, in particular the recent direction of causal dynamical triangulations, quantum Regge calculus, and the "consistent discretization scheme", and by the causal set approach.

All these approaches are presented at an advanced but not over-technical level, so that the reader is offered an introduction to the basic ideas characterizing any given approach as well as an overview of the results it has already achieved and a perspective on its possible development. This overview will make manifest the variety of techniques and ideas currently being used in the field, ranging from continuum/analytic to discrete/combinatorial mathematical methods, from canonical to covariant formalisms, from the most conservative to the most radical conceptual settings.

The final part of the book (Part V) is devoted instead to effective models of Quantum Gravity. By this we mean models that are not intended to be of a fundamental nature, but are likely to provide on the one hand key insights on what sort of features the more fundamental formulation of the theory may possess, and on the other powerful tools for studying possible phenomenological consequences of any Quantum Gravity theory, the future hopefully complete version as well as the current tentative formulations of it. The subject of Quantum Gravity phenomenology is a new and extremely promising area of current research, and gives ground to the hope that in the near future Quantum Gravity research may receive experimental inputs that will complement and direct mathematical insights and constructions.

The aim is to convey to the reader the recent insight that a Quantum Gravity theory need not be forever detached by the experimental realm, and that many possibilities for a Quantum Gravity phenomenology are instead currently open to investigation.

At the end of each part, there is a "Questions & Answers" session. In each of them, the various contributors ask and put forward to each other questions, comments and criticisms to each other, which are relevant to the specific topic covered in that part. The purpose of these Q&A sessions is fourfold: (a) to clarify further subtle or particularly relevant features of the formalisms or perspectives presented; (b) to put to the forefront critical aspects of the various approaches, including potential difficulties or controversial issues; (c) to give the reader a glimpse of the real-life, ongoing debates among scholars working in Quantum Gravity, of their different perspectives and of (some of) their points of disagreement; (d) in a sense, to give a better picture of how science and research (in particular, Quantum Gravity research) really work and of what they really are.

Of course, just as the book as a whole cannot pretend to represent a complete account of what is currently going on in Quantum Gravity research, these Q&A sessions cannot really be a comprehensive list of relevant open issues nor a faithful portrait of the (sometimes rather heated) debate among Quantum Gravity researchers.

What this volume makes manifest is the above-mentioned impressive development that occurred in the field of Quantum Gravity as a whole, over the past, say, 20–30 years. This is quickly recognized, for example, by comparing the range and content of the following contributed papers to the content of similar collective volumes, like *Quantum Gravity 2: a second Oxford symposium*, C. Isham, ed., Oxford University Press (1982), *Quantum structure of space and time*, M. Duff, C. Isham, eds., Cambridge University Press (1982), *Quantum Theory of Gravity, essays in honor of the 60th Birthday of Bryce C DeWitt*, S. D. Christensen, ed., Taylor and Francis (1984), or even the more recent *Conceptual problems of Quantum Gravity*, A. Ashtekar, J. Stachel, eds., Birkhauser (1991), all presenting overviews of the status of the subject at their time. Together with the persistence of the Quantum Gravity problem itself, and of the great attention devoted, currently just as then, to foundational issues alongside the more technical ones, it will be impossible not to notice the greater variety of current approaches, the extent to which researchers have explored beyond the traditional ones, and, most important, the enormous amount of progress and achievements in each of them. Moreover, the very existence of research in Quantum Gravity phenomenology was un-imaginable at the time.

Quantum Gravity remains, as it was in that period, a rather esoteric subject, within the landscape of theoretical physics at large, but an active and fascinating one, and one of fundamental significance. The present volume is indeed a collective report from the frontiers of theoretical physics research, reporting on the latest and most exciting developments but also trying to convey to the reader the sense of intellectual adventure that working at such frontiers implies.

It is my pleasure to thank all those that have made the completion of this project possible. First of all, I gratefully thank all the researchers who have contributed to this volume, reporting on their work and on the work of their colleagues in such an excellent manner. This is a collective volume, and thus, if it has any value, it is solely due to all of them. Second, I am grateful to all the staff at the Cambridge University Press, and in particular to Simon Capelin, for supporting this project since its conception, and for guiding me through its development. Last, I would like to thank, for very useful comments, suggestions and advice, several colleagues and friends: John Baez, Fay Dowker, Sean Hartnoll, Chris Isham, Prem Kumar, Pietro Massignan, and especially Ted Jacobson.

Daniele Oriti

Part I

Fundamental ideas and general formalisms

1

Unfinished revolution

C. ROVELLI

One hundred and forty-four years elapsed between the publication of Copernicus's *De Revolutionibus*, which opened the great scientific revolution of the seventeenth century, and the publication of Newton's *Principia*, the final synthesis that brought that revolution to a spectacularly successful end. During those 144 years, the basic grammar for understanding the physical world changed and the old picture of reality was reshaped in depth.

At the beginning of the twentieth century, General Relativity (GR) and Quantum Mechanics (QM) once again began reshaping our basic understanding of space and time and, respectively, matter, energy and causality – arguably to a no lesser extent. But we have not been able to combine these new insights into a novel coherent synthesis, yet. The twentieth-century scientific revolution opened by GR and QM is therefore still wide open. We are in the middle of an unfinished scientific revolution. Quantum Gravity is the tentative name we give to the "synthesis to be found".

In fact, our present understanding of the physical world at the fundamental level is in a state of great confusion. The present knowledge of the elementary dynamical laws of physics is given by the application of QM to fields, namely Quantum Field Theory (QFT), by the particle-physics Standard Model (SM), and by GR. This set of fundamental theories has obtained an empirical success nearly unique in the history of science: so far there isn't any clear evidence of observed phenomena that clearly escape or contradict this set of theories – or a minor modification of the same, such as a neutrino mass or a cosmological constant.[1] But, the theories in this set are based on badly self-contradictory assumptions. In GR the gravitational field is assumed to be a classical deterministic dynamical field, identified with the (pseudo) Riemannian metric of spacetime: but with QM we have understood that all dynamical fields have quantum properties. The other way around, conventional

[1] Dark matter (not dark energy) might perhaps be contrary evidence.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

QFT relies heavily on global Poincaré invariance and on the existence of a non-dynamical background spacetime metric: but with GR we have understood that there is no such non-dynamical background spacetime metric in nature.

In spite of their empirical success, GR and QM offer a schizophrenic and confused understanding of the physical world. The conceptual foundations of classical GR are contradicted by QM and the conceptual foundation of conventional QFT are contradicted by GR. Fundamental physics is today in a peculiar phase of deep conceptual confusion.

Some deny that such a major internal contradiction in our picture of nature exists. On the one hand, some refuse to take QM seriously. They insist that QM makes no sense, after all, and therefore the fundamental world must be essentially classical. This doesn't put us in a better shape, as far as our understanding of the world is concerned.

Others, on the other hand, and in particular some hard-core particle physicists, do not accept the lesson of GR. They read GR as a field theory that can be consistently formulated in full on a fixed metric background, and treated within conventional QFT methods. They motivate this refusal by insisting than GR's insight should not be taken too seriously, because GR is just a low-energy limit of a more fundamental theory. In doing so, they confuse the details of the Einstein's equations (which might well be modified at high energy), with the new understanding of space and time brought by GR. This is coded in the background independence of the fundamental theory and expresses Einstein's discovery that spacetime is not a fixed background, as was assumed in special relativistic physics, but rather a dynamical field.

Nowadays this fact is finally being recognized even by those who have long refused to admit that GR forces a revolution in the way to think about space and time, such as some of the leading voices in string theory. In a recent interview [1], for instance, Nobel laureate David Gross says: " [...] this revolution will likely change the way we think about space and time, maybe even eliminate them completely as a basis for our description of reality". This is of course something that has been known since the 1930s [2] by anybody who has taken seriously the problem of the implications of GR and QM. The problem of the conceptual novelty of GR, which the string approach has tried to throw out of the door, comes back by the window.

These and others remind me of Tycho Brahe, who tried hard to conciliate Copernicus's advances with the "irrefutable evidence" that the Earth is immovable at the center of the universe. To let the background spacetime go is perhaps as difficult as letting go the unmovable background Earth. The world may not be the way it appears in the tiny garden of our daily experience.

Today, many scientists do not hesitate to take seriously speculations such as extra dimensions, new symmetries or multiple universes, for which there isn't a wit of empirical evidence; but refuse to take seriously the conceptual implications of the physics of the twentieth century with the enormous body of empirical evidence supporting them. Extra dimensions, new symmetries, multiple universes and the like, still make perfectly sense in a pre-GR, pre-QM, Newtonian world, while to take GR and QM seriously *together* requires a genuine reshaping of our world view.

After a century of empirical successes that have equals only in Newton's and Maxwell's theories, it is time to take seriously GR and QM, with their full conceptual implications; to find a way of thinking the world in which what we have learned with QM and what we have learned with GR make sense together – finally bringing the twentieth-century scientific revolution to its end. This is the problem of Quantum Gravity.

1.1 Quantum spacetime

Roughly speaking, we learn from GR that spacetime is a dynamical field and we learn from QM that all dynamical field are quantized. A quantum field has a granular structure, and a probabilistic dynamics, that allows quantum superposition of different states. Therefore at small scales we might expect a "quantum spacetime" formed by "quanta of space" evolving probabilistically, and allowing "quantum superposition of spaces". The problem of Quantum Gravity is to give a precise mathematical and physical meaning to this vague notion of "quantum spacetime".

Some general indications about the nature of quantum spacetime, and on the problems this notion raises, can be obtained from elementary considerations. The size of quantum mechanical effects is determined by Planck's constant $\hbar$. The strength of the gravitational force is determined by Newton's constant G, and the relativistic domain is determined by the speed of light c. By combining these three fundamental constants we obtain the Planck length $l_{\mathrm{P}} = \sqrt{\hbar G/c^3} \sim 10^{-33}$ cm. Quantum-gravitational effects are likely to be negligible at distances much larger than l_{P}, because at these scales we can neglect quantities of the order of G, $\hbar$ or $1/c$.

Therefore we expect the classical GR description of spacetime as a pseudo-Riemannian space to hold at scales larger than l_{P}, but to break down approaching this scale, where the full structure of quantum spacetime becomes relevant. Quantum Gravity is therefore the study of the structure of spacetime at the Planck scale.

1.1.1 Space

Many simple arguments indicate that l_{P} may play the role of a *minimal* length, in the same sense in which c is the maximal velocity and $\hbar$ the minimal exchanged action.

For instance, the Heisenberg principle requires that the position of an object of mass m can be determined only with uncertainty x satisfying $mvx > \hbar$, where v is the uncertainty in the velocity; special relativity requires $v < c$; and according to GR there is a limit to the amount of mass we can concentrate in a region of size x, given by $x > Gm/c^2$, after which the region itself collapses into a black hole, subtracting itself from our observation. Combining these inequalities we obtain $x > l_P$. That is, gravity, relativity and quantum theory, taken together, appear to prevent position from being determined more precisely than the Planck scale.

A number of considerations of this kind have suggested that space might not be infinitely divisible. It may have a quantum granularity at the Planck scale, analogous to the granularity of the energy in a quantum oscillator. This granularity of space is fully realized in certain Quantum Gravity theories, such as loop Quantum Gravity, and there are hints of it also in string theory. Since this is a quantum granularity, it escapes the traditional objections to the atomic nature of space.

1.1.2 Time

Time is affected even more radically by the quantization of gravity. In conventional QM, time is treated as an external parameter and transition probabilities change in time. In GR there is no external time parameter. Coordinate time is a gauge variable which is not observable, and the physical variable measured by a clock is a nontrivial function of the gravitational field. Fundamental equations of Quantum Gravity might therefore not be written as evolution equations in an observable time variable. And in fact, in the quantum-gravity equation *par excellence*, the Wheeler–deWitt equation, there is no time variable t at all.

Much has been written on the fact that the equations of nonperturbative Quantum Gravity do not contain the time variable t. This presentation of the “problem of time in Quantum Gravity”, however, is a bit misleading, since it mixes a problem of classical GR with a specific Quantum Gravity issue. Indeed, *classical* GR as well can be entirely formulated in the Hamilton–Jacobi formalism, where no time variable appears either.

In classical GR, indeed, the notion of time differs strongly from the one used in the special-relativistic context. Before special relativity, one assumed that there is a universal physical variable t, measured by clocks, such that all physical phenomena can be described in terms of evolution equations in the independent variable t. In *special* relativity, this notion of time is weakened. Clocks do not measure a universal time variable, but only the proper time elapsed along inertial trajectories. If we fix a Lorentz frame, nevertheless, we can still describe all physical phenomena in terms of evolution equations in the independent variable x^0, even though this description hides the covariance of the system.

In *general* relativity, when we describe the dynamics of the gravitational field (not to be confused with the dynamics of matter in a given gravitational field), there is *no* external time variable that can play the role of observable independent evolution variable. The field equations are written in terms of an evolution parameter, which is the time coordinate x^0; but this coordinate does not correspond to anything directly observable. The proper time τ along spacetime trajectories cannot be used as an independent variable either, as τ is a complicated non-local function of the gravitational field itself. Therefore, properly speaking, GR does not admit a description as a system evolving in terms of an observable time variable. This does not mean that GR lacks predictivity. Simply put, what GR predicts are relations between (partial) observables, which in general cannot be represented as the evolution of dependent variables on a preferred independent time variable.

This weakening of the notion of time in classical GR is rarely emphasized: after all, in classical GR we may disregard the full dynamical structure of the theory and consider only individual solutions of its equations of motion. A single solution of the GR equations of motion determines "a spacetime", where a notion of proper time is associated to each timelike worldline.

But in the quantum context a single solution of the dynamical equation is like a single "trajectory" of a quantum particle: in quantum theory there are no physical individual trajectories: there are only transition probabilities between observable eigenvalues. Therefore in Quantum Gravity it is likely to be impossible to describe the world in terms of a spacetime, in the same sense in which the motion of a quantum electron cannot be described in terms of a single trajectory.

To make sense of the world at the Planck scale, and to find a consistent conceptual framework for GR and QM, we might have to give up the notion of time altogether, and learn ways to describe the world in atemporal terms. Time might be a useful concept only within an approximate description of the physical reality.

1.1.3 Conceptual issues

The key difficulty of Quantum Gravity may therefore be to find a way to understand the physical world in the absence of the familiar stage of space and time. What might be needed is to free ourselves from the prejudices associated with the habit of thinking of the world as "inhabiting space" and "evolving in time".

Technically, this means that the quantum states of the gravitational field cannot be interpreted like the n-particle states of conventional QFT as living on a given spacetime. Rather, these quantum states must themselves determine and define a spacetime – in the manner in which the classical solutions of GR do.

Conceptually, the key question is whether or not it is logically possible to understand the world in the absence of fundamental notions of time and time evolution, and whether or not this is consistent with our experience of the world.

The difficulties of Quantum Gravity are indeed largely conceptual. Progress in Quantum Gravity cannot be just technical. The search for a quantum theory of gravity raises once more old questions such as: What is space? What is time? What is the meaning of “moving”? Is motion to be defined with respect to objects or with respect to space? And also: What is causality? What is the role of the observer in physics? Questions of this kind have played a central role in periods of major advances in physics. For instance, they played a central role for Einstein, Heisenberg, and Bohr; but also for Descartes, Galileo, Newton and their contemporaries, as well as for Faraday and Maxwell.

Today some physicists view this manner of posing problems as “too philosophical”. Many physicists of the second half of the twentieth century, indeed, have viewed questions of this nature as irrelevant. This view was appropriate for the problems they were facing. When the basics are clear and the issue is problem-solving within a given conceptual scheme, there is no reason to worry about foundations: a pragmatic approach is the most effective one. Today the kind of difficulties that fundamental physics faces has changed. To understand quantum spacetime, physics has to return, once more, to those foundational questions.

1.2 Where are we?

Research in Quantum Gravity developed slowly for several decades during the twentieth century, because GR had little impact on the rest of physics and the interest of many theoreticians was concentrated on the development of quantum theory and particle physics. In the past 20 years, the explosion of empirical confirmations and concrete astrophysical, cosmological and even technological applications of GR on the one hand, and the satisfactory solution of most of the particle physics puzzles in the context of the SM on the other, have led to a strong concentration of interest in Quantum Gravity, and the progress has become rapid. Quantum Gravity is viewed today by many as the big open challenge in fundamental physics.

Still, after 70 years of research in Quantum Gravity, there is no consensus, and no established theory. I think it is fair to say that there isn’t even a single complete and consistent *candidate* for a quantum theory of gravity.

In the course of 70 years, numerous ideas have been explored, fashions have come and gone, the discovery of the Holy Grail of Quantum Gravity has been several times announced, only to be later greeted with much scorn. Of the tentative theories studied today (strings, loops and spinfoams, non-commutative geometry, dynamical triangulations or others), each is to a large extent incomplete and none has yet received a whit of direct or indirect empirical support.

However, research in Quantum Gravity has not been meandering meaninglessly. On the contrary, a consistent logic has guided the development of the research, from the early formulation of the problem and of the major research programs in the 1950s to the present. The implementation of these programs has been laborious, but has been achieved. Difficulties have appeared, and solutions have been proposed, which, after much difficulty, have lead to the realization, at least partial, of the initial hopes.

It was suggested in the early 1970s that GR could perhaps be seen as the low energy limit of a Poincaré invariant QFT without uncontrollable divergences [3]; and today, 30 years later, a theory likely to have these properties – perturbative string theory – is known. It was also suggested in the early 1970s that non-renormalizability might not be fatal for quantum GR [4; 5] and that the Planck scale could cut divergences off nonperturbatively by inducing a quantum discrete structure of space; and today we know that this is in fact the case – ultraviolet finiteness is realized precisely in this manner in canonical loop Quantum Gravity and in some spinfoam models. In 1957 Charles Misner indicated that in the canonical framework one should be able to compute quantum eigenvalues of geometrical quantities [6]; and in 1995, 37 years later, eigenvalues of area and volume were computed – within loop quantum gravity [7; 8]. Much remains to be understood and some of the current developments might lead nowhere. But looking at the entire development of the subject, it is difficult to deny that there has been substantial progress.

In fact, at least two major research programs can today claim to have, if not a complete candidate theory of Quantum Gravity, at least a large piece of it: string theory (in its perturbative and still incomplete nonperturbative versions) and loop quantum gravity (in its canonical as well as covariant – spinfoam – versions) are both incomplete theories, full of defects – in general, strongly emphasized within the opposite camp – and without any empirical support, but they are both remarkably rich and coherent theoretical frameworks, that *might* not be far from the solution of the puzzle.

Within these frameworks, classical and long intractable, physical, astrophysical and cosmological Quantum Gravity problems can finally be concretely treated. Among these: black hole's entropy and fate, the physics of the big-bang singularity and the way it has affected the currently observable universe, and many others. Tentative predictions are being developed, and the attention to the concrete possibility of testing these predictions with observations that could probe the Planck scale is very alive. All this was unthinkable only a few years ago.

The two approaches differ profoundly in their hypotheses, achievements, specific results, and in the conceptual frame they propose. The issues they raise concern the foundations of the physical picture of the world, and the debate between the two approaches involves conceptual, methodological and philosophical issues.

In addition, a number of other ideas, possibly alternative, possibly complementary to the two best developed theories and to one another, are being explored. These include noncommutative geometry, dynamical triangulations, effective theories, causal sets and many others.

The possibility that *none* of the currently explored hypotheses will eventually turn out to be viable, or, simply, none will turn out to be the way chosen by Nature, is very concrete, and should be clearly kept in mind. But the rapid and multi-front progress of the past few years raises hopes. Major well-posed open questions in theoretical physics (Copernicus or Ptolemy? Galileo's parabolas or Kepler's ellipses? How to describe electricity and magnetism? Does Maxwell theory pick a preferred reference frame? How to do the Quantum Mechanics of interacting fields...?) have rarely been solved in a few years. But they have rarely resisted more than a few decades. Quantum Gravity – the problem of describing the quantum properties of spacetime – is one of these major problems, and it is reasonably well defined: is there a coherent theoretical framework consistent with quantum theory and with General Relativity? It is a problem which is on the table since the 1930s, but it is only in the past couple of decades that the efforts of the theoretical physics community have concentrated on it.

Maybe the solution is not far away. In any case, we are not at the end of the road of physics, we are half-way through the woods along a major scientific revolution.

Bibliographical note

For details on the history of Quantum Gravity see the historical appendix in [9]; and, for early history see [10; 11] and [12; 13]. For orientation on current research on Quantum Gravity, see the review papers [14; 15; 16; 17]. As a general introduction to Quantum Gravity ideas, see the old classic reviews, which are rich in ideas and present different points of view, such as John Wheeler 1967 [18], Steven Weinberg 1979 [5], Stephen Hawking 1979 and 1980 [19; 20], Karel Kuchar 1980 [21], and Chris Isham's magisterial syntheses [22; 23; 24]. On string theory, classic textbooks are Green, Schwarz and Witten, and Polchinksi [25; 26]. On loop quantum gravity, including the spinfoam formalism, see [9; 27; 28], or the older papers [29; 30]. On spinfoams see also [31]. On noncommutative geometry see [32] and on dynamical triangulations see [33]. For a discussion of the difficulties of string theory and a comparison of the results of strings and loops, see [34], written in the form of a dialogue, and [35]. On the more philosophical challenges raised by Quantum Gravity, see [36]. Smolin's popular book [37] provides a readable introduction to Quantum Gravity. The expression "half way through the woods" to characterize the present state of fundamental theoretical physics is taken from [38; 39]. My own view on Quantum Gravity is developed in detail in [9].

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2

The fundamental nature of space and time

G. 'T HOOFT

2.1 Quantum Gravity as a non-renormalizable gauge theory

Quantum Gravity is usually thought of as a theory, under construction, where the postulates of quantum mechanics are to be reconciled with those of general relativity, without allowing for any compromise in either of the two. As will be argued in this contribution, this 'conservative' approach may lead to unwelcome compromises concerning locality and even causality, while more delicate and logically more appealing schemes can be imagined.

The conservative procedure, however, must first be examined closely. The first attempt (both historically and logically the first one) is to formulate the theory of 'Quantum Gravity' perturbatively [1; 2; 3; 4; 5], as has been familiar practice in the quantum field theories for the fundamental particles, namely the Standard Model. In perturbative Quantum Gravity, one takes the Einstein–Hilbert action,

$$S = \int \partial^4 x \sqrt{-g}\Big(\frac{R(x)}{\epsilon} + \mathcal{L}^{\text{matter}}(x)\Big) \qquad \epsilon = 16\pi G \tag{2.1}$$

considers the metric to be close to some background value: $g_{\mu\nu} = g^{\text{Bg}}_{\mu\nu} + \sqrt{\epsilon}\, h_{\mu\nu}$, and expands everything in powers of ϵ, or equivalently, Newton's constant G.

Invariance under local coordinate transformations then manifests itself as a local gauge symmetry: $h_{\mu\nu}\rangle h_{\mu\nu} + D_\mu u_\nu + D_\nu u_\mu$, where D_μ is the usual covariant derivative, and $u_\mu(x)$ generates an infinitesimal coordinate transformation. Here one can use the elaborate machinery that has been developed for the Yang–Mills theories of the fundamental particles. After imposing an appropriate gauge choice, all desired amplitudes can be characterized in terms of Feynman diagrams. Usually, these contain contributions of 'ghosts', which are gauge dependent degrees of freedom that propagate according to well-established rules. At first sight, therefore, Quantum Gravity does not look altogether different from a Yang–Mills theory. It appears that at least the difficulties of reconciling quantum mechanics with general coordinate invariance have been dealt with. We understand exactly how the problem

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

of time, of Cauchy surfaces, and of picking physical degrees of freedom, are to be handled in such a formalism. Indeed, unitarity is guaranteed in this formalism, and, in contrast to 'more advanced' schemes for quantizing gravity, the perturbative approach can deal adequately with problems such as: what is the complete Hilbert space of physical states?, how can the fluctuations of the light cone be squared with causality?, etc., simply because at all finite orders in perturbation expansion, such serious problems do not show up. Indeed, this is somewhat surprising, because the theory produces useful amplitudes *at all orders of the perturbation parameter* ϵ.

Yet there is a huge difference with the Standard Model. This 'quantum gauge theory of gravity' is not renormalizable. We must distinguish the *technical* difficulty from the *physical* one. Technically, the 'disaster' of having a non-renormalizable theory is not so worrisome. In computing the $\mathcal{O}(\epsilon^n)$ corrections to some amplitude, one has to establish $\mathcal{O}(\epsilon^n)$ correction terms to the Lagrangian, which are typically of the form $\sqrt{-g}\, R^{n+1}$, where $n+1$ factors linear in the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$ may have been contracted in various possible ways. These terms are necessary to cancel out infinite counter terms of this form, where finite parts are left over. At high orders n, there exist many different expressions of the form R^{n+1}, which will all be needed. This is often presented as a problem, but, in principle, it is not. It simply means that our theory has an infinite sequence of free parameters, not unlike many other theories in science, and it nevertheless gives accurate and useful predictions up to arbitrarily high powers of $G\,E^2$, where E is the energy scale considered. We emphasize that this is actually much better than many of the alternative approaches to Quantum Gravity such as Loop Quantum Gravity, and even string theory presents us with formidable problems when 3-loop amplitudes are asked for. Also, claims [6] that Quantum Gravity effects might cause 'decoherence' at some finite order of $G\,E^2$ are invalid according to this theory.

Physically, however, the perturbative approach fails. The difficulty is *not* the fact that the finite parts of the counter terms can be freely chosen. The difficulty is a combination of two features: (i) perturbation expansion does not converge, and (ii) the expansion parameter becomes large if centre-of-mass energies reach beyond the Planck value. The latter situation is very reminiscent of the old weak interaction theory where a quartic interaction was assumed among the fermionic fields. This Fermi theory was also 'non-renormalizable'.

In the Fermi theory, this problem was solved: the theory was replaced by a Yang–Mills theory with Brout–Englert–Higgs mechanism. This was not just 'a way to deal with the infinities', it was actually an answer to an absolutely crucial question [7]: *what happens at small distance scales?*. At small distance scales, we do not have quartic interactions among fermionic fields, we have a local gauge theory instead. This is actually also the superior way to phrase the problem of

Quantum Gravity: *what happens at, or beyond, the Planck scale?* Superstring theory [8; 9] is amazingly evasive if it comes to considering this question. It is here that Loop Quantum Gravity [10; 11; 12; 13; 14; 15] appears to be the most direct approach. It is an attempt to characterize the local degrees of freedom, but is it good enough?

2.2 A prototype: gravitating point particles in $2 + 1$ dimensions

An instructive exercise is to consider gravity in less than four space-time dimensions. Indeed, removing *two* dimensions allows one to formulate renormalizable models with local diffeomorphism invariance. Models of this sort, having one space- and one time dimension, are at the core of (super)string theory, where they describe the string world sheet. In such a model, however, there is no large distance limit with conventional 'gravity', so it does not give us hints on how to cure non-renormalizable long-distance features by modifying its small distance characteristics. There is also another reason why these two-dimensional models are uncharacteristic for conventional gravity: formally, pure gravity in $d = 2$ dimensions has $\frac{1}{2}d(d-3) = -1$ physical degrees of freedom, which means that an additional scalar field is needed to turn the theory into a topological theory. Conformal symmetry removes one further degree of freedom, so that, if string theory starts with D target space variables, or 'fields', $X^\mu(\sigma, \mathrm{Tr})$, where $\mu = 1, \cdots, D$, only $D - 2$ physical fields remain.

For the present discussion it is therefore more useful to remove just one dimension. Start with gravitationally interacting point particles in two space dimensions and one time. The classical theory is exactly solvable, and this makes it very interesting. Gravity itself, having zero physical degrees of freedom, is just topological; there are no gravitons, so the physical degrees of freedom are just the gravitating point particles. In the large distance limit, where Quantum Mechanical effects may be ignored, the particles are just point defects surrounded by locally flat space-time. The dynamics of these point defects has been studied [16; 17; 18], and the evolution laws during finite time intervals are completely understood. During very long time intervals, however, chaotic behavior sets in, and also, establishing a *complete list* of all distinguishable physical states turns out to be a problem. One might have thought that quantizing a classically solvable model is straightforward, but it is far from that, exactly because of the completeness problem. $2+1$ gravity *without* point particles could be quantized [19; 20], but that is a topological theory, with no local degrees of freedom; all that is being quantized are the boundary conditions, whatever that means.

One would like to represent the (non-rotating) point particles by some scalar field theory, but the problems one then encounters appear to be formidable. Quite generally, in $2+1$ dimensions, the curvature of 2-space is described by defect angles when following closed curves (holonomies). The total defect angle accumulated by a given closed curve always equals the total matter-energy enclosed by the curve. In the classical model, all of this is crystal clear. But what happens when one attempts to 'quantize' it? The matter Hamiltonian density does not commute with any of the particle degrees of freedom, since the latter evolve as a function of time. Thus, anything that moves, is moving in a space-time whose curvature is non-commuting. This is an impediment against a proper formulation of the Hilbert space in question in the conventional manner. *Only eigenstates of the Hamiltonian and the Hamiltonian density can live in a 2-space with precisely defined 2-metric*. Consequently, if we wish to describe physical states in a 2-space with precisely defined metric, these states must be smeared over a period of time that is large compared to the Planck time. We repeat: in a perturbative setting this situation can be handled because the deviations from flat space-time are small, but in a non-perturbative case, we have to worry about the limits of the curvature. The deficit angles cannot exceed the value 2π, and this implies that the Hamilton density must be bounded.

There is, however, an unconventional quantization procedure that seems to be quite appropriate here. We just noted that the Hamiltonian of this theory is unmistakably an angle, and this implies that *time*, its conjugate variable, must become discrete as soon as we quantize. Having finite time jumps clearly indicates in what direction we should search for a satisfactory quantum model: Schrödinger's equation will be a finite difference equation in the time direction. Take that as a modified picture for the small-distance structure of the theory!

How much more complicated will the small-distance structure be in our $3+1$ dimensional world? Here, the Hamiltonian is not limited to be an angle, so, time will surely be continuous. However, if we restrict ourselves to a region where one or more spatial dimensions are taken to be confined, or compactified, taking values smaller than some scale L in Planck units, then it is easy to see that we are back in the $2+1$ dimensional case, the Hamiltonian is again an angle, and time will be quantized. However, the $2+1$ dimensional Newton's constant will scale like $1/L$, and the time quantum will therefore be of order $1/L$ in Planck units. This suggests the following. In finite slabs of 3-space, time is quantized, the states are 'updated' in discretized time steps. If we stitch two equal sized slabs together, producing a slab twice as thick, then updating happens twice as fast, which we interpret as if updating happens alternatingly in one slab and in the other. The total time quantum has decreased by a factor two, but within each slab, time is still quantized in the original units. The picture we get this way is amazingly reminiscent of a computer model, where the computer splits 3-space into slabs of one Planck length thick, and during one Planck time interval every slab is being updated; a stack of N slabs thus requires N updates per Planck unit of time.

2.3 Black holes, causality and locality

The $2+1$ dimensional theory does not allow for the presence of black holes (assuming a vanishing cosmological constant, as we will do throughout). The black hole problem, there, is simply replaced by the restriction that the energy must stay less than the Planck value. In our slab-stack theory (for want of a better name), we see that the energy in every slab is restricted to be less than the Planck value, so any system where one of the linear dimensions is less than L, should have energy less than L in Planck units, and this amounts to having a limit for the total energy that is such that a black hole corresponds to the maximally allowed energy in a given region.

Clearly, black holes will be an essential element in any Quantum Gravity theory. We must understand how to deal with the requirement that the situation obtained after some gravitational collapse can be either described as some superdense blob of mass and energy, or as a geometric region of space-time itself where ingoing observers should be allowed to apply conventional laws of physics to describe what they see.

One can go a long way to deduce the consequences of this requirement. Particles going into a black hole will interact with all particles going out. Of all these interactions, the gravitational one happens to play a most crucial role. Only by taking this interaction into account [21], can one understand how black holes can play the role of resonances in a unitary scattering process where ingoing particles form black holes and outgoing particles are the ones generated by the Hawking process.

Yet how to understand the statistical origin of the Hawking–Bekenstein entropy of a black hole in this general framework is still somewhat mysterious. Even if black hole entropy can be understood in superstring theories for black holes that are near extremality, a deep mystery concerning locality and causality for the evolution laws of Nature's degrees of freedom remains. Holography tells us that the quantum states can be enumerated by aligning them along a planar surface. The slab-stack theory tells us how often these degrees of freedom are updated per unit of time. How do we combine all this in one comprehensive theory, and how can we reconcile this very exotic numerology with causality and locality? May we simply abandon attempts to rescue any form of locality in the $3+1$ dimensional bulk theory, replacing it by locality on the dual system, as is done in the AdS/CFT approach [22; 23] of M-theory?

2.4 The only logical way out: deterministic quantum mechanics

It is this author's opinion that the abstract and indirect formalisms provided by M-theory approaches are unsatisfactory. In particle physics, the Standard Model was superior to the old Fermi theory just because it provided detailed understanding of the small-distance structure. The small-distance structure of the $3+1$ dimensional theory is what we wish to understand. The holographic picture suggests discreteness in space, and the slab-stack theory suggests discreteness in time. Together, they suggest that the ultimate laws of Nature are akin to a cellular automaton [24].

However, our numerology admits far fewer physical states than one (discrete) degree of freedom per unit of bulk volume element. We could start with one degree of freedom for every unit volume element, but then a huge local symmetry constraint would be needed to reduce this to *physical* degrees of freedom which can be limited to the surface. This situation reminds us of *topological* gauge theories. How will we ever be able to impose such strong symmetry principles on a world that is as non-trivial as our real universe? How can we accommodate for the fact that the vast majority of the 'bulk states' of a theory should be made unphysical, like local gauge degrees of freedom?

Let us return to the $2+1$ dimensional case. Suppose that we would try to set up a functional integral expression for the quantum amplitudes. What are the degrees of freedom inside the functional integrand? One would expect these to be the defects in a space-time that is flat everywhere except in the defects. A defect is then characterized by the element of the Poincaré group associated with a closed loop around the defect, the holonomy of the defect. Now this would force the defect to follow a straight path in space-time. It is not, as in the usual functional integral, an arbitrary function of time, but, even inside the functional integral, it is limited to straight paths only. Now this brings us back from the quantum theory to a deterministic theory; only deterministic paths appear to be allowed. It is here that this author thinks we should search for the clue towards the solution to the aforementioned problems.

The topic that we dubbed 'deterministic quantum mechanics' [25; 26] is *not* a modification of standard quantum mechanics, but must be regarded as a special case. A short summary, to be explained in more detail below, is that our conventional Hilbert space is part of a bigger Hilbert space; conventional Hilbert space is obtained from the larger space by the action of some projection operator. The states that are projected out are the ones we call 'unphysical', to be compared with the ghosts in local gauge theories, or the bulk states as opposed to the surface states in a holographic formulation. In the bigger Hilbert space, a basis can be found such that basis elements evolve into basis elements, without any quantum mechanical superposition ever taking place.

One of the simplest examples where one can demonstrate this idea is the harmonic oscillator, consisting of states $|n\rangle,\ n=0,1,\ldots,$ and

$$H|n\rangle = \left(n+\frac{1}{2}\right)|n\rangle. \tag{2.2}$$

If we add to this Hilbert space the states $|n\rangle$ with $n = -1, -2, \ldots$, on which the Hamiltonian acts just as in Eq. (2.2), then our ontological basis consists of the states

$$|\varphi\rangle = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} e^{-in\varphi}|n\rangle, \tag{2.3}$$

which evolve as

$$|\varphi\rangle \underset{t=T}{\longrightarrow} |\varphi + T\rangle, \tag{2.4}$$

provided that $(2N+1)T/2\pi$ is an integer. In the limit $N \to \infty$, time T can be taken to be continuous. In this sense, a quantum harmonic oscillator can be turned into a deterministic system, since, in Eq. (2.4), the wave function does not spread out, and there is no interference. A functional integral expression for this evolution would only require a single path, much as in the case of the $2+1$-dimensional defects as described above. Since φ is periodic, the evolution (2.4) describes a periodic motion with period $T = 2\pi$. Indeed, every periodic deterministic system can be mapped onto the quantum harmonic oscillator provided that we project out the elements of Hilbert space that have negative energy.

In general, any deterministic system evolves according to a law of the form

$$\frac{\partial}{\partial t} q^a(t) = f^a\left(\vec{q}(t)\right) \tag{2.5}$$

(provided that time is taken to be continuous), and in its larger Hilbert space, the Hamiltonian is

$$H = \sum_a f^a p_a, \qquad p_a \overset{\text{def}}{=} -i\frac{\partial}{\partial q^a}, \tag{2.6}$$

where, in spite of the classical nature of the physical system, we defined p_a as *quantum* operators. In this large Hilbert space, one always sees as many negative as positive eigenstates of H, so it will always be necessary to project out states. A very fundamental difficulty is now how to construct a theory where not only the negative energy states can be projected out, but where also the entire system can be seen as a conglomeration of weakly interacting parts (one may either think of neighbouring sectors of the universe, or of weakly interacting particles), such that also in these parts only the positive energy sectors matter. The entire Hamiltonian is conserved, but the Hamilton densities, or the partial Hamiltonians, are not, and interacting parts could easily mix positive energy states with negative energy states. Deterministic quantum mechanics will only be useful if systems can be found where all states in which parts occur with negative energy, can also be projected out. The subset of Hilbert space where all bits and pieces only carry positive energy is only a very tiny section of the entire Hilbert space, and we will have to demonstrate that a theory exists where this sector evolves all by itself, even in the presence of non-trivial interactions.

What kind of mechanism can it be that greatly reduces the set of physical states? It is here that our self-imposed restriction to have strictly deterministic Hamilton equations may now bear fruit. In a deterministic system, we may have *information loss*. In a quantum world, reducing the dimensionality of Hilbert space would lead to loss of unitarity, but in a deterministic world there is no logical impediment that forbids the possibility that two different initial states may both evolve into the same final state.

This gives us a new view on what was once introduced as the 'holographic principle'. According to this principle, the number of independent physical variables in a given volume actually scales with the surface area rather than the volume. This may mean that, in every volume element, information concerning the interior dissipates away due to information loss, while only the information located on the surface survives, possibly because it stays in contact with the outside world.

Information loss forces us to assemble physical states in 'equivalence classes'. Two states are in the same equivalence classes if, in due time, they eventually evolve into the same final state. Equivalence classes may play the role of gauge equivalence classes, and thus we might arrive at a plausible scenario in which the degrees of freedom inside the bulk of some region are reduced to being gauge degrees of freedom, while the physical degrees of freedom are limited to reside on the surface.

Note that, if such a theory can be constructed, the 'primordial' laws of physics may be completely local and causal, but the physical states that figure in the evolution equation (2.5) appear to have a non-local definition. This may be the reason why more direct attempts to interpret quantum mechanical phenomena in terms of realistic theories tend to lead to a mysterious, invisible kind of non-locality, as laid down in the well-known Einstein–Podolsky–Rosen paradox.

2.5 Information loss and projection

How could information loss act as a mechanism to select out only those states where all energies are non-negative? How exactly this works is not understood; however, we do have an instructive but admittedly vague argument, and it is the following. Consider several regions or systems in our universe that are only weakly interacting with one another. With the interaction switched off, they all obey deterministic evolution equations, and therefore, their Hamiltonians, which are of the form (2.6), have positive energy eigenvalues $E(i)_a$ and negative energy eigenvalues $-E(i)_a$, where (i) enumerates the systems and a the eigenvalues. The combination of these systems will again have positive eigenvalues $E^{\text{tot}} = \sum_i E(i)_{a(i)}$ and negative energy eigenvalues $-E^{\text{tot}}$, but interactions must be arranged in such a way that all states where some energies are positive and some are negative are suppressed. The reason why we do allow *all* energies to be negative is that this might describe the physical situation equally well; we then happen to be dealing with the bra states $\langle\psi|$ rather than the kets $|\psi\rangle$.

Let us examine more closely the (weak) interaction between two such systems. Consider a time interval δt_1 for system (1) and δt_2 for system (2). As argued earlier, both systems must be spread over many Planck time units. According to the uncertainty relation, let us assume that

$$\begin{aligned} \frac{1}{2}\,(E(1)+E(2)) &\approx \frac{1}{2(\delta t_1+\delta t_2)}\ ; \\ \frac{1}{2}\,|E(1)-E(2)| &\approx \frac{1}{2|\delta t_1-\delta t_2|}\ . \end{aligned} \tag{2.7}$$

Now, according to Eq. (2.4), uncertainty in time directly reflects uncertainty in the position φ of the system in its periodic orbit. Demanding $E(1)\,E(2) > 0$ corresponds to

$$\begin{aligned} (E(1)+E(2))^2 &> (E(1)-E(2))^2\,, \quad \text{so that} \\ (\delta t_1+\delta t_2)^2 &< (\delta t_1-\delta t_2)^2, \quad \text{or} \\ (\delta\varphi_1+\delta\varphi_2)^2 &< (\delta\varphi_1-\delta\varphi_2)^2. \end{aligned} \tag{2.8}$$

The details concerning the relative position $\delta\varphi_1-\delta\varphi_2$ wash away after a sufficiently large average time interval $\frac{1}{2}\,(\delta t_1+\delta t_2)$. We read off:

$$\delta t_1\,\delta t_2 < 0. \tag{2.9}$$

Thus, the states that we expect to dissipate away due to information loss, are all those states where a positive time lapse δt_1 for one state is associated with a positive time lapse δt_2 for the other state. This may mean that the two states each carry an internal clock. The relative clock speed is controlled by the gravitational potential between the two systems. This potential apparently fluctuates. These fluctuations wash out all information concerning the relative configurations, but the relative clock speeds are always positive.

2.6 The vacuum state and the cosmological constant

We see that if we have a set of different systems which mutually interact only weakly, such as a set of free particles, or a set of disconnected pieces of the universe, either all energies must be selected to be positive, or they all are negative.

This means that there is one very special state where all energies are zero: the vacuum state. Identifying the vacuum state is particularly difficult in our theory, but it seems that the vacuum also poses problems in other approaches. In Loop Quantum Gravity, it is notoriously difficult to say exactly what the vacuum state is in terms of the fundamental loop states that were introduced there. In superstring theory, there are many candidates for the vacuum, all being distinctly characterized by the boundary conditions and the fluxes present in the compactified part of space-time. String theory ends up leaving an entire 'landscape' of vacuum states with no further indication as to which of these to pick. It is of crucial importance in any viable theory of Planck length physics to identify and describe in detail the vacuum state. It appears to be associated with very special fluctuations and correlations of the virtual particles and fields that one wishes to use to describe physical excited states, and the particles in it.

There exists an important piece of information telling us that the vacuum is not just the state with lowest energy. There must exist an additional criterion to identify the vacuum: it is flat – or nearly so. In perturbative gravity, this cannot be understood. The cosmological constant should receive a large finite renormalization counter term from all virtual interactions in the very high energy domain. A superior theory in which the cosmological constant vanishes naturally (or is limited to extremely tiny values) has not yet been found or agreed upon [27]. This should be a natural property of the vacuum state. To see most clearly how strange this situation is, consider the Einstein–Hilbert action,

$$S = \int \sqrt{-g}\left(\frac{1}{16\pi G} R + \frac{\Lambda}{8\pi G}\right), \tag{2.10}$$

Here, the first term describes the response of the total action to any deformation causing curvature. This response is huge, since Newton's constant, which is tiny, occurs in the denominator. In contrast, the second term describes the response of the total action upon scaling. This response is very tiny, since the cosmological constant Λ is extremely small – indeed it was thought to vanish altogether until recently.

In Fig. 2.1, a piece of fabric is sketched with similar properties in ordinary 3-space. Globally, this material allows for stretching and squeezing with relatively little resistance, but changing the ratios of the sides of the large triangle, or its angles, requires much more force. One could build more elaborate structures from these basic triangular units, such that their shapes are fixed, but their sizes not. An engineer would observe, however, that even if the hinges and the rods were made extremely strong and sturdy, resistance against changes of shape would still be rather weak. In the limit where the sizes of the structures are very large compared to those of the hinges, resistance against changes of shape would dwindle.

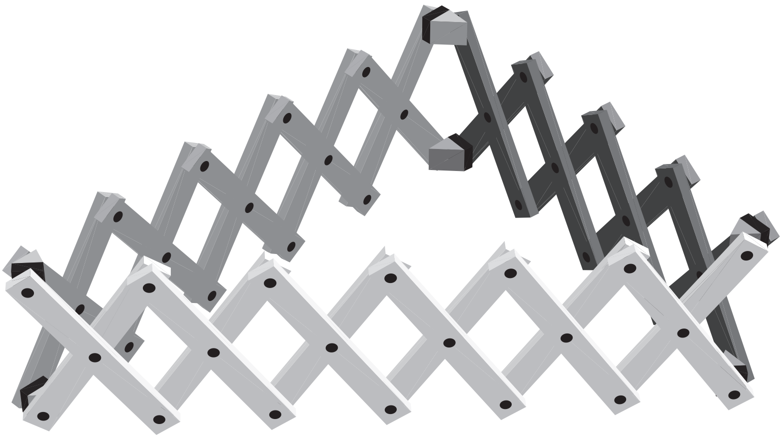

Fig. 2.1. The 'fabric of space-time', with tiny cosmological constant.

Comparing this with the situation in our universe shows in a more tangible way how odd it is that, a term with dimensions as low as the cosmological constant, can nevertheless be so tiny (120 factors of 10) compared with the much higher dimensional Einstein–Hilbert term. This oddity is the main reason why all attempts to find a natural explanation of this feature have failed. Unless one is prepared to accept the anthropic argument ('the universe is like this because all other universes are uninhabitable for intelligent beings'), a more drastic approach will be needed. Here again, we emphasize that, in any more advanced theory for Planck length physics, the definition of what exactly the vacuum state is, will have to require special attention. It could be that one has to *define* that the vacuum state is the one in which 3-space is as flat as it can be. One is then again confronted with the problem of understanding why all other physical states have not only positive energy, but also energy densities that are bounded from below.

Note that, in conventional quantum mechanics, the Hamiltonian plays a dual role: on the one hand it is simply the operator that generates the equations for evolution in time, while on the other hand it *stabilizes* the ground state, or vacuum. Energy conservation prevents small fluctuations from growing, because there are no other states where the total energy vanishes. One-particle states are also stable because there are no other states with matching energy and momentum, and this situation is guaranteed only because all energies are bounded from below. This is why the lower bound on energy is an absolutely vital feature of conventional quantum mechanics. It must be reproduced, whenever an 'underlying' theory is proposed.

2.7 Gauge- and diffeomorphism invariance as emergent symmetries

Most likely, however, the hideously tiny value of the cosmological constant is pointing towards a deeper kind of misunderstanding concerning diffeomorphism invariance in gravity. A remote possibility is suggested by our theory where quantum mechanical effects are generated as an emergent phenomenon in a world that is deterministic at the Planck scale. Information loss leads to a description of physical states forming equivalence classes. As stated, the equivalence classes are very large; when black holes are formed, the equivalence classes assemble on the surface area of the horizon, while the original ontological states are defined in the bulk of 3-space. If information loss forces two states to evolve identically, the states are said to sit in one equivalence class.

Even if one would not buy the idea that there is an underlying deterministic theory, one could suspect that these equivalence classes can be described as *gauge*-equivalence classes. The transition from one element to another element of an equivalence classes is a local gauge transformation. If so, then local gauge invariance will not be a property of the underlying theory, but an emergent phenomenon.

This naturally begs us to question: could diffeomorphism invariance be also just such a symmetry? Could it be that two states that differ from one another just by a local coordinate transformation, sit in one equivalence class, which would mean that they could evolve into the same final state? This might be possible. It would mean that the original, deterministic theory might require a preferred coordinate frame, which however would wash away due to information loss. The preferred coordinate frame might naturally select a flat space-time as a ground state solution, and thus a curvature-free configuration would be selected as the natural vacuum state.

Needless to say, this argument is hopelessly inadequate to solve the cosmological constant problem, but it could serve to shed a different light on it. It illustrates that there may be more, unconventional directions to search for a solution to the problem of reconciling quantum mechanics with general relativity.

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3

Does locality fail at intermediate length scales?

R. D. SORKIN

Assuming that "quantum spacetime" is fundamentally discrete, how might this discreteness show itself? Some of its potential effects are more evident, others less so. The atomic and molecular structure of ordinary matter influences the propagation of both waves and particles in a material medium. Classically, particles can be deflected by collisions and also retarded in their motion, giving rise in particular to viscosity and Brownian motion. In the case of spatio-temporal discreteness, viscosity is excluded by Lorentz symmetry, but fluctuating deviations from rectilinear motion are still possible. Such "swerves" have been described in [1] and [2]. They depend (for a massive particle) on a single phenomenological parameter, essentially a diffusion constant in velocity space. As far as I know, the corresponding analysis for a quantal particle with mass has not been carried out yet, but for massless quanta such as photons the diffusion equation of [1] can be adapted to say something, and it then describes fluctuations of both energy and polarization (but not of direction), as well as a secular "reddening" (or its opposite). A more complete quantal story, however, would require that particles be treated as wave packets, raising the general question of how spatiotemporal discreteness affects the propagation of *waves*. Here, the analogy with a material medium suggests effects such as scattering and extinction, as well as possible nonlinear effects. Further generalization to a "second-quantized field" might have more dramatic, if less obvious, consequences. In connection with cosmology, for example, people have wondered how discreteness would affect the hypothetical inflaton field.

So far, I have been assuming that, although the deep structure of spacetime is discrete, it continues to respect the Lorentz transformations. That this is logically possible is demonstrated [3] by the example of causal set (causet) theory [4; 5; 6; 7]. With approaches such as loop quantum gravity, on the other hand, the status of local Lorentz invariance seems to be controversial. Some people have hypothesized that it would be broken or at least perhaps deformed in such a way that the dispersion relations for light would cease to be those of a massless field. Were this the case,

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

empty space could also resist the passage of particles (a viscosity of the vacuum), since there would now be a state of absolute rest. Moreover, reference [8] has argued convincingly that it would be difficult to avoid $O(1)$ renormalization effects that would lead to different quantum fields possessing different effective light cones. Along these lines, one might end up with altogether more phenomenology than one had bargained for.

As already mentioned, the causal set hypothesis avoids such difficulties, but in order to do so, it has to posit a kinematic randomness, in the sense that a spacetime[1] M may properly correspond only to causets C that could have been produced by a *Poisson process* in M. With respect to an approximating spacetime M, the causet thus functions as a kind of "random lattice". Moreover, the infinite volume of the Lorentz group implies that such a "lattice" cannot be home to a local dynamics. Rather the "couplings" or "interactions" that describe physical processes occurring in the causet are – of necessity – radically nonlocal.

To appreciate why this must be, let us refer to the process that will be the subject of much of the rest of this chapter: propagation of a scalar field ϕ on a background causet C that is well approximated by a Minkowski spacetime $M = \mathbb{M}^d$. To describe such a dynamics, one needs to reproduce within C something like the d'Alembertian operator $\Box$, the Lorentzian counterpart of the Laplacian operator ∇^2 of Euclidean space $\mathbb{E}^3$. Locality in the discrete context, if it meant anything at all, would imply that the action of $\Box$ would be built up in terms of "nearest neighbor couplings" (as in fact ∇^2 can be built up, on either a crystalline or random lattice in $\mathbb{E}^3$). But Lorentz invariance contradicts this sort of locality because it implies that, no matter how one chooses to define nearest neighbor, any given causet element $e \in C$ will possess an immense number of them extending throughout the region of C corresponding to the light cone of e in M. In terms of a Poisson process in M we can express this more precisely by saying that the *probability* of any given element e possessing a limited number of nearest neighbors is vanishingly small. Thus, the other elements to which e must be "coupled" by our box operator will be large in number (in the limit infinite), and in any given frame of reference, the vast majority of them will be remote from e. The resulting "action at a distance" epitomizes the maxim that discreteness plus Lorentz invariance entails nonlocality.

If this reasoning is correct, it implies that physics at the Planck scale must be radically nonlocal. (By Planck scale I just mean the fundamental length scale or volume scale associated with the causet or other discrete substratum.) Were it to be confined to the Planck scale, however, this nonlocality would be of limited phenomenological interest despite its deep significance for the underlying theory. But

[1] In this chapter, "spacetime" will always mean Lorentzian manifold, in particular a continuum.

a little thought indicates that things might not be so simple. On the contrary, it is far from obvious that the kind of nonlocality in question can be confined to any scale, because for any given configuration of the field ϕ, the "local couplings" will be vastly outnumbered by the "nonlocal" ones. How then could the latter conspire to cancel out so that the former could produce a good approximation to $\Box\,\phi$, even for a slowly varying ϕ?

When posed like this, the question looks almost hopeless, but I will try to convince you that there is in fact an answer. What the answer seems to say, though, is that one can reinstate locality only conditionally and to a limited extent. At any finite scale λ, some nonlocality will naturally persist, but the scale λ_0 at which it begins to disappear seems to reflect not only the ultraviolet scale l but also an infrared scale R, which we may identify with the age of the cosmos, and which (in a kind of quantum-gravitational echo of Olber's paradox) seems to be needed in order that locality be recovered at all. On the other hand, (the) spacetime (continuum) as such can make sense almost down to $\lambda = l$. We may thus anticipate that, as we coarse-grain up from l to larger and larger sizes λ, we will reach a stratum of reality in which discontinuity has faded out and spacetime has emerged, but physics continues to be nonlocal. One would expect the best description of this stratum to be some type of nonlocal field theory; and this would be a new sort of manifestation of discreteness: not as a source of fluctuations, but as a source of nonlocal phenomena.

Under still further coarse-graining, this nonlocality should disappear as well, and one might think that one would land for good in the realm of ordinary Quantum Field Theory (and its further coarse-grainings). However, there is reason to believe that locality would fail once again when cosmic dimensions were reached; in fact, the non-zero cosmological constant predicted on the basis of causet theory is very much a nonlocal reflection, on the largest scales, of the underlying discreteness. It is a strictly quantal effect, however, and would be a very different sort of residue of microscopic discreteness than what I'll be discussing here.

These introductory remarks express in a general way most of what I want to convey in this paper, but before getting to the technical underpinnings, let me just (for shortage of space) list some other reasons why people have wanted to give up locality as a fundamental principle of spacetime physics: to cure the divergences of Quantum Field Theory (e.g. [9; 10]); to obtain particle-like excitations of a spin-network or related graph [11]; to give a realistic and deterministic account of Quantum Mechanics (the Bohmian interpretation is both nonlocal and acausal, for example); to let information escape from inside a black hole (e.g. [12]); to describe the effects of hidden dimensions in "brane world" scenarios; to reduce Quantum Gravity to a flat-space Quantum Field Theory via the so called AdS-CFT correspondence; to make room for non-commuting spacetime coordinates.

(This "non-commutative geometry" reason is perhaps the most suggestive in the present context, because it entails a hierarchy of scales analogous to the scales l, λ_0 and R. On the "fuzzy sphere" in particular, the non-commutativity scale λ_0 is the geometric mean between the effective ultraviolet cutoff l and the sphere's radius R.)

3.1 Three D'Alembertians for two-dimensional causets

The scalar field on a causet offers a simple model for the questions we are considering. Kinematically, we may realize such a field simply as a mapping ϕ of the causet into the real or complex numbers, while in the continuum its equations of motion take – at the classical level – the simple form $\Box\,\phi = 0$, assuming (as we will) that the mass vanishes. In order to make sense of this equation in the causet, we "merely" need to give a meaning to the D'Alembertian operator $\Box$. This is not an easy task, but it seems less difficult than giving meaning to, for example, the gradient of ϕ (which for its accomplishment would demand that we define a concept of vector field on a causet). Of course, one wants ultimately to treat the quantum case, but one would expect a definition of $\Box$ to play a basic role there as well, so in seeking such a definition we are preparing equally for the classical and quantal cases.

If we assume that $\Box$ should act linearly on ϕ (not as obvious as one might think!), then our task reduces to the finding of a suitable matrix B_{xy} to play the role of $\Box$, where the indices x, y range over the elements of the causet C. We will also require that B be "retarded" or "causal" in the sense that $B_{xy} = 0$ whenever x is spacelike to, or causally precedes, y. In the first place, this is helpful classically, since it allows one to propagate a solution ϕ forward iteratively, element by element (assuming that the diagonal elements B_{xx} do not vanish). It might similarly be advantageous quantally, if the path integration is to be conducted in the Schwinger–Kel'dysh manner.

3.1.1 First approach through the Green function

I argued above that no matrix B that (approximately) respects the Lorentz transformations can reproduce a local expression like the D'Alembertian unless the majority of terms cancel miraculously in the sum, $\sum_y B_{xy}\phi_y \;=:\; (B\phi)_x$, that corresponds to $\Box\,\phi(x)$.

Simulations by Alan Daughton [13], continued by Rob Salgado [14], provided the first evidence that the required cancellations can actually be arranged for without appealing to anything other than the intrinsic order-structure of the causet. In this approach one notices that, although in the natural order of things one begins with the D'Alembertian and "inverts" it to obtain its Green function G, the result in $1+1$-dimensions is so simple that the procedure can be reversed. In fact, the *retarded* Green function $G(x,y) = G(x-y)$ in $\mathbb{M}^2$ is (with the sign convention $\Box = -\partial^2/\partial t^2 + \partial^2/\partial x^2$) just the step function with magnitude $-1/2$ and supports the future of the origin (the future light cone together with its interior). Moreover, thanks to the conformal invariance of $\Box$ in $\mathbb{M}^2$, the same expression remains valid in the presence of spacetime curvature.

Not only is this continuum expression very simple, but it has an obvious counterpart in the causal set, since it depends on nothing more than the causal relation between the two spacetime points x and y. Letting the symbol $<$ denote (strict) causal precedence in the usual way, we can represent the causet C as a matrix whose elements C_{xy} take the value 1 when $x < y$ and 0 otherwise. The two-dimensional analog G of the retarded Green function is then just $-1/2$ times (the transpose of) this matrix.

From these ingredients, one can concoct some obvious candidates for the matrix B. The one that so far has worked best is obtained by symmetrizing G_{xy} and then inverting it. More precisely, what has been done is the following: begin with a specific region $R \subset \mathbb{M}^2$ (usually chosen to be an order-interval, the diamond-shaped region lying causally between a timelike pair of points); randomly sprinkle N points x_i, $i = 1 \ldots N$ into R; let C be the causet with these points as substratum and the order-relation $<$ induced from $\mathbb{M}^2$; for any "test" scalar field ϕ on R, let $\phi_i = \phi(x_i)$ be the induced "field" on C; build the $N \times N$ matrix G and then symmetrize and invert to get B, as described above; evaluate $B(\phi,\psi) = \sum_{ij} B_{ij}\phi_i\psi_j$ for ϕ and ψ drawn from a suite of test functions on R; compare with the continuum values, $\int d^2x\, \phi(x) \Box \psi(y)\, d^2y$.

For test functions that vanish to first order on the boundary ∂R of R, and that vary slowly on the scale set by the sprinkling density, the results so far exhibit full agreement between the discrete and continuum values [13; 14]. Better agreement than this, one could not have hoped for in either respect: concerning boundary terms, the heuristic reasoning that leads one to expect that inverting a Green function will reproduce a discretized version of $\Box$ leaves open its behavior on ∂R. Indeed, one doesn't really know what continuum expression to compare with: if our fields don't vanish on ∂R, should we expect to obtain an approximation to $\int dxdy\phi(x) \Box \psi(y)$ or $\int dxdy(\nabla\phi(x), \nabla\psi(y))$ or ...? Concerning rapidly varying functions, it goes without saying that, just as a crystal cannot support a sound wave shorter than the interatomic spacing, a causet cannot support a wavelength shorter than l. But unlike with crystals, this statement requires some qualification because the notion of wavelength is frame-dependent. What is a red light for one inertial observer is a blue light for another. Given that the causet can support the red wave, it must be able to support the blue one as well, assuming Lorentz invariance in a suitable sense. Conversely, such paired fields can be used to test the Lorentz invariance of B. To the limited extent that this important test has been done, the results have also been favorable.

On balance, then, the work done on the Green function approach gives cause for optimism that "miracles do happen". However, the simulations have been limited to the flat case, and, more importantly, they do not suffice (as of yet) to establish that the discrete D'Alembertian B is truly frame independent. The point is that although G itself clearly is Lorentz invariant in this sense, its inverse (or rather the inverse of the symmetrized G) will in general depend on the region R in which one works. Because this region is not itself invariant under boosts, it defines a global frame that could find its way into the resulting matrix B. Short of a better analytic understanding, one is unable to rule out this subtle sort of frame dependence, although the aforementioned limited tests provide evidence against it.

Moreover, the Green function prescription itself is of limited application. In addition to two dimensions, the only other case where a similar prescription is known is that of four dimensions *without* curvature, where one can take for G the "link matrix" instead of the "causal matrix".

Interestingly enough, the potential for Lorentz-breaking by the region R does not arise if one works exclusively with retarded functions, that is, if one forms B from the original retarded matrix G, rather than its symmetrization.[2] Unfortunately, computer tests with the retarded Green function have so far been discouraging on the whole (with some very recent exceptions). Since, for quite different reasons, it would be desirable to find a retarded representation of $\Box$, this suggests that we try something different.

3.1.2 Retarded couplings along causal links

Before taking leave of the Green function scheme just described, we can turn to it for one more bit of insight. If one examines the individual matrix elements B_{xy} for a typical sprinkling, one notices first of all that they seem to be equally distributed among positive and negative values, and second of all that the larger magnitudes among them are concentrated "along the light cone"; that is, B_{xy} tends to be small unless the proper distance between x and y is near zero. The latter observation may remind us of a collection of "nearest neighbor couplings", here taken in the only possible Lorentz invariant sense: that of small proper distance. The former observation suggests that a recourse to oscillating signs might be the way to effect the "miraculous cancellations" we are seeking.

[2] One needs to specify a nonzero diagonal for F.

The suggestion of oscillating signs is in itself rather vague, but two further observations will lead to a more quantitative idea. Let a be some point in $\mathbb{M}^2$, let b and c be points on the right and left halves of its past lightcone (a "cone" in $\mathbb{M}^2$ being just a pair of null rays), and let d be the fourth point needed to complete the rectangle. If (with respect to a given frame) all four points are chosen to make a small square, and if ϕ is slowly varying (in the same frame), then the combination $\phi(a) + \phi(d) - \phi(b) - \phi(c)$ converges, after suitable normalization, to $-\Box\,\phi(a)$ as the square shrinks to zero size. (By Lorentz invariance, the same would have happened even if we had started with a rectangle rather than a square.) On the other hand, four other points obtained from the originals by a large boost will form a long skinny rectangle, in which the points a and b (say) are very close together, as are c and d. Thanks to the profound identity, $\phi(a) + \phi(d) - \phi(b) - \phi(c) = \phi(a) - \phi(b) + \phi(d) - \phi(c)$, we will obtain only a tiny contribution from this rectangle – exactly the sort of cancellation we were seeking! By including all the boosts of the original square, we might thus hope to do justice to the Lorentz group without bringing in the unwanted contributions we have been worrying about.

Comparison with the D'Alembertian in one dimension leads to a similar idea, which in addition works a bit better in the causet, where elements corresponding to the type of "null rectangles" just discussed don't really exist. In $\mathbb{M}^1$, which is just the real line, $\Box\,\phi$ reduces (up to sign) to $\partial^2\phi/\partial t^2$, for which a well known discretization is $\phi(a) - 2\phi(b) + \phi(c)$, a, b and c being three evenly spaced points. Such a configuration *does* find correspondents in the causet, for example 3-chains $x < y < z$ such that no element other than y lies causally between x and z. Once again, any single one of these chains (partly) determines a frame, but the collection of all of them does not. Although these examples should not be taken too seriously (compare the sign in equation (3.1) below), they bring us very close to the following scheme.[3]

Imagine a causet C consisting of points sprinkled into a region of $\mathbb{M}^2$, and fix an element $x \in C$ at which we would like to know the value of $\Box\,\phi$. We can divide the ancestors of x (those elements that causally precede it) into "layers" according to their "distance from x", as measured by the number of intervening elements. Thus layer 1 comprises those y which are *linked* to x in the sense that $y < x$ with no intervening elements, layer 2 comprises those $y < x$ with only a single element z such that $y < z < x$, etc. Our prescription for $\Box\,\phi(x)$ is then to take some combination, with alternating signs, of the first few layers, the specific coefficients to be chosen so that the correct answers are obtained from suitably simple test functions. Perhaps the simplest combination of this sort is

[3] A very similar idea was suggested once by Steve Carlip.

$$B\phi(x) = \frac{4}{l^2}\left(-\frac{1}{2}\phi(x) + \left(\sum_1 -2\sum_2 + \sum_3\right)\phi(y)\right), \tag{3.1}$$

where the three sums $\sum$ extend over the first three layers as just defined, and l is the fundamental length-scale associated with the sprinkling, normalized so that each sprinkled point occupies, on average, an area of l^2. The prescription (3.1) yields a candidate for the "discrete D'Alembertian" B which is *retarded*, unlike our earlier candidate based on the symmetrized Green function. In order to express this new B explicitly as a matrix, let $n(x, y)$ denote the cardinality of the order-interval $\langle y, x\rangle = \{z \in C | y < z < x\}$, or in other words the number of elements of C causally between y and x. Then, assuming that $x \geq y$, we have from (3.1),

$$\frac{l^2}{4}B_{xy} = \begin{cases} -\frac{1}{2} & \text{for } x = y \\ 1, -2, 1, & \text{according as } n(x, y) \text{ is } 0, 1, 2, \text{ respectively,} \quad \text{for } x \neq y \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Now let ϕ be a fixed test function of compact support on $\mathbb{M}^2$, and let x (which we will always take to be included in C) be a fixed point of $\mathbb{M}^2$. If we apply B to ϕ we will of course obtain a random answer depending on the random sprinkling of $\mathbb{M}^2$. However, one can prove that the *mean* of this random variable, $\mathbb{E}B\phi(x)$, converges to $\Box\,\phi(x)$ in the continuum limit $l \to 0$:

$$\mathbb{E}\sum_y B_{xy}\phi_y \quad \underset{l\to 0}{\to} \quad \Box\,\phi(x), \tag{3.3}$$

where $\mathbb{E}$ denotes expectation with respect to the Poisson process that generates the sprinkled causet C. [The proof rests on the following facts. Let us limit the sprinkling to an "interval" (or "causal diamond") $\mathfrak{X}$ with x as its top vertex. For test functions that are polynomials of low degree, one can evaluate the mean in terms of simple integrals over $\mathfrak{X}$ – for example the integral $\int du dv/l^2 \exp\{-uv/l^2\}\ \phi(u, v)$ – and the results agree with $\Box\,\phi(x)$, up to corrections that vanish like powers of l or faster.]

In a sense, then, we have successfully reproduced the D'Alembertian in terms of a causet expression that is *fully intrinsic* and therefore automatically *frame-independent*. Moreover, the matrix B, although it introduces nonlocal couplings, does so only on Planckian scales, which is to say, on scales no greater than demanded by the discreteness itself.[4]

[4] It is not difficult to convince oneself that the limit in (3.3) sets in when l shrinks below the characteristic length associated with the function ϕ; or vice versa, if we think of l as fixed, $B\phi$ will be a good approximation to $\Box\,\phi$ when the characteristic length-scale λ over which ϕ varies exceeds l: $\lambda \gg l$. But this means in turn that $(B\phi)(x)$ can be sampling ϕ *in effect* only in a neighborhood of x of characteristic size l. Although B is thoroughly nonlocal at a fundamental level, the scale of its effective nonlocality in application to slowly varying test functions is (in the mean) thus no greater than l.

But is our "discrete D'Alembertian" B really a satisfactory tool for building a field theory on a causet? The potential problem that suggests the opposite conclusion concerns the fluctuations in (3.1), which grow with N rather than dying away. (This growth is indicated by theoretical estimates and confirmed by numerical simulations.) Whether this problem is fatal or not is hard to say. For example, in propagating a classical solution ϕ forward in time through the causet, it might be that the fluctuations in ϕ induced by those in (3.1) would remain small when averaged over many Planck lengths, so that the coarse-grained field would not see them. But if this is true, it remains to be demonstrated. And in any case, the fluctuations would be bound to affect even the coarse-grained field when they became big enough. For the remainder of this paper, I will assume that large fluctuations are not acceptable, and that one consequently needs a different B that will yield the desired answer not only on average, but (with high probability) in each given case. For that purpose, we will have to make more complicated the remarkably simple ansatz (3.2) that we arrived at above.

3.1.3 Damping the fluctuations

To that end, let us return to equation (3.3) and notice that $\mathbb{E}(B\phi) = (\mathbb{E}B)\phi$, where what I have just called $\mathbb{E}B$ is effectively a continuum integral-kernel $\bar{B}$ in $\mathbb{M}^2$. That is to say, when we average over all sprinklings to get $\mathbb{E}B\phi(x)$, the sums in (3.1) turn into integrals and there results an expression of the form $\int \bar{B}(x-y)\phi(y)d^2y$, where $\bar{B}$ is a retarded, continuous function that can be computed explicitly. Incorporating into $\bar{B}$ the δ-function answering to $\phi(x)$ in (3.1), we get for our kernel (when $x > y$),

$$\bar{B}(x-y) = \frac{4}{l^4}\, p(\xi)e^{-\xi} \;-\; \frac{2}{l^2}\delta^{(2)}(x-y), \tag{3.4}$$

where $p(\xi) = 1 - 2\xi + \frac{1}{2}\xi^2$, $\xi = v/l^2$ and $v = \frac{1}{2}||x-y||^2$ is the volume (i.e. area) of the order-interval in $\mathbb{M}^2$ delimited by x and y. The convergence result (3.3) then states that, for ϕ of compact support,

$$\int \bar{B}(x-y)\phi(y)d^2y \quad \underset{l\to 0}{\to} \quad \Box\,\phi(x). \tag{3.5}$$

Notice that, as had to happen, $\bar{B}$ is Lorentz-invariant, since it depends only on the invariant interval $||x-y||^2 = |(x-y)\cdot(x-y)|$.[5]

[5] The existence of a Lorentz-invariant kernel $\bar{B}(x)$ that yields (approximately) $\Box\,\phi$ might seem paradoxical, because one could take the function ϕ itself to be Lorentz invariant (about the origin $x = 0$, say), and for such a ϕ the integrand in (3.5) would also be invariant, whence the integral would apparently have to diverge. This divergence is avoided for compactly supported ϕ, of course, because the potential divergence is cut off where the integrand goes to zero. But what is truly remarkable in the face of the counter-argument just given, is that

Observe, now, that the fundamental discreteness-length has all but disappeared from our story. It remains only in the form of a parameter entering into the definition (3.4) of the integral kernel $\bar{B}$. As things stand, this parameter reflects the scale of microscopic physics from which $\bar{B}$ has emerged (much as the diffusion constants of hydrodynamics reflect atomic dimensions). But nothing in the definition of $\bar{B}$ per se forces us to this identification. If in (3.4) we replace l by a freely variable length, and if we then follow the Jacobian dictum, "Man muss immer umkehren",[6] we can arrive at a modification of the discrete D'Alembertian B for which the unwanted fluctuations are damped out by the law of large numbers.

Carrying out the first step, let us replace $1/l^2$ in (3.4) by a new parameter K. We obtain a new continuum approximation to $\Box$,

$$\bar{B}_K(x-y) = 4K^2\, p(\xi)e^{-\xi} \;-\; 2K\delta^{(2)}(x-y), \tag{3.6}$$

whose associated nonlocality-scale is not l but the length $K^{-1/2}$, which we can take to be much larger than l. Retracing the steps that led from the discrete matrix (3.2) to the continuous kernel (3.4) then brings us to the following causet expression that yields (3.6) when its sprinkling-average is taken:

$$B_K\phi(x) = \frac{4\epsilon}{l^2}\left(-\frac{1}{2}\phi(x) + \epsilon\sum_{y<x} f(n(x,y),\epsilon)\,\phi(y)\right), \tag{3.7}$$

where $\epsilon = l^2 K$, and

$$f(n,\epsilon) = (1-\epsilon)^n\left(1 - \frac{2\epsilon n}{1-\epsilon} + \frac{\epsilon^2 n(n-1)}{2(1-\epsilon)^2}\right). \tag{3.8}$$

For $K = 1/l^2$ we recover (3.1). In the limit where $\epsilon \to 0$ and $n \to \infty$, $f(n,\epsilon)$ reduces to the now familiar form $p(\xi)e^{-\xi}$ with $\xi = n\epsilon$. That is, we obtain in this limit the Montecarlo approximation to the integral $\bar{B}_K\phi$ induced by the sprinkled points. (Conversely, $p(\xi)e^{-\xi}$ can serve as a lazybones' alternative to (3.8).)

Computer simulations show that $B_K\phi(x)$ furnishes a good approximation to $\Box\,\phi(x)$ for simple test functions, but this time one finds that the fluctuations *also* go to zero with l, assuming the physical nonlocality scale K remains fixed as l varies. For example, with $N = 2^9$ points sprinkled into the interval in $\mathbb{M}^2$ delimited by $(t,x) = (\pm 1, 0)$, and with the test functions $\phi = 1, t, x, t^2, x^2, tx$, the fluctuations in $B_K\phi(t=1, x=0)$ for $\epsilon = 1/64$ range from a standard deviation of 0.53 (for $\phi = x^2$) to 1.32 (for $\phi = 1$); and they die out roughly like $N^{-1/2}$

the answer is insensitive to the size of the supporting region. With any reasonable cutoff and reasonably well behaved test functions, the integral still manages to converge to the correct answer as the cutoff is taken to infinity. Nevertheless, this risk of divergence hints at the need we will soon encounter for some sort of infrared cutoff-scale.

6 "One must always reverse direction."

(as one might have expected) when K is held fixed as N increases. The means are accurate by construction, in the sense that they exactly[7] reproduce the continuum expression $\bar{B}_K\phi$ (which in turn reproduces $\Box\,\phi$ to an accuracy of around 1% for $K \gtrsim 200$). (It should also be possible to estimate the fluctuations analytically, but I have not tried to do so.)

In any case, we can conclude that "discretized D'Alembertians" suitable for causal sets do exist, a fairly simple one-parameter family of them being given by (3.7). The parameter ϵ in that expression determines the scale of the nonlocality via $\epsilon = Kl^2$, and it must be $\ll 1$ if we want the fluctuations in $B\phi$ to be small. In other words, we need a significant separation between the two length-scales l and $\lambda_0 = K^{-1/2} = l/\sqrt{\epsilon}$.

3.2 Higher dimensions

So far, we have been concerned primarily with two-dimensional causets (ones that are well approximated by two-dimensional spacetimes). Moreover, the quoted result, (3.3) cum (3.6), has been proved only under the additional assumption of flatness, although it seems likely that it could be extended to the curved case. More important, however, is finding D'Alembertian operators/matrices for four and other dimensions. It turns out that one can do this systematically in a way that generalizes what we did in two dimensions.

Let me illustrate the underlying ideas in the case of four dimensional Minkowski space $\mathbb{M}^4$. In $\mathbb{M}^2$ we began with the D'Alembertian matrix B_{xy}, averaged over sprinklings to get $\bar{B}(x-y)$, and "discretized" a rescaled $\bar{B}$ to get the matrix $(B_K)_{xy}$. It turns out that this same procedure works in four dimensions if we begin with the coefficient pattern $1\ {-3}\ 3\ {-1}$ instead of $1\ {-2}\ 1$.

To see why it all works, however, it is better to start with the integral kernel and not the matrix (now that we know how to pass between them). In $\mathbb{M}^2$ we found $\bar{B}$ in the form of a delta-function plus a term in $p(\xi)\exp(-\xi)$, where $\xi = Kv(x, y)$, and $v(x, y)$ was the volume of the order-interval $\langle y, x\rangle$, or equivalently – in $\mathbb{M}^2$ – Synge's "world function". In other dimensions this equivalence breaks down and we can imagine using either the world function or the volume (one being a simple power of the other, up to a multiplicative constant). Whichever one chooses, the real task is to find the polynomial $p(\xi)$ (together with the coefficient of the companion delta-function term).

To that end, notice that the combination $p(\xi)\exp(-\xi)$ can always be expressed as the result of a differential operator $\mathcal{O}$ in $\partial/\partial K$ acting on $\exp(-\xi)$. But then,

[7] Strictly speaking, this assumes that the number of sprinkled points is Poisson distributed, rather than fixed.

$$\int p(\xi)\exp(-\xi)\phi(x)dx = \int \mathcal{O}\exp(-\xi)\phi(x)dx$$
$$= \mathcal{O}\int \exp(-\xi)\phi(x)dx \equiv \mathcal{O}J. \tag{3.9}$$

We want to choose $\mathcal{O}$ so that this last expression yields the desired results for test functions that are polynomials in the coordinates x^μ of degree two or less. But the integral J has a very simple form for such ϕ. Up to contributions that are negligible for large K, it is just a linear combination of terms of the form $1/K^n$ or $\log K/K^n$. Moreover the only monomials that yield logarithmic terms are (in $\mathbb{M}^2$) $\phi = t^2$, $\phi = x^2$, and $\phi = 1$. In particular the monomials whose D'Alembertian vanishes produce only $1/K$, $1/K^2$ or $O(1/K^3)$, with the exception of $\phi = 1$, which produces a term in $\log K/K$. These are the monomials that we don't want to survive in $\mathcal{O}J$. On the other hand $\phi = t^2$ and $\phi = x^2$ both produce the logarithmic term $\log K/K^2$, and we do want them to survive. Notice further, that the survival of *any* logarithmic terms would be bad, because, for dimensional reasons, they would necessarily bring in an "infrared" dependence on the overall size of the region of integration. Taking all this into consideration, what we need from the operator $\mathcal{O}$ is that it remove the logarithms and annihilate the terms $1/K^n$. Such an operator is

$$\mathcal{O} = \frac{1}{2}(H+1)(H+2) \quad \text{where} \quad H = K\frac{\partial}{\partial K} \tag{3.10}$$

is the homogeneity operator. Applying this to $\exp(-\xi)$ turns out to yield precisely the polynomial $p(\xi)$ that we were led to above in another manner, explaining in a sense why this particular polynomial arises. (The relation to the binomial coefficients, traces back to an identity, proved by Joe Henson, that expresses $(H+1)(H+2)\ \ldots\ (H+n)\exp(-K)$ in terms of binomial coefficients.) Notice finally that $(H+1)(H+2)$ does *not* annihilate $\log K/K$; but it converts it into $1/K$, which can be canceled by adding a delta-function to the integral kernel, as in fact we did. (It could also have been removed by a further factor of $(H+1)$.)

The situation for $\mathbb{M}^4$ is very similar to that for $\mathbb{M}^2$. The low degree monomials again produce terms in $1/K^n$ or $\log K/K^n$, but everything has an extra factor of $1/K$. Therefore $\mathcal{O} = \frac{1}{6}(H+1)(H+2)(H+3)$ is a natural choice and leads to a polynomial based on the binomial coefficients of $(1-1)^3$ instead of $(1-1)^2$. From it we can derive both a causet D'Alembertian and a nonlocal, retarded deformation of the continuum D'Alembertian, as before. It remains to be confirmed, however, that these expressions enjoy all the advantages of the two-dimensional operators discussed above. It also remains to be confirmed that these advantages persist in the presence of curvature (but not, of course, curvature large compared to the nonlocality scale K that one is working with).

It seems likely that the same procedure would yield candidates for retarded D'Alembertians in all other spacetime dimensions.

3.3 Continuous nonlocality, Fourier transforms and stability

In the course of the above reflexions, we have encountered some D'Alembertian matrices for the causet and we have seen that the most promising among them contain a free parameter K representing an effective nonlocality scale or "meso-scale", as I will sometimes call it. For processes occurring on this scale (assuming it is much larger than the ultraviolet scale l so that a continuum approximation makes sense) one would expect to recognize an effective nonlocal theory corresponding to the retarded two-point function $\bar{B}_K(x, y)$. For clarity of notation, I will call the corresponding operator on scalar fields $\underset{K}{\Box}$, rather than $\bar{B}_K$.

Although its nonlocality stems from the discreteness of the underlying causet, $\underset{K}{\Box}$ is a perfectly well defined operator in the continuum, which can be studied for its own sake. At the same time, it can help shed light on some questions that arise naturally in relation to its causet cousin B_{xy}.

One such question (put to me by Ted Jacobson) asks whether the evolution defined in the causet by B_{xy} is stable or not. This seems difficult to address as such except by computer simulations, but if we transpose it to a continuum question about $\underset{K}{\Box}$, we can come near to a full answer. Normally, one expects that if there were an instability then $\underset{K}{\Box}$ would possess an "unstable mode" (quasinormal mode), that is, a spacetime function ϕ of the form $\phi(x) = \exp(ik \cdot x)$ satisfying $\underset{K}{\Box}\, \phi = 0$, with the imaginary part of the wave-vector k being future-timelike.[8]

Now by Lorentz invariance, $\underset{K}{\Box}\, \phi$ must be expressible in terms of $z = k \cdot k$, and it is not too difficult to reduce it to an "Exponential integral" Ei in z. This being done, some exploration in Maple strongly suggests that the only solution of $\underset{K}{\Box} \exp(ik \cdot x) = 0$ is $z = 0$, which would mean the dispersion relation was unchanged from the usual one, $\omega^2 = k^2$. If this is so, then no instabilities can result from the introduction of our nonlocality scale K, since the solutions of $\underset{K}{\Box}\, \phi = 0$ are precisely those belonging to the usual D'Alembertian. The distinction between propagation based on the latter and propagation based on $\underset{K}{\Box}$ would repose only on the different relationship that ϕ would bear to its sources; propagation in empty

[8] One might question whether $\underset{K}{\Box}\, \phi$ is defined at all for a general mode since the integral that enters into its definition might diverge, but for a putative unstable mode, this should not be a problem because the integral has its support precisely where the mode dies out: toward the past.

(and flat) space would show no differences at all. (The massive case might tell a different story, though.)

3.3.1 Fourier transform methods more generally

What we've just said is essentially that the Fourier transform of $\underset{K}{\Box}$ vanishes nowhere in the complex z-plane ($z \equiv k \cdot k$), except at the origin. But this draws our attention to the Fourier transform as yet another route for arriving at a non-local D'Alembertian. Indeed, most people investigating deformations of $\Box$ seem to have thought of them in this way, including for example [9; 10]. They have written down expressions like $\Box \exp(\Box/K)$, but without seeming to pay much attention to whether such an expression makes sense in a spacetime whose signature has not been Wick rotated to $(++++)$. In contrast, the operator $\underset{K}{\Box}$ of this paper was defined directly in "position space" as an integral kernel, not as a formal function of $\Box$. Moreover, because it is retarded, its Fourier transform is rather special .... By continuing in this vein, one can come up with a third derivation of $\underset{K}{\Box}$ as (apparently) the simplest operator whose Fourier transform obeys the analyticity and boundedness conditions required in order that $\underset{K}{\Box}$ be well-defined and retarded.

The Fourier transform itself can be given in many forms, but the following is among the simplest:

$$\underset{K}{\Box}\, e^{ik\cdot x}|_{x=0} = \frac{2z}{i}\int_0^\infty dt\, \frac{e^{itz/K}}{(t-i)^2} \tag{3.11}$$

where here, $z = -k \cdot k/2$.

It would be interesting to learn what operator would result if one imposed "Feynman boundary conditions" on the inverse Fourier transform of this function, instead of "causal" ones.

3.4 What next?

Equations (3.7) and (3.6) offer us two distinct, but closely related, versions of $\Box$, one suited to a causet and the other being an effective continuum operator arising as an average or limit of the first. Both are retarded and each is Lorentz invariant in the relevant sense. How can we use them? First of all, we can take up the questions about wave-propagation raised in the introduction, looking in particular for deviations from the simplified model of [15] based on "direct transmission" from source to sink (a model that has much in common with the approach discussed above under the heading "First approach through the Green function"). Equation (3.7), in particular, would let us propagate a wave-packet through the causet and look for some of the effects indicated in the introduction, like “swerves”, scattering and extinction. These of course hark back directly to the granularity of the causet, but even in the continuum limit the nonlocality associated with (3.6) might modify the field emitted by a given source in an interesting manner; and this would be relatively easy to analyze.

Also relatively easy to study would be the effect of the nonlocality on free propagation in a curved background. Here one *would* expect some change to the propagation law. Because of the retarded character of $\underset{K}{\Box}$, one might also expect to see some sort of induced CPT violation in an expanding cosmos. Because (in a quantal context) this would disrupt the equality between the masses of particles and antiparticles, it would be a potential source of baryon–anti-baryon asymmetry not resting on any departure from thermal equilibrium.

When discreteness combines with spacetime curvature, new issues arise. Thus, propagation of wave-packets in an expanding universe and in a black hole background both raise puzzles having to do with the extreme red shifts that occur in both situations (so-called transplanckian puzzles). In the black hole context, the red shifts are of course responsible for Hawking radiation, but their analysis in the continuum seems to assign a role to modes of exponentially high frequency that arguably should be eschewed if one posits a minimum length. Equation (3.7) offers a framework in which such questions can be addressed without infringing on Lorentz invariance. The same holds for questions about what happens to wave-packets in (say) a de Sitter spacetime when they are traced backward toward the past far enough so that their frequency (with respect to some cosmic rest frame) exceeds Planckian values. Of course, such questions will not be resolved fully on the basis of classical equations of motion. Rather one will have to formulate Quantum Field Theory on a causet, or possibly one will have to go all the way to a quantal field on a quantal causet (i.e. to Quantum Gravity). Nevertheless, a better understanding of the classical case is likely to be relevant.

I will not try to discuss here how to do Quantum Field Theory on a causet, or even in Minkowski spacetime with a nonlocal D’Alembertian. That would raise a whole set of new issues, path-integral vs. operator methods and the roles of unitarity and causality being just some of them.[9] But it does seem in harmony with the aim of this chapter to comment briefly on the role of nonlocality in this connection. As we have seen, the ansatz (3.6) embodies a nonlocal interaction that has survived in the continuum limit, and thus might be made the basis of a nonlocal field theory of the sort that people have long been speculating about.

[9] I will, however, echo a comment made earlier: I suspect that one should not try to formulate a path-integral propagator as such; rather one will work with Schwinger–Kel’dysh paths.

What is especially interesting from this point of view is the potential for a new approach to renormalization theory (say in flat spacetime $\mathbb{M}^d$). People have sometimes hoped that nonlocality would eliminate the divergences of Quantum Field Theory, but as far as I can see, the opposite is true, at least for the specific sort of nonlocality embodied in (3.6). In saying this, I'm assuming that the divergences can all be traced to divergences of the Green function $G(x-y)$ in the coincidence limit $x = y$. If this is correct then one would need to soften the high frequency behavior of G, in order to eliminate them. But a glance at (3.11) reveals that $\underset{K}{\Box}$ has a milder ultraviolet behavior than $\Box$, since its Fourier transform goes to a constant at $z = \infty$, rather than blowing up linearly. Correspondingly, one would expect its Green function to be more singular than that of the local operator $\Box$, making the divergences worse, not better. If so, then one must look to the discreteness itself to cure the divergences; its associated nonlocality will not do the job.

But if nonlocality alone cannot remove the need for renormalization altogether, it might nevertheless open up a new and more symmetrical way to arrive at finite answers. The point is that (3.11) behaves at $z = \infty$ like $1 + O(1/z)$, an expression whose reciprocal has exactly the same behavior! The resulting Green function should therefore also be the sum of a delta-function with a regular[10] function (and the same reasoning would apply in four dimensions). The resulting Feynman diagrams would be finite *except for* contributions from the delta-functions. But these could be removed by hand ("renormalized away"). If this idea worked out, it could provide a new approach to renormalization based on a new type of Lorentz invariant regularization. (Notice that this all makes sense in real space, without the need for Wick rotation.)

3.5 How big is λ_0?

From a phenomenological perspective, the most burning question is one that I cannot really answer here: assuming there are nonlocal effects of the sort considered in the preceding lines, on what length-scales would they be expected to show up? In other words, what is the value of $\lambda_0 = K^{-1/2}$? Although I don't know how to answer this question theoretically,[11] it is possible to deduce bounds on λ_0 if we assume that the fluctuations in individual values of $\Box\,\phi(causet) = B_K\phi$ are

[10] At worst, it might diverge logarithmically on the light cone, but in that case, the residual divergence could be removed by adjusting the Fourier transform to behave like $1 + O(1/z^2)$.

[11] The question of why l_0 would be so much smaller than λ_0 would join the other "large number" (or "hierarchy") puzzles of physics, like the small size of the cosmological constant Λ. Perhaps the ratio λ_0/l would be set dynamically, say "historically" as a concomitant of the large age and diameter of the cosmos (cf. [18]). If a dynamical mechanism doing this could be discovered, it might also help to explain the current magnitude of Λ, either by complementing the mechanism of [16] with a reason why the value about which Lambda fluctuates is so close to zero, or by offering an alternative explanation altogether.

small, as discussed above. Whether such an assumption will still seem necessary at the end of the day is of course very much an open question. Not only could a sum over individual elements of the causet counteract the fluctuations (as already mentioned), but the same thing could result from the sum over causets implicit in Quantum Gravity. This would be a sum of exponentially more terms, and as such it could potentially remove the need for any intermediate nonlocality-scale altogether.

In any case, if we do demand that the fluctuations be elementwise small, then λ_0 is bounded from below by this requirement. (It is of course bounded above by the fact that – presumably – we have not seen it yet.) Although this bound is not easy to analyze, a very crude estimate that I will not reproduce here suggests that we make a small fractional error in $\Box\,\phi$ when (in dimension four)

$$\lambda^2 l^2 R \ll \lambda_0^5 \,, \tag{3.12}$$

where λ is the characteristic length-scale associated with the scalar field. On the other hand, even the limiting continuum expression $\underset{K}{\Box}\,\phi$ will be a bad approximation unless $\lambda \gg \lambda_0$. Combining these inequalities yields $\lambda^2 l^2 R \ll \lambda_0^5 \ll \lambda^5$, or $l^2 R \ll \lambda^3$. For smaller λ, accurate approximation to $\Box\,\phi$ is incompatible with small fluctuations. Inserting for l the Planck length[12] of 10^{-32} cm and for R the Hubble radius, yields $\lambda \sim 10^{-12}$ cm as the smallest wavelength that would be immune to the nonlocality. That this is not an extremely small length, poses the question whether observations already exist that could rule out nonlocality on this scale.[13]

Acknowledgements

It's a pleasure to thank Fay Dowker and Joe Henson for extensive discussions and help on these matters, during their visits to Perimeter Institute. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. This research was partly supported by NSF grant PHY-0404646.

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4

Prolegomena to any future Quantum Gravity

J. STACHEL

4.1 Introduction

"Prolegomena" means "preliminary observations," and my title is meant to recall Kant's celebrated *Prolegomena to Any Future Metaphysics That Can Claim to be a Science*. My words, like his:

> are not supposed to serve as the exposition of an already-existing science, but to help in the invention of the science itself in the first place.

To use another Kantian phrase, I shall discuss some "conditions of possibility" of a quantum theory of gravity, stressing the need for solutions to certain fundamental problems confronting any attempt to apply some method of quantization to the field equations of General Relativity (GR). Not for lack of interest but lack of space-time (S-T), other approaches to Quantum Gravity (QG) are not discussed here (but see [35]).

4.1.1 Background dependence versus background independence

The first problem is the tension between "method of quantization" and "field equations of GR". The methods of quantization of pre-general-relativistic theories[1] have been based on the existence of some fixed S-T structure(s), needed both for the development of the formalism and – equally importantly – for its physical interpretation. This S-T structure provides a *fixed kinematical background* for dynamical theories: the equations for particle or fields must be invariant under all automorphisms of the S-T symmetry group. GR theory, on the other hand, is a *background-independent* theory, without any fixed, non-dynamical S-T structures.

[1] In particular, non-relativistic Quantum Mechanics (QM) based on Galilei–Newtonian S-T, special-relativistic Quantum Field Theory based on Minkowski S-T, and Quantum Field Theories in non-flat Riemannian S-Ts. But see [30] for a discussion of topological QFT.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

Its field equations are invariant under all differentiable automorphisms (diffeomorphisms) of the underlying manifold, which has no S-T structure until a solution of the field equations is specified. In a background-independent theory, there is no kinematics independent of the dynamics.[2]

4.1.2 The primacy of process

GR and Special Relativistic Quantum Field Theory (SRQFT) *do* share one fundamental feature that often is not sufficiently stressed: *the primacy of process over state*.[3] The four-dimensional approach, emphasizing processes in regions of S-T, is basic to both (see, e.g., [11; 22; 23; 6; 7]). Every measurement, classical or quantum, takes a finite time, and thus involves a process. In non-relativistic Quantum Mechanics (QM), one can sometimes choose a temporal slice of S-T so thin that one can speak meaningfully of an "instantaneous measurement" of the state of a system; but even in QM this is not always the case. Continuous quantum measurements are often needed. And this is certainly the case for measurements in SRQFT, and in GR (see, e.g. [4; 22; 23; 27]). The breakup of a four-dimensional S-T region into lower-dimensional sub-regions – in particular, into a one parameter family of three-dimensional *hypersurfaces* – raises another aspect of the problem. It breaks up a process into a sequence of instantaneous states. This is useful, perhaps sometimes indispensable, as a calculational tool in both quantum theory and GR. But no fundamental significance should be attached to such breakups, and results so obtained should be examined for their significance from the four-dimensional, process standpoint (see, e.g. [19; 9]). Since much of this paper is concerned with such

[2] Ashtekar and Lewandowski [2] note that "in interacting [special-relativistic] Quantum Field Theories, there is a delicate relation between quantum kinematics and dynamics: unless the representation of the basic operator algebra is chosen appropriately, typically, the Hamiltonian fails to be well-defined on the Hilbert space;" and go on to suggest that in GR one has the same "problem of choosing the 'correct' kinematical representation" (p. 51). By a "background independent kinematics" for GR they mean a "quantum kinematics for background-independent theories of connections." In making a distinction between "quantum kinematics and dynamics," they evidently have in mind the distinction between the definition of an operator algebra for "position and momenta" operators on some spacelike initial hypersurface and the attendant definition of a Hilbert space of state functions on that hypersurface; and the evolution of this state function from hypersurface to hypersurface induced by a Hamiltonian operator, which has been appropriately defined in terms of these position and momenta. Two comments may help to clarify the difference between their outlook and mine.
(1) In *any* special-relativistic field theory, regardless of the field equations, the basic operator algebra, including the Hamiltonian, must be a representation of the Poincaré group, the fixed isometry group of the background S-T metric; this requirement is what I call a kinematics independent of dynamics. In canonical versions of GR, this algebra emerges from the *field equations*, in particular their division into constraint and evolution operators; and this an example of what I mean by "no kinematics independent of dynamics."
(2) While "kinematical" Hilbert spaces and state functions may be defined on the family of spacelike hypersurfaces, *per se* such state functions are without direct physical significance. They can only serve as aids in the calculation of the probability amplitude for some physical process, which will always involve what Ashtekar and Lewandowski call "dynamics".

[3] Baez [3] emphasizes that both are included in the category of cobordisms. Two manifolds are cobordant if their union is the complete boundary of a third manifold.

breakups, it is important to emphasize this problem from the start, as does Smolin in [31]:

[R]elativity theory and quantum theory each . . . tell us – no, better, they scream at us – that our world is a history of processes. Motion and change are primary. Nothing is, except in a very approximate and temporary sense. How something is, or what its state is, is an illusion. It may be a useful illusion for some purposes, but if we want to think fundamentally we must not lose sight of the essential fact that ‘is’ is an illusion. So to speak the language of the new physics we must learn a vocabulary in which process is more important than, and prior to, stasis (p. 53).

Perhaps the process viewpoint should be considered obvious in GR, but the use of three-plus-one breakups of ST in canonical approaches to QG (e.g. geometrodynamics and loop QG), and discussions of “the problem of time” based on such a breakup, suggest that it is not. The problem is more severe in the case of quantum theory, where the concepts of *state* and state function and discussions of the “collapse of the state function” still dominate most treatments. But, as Bohr and Feynman emphasized, the ultimate goal of any quantum-mechanical theory is the computation of the *probability amplitude* for some *process* undergone by a system. The initial and final states are just the boundaries of the process, marked off by the system’s preparation and the registration of some result, respectively (see [33; 34], which include references to Bohr and Feynman).

In SRQFT, the primary instrument for computation of probability amplitudes is functional integration (see, e.g. [6; 7]). Niedermaier [20] emphasizes the importance of approaches to QG that are:

centered around a functional integral picture. Arguably the cleanest intuition to ‘what quantizing gravity might mean’ comes from the functional integral picture. Transition or scattering amplitudes for nongravitational processes should be affected not only by one geometry solving the gravitational field equations, but by a ‘weighted superposition’ of ‘nearby possible’ off-shell geometries. [A]ll known (microscopic) matter is quantized that way, and using an off-shell matter configuration as the source of the Einstein field equations is in general inconsistent, unless the geometry is likewise off-shell (p. 3).

4.1.3 Measurability analysis

The aim of “measurability analysis”, as it was named in [4], is based on “the relation between formalism and observation” [22; 23]; its aim is to shed light on the physical implications of any formalism: the possibility of formally defining any physically significant quantity should coincide with the possibility of measuring it in principle; i.e. by means of some idealized measurement procedure that is consistent with that formalism. Non-relativistic QM and special relativistic quantum electrodynamics, have both passed this test; and its use in QG is discussed in Section 4.4.

4.1.4 Outline of the chapter

In QM and SRQFT, the choice of classical variables and of methods to describe processes they undergo played a major role in determining possible forms of the transition to quantized versions of the theory, and sometimes even in the content of the quantized theory.[4] Section 4.2 discusses these problems for Maxwell's theory, outlining three classical formalisms and corresponding quantizations. The Wilson loops method, applied to GR, led to the development of a background-independent quantization procedure. Section 4.3 surveys possible choices of fundamental variables in GR, and Section 4.4 discusses measurability analysis as a criterion for quantization. The classification of possible types of initial-value problems in GR is discussed in Sections 4.5 and 4.6. Section 4.7 treats various "mini-" and "midi-superspace" as examples of partially background-dependent S-Ts in GR, and the quantization of asymptotically flat S-Ts allowing a separation of kinematics and dynamics at null infinity. There is a brief Conclusion.

4.2 Choice of variables and initial value problems in classical electromagnetic theory

In view of the analogies between electromagnetism (EM) and GR (see Section 4.3) – the only two classical long-range fields transmitting interactions between their sources – I shall consider some of the issues arising in QG first in the simpler context of EM theory.[5] Of course, there are also profound differences between EM and GR – most notably, the former is background dependent and the latter is not. One important similarity is that both theories are formulated with redundant variables. In any gauge-invariant theory, the number of degrees of freedom equals the number of field variables minus twice the number of gauge functions. For Maxwell's theory, the count is four components of the electromagnetic four-potential A (symbols for geometric objects will often be abbreviated by dropping indices) minus two times one gauge function equals two degrees of freedom. For GR, the count is ten components of the pseudo-metric tensor g minus two times four "gauge" diffeomorphism functions, again equals two. There are two distinct analogies between EM and GR. In the first, A is the analogue of g. In the second, it is the analogue of Γ, the inertio-gravitational connection. In comparisons between

4 In SRQFT, inequivalent representations of the basic operator algebra are possible.

5 This theory is simplest member of the class of gauge-invariant Yang–Mills theories, with gauge group $U(1)$; most of the following discussion could be modified to include the entire class.

gauge fields and GR, the second analogy is usually stressed. Maxwell's theory is a $U(1)$ gauge theory, A is the connection one-form, the analogue of the GR connection one-form; and $F = dA$ is the curvature two-form, the analogue of the GR curvature two-form (see Sections 4.3 and 4.6, for the tetrad formulation of GR).

The first analogy may be developed in two ways. The formulation of EM entirely in terms of the potential four-vector is analogous to the formulation of GR entirely in terms of the pseudo-metric tensor (see Section 4.3): the field equations of both are second order. This analogy is very close for the linearized field equations: small perturbations $h_{\mu\nu}$ of the metric around the Minkowski metric $\eta_{\mu\nu}$ obey the same equations as special-relativistic, gauge-invariant massless spin-two fields, which are invariant under the gauge transformations $h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$ where ξ_ν is a vector field;[6] while A_μ obeys those of a spin-one field, which are invariant under the gauge transformation $A_\kappa \rightarrow A_\kappa + \partial_\kappa \chi$, where χ is a scalar field. The divergence of the left-hand-side of these field equations vanishes identically (in GR this holds for both the exact and linearized equations), so vanishing of the divergence of the right-hand-side (conservation of energy-momentum in GR, conservation of charge in EM) is an integrability condition. This is no accident: invariance and conservation law are related by Noether's second theorem (see Section 4.5).

The formulation of GR in terms of pseudo-metric g and independently defined inertio-gravitational connection Γ is analogous to the formulation of EM in terms of a one form A and a second two-form field G, initially independent of F. The definition of the Christoffel symbols $\{\ \} = \Gamma$ in terms of g and its first derivatives is analogous to the definition of $F = dA$ (see above). The first set of Maxwell equations $dF = 0$ then follows from this definition. Some set of constitutive relations between F and G complete the EM theory. The vacuum relations $F = G$ are analogous to the compatibility conditions $\{\ \} = \Gamma$ in GR. The second set of Maxwell equations: $dG = j$, where j is the charge-current 3-form, are the analogue of the equations $E(\Gamma) = T$ equating the Einstein tensor E to the stress-energy tensor T. This analogy is even closer when GR is also formulated in terms of differential forms (see Section 4.3). Splitting the theory into three-plus-one form (see Section 4.6), is the starting point in EM for quantization in terms of Wilson loops, and in QG for the loop Quantum Gravity (LQG) program (see, e.g., [30]). In some inertial frame in Minkowski space: A splits into the three-vector- and scalar-potentials, $\mathbf{A}$ and ϕ. F and G split into the familiar three vector fields $\mathbf{E}$ and $\mathbf{B}$ and $\mathbf{D}$ and $\mathbf{H}$, respectively; and j splits into the three-current density vector $\mathbf{j}$ and the charge density ρ. In a linear, homogeneous isotropic medium,[7] the constitutive relations are:

6 For the important conceptual distinction between the two see Section 4.7.

7 The rest frame of a material medium is a preferred inertial frame. In the case of the vacuum, a similar split may be performed with respect to any inertial frame.

$$\mathbf{D} = \epsilon\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu\mathbf{H},$$

with $\epsilon\mu = (n/c)^2 s$ ϵ and μ being the dielectric constant and permeability of the medium. Here n is its index of refraction and c is the vacuum speed of light. The second order field equations split into one three-scalar and one three-vector evolution equation:

$$\frac{\partial}{\partial t}(div\mathbf{A}) + (del)^2\phi = \rho, \quad grad\, div\, \mathbf{A} - (del)^2\mathbf{A} - \left(\frac{n}{c}\right)^2 \left(\frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mathbf{j}.$$

Using the gauge freedom to set $div\mathbf{A} = 0$ initially and $(del)^2\phi = \rho$ everywhere, the constraint equation then insures that $div\mathbf{A}$ vanishes everywhere, and the evolution equation reduces to the (three-)vectorial wave equation for $\mathbf{A}$. By judicious choice of gauge, the two degrees of freedom of the EM field have been isolated and embodied in the divergence-free $\mathbf{A}$ field, a local quantity the evolution of which proceeds independently of all other field quantities. In GR, this goal has been attained in only a few exceptional cases (see Section 4.7).

Going over from this second order (Lagrangian) to a first-order (Hamiltonian) formalism, canonical quantization of EM then may take place in either the position-representation; or the unitarily equivalent momentum-representation, leading to a Fock space representation of the free field. Since the asymptotic in- and out-fields always may be treated as free, this representation is useful for describing scattering experiments. In GR, there is no "natural" analogue of an inertial frame of reference; the closest is an arbitrarily selected foliation (global time) and fibration (relative space) (see Section 4.6.1). Geometrodynamics attempts to use the (suitably constrained) three-spatial metric (first fundamental form) of a spacelike foliation as position variables, with the second fundamental form as the corresponding velocities (see Section 4.6.3); but apparently a mathematically rigorous quantization of the theory in this form is impossible (see [2]). LQG takes the Ashtekar three-connection on the hypersurfaces as position variables (see Section 4.5); but rigorous quantization is based on the introduction of loop variables.

The attempt to better understand LQG inspired a similar approach to quantization of the EM field. The integral $\int_C \mathbf{A}$ around a loop or closed curve C in a hyperplane $t = const$ is gauge-invariant.[8] It follows from the definition of $\mathbf{E}$[9] that $\int_C \mathbf{E} = d[\int_C \mathbf{A}]/dt$; so if the $\int_C \mathbf{A}$ are taken as "position" variables, the latter will be the corresponding "velocities". The momenta conjugate to $\int_C \mathbf{A}$ are $\iint_S \mathbf{D}\cdot\mathbf{n}dS$,

[8] It is a non-local, physically significant quantity. In spaces with non-vanishing first Betti number its periods form the basis of the Aharonov–Bohm effect.

[9] If there are topological complications, the periods of $\int_C (grad\phi)$ may also be needed.

where S is any 2-surface bounded by C.[10] The relation between **D** (momentum) and **E** (velocity) is determined by the constitutive relations of the medium, the analogue of the mass in particle mechanics, which relates a particle's momentum and velocity.

In a four-dimensional formulation, the "dual momenta" are the integrals $\iint_S G$ over *any* 2-surface S. This suggests the possibility of extending the canonical loop approach to arbitrary spacelike and null initial hypersurfaces. But it is also possible to carry out a Feynman-type quantization of the theory: a classical S-T path of a such loop is an extremal in the class of timelike world tubes S (oriented 2-surfaces with boundaries) bounded by the loop integral $\int_C \mathbf{A}$ on the initial and final hyperplanes. To quantize, one assigns a probability amplitude $\exp iI(S)$ to each such S, where $I(S)$ is the surface action. The total quantum transition amplitude between the initial and final loops is the sum of these amplitudes over all such 2-surfaces.[11] More generally, loop integrals of the 1-form A for *all possible types* of closed curves C ought to be considered, leading to a Feynman-type quantization that is based on arbitrary spacelike loops. Using null-loops, null-hypersurface quantization techniques might be applicable (see Section 4.6).

The position and momentum-space representations of EM theory are unitarily equivalent; but they are not unitarily equivalent to the loop representation. In order to secure unitary equivalence, one must introduce smeared loops,[12] suggesting that measurement analysis (see the Introduction) might show that ideal measurement of loop variables requires "thickened" four-dimensional regions of S-T around a loop. The implications of measurement analysis for loop quantization of GR also deserve careful investigation (see Section 4.4).

4.3 Choice of fundamental variables in classical GR

One choice is well known: a pseudo-metric and a symmetric affine connection, and the structures derived from them. Much less explored is the choice of the conformal and projective structures (see, e.g., [14], Section 2.1, Geometries). The two choices are inter-related in a number of ways, only some of which will be discussed here.[13]

[10] Reference [32] gives a Lagrangian density for arbitrary constitutive relations. When evaluated on $t = const$, the only term in the Lagrangian density containing a time derivative is $(\partial \mathbf{A}/\partial t) \cdot \mathbf{D}$, from which the expression for the momentum follows. If a non-linear constitutive relation is used, the difference between D and E becomes significant.

[11] See [21; 22; 23].

[12] The loops are "smeared" with a one parameter family of Gaussian functions over the three-space surrounding the loop.

[13] Mathematically, all of these structures are best understood as G-structures of the first and second order; i.e. reductions of the linear frame bundle group $GL(4, R)$ over the S-T manifold with respect to various subgroups (see [28]). The metric and volume structures are first order reductions of the group with respect to the pseudo-orthogonal subgroup $SO(3, 1)$ and unit-determinant subgroup $SL(4, R)$, respectively. The projective structure

4.3.1 Metric and affine connection

The coordinate components of the pseudo-metric[14] field $g_{\mu\nu}$ are often taken as the only set of dynamical variables in GR in second order formulations of the theory. The metric tensor plays a dual role physically.

(i) Through the invariant line element ds between two neighboring events $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ it determines the *chrono-geometry* of S-T (the intervals may be space-like, time-like or null), as manifested in the behavior of ideal rods and clocks. Since ds is not a perfect differential, the proper time between two time-like separated events depends on the path between them.

(ii) Its components also serve as the potentials for the Christoffel symbols, the components of the Levi–Civita connection that determines the *inertio-gravitational field*: (a) directly, through its role in the geodesic equation governing the behavior of freely falling particles (metric geodesics are *extremals* of the interval: shortest for space-like, longest for time-like, or zero-length for null curves); and (b) indirectly, through the role of the Riemann tensor $R_{[\kappa\lambda][\mu\nu]}$ in the equation of geodesic deviation, governing tidal gravitational forces.

According to Einstein's equivalence principle, gravity and inertia are described by a single inertio-gravitational field and at any point a reference frame can always be chosen locally ("free fall"), in which the components of the field vanish. In a four-dimensional formulation of the Newtonian theory as well as in GR, this field is represented by a symmetric linear connection $\Gamma^\kappa_{\mu\nu}$. For this among other reasons, a first order formalism is preferable, taking both pseudo-metric and connection as independent dynamical variables. The connection still describes the inertio-gravitational field through the geodesic equation: affine geodesics, or better affine auto-parallel curves, are the straightest paths in S-T (the connection also determines a preferred affine parameter on these curves). The affine curvature tensor $A^\kappa_{\lambda[\mu\nu]}$, plays a role in the affine equation of geodesic deviation similar to that of the Riemann tensor in the metric equation. The first order field equations can be derived from a Palatini-type variational principle; one set consists of the compatibility conditions between metric and connection, ensuring that the connection is metric: straightest curves coincide with extremals; and the Riemann tensor agrees with the affine curvature tensor. Introducing a tetrad of basis vectors e_I and dual co-basis of one-forms e^I, the relation between tetrad components of metric, connection and curvature tensor may be expressed in various ways. Recent progress in QG has demonstrated the special significance of the one based on the cartan

and the first order prolongation of the volume structure are second order reductions of the frame bundle group. The interrelations between the structures follow from the relations between these subgroups.

[14] Often I shall simply refer to the metric, the Lorentzian signature being understood.

connection (see, e.g. [29]): the chrono-geometry is represented by means of the co-basis of 1-forms: $g = \eta_{IJ} e^I e^J$, where η_{IJ} is the Minkowski metric, and the affine connection and curvature tensor are represented by an $SO(3, 1)$ matrix-valued one-form ω^I_J, and two-form $R^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J$, respectively (see, e.g., [27] or [36]). Starting from this formulation, Ashtekar put the field equations of GR into a form closely resembling that of Yang–Mills theory by defining the "Ashtekar connection", a three-connection on a space-like hypersurface that embodies all the information in the four-connection on the hypersurface (see Section 4.6). Much recent progress in LQG is based on this step.

4.3.2 Projective and conformal structures

Neither metric nor connection are irreducible group theoretically (see the earlier note on G-structures): each can be further decomposed. The metric splits into a conformal, causality-determining structure and a volume-determining structure; the connection splits into a projective, parallel path-determining structure, and an affine-parameter-determining structure. Physically, the conformal structure determines the behavior of null wave fronts and the dual null rays. The projective structure determines the preferred ("straightest") paths of force-free monopole particles.[15] Given a pseudo-Riemannian S-T, the conformal and projective structures determine its metric. Conversely, given conformal and projective structures obeying certain compatibility conditions, the existence of a metric is guaranteed [11]. In GR, these compatibility conditions can be derived from a Palatini-type Lagrangian by taking the conformal, projective, volume-determining and affine parameter-determining structures as independent dynamical variables. There are curvature tensors associated with the conformal and the projective structure; the Weyl or conformal curvature tensor plays an important role in defining the structure of null infinity in asymptotically flat S-Ts, and the projective curvature tensor plays a similar role in defining timelike infinity. This set of structures is currently being investigated as the possible basis of an approach to QG that incorporates the insights of causal set theory (see [35]).[16]

4.4 The problem of Quantum Gravity

In the absence of an accepted theory of QG, measurability analysis (see the Introduction) of various classical dynamical variables in GR (see the previous section)

15 A preferred affine curve, or auto-parallel, curve is parameterized by a preferred affine parameter; a preferred projective path is not so parameterized.

16 A Lagrangian based on the volume-defining and causal structures is cubic in the conformal dynamical variables.

may help delimit the choice of a suitable maximal, independent set. Taking into account the quantum of action should then restrict joint measurability to compatible subsets, which could serve as a basis for quantization. The formal representation of such ideal measurements will require introduction of further, non-dynamical structures on the S-T manifold, such as tetrads, bivector fields, congruences of subspaces, etc., which are then given a physical interpretation in the measurement context (see, e.g., [26] and Sections 4.5 and 4.6 below). This question is closely related to that of initial value problems: possible choices of initial data and their evolution along congruences of subspaces (see Section 4.6). Measurability analysis in GR could be carried out at three levels: metric, connection and curvature (see the previous section).

The pseudo-metric tensor. Measurements of spatial or temporal intervals along some curve; or similar integrals of spatial two-areas and three volumes,[17] or of spatio-temporal four-volumes – or integrals of other similar quantities – could provide information about various aspects of the metric tensor. In a sense, all measurements ultimately reduce to such measurements.[18] The Introduction and Section 4.2 present arguments suggesting that four-dimensional process measurements are fundamental, measurements of apparently lower-dimensional regions actually being measurements of specialized processes approximated by such lower-dimensional regions. Because of its fundamental importance, this question deserves further investigation.

The affine connection. While the inertio-gravitational connection is not a tensor, an appropriately chosen physical frame of reference can be used to define a second, relative inertial connection; and the difference between the inertio-gravitational and the relative inertial connection, like the difference between any two connections is a tensor. So a frame-dependent gravitational tensor can be defined, and might be measurable for example, by deviations of time-like preferred affine inertio-gravitational curves from the preferred purely inertial curves defined with respect to such a frame. Fluctuations around a classical connection, also being tensors, the mean value of classical or quantum fluctuations might also be measurable.

Structures abstracted from the affine connection. Measurement analysis of “smeared” loop integrals of connection one-forms over S-T loops – both spatial and non-spatial – should be done in connection with canonical and non-canonical formulations of LQG. The possibility of similar measurements on the preferred paths

[17] This is especially important in view of the claim that quantized values of spatial two-areas and three-volumes are measurable (see, e.g., [2; 27]); for critical comments on this claim, see [19]). Possible measurability of all two-surface integrals of the curvature two-forms, and not just over spatial two-surfaces, should be investigated.

[18] Kuhlmann 2006 notes, in the context of SRQFT: “[S]pace-time localizations can specify or encode all other physical properties” [17].

of a projective structure, with results that depend only on that structure, should also be studied.

The Riemann or affine curvature tensor. DeWitt [6; 7], and Bergmann and Smith [4] studied the measurability of the components of the linearized Riemann tensor with respect to an inertial frame of reference, and drew some tentative conclusions about the exact theory. Arguing that, in gauge theories, only gauge-invariant quantities should be subject to the commutation rules, they concluded that measurement analysis should be carried out exclusively at the level of the Riemann tensor. However, this conclusion neglects three important factors.

(i) It follows from the compatibility of chrono-geometry and inertio-gravitational field in GR that measurements of the former can be interpreted in terms of the latter. As noted, the interval ds between two neighboring events is gauge invariant, as is its integral along any closed world line. Indeed, all methods of measuring components of the Riemann tensor ultimately depend on measurement of such intervals, either space-like or time-like, which agree (up to a linear transformation) with the corresponding affine parameters on geodesics.

(ii) Introduction of additional geometrical structures on the S-T manifold to model macroscopic preparation and registration devices produces additional gauge-invariant quantities relative to these structures (see [26]).

(iii) While a geometric object may not be gauge-invariant, some non-local integral of it may be. The electromagnetic four-potential, for example, is not gauge invariant, but its loop integrals are (see Section 4.2). Similarly, at the connection level, the holonomies of the set of connection one-forms play an important role in LQG. (see, e.g., [2; 27]).

In both EM and GR, one would like to have a method of loop quantization that does not depend on singling out a family of spacelike hypersurfaces. The various "problems of time" said to arise in the canonical quantization of GR seem to be artifacts of the canonical technique rather than genuine physical problems.[19] The next section discusses some non-canonical possibilities.

Some tensor abstracted from the Riemann tensor, such as the Weyl or conformal curvature tensor. For example, measurability analysis of the Newman–Penrose formalism, based on the use of invariants constructed from the components of the Weyl tensor with respect to a null tetrad (see, e.g., [33], Chapter 7), might suggest new candidate dynamical variables for quantization.

[19] That is, problems that arise from the attempt to attach physical meaning to some global time coordinate introduced in the canonical formalism, the role of which in the formalism is purely as an ordering parameter with no physical significance (see [26; 22; 23]). The real problem of time is the role in QG of the local or proper time, which is a measurable quantity classically.

4.5 The nature of initial value problems in General Relativity

Any initial value problem for a set of hyperbolic[20] partial differential equations on an n-dimensional manifold consists of two parts:

(i) specification of a set of initial data on some submanifold of dimension $d \leq (n-1)$ just sufficient to determine a unique solution; and
(ii) construction of that solution, by showing how the field equations determine the evolution of the initial data along some $(n-d)$-dimensional congruence of subspaces.

The problems can be classified in terms of the *value of d*, the nature of the *initial submanifolds*, characteristic or non-characteristic, and the nature of the $(n-d)$-dimensional *congruence of subspaces*. In GR, there are essentially only two possibilities for d:

- $d = 3$: initial hypersurface(s), with evolution along a vector field (three-plus-one problems);
- $d = 2$: two-dimensional initial surfaces with evolution along a congruence of two-dimensional subspaces (two-plus-two problems).

Below we discuss the possible nature of the initial submanifolds and of the congruence of subspaces.

4.5.1 Constraints due to invariance under a function group

If a system of m partial differential equations for m functions is derived from a Lagrangian invariant up to a divergence under some transformation group depending on q functions of the q independent variables ($q \leq m$), then by Noether's second theorem (see, e.g., [37]) there will be q identities between the m equations. Hence, q of the m functions are redundant when initial data are specified on a (non-characteristic) $(q-1)$-dimensional hypersurface, and the set of m field equations splits into q constraint equations, which need only be satisfied initially, and $(m-q)$ evolution equations. As a consequence of the identities, if the latter are satisfied everywhere, the former will also be.

The ten homogeneous ("empty space") Einstein equations for the ten components of the pseudo-metric field as functions of four coordinates are invariant under the four-function diffeomorphism group. Hence, there are four (contracted Bianchi) identities between them. In the Cauchy or three-plus-one initial value

[20] Initial value problems are well posed (i.e. have a unique solution that is stable under small perturbation of the initial data) only for hyperbolic systems. It is the choice of Lorentz signature for the pseudo-metric tensor that makes the Einstein equations hyperbolic; or rather, because of their diffeomorphism invariance (see Section 4.5.1), only with the choice of an appropriate coordinate condition (e.g. harmonic coordinates) does the system of equations become hyperbolic.

problem on a spacelike hypersurface (see [12]), the ten field equations split into four constraints and six evolution equations. The ten components of the pseudo-metric provide a very redundant description of the field, which as noted earlier has only two degrees of freedom per S-T point. Isolation of these "true" degrees of freedom of the field is a highly non-trivial problem. One approach is to find some kinematical structure, such that they may be identified with components of the metric tensor in a coordinate system adapted to this structure (see, e.g., the discussion in Section 4.6 of the conformal two-structure). Apart from some simple models (see Section 4.7), their complete isolation has not been achieved; but the program is still being pursued, especially using the Feynman approach (see, e.g., [20]). Quantization of the theory may be attempted either after or before isolation of the true observables. In quantization methods before isolation, as in loop Quantum Gravity, superfluous degrees of freedom are first quantized and then eliminated via the quantized constraints (see, e.g., [2]).

Classical GR initial value problems can serve to determine various ways of defining complete (but generally redundant) sets of dynamical variables. Each problem requires introduction of some non-dynamical structures for the definition of such a set, which suggests the need to develop corresponding measurement procedures. The results also provide important clues about possible choices of variables for QG. These questions have been extensively studied for canonical quantization. One can use initial value formulations as a method of defining ensembles of classical particle trajectories, based on specification of half the maximal classical initial data set at an initial (or final) time. The analogy between the probability of some outcome of a process for such an ensemble and the corresponding Feynman probability amplitude (see, e.g., [31]) suggests a similar approach to field theories. In Section 4.2, this possibility was discussed for the loop formulation of electromagnetic theory. The possibility of a direct Feynman-type formulation of QG has been suggested (see, e.g., [6; 7; 20]); and it has been investigated for connection formulations of the theory, in particular for the Ashtekar loop variables. Reisenberger and Rovelli [22; 23] maintain that: "Spin foam models are the path-integral counterparts to loop-quantized canonical theories".[21] These canonical methods of carrying out the transition from classical to quantum theory are based on Cauchy or spacelike hypersurface initial value problems (see Section 4.6.1). Another possible starting point for canonical quantization is the null-hypersurface initial value problem (see Section 4.6.1). Whether analogous canonical methods could be based on two-plus-two initial value problems (see Section 4.6.2) remains to be studied.

[21] See [3] for the analogy between spin foams in GR and processes in quantum theory: both are examples of cobordisms.

4.5.2 Non-dynamical structures and differential concomitants

GR is a *covariant* or *diffeomorphism-invariant* theory, this invariance being defined as invariance under the group of active point diffeomorphisms of the underlying manifold.[22] It is also *generally covariant*, meaning there are no additional intrinsic, non-dynamical background S-T structures in the theory. Such non-dynamical structures as fibrations and foliations of the manifold, subsequently introduced in order to formulate initial value problems for the dynamical variables should be introduced by means of geometrical, coordinate-independent, definitions. In particular, evolution of the dynamical variables should not involve the introduction of a preferred "global time" coordinate.[23] The dynamical fields include the pseudo-metric and inertio-gravitational connection, and any structures abstracted from them (see Section 4.3), so any differential operator introduced to describe their evolution should be independent of metric and connection.[24] In other words these operators should be differential concomitants of the dynamical variables and any non-dynamical structures introduced.[25] The ones most commonly used are the *Lie derivatives* $\mathcal{L}_v\Phi$ of geometric objects Φ with respect to a vector field v, and the *exterior derivatives* $d\omega$ of p-forms ω (see, e.g., [36], Chapter 2).[26] Various combinations and generalizations of both, such as the Schouten–Nijenhuis and Frlicher–Nijenhuis brackets, have been – or could be – used in the formulation of various initial value problems.

4.6 Congruences of subspaces and initial-value problems in GR

Initial value problems in GR involve:

(1) (a) choice of initial submanifold(s) and of complementary congruence(s) of subspaces,[27] and (b) choice of differential concomitant(s) to describe the evolution of the initial submanifold(s) along the congruence of complementary subspaces;
(2) (a) choice of a set of dynamical variables, usually related to the pseudo-metric and the affine connection, and their split-up by projection onto the initial submanifold(s) and the complementary subspace(s), and (b) choice of differential concomitants to describe their evolution;

22 It is trivially true that all physical results are independent of passive changes of the coordinate system.

23 Subsequent introduction of a coordinate system adapted to some geometrical structure is often useful for calculations. But coordinate-dependent descriptions of an initial value problem implicitly introduce these structures. But doing tacitly what should be done explicitly often creates confusion.

24 If the conformal and projective structures are taken as primary dynamical variables, the operators should be independent of these structures.

25 A differential concomitant of a set of geometric objects is a geometric object formed from algebraic combinations of the objects in the set and their partial derivatives.

26 Or, equivalently, the "curl" of a totally antisymmetric covariant tensor and the "divergence" of its dual contravariant tensor density.

27 "Complementary" in the sense that the total tangent space at any point can be decomposed into the sum of the tangent spaces of the initial sub-manifold and of the complementary subspace.

(3) a break-up of the field equations into constraint equations on the initial submanifold(s) and evolution equations along the congruence(s) of complementary subspaces.

The non-dynamical steps (1a), (1b) and (2a) will be discussed in this subsection, the dynamical ones (2b) and (3) in the next.

As discussed above, in GR there are only two basic choices for step (1a): three-plus-one or two-plus-two splits.[28] But two further choices are possible: a congruence of subspaces may be holonomic or non-holonomic; and some submanifold(s) may or may not be *null*.

In the three-plus-one case, a sufficiently smooth vector field is always holonomic (curve-forming); but in the two-plus-two case, the tangent spaces at each point of the congruence of two-dimensional subspaces may not fit together holonomically to form submanifolds.

In any theory involving a pseudo-metric (or just a conformal structure), the initial submanifold(s) or the complementary subspace(s) may be *null*, i.e. tangent to the null cone. A null tangent space of dimension p always includes a unique null direction, so the space splits naturally into $(p - 1)$- and 1-dimensional subspaces. The choice of the $(p - 1)$-dimensional subspace is not-unique but it is always space-like.

A non-null tangent space of dimension p in a pseudo-metric space of dimension n has a unique *orthogonal* tangent space of dimension $(n - p)$; so there are orthogonal projection operators onto the p- and $(n - p)$-dimensional subspaces. The evolution of initial data on a space-like p-dimensional submanifold is most simply described along a set of $(n - p)$-orthonormal vectors spanning the orthogonal congruence of subspaces (or some invariant combination of them (see the next subsection). Otherwise, *lapse* and *shift functions* must be introduced (see Sections 4.6.1 and 4.6.2), which relate the congruence of subspaces actually used to the orthonormal congruence.

By definition, null vectors are self-orthogonal, so construction of an orthonormal subspace fails for null surface-elements. And since there is no orthonormal, the null-initial value problem is rather different (see the next subsections). A similar analysis of two-plus-two null versus non-null initial value problems has not been made, but one would expect similar results.

4.6.1 Vector fields and three-plus-one initial value problems

In the Cauchy problem, the use of a unit vector field n normal to the initial hypersurface leads to the simplest formulation of the Cauchy problem. Lie derivatives w.r.t. this field $\mathcal{L}_n\Phi$ are the natural choice of differential concomitants acting on the

[28] Various sub-cases of each arise from possible further breakups, and I shall mention a few of them below.

chosen dynamical variables Φ in order to define their "velocities" in the Lagrangian and their "momenta" in the Hamiltonian formulation of the initial-value problem. Their evolution in the unit normal direction can then be computed using higher order Lie derivatives. If $\mathcal{L}_v$ with respect to another vector field v is used, the relation between v and n must be specified in terms of the lapse function ρ and the shift vector σ, with

$$v = \rho n + \sigma, \; n \cdot \sigma = 0.$$

There is a major difficulty associated with the Cauchy problem for the Einstein equations. The initial data on a space-like hypersurface, basically the first and second fundamental forms of the hypersurface, are highly redundant and subject to four constraint equations (see Section 4.5), which would have to be solved in terms of a pair of freely specifiable initial "positions" and "velocities" of the two "true observables"; their evolution would then be uniquely determined by the evolution equations. Only in a few highly idealized cases, notably for cylindrical gravitational waves (see Section 4.7), has this program been carried out using only locally-defined quantities. In general, on a spacelike hypersurface, quantities expressing the degrees of freedom and the equations governing their evolution are highly non-local and can only be specified implicity; for example, in terms of the conformal two structure (see [8]).

Things are rather better for null hypersurface and two-plus-two initial value problems. By definition, on a characteristic hypersurface of a set of hyperbolic partial differential equations no amount of initial data suffices to determine a unique solution. In GR, the characteristics are the null hypersurfaces, and data must be specified on a *pair* of intersecting null hypersurfaces in order to determine a unique solution in the S-T region to the future of both (see, e.g., [8]). There is a sort of "two-for-one" tradeoff between the initial data needed on a single Cauchy hypersurface and such a pair of null hypersurfaces. While "position" and "velocity" variables must be given on a spacelike hypersurface, only "position" variables need be given on the two null hypersurfaces. Various approaches to null hypersurface quantization have been tried. For example, one of the two null hypersurfaces may be chosen as future or past null infinity $\Im^{\pm}$ (read "scri-plus" or "scri-minus"; for their use in asymptotic quantization, see Section 7) and combined with another finite null hypersurface [13]. As noted above, a null hypersurface is naturally fibrated by a null vector field, and the initial data can be freely specified in a rather "natural" way on a family of transvecting space-like two surfaces: the projection of the pseudo-metric tensor onto a null hypersurface is a degenerate three-metric of rank two, which provides a metric for these two-surfaces. Owing to the halving of initial data (discussed above), only two quantitities per point of each initial null hypersurface (the "positions") need be specified, leading to considerable simplification of the constraint problem; the price paid is the need to specify initial data on two intersecting null hypersurfaces. One way to get these hypersurfaces is to start from a spacelike two-surface and drag it along two independent congruences of null directions, resulting in two families of spacelike two-surfaces, one on each of the two null hypersurfaces. The initial data can be specified on both families of two-surfaces, generating a double-null initial value problem. But the same data could also be specified on the initial spacelike two-surface, together with all of its Lie derivatives with respect to the two congruences of null vectors. This remark provides a natural transition to two-plus-two initial value problems.

4.6.2 Simple bivector fields and two-plus-two initial value problems

In the two-plus-two case, one starts from a space-like two-manifold, on which appropriate initial data may be specified freely (see [8]); the evolution of the data takes places along a congruence of time-like two surfaces that is either orthonormal to the initial submanifold, or is related to the orthonormal subspace element by generalizations of the lapse and shift functions. The congruence is holonomic, and a pair of commuting vector fields[29] spanning it may be chosen, and evolution off the initial two-manifold studied using Lie derivatives w.r.t. the two vector fields. They may be chosen either as one time-like and one space-like vector, which leads to results closely related to those of the usual Cauchy problem;[30] or more naturally as two null vectors, which, as noted above, leads to results closely related to the double-null initial value problem. It is also possible entirely to avoid such a breakup of the two-surfaces by defining a differential concomitant that depends on the metric of the two-surface elements.

4.6.3 Dynamical decomposition of metric and connection

A p-dimensional submanifold in an n-dimensional manifold can be "rigged" at each point with a complementary $(n - p)$-dimensional subspace "normal" to it.[31] Every co- or contra-variant vector at a point of the surface can be uniquely decomposed into tangential and normal components; and hence any tensor can be similarly decomposed.

Metric: the concept of "normal subspace" may now be identified with "orthogonal subspace",[32] the metric tensor g splits into just two orthogonal components[33]

[29] They are chosen to commute, so that all results are independent of the order, in which dragging along one or the other vector field takes place.
[30] If one drags the space-like two-surface first with the space-like vector field, one gets an initial space-like hypersurface.
[31] The word *normal* here is used without any metrical connotation. *Transvecting* would be a better word, but I follow the terminology of Weyl.
[32] This identification excludes the case of null submanifolds.
[33] Here again, I avoid the use of indices where their absence is not confusing.

$$g = {}'g + {}''g \qquad {}'g \cdot {}''g = 0,$$

where $'g$ refers to the p-dimensional submanifold, and $''g$ refers to the $(n - p)$-dimensional orthogonal rigging subspace. The properties of these subspaces, including whether they fit together holonomically to form submanifolds, can all be expressed in terms of $'g$, $''g$ and their covariant derivatives; and all non-null initial value problems can be formulated in terms of such a decomposition of the metric. It is most convenient to express $'$g in covariant form, in order to extract the two dynamical variables from it, and to express $''$g in contravariant form, in order to use it in forming the differential concomitant describing the evolution of the dynamical variables. Note that $''$g is the pseudo-rotationally invariant combination of any set of pseudo-orthonormal basis vectors spanning the time-like subspace, and one may form a similarly invariant combination of their Lie derivatives.[34] In view of the importance of the analysis of the affine connection and curvature tensors in terms of one- and two-forms, respectively, in carrying out the analysis at the metric level, it is important to include representations based on tetrad vector fields and the dual co-vector bases, spanning the p-dimensional initial surface and the $(n - p)$-dimensional rigging space by corresponding numbers of basis vectors.

Connection: an n-dimensional affine connection can be similarly decomposed into four parts with respect to a p-dimensional submanifold and complementary "normal" $(n - p)$-dimensional subspace (see the earlier note). Using the n-connection consider an infinitesimal parallel displacement in a direction tangential to the submanifold. The four parts are as follows.

(i) *The surface or* (t, t) *affine connection*. The p-connection on the submanifold that takes a tangential (t) vector into the tangential (t) component of the parallel-displaced vector.
(ii) *The longitudinal or* (t, n) *curvature*.[35] The mapping taking a tangential (t) vector into the infinitesimal normal (n) component of its parallel-displaced vector.
(iii) *The* (n, n) *torsion*.[36] The linear mapping taking a normal (n) vector into the infinitesimal normal (n) component of its parallel-displaced vector.
(iv) *The transverse or* (n, t) *curvature*. The linear mapping that taking a normal (n) vector into the infinitesimal tangential (t) component of its parallel-displaced vector.

One gets a similar decomposition of the matrix of connection one-forms by using covectors. These decompositions of metric and connection can be used to

[34] The simple multivector formed by taking the antisymmetric exterior product of the basis vectors is also invariant under a pseudo-rotation of the basis, and the exterior product of their Lie derivatives is also invariant and may also be used.
[35] The use of "curvature" here is a reminder of its meaning in the Frenet–Serret formulas for a curve, and has nothing to do with the Riemannian or affine curvature tensors.
[36] Note this use of "torsion" has nothing to do with an asymmetry in the connection. All connections considered in this paper are symmetric.

investigate (3 + 1) and (2 + 2) decompositions of the first order form of the field equations and of the compatibility conditions between metric and affine connection (see Sections 4.3 and earlier in 4.6), and in first order formulations of initial value problems. If the n-connection is metric, then "normal" has the additional meaning of "orthogonal" (see discussion above). The (t, t) surface affine connection is (uniquely) compatible with the surface metric; the (t, n) (n, t) curvatures are equivalent; and the (n, n) torsion reduces to an infinitesimal rotation. On a hypersurface ($p = n - 1$), the torsion vanishes, and the (t, n) and (n, t) curvatures are equivalent to the second fundamental form of the hypersurface.

The Ashtekar connection combines the (t, t) and (n, t) curvatures into a single three-connection. Extension of the Ashtekar variables, or some generalization of them, to null hypersurfaces is currently under investigation.[37] In the two-plus-two decomposition, there is a pair of second fundamental forms and the (n, n) rotation is non-vanishing. For a formulation of the two-plus-two initial value problem when the metric and connection are treated as independent before imposition of the field equations, see [25]. Whether some analogue of the Ashtekar variables can be usefully introduced in this case remains to be studied.

4.7 Background space-time symmetry groups

The isometries of a four-dimensional pseudo-Riemannian manifold are characterized by two integers: the dimension $m \leq 10$ of its isometry group (i.e. its group of automorphisms or motions) and the dimension $o \leq min(4, m)$ of this group's highest-dimensional orbits (see, e.g., [36; 15]). There are two extreme cases.

The maximal symmetry group: $(m = 10, o = 4)$. Minkowski S-T is the unique Ricci-flat S-T in this group. Its isometry group is the Poincaré or inhomogeneous Lorentz group, acting transitively on the entire S-T manifold. Special-relativistic field theories involving field equations that are invariant under this symmetry group are the most important example of background-dependent theories (see Introduction). At the other extreme is

The class of generic metrics: $(m = 0, o = 0)$. These S-Ts have no non-trivial isometries. The class of all solutions to a set of covariant field equations (see Section 4.5.2) will include a subclass – by far the largest – of generic metrics.[38]

37 For a review of some results of a generalization based on null hypersurfaces, see [24]. D'Inverno and coworkers have researched null Ashtekar variables.

38 This global, active diffeomorphism group should not be confused with the groupoid of passive, local coordinate transformations. Nor must the trivial freedom to carry out active diffeomorphisms acting on all structures on the manifold, including whatever fixed background metric field (such as the Minkowski metric) may be present, be confused with the existence of a subgroup of such diffeomeorphisms that constitutes the isometry group of this background metric.

4.7.1 Non-maximal symmetry groups and partially fixed backgrounds

Covariant theories not involving any background S-T structures, such as GR, are called generally covariant, background-independent theories (see Section 4.5.2). We shall say a theory is a *partially fixed background theory* if the metric solutions to a background-independent theory are further required to preserve some fixed, non-maximal isometry group. These solutions belong to some class between the two extremes discussed above. Although the overwhelming majority of solutions to the Einstein equations must be generic, no generic solution is known. Only the imposition of a partially fixed background isometry group enables construction of explicit solutions (see, e.g., [36]). The background-dependent isometry group determines a portion of the pseudo-metric tensor field non-dynamically, and the remaining, unrestricted portion obeys a reduced set of dynamical field equations. For each isometry group one must determine how much dynamical freedom remains. Considerable work has been done on the quantization of two classes of such solutions.

(i) The "*mini-superspace*" cosmological solutions, in which the isometry group imposed is so large that only functions of one parameter (the "time") are subject to dynamical equations. Quantization here resembles that of a system of particles rather than fields, and does not seem likely to shed too much light on the generic case.

(i) The "*midi-superspace*" solutions, notably the cylindrical wave metrics (see [5]), for which sufficient freedom remains to include both degrees of freedom of the gravitational field. In an appropriately adapted coordinate system, they can be isolated and represented by a pair of "scalar" fields obeying non-linear, coupled scalar wave equations in two-dimensional flat S-T. In addition to static and stationary fields, the solutions include gravitational radiation fields having both states of polarization. Their quantization can be carried out as if they were two-dimensional fields. But, of course, the remaining portions of the metric must be constructed and diffeomorphism invariance of all results carefully examined, as well as possible implications for the generic case. Niedermaier in [20] summarizes the work done on Feynman path quantization of such models.

Marugan and Montejo have discussed quantization of gravitational plane waves, and Stephani *et al*. in [36] discuss solutions to the Einstein equations having groups of motions with null and non-null orbits, so it should be possible to study the quantization of such metrics in a systematic way.

4.7.2 Small perturbations and the return of diffeomorphism invariance

While the fiber manifold consisting of all four-metrics over a base manifold is itself a manifold, the space of all four-geometries is not.[39] It is a *stratified manifold*,

[39] The space of all metrics divides into equivalence classes under the diffeomorphism group, suitably restricted for each subclass of metrics having a common isometry group. Each equivalence class corresponding to a

partitioned into slices; each of which consists of all geometries having the same isometry group. But, unless it is restricted to lie within some isometry group, the smallest perturbation of a geometry with non-trivial isometry group takes the resulting geometry into the generic slice of the stratified manifold. This observation is often neglected; in particular, when perturbation-theoretic quantization techniques developed for special relativistic field theories are applied to perturbations of the Minkowski solution in GR. Infinitesimal diffeomorphisms of such perturbations cannot be treated as pure gauge transformations on the fixed background Minkowki S-T, but modify the entire causal and inertio-gravitational structure (see, e.g., [10], Chapter 21). This is the fundamental reason for the problems that arise in formally applying special relativistic quantization techniques to such perturbations.

4.7.3 Asymptotic symmetries

An important class of solutions to the field equations, while lacking global symmetries, has a group of asymptotic symmetries as infinity is approached along null directions, which permits their asymptotic quantization (see [15], also [17], Section VI, and [1]). Imposition of certain conditions on the behavior of the Weyl tensor in the future or past null limit allows conformal compactification of this class of S-Ts by adjoining boundary null hypersurfaces, $\Im^{\pm}$, to the S-T manifold. Both $\Im^{\pm}$ have a symmetry group that is independent of particular dynamical solutions to the field equations in this class. Thus, on $\Im^{\pm}$ there is a separation of kinematics and dynamics, and a more or less conventional quantization based on this asymptotic symmetry group can be carried out. "More or less" because the asymptotic symmetry group, the Bondi–Metzner–Sachs (BMS) group, is not a finite-parameter Lie group like the Poincaré group usually used to introduce gravitons in the linear approximation, but includes four so-called "supertranslation", functions that depend on two "angular" variables. Nevertheless, asymptotic gravitons with two states of polarization may be defined as representations of the BMS group, no matter how strong the interior gravitational field [1].

4.8 Conclusion

This paper has discussed only a few possible approaches to quantization of the field equations of GR. In spite of its emphasis on background-independent techniques, it is rather conservative, ignoring such promising avenues of research as

single four-geometry, or physical S-T. The quotient space (see [4]) of the space of all metrics by the (suitably restricted) diffeomorphism group is a four-dimensional superspace (for three-dimensional superspace see work by Fischer), which is a stratified manifold.

causal set theory, causal dynamic triangulations, twistor theory; and attempts to derive S-T structures as emerging from radically different underlying entities, such as the symmetries of coherent states in quantum information theory (such theories are reviewed elsewhere in this volume). It is by no means certain that any of the conservative approaches will lead to a fruitful fusion of quantum theory and GR – indeed, it is even probable that they will not. But until some approach has been developed leading to a consensus in the QG community, every approach deserves to be explored to its limits, if only to draw from the limited successes and ultimate failure of each such attempt, lessons for the formulation of better alternative approaches.

Acknowledgements

I thank Mihaela Iftime and Christian Wuthrich [38] for reading an earlier draft of this paper and each making several valuable suggestions for improvements adopted in this version.

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5

Spacetime symmetries in histories canonical gravity

N. SAVVIDOU

5.1 Introduction

5.1.1 The principles of General Relativity

The construction of a quantum theory of gravity is a major ambition of modern physics research. However, the absence of any direct experimental evidence implies that we do not have any empirical point of reference about the principles that will underlie this theory. We therefore have to proceed mainly by theoretical arguments, trying to uncover such principles from the structure of the theories we already possess.

Clearly, the most relevant theory for this purpose is General Relativity, which provides the classical description of the gravitational field. General Relativity is essentially based on two principles, uncovered by Einstein after the continuous effort of seven years. The first one asserts the importance of the spacetime description: all gravitational phenomena can be expressed in terms of a Lorentzian metric on a four-dimensional manifold. The second one is the principle of general covariance: the Lorentzian metric is a dynamical variable, its equations of motion preserve their form in all coordinate systems of the underlying manifold.

The first principle defines the *kinematics* of General Relativity. It identifies the basic variables that are employed in the theory's mathematical description, and determines their relation to physical quantities measured in experiments. This principle implies that General Relativity is a geometric theory. It refers primarily to the relations between spacetime events: the metric determines their distance and causal relation. The gravitational 'force' is reduced to nothing but a phenomenological consequence of the non-trivial geometry of spacetime; distance and causal structure, i.e. light-cones, are the fundamental concepts.

The principle of general covariance refers to the *dynamics* of the theory. Einstein had by 1912 completed the identification of the new theory's kinematical structure. He believed that the spacetime geometry was a dynamical quantity, its curvature

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

determined by its interaction with matter, and he tried to determine its equations of motion. Remarkably, these turned out to be almost uniquely fixed by a symmetry requirement, which became known as the principle of general covariance: the equations of motion ought to retain their form in any coordinate system associated to the manifold. In modern language, one would say that the equations of motion ought to be invariant under the action of the group $Diff(M)$ of passive diffeomorphisms on the spacetime manifold M. If, however, the equations of motion are invariant under the action of $Diff(M)$, they cannot contain any non-dynamical fields, for the latter do not remain invariant under the action of the $Diff(M)$ group. Hence, general covariance implies that the theory of gravity ought to be *background independent*, i.e. no fixed externally imposed structures are to be used in the formulation of the theory's laws of motion.

The standard quantization procedures applied to General Relativity seem to contradict its basic principles. Quantum theory is fundamentally canonical: the Hilbert space refers to the properties of a system at a single moment of time, hence, manifest covariance is lost at the first step.

More importantly, the canonical commutation relations are defined on a 'spacelike' surface, however, a surface is spacelike with respect to some particular spacetime metric g – which is itself a quantum observable that is expected to fluctuate. The prior definability of the canonical commutation relations is not merely a mathematical requirement. In a generic quantum field theory they implement the principle of *microcausality*: namely that field observables that are defined in spacelike separated regions commute. However, if the notion of spacelikeness is also dynamical, it is not clear in what way this relation will persist.

The canonical treatment of Quantum Gravity introduces a spacelike foliation that enters the quantum description. However, the physical predictions should be independent of the choice of this foliation. This is part of the famous 'problem of time', as are attempts to understand the spacetime diffeomorphism group in this context.

In one or another form the aforementioned problems persist in the major programmes towards Quantum Gravity, namely canonical quantization and spacetime (perturbative) quantization – see [12] for a related discussion.

The histories framework is motivated by the belief that it would be prudent to preserve the basic principles of General Relativity in our attempts to quantize gravity. This reason conveys the importance of a genuine spacetime description of physical events.

5.1.2 The histories theory programme

The fundamental entity of the theory is the notion of a history: it corresponds to the specification of information about the state of a system, at different moments of time. The development of the Histories Projection Operator (HPO) approach in particular, showed that it is characterised by two distinctive features. First, a history is a temporally extended object that it is represented quantum mechanically by a *single* projection operator, on a suitably constructed Hilbert space [12]. Second, the theory possesses a novel temporal structure [21], since time is implemented by two distinct parameters, one of which refers to the kinematics of the theory, while the other refers to its dynamical behavior. At the classical level the two parameters coincide for all histories that correspond to the solutions of the equations of motion.

Hence, the HPO theory is endowed with a rich kinematical structure. In the case of General Relativity this results in the fact that different 'canonical' descriptions of the theory, corresponding to different choices of spacelike foliation, coexist in the space of histories and may be related by a properly defined transformation. This allows the preservation of the spacetime description of the theory, even if one chooses to work with canonical variables.

The General Relativity histories theory suggests a quantum mechanical treatment of the full Lorentzian metric. Other programmes also put emphasis on the spacetime description, namely the causal set approach [6], and the Lorentzian dynamical triangulations [1]. The twistor programme has the same avowed aims. The HPO formalism, however, allows the incorporation of other theories, enriching them with a spacetime kinematical description, while preserving the main features of their dynamical behavior.

A histories-based quantisation of General Relativity, like the canonical theory, has to address the issue of defining an appropriate Hamiltonian constraint operator. Loop quantum gravity has made the greatest progress so far in the construction of such an operator, therefore it would be very interesting to exploit a histories version of this theory.

5.2 History Projection Operator theory

5.2.1 Consistent histories theory

The consistent histories formalism was originally developed by Griffiths [9] and Omnés [17; 18], as an interpretation of quantum theory for closed systems.

Gell-Mann and Hartle [8] elaborated this scheme in the case of quantum cosmology – the Universe being regarded as a closed system. They emphasised in particular that a theory of Quantum Gravity that is expected to preserve the spacetime character of General Relativity would need a quantum formalism in which the irreducible elements are temporally extended objects, namely histories.

The basic object in the consistent histories approach is a history

$$\alpha := (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, ..., \hat{\alpha}_{t_n}), \tag{5.1}$$

which is a time-ordered sequence of properties of the physical system, each one represented by a single-time projection operator on the standard Hilbert space. The emphasis is given on histories rather than states at a single time.

The probabilities and the dynamics are contained in the decoherence functional, a complex-valued function on the space of histories

$$d_{H,\rho}(\alpha, \beta) = tr(\tilde{C}^{\dagger}_{\alpha}\rho_{t_0}C_{\beta}), \tag{5.2}$$

where ρ_{t_0} is the initial quantum state and where

$$\tilde{C}_{\alpha} := U(t_0, t_1)\hat{\alpha}_{t_1}U(t_1, t_2)...U(t_{n-1}, t_n)\hat{\alpha}_{t_n}U(t_n, t_0) \tag{5.3}$$

is the class operator that represents the history α.

When a set of histories satisfies a decoherence condition, $d_{\mathcal{H},\rho}(\alpha\,, \beta) = 0$ then $\alpha\,, \beta$ are in the consistent set, which means that we have zero interference between different histories, and then it is possible to consistently assign probabilities to each history in that set; it is called a *consistent set*.

Then we can assign probabilities to each history in the consistent set

$$d_{\mathcal{H},\rho}(\alpha\,, \alpha) = Prob(\alpha; \rho_{t_0}) = tr(\tilde{C}^{\dagger}_{\alpha}\rho_{t_0}C_{\alpha}). \tag{5.4}$$

One of the aims of the histories formalism is to provide a generalised quantum mechanics definition, so that, one may deal with systems possessing a non-trivial causal structure, including perhaps Quantum Gravity. In particular, Hartle has provided examples of how this procedure would work, based mainly on a path integral expression of the decoherence functional [10].

5.2.2 HPO formalism – basics

In the History Projection Operator (HPO) approach to consistent histories theory the emphasis is given on the *temporal* quantum logic. Thus it offers the possibility of handling the ideas of space and time in a significantly new way within the quantum theory.

A history is represented by a tensor product of projection operators

$$\hat{\alpha} := \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes ... \otimes \hat{\alpha}_{t_n}, \tag{5.5}$$

each operator $\hat{\alpha}_{t_i}$ being defined on a copy of the single-time Hilbert space $\mathcal{H}_{t_i}$ at that time t_i and corresponding to some property of the system at the same time indicated by the t-label. Therefore – in contrast to the sum over histories formalism – a history is *itself* a genuine projection operator defined on the history Hilbert space $\mathcal{V}_n$, which is a tensor product of the single-time Hilbert spaces

$$\mathcal{V}_n := \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes ... \otimes \mathcal{H}_{t_n}. \tag{5.6}$$

In order to define continuous time histories, we do not take the continuous limit of the tensor product of Hilbert spaces, as it cannot be properly defined. The *history group*, which is a generalised analogue of the canonical group of standard quantum theory was employed [13; 15], in order to construct the continuous-time history Hilbert space.

For example, for a particle moving on a line the single-time canonical commutation relations, e.g.

$$[\hat{x}, \hat{p}] = i\hbar \tag{5.7}$$

become the history group that described by the following history commutation relation, defined at unequal moments of time

$$[\hat{x}_t, \hat{p}_{t'}] = i\hbar\delta(t - t'), \tag{5.8}$$

the operator x_t refers to the position of the particle at a specific fixed moment of time t. The key idea in the definition of the history group is that the spectral projectors of the generators of its Lie algebra represent propositions about phase space observables of the system.

The notion of a '*continuous* tensor product' – and hence '*continuous* temporal logic' – arises via a representation of the history algebra. In order to describe discrete-time histories we have to replace the delta function, on the right-hand side of Eq. (5.8), with the Kronecker delta.

Propositions about histories of the system are associated with projectors on history Hilbert space. As we shall see in the following section, the temporal structure that was later introduced [21], allowed the interpretation of the index t as the index that does not refer to dynamics – it is *not* the parameter of time evolution – it is the label of the temporal quantum logic, in the sense that it refers to the time a proposition about momentum or position is asserted.

It is important to remark that physical quantities are naturally time-averaged in this scheme. The smeared form of the history algebra

$$[\hat{x}_f, \hat{x}_g] = 0 = [\hat{p}_f, \hat{p}_g] \tag{5.9}$$

$$[\hat{x}_f, \hat{p}_g] = i\hbar(f, g), \tag{5.10}$$

where: $(f, g) = \int_{-\infty}^{\infty} dt \;\; f(t)g(t)$, resembles that of a one-dimensional quantum field theory and therefore techniques from quantum field theory may be used in the study of these representations. Analogous versions of the history group have been studied for field theories [22; 14; 7], while the relation to the representations of the related canonical theories has been studied in [2].

The existence of a properly defined Hamiltonian operator H is proved to uniquely select the physically appropriate representation of the history algebra, therefore the definition of the time-averaged energy operator H is crucial for the formalism.

5.2.3 Time evolution – the action operator

The introduction of the history group allowed the definition of continuous-time histories; however, any notion of dynamics was lost and the theory was put on hold. The situation changed after the introduction of a new idea concerning the notion of time: the distinction between dynamics and kinematics corresponds to the mathematical distinction between the notion of 'time evolution' from that of 'time ordering' or 'temporal logic time'. The distinction proved very fruitful, especially for the histories General Relativity theory.

The crucial step in the identification of the temporal structure was the definition of the action operator S [21], a quantum analogue of the Hamilton–Jacobi functional, written for the case of a one-dimensional simple harmonic oscillator as

$$S_\kappa := \int_{-\infty}^{+\infty} dt\ (p_t \dot{x}_t - \kappa(t) H_t), \tag{5.11}$$

where $\kappa(t)$ is an appropriate test function. The results can be generalised appropriately for other systems.

The first term of the action operator S_κ is identical to the kinematical part of the classical phase space action functional. This 'Liouville' operator is formally written as

$$V = \int_{-\infty}^{\infty} dt\ (p_t \dot{x}_t) \tag{5.12}$$

so that $S_\kappa = V - H_\kappa$. The 'average-energy' operator

$$\hat{H}_\kappa = \int_{-\infty}^{\infty} dt\, \kappa(t) \hat{H}_t; \quad H_t = \frac{p_t^2}{2m} + \frac{m\omega^2}{2} x_t^2$$

is also smeared in time by smearing functions $\kappa(t)$. The Hamiltonian operator may be employed to define *Heisenberg* picture operators for the smeared operators like x_f

$$\hat{x}_f(s) := e^{\frac{i}{\hbar} s \hat{H}}\ \hat{x}_f\ e^{-\frac{i}{\hbar} s \hat{H}}$$

where $f = f(t)$ is a smearing function. Hence $\hat{H}_\kappa$ generates transformations with respect to the Heisenberg picture parameter s, therefore, s *is the time label as it appears in the implementation of dynamical laws*

$$e^{\frac{i}{\hbar} \tau \hat{H}}\ \hat{x}_f(s)\ e^{-\frac{i}{\hbar} \tau \hat{H}} = \hat{x}_f(s+\tau).$$

The novel feature in this construction is the definition of the 'Liouville' operator $\hat{V}$, which generates transformations with respect to the time label t as it appears in the history algebra, hence, t *is the label of temporal logic or the label of kinematics*

$$e^{\frac{i}{\hbar}\tau\hat{V}}\,\hat{x}_f(s)e^{-\frac{i}{\hbar}\tau\hat{V}} = \hat{x}_{f'}(s)\,, \quad f'(t) = f(t+\tau).$$

We must emphasise the distinction between the notion of time evolution from that of logical time-ordering. The latter refers to the *temporal ordering* of logical propositions in the consistent histories formalism. The corresponding parameter t does not coincide with the notion of physical time – as it is measured for instance by a clock. It is an abstraction, which keeps from physical time only its *ordering* properties, namely that it designates the sequence at which different events happen – the same property that it is kept by the notion of a time-ordered product in quantum field theory. Making this distinction about time, it is natural to assume that in the HPO histories one may not use the same label for the time evolution of physical systems and the time-ordering of events. The former concept incorporates also the notion of a clock, namely it includes a *measure* of time duration, as something distinct from temporal ordering.

The realisation of this idea on the notion of time was possible in this particular framework because of the logical structure of the theory, as it was originally introduced in the consistent histories formalism and as it was later recovered as temporal logic in the HPO scheme. One may say then that the definition of these two operators, V and H, implementing time translations, signifies the *distinction* between the kinematics and the dynamics of the theory.

However, a crucial result of the theory is that $\hat{S}_\kappa$ *is the physical generator of the time translations in histories theory*, as we can see from the way it appears in the decoherence functional and hence the physical predictions of the theory.

Relativistic quantum field theory

In the classical histories theory, the basic mathematical entity is the space of differentiable paths $\Pi = \{\gamma \mid \gamma : R \to \Gamma\}$, taking their value in the space Γ of classical states. The key idea in this new approach to classical histories is contained in the symplectic structure on this space of temporal paths. In analogy to the quantum case, there are generators for two types of time transformation: one associated with classical temporal logic, and one with classical dynamics. One significant feature is that the paths corresponding to solutions of the classical equations of motion are determined by the requirement that they remain invariant *under the symplectic transformations generated by the action*.

Starting from the field theory analogue of the Eq. (5.7), the relativistic analogue of the two types of time translation in a non-relativistic history theory is the existence of two distinct *Poincaré groups*. The 'internal' Poincaré group is analogous to the one in the standard canonical quantisation scheme. However, the 'external' one is a novel object: it is similar to the group structure that arises in a *Lagrangian* description. In particular, it *explicitly performs changes of foliation*. It has been shown that even though the representations of the history algebra are foliation dependent, the physical quantities (probabilities) *are not*.

5.3 General Relativity histories

The application of the ideas of continuous-time histories led to a 'covariant' description of General Relativity in terms of a Lorentzian metric g and its 'conjugate momentum' tensor π, on a spacetime manifold M with the topology $\Sigma \times R$ [23; 24]. We define the covariant history space $\Pi^{cov} = T^{*}\mathrm{LRiem}(M)$ as the cotangent bundle of the space of all Lorentzian, globally hyperbolic four metrics on M, and where $\mathrm{LRiem}(M)$ is the space of all Lorentzian four-metrics.

Π^{cov} is equipped with a symplectic structure, or else with the covariant Poisson brackets algebra on Π^{cov},

$$\{g_{\mu\nu}(X),\ g_{\alpha\beta}(X')\} = 0 = \{\pi^{\mu\nu}(X),\ \pi^{\alpha\beta}(X')\}$$
$$\{g_{\mu\nu}(X),\ \pi^{\alpha\beta}(X')\} = \delta^{\alpha\beta}_{(\mu\nu)}\, \delta^4(X, X'),$$

where $\delta^{\alpha\beta}_{(\mu\nu)} := \frac{1}{2}(\delta_{\mu}{}^{\alpha}\delta_{\nu}{}^{\beta} + \delta_{\mu}{}^{\beta}\delta_{\nu}{}^{\alpha})$. The physical meaning of π can be understood after the $3+1$ decomposition of M in which it will be related to the canonical conjugate momenta.

5.3.1 Relation between spacetime and canonical description

The representation of the group Diff(M)

The relation between the spacetime diffeomorphism algebra, and the Dirac constraint algebra has long been an important matter for discussion in Quantum Gravity. It is very important that, in this new construction, the two algebras appear together in an explicit way: the classical theory contains realisations of both the spacetime diffeomorphism group *and* the Dirac algebra.

The history space Π^{cov} carries a symplectic action of the *Diff*(M) group of spacetime diffeomorphisms, with the generator defined for any vector field W on M as $V_W := \int d^4X\, \pi^{\mu\nu}(X)\, \mathcal{L}_W g_{\mu\nu}(X)$, where $\mathcal{L}_W$ denotes the Lie derivative with respect to W. The functions V_W satisfy the Lie algebra of *Diff*(M)

$$\{V_{W_1}, V_{W_2}\} = V_{[W_1, W_2]},$$

where $[W_1, W_2]$ is the Lie bracket between vector fields W_1, W_2 on M.

The spacetime description presented is *kinematical*, in the sense that we do not start from a Lagrangian formalism and deduce from this the canonical constraints.

We start instead from the histories canonical General Relativity and we show that this formalism is augmented by a spacetime description that carries a representation of the *Diff(M)* group.

In the standard canonical formalism we introduce a spacelike foliation $\mathcal{E} : R \times \Sigma \rightarrow M$ on M, with respect to a fixed Lorentzian four-metric g. Then the spacelike character of the foliation function implies that the pull-back of the four metric on a surface Σ is a Riemannian metric with signature $+ + +$. In the histories theory we obtain a path of such Riemannian metrics $t \mapsto h_{ij}(t, \underline{x})$, each one defined on a copy of Σ_t with the same t label. However, a foliation cannot be spacelike with respect to *all* metrics g and in general, for an arbitrary metric g the pullback of a metric $\mathcal{E}^* g$ *is not* a Riemannian metric on Σ.

This point reflects a major conceptual problem of Quantum Gravity: the notion of ‘spacelike’ has no *a priori* meaning in a theory in which the metric is a non-deterministic dynamical variable; in the absence of deterministic dynamics, the relation between canonical and covariant variables appears rather puzzling. In Quantum Gravity, especially, where one expects metric fluctuations the notion of spacelikeness is problematic.

In histories theory this problem is addressed by introducing the notion of a *metric dependent* foliation $\mathcal{E}[\cdot]$, defined as a map $\mathcal{E}[g] : \text{LRiem}(M) \mapsto FolM$, that assigns to each Lorentzian metric a foliation that is *always* spacelike with respect to that metric. Then we use the metric dependent foliation $\mathcal{E}[g]$ to define the canonical decomposition of the metric g with respect to the canonical three-metric h_{ij}, the lapse function N and the shift vector N^i. Defined in this way h_{ij} is *always* a Riemannian metric, with the correct signature. In the histories theory therefore, *the* $3 + 1$ *decomposition preserves the spacetime character of the canonical variables*, a feature that we expect to hold in a theory of Quantum Gravity.

The introduction of the metric-dependent foliation allows the expression of the symplectic form Ω in an equivalent canonical form, on the space of canonical General Relativity histories description Π^{can}, by introducing conjugate momenta for the three-metric π^{ij}, for the lapse function p and for the shift vector p_i. Thus we prove that there exists an equivalence between the covariant history space Π^{cov} and the space of paths on the canonical phase space $\Pi^{can} = \times_t (T^*\text{Riem}(\Sigma_t) \times T^*Vec(\Sigma_t) \times T^*C^\infty(\Sigma_t))$, where $\text{Riem}(\Sigma_t)$ is the space of all Riemannian three-metrics on the surface Σ_t, $Vec(\Sigma_t)$ is the space of all vector fields on Σ_t, and $C^\infty(\Sigma_t)$ is the space of all smooth scalar functions on Σ_t.

Canonical description

The canonical history space of General Relativity Π^{can} is a suitable subset of the Cartesian product of copies of the phase space Γ of standard canonical General

Relativity $\Pi^{can} \subset \times_t \Gamma_t$, where $\Gamma_t = \Gamma(\Sigma_t)$. A history is a smooth map $t \mapsto (h_{ij}, \pi^{kl}, N^i, p_i, N, p)(t, \underline{x})$.

We then obtain the history version of the canonical Poisson brackets from the covariant Poisson brackets, for instance

$$\{h_{ij}(t, \underline{x}) \,, \pi^{kl}(t', \underline{x}')\} = \delta_{(ij)}{}^{kl}\, \delta(t, t')\, \delta^3(\underline{x}, \underline{x}').$$

5.3.2 Invariance transformations

The generators of the *Diff(M)* group, act on the spacetime variables in a natural way, generating spacetime diffeomorphisms

$$\{\, g_{\mu\nu}(X), V_W\,\} = \mathcal{L}_W g_{\mu\nu}(X)$$
$$\{\, \pi^{\mu\nu}(X), V_W\,\} = \mathcal{L}_W \pi^{\mu\nu}(X).$$

The coexistence of the spacetime and the canonical variables allows one to write the history analogue of the canonical constraints. The canonical description leads naturally to a one-parameter family of super-Hamiltonians $t \mapsto \mathcal{H}_\perp(t, \underline{x})$ and super-momenta $t \mapsto \mathcal{H}_i(t, \underline{x})$, that they satisfy a history version of the Dirac algebra. We can also write the constraints in a covariant form [23; 24].

Equivariance condition

The explicit relation between the *Diff(M)* group and the canonical constraints is realised by an important mathematical restriction on the foliation, the *equivariance condition*. This condition follows from the requirement of general covariance, namely that the description of the theory ought to be invariant under changes of coordinate systems implemented by spacetime diffeomorphisms.

A metric-dependent foliation functional $\mathcal{E}$: LRiem$(M) \to Fol(M)$ is defined as an *equivariant foliation* if it satisfies the mathematical condition

$$\mathcal{E}[f^* g] = f^{-1} \circ \mathcal{E}[g], \tag{5.13}$$

for all metrics g and $f \in \textit{Diff}(M)$. The interpretation of this condition is as follows: if we perform a change of the coordinate system of the theory under a spacetime diffeomorphism, then the expressions of the objects defined in it will change, and so will the foliation functional $\mathcal{E}[g]$ and the four-metric g. However, the change of the foliation must be compensated by the change due to its functional dependence on the metric g. This is essentially the *passive interpretation* of spacetime diffeomorphisms: the foliation functional ‘looks the same’ in all coordinate systems.

Relation between the invariance groups

One of the deepest issues to be addressed in canonical gravity is the relation of the algebra of constraints to the spacetime diffeomorphisms group. The canonical constraints depend on the $3+1$ decomposition and hence on the foliation.

The equivariance condition manifests a striking result both in its simplicity and its implications: the action of the spacetime *Diff(M)* group *preserves* the set of the constraints, in the sense that it transforms a constraint into another of the same type but of different argument. Hence, the choice of an equivariance foliation implements that histories canonical field variables related by spacetime diffeomorphisms are physically equivalent. Furthermore this result means also that the group *Diff(M)* is represented in the space of the true degrees of freedom. Conversely, the space of true degrees of freedom is invariant under *Diff(M)*.

Hence, the requirement of the physical equivalence of different choices of time direction is satisfied by means of the equivariance condition.

5.3.3 Reduced state space

General Relativity is a parameterised system in the sense that it has vanishing Hamiltonian on the reduced phase space due to the presence of first class constraints.

In the histories framework we define the history constraint surface $C_h = \{t \mapsto C, t \in R\}$ as the space of maps from the real line to the single-time constraint surface C of canonical General Relativity. The reduced state space is obtained as the quotient of the history constraint surface, with respect to the action of the constraints.

The Hamiltonian constraint is defined as $H_\kappa = \int dt\, \kappa(t) h_t$, where $h_t := h(x_t, p_t)$ is first-class constraint. For all values of the smearing function $\kappa(t)$, the history Hamiltonian constraint H_κ generates canonical transformations on the history constraint surface.

It has been shown [23; 24] that the history reduced state space Π_{red} is a symplectic manifold that can be identified with the space of paths on the canonical reduced state space $\Pi_{red} = \{t \mapsto \Gamma_{red}, t \in R\}$. Therefore the histories reduced state space is *identical* to the space of paths on the canonical reduced state space. Consequently the time parameter t also exists on Π_{red}, and the notion of *time ordering* remains on the space of the true degrees of freedom. This last result is in contrast to the standard canonical theory where there exists ambiguity with respect to the notion of time after reduction.

Moreover, the action functional S commutes weakly with the constraints, so it can be projected on the reduced state space. It then serves its role in determining the equations of motion [23; 24].

A function on the full state space represents a physical observable if it is projectable into a function on Π_{red}. Hence, it is necessary and sufficient that it commutes with the constraints on the constraint surface.

Contrary to the canonical treatments of parameterised systems, the classical equations of motion of the histories theory are *explicitly realised* on the reduced state space Π_{red}. Indeed, the equations of motion are the paths on the phase space that remain invariant under the symplectic transformations generated by the action functional projected on Π_{red}

$$\{\tilde{S}, F_t\}(\gamma_{cl}) = 0,$$

where F_t is a functional of the field variables and it is constant in t. The path γ_{cl} is a solution of the equations of motion, therefore it corresponds to a spacetime metric that is a solution of the Einstein equations.

The canonical action functional S is also diffeomorphic-invariant

$$\{V_W, S\} = 0. \tag{5.14}$$

This is a significant result: it leads to the conclusion that *the dynamics of the histories theory is invariant under the group of spacetime diffeomorphisms*.

The parameter with respect to which the orbits of the constraints are defined, is not in any sense identified with the physical time t. In particular, one can distinguish the paths corresponding to the equations of motion by the condition $\{F, \tilde{V}\}_{\gamma_{cl}} = 0$.

In standard canonical theory, the elements of the reduced state space are all solutions to the classical equations of motion. In histories canonical theory, however, an element of the reduced state space is a solution to the classical equations of motion only if it also satisfies the above condition. The reason for this is that the histories reduced state space Π_{red} contains a much larger number of paths, essentially all paths on Γ_{red}. For this reason, histories theory may naturally describe observables that commute with the constraints but which are not solutions to the classical equations of motion.

This last point should be particularly emphasised because of its possible corresponding quantum analogue. We know that in quantum theory, paths may be realised that *are not* solutions to the equations of motion. The histories formalism, in effect, distinguishes between instantaneous laws [16] (namely constraints), and dynamical laws (equations of motion). Hence, it is possible to have a quantum theory for which the instantaneous laws are satisfied, while the classical dynamical laws are not. This distinction is present, for example, in the history theory of the quantised electromagnetic field [7], where all physical states satisfy the Gauss law exactly; however, electromagnetism field histories are possible which do not satisfy the dynamical equations.

5.4 A spacetime approach to Quantum Gravity theory

5.4.1 Motivation

The histories approach to General Relativity suggests a new, spacetime-focussed, approach to Quantum Gravity, characterized by two features that are not implemented in the existing Quantum Gravity schemes.

First, the Lorentzian metric is quantized, analogous to the 'external' quantum field in the history approach to scalar quantum field theory [22; 14]. This contrasts with conventional canonical Quantum Gravity where only a spatial three-metric is quantised. Second, the history scheme incorporates general covariance via a manifest representation of the spacetime diffeomorphism group.

The canonical quantisation scheme was originally developed with the hope of providing a background independent formulation of Quantum Gravity. The general procedure involves (i) a $3+1$ splitting of spacetime; (ii) the construction of a suitable Hilbert space to accommodate the basic kinematical quantities of the theory; and (iii) the definition of self-adjoint operators that represent the Hamiltonian constraints. The imposition of the constraints on the state vectors then projects out the physical degrees of freedom.

The canonical treatment of Quantum Gravity introduces a spacelike foliation that enters the quantum description. However, the physical predictions should be independent of the choice of this foliation. This is part of the famous 'problem of time', as are attempts to understand the spacetime diffeomorphism group in this context. These issues are significantly addressed by the histories formulation with its genuine spacetime description of physical quantities.

The definition of the history group provides the HPO formalism with a quantisation scheme that follows the general lore of canonical quantisation, providing however a fully covariant description – see for example the quantum treatment of minisuperspace models in [3].

The obvious technical problem in a histories-based quantisation is the rigorous implementation of the dynamics by a history analogue of the Hamiltonian constraint operator. As in standard canonical theory, the classical expression is non-quadratic – indeed non-polynomial – in the field variables, and so the construction of an operator for the Hamiltonian constraint seems a hopeless task using conventional methods. For this reason, we intend to exploit the basic ideas of loop quantum gravity, which has been a promising approach for the construction this operator.

5.4.2 Towards a histories analogue of loop quantum gravity

Loop quantum gravity is a successful canonical theory in many respects. The basic algebra is defined with reference to objects that have support on loops in the three-dimensional surface Σ.

The first step in the application of the histories formalism to loop quantum gravity is to develop the histories analogue of the connection formalism of General Relativity. The original formulation of the programme involved the consideration of self-dual $SL(2, \mathbf{C})$ connections on spacetime, together with a field of tetrads for a Lorentzian metric [26; 4]. However, the mainline approach in loop quantum gravity finds it more convenient to employ in quantisation a real $SU(2)$ connection, proposed by Barbero [5]. The Barbero connection may be obtained as a variable in a state space, extending that of canonical General Relativity; or it may be obtained from a Lagrangian action (the Holst Lagrangian by [11]). However, the latter procedure involves gauge fixing, and it is not clear whether the connection may be defined in its absence – see [19; 20] for related discussions.

The histories description for classical gravity in term of the Holst Lagrangian has been developed in [25]. The basic variables at the covariant level is an $SL(2, \mathbf{C})$ connection and a field of tetrads on spacetime M, together with their conjugate variables. The corresponding history space carries a symplectic action of the group $Diff(M)$ of spacetime diffeomorphisms. The introduction of an equivariant foliation functional allows the translation of the spacetime description into that of an one-parameter family of canonical structures. The results of the metric-based theory can be fully reproduced in this construction: the set of constraints corresponding to the Holst Lagrangian is invariant under the action of the spacetime $Diff(M)$ group. Hence the generators of the spacetime diffeomorphisms group can also be projected onto the reduced state space.

The next step would involve choosing the basic variables for quantisation. Following the spirit of loop quantum gravity, we may try to identify a loop algebra, and then construct a histories Hilbert space by studying its representation theory. The obvious place to start would be the loop algebra corresponding to the spacetime $SL(2, \mathbf{C})$ connection of the covariant description. This, however, would involve a representation theory for loop variables with a non-compact gauge group, which to the best of our knowledge has not yet been fully developed. Moreover, we would have to identify a new role for the tetrad fields, because at this level they commute with the connection variables.

It may be more profitable to work with ‘internal’ fields, namely the ones that correspond to one-parameter families of the standard canonical variables. This would allow the consideration of connections with compact gauge group. However, a complication arises, because of the gauge-dependence of the definition of the Barbero connection. A gauge-fixing condition, at this level, breaks the background independence of the theory. In [25] we show that a connection sharing all properties of the Barbero connection can be defined in a gauge invariant way, albeit in a larger space than the one usually employed.

A history quantisation may be therefore envisioned that will employ variables defined with support on a two-dimensional cylinder – giving a history analogue of the T_0 variables – and a three-dimensional space $S \times R$ (where S is a spatial two-surface), as a history analogue of the T_1 variables. The price of gauge-invariance is that additional canonical variables have to be quantised: they correspond to a unit timelike vector field that determines the possible ways that the group $SU(2)$ can be embedded into $SL(2, C)$ – see [14] for the quantisation of a related structure in the context of quantum field theories.

The above is a potential point of divergence between histories quantisation and the canonical loop approach, which is necessary in light of the strong restrictions placed by our requirement of full spacetime general covariance. At present the research is focussed on finding a proper algebra for quantisation. An interesting possibility is that the histories formalism may provide spacetime geometric operators: for example, an operator for spacetime volume; or 'length' operators that distinguish between spacelike and timelike curves.

Acknowledgement

The preparation of this publication was supported by the EP/C517687 EPSRC grant.

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6

Categorical geometry and the mathematical foundations of Quantum Gravity

L. CRANE

6.1 Introduction

The mathematical structure of a theory is a very abstract collection of assumptions about the nature of the sphere of phenomena the theory studies. Given the great cultural gap which has opened between mathematics and physics, it is all too easy for these assumptions to become unconscious.

General Relativity (GR) is a classical theory. Its mathematical foundation is a smooth manifold with a pseudometric on it. This entails the following assumptions.

(i) Spacetime contains a continuously infinite set of pointlike events which is independent of the observer.
(ii) Arbitrarily small intervals and durations are well defined quantities. They are either simultaneously measurable or must be treated as existing in principle, even if unmeasurable.
(iii) At very short distances, special relativity becomes extremely accurate, because spacetime is nearly flat.
(iv) Physical effects from the infinite set of past events can all affect an event in their future, consequently they must all be integrated over.

The problem of the infinities in Quantum General Relativity is intimately connected to the consequences of these assumptions.

In my experience, most relativists do not actually believe these assumptions to be reasonable. Nevertheless, any attempt to quantize relativity which begins with a metric on a three or four dimensional manifold, a connection on a manifold, or strings moving in a geometric background metric on a manifold, is in effect making them.

Philosophically, the concept of a continuum of points is an idealization of the principles of classical physics applied to the spacetime location of events. Observations can localize events into regions. Since classically all observations can

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

be performed simultaneously and with arbitrary accuracy, we can create infinite sequences of contracting regions, which represent points in the limit.

Relativity and Quantum Mechanics both create obstacles to this process. Determinations of position in spacetime cannot be arbitrarily precise, nor can they be simultaneously well defined.

Unfortunately the classical continuum is thousands of years old and is very deeply rooted in our education. It tends to pass under the radar screen.

I often suspect that quantum physicists are suspicious of mathematics because so much of it seems wrong to them. I think the solution is more mathematics rather than less.

One often hears from quantum field theorists that the continuum is the limit of the lattice as the spacing parameter goes to 0. It is not possible to obtain an uncountable infinite point set as a limit of finite sets of vertices, but categorical approaches to topology do allow us to make sense of that statement, in the sense that topoi of categories of simplicial complexes are limits of them.

I have become convinced that the extraordinary difficulty of quantizing gravity is precisely due to the omnipresence of the numerated assumptions. For this reason, this chapter will explore the problem of finding the appropriate mathematical concept of spacetime in which a quantum theory of GR could be constructed.

Now, although it is not well known among physicists of any stripe, mathematicians have developed very sophisticated foundations both for topology and the geometry of smooth manifolds, in which an underlying point set is not required.

We will be interested in two related lines of development here; higher category theory and topos theory. Over the past several years, it has become clear that these mathematical approaches have a number of close relationships with interesting new models for Quantum Gravity, and also with foundational issues in Quantum Mechanics which will have to be faced in QGR.

In this chapter, I hope to introduce these ideas to the relativity community. The most useful approach seems to be to begin with a non-technical introduction to the mathematical structures involved, followed by a survey of actual and potential applications to physics.

6.2 Some mathematical approaches to pointless space and spacetime

6.2.1 Categories in quantum physics; Feynmanology

Although categorical language is not explicitly familiar in physics, quantum field theory is in fact dominated by the theory of tensor categories under a different name. A category is a mathematical structure with objects and maps between them called morphisms [1]. A tensor category has the further structure of a product which allows us to combine two objects into a new one.

If we write out a general morphism in a tensor category, we get arrows starting from sets of several objects and ending in different sets of several objects, where we think of maps into and out of the tensor product of objects as maps into and out of their combination. When we compose these, we get exactly Feynman graphs.

The objects in the category are the particles (or more concretely their internal Hilbert spaces), and the vertices are the tensor morphisms.

The resemblance is not accidental. The Kronecker product which tells us how to combine the Hilbert spaces of subsystems is just what mathematicians call the tensor product.

The representations of a Lie group form a tensor category, in which the morphisms are the maps which intertwine the group action. This is equivalent to the prescription in Feynmanology that we include all vertices not excluded by the symmetries of the theory.

The physicist reader can substitute for the idea of a categorical space the idea that the spacetime is actually a superposition of Feynman graphs, which we can think of as a vacuum fluctuation.

The Feynmanological point of view has been developed for the BC model under the name of group field theory [2]. The 4-simplices of the triangulation are treated as vertices in this point of view, and the 3-simplices as particles.

The categorical language is much more developed, and connected to more mathematical examples. I hope I will be forgiven for staying with it.

6.2.2 Grothendieck sites and topoi

The next mathematical ideas we shall consider trace back to the work of Alexander Grothendieck, perhaps the deepest mathematical thinker who ever lived. Much of his work was only appreciated after several decades, his deepest ideas are still not fully understood.

Grothendieck made the observation that the open sets of a topological space could be considered as the objects of a category, with a morphism between two objects if the first was contained in the second. He called this the site of the space. This was motivated by the observation that presheaves over the space are the same as functors from the site to the category of whatever type of fiber the sheaf is supposed to have. Since the constructions of topology and geometry can be reformulated in terms of presheaves (a bundle, for example can be replaced with the presheaf of its local sections), this opened the way to a far ranging generalization of topology and geometry, in which general categories play the role of spaces.

Grothendieck also realized that rather than the site itself, the central object of study was the category of presheaves over it (or functors into the category of sets), which he called its topos [3].

Topoi also have an axiomatic definition, which amounts to the idea that they are categories in which all the normal constructions done on sets have analogs. It was then proven that every abstract topos is the topos of some site [3; 4].

For this reason, the objects in a topos can be thought of either as abstract sets, or variable or relative sets.

One of the interesting aspects of topos theory is that the objects in a topos can inherit structure from the objects in the category which is its site.

An important example is synthetic differential geometry [5], the study of the topos over the site category of smooth rings, or "analytic spaces" (there are several variants).

Objects in this topos inherit a notion of differential and integral calculus. The object in this category which corresponds to the real numbers has infinitesimal elements. It is much more convenient to treat infinitesimals in a setting where not everything is determined by sets of elements. The result is that the calculus techniques of physicists which mathematicians are forever criticizing suddenly become rigorous.

A topos is a more subtle replacement of the notion of space than a category. It is a category of maps between categories, so it has the character of a relative space. In this paper, we are exploring the possibility that the relativity of objects in a topos could be a model for the relativity of the state of a system to the observer.

6.2.3 Higher categories as spaces

The idea that topology and geometry are really about regions and maps between them rather than sets of points, has been a subtle but widespread influence in mathematics.

A mathematical object with many objects and maps between them is a category [1]. There are many approaches to regarding a category as a kind of space.

Mathematicians have extended the idea of a category to an n-category. A 2-category has objects, maps and maps between maps, known of as homotopies or 2-morphisms. An n-category has $1, 2 \ldots n$ morphisms [6].

The simplest situation in which a higher category can be thought of as a kind of space is the case of a simplicial complex.

A simplicial complex is a set of points, intervals, triangles tetrahedra, etc., referred to as n-simplices, where n is the dimension. The faces of the n-simplices are identified with $n-1$ simplices, thus giving a discrete set of gluing rules. Faces are defined combinatorially as subsets of vertices. The whole structure is given by discrete combinatorial data.

A simplicial complex is thus a discrete combinatorial object. It does not contain a sets of internal points. These can be added to form the geometric realization of a simplicial complex, but that is usually not done.

Because the vertices of a simplex are ordered, which fixes an orientation on each of its faces of all dimensions, it is natural to represent it as a higher category. The vertices are the objects, the edges are 1-morphisms, the triangles 2-morphisms, etc.

For many purposes, simplicial complexes are just as good as topological spaces or manifolds. Physicists who like to do physics on a lattice can generalize to curved spacetime by working on a simplicial complex.

There is also a notion of the topology of a simplicial complex including cohomology and homotopy theory. A celebrated theorem states that the categories of homotopy types of simplicial complexes and of topological spaces are equivalent [7].

A naive first approach to quantum spacetime would say that at the Planck scale spacetime is described by a simplicial complex, rather than a continuum. This point of view would nicely accommodate the state sum models for Quantum Gravity, and the categorical language would allow a very elegant formulation of them, as we shall discuss below. The richness of the connections between category theory and topology allows for more sophisticated versions of this, in which simplicial complexes appear relationally, i.e. the information flowing between two regions forms a simplicial complex. We will discuss physical approaches to this below.

Another way to relate categories to simplicial complexes is the construction of the nerve of a category, which is a simplicial complex which expresses the structure of the category. The nerve is constructed by assigning an n-simplex to each chain of $n+1$ composible morphisms in the category. The $n-1$ faces are each given by composing one successive pair of morphisms to form an n-chain.

The simplicial complex so formed is a generalization of the classifying space of a group. A group is a category with one object and all morphisms invertible.

There are also constructions which associate a category to a cellular or cubical complex.

The various descriptions of spaces by categories also extend to descriptions of maps between spaces as functors between categories.

Since the setting of a Yang–Mills or Kaluza–Klein theory is a projection map between manifolds, these have categorical generalizations which include more possibilities than the manifold versions.

One very interesting aspect of topos theory is the change in the status of points. A topos does not have an absolute set of points; rather, any topos can have points in any other topos. This was originally discovered by Grothendieck in algebraic geometry [3], where the topoi are called schemes. We shall discuss physical implications of this below.

6.2.4 Stacks and cosmoi

As we shall see in the next section, both higher categorical and topos theoretical notions of space have strong connections to ideas in Quantum Gravity. For various reasons, it seems desirable to form a fusion of the two; that is, to form relational versions of higher categories.

Interestingly, this was the goal of the final work of Grothendieck on stacks, which he did not complete. Much of this has been worked out more recently by other authors [8].

The maps between two categories form a category, not merely a set. This is because of the existence of natural transformations between the functors. Similarly the morphisms between two 2-categories form a 2-category, etc. The analog of sheaves over sites for 2-categories are called stacks. Much as the case of sheaves, these are equivalent to 2-functors. Incidently the word Grothendieck chose for a stack in French is champs, the same as the French word for a physical field.

One can also investigate the 2-categorical analog for a topos, which is a 2-category with an analogous structure to the 2-category of all "small" categories. This has been defined under the name of a cosmos [9].

An interesting class of examples of stacks are the gerbes [10], which have attracted interest in string theory and 2-Yang–Mills theory [11]. Theories with gerbe excitations would generalize naturally into a 2-categorical background spacetime.

6.3 Physics in categorical spacetime

The ideal foundation for a quantum theory of gravity would begin with a description of a quantum mechanical measurement of some part of the geometry of some region; proceed to an analysis of the commutation relations between different observations, and then hypothesize a mathematical structure for spacetime which would contain these relations and give General Relativity in a classical limit.

We do not know how to do this at present. However, we do have a number of approaches in which categorical ideas about spacetime fit with aspects of geometry and quantum theory in interesting ways. We shall present these, and close with some ideas about how to achieve a synthesis.

6.3.1 The BC categorical state sum model

The development of the Barrett–Crane model for Quantum General Relativity [12; 13] begins by substituting a simplicial complex for a manifold. It is possible to adopt the point of view that this is merely a discrete approximation to an underlying continuous geometry located on a triangulation of the manifold. That was never my motivation. Rather considerations of the Planck scale cutoff and the limitations of information transfer in General Relativity suggested that discrete geometry was more fundamental.

In any event, the problem of quantizing the geometry on a simplicial complex has proved to be much more tractable than the continuum version.

The bivectors assigned by the geometry to the triangles of the complex can be identified with vectors in the dual of the Lorentz algebra, and hence have a very well understood quantization using the Kostant–Kirillov approach [14]. The quantum theory reduces to a careful combination of the unitary representations of the Lorentz algebra due to Gelfand [15; 16], and of intertwining operators between them.

We tensor together the representations corresponding to the assignments of area variables to the faces, then take the direct sum over all labellings. The resultant expression is what we call a categorical state sum.

The expression obtained for the state sum on any finite simplicial complex has been shown to be finite [17].

In addition, the mathematical form of the state sum is very elegant from the categorical point of view. If we think of the simplicial complex as a higher category, and the representations of the Lorentz group as objects in a tensor category (which is really a type of 2-category), then the state sum is a sum over the functors between them.

The BC model is expressed as the category of functors between a spacetime category and a field category, the field category being a suitable subcategory of the unitary representations of the Lorentz algebra. This suggests a general procedure for connecting more sophisticated categorical approaches to spacetime to Quantum Gravity. Namely, we could examine the category of functors from whatever version of spacetime category we are studying to the representation category of the Lorentz algebra in order to put in the geometric variables.

It is not necessary for the simplicial complex on which we define the BC model to be equivalent to a triangulation of a manifold. A 4D simplicial complex in general has the topology of a manifold with conical singularities. There has been some work interpreting the behavior of the model near a singular point such as a particle, with interesting results [18; 19]. The singularities conic over genus 1 surfaces reproduce, at least in a crude first approximation, the bosonic sector of the standard model, while the higher genus singularities decouple at low energy, with interesting early universe implications. The possibility of investigating singular points would not arise in any theory formulated on a manifold.

The BC model has not yet gained general acceptance as a candidate for quantum General Relativity. The fundamental problem is the failure of attempts to find its classical limit.

I want to argue that the work done to date on the classical limit of the model, my own included, has been based on a misconception.

A categorical state sum model is not a path integral, although it resembles one in many aspects. Rather the geometry of each simplex has been quantized separately, and the whole model represented on a constrained tensor product of the local Hilbert spaces.

For this reason the terms in the CSS are not classical histories, but rather quantum states. It is not really surprising, then, that the geometric variables on them do not have simultaneous sharp values, or that they can contain singular configurations. Attempting to interpret them as classical is analogous to confusing the zitterbewegung of the electron with a classical trajectory.

In order to construct the classical limit of the BC model, it is necessary to study the problem of the emergence of a classical world in a quantum system. Fortunately, there has been great progress on this in recent years in the field variously known as consistent histories or decoherence.

The decoherent or consistent histories program has recently been interpreted as indicating that quantum measurements should be considered as occurring in a topos.

In the next sections, we shall briefly review the ideas of consistent histories and decoherence, and explain how they lead to topos theory. Then we shall discuss how to apply these ideas to the BC model.

6.3.2 Decoherent histories and topoi

The consistent histories/decoherence approach to the interpretation of Quantum Mechanics is concerned with the problem of how classical behavior emerges in a suitable approximation in a quantum system [20].

We have to begin by coarse graining the system to be studied by decomposing its Hilbert space into a sum of subspaces described as the images of orthogonal projections. A history is a sequence of members of the set of projections at a sequence of times.

Next we need to define the decoherence functional D. It is the trace of the product of the first series of projections time reversed, the density matrix of the original state of the system, and the product of the first series of projections:

$$D(H_1, H_2) = tr(H_1{}^* \rho H_2).$$

Classical behavior occurs if the decoherence functional is concentrated on the diagonal, more precisely if there is a small decoherence parameter η such that

$$D(H_1, H_2) = o(\eta) \text{ if } H_1 \neq H_2.$$

This implies that states described by histories from the chosen set do not interfere significantly. This implies classical behavior.

The next property of D to prove is that it concentrates near histories which correspond to solutions of the equations of motion. This is a way of affirming the correspondence principle for the system.

Since consistency is not perfect, we must think of the classical limit as appearing in the limit of coarse grainings.

Decoherence, the second half of the program, is an extremely robust mechanism causing histories to become consistent. When the variables correspond to typical macroscopic quantities decoherence occurs extremely quickly.

The central observation of the decoherence program is that classical systems can never be effectively decoupled from their environment.

For instance, a piston in a cylinder containing a very dilute gas might experience a negligible force. Nevertheless, the constant collisions with gas molecules would cause the phase of the piston, treated as a quantum system, to vary randomly and uncontrollably.

Since it is not possible to measure the phases of all the molecules, the determinations an observer could make about the position of the piston would be modelled by projection operators whose images include an ensemble of piston states with random phases, coupled to gas molecule states.

This effect causes pistons (or any macroscopic body) to have diagonal decoherence functionals to a high degree of accuracy, and hence to behave classically.

The definition of a classical system as one which cannot be disentangled is a very useful one. It has enabled experiments to be designed which study systems which are intermediate between classical and quantum behavior [20].

When we observe a system, it is not possible to say exactly what set of consistent histories we are using. It is more natural to think that we are operating in a net of sets of consistent histories simultaneously.

We then expect that the result of an observation will be consistent if we pass from one set of consistent histories to a coarse graining of it.

The idea has been studied that this means that the results of experiments should be thought of as taking values in a topos [21]. The category whose objects are sets of consistent histories and whose morphisms are coarse grainings can be thought of as a site, and the results of experiments take place in presheaves over it.

In my view, the implications of this idea should be studied for physical geometry. Does it mean, for example, that the physical real numbers contain infinitesimals?

6.3.3 Application of decoherent histories to the BC model

This section is work in progress.

We would like to explore classical histories in the BC model. The goal of this is to show that consistent histories exist for the model which closely approximate the geometry of pseudo-Riemannian manifolds, and that the decoherence functional concentrates around solutions of Einstein's equation.

The natural choice for macroscopic variables in the BC model would be the overall geometry of regions composing a number of simplices in the underlying complex of the model. It is easier to choose the regions themselves to be simplices which we call large to distinguish them from the fundamental simplices of which they are composed.

The program for showing that the geometric data on the internal small simplices decoheres the overall geometry of the large ones involves two steps.

In the first, we use microlocal analysis to construct a basis of states in which all the geometrical variables of the large simplices are simultaneously sharp to a small inaccuracy. These would combine to give a set of projection operators whose images correspond to pseudo-Riemannian geometries on the complex, now thought of as a triangulated manifold.

This problem is mathematically similar to finding a wavepacket for a particle. The symplectic space for the tetrahedron turns out to be equivalent to the symplectic structure on the space of Euclidean quadrilaterals in the Euclidean signature case, and to have an interesting hyper-Kahler structure in the case of the Lorentzian signature. This allows us to use powerful mathematical simplifications, which make me believe the problem is quite solvable.

The second step would be to show the decoherence functional which arises from averaging over the small variables causes the large variables to decohere, and that the decoherence functional concentrates around solutions of Einstein's equation.

This is quite analogous to known results for material systems such as the piston.

The existence of a Brownian motion approximation for the internal variables makes me hopeful that this will work out, similarly to the case of the piston, where an ideal gas approximation is the key to the calculation.

A more challenging problem would be to work out the topos theoretic interpretation of the decoherence program in the case of the BC model.

The site of this topos would be the category whose objects are the "large" triangulations, and whose morphisms are coarse grainings.

One could then apply the ideas about modelling quantum observation in a topos described above to the BC model. This would amount to the construction of a 2-stack, since the BC model itself is 2-categorical.

This would give us a setting to ask the question: “what does one region in a spacetime, treated as classical, observe of the geometry of another part?”.

This problem was suggested to me by Chris Isham.

6.3.4 Causal sites

As we explained above, the site of a topological space X is a category whose objects are the open sets of X and whose morphisms are inclusions. The whole construction of a site rests on the relationship of inclusion, which is a partial order on the set of open subsets. This change of starting point has proven enormously productive in Mathematics.

In Physics up to this point, the topological foundations for spacetime have been taken over without alteration from the topological foundation of space. In General Relativity, a spacetime is distinguished from a four-dimensional space only by the signature of its metric.

Categorical concepts of topology are richer and more flexible than point sets, however, and allow specifically spacetime structures to become part of the topological foundation of the subject.

In particular, regions in spacetime, in addition to the partial order relation of inclusion, have the partial order relation of causal priority, defined when every part of one region can observe every part of the other.

The combination of these two relations satisfy some interesting algebraic rules. These amount to saying that the compact regions of a causal spacetime are naturally the objects of a two category, in much the same way that open sets form a site.

This suggests the possibility of defining a spacetime directly as a higher categorical object in which topology and causality are unified, a topodynamics to join geometrodynamics.

Recently, Dan Christensen and I implemented this proposal by giving a definition of causal sites and making an investigation of their structure [22].

We began by axiomatizing the properties of inclusion and causal order on compact regions of a strongly causal spacetime, then looked for more general examples not directly related to underlying point sets.

The structure which results is interesting in a number of ways. There is a natural 2-categorical formulation of causal sites. Objects are regions, 1-morphisms are causal chains, defined as sequences of regions each of which is causally prior to the next, and 2-morphisms are inclusions of causal chains, rather technically defined.

We think of causal chains as idealizations of observations, in which information can be retransmitted.

We discovered several interesting families of examples. One family was constructed by including a cutoff minimum spacetime scale. These examples have the interesting property that the set of causal chains between any two regions has a maximal length. This length can be interpreted as the duration of a timelike curve, and can very closely approximate the durations in a classical causal spacetime.

Since the pseudo-metric of a spacetime can be recovered from its timelike durations, the 2-categorical structure of a causal site can contain not only the topology of a spacetime, but also its geometry.

We also discovered that any two causally related regions have a relational tangent space, which describes the flow of information between them. This space has the structure of a simplicial complex, as opposed to a causal site itself, which has a bisimplicial structure because of the two relations on it. In category theoretic terms, the spacetime is a 2-category, but relationally it is a category.

An interesting feature of causal sites is that regions have relational points, i.e. regions which appear to another region to be indivisible, but perhaps are not absolutely so.

We hope that this feature may make causal sites useful in modelling the theory of observation in General Relativity, in which only a finite amount of information can flow from one region to another [23], so that an infinite point set is not observably distinguished.

We also think it an interesting echo of the relational nature of points in topos theory.

If infinite point sets cannot be observed, then according to Einstein's principle, they should not appear in the theory. Causal sites are one possible way to implement this.

6.3.5 The 2-stack of Quantum Gravity? Further directions

At this point, we have outlined two approaches to categorical spacetime, which include geometric information corresponding to the metric structure in General Relativity in two different ways.

In the Barrett–Crane model, the data which express the geometry are directly quantum in nature. The geometric variables are given by assigning unitary representations of the Lorentz algebra to the 2-faces or triangles of a simplicial complex. These are Hilbert spaces on which operators corresponding to elements of the Lorentz algebra act, thus directly quantizing the degrees of freedom of the bivector, or directed area element, which would appear on the 2-face if it had a classical geometry, inherited from an embedding into Minkowski space.

This model also has a natural functorial expression, as we mentioned above.

On the other hand, in the causal sites picture spacetime is represented as a family of regions, with two related partial orders on them. Mathematically this can be expressed by regarding the regions as the vertices of a bisimplicial set. Bisimplicial sets are one mathematical approach to 2-categories [24]. This is also an expression of the topological structure of the spacetime, although a more subtle one than a simplicial complex, which could arise from a triangulation of a manifold.

In some interesting examples, the **classical** geometry of a spacetime is naturally included in this bisimplicial complex, measured by the lengths of maximal causal chains. The approximation of the geometry by a causal set [25] can not be as precise, since a causal site has minimal regions which can be adapted to the direction of a path.

Now how could these two picture be synthesized?

One element which has not been included so far in the structure of a causal site is local symmetry. It is clear that this would have to appear in a fully satisfactory development of the theory, since the local symmetries of spacetime are so physically important.

Including local symmetry in the structure of a causal site seems a natural direction to study in linking the causal sites picture to the BC model, since the geometrical variables of the BC model are representations of the Lorentz group.

The fundamental variables of a causal site have a yes/no form: region A either is or is not in the causal past of region B.

We could attempt to quantize a causal site by replacing the definite causal relations by causality operators. We can now define a 2-dimensional Hilbert space H(A,B) for each pair of regions with a basis representing the yes and no answers to the causal relatedness question. This corresponds to a gravitational experiment in which an observer at B sees or fails to see an event at A. The totality of such experiments should define a quantum geometry on the site in the cases discussed above with bounds on chain length, since the metric can be effectively reconstructed from the classical answers.

In the presence of an action of local symmetry on the regions of the site, the tensor product of the spaces H(A,B) would decompose into representations of the local symmetry group.

If this led to the reappearance of the BC model on the relative tangent space between two regions in a site, it would create a setting in which the idea of the BC model as describing the geometry of one region as observed by another could be realized.

The physical thought is that since only a finite amount of information can pass from A to B in General Relativity, the set of vertex points in a relative BC model could include all the topology of A which B could detect.

The idea of constructing a topos version of the BC model using decoherent histories also points to a BC model which varies depending on which classical observer the model is observed by.

Both of these ideas (neither implemented yet, and neither easy) seem to hint at a simultaneously higher categorical and topos theoretical description of quantum spacetime which would fulfill the physical idea of a relational spacetime.

Perhaps there is an as yet un-guessed construction of a 2-stack which will provide a synthesis of these ideas. Einstein's relational ideas may find their final form in the mathematical ideas of Grothendieck.

Acknowledgements

The idea of topos theory arising in quantum theory in general and Quantum Gravity in particular is something I learned from Chris Isham. Much of the higher category theory in this paper was strongly influenced by working with Dan Christensen during my visit to the University of Western Ontario. I learned about Grothendieck's work during my visit to Montpellier where I was invited by Philippe Roche. I benefited from conversations about topos theory with Carlos Contou-Carrere while I was there. I also had many interesting conversations with Marni Sheppeard at both places. The BC model, of course, is joint work with John Barrett. This work is supported by a grant from FQXi.

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7

Emergent relativity

O. DREYER

7.1 Introduction

This chapter wants to be two things. On the one hand it wants to review a number of approaches to the problem of Quantum Gravity that are new and have not been widely discussed so far. On the other hand it wants to offer a new look at the problem of Quantum Gravity. The different approaches can be organized according to how they answer the following questions: Is the concept of a spacetime fundamental? Is a background time used? Are Einstein's equations assumed or derived? (See figure 7.1.)

In string theory, loop quantum gravity, and most other approaches reviewed in this book spacetime plays a fundamental role. In string theory a given spacetime is used to formulate the theory, in loop quantum gravity one tries to make sense of quantum superpositions of spacetimes. It is these spacetimes in the fundamental formulation of the theory that are directly related to the spacetime we see around us. In this broad sense these approaches treat spacetime as something fundamental. Here we want to focus our attention on approaches that take a different view. In these approaches spacetime emerges from a more fundamental theory.

The next questions concern the role of time. The models that we will be looking at will all have some sort of given time variable. They differ though in the way they treat this time variable. One attitude is to use this time variable in the emergent theory. The goal of Quantum Gravity in this context could then be to find a massless spin two particle in the excitation spectrum of the Hamiltonian corresponding to the given time. We will see in section 7.2.1 a solid state physics inspired approach due to G. Volovik that takes this point of view.

The other possible attitude towards the background time is that it is just a fiducial parameter that is not important for the emergent physics. If one takes this view then there is one more question: what is the role of the Einstein equations? In section 7.2.2 we will see a quantum information theory inspired model by S. Lloyd that

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

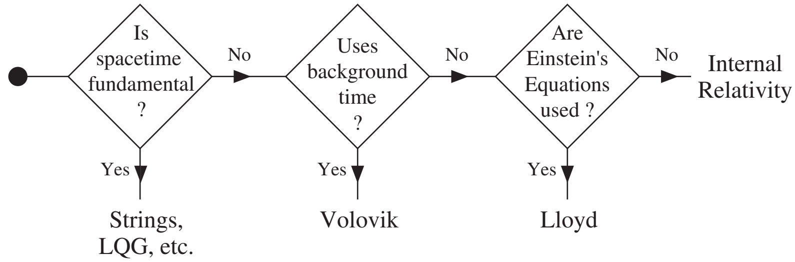

Fig. 7.1. Choices on the road to Quantum Gravity.

uses the Einstein equations to formulate the theory. The other possibility is to argue for why the Einstein equations hold true. In section 7.3 we will show how such an argument can be made. We call this approach Internal Relativity.

7.2 Two views of time

In this section we review two approaches to Quantum Gravity that differ in the way they view time. The first approach comes from solid state physics; the second comes from quantum information theory.

7.2.1 Fermi points

In this section we are interested in the low energy behavior of quantum mechanical Fermi liquids. It turns out that this behavior does not depend on the details of the model but is rather described by a small number of universality classes. Which universality class a given model falls into is determined by the topology of the energy spectrum in momentum space. The best known class is that of a simple Fermi surface (see figure 7.2**A**a). In an ideal free Fermi gas the Fermi surface is the boundary in momentum space between the occupied and unoccupied states. If p_{F} is the corresponding momentum then the energy spectrum is given by

$$E(\boldsymbol{p}) = v_{\mathrm{F}}(|\boldsymbol{p}| - p_{\mathrm{F}}). \tag{7.1}$$

In addition to these fermionic degrees of freedom there are also bosonic excitations given by oscillations of the Fermi surface itself. The dynamics of the fermionic and bosonic degrees of freedom is described by the Landau theory of Fermi liquids.

The other well known situation is that of a fully gapped system (see figure 7.2**A**b). In this case the next available energy level above the Fermi surface is everywhere separated from it by a non-zero amount Δ. This situation is encountered in superfluids and superconductors.

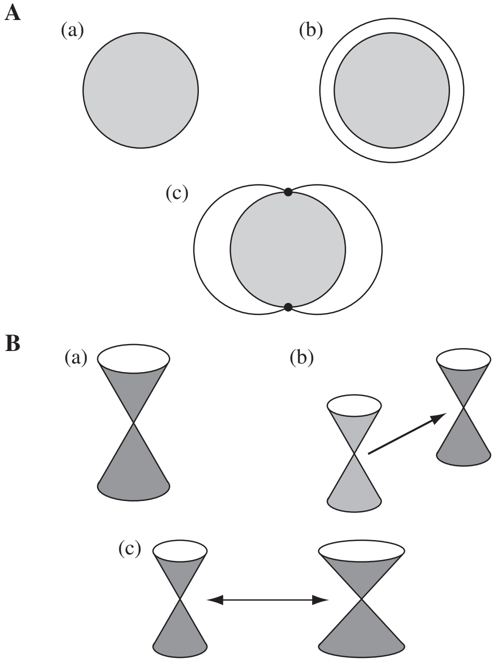

Fig. 7.2. **A** Possible momentum space topologies for a Fermi liquid. (a) a Fermi surface, (b) a fully gapped system, and (c) a system with Fermi points. **B** The possible excitations of a system with Fermi points. (a) The light cone of the emergent fermions, (b) moving the Fermi points corresponds to gauge degrees of freedom, (c) shape changes of the light cone give a kind of emergent gravity.

Most interesting for us is the situation when the gap Δ is not uniform but vanishes at certain points (see figure 7.2**A**c). These points are called *Fermi points*. It is the low energy behavior of this universality class that shows the kind of excitations we see around us: fermions, gauge fields, and even gravity. This happens because a Fermi point is a stable feature that is insensitive to small perturbations (see [1; 2] for more details). Its presence is protected by topology. The Fermi point itself represents a singularity in the Fermi propagator G. Its inverse $\mathcal{G}$ has a zero at the Fermi point. If we think of a small sphere S^3 centered at the Fermi point then $\mathcal{G}$ defines a map

$$\mathcal{G} : S^3 \longrightarrow \mathrm{GL}(N, \mathbb{C}), \tag{7.2}$$

where N is the number of components of the fundamental fermions including internal indices. Thus $\mathcal{G}$ defines an element in $\pi_3(\mathrm{GL}(N, \mathbb{C}))$, the third homotopy group of $\mathrm{GL}(N, \mathbb{C})$. If this homotopy class is non-trivial the Fermi point can not be removed by a small perturbation.

Since the inverse propagator $\mathcal{G}$ vanishes at the Fermi point it has the following expansion near the Fermi point:

$$\mathcal{G}(p) = \sigma^a e_a^\mu (p_\mu - p_\mu^0), \tag{7.3}$$

where we have for concreteness assumed that we have two spin components so that the Pauli matrices σ^a, $a = 0, \dots, 3$, can be used as a basis. Given that the Fermi points can not disappear the effect small perturbations can have is rather restricted. It can move the position of the Fermi point (see figure 7.2**B**b) or it can change the shape of the light cone (see figure 7.2**B**c). The parameters e_a^μ and p_μ^0 appearing in the above expansion thus become dynamic. We can infer the physical meaning of these new dynamical degrees of freedom by looking at the energy spectrum. The spectrum is determined by the zero of $\mathcal{G}$. Here we obtain

$$g^{\mu\nu}(p_\mu - p_\mu^0)(p_\nu - p_\nu^0) = 0, \tag{7.4}$$

where

$$g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu. \tag{7.5}$$

The change in the shape of the light cone can thus be identified with a changing metric $g^{\mu\nu}$ and the change of the position of the Fermi point gives rise to an electromagnetic field A_μ:

$$A_\mu = p_\mu^0. \tag{7.6}$$

We thus see that the low energy physics of a Fermi liquid with a Fermi point possesses all the kind of excitations that we see around us, i.e. fermions, gauge fields, and dynamics. Unfortunately the mass of the graviton is not generically zero. Instead the parameters of the model have to be chosen in a special way to make the mass vanishingly small.

7.2.2 Quantum computation

A completely different approach is the one proposed by S. Lloyd [3]. For him the universe is one giant quantum computation. The problem of Quantum Gravity is then to show how a quantum computation gives rise to a spacetime.

A quantum computation is given by a unitary operator U acting on the Hilbert space of our system. Here we take this system to be N qubits. The Hilbert space is thus

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes N}. \tag{7.7}$$

We can decompose U into quantum gates U_l, $l = 1, \dots, n$, that are acting on two qubits at a time:

$$U = U_n \cdots U_1. \tag{7.8}$$

It is here that a discrete form of background time makes its appearance. The individual U_ls appear in a definite order given by the parameter l. We will see though that this time is not related to the time as perceived by an observer in the model. Without restriction we can assume that the U_ls have the form

$$U_l = e^{-i\theta_l P} \tag{7.9}$$

$$= (1 - P) + e^{-i\theta_l} P, \tag{7.10}$$

for a projection operator $P = P^2$. We can represent such U_ls as in figure 7.3**A**. The two parts of equation 7.10 can be given a physical interpretation. In the subsystem that P projects onto the two qubits that U is acting on scatter. This results in a phase shift of θ_l. In the orthogonal subspace the two qubits do not scatter. Here there is no phase shift. The whole unitary U can now be written as follows

$$U = \prod_{l=1}^{n} ((1 - P_l) + e^{-i\theta_l} P_l) \tag{7.11}$$

$$= \sum_{b_1,\ldots,b_n \in \{0,1\}} e^{-i \sum_{l=1}^{n} b_l \theta_l} P_n(b_n) \cdots P_1(b_1), \tag{7.12}$$

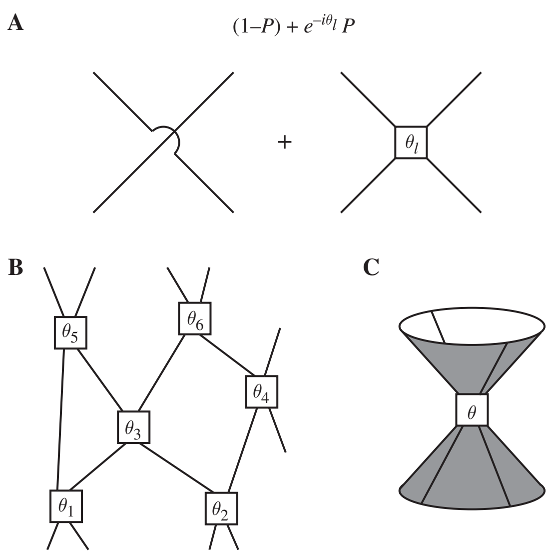

Fig. 7.3. **A** The unitary U_l consists of two parts. On the right the two qubits scatter off each other, giving rise to a phase θ_l. On the right the qubits miss each other. **B** The different U_l, $l = 1, \ldots, n$, give rise to 2^n different possible computational histories. Each history consists of a causal set and a set of phases θ_l. **C** The two incoming and outgoing qubits give four directions on the light cone at a node of the causal set. These four null directions determine four of the ten components of the metric.

where we have denoted $(1 - P_l)$ by $P_l(0)$ and P_l by $P_l(1)$. Given our interpretation of the individual U_ls we can represent the terms in the sum (7.12) as causal sets as in figure 7.3**B**. We will call such a causal set together with the angles θ_l a computational history. Any quantum computation can thus be interpreted as a superposition of computational histories.

The next step is to interpret each computational history as a discrete spacetime. To see this we embed all the histories into one manifold $\mathcal{M}$. The lines of the calculation that run between the scattering events are identified with null geodesics of the metric. At each node of the causal set we have four vectors that lie on the light cone at that point (see figure 7.3**C**). It follows that at this point four of the ten components of the metric are given. To find the remaining six we use the Einstein equations in their Regge form. Before we can use Regge calculus we have to turn our causal sets into simplicial lattices. The added lines will in general no longer be null. The metric will be fully determined once the lengths of all these additional lines are specified.

We will choose the lengths of the new lines in such a way that the Einstein equations

$$\frac{\delta I_{\mathrm{G}}}{\delta g} + \frac{\delta I_{\mathrm{M}}}{\delta g} = 0, \tag{7.13}$$

are satisfied. Here I_{G} is the gravitational action in its Regge calculus form and I_{M} is the matter action which is a function of the θ_ls and the metric (i.e. the length of the lines). Given a quantum computation we arrive at a superposition of discrete spacetimes. Since this is a quantum superposition one still has to argue how the classical limit is achieved. Note though that the task is easier in this setup since all the computational histories are embedded into one manifold $\mathcal{M}$. There is no problem in identifying points in the different histories as there is in other approaches to Quantum Gravity.

One problem that remains is the universality of the above construction. We have assigned spacetimes to all quantum computations. It is not clear what the meaning of this spacetime picture is for a generic quantum computation. The question arises of what the right calculation is.

7.3 Internal Relativity

In the previous section we have encountered two approaches to Quantum Gravity in which the metric emerges. In the view proposed by Volovik, gravity emerges as a massless spin two excitation of a Fermi system with a Fermi point. We are able to find gravity using the background time the theory is formulated in. In the computational universe there is also a background time but it plays no role in the spacetime constructed from the computational histories. In this construction the Einstein equations are used. The approach we want to propose here is similar in that only internally available information is used to reconstruct a spacetime. It differs in that the Einstein equations are not used but are to be derived.

7.3.1 Manifold matter

The most important ingredient in our construction are coherent degrees of freedom. It is these degrees of freedom that provide the glue that makes the manifold. Without them there is no notion of causality. Given two such coherent degrees of freedom we can identify a point by the intersection of the two. Our manifold will consist of points of this kind.

An example of coherent degrees of freedom is provided by a simple spin model from solid state physics. The XY-model is given by the Hamiltonian

$$H = \sum_{i=1}^{N} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+), \tag{7.14}$$

where $\sigma^\pm = \sigma^x \pm i\sigma^y$, and the σs are the Pauli matrices. This model can be solved exactly using the Jordan–Wigner transformation [4]. One obtains a free fermionic model described by the Hamiltonian

$$H = \sum_{k=1}^{N} \varepsilon(k) f_k^\dagger f_k, \tag{7.15}$$

where f, and $f^\dagger$ are the annihilation and creation operators for the fermions and $\varepsilon(k)$ is the energy

$$\varepsilon(k) = 16\pi \cos \frac{2\pi}{N} k. \tag{7.16}$$

One ground state can be obtained by half filling the Fermi sea. The excitations then have a linear dispersion relation given by

$$\Delta\varepsilon = 16\pi J_\perp \frac{2\pi}{N} \Delta k \equiv v_{\mathrm{F}} \Delta k. \tag{7.17}$$

It is excitations like these that play the role of our coherent degrees of freedom. The above example is too simple to stand in as a model for our world. A far more interesting example has recently been proposed by X.-G. Wen [5]. Although it is also built with simple spins it has both fermions and gauge interactions in its low energy limit. The particles of this model make for far more interesting coherent degrees of freedom that we can use in our construction.

Compare this notion with what we have seen in the computational universe. The coherent degrees of freedom are the lines in the computational graph, i.e. the qubits, and the points are the places they interact, i.e. the quantum gates. Compare also the article by F. Markopoulou in this volume in which coherent degrees of freedom are described by noiseless subsystems, a quantum information theoretic notion. Thus we have the following correspondences:

$$\begin{gathered}\text{coherent degree of freedom}\\ \equiv\\ \text{noiseless subsystem}\\ \equiv\\ \text{qubits in computational history.}\end{gathered}$$

The correspondences for the points of the manifold are thus:

$$\begin{gathered}\text{points of manifold}\\ \equiv\\ \text{intersections of coherent degrees of}\\ \text{freedom/noiseless subsystems}\\ \equiv\\ \text{quantum gates.}\end{gathered}$$

We want to stress one point here that all the proposals so far have in common. For all of them spacetime and matter arise together. They can not be separated. We will see in section 7.3.4 that it is here that the cosmological constant problem is solved.

7.3.2 Metric from dynamics

Having introduced our manifold as the set of coincidences of coherent degrees of freedom we now want to endow this set with a metric structure. How are we going to go about this? It is clear that there is one thing that we can not do. We can not use the background structure to introduce notions of distance or time. That means that the lattice our theory is defined on and the background time can not be used for this purpose. Instead what we will have to do is to use only notions that are internally available in the system. These are again our coherent degrees of freedom.

In our current system of units we are using light and cesium atoms to define what we mean by space and time. In the language used in this article we would say that we are using coherent degrees of freedom to arrive at metric notions. In our above example a spin wave could play the role that is played by light for us. Since we only allow access to such internal information it is not hard to see that the world will look relativistic to observers in the system.

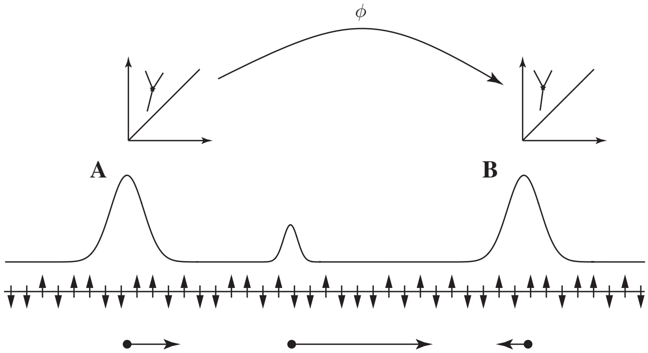

Fig. 7.4. A view of the system that is not available to observers confined inside the system. The observers **A** and **B** have no way of telling what their motion is with respect to the lattice. This is why both observers assign the same speed to the excitation. There exists a map ϕ between the two coordinate systems given by the mapping of physical events onto each other. This map ϕ will have the property that it maps light onto light. We find then that this map ϕ must be a Poincaré transformation.

Since the observers have no access to the underlying model they can not tell whether they are moving with respect to it. They will thus all assign the same speed to the coherent degree of freedom. The only transformation between their respective coordinate representations is then a Poincaré transformation since this is the only transformation that leaves the speed of the excitations unchanged (see figure 7.4).

It is in this sense that our approach is background independent. It is only through the dynamics of the system and the emergent coherent degrees of freedom that we arrive at metric notions.

This is again to be compared with the computational universe. The qubits are defined to be null just as the coherent degrees of freedom in our approach are null. The same is true for the noiseless subsystems of F. Markopoulou.

7.3.3 The equivalence principle and the Einstein equations

We now want to look at why our approach includes more then just flat Minkowski space. Having introduced metric notions we now want to proceed to define notions like mass and energy. It is here that we will see that the presence of a massive body will have an influence on the spacetime surrounding it.

When defining the mass of a body we have to do it in an internal or background independent way. One such way was described by E. Mach [6]. If one takes two masses m_1 and m_2 and makes them collide, the changes in velocity Δv_1 and Δv_2 will be related by

$$\frac{m_1}{m_2} = -\frac{\Delta v_2}{\Delta v_1}. \tag{7.18}$$

Given one standard mass this relation can be used to define all other masses. Note that this definition of inertial mass is completely relational. Note also that for this definition to work the theory can not be free. We need interactions for the two masses to bounce off of each other. It is here that things become interesting. To have a notion of mass for our coherent degrees of freedom they have to interact. But it is these same degrees of freedom that we have used to define our notions of spacetime. That means that the metric changes because of the presence of a massive object.

This connection between inertial and gravitational mass is well known and goes under the name of equivalence principle. We have argued that the equivalence principle follows from background independence. We want to go one step further and make the following conjecture.

Conjecture When notions of distance, time, mass, energy, and momentum are defined in a completely internal way the Einstein equations hold.

Let us call this approach to the problem of Quantum Gravity *Internal Relativity* to stress the internal background independent point of view.

7.3.4 Consequences

Our point of view sheds light on two long standing puzzles: the cosmological constant problem and the problem of time. Here we want to describe shortly how these problems dissolve when spacetime and matter are not treated separately.

This cosmological problem arises when one views quantum field theory as a theory describing fields living on a curved spacetime. This view runs into a serious problem when one considers the effect the quantum fields should have on spacetime. Since all the modes of the quantum field have a zero energy of $\pm 1/2\hbar\omega$, one expects a contribution to the vacuum energy on the order of

$$\int^{\Xi} d\omega\, \hbar\omega^3 \sim \hbar\, \Xi^4, \tag{7.19}$$

where Ξ is some high energy cut-off. If one takes this cut-off to be the Planck energy the vacuum energy is some 123 orders of magnitude away from the observed value of the cosmological constant, making this the worst prediction in theoretical physics.

We see that the root of the cosmological constant problem lies in the fact that we have treated spacetime and matter as separate objects. If we treat quantum fields as living on a spacetime, then we will encounter the cosmological constant problem.

If, on the other, hand we realize that it is only through the excitations described by the quantum fields that a spacetime appears in the first place, the above argument can not be given and the cosmological constant problem disappears.

The problem of time appears when one tries to quantize the gravitational field on its own. Because the Hamiltonian vanishes there is no notion of time evolution left. In our approach it does not make sense to treat the gravitational field without matter. To do so means stepping into the "problem of time" trap.

7.4 Conclusion

In this chapter we have tried to review a number of approaches to the problem of Quantum Gravity in which spacetime is emergent. We have seen that even when spacetime is not fundamental there are still a number of choices to be made. The first choice to be made concerns the role of time. Is the background time to be used or is it more like a fiducial parameter?

An example where the background time is used is Volovik's theory of Fermi liquids with a Fermi point. The quest here is for a theory that has a massless spin two particle in its spectrum. We have seen that Volovik comes close. It is the mass of the graviton that is the problem. Generically it will not vanish. It is interesting though that this model reproduces a lot of the physics we see around us, including fermions and gauge excitations.

As an example where the role of time is different we have seen Lloyd's computational universe. The discrete time labeling the individual quantum gates U_l, $l = 1, \dots, n$, is not used in the construction of the spacetime metrics of the computational histories. Note how the questions changes here. One is no longer looking for a massless spin two excitation. In the context where the whole spacetime metric is to be defined it would not even be clear what a massless spin two excitation would mean. Instead one looks for the whole metric using the Einstein equations.

This attempt is also not without problems. Given *any* quantum computation one can construct computational histories with a corresponding spacetime interpretation. The question of the meaning of these metrics then arises. Why is there a spacetime interpretation to a calculation that factorizes large integers?

The proposals reviewed here were all presented at a workshop at the Perimeter Institute in Canada.[1] We have not discussed approaches that are included in this volume through the contributions of participants to the workshop. See R. Loll, F. Markopoulou, and the string theorists who have also ventured into the realm of emergent spacetime.

[1] Recordings of the talks can be found on the website of the Perimeter Institute at www.perimeterinstitute.ca.

In addition to the proposals presented at the workshop we have also discussed a novel approach which differs from the computational universe mainly in that it does not use the Einstein equations. We instead argued that they are a result of the internal and background independent approach.

The main ingredient are coherent degrees of freedom. These play the role of matter but they are also used to define notions of space and time. It is because they play this dual role that the equivalence principle and also the Einstein equations are true.

In this approach there is no notion of spacetime without matter. Tearing apart spacetime and matter by viewing the latter as living on the former creates deep problems like the cosmological constant problem and the problem of time. Here we avoid these problems.

This view also goes well with a new view of quantum mechanics [7]. In this view of quantum mechanics a notion like position is only applicable to large quantum systems and is not fundamental. Given such a view, it is only natural that a spacetime emerges and is not included as a basic building block.

In recent years we have seen a number of new approaches to the problem of Quantum Gravity come very close to the stated goal. Using methods and ideas foreign to the more traditional approaches they were able to make progress where others got stuck. Maybe we will soon have not just one quantum theory of gravity but several to choose from. To decide which one is the right one will then require recourse to experiment. What an exciting possibility.

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8

Asymptotic safety

R. PERCACCI

8.1 Introduction

The problems of perturbative Quantum Field Theory (QFT) in relation to the UV behaviour of gravity have led to widespread pessimism about the possibility of constructing a fundamental QFT of gravity. Instead, we have become accustomed to thinking of General Relativity (GR) as an effective field theory, which only gives an accurate description of gravitational physics at low energies. The formalism of effective field theories provides a coherent framework in which quantum calculations can be performed even if the theory is not renormalizable. For example, quantum corrections to the gravitational potential have been discussed by several authors; see [5] and references therein. This continuum QFT description is widely expected to break down at very short distances and to be replaced by something dramatically different beyond the Planck scale. There is, however, no proof that continuum QFT will fail, and the current situation may just be the result of the lack of suitable technical tools. Weinberg [46] described a generalized, nonperturbative notion of renormalizability called "asymptotic safety" and suggested that GR may satisfy this condition, making it a consistent QFT at all energies. The essential ingredient of this approach is the existence of a Fixed Point (FP) in the Renormalization Group (RG) flow of gravitational couplings. Several calculations were performed using the ϵ-expansion around $d = 2$ dimensions, supporting the view that gravity is asymptotically safe [17; 10; 20]. However, the continuation to four dimensions ($\epsilon \to 2$) was questionable and this line of research slowed down for some time. It was revived by Reuter [36] who calculated the gravitational beta functions directly in $d = 4$ dimensions, using a truncation of an Exact Renormalization Group Equation (ERGE). Matter couplings were considered by Dou & Percacci [13]; then Souma [41] found that these beta functions admit a non-Gaussian FP. Further work [22; 23; 34; 11] strongly supports the view that this FP is not a mere artifact of the approximations made. An extensive review of this subject can be found in [29].

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

In section 8.2 I introduce the general idea of asymptotic safety; the reader is referred to [46] for a more detailed discussion. In section 8.3 I describe some peculiarities of the gravitational RG, which derive from the dual character of the metric as a dynamical field and as definition of lengths. Recent evidence for a FP, coming mainly from the ERGE, is reviewed in section 8.4. Some relations to other approaches to Quantum Gravity are briefly mentioned in section 8.5.

8.2 The general notion of asymptotic safety

The techniques of effective QFT have been recognized as being of great generality and are now quite pervasive in particle physics. An effective field theory is described by an effective action Γ_k which can be thought of as the result of having integrated out all fluctuations of the fields with momenta larger than k. We need not specify here the physical meaning of k: for each application of the theory one will have to identify the physically relevant variable acting as k (in particle physics it is usually some external momentum). One convenient definition of Γ_k that we shall use here is as follows. We start from a ("bare") action $S[\phi_A]$ for multiplets of quantum fields ϕ_A, describing physics at an energy scale k_0. We add to it a term $\Delta S_k[\phi_A]$, quadratic in the ϕ_A, which in Fourier space has the form: $\Delta S_k[\phi] = \int d^d q \phi_A R_k^{AB}(q^2)\phi_B$. The kernel $R_k^{AB}(q^2)$, henceforth called the cutoff function, is chosen in such a way that the propagation of field modes $\phi_A(q)$ with momenta $q < k$ is suppressed, while field modes with momenta $k < q < k_0$ are unaffected. We formally define a k-dependent generating functional of connected Green functions

$$W_k[J^A] = -\log \int (d\phi_A) \exp\left(-S[\phi_A] - \Delta S_k[\phi_A] - \int J^A \phi_A\right) \tag{8.1}$$

and a modified k-dependent Legendre transform

$$\Gamma_k[\phi_A] = W_k[J^A] - \int J^A \phi_A - \Delta S_k[\phi_A]\,, \tag{8.2}$$

where ΔS_k has been subtracted. The "classical fields" $\frac{\delta W_k}{\delta J^A}$ are denoted again ϕ_A for notational simplicity. This functional interpolates continuously between S, for $k = k_0$, and the usual effective action $\Gamma[\phi_A]$, the generating functional of one-particle irreducible Green functions, for $k = 0$. It is similar in spirit, but distinct from, the Wilsonian effective action. In the following we will always use this definition of Γ_k, but much of what will be said should be true also with other definitions.

In the case of gauge theories there are complications due to the fact that the cutoff interferes with gauge invariance. One can use a background gauge condition, which circumvents these problems by defining a functional of two fields, the background field and the classical field; the effective action Γ_k is then obtained by identifying these fields. See [30] or [36] for the case of gravity.

The effective action $\Gamma_k[\phi_A]$, used at tree level, gives an accurate description of processes occurring at momentum scales of order k. In general it will have the form $\Gamma_k(\phi_A, g_i) = \sum_i g_i(k)\mathcal{O}_i(\phi_A)$, where g_i are running coupling constants and $\mathcal{O}_i$ are all possible operators constructed with the fields ϕ_A and their derivatives, which are compatible with the symmetries of the theory. It can be thought of as a functional on $\mathcal{F} \times \mathcal{Q} \times R^+$, where $\mathcal{F}$ is the configuration space of the fields, $\mathcal{Q}$ is an infinite dimensional manifold parametrized by the coupling constants, and R^+ is the space parametrized by k. The dependence of Γ_k on k is given by $\partial_t \Gamma_k(\phi_A, g_i) = \sum_i \beta_i(k)\mathcal{O}_i(\phi_A)$ where $t = \log(k/k_0)$ and $\beta_i(g_j, k) = \partial_t g_i$ are the beta functions.

Dimensional analysis implies the scaling property

$$\Gamma_k(\phi_A, g_i) = \Gamma_{bk}(b^{d_A}\phi_A, b^{d_i} g_i) \ , \tag{8.3}$$

where d_A is the canonical dimension of ϕ_A, d_i is the canonical dimension of g_i, and $b \in R^+$ is a positive real scaling parameter.[1] One can rewrite the theory in terms of dimensionless fields $\tilde{\phi}_A = \phi_A k^{-d_A}$ and dimensionless couplings $\tilde{g}_i = g_i k^{-d_i}$. A transformation (8.3) with parameter $b = k^{-1}$ can be used to define a functional $\tilde{\Gamma}$ on $(\mathcal{F} \times \mathcal{Q} \times R^+)/R^+$:

$$\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_i) := \Gamma_1(\tilde{\phi}_A, \tilde{g}_i) = \Gamma_k(\phi_A, g_i) \ . \tag{8.4}$$

Similarly, $\beta_i(g_j, k) = k^{d_i} a_i(\tilde{g}_j)$ where $a_i(\tilde{g}_j) = \beta_i(\tilde{g}_j, 1)$. There follows that the beta functions of the dimensionless couplings,

$$\tilde{\beta}_i(\tilde{g}_j) \equiv \partial_t \tilde{g}_i = a_i(\tilde{g}_j) - d_i \tilde{g}_i \tag{8.5}$$

depend on k only implicitly via the $\tilde{g}_j(t)$.

The effective actions Γ_k and $\Gamma_{k-\delta k}$ differ essentially by a functional integral over field modes with momenta between k and $k - \delta k$. Such integration does not lead to divergences, so the beta functions are automatically finite. Once calculated at a certain scale k, they are automatically determined at any other scale by dimensional analysis. Thus, the scale k_0 and the "bare" action S act just as initial conditions: when the beta functions are known, one can start from an arbitrary initial point on $\mathcal{Q}$ and follow the RG trajectory in either direction. The effective action Γ_k at any scale k can be obtained integrating the flow. In particular, the UV behaviour can be studied by taking the limit $k \to \infty$.

It often happens that the flow cannot be integrated beyond a certain limiting scale Λ, defining the point at which some "new physics" has to make its appearance. In

[1] We assume that the coordinates are dimensionless, as is natural in curved space, resulting in unconventional canonical dimensions. The metric is an area.

this case the theory only holds for $k < \Lambda$ and is called an "effective" or "cutoff" QFT. It may happen, however, that the limit $t \to \infty$ can be taken; we then have a self-consistent description of a certain set of physical phenomena which is valid for arbitrarily high energy scales and does not need to refer to anything else outside it. In this case the theory is said to be "fundamental".

The couplings appearing in the effective action can be related to physically measurable quantities such as cross-sections and decay rates. Dimensional analysis implies that aside from an overall power of k, such quantities only depend on dimensionless kinematical variables X, like scattering angles and ratios of energies, and on the dimensionless couplings $\tilde{g}_i$ (recall that usually k is identified with one of the momentum variables). For example, a cross-section can be expressed as $\sigma = k^{-2}\tilde{\sigma}(X, \tilde{g}_i)$. If some of the couplings $\tilde{g}_i$ go to infinity when $t \to \infty$, also the function $\tilde{\sigma}$ can be expected to diverge. A sufficient condition to avoid this problem is to assume that in the limit $t \to \infty$ the RG trajectory tends to a FP of the RG, i.e. a point $\tilde{g}_*$ where $\tilde{\beta}_i(\tilde{g}_*) = 0$ for all i. The existence of such a FP is the first requirement for asymptotic safety. Before discussing the second requirement, we have to understand that one needs to impose this condition only on a subset of all couplings.

The fields ϕ_A are integration variables, and a redefinition of the fields does not change the physical content of the theory. This can be seen as invariance under a group $\mathcal{G}$ of coordinate transformations in $\mathcal{F}$. There is a similar arbitrariness in the choice of coordinates on $\mathcal{Q}$, due to the freedom of redefining the couplings g_i. Since, for given k, Γ_k is assumed to be the "most general" functional on $\mathcal{F} \times \mathcal{Q}$ (in some proper sense), given a field redefinition $\phi' = \phi'(\phi)$ one can find new couplings g'_i such that

$$\Gamma_k(\phi'_B(\phi_A), g_i) = \Gamma_k(\phi_A, g'_i)\ . \tag{8.6}$$

At least locally, this defines an action of $\mathcal{G}$ on $\mathcal{Q}$. We are then free to choose a coordinate system which is adapted to these transformations, in the sense that a subset $\{g_{\hat{\imath}}\}$ of couplings transform nontrivially and can be used as coordinates in the orbits of $\mathcal{G}$, while a subset $\{g_{\bar{\imath}}\}$ are invariant under the action of $\mathcal{G}$ and define coordinates on $\mathcal{Q}/\mathcal{G}$. The couplings $g_{\hat{\imath}}$ are called redundant or inessential, while the couplings $g_{\bar{\imath}}$ are called essential. In an adapted parametrization there exists, at least locally, a field redefinition $\bar{\phi}(\phi)$ such that using (8.6) the couplings $g_{\hat{\imath}}$ can be given fixed values $(g_{\hat{\imath}})_0$. We can then define a new action $\bar{\Gamma}$ depending only on the essential couplings:

$$\bar{\Gamma}_k(\bar{\phi}_A, g_{\bar{\imath}}) := \Gamma_k(\bar{\phi}_A, g_{\bar{\imath}}, (g_{\hat{\imath}})_0) = \Gamma_k(\phi_A; g_{\bar{\imath}}, g_{\hat{\imath}}). \tag{8.7}$$

Similarly, the values of the redundant couplings can be fixed also in the expressions for measurable quantities, so there is no need to constrain their RG flow in any way: they are not required to flow towards an FP.

For example, the action of a scalar field theory in a background $g_{\mu\nu}$,

$$\Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}) = \int d^4x\sqrt{g}\left[\frac{Z_\phi}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda_2\phi^2 + \lambda_4\phi^4 + \cdots\right] \quad (8.8)$$

has the scaling invariance

$$\Gamma_k(c\phi, g_{\mu\nu}; c^{-2}Z_\phi, c^{-2i}\lambda_{2i}) = \Gamma_k(\phi, g_{\mu\nu}; Z_\phi, \lambda_{2i}), \quad (8.9)$$

which is a special case of (8.6). There exists an adapted coordinate system where Z is inessential and $\bar{\lambda}_{2i} = \lambda_{2i}Z_\phi^{-i}$ are the essential coordinates. A transformation with $c = \sqrt{Z_\phi}$ then leads to $Z_\phi = 1$, leaving the essential couplings unaffected.

A comparison of (8.4) and (8.7) shows that k behaves like a redundant coupling. In ordinary QFTs, it is generally the case that for each multiplet of fields ϕ_A there is a scaling invariance like (8.9) commuting with (8.3). One can use these invariances to eliminate simultaneously k and one other redundant coupling per field multiplet; the conventional choice is to eliminate the wave function renormalization Z_A. No conditions have to be imposed on the RG flow of the Z_As, and the anomalous dimensions $\eta_A = \partial_t \log Z_A$, at an FP, can be determined by a calculation. More generally, (8.3) and (8.6) can be used to eliminate simultaneously the dependence of Γ_k on k and on the inessential couplings, and to define an effective action $\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_{\bar{\imath}})$, depending only on the dimensionless essential couplings $\tilde{g}_{\bar{\imath}} = g_{\bar{\imath}}k^{-d_{\bar{\imath}}}$. It is only on these couplings that one has to impose the FP condition $\partial_t\tilde{g}_{\bar{\imath}} = 0$.

We can now state the second requirement for asymptotic safety. Denote $\tilde{\mathcal{Q}} = (\mathcal{Q} \times R^+)/(\mathcal{G} \times R^+)$ the space parametrized by the dimensionless essential couplings $\tilde{g}_{\bar{\imath}}$. The set $\mathcal{C}$ of all points in $\tilde{\mathcal{Q}}$ that flow towards the FP in the UV limit is called the UV critical surface. If one chooses an initial point lying on $\mathcal{C}$, the whole trajectory will remain on $\mathcal{C}$ and will ultimately flow towards the FP in the UV limit. Points that lie outside $\mathcal{C}$ will generally flow towards infinity (or other FPs). Thus, demanding that the theory lies on the UV critical surface ensures that it has a sensible UV limit. It also has the effect of reducing the arbitrariness in the choice of the coupling constants. In particular, if the UV critical surface is finite dimensional, the arbitrariness is reduced to a finite number of parameters, which can be determined by a finite number of experiments. Thus, a theory with an FP and a finite dimensional UV critical surface has a controllable UV behaviour, and is predictive. Such a theory is called "asymptotically safe".

A perturbatively renormalizable, asymptotically free field theory such as QCD is a special case of an asymptotically safe theory. In this case the FP is the Gaussian FP, where all couplings vanish, and the critical surface is spanned, near the FP, by the couplings that are renormalizable in the perturbative sense (those with dimension $d_{\bar{\imath}} \geq 0$).

The requirement of renormalizability played an important role in the construction of the Standard Model (SM) of particle physics. Given that the SM is not a complete theory, and that some of its couplings are not asymptotically free, nowadays it is regarded as an effective QFT, whose nonrenormalizable couplings are suppressed by some power of momentum over cutoff. On the other hand, any theory that includes both the SM and gravity should better be a fundamental theory. For such a theory, the requirement of asymptotic safety will have the same significance that renormalizability originally had for the SM.

8.3 The case of gravity

We shall use a derivative expansion of Γ_k:

$$\Gamma_k(g_{\mu\nu}; g_i^{(n)}) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) \mathcal{O}_i^{(n)}(g_{\mu\nu}), \tag{8.10}$$

where $\mathcal{O}_i^{(n)} = \int d^d x \sqrt{g} \mathcal{M}_i^{(n)}$ and $\mathcal{M}_i^{(n)}$ are polynomials in the curvature tensor and its derivatives containing $2n$ derivatives of the metric; i is an index that labels different operators with the same number of derivatives. The dimension of $g_i^{(n)}$ is $d_n = d - 2n$. The first two polynomials are just $\mathcal{M}^{(0)} = 1$, $\mathcal{M}^{(1)} = R$. The corresponding couplings are $g^{(1)} = -Z_g = -\frac{1}{16\pi G}$, $g^{(0)} = 2Z_g\Lambda$, Λ being the cosmological constant. Newton's constant G appears in Z_g, which in linearized Einstein theory is the wave function renormalization of the graviton. Neglecting total derivatives, one can choose as terms with four derivatives of the metric $\mathcal{M}_1^{(2)} = C^2$ (the square of the Weyl tensor) and $\mathcal{M}_2^{(2)} = R^2$. We also note that the coupling constants of higher derivative gravity are not the coefficients $g_i^{(2)}$ but rather their inverses $2\lambda = (g_1^{(2)})^{-1}$ and $\xi = (g_2^{(2)})^{-1}$. Thus,

$$\Gamma_k^{(n\leq 2)} = \int d^d x \sqrt{g} \left[2Z_g\Lambda - Z_g R + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 \right]. \tag{8.11}$$

As in any other QFT, Z_g can be eliminated from the action by a rescaling of the field. Under constant rescalings of $g_{\mu\nu}$, in d dimensions,

$$\Gamma_k(g_{\mu\nu}; g_i^{(n)}) = \Gamma_{bk}(b^{-2} g_{\mu\nu}; b^{d-2n} g_i^{(n)}). \tag{8.12}$$

This relation is the analog of (8.9) for the metric, but also coincides with (8.3), the invariance at the basis of dimensional analysis; fixing it amounts to a choice of unit of mass. This is where gravity differs from any other field theory [33; 35]. In usual QFTs such as (8.8), one can exploit the two invariances (8.3) and (8.9) to eliminate simultaneously k and Z from the action. In the case of pure gravity there is only one such invariance and one has to make a choice.

If we choose k as unit of mass, we can define the effective action,

$$\tilde{\Gamma}(\tilde{g}_{\mu\nu}; \tilde{Z}_g, \tilde{\Lambda}, \ldots) = \Gamma_1(\tilde{g}_{\mu\nu}; \tilde{Z}_g, \tilde{\Lambda}, \ldots) = \Gamma_k(g_{\mu\nu}; Z_g, \Lambda, \ldots), \tag{8.13}$$

where $\tilde{g}_{\mu\nu} = k^2 g_{\mu\nu}$, $\tilde{Z}_g = \frac{Z_g}{k^2} = \frac{1}{16\pi\tilde{G}}$, $\tilde{\Lambda} = \frac{\Lambda}{k^2}$, etc. There is then no freedom left to eliminate Z_g. Physically measurable quantities will depend explicitly on $\tilde{Z}_g$, so by the arguments of section 8.2, we have to impose that $\partial_t \tilde{Z}_g = 0$, or equivalently $\partial_t \tilde{G} = 0$, at a FP.

Alternatively, one can use (8.12) to set $Z_g = 1$: this amounts to working in Planck units. Then we can define a new action:[2]

$$\Gamma'_{k'}(g'_{\mu\nu}; \Lambda', \ldots) = \Gamma_{k'}(g'_{\mu\nu}; \Lambda', 1, \ldots) = \Gamma_k(g_{\mu\nu}; \Lambda, Z_g, \ldots), \tag{8.14}$$

where $g'_{\mu\nu} = 16\pi Z_g g_{\mu\nu}$, $\Lambda' = \frac{1}{16\pi Z_g}\Lambda$, $k' = \sqrt{\frac{1}{16\pi Z_g}}k$, etc., are the metric, cosmological constant and cutoff measured in Planck units. In this case, the dependence on G disappears; however, the beta functions and measurable quantities will depend explicitly on k'.

In theories of gravity coupled to matter, the number of these scaling invariances is equal to the number of field multiplets, so the situation is the same as for pure gravity. (Without gravity, it is equal to the number of field multiplets plus one, due to dimensional analysis.) The situation can be summarized by saying that when the metric is dynamical, k should be treated as one of the couplings, and that there exist parametrizations where k is redundant or G is redundant, but not both.

Scale invariance is usually thought to imply that a theory contains only dimensionless parameters, and the presence at a FP of nonvanishing dimensionful couplings may seem to be at odds with the notion that the FP theory is scale–invariant. This is the case if only the fields are scaled, and not the couplings. In an asymptotically safe QFT, scale invariance is realized in another way: all dimensionful couplings scale with k as required by their canonical dimension. In geometrical terms, the RG trajectories in $\mathcal{Q}$ lie asymptotically in an orbit of the transformations (8.3) and (8.6). This also has another consequence. At low momentum scales $p \ll \sqrt{Z_g}$ the couplings are not expected to run and the terms in the action (8.11) with four derivatives are suppressed relative to the term with two derivatives by a factor p^2/Z_g. On the other hand in the FP regime, if we evaluate the couplings at $k = p$, the running of Z_g exactly compensates the effect of the derivatives: both terms are of order p^4. From this point of view, *a priori* all terms in (8.10) could be equally important.

From the existence of a FP for Newton's constant there would immediately follow two striking consequences. First, the cutoff measured in Planck units would

[2] Note that to completely eliminate Z_g from the action one has to scale the whole metric, and not just the fluctuation, as is customary in perturbation theory.

be bounded. This is because the cutoff in Planck units, $k' = k\sqrt{G}$, is equal to the square root of Newton's constant in cutoff units, $\sqrt{\tilde{G}}$. Since we have argued that the latter must have a finite limit at a FP, then also the former must do so. This seems to contradict the notion that the UV limit is defined by $k \to \infty$. The point is that only statements about dimensionless quantities are physically meaningful, and the statement "$k \to \infty$" is meaningless until we specify the units. In a fundamental theory one cannot refer to any external "absolute" quantity as a unit, and any internal quantity which is chosen as a unit will be subject to the RG flow. If we start from low energy ($k' \ll 1$) and we increase k, k' will initially increase at the same rate, because in this regime $\partial_t G \approx 0$; however, when $k' \approx 1$ we reach the FP regime where $G(k) \approx \tilde{G}_*/k^2$ and therefore k' stops growing.

The second consequence concerns the graviton anomalous dimension, which in d dimensions is $\eta_g = \partial_t \log Z_g = \partial_t \log \tilde{Z}_g + d - 2$. Since we have argued that $\partial_t \tilde{Z}_g = 0$ at a gravitational FP, if $\tilde{Z}_{g*} \neq 0$ we must have $\eta_{g*} = d - 2$. The propagator of a field with anomalous dimension η behaves like $p^{-2-\eta}$, so one concludes that at a nontrivial gravitational FP the graviton propagator behaves like p^{-d} rather than p^{-2}, as would follow from a naive classical interpretation of the Einstein–Hilbert action. Similar behaviour is known also in other gauge theories away from the critical dimension, see e.g. [21].

8.4 The Gravitational Fixed Point

I will now describe some of the evidence that has accumulated in favour of a nontrivial Gravitational FP. Early attempts were made in the context of the ϵ-expansion around two dimensions ($\epsilon = d - 2$), which yields

$$\beta_{\tilde{G}} = \epsilon \tilde{G} - q \tilde{G}^2. \tag{8.15}$$

Thus there is a UV-attractive FP at $\tilde{G}_* = \epsilon/q$. The constant $q = \frac{38}{3}$ for pure gravity [46; 20], see [1] for two-loop results. Unfortunately, for a while it was not clear whether one could trust the continuation of this result to four dimensions ($\epsilon = 2$).

Most of the recent progress in this approach has come from the application to gravity of the ERGE. It was shown by Wetterich [47] that the effective action Γ_k defined in (8.2) satisfies the equation

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\frac{\delta^2 \Gamma_k}{\delta \phi_A \delta \phi_B} + R_k^{AB} \right)^{-1} \partial_t R_k^{BA}, \tag{8.16}$$

where STr is a trace over momenta as well as over particle species and any spacetime or internal indices, including a sign -1 for fermionic fields and a factor 2 for complex fields. In the case of gauge theories, the ghost fields have to be included among the ϕ_A.

Comparing the r.h.s. of the ERGE with the t-derivative of (8.10) one can extract the beta functions. Note that in general the cutoff function R_k may depend on the couplings and therefore the term $\partial_t R_k$ in the r.h.s. of (8.16) contains the beta functions. Thus, extracting the beta functions from the ERGE implies solving an equation where the beta functions appear on both sides. At one loop, the effective action Γ_k is $\mathrm{Tr}\log\frac{\delta^2(S+\Delta S_k)}{\delta\phi\delta\phi}$; it satisfies an equation which is formally identical to (8.16) except that in the r.h.s. the running couplings $g_i(k)$ are replaced everywhere by the "bare" couplings $g_i(k_0)$, appearing in S. We will call "one–loop beta functions" those extracted from the ERGE ignoring the derivatives of the couplings that may appear in the r.h.s. of (8.16).

It is usually impossible to get the beta functions for all couplings, so a common procedure is to consider a truncation of the theory where the effective action Γ_k contains only a (finite or infinite) subset of all possible terms. In these calculations there is no small parameter to tell us what terms can be safely neglected, so the choice of truncation has to be motivated by physical insight. On the other hand, in this way one can obtain genuine nonperturbative information. This and other similar ERGEs have been applied to a variety of problems. One can reproduce the universal one loop beta functions of familiar theories, and in more advanced approximations the results are quantitatively comparable to those obtainable by other methods. See [3; 4; 30] for reviews.

The simplest way to arrive at a Gravitational FP in four dimensions, avoiding the technical complications of graviton propagators, is through the contributions of matter loops to the beta functions of the gravitational couplings. Thus, consider gravity coupled to n_{S} scalar fields, n_{D} Dirac fields, n_{M} gauge (Maxwell) fields, all massless and minimally coupled. A priori, nothing is assumed about the gravitational action. For each type of field ϕ_A we choose the cutoff function in such a way that $P_k(\Delta^{(A)}) = \Delta^{(A)} + R_k(\Delta^{(A)})$, where $\Delta^{(S)} = -\nabla^2$ on scalars, $\Delta^{(D)} = -\nabla^2 + \frac{R}{4}$ on Dirac fields and $\Delta^{(M)} = -\nabla^2\delta^\mu_\nu + R^\mu{}_\nu$ on Maxwell fields in the gauge $\alpha = 1$. Then, the ERGE is simply

$$\partial_t\Gamma_k = \sum_{A=S,D,M} \frac{n_A}{2}\mathrm{STr}_{(A)}\left(\frac{\partial_t P_k}{P_k}\right) - n_{\mathrm{M}}\mathrm{Tr}_{(S)}\left(\frac{\partial_t P_k}{P_k}\right), \tag{8.17}$$

where $\mathrm{STr} = \pm\mathrm{Tr}$ depending on the statistics, and the last term comes from the ghosts. Using integral transforms and the heat kernel expansion, the trace of a function f of Δ can be expanded as

$$\mathrm{Tr} f(\Delta) = \sum_{n=0}^{\infty} Q_{2-n}(f) B_{2n}(\Delta), \tag{8.18}$$

where the heat kernel coefficients $B_{2n}(\Delta)$ are linear combinations of the $\mathcal{O}_i^{(n)}$, $Q_n(f) = (-1)^n f^{(n)}(0)$ for $n \leq 0$ and $Q_n(f)$ are given by Mellin transforms of f for $n > 0$.[3] In this way one can write out explicitly the r.h.s. of (8.17) in terms of the $\mathcal{O}_i^{(n)}$ and read off the beta functions.

When $N \to \infty$, this is the dominant contribution to the gravitational beta functions, and graviton loops can be neglected [43; 40; 34]. The functions $a_i^{(n)}$ defined in (8.5) become numbers; with the so-called optimized cutoff function $R_k(z) = (k^2 - z)\theta(k^2 - z)$, discussed in [25; 26], they are

$$a^{(0)} = \frac{n_S - 4n_D + 2n_M}{32\pi^2}\,, \qquad a^{(1)} = \frac{n_S + 2n_D - 4n_M}{96\pi^2}\,,$$
$$a_1^{(2)} = \frac{6n_S + 36n_D + 72n_M}{11520\pi^2}\,, \qquad a_2^{(2)} = \frac{10n_S}{11520\pi^2}\,,$$

while $a_i^{(n)} = 0$ for $n \geq 3$. The beta functions (8.5) are then

$$\partial_t \tilde{g}_i^{(n)} = (2n - 4)\tilde{g}_i^{(n)} + a_i^{(n)}. \tag{8.19}$$

For $n \neq 2$ this leads to an FP

$$\tilde{g}_{i*}^{(n)} = \frac{a_i^{(n)}}{4 - 2n}\,, \tag{8.20}$$

in particular we get

$$\tilde{\Lambda}_* = -\frac{3}{4}\frac{n_S - 4n_D + 2n_M}{n_S + 2n_D - 4n_M}\,, \quad \tilde{G}_* = \frac{12\pi}{-n_S - 2n_D + 4n_M}\,. \tag{8.21}$$

For $n = 2$, one gets instead $g_i^{(2)}(k) = g_i^{(2)}(k_0) + a_i^{(2)} \ln(k/k_0)$, implying asymptotic freedom for the couplings λ and ξ of (8.11). Remarkably, with this cutoff all the higher terms are zero at the FP. The critical exponents are equal to the canonical dimensions of the $g^{(n)}$s, so Λ and G are UV-relevant (attractive), λ and ξ are marginal and all the higher terms are UV-irrelevant. Note that in perturbation theory G would be UV-irrelevant (nonrenormalizable). At the nontrivial FP the quantum corrections conspire with the classical dimensions of Λ and G to reconstruct the dimensions of $g^{(0)}$ and $g^{(1)}$. This does not happen at the Gaussian FP, where the transformation between $\tilde{G}$ and $\tilde{g}^{(1)}$ is singular.

Using the same techniques, the one loop beta functions for gravity with the action (8.11) have been calculated by Codello & Percacci [11]. The beta functions for λ and ξ agree with those derived in the earlier literature on higher derivative gravity [16; 2; 12]. These couplings tend logarithmically to zero with a fixed ratio $\omega = -3\lambda/\xi \to \omega_* = -0.023$. The beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ differ from the ones that were given in the earlier literature essentially by the first two terms of

[3] This technique is used also in some noncommutative geometry models, see [9].

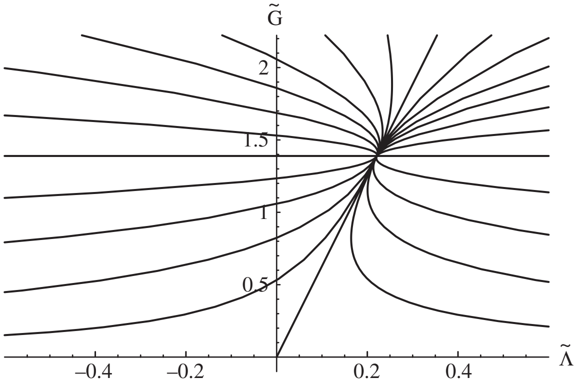

Fig. 8.1. The flow in the upper $\tilde{\Lambda}$–$\tilde{G}$ plane for pure gravity with higher derivative terms at one loop, eq. (8.22). All other couplings are set to zero. The nontrivial FP at (0.221,1.389) is UV-attractive with eigenvalues (−4, −2), the one in the origin is UV-attractive along the $\tilde{\Lambda}$ axis with eigenvalue −2 and repulsive in the direction of the vector $(1/2\pi, 1)$ with eigenvalue 2.

the expansion (8.18). In a conventional calculation of the effective action these terms would correspond to quartic and quadratic divergences, which are normally neglected in dimensional regularization, but are crucial in generating a nontrivial FP. Setting the dimensionless couplings to their FP-values, one obtains:

$$\beta_{\tilde{\Lambda}} = 2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_* \tilde{G}\tilde{\Lambda}\ , \quad \beta_{\tilde{G}} = 2\tilde{G} - q_* \tilde{G}^2, \tag{8.22}$$

where $q_* \approx 1.440$. This flow is qualitatively identical to the flow in the $N \to \infty$ limit, and is shown in fig. 8.1.

In order to appreciate the full nonperturbative content of the ERGE, let us consider pure gravity in the Einstein–Hilbert truncation, i.e. neglecting terms with $n \geq 2$. In a suitable gauge the operator $\frac{\delta^2 \Gamma_k}{\delta g_{\mu\nu} \delta g_{\rho\sigma}}$ is a function of $-\nabla^2$ only. Then, rather than taking as Δ the whole linearized wave operator, as we did before, we use (8.18) with $\Delta = -\nabla^2$. In this way we retain explicitly the dependence on Λ and R. Using the optimized cutoff, with gauge parameter $1/\alpha = Z$, the ERGE gives

$$\beta_{\tilde{\Lambda}} = \frac{-2(1-2\tilde{\Lambda})^2\tilde{\Lambda} + \frac{36-41\tilde{\Lambda}+42\tilde{\Lambda}^2-600\tilde{\Lambda}^3}{72\pi}\tilde{G} + \frac{467-572\tilde{\Lambda}}{288\pi^2}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}}, \tag{8.23}$$

$$\beta_{\tilde{G}} = \frac{2(1-2\tilde{\Lambda})^2\tilde{G} - \frac{373-654\tilde{\Lambda}+600\tilde{\Lambda}^2}{72\pi}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}}. \tag{8.24}$$

This flow is shown in fig. 8.2.

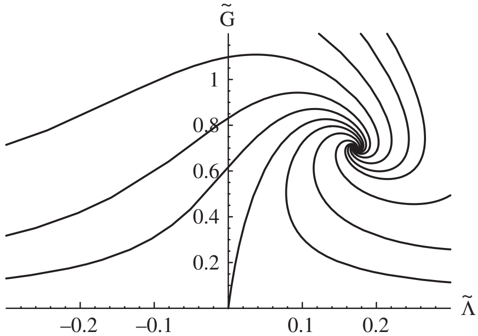

Fig. 8.2. The flow in the Einstein–Hilbert truncation, see eqs. (8.23) and (8.24). The nontrivial FP at $\tilde{\Lambda} = 0.171$, $\tilde{G} = 0.701$ is UV-attractive with eigenvalues $-1.69 \pm 2.49i$. The Gaussian FP is attractive along the $\tilde{\Lambda}$-axis with eigenvalue -2 and repulsive in the direction (0.04, 1.00) with eigenvalue 2.

Lauscher & Reuter [22] and Reuter & Saueressig [37] have studied the gauge- and cutoff-dependence of the FP in the Einstein–Hilbert truncation. The dimensionless quantity $\Lambda' = \Lambda G$ (the cosmological constant in Planck units) and the critical exponents have a reassuringly weak dependence on these parameters. This has been taken as a sign that the FP is not an artifact of the truncation. Lauscher & Reuter [23] have also studied the ERGE including a term R^2 in the truncation. They find that in the subspace of $\tilde{\mathcal{Q}}$ spanned by $\tilde{\Lambda}$, $\tilde{G}$, $1/\xi$, the non-Gaussian FP is very close to the one of the Einstein–Hilbert truncation, and is UV-attractive in all three directions. More recently, the FP has been shown to exist if the Lagrangian density is a polynomial in R of order up to six (Codello, Percacci and Rahmede, in preparation). In this truncation the UV critical surface is three dimensional.

There have been also other generalizations. Niedermaier [28] considered the RG flow for dimensionally reduced $d = 4$ gravity, under the hypothesis of the existence of two Killing vectors. This subsector of the theory is parametrized by infinitely many couplings, and has been proved to be asymptotically safe.

Matter couplings have been considered by Percacci & Perini [31; 32]. Consider the general action

$$\Gamma_k(g_{\mu\nu}, \phi) = \int d^4x \sqrt{g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi^2) + F(\phi^2) R \right), \qquad (8.25)$$

where V and F are arbitrary functions of ϕ^2, analytic at $\phi^2 = 0$. This action has a so-called Gaussian-Matter FP, meaning that only the coefficients of the ϕ-independent terms in (8.25) (namely $g^{(0)}$ and $g^{(1)}$) are nonzero. The critical surface has dimension four and there are no marginal operators. In the presence of other, minimally coupled matter fields, the dimension of the critical surface can be larger, and it is easy to find theories where a polynomial potential in ϕ is renormalizable and asymptotically free. Thus, gravity seems to provide a solution to the so-called triviality problem of scalar field theory.

It is tempting to speculate with Fradkin & Tseytlin [16] that in the presence of gravity all matter interactions are asymptotically free. One-loop calculations reported in [8; 39] indicate that this may be the case also for gauge and Yukawa interactions. Then, in studying the FP, it would be consistent to neglect matter interactions, as we did in the $1/N$ expansion. If this is the case, it may become possible to show asymptotic safety for realistic unified theories including gravity and the SM.

For the time being, the Gravitational FP has been found with a number of different approximations: the $2 + \epsilon$ expansion, the $1/N$ expansion, polynomial truncations with a variety of cutoffs and gauges, the two Killing vector reduction and the most general four-derivative gravity theory at one loop. The fact that all these methods yield broadly consistent results should leave little doubt about the existence of a nontrivial FP with the desired properties.

8.5 Other approaches and applications

In this final section we briefly comment on the relation of asymptotic safety to other approaches and results in Quantum Gravity.

Gravity with the Einstein–Hilbert action has been shown by Goroff & Sagnotti [18] and van de Ven [45] to be perturbatively nonrenormalizable at two loops. Stelle [42] proved that the theory with action (8.11) and $\Lambda = 0$ is perturbatively renormalizable: all divergences can be absorbed into redefinitions of the couplings. In general, asymptotic safety does not imply that in the UV limit only a finite number of terms in (8.10) survive: there could be infinitely many terms, but there would be relations between their coefficients in such a way that only a finite number of parameters would be left free. At one loop or in the large-N limit, the ERGE predicts that the UV critical surface can be parametrized by the four couplings $\tilde{\Lambda}$, $\tilde{G}$, λ and ξ, the first two being nonzero at the FP and UV-relevant, the latter two being asymptotically free and marginal. Thus, at least in some approximations, asymptotic safety implies that near the FP quantum corrections to the action (8.11) will not generate new terms when one takes the UV limit. This is very similar to the result of Stelle. The main difference lies therein, that the perturbative proof holds at the Gaussian FP while the statement of asymptotic safety holds near the non-Gaussian one. According to the ERGE, the Gaussian FP is unstable, and moving by an infinitesimal amount towards positive $\tilde{G}$ (even with $\tilde{\Lambda} = 0$) would cause the system to be dragged in the direction of the repulsive eigenvector towards the non-Gaussian FP (see fig. 8.1). It is unclear whether in a more accurate description it will still be possible to describe the UV limit of the theory by an action containing finitely many terms.

We now come to other nonperturbative approaches to Quantum Gravity. Monte Carlo simulations of Quantum Gravity have found evidence of a phase transition which can be related to the existence of a Gravitational FP. Hamber & Williams [19] review various results and arguments, mainly from quantum Regge calculus, supporting the claim that the mass critical exponent ν is equal to 1/3. In a theory with a single coupling constant $\tilde{G}$ we have $-1/\nu = \beta'_{\tilde{G}}(\tilde{G}_*)$, so for a rough comparison we can solve (8.24) with $\tilde{\Lambda} = 0$, finding an FP at $\tilde{G}_* = 1.21$ with $\beta'(\tilde{G}_*) \approx -2.37$. The agreement is numerically not very good for a universal quantity, but it might perhaps be improved by taking into account the flow of the cosmological constant.

In the so-called causal dynamical triangulation approach, recent numerical simulations have produced quantum worlds that exhibit several features of macroscopic four-dimensional spacetimes (see Ambjørn, Jurkiewicz and Loll's contribution to this volume). In particular they have also studied diffusion processes in such quantum spacetimes and found that the spectral dimension characterizing them is close to two for short diffusion times and to four for long diffusion times. This agrees with the expectation from asymptotic safety and can be seen as further independent evidence for a gravitational FP, as we shall mention below.

The physical implications of a Gravitational FP and, more generally, of the running of gravitational couplings, are not yet well understood. First and foremost, one would expect asymptotic safety to lead to new insight into the local, short-distance structure of a region of spacetime. The boundedness of the cutoff in Planck units, derived in section 8.3, would be in accord with the widely held expectation of some kind of discrete spacetime structure at a fundamental level. In particular, it may help understand the connection to theories such as loop Quantum Gravity, which predict that areas and volumes have quantized values. However, the discussion in section 8.3 should make it clear that the issue of a minimal length in Quantum Gravity may have limited physical relevance, since the answer depends on the choice of units.

Another point that seems to emerge is that the spacetime geometry cannot be understood in terms of a single metric: rather, there will be a different effective metric at each momentum scale. This had been suggested by Floreanini & Percacci [14; 15], who calculated the scale dependence of the metric using an effective potential for the conformal factor. Such a potential will be present in the effective action Γ_k before the background metric is identified with the classical metric (as mentioned in section 8.2). A scale dependence of the metric has also been postulated by Magueijo & Smolin [27] in view of possible phenomenological effects. Lauscher & Reuter [24] have suggested the following picture of a fractal spacetime.

Dimensional analysis implies that in the FP regime $\langle g_{\mu\nu}\rangle_k = k^{-2}(\tilde{g}_0)_{\mu\nu}$, where $\tilde{g}_0$, defined as in (8.13), is a fiducial dimensionless metric that solves the equations of motion of Γ_{k_0}. For example, in the Einstein–Hilbert truncation, the effective metric $\langle g_{\mu\nu}\rangle_k$ is a solution of the equation $R_{\mu\nu} = \Lambda_k g_{\mu\nu}$, so

$$\langle g_{\mu\nu}\rangle_k = \frac{\Lambda_{k_0}}{\Lambda_k}\langle g_{\mu\nu}\rangle_{k_0} \approx \left(\frac{k_0}{k}\right)^2 \langle g_{\mu\nu}\rangle_{k_0} = k^{-2}(\tilde{g}_0)_{\mu\nu}, \tag{8.26}$$

where $\approx$ means "in the FP regime". The fractal spacetime is described by the collection of all these metrics.

A phenomenon characterized by an energy scale k will "see" the effective metric $\langle g_{\mu\nu}\rangle_k$. For a (generally off-shell) free particle with four-momentum p_μ it is natural to use $k \propto p$, where $p = \sqrt{(\tilde{g}_0)^{\mu\nu} p_\mu p_\nu}$. Its inverse propagator is then $\langle g^{\mu\nu}\rangle_p p_\mu p_\nu$. At low energy $\langle g_{\mu\nu}\rangle_k$ does not depend on k and the propagator has the usual p^{-2} behaviour; in the FP regime, (8.26) implies instead that it is proportional to p^{-4}. Its Fourier transform has a short-distance logarithmic behaviour which is characteristic of two dimensions, and agrees with the aforementioned numerical results on the spectral dimension in causal dynamical triangulations. This agreement is encouraging, because it suggests that the two approaches are really describing the same physics. When applied to gravitons in four dimensions (and only in four spacetime "dimensions") it also agrees with the general prediction, derived at the end of section 8.3, that $\eta_g = 2$ at a nontrivial Gravitational FP.

The presence of higher derivative terms in the FP action raises the old issue of unitarity: as is well-known, the action (8.11) describes, besides a massless graviton, particles with Planck mass and negative residue (ghosts). From a Wilsonian perspective, this is clearly not very significant: to establish the presence of a propagator pole at the mass $m_{\rm P}$ one should consider the effective action Γ_k for $k \approx m_{\rm P}$, which may be quite different from the FP action. Something of this sort is known to happen in the theory of strong interactions: at high energy they are described by a renormalizable and asymptotically free theory (QCD), whose action near the UV (Gaussian) FP describes quarks and gluons. Still, none of these particles appears in the physical spectrum.

As in QCD, matching the UV description to low energy phenomena may turn out to be a highly nontrivial issue. A change of degrees of freedom could be involved. From this point of view one should not assume a priori that the metric appearing in the FP action is "the same" metric that appears in the low energy description of GR. Aside from a field rescaling, as discussed in section 8.2, a more complicated functional field redefinition may be necessary, perhaps involving the matter fields, as exemplified in [44]. Unless at some scale the theory was purely topological, it will always involve a metric and from general covariance arguments it will almost unavoidably contain an Einstein–Hilbert term. This explains why the

Einstein–Hilbert action, which describes GR at macroscopic distances, may play an important role also in the UV limit, as the results of section 8.4 indicate. With this in mind, one can explore the consequences of a RG running of gravitational couplings also in other regimes.

Motivated in part by possible applications to the hierarchy problem, Percacci [35] considered a theory with an action of the form (8.25), in the intermediate regime between the scalar mass and the Planck mass. Working in cutoff units (8.13), it was shown that the warped geometry of the Randall–Sundrum model can be seen as a geometrical manifestation of the quadratic running of the mass.

For applications to black hole physics, Bonanno & Reuter [6] have included Quantum Gravity effects by substituting G with $G(k)$ in the Schwarzschild metric, where $k = 1/r$ and r is the proper distance from the origin. This is a gravitational analogue of the Ühling approximation of QED. There is a softening of the singularity at $r = 0$, and it is predicted that the Hawking temperature goes to zero for Planck mass black holes, so that the evaporation stops at that point.

In a cosmological context, it would be natural to identify the scale k with a function of the cosmic time. Then, in order to take into account the RG evolution of the couplings, Newton's constant and the cosmological constant can be replaced in Friedman's equations by the effective Newton's constant and the effective cosmological constant calculated from the RG flow. With the identification $k = 1/t$, where t is cosmic time, Bonanno & Reuter [7] have applied this idea to the Planck era, finding significant modifications to the cosmological evolution; a more complete picture extending over all of cosmic history has been given in [38]. It has also been suggested that an RG running of gravitational couplings may be responsible for several astrophysical or cosmological effects. There is clearly scope for various interesting speculations, which may even become testable against new cosmological data.

Returning to the UV limit, it can be said that asymptotic safety has so far received relatively little attention, when compared to other approaches to Quantum Gravity. Establishing this property is obviously only the first step: deriving testable consequences is equally important and may prove an even greater challenge. Ultimately, one may hope that asymptotic safety will play a similar role in the development of a QFT of gravity as asymptotic freedom played in the development of QCD.

8.6 Acknowledgements

I wish to thank R. Floreanini, D. Dou, D. Perini, A. Codello and C. Rahmede for past and present collaborations, and M. Reuter for many discussions.

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9

New directions in background independent Quantum Gravity

F. MARKOPOULOU

9.1 Introduction

The different approaches to Quantum Gravity can be classified according to the role that spacetime plays in them. In particular, we can ask two questions of each approach. (1) Is spacetime geometry and General Relativity fundamental or emergent? (2) Is spacetime geometry, if present, dynamical or fixed?

Reviewing the different approaches we find that they split into four categories. First, there are the Quantum Field Theory-like approaches, such as string theory and its relatives. Here General Relativity is to be an emergent description; however, the spacetime that appears in the initial formulation of the theory is fixed and not dynamical. Next are the so-called background independent approaches to Quantum Gravity, such as loop quantum gravity, spin foams, causal sets and causal dynamical triangulations. Geometry and gravity here are fundamental, except quantum instead of classical. These approaches implement background independence by some form of superposition of spacetimes, hence the geometry is not fixed. Third, there are condensed matter approaches (see [37]). While it is clear that relativity is to be emergent, there is confusion on question (2) above. These are condensed matter systems, so it seems clear that there is a fixed spacetime in which the lattice lives; however, it can be argued that it is an auxiliary construction, an issue we shall not resolve here.

Our main focus in this chapter is a new, fourth, category that is currently under development and constitutes a promising and previously unexplored direction in background independent Quantum Gravity. This is *pre-geometric* background independent approaches to Quantum Gravity. These approaches start with an underlying microscopic theory of quantum systems in which no reference to a spatiotemporal geometry is to be found. Both geometry and hence gravity are emergent. The geometry is defined intrinsically using subsystems and their interactions. The geometry is subject to the dynamics and hence itself dynamical. This

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti.
Published by Cambridge University Press.

has been claimed to be the case, in different systems, by Dreyer ([8] and this volume), Lloyd [19], Kribs and Markopoulou [16], and Konopka, Markopoulou and Smolin [14].

As can be seen from the above, this new direction is in fact orthogonal to all previous approaches and so it comes with its own set of promises and challenges. We shall discuss these but we also wish to outline the choices involved in the answers to our two questions above. It is normally difficult to have an overview of the choices involved in picking different directions in Quantum Gravity because the mathematical realizations are intricate and all different. Luckily, for the present purposes, we find that we can base the discussion on the formalism of *Quantum Causal Histories* (QCH), a locally finite directed graph of finite-dimensional quantum systems.[1]

A QCH, depending on the physical interpretation of its constituents, can model a discrete analog of Quantum Field Theory, a traditional, quantum geometry based background independent system, or the new, pre-geometric background independent theories. This will allow us to keep an overview of the forks on the road to Quantum Gravity. It will also be ideal for analyzing the newest kind of background independent systems and obtaining some first results on their effective properties. In particular, we shall see how one can extract conserved quantities in pre-geometric systems using a straightforward map between a QCH and a quantum information processing system.

The outline of this chapter is as follows. In section 9.2 we give the definition of a Quantum Causal History, together with a simple example, locally evolving networks in subsection 9.2.1. At this point we have not restricted ourselves to any particular physical interpretation of the QCH and the options are listed in 9.2.2. In section 9.3 we give the necessary definitions of Background Independence. The following three sections contain three distinct physical interpretations of a QCH: as a discrete analog of Quantum Field Theory (a background dependent theory) in section 9.4, a quantum geometry theory in section 9.5 with a discussion of advantages and challenges (9.5.1) and finally the new type of background independent systems in section 9.6. Their advantages and challenges are discussed in 9.6.2. In section 9.6.3, we map a QCH to a quantum information processing system and use this to derive conserved quantities with no reference to a background spacetime, complete with a simple example of such conserved quantities. We conclude with a brief discussion of these new directions in section 9.7.

[1] The finiteness is a simple implementation of the expectation that there really are only a finite number of degrees of freedom in a finite volume, arguments for which are well-known and we have reviewed them elsewhere [25].

9.2 Quantum Causal Histories

A Quantum Causal History is a locally finite directed graph of finite-dimensional quantum systems. We start by giving the properties of the directed graph and the assignment of quantum systems to its vertices and appropriate operators to its edges. The addition of three axioms ensures that the properties of a given graph are reflected in the flow of physical information in the corresponding quantum operators and completes the definition of a Quantum Causal History.[2]

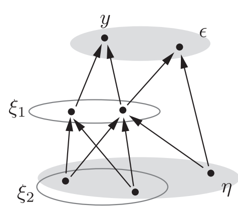
Let Γ be a directed graph with vertices $x \in V(\Gamma)$ and directed edges $e \in E(\Gamma)$. The *source* $s(e)$ and *range* $r(e)$ of an edge e are, respectively, the initial and final vertex of e. A (finite) path $w = e_k \cdots e_1$ in Γ is a sequence of edges of Γ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i < k$. If $s(w) = r(w)$ then we say w is a *cycle*. We require that Γ has no cycles.

If there exists a path w such that $s(w) = x$ and $r(w) = y$ let us write $x \leq y$ for the associated partial ordering. We call such vertices *related*. Otherwise, they are *unrelated*. We use $x \sim z$ to denote that x and z are unrelated. Given any $x \leq y$, we require that there are finitely many $z \in V(\Gamma)$ such that $x \leq z \leq y$. This is the condition of *local finiteness*.

Definition 1 *Parallel set, complete source, complete range, complete pair.*

A parallel set $\xi \subseteq E(\Gamma)$ *is defined by the property that* $x \sim y$ *whenever* $x, y \in \xi$. *A parallel set* ξ *is a* complete source *of* x *if all paths* w *with* $r(w) \equiv x$ *have* $s(w) \in \xi$. *Conversely, a parallel set* ζ *is a* complete range *of* x *if all paths* w *with source* $s(w) \equiv x$ *have range in* ζ, $r(w) \in \zeta$. *Two parallel sets* ξ *and* ζ *are a* complete pair *if all paths* w *that start in* ξ $s(w) \in \xi$ *end up in* ζ, $r(w) \in \zeta$ *and the reverse.*

For example, in the directed graph



ξ_1 is a complete source for y while the parallel sets ξ_2 and η are not. The sets η and ϵ are a complete pair.

We now wish to associate quantum systems to the graph. The construction of a Quantum Causal History starts with a directed graph Γ and assigns to every

[2] The abstract form of a Quantum Causal History based on a directed graph that we follow here was given by Kribs [15], based on the original definition in [22] and [11].

vertex $x \in V(\Gamma)$ a finite-dimensional Hilbert space $\mathcal{H}(x)$ and/or a matrix algebra $\mathcal{A}(\mathcal{H}(x))$ (or $\mathcal{A}(x)$ for short) of operators acting on $\mathcal{H}(x)$. It is best to regard the algebras as the primary objects, but we will not make this distinction here.

If two vertices, x and z, are unrelated, their joint state space is

$$\mathcal{H}\,(x \cup z) = \mathcal{H}(x) \otimes \mathcal{H}(y). \tag{9.1}$$

If vertices x and y are related, let us for simplicity say by a single edge e, we shall think of e as a *change* of the quantum systems of the source of e into a new set of quantum systems (the range of e). It is then natural to assign to each $e \in E(\Gamma)$ a *completely positive map* Φ_e:

$$\Phi_e : \mathcal{A}(s(e)) \longrightarrow \mathcal{A}(r(e)), \tag{9.2}$$

where $\mathcal{A}(x)$ is the full matrix algebra on $\mathcal{H}(x)$. Completely positive maps are commonly used to describe evolution of open quantum systems and generally arise as follows (see, for example, [27]).

Let $\mathcal{H}_{\rm S}$ be the state space of a quantum system in contact with an environment $\mathcal{H}_{\rm E}$ (here $\mathcal{H}_{\rm S}$ is the subgraph space and $\mathcal{H}_{\rm E}$ the space of the rest of the graph). The standard characterization of evolution in open quantum systems starts with an initial state in the system space that, together with the state of the environment, undergoes a unitary evolution determined by a Hamiltonian on the composite Hilbert space $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm E}$, and this is followed by tracing out the environment to obtain the final state of the system.

The associated evolution map $\Phi : \mathcal{A}(\mathcal{H}_{\rm S}) \rightarrow \mathcal{A}(\mathcal{H}_{\rm S})$ between the corresponding matrix algebras of operators on the respective Hilbert spaces is necessarily completely positive (see below) and trace preserving. More generally, the map can have different domain and range Hilbert spaces. Hence the operational definition of quantum evolution Φ from a Hilbert space $\mathcal{H}_1$ to $\mathcal{H}_2$ is as follows.

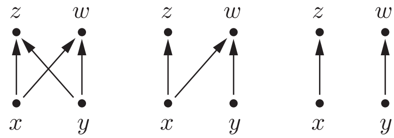
Definition 2 *Completely positive (CP) operators. A* completely positive *operator* Φ *is a linear map* $\Phi : \mathcal{A}(\mathcal{H}_1) \longrightarrow \mathcal{A}(\mathcal{H}_2)$ *such that the maps*

$$id_k \otimes \Phi : M_k \otimes \mathcal{A}(\mathcal{H}_1) \rightarrow M_k \otimes \mathcal{A}(\mathcal{H}_2) \tag{9.3}$$

are positive for all $k \geq 1$.

Here we have written M_k for the algebra $\mathcal{A}(\mathbb{C}^k)$.

Consider vertices x, y, z and w in Γ. There are several possible connecting paths, such as

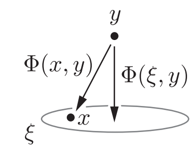


We need the quantum evolution from $\mathcal{A}(x \cup y)$ to $\mathcal{A}(z \cup w)$ to reflect the underlying graph configuration (the quantum operators should distinguish between the above diagrams). The following definition ensures this.

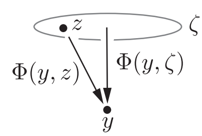
Definition 3 *A* Quantum Causal History *consists of a simple matrix algebra* $\mathcal{A}(x)$ *for every vertex* $x \in V(\Gamma)$ *and a completely positive map* $\Phi(x, y) : \mathcal{A}(y) \to \mathcal{A}(x)$ *for every pair of related vertices* $x \leq y$*, satisfying the following axioms.*

Axiom 1: Extension.

Let ξ be the complete source of y and $x \in \xi$. For any such y, there exists a homomorphism $\Phi(\xi, y) : \mathcal{A}(y) \to \mathcal{A}(\xi)$ such that the reduction of $\Phi(\xi, y)$ to $\mathcal{A}(y) \to \mathcal{A}(x)$ is $\Phi(x, y)$.



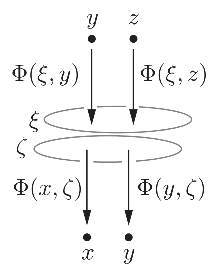
Similarly, for the reflected diagram on the right for ζ a complete range of y. The adjoint of $\Phi(y, \zeta)$ is a homomorphism while its reduction to $y \to z$ is $\Phi(y, z)$.



Axiom 2: Commutativity of unrelated vertices.

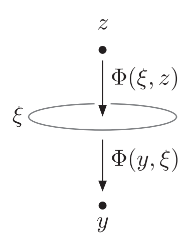
If $x \sim z$ and ξ is a complete source of both y and z, then the images of $\Phi(\xi, z)$ and $\Phi(\xi, y)$ in $\mathcal{A}(\xi)$ commute.

Similarly, on the right, the images of $\Phi^{\dagger}(x, \zeta)$ and $\Phi^{\dagger}(y, \zeta)$ in $\mathcal{A}(\zeta)$ commute.



Axiom 3: Composition.

If ξ is a complete source of z and a complete range of y, then $\Phi(y, z) = \Phi(y, \xi) \circ \Phi(\xi, z)$. Similarly for the reverse direction.



Note that completely positive maps between algebras go in the reverse direction to the edges of the graph. This is as usual for maps between states (forward) and between operators (pullbacks).

The above axioms ensure that the actual relations between the vertices of a given graph are reflected in the operators of the QCH.[3] Furthermore, as shown in [11], if we are given the CP maps on the edges, these axioms mean that *unitary* operations will be found at the right places: interpolating between *complete pairs*. When ξ and ζ are a complete pair, we can regard the subgraph that interpolates between ξ and ζ as the evolution of an isolated quantum system. We would expect that in this case the composite of the individual maps between ξ and ζ is unitary and indeed the above axioms ensure that this is the case.

9.2.1 Example: locally evolving networks of quantum systems

Possibly the most common objects that appear in background independent theories are networks. Network-based, instead of metric-based, theories are attractive implementations of the relational content of diffeomorphism invariance: it is the connectivity of the network (relations between the constituents of the universe) that matter, not their distances or metric attributes. We shall use a very simple network-based system as a concrete example of a QCH.

We start with a network $\mathcal{S}$ of $n = 1, \ldots, N$ nodes, each with three edges attached to it, embedded in a topological three-dimensional space Σ (no metric on Σ). The network $\mathcal{S}$ is not to be confused with the graph Γ, it is changes of $\mathcal{S}$ that will give rise to Γ. A map from $\mathcal{S}$ to a quantum system can be made by associating a finite-dimensional state space $\mathcal{H}_n$ to each minimal piece of $\mathcal{S}$, namely, one node and three open edges:

$\mathcal{H}_n =$  (9.4)

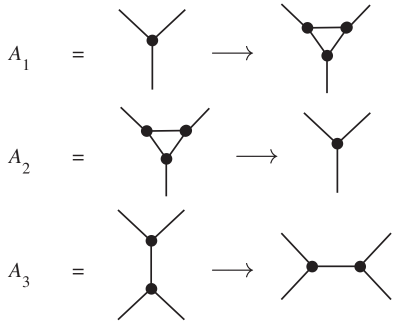
Two such pieces of $\mathcal{S}$ with no overlap are unrelated and thus the state space of the entire network $\mathcal{S}$ is the tensor product over all the constituents,

$$\mathcal{H}_{\mathcal{S}} = \bigotimes_{n \in \mathcal{S}} \mathcal{H}_n, \tag{9.5}$$

and the state space of the theory is

$$\mathcal{H} = \bigoplus_{\mathcal{S}_i} \mathcal{H}_{\mathcal{S}_i}, \tag{9.6}$$

[3] Very interesting recent results of Livine and Terno [18] further analyze and constrain the allowed graph structure to take into account the quantum nature of the physical information flow represented.



(9.7)

Fig. 9.1. The three generators of evolution on the network space $\mathcal{H}$. They are called expansion, contraction and exchange moves.

where the sum is over all topologically distinct embeddings of all such networks in Σ with the natural inner product $\langle \mathcal{S}_i|\mathcal{S}_{i'}\rangle = \delta_{\mathcal{S}_i \mathcal{S}_{i'}}$.

Local dynamics on $\mathcal{H}$ can be defined by excising pieces of $\mathcal{S}$ and replacing them with new ones with the same boundary [21; 26]. The generators of such dynamics are given graphically in Fig. 9.1. Given a network $\mathcal{S}$, application of A_i results in

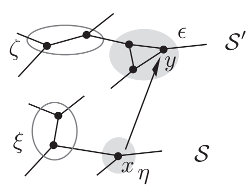
$$\hat{A}_i|\mathcal{S}\rangle = \sum_\alpha |\mathcal{S}'_{\alpha i}\rangle, \tag{9.8}$$

where $\mathcal{S}'_{\alpha i}$ are all the networks obtained from $\mathcal{S}$ by an application of one move of type i ($i = 1, 2, 3$). Together with the identity $\mathbf{1}$, these moves generate the *evolution algebra*

$$\mathcal{A}_{\text{evol}} = \{\mathbf{1}, A_i\}, \quad i = 1, 2, 3 \tag{9.9}$$

on $\mathcal{H}$.

Finally, changing the network $\mathcal{S}$ by the above local moves produces a directed graph Γ. The vertices of $\mathcal{S}$ are also the vertices of Γ. The generator moves correspond to complete pairs and hence unitary operators, however, the operators between individual vertices are CP and the resulting system of locally evolving networks is a Quantum Causal History. For example, in this change of $\mathcal{S}$ to $\mathcal{S}'$



we have operated with A_3 between complete pair sets ξ and ζ and with A_1 between complete pairs η and ϵ. The map from x to y is a CP map.

9.2.2 The meaning of Γ

At this stage we have said nothing about the physical interpretation of Γ or the individual quantum systems $\mathcal{A}(x)$ on its vertices. While Γ has the same properties as a causal set,[4] i.e. the discrete analog of a Lorentzian spacetime, it does not have to be one. For example, in the circuit model of quantum computation, a *circuit*, that is, a collection of gates and wires also has the same properties as Γ and simply represents a sequence of information transfer which may or may not be connected to spatiotemporal motions (see [27], p. 129).

We shall use this flexibility of the QCH to illustrate both the difference between a background dependent and a background independent system as well as the distinction between background independent theories of quantum geometry and a new set of pre-geometric theories that have been recently proposed. In what follows, we shall see that three different interpretations of Γ and the $\mathcal{A}(x)$s give three different systems. (1) A discrete version of algebraic Quantum Field Theory, when Γ is a discretization of a Lorentzian spacetime and $\mathcal{A}(x)$ is matter on it. (2) A causal spin foam, i.e. a background independent theory of quantum geometry. Here Γ is a locally finite analog of a Lorentzian spacetime and the $\mathcal{A}(x)$ contain further quantum geometric degrees of freedom. Such a theory is background independent when we consider a quantum superposition of all Γs. (3) A pre-geometric background independent theory, when neither Γ nor the $\mathcal{A}(x)$s have geometric information. The possibility that such a system, with a single underlying graph Γ may be background independent has only recently been raised and explored.

We shall discuss each of these three possibilities in detail in the rest of this chapter, starting with the necessary definitions of background independence, next.

9.3 Background independence

Background independence (BI) is thought to be an important part of a quantum theory of gravity since it is an important part of the classical theory.[5] Background independence in General Relativity is the fact that physical quantities are invariant under spacetime diffeomorphisms. There is no definite agreement on the form that BI takes in Quantum Gravity. Stachel gives the most concise statement of background independence: "In a background independent theory there is no kinematics independent of dynamics."

In this chapter, we shall need to discuss specific aspects of background independence and to aid clarity we give the following definitions that we shall use.

[4] See [6].
[5] See [7; 36; 35].

Definition 4 *Background independence I (BI-I): a theory is background independent if its basic quantities and concepts do not presuppose the existence of a given background spacetime metric.*

All well-developed background independent approaches to Quantum Gravity such as loop quantum gravity [33][34], causal sets [6], spin foams [31][32] [26][3][29] and Oriti, this volume, causal dynamical triangulations [1] [2], or quantum Regge calculus [30], implement background independence as a special case of the above by quantum analogy to the classical theory:

Definition 5 *Background independence II (BI-II): a background independent theory of quantum geometry is characterized by (a) quantum geometric microscopic degrees of freedom or a regularization of the microscopic geometry and (b) a quantum sum-over-histories of the allowed microscopic causal histories (or equivalent histories in the Riemannian approaches).*

Recently, new approaches to Quantum Gravity have been proposed that satisfy BI-I but not BI-II: the computational universe [19], internal relativity ([8] and Dreyer, this volume) and quantum graphity [14]. More specifically, Dreyer advocates the following.

Definition 6 *Background independence (Dreyer): a theory is background independent if all observations are* internal*, i.e. made by observers inside the system.*

Note that this is a natural condition for a cosmological theory as has also been pointed out in [23].

In summary, what constitutes a background independent theory is a question that is currently being revisited and new, on occasion radical, suggestions have been offered. These are opening up new exciting avenues in Quantum Gravity research and will be our focus in this chapter. In order to discuss them in some detail, however, we shall give examples of each in the unifying context of QCH.

9.4 QCH as a discrete Quantum Field Theory

There is substantial literature in Quantum Gravity and high energy physics that postulates that in a finite region of the universe there should be only a finite number of degrees of freedom, unlike standard Quantum Field Theory where we have an infinite number of degrees of freedom at each spacetime point. This is supported by Bekenstein's argument, the black hole calculations in both string theory and loop quantum gravity and is related to holographic ideas.

It has been suggested that such a locally finite version of Quantum Field Theory should be implemented by a many-Hilbert space theory (as opposed to the single

Fock space for the entire universe in Quantum Field Theory). A QCH on a causal set is exactly such a locally finite Quantum Field Theory. This can be seen most clearly by formulating QCH as a locally finite analog of algebraic Quantum Field Theory. Algebraic Quantum Field Theory is a general approach to Quantum Field Theory based on algebras of local observables, the relations among them, and their representations [10]. A QCH provides a similar discrete version as follows.

Let Γ be a causal set. This is a partial order of events, the locally finite analog of a Lorentzian spacetime. Two events are causally related when $x \leq y$ and spacelike otherwise. A parallel set ξ is the discrete analog of a spacelike slice or part of a spacelike slice. The causal relation $\leq$ is transitive.

An algebraic Quantum Field Theory associates a von Neumann algebra to each causally complete region of spacetime. This generalizes easily to a directed graph. The following definitions are exactly the same as for continuous spacetime. For any subset $X \subset \Gamma$, define the *causal complement* as

$$X' := \{y \in \Gamma \mid \forall x \in X : x \sim y\}$$

the set of events which are spacelike to all of X. The *causal completion* of X is X'', and X is *causally complete* if $X = X''$. A causal complement is always causally complete (i.e. $X''' = X'$).

In the most restrictive axiomatic formulation of algebraic Quantum Field Theory there is a von Neumann algebra $\mathcal{A}(X)$ for every causally complete region. These all share a common Hilbert space. Whenever $X \subseteq Y$, $\mathcal{A}(X) \subseteq \mathcal{A}(Y)$. For any causally complete region X, $\mathcal{A}(X')$ is $\mathcal{A}(X)'$, the commutant of $\mathcal{A}(X)$. The algebra associated to the causal completion of $X \cup Y$ is generated by $\mathcal{A}(X)$ and $\mathcal{A}(Y)$.[6]

In our discrete version, only a finite amount of structure should be entrusted to each event. In other words, each von Neumann algebra should be a finite-dimensional matrix algebra. In von Neumann algebra terms, these are finite type I factors. Not surprisingly, simple matrix algebras are much easier to work with than type III von Neumann factors. Using the (unique) normalized trace, any state is given by a density matrix. Recall that the adjoint maps $\Phi^{\dagger}(x, y)$ in a Quantum Causal History are the induced maps on density matrices.

So, we see that the obvious notion of an algebraic Quantum Field Theory on a causal set, with the physically reasonable assumption of finite algebras on events, gives the structure of a QCH. This means that the structure of a QCH encompasses

[6] Some of the standard arguments about the properties of the local von Neumann algebras are valid for causal sets; some are not. The algebras should all be simple (i.e. von Neumann factors) because the theory would otherwise have local superselection sectors. For continuous spacetime it is believed that the local algebras should be type III_1 hyperfinite factors; however, the reasoning involves the assumption that there exists a good ultraviolet scaling limit. This does not apply here; the small-scale structure of a causal set is discrete and not self similar at all.

a reasonable notion of a Quantum Field Theory, and hence is capable of describing matter degrees of freedom.

This framework may be a good one to investigate questions such as the trans-planckian mode problem that arises in attempts at a locally finite Quantum Field Theory in an expanding universe (for example in [9]). For the purposes of Quantum Gravity, this is a background *dependent* theory: Γ is fixed, we only follow the dynamics of the $\mathcal{A}(x)$s on the Γ which does not affect Γ itself.

9.5 Background independent theories of quantum geometry

The traditional path to a background independent candidate quantum theory of gravity is to consider a quantum superposition of geometries. This is the case in loop quantum gravity, quantum Regge calculus and causal sets and more recently spin foams and causal dynamical triangulations.

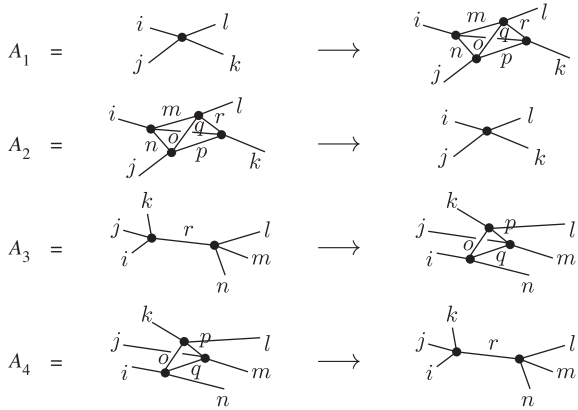
These are realizations of BI-II theories and can be illustrated by a QCH in a straightforward way: Γ will be a causal set, namely, a partial order of events that are causally related when $x \leq y$ and spacelike when $x \sim y$. To each event x, we shall associate an elementary space of quantum geometrical degrees of freedom that are postulated to exist at Planck scale. The theory provides a *sum*-over-all Γ amplitude to go from an initial to a final quantum geometry state. For example, this can be done as in *causal spin networks* [26; 21].

Spin networks are graphs with directed edges labeled by representations of $SU(2)$. Reversing the direction of an edge means taking the conjugate representation. A node in the graph represents the possible channels from the tensor product of the representations $\rho_{e_{\text{in}}}$ on the incoming edges e_{in} to the tensor product of the representations on the outgoing ones, $\rho_{e_{\text{out}}}$, i.e. it is the linear map

$$\iota : \bigotimes_{e_{\text{in}}} \rho_{e_{\text{in}}} \to \bigotimes_{e_{\text{out}}} \rho_{e_{\text{out}}}. \tag{9.10}$$

Such a map ι is called an *intertwiner*. The intertwiners on a node form a finite-dimensional vector space. Hence, a subgraph in the spin network containing one node x corresponds to a Hilbert space $\mathcal{H}(x)$ of intertwiners. Two spacelike events are two independent subgraphs, and the joint Hilbert space is $\mathcal{H}(x \cup y) = \mathcal{H}(x) \otimes \mathcal{H}(y)$ if they have no common edges, or $\mathcal{H}(x \cup y) = \sum_{\rho_1,\ldots\rho_n} \mathcal{H}(x) \otimes \mathcal{H}(y)$, if x and y are joined in the spin network graph by n edges carrying representations $\rho_1, \ldots \rho_n$.

Given an initial spin network, to be thought of as modelling a quantum “spatial slice”, Γ is built by repeated application of local moves, local changes of the spin network graph. Each move is a causal relation in the causal set. The standard set of local generating moves for 4-valent spin networks is given by the following four operators.



Note that the new subgraph has the same boundary as the original one and therefore corresponds to the same vector space of intertwiners.[7,8] A move A_i is a unitary operator from a state $|S\rangle$ to a new one $|S'\rangle$ in $\mathcal{H}$.

A path integral quantum theory of gravity is then obtained from the superposition of all possible $\mathcal{G}$s, leading to an amplitude of the form

$$A_{S_{\text{in}} \to S_{\text{out}}} = \sum_{\partial\Gamma = S_{\text{in}} \to S_{\text{out}}} \prod_{\text{moves} \in \Gamma} A_i(\text{move}) \tag{9.11}$$

to go from initial spin network S_{in} to final spin network S_{out}.

9.5.1 Advantages and challenges of quantum geometry theories

Particular realizations of quantum geometry theories, such as loop quantum gravity, spin foams or CDT, amount to quantizations, canonical or path integral, of General Relativity. A specific quantization procedure will result in specific elementary state spaces and evolution operators. The advantage of this is clear: one follows the well-tested path to a new theory via the quantization of the classical one, a method that has been successful with all other theories that we have tried.

Nonetheless, progress has been difficult, precisely because of the Background Independence of the classical theory, a feature that distinguishes it from all other theories that we have successfully quantized. The equations of General Relativity

7 Spin networks were originally defined by Penrose as trivalent graphs with edges labeled by representations of $SU(2)$. Later, in Loop Quantum Gravity, spin networks were shown to be the basis states for the spatial geometry states. The kinematical quantum area and volume operators, in the spin network basis, have discrete spectra, and their eigenvalues are functions of the labels on the spin network.

8 One uses 4-valent networks and moves for $SU(2)$ spin networks, instead of the simpler 3-valent ones we used in Fig.9.1 because 3-valent $SU(2)$ intertwiner spaces are one-dimensional and thus trivial.

Also note that there is no preferred foliation in this model. The allowed moves change the network locally and any foliation consistent with the causal set (i.e. that respects the order the moves occurred) is possible. This is a discrete analog of multifingered time evolution. For more details, see [21].

are invariant under the diffeomorphism group of the manifold under investigation. A canonical analysis reveals that this means that the system is completely constrained: instead of generating time evolution, the Hamiltonian vanishes on solutions. That means that in the description above, any intuition we may have of the $\mathcal{G}$ as describing changes of the network in time is incorrect, instead it represents a *projector* from the kinematical spin network states to the physical solutions. This fact makes it especially hard to tackle questions of physical importance such as the emergence of the classical low energy limit, i.e. the recovery of the classical theory from the Quantum Gravity candidate.

Without going in detail into specific issues that arise in each of the BI approaches to Quantum Gravity, one can get an idea of the problems that one encounters in the quest for the low energy limit of background independent theories, especially issues specific to BI systems by comparing our example to a condensed matter system. The graph $\mathcal{G}$ plays the role of the lattice, while the $\mathcal{H}_n$s are the microscopic quantum degrees of freedom. The low energy problem is analogous to describing the macroscopic behavior emergent from a many-body system in condensed matter physics. Building on that analogy, there has been work, for example, on the application of renormalization group methods to such BI systems [24; 28; 17; 20; 5].

There are, of course, technical obstacles such as the irregular nature of the lattices, the often complicated calculations involving the microscopic variables (usually group representations) and the lack of experimental controls, readily available in standard condensed matter systems. But there are also problems specific to BI systems.

- *Dynamics*. The low energy behavior of a physical system depends on its dynamics.

  Causal dynamical triangulations (CDT) is a clear demonstration of this basic fact of physical systems in Quantum Gravity. Both CDT and euclidean dynamical triangulations (DT) start with building blocks of the same dimensionality, four-simplices. They differ in the dynamics. In the continuum limit, CDT finds Hausdorff and heat dimensions near $3+1$, while the euclidean theory ends up either with effective dimension of two or infinite. Dynamics is notoriously difficult to implement in most background independent approaches, which makes it tempting to draw conclusions about the physical content of a theory before we have taken dynamics into account. For example, spin foam models often relate the valence of the nodes in the spin foam 2-complex to the dimensionality of the system and much of the analysis of specific models involves analyzing the properties of a single building block without considering the entire path-integral. This is analogous to considering a spin system in condensed matter physics and inferring properties of its continuum limit by looking at the spins, independently of the hamiltonian. The Ising model in two dimensions and string networks [38] have precisely the same building blocks and kinematics, square lattices of spins, but different dynamics. The resulting effective theories could not be more different. In the field of Quantum Gravity itself, the example of CDT vs DT shows us how little trust we should put in properties of the microscopic constituents surviving to the low-energy theory.

We must conclude that any method we may use to analyze the low-energy properties of a theory needs to take the dynamics into account.

- *Observables*. Using the analogy between the graphs $\mathcal{G}$ of our theory and a condensed matter system, we may consider applying condensed matter methods to the graphs, such as a real space renormalization (coarse-graining the graph). However, careful inspection of the real space renormalization method in ordinary systems shows that implicit in the method is the fact that, coarse-graining the lattice spacing coarse-grains the observables. In BI systems, the best we can do is relational observables and there is no direct relationship between BI observables and the lattice or the history. Hence, the physical meaning of coarse-graining a graph is unclear.

  In theories of regularized geometries, such as CDT, there is a somewhat different issue. The continuum limit observables that have been calculated so far are averaged ones, such as the Hausdorff or heat dimensions. One still needs to find localized observables in order to compare the predictions of the theory to our world.
- *(Lack of) symmetries*. We should clarify that when we use the term *low-energy* it is only by analogy to ordinary physics and both *energy* and *low* are ill-defined. The definition of energy needs a timelike Killing vector field, clearly not a feature of a BI theory. A notion of scale is necessary to compare *low* to *high*. Outside CDT, it is not clear how scale enters BI systems.

Note that all of the above issues are really different aspects of the question of dynamics in background independent theories.

9.6 Background independent pre-geometric systems

Is it possible to have a system that satisfies the definition of BI-I in section 9.3 but does not take the form of quantum geometry as in BI-II? Even if this is possible, would such an object be of relevance in Quantum Gravity research? The answer to both of these questions is not only yes, but it constitutes an entire new direction in Quantum Gravity with a new set of exciting ideas.

First, let us note that the example system of section 9.2.1 viewed as a quantum information processing system is BI-I in the obvious sense: it describes a network of quantum systems and makes no reference to any spatiotemporal geometry. More precisely, one can ask what a quantum information processing system (a quantum computer) and our locally evolving networks have in common? The answer is that they are the same mathematical structure, tensor categories of finite-dimensional vector spaces with arrows that are unitary or CP operators. This is simply the mathematics of finite-dimensional quantum systems. What is interesting for us is that this mathematics contains no reference to any background spacetime that the quantum systems may live in and hence it is an example of BI-I.

In the past two years, a number of BI-I systems have been put forward: Dreyer's internal relativity ([8] and Dreyer, this volume), Lloyd's computational universe [19], emergent particles from a QCH [16] and Quantum Graphity [14]. All of these can be easily written as a QCH (with a single Γ and no geometric information on the state spaces, hence BI-I), so we shall continue the discussion in the more general terms of a pre-geometric QCH, just as it was defined in section 9.2.

9.6.1 The geometrogenesis picture

Let us consider a simple scenario of what we may expect to happen in a BI theory with a good low energy limit. It is a factor of about 20 orders of magnitude from the physics of the Planck scale described by the microscopic theory to the standard subatomic physics. By analogy with all other physical systems we know, it is reasonable to expect that physics at the two scales decouples to a good approximation. We can expect at least one phase transition interpolating between the microscopic BI phase and the familiar one in which we see dynamical geometry. We shall use the word *geometrogenesis* for this phase transition.

This picture implements the idea that spacetime geometry is a derivative concept and only applies in an approximate emergent level. More specifically, this is consistent with the relational principle that spatial and temporal distances are to be defined internally, by observers inside the system. This is the physical principle that led Einstein to special and General Relativity. The geometrogenesis picture implies that the observers (subsystems), as well as any excitations that they may use to define such spatiotemporal measures, are only applicable at the emergent geometric phase.

The breakthrough realization ([8] and Dreyer, this volume, [19]) is that the inferred geometry will necessarily be dynamical, since the dynamics of the underlying system will be reflected in the geometric description. This is most clearly stated by Dreyer who observes that since the same excitations of the underlying system (characterizing the geometrogenesis phase transition) and their interactions will be used to define *both* the geometry and the energy-momentum tensor $T_{\mu\nu}$. This leads to the following Conjecture on the role of General Relativity.

If the assignment of geometry and $T_{\mu\nu}$ from the same excitations and interactions is done consistently, the geometry and $T_{\mu\nu}$ will not be independent but will satisfy Einstein's equations as identities.

What is being questioned here is the separation of physical degrees of freedom into matter and gravitational ones. In theories with a fixed background, such as

Quantum Field Theory, the separation is unproblematic, since the gravitational degrees of freedom are not really free and do not interact with the matter. In the classical background independent theory, General Relativity, we are left with an intricate non-linear relation between the two sets: the Einstein equations. As the practitioners of canonical Quantum Gravity know well, cleanly extracting dynamical gravitational degrees of freedom from the matter is fraught with difficulties. If such a clean separation could be achieved, canonical Quantum Gravity would have succeeded at least two decades ago.

The new direction unifies matter and gravity in the pre-geometric phase and provides a path towards *explaining* gravity rather than just quantizing it.

9.6.2 Advantages and challenges of pre-geometric theories

Such a radical move raises, of course, numerous new questions. Because of the short time that this direction has been pursued, the advantages and the challenges here are not as well-studied as in the case of quantum geometry, we shall, however, list some here.

The main advantage in practical terms is that this approach allows for ordinary quantum dynamics in the pre-spacetime theory, instead of a quantum constraint, potentially providing a way out of the issues listed in section 9.5.1. If successful, it promises a deeper understanding of the origin of gravity, usually beyond the scope of quantum geometry theories.

The obvious challenges are as follows.

- *Time.* Does the ordinary dynamics of the pre-geometric phase amount to a background time? Keep in mind that there are strict observational limits on certain kinds of background time [12]. Recent work indicates that the answer is not clear. There are several possible mechanisms that may wipe out any signature of the pre-geometric time when we go through the phase transition ([19; 14], Dreyer, this volume).
- *Geometry.* How can we get geometry out if we do not put it in? Presumably, most pre-spacetime systems that satisfy the QCH definition will not have a meaningful geometric phase. Will we need a delicate fine-tuning mechanism to have a geometric phase or is there a generic reason for its existence?

  A variety of ways that geometry can arise have been proposed: dispersion relations at the Fermi point [37] (see also Dreyer, this volume), symmetries of the emergent excitations [16], free excitations [8], restrictions on the properties of the graph Γ [19] or emergent symmetries of the ground state [14]. It is promising that most of these point towards generic mechanisms for the presence of a regular geometry.

9.6.3 Conserved quantities in a BI system

Admittedly, we only have guesses as to the microscopic theory and very limited access to experiment. Additionally, phase transitions are not very well understood even in ordinary lab systems, let alone phase transitions of background independent systems. In spite of these issues, we find that the geometrogenesis picture suggests a first step towards the low energy physics that we can take.

A typical feature of a phase transition is that the degrees of freedom that characterize each of the two phases are distinct (e.g. spins vs spin waves in a spin chain or atoms vs phonons in solid state systems), with the emergent degrees of freedom being collective excitations of the microscopic ones. In our example, the vector spaces on graphs contain the microscopic degrees of freedom and operators in $\mathcal{A}_{\text{evol}}$ are the microscopic dynamics. Is there a way to look for collective excitations of these that are long-range and coherent so that they play a role in the low-energy phase?

We find that this is possible, at least in the idealized case of conserved (rather than long-range) quantities in a background independent system such as our example. The method we shall use, *noiseless subsystems*, is borrowed from quantum information theory, thanks to the straightforward mapping between locally finite BI theories and quantum information processing systems which we described above. We are then suggesting a new path to the effective theory of a background independent system. The basic strategy is to begin by identifying effective coherent degrees of freedom and use these and their interactions to characterize the effective theory. If they behave as if they are in a spacetime, we have a spacetime.

In [16], we found that the field of quantum information theory has a notion of coherent excitation which, unlike the more common ones in Quantum Field Theory and condensed matter physics, makes no reference to a background geometry and can be used on a BI system. This is the notion of a *noiseless subsystem* (NS) in quantum error correction, a subsystem protected from the noise, usually thanks to symmetries of the noise [39; 13]. Our observation is that passive error correction is analogous to problems concerned with the emergence and stability of persistent quantum states in condensed matter physics. In a Quantum Gravity context, the role of noise is simply the fundamental evolution and the existence of a noiseless subsystem means a coherent excitation protected from the microscopic Planckian evolution, and thus relevant for the effective theory.

Definition 7 Noiseless subsystems. *Let* Φ *be a quantum channel on* $\mathcal{H}$ *and suppose that* $\mathcal{H}$ *decomposes as* $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$*, where A and B are subsystems and* $\mathcal{K} = (\mathcal{H}^A \otimes \mathcal{H}^B)^\perp$*. We say that B is* noiseless *for* Φ *if*

$$\forall\sigma^A\,\forall\sigma^B,\ \exists\tau^A\ :\ \Phi(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B. \tag{9.12}$$

Here we have written σ^A (resp. σ^B) for operators on $\mathcal{H}^A$ (resp. $\mathcal{H}^B$), and we regard $\sigma = \sigma^A \otimes \sigma^B$ as an operator that acts on $\mathcal{H}$ by defining it to be zero on $\mathcal{K}$.

In general, given $\mathcal{H}$ and Φ, it is a non-trivial problem to find a decomposition that exhibits a NS. Much of the relevant literature in quantum information theory is concerned with algorithmic searches for a NS given $\mathcal{H}$ and Φ. However, if we apply this method to the example theory of 9.2.1, it is straightforward to see that it has a large conserved sector.[9]

Noiseless subsystems in our example theory.

Are there any non-trivial noiseless subsystems in $\mathcal{H}$? There are, and they are revealed when we rewrite $\mathcal{H}_S$ in eq. (9.6) as

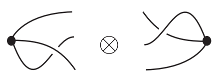
$$\mathcal{H}_S = \mathcal{H}_S^{n'} \otimes \mathcal{H}_S^{B}, \tag{9.13}$$

where $\mathcal{H}_S^{n'} := \bigotimes_{n' \in S} \mathcal{H}^{n'}$ contains all *unbraided* single node subgraphs in S (the prime on n serves to denote unbraided) and $\mathcal{H}_S^b := \bigotimes_{b \in S} \mathcal{H}_b$ are state spaces associated to braidings of the edges connecting the nodes. For the present purposes, we do not need to be explicit about the different kinds of braids that appear in $\mathcal{H}_S^b$.

The difference between the decomposition (9.6) and the new one (9.13) is best illustrated with an example (details can be found in [4]). Given the state

(9.14)

eq. (9.6) decomposes it as



(9.15)

while (9.13) decomposes it to

(9.16)

With the new decomposition, one can check that operators in $\mathcal{A}_{\text{evol}}$ can only affect the $\mathcal{H}_S^{n'}$ and that $\mathcal{H}_S^b$ is *noiseless under* $\mathcal{A}_{\text{evol}}$. This can be checked explicitly by showing that the actions of braiding of the edges of the graph and the evolution moves commute.

We have shown that braiding of graph edges are unaffected by the usual evolution moves. Any physical information contained in the braids will propagate coherently under $\mathcal{A}_{\text{evol}}$. These are effective coherent degrees of freedom.[10]

Note that this example may appear simple but the fact that the widely used system of locally evolving graphs exhibits broken ergodicity ($\mathcal{H}$ splits into sectors, characterized by

9 The noiseless subsystem method (also called decoherence-free subspaces and subsystems) is the fundamental passive technique for error correction in quantum computing. In this setting, the operators Φ are called the *error* or *noise* operators associated with Φ. It is precisely the effects of such operators that must be mitigated for in the context of quantum error correction. The basic idea in this setting is to (when possible) encode initial states in sectors that will remain immune to the deleterious effects of the errors Φ associated with a given channel.

The term "noiseless" may be confusing in the present context: it is not necessary that there is a noise in the usual sense of a given split into system and environment. As is clear from the definition above, simple evolution of a dynamical system is all that is needed, the noiseless subsystem is what evolves coherently under that evolution.

10 The physical interpretation of the braids is beyond the scope of this paper. See [4], for an interpretation of the braids as quantum numbers of the standard model.

their braiding content, and $\mathcal{A}_{\text{evol}}$ cannot take us between sectors) went unnoticed prior to the introduction of the NS method.

Before closing, we would like to point out some of the subtleties of Background Independence that, not surprisingly, arise here. Our original motivation to search for conserved quantities was that they can be thought of as a special case of emergent long-range propagating degrees of freedom, where the lifetime of the propagating ones is infinite (and so tell us something about the geometric phase of the theory). Noiseless subsystems can only deal with this case because it only looks at the symmetries of the microscopic dynamics. Presumably, what we need is to weaken the notion of a noiseless subsystem to "approximately conserved" so that it becomes long-range rather than infinite. Long-range, however, is a comparative property and to express it we need a way to introduce scale into our system. It is unclear at this point whether it is possible to introduce a scale in a pre-geometric theory without encountering the problems listed in section 9.5.1.

9.7 Summary and conclusions

In this article we started with the traditional background independent approaches to Quantum Gravity which are based on quantum geometric/gravitational degrees of freedom. We saw that, except for the case of causal dynamical triangulations, these encounter significant difficulties in their main aim, i.e. deriving General Relativity as their low energy limit. We then suggested that General Relativity should be viewed as a strictly effective theory coming from a fundamental theory with no geometric degrees of freedom (and hence background independent in the most direct sense).

The basic idea is that an effective theory is characterized by effective coherent degrees of freedom and their interactions. Having formulated the pre-geometric BI theory as a quantum information theoretic processor, we were able to use the method of noiseless subsystems to extract such coherent (protected) excitations.

The geometrogenesis picture leads one to reconsider the role of microscopic quantum geometric degrees of freedom traditionally present in background independent theories. It appears unnatural to encounter copies of the geometry characteristic of the macroscopic phase already present in the microscopic phase, as is the case, for example, when using quantum tetrahedra in a spin foam. Instead, one can start with a pre-geometric theory and look for the effective coherent degrees of freedom along the lines described. Spacetime is to be inferred by them internally, namely, using only operations that are accessible to parts of the system.

This is very promising for three reasons. (1) The emphasis on the effective coherent degrees of freedom addresses directly and in fact uses the dynamics.

The dynamics is physically essential but almost impossible to deal with in other approaches. (2) A truly effective spacetime has novel phenomenological implications not tied to the Planck scale which can be tested and rejected if wrong. (3) A pre-spacetime background independent quantum theory of gravity takes us away from the concept of a quantum superposition of spacetimes which can be easily written down formally but has been impossible to make sense of physically in any approach other than causal dynamical triangulations.

Some of the more exciting possibilities we speculated on included solving the problem of time and *deriving* the Einstein equations. Clearly this direction is in its beginning, but the basic message is that taking the idea that General Relativity is an effective theory seriously involves rethinking physics without spacetime. This opens up a whole new set of possibilities and opportunities.

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Questions and answers

- **Q - L. Crane - to C. Rovelli**:

  You say in your paper that we need to think about replacing the classical spacetime continuum. The GFT picture seems to suggest that many parallel discrete "leaves" of spacetime exist, and that their effects superimpose in an observer dependent way. Have you thought along these lines? (Mathematicians call such a structure a "site", and constructing objects over a site is called topos theory.)

  – **A - C. Rovelli**:

    No, I haven't. But I take this as a very interesting suggestion. I fully agree that the GFT picture is strongly suggestive, and points to "something". I also understand that categories and topoi might be a valuable language here, but I do not have the expertise needed to take full advantage of these, I think.

- **Q - D. Oriti - to C. Rovelli**:

  You mention the possibility of a quantum granularity of space as a consequence of a proper quantum mechanical treatment of the gravitational field, and the fact that this sort of granularity is indeed realised in the spectrum of some geometric observables in loop quantum gravity, and also hinted at in some string theory models. However, it is not obvious to me what sort of discreteness we should really expect from a quantum theory of spacetime just by looking at the quantum mechanical systems we know of. On the one hand, in fact, we have systems like the hydrogen atom with its discrete energy spectrum, while on the other hand we have quantum field theories, where spectra of observables are continuous but the quantum discreteness is present in the availability of a Fock space description of their state space, i.e. in the possibility of describing them as made out of discrete fundamental constituents. In the case of Quantum Gravity therefore one can equally well expect to obtain granularity in the form of either discrete spectra for geometric observables or in the form of some sort of fundamental "quanta or atoms" of space, whose related observables are however continuous. The first scenario seems to be realised in $SU(2)$-based Loop

Quantum Gravity, while the second seems to me to correspond to the picture we get from Lorentzian group field theories. What is you point of view on this?

– **A - C. Rovelli**:

Discreteness of observable spectra is not a property of quantum theories with a finite number of degrees of freedom in contrast to quantum field theory. There are observables with discrete spectra in quantum field theory. The particle number (energy) at a given wavelength, for instance, which is the observable most routinely measured in particle physics, has a discrete spectrum, and is responsible for the particle-like discreteness of the quantum fields. To know if an observable is discrete or not, we must solve the spectral problem of the corresponding operator. Thus, I do not understand the idea of "quanta" with continuous observables. In loop quantum gravity, the granularity of spacetime is not hypothetically assumed in analogy with non-relativistic quantum theory. It is the result of a calculation: the computation of the spectra of a class of operators describing the geometry of spacetime. I find group field theory an extremely fascinating and promising direction of investigation. But, as far as I understand, it is still far too poorly understood, especially in the Lorentzian, where the match with the canonical theory is particularly unclear, to be taken as a credible new independent paradigm.

- **Q - L. Crane - to G. 't Hooft**:

1. I hope you don't mind my asking you something pertaining to your earlier work, but relevant to the present discussion. Do you think of dimensional regularization as a particularly effective trick, or do you believe that it is a hint as to the fine structure of spacetime; in particular have you thought about the possibility of quantum spacetime having a non-integral Hausdorff dimension, distinct from its topological dimension?

2. If we are to think of information loss as fundamental, shouldn't that be observer dependent, leading to a relational spacetime structure?

3. What do you think of the proposal of Ng, relating information loss to the limitations of quantum measurement at a distance? He seems to reproduce the Bekenstein bound.

– **A - G. 't Hooft**:

1. We thought of such a possibility. As far as the real world is concerned, dimensional regularization is nothing but a trick. What's more, the trick can only be rigorously applied in the context of the usual perturbation expansion with respect to the coupling constant(s). Veltman once thought there might be real physics in non-integer dimensions, but he never got anywhere with that. However, there is also the world of mathematical physics. As far as I know, it seems to be not quite well-defined what fractional dimensions mean, when going beyond perturbation expansion. Worlds with fractional

Hausdorff dimensions would break Lorentz invariance and translation invariance, so that much of the beauty and simplicity of dimensional regularization would get lost. I sometimes tried to speculate that rigorous, finite definitions of functional integrals can be given in complex dimensions, since all terms in the perturbation expansion are finite, but did not succeed. I do know what *negative* dimensions mean: a negative-dimension coordinate is an anticommuting coordinate, or equivalently, a negative dimension coordinate replaces an integration by a differentiation; differentiation is the inverse of integration.

2. Absolutely. In particular this is an important point for black holes. An ingoing observer has different pieces of information at his/her disposal than an observer who stays outside. In view of my conjecture that the quantum states associated to the primordial basis are informational equivalence classes, this means that the transformation from a black hole horizon to flat space is not a transformation in Hilbert space.

3. That is an interesting view, but "measurement" does not play a prominent role in my proposals. "Measurement" requires a measuring device, which to my mind is an unnecessary complication when it comes to model building.

- **Q - L. Crane - to R. Sorkin**:

  1. How do you think of the points of a causet? Are they just part of an approximation to some more subtle spactime structure which has a discrete aspect, small regions treated as pointlike, or an actual hypothesis about physical spacetime?

  2. If the points in causets are quantum events, isn't there a superposition principle? In other words, shouldn't we be modelling spacetime as a quantum superposition of an ensemble of causets, rather than just one? Could this allow symmetry to be restored in the average, and wouldn't this be an attractive alternative to the loss of locality?

  – **A - R. Sorkin**:

  1. The answer is your third alternative: "an actual hypothesis about physical spacetime". The elements of the causet are meant to be constituents of spacetime that really exist (or better "happen"). Of course, the causet language would be more fundamental than the geometric, spacetime language, and as such, it would retain its validity in extreme conditions where a spacetime description would no longer make sense, inside a black hole for example.

  2. Quantum mechanically, it should indeed be true that spacetime is something like an ensemble of causets. (Or, as one might express it, reality must be a quantal causet, not a classical one.) I don't know whether this could produce symmetry in the average, but one of the central messages of my article

was that the question is spurious because an average Lorentz symmetry is already manifested by the *individual* causet.

Perhaps, however, you are implicitly asking whether quantum effects could produce a type of "averaging" that would remove the need for an intermediate nonlocality scale, or at least lower that scale down toward the Planck length. That is an important question, but as far as I can see, we don't yet have the tools for answering it.

- **Q - D. Oriti - to R. Sorkin**:

1. I would like to draw your attention to the perspective offered on the issues you raise by Deformed Special Relativity models. On the one hand, it seems to me that they are a counterexample to your statement that the deformation of the dispersion relation for matter or gauge fields would necessarily imply the existence of a state of absolute rest, i.e. a violation of Lorenz invariance. No such state exists in DSR theories, which have a full 10-dimensional symmetry group in 4d, despite the deformation of dispersion relations that some of these models predict.

2. DSR models also seem good candidates for the effective dynamics of matter fields in discrete approaches to Quantum Gravity like spin foam models or group field theories. On the other hand, and exactly because they are fully Lorentz invariant in the above sense, DSR models, which are closely related to non-commutative geometry, seem to confirm your conclusion that "discreteness plus Lorentz invariance implies non-locality". Indeed, as you say regarding non-commutative geometry-based models, they seem to suggest that, at least in some cases, the modifications coming from Quantum Gravity to usual flat space field theories can be encoded in non-local field theory formulations. Also, the existence of $^*two^*$ scales of deformation of usual flat space physics, related to a minimal length scale and a maximal length scale (the cosmological constant), has been suggested as natural in the context of so called "doubly deformed (or triply) special relativity".

3. It would be very interesting in this respect to obtain the dispersion relation for some matter field propagating on a causal set and then compare this with those studied in DSR models.

4. Do you expect a phenomenon like UV/IR mixing in any field theory on causal sets, according to the recent results on the D'Alembertian on a causal set that you have described?

5. How does the non-locality of causal sets compare, in your opinion, with that suggested by Markopoulou, that identifies the discrepancy between macroscopic (i.e. metric-induced) locality and microscopic locality defined in terms of nearest-neighbor relations on the underlying graph (which is not, in itself, a geometric object)?

– **A - R. Sorkin**:

1. It has never been clear to me whether or not these models really live up to their claim to respect Lorentz invariance. Perhaps if I understood them better, I could decide, but the proponents of the idea seem to disagree among themselves about questions like whether the dispersion relations are even modified for a single particle, say a photon. What is clear, I think, is that (contrary to what you write above) these theories do not admit a 10-dimensional symmetry *group*. Instead they have a Hopf algebra, maybe a "quantum group". Does this really entail the physical equivalence of different reference frames in the sense required by the Michelson–Morley experiment, etc?

2. It's striking that this same conclusion (nonlocality) has emerged from such apparently different trains of thought. But why do you name discreteness as an input to the DSR models? Is the point that modified dispersion relations wouldn't really arise except as an effect of an underlying discreteness? In any case, in the causet case *three* length scales seem to arise: not only a UV discreteness scale and an IR scale (as Lambda), but (modulo the caveats in my article) an intermediate nonlocality scale. A similar triplet of scales is seen in the "fuzzy sphere".

3. Agreed. As I wrote in the article, the story for a massless scalar field (the only case under control so far) seems to be that the dispersion relations are *unchanged* from those of the continuum. However, this conclusion refers to causets well approximated by Minkowski spacetime. It would indeed be very interesting to work out the dispersion relations in the presence of curvature, say in de Sitter, for starters.

4. I've expected such "mixing" all along as a concomitant of the non-locality implied by Lorentz invariance plus discreteness. One can see for example, that an IR cutoff of any sort sets an upper bound to the degree of boosting that can have meaning (thus a maximum velocity very slightly less than that of light). What the recent results on the D'Alembertian add is the implication that nonlocality might show up well before you reach the Planck length.

5. The main differences spring from the spatio-*temporal* and causal character of causets as opposed to the purely spatial and "topological" character of the graphs Fotini is working with. The nonlocality I'm talking about ("nonlocal links") predominates on microscopic scales to the extent that locality loses all meaning there. In contrast, Fotini's graphs still have a microscopic form of locality if I understand correctly (a relatively small number of nearest neighbors), and the "nonlocal links" are meant as a small perturbation. Also, the causet nonlocality is present whether or not there's any slight mismatch between the macroscopic light cones and the microscopic order-relation, whereas the graphical nonlocality by definition violates macroscopic

causality if it is present at all. (So for example, it probably would not make sense to try to identify the causet nonlocality with the "violations of local causality" manifested in the Aspect experiment.)

- **Q - L. Crane - to N. Savvidou**:

  Do you know of any approach to the application of the histories picture to gravity which doesn't assume a global spacelike foliation? Isn't decoherence a local process?

  – **A - N. Savvidou**:

    There is no paper yet treating foliations that are not globally spacelike. It is, however, in principle possible to do so in the histories formalism. One would have to suitably redefine the foliation functional employed there. For example, in a spacetime that involves topology change, the foliation functional may be defined with a dependence on the topology of the spatial slice. The main problem in pursuing such an approach would be to ensure the proper definability (and interpretation) of the canonical constraints. Concerning decoherence: in the most general case decoherence refers to the probabilistic behaviour of histories and as such it primarily refers to the state of the quantum system (or the decoherence functional in the histories approach). For this reason, it is a priori a global rather than a local concept. In my opinion, even decoherence from the environment cannot be said to be a local process, because it involves a separation into system and environment which is not given a priori (especially in cosmology). Moreover, in gravity the true degrees of freedom are non-local functionals of the spacetime fields (because of the spatial diffeomorphism constraint) and in a theory of Quantum Gravity even the definition of the notion of local process is problematic – at least before we know that a specific history (4-metric) has been realized.

- **Q - J. Henson - to L. Crane**:

  The contribution in this volume of Collins, Perez and Sudarsky calls for a "physical regularisation" of Quantum Field Theory on Minkowski space, meaning one that can be imposed without destroying the symmetries that we observe at low energies. Some discussion of this is also given in Sorkin's contribution. Does the causal sites idea give new insights in this direction?

  – **A - L. Crane**:

    Causal sites are a new type of mathematical structure whose possibilities have not been explored. I believe some of these possibilities are closely related to ideas arising from renormalization.

    One possibility I have been exploring is imposing a fixed topological dimension via the condition that every cover of a site admit a refinement cover all of whose $n+2$ fold intersections are empty. This is a well known property of n-manifolds. Call such a cover good.

Now a good cover has the combinatorial structure of a simplicial complex.

Thus an "n-dimensional" site would be describable as a partially ordered set of simplicial n-complexes. This would begin to make contact with the spin foam models for gravity.

An interesting fact about causal sites is that their Hausdorff dimension could easily not equal their topological dimension. Here by Hausdorff dimension I mean a measure of the growth in the number of regions of a given diameter, whose actual definition is rather technical. This would depend on the extent to which different maximally refined covers shared larger regions.

A causal site with such properties would be a natural setting for dimensional regularization. Given the importance of dimensional regularization in particle physics, this seems to be a candidate for a "physical regularization."

I am in the process of studying a gedanken experiment for Quantum Gravity which may shed light on this question. Consider a finite region in which a reproducible state of Quantum Gravity can be created, surrounded by observers that can be considered to live in Minkowski space. If we create the state of Quantum Gravity, and probe it with rockets which set off bursts of light at set times, we can consider the times and angles where the observers see the bursts as measurements of the quantum state.

Now it is quite possible that a single observer will see a given burst either as a set of images, as in the case of gravitational lensing, or not at all, in the case of an event horizon. Let us ignore these possibilities for the moment.

If the state in the region is essentially a single classical state, the simultaneous observation of bursts will give consistent identifications of the apparent Minkowski pasts of the observers, giving a single set of regions which would amount to an approximate description of a manifold.

Now suppose, for simplicity, that the state is a superposition of two classical metrics. The observation of the burst at a certain angle and time by one observer would appear correlated sometimes with one apparent region and sometimes with another to a second observer.

Apparent subregions of the experimental region would appear in "sheets" that could not be directly compared.

The set of all observable subregions would appear as a sheaf over the set of all observers. Consistency relationships between observers would enrich this to a site of observers. The observable subregions would also fiber over the set of metrics on the observed region. Subregions corresponding to different metrics would have no privileged identification, since different observers would relate them differently.

If we take the position that geometry in Quantum Gravity means observed geometry (necessary if we want an operational view of quantum mechanics to hold), these considerations force us to replace the underlying manifold by something rather like a topos. The categorical structure of a causal site could accommodate that.

On the other hand, if a subregion in one metric appeared to be inside a larger subregion of another to all observers, we could consider it to be contained.

This picture leads to a causal site in which the ratio of numbers of regions of different sizes could be quite different from the usual scaling property in a manifold. Perhaps Quantum Gravity leads to dimensional regularization at lower energies as well as a Planck scale cutoff. Quantum Field Theory would include contributions from Feynman diagrams with vertices in positions not simultaneously meaningful.

I think adhering to the traditional picture of an absolute background manifold in analysing this experiment would be very awkward. It would be necessary to pick some arbitrary correspondence between the observed space-time regions corresponding to each pair of classical metrics supported by the quantum state. These correspondences would not affect the result of any experiment. Einstein's principle suggests we discard them.

- **Q - D. Oriti - to O. Dreyer**:

1. If I understand correctly, in Volovik's approach the non-zero mass of the effective graviton and the failure to achieve full general covariance is the result, in the end, of the non-relativistic nature of the fundamental system he deals with, i.e. the fermionic gas/liquid; can you please clarify how exactly the presence of a background absolute time in the fundamental system is associated to this lack of general covariance in the effective theory? Also, it seems to me that Volovik's approach relies on a fundamental time variable only because of the specific choice of the physical system (here a non-relativistic and background dependent one) whose effective dynamics one studies, but that his general idea of space-time and General Relativity as emerging from some sort of condensed matter system in a specific phase does not really depend on this. If this is true, then his approach and ideas could be applied to fully background independent systems like for example matrix models and group field theories where one could hope of not ending up with any failure of general covariance in the condensed phase. What is your opinion on this?

2. In your contribution, you didn't mention explicitly the idea by Ted Jacobson of the "Einstein's equations as an equation of state", that seems to me very much related to the type of ideas you nicely reviewed. Where would it fit within your scheme of "emergent gravity" approaches?

3. What is the role of the background time or temperature (in a statistical mechanics setting) of the spin system in your 'Internal Relativity' model? It

would seem to me that they would play a crucial role in the definition and properties of the excitations you want to use to reconstruct spacetime. How do your excitations differ in this respect from those emerging in Fermi liquids?

4. I am not understanding how you reconstruct Poincaré symmetry and thus Minkowski space out of the coherent excitations you identified; in particular I do not understand how the the fact that their speed is left unchanged can suffice to identify the Poincaré group. How do you reconstruct the dimensionality of your space, in the first place? How do you realize, in terms of excitations only, that you are using the Poincaré group as opposed to, say, the conformal group SO(4,1), which has the same dimensionality? How do you see that you are not using a non-linear realization of the Poincaré group, or a non-commutative version of the same, as for example in Deformed Special Relativity models? Can you please sketch in slightly more detail the argument?

– **A - O. Dreyer**:

1. It is not the presence of a background time that is the problem. In fact I am proposing that one can get a background independent emergent theory although the fundamental theory has a background time. The problem is in the way gravity appears in Volovik's model. For him Quantum Gravity is the search for a massless spin 2 excitation. Now usually such an excitation does not arise naturally. It is usually very hard to get rid of the longitudinal modes. This is why Volovik has to tune one parameter of the theory so that the mass of the graviton becomes negligible. It is not clear whether starting from a manifestly background independent theory will cure this problem. The basic character of the modes would seem to be untouched.

2. On a superficial level one could take the phrase "it may be no more appropriate to quantize the Einstein equation than it would be to quantize the wave equation for sound in air." from the introduction of T. Jacobson's article as the motto of my approach. In my approach gravity is part of the low energy emergent physics and not a part of the more fundamental underlying theory. Quantizing the gravitational field thus does not give the fundamental theory. The more detailed question of how Jacobson's derivation of the Einstein equations relates to the proposal here is a more interesting but also more difficult question. To answer it one has to identify horizons and then find expressions for entropy, heat and temperature in terms of the underlying theory of quantum spins. The state of the theory right now does not allow for this.

3. It may be best to answer the first part of this question together with question 4.

The excitations that I am considering here do not differ from the excitations in a Fermi liquid. The reason why I am discussing them separately is because of how gravity emerges. In the Fermi liquid example gravity emerges as a genuine spin 2 excitation. I, on the other hand, am not looking for such a spin 2 excitation. Apart from this difference the excitations in the Fermi liquid would do just fine for my purpose.

4. The argument leading to Poincaré invariance and Minkowski space is indeed somewhat sketchy so let me try to expand on it a little. The original idea was to use the coherent excitations of the spin model to define the light cones of the emergent theory. The linear dispersion of the excitations then ensured the constancy of the speed of light and thus the emergence of relativity.

It might be worthwhile to make a little detour and look at the history of special relativity. When Lorentz introduced the transformations that now carry his name he was looking at the Maxwell equations and asked how one would actually measure quantities like length and time. As was discovered by Heavyside the field of a charge moving with velocity v is no longer spherically symmetric. Instead it is an ellipsoid whose one side is compressed by the now well known factor $\gamma = \sqrt{1 - v^2/c^2}$. From this observation Lorentz argued that physical bodies like measuring rods will be compressed by the same factor. The conclusion is thus that a world described by Maxwell equations will look internally like Minkowski space. What we are proposing is to adopt exactly this kind of attitude towards relativity. Minkowski space is thus not, as Einstein proposed it, a background on which matter propagates but is itself a consequence of the behavior of matter. Matter and geometry are thus inseparable. One implies the other and vice versa.

Where we deviate from Lorentz is that we use a quantum mechanical model instead of the classical Maxwell equations. A more interesting model than the one presented here is a model presented by Levin and Wen (hep-th/0507118). This model has fermions and photons as low energy excitations and their interactions are described by QED. We thus find the same situation as the one described by Lorentz only that now we are dealing with a quantum theory.

- **Q - D. Oriti - to R. Percacci**:

1. What is your take on the issue of continuum versus discrete picture of spacetime, coming from a renormalization group perspective? If gravity is asymptotically safe, would it imply that a continuum description of spacetime is applicable at all scales, or one can envisage a role of discrete spacetime structures even in this case? How would a breakdown of the continuum description show up in the ERG approach?

2. What differences, in formalism and results, can one expect in the ERG approach, if one adopts a 1st order (e.g. Palatini) or BF-like (e.g. Plebanski) description of gravity?

3. You mention that the results of the ERG seem to point out that spacetime structure cannot be described in terms of a single metric for any momentum scale. How would one notice, in the RG approach, that it cannot be described by a metric field at all, but that a description in terms of connections or even a nonlocal one would be more appropriate, say, at the Planck scale?

4. Can you please comment on the possibility of extending the ERG approach to the Lorentzian signature or to the case of dynamical space topology?

– **A - R. Percacci**:

1. First of all it should be said that the renormalization group can be realized both in continuum and discrete formulations and is likely to play a role in Quantum Gravity in either case. It should describe the transition from physics at the "lattice" or UV cutoff scale down to low energies.

Then, one has to bear in mind that when one formulates a Quantum Field Theory in the continuum but with a cutoff Λ, it is impossible to resolve points closer that $1/\Lambda$, so the continuum should be regarded as a convenient kinematical framework that is devoid of physical reality. If the asymptotic safety program could be carried through literally as described, it would provide a consistent description of physics down to arbitrarily short length scales, and in this sense the continuum would become, at least theoretically, a reality.

Of course, it would be impossible to establish experimentally the continuity of spacetime in the mathematical sense, so this is not a well-posed physical question. What is in principle a meaningful physical question, and may become answerable sometimes in the future, is whether spacetime is continuous down to, say, one tenth of the Planck length. But even then, the answer may require further qualification. Recall that in order to define a distance one has to specify a unit of length. Units can ultimately be traced to some combination of the couplings appearing in the action. For example, in Planck units one takes the square root of Newton's constant as a unit of length. Because the couplings run, when the cutoff is sent to infinity the distance between two given points could go to zero, to a finite limit or to infinity depending on the asymptotic behaviour of the unit. In principle it seems possible that spacetime looks discrete in certain units and continuous in others. Then, even if asymptotic safety was correct, it need not be in conflict with models where spacetime is discrete.

2. Writing the connection as the sum of the Levi–Civita connection and a three-index tensor Φ, one can always decompose an action for independent connection and metric into the same action written for the Levi–Civita connection, plus terms involving Φ. The effects due to Φ will be similar to those of a matter field. In the case when the action is linear in curvature, and possibly quadratic in torsion and non-metricity, up to a surface term

the action for Φ is just a mass term, implying that Φ vanishes on shell. In this case one expects the flow to be essentially equivalent to that obtained in the Einstein–Hilbert truncation plus some matter fields, although this has not been explicitly checked yet. The presence of a mass for Φ of the order of the Planck mass suggests that a decoupling theorem is at work and that Φ (or equivalently the connection) will become propagating degrees of freedom at the Planck scale. This is indeed the case when the action involves terms quadratic in curvature (which can be neglected at low energies). Then the field Φ propagates, and has quartic self-interactions. There will be new couplings, that may influence the running of Newton's constant, for example. But again, this should be equivalent to fourth-order gravity plus matter.

3. I do in fact expect that an independent connection will manifest itself at the Planck scale, as I have indicated in my answer to another question, though I don't think that this will be forced upon us by the ERG.

The scale-dependence of the metric could manifest itself as violations of the equivalence principle, or perhaps as Lorentz-invariance violations or deformations of the Lorentz group. There is much work to be done to understand this type of phenomenology. Even more radically, it is possible that gravity is just the "low energy" manifestation of some completely different physics, as suggested in the article by Dreyer. This would probably imply a failure of the asymptotic safety programme, for example a failure to find a fixed point when certain couplings are considered.

4. So far the ERG has been applied to gravity in conjunction with the background field method. Calculations are often performed in a convenient background, such as (Euclidean) de Sitter space, but the beta functions obtained in this way are then completely general and independent of the background metric and spacetime topology. The choice of a background is merely a calculational trick. It is assumed that the beta functions are also independent of the signature of the background metric, although this point may require further justification. One should also stress in this connection that the use of the background field method and of the background field gauge does not make this a "background-dependent" approach. On the contrary, when properly implemented it guarantees that the results are background-independent.

- **Q - F. Girelli - to R. Percacci**:

  Could an asymptotically safe theory be regarded as an approximation to another more fundamental theory, or does it have to be regarded as a self-contained fundamental theory?

– **A - R. Percacci**:

The asymptotic safety programme is very closely related to the formalism of effective field theories and both possibilities can be envisaged. If a fixed point with the desired properties did exist, then mathematically it would be possible to take the limit $k \to \infty$ and one could call this a fundamental theory. It would do for gravity what the Weinberg–Salam model originally did for electroweak interactions. However, experience shows that today's fundamental theory may become tomorrow's effective theory. The renormalizability of the Weinberg–Salam model was important in establishing it as a viable theory but nowadays this model is widely regarded as an effective theory whose nonrenormalizable couplings are suppressed by powers of momentum over some cutoff. In a distant future, the same could happen to an asymptotically safe theory of gravity.

To understand this point better, notice that in order to hit the fixed point as $k \to \infty$, one would have to place the initial point of the flow in the critical surface with "infinite precision". In the case of the standard model, where the use of perturbative methods is justified, this corresponds to setting all couplings with negative mass dimension *exactly* equal to zero. Even assuming that the property of asymptotic safety could be firmly established theoretically, because measurements are always imprecise, it is hard to see how one could ever establish experimentally that the world is described by such a theory. One could say at most that experiments are compatible with the theory being fundamental.

On the other hand suppose that the theory requires drastic modification at an energy scale of, say, a billion Planck masses, perhaps because of the existence of some presently unknown interaction. Then at the Planck scale one would expect the dimensionless couplings of the theory ($\tilde{g}_i$) to lie off the critical surface by an amount of the order of some power of one in a billion. Suppose we follow the flow in the direction of decreasing energies starting from a scale which is much larger than one, and much less than a billion Planck masses. Since the fixed point is IR-attractive in all directions except the ones in the critical surface, starting from a generic point in the space of coupling constants, the theory will be drawn quickly towards the critical surface. Going towards the infrared, the flow at sub–Planckian scales will then look as if it had originated from the fixed point, up to small deviations from the critical surface which may be hard or impossible to measure.

Thus, the formalism can accommodate both effective and fundamental theories of gravity. The most important point is that asymptotic safety would allow us to push QFT beyond the Planck scale, up to the next frontier, wherever that may be.

- **Q - L. Crane - to F. Markopoulou**:

1. Since you are looking at finite nets of finite dimensional vector spaces, while all the unitary representations of the Poincaré group are infinite dimensional, how will you implement Poincaré invariance?

2. I do not see how Poincaré invariance automatically will lead to approximate Minkowski space localization. For instance, a QFT on the group manifold of the Poincaré group could easily have Poincaré group invariance, but there is no homomorphism to the Minkowski space (it is canonically a subspace, not a quotient) so no invariant way of assigning localization in Minkowski space to excitations.

  – **A - F. Markopoulou**:

  1. I am expecting to find approximate Poincaré invariance only.

  2. Your statement is correct. In our scheme, the (approximate) Poincaré invariance of the excitations is a necessary condition for an effective Minkowski space, not a sufficient one.

- **Q - J. Henson - to F. Markopoulou**:

1. When referring to the QCH on a graph Γ as a basis for a theory with no fundamental variables which we would think of as geometrical, you say that "It is important to note that the effective degrees of freedom will not have a causal structure directly related to Γ". The braid example shows that the effective causal structure in the sub-system can indeed be more trivial than that on the graph. But consider a directed graph made up of two chains which were otherwise unrelated. Because of the axioms of the QCH, degrees of freedom in the system represented by one chain would never affect the other. How do you interpret this situation, which would naively look like two causally disconnected universes? It seems that the graph order puts some limits on causality even if you intend to derive it at an effective level. If you do not want any such restriction on the effective causality, the only graph possible is a single chain, and we are back to a standard discrete-time quantum system (but a completely general one). So, in general, why is the "microcausality" necessary when there is no "microgeometry"? (I have in mind condensed state systems in which an effective relativistic dynamics can arise from a non-relativistic system, where the "microcausality" is trivial.)

2. You explain what you mean by a group-invariant noiseless subsystem, and what you would interpret as Poincaré invariance. This applies in the case in which the subsystem is strictly noiseless, but in the full theory there will come a point at which the Planckian dynamics becomes relevant, with its different

notion of locality. What is your attitude to symmetries, for example, to local Lorentz invariance, coming from these considerations? Do you expect the novel phenomenology you are searching for to be Lorentz violating (as in the conceptually similar condensed state matter models I mentioned), or is this unclear? If so, what about the arguments against Lorentz violation coming from effective field theory?

– **A - F. Markopoulou**:

1. An example of microcausality and macrocausality that are what I called "directly related" is when the macro-system is obtained by coarse-graining the microscopic one. It is well-known that coarse-graining is a method with great limitations. For example, the classical 2d Ising system can be solved by coarse graining, but we would not use this method to extract spin waves from a spin chain. The noiseless subsystem construction illustrates this and adds another level of complication in the relationship between micro and macro that can be traced to entanglement. This does not mean that there will be no relation whatsoever between the micro and macro systems and I agree that your example should constitute a constraint for the effective theory. The microcausality is necessary simply because it is present in any dynamical system. However, it does not need to have a geometric form and your example of the condensed matter system is exactly an instance of this.

2. I expect violation of Lorentz invariance. As we have learned in recent years, there is a variety of ways to break Lorentz invariance with distinguishable experimental signatures. We do not yet know what kind of violations our scenario leads to. This is a question to be investigated in a specific model implementation of the mechanism outlined here, such as Quantum Graphity. As for effective field theory arguments, it is not clear that they are constraining. EFT relies on assumptions such as CPT invariance that may or may not hold in the quantum theory of gravity.

• **Q - D. Oriti - to F. Markopoulou**:

If the Einstein equations emerge as identities between the geometric degrees of freedom and the matter degrees of freedom, both identified with coherent excitations of an underlying discrete and pre-geometric system, there is no room in the theory (and in the world) for anything like "off-shell" propagation of gravity degrees of freedom, i.e. for purely 'quantum' or virtual propagating gravitational fluctuations, or geometric fluctuations of spacetime. Is this what you expect? Why? Or do you expect this to be true only if the underlying quantum pre-geometric system is in some sense "in equilibrium", so that the Einstein equations would represent something like an equation of state à la Jacobson, that are however violated when the system is even slightly out of equilibrium?

– **A - F. Markopoulou**:
Yes, we do not expect pure quantum gravitational excitations. In fact, this leads to predictions for measurable outcomes, such as the absence of tensorial modes in the CMB. There are other approaches with the same feature, such as Lloyd's computational universe and Dreyer's internal relativity and it is a current joint project to characterize the observable consequences of precisely this point.

Part II

String/M-theory

10

Gauge/gravity duality

G. HOROWITZ AND J. POLCHINSKI

10.1 Introduction

Assertion: hidden within every non-Abelian gauge theory, even within the weak and strong nuclear interactions, is a theory of Quantum Gravity.

This is one implication of AdS/CFT duality. It was discovered by a circuitous route, involving in particular the relation between black branes and D-branes in string theory. It is an interesting exercise, however, to first try to find a path from gauge theory to gravity as directly as possible. Thus let us imagine that we know a bit about gauge theory and a bit about gravity but nothing about string theory, and ask, how are we to make sense of the assertion?

One possibility that comes to mind is that the spin-two graviton might arise as a composite of two spin-one gauge bosons. This interesting idea would seem to be rigorously excluded by a no-go theorem of Weinberg & Witten [41]. The Weinberg–Witten theorem appears to assume nothing more than the existence of a Lorentz-covariant energy momentum tensor, which indeed holds in gauge theory. The theorem does forbid a wide range of possibilities, but (as with several other beautiful and powerful no-go theorems) it has at least one hidden assumption that seems so trivial as to escape notice, but which later developments show to be unnecessary. The crucial assumption here is that the graviton moves in the same spacetime as the gauge bosons of which it is made!

The clue to relax this assumption comes from the holographic principle [21; 38], which suggests that a gravitational theory should be related to a non-gravitational theory in *one fewer* dimension. In other words, we must find within the gauge theory not just the graviton, but a fifth dimension as well: the physics must be local with respect to some additional hidden parameter. Several hints suggest that the role of this fifth dimension is played by the *energy scale* of the gauge theory. For example, the renormalization group equation is local with respect to energy: it is

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

a nonlinear evolution equation for the coupling constants as measured at a given energy scale.[1]

In order to make this precise, it is useful to go to certain limits in which the five-dimensional picture becomes manifest; we will later return to the more general case. Thus we consider four-dimensional gauge theories with the following additional properties.

- *Large N_c.* While the holographic principle implies a certain equivalence between four- and five-dimensional theories, it is also true that in many senses a higher dimensional theory has more degrees of freedom; for example, the one-particle states are labeled by an additional momentum parameter. Thus, in order to find a fifth dimension of macroscopic size, we need to consider gauge theories with many degrees of freedom. A natural limit of this kind was identified by [20]: if we consider $SU(N_c)$ gauge theories, then there is a smooth limit in which N_c is taken large with the combination $g^2_{\rm YM} N_c$ held fixed.
- *Strong coupling.* Classical Yang–Mills theory is certainly not the same as classical general relativity. If gravity is to emerge from gauge theory, we should expect that it will be in the limit where the gauge fields are strongly quantum mechanical, and the gravitational degrees of freedom arise as effective classical fields. Thus we must consider the theory with large 't Hooft parameter $g^2_{\rm YM} N_c$.
- *Supersymmetry.* This is a more technical assumption, but it is a natural corollary to the previous one. Quantum field theories at strong coupling are prone to severe instabilities; for example, particle–antiparticle pairs can appear spontaneously, and their negative potential energy would exceed their positive rest and kinetic energies. Thus, QED with a fine structure constant much greater than 1 does not exist, even as an effective theory, because it immediately runs into an instability in the ultraviolet (known as the Landau pole). The Thirring model provides a simple solvable illustration of the problem: it exists only below a certain critical coupling [10]. Supersymmetric theories however have a natural stability property, because the Hamiltonian is the square of a Hermitean supercharge and so bounded below. Thus it is not surprising that most examples of field theories with interesting strong coupling behavior (i.e. dualities) are supersymmetric. We will therefore start by assuming supersymmetry, but after understanding this case we can work back to the nonsupersymmetric case.

We begin with the most supersymmetric possibility, $\mathcal{N} = 4$ $SU(N_c)$ gauge theory, meaning that there are four copies of the minimal $D = 4$ supersymmetry algebra. The assumption of $\mathcal{N} = 4$ supersymmetry has a useful bonus in that the beta function vanishes, the coupling does not run. Most gauge theories have running couplings, so that the strong coupling required by the previous argument persists only in a very narrow range of energies, becoming weak on one side and blowing up on the other. In the $\mathcal{N} = 4$ gauge theory the coupling remains

[1] This locality was emphasized to us by Shenker, who credits it to Wilson.

strong and constant over an arbitrarily large range, and so we can have a large fifth dimension.

The vanishing beta function implies that the classical conformal invariance of the Yang–Mills theory survives quantization: it is a conformal field theory (CFT). In particular, the theory is invariant under rigid scale transformations $x^\mu \to \lambda x^\mu$ for $\mu = 0, 1, 2, 3$. Since we are associating the fifth coordinate r with energy scale, it must transform inversely to the length scale, $r \to r/\lambda$. The most general metric invariant under this scale invariance and the ordinary Poincaré symmetries is

$$ds^2 = \frac{r^2}{\ell'^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{\ell^2}{r^2}dr^2 \tag{10.1}$$

for some constants ℓ and ℓ'; by a multiplicative redefinition of r we can set $\ell' = \ell$. Thus our attempt to make sense of the assertion at the beginning has led us (with liberal use of hindsight) to the following conjecture: $D = 4$, $\mathcal{N} = 4$, $SU(N_c)$ gauge theory is equivalent to a gravitational theory in five-dimensional anti-de Sitter (AdS) space. Indeed, this appears to be true. In the next section we will make this statement more precise, and discuss the evidence for it. In the final section we will discuss various lessons for Quantum Gravity, generalizations, and open questions.

10.2 AdS/CFT duality

Let us define more fully the two sides of the duality.[2] The gauge theory can be written in a compact way by starting with the $D = 10$ Lagrangian density for an $SU(N_c)$ gauge field and a 16 component Majorana–Weyl spinor, both in the adjoint ($N_c \times N_c$ matrix) representation:

$$\mathcal{L} = \frac{1}{2g_{\text{YM}}^2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\text{Tr}(\bar{\psi}\gamma^\mu D_\mu\psi). \tag{10.2}$$

This Lagrangian preserves 16 supersymmetries, the smallest algebra in $D = 10$. Now dimensionally reduce to $D = 4$, meaning that we define all fields to be independent of the coordinates $x^4, \ldots, x^9$. The ten-dimensional gauge field separates into a four-dimensional gauge field and six scalars φ^i, and the ten-dimensional spinor separates into four four-dimensional Weyl spinors.

On the other side of the duality, we must consider not just gravity but its supersymmetric extension, to match what we have in the gauge theory. The necessary theory is IIB supergravity. This theory too is most naturally formulated in $D = 10$,

[2] This subject has a vast literature, and so we will be able to cite only a few particularly pertinent references. We refer the reader to the review [1] for a more complete treatment.

where its fields includes the metric, two scalars Φ and C, two two-form potentials B_{MN} and C_{MN}, a four-form potential C_{MNPQ} whose five-form field strength is self-dual, and fermionic partners (including the gravitino) as required by supersymmetry. This ten-dimensional theory has a solution with spacetime geometry $AdS_5 \times S^5$. In fact, one finds that it is this full ten-dimensional theory that arises in the strong-coupling limit of the gauge theory. There emerges not only the fifth dimension required by holography, but five more. The additional five dimensions, which can be thought of as arising from the scalars φ^i, form a compact five-sphere.

On both sides of the duality we have started in $D = 10$, because this is the natural dimensionality for this supersymmetry algebra. On the gauge side, however, this was just a device to give a compact description of the Lagrangian; the field theory lives in four dimensions. On the gravity side, the quantum theory is fully ten-dimensional, not just a dimensional reduction. These statements follow from comparison of the space of states, or from the original Maldacena argument, as we will shortly explain.

The claim that a four-dimensional gauge theory gives rise to a ten-dimensional gravitational theory is remarkable. One sign that it is not completely crazy comes from comparing the symmetries. The $D = 4$, $\mathcal{N} = 4$, $SU(N_c)$ super-Yang–Mills theory has an $SO(4, 2)$ symmetry coming from conformal invariance and an $SO(6)$ symmetry coming from rotation of the scalars. This agrees with the geometric symmetries of $AdS_5 \times S^5$. On both sides there are also 32 supersymmetries. Again on the gravitational side these are geometric, arising as Killing spinors on the $AdS_5 \times S^5$ spacetime. On the gauge theory side they include the 16 "ordinary" supersymmetries of the $\mathcal{N} = 4$ algebra, and 16 additional supersymmetries as required by the conformal algebra.

The precise (though still not fully complete) statement is that the IIB supergravity theory in a space whose geometry is asymptotically $AdS_5 \times S^5$ is dual to the $D = 4, \mathcal{N} = 4$, $SU(N_c)$ gauge theory. The metric (10.1) describes only a Poincaré patch of AdS spacetime, and the gauge theory lives on $\mathbf{R}^4$. It is generally more natural to consider the fully extended global AdS space, in which case the dual gauge theory lives on $S^3 \times \mathbf{R}$. In each case the gauge theory lives on the conformal boundary of the gravitational spacetime ($r \to \infty$ in the Poincaré coordinates), which will give us a natural dictionary for the observables.

The initial checks of this duality concerned perturbations of $AdS_5 \times S^5$. It was shown that all linearized supergravity states have corresponding states in the gauge theory [42]. In particular, the global time translation in the bulk is identified with time translation in the field theory, and the energies of states in the field theory and string theory agree. For perturbations of $AdS_5 \times S^5$, one can reconstruct the background spacetime from the gauge theory as follows. Fields on S^5 can be decomposed into spherical harmonics, which can be described as symmetric traceless tensors on $\mathbf{R}^6$: $T_{i\cdots j}X^i\cdots X^j$. Restricted to the unit sphere one gets a basis of functions. Recall that the gauge theory has six scalars and the $SO(6)$ symmetry of rotating the φ^i. So the operators $T_{i\cdots j}\varphi^i\cdots\varphi^j$ give information about position on S^5. Four of the remaining directions are explicitly present in the gauge theory, and the radial direction corresponds to the energy scale in the gauge theory.

In the gauge theory the expectation values of local operators (gauge invariant products of the $\mathcal{N}=4$ fields and their covariant derivatives) provide one natural set of observables. It is convenient to work with the generating functional for these expectation values by shifting the Lagrangian density

$$\mathcal{L}(x) \to \mathcal{L}(x) + \textstyle\sum_I J_I(x)\mathcal{O}_I(x), \tag{10.3}$$

where $\mathcal{O}_I$ is some basis of local operators and $J_I(x)$ are arbitrary functions. Since we are taking products of operators at a point, we are perturbing the theory in the ultraviolet, which according to the energy–radius relation maps to the AdS boundary. Thus the duality dictionary relates the gauge theory generating functional to a gravitational theory in which the boundary conditions at infinity are perturbed in a specified way [16; 42]. As a further check on the duality, all three-point interactions were shown to agree [28].

The linearized supergravity excitations map to gauge invariant states of the gauge bosons, scalars, and fermions, but in fact only to a small subset of these; in particular, all the supergravity states live in special small multiplets of the superconformal symmetry algebra. Thus the dual to the gauge theory contains much more than supergravity. The identity of the additional degrees of freedom becomes particularly clear if one looks at highly boosted states, those having large angular momentum on S^5 and/or AdS_5 [5; 17]. The fields of the gauge theory then organize naturally into one-dimensional structures, coming from the Yang–Mills large-N_c trace: they correspond to the excited states of strings. In some cases, one can even construct a two dimensional sigma model directly from the gauge theory and show that it agrees (at large boost) with the sigma model describing strings moving in $AdS_5 \times S^5$ [27].

Thus, by trying to make sense of the assertion at the beginning, we are forced to "discover" string theory. We can now state the duality in its full form [30].

Four-dimensional $\mathcal{N}=4$ supersymmetric $SU(N_c)$ gauge theory is equivalent to IIB string theory with $AdS_5 \times S^5$ boundary conditions.

The need for strings (though not the presence of gravity!) was already anticipated by 't Hooft [20], based on the planar structure of the large-N_c Yang–Mills perturbation theory; the AdS/CFT duality puts this into a precise form. It also fits with the existence of another important set of gauge theory observables, the one-dimensional Wilson loops. The Wilson loop can be thought of as creating a string at the AdS_5 boundary, whose world-sheet then extends into the interior [31; 35].

We now drop the pretense of not knowing string theory, and outline the original argument for the duality in [30]. Maldacena considered a stack of N_c parallel D3-branes on top of each other. Each D3-brane couples to gravity with a strength proportional to the dimensionless string coupling g_s, so the distortion of the metric by the branes is proportional to $g_s N_c$. When $g_s N_c \ll 1$ the spacetime is nearly flat and there are two types of string excitations. There are open strings on the brane whose low energy modes are described by a $U(N_c)$ gauge theory. There are also closed strings away from the brane. When $g_s N_c \gg 1$, the backreaction is important and the metric describes an extremal black 3-brane. This is a generalization of a black hole appropriate for a three dimensional extended object. It is extremal with respect to the charge carried by the 3-branes, which sources the five form F_5. Near the horizon, the spacetime becomes a product of S^5 and AdS_5. (This is directly analogous to the fact that near the horizon of an extremal Reissner–Nordstrom black hole, the spacetime is $AdS_2 \times S^2$.) String states near the horizon are strongly redshifted and have very low energy as seen asymptotically. In a certain low energy limit, one can decouple these strings from the strings in the asymptotically flat region. At weak coupling, $g_s N_c \ll 1$, this same limit decouples the excitations of the 3-branes from the closed strings. Thus the low energy decoupled physics is described by the gauge theory at small g_s and by the $AdS_5 \times S^5$ closed string theory at large g_s, and the simplest conjecture is that these are the same theory as seen at different values of the coupling.[3] This conjecture resolved a puzzle, the fact that very different gauge theory and gravity calculations were found to give the same answers for a variety of string–brane interactions.

In the context of string theory we can relate the parameters on the two sides of the duality. In the gauge theory we have g^2_{YM} and N_c. The known D3-brane Lagrangian determines the relation of couplings, $g^2_{\mathrm{YM}} = 4\pi g_s$. Further, each D3-brane is a source for the five-form field strength, so on the string side N_c is determined by $\int_{S^5} F_5$; this integrated flux is quantized by a generalization of Dirac's argument for quantization of the flux $\int_{S^2} F_2$ of a magnetic monopole. The supergravity field equations give a relation between this flux and the radii of curvature of the AdS_5 and S^5 spaces, both being given by

$$\ell = (4\pi g_s N_c)^{1/4} \ell_s. \tag{10.4}$$

[3] The $U(1)$ factor in $U(N_c) = SU(N_c) \times U(1)$ also decouples: it is Abelian and does not feel the strong gauge interactions.

Here ℓ_s is the fundamental length scale of string theory, related to the string tension μ by $\mu^{-1} = 2\pi\ell_s^2$. Notice that the spacetime radii are large in string units (and so the curvature is small) precisely when the 't Hooft coupling $4\pi g_s N_c = g_{YM}^2 N_c$ is large, in keeping with the heuristic arguments that we made in the introduction. It is also instructive to express the AdS radius entirely in gravitational variables. The ten-dimensional gravitational coupling is $G \sim g_s^2 \ell_s^8$, up to a numerical constant. Thus

$$\ell \sim N_c^{1/4} G^{1/8}, \quad G \sim \frac{\ell^8}{N_c^2} . \tag{10.5}$$

In other words, the AdS radius is $N_c^{1/4}$ in Planck units, and the gravitational coupling is N_c^{-2} in AdS units.

10.3 Lessons, generalizations, and open questions

10.3.1 Black holes and thermal physics

The fact that black holes have thermodynamic properties is one of the most striking features of classical and Quantum Gravity. In the context of AdS/CFT duality, this has a simple realization: in the dual gauge theory the black hole is just a hot gas of gauge bosons, scalars, and fermions, the gauge theory degrees of freedom in equilibrium at the Hawking temperature.

A black hole in AdS_5 is described by the Schwarzschild AdS geometry

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1 - \frac{r_0^2}{r^2}\right) dt^2 + \left(\frac{r^2}{\ell^2} + 1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3 . \tag{10.6}$$

Denoting the Schwarzschild radius by r_+, the Hawking temperature of this black hole is $T_H = (\ell^2 + 2r_+^2)/2\pi r_+ \ell^2$. When $r_+ \gg \ell$, the Hawking temperature is large, $T_H \sim r_+/\ell^2$. This is quite different from a large black hole in asymptotically flat spacetime which has $T_H \sim 1/r_+$. The gauge theory description is just a thermal state at the same temperature T_H.

Let us compare the entropies in the two descriptions. It is difficult to calculate the field theory entropy at strong coupling, but at weak coupling, we have of order N_c^2 degrees of freedom, on a three sphere of radius ℓ at temperature T_H and hence

$$S_{YM} \sim N_c^2 T_H^3 \ell^3 . \tag{10.7}$$

On the string theory side, the solution is the product of (10.6) and an S^5 of radius ℓ. So recalling that $G \sim g_s^2 \ell_s^8$ in ten dimensions and dropping factors of order unity, the Hawking–Bekenstein entropy of this black hole is

$$S_{BH} = \frac{A}{4G} \sim \frac{r_+^3 \ell^5}{g_s^2 \ell_s^8} \sim \frac{T_H^3 \ell^{11}}{g_s^2 \ell_s^8} \sim N_c^2 T_H^3 \ell^3 \tag{10.8}$$

where we have used (10.4) in the last step. The agreement with (10.7) shows that the field theory has enough states to reproduce the entropy of large black holes in AdS_5.

On the gauge theory side, the scaling of the entropy as $T_{\rm H}^3$ is just dimensional analysis for a massless field theory in 3+1 dimensions. That the ten-dimensional string theory produces the same behavior is a surprising consequence of the AdS geometry. The factor of N_c^2 similarly just counts explicit degrees of freedom on the gauge theory side, while on the string side it comes from the scaling of the horizon area.

Putting in all the numerical factors one finds that $S_{\rm BH} = \frac{3}{4} S_{\rm YM}$ [15]. The numerical disagreement is not surprising, as the Yang–Mills calculation is for an ideal gas, and at large $g_{\rm s}$ the Yang–Mills degrees of freedom are interacting. Thus one expects a relation of the form $S_{\rm BH} = f(g_{\rm s} N_c) S_{\rm YM,ideal}$, where $f(0) = 1$; the above calculation implies that $f(\infty) = \frac{3}{4}$. We do not yet have a quantitative understanding of the value $\frac{3}{4}$, but the first correction has been calculated both at weak and strong coupling and is consistent with $f(g_{\rm s} N_c)$ interpolating in a rather smooth way.

Hawking & Page [18] showed that for thermal AdS boundary conditions there is a phase transition: below a transition temperature of order $1/\ell$ the dominant configuration is not the black hole but a gas of particles in AdS space. The low temperature geometry has no horizon and so its entropy comes only from the ordinary statistical mechanics of the gas. The same transition occurs in the gauge theory [43]. The $\mathcal{N} = 4$ gauge theory on S^3 has an analog of a confinement transition. At low temperature one has a thermal ensemble of gauge-invariant degrees of freedom, whose entropy therefore scales as N_c^0, and at high temperature one has the N_c^2 behavior found above – the same scalings as on the gravitational side.

There is another test one can perform with the gauge theory at finite temperature. At long wavelengths, one can use a hydrodynamic approximation and think of this as a fluid (for a recent overview see [25]). It is then natural to ask: what is the speed of sound waves? Conformal invariance implies that the stress energy tensor is traceless, so $p = \rho/3$ which implies that $v = 1/\sqrt{3}$. The question is: can you derive this sound speed from the AdS side? This would seem to be difficult since the bulk does not seem to have any preferred speed other than the speed of light. But recent work has shown that the answer is yes.

The AdS/CFT duality also gives an interesting perspective on the black hole membrane paradigm [40]. The black hole horizon is known to have many of the properties of a dissipative system. On the dual side it *is* a dissipative system, the hot gauge theory. One can thus compute such hydrodynamic quantities such as the shear viscosity. These are hard to check since they are difficult to calculate directly in the strongly coupled thermal gauge theory, but, rather remarkably, the numerical agreement with the observed properties of the real quark–gluon plasma at RHIC is better than for conventional field theory calculations (for a discussion see [7]).

There is also a field theory interpretation of black hole quasinormal modes [22]. A perturbation of the black hole decays with a characteristic time set by the imaginary part of the lowest quasinormal mode. This should correspond to the timescale for the gauge theory to return to thermal equilibrium. One can show that the quasinormal mode frequencies are poles in the retarded Green's function of a certain operator in the gauge theory. The particular operator depends on the type of field used to perturb the black hole [26].

Finally, consider the formation and evaporation of a small black hole in a spacetime which is asymptotically $AdS_5 \times S^5$. By the AdS/CFT correspondence, this process is described by ordinary unitary evolution in the gauge theory. So black hole evaporation does not violate quantum mechanics: information is preserved. This also provides an indirect argument against the existence of a "bounce" at the black hole singularity, because the resulting disconnected universe would presumably carry away information.

10.3.2 Background independence and emergence

The AdS/CFT system is entirely embedded in the framework of quantum mechanics. On the gauge theory side we have an explicit Hamiltonian, and states which we can think of as gauge invariant functionals of the fields. Thus the gravitational theory on the other side is quantum mechanical as well. In particular the metric fluctuates freely except at the AdS boundary. One is not restricted to perturbations about a particular background.

This is clearly illustrated by a rich set of examples which provide a detailed map between a class of nontrivial asymptotically $AdS_5 \times S^5$ supergravity solutions and a class of states in the gauge theory [29]. These states and geometries both preserve half of the supersymmetry of $AdS_5 \times S^5$ itself. On the field theory side, one restricts to fields that are independent of S^3 and hence reduce to $N_c \times N_c$ matrices. In fact, all the states are created by a single complex matrix, so can be described by a one-matrix model. This theory can be quantized exactly in terms of free fermions, and the states can be labeled by a arbitrary closed curve (the Fermi surface) on a plane. On the gravity side, one considers solutions to ten dimensional supergravity involving just the metric and self-dual five form F_5. The field equations are simply $dF_5 = 0$ and

$$R_{MN} = F_{MPQRS}F_N{}^{PQRS}. \tag{10.9}$$

There exists a large class of stationary solutions to (10.9), which have an $SO(4) \times SO(4)$ symmetry and can be obtained by solving a linear equation. These solutions are nonsingular, have no event horizons, but can have complicated topology.

They are also labeled by arbitrary closed curves on a plane. This provides a precise way to map states in the field theory into bulk geometries. Only for some "semi-classical" states is the curvature below the Planck scale everywhere, but the matrix/free fermion description readily describes all the states, of all topologies, within a single Hilbert space.

Thus the gauge theory gives a representation of Quantum Gravity that is background independent almost everywhere – that is, everywhere except the boundary. Conventional string perturbation theory constructs string amplitudes as an asymptotic expansion around a given spacetime geometry; here we have an exact quantum mechanical construction for which the conventional expansion generates the asymptotics. All local phenomena of Quantum Gravity, such as formation and evaporation of black holes, the interaction of quanta with Planckian energies, and even transitions that change topology, are described by the gauge theory. However, the boundary conditions do have the important limitation that most cosmological situations, and most compactifications of string theory, cannot be described; we will return to these points later.

To summarize, AdS/CFT duality is an example of emergent gravity, emergent spacetime, and emergent general coordinate invariance. But it is also an example of emergent strings! We should note that the terms "gauge/gravity duality" and "gauge/string duality" are often used, both to reflect these emergent properties and also the fact that (as we are about the see) the duality generalizes to gravitational theories with certain other boundary conditions, and to field theories that are not conformally invariant.

Let us expand somewhat on the emergence of general coordinate invariance. The AdS/CFT duality is a close analog to the phenomenon of emergent gauge symmetry (e.g. [11; 4]). For example, in some condensed matter systems in which the starting point has only electrons with short-ranged interactions, there are phases where the electron separates into a new fermion and boson,

$$e(x) = b(x) f^{\dagger}(x). \tag{10.10}$$

However, the new fields are redundant: there is a gauge transformation $b(x) \to e^{i\lambda(x)} b(x)$, $f(x) \to e^{i\lambda(x)} f(x)$, which leaves the physical electron field invariant. This new gauge invariance is clearly emergent: it is completely invisible in terms of the electron field appearing in the original description of the theory.[4] Similarly, the gauge theory variables of AdS/CFT are trivially invariant under the bulk diffeomorphisms, which are entirely invisible in the gauge theory (the gauge theory fields do transform under the asymptotic symmetries of $AdS_5 \times S^5$, but

[4] This "statistical" gauge invariance is not to be confused with the ordinary electromagnetic gauge invariance, which does act on the electron.

these are ADM symmetries, not gauge redundancies). Of course we can always in general relativity introduce a set of gauge-invariant observables by setting up effectively a system of rods and clocks, so to this extent the notion of emergence is imprecise, but it carries the connotation that the dynamics can be expressed in a simple way in terms of the invariant variables, as is the case in AdS/CFT.[5]

10.3.3 Generalizations

Thus far we have considered only the most well-studied example of gauge/gravity duality: $D = 4$, $\mathcal{N} = 4$, Yang–Mills $\Leftrightarrow$ string theory with $AdS_5 \times S^5$ boundary conditions. Let us now ask how much more general this phenomenon is (again, for details see the review [1]).

First, we imagine perturbing the theory we have already studied, adding additional terms (such as masses for some of the fields) to the gauge theory action. This is just a special case of the modification (10.3), such that the functions $J_I(x) = g_I$ are independent of position. Thus we already have the dictionary, that the dual theory is given by IIB string theory in a spacetime with some perturbation of the $AdS_5 \times S^5$ boundary conditions.

In general, the perturbation of the gauge theory will break conformal invariance, so that the physics depends on energy scale. In quantum field theory there is a standard procedure for integrating out high energy degrees of freedom and obtaining an effective theory at low energy. This is known as renormalization group (RG) flow. If one starts with a conformal field theory at high energy, the RG flow is trivial. The low energy theory looks the same as the high energy theory. This is because there is no intrinsic scale. But if we perturb the theory, the RG flow is nontrivial and we obtain a different theory at low energies. There are two broad possibilities: either some degrees of freedom remain massless and we approach a new conformal theory at low energy, or all fields become massive and the low energy limit is trivial.

Since the energy scale corresponds to the radius, this RG flow in the boundary field theory should correspond to radial dependence in the bulk. Let us expand a bit on the relation between radial coordinate and energy (we will make this argument in Poincaré coordinates, since the perturbed gauge theories are usually studied on $\mathbf{R}^4$). The AdS geometry (10.1) is *warped*: in Poincaré coordinates, the four flat dimensions experience a gravitational redshift that depends on a fifth coordinate, just as in Randall–Sundrum compactification. Consequently the conserved Killing

5 Note that on the gauge theory side there is still the ordinary Yang–Mills gauge redundancy, which is more tractable than general coordinate invariance (it does not act on spacetime). In fact in most examples of duality there are gauge symmetries on both sides and these are unrelated to each other: the duality pertains only to the physical quantities.

momentum p_μ (Noether momentum in the gauge theory) is related to the local inertial momentum $\tilde{p}_\mu$ by

$$p_\mu = \frac{r}{\ell}\tilde{p}_\mu. \tag{10.11}$$

A state whose local inertial momenta are set by the characteristic scale $1/\ell$ therefore has a Killing momentum $p_\mu \sim r/\ell^2$, displaying explicitly the mapping between energy/momentum scale and radius.

Given a perturbation that changes the boundary conditions, AdS is no longer a solution and we must solve Einstein's equation to find the correct solution. Just as in the gauge theory there are two possibilities: either we approach a new AdS solution at small radius (with, in general, a different radius of curvature), or the small radius geometry is cut off in such a way that the warp factor (which is r/ℓ in AdS spacetime) has a lower bound. The former clearly corresponds to a new conformal theory, while the latter would imply a mass gap, by the argument following eq. (10.11). In the various examples, one finds that the nature of the solution correctly reflects the low energy physics as expected from gauge theory arguments; there is also more detailed numerical agreement [13]. So the classical Einstein equation knows a lot about RG flows in quantum field theory.

A notable example is the case where one gives mass to all the scalars and fermions, leaving only the gauge fields massless in the Lagrangian. One then expects the gauge theory to flow to strong coupling and produce a mass gap, and this is what is found in the supergravity solution. Further, the gauge theory should confine, and indeed in the deformed geometry a confining area law is found for the Wilson loop (but still a perimeter law for the 't Hooft loop, again as expected). In other examples one also finds chiral symmetry breaking, as expected in strongly coupled gauge theories [24].

As a second generalization, rather than a deformation of the geometry we can make a big change, replacing S^5 with any other Einstein space; the simplest examples would be S^5 identified by some discrete subgroup of its $SO(6)$ symmetry. The product of the Einstein space with AdS_5 still solves the field equations (at least classically), so there should be a conformally invariant dual. These duals are known in a very large class of examples; characteristically they are quiver gauge theories, a product of $SU(N_1) \times \cdots \times SU(N_k)$ with matter fields transforming as adjoints and bifundamentals (one can also get orthogonal and symplectic factors).

As a third generalization, we can start with Dp-branes for other values of p, or combinations of branes of different dimensions. These lead to other examples of gauge–gravity duality for field theories in various dimensions, many of which are nonconformal. The case $p = 0$ is the BFSS matrix model, although the focus in that case is on a different set of observables, the scattering amplitudes for the D0-branes themselves. A particularly interesting system is D1-branes plus D5-branes, leading to the near-horizon geometry $AdS_3 \times S^3 \times T^4$. This case has at least one advantage over $AdS_5 \times S^5$. The entropy of large black holes can now be reproduced exactly, including the numerical coefficient. This is related to the fact that a black hole in AdS_3 is a BTZ black hole which is locally AdS_3 everywhere. Thus when one extrapolates to small coupling, one does not modify the geometry with higher curvature corrections.

We have discussed modifications of the gauge theory's Hamiltonian, its spectrum, and even its dimensionality. Many of these break the theory's conformal symmetry and some or all of its supersymmetry (with all of it broken the stability is delicate, but possible). Thus we can relax the assumption of supersymmetry, as promised earlier. If we start with a nonsupersymmetric gauge theory, do we get a gravitational theory without supergravity (and maybe without strings)? Apparently not. When we change the dynamics of the gauge theory, we do *not* change the local dynamics of the gravitational theory, i.e. its equation of motion, but only its boundary conditions at AdS infinity. In all known examples where a macroscopic spacetime and gravitational physics emerge from gauge theory, the local dynamics is given by string theory. This is consistent with the lore that string theory has no free parameters, the local dynamical laws are completely fixed. This was the conclusion when string theory was first constructed as an expansion around a fixed spacetime, and it has not been altered as the theory has been rediscovered in various dual forms; it is one of the principal reasons for the theory's appeal.

Let us also relax the other assumptions from the introduction, large 't Hooft coupling and large N_c. The AdS radius $\ell = (g^2_{\rm YM} N_c)^{1/4} \ell_{\rm s} \sim N_c^{1/4} G^{1/8}$ becomes small compared to the string size when the 't Hooft coupling is small, and comparable to the Planck scale when N_c is not large. This is consistent with our argument that we needed strong coupling and large N_c in order to see macroscopic gravity. However, string theory remains well-defined on spaces of large curvature, so the string dual should still make sense; hence our assertion that even the strong and weak nuclear interactions can be written as string theories, though in highly curved spaces.[6]

In more detail, consider first varying the 't Hooft coupling. The string world-sheet action in $AdS_5 \times S^5$ is proportional to $\ell^2/\ell_{\rm s}^2 = (g^2_{\rm YM} N_c)^{1/2}$. This is large when the 't Hooft coupling is large, so the world-sheet path integral is then nearly gaussian (i.e. weakly coupled). On the other hand when the 't Hooft coupling is small the string world-sheet theory is strongly coupled: the cost of living on a space of high curvature is strong world-sheet coupling. This limits one's ability to calculate, though in the case of $AdS_5 \times S^5$ there is enough symmetry that one might ultimately be able to solve the world-sheet theory completely [6].

6 There have been proposals that a five-dimensional picture is phenomenologically useful even for real QCD; see the recent papers [12; 9], and references therein.

Now consider varying N_c. From eq. (10.5) the gauge theory expansion parameter $1/N_c^2$ matches the gravitational loop expansion parameter G, so we can expect an order-by-order matching. In fact, there are various indications that the duality remains true even at finite values of N_c, and not just as an expansion in $1/N_c^2$. A striking example is the "string exclusion principle" [32]. We have noted that the wave functions of the gravity states on S^5 arises in the gauge theory from traces of products of the φ^i. However, these fields are $N_c \times N_c$ matrices, so the traces cease to be independent for products of more than N_c fields: there is an upper bound

$$J/N_c \leq 1 \tag{10.12}$$

for the angular momentum on S^5. From the point of view of supergravity this is mysterious, because the spherical harmonics extend to arbitrary J. However, there is an elegant resolution in string theory [33]. A graviton moving sufficiently rapidly on S^5 will blow up into a spherical D3-brane (this growth with energy is a characteristic property of holographic theories), and $J = N_c$ is the largest D3-brane that will fit in the spacetime. Thus the same bound is found on both sides of the duality, and this is a nonperturbative statement in N_c: it would be trivial in a power series expansion around $1/N_c = 0$.

10.3.4 Open questions

An obvious question is, to what extent is the AdS/CFT duality proven?

We should first note that this duality is itself our most precise definition of string theory, giving an exact construction of the theory with $AdS_5 \times S^5$ boundary conditions or the various generalizations described above. This does not mean that the duality is a tautology, because we have a great deal of independent information about string theory, such as its spectrum, its low energy gravitational action, the weak coupling expansion of its amplitudes, and so on: the gauge theory must correctly reproduce these. Thus the duality implies a large number of precise statements, for example about the amplitudes in the strongly coupled gauge theory at each order in $1/N_c$ and $1/\sqrt{g_{\rm YM}^2 N_c}$.[7]

What has been proven is much less. The original Maldacena argument above makes the duality very plausible but of course makes many assumptions. The quantitative tests are largely restricted to those quantities that are required by supersymmetry to be independent of the coupling. This is not to say that the agreement follows from supersymmetry alone. For example, supersymmetry requires the

[7] We should note that there are also purely field theoretic dualities, where both sides presumably have a precise definition, and whose status is very similar to that of AdS/CFT duality. The simplest example again involves the $D = 4$, $\mathcal{N} = 4$, Yang–Mills theory but in a different part of its parameter space, $g_{\rm YM}^2 \to \infty$ at fixed N_c. The Maldacena duality relates this field-theoretic duality to the S-duality of the IIB string theory.

states to lie in multiplets, but the number of multiplets (as a function of their $SO(6)$ charges) is not fixed, and the fact that it agrees for each value of the charges is a strong dynamical statement – recall in particular that the string exclusion principle must enter to make the range of charges match.

In many ways the more impressive tests are the more qualitative ones. The point has often been made that the claim that a 10-dimensional string theory is the same as a four-dimensional field theory is so audacious that if it were incorrect this should be easy to show. Instead we find, as we look at a wide variety of situations, that the qualitative physics is exactly what we would expect. We have noted some of these situations above: the appearance of string-like states in the gauge theory at large boost, the matching of the confining transition with the Hawking–Page transition and with the correct N_c scaling on each side, the hydrodynamic properties, the matching of the deformed geometries with the RG flows and the expected low energy physics be it conformal, massive, confining, chiral symmetry-breaking, and so on. For the confining theories, with all conformal and supersymmetries broken, one can calculate the results of high energy scattering processes. The results differ from QCD because the theory is different, but the differences are qualitatively just those that would be expected [34].

Finally, we mention a very different kind of quantitative test. Statements about strongly coupled gauge theory can be tested directly by simulation of the theory. The range of tests is limited by the computational difficulty, but some positive results have been reported [2; 19].

In summary, we see convincing reason to place AdS/CFT duality in the category of true but not proven. Indeed, we regard it on much the same footing as such mathematical conjectures as the Riemann hypothesis. Both provide unexpected connections between seemingly different structures (and speaking as physicists we find a connection between gauge theory and gravity even more fascinating than one between prime numbers and analytic functions), and each has resisted either proof or disproof in spite of concentrated attention. In either case it may be that the final proof will be narrow and uninstructive, but it seems more likely that the absence of a proof points to the existence of important new concepts to be found.

As another open question, the dictionary relating spacetime concepts in the bulk and field theory concepts on the boundary is very incomplete, and still being developed. For example, while we know how to translate certain states of the CFT into bulk geometries, we do not yet know the general condition on the state in order for a semiclassical spacetime to be well defined.

A related issue is a more precise understanding of the conservation of information in black hole decay. The AdS/CFT duality implies that we can find an S-matrix by passing to the gauge theory variables, but there should be some prescription directly in the gravitational theory. The black hole information problem can be understood as a conflict between quantum mechanics and locality. In the context of emergent spacetime it is not surprising that it is locality that yields, but we would like to understand the precise manner in which it does so.

A big open question is how to extend all this from AdS boundary conditions to spacetimes that are more relevant to nature; we did find some generalizations, but they all have a causal structure similar to that of AdS. Again, the goal is a precisely defined nonperturbative construction of the theory, presumably with the same features of emergence that we have found in the AdS/CFT case. A natural next step might seem to be de Sitter space. There were some attempts along these lines, for example [37; 44], but there are also general arguments that this idea is problematic [39]. In fact, this may be the wrong question, as constructions of de Sitter vacua in string theory (beginning with [36; 23]) always seem to produce states that are only metastable (see [14], for further discussion, and [3], for an alternate view). As a result, cosmology will produce a chaotic state with bubbles of all possible metastable vacua [8]. The question is then the nonperturbative construction of states of this kind. The only obvious spacetime boundaries are in the infinite future, in eternal bubbles of zero cosmological constant (and possibly similar boundaries in the infinite past). By analogy these would be the location of the holographic dual variables [39].

In conclusion, the embedding of Quantum Gravity in ordinary gauge theory is a remarkable and unexpected property of the mathematical structures underlying theoretical physics. We find it difficult to believe that nature does not make use of it, but the precise way in which it does so remains to be discovered.

Acknowledgments

This work was supported in part by NSF grants PHY99-07949, PHY02-44764, and PHY04-56556.

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11

String theory, holography and Quantum Gravity

T. BANKS

11.1 Introduction

It is the opinion of this author that many theories of Quantum Gravity have already been discovered, but that the one which applies to the real world still remains a mystery. The theories I am referring to all go under the rubric of M/string-theory, and most practitioners of this discipline would claim that they are all “vacuum states of a single theory". The model for such a claim is a quantum field theory whose effective potential has many degenerate minima, but I believe this analogy is profoundly misleading.

Among these theories are some which live in asymptotically flat space-times of dimensions between 11 and 4. The gauge invariant observables of these theories are encoded in a scattering matrix.[1] All of these theories are exactly supersymmetric, a fact that I consider to be an important clue to the physics of the real world. In addition, they all have continuous families of deformations. These families are very close to being analogs of the moduli spaces of vacuum states of supersymmetric quantum field theory. They all have the same high energy behavior, and one can create excitations at one value of the moduli which imitate the physics at another value, over an arbitrarily large region of space. Except for the maximally supersymmetric case, there is no argument that all of these models are connected by varying moduli in this way. One other feature of these models is noteworthy. Some of them are related to others by compactification, e.g. the same low energy Lagrangian appears on $R^{1,10-D} \times T^D$, for various values of D. It is always the case that the models with more compact dimensions have *more* fundamental degrees of freedom.

[1] In four dimensions, the gravitational scattering matrix has familiar infrared divergences. It is believed by many that this is a technical problem, which is more or less understood. There can also be problems with confining gauge theories, whose resolution in a purely S-matrix context is somewhat obscure. String perturbation theory for four dimensional compactifications instructs us to compute gauge boson scattering amplitudes, which probably do not exist.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

This looks peculiar to someone used to the rules of local field theory. Typically, compactification reduces the number of degrees of freedom by imposing periodicity conditions on fields, though in gauge theories we can have a mild increase in the number of degrees of freedom. Pure gauge modes on a non-compact space can become gauge invariant modes on a compact one because the would be gauge function is not a well defined function on the compact manifold. In string theory we have a vast new class of states coming from *p-branes* wrapping p-cycles of the compactification manifold; p-branes are p-dimensional extended objects. In field theory, such objects exist as coherent states of the fundamental degrees of freedom. Compactifying the field degrees of freedom automatically includes the wrapped brane configurations. In string theory they must be treated as new fundamental degrees of freedom (elementary strings are only the simplest example).

There is a purely gravitational indication of the increase in the number of degrees of freedom upon compactification. The entropy, or logarithm of the density, of uncharged black hole states in an asymptotically flat space-time of $d \geq 4$ dimensions, behaves as

$$S(E) \sim E^{\frac{d-2}{d-3}}.$$

It increases more rapidly in lower dimension. For certain supersymmetric charged black holes, the entropy can be calculated in terms of wrapped brane states like those discussed above [1; 2; 3], so these remarks are related to each other.

The other class[2] of well understood theories of Quantum Gravity resides in asymptotically anti-deSitter (AdS) spaces, of varying dimension. The exact quantum theory of such space-times is determined by a quantum field theory which lives on the conformal boundary of AdS_d, $R \times S^{d-2}$, where d is the dimension of the AdS space. The radius of the sphere is the same as the radius of curvature of the AdS space. In every known case, the allowed values of this radius in Planck units are discrete. The correlation functions of this boundary field theory, are the analogs of the S-matrix in asymptotically flat space-time. The field theory is generally a renormalizable (i.e. relevant) perturbation of a conformal field theory (CFT). If the falloff of the geometry towards AdS is sufficiently rapid, then the theory is conformally invariant. A renormalizable field theory which is not conformally invariant looks like a non-homogeneous perturbation of the AdS geometry, which does not fall off sufficiently fast to be a normalizable excitation of the AdS background.

Some of these theories contain continuous parameters, but here these parameters refer to deformation of the Lagrangian of the boundary field theory – *lines of*

[2] There is yet a third class, *linear dilaton asymptotics*, which could be understood in terms of little string theories [4], if we really knew what those were. Certain properties of these systems can be worked out, and they seem to be qualitatively similar to AdS/CFT, but also share some features of flat space.

fixed points in the parlance of the renormalization group. The high energy behavior at different parameter values is different, and one cannot create large internal regions with different values of the parameter. This illustrates the so called UV/IR connection [5; 6]: the behavior near the boundary of AdS space corresponds to the ultra-violet behavior of the underlying field theory.

Given a field theory which is supposed to represent an asymptotically AdS space-time, the AdS radius in Planck units can be read off from the asymptotic spectrum of operator dimensions, by comparing the black hole entropy formula to that of the CFT. It is easy to find theories where this radius is large. However, this is not sufficient for the theory to have a valid low energy description in terms of gravity coupled to a finite number of other fields and perhaps compactified. The condition for a long wavelength bulk field theory description is that there is a regime of dimensions, starting at the stress tensor, where the degeneracy of operators with a given dimension grows only like a power. There must be a large parameter ($g^2 N$ plays this role for $AdS_5 \times S^5$) which controls the transition to a normal CFT regime with exponential dimensional degeneracies. In addition, all operators with $s > 2$ must have dimensions which go to infinity when this parameter is large. The only examples where we have reliable evidence that this is true, so that we can imagine taking the asymptotically flat limit, become supersymmetric as the parameter which controls the gravity approximation is taken to infinity. We again see that asymptotically flat space seems to require the restoration of SUSY.

AdS/CFT also gives us a way of assigning rigorous meaning to certain features of a bulk effective potential. AdS theories can make sense at a maximum of the effective potential, if the tachyon mass obeys the Breitenlohner–Friedman [7; 8] (BF) bound. In CFT language, the tachyon is dual to a relevant perturbation of the CFT. There are BPS domain walls connecting supersymmetric BF allowed maxima, with other SUSic AdS minima. It has been shown that these domain walls are the classical gravity approximation to the renormalization group flow between two boundary CFTs. In the naive bulk field theory interpretation, a domain wall would separate two ”vacua" of the same set of degrees of freedom. The RG interpretation shows that instead, the IR end of the flow (the minimum) is describing a field theory with fewer degrees of freedom and a different Hamiltonian (the high energy spectra of the two systems are different). The relation between the two “states” is similar to that between any AdS theory and an exact description of branes in asymptotically flat space-time. One is obtained from the other by taking an infrared limit and throwing away most of the degrees of freedom. The relation is unidirectional, quite unlike that between vacua in field theory. Note finally, that, as in all RG flows, features of the effective potential between the maximum and minimum are “scheme dependent” and have no particular invariant meaning in the boundary field theory.

This lightning review of results from string theory[3] was supposed to make the reader realize that existing forms of string theory are disconnected from each other, and that the unique features of each example depend on the asymptotic geometry of space-time. We should expect the same to be true in more complicated situations, and should be particularly wary of cases, like Big Bangs and Crunches, and de Sitter space, where the boundaries are not all under control. It is my opinion that the study of cosmology and/or de Sitter space, requires us to go beyond conventional string theory. Unless one believes in Big Bounce scenarios, in which there is an asymptotically infinite past, then cosmology cannot be described by a real S-matrix. There is an initial space-like hypersurface, a finite proper time in the past of all observers. The initial state describing this configuration may be uniquely determined, from first principles (I call this an S-vector scenario), or must be chosen at random subject to some constraints. In neither case does conventional string theory *have* to be a valid description near the Big Bang. Attempts to apply conventional string theory techniques to model cosmologies have not met with success.

Similarly, if the final state of the universe is a stable, asymptotically de Sitter space, then the scattering boundary conditions of string theory are not applicable either (though something approximating them for small Λ might be appropriate).[4]

I believe that, in order to formulate a more general theory of Quantum Gravity, which will enable us to cope with cosmological situations, we must find a description analogous to the local field theory description of classical gravitation. This formalism works with non-gauge invariant quantities because there are no gauge invariant local observables in diffeomorphism invariant theories. The quantum formalism I propose will be similar, and will be tied to a fixed physical gauge, which in the semi-classical limit should be thought of as the coordinate system of a given time-like observer. I will use the word observer to denote a large quantum system with a wealth of observables whose quantum fluctuations are exponentially small as a function of a macroscopic volume parameter. Systems well described by (perhaps cut-off) quantum field theory provide us with many examples of observers in this sense. Note that an observer need have neither gender nor consciousness.

Local physics in generally covariant theories is either gauge variant or defined by a given classical background. There can be no gauge invariant quantum notion of locality in the quantum theory of gravity. But there is no reason why we cannot introduce local or quasi-local concepts which are tied to a particular reference frame/gauge choice. Indeed, all local physics in the real world is based on the existence of a regime where we can have large classical objects, which do not collapse

3 A more extensive discussion of string dualities and AdS/CFT can be found in the chapter by Horowitz and Polchinski and in the references cited there.

4 I am giving short shrift here to the idea of dS/CFT [9; 10; 11]. I do not believe this formalism actually makes sense, but it deserves more of a discussion than I have space for here.

into black holes. We cannot take a limit where such objects become infinitely large, without going off to the boundaries of an infinite space. So we might expect a local formulation of Quantum Gravity to have an ineluctably approximate nature. We will see that this is the case.

The clue to the nature of a local formulation of Quantum Gravity is the covariant entropy bound [12; 13; 14; 15; 16] for causal diamonds. A causal diamond in a Lorentzian space-time is the intersection of the interior of the backward light cone of a point P, with that of the forward light cone of a point Q in the causal past of P. The boundary of the causal diamond is a null surface, and the *holographic screen* of the diamond is the maximal area space-like $d-2$ surface on the boundary. The covariant entropy bound says that the entropy which flows through the future boundary of the diamond is bounded by one quarter of the area of this surface, in Planck units. For sufficiently small time-like separation between P and Q, this area is always finite, and its behavior as a function of the time-like separation is an indicator of the asymptotic structure of the space-time. In particular, for a future asymptotically de Sitter space-time, with a Big Bang as its origin, the area approaches a maximal value, equal to four times the Gibbons–Hawking de Sitter entropy.

In Quantum Mechanics, entropy is $-\mathrm{tr}\rho\ln\rho$, where ρ is the density matrix of the system. Infinite systems can have density matrices of finite entropy. However, this is usually a consequence of the existence of special operators, like a Hamiltonian: the archetypal case being a thermal density matrix. Fischler and the present author suggested that generally covariant theories have no such canonical operators (the problem of time) and that the only general assumption one could make about the density matrix implicit in the covariant entropy bound was that it is proportional to the unit matrix. In other words, *the number of quantum states associated with a small enough causal diamond is always finite.* This conjecture is in accord with our intuition about simple stationary systems in asymptotically flat and AdS space-times. The maximal entropy configurations localized within a given area are black holes. One cannot add more quantum states to a localized system without making both its mass and its area grow. In this case, the entropy bound counts the number of states.

A finite quantum system cannot contain machines which can make infinitely precise measurements on other parts of the system. Thus, the finite state space hypothesis implies an irreducible ambiguity in the physics of a local region of space-time. We cannot expect its (generally time dependent) Hamiltonian, nor any other operator to have a precise mathematical definition,[5] since there is no way, even in principle, to measure properties of the region with infinite accuracy.

[5] More properly: any precise mathematical definition will include elements which cannot be verified by measurement, and are thus gauge artifacts. There will be a class of Hamiltonians which give the same physics, within the ineluctable error associated with the finite size of the region.

Since the holographic screen is central to these ideas, it is natural to take its geometry for a given causal diamond to be the primordial dynamical variable of Quantum Gravity. Consider an infinitesimal area element on the screen. The Cartan–Penrose (CP) equation gives us a way to specify the holographic screen element associated with this area, in terms of a *pure spinor*. This is a commuting classical spinor satisfying

$$\bar{\psi}\gamma^{\mu}\psi\gamma_{\mu}\psi = 0.$$

The equation implies that $\bar{\psi}\gamma^{\mu}\psi$ is a null vector,n^{μ}, and the non-vanishing components of $\bar{\psi}\gamma^{\mu_1\cdots\mu_k}\psi$, for all k lie in a $d-2$ hyperplane transverse to n^{μ}. We will call $\tilde{n}^{\mu}$ the reflected null vector transverse to the same surface and satisfying $n^{\mu}\tilde{n}_{\mu} = -2$. Actually, the CP equation is homogeneous and the rescaling $\psi \to \lambda\psi$ is considered a gauge equivalence at the classical level. The classical CP equation specifies only the local orientation of the holoscreen and of null directions passing through it. The non-vanishing components of the pure spinor

$$S = \gamma^{\mu}n_{\mu}\gamma^{\nu}\tilde{n}_{\nu}\psi,$$

transform like an $SO(d-2)$ spinor under rotations transverse to n^{μ}.

Thus, the full conformal structure of the holoscreen is encoded in an element, $S_a(\sigma)$, of the spinor bundle over the holoscreen. S_a are the real components of this spinor, and represent the independent components of a covariant spinor satisfying the CP equation.

As might be expected from the Bekenstein–Hawking formula, the classical notion of area is only obtained after quantization of the spinor variables. If k specifies a *pixel* on the holoscreen, then we quantize $S_a(k)$ by postulating

$$[S_a(k), S_b(k)]_+ = \delta_{ab}.$$

This rule is $SO(d-2)$ invariant, and assigns a finite number of states to the pixel. It also breaks the projective invariance of the CP equation down to the Z_2, $S_a \to -S_a$. We will keep this as a gauge invariance of the formalism, which will turn out to be fermion parity, $(-1)^F$.

The S_a operators for independent pixels should commute, but we can use this gauge invariance to perform a Klein transformation and cast the full operator algebra of the holoscreen as

$$[S_a(k), S_b(l)]_+ = \delta_{ab}\delta_{kl}.$$

We have used the word pixel, and discrete labeling to anticipate the fact that the requirement of a finite number of states forces us to discretize the geometry of the holoscreen of a finite area causal diamond. The labels k, l run over a finite set of integers. Note that the operator algebra is invariant under a larger group of transformations than $SO(d-2)$. It is not reasonable to associate every linear combination of $S_a(n)$ with pixels, or small areas on the holoscreen. Rather, we should think of the discretization of the holoscreen topology to occur through the replacement of its algebra of functions with a finite dimensional algebra. Different linear combinations of the $S_a(k)$ correspond to operators associated with different bases of the finite function algebra. If the function algebra were abelian, we would have a standard geometrical discretization of the surface (e.g. a triangulation of a two surface) and we could choose a special basis for the algebra consisting of functions which were non-vanishing on only a single pixel.

At least in the case where we want to preserve exact continuous symmetries,[6] this will lead us into the simple case of non-commutative geometry, called *fuzzy geometry*. The function algebras for spherical holoscreens will be finite dimensional matrix algebras. Here the notion of a pixel is only an approximate one, similar to the localization of quantum Hall states within a Larmor radius of a point.

If we go to the particular basis where $S_a(n)$ represents a single pixel, we see a connection between this formalism and supersymmetry. The algebra of operators for a pixel is precisely the supersymmetry algebra for a massless supermultiplet with fixed momentum. We thus claim that the degrees of freedom specifying the orientation of a pixel on the holographic screen of a causal diamond are the states of a massless superparticle which emerges from (or enters into) the diamond through that pixel. In an asymptotically flat space, the limit of large causal diamonds should approach null infinity. The number of degrees of freedom becomes infinite, and in particular, we expect the pixel size to shrink to zero, relative to the area of the holoscreen. Thus, we should associate the pixel with a particular outgoing null direction $(1, \Omega)$ at null infinity. We will see later that the overall scale of the massless momentum can also be encoded in the algebra of operators. This observation is, I believe, an indication that the formalism automatically generates supersymmetric theories in asymptotically flat space. Indeed, when the SUSY algebra is large enough to force us to include the gravitino in the multiplet, we already know that the dynamics must be exactly supersymmetric. The conjecture that *all* asymptotically flat theories of Quantum Gravity must be Super Poincaré invariant is called Cosmological SUSY Breaking (CSB)[17].

11.2 Dynamical constraints

The time evolution operator describing dynamics for a given observer cannot be a gauge invariant operator. In a generally covariant theory there cannot be a canonical

6 If we are trying to model causal diamonds in a space-time with an asymptotic symmetry group, it is reasonable to restrict attention to diamonds which are invariant under as much of that group as possible. Remember that the choice of finite causal diamonds is a gauge choice.

definition of local evolution. This is the familiar problem of time, which has been discussed endlessly by would-be quantizers of gravity. However, time evolution is constrained by the requirement that two observers whose causal diamonds overlap, should have a consistent description of the overlap. A simple example of these constraints is the consistency of a single observer's description of two nested causal diamonds describing overlapping time intervals in its history.

For simplicity, we describe these constraints for a Big Bang space-time, beginning with the constraints for a given observer. In Big Bang space-times, it is convenient to start all causal diamonds on the Big Bang hypersurface. Successive diamonds contain exactly one pixel of extra information, that is, one extra copy of the S_a algebra. The Hilbert space $\mathcal{H}(k, \mathbf{x})$,[7] of the kth diamond has entropy $k \ln \dim \mathcal{K}$, where $\mathcal{K}$ is the irreducible representation of the spinor algebra. In this Hilbert space there exists a sequence of unitary time evolution operators $U_k(i) \quad i = 1 \ldots k$. One can choose to interpolate between these by some continuous evolution, but it is unlikely to lead to observable consequences. Note that the discretization of time implicit here is not uniform. In typical expanding FRW universes, entropy grows like t^{d-2}, so the time cut-off gets smaller as the causal diamond expands.

The consistency condition for a single observer is that

$$U_k(i) = U_p(i) \otimes V_{pk}(i)$$

whenever $k \geq p \geq i$. The operator $U_p(i)$ depends only on the S_a operators in $\mathcal{H}(i, \mathbf{x})$, and $V_{pk}(i)$ commutes with all those variables. That is, the evolution of the degrees of freedom accessible in the pth causal diamond is consistently described by the observer at all times after the pth time step. The new degrees of freedom added after the pth step do not interact with those inside the pth causal diamond until later times. This condition incorporates the notion of *particle horizon*, usually derived from micro-causality, into our holographic theory. The unitary operator $V_{pk}(i)$ represents the evolution of degrees of freedom outside the particle horizon at time p, which have come into the horizon by time k. The consistency condition guarantees that, in the history of a given observer, degrees of freedom evolve independently until "causality" allows them to interact.

This consistency condition is easy to satisfy, and certainly does not guarantee that the dynamics resembles space-time physics when the observer's horizon area is large. More of the structure of space-time can be built in to the theory by postulating, at time zero, a spatial lattice with the topology of Euclidean space. The topology, but not the geometry, of this lattice, is associated with that of spatial

[7] The additional label **x** in this notation refers to a family observers whose initial spatial position **x** lives on a lattice which specifies the topology of non-compact spatial slices. We will introduce this lattice below.

slices of space-time (recall always that our formalism is constructed in a fixed but arbitrary physical gauge). We postulate that this topology does not change with time.[8] We attach a sequence of observer Hilbert spaces and evolution operators to each point of the lattice. In the Big Bang cosmology, which we are using as an example, it is convenient to choose "equal area time slicing", where the dimension of the kth Hilbert space at each point $\mathbf{x}$ is the same.

For each pair of points on the lattice and each time, we define an *overlap Hilbert space*, $\mathcal{O}(\mathbf{x}, \mathbf{y}, t)$, which is a tensor factor of both $\mathcal{H}(t, \mathbf{x})$ and $\mathcal{H}(t, \mathbf{y})$, and require that the dynamics imposed on this tensor factor by the two individual observers is the same.[9] The idea behind this condition comes from a geometrical notion. The intersection of two causal diamonds is not a causal diamond, but it does contain a maximal causal diamond. The physics in that maximal diamond should not depend on whether it is observed at some later time by one or the other of the favored observers in our gauge. We insist that for nearest neighbor points on the lattice, at time t, the overlap Hilbert space has dimension $(\dim\mathcal{K})^{t-1}$. This defines the spacing of our lattice such that moving over one lattice spacing decreases the overlap by one unit of area. Note that this is the same as the time spacing and, like it, the spatial resolution goes to zero as the area grows.

Consider a point $\mathbf{x}$ on the lattice, and a path emanating from it, whose lattice distance from $\mathbf{x}$ increases monotonically. We require that the dimension of the overlap Hilbert space at fixed time, decrease monotonically along the path. We also insist that, as our notation suggests, the dimension of the overlap depends only on the endpoints of a path, not on the path itself.

These conditions are incredibly complicated, but seem to incorporate a minimal sort of framework for a unitary theory of Quantum Gravity. We have, in effect, constructed a quantum version of a coordinate system on a Lorentzian manifold, built from the trajectories of a group of time-like observers. The two rules that we use are equal area time-slicing, and space-time resolution defined in terms of a minimal difference in the size of the holographic screens at nearest neighbor space-time points. Given any Big Bang space-time whose expansion continues forever, we could set up such a coordinate system. The complicated consistency conditions

[8] This may disturb readers familiar with claims for topology change in string theory. Here we are discussing the topology of non-compact dimensions of space-time. I believe that the real lesson about topology change coming from the duality revolution is that the topology of compact manifolds is entirely encoded in quantum numbers, which are measured in scattering experiments in the non-compact dimensions. In different limits of the parameter space, the quantum numbers can be interpreted in terms of the topologies of different compact space-times. The way to incorporate that lesson into the present formalism is to complicate the algebra of operators for a given pixel, to incorporate information about the compactification. That is, the holographic screen in the non-compact dimensions, contains operators which describe the compactification. This is the way the compact factors X in $AdS_d \times X$ are described in AdS/CFT.

[9] It may be sufficient to require that the two sequences of evolution operators are related by a unitary transformation on $\mathcal{O}$.

we have postulated are the analog of the Dirac–Schwinger commutation relations for the Wheeler–DeWitt operator in canonical approaches to Quantum Gravity.

There is, at the present time, only one known solution to these conditions. If we insist that the time dependent Hamiltonian of a given observer, is chosen independently at each instant from a certain random ensemble [18],[10] and choose the overlap so that $\mathcal{O}(\mathbf{x}, \mathbf{y}, t) = \mathcal{H}(\mathbf{x}, t - D)$, where D is the minimal lattice path length between $\mathbf{x}$ and $\mathbf{y}$, then all the consistency conditions are satisfied, and all the scaling relations of the flat FRW space-time with equation of state $p = \rho$ are satisfied by this rather explicit quantum system. This is the correct quantum description of the *dense black hole fluid* that was postulated in [19; 20; 21]. By construction, it is a cosmology that saturates the covariant entropy bound at all times.

The heuristic picture of a dense black hole fluid is based on the idea that at any given time, all the degrees of freedom in a horizon volume have coalesced to form a single black hole. An instantaneous distribution of black holes at relative separations of order of their Schwarzschild radii, have the energy/entropy relation of a $p = \rho$ fluid. If they continually coalesce to make larger, horizon filling black holes, always separated by about a horizon scale, then we indeed have an equilibrium system with equation of state $p = \rho$. The random Hamiltonian model described in the previous paragraph is an explicit quantum system which has many of the properties derived from this heuristic picture.

The concept of an observer does not make sense in the $p = \rho$ background, because all degrees of freedom in any causal diamond are always in intense interaction, and there are no isolated sub-systems with a large number of semi-classical observables. The idea of holographic cosmology is that the universe we live in began as close as possible to the $p = \rho$ system, consistent with the *observerphilic principle*: it is the maximally entropic solution of the consistency conditions we have outlined, which does not quickly collapse back into the $p = \rho$ phase, and allows for the existence of what we have called observers over very long time periods. Of course, we might want to strengthen our demands, and insist on some sort of criterion that guaranteed the existence of intelligent living organisms – observers in the more colloquial sense. Such restrictions are fine as long as we do not make claims that go beyond our abilities to actually do the calculations involved in guaranteeing or ruling out life. It's also obvious that we want to make the weakest assumptions of this kind that give correct answers. It may be that the $SU(1, 2, 3)$ gauge group of the standard model is only explainable because it leads to the type

[10] The existence and nature of this ensemble is based on the properties of quadratic fermionic systems with random one body Hamiltonians. In this way, the choice of holographic pixels as the fundamental variables of Quantum Gravity, enters directly into the formulation of holographic cosmology.

of chemistry and biology we know and love, but it would surely be more satisfying to derive it from a less restrictive assumption.

We do not even have a precise mathematical definition of our much less restrictive observerphilic principle. However, if we make some assumptions, we can see some of its consequences. Assume that the observerphilic part of the universe will eventually evolve to be an FRW space-time. The $p = \rho$ universe is infinite. Obviously, in entropic terms, it is preferable for the low entropy, observerphilic part of the universe to involve as few degrees of freedom of the full system as possible. In particular, a finite number is infinitely more probable than an infinite number. This principle thus predicts that the cosmology of the observerphilic part of the universe should have causal diamonds with bounded area. There are two ways to achieve this: the observerphilic part of the universe could end in a Big Crunch, or asymptote to dS space. The life-time of observers in an asymptotically dS space of given horizon size, is exponentially longer than in a Big Crunch space-time with the same size maximal causal diamond.[11] Thus, if one looks for an observer in a $p = \rho$ universe sprinkled with observerphilic regions of various sizes and types, one is more likely to find it in an asymptotically dS region. Note that, unlike the anthropic principle, the observerphilic principle lends itself to simple calculations of probabilities, and makes no assumptions about particular biological structures, or the nature of low energy particle physics, except that it is well described by quantum field theory. It predicts that an observerphile will want to search for the objects of his or her affection in an asymptotic de Sitter universe, with the maximal value of the cosmological constant consistent with whatever version of observers he/she wants to insist on.

It is amusing that the necessity that a locally FRW region should be asymptotically dS can also be derived directly from General Relativity [19; 20; 21], by assuming that there is, for all times, a consistent interface between a particle horizon sized bubble of normal universe, and the $p = \rho$ background. In all cases where the particle horizon expands indefinitely, the parallel components of the Israel junction condition show that we can only make an interface between the two FRW systems if the coordinate volume of the less stiff fluid shrinks with time. By contrast, we can match the future cosmological horizon of a single observer in an asymptotically dS space, to a marginally trapped surface embedded in the $p = \rho$ background. Black holes in the $p = \rho$ background cannot decay, because the background space-time already saturates the maximum entropy bound. I view this as evidence that a complete quantum theory of a $p = \rho$ background sprinkled

[11] In an asymptotically dS space, observers are destroyed by thermal nucleation of bursts of radiation, or black holes, at their position. The probability of such processes vanishes exponentially with the dS radius.

with observerphilic defects, will derive asymptotic de Sitter spaces as the only kind of stable defect.

Our maximum entropy principle can also explain why empty de Sitter space (or e.g. some of the expanding portion of it) is not the most probable state of the universe. From the point of view of small causal diamonds near the Big Bang, the empty dS universe is not the most general state which can evolve into the empty dS universe in the asymptotic future. In [19; 20; 21] it was conjectured that instead the most probable state looked like the dense, $p = \rho$, fluid, over most of the coordinate volume that would eventually evolve into the static patch of dS space. The normal region of the universe is initially a sort of percolation cluster of linked regions where the initial black hole size was insufficiently large to merge with black holes in neighboring particle horizon volumes. Instead, these black holes rapidly decay into radiation. On equal area time slices, the radiation dominated regions grow in physical volume, relative to the $p = \rho$ regions, and the universe eventually undergoes a phase transition to a point where it is best described as a non-relativistic gas of black holes (the former $p = \rho$ regions) in a matter dominated background created by their average density. The whole construction can be embedded in an infinite $p = \rho$ background, if the normal part of the universe is asymptotically dS.

The distribution of matter in this universe is forced to be quite uniform. If it were not, black hole collisions would rapidly form large black holes and the universe would relax back into the uniform $p = \rho$ phase of constantly merging black holes. Unfortunately, we do not understand the phase transition between dense and dilute black hole fluids very well, and it has so far been impossible to obtain quantitative information about the matter distribution or the size of the particle horizon at the time of the transition. This is the biggest unsolved problem in holographic cosmology. It determines almost all of the parameters which go into the estimate of the observational consequences of the model. These include the question of whether the observed CMB fluctuations have their origin during an inflationary era[12] or during the $p = \rho$ era, and the overall amplitude of the latter fluctuations.

Furthermore, a better understanding of this phase transition may provide an explanation of the origin of the thermodynamic arrow of time, which Penrose [22] has emphasized as a key unsolved problem of modern cosmology.[13] The holographic formalism has a built in arrow of time, which comes from the way it enforces the existence of particle horizons. However, this is not obviously linked to

[12] Holographic cosmology requires a brief period of inflation, which may be as short as ten e-foldings, to explain the correlations in the CMB over our entire particle horizon. However, in principle it provides alternative explanations for all of the other cosmological conundra solved by inflation and a possible alternative origin for the CMB fluctuations. The observational signature of the fluctuations generated in the $p = \rho$ era is an exactly scale invariant spectrum, with sharp cutoffs in both the UV and the IR.

[13] Penrose rejects the claim that standard inflationary arguments solve this problem, and I agree with him.

the idea of a low entropy beginning of the universe, and the uniform $p = \rho$ solution of holographic cosmology provides a counterexample. In that system, there is an arrow of time, but the system maximizes the entropy available to it at all times. As a consequence, it does not contain local observers, but even if it did, a local observer would not perceive a thermodynamic arrow of time.

However, we have seen at an intuitive level that the requirement that a normal region of the universe exists at all, and does not immediately subside into the dense black hole fluid, puts strong constraints on fluctuations in the matter density. Perhaps when we understand these constraints in a quantitative manner, they will explain the low entropy of the early universe.

11.3 Quantum theory of de Sitter space

The cosmology of the previous section leads one to study the idealized problem of de Sitter space-time as the ultimate endpoint, toward which the universe (or at least the only part of it we will ever observe) is tending. The initial approach taken by string theorists interested in particle phenomenology was to look for models of string theory in asymptotically flat space. Arguments based on the locality of quantum field theory (and the presumption that a similarly local formulation of string theory should exist) suggested that such a theory should be adequate for understanding the masses and interactions of particles below the Planck energy.

This program has run into difficulty because no one has found an asymptotically flat form of string theory, which is not exactly supersymmetric. All attempts to break supersymmetry lead to, *at the very least*, a breakdown of string perturbation theory, and clear indications that the geometry of the resulting space-time is not asymptotically flat.

I believe that the breaking of supersymmetry in the real world is intimately connected with the fact that the real world is not asymptotically flat, but instead asymptotically de Sitter [17]. The phenomenology of particle physics should thus be derivable from a theory of eternal de Sitter space. The holographic entropy bound, in the strong form conjectured by Fischler and the present author, indicates that this is a quantum theory with a finite number of states, and cannot fit directly into the existing formalism of string theory.[14] It also implies that if such a theory exists then dS space is stable.

The general formalism described above indicates that the variables for describing de Sitter space should be fermions which are a section of the spinor bundle over a pixelated cosmological horizon. The natural $SU(2)$ invariant pixelation of the S^2

[14] Except by finding a subset of states in an asymptotically flat or anti-de Sitter string theory, which is approximately described by de Sitter space, and decouples from the rest of the system.

de Sitter horizon is the fuzzy sphere. The spinor bundle over the fuzzy sphere is the set of complex $N \times N + 1$ matrices, transforming in the $[N] \otimes [N+1] = [2] \oplus \cdots [2N]$ dimensional representation of $SU(2)$. We postulate the invariant commutation relations

$$[\psi_i^A, (\psi^\dagger)_B^j] = \delta_i^j \delta_B^A.$$

The logarithm of the dimension of the Hilbert space of this system is $N(N+1)$ $\ln 2 \to \pi(RM_P)^2$, which indicates that we should identify $N\sqrt{\ln 2} = \sqrt{\pi} RM_P$ in the large N limit.

To get a better idea of what the Hamiltonian for dS space should look like, we use the semi-classical results of Gibbons and Hawking [23], and work that followed it, as experimental data. The natural Hamiltonian, H, should be the generator of static translations for a time-like geodesic observer. The density matrix is thermal for this system, with inverse temperature $\beta_{\mathrm{dS}} = 2\pi R$. Note that, at first glance, this seems to contradict the assumption of Banks and Fischler, that the density matrix is proportional to the unit matrix. Indeed, finite entropy for a thermal density matrix does not imply a finite number of states, unless the Hamiltonian is bounded from above.

That the Hamiltonian is so bounded follows from the fact that black holes in de Sitter space have a maximum mass, the Nariai mass [24]. The Schwarzschild–de Sitter metric is

$$ds^2 = (1 - \frac{2M}{r} - \frac{r^2}{R^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} - \frac{r^2}{R^2})} + r^2 d\Omega^2.$$

The equations for cosmological and black hole horizons, $R_\pm$ are

$$(r - R_+)(r - R_-)(r + R_+ + R_-) = 0,$$

$$R_+ R_-(R_+ + R_-) = 2MR^2,$$

$$R^2 = (R_+ + R_-)^2 - R_+ R_-.$$

These have a maximal solution when $R_+ = R_- = \sqrt{\frac{2}{3}} R$. Note that as the black hole mass is increased, its entropy increases, but the total entropy decreases. We interpret this as saying that states with entropy localized along the world line of the static observer are states where the system is frozen into a special configuration. The generic state of the system is the thermal de Sitter vacuum ensemble.

In fact, the Nariai estimate is a wild overestimate of the maximal eigenvalue of the static Hamiltonian. This follows from the fact that black holes decay into the vacuum. Indeed, even elementary charged particles decay in de Sitter space. The solution of Maxwell's equations corresponding to an electron in dS space has a compensating positive charge density spread over the horizon of the electron's causal diamond. There is a small but finite quantum tunneling amplitude for this charge to materialize as a positron, and annihilate the electron. The decay products will move out through the de Sitter horizon and the state will become identical to the vacuum ensemble. Every localized system in de Sitter space has a finite life-time.

Thus, all of the eigenstates of the static Hamiltonian must be states of the vacuum ensemble. Classically these all have zero energy. Quantum mechanically we envision them as being spread between 0 and something of order the dS temperature, with a density $e^{-\pi(RM_P)^2}$. A random Hamiltonian with these spectral bounds, acting on a generic initial state, will produce a state where correlation functions of simple operators are practically indistinguishable from thermal correlation functions at the de Sitter temperature. In other words, the hypothesized spectrum could explain the thermal nature of dS space.

There is a further piece of semi-classical evidence for this picture of the static spectrum. The Coleman–DeLucia instanton [25] for transitions between two dS spaces with different radii, indicates that the ratio of transition probabilities is

$$\frac{P_{1\to 2}}{P_{2\to 1}} \sim e^{-\Delta S}.$$

This is in accord with the principle of detailed balance, if the free energy of both these systems is dominated by their entropy. The condition for this is that the overwhelming majority of states have energies below the de Sitter temperature. Note that in this case, the thermal density matrix is essentially the unit matrix as $R \to \infty$, and the Gibbons–Hawking ansatz agrees qualitatively with that of Banks and Fischler.

There are still two peculiar points to be understood. If the static energy is bounded by the dS temperature, then what are the energies we talk about in everyday life? In addition to this, the semi-classical evidence that the vacuum of dS space is thermal seems to suggest a thermal ensemble with precisely these everyday energies in the exponent. The fact that, in the classical limit of $R \to \infty$, the density matrix is actually proportional to the unit matrix, suggests an answer to both questions. Imagine that there is an operator P_0, whose eigenspaces all have the form

$$|p_0\rangle \otimes |v_{p_0}\rangle,$$

where $|v_{p_0}\rangle$ is any vector in a certain tensor factor of the Hilbert space, associated with the eigenvalue p_0. Suppose further that the dimension of the tensor factor is $e^{-2\pi R p_0}$. Then the probability of finding a given p_0 eigenvalue, with a density matrix $\rho \sim 1$, will be precisely a Boltzmann factor of p_0.

There is a class of semi-classical eigenspaces of p_0 for which we can check both the entropy and the energy. These are black holes, if we identify the mass parameter of the dS–Schwarzschild metric with the P_0 eigenvalue. Of course, extant quantum field theory calculations which demonstrate that we have a thermal ensemble of ordinary energies in de Sitter space, refer only to energies much smaller than the maximal black hole mass. We are led to conjecture the above relation between the entropy deficit (relative to the vacuum) of a p_0 eigenspace, and the eigenvalue, only to leading order in the ratio of the black hole mass to the Nariai mass. Remarkably, this prediction is valid [26]!

It is easy to construct a Hamiltonian out of the fermionic pixel operators introduced above, which reproduces the spectrum of black holes in dS space. One works in the approximation where the vacuum eigenstates are all exactly degenerate, so that black holes are stable. The vacuum density matrix is just the unit matrix. Black hole states are simply states in which we break the fermionic matrix ψ_i^A into four blocks, and insist that $\psi_m^D|BH\rangle = 0$, for matrix elements in the lower off diagonal block. A clumsy but explicit formula for the Hamiltonian P^0 can be constructed [26].

Some insight into the Hamiltonian P^0 is gained by remembering that global symmetry generators in General Relativity are defined on space-like or null boundaries. The way in which dS space converges to Minkowski space is that the causal diamond of a single observer approaches the full Minkowski geometry. The future and past cosmological horizons of the observer converge to future and past infinity in asymptotically flat space. Our basic proposal for the definition of observables in de Sitter space [27] is that there is an approximate S-matrix, S_R which, as $R \to \infty$, approaches the S-matrix of asymptotically flat space. S_R refers only to localizable processes in a single horizon volume. As in any such limiting situation, we may expect that S_R is not unique, and it is important to understand what aspects of it are universal for large R. We will argue later that for scattering processes whose center of mass energy is fixed as $R \to \infty$, the non-universal features may fall off like $e^{-(RM_P)^{3/2}}$.

The geometry of the future cosmological horizon is the $v \to 0$ limit of:

$$ds^2 = R^2(-du dv + d\Omega^2),$$

and the static Hamiltonian is associated with the Killing vector

$$(u\partial_u - v\partial_v).$$

Here, $d\Omega^2$ is the round metric on the 2-sphere. By contrast, future infinity in asymptotically flat space, is the $v \to 0$ limit of

$$ds^2 = \frac{-du dv + d\Omega^2}{v^2}.$$

Observables are insensitive to the infinite volume of this space, because they are covariant under the conformal group $SO(1, 3)$, which is identified with the Lorentz group. The Poincaré Hamiltonian P_0 is associated with the Killing vector ∂_u.

Our proposal is that this Poincaré Hamiltonian is the generator with the same symbol that we discussed above. This is obviously the right identification for black holes of size much smaller than the Nariai hole. We can introduce this generator in dS space, where it is no longer a Killing vector. A cartoon of the algebra of these two generators is

$$\begin{aligned} H &= \frac{1}{R}(u\partial_u - v\partial_v), \\ P_0 &= R\partial_u, \\ [H, P_0] &= \frac{1}{R}P_0. \end{aligned}$$

This incorporates our knowledge of the physical bounds on the spectra of these two generators, if we impose an order 1 cut off on the spectra of the boost and partial derivative operators. It also gives us a hint at why P_0 eigenstates, with eigenvalue small compared to R, are approximately stable under the time evolution defined by H. P_0 is an approximately conserved quantum number, which resolves part of the huge degeneracy of the static vacuum ensemble. The challenge of building a theory of de Sitter space consists in constructing models for P_0, H, and a system of equations determining S_R, which are compatible with the above remarks.

As a first step, we should try to understand how to construct "particle states" in dS space. We first analyze this in terms of low energy effective field theory. If M is the cut–off scale, then the entropy of a field theory in a given horizon volume is

$$S_{\rm FT} \sim (MR)^3.$$

However, a typical state in this ensemble will have energy

$$M^4 R^3.$$

The Schwarzschild radius corresponding to this is

$$M^4 R^3 / M_{\rm P}^2,$$

and the condition for the validity of the field theory approximation is

$$M^4 R^3 < M_{\rm P}^2 R,$$

or

$$M < \sqrt{\frac{M_{\rm P}}{R}}.$$

Thus, there are of order $(RM_{\rm P})^{3/2}$ field theoretic degrees of freedom in a horizon volume. If the field theory has a particle description this corresponds to of order $(RM_{\rm P})^{3/2}$ particles.

This description gets the counting right, but conflicts with the experimental fact that we can excite momenta much higher than this cut off in the laboratory. We will see that the fermionic pixel variables suggest a more flexible way for the particle interpretation to emerge from the formalism.

We should also note that, although we will continue to concentrate on the description appropriate to a given causal diamond, this estimate allows us to understand how the global coordinate description of dS space might emerge in the large R limit. The total entropy of dS space is of order $(RM_{\rm P})^2$. This means that there are enough degrees of freedom to account for $(RM_{\rm P})^{1/2}$ commuting copies of the field theory variables allowed in a given horizon volume. In global coordinates, at early and late times, the number of independent horizon volumes seems to grow without bound. However, if we imagine filling each of those volumes with a generic field theoretic state, then the extrapolation into either the past or the future leads to a space-like singularity before the minimal volume sphere is reached. We interpret this as saying that the general field theoretic state in very late or very early time dS space, does not correspond to a state in the quantum theory of dS space. Only when the absolute value of the global time is small enough that there are at most $(RM_{\rm P})^{1/2}$ horizon volumes, does a generic field theory state (with cut-off $\sqrt{\frac{M_{\rm P}}{R}}$) correspond to a state in Quantum Gravity. At later times, most horizon volumes must be empty. As $RM_{\rm P} \to \infty$, these restrictions become less important. The conventional formalism of quantum field theory in dS space is the singular limit $RM_{\rm P} \to \infty$ with R kept finite in units of particle masses. If particle masses in Planck units do not approach constant values at $RM_{\rm P} \to \infty$, then this limit does not make any sense. In particular, if SUSY is restored in this limit, the splittings in supermultiplets do not approach constant values.

Here is the way to reproduce the field theory state counting in terms of fermionic pixel variables. Write the fermionic matrix in terms of blocks of size $M \times M + k$ with $k = 0, 1$ and $M \sim \sqrt{N}$. The states in a given horizon volume are associated with fermionic variables along a block diagonal, as follows

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & M \\ M & 1 & 2 & \ldots & M-1 \\ M-1 & M & 1 & \ldots & M-2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 3 & 4 & 5 & \ldots & 2 \\ 2 & 3 & 4 & \ldots & 1 \end{pmatrix}.$$

Each integer from 1 to M denotes states in one of M horizon volumes. The states in a given block correspond to "single particle states", and the multiple blocks with the same label are multiple particles in a single horizon volume, much as in Matrix Theory [28]. Work in progress [29] will show that the single particle states in this description indeed correspond to states of a massless $N = 1$ supermultiplet described at null infinity. The sphere at null infinity is fuzzy [30] for finite R, corresponding to the cut off on particle momenta in the field theory discussion. As in Matrix Theory, the size of the block representing a single particle measures its longitudinal momentum along a particular null direction, while the state of the fermionic variables corresponding to the block determines the spin and angular wave functions of the particle on the sphere at null infinity. The Matrix Theory formalism was contructed on a light front, and all longitudinal momenta were aligned in the same direction. Here, each particle carries its own longitudinal direction, which identifies the pixel via which it enters or exits the holographic screen.

This formalism is more flexible than cut-off field theory. It can describe particles of large momentum in a given horizon volume by making one of the blocks in a given horizon much larger than M. The price for this is paid by having fewer particles, or by forcing the other particles to have low momentum.

One of the intriguing features of this conjecture about the way in which super-Poincaré invariant particle physics will emerge from the formulation of the quantum theory of dS space in terms of fermionic pixels, is the natural appearance of the variable $M = \sqrt{N}$. This is the variable which controls the fuzziness of the geometry of the momentum space spheres of individual particles. We hope to show [29] that the theory becomes super-Poincaré invariant in the limit $M \to \infty$, with corrections of order $\frac{1}{M}$. In particular, this would imply that

$$[P_0, Q_\alpha] \sim o(\frac{1}{M}).$$

In spontaneously broken low energy SUGRA, the superpartner of a given particle is a two particle state with an extra zero-momentum gravitino. The splitting between these states is just the gravitino mass. Thus, taken at face value, the above equation says that the gravitino mass scales like $\frac{1}{M} \sim \Lambda^{1/4}$. This is the scaling relation I postulated under the name Cosmological SUSY Breaking [17]. It predicts superpartner masses in the TeV energy range and has numerous implications for low energy physics and dark matter.

11.4 Summary

String theory has provided us with extensive evidence for the existence of models of Quantum Gravity in asymptotically flat and AdS space-times. In the AdS case it has given us a construction of these models as quantum field theories on the conformal boundary of space-time. Remarkably, all the asymptotically flat models are exactly supersymmetric, and all the well understood AdS models with curvature small enough for the SUGRA approximation to be valid have SUSY restored asymptotically on the boundary of space-time. All these models are holographic in that they describe space-time in terms of variables defined on a holographic screen at infinity.

More realistic models of Quantum Gravity, which take into account cosmology, need a more flexible and local version of holography. General arguments show that the underlying variables of such a local formulation cannot be gauge invariant. I described a proposal for a general quantum space-time as a network of Hilbert spaces and evolution operators. Each Hilbert space was to be thought of as the representation of physics in a particular causal diamond in space-time. The holographic principle is implemented by relating the dimension of the Hilbert space to the area of the holographic screen of the causal diamond. This was made more precise by describing the operator algebra in terms of operators representing pixels of the holographic screen. The Cartan–Penrose equation leads to a description of these variables as elements of the $SO(d-2)$ spinor bundle over the screen, where d is the space-time dimension encoded in the topology of the network of Hilbert spaces.

We saw that quantization of these spinor variables identified the states of a pixel as the states of a massless super-particle. Compact dimensions of space could be incorporated by enlarging the algebra of spinor operators at each pixel to include central charges corresponding to Kaluza–Klein momenta, or brane wrapping numbers on topological cycles of the internal manifold. This is precisely the data about compact geometry that is invariant under topology changing string dualities. Thus, the holographic formulation provides a rationale for not just gravity, but supergravity, as the natural outcome of quantum geometry.

The holographic formulation of Quantum Gravity provided an explicit model of a quantum system corresponding to a classical cosmology: a flat FRW universe with equation of state $p = \rho$. This universe saturates the holographic entropy bound at all times. It has a heuristic description as a dense black hole fluid, and does not resemble our universe. An heuristic description of our own universe as a collection of defects in the $p = \rho$ background, maximizing the entropy subject to the constraints of the existence of observers (in a fairly well-defined mathematical sense) seems to account for many facts about cosmology. It also leads to the prediction that the universe is future asymptotically de Sitter, with a de Sitter radius as small as permitted by environmental constraints like the existence of galaxies.

I also described the beginnings of a holographic theory of eternal de Sitter space, which might be the appropriate arena for discussing non-cosmological particle physics. I proposed tentative identifications of black hole, and particle states in terms of the spinor variables on the cosmological horizon. The geometry of the horizon was a fuzzy sphere of the de Sitter radius, but the geometry of the momentum space of a single particle had a more severe cut off, scaling like the square root of the de Sitter radius. This suggests that if the infinite radius limit is super-Poincaré invariant, the gravitino mass will scale like $\Lambda^{1/4}$.

The holographic approach to the quantum theory of gravity incorporates insights from string theory about the importance of supersymmetry and the holographic principle to the definition of the quantum generalization of a Lorentzian geometry. It has not yet made explicit contact with string theory. The route toward such contact branches into two: kinematics and dynamics. The first step is to show how the kinematical variables $S_a(n)$ of the causal diamond approach, converge to the natural asymptotic variables of the boundary description of string theory: Fock spaces of scattering states for asymptotically flat space-times, and conformal fields for asymptotically AdS space times. The second is to relate the boundary dynamics to the consistency condition of the causal diamond approach. In the AdS case the problem is essentially kinematic. Once we have established that the boundary variables satisfy the locality axiom of field theory, the dynamics must be that of a CFT. The relation to the causal diamond approach will help us to understand how to describe local processes, and the inevitable gauge dependence of any such description, in AdS/CFT.

For the asymptotically flat case we only have a non-perturbative dynamical principle for those space-times in which Matrix Theory applies. Even there one must take a difficult large N limit to establish the symmetry properties of the boundary theory. It would be more attractive to have a non-perturbative equation which determined the super-Poincaré invariant S-matrix directly. In ancient times it was shown that unitarity, holomorphy, and some information about high energy behavior, completely determined the scattering matrix in perturbation theory, up to local counterterms. Even in the maximally symmetric case of eleven dimensions, these principles do not seem to uniquely determine the counterterms. They also suffer from a lack of elegance and a vagueness of definition. One can hope that the consistency condition of the causal diamond formulation can lead to an elegant and precise form of the holomorphy of the S-matrix, which will completely determine it. The first, kinematic, step of relating the causal diamond formalism to Fock space will be discussed in [31].

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12

String field theory

W. TAYLOR

12.1 Introduction

In the early days of the subject, string theory was understood only as a perturbative theory. The theory arose from the study of S-matrices and was conceived of as a new class of theory describing perturbative interactions of massless particles including the gravitational quanta, as well as an infinite family of massive particles associated with excited string states. In string theory, instead of the one-dimensional world line of a pointlike particle tracing out a path through space-time, a two-dimensional surface describes the trajectory of an oscillating loop of string, which appears pointlike only to an observer much larger than the string.

As the theory developed further, the need for a nonperturbative description of the theory became clear. The M(atrix) model of M-theory, and the AdS/CFT correspondence, each of which is reviewed in another chapter of this volume, are nonperturbative descriptions of string theory in space-time backgrounds with fixed asymptotic forms. These approaches to string theory give true nonperturbative formulations of the theory, which fulfill in some sense one of the primary theoretical goals of string theory: the formulation of a nonperturbative theory of Quantum Gravity.

There are a number of questions, however, which cannot – even in principle – be answered using perturbative methods or the nonperturbative M(atrix) and AdS/CFT descriptions. Recent experimental evidence points strongly to the conclusion that the space-time in which we live has a small but nonzero positive cosmological constant. None of the existing formulations of string theory can be used to describe physics in such a space-time, however, existing tools in string theory and field theory suggest that string theory has a large number of metastable local minima with positive cosmological constants. The term "string landscape" (see, e.g., [35]) is often used to describe the space of string theory configurations which includes all these metastable local minima. We currently have

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti.
Published by Cambridge University Press.

no tools to rigorously define this space of string theory configurations, however, or to understand the dynamics of string theory in a cosmological context – a formalism capable of describing the string landscape would presumably need to be a background-independent formulation of the theory such as string field theory.

The traditional perturbative approach to string theory involves constructing a field theory on the two-dimensional string "world-sheet" Σ, which is mapped into the "target" space-time through a function $X : \Sigma \to$ space-time; this function is locally described by a set of coordinates X^μ. The theory on the world-sheet is quantized, and the excitations of the resulting string become associated with massless and massive particles moving in space-time. The states of the string live in a Fock space similar to the state space of a quantized simple harmonic oscillator. The ground state of the string at momentum p, denoted $|p\rangle$, is associated with a space-time scalar particle[1] of momentum p. There are two kinds of raising operators acting on the single-string Fock space, analogous to the raising operator $a^\dagger$ which adds a unit of energy to a simple harmonic oscillator. The operators $\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger$ and $\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^\dagger$ each add a unit of excitation to the nth oscillation modes of the μ coordinate of the string. There are two operators for each n because there are two such oscillation modes, which can be thought of as sin and cos modes or as right- and left-moving modes. The excited states of the string correspond to different particles in space-time. For example, the state

$$(\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1} + \alpha^\nu_{-1}\tilde{\alpha}^\mu_{-1})|p\rangle \tag{12.1}$$

corresponds to a symmetric spin 2 particle of momentum p. These states satisfy a physical state condition $p^2 = 0$, so that this excitation state of the string can be associated with a quantum of the gravitational field – a graviton. Acting with more raising operators on the string state produces a series of more and more highly excited strings corresponding to a tower of massive particle states in space-time. In perturbative string theory, interactions between the massless and massive particles of the theory are computed by calculating correlation functions on the string world-sheet using techniques of two-dimensional conformal field theory.

The basic idea of string field theory is to reformulate string theory in the target space-time, rather than on the world sheet, as an off-shell theory of the infinite number of fields associated with the states in the string Fock space. The degrees of freedom in string field theory are encoded in a "string field", which can be thought of in several equivalent ways. Conceptually, the simplest way to think of a string field is as a functional $\Psi[X(\sigma)]$, which associates a complex number with every

[1] Actually, this ground state is associated with a scalar tachyon field describing a particle with negative mass squared $m^2 < 0$. The presence of such a tachyon indicates that the vacuum around which the theory is being expanded is unstable. This tachyon is removed from the spectrum when we consider supersymmetric string theory.

possible configuration $X(\sigma)$ of a one-dimensional string with coordinate σ. This is the natural generalization to a string of the standard quantum mechanical wave function $\psi(x)$, which associates a complex number with every possible position x of a pointlike particle in space. Mathematically, however, dealing directly with functionals like $\Psi[X(\sigma)]$ is difficult and awkward. In most cases it is more convenient to use a Fock space representation of the the string field. Just as a wave function $\psi(x) \in \mathcal{L}^2(\mathbf{R})$ for a single particle can be represented in a basis of harmonic oscillator eigenstates $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ through $\psi(x) \to \sum_n c_n|n\rangle$, the string field $\Psi[X(\sigma)]$ representing a string moving in D space-time dimensions can be equivalently represented in the string Fock space through

$$\Psi = \int d^D p[\phi(p)|p\rangle + g_{\mu\nu}(p)(\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1} + \alpha^\nu_{-1}\tilde{\alpha}^\mu_{-1})|p\rangle + \cdots] \tag{12.2}$$

where the sum includes contributions from the infinite tower of massive string states. Because in this case the states carry a continuously varying momentum, the coefficient of each state, which was just a constant c_n in the case of the harmonic oscillator, becomes a field in space-time written in the Fourier representation. Thus, we see that the string field contains within it an infinite family of space-time fields, including the scalar field ϕ, the graviton field (metric) $g_{\mu\nu}$, and an infinite family of massive fields.

String field theory is defined by giving an action functional $\mathcal{L}(\Psi)$ depending on the string field. When written in terms of the individual component fields $\phi(x)$, $g_{\mu\nu}(x), \ldots$, this then gives a fairly conventional-looking action for a quantum field theory, although the number of fields is infinite and the interactions may contain higher derivatives and appear nonlocal. To be a consistent description of a known perturbative string theory, the action must be chosen carefully so that the perturbative string field theory diagrams precisely reproduce the string amplitudes computed from the perturbative string theory. This requirement puts a highly constraining algebraic structure on the theory [42; 12; 13]. Generally, it is necessary to include an infinite series of terms in the action to meet this requirement, although in the case of the bosonic open string Witten has given an elegant formulation of string field theory which includes only cubic interaction terms for the string field Ψ. We will describe this simplest and best-understood string field theory in the next section.

Once a string field theory has been defined through an action, the next question is whether it can be used as a tool to usefully compute new results in string theory which extend beyond those accessible to the perturbative formulation of the theory. Although work on string field theory began over 30 years ago, until 7 years ago there was no clear example of a calculation in which string field theory gave results which go beyond perturbation theory. In 1999, however, Ashoke Sen [30] made an insightful conjecture that two distinct open string backgrounds, one with a space filling D-brane and one without, could be explicitly realized as different solutions of the same open string field theory. Subsequent work on this conjecture has brought new impetus to the study of string field theory, and has conclusively demonstrated the nonperturbative background-independence of the theory. Despite these advances, however, there are still enormous technical challenges for the theory. The theory is not completely well-defined even at the classical level, and a full definition of the quantum theory seems very difficult. Analytic calculations are difficult and involve subtle issues of limits and divergences, and numerical computations, while possible in many cases, are cumbersome and often difficult to interpret. Even for the simpler open string field theory many conceptual challenges exist, and although there has been recent progress on formulating closed string field theories, using these theories to describe the landscape of string vacua is still well beyond our technical capacity.

In the remainder of this paper we describe in some further detail the state of knowledge in this subject. In section 12.2 we give a somewhat more explicit description of Witten's open bosonic string field theory; we describe the recent work in which this theory was shown to describe distinct string backgrounds, and we discuss some outstanding issues for this theory. In section 12.3 we review the state of the art in closed string field theory. Section 12.4 contains a summary of successes and challenges for this formulation of string theory and some speculation about possible future directions for this area of research

12.2 Open string field theory (OSFT)

We now introduce the simplest covariant string field theory. A very simple cubic form for the off-shell open bosonic string field theory action was proposed by Witten [38]. In subsection 12.2.1 we briefly summarize the string field theory described by this action. In subsection 12.2.2 we review the recent work applying this theory to the study of Sen's conjecture and discuss the progress which has been made. For a more detailed review of this subject see [37]. In subsection 12.2.3 we discuss some problems and outstanding issues for open string field theory.

It is useful to recall here the difference between open and closed strings. A closed string forms a one-dimensional loop. Parameterizing the string by $\sigma \in [0, 1]$ we form a closed string by identifying the endpoints $\sigma = 0, \sigma = 1$. Because fields on a closed string take periodic boundary conditions, there are separate right- and left-moving modes. This is what allows us to construct a graviton state from a closed string as in (12.1). An open string, on the other hand, has Dirichlet ($X = 0$) or Neumann ($\partial_\sigma X = 0$) boundary conditions at the endpoints, and therefore only has one set of oscillation modes, which are associated with a single family of raising operators α^μ_{-n}. For the bosonic open string, the string field can then be expanded as

$$\Psi = \int d^{26}p \, \left[\varphi(p) \, |p\rangle + A_\mu(p) \, \alpha^\mu_{-1} |p\rangle + \cdots \right] . \tag{12.3}$$

The leading fields in this expansion are a space-time tachyon field $\varphi(p)$ and a massless space-time vector field $A_\mu(p)$.

12.2.1 Witten's cubic OSFT action

The action proposed by Witten for the open bosonic string field theory takes the simple cubic form

$$S = -\frac{1}{2} \int \Psi \star Q\Psi - \frac{g}{3} \int \Psi \star \Psi \star \Psi \, . \tag{12.4}$$

In this action, g is the (open) string coupling constant. The field Ψ is the open string field. Abstractly, this field can be considered to take values in an algebra $\mathcal{A}$. Associated with the algebra $\mathcal{A}$ there is a star product

$$\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{12.5}$$

The algebra $\mathcal{A}$ is graded, such that the open string field has degree $G = 1$, and the degree G is additive under the star product ($G_{\Psi \star \Phi} = G_\Psi + G_\Phi$). There is also an operator

$$Q : \mathcal{A} \to \mathcal{A}, \tag{12.6}$$

called the BRST operator, which is of degree one ($G_{Q\Psi} = 1 + G_\Psi$). String fields can be integrated using

$$\int : \mathcal{A} \to \mathbf{C} \, . \tag{12.7}$$

This integral vanishes for all Ψ with degree $G_\Psi \neq 3$. Thus, the action (12.4) is only nonvanishing for a string field Ψ of degree 1. The action (12.4) thus has the general form of a Chern–Simons theory on a 3-manifold, although for string field theory there is no explicit interpretation of the integration in terms of a concrete 3-manifold.

The elements $Q, \star, \int$ that define the string field theory are assumed to satisfy the following axioms.

(a) Nilpotency of Q: $Q^2 \Psi = 0, \quad \forall \Psi \in \mathcal{A}$.

(b) $\int Q\Psi = 0, \quad \forall \Psi \in \mathcal{A}$.

(c) Derivation property of Q:
$$Q(\Psi \star \Phi) = (Q\Psi) \star \Phi + (-1)^{G_\Psi} \Psi \star (Q\Phi), \quad \forall \Psi, \Phi \in \mathcal{A}.$$

(d) Cyclicity: $\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi, \quad \forall \Psi, \Phi \in \mathcal{A}$.

(e) Associativity: $(\Phi \star \Psi) \star \Xi = \Phi \star (\Psi \star \Xi), \quad \forall \Phi, \Psi, \Xi \in \mathcal{A}$.

When these axioms are satisfied, the action (12.4) is invariant under the gauge transformations

$$\delta\Psi = Q\Lambda + \Psi \star \Lambda - \Lambda \star \Psi\,, \tag{12.8}$$

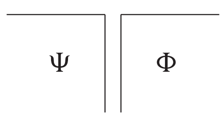
for any gauge parameter $\Lambda \in \mathcal{A}$ with degree 0.

When the string coupling g is taken to vanish, the equation of motion for the theory defined by (12.4) simply becomes $Q\Psi = 0$, and the gauge transformations (12.8) simply become

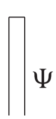
$$\delta\Psi = Q\Lambda\,. \tag{12.9}$$

This structure at $g = 0$ is precisely what is needed to describe a free bosonic string in the BRST formalism, where physical states live in the cohomology of the BRST operator Q, which acts on the string Fock space.[2] The motivation for introducing the extra structure in (12.4) was to find a simple interacting extension of the free theory, consistent with the perturbative expansion of open bosonic string theory.

Witten presented this formal structure and argued that all the needed axioms are satisfied when $\mathcal{A}$ is taken to be the space of string fields of the form (12.3). In this realization, the star product $\star$ acts on a pair of functionals Ψ, Φ by gluing the right half of one string to the left half of the other using a delta function interaction.

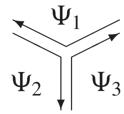


Similarly, the integral over a string field corresponds to gluing the left and right halves of the string together with a delta function interaction.



Combining these pictures, the three-string vertex $\int \Psi_1 \star \Psi_2 \star \Psi_3$ corresponds to a three-string overlap.

[2] For a detailed introduction to BRST string quantization, see [26]



While these pictures may seem rather abstract, they can be given explicit meaning in terms of the oscillator raising and lowering operators α_n^μ [7; 22; 28; 18]. Given an explicit representation of the terms in the string field action in terms of these raising and lowering operators, the contribution to the action from any set of component fields in the full string field can be worked out. The quadratic terms for the string fields $\varphi(p)$, $A_\mu(p)$ are the standard kinetic and mass terms for a tachyon field and a massless gauge field. The massive string fields similarly have kinetic terms and positive mass squared terms. The interaction terms for the component fields coming from the term $\int \Psi \star \Psi \star \Psi$ in the action, however, seem more exotic from the point of view of conventional field theory. These terms contain exponentials of derivatives, which appear as nonlocal interactions from the point of view of field theory. For example, the cubic interaction term for the scalar tachyon field $\varphi(p)$ takes the momentum space form

$$\int d^{26}pd^{26}q\ \frac{\kappa g}{3}\ e^{(\ln 16/27)(p^2+q^2+p\cdot q)}\varphi(-p)\varphi(-q)\varphi(p+q), \tag{12.10}$$

where κ is a constant. There are similar interaction terms between general sets of 3 component fields in the string field.

The appearance of an infinite number of fields and arbitrary numbers of derivatives (powers of momentum) in the action make the target space string field theory into a very unusual field theory. There are a number of obstacles to having a complete definition of this theory as a quantum field theory. Even at the classical level, it is not clear precisely what range of fields is allowed for the string field. In particular, owing to the presence of ghosts, there is no positive definite inner product on the string Fock space, so there is no natural finite norm condition to constrain the class of allowed string fields. Determining precisely what normalization condition should be satisfied by physical states is an important problem which may need to be solved to make substantial progress with the theory as a nonperturbative formulation of string theory. Beyond this issue the unbounded number of derivatives makes even the classical time-dependence of the string field difficult to pin down. The string field seems to obey a differential equation of infinite order, suggesting an infinite number of boundary conditions are needed. Some recent progress on these problems has been made [20; 11; 6], but even in this simplest case of Witten's open cubic bosonic string field theory, it seems clear that we are far from a complete understanding of how the theory should be defined. Despite these difficulties, however, the action (12.4) gives rise to a well-defined perturbative theory which can be used to calculate scattering amplitudes of on-shell string states associated with particles in the string Fock space. Furthermore, it was shown that these amplitudes agree with the perturbative formulation of string theory, as desired [16; 17; 43].

12.2.2 The Sen conjectures

Despite our limited understanding of the full definition of quantum string field theory, in the past few years a great deal of progress has been made in understanding the nature of the classical open string field theory described in the previous subsection.

One apparent problem for the open bosonic string and the associated string field theory is the open string tachyon. This tachyon indicates that the vacuum of the theory is unstable and can decay. Ashoke Sen [30] conjectured that a precise understanding of the nature of this instability and decay process could be attained through open string field theory. He argued that the unstable vacuum is one with a space-filling “D-brane” carrying positive energy density. D-branes have been a major subject of study in string theory over the last decade. D-branes are higher-dimensional extended objects on which open strings can end. In supersymmetric string theories, D-branes of some dimensions can be stable and supersymmetric. In the bosonic string theory, however, all D-branes are unstable. Sen suggested that the instability of the space-filling D-brane in bosonic string theory is manifested by the open bosonic string tachyon. He further suggested that string field theory should contain another nonzero field configuration Ψ_* which would satisfy the classical equation of motion $Q\Psi_* + g\Psi_* \star \Psi_* = 0$. Sen argued that this nontrivial vacuum field configuration should have several specific properties. It should have a vacuum energy which is lower than the initial unstable vacuum by precisely the volume of space-time times the energy density (tension) T of the unstable D-brane. The stable vacuum should also have no open string excitations. This latter condition is highly nontrivial and states that at the linearized level all open string fluctuations around the nontrivial vacuum become unphysical. To realize this change of backgrounds, the degrees of freedom of the theory must reorganize completely in going from one background to another. The ability of a single set of degrees of freedom to rearrange themselves to form the physical degrees of freedom associated with fluctuations around different backgrounds is perhaps the most striking feature of background-independent theories, and presents the greatest challenge in constructing and understanding such theories.

Following Sen’s conjectures, a substantial body of work was carried out which confirmed these conjectures in detail. A primary tool used in analyzing these conjectures using string field theory was the notion of “level truncation”. The idea of level truncation is to reduce the infinite number of string fields to a finite number by throwing out all fields above a fixed mass cutoff. By performing such a truncation and restricting attention to the constant modes with $p = 0$, the infinite number of string field component equations reduces to a finite system of cubic equations. These equations were solved numerically at various levels of truncation, and confirmed to 99.99% accuracy the conjecture that there is a nontrivial vacuum solution with the predicted energy [33; 19; 14; 36]. The conjecture that the nontrivial vacuum has no physical open string excitations was also tested numerically and found to hold to high accuracy [9; 8]. The effective potential $V(\varphi)$ for the tachyon field can be computed using this approach; this potential is graphed in Figure 12.1. This figure clearly illustrates the unstable perturbative vacuum as well as the stable nonperturbative vacuum.

The results of numerical analysis have confirmed Sen's conjectures very clearly. Perhaps the most important consequence of this confirmation is that we have for the first time concrete evidence that string field theory can describe multiple disconnected[3] string vacua in terms of a common set of variables. This is in principle the kind of construction which is needed to describe the disparate string vacua of the closed string landscape. Indeed, Figure 12.1 can be seen as a piece of the "open string landscape". To extrapolate from the results achieved so far in classical open string field theory to the picture we desire of a set of independent solutions of a quantum closed string field theory, however, a number of significant further steps must be taken. We discuss some of the issues which must be resolved in the following subsection.

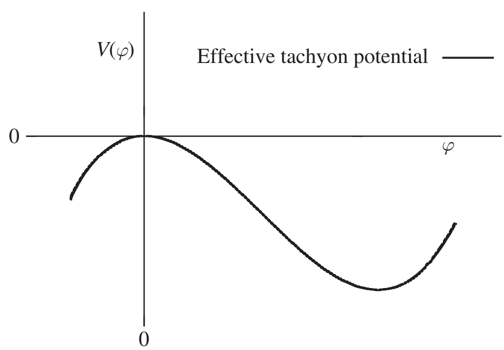


Fig. 12.1. The effective tachyon potential in Open String Field Theory.

[3] By disconnected we mean that there is no continuous family of vacuum solutions interpolating between the distinct vacua.

12.2.3 Outstanding problems and issues in OSFT

In order to improve our understanding of OSFT so that we can better understand the space of solutions of the theory, one very important first step is to develop analytic tools to describe the nontrivial open string vacuum described in the previous subsection. One approach to this problem was to try to reformulate string field theory around this vacuum using "vacuum string field theory" [27]. This approach led to the development of some powerful analytic tools for understanding the star algebra and projectors in the theory; recently these tools were used to make an important step forward by Schnabl [29], who has found an analytic form for the nontrivial vacuum of Witten's open string field theory. The presentation of this vacuum state has interesting analytic properties related to Bernoulli numbers. It seems to have a part which is well-behaved under level truncation, and another part which involves an infinite sequence of massive string fields. The second part of this state has vanishing inner product with all states which appear in level truncation, and is not yet completely understood (for further discussion of this state see [23]). This construction seems to be a promising step towards developing analytic machinery to describe solutions of classical string field theory; it seems likely that in the reasonably near future this may lead to significant new developments in this area.

Another important issue, relevant for understanding string field theory analytically and for describing a disparate family of solutions to the theory, even at the classical level, is the problem of field redefinitions. The issue here is that the fields appearing in the string field, such as φ and A_μ, are only identified at linear order with the usual space-time fields of conformal field theory. At higher order, these fields are related by a highly nontrivial field redefinition which can include arbitrary numbers of derivatives [15]. For example, the SFT A_μ (after integrating out the massive fields) is related to the CFT $\tilde{A}_\mu$ by a field redefinition

$$\tilde{A}_\mu = A_\mu + \alpha A^2 A_\mu + \beta A^2 \partial^2 A_\mu + \cdots \tag{12.11}$$

where arbitrarily complicated terms appear on the RHS [5]. Because of these field redefinitions, simple physical properties such as turning on a constant deformation A_μ, corresponding to the simple translation of a D-brane in flat space in a dual picture, are difficult to understand in the variables natural to SFT [33; 34]. Similar field redefinitions, involving arbitrary numbers of time derivatives, take a reasonably well-behaved time-dependent tachyon solution which classically rolls down the hill depicted in Figure 12.1 in the CFT description of a string field theory solution which has wild exponentially increasing oscillations [20; 6]. These field redefinitions make it very difficult to interpret simple physical properties of a system in the variables natural to string field theory. This is a generic problem for background-independent theories, but some systematic way of dealing with these different descriptions of physics needs to be found for us to sensibly interpret and analyze multiple vacua within a single formulation of string field theory.

Closely related to the issue of field redefinitions is the issue of gauge fixing. To perform explicit calculations in string field theory, the infinite gauge symmetry (12.8) must be fixed. One standard approach to this is the "Feynman–Siegel" gauge, where all states are taken to be annihilated by a certain ghost field. For string fields near $\Psi = 0$ this is a good gauge fixing. For larger string fields, however, this gauge fixing is not valid [10]. Some string field configurations have no representative in this gauge, and some have several (Gribov ambiguities). If for example one tries to continue the potential graphed in Figure 12.1 to negative φ much below the perturbative unstable vacuum or to positive φ much past the stable vacuum, the calculation cannot be done in Feynman–Siegel gauge. Currently no systematic way of globally fixing the gauge is known. This issue must be better understood to fully analyze the space of vacua classically and to define the quantum theory. For example, it should be possible in principle to describe a two-D-brane state in the Witten OSFT starting in the background with a single D-brane. This would correspond to a configuration satisfying the equation of motion, but with energy above the perturbative vacuum by the same amount as the stable vacuum is below it. In this 2 D-brane vacuum there would be 4 copies of each of the perturbative open string states in the original model. No state of this kind has yet been found, and it seems likely that such a state cannot be identified without a better approach to global gauge fixing. It is interesting to note that the analytic solution by Schnabl uses a different gauge choice than the Feynman–Siegel gauge; it will be interesting to see if this gauge has better features with regard to some of the problems mentioned here.

The open string field theory we have discussed here is a theory of bosonic strings. Attempting to quantize this theory is problematic because of the bosonic closed string tachyon, which leads to divergences and which is still poorly understood.[4] To discuss the quantum theory we should shift attention to supersymmetric open string field theory, which is tachyon free. Witten's approach to describing OSFT through a cubic action encounters problems for the superstring due to technical issues with "picture changing" operators. Although it may be possible to resolve these issues in the context of Witten's cubic formulation [1], an approach which may be more promising was taken by Berkovits [2; 3], where he developed an alternative formulation of the open superstring field theory. This formulation is more like a Wess–Zumino–Witten model than the Chern–Simons model on which (12.4) is based. The action has an infinite number of terms but can be written in closed form. Some analysis of this model using level truncation (see [21] for

[4] Recent work suggests, however, that even this tachyon may condense to a physically sensible vacuum [39; 40]

a review) gives evidence that this framework can be used to carry out a parallel analysis to that of the bosonic theory, and that disconnected open superstring vacua can be described using this approach, at least numerically. At the classical level, the same problems of field redefinition, lack of analytic tools, and gauge fixing must be tackled. But in principle this is a promising model to extend to a quantum theory. In principle, a complete quantum theory of open strings must include closed strings, since closed strings appear as intermediate states in open string scattering diagrams (indeed in some sense this is how closed strings were first discovered, as poles in open string scattering amplitudes). It should then in principle be possible to compute closed string scattering amplitudes using OSFT. A much more challenging problem, however, is turning on nonperturbative deformations of closed string fields in the open string language. The simple version of this would be to deform a modulus such as the dilaton by a constant value. Much more challenging would be to identify topologically distinct closed string vacua as quantum states in a single OSFT. Such a construction is well beyond any tools currently available. Since open string field theory seems better understood than closed string field theory this is perhaps a goal worth aiming at. In the next section, however, we describe the current state of direct constructions of closed string field theory.

12.3 Closed string field theory

A direct formulation of closed string field theory is more complicated than the theory for open strings. In closed string field theory, the string field $\Psi[X(\sigma)]$ has a field expansion (12.2) analogous to the open string field expansion (12.3). Writing an action for this string field which reproduces the perturbative amplitudes of conformal field theory is, however, much more complicated even in the bosonic theory than the simple Witten action (12.4).

Using a generalization of the BRST formalism, Zwiebach [42] developed a systematic way of organizing the terms in a closed bosonic string field theory action. Unlike the Witten action, which has only cubic interactions, Zwiebach's closed string field theory action contains interaction terms at all orders. The key to organizing this action and making sure that it reproduces the standard closed string perturbative expansion from CFT was finding a way of systematically cutting apart Riemann surfaces (using "Strebel differentials") so that each Riemann surface can be written in a unique way in terms of propagators and vertices. This approach is based very closely on the geometry of the string world-sheet and it seems to give a complete formulation of the bosonic theory, at least to the same extent that Witten's theory describes the open bosonic string.

In closed string field theory there are massless fields corresponding to marginal deformations of the closed string background. Such deformations include a modification of the string coupling, which is encoded in the dilaton field $\phi(x)$ through $g = e^{\phi}$. For closed string field theory to be background independent, it needs to be the case that turning on these marginal deformations can be accomplished by simply turning on the fields in the SFT. For example, it must be the case that the string field theory defined with string coupling g has a background described by a certain field configuration Ψ', such that expanding the theory around this background gives a theory equivalent to the SFT defined in a background with a different string coupling g'. This background independence was shown for infinitesimal marginal deformations by Sen and Zwiebach [31; 32]. This shows that closed string field theory is indeed background independent. It is more difficult, however, to describe a finite marginal deformation in the theory. This problem is analogous to the problem discussed in open string field theory of describing a finite marginal deformation of the gauge field or position of a D-brane, and there are similar technical obstacles to resolving the problem. This problem was studied for the dilaton and other marginal directions by Yang and Zwiebach [39; 40]. Presumably similar techniques should resolve this type of marginal deformation problem in both the open and closed cases. A resolution of this would make it possible, for example, to describe the moduli space of a Calabi–Yau compactification using closed string field theory. One particularly interesting question is whether a deformation of the dilaton to infinite string coupling, corresponding to the M-theory limit, can be described by a finite string field configuration; this would show that the background-independence of string field theory includes M-theory.

To go beyond marginal deformations, however, and to identify, for example, topologically distinct or otherwise disjoint vacua in the theory is a much greater challenge. Recently, however, progress has been made in this direction also using closed string field theory. Zwiebach's closed bosonic string field theory can be used to study the decay of a closed string tachyon in a situation parallel to the open string tachyon discussed in the previous section. It has been shown [24] that the first terms in the bosonic closed string field theory give a nonperturbative description of certain closed string tachyons in accord with physical expectations. The situation here is more subtle than in the case of the open string tachyon, since the tachyon occurs at a point in space where special "twisted" modes are supported, and the tachyon lives in these twisted modes, but as the tachyon condenses, the process affects physics in the bulk of space-time further and further from the initial twisted modes. This makes it impossible to identify the new stable vacuum in the same direct way as was done in OSFT, but the results of this analysis do suggest that closed string field theory correctly describes this nonperturbative process and should be capable of describing disconnected vacua. Again, however, presumably similar complications of gauge choice, field redefinitions, and quantum definition will need to be resolved to make progress in this direction.

Because of the closed string bulk tachyon in the bosonic theory, which is not yet known to condense in any natural way, the bosonic theory may not be well-defined quantum mechanically. Again, we must turn to the supersymmetric theory. Until recently, there was no complete description of even a classical supersymmetric closed string field theory. The recent work of Okawa and Zwiebach [25] and of Berkovits, Okawa and Zwiebach [4], however, has led to an apparently complete formulation of a classical string field theory for the heterotic string. This formulation combines the principles underlying the construction by Berkovits of the open superstring field theory with the moduli space decomposition developed by Zwiebach for the bosonic closed string field theory. Interestingly, for apparently somewhat technical reasons, the approach used in constructing this theory does not work in any natural way in the simpler type II theory. The action of the heterotic superstring field theory has a Wess–Zumino–Witten form, and contains an infinite number of interactions at arbitrarily high orders. The development of a SUSY CSFT makes it plausible for the first time that we could use a background-independent closed string field theory to address questions of string backgrounds and cosmology. Like the open bosonic theory discussed in the previous section, this closed string field theory can be defined in level truncation to give a well-defined set of interaction terms for a finite number of fields, but it is not known in any precise way what the allowed space of fields should be or how to quantize the theory. These are important problems for future work in this area.

12.4 Outlook

We have reviewed here the current state of understanding of string field theory and some recent developments in this area. String field theory is currently the only truly background-independent approach to string theory. We have reviewed some recent successes of this approach, in which it was explicitly shown that distinct vacua of open string field theory, corresponding to dramatically different string backgrounds, appear as solutions of a single theory in terms of a single set of degrees of freedom. While much of the work concretely confirming this picture in string field theory was numerical, it seems likely that further work in the near future will provide a better analytic framework for analyzing these vacua, and for understanding how open string field theory can be more precisely defined, at least at the classical level.

We described open string field theory in some detail, and briefly reviewed the situation for closed string field theory. While gravity certainly requires closed strings, it is not yet clear whether we are better off attempting to directly construct closed string field theory by starting with the closed string fields in a fixed gravity background, or, alternatively, starting with an open string field theory and working with the closed strings which arise as quantum excitations of this theory. On the one hand, open string field theory is better understood, and in principle includes all of closed string physics in a complete quantum formulation. But on the other hand, closed string physics and the space of closed superstring vacua seems much closer in spirit to closed string field theory. Recent advances in closed superstring field theory suggest that perhaps this is the best direction to look in if we want to describe cosmology and the space of closed string vacua using some background-independent formulation of string theory along the lines of SFT.

We reviewed some concrete technical problems which need to be addressed for string field theory, starting with the simpler OSFT, to make the theory better defined and more useful as a tool for analyzing classes of solutions. Some problems, like gauge fixing and defining the space of allowed states, seem like particular technical problems which come from our current particular formulation of string field theory. Until we can solve these problems, we will not know for sure whether SFT can describe the full range of string backgrounds, and if so how. One might hope that these problems will be resolved as we understand the theory better and can find better formulations. One hope may be that we might find a completely different approach which leads to a complementary description of SFT. For example, the M(atrix) model of M-theory can be understood in two ways: first as a quantum system of D0-branes on which strings moving in 10 dimensions end, and second as a regularized theory of a quantum membrane moving in 11 dimensions. These two derivations give complementary perspectives on the theory; one might hope for a similar alternative approach which would lead to the same structure as SFT, perhaps even starting from M-theory, which might help elucidate the mathematical structure of the theory.

One of the problems we have discussed, however, seems generic to all background-independent theories. This is the problem of field redefinitions. In any background-independent theory which admits numerous solutions corresponding to different perturbative backgrounds, the natural degrees of freedom of each background will tend to be different. Thus, in any particular formulation of the theory, it becomes extremely difficult to extract physics in any background whose natural variables are different. This problem is already very difficult to deal with at the classical level. Relating the degrees of freedom of Witten's classical open string field theory to the natural fields of conformal field theory in order to describe familiar gauge physics, open string moduli, or the dynamical tachyon condensation process makes it clear that simple physics can be dramatically obscured by the choice of variables natural to string field theory. This problem becomes even more challenging when quantum dynamics are included. QCD is a simple example of this; the physical degrees of freedom we see in mesons and baryons are very difficult to describe precisely in terms of the natural degrees of freedom (the quarks and gluons) in which the fundamental QCD Lagrangian is naturally written. Background independent Quantum Gravity seems to be a similar problem, but orders of magnitude more difficult.

Any quantum theory of gravity which attempts to deal with the landscape of string vacua by constructing different vacua as solutions of a single theory in terms of a single set of degrees of freedom will face this field-redefinition problem in the worst possible way. Generally, the degrees of freedom of one vacuum (or metastable vacuum) will be defined in terms of the degrees of freedom natural to another vacuum (or metastable vacuum) through an extremely complicated, generically quantum, field redefinition of this type. This presents a huge obstacle to achieving a full understanding of quantum cosmology. This obstacle is very concrete in the case of string field theory, where it will make it difficult to describe the landscape of string vacua in the language of a common theory. It is also, however a major obstacle for any other attempt to construct a background-independent formulation of Quantum Gravity (such as loop quantum gravity or other approaches reviewed in this book). Only the future will tell what the best means of grappling with this problem may be, or if in fact this is the right problem to pose. Perhaps there is some radical insight not yet articulated which will make it clear that we are asking the wrong questions, or posing these questions in the wrong way.

Two more fundamental issues which must be confronted if we wish to use string field theory to describe cosmology are the issues of observables and of boundary conditions and initial conditions. These are fundamental and unsolved issues in any framework in which we attempt to describe quantum physics in an asymptotically de Sitter or metastable vacuum. As yet there are no clear ways to resolve these issues in SFT. One interesting possibility, however, is that by considering string field theory on a space-time with all spatial directions compactified, these issues could be somewhat resolved. In particular, one could consider quantum OSFT on an unstable D-brane (or a brane/antibrane pair for the supersymmetric OSFT or the closed heterotic SFT without D-branes) on the background $T^9 \times \mathbf{R}$. The compactification provides an IR cutoff, and by putting in UV cutoffs through level truncation and a momentum cutoff, the theory could be approximated by a finite number of quantum mechanical degrees of freedom. This theory could be studied analytically, or, like lattice QCD, one could imagine simulating this theory and getting some approximation of cosmological dynamics. If SFT is truly background independent, quantum excitations of the closed strings should have states corresponding to other compactification topologies, including for example $T^3 \times X$ where X is any flux compactification of the theory on a Calabi–Yau or other 6D manifold. Quantum fluctuations should also allow the T^3 to contain inflating regions where the energy of X is positive, and one could even imagine eternal inflation occurring in such a region, with bubbles of other vacua branching off to populate the string landscape. Or one could imagine some other dynamics occurring, demonstrating that the landscape picture is incorrect. It is impractical with our current understanding to implement such a computation, and presumably the detailed physics of any inflating region of the universe would require a prohibitive number of degrees of freedom to describe. Nonetheless, if we can sensibly quantize open superstring field theory, or a Closed String Field Theory, on T^9 or another completely compact space, it may in some sense be the best-defined background independent formulation of string theory in which to grapple with issues of cosmology.

Acknowledgements

This work was supported in part by the DOE under contract #DE-FC02-94ER40818, and in part by the Stanford Institute for Theoretical Physics (SITP). The author would also like to thank Harvard University for hospitality during part of this work. Thanks to Barton Zwiebach for helpful discussions and comments on an earlier version of this manuscript.

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Questions and answers

- **Q - D. Oriti - to G. Horowitz and J. Polchinski**:
  1. On the basis of the AdS/CFT duality, and assuming that this can be generalised to other spacetimes and other type of gauge field theories, as you say, it seems to me that there are three possible attitudes one can take towards the nature of spacetime: if one takes a realistic standpoint, one is forced to choose between the reality of the 4d flat spacetime on which the gauge theory lives and consequently interpret the 10d spacetime of the Quantum Gravity theory as an auxiliary construction one can use to study the first, and the reality of the 10d spacetime where gravity propagates. In the first case, that appears to me the point of view currently taken in the application of the AdS/CFT correspondence to nuclear physics, it would seem to me that the physical problem of Quantum Gravity would remain open since one would still have to explain the origin and nature of the 4d flat spacetime on which the gauge theory lives. In the second case, one would have indeed understood an important sector of Quantum Gravity, but we would be left with the problem of compactification to the 4d spacetime we experience, and at the same time be left without any physical reason to apply the AdS/CFT correspondence to physical 4d gauge theories in the lab, or will have to resort to some version of brane world scenario whose status within string theory is not clear to me. The third option of course is to decline to take any realistic standpoint at all on the AdS/CFT and take it as an astonishing and intriguing, but purely formal mathematical construction, which suggests that a theory of Quantum Gravity (whatever that is in the end) can be formulated as an ordinary gauge theory of some sort (not living in physical spacetime), but whose physical significance is not yet understood. Can you please point out whether and in which respect my understanding of the situation is limited or mistaken, and what is your point of view on the above?
  2. What can we infer on the nature and Quantum Gravity origin of the cosmological constant from the AdS/CFT correspondence?

3. You state that the AdS/CFT correspondence provides a background independent formulation of Quantum Gravity in terms of the dual gauge theory, for given boundary. Let me understand better the statement, given that background independence is such a crucial notion in all Quantum Gravity approaches, including of course string theory. If I was to re-phrase the quantum dynamics of the gravitational degrees of freedom as encoded in the AdS/CFT, i.e. as described by the dual gauge theory, in the form of a path integral for Quantum Gravity, should I think of it as given by a sum over all possible geometries, at fixed topology, for given boundary conditions (the flat 4d Minkowski geometry), or rather by a sum over all possible geometries, at fixed topology, for given boundary conditions *and* given asymptotic behaviour of the geometries? In the first case, indeed, one would have a full definition of the gravitational path integral, for given boundary, while the second would incorporate also a (admittedly mild) restriction on the configurations summed over.

– **A - G. Horowitz and J. Polchinski**:
  1. We disagree that your options are mutually exclusive. Certainly the third option is true: AdS/CFT and other dualities are statements about mathematical physics, which can be used to derive relations between the spectrum, amplitudes, and other physical properties of the two sides of the duality. However, we disagree with the premise that only one side of a duality can be "real". In electric–magnetic duality in quantum field theory, both the electrical charges and the magnetic charges are "real". There is simply one classical limit described in terms of electrically charged fields, and the quantum theory constructed as a path integral over such fields, and another classical limit described in terms of magnetically charged fields, and the quantum theory constructed as a path integral over these fields. One is just making a change of variables, neither description is more "real". In the AdS/CFT case, the situation may not be so symmetric, in that for now the gauge side has an exact description and the string/gravity side only an approximate one: we might take the point of view that strings and spacetime are "emergent" and that the ultimate precise description of the theory will be in variables closer to the CFT form. "Emergent", however, is not the opposite of "real": most phenomena in nature are emergent, but nevertheless real. In particular, since we experience gravity, it would be this emergent description that is real to us.
  2. At the moment, AdS/CFT does not shed any light on the cosmological constant. The best explanation that string theory can provide at the moment comes from the large number of classical vacua (the "landscape").
  3. Neither of your descriptions is correct since the path integral in the bulk includes a sum over topologies as well as metrics. In terms of the boundary

condition on the metric, it is presumably only the leading order behavior (yielding a flat metric on the boundary) and not the rate of approach that needs to be specified. (To be sure, one would need a more rigorous definition of the gravitational functional integral.)

- **Q - L. Crane - to T. Banks**:
  Supersymmetry and holography can both be thought of as approaches to resolving the problem of the ultraviolet divergences in QFT. Susy suggests that the divergences are removed by exact cancellations between Feynman graphs involving superpartners. Holography suggests that short wavelength disturbances do not propagate because of small black holes.
  Superficially at least, these are very different ideas. What do you think of the proposal that given holography susy is no longer necessary? It would mean a greater range of possibilities to study. Would you agree that the development of supersymmetric models has been rather disappointing?
  - **A - T. Banks**:
    1. As I tried to explain in my contribution, I believe that supersymmetry will eventually be seen to follow from holography. The natural variables describing a pixel on the holographic screen of a causal diamond are spinors, which have the algebra of a supersymmetric massless particle which enters or leaves the diamond through that pixel.
    2. The only string theories (this is true also of AdS/CFT models which have no weakly coupled string theory limit) which have low curvature (compared to the string scale) space-times are exactly supersymmetric in the ultraviolet. In asymptotically flat space-times the only known consistent string theories are exactly supersymmetric.
    3. I don’t think supersymmetric model building is that disappointing. SUSY is the simplest way to unify the standard model couplings, and the simplest way to explain the hierarchy between the electroweak scale and the Planck scale. The exactly supersymmetric standard model is quite elegant and has only one extra parameter compared to the standard model. The real problem is with supersymmetry breaking, which we don’t understand. There are millions of ways to break supersymmetry in effective field theory, and the thing we usually call the supersymmetric standard model is just a parametrization of all the possible ways it could be broken. That’s why it has so many parameters. My own belief is that the mechanism for SUSY breaking is deeply related to Quantum Gravity, and the value of the cosmological constant, and that once we find the correct theory of it, it will be relatively unique.
    4. Last but most important: LHC will soon turn on. A lot of questions about supersymmetry and its breaking will be answered by that machine. My own ideas can probably be quite definitively ruled out by LHC experiments

(proving that they are right is harder). Many other suggestions could be ruled out as well. Perhaps definitive insight into the relevance of SUSY to the real world and the mechanism for its breaking will be gained. I will wait for the results of this machine before I decide whether to be disappointed by supersymmetric models.

- **Q - D. Oriti - to W. Taylor**:
  Can you please clarify and comment on the relation between string field theory, as currently understood, and matrix models for 2d Quantum Gravity coupled to scalar matter? Matrix models would indeed seem to me just a definition of something like a "simplicial string field theory", in that they define in perturbative expansion a sum over simplicial worldsheets of arbitrary topology, and they are reasonably successful in reproducing continuum worldsheet gravity, for what I know. Where do they instead fail in realizing the aims of string field theory?
  - **A - W. Taylor**:
    Indeed, matrix models for 2D Quantum Gravity coupled to scalar matter are non-perturbative formulations of string theory in certain backgrounds which achieve some of the goals of string field theory. One substantial limitation of these models is that they cannot be solved for matter with central charge $c > 1$, so existing techniques for these models are not applicable to physically interesting string theories, such as the superstring with central charge $c = 10$ or critical bosonic string with central charge $c = 26$. There has been interesting progress recently in understanding new features of these matrix models related to recent work in string field theory. For now, however, these models are at best "toy models" of the physics which would hopefully be captured by a complete string field theory for critical strings.

Part III

Loop quantum gravity and spin foam models

13

Loop quantum gravity

T. THIEMANN

13.1 Introduction

The modern version of canonical Quantum Gravity is called loop quantum gravity (LQG), see [1; 2] for textbooks and [3; 4; 5; 6] for recent reviews. At present, there is no other canonical approach to Quantum Gravity which is equally well developed. LQG is a Quantum Field Theory of geometry and matter which is background independent and takes fully into account the backreaction of (quantum) matter on (quantum) geometry. Background independence means that there is no preferred spacetime metric available, rather the metric is a dynamical entity[1] which evolves in tandem with matter, classically according to the Einstein equations. These precisely encode the backreaction. This is therefore an entirely new type of QFT which is radically different from ordinary QFT. One could even say that the reason for the fact that today there is not yet an established theory of Quantum Gravity is rooted in the background dependence of ordinary QFT. Therefore ordinary QFT (quantum mechanics) violates the background independence of classical GR while classical GR violates the quantum principle of QFT. This is the point where the two fundamental principles of modern physics collide. LQG tries to overcome this obstacle by constructing a background independent QFT.

In order to see in more detail where the background metric finds its way into the very definition of an ordinary QFT, recall the fundamental locality axiom of the algebraic approach [7]. There one deals with nets of local algebras $\mathfrak{A}(\mathcal{O})$ defined over regions $\mathcal{O}$ of a spacetime (M, g_0) where M is a differential manifold and g_0 a Lorentzian metric on M. The locality axiom now demands that if $\mathcal{O}, \ \mathcal{O}'$ are spacelike separated with respect to g_0 (that is, no causal geodesics of (M, g_0) can connect points of $\mathcal{O}, \ \mathcal{O}'$) then the elements of the two algebras $\mathfrak{A}(\mathcal{O}), \mathfrak{A}(\mathcal{O}')$

[1] Describing an infinite number of physical degrees of freedom.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

(anti)commute. We see that *without g_0 we do not even know what an ordinary QFT is because we do not even know the algebra of the field operators!*

Let us now contrast this with the situation of Quantum Gravity: there g_0 is not available, hence we do not know what the causal structure is, what the lightcones are, what geodesics are, what spacelike separated means, etc. Even worse, the metric is not only a dynamical quantity, it even becomes an operator. Hence, even if we are in a semiclassical regime where the expectation value of the metric operator is close to a given classical metric, the lightcones are fuzzy due to the fluctuations of the metric operator. Still worse, in extreme astrophysical (black holes) or cosmological (big bang) situations there simply is no semiclassical regime and the fluctuations become so large that the very notion of a metric entirely disappears. This is the reason why any perturbative approach, based on a split of the metric as $g = g_0 + \delta g$ where g_0 is a background metric and δg is a fluctuation and where one constructs an ordinary QFT of δg on the background g_0, cannot correctly describe a regime where it no longer makes sense to speak of any g_0. Notice that the split $g = g_0 + \delta g$ breaks background independence and diffeomorphism covariance simultaneously, so the resulting theory has at most the Killing symmetries of g_0. Of course, we know that GR is a nonrenormalizable theory and hence it is generally accepted that the perturbative approach makes no sense (at most as an effective theory). However, our argument also applies to the currently background dependent formulation of string theory which is believed to be a renormalisable, perturbative 2D QFT with a 10D or 11D target space interpretation of gravitons (and matter) propagating on a spacetime (M, g_0): this background dependent theory will at most capture a semiclassical regime of full Quantum Gravity where the expectation value of the metric operator is close to g_0 and the fluctuations are small.

Finally, let us mention a very interesting recent development within the algebraic approach: there a new, functorial definition of a generally covariant QFT [8] has recently been developed which essentially describes all ordinary QFTs on given backgrounds simultaneously. This formulation is therefore background independent by definition and can also describe Quantum Gravity at least perturbatively ("just" develop all perturbative graviton QFTs on all possible backgrounds). We expect, however, that this formulation and LQG will again drastically differ precisely when there is *no* classical (smooth) background metric at all, rather than *some* background metric. It will be very interesting to compare the two Ansätze in regimes where they are both valid.

This chapter has two sections. In the first we outline the canonical quantisation programme. In the second we apply it to GR thereby sketching a status report of LQG.

13.2 Canonical quantisation of constrained systems

It is well known [9] that General Relativity (geometry and matter) can be described as a Hamiltonian system with first class constraints.[2] The corresponding canonical formulation is due to Arnowitt, Deser and Misner (ADM) and is employed crucially and quite successfully in numerical General Relativity. We now briefly describe how to quantise such systems, focussing on the structural elements and choices that one has to make. For more details see [1; 2].

I. *The classical algebra $\mathfrak{P}$ of elementary observables.*
The first step is the choice of a *subalgebra of the Poisson algebra of smooth functions on the phase space $\mathcal{M}$. This means the following.

1. The elements of $\mathfrak{P}$ separate the points of $\mathcal{M}$, i.e. for all $m, m' \in \mathcal{M}$, $m \neq m'$ there exists $f \in \mathfrak{P}$ such that $f(m) \neq f(m')$. In particular, the elements of $\mathfrak{P}$ are globally defined.[3]
2. $\mathfrak{P}$ is closed under complex conjugation, that is $\overline{f} \in \mathfrak{P}$ for all $f \in \mathfrak{P}$.
3. $\mathfrak{P}$ is a Poisson subalgebra of $C^\infty(\mathcal{M})$, that is $\{f, f'\} \in \mathfrak{P}$ and $zf + z'f' \in \mathfrak{P}$ for all $f', f' \in \mathfrak{P}$ and $z, z' \in \mathbb{C}$. Here we consider $(M, \{.,.\})$ as a real symplectic manifold.

Notice that the choice of $\mathfrak{P}$ is far from being unique and will be guided by practical and physical considerations for the system at hand. Usually, if possible, one chooses $\mathfrak{P}$ such that the Poisson algebra is simple and that its elements transform simply under the gauge motions generated by the constraints.

II. *The quantum algebra $\mathfrak{A}$ of elementary observables.*
We define an abstract *algebra, whose elements we denote by $a, b, c, \ldots$ consisting of the finite linear combinations of all the finite sequences $(f_1, \ldots, f_n)$ of elements $f_k \in \mathfrak{P}$ equipped with the following algebraic operations

$$
\begin{aligned}
&(f_1, \ldots, f_n) \cdot (f'_1, \ldots, f'_{n'}) := (f_1, \ldots, f_n, f'_1, \ldots, f'_{n'}) \\
&(f_1, \ldots, f_n)^* := (\overline{f_n}, \ldots, \overline{f_1}).
\end{aligned}
\tag{13.1}
$$

We divide this algebra by the two sided ideal generated by elements of the following form

$$
\begin{aligned}
&(zf) - z(f), \quad (f + f') - (f) - (f') \\
&(f, f') - (f', f) - i\hbar(\{f, f'\}).
\end{aligned}
\tag{13.2}
$$

The result is the quantum algebra of elementary observables $\mathfrak{A}$. If it is not yet unital we add a unit by $\mathbf{1}\cdot a := a\cdot\mathbf{1} := a$ for all $a \in \mathfrak{A}$.

[2] That means that the Hamiltonian vector fields of the constraints are tangential to the constraint hypersurface in phase space [10].

[3] That might not be possible without over-coordinatising $\mathcal{M}$, e.g. if $\mathcal{M}$ is topologically non-trivial. In that case one embeds $\mathcal{M}$ into a topologically trivial phase space and imposes the (non-linear) embedding relations as additional constraints. We assume that this has been done for what follows.

Notice that in general the functions f will be unbounded on $\mathcal{M}$ and thus will be promoted to unbounded operators later on, which will raise inconvenient domain questions. One could use bounded functions instead but this usually comes at the price of complicating the Poisson relations and thus the representation theory of $\mathfrak{A}$. An exception is when $\mathfrak{P}$ is a real Poisson Lie algebra in which case we can pass to the unique associated Weyl C^*-algebra generated by unitary operators in any representation.

III. *Representation theory of* $\mathfrak{A}$.

The next step is the choice of a representation π of the elements a of $\mathfrak{A}$ by (unbounded) operators $\pi(a)$ on a Hilbert space $\mathcal{H}$. We will not enter into the discussion of the most general representation but describe an important and large subclass arising from positive linear functionals ω on the $^*-$algebra $\mathfrak{A}$, that is, $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{A}$. The associated, so-called GNS representation is constructed as follows. Consider the set $\mathfrak{I}_\omega := \{a \in \mathfrak{A}; \quad \omega(a^*a) = 0\}$. One can show that this defines a left ideal and therefore the natural operations $[a]+[b] := [a+b], \quad [a]\cdot[b] := [a\cdot b]$ on the classes $[a] := \{a+b,\ b \in \mathfrak{I}_\omega\}$ are well defined. We define $\Omega_\omega := [\mathbf{1}]$ and $\pi_\omega(a) := [a]$. The Hilbert space $\mathcal{H}_\omega$ is the completion of the vector space $\mathfrak{A}/\mathfrak{I}_\omega$ in the following inner product

$$< [a], [b] >=< \pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega >:= \omega(a^*b). \tag{13.3}$$

Notice the double role of $\mathfrak{A}$ as a Hilbert space and as a space of operators. The vector Ω_ω is automatically cyclic in this representation and obviously there are no domain questions: all operators are densely defined on the dense subspace $\mathfrak{A}/\mathfrak{I}_\omega$. The GNS data $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ are uniquely determined by ω up to unitary equivalence.

The choice of ω is again far from unique and will be guided by physical input. For instance, it may be true that a subset of the constraints generates a Poisson Lie group $\mathfrak{G}$. One then has a natural action of $\mathfrak{G}$ on $\mathfrak{P}$ via $f \mapsto \alpha_g(f)$ where α_g denotes the Hamiltonian flow corresponding to $g \in \mathfrak{G}$. For instance in the case of the free Maxwell field

$$\alpha_g(f) = \exp(\{\int d^3x \Lambda \partial_a E^a, .\}) \cdot f, \tag{13.4}$$

where $g = \exp(i\Lambda)$ is a local $U(1)$ gauge transformation and E denotes the electric field. The action α is obviously a Poisson automorphism and extends to $\mathfrak{A}$ via $\alpha_g((f_1, .., f_n)) := (\alpha_g(f_1), .., \alpha_g(f_n))$. In this situation it is natural to look for states ω which are $\mathfrak{G}$-invariant, that is $\omega \circ \alpha_g = \omega$ for all $g \in \mathfrak{G}$ because the following representation of the gauge group $\mathfrak{G}$ on $\mathcal{H}_\omega$ is automatically unitary: $U_\omega(g)\pi_\omega(a)\Omega_\omega := \pi_\omega(a)\Omega_\omega$.

Further criteria are the irreducibility of the representation. All we know is that the vector Ω_ω is cyclic. Irreducibility means that all vectors are cyclic. If a representation is not irreducible then the Hilbert space is a direct sum of irreducible subspaces and no observables exist which map between these sectors, they are superselected. Hence the physically interesting information is realised already in one of those sectors.

In what follows we will denote the Hilbert space $\mathcal{H}_\omega$ by $\mathcal{H}_{\text{Kin}}$ in order to indicate that it is a Hilbert space of kinematical states, i.e. the constraints have not yet been implemented and the states are therefore not gauge invariant.

IV. *Implementation of the constraints.*

The crucial question is whether the constraints can be realised in this representation as densely defined and closable (the adjoint is also densely defined) operators. This is non-trivial, especially in field theories such as General Relativity due to the following reasons.

1. The constraints are usually defined as functions of certain limits of elements of $\mathfrak{P}$. For instance, if $\mathcal{M}$ is a cotangent bundle then $\mathfrak{P}$ consists of smeared configuration and momentum variables, say $S(q) := \int_\Sigma d^3x S^{ab} h_{ab}$, $\mathcal{P}(s) := \int_\Sigma d^3x s_{ab} \pi^{ab}$ for GR where s, S are smooth, symmetric tensor (densities) of compact support. However, the (smeared) constraints of GR are not polynomials of $P(s)$, $S(q)$, rather they are non-polynomial expressions of the local functions $h_{ab}(x)$, $\pi^{ab}(x)$ and their first and second derivatives. Obviously, one can get those functions by taking a limit in which S^{ab}, s_{ab} become Dirac distributions, however, since only the smeared fields are defined as operators on $\mathcal{H}_{\text{Kin}}$, it is a highly non-trivial question whether the constraints are densely defined at all. Technically, the un-smeared fields become operator valued distributions and it is difficult to make sense out of products of those located at the same points. Thus, one may be facing ultraviolet problems.
2. Notice that all but the at most linear functions face the so called operator ordering problem: It makes a difference whether we identify the function $f_1 f_2 \in C^\infty(\mathcal{M})$ (which does not belong to $\mathfrak{P}$) with (f_1, f_2) or (f_2, f_1) in $\mathfrak{A}$. If f_1, f_2 are real valued, then one may choose a symmetric ordering $[(f_1, f_2) + (f_2, f_1)]/2$, however, it is not possible to rescue all the classical relations to the quantum level, at least in irreducible representations, which is the content of the famous Groenewald–van Howe theorem [11]. This may be an obstacle especially for constraint quantisation, because we may pick up what are called anomalies: While the classical constraints form a closed subalgebra (possibly with structure functions), the quantum constraints may not. This could imply that the physical Hilbert space, discussed below, is too small.

V. *Solving the constraints and physical Hilbert space.*

Let us assume that we are given some set of real valued constraints C_I where I takes a range in some index set and suppose that they form a first class system, that is, $\{C_I, C_J\} = f_{IJ}{}^K C_K$ where $f_{IJ}{}^K$ may be non-trivial, real valued functions on phase space. This is precisely the situation in GR where the index set stands for some countable system of smearing functions $I = (N, \vec{N})$ called lapse and shift functions. Suppose that we have successfully quantised the constraints and structure functions as operators $\hat{C}_I$, $\hat{f}_{IJ}{}^K$ on $\mathcal{H}_{\text{Kin}}$ as specified in step IV. The first possible problem is that the point zero is not contained in the spectrum of some of the $\hat{C}_I$ in which case the physical Hilbert space is empty. In that case the quantisation of those operators or the kinematical Hilbert space is invalid and must be changed. Let us assume that this problem has been circumvented. If the point zero is not contained in the point spectrum of all the $\hat{C}_I$, there is no non-trivial solution $\Psi \in \mathcal{H}_{\text{Kin}}$ to the system of quantum constraint equations $\hat{C}_I\Psi = 0$ for all I which is the quantum analogue of the classical system of constraint equations $C_I = 0$ for all I (because this would mean that Ψ is a common zero eigenvector). For instance, the operator id/dx on $L_2(\mathbb{R}, dx)$ has spectrum $\mathbb{R}$ but none of the formal "eigenvectors" $\exp(-ikx)$ with eigenvalue k is normalizable. Thus, the solution to the constraints has to be understood differently, namely in a generalised sense. This comes at the price that the solutions must be given a new Hilbert space inner product with respect to which they are normalisable.

We will now present a method to solve all the constraints and to construct an inner product induced from that of $\mathcal{H}_{\text{Kin}}$ in a single stroke, see [12] and [13] for more details. Consider the Master constraint

$$\mathbf{M} := \sum_{IJ} C_I \; K^{IJ} \; C_J \tag{13.5}$$

where K^{IJ} is a positive definite matrix which may depend non-trivially on the phase space and which decays sufficiently fast so that $\mathbf{M}$ is globally defined and differentiable on $\mathcal{M}$. It is called the Master constraint because obviously $\mathbf{M} = 0 \;\Leftrightarrow\; C_I = 0 \;\forall I$. The concrete choice of K^{IJ} is further guided by possible symmetry properties that $\mathbf{M}$ is supposed to have and by the requirement that the corresponding Master constraint operator $\widehat{\mathbf{M}}$ is densely defined on $\mathcal{H}_{\text{Kin}}$. As a first check, consider the case that the point zero is only contained in the point spectrum of every $\hat{C}_I$ and define $\widehat{\mathbf{M}} := \sum_I K^I \hat{C}_I^\dagger \hat{C}_I$ where $K^I > 0$ are positive numbers. Obviously, $\hat{C}_I\Psi = 0$ for all I implies $\widehat{\mathbf{M}}\,\Psi = 0$. Conversely, if $\widehat{\mathbf{M}}\,\Psi = 0$ then $0 =< \Psi, \widehat{\mathbf{M}}\,\Psi >= \sum_I K^I \; ||\hat{C}_I\Psi||^2$ implies $\hat{C}_I\Psi = 0$ for all I. Hence, in the simplest case, the single Master constraint contains the same information as the system of all constraints.

Let us now consider the general case and assume that $\widehat{\mathbf{M}}$ has been quantised as a positive self-adjoint operator on $\mathcal{H}_{\text{Kin}}$.[4] Then it is a well known fact that the Hilbert space $\mathcal{H}_{\text{Kin}}$ is unitarily equivalent to a direct integral of Hilbert spaces subordinate to $\widehat{\mathbf{M}}$, that is,

$$\mathcal{H}_{\text{Kin}} \cong \int_{\mathbb{R}^+}^{\oplus} d\mu(\lambda)\; \mathcal{H}^{\oplus}(\lambda) =: \mathcal{H}^{\oplus}_{\mu,N}. \tag{13.6}$$

Here the Hilbert spaces $\mathcal{H}^{\oplus}(\lambda)$ are induced from $\mathcal{H}_{\text{Kin}}$ and by the choice of the measure μ and come with their own inner product. One can show that the measure class $[\mu]$ and the function class $[N]$, where $N(\lambda) = \dim(\mathcal{H}^{\oplus}(\lambda))$ is the multiplicity of the "eigenvalue" λ, are unique[5] and in turn determine $\widehat{\mathbf{M}}$ uniquely up to unitary

[4] Notice that $\widehat{\mathbf{M}}$ is naturally quantised as a positive operator and that every positive operator has a natural self-adjoint extension, the so-called Friedrichs extension [14].

[5] Two measures are equivalent if they have the same measure zero sets. Two measurable functions are equivalent if they agree up to measure zero sets.

equivalence. Every element $\Psi \in \mathcal{H}_{\text{Kin}}$ can be thought of as the collection of "Fourier coefficients" $(\hat{\Psi}(\lambda))_{\lambda\in\mathbb{R}^+}$ where $\Psi(\lambda) \in \mathcal{H}^{\oplus}(\lambda)$ and

$$< \Psi, \Psi' >_{\mathcal{H}_{\text{Kin}}} = \int d\mu(\lambda) \ < \hat{Psi}(\lambda), \hat{Psi}'(\lambda) >_{\mathcal{H}^{\oplus}(\lambda)} . \tag{13.7}$$

The point of the Fourier representation (13.6) is of course that it is adapted to $\widehat{\mathbf{M}}$, namely $\widehat{\mathbf{M}}$ acts diagonally: $\widehat{\mathbf{M}}(\hat{\Psi}(\lambda)) = (\lambda\hat{\Psi}(\lambda))$. It follows that the physical Hilbert space is given by

$$\mathcal{H}_{Phys} = \mathcal{H}^{\oplus}(0). \tag{13.8}$$

Three remarks are in order.

1. While the representative μ is irrelevant, the representative N is crucial and requires further physical input. For instance, if the point zero is of measure zero (lies entirely in the continuous spectrum of $\widehat{\mathbf{M}}$) then we may choose the representative N such that $N(0) = 0$ which would mean that the physical Hilbert space is trivial. This is certainly not what one wants. The input required is that we want an irreducible representation of the algebra of Dirac observables (gauge invariant functions), which are automatically fibre preserving, on $\mathcal{H}_{\text{Phys}}$. This can be shown to drastically reduce the freedom in the choice of N.
2. It may happen that the spectrum of $\widehat{\mathbf{M}}$ does not contain the point zero at all in which case the physical Hilbert space again would be trivial. This can be the consequence of an anomaly. In this case it turns out to be physically correct to replace $\widehat{\mathbf{M}}$ with $\widehat{\mathbf{M}}' := \widehat{\mathbf{M}} - \min(\text{spec}(\widehat{\mathbf{M}}))\mathbf{1}$ provided that the "normal ordering constant" is finite and vanishes in the classical limit, that is, $\lim_{\hbar\to 0} \min(\text{spec}(\widehat{\mathbf{M}})) = 0$, so that $\widehat{\mathbf{M}}'$ is a valid quantisation of $\mathbf{M}$. Finiteness and the question whether $\widehat{\mathbf{M}}$ is densely defined at all crucially depends on the choice of K^{IJ}.
3. To see how an anomaly may arise, especially in the case of structure functions, suppose that $\hat{C}_I, \ \ \hat{f}_{IJ}{}^K$ are symmetric operators. Then the classical relation $\{C_I, C_J\} = f_{IJ}{}^K \ \ C_K$ is replaced by the quantum relation

$$\begin{aligned}[][\hat{C}_I, \hat{C}_J] &= i\hbar(\hat{f}_{IJ}{}^K \ \ \hat{C}_K + \hat{C}_K \hat{f}_{IJ}{}^K)/2 \\ &= i\hbar\hat{f}_{IJ}{}^K \ \ \hat{C}_K + -\frac{\hbar^2}{2}\frac{[\hat{C}_K, \hat{f}_{IJ}{}^K]}{i\hbar}\end{aligned} \tag{13.9}$$

where the symmetric ordering on the right hand side is a consequence of the antisymmetry of the commutator. It follows that any (generalised) solution Ψ of $\hat{C}_I\Psi = 0$ for all I automatically satisfies also

$$\hbar^2([\hat{C}_K, \hat{f}_{IJ}{}^K]/(i\hbar))\Psi = 0$$

for all I, J. However, the classical limit of that operator is $\hbar^2\{C_K, f_{IJ}{}^K\}$ which might be non-vanishing, not even on the constraint surface. This means that the physical Hilbert space is constrained more than the physical phase space and thus is not a proper quantisation of the classical system. We see in particular that in order to avoid anomalies, one should not order the constraints symmetrically unless $\{C_K, f_{IJ}{}^K\} = 0$, which is not the case in GR.

VI. *Dirac observables and the problem of time*.
Classically, (weak) Dirac observables are defined by $\{C_I, O\}_{\mathbf{M}=0} = 0$ for all I. It is easy to check that this system of conditions is equivalent to a single relation, namely $\{O, \{O, \mathbf{M}\}\}_{\mathbf{M}=0} = 0$. There is a formal but rather natural way to construct them [15; 16]. Consider a system of phase space functions T_I such that the matrix $A_{IJ} := \{C_I, T_J\}$ is at least locally invertible and define the equivalent set of constraints $C'_I := (A^{-1})_{IJ} C_J$. Remarkably, these constraints have weakly commuting Hamiltonian vector fields X_I. It is then tedious but straightforward to check that for any function f on phase space the function

$$F^{\tau}_{f,T} := \sum_{\{n_J\}} \prod_I \frac{(\tau_I - T_I)^{n_I}}{n_I!} \prod_I X_I^{n_I} \cdot f \tag{13.10}$$

is a weak Dirac observable. Here the sum runs over all sequences $\{n_I\}$ of non-negative integers. The physical interpretation of (13.10) is as follows. The constraint surface $\overline{\mathcal{M}}$ of the unconstrained phase space can be thought of as a fibre bundle with base given by the physical phase $[\mathcal{M}] = \{[m];\ m \in \overline{\mathcal{M}}\}$, where $[m] := \{m' \in \overline{\mathcal{M}};\ m' \sim m\}$ denotes the gauge orbit through m while the fibre above $[m]$ are the points of the subset $[m] \subset \overline{\mathcal{M}}$. By assumption, the functions T_I are local coordinates in the fibres above each point, that is, given $m \in \overline{\mathcal{M}}$ we may coordinatise it by $m \mapsto ([m], T(m))$. Hence we have a local trivialisation of the bundle. The gauge condition $T(m) = \tau$ for a value τ in the range of T now fixes a unique point $m_T(\tau, [m])$ in the fibre above $[m]$ and at that point $F^{\tau}_{f,T}$ obviously assumes the value $f(m_T(\tau, [m]))$. Since $F^{\tau}_{f,T}$ is gauge invariant, we have $F^{\tau}_{f,T}(m) = f(m_T(\tau, [m]))$ for all $m \in \overline{\mathcal{M}}$. It follows that $F^{\tau}_{f,T}$ only depends on $[\mathcal{M}]$ for all values of τ and its value at $p \in [\mathcal{M}]$ is the value of f at the point $m \in \overline{\mathcal{M}}$ with local coordinates $([m] = p, T(m) = \tau)$.
The functional (13.10) is what one calls a relational observable: none of the functions f, T_I is gauge invariant and therefore not observable. Only $F^{\tau}_{f,T}$ is observable. This is precisely what happens in physics: consider the example of a relativistic particle. Like GR, the relativistic particle has no Hamiltonian, only a Hamiltonian constraint which in this case is the mass shell constraint $C = (p^2 + m^2)/2 = 0$. It arises because the classical action is reparameterisation invariant. None of the coordinates X^{μ} of the particle is gauge invariant and thus observable. What is observable is the trajectory of the particle, that is, its graph. It can be implicitly described by $P^0 X^a - P^a X^0 = const.$ or explicitly by $F^{\tau}_{X^a, X^0} = X^a + (\tau - X^0) P^a / P^0$.
Relational observables also solve the Problem of Time: since the vector fields X_I are weakly commuting it is easy to see that $f \mapsto F^{\tau}_{f,T}$ is a Poisson automorphism among f which satisfy $\{f, T_I\} = 0$ for all I. Therefore, the multifingered physical time automorphism $F^{\tau}_{\cdot,T}$ has canonical generators defined by $\{H_I(\tau), F^{\tau}_{f,T}\} =$

$\partial F^{\tau}_{f,T}/\partial\tau_I$. They define physical Hamiltonians, i.e. Dirac observables. For the relativistic particle it is easy to see that this generator coincides with $\sqrt{\delta^{ab}p_a p_b + m^2}$ as expected.
Unfortunately, for sufficiently complicated dynamical systems such as GR, the expressions (13.10) are rather complex and formal in the sense that little is known about sufficient criteria for the convergence of the series involved in (13.10) and whether they can be quantised on $\mathcal{H}_{\text{Kin}}$ crucially depends on a judicious choice of the T_I. However, at least in principle there is a guideline to address the Problem of Time.

This concludes the outline of the canonical quantisation programme for arbitrary constrained systems. We will apply it in the next section to General Relativity.

13.3 Loop quantum gravity

The classical canonical framework was developed by ADM in the 1960s and in the previous subsection we have outlined the canonical quantisation algorithm. Hence we should now start to systematically apply it to GR. Unfortunately this is not directly possible because for the ADM formulation it has not been possible to find background independent representations of the algebra $\mathfrak{P}$ generated from the functions $S(q)$, $P(s)$ discussed in the previous subsection which support the constraints. Therefore, the canonical programme was stuck for decades until the mid 1980s and all the results obtained before that date are at best formal. Without a representation one cannot tell whether the algebraic objects that one is dealing with are densely defined at all, what the spectra of operators are, whether formal solutions to the constraint equations are indeed generalised eigenvectors, etc. For instance, the function $x \mapsto \exp(kx)$, $k \in \mathbb{R} - \{0\}$ certainly is formally an eigenfunction of the operator id/dx on $L_2(\mathbb{R}, dx)$, however, it is neither a proper eigenvector (since it is not normalisable) nor a generalised eigenvector because it cannot appear in the spectral resolution of the self-adjoint operator id/dx (because $\exp(kx)$ has a formally imaginary eigenvalue). Hence a representation is indispensable in order to construct a viable theory.

13.3.1 New variables and the algebra $\mathfrak{P}$

Progress was made due to a switch to new canonical variables [17; 18] which we now describe. We will be brief, the interested reader can find the details[6] in [1; 2].

[6] As a historical aside, it was believed that the theory for $\iota = \pm i$ is distinguished because the Hamiltonian constraint (13.19) then simplifies and even becomes polynomial after multiplying by $\sqrt{|\det(E)|}$. Unfortunately the representation theory for this theory could never be made sense of because the connection then is complex valued and one obtains the non-polynomial reality conditions $A + \overline{A} = 2\Gamma$. It was then believed that one should

We consider spacetime manifolds M which are diffeomorphic to $\mathbb{R}\times\Sigma$ where Σ is a 3D manifold of arbitray topology. Consider any principal $SU(2)$ bundle[7] $\mathcal{P}$ over Σ and denote by A the pull back by local sections of a connection on P. Likewise, consider an associated, under the adjoint representation of $SU(2)$, vector bundle whose local sections are $su(2)$ valued vector densities E. We now consider a new phase space with the following symplectic structure

$$\{E^a_j(x), A^k_b(y)\} = \kappa\iota\delta^a_b\delta^k_j\delta(x, y) \tag{13.11}$$

where $\kappa = 8\pi G_N$, G_N is Newton's constant and $\iota > 0$ is any positive real number [19; 20]. All other brackets vanish.

We now want to establish the connection with the ADM formulation. Consider a Dreibein whose local sections are $su(2)$ valued one forms e. Then $q_{ab} = \delta_{ij}e^i_a e^j_b$ and $E^a_j = |\det(e)|e^a_j$ where $e^a_j e^j_b = \delta^a_b$, $e^a_j e^k_a = \delta^k_j$ and q is the pull back of the spacetime metric to Σ. Next, denote the pull back by local sections of the spin connection[8] associated with e by Γ. Then $A^j_a = \Gamma^j_a + \iota K_{ab}e^b_j$ where K_{ab} is the pull back to Σ of the extrinsic curvature of the foliation $M \cong \mathbb{R}\times\Sigma$.

The non-trivial result is now the following. We may use the relations just displayed to invert $q_{ab} = H_{ab}(A, E)$, $P^{ab} = \Pi^{ab}(A, E)$ where H, Π are functions which can be easily derived by the reader from the definition $P^{ab} = \sqrt{\det(q)}[q^{ac}q^{bd} - q^{ab}q^{cd}]K_{cd}$ of the momentum conjugate[9] to q. Define the Gauss constraint

$$C_j := \partial_a E^a_j + \epsilon_{jkl}A^k_a E^a_l. \tag{13.12}$$

Then one can show that the Poisson brackets of H, Π, using the symplectic structure (13.11) are precisely the Poisson brackets of q, P up to terms which vanish when $C_j = 0$. In other words, the phase space spanned by A, E with symplectic structure (13.11) and constraint (13.12) imposed is precisely the ADM phase space of geometrodynamics. Notice that H, Π are Dirac observables with respect to the Gauss constraint.

We can now define the algebra $\mathfrak{P}$ of elementary observables based on A, E. The analogy with an $SU(2)$ gauge theory familiar from the electroweak theory naturally suggests the use of techniques which are standard in (lattice) gauge theory. To that end, let s, S be a one- and two-dimensional submanifold of Σ respectively, which

use real valued ι and multiply the Hamiltonian constraint by a sufficiently high power of $\sqrt{|\det(E)|}$ in order to make it polynomial. The work of [25; 26; 27; 28; 29; 30; 31; 32] showed that the representation for $\mathfrak{A}$ supports the Hamiltonian constraint only if one uses the non-polynomial form. The reason for this is again background independence.

7 One can show that principal $SU(2)$ bundles over 3D Σ are necessarily trivial but we will not need this in what follows.

8 It is defined by the covariant constance of e, i.e. $D_a e^j_b = 0$.

9 I.e. $\{P^{ab}(x), q_{cd}(y)\} = 16\pi G_N \delta^a_{(c}\delta^b_{d)}\delta(x, y)$.

will play the role of labelling smearing functions. Furthermore, let k be an $su(2)$ valued, smooth function on Σ. We define holonomy and flux functions

$$A(s) := \mathcal{P}\exp(\int_p A), \quad E_k(S) := \int_S \mathrm{Tr}(k\ *E) \tag{13.13}$$

where $*E$ is the metric independent, pseudo-two form dual to E. Now consider an arbitrary, finite collection of paths s. Their union forms a finite[10] graph γ and we may compose the paths s from the edges of the resulting graph. We now call a function cylindrical over a graph provided it is a complex valued function of the $A(s)$ where s runs through the edges of the graph. The cylindrical functions form an Abelean *-algebra which we denote by Cyl. Next, denote by $Y_{k,S}$ the Hamiltonian vector field of $E_k(S)$. Then $\mathfrak{P}$ is defined as the Lie algebra of cylindrical functions f and vector fields v equipped with the following Lie bracket $[(f, v), (f', v')] = (v[f'] - v'[f], [v, v'])$. The most important building block in that algebra is

$$Y_{k,S}[A(s)] = \{E_{k,S}, A(s)\} = \iota\kappa A(s_1)k(s \cap S)A(s_2) \tag{13.14}$$

where $s = s_1 \circ s_2$ and we have assumed that $s \cap S$ is precisely one point, the general case being similar.

13.3.1.1 The quantum algebra $\mathfrak{A}$ and its representations

The corresponding $\mathfrak{A}$ is defined by formally following the procedure of section 13.2. We now consider its representation theory. Since we are dealing with a field theory the representation theory of $\mathfrak{A}$ will be very rich so we have to downsize it by imposing additional physical requirements. The natural requirement is that the representation derives from a state invariant under the automorphisms of the bundle P. Locally these automorphisms can be identified with the semidirect product $\mathfrak{G} := \mathcal{G} \rtimes \mathrm{Diff}(\Sigma)$ of local $SU(2)$ gauge transformations and spatial diffeomorphisms. The requirement of $\mathfrak{G}$-invariance is natural because both groups are generated canonically, that is by the exponential of the respective Hamiltonian vector fields, from the Gauss constraint and spatial diffeomorphism constraint respectively. For instance we have with $C(\Lambda) = \int_\Sigma d^3x\ \Lambda^j C_j$ and $C(\vec{N}) = \int_\Sigma d^3x\ N^a C_a$ that

$$\begin{aligned} \alpha_{g_\Lambda}(A(s)) &:= \exp(X_{C(\Lambda)}) \cdot A(s) = g_\Lambda(b(s)A(s)g_\Lambda(f(s)) \\ \alpha_{\varphi_{\vec{N}}}(A(s)) &:= \exp(X_{C(\vec{N})}) \cdot A(s) = A(\varphi_{\vec{N}}(s)), \end{aligned} \tag{13.15}$$

where $g_\Lambda = \exp(\Lambda)$ and $\varphi_{\vec{N}}$ is the diffeomorphism defined by the integral curves of the vector field $\vec{N}$. Here, $b(s)$, $f(s)$ respectively denote beginning and final point of the path s and X_F denotes the Hamiltonian vector field of F.

[10] Technically paths and surfaces must be semi-analytic and compactly supported in order for that to be true [1; 2; 21] but we will not go into these details here.

Hence both gauge groups act naturally by Poisson automorphisms on $\mathfrak{P}$ which lifts to $\mathfrak{A}$.

This brings us precisely into the situation of the previous subsection. The non-trivial result is now [21] as follows.

There exists a unique $\mathfrak{G}$-invariant state ω on the holonomy flux *-algebra $\mathfrak{A}$ which is uniquely defined by the relations

$$\omega(fY_{k,S}) = 0, \quad \omega(f) = \int_{SU(2)^N} d\mu_H(g_1)\ldots d\mu_H(g_N)\; f_\gamma(g_1,\ldots,g_N) \tag{13.16}$$

where $f(A) = f_\gamma(A(s_1),\ldots,A(s_N))$ is a function cylindrical over the graph $\gamma = \cup_{k=1}^N s_k$. The corresponding GNS Hilbert space can be shown to be a certain L_2 space over a space of distributional connections in which the $\pi_\omega(Y_{l,S})$ act as self-adjoint derivation operators while the $\pi_\omega(A(s))$ are simply $SU(2)$ valued multiplication operators. In this space the space of smooth connections of *every* bundle is densely embedded, hence the choice of the initial bundle is measure theoretically irrelevant.[11] The representation (13.16) had been constructed before [22; 23] by independent methods which were guided by background independence.

This result is somewhat surprising because usually one gets uniqueness of representations in field theory only by invoking dynamical information such as a specific Hamiltonian. In our case, this information is brought in through $\mathfrak{G}$-invariance. The result is significant because it says that LQG is defined in terms of a preferred representation in which $\mathfrak{G}$ is unitarily implemented. In particular, there are no anomalies as far as $\mathfrak{G}$ is concerned.

13.3.1.2 Implementation and solution of the constraints

The Gauss constraint simply asks that the L_2 functions be invariant under local $SU(2)$ gauge transformations and can be trivially solved by choosing the f_γ to be the gauge invariant functions familiar from lattice gauge theory.

Let us therefore turn to the other two constraints. The spatial diffeomorphism group is unitarily implemented as $U(\varphi)\Psi = \alpha_\varphi(\Psi)$ and the invariance condition amounts to $\alpha_\varphi(\Psi) = \Psi$ for all $\varphi \in \mathrm{Diff}(\Sigma)$. One can easily show that this eigenvalue equation has only one (normalisable) solution $\Psi = 1$ (and constant multiples). It follows that most of the solutions are distributions (generalised eigenvectors). They can be found by the methods displayed in the previous subsection and we will restrict ourselves here to displaying the result, see [24] for more details.

The Hilbert space has a distinguished orthonormal basis $T_n,\; n = (\gamma, D)$, the so-called spin network functions. They are labelled by a graph and certain discrete additional quantum numbers D whose precise form is not of interest here. We have

[11] In fact, the space of classical connections in any bundle is of measure zero, similar to that of the space of classical free scalar fields in any Fock space Gaussian measure.

$U(\varphi)T_n = T_{\varphi(n)},\ \varphi((\gamma, D)) = (\varphi(\gamma), D)$. We define the (generalised) knot classes $[n] := \{\varphi(n);\ \varphi \in \mathrm{Diff}(\Sigma)\}$ and with it the distributions

$$l_{[n]}(T_{n'}) := \chi_{[n]}(n') = \delta_{[n],[n']} \tag{13.17}$$

where χ_B denotes the characteristic function of the set B. The solution space consists of the linear span of the distributions (13.17) which can be given a Hilbert space structure $\mathcal{H}_{\mathrm{Diff}}$ by completing it in the scalar product

$$\langle l_{[n]}, l_{[n']}\rangle_{\mathrm{Diff}} := l_{[n']}(T_n). \tag{13.18}$$

Let us now turn to the final Hamiltonian or Wheeler–DeWitt constraint which in terms of A, E takes the form

$$C = |\det(E)|^{-1/2}\ \mathrm{Tr}([(1+\iota^2)[K_a, K_b] - F_{ab}]\ [E^a, E^b]) \tag{13.19}$$

where $\iota K(A, E) = A - \Gamma(E)$ and F is the curvature of A. It is obvious that (13.19) presents a challenge for the representation $\mathcal{H}_{Kin}$ because it is a non-polynomial function of the *unsmeared* functions E which become operator valued distributions. Indeed, in order to define the smeared Hamiltonian constraint $C(\alpha) = \int_\Sigma d^3x\alpha C$ we must proceed entirely differently from the Gauss or spatial diffeomorphism constraint because it does not generate a Lie algebra due to the structure functions involved. One can proceed as follows: one point splits (regularises) the constraint (13.19), thus arriving at a well defined operator $\hat{C}_\epsilon(N)$ and then takes the limit $\epsilon \to 0$ in a suitable operator topology. The operator topology that naturally suggests itself is a weak topology based on the space $\mathcal{H}_{\mathrm{Diff}}$ viewed as a space of linear functionals over (a dense subspace of) $\mathcal{H}_{\mathrm{Kin}}$. It turns out that the limit exists in this topology precisely due to spatial diffeomorphism invariance of the distributions $l_{[n]}$. In a technically precise sense, the group $\mathrm{Diff}(\Sigma)$ swallows the ultraviolet regulator because in a background independent framework there is no meaning to the notion of "short" distance behaviour. One can also show that the commutator $[\hat{C}(N), \hat{C}(N')]$ is non-vanishing but that its dual[12] annihilates $\mathcal{H}_{\mathrm{Diff}}$. As one can show [9], also the classical Poisson bracket $\{C(N), C(N')\}$ vanishes on the constraint surface defined by the spatial diffeomorphism constraint, hence we get a consistent constraint algebra. However, the disadvantage of this procedure [25; 26; 27; 28; 29; 30; 31; 32] is that one does not have access to a physical inner product.

The more elegant solution uses the Master constraint technique outlined in the previous subsection. Recall the relation $\{C(\vec{N}), C(N)\} \propto C(\vec{N}[N])$ which says

[12] Given a Hilbert space $\mathcal{H}$ with dense subspace Φ on which a operator A is defined together with its adjoint, the dual A' on the space Φ^* of linear functionals l on Φ is defined by $(A'l)[f] := l(A^\dagger f)$ for all $f \in \Phi$.

that $\hat{C}(N)$ cannot be defined on $\mathcal{H}_{\mathrm{Diff}}$ because it must not leave that space invariant in any non-anomalous representation. Now consider the Master constraint

$$\mathbf{M} := \int_{\Sigma} d^3x \; \frac{C^2}{\sqrt{|\det(E)|}}. \tag{13.20}$$

Owing to the judicious choice of the "matrix" $K \propto |\det(E)|^{-1/2}$ the function $\mathbf{M}$ *is* spatially diffeomorphism invariant. It therefore can be represented directly on the Hilbert space $\mathcal{H}_{\mathrm{Diff}}$ and can be solved by the direct integral method of the previous subsection.

Thus we arrive at the physical Hilbert space $\mathcal{H}_{\mathrm{Phys}}$, which, however, is rather implicitly defined via the spectral resolution of the operator $\widehat{\mathbf{M}}$. The operator $\widehat{\mathbf{M}}$ is rather complicated as one might expect and hence its spectrum cannot be determined in closed form, although simple, normalisable (in the inner product of $\mathcal{H}_{\mathrm{Diff}}$) solutions are already known.

13.3.2 Outstanding problems and further results

In the previous subsection we have restricted our attention to the gravitational degrees of freedom but similar results also hold for the matter content of the (supersymmetric extension of) the standard model. In order to perform calculations of physical interest and to make contact with the well established framework of QFT on curved spacetimes (e.g. the physics of the standard model at large physical scales) it is mandatory to develop approximation schemes both for the physical inner product and for the Dirac observables that are in principle available as displayed in section 13.2. Also it is possible that what we have arrived at is a theory whose classical limit is not GR but rather a completely different sector, similar to the different phases that one can get in statistical physics or Euclidean QFT. Hence it is necessary to develop semiclassical tools in order to establish the correct classical limit. There is work in progress on both fronts: the spin foam models [33] that have been intensively studied can be viewed as avenues towards approximation schemes for the physical inner product. Furthermore, coherent (minimal uncertainty) states for background independent theories of connections have already been constructed at the level of $\mathcal{H}_{\mathrm{Kin}}$ [34; 35; 36; 37; 38; 39; 40; 41] and one now has to lift them to the level of $\mathcal{H}_{\mathrm{Diff}}$ and $\mathcal{H}_{\mathrm{Phys}}$ respectively.

Next, within LQG it has been possible to identify a black hole sector [42] which encompasses all black holes of astrophysical interest (Schwarzschild–Reissner–Nordstrom–Kerr–Newman family) and a careful analysis has identified the microscopic origin of the black hole entropy as punctures of the knots labelling physical states (plus the labels D) with the horizon. The entropy counting for large black holes results in the Bekenstein–Hawking value if the parameter ι assumes a definite, universal value. This is possible because in LQG geometrical operators such as volumes, areas and lengths of regions, surfaces (such as the horizon) and curves have discrete spectrum [43; 44; 45; 46] with a gap away from zero (otherwise the entropy would be infinite).[13] One may speculate whether the discreteness of the spectra hints at a combinatorical, distributional Planck scale structure of spacetime with no meaning to notions like smoothness or metrics which should emerge only on large scales.

Finally, certain minisuperspace approximations to LQG have been developed in order to perform approximate quantum cosmology [48]. The results obtained are subject to the usual restriction that a truncated model might not display the true behaviour of the full theory due to artificial suppression of degrees of freedom which might have large fluctuations in the full theory. See e.g. [49; 50] where it is shown that (big bang) singularity avoidance of the models is due to a mechanism which is only available in those models. On the other hand, as shown, in the full theory singularity avoidance could possibly be obtained by a more subtle feature of LQG. In any case, the models indicate that LQG indeed might be able to resolve the singularities of full GR.

To summarise: LQG is a mathematically rigorous approach to Quantum Gravity which is conceptually clear and simple. It only uses the principles of General Relativity and Quantum Mechanics and no experimentally unverified assumptions. It is fully background independent as every Quantum Gravity theory must be. Now tools have have been developed that enable one to make contact with experiment and thus to falsify the theory.

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[13] Actually, there are two unequivalent Volume Operators, one due to Rovelli and Smolin (RS) and the other one due to Ashtekar and Lewandowski (AL) [43; 44; 45; 46] which are both derived using background independent techniques from the fundamental flux operator. In a recent non-trivial consistency check [47] the RS and AL volume operators have been shown to be inconsistent and consistent respectively with the flux operator. This is a first example for an analysis which uses internal mathematical consistency in order to improve the degree of uniqueness of LQG.

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14

Covariant loop quantum gravity?

E. LIVINE

14.1 Introduction

In recent years, loop quantum gravity (LQG) has become a promising approach to Quantum Gravity (see e.g. [1; 2] for reviews). It has produced concrete results such as a rigorous derivation of the kinematical Hilbert space with discrete spectra for areas and volumes, the resulting finite isolated horizon entropy counting and regularization of black hole singularities, a well-defined framework for a (loop) quantum cosmology, and so on. Nevertheless, the model still has to face several key issues: a well-defined dynamics with a semi-classical regime described by Newton's gravity law and General Relativity, the existence of a physical semi-classical state corresponding to an approximately flat space-time, a proof that the no-gravity limit of LQG coupled to matter is standard quantum field theory, the Immirzi ambiguity, etc. Here, we address a fundamental issue at the root of LQG, which is necessarily related to these questions: *why the* SU(2) *gauge group of loop quantum gravity?* Indeed, the compactness of the SU(2) gauge group is directly responsible for the discrete spectra of areas and volumes, and therefore is at the origin of most of the successes of LQG: what happens if we drop this assumption?

Let us start by reviewing the general structure of LQG and how the SU(2) gauge group arises. In a first order formalism, General Relativity (GR) is formulated in term of tetrad e which indicates the local Lorentz frame and a Lorentz connection ω which describes the parallel transport. The theory is invariant under local Lorentz transformations and (space-time) diffeomorphisms.

The complex formulation of LQG is equivalent to that first order formalism. It is a canonical formulation based on a splitting of the space-time as a spatial slice evolving in time. The canonical variables are the Ashtekar variables: a self-dual complex connection A^{Ash} and its conjugate triad field E. The theory is invariant under the Lorentz group $\mathrm{SL}(2,\mathcal{C})$ (seen as the complexified SU(2) group) and under space-time diffeomorphisms. In these variables, GR truly looks like a

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

SU(2) gauge theory. The difficulty comes from reality constraints expressing that the imaginary part of the triad field E vanishes and that the real part of the connection A^{Ash} is actually a function of the E. More precisely, on one hand, keeping the metric real under the Hamiltonian flow requires that $\text{Re}\, E\nabla[E, E] = 0$ and, on the other hand, the real part of $A^{\text{Ash}} = \Gamma(E) + iK$ is the spin-connection $\Gamma(E)$ while its imaginary part is the extrinsic curvature. Such constraints must be taken into account by the measure of the space of connection and render the quantization complicated.

The real formulation of LQG came later as a way to avoid the reality constraint issue and has now become the standard formulation of LQG. It uses the real Ashtekar–Barbero connection $A^{\gamma} = \Gamma(E) + \gamma K$ and its conjugate triad field E. Here γ is called the *Immirzi parameter* and is an arbitrary real parameter. The theory is derived from the original first order GR formulation in a particular (partial) gauge fixing, the *time gauge*, which breaks the local Lorentz invariance down to a local SU(2) gauge invariance. The theory then has a compact gauge invariance, is free from complicated reality conditions and its Hamiltonian (constraint) can be regularized and quantized. Nevertheless, it appears as the result of a gauge fixing. The natural question is whether this affects the quantization or not: can we trust all the results of the real LQG formulation? As we will see, considering SU(2) as the gauge group of GR instead of the non-compact Lorentz group is related to several issues faced by the standard formulation of LQG.

- Since we have chosen a particular gauge fixing, should not we take it into account in the measure on the phase space through a Faddeev–Popov determinant? Would it not change the spectrum of the observables of the theory? Moreover, does choosing the time gauge constrain us to a specific class of measurements?
- The Ashtekar–Barbero connection A^{γ}, on the spatial slice, is not the pull-back of a space-time connection [3], since one can show that its holonomy on the spatial slice depends on the embedding on that slice in space-time. This is true unless the Immirzi parameter is taken as equal to the purely imaginary values $\gamma = \pm i$ corresponding to the original self-dual Ashtekar connection. From that point of view, the real connection A^{γ} can not be considered as a genuine gauge field and SU(2) can not be viewed as the gauge group of gravity.
- The complex LQG formalism has a simple polynomial Hamiltonian constraint. On the other hand, the real LQG formulation has an extra non-polynomial term. In fact, it seems we trade the reality condition problem with the issue of a more complicated Hamiltonian.
- There is a discrepancy with the standard spin foam models for GR. Spin foam models have been introduced as discretization of the GR path integral seen as a constrained topological theory [4]. They naturally appear as the space-time formalism describing the evolution and dynamics of the LQG canonical theory. Nevertheless, they use the Lorentz

group as gauge group and therefore the quantum states of quantum geometry are spin networks for the Lorentz group [5] instead of the standard SU(2) spin networks of LQG.

- In three space-time dimension, the standard loop gravity quantization of 3d gravity has as gauge group the full Lorentz group and not only the little group of spatial rotations. Indeed, in three space-time dimensions, the gauge group is always the Lorentz group, SO(3) in the Riemannian version [6] and SO(2, 1) in the Lorentzian theory [7]. This allows a precise matching between the LQG framework and the spin foam quantization for 3d gravity.
- Finally, the real LQG formulation faces the issue of the Immirzi ambiguity: γ is an arbitrary unfixed parameter. It enters the spectrum of geometrical observables such as areas and volumes (at the kinematical level). It is usually believed that black hole entropy calculations should fix this ambiguity by requiring a precise match with the semi-classical area–entropy law. More recently, γ has been argued to be related to parity violation when coupling fermions to gravity. Nevertheless, at the level of pure gravity, there still lacks a clear understanding of the physical meaning of γ: it does not change the classical phase space and canonical structure but leads to unitarily inequivalent quantization (at the kinematical level). We can not forget the possibility that this dependence on γ might only be due to the choice of the time gauge.

Here, we review a Lorentz covariant approach to loop quantum gravity, which has been coined *covariant loop quantum gravity*. It is based on an explicit canonical analysis of the original Palatini action for GR without any time gauge, first performed by Alexandrov [8]. The canonical variables are a Lorentz connection and its conjugate triad (a 1-form valued in the Lorentz algebra). The states of quantum geometry are Lorentz spin networks which reduce in a particular case to the standard SU(2) spin networks.

The main difference with the standard LQG is a continuous spectrum for areas at the kinematical level. The main advantages of the formalism is that the Immirzi ambiguity disappears and it becomes possible to make contact between the canonical theory and spin foam models. The main drawbacks of the approach are a non-compact gauge group and a non-commutative connection. Finally, there is still a lot of work left in order to precisely define the framework: rigorously define and study the Hilbert space (the problem is to deal with the non-commutativity of the connection) and derive the dynamics of the theory (quantize the Hamiltonian constraint and compare to the standard spin foam models).

14.2 Lorentz covariant canonical analysis

In a first order formalism, GR is formulated in terms of the space-time connection $\omega = \omega_\mu^{IJ} J_{IJ} dx^\mu$, defined as a $\mathfrak{so}(3,1)$-valued 1-form, and the tetrad field $e^I = e^I_\mu dx^\mu$. The space-time is a four-dimensional Lorentzian manifold $\mathcal{M}$ with signature $(- + ++)$; $I, \ldots$ are internal indices living in the tangent Minkowski space; η_{IJ} is the flat metric and J_{IJ} are the Lorentz generators; $\mu, \ldots$ are space-time indices. The *Palatini–Holst* action is [9]:

$$S[\omega, e] = \int_{\mathcal{M}} \left[\phi 12\, \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) - \phi 1\gamma\, e^I \wedge e^J \wedge F_{IJ}(\omega)\right], \quad (14.1)$$

where $F(\omega) = d\omega + \omega \wedge \omega$ is the curvature tensor of the connection ω. The metric is defined from the tetrad field as $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$. The first term of the previous action is the standard Palatini action. Its equations of motion are equivalent to the usual Einstein equations when the tetrad is non-degenerate. The second term actually has no effect on the equations of motion and thus does not matter at the classical level. The coupling constant γ is the Immirzi parameter.

The difficulty in the canonical analysis comes from the second class constraints. Indeed, the canonically conjugate variable to the connection ω_a^{IJ} is $\pi^a_{IJ} = \epsilon^{abc}\epsilon_{IJKL} e^K_b e^L_c$. These variables are unfortunately not independent and they satisfy the *simplicity constraints*:

$$\forall a, b,\ \epsilon^{IJKL} \pi^a_{IJ} \pi^b_{KL} = 0. \quad (14.2)$$

These constraints are the non-trivial part of the canonical structure. Nevertheless, Holst showed in [9] that, in the time gauge $e_{a0} = 0$, the tetrad e reduces to a triad field E, the simplicity constraints do not appear and we recover the canonical phase space and constraints of the real formulation of LQG with the Ashtekar–Barbero connection $A^{(\gamma)}$ conjugated to E and the Immirzi parameter γ.

The natural question is: how did the simplicity constraints go away? Barros pushed Holst's analysis further and showed it is possible to solve these constraints explicitly [10]. The phase space is parameterized by two couples of conjugate variables (A, E) and (χ, ζ). The first couple of canonical variables is a generalization of the Ashtekar–Barbero connection and triad. The new variable χ is the time normal (or internal time direction) defined as the normalized space component (in the internal indices) of the time component of the tetrad field: $\chi^i = -e^{0i}/e^{00}$. Finally, it is possible to gauge fix the boost part of the Lorentz gauge symmetry by fixing $\chi = 0$. This is the *time gauge*. In this frame, we exactly retrieve the variables and constraints of LQG. However, the price is the loss of the explicit Lorentz covariance of the theory.

14.2.1 Second class constraints and the Dirac bracket

The strategy of *covariant loop gravity* is to compute the whole set of second class constraints, derive the associated Dirac bracket and then quantize the theory. Here, we follow the canonical analysis of [8].

We start with a space-time $\mathcal{M} \sim \mathbb{R} \times \Sigma$ where we distinguish the time direction from the three space dimensions. We decompose the tetrad field e^I as:

$$\begin{aligned} e^0 &= N\mathrm{d}t + \chi_i E^i_a \mathrm{d}x^a \\ e^i &= E^i_a N^a \mathrm{d}t + E^i_a \mathrm{d}x^a, \end{aligned} \tag{14.3}$$

where $i = 1...3$ is an internal index (space components of I) and a is the space index labelling the coordinates x^a. N and N^a are respectively the *lapse* and the *shift*. χ^i indicates the deviation of the normal to the canonical hypersurface from the time direction: the time normal is defined as the normalised time-like 4-vector $\chi = (1, \chi_i)/\sqrt{1 - |\vec{\chi}|^2}$.

Let's call $X, Y, \ldots = 1 \ldots 6$ $\mathfrak{sl}(2, \mathcal{C})$-indices labelling antisymmetric couples $[IJ]$. We define new connection/triad variables valued in $\mathfrak{sl}(2, \mathcal{C})$ instead of the standard $\mathfrak{su}(2)$ of LQG. The Lorentz connection A^X_a is:

$$A^X = (\phi 12\omega^{0i}, \phi 12\epsilon^i_{\ jk}\omega^{jk}). \tag{14.4}$$

Then we define a "rotational" triad and a boost triad,

$$R^a_X = (-\epsilon_i^{\ jk} E^i_a \chi_k, E^i_a), \qquad B^a_X = (\star R^a)_X = (E^i_a, \epsilon_i^{\ jk} E^i_a \chi_k), \tag{14.5}$$

where $\star$ is the Hodge operator on $sl(2, \mathcal{C})$ switching the boost and rotation part of the algebra. We further define the actual projectors on the boost and rotation sectors of $\mathfrak{sl}(2, \mathcal{C})$, $(P_R)^X_Y = R^X_a R^a_Y$, $(P_B)^X_Y = B^X_a B^a_Y$:

$$P_R = \begin{pmatrix} -\frac{(\delta^b_a \chi^2 - \chi_a \chi^b)}{1-\chi^2} & -\frac{\epsilon_a^{\ bc}\chi_c}{1-\chi^2} \\ -\frac{\epsilon_a^{\ bc}\chi_c}{1-\chi^2} & \frac{\delta^b_a - \chi_a \chi^b}{1-\chi^2} \end{pmatrix}, \quad P_B = \mathrm{Id} - P_R, \quad P_B P_R = 0.$$

P_R projects on the subspace $\mathfrak{su}(2)_\chi$ generating the rotations leaving the vector χ invariant, while P_B projects on the complementary subspace. The action then reads:

$$S = \int dt d^3x \, \left(\left(B^a_X - \phi 1\gamma R^a_X \right) \partial_t A^X_a + \Lambda^X \mathcal{G}_X + \mathcal{N}^a \mathcal{H}_a + \mathcal{N}\mathcal{H} \right). \tag{14.6}$$

The phase space is thus defined with the Poisson bracket,

$$\left\{ A^X_a(x), \left(B^b_Y - \phi 1\gamma R^b_Y \right)(y) \right\} = \delta^X_Y \delta^b_a \delta^{(3)}(x, y). \tag{14.7}$$

$\Lambda^X, \mathcal{N}^a, \mathcal{N}$ are Lagrange multipliers enforcing the first class constraints:

$$\begin{aligned} \mathcal{G}_X &= \mathcal{D}_A \left(B_X - \phi 1\gamma R_X \right), \\ \mathcal{H}_a &= -\left(B^b_X - \phi 1\gamma R^b_X \right) F^X_{ab}(A), \\ \mathcal{H} &= \phi 11 + \frac{1}{\gamma^2} \left(B - \phi 1\gamma R \right) \left(B - \phi 1\gamma R \right) F(A). \end{aligned} \tag{14.8}$$

However, in contrast to the usual LQG framework, we also have *second class* constraints:

$$\phi^{ab} = (\star R^a)^X R^b_X = 0, \qquad \psi^{ab} \approx RR\mathcal{D}_A R. \tag{14.9}$$

The constraint $\phi = 0$ is the simplicity constraint. The constraint $\psi = 0$ comes from the Poisson bracket $\{\mathcal{H}, \phi\}$ and is required in order that the constraint $\phi = 0$ is preserved under gauge transformations (generated by $\mathcal{G}, \mathcal{H}_a, \mathcal{H}$) and in particular under time evolution. ψ corresponds to the reality constraint Re $E\nabla[E, E] = 0$ of complex LQG.

To solve the second class constraints, we define the *Dirac bracket* $\{f, g\}_D = \{f, g\} - \{f, \varphi_r\}\Delta^{-1}_{rs}\{\varphi_s, g\}$ where the Dirac matrix $\Delta_{rs} = \{\varphi_r, \varphi_s\}$ is made of the Poisson brackets of the constraints $\varphi = (\phi, \psi)$. Following [8; 11], one then checks that the algebra of the first class constraints is not modified. Defining smeared constraints, we find the following Dirac brackets:

$$\mathcal{G}(\Lambda) = \int_\Sigma \Lambda^X \mathcal{G}_X, \quad \mathcal{H}(N) = \int_\Sigma N\mathcal{H}, \quad \mathcal{D}(\vec{N}) = \int_\Sigma N^a(\mathcal{H}_a + A^X_a \mathcal{G}_X),$$

$$\begin{aligned}
&\Big\{\mathcal{G}(\Lambda_1), \mathcal{G}(\Lambda_2)\Big\}_D = \mathcal{G}([\Lambda_1, \Lambda_2]), && \Big\{\mathcal{D}(\vec{N}), \mathcal{D}(\vec{M})\Big\}_D = -\mathcal{D}([\vec{N}, \vec{M}]),\\
&\Big\{\mathcal{D}(\vec{N}), \mathcal{G}(\Lambda)\Big\}_D = -\mathcal{G}(N^a \partial_a \Lambda), && \Big\{\mathcal{D}(\vec{N}), \mathcal{H}(N)\Big\}_D = -\mathcal{H}(\mathcal{L}_{\vec{N}} N),\\
&\Big\{\mathcal{H}(N), \mathcal{G}(\Lambda)\Big\}_D = 0, && \Big\{\mathcal{H}(N), \mathcal{H}(M)\Big\}_D = \mathcal{D}(\vec{K}) - \mathcal{G}(K^b A_b),
\end{aligned}$$

$$\begin{aligned}
&[\Lambda_1, \Lambda_2]^X = f^X_{YZ} \Lambda^Y_1 \Lambda^Z_2, && [\vec{N}, \vec{M}]^a = N^b \partial_b M^a - M^b \partial_b N^a,\\
&\mathcal{L}_{\vec{N}} N = N^a \partial_a N - N \partial_a N^a, && K^b = (N\partial_a M - M\partial_a N) R^a_X R^b_Y g^{XY},
\end{aligned}$$

where f^X_{YZ} are the structure constant of the algebra $\mathfrak{sl}(2, \mathcal{C})$. With $A \in \{1, 2, 3\}$ boost indices and $B \in \{4, 5, 6\} \sim \{1, 2, 3\}$ rotation indices, we have $f^A_{AA} = f^A_{BB} = f^B_{AB} = 0$ and $f^A_{AA} = -f^A_{AB} = -f^B_{BB}$ given by the antisymmetric tensor ϵ.

The $\mathcal{G}$s generate SL$(2, \mathcal{C})$ gauge transformations. The vector constraint $\mathcal{H}_a$ generates spatial diffeomorphisms on the canonical hypersurface invariant Σ. Finally, the scalar constraint $\mathcal{H}$ is called the Hamiltonian constraint and generates the (time) evolution of the canonical variables.

14.2.2 The choice of connection and the area spectrum

As shown in [8; 11; 12], although the triad field R is still commutative for the Dirac bracket, the properties of the connection A change drastically: it is not canonically conjugated to the triad and it does not commute with itself. Nevertheless, one should keep in mind that when using the Dirac bracket the original canonical variables lose their preferred status and we should feel free to identify better suited variables. Following [12], we do not modify the triad R but we look for a new connection $\mathcal{A}$ satisfying the following natural criteria.

- $\mathcal{A}$ must be a Lorentz connection i.e. it should behave correctly under the Gauss law $\mathcal{G}$:

$$\{\mathcal{G}(\Lambda), \mathcal{A}_a^X\}_D = \partial_a \Lambda^X - [\Lambda, A_a]^X = \partial_a \Lambda^X - f_{YZ}^X \Lambda^Y A_a^Z. \tag{14.10}$$

- $\mathcal{A}$ must be a 1-form and therefore properly transform under spatial diffeomorphisms:

$$\{\mathcal{D}(\vec{N}), \mathcal{A}\}_D = \mathcal{A}_b^X \partial_a N^b - N^b \partial_a \mathcal{A}_b^X. \tag{14.11}$$

- $\mathcal{A}$ must be conjugated to the triad R. This is required in order that the area operators $Area_{\mathcal{S}} \sim \int_{\mathcal{S}} d^2x \sqrt{n_a n_b R_X^a R^{bX}}$ (with n_a the normal to the surface $\mathcal{S}$) be diagonalized in the spin network basis resulting from a loop quantization. This condition reads:

$$\{\mathcal{A}_a^X(x), R_Y^b(y)\}_D \propto \delta_a^b \delta^{(3)}(x, y). \tag{14.12}$$

We obtain a 2-parameter family of such connections $\mathcal{A}(\lambda, \mu)$ [12]:

$$\begin{aligned}\mathcal{A}_a^X(\lambda, \mu) \quad &= \quad A_a^X + \frac{1}{2}(\gamma + \lambda - \mu\star)\, P_R \phi(\gamma - \star)1 + \gamma^2\, [B_a, \mathcal{G}]^X \\ &\quad + (\lambda + (1-\mu)\star)\left(P_R \star A_a^X + \Theta_a^X(R)\right),\end{aligned} \tag{14.13}$$

with

$$\Theta_a^X(R) = \Theta_a^X(\chi) = \left(-\frac{\epsilon^{ijk}\chi_j \partial_a \chi_k}{1-\chi^2}, \frac{\partial_a \chi^i}{1-\chi^2}\right).$$

Their commutation relation with the triad are very simple:

$$\{\mathcal{A}_a^X(\lambda, \mu), B_Y^b\}_D = \delta_a^b\, [(\mu - \lambda\star)\, P_B]_Y^X \tag{14.14}$$

$$\{\mathcal{A}_a^X(\lambda, \mu), P_B\}_D = \{\mathcal{A}_a^X(\lambda, \mu), \chi\}_D = 0. \tag{14.15}$$

Despite this, the bracket $\{\mathcal{A}, \mathcal{A}\}_D$ remains complicated. From there, the loop quantization chooses functions of $\mathcal{A}$ (Wilson loops and spin networks) as wave functions and raises the triads B, R to derivation operators. Each connection $\mathcal{A}(\lambda, \mu)$ will lead to a non-equivalent quantization. We can then compute the action of an area operator on a $\mathcal{A}(\lambda, \mu)$ Wilson line and we find [12; 13]:

$$Area_{\mathcal{S}} \sim l_P^2 \sqrt{(\lambda^2 + \mu^2) C(\mathfrak{su}(2)_\chi) - \mu^2 C_1(\mathfrak{sl}(2, \mathcal{C})) + \lambda\mu C_2(\mathfrak{sl}(2, \mathcal{C}))},$$

where $C(su(2)_\chi) = \vec{J} \cdot \vec{J}$ is the Casimir operator of $\mathfrak{su}(2)_\chi$ (stabilizing the vector χ), $C_1(sl(2, \mathcal{C})) = T^X T_X = \vec{J}^2 - \vec{K}^2$ and $C_2(sl(2, \mathcal{C})) = (\star T)^X T_Y = \vec{J} \cdot \vec{K}$ are the two (quadratic) Casimirs of $\mathfrak{sl}(2, \mathcal{C})$. Since the algebra $\mathfrak{su}(2)$ enters the formula, one could think at first that this area spectrum is not Lorentz invariant. However, one must not forget that χ enters the formula and gets rotated under Lorentz transformations. Thus we see two alternatives.

(i) Either we work with functionals of the connection $\mathcal{A}$. Then a basis of quantum states is provided by spin networks for the Lorentz group. These are labelled by unitary representations of $\mathfrak{sl}(2, \mathcal{C})$, they diagonalize $C_1(\mathfrak{sl})$ and $C_2(\mathfrak{sl})$, but they do not diagonalize $C(\mathfrak{su})$. Therefore they do not diagonalize the area operator.
(ii) Or we work with functionals of both the connection $\mathcal{A}$ and the time normal field χ. This is possible when $\mathcal{A}$ and χ commute (see (14.15)). It is possible to introduce *projected spin networks*, which project on given eigenvalues of $C(\mathfrak{su})$ and therefore diagonalize the area operator. We will discuss the details of these states later.

In the following, we will work with the latter alternative. Then the irreducible unitary representations (of the principal series) of $\mathfrak{sl}(2, \mathcal{C})$ are labelled by a couple of numbers ($n \in \mathbb{N}$, $\rho \geq 0$). The Casimir's values are then:

$$C_1 = n^2 - \rho^2 - 1, \quad C_2 = 2n\rho, \quad C = j(j+1), \quad \text{with } j \geq n. \tag{14.16}$$

The restriction $j \geq n$ comes from the decomposition of the $\mathfrak{sl}(2, \mathcal{C})$ representations on $\mathfrak{su}(2)$ irreducible representations. Moreover this condition ensures that the area eigenvalues are all real (and positive) for any value of (λ, μ). This is a nice consistency check. Note however that, since the formula involves the real parameter ρ, we lose the discreteness of the spectrum, which was a key result of LQG!

Now, it seems that we do not have any preferred choice of connection, and therefore no rigorous prediction on the area spectrum. This would be an extra ambiguity besides the choice of the Immirzi parameter γ. Instead, we choose to impose further constraints on the connection $\mathcal{A}(\lambda, \mu)$ and two criteria naturally appear.

(i) We require that the connection behaves properly under *space-time* diffeomorphisms, generated by $\mathcal{H}_a$ and $\mathcal{H}$.
(ii) We require that the connection be commutative, i.e that $\{\mathcal{A}, \mathcal{A}\}_D$ vanishes.

Unfortunately, these two conditions are not compatible. As we will see in the next sections, the first choice corresponds to the only unique choice of a *covariant connection* and is the one used by the proposed Covariant LQG. Very interestingly, the area spectrum for this covariant connection does not depend on the Immirzi parameter γ. While this resolves the Immirzi ambiguity, it is still complicated to quantize the theory due to the non-commutativity of the connection. On the other hand, the second criteria leads to a unique commutative Lorentz extension of the Ashtekar–Barbero connection. It allows us to recover the $\mathfrak{su}(2)$ structure and area spectrum and Immirzi ambiguity of the real formulation of LQG.

This raises the issue of the space-time covariance of the standard formulation of LQG based on the Ashtekar–Barbero connection. Although there is no doubt that $\mathcal{H}_a$ and $\mathcal{H}$ satisfy the same algebra as the generators of the space-time diffeomorphisms, the action of $\mathcal{H}$ on the connection is not the usual one. This means that this connection is not a space-time 1-form and thus does not have a clear geometric interpretation. Although it is not clear to what extent this is a problem, we expect this to be an obstacle when studying the quantum dynamics of the theory.

14.3 The covariant connection and projected spin networks

14.3.1 A continuous area spectrum

As shown in [12; 13], there is a unique space-time connection, i.e which transforms as a 1-form under space-time diffeomorphism generated by the constraints $\mathcal{H}_a, \mathcal{H}$. It is actually the unique connection which is equal to the original connection A on the constrained surface $\mathcal{G}^X = \mathcal{H}_a = \mathcal{H} = 0$. It corresponds to the choice $(\lambda, \mu) = (0, 1)$ and we will simply write $\mathcal{A}$ for $\mathcal{A}(0, 1)$ in the following sections. Its brackets with the triad are:

$$\{\mathcal{A}_a^X, B_Y^b\}_D = \delta_a^b (P_B)_Y^X, \quad \{\mathcal{A}_a^X, (P_B)_Z^Y\}_D = 0. \tag{14.17}$$

The first bracket says that only the boost part of the connection seems to matter. The second relation is also very important and states that the field χ commutes with both the connections and can thus be treated as an independent variable. Then, following the results of the previous section, it turns out that the area spectrum does not depend on the Immirzi parameter at all and is given by the boost part of the $\mathfrak{sl}(2, \mathcal{C})$ Casimir:

$$Area \sim l_P^2 \sqrt{C(\mathfrak{su}(2)_\chi) - C_1(\mathfrak{sl}(2, \mathcal{C}))} = l_P^2 \sqrt{j(j+1) - n^2 + \rho^2 + 1}.$$

Interestingly, this spectrum is *not* the standard $\sqrt{j(j+1)}$ $\mathfrak{su}(2)$-Casimir area spectrum, but it contains a term coming from the Lorentz symmetry which makes it *continuous*.

The problem with this connection is that it is non-commutative. Indeed, the bracket $\{\mathcal{A}^X, \mathcal{A}^Y\}_D$ does not vanish and turns out to be complicated. At least, it is possible to prove that it does not depend on the Immirzi parameter. Actually it was shown [13] that this complicated bracket was due to the fact that the rotational part of $\mathcal{A}$ was not independent from the triad field but equal to the spin-connection:

$$P_R \mathcal{A}_a^X = \Gamma(R)_a^X \sim [R, \partial R] + RR[R, \partial R].$$

The explicit expression can be found in [13; 14; 15]. This relation is reminiscent of the reality constraint of the complex LQG formulation where the real part of the self-dual connection is a function of the triad E and is constrained to be the spin-connection $\Gamma(E)$. Moreover, it turns out that both the rotation and the boost parts of the connection are commutative:

$$\{(P_R\mathcal{A})^X, (P_R\mathcal{A})^Y\}_D = \{(P_B\mathcal{A})^X, (P_B\mathcal{A})^Y\}_D = 0. \tag{14.18}$$

At the end of the day, the non-commutativity of the connection comes from the facts that $P_B\mathcal{A}$ is canonically conjugate to the (boost) triad ($B = \star R$) and that the other half of the connection $P_R\mathcal{A}$ is a function of the triad. It thus seems as if this non-commutativity comes from taking into account the reality constraints.

14.3.2 Projected spin networks

In order to talk about the quantum theory and the area spectrum, we should precisely define the Hilbert space and our quantum states of space(-time) geometry. Since geometric observables (such as the area) involve χ and that χ commutes with $\mathcal{A}$, it is natural to consider functionals $f(\mathcal{A}, \chi)$ as wave functions for the quantum geometry. Then requiring gauge invariance under the Lorentz group SL$(2, \mathcal{C})$ reads:

$$\forall g \in \mathrm{SL}(2,\mathcal{C})\; f(\mathcal{A}, \chi) = f({}^g\mathcal{A} = g\mathcal{A}g^{-1} + g\partial g^{-1}, g.\chi). \tag{14.19}$$

Assuming that χ is time-like everywhere (i.e. the canonical hypersurface is space-like everywhere) and that all the fields are smooth, we can do a smooth gauge transformation to fix χ to $\chi_0 = (1, 0, 0, 0)$ everywhere. Thus the wave function is entirely defined by its section $f_{\chi_0}(\mathcal{A}) = f(\mathcal{A}, \chi_0)$ at $\chi = \chi_0$ constant:

$$f(\mathcal{A}, \chi) = f_{\chi_0}({}^g\mathcal{A}) \quad \text{for all } g \text{ such that } g.\chi = \chi_0.$$

Then f_{χ_0} has a residual gauge invariance under SU$(2)_{\chi_0}$. We are actually considering functionals of the Lorentz connection $\mathcal{A}$ which are not invariant under the full Lorentz group SL$(2, \mathcal{C})$ but only under the *compact* group of spatial rotations (defined through the field χ).

To proceed to a loop quantization, we introduce *cylindrical functionals* which depend on the fields $\mathcal{A}, \chi$ through a finite number of variables. More precisely, given a fixed oriented graph Γ with E links and V vertices, a cylindrical function depends on the holonomies $U_1, \ldots, U_E \in$ SL$(2, \mathcal{C})$ of $\mathcal{A}$ along the edges of Γ and on the values $\chi_1, \ldots, \chi_V$ of χ at the vertices of the graph. The gauge invariance then reads:

$$\forall k_v \in \mathrm{SL}(2,\mathcal{C}),\; \phi(U_e, \ldots, \chi_v, ..) = \phi(k_{s(e)}U_e k_{t(e)}^{-1}, \ldots, k_v.\chi_v, \ldots), \tag{14.20}$$

where $s(e), t(e)$ denote the source and target vertices of an edge e. As previously, such an invariant function is fully defined by its section $\phi_{\chi_0}(U_1, \ldots, U_E)$ at constant $\chi_1 = \cdots = \chi_V = \chi_0$. The resulting function ϕ_{χ_0} is invariant under $(\mathrm{SU}(2)_{\chi_0})^V$: we effectively reduced the gauge invariance from the non-compact SL$(2, \mathcal{C})^V$ to the compact $(\mathrm{SU}(2)_{\chi_0})^V$.

Physically, the field χ describes the embedding of the hypersurface Σ in the space-time $\mathcal{M}$. From the point of view of the cylindrical functionals, the embedding is defined only at a finite number of points (the graph's vertices) and is left fuzzy everywhere else. At these points, the normal to the hypersurface is fixed to the value χ_v and the symmetry thus reduced from $\mathrm{SL}(2,\mathcal{C})$ to $\mathrm{SU}(2)_{\chi_v}$.

Since the gauge symmetry is compact, we can use the Haar measure on $\mathrm{SL}(2,\mathcal{C})$ to define the scalar product on the space of wave functions:

$$\begin{aligned}\langle\phi|\psi\rangle &= \int_{\mathrm{SL}(2,\mathcal{C})^E}\prod_e dg_e\,\bar{\phi}(g_e,\chi_v)\psi(g_e,\chi_v)\\ &= \int_{\mathrm{SL}(2,\mathcal{C})^E}\prod_e dg_e\,\bar{\phi}_{\chi_0}(g_e)\psi_{\chi_0}(g_e).\end{aligned} \tag{14.21}$$

The Hilbert space H_Γ is finally defined as the space of L^2 cylindrical functions with respect to this measure. A basis of this space is provided by the *projected spin networks* [14; 16]. Following the standard construction of spin networks, we choose one (irreducible unitary) $\mathrm{SL}(2,\mathcal{C})$ representation $\mathcal{I}_e = (n_e,\rho_e)$ for each edge $e\in\Gamma$. However, we also choose one SU(2) representation $j_e^{(v)}$ for each link e at each of its extremities v. Moreover, we choose an SU(2) intertwiner i_v for each vertex instead of an $\mathrm{SL}(2,\mathcal{C})$ intertwiner. This reflects that the gauge invariance of the cylindrical function is $\mathrm{SU}(2)^V$.

Let's call $R^{(n,\rho)}$ the Hilbert space of the $\mathrm{SL}(2,\mathcal{C})$ representation $\mathcal{I}=(n,\rho)$ and V^j the space of the SU(2) representation j. If we choose a (time) normal $x\in \mathrm{SL}(2,C)/\mathrm{SU}(2)$ and consider the subgroup $\mathrm{SU}(2)_x$ stabilizing x, we can decompose $R^{\mathcal{I}}$ onto the irreducible representations of $\mathrm{SU}(2)_x$:

$$R^{(n,\rho)} = \bigoplus_{j\geq n} V^j_{(x)}. \tag{14.22}$$

Let's call $P^j_{(x)}$ the projector from $R^{(n,\rho)}$ onto $V^j_{(x)}$:

$$P^j_{(x)} = \Delta_j\int_{SU(2)_x} dg\,\overline{\zeta}^j(g)D^{(n,\rho)}(g), \tag{14.23}$$

where $\Delta_j = (2j+1)$ is the dimension of V^j, the integration is over $\mathrm{SU}(2)_x$, $D^{\mathcal{I}}(g)$ is the matrix representing the group element g acting on $R^{\mathcal{I}}$ and ζ^j is the character of the j-representation. To construct a projected spin network, we insert this projector at the end vertices of every link which allows us to glue the Lorentz holonomies to the SU(2) intertwiners. The resulting functional is:

$$\phi^{(\mathcal{I}_e, j_e, i_v)}(U_e, \chi_v) \;=\; \prod_v i_v \left[\bigotimes_{e \leftrightarrow v} |\mathcal{I}_e \chi_v j_e^{(v)} m_e\rangle \right] \prod_e \langle \mathcal{I}_e \chi_{s(e)} j_e^{(s(e))} m_e | D^{\mathcal{I}_e}(U_e) | \mathcal{I}_e \chi_{t(e)} j_e^{(t(e))} m_e \rangle, \tag{14.24}$$

with an implicit sum over the ms. $|\mathcal{I} x j m\rangle$ is the standard basis of $V^j_{(x)} \hookrightarrow R^{\mathcal{I}}$ with m running from $-j$ to j. In short, compared to the usual spin networks, we trace over the subspaces $V^j_{(\chi)}$ instead of the full spaces $R^{\mathcal{I}}$.

Using these projected spin networks allows us to project the Lorentz structures on specific fixed SU(2) representations. This allows us to diagonalize the area operators. Considering a surface $\mathcal{S}$ intersecting the graph Γ only on one edge e at the level of a (possibly bivalent) vertex, its area operator $Area_{\mathcal{S}}$ will be diagonalized by the projected spin network basis with the eigenvalues given above:

$$Area_{\mathcal{S}}\, |\phi_\Gamma^{(\mathcal{I}_e, j_e, i_v)}\rangle \;=\; l_P^2 \sqrt{j_e(j_e+1) - n_e^2 + \rho_e^2 + 1}\, |\phi_\Gamma^{(\mathcal{I}_e, j_e, i_v)}\rangle.$$

The procedure is now simple. Given a graph Γ and a set of surfaces, in order to have a spin network state diagonalizing the area operators associated to all these surfaces, we simply need to project that spin network state at all the intersections of the surfaces with Γ. If we want to obtain quantum geometry states diagonalizing the areas of all the surfaces in the hypersurface Σ, we would need to consider a "infinite refinement limit" where we project the spin network state at all points of its graph Γ. Such a procedure is described in more details in [13; 14]. However, from the point of view that space-time is fundamentally discrete at microscopic scales, it sounds reasonable to be satisfied with quantum geometry states that diagonalize the areas of a discrete number of surfaces (intersecting the graph at the points where we have projected the spin network states). This is consistent with the picture that considering a projected spin network state, the embedding of the hypersurface Σ into the space-time is only well-defined at the vertices of the graph, where we know the time normal χ: at all other points, Σ remains fuzzy and so must be the surfaces embedded in Σ.

14.3.3 Simple spin networks

Up to now, we have described quantum states $f(\mathcal{A})$ and the action of triad-based operators on them. We should also define the action of connection-based operators. We normally expect that $\mathcal{A}$ would act by simple multiplication of a wave function $f(\mathcal{A})$. Unfortunately, in our framework, $\mathcal{A}$ does not commute with itself, so this naïve prescription does not work. The point is that, owing to the second class constraint, the rotation part of the connection $P_R\mathcal{A}$ is constrained and must be equal to the spin connection $\Gamma[R]$ defined by the triad R. This reflects the reality constraints of LQG. The natural way out is that we would like wave functions which do *not* depend on $P_R\mathcal{A}$ but only on $P_B\mathcal{A}$. On such a state, the operator $\widehat{P_R\mathcal{A}}$ will be defined as $\Gamma[\widehat{R}]$, while $\widehat{P_B\mathcal{A}}$ will act simply by multiplication. This is consistent with the Dirac bracket since $P_B\mathcal{A}$ commutes with itself, $\{P_B\mathcal{A}^X, P_B\mathcal{A}^Y\}_D = 0$.

One way to achieve this using projected spin networks is to consider the case where we project on the trivial SU(2) representation $j = 0$. These are called *simple spin networks*. To start with, a simple representation $\mathcal{I} = (n, \rho)$ of the Lorentz group is defined such that $C_2(\mathcal{I}) = 2n\rho$ vanishes: we only consider representations of the type $(n, 0)$ and $(0, \rho)$. Then for a representation $\mathcal{I}$ to contain a SU(2)-invariant vector (corresponding to the $j = 0$ sector), we must necessarily have $n = 0$. Therefore, simple spin networks use simple Lorentz representations of the continuous type $\mathcal{I}_{\text{simple}} = (0, \rho)$. A simple spin network is defined by the assignment of such representations $(0, \rho_e)$ to each edge e of the graph Γ. Since SU(2)-intertwiners are trivial for the trivial representation $j = 0$, the functional then reads:

$$\phi^{(\rho_e)}(U_e, \chi_v) = \prod_e \langle (0, \rho_e)\chi_{s(e)} j = 0 | U_e | (0, \rho_e)\chi_{t(e)} j = 0\rangle. \tag{14.25}$$

Let's point out that in this special case of projected spin networks, we can consider open graphs (with "one-valent" vertices).

Simple spin networks are such that $\phi_\Gamma(U_e[\mathcal{A}])$ does not depend on $P_R\mathcal{A}$ at the vertices v of the graph Γ. In particular, considering two simple spin networks ϕ, ϕ' based on two graphs Γ and Γ' which only intersect at mutual vertices, then $\phi_\Gamma(U_e[\mathcal{A}])$ and $\phi'_{\Gamma'}(U_e[\mathcal{A}])$ commute.

From there, we have two alternatives. Either we consistently project the spin network states onto $j = 0$ at every point of the graph so that they completely solve the second class constraints. Or we can keep working with the present simple spin networks who only solve the second class constraints at a discrete level. At the end of the day, it will be these same simple spin networks which appear as kinematical geometry states in the spin foam quantization, as we will see later.

To summarize, we proved that the second class constraints are taken into account at the quantum level by restricting the previous projected spin networks to be simple. The final area spectrum taking into account the Lorentz gauge invariance and all the (kinematical) constraints is purely continuous:

$$Area_{\mathcal{S}}\,|\phi_\Gamma^{(\rho_e)}\rangle = l_P^2\sqrt{\rho_e^2 + 1}\,|\phi_\Gamma^{(\rho_e)}\rangle, \tag{14.26}$$

for a surface $\mathcal{S}$ intersecting Γ on the edge e. Nevertheless, the shift $\rho^2 \to \rho^2 + 1$ still leads to a non-vanishing minimal area l_P^2.

14.4 Going down to SU(2) loop gravity

As we have said earlier, there is a unique commutative Lorentz connection, which we will denote $\mathbf{A}$, and which corresponds to the choice $(\lambda, \mu) = (-\gamma, 0)$. It satisfies the following commutation relations:

$$\{\mathbf{A}, \mathbf{A}\}_D = 0, \qquad \{\mathbf{A}_a^X, R_Y^b\}_D = \gamma \delta_a^b (P_R)_Y^X. \tag{14.27}$$

Intuitively, while $\mathcal{A}$ was a pure boost connection, $\mathbf{A}$ is a purely rotational connection. More precisely, $\mathbf{A}$ can be simply expressed in terms of the original connection A and the time normal field χ:

$$\left| \begin{array}{lcl} P_R\mathbf{A} & = & P_R(1-\gamma\star)A - \gamma\Theta, \\ P_B\mathbf{A} & = & \star\Theta(\chi) = \star(\chi \wedge \partial\chi) \end{array} \right. , \tag{14.28}$$

where $\Theta(\chi)$ was introduced earlier in eqn. (14.13). From this expression, it is clear that $\mathbf{A}$ is commutative and that $P_B\mathbf{A}$ is not an independent variable. Actually, in the time gauge where the field χ is taken as a constant equal to χ_0, $\mathbf{A}$ reduces to the SU(2)-connection of the real Ashtekar–Barbero formalism. Then $\mathbf{A}$ is the natural Lorentz extension of that SU(2)-connection [13]. Finally, the area spectrum for this connection reproduces exactly the standard spectrum:

$$Area_{\mathcal{S}} \sim l_P^2 \sqrt{C(\mathfrak{su}(2)_\chi)} = l_P^2 \sqrt{j(j+1)}. \tag{14.29}$$

In order to completely recover LQG, we still need to take care of the second class constraints. To faithfully represent the Dirac bracket, we would indeed like wave functions which do not depend on the boost part of the connection $P_B\mathbf{A}$. We take this into account in the scalar product. Instead of using the SL$(2, \mathcal{C})$-Haar measure, we can restrict ourselves to the SU(2) subgroup. More precisely, we define the scalar product using the $\chi = \chi_0$ section of the wave functions:

$$\langle f|g\rangle = \int_{[SU_{\chi_0}(2)]^E} dU_e\, \overline{f_{\chi_0}(U_e)} g_{\chi_0}(U_e). \tag{14.30}$$

We are considering the usual scalar product of LQG, and a basis is given by the standard SU(2) spin networks. Nevertheless, it is now possible to go out of the time gauge and describe the wave functions for arbitrary χ fields. More precisely, if $f_{\chi_0}(A)$ is given by a SU(2) spin network, then $f(A, \chi)$ is a projected spin network in the "infinite refinement" limit. The interested reader will find more details in [13; 14].

14.5 Spin foams and the Barrett–Crane model

Up to now, we have described the kinematical structure of covariant loop (quantum) gravity. We still need to tackle the issue of defining the dynamics of the theory. On one hand, one can try to regularize and quantize *à la* Thiemann the action of the Hamiltonian constraint either on the covariant connection $\mathcal{A}$ or the commutative connection $\mathbf{A}$. In this case, we will naturally have to study the volume operator and face the usual ambiguities of LQG. On the other hand, one can turn to the *spin foam* formalism. Spin foams have evolved independently but in parallel to LQG. Inspired from state sum models, they provide well-defined path integrals for "almost topological" theories, which include gravity-like theories. Moreover, they use the same algebraic and combinatorial structures as LQG. In particular, spin networks naturally appear as the kinematical states of the theory. From this perspective, spin foams allow a covariant implementation of the LQG dynamics and a rigorous definition of the physical inner product of the theory.

In three space-time dimensions, pure gravity is described by a BF theory and is purely topological. The spin foam quantization is given by the Ponzano–Regge model [17]. Its partition function defines the projector onto the gravity physical states, i.e wave functions on the moduli space of flat Lorentz connections.

In four space-time dimensions, it turns out that General Relativity can be recast as a *constrained BF theory*. One can quantize the topological BF theory as a spin foam model and then impose the extra constraints directly on the partition function at the quantum level (e.g [18]). For 4d gravity, this leads to the Barrett–Crane model [19]. There are of course ambiguities in the implementation of the constraints, which lead to different versions of this model. We show below that the Barrett–Crane model provides a dynamical framework for covariant LQG.

14.5.1 Gravity as a constrained topological theory

Let us start with the Plebanski action:

$$S[\omega, B, \phi] = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}[\omega] - \frac{1}{2}\phi_{IJKL}\, B^{KL} \wedge B^{IJ} \right], \qquad (14.31)$$

where ω is the $\mathfrak{so}(3,1)$ connection, $F[\omega] = d_\omega \omega$ its curvature, B a $\mathfrak{so}(3,1)$-valued 2-form and ϕ a Lagrange multiplier satisfying $\phi_{IJKL} = -\phi_{JIKL} = -\phi_{IJLK} = \phi_{KLIJ}$ and $\phi_{IJKL}\epsilon^{IJKL} = 0$. The equations of motion are:

$$dB + [\omega, B] = 0, \quad F^{IJ}(\omega) = \phi^{IJKL} B_{KL}, \quad B^{IJ} \wedge B^{KL} = e\,\epsilon^{IJKL},$$

with $e = \frac{1}{4!}\epsilon_{IJKL}B^{IJ} \wedge B^{KL}$. When $e \neq 0$, the constraint on B is equivalent to the simplicity constraint, $\epsilon_{IJKL}B^{IJ}_{ab}B^{KL}_{cd} = \epsilon_{abcd}e$.

This constraint is satisfied if and only if there exists a real tetrad field $e^I = e^I_a dx^a$ such that either $B = \pm e \wedge e$ (sector $I_\pm$) or $B = \pm \star (e \wedge e)$ (sector $II_\pm$). These four sectors are due to the symmetry of the constraints under $B \to (\star B)$. The $\star$ operation allows us to switch the sectors: $I_+ \to II_+ \to I_- \to II_- \to I_+$. Restricting ourselves to the II_+ sector, the action reduces to $S = \int \star(e \wedge e) \wedge F$ and we recover General Relativity in the first order formalism. A first remark is that we still have to get rid of the $I_\pm$ and II_- sectors in the path integral at the quantum level. These are respectively related to the chirality of the 3-volume and to the issue of time orientation [21]. A second remark is that taking a more general constraint on the ϕ field, for instance $a\phi_{IJ}{}^{IJ} + b\phi_{IJKL}\epsilon^{IJKL} = 0$, we recover the Palatini–Holst action for General Relativity with Immirzi parameter [20].

14.5.2 Simple spin networks again

The spin foam strategy is first to discretize and quantize the topological BF theory as a state sum model, then to impose the B-constraints on the discretized partition function.

In order to discretize the path integral, we choose a triangulation (or more generally a cellular decomposition) of the 4d space-time gluing 4-simplices together. We then associate the B field to triangles, $B^{IJ}(t) = \int_t B^{IJ}$, and the connection curvature to the dual surfaces. The simplicity constraint of the 2-form, $\epsilon_{IJKL} B^{IJ}_{ab} B^{KL}_{cd} = e\epsilon_{abcd}$, is then translated to the discrete setting. For any two triangles t, t', we have:

$$\epsilon_{IJKL}\, B^{IJ}(t)\, B^{KL}(t') = \int_{t,t'} ed^2\sigma \wedge d^2\sigma' = V(t,t'),$$

where $V(t, t')$ is the 4-volume spanned by the two triangles. In particular, for any two triangles which share an edge, we have:

$$\epsilon_{IJKL}\, B^{IJ}(t)\, B^{KL}(t') = 0. \tag{14.32}$$

These are the Barrett–Crane constraints which are implemented at the quantum level. More precisely, we associate a copy of the $\mathfrak{sl}(2, \mathcal{C})$-algebra to each triangle t and we quantize the $B^{IJ}(t)$s as the Lorentz generators J^{IJ}_t. For a given triangle t, the previous constraint for $t = t'$ becomes $\epsilon_{IJKL} J^{IJ}_t J^{KL}_t = 0$, which is the vanishing of the second Casimir $C_2(\mathfrak{sl}(2, \mathcal{C})) = 0$. This means that the representation $\mathcal{I}_t$ associated to a triangle t must be *simple*: either $(n_t, 0)$ or $(0, \rho_t)$. The first Casimir $C_1 = J_{IJ} J^{IJ}$ gives the (squared) area of the triangle. For the discrete series, $C_1(n, 0) = -n^2 + 1$ is negative and the triangle is time-like. For the continuous series, $C_1(0, \rho) = \rho^2 + 1$ is positive and the triangle is space-like. Thus we recover the same simplicity of the $\mathfrak{sl}(2, \mathcal{C})$ representations as in covariant LQG.

The only difference is that we only consider space-like triangles in the canonical framework, and therefore only obtain the $(0, \rho)$ representations. The time-like representations would naturally appear in the canonical setting if considering a time-like normal χ (e.g. [15]). In the following, we will restrict ourselves to the $(0, \rho)$ representations.

Coupling between different triangles happens at the level of tetrahedra: to each tetrahedron is associated an intertwiner between the representations attached to its four triangles. Solving the constraints $\epsilon_{IJKL} J_t^{IJ} J_{t'}^{KL} = 0$ for every couple of triangles (t, t') of the tetrahedron leads to a unique intertwiner. This Barrett–Crane intertwiner $I_{BC} : \otimes_{t=1}^4 R^{(0,\rho_t)} \to \mathcal{C}$ is the only SU(2)-invariant intertwiner:

$$I_{BC} = \int_{\mathrm{SL}(2,\mathcal{C})/\mathrm{SU}(2)} d\chi \bigotimes_{t=1}^{4} \langle (0, \rho_t)\, \chi\, j = 0| \,. \tag{14.33}$$

We recover the intertwiner structure of the simple spin networks introduced for covariant LQG. More precisely, the quantum geometry states associated to any space-like slice of the triangulation in the Barrett–Crane model are simple spin networks [13; 21].

This makes the link between the kinematical states of the canonical theory and the spin foam states. Then the transition amplitudes of the Barrett–Crane model can be translated to the canonical context and considered as defining the dynamics of Covariant LQG.

14.5.3 The issue of the second class constraints

In the previous spin foam quantization, we discretized and quantized the path integral for General Relativity. We have dealt with the simplicity constraint $B.(\star B) = 0$ by imposing on the path integral. A priori, this corresponds to the simplicity constraint (14.9), $\phi = R.(\star R) = 0$ of the canonical analysis. However, it seems that we are missing the other second class constraint $\psi \sim RR\mathcal{D}_A R$. The ψ constraints are essential to the computation of the Dirac bracket: shouldn't we discretize them too and include them in the spin foam model?

The spin foam point of view is that we have already taken them into account. Indeed, the ψ are secondary constraints, coming from the Poisson bracket $\mathcal{H}, \phi$: at first, $\phi = 0$ is only imposed on the initial hypersurface and we need $\psi = 0$ to ensure we keep $\phi = 0$ under the Hamiltonian evolution. On the other hand, the Barrett–Crane model is fully covariant and $\phi = 0$ is directly imposed on all the space-time structures: we have projected on $\phi = 0$ at all stages of the evolution (i.e. on all hypersurfaces). The Barrett–Crane construction ensures that a simple spin network will remain a simple spin network under evolution. In this sense, we do not need the secondary constraints ψ. It would nevertheless be interesting to check that a discretized version of ψ vanishes on the Barrett–Crane partition function.

14.6 Concluding remarks

Starting with the canonical analysis of the Palatini–Holst action, we have shown how the second class constraints are taken into account by the Dirac bracket. Requiring a good behavior of the Lorentz connection under Lorentz gauge transformations and space diffeomorphisms, we obtain a two-parameter family of possible connection variables. Requiring that the connection further behaves as a 1-form under space-time diffeomorphisms, we obtain a unique covariant connection. This leads to a covariant LQG with "simple spin networks" (for the Lorentz group), a continuous area spectrum and an evolution dictated by the Barrett–Crane spin foam model. The theory turns out to be independent of the Immirzi parameter. The main obstacle to a full quantization is the non-commutativity of this connection. This can be understood as reflecting the reality conditions of the complex formulation of LQG. On the other hand, there exists a unique commutative connection. It turns out to be a generalization of the Ashtekar–Barbero connection of the real formulation of LQG. We further recover the $SU(2)$ spin networks, the standard discrete area spectrum and the usual Immirzi ambiguity.

It seems that covariant LQG could help address some long-standing problems of the standard formulation of LQG, such as the Immirzi ambiguity, the issue of the Lorentz symmetry, the quantization of the Hamiltonian constraint and how to recover the space-time diffeomorphisms at the quantum level.

Finally, a couple of issues which should be addressed within the covariant LQG theory to ground it more solidly are:

- a study of the 3-volume operator acting on simple spin networks;
- a derivation of the spin foam amplitudes from the covariant LQG Hamiltonian constraint, possibly following the previous work in 3d gravity [22].

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15

The spin foam representation of loop quantum gravity

A. PEREZ

15.1 Introduction

The problem of background independent Quantum Gravity is the problem of defining a Quantum Field Theory of matter and gravity in the absence of an underlying background geometry (see Chapter 1 by Rovelli). Loop quantum gravity (LQG) is a promising proposal for addressing this difficult task. Its main predictions and underlying mathematical structure are described in Chapter 13 by Thiemann. Despite the steady progress of the field, dynamics remains to a large extend an open issue in LQG. Here we present the main ideas behind a series of proposals for addressing the issue of dynamics. We refer to these constructions as the *spin foam representation* of LQG. This set of ideas can be viewed as a systematic attempt at the construction of the path integral representation of LQG.

The *spin foam representation* is mathematically precise in $2+1$ dimensions, so we will start this chapter by showing how it arises in the canonical quantization of this simple theory (more about $2+1$ gravity can be found in Chapter 16 by Freidel). This toy model will be used to precisely describe the true geometric meaning of the histories that are summed over in the path integral of generally covariant theories.

In four dimensions similar structures appear. We call these constructions *spin foam models* as their definition is incomplete in the sense that at least one of the following issues remains unclear: (1) the connection to a canonical formulation, and (2) regularization independence (renormalizability). In the second part of this chapter we will describe the definition of these models, emphasizing the importance of these open issues.

15.2 The path integral for generally covariant systems

LQG is based on the canonical (Hamiltonian) quantization of general relativity whose gauge symmetry is diffeomorphism invariance. In the Hamiltonian formulation the presence of gauge symmetries [1] gives rise to relationships among the

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti.
Published by Cambridge University Press.

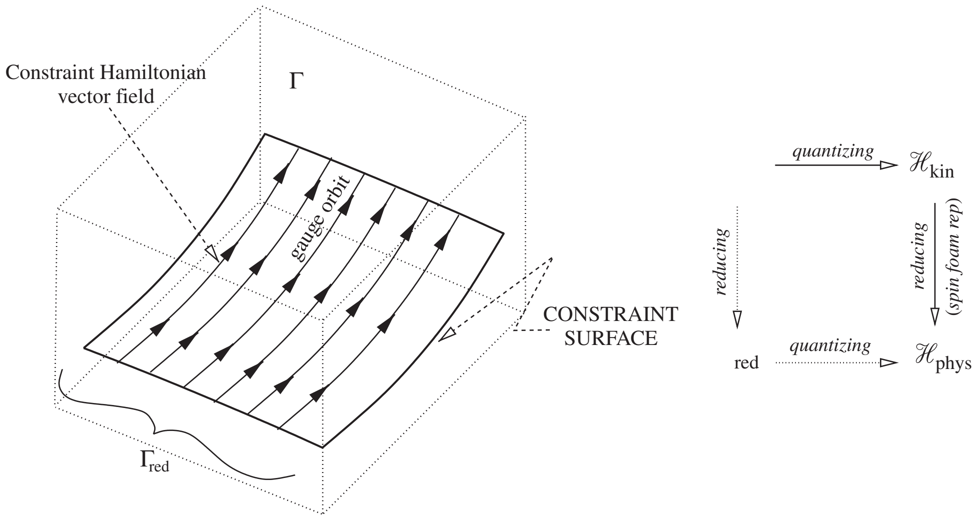


Fig. 15.1. On the left: the geometry of phase space in gauge theories. On the right: the quantization path of LQG (continuous arrows).

phase space variables – schematically $C(p, q) = 0$ for $(p, q) \in \Gamma$ – which are referred to as *constraints*. The constraints restrict the set of possible states of the theory by requiring them to lie on the constraint hyper-surface. In addition, through the Poisson bracket, the constraints generate motion associated with gauge transformations on the constraint surface (see Fig. 15.1). The set of physical states (the so called reduced phase space Γ_{red}) is isomorphic to the space of orbits, i.e. two points on the same gauge orbit represent the same state in Γ_{red} described in different gauges (Fig. 15.1).

In general relativity the absence of a preferred notion of time implies that the Hamiltonian of gravity is a linear combination of constraints. This means that Hamilton equations cannot be interpreted as time evolution and rather correspond to motion along gauge orbits of general relativity. In generally covariant systems conventional time evolution is pure gauge: from initial data satisfying the constraints one recovers a spacetime by selecting a particular one-parameter family of gauge-transformations (in the standard ADM context this amounts to choosing a particular lapse function $N(t)$ and shift $N^a(t)$).

From this perspective the notion of spacetime becomes secondary and the dynamical interpretation of the the theory seems problematic (in the quantum theory this is referred to as the "problem of time"). A possible reason for this apparent problem is the central role played by the spacetime representation of classical gravity solutions. However, the reason for this is to a large part due to the applicability of the concept of *test observers* (or more generally test fields) in classical general relativity.[1] Owing to the fact that this idealization is a good approximation to the (classical) process of observation the notion of spacetime is useful in classical gravity.

As emphasized by Einstein with his hole argument (see [2] for a modern explanation) only the information in relational statements (independent of any spacetime representation) have physical meaning. In classical gravity it remains useful to have a spacetime representation when dealing with idealized test observers. For instance to solve the geodesic equation and then ask diff-invariant-questions such as: what is the proper time elapsed on particle 1 between two successive crossings with particle 2? However, in the classical theory the advantage of the spacetime picture becomes, by far, less clear if the test particles are replaced by real objects coupling to the gravitational field.[2]

However, this possibility is no longer available in Quantum Gravity where at the Planck scale ($\ell_{\mathrm{p}} \approx 10^{-33}$cm) the quantum fluctuations of the gravitational field become so important that there is no way (not even in principle[3]) to make observations without affecting the gravitational field. In this context there cannot be any, *a priori*, notion of time and hence no notion of spacetime is possible at the fundamental level. A spacetime picture would only arise in the semi-classical regime with the identification of some subsystems that approximate the notion of test observers.

What is the meaning of the path integral in the background independent context? The previous discussion rules out the conventional interpretation of the path integral. There is no meaningful notion of transition amplitude between states at different times $t_1 > t_0$ or equivalently a notion of "unitary time evolution" represented by an operator $U(t_1 - t_0)$. Nevertheless, a path integral representation of generally covariant systems arises as a tool for implementing the constraints in the quantum theory as we argue below.

Because of the difficulty associated with the explicit description of the reduced phase space Γ_{red}, in LQG one follows Dirac's prescription. One starts by quantizing unconstrained phase space Γ, representing the canonical variables as self-adjoint operators in a kinematical Hilbert space $\mathscr{H}_{\mathrm{kin}}$. Poisson brackets are replaced by

[1] Most (if not all) of the textbook applications of general relativity make use of this concept together with the knowledge of certain exact solutions. In special situations there are even preferred coordinate systems based on this notion which greatly simplify interpretation (e.g. co-moving observers in cosmology, or observers at infinity for isolated systems).

[2] In this case one would need first to solve the constraints of general relativity in order to find the initial data representing the self-gravitating objects. Then one would have essentially two choices: (1) fix a lapse $N(t)$ and a shift $N^a(t)$, evolve with the constraints, obtain a spacetime (out of the data) in a particular gauge, and finally ask the diff-invariant-question; or (2) try to answer the question by simply studying the data itself (without t-evolution). It is far from obvious whether the first option (the conventional one) is any easier than the second.

[3] In order to make a Planck scale observation we need a Planck energy probe (think of a Planck energy photon). It would be absurd to suppose that one can disregard the interaction of such a photon with the gravitational field treating it as a test photon.

commutators in the standard way, and the constraints are promoted to self-adjoint operators (see Fig. 15.1). If there are no anomalies the Poisson algebra of classical constraints is represented by the commutator algebra of the associated quantum constraints. In this way the quantum constraints become the infinitesimal generators of gauge transformations in $\mathscr{H}_{\rm kin}$. The physical Hilbert space $\mathscr{H}_{\rm phys}$ is defined as the kernel of the constraints, and hence to gauge invariant states. Assuming for simplicity that there is only one constraint we have

$$\psi \in \mathscr{H}_{\rm phys} \quad \text{iff} \quad \exp[iN\hat{C}] \mid \psi\rangle = \mid \psi\rangle \quad \forall \quad N \in \mathbb{R},$$

where $U(N) = \exp[iN\hat{C}]$ is the unitary operator associated with the gauge transformation generated by the constraint C with parameter N. One can characterize the set of gauge invariant states, and hence construct $\mathscr{H}_{\rm phys}$, by appropriately defining a notion of 'averaging' along the orbits generated by the constraints in $\mathscr{H}_{\rm kin}$. For instance if one can make sense of the *projector*

$$P : \mathscr{H}_{\rm kin} \to \mathscr{H}_{\rm phys} \quad \text{where} \quad P := \int dN\, U(N). \tag{15.1}$$

It is apparent from the definition that for any $\psi \in \mathscr{H}_{\rm kin}$ then $P\psi \in \mathscr{H}_{\rm phys}$. The path integral representation arises in the representation of the unitary operator $U(N)$ as a sum over gauge-histories in a way which is technically analogous to a standard path integral in quantum mechanics. The physical interpretation is however quite different as we will show in Section 15.3.4. The spin foam representation arises naturally as the path integral representation of the field theoretical analog of P in the context of LQG. Needless to say, many mathematical subtleties appear when one applies the above formal construction to concrete examples (see [3]).

15.3 Spin foams in 3d Quantum Gravity

Here we derive the spin foam representation of LQG in a simple solvable example: $2+1$ gravity. For the definition of spin foam models directly in the covariant picture see the chapter by Freidel, and for other approaches to 3d Quantum Gravity see Carlip's book [5].

15.3.1 The classical theory

Riemannian gravity in three dimensions is a theory with no local degrees of freedom, i.e. a topological theory. Its action (in the first order formalism) is given by

$$S[e, \omega] = \int_M \mathrm{Tr}(e \wedge F(\omega)), \tag{15.2}$$

where $M = \Sigma \times \mathbb{R}$ (for Σ an arbitrary Riemann surface), ω is an $SU(2)$-connection and the triad e is an $su(2)$-valued 1-form. The gauge symmetries of the action are the local $SU(2)$ gauge transformations

$$\delta e = [e, \alpha], \qquad \delta\omega = d_\omega \alpha, \tag{15.3}$$

where α is a $su(2)$-valued 0-form, and the "topological" gauge transformation

$$\delta e = d_\omega \eta, \qquad \delta\omega = 0, \tag{15.4}$$

where d_ω denotes the covariant exterior derivative and η is a $\mathtt{su}(2)$-valued 0-form. The first invariance is manifest from the form of the action, while the second is a consequence of the Bianchi identity, $d_\omega F(\omega) = 0$. The gauge symmetries are so large that all the solutions to the equations of motion are locally pure gauge. The theory has only global or topological degrees of freedom.

Upon the standard $2+1$ decomposition, the phase space in these variables is parametrized by the pull back to Σ of ω and e. In local coordinates one can express them in terms of the two-dimensional connection A_a^i and the triad field $E_j^b = \epsilon^{bc} e_c^k \delta_{jk}$ where $a = 1, 2$ are space coordinate indices and $i, j = 1, 2, 3$ are $su(2)$ indices. The Poisson bracket is given by

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^{\ b}\, \delta^i_{\ j}\, \delta^{(2)}(x, y). \tag{15.5}$$

Local symmetries of the theory are generated by the first class constraints

$$D_b E_j^b = 0, \quad F_{ab}^i(A) = 0, \tag{15.6}$$

which are referred to as the Gauss law and the curvature constraint respectively. This simple theory has been quantized in various ways in the literature [5], here we will use it to introduce the spin foam representation.

15.3.2 Spin foams from the Hamiltonian formulation

The physical Hilbert space, $\mathscr{H}_{\text{phys}}$, is defined by those "states in $\mathscr{H}_{\text{kin}}$" that are annihilated by the constraints. As discussed in the chapter by Thiemann (see also [2; 4]), spin network states solve the Gauss constraint – $\widehat{D_a E_i^a}|s\rangle = 0$ – as they are manifestly $SU(2)$ gauge invariant. To complete the quantization one needs to characterize the space of solutions of the quantum curvature constraints $\widehat{F}_{ab}^i$, and to provide it with the physical inner product. As discussed in Section 15.2 we can achieve this if we can make sense of the following formal expression for the generalized projection operator P:

$$P = \int D[N] \exp(i \int_\Sigma \mathrm{Tr}[N \widehat{F}(A)]) = \prod_{x \subset \Sigma} \delta[\widehat{F(A)}], \tag{15.7}$$

where $N(x) \in su(2)$. Notice that this is just the field theoretical analog of equation (15.1). P will be defined below by its action on a dense subset of test-states called the cylindrical functions $\mathrm{Cyl} \subset \mathscr{H}_{\mathrm{kin}}$ (see the chapter by Thiemann). If P exists then we have

$$\langle sPU[N], s'\rangle = \langle sP, s'\rangle \;\forall\; s, s' \in \mathrm{Cyl},\; N(x) \in su(2) \tag{15.8}$$

where $U[N] = \exp(i \int \mathrm{Tr}[iN\hat{F}(A)])$. P can be viewed as a map $P : \mathrm{Cyl} \to K_F \subset \mathrm{Cyl}^{\star}$ (the space of linear functionals of Cyl) where K_F denotes the kernel of the curvature constraint. The physical inner product is defined as

$$\langle s', s\rangle_p := \langle s'P, s\rangle, \tag{15.9}$$

where $\langle, \rangle$ is the inner product in $\mathscr{H}_{\mathrm{kin}}$, and the physical Hilbert space as

$$\mathscr{H}_{\mathrm{phys}} := \overline{\mathrm{Cyl}/J} \quad \text{for} \quad J := \{s \in \mathrm{Cyl} \;\text{ s.t. }\; \langle s, s\rangle_p = 0\}, \tag{15.10}$$

where the bar denotes the standard Cauchy completion of the quotient space in the physical norm.

One can make (15.7) a rigorous definition if one introduces a regularization. A regularization is necessary to avoid the naive UV divergences that appear in QFT when one quantizes non-linear expressions of the canonical fields such as $F(A)$ in this case (or those representing interactions in standard particle physics). A rigorous quantization is achieved if the regulator can be removed without the appearance of infinities, and if the number of ambiguities appearing in this process is under control (more about this in Section 15.4.1). We shall see that all this can be done in the simple toy example of this section.

We now introduce the regularization. Given a partition of Σ in terms of two-dimensional plaquettes of coordinate area ϵ^2 (Fig. 15.2) one can write the integral

$$F[N] := \int_{\Sigma} \mathrm{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_{p} \epsilon^2 \mathrm{Tr}[N_p F_p] \tag{15.11}$$

as a limit of a Riemann sum, where N_p and F_p are values of the smearing field N and the curvature $\epsilon^{ab} F^i_{ab}[A]$ at some interior point of the plaquette p and ϵ^{ab} is

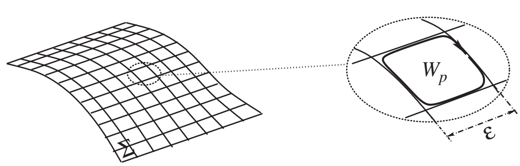


Fig. 15.2. Cellular decomposition of the space manifold Σ (a square lattice of size ϵ in this example), and the infinitesimal plaquette holonomy $W_p[A]$.

the Levi–Civita tensor. Similarly the holonomy $W_p[A]$ around the boundary of the plaquette p (see Fig. 15.2) is given by

$$W_p[A] = \mathbb{1} + \epsilon^2 F_p(A) + \mathcal{O}(\epsilon^2). \tag{15.12}$$

The previous two equations imply that $F[N] = \lim_{\epsilon\to 0} \sum_p \mathrm{Tr}[N_p W_p]$, and lead to the following definition: given $s, s' \in \mathrm{Cyl}$ (think of *spin network* states) the physical inner product (15.9) is given by

$$\langle s' P, s \rangle := \lim_{\epsilon \to 0} \ \langle s \prod_p \int dN_p \exp(i\mathrm{Tr}[N_p W_p]), s\rangle. \tag{15.13}$$

The partition is chosen so that the links of the underlying spin network graphs border the plaquettes. One can easily perform the integration over the N_p using the identity (Peter–Weyl theorem)

$$\int dN \exp(i\mathrm{Tr}[NW]) = \sum_j (2j+1)\ \mathrm{Tr}[\overset{j}{\Pi}(W)], \tag{15.14}$$

where $\overset{j}{\Pi}(W)$ is the spin j unitary irreducible representation of $SU(2)$. Using the previous equation

$$\langle s' P, s \rangle := \lim_{\epsilon \to 0} \ \prod_p^{n_p(\epsilon)} \sum_{j_p} (2j_p+1)\ \langle s'\ \mathrm{Tr}[\overset{j_p}{\Pi}(W_p)]), s\rangle, \tag{15.15}$$

where the spin j_p is associated with the pth plaquette, and $n_p(\epsilon)$ is the number of plaquettes. Since the elements of the set of Wilson loop operators $\{W_p\}$ commute, the ordering of plaquette-operators in the previous product does not matter. The limit $\epsilon \to 0$ exists and one can give a closed expression for the physical inner product. That the regulator can be removed follows from the orthonormality of $SU(2)$ irreducible representations which implies that the two spin sums associated with the action of two neighboring plaquettes collapses into a single sum over the action of the *fusion* of the corresponding plaquettes (see Fig 15.3). One can also show that it is finite,[4] and satisfies all the properties of an inner product [6].

[4] The physical inner product between spin network states satisfies the following inequality

$$\left|\langle s, s'\rangle_p\right| \leq C \textstyle\sum_j (2j+1)^{2-2g},$$

for some positive constant C. The convergence of the sum for genus $g \geq 2$ follows directly. The case of the sphere $g = 0$ and the torus $g = 1$ can be treated individually [6].

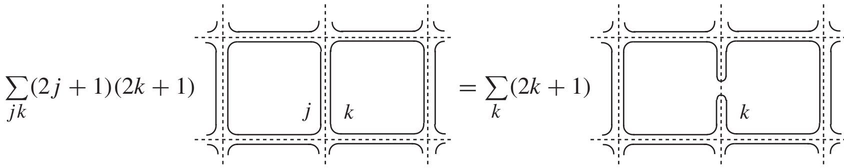
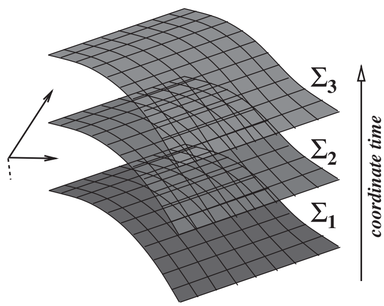


Fig. 15.3. In two dimensions the action of two neighboring plaquette-sums on the vacuum is equivalent to the action of a single larger plaquette action obtained from the fusion of the original ones. This implies the trivial *scaling* of the physical inner product under refinement of the regulator and the existence of a well defined limit $\epsilon \to 0$.

15.3.3 The spin foam representation

Each $\mathrm{Tr}[\overset{j_p}{\Pi}(W_p)]$ in (15.15) acts in $\mathscr{H}_{\mathrm{kin}}$ by creating a closed loop in the j_p representation at the boundary of the corresponding plaquette (Figs. 15.4 and 15.6). Now, in order to obtain the spin foam representation we introduce a non-physical (coordinate time) as follows. Instead of working with one copy of the space manifold Σ we consider $n_p(\epsilon)$ copies as a discrete foliation $\{\Sigma_p\}_{p=1}^{n_p(\epsilon)}$. Next we represent each of the $\mathrm{Tr}[\overset{j_p}{\Pi}(W_p)]$ in (15.15) on the corresponding Σ_p. If one inserts the partition of unity in $\mathscr{H}_{\mathrm{kin}}$ between the slices, graphically

$$\mathbb{1} = \sum_{\gamma \subset \Sigma, \{j\}_\gamma} |\gamma, \{j\}\rangle\langle\gamma, \{j\}| \tag{15.16}$$



where the sum is over the complete basis of spin network states $\{|\gamma, \{j\}\rangle\}$ – based on all graphs $\gamma \subset \Sigma$ and with all possible spin labeling – one arrives at a sum over spin-network histories representation of $\langle s, s'\rangle_p$. More precisely, $\langle s', s\rangle_p$ can be expressed as a sum over amplitudes corresponding to a series of transitions that can be viewed as the "time evolution" between the "initial" spin network s' and the "final" spin network s. This is illustrated in the two simple examples of Figs. 15.5 and 15.7); on the r.h.s. we illustrate the continuum spin foam picture obtained when the regulator is removed in the limit $\epsilon \to 0$.

Spin network nodes evolve into edges while spin network links evolve into two-dimensional faces. Edges inherit the intertwiners associated with the nodes and faces inherit the spins associated with links. Therefore, the series of transitions can be represented by a 2-complex whose 1-cells are labelled by intertwiners and

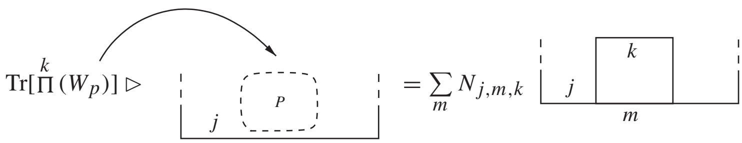


Fig. 15.4. Graphical notation representing the action of one plaquette holonomy on a spin network state. On the right is the result written in terms of the spin network basis. The amplitude $N_{j,m,k}$ can be expressed in terms of Clebsch–Gordan coefficients.

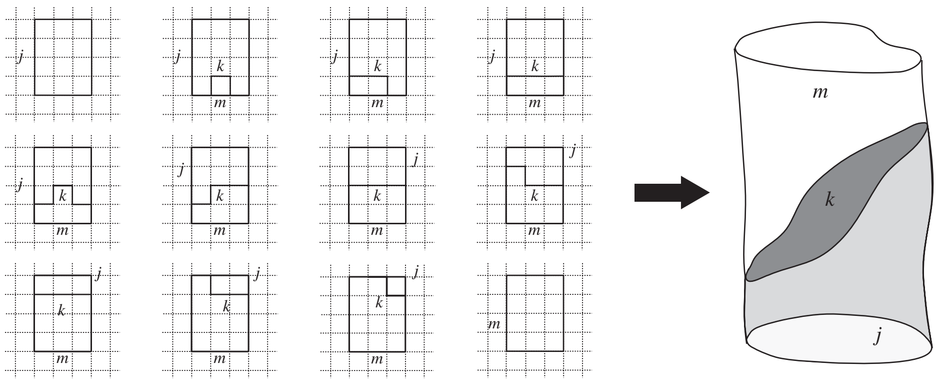


Fig. 15.5. A set of discrete transitions in the loop-to-loop physical inner product obtained by a series of transitions as in Fig. 15.4. On the right, the continuous spin foam representation in the limit $\epsilon \to 0$.

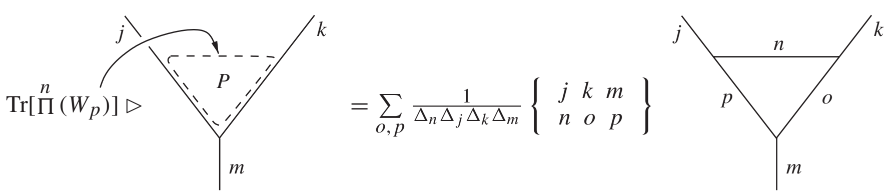


Fig. 15.6. Graphical notation representing the action of one plaquette holonomy on a *spin network* vertex. The object in brackets ({}) is a $6j$-symbol and $\Delta_j := 2j + 1$.

whose 2-cells are labelled by spins. The places where the action of the plaquette loop operators create new links (Figs. 15.6 and 15.7) define 0-cells or vertices. These foam-like structures are the so-called spin foams. The spin foam amplitudes are purely combinatorial and can be explicitly computed from the simple action of the loop operator in $\mathscr{H}_{\text{kin}}$. The physical inner product takes the standard

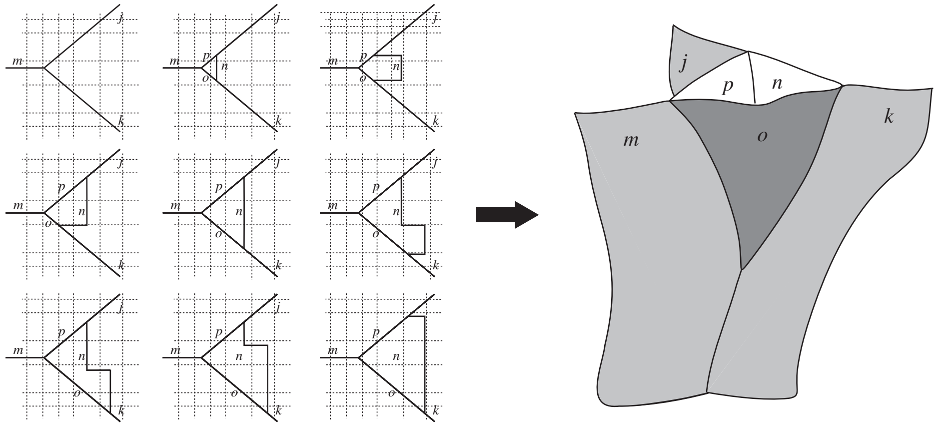
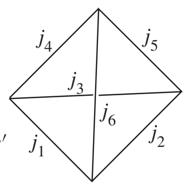


Fig. 15.7. A set of discrete transitions representing one of the contributing histories at a fixed value of the regulator. On the right, the continuous spin foam representation when the regulator is removed.

Ponzano–Regge form when the spin network states s and s' have only 3-valent nodes. Explicitly,

$$\langle s, s' \rangle_p = \sum_{F_{s \to s'}} \prod_{f \subset F_{s \to s'}} (2j_f + 1)^{\frac{\nu_f}{2}} \prod_{v \subset F_{s \to s'}}$$  , (15.17)

where the sum is over all the spin foams interpolating between s and s' (denoted $F_{s \to s'}$, see Fig. 15.10), $f \subset F_{s \to s'}$ denotes the faces of the spin foam (labeled by the spins j_f), $v \subset F_{s \to s'}$ denotes vertices, and $\nu_f = 0$ if $f \cap s \neq 0 \wedge f \cap s' \neq 0$, $\nu_f = 1$ if $f \cap s \neq 0 \vee f \cap s' \neq 0$, and $\nu_f = 2$ if $f \cap s = 0 \wedge f \cap s' = 0$. The tetrahedral diagram denotes a $6j$-symbol: the amplitude obtained by means of the natural contraction of the four intertwiners corresponding to the 1-cells converging at a vertex. More generally, for arbitrary spin networks, the vertex amplitude corresponds to $3nj$-symbols, and $\langle s, s' \rangle_p$ takes the same general form.

Even though the ordering of the plaquette actions does not affect the amplitudes, the spin foam representation of the terms in the sum (15.17) is highly dependent on that ordering. This is represented in Fig. 15.8 where a spin foam equivalent to that of Fig. 15.5 is obtained by choosing an ordering of plaquettes where those of the central region act first. One can see this freedom of representation as an analogy of the gauge freedom in the spacetime representation in the classical theory.

One can in fact explicitly construct a basis of $\mathscr{H}_{\text{phys}}$ by choosing a linearly independent set of representatives of the equivalence classes defined in (15.10).

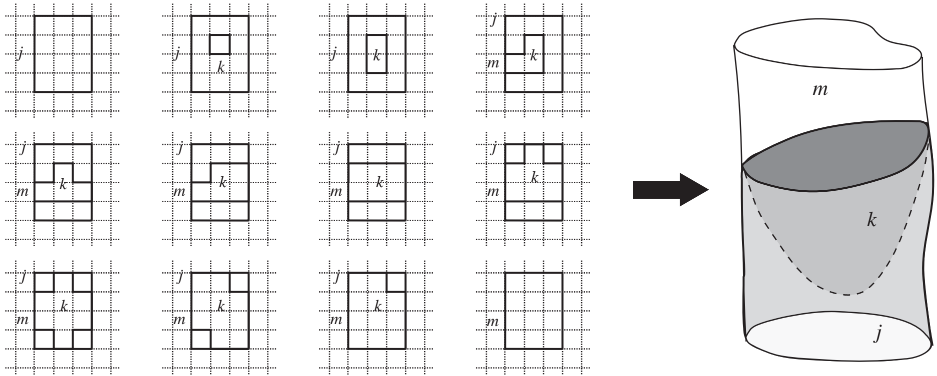


Fig. 15.8. A different representation of the transition of Fig. 15.5. This spin foam is obtained by a different ordering choice in (15.15).

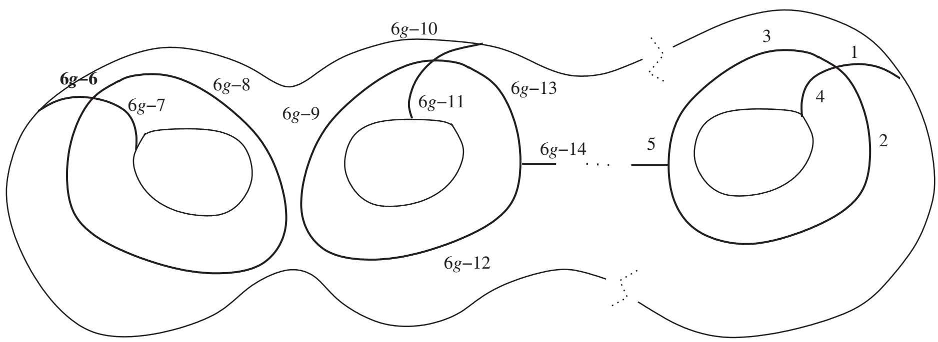


Fig. 15.9. A spin-network basis of physical states for an arbitrary genus g Riemann surface. There are $6g-6$ spins labels (recall that 4-valent nodes carry an intertwiner quantum number).

One such basis is illustrated in Fig. 15.9. The number of quantum numbers necessary to label the basis element is $6g-6$, corresponding to the dimension of the moduli space of $SU(2)$ flat connections on a Riemann surface of genus g. This is the number of degrees of freedom of the classical theory. In this way we arrive at a fully combinatorial definition of the standard $\mathscr{H}_{\rm phys}$ by reducing the infinite degrees of freedom of the kinematical phase space to finitely many by the action of the generalized projection operator P.

15.3.4 Quantum spacetime as gauge-histories

What is the geometric meaning of the spin foam configurations? Can we identify the spin foams with "quantum spacetime configurations"? The answer to the above questions is, strictly speaking, in the negative in agreement with our discussion at

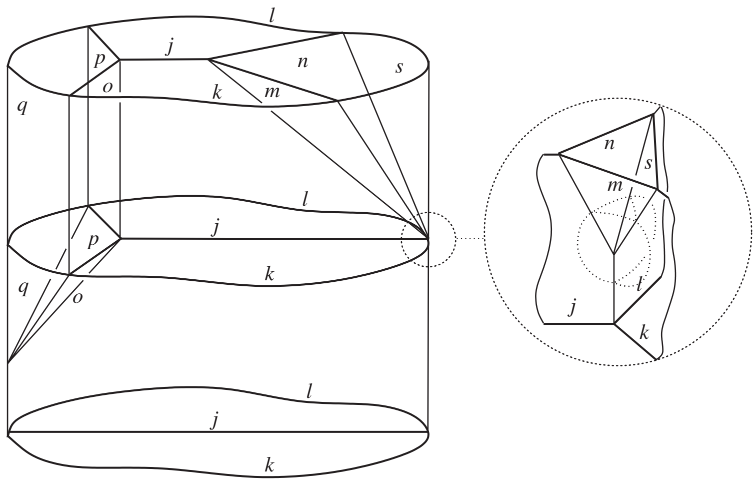


Fig. 15.10. A spin foam as the "colored" 2-complex representing the transition between three different spin network states. A transition vertex is magnified on the right.

the end of Section 15.2. This conclusion can be best illustrated by looking first at the simple example in $2+1$ gravity where $M = S^2 \times \mathbb{R}$ ($g = 0$). In this case the spin foam configurations appearing in the transition amplitudes look locally the same as those appearing in the representation of P for any other topology. However, a close look at the physical inner product defined by P permits one to conclude that the physical Hilbert space is one dimensional – the classical theory has zero degree of freedom and so there is no non-trivial Dirac observable in the quantum theory. This means that the sum over spin foams in (15.17) is nothing else but a sum over pure gauge degrees of freedom and hence no physical interpretation can be associated to it. The spins labelling the intermediate spin foams do not correspond to any measurable quantity. For any other topology this still holds true, the true degrees of freedom being of a global topological character. This means that in general (even when local excitations are present as in 4d) the spacetime geometric interpretation of the spin foam configurations is subtle. This is an important point that is often overlooked in the literature: one cannot interpret the spin foam sum of (15.17) as a sum over geometries in any obvious way. Its true meaning instead comes from the averaging over the gauge orbits generated by the quantum constraints that defines P – recall the classical picture Fig. 15.1, the discussion around eq. (15.1), and the concrete implementation in $2+1$ where $U(N)$ in (15.8) is the unitary transformation representing the orbits generated by F. Spin foams represent a gauge history of a kinematical state. A sum over gauge histories is what defines P as a means for extracting the true degrees of freedom from those which are encoded in the kinematical boundary states.

Here we studied the interpretation of the spin foam representation in the precise context of our toy example; however, the validity of the conclusion is of general character and holds true in the case of physical interest four dimensional LQG. Although, the quantum numbers labelling the spin foam configurations correspond to eigenvalues of kinematical geometric quantities such as length (in $2+1$) or area (in $3+1$) LQG, their physical meaning and measurability depend on dynamical considerations (for instance the naive interpretation of the spins in $2+1$ gravity as quanta of physical length is shown here to be of no physical relevance). Quantitative notions such as time, or distance as well as qualitative statements about causal structure or time ordering are misleading (at best) if they are naively constructed in terms of notions arising from an interpretation of spin foams as quantum spacetime configurations.[5]

15.4 Spin foam models in four dimensions

We have studied $2+1$ gravity in order to introduce the qualitative features of the spin foam representation in a precise setting. Now we discuss some of the ideas that are pursued for the physical case of $3+1$ LQG.

Spin foam representation of canonical LQG

There is no complete construction of the physical inner product of LQG in four dimensions. The spin foam representation as a device for its definition was originally introduced in the canonical formulation by Rovelli [2]. In four-dimensional LQG difficulties in understanding dynamics are centered around understanding the space of solutions of the quantum scalar constraint $\hat{S}$ (see Chapter 13 by Thiemann). The physical inner product formally becomes

$$\langle Ps, s' \rangle_{\text{diff}} = \int D[N] \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle \left[\int_{\Sigma} N(x) \widehat{S}(x) \right]^n s, s' \right\rangle_{\text{diff}}, \qquad (15.18)$$

where $\langle\ ,\ \rangle_{\text{diff}}$ denotes the inner product in the Hilbert space of diff-invariant states, and the exponential in (the field theoretical analog of) (15.1) has been expanded in powers.

Smooth loop states are naturally annihilated by $\widehat{S}$ (independently of any quantization ambiguity [9; 10]). Consequently, $\widehat{S}$ acts only on spin network nodes. Generically, it does so by creating new links and nodes modifying the underlying graph of the spin network states (Fig. 15.11).

[5] The discussion of this section is a direct consequence of Dirac's perspective applied to the *spin foam representation*.

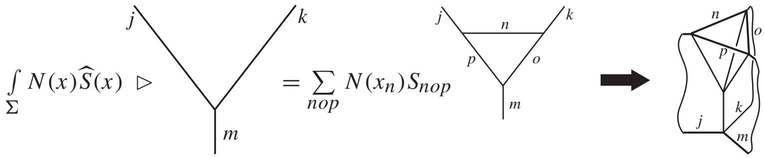


Fig. 15.11. The action of the scalar constraint and its spin foam representation. $N(x_n)$ is the value of N at the node and S_{nop} are the matrix elements of $\widehat{S}$.

In a way that is qualitatively similar to what we found in the concrete implementation of the curvature constraint in $2+1$ gravity, each term in the sum (15.18) represents a series of transitions – given by the local action of $\widehat{S}$ at spin network nodes – through different spin network states interpolating the boundary states s and s' respectively. The action of $\hat{S}$ can be visualized as an "interaction vertex" in the 'time' evolution of the node (Fig. 15.11). As in $2+1$ dimensions, eq. (15.18) can be pictured as the sum over "histories" of spin networks pictured as a system of branching surfaces described by a 2-complex whose elements inherit the representation labels on the intermediate states (see Fig. 15.10). The value of the "transition" amplitudes is controlled by the matrix elements of $\widehat{S}$.

Spin foam representation in the Master Constraint Program

The previous discussion is formal. One runs into technical difficulties if one tries to implement the construction of the $2+1$ gravity in this case. The main reason for this is the fact that the constraint algebra does not close with structure constants in the case of $3+1$ gravity.[6] In order to circumvent this problem (see the chapter by Thiemann) Thiemann recently proposed to impose one single *master* constraint defined as

$$M = \int_\Sigma dx^3 \, \frac{S^2(x) - q^{ab} V_a(x) V_b(x)}{\sqrt{\det q(x)}}, \tag{15.19}$$

where q^{ab} is the space metric and $V_a(x)$ is the vector constraint. Using techniques developed by Thiemann, this constraint can indeed be promoted to a quantum operator acting on $\mathscr{H}_{\rm kin}$. The physical inner product could then be defined as

$$\langle s, s' \rangle_p := \lim_{T\to\infty} \left\langle s, \int_{-T}^{T} dt \; e^{it\widehat{M}} s' \right\rangle. \tag{15.20}$$

[6] In $2+1$ gravity the constraint algebra correspond to the Lie algebra of $ISO(3)$ (isometries of Euclidean flat spacetime). There are no local degrees of freedom and the underlying gauge symmetry has a non-dynamical structure. In $3+1$ gravity the presence of gravitons changes that. The fact that the constraint algebra closes with structure functions means that the gauge symmetry structure is dynamical or field dependent. This is the key difficulty in translating the simple results of $2+1$ into $3+1$ dimensions.

A spin foam representation of the previous expression is obtained by splitting the t-parameter in discrete steps and writing

$$e^{it\widehat{M}} = \lim_{n\to\infty} [e^{it\widehat{M}/n}]^n = \lim_{n\to\infty} [1 + it\widehat{M}/n]^n. \qquad (15.21)$$

The *spin foam representation* follows from the fact that the action of the basic operator $1+it\widehat{M}/n$ on a spin network can be written as a linear combination of new spin networks whose graphs and labels have been modified by the creation of new nodes (in a way qualitatively analogous to the local action shown in Fig. 15.11). An explicit derivation of the physical inner product of 4d LQG along these lines is under current investigation.

Spin foam representation: the covariant perspective

In four dimensions the spin foam representation of LQG has also been motivated by lattice discretizations of the path integral of gravity in the covariant formulation (for recent reviews see [7; 8] and Chapter 16 by Freidel). In four dimensions there are two main lines of approach; both are based on classical formulations of gravity based on modifications of the BF-theory action.

The first approach is best represented by the Barrett–Crane model [11] and corresponds to the quantization attempt of the classical formulation of gravity based on the Plebanski action

$$S[B, A, \lambda] = \int \mathrm{Tr}\,[B \wedge F(A) + \lambda\; B \wedge B]\,, \qquad (15.22)$$

where B is an $so(3, 1)$-valued two-form λ is a Lagrange multiplier imposing a quadratic constraint on the Bs whose solutions include the sector $B = \star(e \wedge e)$, for a tetrad e, corresponding to gravity in the tetrad formulation. The key idea in the definition of the model is that the path integral for BF-theory, whose action is $S[B, A, 0]$,

$$P_{\mathrm{topo}} = \int\; D[B]D[A]\,\exp\left[i\int \mathrm{Tr}\,[B \wedge F]\right] \qquad (15.23)$$

can be defined in terms of spin foams by a simple generalization of the construction of Section 15.3 [13]. Notice that the formal structure of the action $S[B, A, 0]$ is analogous to that of the action of $2+1$ gravity (15.2) (see [12]). The Barrett–Crane model aims at providing a definition of the path integral of gravity formally written as

$$P_{\mathrm{GR}} = \int\; D[B]D[A]\;\delta\,[B \to \star(e \wedge e)]\;\exp\left[i\int \mathrm{Tr}\,[B \wedge F]\right], \qquad (15.24)$$

where the measure $D[B]D[A]\delta[B \to \star(e\wedge e)]$ restricts the sum in (15.23) to those configurations of the topological theory satisfying the constraints $B = \star(e \wedge e)$ for some tetrad e. The remarkable fact is that the constraint $B = \star(e \wedge e)$ can be directly implemented on the spin foam configurations of P_{topo} by appropriate restriction on the allowed spin labels and intertwiners. All this is possible if a regularization is provided, consisting of a cellular decomposition of the spacetime manifold. The key open issue is, however, how to get rid of this regulator. A proposal for a regulator independent definition is that of the *group field theory formulation* presented in Chapter 17 by Oriti.

A second proposal is the one recently introduced by Freidel and Starodubtsev [14] based on the formulation McDowell–Mansouri action of Riemannian gravity given by

$$S[B, A] = \int \text{Tr}[B \wedge F(A) - \frac{\alpha}{4} B \wedge B\gamma_5], \tag{15.25}$$

where B is an $so(5)$-valued two-form, A an $so(5)$ connection, $\alpha = G\Lambda/3 \approx 10^{-120}$ a coupling constant, and the γ_5 in the last term produces the symmetry breaking $SO(5) \to SO(4)$. The idea is to define P_{GR} as a power series in α, namely

$$P_{\text{GR}} = \sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{4^n n!} \int D[B]D[A](\text{Tr}[B \wedge B\gamma_5])^n \exp\left[i \int \text{Tr}[B \wedge F]\right]. \tag{15.26}$$

Notice that each term in the sum is the expectation value of a certain power of Bs in the well understood topological BF field theory. A regulator in the form of a cellular decomposition of the spacetime manifold is necessary to give a meaning to the former expression. Because of the absence of local degrees of freedom of BF-theory it is expected that the regulator can be removed in analogy to the $2+1$ gravity case. It is important to show that removing the regulator does not produce an uncontrollable set of ambiguities (see remarks below regarding renormalizability).

15.4.1 The UV problem in the background independent context

In the spin foam representation, the functional integral for gravity is replaced by a sum over amplitudes of combinatorial objects given by foam-like configurations (spin foams). This is a direct consequence of the background independent treatment of the gravitational field degrees of freedom. As a result there is no place for the UV divergences that plague standard Quantum Field Theory. The combinatorial nature of the fundamental degrees of freedom of geometry appears as a regulator of all the interactions. This seem to be a common feature of all the formulations referred to in this chapter. Does it mean that the UV problem in LQG is resolved? The answer to this question remains open for the following reason. All the definitions of spin foams models require the introduction of some kind of regulator generically represented by a space (e.g. in the canonical formulation of $2+1$ gravity or in the master-constraint program) or spacetime lattice (e.g. in the Barrett–Crane model or in the Freidel–Starodubtsev prescription). This lattice plays the role of a UV regulator in more or less the same sense as a UV cut-off (Λ) in standard QFT. The UV problem in standard QFT is often associated with divergences in the amplitudes when the limit $\Lambda \to \infty$ is taken. The standard renormalization procedure consists of taking that limit while appropriately tuning the bare parameters of the theory so that UV divergences cancel to give a finite answer. Associated to this process there is an intrinsic ambiguity as to what values certain amplitudes should take. These must be fixed by appropriate comparison with experiments (renormalization conditions). If only a finite number of renormalization conditions are required the theory is said to be renormalizable. The ambiguity of the process of removing the regulator is an intrinsic feature of QFT.

The background independent treatment of gravity in LQG or the spin foam models we have described here do not escape these general considerations (see [15]). Therefore, even though no UV divergences can arise as a consequence of the combinatorial structure of the gravitational field, the heart of the UV problem is now to be found in the potential ambiguities associated with the elimination of the regulator. This remains an open problem for all the attempts of quantization of gravity in $3+1$ dimensions. The problem takes the following form in each of the approaches presented in this chapter.

- The removal of the regulator in the $2+1$ case is free of ambiguities and hence free of any UV problem (see [15]).
- In the case of the master constraint program one can explicitly show that there is a large degree of ambiguity associated to the regularization procedure [15]. It remains to be shown whether this ambiguity is reduced or disappears when the regulators are removed in the definition of P.
- The Barrett–Crane model is discretization dependent. No clear-cut prescription for the elimination of the triangulation dependence is known.
- The Freidel–Starodubtsev prescription suffers (in principle) from the ambiguities associated with the definition of the expectation value of the B-monomials appearing in (15.26) before the regulator is removed.[7] It is hoped that the close relationship with a topological theory might cure these ambiguities although this remains to be shown.

Progress in the resolution of this issue in any of these approaches would represent a major breakthrough in LQG.

[7] There are various prescriptions in the literature on how to define these monomials. They are basically constructed in terms of the insertion of appropriate sources to construct a BF generating function. All of them are intrinsically ambiguous, and the degree of ambiguity grows with the order of the monomial. The main source of ambiguity resides in the issue of where in the discrete lattice to act with the functional source derivatives.

Acknowledgement

I would like to thank M. Mondragon for careful reading of the manuscript and D. Oriti for his effort and support in this project.

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16

Three-dimensional spin foam Quantum Gravity

L. FREIDEL

16.1 Introduction

Loop quantum gravity provides a background independent approach to Quantum Gravity. In this context the kinematical Hilbert space is spanned by spin networks (graph labeled by Lorentz group representations) which are eigenstates of geometrical operators. The dynamics of such theories is encoded in a set of transitions amplitudes between initial and final spin network states which carry the information about the physical inner product of the theory. Generically, these amplitudes are constructed in terms of spin foam models which are local state sum models associated with a sum of colored 2-complexes interpolating between initial and final spin network states.

There are many important questions that need to be addressed in this framework such as the proper choice of the dynamics, the construction and interpretation of the spin foam amplitude, the coupling to matter and the description of the semi-classical regime of such a theory. The contributions of D. Oriti and A. Perez in this volume address some of these issues.

In this contribution we will focus on the simple case of three-dimensional gravity and its quantization via spin foam models. The advantage of using the spin foam framework is twofold. First, this framework is not specifically tailored to three dimensions, unlike Chern–Simons quantization for instance, and some of the lessons and techniques used there can be useful for the more realistic four-dimensional case. Second, this way of quantizing gravity agrees with other quantizations when they apply but is in general applicable to a larger class of problems (like Lorentzian gravity and computation of topology changing amplitudes).

The proper way to encode the dynamic of 3d gravity in terms of spin foam model was discovered a long time ago by Ponzano and Regge [1]. In the following we summarize a set of recent results [2; 3; 4; 5] concerning the Ponzano–Regge

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti.
Published by Cambridge University Press.

model. We show how this definition can be related to the discretization of the gravity action, we then show that a residual gauge symmetry still present in the system should be gauge fixed and that this eliminates unwanted symmetry; we then describe the coupling of matter to 3d gravity and show how it can be effectively described in terms of a non-commutative braided quantum field theory.

16.2 Classical gravity and matter

In the first order formalism, 3d gravity is described in term of a frame field $e^i_\mu dx^\mu$ and a spin connection $\omega^i_\mu dx^\mu$. They are both valued in the Lie algebra $\mathfrak{so}(3)$ for the Euclidean theory, while they would be in $\mathfrak{so}(2,1)$ in the Lorentzian theory. Both indices i and μ run from 0 to 2. The action is defined as:

$$S[e,\omega] = \frac{1}{16\pi G} \int e^i \wedge F_i[w], \tag{16.1}$$

where $F \equiv d\omega + \omega \wedge \omega$ is the curvature tensor of the 1-form ω. The equation of motion for pure gravity then simply imposes that the connection is flat,

$$F[\omega] = 0.$$

The second equation of motion imposes that the torsion vanishes, $T = d_\omega e = 0$. Spinless particles are introduced as a source of curvature (the spin would be introduced as a source of torsion):

$$F^i[\omega] = 4\pi G p^i \delta(x).$$

Outside the particle, the space-time remains flat and the particle simply creates a conical singularity with deficit angle related to the particle's mass [6; 7]:

$$\theta = \kappa m, \quad \kappa \equiv 4\pi G. \tag{16.2}$$

This deficit angle gives the back reaction of the particle on the space-time geometry.

Since the deficit angle is obviously bounded by 2π, we have a maximal mass (for a point-particle) which defines the Planck mass:

$$m \leq m_{\max} = \frac{2\pi}{\kappa} = \frac{1}{2G} = m_{\mathrm{P}}.$$

The fact that the Planck mass m_{P} in three space-time dimensions depends only on the Newton constant G and does not depend on the Planck constant $\hbar$ is an essential feature of 3d Quantum Gravity and explains why 3d gravity possesses such surprising features as an ADM energy bounded from below and above.

16.3 The Ponzano–Regge model

The Ponzano–Regge model provides a full quantization of 3d gravity as a state sum model. It can be considered as a discretization of the path integral of the action (16.1),

$$Z = \int De D\omega e^{iS[e,\omega]}.$$

Nevertheless, since the theory is topological, the discretized path integral actually provides a quantization of the continuum theory.

More specifically, we consider a triangulation Δ of a 3d manifold M, made of vertices, edges e and faces f. We can work more generically with a cellular decomposition. The triad field e^i_μ is discretized as Lie algebra elements $X_e \in \mathfrak{so}(3)$ attached to each edge $e \in \Delta$ while the connection ω^i_μ is defined through SO(3) group elements $g_f \in \mathrm{SO}(3)$ attached to each face f or equivalently attached to each dual edge $e^* \in \Delta^*$. We can further define the holonomy G_e around each edge e and discretize the action as:

$$S[X_e, g_f] = \frac{1}{16\pi G} \sum_e \mathrm{tr}(X_e G_e), \quad G_e \equiv \overrightarrow{\prod_{\partial f \ni e}} g_f. \tag{16.3}$$

Using the following formula giving the δ-distribution on the group[1] SO(3):

$$\delta(G) = \int_{\mathfrak{so}(3)} \frac{d^3X}{8\pi} e^{\frac{1}{2}\mathrm{tr}(XG)}, \tag{16.4}$$

we can integrate the triad variables X_e and write the partition function as a product of δ-functions imposing the flatness of the connection:

$$Z = \int \prod_e \mathrm{d}X_e \prod_f \mathrm{d}g_f \, e^{\frac{1}{2}\sum_e \mathrm{tr}(X_e G_e)} \tag{16.5}$$

$$= \int \prod_f \mathrm{d}g_f \prod_e \delta(G_e), \tag{16.6}$$

with $\mathrm{d}g$ the normalized Haar measure and $\mathrm{d}X \equiv d^3X/8\pi$. Re-expanding $\delta(G)$ in terms of the SU(2)-characters:

$$\delta(g) = \sum_{j \in \mathcal{N}} d_j \chi_j(g),$$

[1] In this chapter we work with $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, the delta function on SO(3) is related to the delta function on SU(2) by $\delta_{\mathrm{SO}(3)}(g) = (\delta_{\mathrm{SU}(2)}(g) + \delta_{\mathrm{SU}(2)}(-g))$.

with $d_j = 2j + 1$, we can finally integrate out the g_f variables and express the partition function as a state sum model with sole variables the representation labels j_e attached to each edge of the triangulation:

$$Z_\Delta = \sum_{\{j_e\}} \prod_e d_{j_e} \prod_t \left\{ \begin{array}{ccc} j_{e_1} & j_{e_2} & j_{e_3} \\ j_{e_3} & j_{e_5} & j_{e_6} \end{array} \right\}, \tag{16.7}$$

the group integration gives a product of SO(3) $\{6j\}$ symbols corresponding to every tetrahedra of the triangulation. This is the partition function originally defined by Ponzano and Regge. Z_Δ is independent of the triangulation Δ and depends only on the topology of the 3d manifold $\mathcal{M}$. Moreover, as we will see, it is finite after proper gauge fixing of the diffeomorphism symmetry [2; 3; 8].

If we consider a triangulation Δ with boundary $\partial\Delta_{\text{in}} \cup \partial\Delta_{\text{out}}$, we define the Quantum Gravity amplitude in the same way except that we do not sum over the boundary spins. The amplitude depends on the boundary spin networks dual to the boundary triangulation and this gives us the Quantum Gravity transition amplitude $Z_\Delta(j_{\text{in}}, j_{\text{out}})$. If we specialize to the cylinder topologies $M = [0, 1] \times \Sigma_2$, $Z_\Delta(j_{\text{in}}, j_{\text{out}})$ is actually the projector on the flat connections (on Σ_2), which is the expected projector onto the physical states in the continuum theory [9].

The partition function expressed as above is purely algebraic and the Newton constant G for gravity does not appear at all. This is expected from the continuum theory since G can be absorbed in a renormalization of the triad field e. G re-appears when we express physical lengths and distances in terms of the representation labels and the Planck length:

$$l = j\, l_{\text{P}} = j\, \hbar G.$$

This clearly appears in the semi-classical behavior of the $\{6j\}$ symbols. Indeed, for large spins js, the $\{6j\}$ symbol is up to a normalization factor the cosine of the Regge action S_{Regge} for a tetrahedron with edge lengths $j \times l_{\text{P}}$ [10; 11; 12].

16.3.1 Gauge symmetry

The construction of the Ponzano–Regge is quite formal since the summations over spins that appear in the definition of the partition function (16.7) are badly divergent. Namely if one put a cutoff on the spins $j < \Lambda$ then one would expect the partition function to scale as Λ^{3V} where V is the number of vertices in the triangulation. This fact has, for a long time, prevented a deeper understanding of this model. The key point made in [8; 2] is the understanding that these divergences are mainly due to the presence of gauge symmetry: these divergences just express the fact that the volume of the gauge group is not finite. Clearly 3d gravity is invariant under usual gauge transformation. The gauge symmetries of the continuum action (16.1) are the Lorentz gauge symmetry

$$\omega \to g^{-1} dg + g^{-1} \omega g, \quad e \to g^{-1} e g, \tag{16.8}$$

locally parameterized by a group element g, and the translational symmetry locally parameterized by a Lie algebra element ϕ

$$\omega \to \omega, \quad e \to e + d_\omega \phi \tag{16.9}$$

and which holds due to the Bianchi identity $d_\omega F = 0$. The combination of these symmetries is equivalent on-shell to diffeomorphism symmetry.

The discrete action (16.3) is invariant under discrete Lorentz gauge transformation acting at each tetrahedra. This is the analog of the usual gauge symmetry of lattice gauge theory. Since we consider here Euclidean gravity, the Lorentz group is a compact group and this gauge symmetry is taken into account by using the normalized Haar measure in (16.6).

Remarkably the discrete action (16.3) is also invariant under a discrete version of the translational symmetry. Namely, it is possible to define a covariant derivative $\nabla_e \Phi$ acting on Lie algebra elements Φ_v associated to vertices of the triangulation, such that the variation

$$\delta X_e = \nabla_e \Phi \tag{16.10}$$

leaves the action (16.3) invariant. The discrete covariant derivative reduces to the usual derivative $\nabla_e \Phi \sim \Phi_{s_e} - \Phi_{t_e}$ when the gauge field is Abelian and the symmetry is due to the discrete Bianchi identity.

Since this symmetry is non-compact we need to gauge fix it in order to define the partition function and expectation values of observables. A natural gauge fixing consists of choosing a collection of edges T which form a tree (no loops) and which is maximal (connected and which goes through all vertices). We then arbitarily fix the value of X_e for all edges $e \in T$. In the continuum this gauge fixing amounts to choosing a vector field v (the tree) and fixing the value of $e^i_\mu v^\mu$, that is to chose an "axial" gauge.

Taking this gauge fixing and the Faddev–Popov determinant into account in the derivation (16.6, 16.7) we obtain the gauge fixed Ponzano–Regge model

$$Z_{\Delta,T,j^0} = \sum_{\{j_e\}} \prod_e d_{j_e} \prod_{e \in T} \frac{\delta_{j_e, j_e^0}}{(d_{j_e^0})^2} \prod_t \left\{ \begin{array}{ccc} j_{e_1} & j_{e_2} & j_{e_3} \\ j_{e_3} & j_{e_5} & j_{e_6} \end{array} \right\}. \tag{16.11}$$

As a consistency test it can be shown that $Z_{\Delta,T,j^0} = Z^{\mathrm{GF}}_\Delta$ is independent of the choice of maximal tree T and gauge fixing parameter j^0.

16.4 Coupling matter to Quantum Gravity

In order to couple particles to matter fields we first construct the coupling of gravity to Feynman integrals since, as we are going to see, there is a natural and unambiguous way to couple the Ponzano–Regge model to Feynman integrals.

We use the fact that Feynman integrals can be written as a worldline integral [13], that is if Γ is a (closed for simplicity) Feynman graph its Feynman integral is given by

$$I_\Gamma(e) = \int \mathcal{D}\lambda_\Gamma \mathcal{D}x_\Gamma \mathcal{D}p_\Gamma \, e^{iS_\Gamma}, \tag{16.12}$$

where

$$S_\Gamma(x_\Gamma, p_\Gamma, \lambda_\Gamma) = \frac{1}{2}\sum_{e\in\Gamma}\int_e \mathrm{d}\tau \, \mathrm{tr}\left(p_e e_t - \frac{\lambda_e}{2}(p_e^2 - \mu_e^2)\right). \tag{16.13}$$

λ is a Lagrange multiplier field which is the worldline frame field and is restricted to be always positive (the metric on the worldline is given by $ds^2 = \lambda^2 d\tau^2$); x_Γ denotes the embedding of the graph Γ into spacetime and $e_t = e_\mu \dot{x}^\mu$, $p_e = p_e^i \sigma_i$ are Lie algebra elements. We want to compute the coupling of matter to gravity that is the expectation value

$$I_\Gamma = \int De D\omega \, e^{iS[e,\omega]} \, I_\Gamma(e).$$

Note that when we perform the integral over all geometry we effectively integrate out all possible embedding of the graph Γ. So the integral over all embedding x_Γ is redundant and can be dropped: I_Γ is also equal to the previous integral but with a fixed graph. What is happening is that the presence of a fixed graph breaks diffeomorphism symmetry along the graph, these gauge degrees of freedom are becoming dynamical and play the role of the particle position.

Since we have seen that there is a residual action of translational (which equal diffeomorphism on-shell) symmetry in the Ponzano–Regge model a similar phenomena will happen and the coupling of a fixed Feynman graph to the Ponzano–Regge model will effectively contain the sum over embeddings and give rise to the right dynamics.

In order to couple particles to our discrete gravity action (16.3) let us consider the triangulation Δ and insert particles along the edges of a Feynman graph $\Gamma \subset \Delta$, where the edges of Γ are edges of the triangulation. The discrete action describing coupling of gravity to this graph is then

$$S_{P_\Gamma} = \sum_{e\in\Gamma} \mathrm{tr}\left(\kappa X_e P_e + \lambda_e (P_e^2 - M_e^2)\right). \tag{16.14}$$

We see that the addition of this action inserts particles by modifying the flatness condition. The holonomy around a particle is not constrained to be the identity but is now constrained to be in the conjugacy class of h_{m_e}, where $m_e = 4\pi G\tilde{m}_e$ is the deficit angle created by the particle of mass $\tilde{m}_e$ and h_{m_e} is the element of the Cartan subgroup corresponding to the rotation of angle $2m_e$:

$$h_m \equiv \begin{pmatrix} e^{im} & 0 \\ 0 & e^{-im} \end{pmatrix}.$$

The deficit angle is related to the parameter M_e in the discrete action by

$$\kappa M_e = \sin m_e.$$

Then the corresponding quantum amplitude

$$I_\Delta(\Gamma) = \int \prod_e \mathrm{d}X_e \prod_f \mathrm{d}g_f \prod_{e\in\Gamma} \mathrm{d}\lambda_e \prod_{e\in\Gamma} \frac{\mathrm{d}^3 P_e}{4\pi^2}\; e^{\frac{1}{2}\sum_e \mathrm{tr}(X_e G_e) - iS_{P_\Gamma}(X_e, P_e, \lambda_e)}$$

is given by

$$I_\Delta(\Gamma) = \int \prod_f \mathrm{d}g_f \prod_{e\in\Gamma} \tilde{K}_{m_e}(G_e) \prod_{e\notin\Gamma} \delta(G_e). \tag{16.15}$$

$\tilde{K}_m(g)$ is a function on SO(3) which is invariant under conjugation and defined in terms of the momenta $2i\kappa\vec{P}(g) \equiv \mathrm{tr}(g\vec{\sigma})$ given by the projection of g on Pauli matrices.

$$\tilde{K}_m(g) = \frac{i\kappa^2}{(\kappa^2 P^2(g) - \sin^2 m - i\epsilon)}. \tag{16.16}$$

Since this is a class function we can expand it in terms of characters. We have the identity

$$\tilde{K}_m(g) = \sum_{j\in\mathbb{N}} K_m(j)\chi_j(g), \tag{16.17}$$

where $\chi_j(h_m)$ is the character of h_m in the j-representation:

$$\chi_j(h_m) = \frac{\sin(2j+1)m}{\sin m},$$

and

$$K_m(j) = \frac{2i\kappa^2 e^{id_j(m+i\epsilon)}}{\cos m}.$$

It is interesting to note that this is essentially the usual Feynman propagator evaluated on a discrete lattice, that is if $k_m(x)$ is the Feynamn propagator solution of $(\Box + m^2)k_m(x) = -i\delta(x)$, then $\frac{K_m(j)}{d_j} = \frac{\kappa^3}{4\pi}\frac{k_m(\kappa d_j)}{\cos m}$.

Note also that since $\mathrm{Re}\, G_m(j) = -\dfrac{2\kappa^2 \sin m}{\cos m}\chi_j(h_m)$ we have that

$$-\,Re(\tilde{G}_m(g)) = \pi\delta\left(P^2(g) - \frac{\sin m^2}{\kappa^2}\right) = \left(2\kappa^2 \frac{\sin m}{\cos m}\right)\delta_m(g), \qquad (16.18)$$

where $\delta_m(g)$ is a distribution on SO(3) which fixes g to be in the conjugacy class labelled by m:

$$\int_{\mathrm{SO(3)}} \mathrm{d}g f(g)\delta_m(g) = \int_{\mathrm{SO(3)}/\mathcal{U}(1)} \mathrm{d}x f(x h_m x^{-1}).$$

This is is the Hadamard propagator.

Using the character decomposition we can eventually re-write $I_\Delta(\Gamma)$ in terms of the $\{6j\}$ symbols:

$$I_\Delta(\Gamma) = \sum_{\{j_e\}} \prod_{e\notin\Gamma} d_{j_e} \prod_{e\in\Gamma} K_{j_e}(h_{m_e}) \prod_t \left\{ \begin{array}{ccc} j_{e_1} & j_{e_2} & j_{e_3} \\ j_{e_3} & j_{e_5} & j_{e_6} \end{array} \right\}. \qquad (16.19)$$

This expression makes clear that the insertion of particles on the graph Γ corresponds to computing the expectation value of an observable $\mathcal{O}^{\Gamma}_{m_e}$ in the topological state sum:

$$\mathcal{O}^{\Gamma}_{m_e}(j_e) = \prod_{e\in\Gamma} \frac{K_{j_e}(h_{m_e})}{d_{j_e}}.$$

Once again, the Quantum Gravity amplitude $I_\Delta(\Gamma)$ is purely algebraic and the Newton constant G only appears as a unit in order to translate the algebraic quantities j, m into the physical length $l = j l_{\mathrm{P}} = j\hbar G$ and the physical mass $\tilde{m} = m\kappa = \theta/4\pi G$. Note that in our derivation we have encountered no ambiguity in constructing the off-shell amplitudes, the final expression agrees with the one proposed in ([4]) but differs with the one in ([14]).

As in the vacuum case the amplitude (16.19) should be properly gauge fixed, this is done similarly by inserting in the expectation value the observable

$$\prod_{e\in T} \frac{\delta_{j_e, j_e^0}}{(d_{j_e^0})^2} \qquad (16.20)$$

where T is a tree touching every vertex of Δ which is not a vertex of Γ. Note that we should not gauge fix vertices touching Γ since now the gauge degrees of freedom at the location of Γ are dynamical entities corresponding to the particle location.

The gauge fixed partition function $I_\Delta(\Gamma)$ can be shown to be independent of the triangulation Δ and the gauge fixing ([3]) and only depends on the topology of $(\mathcal{M}, \Gamma)$. This means that we can trivially take the limit of infinitely fine triangulations and that the Ponzano–Regge model corresponds to an effective continuum theory even if it is originally described in terms of a discrete structure.

16.4.1 Mathematical structure

We have seen that the gauge fixing removes the redundant gauge degree of freedom and the corresponding infinities. We can ask now whether the gauge fixed partition function is always finite and what type of invariant it computes.

For instance it has been shown in [2] that the Ponzano–Regge invariant computed for a cylinder manifold $M = \Sigma_g \times I$, where Σ_g is a surface of genus g, is finite after gauge fixing and computes the projector onto the physical states, that is the space the flat connections on Σ_2 [9].

More generally if we consider a manifold M with a boundary and with an inserted graph, and we fix the deficit angle around the edges of the graph (that is we computed the gauge fixed partition function (16.19) with the insertion of the Hadamard propagator $Re(K_m)$). Then, as shown in [3], the Ponzano–Regge model is finite provided that the complement of the graph in M admits only one flat connection with the prescribed deficit angles. The Ponzano–Regge model is then understood as a an invariant providing a measure on the space of flat connection [3]; this measure is known as the Reidemeister torsion (see also [15]).

Moreover, it is known that at the classical level this 3d gravity with zero cosmological constant can be formulated as a Chern–Simons theory for the Poincaré group. When the gauge group of the Chern–Simons theory is compact there exists a notion of Chern–Simons quantization given by the Witten–Reshetikhin–Turaev invariant associated to quantum groups.

When the gauge group is non-compact only some Hamiltonian versions of Chern–Simons quantization were known. In [3] it has been shown that the Chern–Simons quantization can be expanded to the case of the Poincaré group and that the Ponzano–Regge invariant is equivalent to the Chern–Simons quantization. Namely one can show that the Ponzano–Regge invariant can be expressed as a Witten–Reshetikhin–Turaev invariant based on the Drinfeld double, which is a κ deformation of the Poincaré group.

16.5 Quantum Gravity Feynman rules

Now that we have obtained the Feynman rules for scalar matter coupled to gravity we would like to show that these amplitudes can be understood in terms of Feynman diagrams of an effective field theory which effectively describe the coupling of matter field to 3d gravity. As we already stressed, the expression being purely algebraic and dependent on a triangulation of our spacetime seems at first sight quite remotely connected to a usual Feynman diagram evaluation. In order to show that (16.19) can indeed be reinterpreted as a Feynman diagram evaluation we first restrict ourselves to the case where the ambient manifold is of trivial topology, that is $\mathcal{M} = S^3$, and also to the case where Γ is planar (we will come back later to the case of non-planar diagrams). In this case we can get rid of the triangulation dependence[2] and rewrite $I_\Delta[\Gamma]$ (16.19) purely in terms of the Feynman diagram data [4]:

$$I_\Delta(\Gamma) = \int \prod_{v\in\Gamma} \frac{\mathrm{d}X_v}{8\pi\kappa^3} \int \prod_{e\in\Gamma} \mathrm{d}g_e\, \tilde{K}_{m_e}(g_e) \prod_{v\in\Gamma} e^{\frac{1}{2\kappa}\operatorname{tr}(X_v G_v)}, \tag{16.21}$$

where the product is over the vertices v and edges e of Γ. The integral is over one copy of $\mathcal{R}^3$ for each vertex $X_v \equiv X_v^i \sigma_i$ and one copy of SO(3) (our deformed momentum space) for each edge and

$$G_v = \overrightarrow{\prod_{e\supset v}} g_e^{\epsilon_v(e)}, \tag{16.22}$$

with $\epsilon_v(e) = \pm 1$ depending on whether the edge e is incoming or outgoing and the product respect the cyclic ordering of edges which is well defined for planar diagrams.

We see that the main effect of Quantum Gravity is twofold. First the mass gets renormalized $m \to \sin\kappa m/\kappa$ and then the momentum space is no longer flat space but a homogeneously curved space: S^3 in the Euclidean case, or AdS^3 in the Lorentzian case. Equivalently the momentum $\vec{P}(g) \equiv \frac{1}{2i\kappa}\operatorname{tr}(g\vec{\sigma})$ is restricted to satisfy the bound $\kappa|P| < 1$.

The expression (16.21) looks almost like a Feynman diagram except that the Fourier kernel $\exp(\operatorname{tr}(X_v G_v)/2\kappa)$ entangles the edge momenta in a non-trivial way.

16.5.1 QFT as the semi-classical limit of QG

We can now take the $\kappa \to 0$ limit of the Quantum Gravity amplitudes (16.21). This corresponds to the limit $G_N \to 0$ in which the coupling to gravity becomes negligible. While taking this limit one should keep the physical mass $\tilde{m} = m/\kappa$ finite, this amounts to sending the angle m to 0. In this limit, we are considering small perturbations of SO(3) around the identity of the group:

$$g = e^{i\kappa\vec{p}.\vec{\sigma}} = 1 + \kappa(\vec{p}.\vec{\sigma}) + O(\kappa^2) \tag{16.23}$$

$$\Rightarrow\ g_1 g_2 = 1 + \kappa(\vec{p}_1 + \vec{p}_2).\vec{\sigma} + O(\kappa^2). \tag{16.24}$$

[2] Provided we choose the simplest embedding of Γ into S^3, that is one in which Γ is drawn on the surface of a sphere $S^2 \subset S^3$ without any crossings.

Therefore, at first order in κ, we are in an *Abelian limit* and the integral over SO(3) is approximately an integral over $\mathcal{R}^3$:

$$\int \mathrm{d}g \sim \kappa^3 \int_{\mathcal{R}^3} \frac{d^3\vec{p}}{2\pi^2}.$$

The amplitude (16.21) becomes

$$I_\Gamma \sim \kappa^{3|e_\Gamma|} \int \prod_{v\in\Gamma} d\vec{x}_v \int \prod_{e\in\Gamma} d\vec{p}_e K^0_{m_e}(\vec{p}_e) \prod_e e^{i\vec{p}_e.(\vec{x}_{t(e)}-\vec{x}_{s(e)})}, \tag{16.25}$$

where we integrate over variables x_v attached to each vertex of the graph Γ with $s(e), t(e)$ being respectively the source and target vertices of the oriented edge e. $\tilde{K}^0_m(\vec{p})$ is the Feynman propagator:

$$\tilde{K}^0_m(\vec{p}) = \int_0^{+\infty} \mathrm{d}T\, e^{-iT(p^2-m^2)}.$$

The amplitude (16.25) is actually the standard Feynman diagram evaluation of quantum field theory (for a massive scalar field).

We can equivalently take the limit $\kappa \to 0$ directly in the spin foam expression (16.19). Since the lengths are expressed in κ units as $l = \kappa j$, keeping l finite will send the representation label j to infinity: it is the *asymptotic limit* of spin foam amplitudes. More precisely, we can replace the sum over j by an integral over l:

$$\sum_j \sim \frac{1}{\kappa} \int_0^\infty \mathrm{d}l,$$

and replace the $6j$-symbol in the expression (16.19) by its asymptotics. This gives an expression of usual Feynman integrals as an expectation value of certain observable values in an asymptotic state sum model. The role of this state sum model is to provide the right measure of integration of a collection of points in flat space expressed in terms of invariant relative length, this has been shown in ([16]) (see also [17]).

What is quite remarkable is the fact that the full amplitude can be also interpreted as a Feynman diagram amplitude provided we introduce a non-trivial $\star$-product.

16.5.2 Star product

As we have seen previously the momentum space that appears in the Quantum Gravity amplitude (16.21) is an element of the SU(2) group. It is then natural to introduce a notion of plane waves defined to be

$$E_g(\vec{X}) \equiv e^{\frac{1}{2\kappa}\mathrm{tr}(Xg)} \tag{16.26}$$

where $X = X_i \sigma^i$. The group elements can be concretely written as

$$g = (P_4 + \imath\kappa P^i \sigma_i), \quad P_4^2 + \kappa^2 P^i P_i = 1, \, P_4 \geq 0. \tag{16.27}$$

We restrict ourselves to the "northern hemisphere" of SU(2) $P_4 > 0$ since this is enough to label SO(3) elements, and the plane waves are simply $E_g(X) = e^{i \vec{P}(g)\cdot\vec{X}}$.

We define a non-commutative $\star$-product on $\mathcal{R}^3$ which is defined on plane waves by

$$(E_{g_1} \star E_{g_2})(X) \equiv E_{g_1 g_2}(X). \tag{16.28}$$

This $\star$-product can be more explicitly written in terms of the momenta as

$$e^{\imath \vec{P}_1\cdot\vec{X}} \star e^{\imath \vec{P}_2\cdot\vec{X}} = e^{\imath(\vec{P}_1 \oplus \vec{P}_2)\cdot\vec{X}}, \tag{16.29}$$

where

$$\vec{P}_1 \oplus \vec{P}_2 = \sqrt{1-\kappa^2|\vec{P}_2|^2}\, \vec{P}_1 + \sqrt{1-\kappa^2|\vec{P}_1|^2}\, \vec{P}_2 \tag{16.30}$$

$$-\kappa \vec{P}_1 \times \vec{P}_2, \tag{16.31}$$

and $\times$ is the 3d vector cross product. By linearity this star product can be extended to any function of $\mathbb{R}^3$ which can be written as a linear combination of plane waves. It can also be extended to any polynomial function of X by taking derivatives of E_g with respect to P around $P = 0$. Using this, it can be easily shown that this star product describes a non-commutative spacetime with the non-commutative coordinates satisfying

$$[X_i, X_j] = i\kappa\epsilon_{ijk} X_k,$$
$$[X_i, P_j] = i\sqrt{1-\kappa^2 P^2}\, \delta_{ij} - i\kappa\epsilon_{ijk} P_k. \tag{16.32}$$

The non-commutativity of the space time is directly related to the fact that momentum space is curved. Indeed in a quantum mechanics $X \sim i\partial_P$ the coordinate is a derivation on momentum space, and derivatives of a curved space do not commute.[3] That a non-commutative spacetime structure arises in the quantization of 3d gravity was first proposed by 't Hooft [19], although the details are different. The existence of plane waves pairing $\mathcal{R}^3$ with SO(3) allows us to develop a new Fourier transform [4; 18] $F : C(\mathrm{SO}(3)) \to C_\kappa(\mathcal{R}^3)$ mapping functions on the group to functions on $\mathcal{R}^3$ having momenta bounded by $1/\kappa$:

$$\phi(X) = \int \mathrm{d}g\, \tilde{\phi}(g) e^{\frac{1}{2\kappa}\mathrm{tr}(Xg)}. \tag{16.33}$$

[3] The left $\star$-multiplication by X is realized as a right invariant derivative on momentum space S^3.

The inverse group Fourier transform is then explicitly written

$$\tilde{\phi}(g) = \int_{\mathcal{R}^3} \frac{d^3X}{8\pi\kappa^3}\, \phi(X) \star e^{\frac{1}{2\kappa}\mathrm{tr}(Xg^{-1})}$$
$$= \int_{\mathcal{R}^3} \frac{\mathrm{d}^3X}{8\pi\kappa^3}\, \phi(X)\sqrt{1-\kappa^2 P^2(g)}\, e^{\frac{1}{2\kappa}\mathrm{tr}(Xg^{-1})}. \tag{16.34}$$

This Fourier transform intertwines the $\star$-product with the group convolution product $\bullet$

$$\widetilde{\phi_1 \star \phi_2}(g) = \tilde{\phi}_1 \bullet \tilde{\phi}_2(g). \tag{16.35}$$

Finally this Fourier transform is an isometry between $L^2(\mathrm{SO}(3))$ and $C_\kappa(\mathcal{R}^3)$ equipped with the norm

$$||\phi||_\kappa^2 = \int \frac{\mathrm{d}X}{8\pi\kappa^3}\, \phi \star \phi(X). \tag{16.36}$$

The non-commutative space-time structure and the fact that the space of fields $C_\kappa(\mathcal{R}^3)$ have bounded momenta expresses the fact that there exists a minimal length scale accessible in the theory. This is clear if one looks at the non-commutative delta function defined by

$$\delta_0 \star \phi(X) = \phi(0)\delta_0(X). \tag{16.37}$$

It is given by

$$\delta_0(X) = 2\kappa \frac{J_1\left(\frac{|X|}{\kappa}\right)}{|X|}, \tag{16.38}$$

with J_1 the first Bessel function, it is clear that $\delta_0(X)$ is concentrated around $X = 0$ but has a non-zero width.

16.6 Effective non-commutative field theory

Now that we are equipped with this star product we can write the Fourier kernel of (16.21) as a product

$$e^{\frac{1}{2\kappa}\mathrm{tr}(X_v G_v)} = \bigstar_{\partial e \in v}\, e^{\frac{\epsilon_v(e)}{2\kappa}\mathrm{tr}(X_v g_e)} \tag{16.39}$$

and the amplitude (16.21) reads

$$I_\Gamma = \int \prod_{v\in\Gamma} \frac{\mathrm{d}X_v}{8\pi\kappa^3} \prod_{e\in\Gamma} \mathrm{d}g_e\, \tilde{K}_{m_e}(g_e) \prod_{v\in\Gamma}\left(\bigstar_{v\in\Gamma}\, e^{\frac{\epsilon_v(e)}{2\kappa}\mathrm{tr}(X_v g_e)}\right). \tag{16.40}$$

The effective Feynman propagator is given by

$$K_m(X) = i\int \mathrm{d}g \frac{e^{\frac{1}{2\kappa}\mathrm{tr}(Xg)}}{P^2(g) - \left(\frac{\sin\kappa m}{\kappa}\right)^2}. \tag{16.41}$$

From the expression (16.40) it is now clear that I_Γ is the Feynman diagram evaluation of a non-commutative field theory based on the previous $\star$-product. This statement is true for all possible diagrams Γ even if we should keep in mind that the equality $I_\Gamma = I_\Delta(\Gamma)$ between the integral (16.40) and the Quantum Gravity amplitude (16.19) has been established only for planar diagrams so far.

More precisely let us consider the case where we have particles of only one type, so all masses are taken equal, $m_e \equiv m$. Having different masses would only require us to introduce more fields and would not modify the overall picture in any way. Let us now consider the sum over trivalent graphs:

$$\sum_{\Gamma \text{trivalent}} \frac{\lambda^{|v_\Gamma|}}{S_\Gamma} I_\Gamma, \tag{16.42}$$

where λ is a coupling constant, $|v_\Gamma|$ is the number of vertices of Γ and S_Γ is the symmetry factor of the graph.

The main point is that this sum can be obtained from the perturbative expansion of a non-commutative field theory given explicitly by:

$$S = \frac{1}{8\pi\kappa^3} \int d^3x \left[\frac{1}{2}(\partial_i\phi \star \partial_i\phi)(x) - \frac{1}{2}\frac{\sin^2 m\kappa}{\kappa^2}(\phi \star \phi)(x) \right.$$
$$\left. + \frac{\lambda}{3!}(\phi \star \phi \star \phi)(x) \right], \tag{16.43}$$

where the field ϕ is in $\mathbb{C}_\kappa(\mathcal{R}^3)$. Its momentum has support in the ball of radius κ^{-1}. We can write this action in momentum space

$$S(\phi) = \frac{1}{2} \int dg \left(P^2(g) - \frac{\sin^2 \kappa m}{\kappa^2} \right) \widetilde{\phi}(g)\widetilde{\phi}(g^{-1})$$
$$+ \frac{\lambda}{3!} \int dg_1 dg_2 dg_3\, \delta(g_1g_2g_3)\, \widetilde{\phi}(g_1)\widetilde{\phi}(g_2)\widetilde{\phi}(g_3), \tag{16.44}$$

from which it is straightforward to read the Feynman rules and show our statement. Remarkably, this non-commutative field theory was first considered by Imai and Sasakura in [20] in an attempt to construct a non-commutative but relativistically invariant quantum field theory.

The interaction term written in momentum space shows clearly that the momentum addition rule becomes non-linear, in order to preserve the condition that momenta is bounded. At the interaction vertex the momentum conservation reads:

$$0 = P_1 \oplus P_2 \oplus P_3 = P_1 + P_2 + P_3 \tag{16.45}$$
$$- \kappa(P_1 \wedge P_2 + P_2 \wedge P_3 + P_3 \wedge P_1) + \mathcal{O}(\kappa^2). \tag{16.46}$$

From this identity, it appears that the momenta is non-linearly conserved, and the non-conservation is stronger when the momenta are non-collinear. The natural interpretation is that part of the energy involved in the collision process is absorbed by the gravitational field, this effect prevents any energy involved in a collision process being larger than the Planck energy. This phenomena is simply telling us that when we have a high momentum transfer involved in a particle process, one can no longer ignore gravitational effects which do modify how the energy is transferred.

The non-commutative field theory action is symmetric under a κ-deformed action of the Poincaré group. If we denote by Λ the generators of Lorentz transformations and by $T_{\vec{a}}$ the generators of translations, it appears that the action of these generators on one-particle states is undeformed:

$$\Lambda \cdot \tilde{\phi}(g) = \tilde{\phi}(\Lambda g \Lambda^{-1}) = \tilde{\phi}(\Lambda \cdot P(g)), \tag{16.47}$$

$$T_{\vec{a}} \cdot \tilde{\phi}(g) = e^{i\vec{P}(g)\cdot\vec{a}}\tilde{\phi}(g). \tag{16.48}$$

The non-trivial deformation of the Poincaré group appears at the level of multi-particle states. Only the action of the translations is truly deformed :

$$\Lambda \cdot \tilde{\phi}(P_1)\tilde{\phi}(P_2) = \tilde{\phi}(\Lambda \cdot P_1)\tilde{\phi}(\Lambda \cdot P_2), \tag{16.49}$$

$$T_{\vec{a}} \cdot \tilde{\phi}(P_1)\tilde{\phi}(P_2) = e^{i\vec{P}_1 \oplus \vec{P}_2 \cdot \vec{a}}\tilde{\phi}(P_1)\tilde{\phi}(P_2). \tag{16.50}$$

We would like to interpret the previous field theory as the effective field theory describing the dynamics of matter in Quantum Gravity after integration of the gravitational degrees of freedom. Before doing so we need to extend our results to the case of non-planar diagrams.

16.7 Non-planar diagrams

It turns out that the Quantum Gravity expression (16.21) is not equivalent to the integral (16.40) when the diagram Γ is not planar.

This should not come too much as a surprise since (16.21) depends not only on the topology of Γ but also on the embedding of Γ into $\mathbb{R}^3$. Moreover in order to define the integral we have to choose a cyclic ordering at the vertices of the graph which is unambiguously defined only in the case of planar graphs.

To understand where the problem is rooted let's recall that even when the matter is non-interacting there is still a non-trivial S-matrix due to the presence of the gravitational field [21; 22]. This can be seen at a semiclassical level when we are looking at the scattering of a field in the presence of a gravitational field (a conical singularity) created by a massive particle. This translates [23; 24] at the quantum level into the fact that a non-trivial braiding factor arises when computing the Quantum Gravity amplitude, this braiding factor is naturally included in the

Ponzano–Regge model as shown in [2]. In [4] we have shown that this result can be extended to an arbitrary Feynman diagram Γ embedded in $\mathbb{R}^3$.

More precisely this means that we can evaluate explicitly the amplitude of the non-planar diagram coupled to Quantum Gravity in terms of a set of local Feynman rules provided we add to the usual Feynman rules an additional one for each crossing of the diagrams. The set of Feynman rules is summarized in Fig.16.1. For each edge of Γ we insert a propagator $\tilde{K}(g)$, for each trivalent vertex we insert a conservation rule $\delta(g_1 g_2 g_3)$ where g_i labels the incoming group valued momenta at the vertex, and for each crossing of the diagram we associate a weight $\delta(g_1 g_2 {g'_1}^{-1} {g_2}^{-1})$ where g_2 is labeling the edge which is over crossing and the g_1s are labelling the edge undercrossing (see Fig.16.1). The Feynman diagram amplitude for a closed Feynman diagram is then obtained by integrating over all group momenta.

This completes the description of the Feynman rules and it can be easily shown that these rules do not depend on the choices of projection and representative edges.

These Quantum Gravity Feynman rules are exactly the Feynman rules of the non-commutative field theory introduced above provided that the field entering the definition of the action (16.49) obeys a *non-trivial* statistics. Indeed when we compute the Feynman amplitude from field theory one first has to expand the expectation value of a product of free field in terms of two point functions using the Wick theorem. In order to do this operation we first need to exchange the order of Fourier modes $\tilde{\phi(g)}$ before using the Wick theorem. If the diagram is planar no exchange of Fourier mode is needed but such exchanges are necessary in the non-planar case. The specification of the rules of exchange of Fourier modes is a

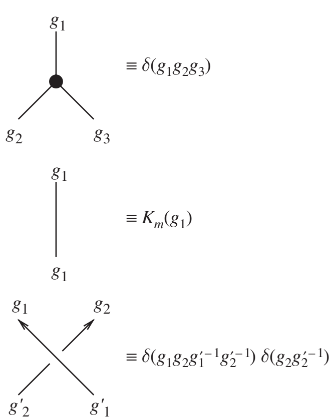

Fig. 16.1. Feynman rules for particles propagation in the Ponzano–Regge model.

choice of statistics. In order to reproduce the Quantum Gravity amplitudes we need to choose a non-trivial statistics where the Fourier modes of the fields are assumed to obey the exchange relation:

$$\tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2). \tag{16.51}$$

This exchange relation is in fact naturally determined by our choice of star product and the duality between space and time (plane waves). Indeed, let us look at the product of two identical fields:

$$\phi \star \phi\,(X) = \int dg_1 dg_2\, e^{\frac{1}{2\kappa}\mathrm{tr}(Xg_1g_2)} \widetilde{\phi}(g_1)\widetilde{\phi}(g_2). \tag{16.52}$$

We can 'move' $\tilde{\phi}(g_2)$ to the left by making the following change of variables $g_1 \to g_2$ and $g_2 \to g_2^{-1}g_1g_2$, the star product reads

$$\phi \star \phi\,(X) = \int dg_1 dg_2\, e^{\frac{1}{2\kappa}\mathrm{tr}(Xg_1g_2)} \widetilde{\phi}(g_2)\widetilde{\phi}(g_2^{-1}g_1g_2). \tag{16.53}$$

The identification of the Fourier modes of $\phi \star \phi\,(X)$ leads to the exchange relation (16.51).

This commutation relation is exactly the one arising from the braiding of two particles coupled to Quantum Gravity. This braiding was first proposed in [24] and computed in the spin foam model in [2]. It is encoded into a braiding matrix

$$R \cdot \tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2). \tag{16.54}$$

This is the R matrix of the κ-deformation of the Poincaré group [24]. We see that the non-trivial statistics imposed by the study of our non-commutative field theory is related to the braiding of particles in three spacetime dimensions. This non-trivial braiding accounts for the non-trivial gravitational scattering between two matter particles. Such field theories with non-trivial braided statistics are usually simply called braided non-commutative field theories and were first introduced in [25].

16.8 Generalizations and conclusion

Our results naturally extend to the Lorentzian theory. Although a direct derivation of the spin foam model from the continuum theory is still lacking, a Lorentzian version of the Ponzano–Regge model has been written down [26; 27] and the topological state sum is formulated in terms of the $\{6j\}$ symbols of $\mathrm{SU}(1, 1)$. One can already apply existing gauge fixing techniques [2; 28] to regularize the amplitudes based on a non-compact gauge group. Moreover, particles are once again inserted as topological defects creating conical singularities and a similar (almost identical) effective non-commutative field theory can be derived from the spin foam amplitudes.

The Lorentzian version of the Ponzano–Regge model is expressed in terms of the $\{6j\}$ symbols of the non-compact group SO(2, 1) [26]. Holonomies around particles are SO(2, 1) group elements parametrized as

$$g = P_4 + i\kappa P_i \tau^i \text{with } P_4^2 + \kappa^2 P_i P^i = 1, \text{ and } P_4 \geq 0, \tag{16.55}$$

with the metric $(+--)$ and the $\mathfrak{su}(1,1)$ Pauli matrices, $\tau_0 = \sigma_0, \tau_{1,2} = i\sigma_{1,2}$. Massive particles correspond to the $P_i P^i > 0$ sector. They are described by elliptic group elements, $P_4 = \cos\theta$, $\kappa|P| = \sin\theta$. The deficit angle is given by the mass, $\theta = \kappa m$. All the mathematical relations of the Riemannian theory are translated to the Lorentzian framework by performing the transformation

$$P_0 \to P_0,\ P_1 \to i P_1,\ P_2 \to i P_2. \tag{16.56}$$

Note that this transformation differs from a usual Wick rotation (which rotates P_0 only).

The propagator remains given by the formula (16.41). The momentum space is now $\mathrm{AdS}^3 \sim \mathrm{SO}(2,1)$. The addition of momenta is deformed accordingly to the formula (16.31). We similarly introduce a group Fourier transform F : $C(\mathrm{SO}(2,1)) \to C_\kappa(\mathcal{R}^3)$ and a $\star$-product dual to the convolution product on SO(2, 1). Finally we derive the effective non-commutative field theory with the same expression (16.43) as in the Riemannian case.

It is also possible, in the context of Euclidean gravity, to take into account a non-zero cosmological constant Λ. The corresponding model is the Turaev–Viro model [29] based on $\mathcal{U}_q(\mathrm{SU}(2))$, where q is on the unit circle for a positive cosmological constant and q is real for a negative cosmological constant. For a positive cosmological constant, Λ provides a maximal length scale. We wrote the explicit Feynman rules corresponding to this spin foam model in [4] and showed that we obtain a spherical or hyperboloid state sum based on the propagators on the 3-sphere or the 3-hyperboloid respectively depending on the sign of Λ. Further work is needed to analyze the details of these models and extend the results to the Lorentzian case.

To sum up, we have shown how the Ponzano–Regge spin foam model can be properly gauge fixed in order to provide a proper definition of 3d euclidean Quantum Gravity. We have seen how this model can be naturally coupled to matter and that the corresponding 3d Quantum Gravity amplitudes are actually the Feynman diagram evaluations of a braided and non-commutative QFT. This effective field theory describes the dynamics of the matter field after integration of the gravitational degrees of freedom. The theory is invariant under a κ-deformation of the Poincaré algebra, which acts non-trivially on many-particle states. This is an explicit realization of a QFT in the framework of deformed special relativity (see e.g. [30]), which implements from first principles the original idea of Snyder [31] of using a curved momentum space to regularize the Feynman diagrams.

A deeper study of the meaning of the braided non-commutative field theory that arises in this study is needed. Above all it will be important to understand if such a theory admits a Hamiltonian description and describes (or not) a unitary theory.

Finally one would like to understand what type of result obtained here can be extended to four dimensions. It has been shown for instance that we can express the standard 4d QFT Feynman graphs as expectation values of certain observables in a 4d topological spinfoam model (see e.g. [32]). The corresponding spin foam model provides the semi-classical limit of QG and can be identified as the zeroth order of an expansion in term of the inverse Planck mass κ of the full QG spin foam amplitudes [33]. QG effects would then appear as deformations of the Feynman graph evaluations and QG corrections to the scattering amplitudes could be computed order by order in κ.

Acknowledgements

My principal acknowledgment goes to D. Oriti, the editor of this volume, for his encouragement, his understanding and infinite patience. I also would like to thank my collaborators D. Louapre and E. Livine for the work presented here.

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17

The group field theory approach to Quantum Gravity

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17.1 Introduction and motivation

Group field theories (GFTs) [1; 2] were developed at first as a generalization of matrix models for 2d Quantum Gravity to 3 and 4 spacetime dimensions to produce a lattice formulation of topological theories. More recently, they have been developed further in the context of spin foam models for Quantum Gravity, as a tool to overcome the limitations of working with a fixed lattice in the non-topological case. In our opinion, however, GFTs should be seen as a fundamental formulation of Quantum Gravity and not just as an auxiliary tool. The bottom line of this perspective, here only tentatively outlined and still to be fully realized, hopefully, after much more work, can be summarized as follows: GFTs are quantum field theories **of** spacetime (as opposed to QFTs *on* spacetime), that describe the dynamics of both its topology and geometry in local, simplicial, covariant, algebraic terms, and that encompass ideas and insights from most of the other approaches to non-perturbative Quantum Gravity. We have just began to explore the structure of these models, but there is already some evidence, in our opinion, that in the GFT framework lies the potential for important developments.

The idea of defining a quantum field theory *of* geometry, i.e. a QFT on superspace (the space of 3-geometries) for given spatial topology, say S^3, was already explored in the past [3; 4; 5]. The context was then a global or "quantum cosmology" one. Such a theory would produce, in its perturbative expansion, a sum over different topologies each corresponding to a possible Feynman graph, i.e. to a possible interaction process for "universes" represented by the basic 3-sphere. The spatial topology change would be limited therefore to a changing number of disjoint copies of S^3. The field would represent a second quantization of the canonical wave function on superspace, here describing the "one-particle sector" of the theory.[1] The quantum amplitude for each Feynman graph, corresponding to

[1] The 3-metric being itself a field, this second quantization of what is already a field theory was dubbed "third quantization".

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti.
Published by Cambridge University Press.

a particular spacetime topology with n boundary components, would be given by a sum-over-histories quantization of gravity on the given topology with the usual exponential-of-action amplitude for each history. The difficulties in making mathematical sense of the continuum path integral itself are well-known, and it is a safe guess that the technical difficulties in turning this third quantization idea into a mathematically rigorous framework in the continuum are even more formidable. Also, such a cosmological setting presents notorious interpretation problems. The general idea, however, is appealing, as it would provide a natural mechanism for implementing topology change within a covariant sum-over-histories quantization of gravity. In particular one could imagine that the interpretation issues, if not the technical difficulties, would be made easier if it was possible to implement the above ideas in a **local** framework, for example by generalizing the superspace construction to *open chunks of the universe*, for example 3-balls, and then describing in a third quantized language the interaction of these local pieces of the universe generating dynamically the whole universe and spacetime in their evolution. Again, however, the continuum setting seems to prevent a rigorous realization of these ideas. By turning to a **simplicial** description of spacetime, the group field theory formalism gives a mathematically better defined realization of these appealing ideas, and allows for an easier physical interpretation, being based on an intrinsically *local* picture of the evolution of geometry and topology.

17.2 The general formalism

The geometry of a simplicial space (a triangulation) is fully characterized by a countable, if not finite, number of variables, i.e. superspace becomes discrete. Also, every closed D-dimensional simplicial complex can be obtained by gluing fundamental D-dimensional building blocks, each with the topology of a D-ball, along their boundaries (given by (D − 1)-simplices). A **local**, and thus more physically sensible, realization of the idea of a field theory on superspace is then possible, by considering first a wave function associated to each (D − 1)-dimensional simplicial building block of space (if spacetime is D-dimensional), and then second quantizing it. The quantum geometry of a larger spatial simplicial complex will be encoded in the tensor product of such wave functions/operators for the individual building blocks forming them. How does one characterize the geometry of each simplicial building block, and thus of the full simplicial complex? Here group field theories follow the path traced by loop quantum gravity, and describe quantum geometry in terms of group and representation variables. This descends [1; 9; 10] from the classical description of gravity in terms of connection variables valued in the Lie algebra of the Lorentz group of the appropriate dimension, discretized to give elementary group valued parallel transports along paths in the (dual of the) simplicial complex, or equivalently in terms of Lie algebra-valued (D − 2)-forms as in BF-like formulations of gravity, discretized to give the volumes of the (D−2)-dimensional cells of the simplicial complex, labeled by irreducible representations of the Lorentz group. The equivalence between these two sets of variables is given by the harmonic analysis on the group manifold that expresses their conjugate nature. More concretely, the field is a $\mathbb{C}$-valued function of D group elements, for a generic group G, one for each of the D boundary (D − 2)-faces of the (D − 1)-simplex the field corresponds to:

$$\phi(g_1, g_2, ..., g_D) : G^{\otimes D} \rightarrow \mathbb{C}.$$

The order of the arguments in the field corresponds to a choice of orientation for the (D − 1)-simplex it represents; therefore it is natural to impose the field to be invariant under even permutations of its arguments (that do not change the orientation) and to turn into its own complex conjugate under odd permutations; this choice ensures that only orientable complexes are generated in the Feynman expansion of the field theory [17]. Other symmetry properties can also be considered [7]. The closure of the D (D − 2)-faces to form a (D − 1)-simplex is expressed algebraically by the invariance of the field under diagonal action of the group G on the D arguments of the field: $\phi(g_1, ..., g_D) = \phi(g_1 g, ..., g_D g)$, which is also imposed [6; 13]. This is the simplicial counterpart of the Lorentz gauge invariance of 1st order gravity. The mode expansion gives:

$$\phi(g_i) = \sum_{J_i, \Lambda, k_i} \phi_{k_i}^{J_i \Lambda} \prod_i D_{k_i l_i}^{J_i}(g_i) C_{l_1 ... \, l_4}^{J_1 ... \, J_4 \Lambda},$$

with the Js labeling representations of G, the ks vector indices in the representation spaces, and the Cs being intertwiners of the group G, an orthonormal basis of which is labeled by an extra parameter Λ. Group variables represent configuration space, while the representation parameters label the corresponding momentum space. Geometrically, the group variables, as said, represent parallel transport of a connection along elementary paths dual to the (D − 2)-faces, while the representations J can be put in correspondence with the volumes of the same (D − 2)-faces, the details of this correspondence depending on the specific model [9; 10]. The first quantization of a geometric (D − 1)-simplex in terms of these variables was performed in great detail in the 3- and 4-dimensional case in [6], but a similar analysis is lacking in higher dimensions. A simplicial space built out of N such (D − 1)-simplices is then described by the tensor product of N such wave functions, at the 1st quantized level, with suitable constraints implementing their gluing, i.e. the fact that some of their (D − 2)-faces are identified. For example, a state describing two (D − 1)-simplices glued along one common (D − 2)-face would be represented by: $\phi_{k_1 k_2 ... k_D}^{J_1 J_2 .. J_D \Lambda} \phi_{\tilde{k}_1 k_2 ... \tilde{k}_D}^{\tilde{J}_1 J_2 ... \tilde{J}_D \tilde{\Lambda}}$, where the gluing is along the face labeled by the representation J_2, and effected by the contraction of the corresponding vector indices (of course, states corresponding to disjoint (D − 1)-simplices are also allowed). The corresponding state in configuration variables is: $\int dg_2 \phi(g_1, g_2, ..., g_D)\phi(\tilde{g}_1, g_2, ..., \tilde{g}_D)$. We see that states of the theory are then labeled, in momentum space, by *spin networks* of the group G (see chapter 13 by Thiemann and chapter 15 by Perez). The second quantization of the theory promotes these wave functions to operators, and the field theory is specified by a choice of action and by the definition of the quantum partition function. The partition function is then expressed perturbatively in terms of Feynman diagrams, as we are going to discuss. This implicitly assumes a description of the dynamics in terms of creation and annihilation of (D − 1)-simplices, whose interaction generates a (discrete) spacetime as a particular interaction process (Feynman diagram) [7]. This picture has not been worked out in detail yet, and no clear Fock structure on the space of states has been constructed. Work on this is in progress [11].

Spacetime, represented by a D-dimensional simplicial complex, emerges in perturbative expansion as a particular interaction process among (D − 1)-simplices, described as an ordinary QFT Feynman diagram. It is then easy to understand the choice of classical field action in group field theories. This action, in configuration space, has the general structure:

$$S_D(\phi, \lambda) = \frac{1}{2}\left(\prod_{i=1}^{D}\int dg_i d\tilde{g}_i\right)\phi(g_i)\mathcal{K}(g_i\tilde{g}_i^{-1})\phi(\tilde{g}_i) + \frac{\lambda}{(D+1)!}\left(\prod_{i\neq j=1}^{D+1}\int dg_{ij}\right)\phi(g_{1j})...\phi(g_{D+1j})\,\mathcal{V}(g_{ij}g_{ji}^{-1}), \quad (17.1)$$

where the choice of kinetic and interaction functions $\mathcal{K}$ and $\mathcal{V}$ define the specific model. The interaction term describes the interaction of D + 1 (D − 1)-simplices to form a D-simplex by gluing along their (D − 2)-faces (arguments of the fields). The nature of this interaction is specified by the choice of function $\mathcal{V}$. The (quadratic) kinetic term involves two fields each representing a given (D − 1)-simplex seen from one of the two D-simplices (interaction vertices) sharing it, so that the choice of kinetic functions $\mathcal{K}$ specifies how the information and therefore the geometric degrees of freedom corresponding to their D (D − 2)-faces are propagated from one vertex of interaction (fundamental spacetime event) to another. What we have then is an almost ordinary field theory, in that we can rely on a fixed background metric structure, given by the invariant Killing–Cartan metric, and the usual splitting between kinetic (quadratic) and interaction (higher order) terms in the action, that will later allow for a straightforward perturbative expansion. However, the action is also *non-local* in that the arguments of the D + 1 fields in the interaction term are not all simultaneously identified, but only pairwise. This is certainly a complication with respect to usual field theories in Minkowski space, but it may simplify somehow renormalization issues, since it means that, even if the interaction is of order D + 1, in terms of number of fields involved, it is still quadratic in terms of the individual arguments of the fields. No detailed analysis of the equations of motion following from the above action in any specific GFT model has been carried out to date, but work on this is in progress [12]. These equations define the classical dynamics of the field theory, they would allow the identification of classical background configurations around which to expand in a semi-classical perturbation expansion, etc. However, what is their meaning from the point of view of Quantum Gravity, in light of the geometric interpretation of the GFT as a *local simplicial "third quantization" of gravity*? The answer is simple if striking: just as the Klein–Gordon equation gives at the same time the classical dynamics of a (free) scalar field theory and the quantum dynamics for the first quantized (free) theory, the classical GFT equations encode fully the quantum dynamics of the underlying (simplicial) canonical Quantum Gravity theory. Solving the above equations then means identifying Quantum Gravity wave functions satisfying **all** the Quantum Gravity constraints!

Another issue that still needs a careful investigation is that of the classical symmetries of the above action. Some of them, holding regardless of the specific choice of kinetic and interaction operators, are the above-mentioned "closure" symmetry imposed on each field: $\phi(g_i) = \phi(g_i g)$, $\forall g \in G$, encoded in the symmetry property of the kinetic and vertex operators: $\mathcal{K}(g_i \tilde{g}_i^{-1}) = \mathcal{K}(g g_i \tilde{g}_i^{-1} g')$, $\mathcal{V}(g_{ij} \tilde{g}_{ji}^{-1}) = \mathcal{V}(g_i g_{ij} \tilde{g}_{ji}^{-1} g_j)$, and the global symmetry of the action under: $\phi(g_i) \to \phi(g g_i)$ $\forall g \in G$. Additional symmetries may be present depending on the specific model, and would correspond to specific symmetries of the classical discrete theories being quantized. The identification of such GFT analogs of the classical symmetries is no easy task. [2]

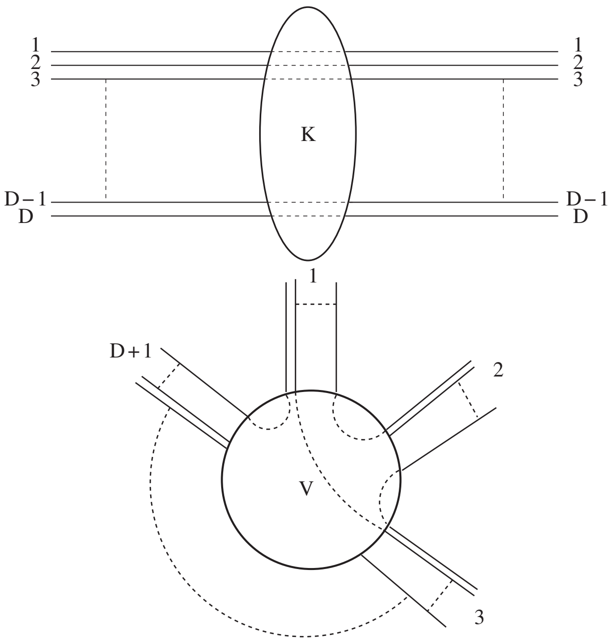
Most of the work up to now has focused on the perturbative aspects of quantum GFTs, i.e. the expansion in Feynman diagrams of the partition function and the properties of the resulting Feynman amplitudes:

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^N}{sym[\Gamma]} Z(\Gamma),$$

where N is the number of interaction vertices in the Feynman graph Γ, $sym[\Gamma]$ is a symmetry factor for the diagram and $Z(\Gamma)$ the corresponding Feynman amplitude.

[2] Already in the simpler example of GFT formulations of BF theories, characteristic symmetries as translation or topological symmetries that can be correctly identified at the level of the GFT Feynman amplitudes, do not correspond to the above obvious symmetries of the GFT action [16].

The Feynman amplitudes can be constructed easily after identification of the propagator and the vertex amplitude. Each edge of the Feynman diagram is made of D strands, one for each argument of the field, and each one is then re-routed at the interaction vertex, with the combinatorial structure of an D-simplex, following the pairing of field arguments in the vertex operator. This is shown diagrammatically as follows.



Each strand goes through several vertices, coming back to where it started, for closed Feynman diagrams, and therefore identifies a 2-cell. Each Feynman diagram Γ is then a collection of 2-cells (faces), edges and vertices, i.e. a 2-complex, that, because of the chosen combinatorics for the arguments of the field in the action, is topologically dual to a D-dimensional simplicial complex [7; 17]. Clearly, the resulting complexes/triangulations can have arbitrary topology, each corresponding to a particular *scattering process* of the fundamental building blocks of space, i.e. $(D-1)$-simplices. The D-dimensional triangulation dual to the 2-complex, arising as a GFT Feynman diagram, would not necessarily be a simplicial *manifold*, as the data in the GFT Feynman diagrams do not constrain the neighborhoods of simplices of dimensions from $(D-3)$ downwards to be spheres. In the general case, the resulting simplicial complex, obtained by gluing D-simplices along their $(D-1)$-faces, would correspond to a *pseudo-manifold*, i.e. to a manifold with *conical singularities* [7; 17; 37]. A precise set of conditions under which the GFT Feynman diagrams correspond to manifolds is identified and discussed at length in [17]. All the relevant conditions can be checked algorithmically on any given Feynman diagram. It is not clear, at present, whether one can construct suitably constrained GFT models satisfying these conditions, thus generating only manifold-like complexes in their Feynman expansion.

Each strand carries a field variable, i.e. a group element in configuration space or a representation label in momentum space. Therefore in momentum space each Feynman diagram is given by a spin foam (a 2-complex with faces labeled by representation variables), and each Feynman amplitude (a complex function of the representation labels, obtained by contracting vertex amplitudes with propagator functions) by a spin foam model (see chapter 15 by Perez):

$$Z(\Gamma) = \sum_{J_f} \prod_f A(J_f) \prod_e A_e(J_{f|e}) \prod_v A_v(J_{f|v}).$$

As in all spin foam models, the representation variables have a geometric interpretation (edge lengths, areas, etc.) (see [9; 10]) and so each of these Feynman amplitudes corresponds to a definition of a sum-over-histories for discrete Quantum Gravity on the specific triangulation dual to the Feynman diagram, although the quantum amplitudes for each geometric configuration are not necessarily given by the exponential of a discrete gravity action. For more on the quantum geometry behind spin foam models we refer to the literature [9; 10; 28]. One can show that the inverse is also true: any local spin foam model can be obtained from a GFT perturbative expansion [13; 2]. This implies that the GFT approach *subsumes* the spin foam approach at the perturbative level, while at the same time going beyond it, since there is of course much more in a QFT than its perturbative expansion. The sum over Feynman diagrams gives then a sum over spin foams (histories of the spin networks on the boundary in any scattering process), and equivalently a sum over triangulations, augmented by a sum over algebraic data (group elements or representations) with a geometric interpretation, assigned to each triangulation. Expectation values of GFT observables can also be evaluated perturbatively. These are given [2] by gauge invariant combinations of the basic field operators that can be constructed in momentum space using spin networks according to the formula

$$O_{\Psi=(\gamma,j_{\mathrm{e}},i_{\mathrm{v}})}(\phi) = \left(\prod_{(ij)} \int dg_{ij} dg_{ji}\right) \Psi_{(\gamma,j_{\mathrm{e}},i_{\mathrm{v}})}(g_{ij} g_{ji}^{-1}) \prod_i \phi(g_{ij}),$$

where $\Psi_{(\gamma,j_{\mathrm{e}},i_{\mathrm{v}})}(g)$ identifies a spin network functional for the spin network labeled by a graph γ with representations j_{e} associated to its edges and intertwiners i_{v} associated to its vertices, and g_{ij} are group elements associated to the edges (ij) of γ

that meet at the vertex i. In particular, the transition amplitude (probability amplitude for a certain scattering process) between certain boundary data represented by two spin networks, of arbitrary combinatorial complexity, can be expressed as the expectation value of the field operators having the same combinatorial structure of the two spin networks [2]:

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \mathcal{D}\phi \, O_{\Psi_1} \, O_{\Psi_2} \, e^{-S(\phi)} = \sum_{\Gamma/\partial\Gamma = \gamma_{\Psi_1} \cup \gamma_{\Psi_2}} \frac{\lambda^N}{sym[\Gamma]} \, Z(\Gamma),$$

where the sum involves only 2-complexes (spin foams) with boundary given by the two spin networks chosen.

The above perturbative expansion involves therefore two very different types of sums: one is the sum over geometric data (group elements or representations of G) which is the GFT analog of the integral over momenta or positions of usual QFT; the other is the sum over Feynman diagrams. This includes a sum over all triangulations for a given topology and a sum over all topologies (since all possible gluings of D-simplices and face identifications are present by construction in the GFT Feynman expansion). Both sums are potentially divergent. First of all the naive definition of the Feynman amplitudes implies a certain degree of redundancy, resulting from the symmetries of the defining GFT. A proper gauge fixing of these symmetries, especially those whose group is non-compact, is needed to avoid divergences [16]. Even after gauge fixing, the sum over geometric data has a potential divergence for every "bubble" of the GFT Feynman diagram, i.e. for every closed collection of 2-cells. This is the GFT analog of loop divergences of the usual QFT. Of course, whether the GFT amplitudes are divergent or not depends on the specific model.[3] In general a regularization and perturbative renormalization procedure would be needed, but no systematic study of GFT renormalization has been carried out to date, despite its obvious importance. The sum over Feynman diagrams, on the other hand, is most certainly divergent. This is not surprising. The sum over Feynman diagrams gives a sum over *all* triangulations for *all* topologies, each weighted by a (discrete) Quantum Gravity sum-over-histories. That such a sum can be defined constructively thanks to the simplicial and QFT setting is already quite an achievement, and to ask for it to be finite would be really too much! Also, from the strictly QFT perspective, it is to be expected that the expansion in Feynman diagrams of a QFT would produce at most an asymptotic series and not a convergent one. This is the case for all the interesting QFTs we know of. What makes the usual QFT perturbative expansion useful in spite of its divergence

[3] For example, while the most natural definition of the group field theory for the Barrett–Crane spin foam model [7], presents indeed bubble divergences, a simple modification of it [18; 19], possesses *finite* Feynman amplitudes, i.e. it is *perturbatively finite* without the need for any regularization.

is the simple fact that it has a clear physical meaning, i.e. we know what it means to compute a transition amplitude up to a given order. In the GFT case this means providing a clear physical interpretation for the coupling constant λ. This can be done, actually, in more than one way. First of all, defining $\alpha = \lambda^{\frac{1}{D-1}}$ and redefining $\tilde{\phi} = \alpha\phi$, we can recast the GFT action in the form $S_\lambda[\phi] = \frac{1}{\alpha^2} S_{\lambda=1}[\tilde{\phi}]$. One can then perform a loop expansion of the GFT partition function, that is an expansion in the parameter α, instead of a perturbative expansion in the coupling constant. This gives, for a generic transition amplitude between two boundary states Ψ_1 and Ψ_2: $\langle \Psi_1 \mid \Psi_2 \rangle_\alpha = \frac{1}{\alpha^2} \sum_{i=0}^{\infty} \alpha^{2i} \langle \Psi_1 \mid \Psi_2 \rangle_i$, where $\langle \Psi_1 \mid \Psi_2 \rangle_i$ is a sum over Feynman diagrams with i loops. The point here is to realize that adding a loop to a given Feynman diagram is equivalent [2] to adding a handle to the simplicial complex dual to it. This means that the parameter $\alpha = \lambda^{\frac{1}{D-1}}$ governs the strength of topology changing processes in the GFT perturbative expansion. This interpretation can also be confirmed by analyzing the Schwinger–Dyson equations for a generic GFT [2]. A different perspective on the physical meaning of λ is obtained by noticing that λ weights somehow the "size" of the spacetimes emerging in the GFT perturbative expansion, assuming that the number of D-simplices is a measure of the D-volume of spacetime. For example, if $Z(\Gamma) = e^{iS(\Gamma)}$, with $S(\Gamma)$, say, the Regge action for pure gravity with no cosmological constant on a triangulation (dual to Γ) with fixed edge lengths, then one could define $\lambda = e^{i\Lambda}$, and thus rewrite the GFT partition function as: $Z = \sum_\Gamma \frac{1}{sym(\Gamma)} e^{i(S(\Gamma)+\Lambda V(\Gamma))}$. Then Λ would play the role of a bare cosmological constant. Indeed this would be exactly the expression for a dynamical triangulations model [14]. This heuristic argument can be made rigorous in a tensor model [15; 17], a special case of the GFT formalism. The two proposed interpretations for λ are compatible with each other, and finding a clear link between the two would mean linking the value of the bare, and then of the renormalized, cosmological constant to the presence of spatial topology change. This would realize rigorously one of the initial aims of a "third quantization" formalism [3; 4; 5].

Let us now remark once more on the connection between GFT and canonical Quantum Gravity. As already mentioned, the classical GFT equations of motion encode the full quantum dynamics of the corresponding first quantized theory; this is a simplicial Quantum Gravity theory whose kinematical quantum states are labeled by D-valent spin networks for the group G. One may want to give a covariant or sum-over-histories definition of the canonical inner product (encoding the full dynamics of the quantum theory, and the action of the Hamiltonian constraint operator, see the chapters by Thiemann and Perez) for a simplicial version of loop quantum gravity based on such states. The restriction of the GFT perturbative expansion to *tree level*, involving indeed only *classical information*, for given boundary spin network observables [2], can be considered as the GFT definition of such a canonical inner product, *if the resulting 2-point functions result in being real and positive*, as for example those of the BF or Barrett–Crane models. The definition is well posed, because at tree level every single amplitude $Z(\Gamma)$ is finite whatever the model considered due to the absence of infinite summation. Moreover, it possesses all the properties one expects from a canonical inner product: (1) it involves a sum over Feynman diagrams, and therefore triangulations, with the cylindrical topology $S^{D-1} \times [0, 1]$, for closed spin networks Ψ_i associated with the two boundaries, as is easy to verify; (2) it is real and positive, but not strictly positive; it has a non-trivial kernel that can be shown [2] to include all solutions of the classical GFT equations of motion, as expected. This means that the physical Hilbert space for canonical spin network states can be constructed, using the GNS construction, from the kinematical Hilbert space of all spin network states by quotienting out those states belonging to this kernel. This represent a concrete testable proposal for completing the definition of a loop formulation of Quantum Gravity, and a proof of the usefulness of GFT ideas and techniques. At the same time, it shows that the GFT formalism contains much more than any canonical quantum theory of gravity, given that the last is fully contained at the "classical" level only of the former.

17.3 Some group field theory models

Let us now discuss some specific GFT models. The easiest example is the straightforward generalization of matrix models for 2d Quantum Gravity to a GFT [20], given by the action:

$$S[\phi] = \int_G dg_1 dg_2 \frac{1}{2}\phi(g_1, g_2)\phi(g_1, g_2) + \frac{\lambda}{3!}\int dg_1 dg_2 dg_3 \phi(g_1, g_2)\phi(g_1, g_3)\phi(g_2, g_3) \tag{17.2}$$

where G is a generic compact group, say $SU(2)$, and the symmetries mentioned above are imposed on the field ϕ implying, in this case: $\phi(g_1, g_2) = \tilde{\phi}(g_1 g_2^{-1})$. The relation with matrix models is apparent in momentum space, expanding the field in representations j of G to give:

$$S[\phi] = \sum_j dim(j)\left(\frac{1}{2}tr(\tilde{\phi}_j^2) + \frac{\lambda}{3!}tr(\tilde{\phi}_j^3)\right) \tag{17.3}$$

where the field modes $\tilde{\phi}_j$ are indeed matrices with dimension $dim(j)$, so that the action is given by a sum of matrix models actions for increasing dimensions, or, better, by a single matrix model in which the matrix dimension has been turned from a parameter into a dynamical variable. The Feynman amplitudes are given by $Z(\Gamma) = \sum_j dim(j)^{2-2g(\Gamma)}$, so the GFT above gives a quantization of BF theory (with gauge group G) on a closed triangulated surface, dual to Γ, of genus $g(\Gamma)$, augmented by a sum over all such surfaces [20]. A similar quantization of 2d gravity would use $G = U(1)$, a restriction on the representations, and additional data encoding bundle information [21].

The extension to higher dimensions can proceed in two ways. In [15] the first "tensor model", for an $N \times N \times N$ tensor ϕ was introduced:

$$S[\phi] = \sum_{\alpha_i} \left(\frac{1}{2} \phi_{\alpha_1\alpha_2\alpha_3} \phi_{\alpha_1\alpha_2\alpha_3} + \frac{\lambda}{4!} \phi_{\alpha_1\alpha_2\alpha_3} \phi_{\alpha_3\alpha_4\alpha_5} \phi_{\alpha_5\alpha_2\alpha_6} \phi_{\alpha_6\alpha_4\alpha_1} \right),$$

which generates both manifold- and pseudo-manifold-like 3d simplicial complexes [15; 17]. This is turned easily into a GFT by a straightforward generalization of the 2d case. The following kinetic and vertex terms:

$$\mathcal{K}(g_i, \tilde{g}_i) = \int_G dg \prod_i \delta(g_i \tilde{g}_i^{-1} g),$$

or

$$\mathcal{V}(g_{ij}, g_{ji}) = \prod_i \int_G dg_i \prod_{i<j} \delta(g_i g_{ij} g_{ji}^{-1} g_j^{-1}),$$

where the integrals impose the gauge invariance under the action of G, give the GFT quantization of BF theories, for gauge group G, in any dimension [22; 23]. In particular, in three dimensions, the choice [22] $G = SO(3)$ or $G = SO(2,1)$ provides a quantization of 3D gravity in the Euclidean and Minkowskian signatures, respectively, and the so-called Ponzano–Regge spin foam model, while the choice of the quantum group $SU(2)_q$ gives the Turaev–Viro topological invariant. The action is then:

$$\begin{aligned} S[\phi] = &\prod_i \int_G \phi(g_1, g_2, g_3)\phi(g_1, g_2, g_3) \\ &+ \frac{\lambda}{4!} \prod_{i=1}^{6} \int_G dg_i \; \phi(g_1, g_2, g_3)\phi(g_3, g_4, g_5)\phi(g_5, g_2, g_6)\phi(g_6, g_4, g_1). \end{aligned} \tag{17.4}$$

Lots is known about the last model (see chapter 16 by Freidel). Here, we mention only one result that is of interest for the general issue of GFT renormalization. This is the proof [25] that a simple modification of the GFT above gives a model whose perturbative expansion is Borel summable. The modification amounts to adding another vertex term to the original one, given by:

$$+\frac{\lambda\,\delta}{4!}\prod_{i=1}^{6}\int dg_i\,[\phi(g_1,g_2,g_3)\phi(g_3,g_4,g_5)\phi(g_4,g_2,g_6)\phi(g_6,g_5,g_1)]\quad |\,\delta\,|<1. \tag{17.5}$$

The new term corresponds simply to a slightly different recoupling of the group/representation variables at each vertex of interaction, geometrically to the only other possible way of gluing four triangles to form a closed surface. This result is interesting for more than one reason: (1) it shows that it *is* possible to control the sum over triangulations of all topologies appearing in the GFT perturbative expansion; (2) even if it has no clear physical interpretation yet from the Quantum Gravity point of view, it is indeed a very mild modification, and most importantly one likely to be forced upon us by renormalization group arguments, that usually require us to include in the action of our field theory all possible terms that are compatible with the symmetries. The restriction of the 3D Boulatov model for a real field to the homogeneous space $SO(3)/SO(2)\simeq S^2$ [25], and with the global $SO(3)$ invariance having been dropped, gives a generalization of the tensor model (17.3) with action:

$$S[\phi]=\sum_{j_i,\alpha_i}\frac{1}{2}\phi^{j_1j_2j_3}_{\alpha_1\alpha_2\alpha_3}\phi^{j_1j_2j_3}_{\alpha_1\alpha_2\alpha_3}+\frac{\lambda}{4!}\sum_{j_i,\alpha_i}\phi^{j_1j_2j_3}_{\alpha_1\alpha_2\alpha_3}\phi^{j_3j_4j_5}_{\alpha_3\alpha_4\alpha_5}\phi^{j_5j_2j_6}_{\alpha_5\alpha_2\alpha_6}\phi^{j_6j_4j_1}_{\alpha_6\alpha_4\alpha_1}$$

where the indices α_i run over a basis of vectors in the representation space j_i, and its partition function is: $Z=\sum_\Gamma\frac{(-\lambda)^{n_\Gamma}}{\mathrm{sym}(\Gamma)}\sum_{j_f}\prod_f(2j_f+1)$, with f being the faces of the 2-complex/Feynman graph, which is divergent and has to be regularized. There are three ways of doing it, all a tensor model as a result: (1) simply dropping the sum over the representations j_f by fixing them to equal a given J; (2) placing a cut-off on the sum by restricting $j_i<N$, obtaining $Z=\sum_\Gamma\frac{(-\lambda)^{n_v(\Gamma)}}{sym(\Gamma)}[(N+1)^2]^{n_f(\Gamma)}$; (3) equivalently, but more elegantly, by defining the model not on S^2 but on the non-commutative 2-sphere S^2_N, which also carries a representation of $SU(2)$ but implies a bounded decomposition in spherical harmonics (labeled by $j<N$), thus giving the same result for the partition function. We recognize in the above result the partition function for the tensor model (17.3) and for a dynamical triangulations model [14].

Let us now discuss the 4D case. Here GFT model building has followed the development of spin foam models for 4D Quantum Gravity (see chapter 15 by Perez). The guiding idea has been the fact that *classical* gravity can be written as a constrained version of a BF theory for the Lorentz group. The Barrett–Crane spin foam models in fact [8] amount roughly to a restriction of spin foam models for BF theories to involve only *simple* representations of the Lorentz group ($SO(4)$

or $SO(3,1)$) [8; 9; 10]. This restriction can be imposed at the GFT level, starting from the GFT describing 4D BF theory, by projecting down the arguments of the field from $G = SO(4)$ ($SO(3,1)$) to the homogeneous space $SO(4)/SO(3) \simeq S^3$ ($SO(3,1)/SO(3)$ or $SO(3,1)/SO(2,1)$ in the Lorentzian case), exploiting the fact that only simple representations of G appear in the harmonic decomposition of functions on these spaces. The GFT action is then defined [7] as:

$$S[\phi] = \frac{1}{2}\left(\prod_i \int_{SO(4)} dg_i\right) P_g P_h \phi(g_1, g_2, g_3, g_4) P_g P_h \phi(g_1, g_2, g_3, g_4)$$
$$+ \frac{\lambda}{5!}\left(\prod_{i=1}^{10} \int_{SO(4)} dg_i\right) \big[P_g P_h \phi(g_1, g_2, g_3, g_4) P_g P_h \phi(g_4, g_5, g_6, g_7)$$
$$P_g P_h \phi(g_7, g_8, g_3, g_9) P_g P_h \phi(g_9, g_5, g_2, g_{10}) P_g P_h \phi(g_{10}, g_8, g_6, g_1)\big] \tag{17.6}$$

where the projection $P_h \phi(g_i) = \prod_i \int_{SO(3)} dh_i \, \phi(g_i h_i)$ from the group to the homogeneous space imposes the wanted constraints on the representations, and the projection $P_g \phi(g_i) = \int_{SO(4)} dg \, \phi(g_i g)$ ensures that gauge invariance is maintained. Different variations of this model, resulting in different edge amplitudes A_e, can be constructed [18; 19; 9; 10] by inserting the two projectors P_h and P_g in the action in different combinations. The corresponding Feynman amplitudes are:

$$Z(\Gamma) = \sum_{J_f} \prod_f \dim(J_f) \prod_e A_e(J_{f|e}) \prod_v \mathcal{V}_{\mathrm{BC}}(J_{f|v}), \tag{17.7}$$

where $\dim(J_f)$ is the measure for the representation J_f, labeling the faces of the 2-complex/Feynman graph, entering the harmonic decomposition of the delta function on the group, and the function $\mathcal{V}_{\mathrm{BC}}(J_{f|v})$, depending on the ten representations labeling the ten faces of Γ incident to the same vertex v is the so-called Barrett–Crane vertex [8; 9; 10].

The above Feynman amplitudes can be justified in various ways, e.g. starting from a discretization of classical BF theory and a subsequent imposition of the constraints [9; 10], and there is a good consensus on the fact that the Barrett–Crane vertex amplitude captures at least some of the properties needed by a spin foam description of 4D Quantum Gravity. Also [27], for configurations corresponding to non-degenerate simplicial geometries the asymptotic limit of the Barrett–Crane amplitude $\mathcal{V}_{\mathrm{BC}}(J)$ is proportional to the cosine of the Regge action, i.e. a correct discretization of General Relativity.

All the above models share the following properties: (1) their Feynman amplitudes are real; (2) no unique orientation for the (various elements of the) triangulation dual to any Feynman graph can be reconstructed from the amplitude associated with it; (3) in Quantum Gravity models, the asymptotic limit of the vertex amplitude gives (in the non-degenerate sector) the *cosine* of the Regge action instead of the exponential of it. These properties suggest the interpretation [24] of the corresponding models as defining the Quantum Gravity analog of the Hadamard function for a relativistic particle, and, as said, are the wanted properties if we seek a GFT definition of the canonical/Hamiltonian inner product.[4] However, there are several reasons why one may want to go beyond this type of structure. (1) From the point of view of a field theory on the simplicial superspace we are advocating here, the most natural object one would expect a GFT to define with its 2-point functions is not a canonical inner product, solution of the Hamiltonian constraint, but a Green function for it. This is what happens in ordinary QFT, for the free theory, and in the formal context of continuum third quantization for Quantum Gravity, where the (free theory) Feynman amplitudes correspond to the usual path integral for Quantum Gravity, with amplitude given by the exponential of the GR action [3; 4; 5], which is a Green function for the Hamiltonian constraint, and not a solution of the same, in each of its arguments. (2) The orientation of the GFT 2-complexes can be given, for Lorentzian models, a *causal* interpretation [28; 29], and thus the orientation independence of the usual models suggests that one should be able to construct other types of models defining *causal* Quantum Gravity transition amplitudes [28; 29] and corresponding GFTs. (3) No clear meaning can be given from the Hamiltonian/canonical perspective to the GFT amplitudes for Feynman graphs beyond the tree level, when spatial topology change is present. For all these reasons one would like to have a more general class of GFT models that do depend on the orientation of the GFT Feynman graphs, that can be interpreted consistently as analogs of causal transition amplitudes of QFT, that are in more direct contact with usual path integral formulations of (simplicial) gravity, and that reduce to the above type of models when suitably restricted. A class of models that achieves this was constructed in [24]. Here a generalized version of the GFT formalism was defined, for a field $\phi(g_i, s_i) : (G \times \mathbb{R})^{\otimes 4} \to \mathbb{C}$:

$$S_{\text{gen}} = \sum_{\mu,\alpha} \frac{1}{4} \prod_{i=1}^{4} \int dg_i \int_{\mathbb{R}} ds_i \left\{ \phi^{-\mu\alpha}(g_i, s_i) \left[\prod_i \left(-i\mu\alpha\partial_{s_i} + \nabla_i \right) \right] \phi^{\mu\alpha}(g_i, s_i) \right\}$$

$$+ \sum_{\mu} \sum_{\alpha_i} \frac{\lambda_{\{\alpha_i,\mu\}}}{5!} \prod_{i \neq j=1}^{5} \int_G dg_{ij} \int_{\mathbb{R}} ds_{ij} \left\{ P_h\phi^{\mu\alpha_1}(g_{1j}, s_{1j}) P_h\phi^{\mu\alpha_2}(g_{2j}, s_{2j}) \right.$$

$$\left. \ldots P_h\phi^{\mu\alpha_5}(g_{5j}, s_{5j}) \prod \theta(\alpha_i s_{ij} + \alpha_j s_{ji}) K\left(g_{ij}, g_{ji}; \mu(\alpha_i s_{ij} + \alpha_j s_{ji})\right) \right\},$$

[4] In other words, the Feynman amplitudes of these GFT models would correspond not to a simplicial version of the path integral formalism for Quantum Gravity, but to the symmetrized version of the same over opposite spacetime orientations, that indeed gives a path integral definition of solutions of the Hamiltonian constraint operator of canonical Quantum Gravity [26].

where: $g_i \in G$, $s_i \in \mathbb{R}$, $\mu = \pm 1$ and $\alpha_i = \pm 1$ are orientation data that allow one to reconstruct the orientation of the Feynman graph from the **complex** amplitude associated to it, $\phi^+(g_i, s_i) = \phi(g_1, s_1; ..., g_4, s_4)$ and $\phi^-(g_i, s_i) = \phi^\dagger(g_i, s_i)$, P_h is the projector imposing invariance under the $SO(3)$ subgroup, ∇ is the D'Alembertian operator on the group G, $\theta(s)$ is the step function and $K(g, s)$ is the evolution kernel for a scalar particle on the group manifold G with evolution parameter s. The field is assumed invariant under the diagonal action of G as described above. The form of the kinetic and vertex operator impose a non-trivial dependence on the orientation data in fully covariant way. The resulting Feynman amplitudes [24] have all the properties wanted, being complex and orientation-dependent, and have the natural interpretation as analogs of Feynman transition amplitudes for Quantum Gravity [24; 29]. Also, when expressed in terms of the variables conjugate to the s_i, the amplitude for each vertex is given by the exponential of the Regge action in first order formalism, times an appropriate measure factor [24]. It remains to be proven that this also holds for the amplitude associated to the whole Feynman graph [30; 31]. Other models based on the same formalism and same type of field, but differing, for example, in the expression for the vertex term can also be constructed, and share similar properties [31].

Other types of GFTs have been constructed in the literature, ranging from a Boulatov-like model for 3d gravity based on the quantum group $DSU(2)$ [32], with links to models of 3d Quantum Gravity coupled to matter mentioned below, to a modified version [33] of the GFTs for the Barrett–Crane models, with a tunable extra coupling among the 4-simplices and a possible use in the renormalization of spin foam models. For all this we refer to the literature. We refer to the literature also for the recent construction of group field theory models for Quantum Gravity coupled to matter fields of any mass and spin in 3d [34; 35; 36], for work in progress concerning the 4d case (coupling of Quantum Gravity and gauge fields, of topological gravity and strings, etc.), and for the proposal of re-interpreting the conical singularities appearing in non-manifold-like Feynman graphs of GFTs as matter fields [37].

17.4 Connections with other approaches

We would like to recapitulate here some links to other approaches, and sketch a (rather speculative, at present) broader picture of GFTs as a *generalized formalism for Quantum Gravity*, in which other discrete approaches can be subsumed.

GFTs seek to realize a **local simplicial third quantization of gravity**, with discrete gravity path integrals as Feyman amplitudes and a sum over simplicial spacetimes of all topologies realized as a Feynman expansion. What is the exact relationship with the more traditional path integral quantizations of simplicial gravity: quantum Regge calculus (see chapter 19 by Williams) and dynamical triangulations (see chapter 18 by Ambjørn *et al.*)? The first of the above uses a fixed triangulation of spacetime, and thus should be reproduced at the level of the GFT Feynman amplitudes for a given Feynman diagram. Given the geometric interpretation of the GFT variables [28; 9; 10], each amplitude should correspond to a first order path integral quantization of discrete gravity, i.e. treating on equal footing $(D-2)$-volumes and dihedral angles (equivalently, appropriate parallel transports of a Lorentz connection) as fundamental variables, as opposed to the second order formulation of traditional Regge calculus in terms of edge lengths. This, however, may be consider a somewhat minor difference. The main issue to be clarified in order to establish a clear link with the quantum Regge calculus approach has to do with the fact that the quantum amplitudes of the latter approach are given by the exponential of the Regge action for discrete gravity, while in the most studied spin foam models the connection between the quantum amplitudes and the Regge action is clear only in a particular regime and rather involved. However, it seems plausible that the new generalized models of [24], or a suitably modification of the same, can indeed give amplitudes with the same structure as in quantum Regge calculus, with a measure being uniquely determined by the choice of GFT action, thus clarifying the connection with discrete gravity and at the same time subsuming the quantum Regge calculus approach within the GFT formalism. The same type of amplitudes is needed also to establish a solid link with the dynamical triangulations approach, where the Regge action weights this time the combinatorial structure of the triangulation itself, which is treated as the only true dynamical variable within a sum over all possible triangulations of a given topology. The dynamical triangulations approach would then once more arise as a subsector of the GFT formalism, if one could find the right way of trivializing the extra structure associated to each triangulation (thus dropping the sum over geometric data). Of course, more work would be needed then to impose the extra conditions (fixed slicing structure, absence of baby universe nucleation, etc) that seem to be needed in the modern version of the approach (see chapter 18 by Ambjørn *et al.*) to have a good continuum limit. Work on this is in progress [30].

It is well known that a covariant path integral quantization is more general than the corresponding canonical/Hamiltonian one, and that this is even more true in a third quantization formalism with its sum over topologies. One expects to be able to reproduce from a GFT the results of a canonical Quantum Gravity with group elements and group representations as basic variables, and spin networks as quantum states, i.e. loop quantum gravity. We have discussed above how this can indeed be realized [2]. The main differences between the particular version of the LQG formalism that the GFT approach reproduces, and the traditional one (see the chapter by Thiemann), are: (1) the spin networks appearing as boundary states or observables in GFTs are inherently adapted to a simplicial context in that they are always D-valent in D spacetime dimensions, being dual to appropriate (D − 1)-triangulations, while the spin networks arising in the *continuum* loop quantum gravity approach are of arbitrary valence; (2) the group used to label these states and their histories in the GFT case is the Lorentz group of the corresponding dimension (e.g. in dimension 4 and Minkowskian signature, the non-compact group $SO(3, 1)$), while LQG uses $SU(2)$ spin networks. The first of these differences is not so crucial, since on the one hand any higher-valent spin network in LQG can be decomposed into lower-valent ones, and on the other hand any coarse graining procedure approximating simplicial structures with continuum ones would likely remove any restriction on the valence. The second difference is more troublesome, and establishing an explicit connection between the fully covariant GFT spin networks and $SU(2)$ ones is no easy task. However, lots of work has already been done on this issue [38] (see chapter 14 by Livine) and can be the starting point for (1) establishing a well-defined canonical formalism from the GFT structures first, and then (2) linking (more appropriately, reducing, probably through some sort of gauge fixing) this formalism to that of traditional LQG.

A fourth approach that can be linked to the GFT one is the causal set approach (see chapter 21 by Henson). Recent work on spin foam models and GFT [28; 29; 24] has shown how the GFT Feynman amplitudes can be re-written as models of causal evolution of spin networks [39], by a correct implementation of causality requirements. A key step in doing this is the causal interpretation, in the Lorentzian context, of the GFT Feynman graph, this being a *directed graph*, i.e. a diagrams with “directions” or arrows labeling its edges, thus endowed with an orientation. In this interpretation, the vertices of the graph, i.e. the elementary GFT interactions, dual to D-simplices, are the fundamental spacetime events, and the links of the graph each connecting two such vertices, dual to (D − 1)-simplices and corresponding to elementary propagation of degrees of freedom in the GFTs, represent the fundamental causal relations between spacetime events. A directed graph differs from a causal set for just one, albeit important, property: it possibly includes closed loops of arrows. This, from the point of view of causal set theory, is a violation of causality, the microscopic discrete equivalent of a closed timelike loop in General Relativity, forbidden in the basic axioms defining the approach. No such restriction is imposed, a priori, on the corresponding GFT structures. There are several possible attitudes towards this issue from the GFT perspective: (1) it is possible that such configurations are not relevant for the continuum approximation, i.e. they give a negligible contribution to the sum under the appropriate coarse graining procedure; (2) in the specific GFT models that will turn out to be of most interest for Quantum Gravity, Feynman graphs possessing such “closed timelike loops” may end up being assigned quantum amplitudes that suppress them; (3)

one may be able to give a purely field-theoretic interpretation of such loops in the GFT context and then identify some sort of "superselection rules" forbidding them; (4) finally, one may decide that there is no fundamental reason to ban such configuration, and find instead the way to interpret them physically and study their observable consequences. Finally, there is one more difference with causal sets: due again to the simplicial setting, the GFT Feynman diagrams have vertices of finite and fixed valence depending on the spacetime dimension, while the causal set vertices have none. Once more, it is well possible that one has to welcome such restriction because it results in one more sign of a fundamental spacetime discreteness, that may be attractive from both philosophical and physical reasons. It is also possible that such restriction on valence will be removed automatically in the study of the continuum approximation of the GFT discrete spacetimes, because of coarse graining procedures or of renormalization group arguments (e.g. inclusion of more interaction terms in the GFT action).

The GFT formalism is therefore able to encompass several other approaches to Quantum Gravity, each carrying its own set of ideas and techniques. Strengthening the links with these other approaches will be, in our opinion, of great importance for the further development of the GFT framework itself, but also for progress on the various open issues that such other approaches still face.

17.5 Outlook

Let us summarize. The group field theory approach aims to describe the dynamics of both spacetime geometry and topology down to the Planck scale, in a background independent and non-perturbative way (even if at present almost only the perturbation expansion around the "complete vacuum" is well understood), using a field-theoretic formalism. In essence a GFT is a field theory over a group manifold, as for the mathematical formulation, and at the same time a field theory over a simplicial superspace (space of geometries), as for the physical interpretation. It corresponds to a *local* third quantization of gravity, in which the "quanta" being created and annihilated are not universes, as in the traditional approach, but appropriately defined chunks of space. What is particularly attractive, in our opinion, about this approach is the combination of *orthodoxy* in the mathematical language used and of *radicalness* in the ideas that this language expresses. On the one hand, in fact, GFTs are almost ordinary field theories, defined on a group manifold with fixed metric and topology, and thus, formally speaking, background dependent. This means that GFTs allow, at least in principle, one to tackle any of the traditional questions in Quantum Gravity using techniques and ideas from QFT, thus making use of the vast body of knowledge and methods developed in a background dependent context that appeared for long time not directly applicable to Quantum Gravity research. On the other hand, the overall picture of spacetime and of gravity that this approach is based on is definitely radical and suggests the following. There exist fundamental building blocks or atoms of space, which can be combined to give rise to all sorts of geometry and topology of space. At the perturbative level spacetime is the discrete (virtual) history of creation/annihilation of these fundamental atoms; it has no *real* existence, at least no more real existence in itself than each of the infinite possible interaction processes corresponding to individual Feynman diagrams in any field theory. The description of this evolution is necessarily background independent (from the point of view of spacetime) because spacetime itself is built from the bottom up and all of spacetime information has to be reconstructed from the information carried by the "atoms" and thus by the Feynman diagrams. At the non-perturbative level, for what we can see given the present status of the subject, spacetime is simply not there, given that the non-perturbative properties of Quantum Gravity would be encoded necessarily either in the GFT action, and in the resulting equations of motion, or in the GFT partition function, and the related correlation functions, to be studied non-perturbatively, neither of which need any notion of spacetime to be defined or analyzed. Spacetime information is thus necessarily encoded in structures that do not use *per se* a notion of spacetime. Finally, there would be a *fundamental discreteness* of spacetime and a key role for *causality*, in the pre-geometric sense of *ordering* (so that it would probably be better to talk about "pre-causality"). Many of these ideas had been proposed several times in the past, and occur in more than one other approach to Quantum Gravity, but the GFT formalism brings all of them together within a unique framework and, as said, expresses them in a rather conventional and powerful language.

Let us sketch some examples of how traditional field theoretic methods can be used to tackle within a new perspective some crucial open issue in Quantum Gravity research. We have already mentioned some of these examples. The long-standing problem of solving the Hamiltonian constraint equation of canonical Quantum Gravity can be identified with the task of solving the classical GFT equations of motion. The other long-standing issue of defining a canonical inner product for Quantum Gravity states is turned into the task of analyzing the tree level truncation of the (perturbative expansion of the) appropriate GFT. Also, the perturbation theory around such Quantum Gravity states would be governed, according to the above results, by the approximation of the GFT partition function around its classical solutions, and this suggests a new strategy for investigating the existence of gravitons (propagating degrees of freedom) in specific GFT/spin foam models. The most outstanding open issue that most of the discrete non-perturbative approaches to Quantum Gravity still face is, however, that of the continuum approximation. This problem has been formulated and tackled in a variety of ways. Obviously, given the role that formalisms like dynamical triangulations, quantum Regge calculus, causal sets or loop quantum gravity can play within the group field theory framework, the various techniques developed for them can be adapted to the GFTs. However, the field theory language that is at the forefront of the GFT approach suggests once more new perspectives. Let us sketch them briefly.

The continuum approximation issue can be seen as the search for the answer to two different types of questions. (a) What is the best procedure to approximate a discrete spacetime, e.g. a simplicial complex, with a continuum manifold, and to obtain some effective quantum amplitude for each geometric configuration from the underlying fundamental discrete model? In the context of spin foam models, this amounts to devising a background independent procedure for "coarse graining" the spin foam 2-complexes and the corresponding amplitudes [40; 41] to obtain a smooth approximation of the same. (b) If a continuum spacetime or space are nothing else than some sort of "condensate" of fundamentally discrete objects, as in some "emergent gravity" approaches (see chapter 7 by Dreyer and chapter 9 by Markopoulou) and, as suggested by condensed matter analog models of gravity [42; 43], what are these fundamental constituents? What are their properties? What kind of (necessarily background independent) model can describe them and the whole process of "condensation"? What are the effective hydrodynamic variables and what is their dynamics in this "condensed or fluid phase"? How does it compare to GR?

For what concerns the first (set of) question(s), the GFT approach offers a potentially decisive reinterpretation: since spin foam are nothing else than Feynman diagrams of a GFT, and that spin foam models are nothing else than their corresponding Feynman amplitudes, the coarse graining of a spin foam model [40; 41], is exactly the *perturbative renormalization* of the corresponding GFT. On the one hand this suggests that one deal with the problem of continuum approximation of spin foams using all the perturbative and non-perturbative renormalization group techniques from ordinary field theory adapted to the GFT case. On the other hand gives a further justification for the idea, proposed in [41], that the Connes–Kreimer Hopf algebra of renormalization developed for QFT could be the right type of formalism to use in such a Quantum Gravity context.

As for the second (set of) question(s), the GFT approach identifies uniquely the basic building blocks of a quantum space, those that could be responsible for the kind of "condensation" process or the transition to a fluid phase at the root of the emergence of a smooth spacetime in some approximation and physical regime, and gives a precise prescription for their classical and quantum dynamics, that can now be investigated. From this perspective, it is best interpreted as a theory of "pre-geometry" in the sense discussed in the chapters by Markopoulou and Dreyer. In particular, one could develop a statistical mechanics picture for the dynamics of the GFT "atoms" of space, and then the above idea of a "condensation" or in general of the possibility of an hydrodynamic description could be tested in specific GFT models, and in very concrete and precise terms. A more detailed discussion of the possible development of GFTs along these lines can be found in [44].

Whether any of the above ideas will be realized, or other, not yet imagined, possibilities for development will become manifest in the near future, only further work will tell. In our opinion, however, it is already clear that the GFT approach can be the right framework for investigating the most fundamental questions about Quantum Gravity.

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Questions and answers

- **Q - L. Crane - to T. Thiemann**:
  In order to apply the canonical approach to General Relativity, it is necessary to choose a spacelike foliation of the spacetime. Is it important that a general spacetime does not admit such a foliation? For example, spacetimes with black holes in them do not admit such foliations, or at least not ones with physical time functions and constant topology. Does this manifest itself indirectly in some of the problems of the LQG approach?
  - **A - T. Thiemann**:
    By a well known theorem due to Geroch, every globally hyperbolic spacetime admits a foliation by spacelike hypersurfaces. Global hyperbolicity is a physical requirement that is motivated by being able to have a well posed initial value formulation of General Relativity. Hence, classically there is absolutely no loss in making this assumption. In particular, spacetimes with black holes are certainly globally hyperbolic, in fact the black hole theorems due to Penrose and Hawking have global hyperbolicity in their assumptions (for Schwarzschild use Kruskal coordinates to see it explicitly).
    LQG starts from this classical framework and so one may think that it cannot deal with topology change. However, very beautifully this is not the case: vectors in the LQG Hilbert space are superpositions of spin network states. These describe polymerlike excitations of the gravitational field on finite graphs. Consider the volume operator of LQG associated with some spatial region. If that region has empty intersection with the given graph then the volume vanishes. Physically this means that the given state assigns no volume to that region, i.e. that there is a hole in that hypersurface. Hence we see that topology change is all over the place in LQG. The reason why this happens is that in order to mathematically define the classical Einstein equations we must assume that the metric is everywhere non-degenerate. However, that requirement can be totally relaxed in the quantum formulation. Notice that

the holes can be seen only when probing the geometry with regions which are "smaller" than the scale of the graph. Macroscopically the geometry therefore remains non degenerate because a semiclassical state necessarily is based on very "fine" graphs.

In conclusion, there are absolutely no problems in LQG associated with that type of question.

- **Q - R. Percacci - to T. Thiemann**:
  LQG can be seen as an attempt to directly "quantize Einstein's theory". As discussed in Burgess' contribution, Einstein's theory can be seen as a low energy effective field theory and one would expect that the gravitational dynamics gets modified at very high energies. For example, higher derivative terms could appear in the action. To what extent could one hope to generalize the results of LQG for these more general actions?
  - **A - T. Thiemann**:
    The semiclassical limit of LQG is the Einstein–Hilbert term. The correction terms of higher power in $\hbar$ or rather ℓ_P^2 can indeed be interpreted as higher derivative terms of the type that Burgess is discussing. The important point is that this interpretation holds only when using the equations of motion of the Einstein–Hilbert term. This is necessary in order to substitute the canonical momenta of the canonical theory by the covariantly defined extrinsic curvature which supplies the higher covariant derivatives. The real question is why one does not quantize higher derivative actions directly. The answer is very simple: one could, but unless the additional terms are topological, i.e. are at least on shell equal to total derivatives, one changes the number of degrees of freedom of the theory. Let us discuss a simple example, an R^2 term. Even after performing an integration by parts, this term will depend on time derivatives of the spatial metric up to third if not fourth order. Thus, in order to solve the equations of motion, one needs to specify initial data involving the spatial metric together with its velocity, acceleration and possibly time derivatives of third order. Thus, even at linear order the theory does not only have the familiar two polarization degrees of freedom of gravitational waves but in fact more. Notice that this is a purely classical observation and in the literature is well known as generalized Ostrogadsky method. See e.g. the book by Tuytin on constrained systems or recent papers by Woodard. Hence, as in Yang–Mills theories, higher derivative effective actions are never to be thought of as classical starting points for quantization but rather as effective tools or vehicles in order to do calculations such as only computing tree diagrams of the effective theory rather than doing all loop orders of the fundamental theory. This is the same in the Lagrangian and in the Hamiltonian approach. In summary, there is total agreement in the two approaches.
- **Q - R. Percacci - to T. Thiemann**:

I partly disagree with your answer.
1. If the starting point of quantization (the “bare” action) contained other Planck-mass degrees of freedom beyond the graviton, that would not really be a problem because we do not have access to those energies and we cannot check. For now, all we know is that the theory must describe a massless graviton at low energies.
2. The reason why higher derivative terms are not present in the bare action in Yang–Mills theories is that they are not renormalizable in perturbation theory, and perturbation theory works in that case. In the case of gravity we both seem to think that it is necessary to go beyond perturbation theory. But then, one cannot apply the familiar power counting arguments and it is not so clear what criteria can be used to determine the bare action. The Einstein–Hilbert or Palatini action is a good starting point but I do not expect it to be the whole story.
Instead of trying to guess the bare action and then derive low energy physics from it in a “top down” fashion, the Wilsonian approach may provide a method of determining it starting from below. Start by assuming an effective field theory point of view with an action containing all terms that are compatible with diffeomorphism invariance. As discussed in Burgess’ contribution, this allows one to consistently talk of quantum field theories of gravity. One would then calculate the beta functions and see where the flow leads to when the energy tends to infinity. If the limit can be taken, that is the bare action. Thus, at least in principle, this “bottom up” approach can be used to determine the bare action.
I agree with you that if the bare action contained higher derivative terms life would be messy, but that would be our problem and not a fundamental inconsistency in the physical laws. Perhaps if we are lucky the bare action will still look reasonably simple after some field redefinition, as discussed by Taylor.

– **A - T. Thiemann**:
  I also do with your response.
  1. I do not buy that. Usually we say that we cannot see a particle species at low energies because its rest mass is too high and hopefully there are decay channels that allow for its decay into lighter particles. Nobody has shown that such decay channels exist for higher derivative theories nor has it been shown that the effective rest mass of the additional degrees of freedom is of Planck size, it may well be much lower. Next, concepts such as rest masses and energies are (Minkowski) background dependent, LQG does not allow for such notions a priori and hence usual intuition may very well be completely misleading. Finally, notice that very massive particles have the tendency to vastly increase the value of the cosmological constant through

their vacuum fluctuations which is in conflict with observation unless one starts fine tuning.
2. The case of QCD is a counter example of your statement. QCD is perturbatively renormalizable; however, perturbation theory is not applicable to the most important phenomena such as confinement. Your statement is obviously inconclusive and we seem to have arrived at a point where only experiments may be able to decide. Here I want to remind of an analysis due to T. Damour *et al*. who numerically showed that within a 15 parameter space of generalized actions the pure Einstein–Hilbert term is by far the most natural choice when comparing with experiment. I am familiar with the Wilsonian approach and of course I completely agree with it.
I think we do not disagree on the point that the effective Lagrangean contains higher derivative terms. However, what I want to say is that in a Hamiltonian approach such as LQG the treatment of the higher derivative action as a fundamental action would be different from what one does usually in the Lagrangean counter term framework index quantization path integral. In the latter approach, these counter terms do not modify the number of degrees of freedom, while in the former they would do. You can see this plainly by looking at how block spin transformations generate additional effective terms. You always integrate out high momentum degrees of freedom with respect to the naive action, you never change the number of degrees of freedom in the path integral measure (in Yang–Mills theories you only use a measure depending on the connection but not its higher (covariant) derivatives). In the Hamiltonian approach you would have to face more degrees of freedom. In order to reconcile both approaches, you use the equations of motion of the naive (first order) action in order to turn higher derivative terms into lower derivative terms. I do not care if life is messy, I wanted to point out that the treatment of effective actions as fundamental Lagrangeans in canonical treatments is inconsistent with the usual treatment. This is how I interpreted your question.

- **Q - R. Percacci - to E. Livine**:
  Could you elaborate further on the physical significance of the continuous vs. discrete spectrum of the area operator?

  - **A - E. Livine**:
    Loop quantum gravity (LQG) formulates gravity as a gauge theory based on the compact group SU(2). The Casimir of SU(2) gives the area spectrum. We then get a discrete spectrum. On the other hand, covariant loop quantum gravity (CLQG) has the non-compact Lorentz group as gauge group and obtains a continuous area spectrum. In three space-time dimensions, the gauge group is actually the Lorentz group, which gives a discrete length

spectrum in the Riemannian theory and a continuous spectrum (for space-like intervals) in the Lorentzian case. In four space-time dimensions, the gauge group of LQG is truly the complexification of SU(2) and the reality conditions might actually select a non-compact section of the complex group from which we would then derive a contiuous spectrum. Finally, these results are only at the kinematical level. They do not use the physical Hilbert space and inner-product, so we can not be sure of their physical relevance. Actually, the area operator is itself only defined in the kinematical Hilbert space (not invariant under diffeomorphism and not in the kernel of the Hamiltonian constraint) and we have not been able to lift it to a physical operator acting on physical state. Nevertheless, in three space-time dimensions, work by Noui & Perez (2004) suggests that we can construct a physical length operator by introducing particles in the theory and we then recover the kinematical results i.e a continuous length spectrum for the Lorentzina theory. The issue is, however, still open in four space-time dimensions.

- **Q - L. Crane - to D. Oriti**:
  It seems an awful shame to get to the point where each Feynman diagram in a GFT model is finite, then to describe the final theory as an infinite sum of such terms. Have you ever thought of the possibility that by specifying the structure of the observer including its background geometry we limit the number of simplicial complexes we need to sum over, or at least make most of the contributions small, thereby rendering the answer to any genuinely physical question finite?
  - **A - D. Oriti**:
    I agree. I would be careful in distinguishing the "definition of the theory", given by its partition function (or its transition amplitudes), and the quantities that, in the theory itself, corresponds to physical observables and are thus answers to physical questions. The partition function itself may be defined, in absence of a better way, through its perturbative expansion in Feynman diagrams, and thus involve an infinite sum that is most likely beyond reach of practical computability, and most likely divergent. However, I do believe that, once we understand the theory better, the answer to physical questions will require only finite calculations. This can happen in three ways, I think. As you suggest, the very mathematical formulation of the question, involving maybe the specification of an observer or of a reference frame, or referring to a finite spacetime volume only, or some other type of physical restriction, will allow or even force us to limit the sum over graphs to a finite number of them, thus making the calculation finite. Another possibility is that, as in ordinary QFT, the answer to a physical question (e.g. the result

of some sort of scattering process, thus the corresponding transition amplitude) will require the calculation only to a finite order in perturbation theory. This means obtaining only approximate answers, but it may well be good enough for all practical purposes (again, this is the case in ordinary QFT). In order for this possibility to be realized, of course, one needs to clarify further the physical interpretation of the GFT coupling constant, beyond what is already known. Last, the infinite sums appear in the perturbative expansion of the full, microscopic, partition function; it is possible that, after more work, and with a deeper understanding of the GFT formalism, one will be able to obtain effective theories adapted to a more macroscopic context, e.g. suitable to study some specific phase of the theory (like the "condensed" one corresponding to a continuum approximation of spacetime), from the microscopic GFT; if this is the case the infinite sums of the perturbative expansion of the microscopic GFT will not be directly relevant for answering questions in this phase/approximation, and these questions may instead require only finite calculations in the effective theory.

Part IV

Discrete Quantum Gravity

18

Quantum Gravity: the art of building spacetime

J. AMBJØRN, J. JURKIEWICZ AND R. LOLL

18.1 Introduction

What is more natural than constructing space from elementary geometric building blocks? It is not as easy as one might think, based on our intuition of playing with Lego blocks in three-dimensional space. Imagine the building blocks are d-dimensional flat simplices all of whose side lengths are a, and let $d > 2$. The problem is that if we glue such blocks together carelessly we will *with probability one* create a space of no extension, in which it is possible to get from one vertex to any other in a few steps, moving along the one-dimensional edges of the simplicial manifold we have created. We can also say that the space has an extension which remains at the "cut-off" scale a. Our intuition coming from playing with Lego blocks is misleading here because it presupposes that the building blocks are embedded geometrically faithfully in Euclidean $\mathbb{R}^3$, which is not the case for the intrinsic geometric construction of a simplicial space.

By contrast, let us now be more careful in our construction work by assigning to a simplicial space $\mathcal{T}$ – which we will interpret as a (Euclidean) spacetime – the weight $e^{-S(\mathcal{T})}$, where $S(\mathcal{T})$ denotes the Einstein action associated with the piecewise linear geometry uniquely defined by our construction.[1] As long as the (bare) gravitational coupling constant G_N is large, we have the same situation as before. However, upon lowering G_N we will eventually encounter a phase transition beyond which the geometry is no longer crumpled into a tiny ball, but maximally extended. Such a geometry is made out of effectively one-dimensional filaments[2] which can branch out, and are therefore called *branched polymers* or *trees* [4; 2]. The transition separating the two phases [13; 14] is of first order, which implies that there is no smooth change between the two pathological types of minimally or maximally extended "universes".

1 There exists a natural, coordinate-independent definition of the Einstein action for piecewise linear geometries called the Regge action.
2 The d-dimensional building blocks are arranged such that $(d-1)$ "transverse" dimensions have a size of only a few lattice spacings.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

In order for the sum over geometries to produce a quantum theory of gravity in which classical geometry is reproduced in a suitable limit, we therefore need a different principle for selecting the geometries to be included in this sum. Below we will introduce such a principle: our prescription will be to sum over a class of (Euclidean) geometries which are in one-to-one correspondence with Lorentzian, *causal* geometries. At the discretized level, where we use a specific set of building blocks and gluing rules to constructively define the path integral, we call these geometries *causal dynamical triangulations* (CDT) [5; 6; 8; 7].

Before discussing CDT in more detail let us comment on the nature of the geometries contributing to the path integral. It is important to emphasize that in a *quantum* theory of gravity a given spacetime geometry as such has no immediate physical meaning. The situation is really the same as in ordinary quantum field theory or even quantum mechanics, where individual field configurations $\phi(x,t)$ or particle paths $x(t)$ are *not* observable. Only certain expectation values related to the fields or paths can be observed in experiments. This does not mean there cannot exist limits in which it is appropriate to talk about a particular field configuration or the path of a particle in an approximate sense. In the case of our actual universe, down to the smallest distances that have been probed experimentally, it certainly does seem adequate to talk about a fixed classical spacetime geometry. Nevertheless, at sufficiently small distances it will no longer make sense to ask classical questions about spacetime, at least if we are to believe in the principles of conventional quantum theory.

By way of illustration let us discuss the situation for the ordinary harmonic oscillator (or the free particle) and consider the path integral from (x_1, t_1) to (x_2, t_2). Precisely for the harmonic oscillator (or the free particle) the decomposition

$$x(t) = x_{\rm cl}(t) + y(t), \quad y(t_1) = y(t_2) = 0, \tag{18.1}$$

leads to an exact factorization of the path integral, because the action satisfies

$$S(x) = S(x_{\rm cl}) + S(y). \tag{18.2}$$

This implies that the classical path $x_{\rm cl}(t)$ contributes to the path integral with the classical action, and $y(t)$ with quantum fluctuations independent of this classical part. Taking the classical trajectory to be macroscopic one obtains the picture of a macroscopic path dressed with small quantum fluctuations; small because they are independent of the classical motion. An explicit Euclidean calculation yields the result

$$\left\langle \int_0^T {\rm d}t \; y^2(t) \right\rangle = \frac{\hbar}{2m\omega^2} \left(\omega T \tanh^{-1} \omega T - 1\right) \tag{18.3}$$

as a function of the oscillator frequency ω and mass m. Let us now consider a situation where we have chosen the "system size", i.e. $x_{\rm cl}(t)$, to be macroscopic.

According to (18.3), the quantum fluctuations around this path can then be considered small since $\hbar$ is small.

This is more or less the picture we envisage for our present-day universe in Quantum Gravity: the universe is of macroscopic size, governed by the classical equations of motion (the analog of choosing "by hand" (x_1, t_1) and (x_2, t_2) to be macroscopic in the example above), and the small quantum fluctuations are dictated by the gravitational coupling constant (times $\hbar/c^3$).

A given configuration $x(t)$ in the path integral for the quantum-mechanical particle is (with probability one) a continuous, nowhere differentiable path, which moreover is fractal with Hausdorff dimension two, as we know from the rigorous construction of the Wiener measure on the set of parametrized paths. In the case of Quantum Gravity we do not have a similar mathematically rigorously defined measure on the space of geometries, but it is natural to expect that *if* it exists, a typical geometry in the path integral will be continuous, but nowhere differentiable. By analogy, the piecewise linear geometries seem a good choice if we want to approximate the gravitational path integral by a set of geometries and subsequently take a limit where the approximation (the cut-off) is removed. Moreover, such simplicial manifolds possess a natural, geometric and coordinate-independent implementation of the Einstein–Hilbert action. With all local curvature degrees of freedom present (albeit in a discretized fashion), we also expect them to be suitably "dense" in the set of all continuous geometries.

The spirit is very much that of the standard lattice formulation of quantum field theory where (flat) spacetime is approximated by a hypercubic lattice. The ultraviolet cut-off in such field theories is given by the lattice spacing, i.e. the length of all one-dimensional lattice edges. We can in a similar and simple manner introduce a *diffeomorphism-invariant* cut-off in the sum over the piecewise linear geometries by restricting it to the *building blocks* mentioned earlier. A natural building block for a d-dimensional spacetime is a d-dimensional equilateral simplex with side-length a, and the path integral is approximated by performing the sum over all geometries (of fixed topology[3]) which can be obtained by gluing such

[3] In classical General Relativity there is no motivation to consider spacetimes whose spatial topology changes in time, since their Lorentzian structure is necessarily singular. There is an interesting and long-standing discussion about whether one should include topology changes in a *quantum* theory of gravity. However, even in the case of two-dimensional Euclidean Quantum Gravity, where the classification of topology changes is simple, the summation over topologies has never been defined non-perturbatively in a satisfactory way, despite many attempts, in particular, in so-called non-critical string theory. (However, see [24; 25; 26] for how one may improve the convergence of the sum in two-dimensional *Lorentzian* Quantum Gravity by invoking not just the topological, but the causal, geometric structure of spacetime.) The situation becomes worse in higher dimensions. For instance, four-dimensional topologies are not classifiable, so what does it mean to sum over them in the path integral? The problem – even in dimension two – is that there are many more geometries of complicated topology than there are of simple topology, with the consequence that any sum over geometries will be (i) completely dominated by these complicated topologies, and (ii) plainly divergent in a way which (until now) has made it impossible to define the theory non-perturbatively in an unambiguous and physically satisfactory manner. In higher dimensions these problems are totally out of control.

building blocks together, each geometry weighted appropriately (for example, by e^{-S}, where S is the Einstein–Hilbert action). Afterwards we take the limit $a \to 0$. For a particular choice of the bare, dimensionless coupling constants one may be able to obtain a continuum limit, and thus extract a continuum theory. For other values, if the sum exists at all (possibly after renormalization), one will merely obtain a sum which has no continuum interpretation. This situation is precisely the same as that encountered in ordinary lattice field theory in flat spacetime.

As mentioned earlier it has, up to now, not been possible to define constructively a Euclidean path integral for gravity in four dimensions by following the philosophy just outlined. One simply has not succeeded in identifying a continuum limit of the (unrestricted) sum over Euclidean building blocks. Among the reasons that have been advanced to explain this failure, it is clear that the *entropy* of the various geometries plays an important role. We have already pointed out that the crumpled geometries of no extension dominate the space of all continuous geometries whenever the dimension of spacetime is larger than two. There is nothing wrong with this a priori; the path integral of any quantum field theory is dominated completely by wild UV-field fluctuations. However, in the case of *renormalizable* quantum field theories there exists a well-defined limiting procedure which allows one to extract "continuum" physics by fine-tuning the bare coupling constants of the theory. An analogous procedure in Euclidean Quantum Gravity still has not been found, and adding (bosonic) matter does not improve the situation. Instead, note that the Einstein–Hilbert action has a unique feature, namely, it is unbounded from below. The transition between the crumpled and the branched-polymer geometries can be seen as a transition from a phase where the entropy of configurations dominates over the action to a phase where the unboundedness of the Euclidean action becomes dominant.[4] The impossibility of finding a continuum limit may be seen as the impossibility of balancing the entropy of configurations against the action. We need another guiding principle for selecting Euclidean geometries in the path integral in order to obtain a continuum limit, and it is such a principle we turn to next.

18.2 Defining CDT

It has been suggested that the signature of spacetime may be explained from a dynamical principle [16]. Being somewhat less ambitious, we will assume it has Lorentzian signature and accordingly change our perspective from the Euclidean

[4] Although the action is not unbounded below in the regularized theory, this feature of the continuum action nevertheless manifests itself in the limit as the (discretized) volume of spacetime is increased, eventually leading to the above-mentioned phase transition at a particular value of the bare gravitational coupling constant. Remarkably, a related phenomenon occurs in bosonic string theory. If the world-sheet theory is regularized non-perturbatively in terms of triangulations (with each two-dimensional world-sheet glued from fundamental simplicial building blocks), the tachyonic sickness of the theory manifests itself in the form of surfaces degenerating into branched polymers [1] .

formulation of the path integral discussed in the previous section to a Lorentzian formulation, motivated by the uncontroversial fact that our universe has three space and one time dimension. A specific rotation to Euclidean signature introduced below will be needed in our set-up as a merely technical tool to perform certain sums over geometries. Unlike in flat spacetime there are no general theorems which would allow us to relate the Euclidean and Lorentzian quantum field theories when dealing with Quantum Gravity.

Consider now a connected space-like hypersurface in spacetime. Any classical evolution in general relativity will leave the topology of this hypersurface unchanged, since otherwise spacetime would contain regions where the metric is degenerate. However, as long as we do not have a consistent theory of Quantum Gravity we do not know whether such degenerate configurations should be included in the path integral. We have already argued that the inclusion of arbitrary spacetime topologies leads to a path integral that has little chance of making sense. One might still consider a situation where the overall topology of spacetime is fixed, but where one allows "baby universes" to branch off from the main universe, without permitting them to rejoin it and thus form "handles". Apart from being a rather artificial constraint on geometry, such a construction is unlikely to be compatible with unitarity. We will in the following take a conservative point of view and only sum over geometries (with Lorentzian signature) which permit a foliation in (proper) time and are causally well-behaved in the sense that no topology changes are allowed as a function of time. In the context of a formal continuum path integral for gravity, similar ideas have earlier been advanced by Teitelboim [33; 34].

Of the diffeomorphism-invariant quantities one can consider in the quantum theory, we have chosen a particular *proper-time propagator*, which can be defined constructively in a transparent way. We are thus interested in defining the path integral

$$G(g(0), g(T); T) = \int_{g(0)}^{g(T)} \mathcal{D}g \; \mathrm{e}^{iS[g]} \tag{18.4}$$

over Lorentzian geometries on a manifold $\mathcal{M}$ with topology $\Sigma \times [0, 1]$, where Σ is a compact, connected three-dimensional manifold. The geometries included in the path integral will be such that the induced boundary three-geometries $g(0)$ and $g(T)$ are space-like and separated by a time-like geodesic distance T, with T an external (diffeomorphism-invariant) parameter.

We now turn to the constructive definition of this object in terms of building blocks. The discretized analogue of an infinitesimal proper-time "sandwich" in the continuum will be a finite sandwich of thickness $\Delta t = 1$ (measured in "building block units" a) of topology $\Sigma \times [0, 1]$ consisting of a single layer of four-simplices. This layer has two spacelike boundaries, corresponding to two slices of constant (integer) "proper time" t which are one unit apart. They form two three-dimensional piecewise flat manifolds of topology Σ and consist of purely spacelike tetrahedra. By construction, the sandwich interior contains no vertices, so that any one of the four-simplices shares k of its vertices with the initial spatial slice and $5 - k$ of them with the final spatial slice, where $1 \leq k \leq 4$. To obtain extended spacetimes, one glues together sandwiches pairwise along their matching three-dimensional boundary geometries. We choose each four-simplex to have time-like links of length-squared a_{t}^2 and space-like links of length-squared a_{s}^2, with all of the latter located in spatial slices of constant integer t.

Each spatial tetrahedron at time t is therefore shared by two four-simplices (said to be of type (1,4) and (4,1)) whose fifth vertex lies in the neighbouring slice of constant time $t - 1$ and $t + 1$ respectively. In addition we need four-simplices of type (2,3) and (3,2) which share one link and one triangle with two adjacent spatial slices, as illustrated in Fig. 18.1 (see [8] for details). The integer-valued proper time t can be extended in a natural way to the interiors of the four-simplices, leading to a global foliation of any causal dynamically triangulated spacetime into piecewise flat (generalized) triangulations for any constant real value of t [15]. Inside each building block this time coincides with the proper time of Minkowski space. Moreover, it can be seen that in the piecewise linear geometries the mid-points of all spatial tetrahedra at constant time t are separated a fixed time-like geodesic distance (in lattice units a_{t}, a_{s}) from the neighbouring hypersurfaces at $t - 1$ and $t + 1$. It is in this sense that the "link distance" t, i.e. counting future-oriented time-like links between spatial slices is a discretized analogue of their proper-time distance.

Let us furthermore assume that the two possible link lengths are related by

$$a_{\mathrm{t}}^2 = -\alpha a_{\mathrm{s}}^2. \tag{18.5}$$

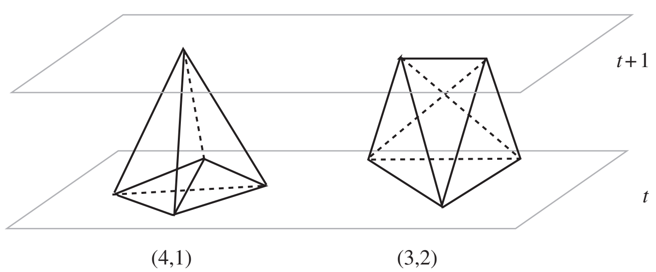

Fig. 18.1. The two fundamental building blocks of causal dynamically triangulated gravity. The flat four-simplex of type (4,1) on the left has four of its vertices at time t and one at time $t+1$, and analogously for the (3,2)-simplex on the right. The "gap" between two consecutive spatial slices of constant integer time is filled by copies of these simplicial building blocks and their time-reversed counterparts, the (1,4)- and the (2,3)-simplices.

All choices $\alpha > 0$ correspond to Lorentzian and all choices $\alpha < -7/12$ to Euclidean signature, and a Euclideanization of geometry is obtained by a suitable analytic continuation in α (see [8] for a detailed discussion of this "Wick rotation" where one finds $S_{\mathrm{E}}(-\alpha) = i S_{\mathrm{L}}(\alpha)$ for $\alpha > 7/12$).

Setting $\alpha = -1$ leads to a particularly simple expression for the (Euclidean) Einstein–Hilbert action of a given triangulation $\mathcal{T}$ (since all four-simplices are then identical geometrically), namely,

$$S_{\mathrm{E}}(\mathcal{T}) = -k_0 N_0(\mathcal{T}) + k_4 N_4(\mathcal{T}), \tag{18.6}$$

with $N_i(\mathcal{T})$ denoting the number of i-dimensional simplices in T. In (18.6), k_0 is proportional to the inverse (bare) gravitational coupling constant, $k_0 \sim 1/G_N$, while k_4 is a linear combination of the cosmological and inverse gravitational coupling constants. The action (18.6) is calculated from Regge's prescription for piecewise linear geometries. If we take $\alpha \neq -1$ the Euclidean four-simplices of type (1,4) and type (2,3) will be different and appear with different weights in the Einstein–Hilbert action (see [8]). For our present purposes it is convenient to use the equivalent parametrization

$$S_{\mathrm{E}}(\mathcal{T}) = -k_0 N_0(\mathcal{T}) + k_4 N_4(\mathcal{T}) + \Delta(2N_{14}(\mathcal{T}) + N_{23}(\mathcal{T})), \tag{18.7}$$

where $N_{14}(\mathcal{T})$ and $N_{23}(\mathcal{T})$ denote the combined numbers in $\mathcal{T}$ of four-simplices of types $(1, 4)$ and $(4, 1)$, and of types $(2, 3)$ and $(3, 2)$, respectively. The explicit map between the parameter Δ in eq. (18.7) and α can be readily worked out [10]. For the simulations reported here we have used Δ in the range 0.4–0.6.

The (Euclidean) discretized analogue of the continuum proper-time propagator (18.4) is defined by

$$G_{k_0,k_4,\Delta}(\mathcal{T}^{(3)}(0), \mathcal{T}^{(3)}(T), T) = \sum_{\mathcal{T}\in\mathcal{T}_T} \frac{1}{C_{\mathcal{T}}}\, e^{-S_E(\mathcal{T})}, \tag{18.8}$$

where the summation is over the set $\mathcal{T}_T$ of all four-dimensional triangulations of topology $\Sigma^3 \times [0, 1]$ (which we in the following always choose to be S^3) and T proper-time steps, whose spatial boundary geometries at proper times 0 and T are $\mathcal{T}^{(3)}(0)$ and $\mathcal{T}^{(3)}(T)$. The order of the automorphism group of the graph $\mathcal{T}$ is denoted by $C_{\mathcal{T}}$. The propagator can be related to the quantum Hamiltonian conjugate to t, and in turn to the transfer matrix of the (Euclidean) statistical theory [8].

It is important to emphasize again that we rotate each configuration to a Euclidean "spacetime" simply in order to perform the summation in the path integral, and that this is made possible by the piecewise linear structure of our geometry and the existence of a proper-time foliation. Viewed from an inherently Euclidean perspective there would be no motivation to restrict the sum over geometries to "causal" geometries of the kind constructed above. We also want to stress that the use of piecewise linear geometries has allowed us to write down a (regularized) version of (18.4) using only geometries, not metrics (which are of course not diffeomorphism-invariant), and finally that the use of building blocks has enabled the introduction of a diffeomorphism-invariant cut-off (the lattice link length a).

18.3 Numerical analysis of the model

While it may be difficult to find an explicit analytic expression for the full propagator (18.8) of the four-dimensional theory, Monte Carlo simulations are readily available for its analysis, employing standard techniques from Euclidean dynamically triangulated Quantum Gravity [3]. Ideally one would like to keep the *renormalized*[5] cosmological constant Λ fixed in the simulation, in which case the presence of the cosmological term $\Lambda \int \sqrt{g}$ in the action would imply that the four-volume V_4 fluctuated around $\langle V_4 \rangle \sim \Lambda^{-1}$. However, for simulation-technical reasons one fixes instead the number N_4 of four-simplices (or[6] the four-volume V_4) from the outset, working effectively with a cosmological constant $\Lambda \sim V_4^{-1}$.

18.3.1 The global dimension of spacetime

A "snapshot", by which we mean the distribution of three-volumes as a function of the proper time $0 \leq t \leq T$ for a spacetime configuration randomly picked from the Monte Carlo-generated geometric ensemble, is shown in Fig. 18.2. One observes a "stalk" of essentially no spatial extension (with spatial volumes close to the minimal triangulation of S^3 consisting of five tetrahedra) expanding into a universe of genuine "macroscopic" spatial volumes, which after a certain time $\tau \leq T$ contracts again to a state of minimal spatial extension. As we emphasized earlier, a single such configuration is unphysical, and therefore not observable. However, a more systematic analysis reveals that fluctuations around an overall "shape" similar to the one of Fig. 18.2 are relatively small, suggesting the existence of a *background geometry* with relatively small quantum fluctuations superimposed. This is precisely the scenario advocated in Section 18.1 and is rather remarkable, given that our formalism is background-independent. Our first major goal is to verify quantitatively that we are indeed dealing with an approximate

5 For the relation between the bare (dimensionless) cosmological constant k_4 and the renormalized cosmological constant Λ see [4].

6 For fixed α (or Δ) one has $\langle N_{14} \rangle \propto \langle N_{23} \rangle \propto \langle N_4 \rangle$. V_4 is given as (see [8] for details): $V_4 = a_s^4 (N_{14} \sqrt{8\alpha + 3} + N_{23} \sqrt{12\alpha + 7})$. We set $a_s = 1$.

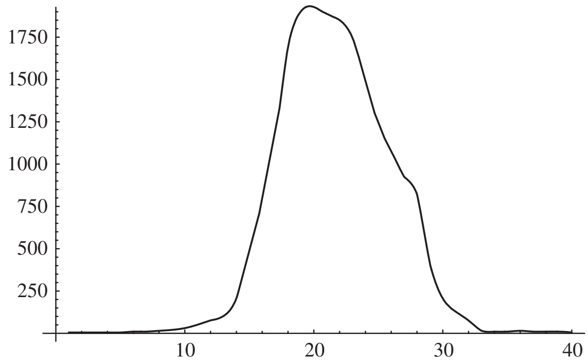

Fig. 18.2. Snapshot of a "typical universe" consisting of approximately 91 000 four-simplices as it appears in the Monte Carlo simulations at a given "computer time". We plot the three-volume at each integer step in proper time, for a total time extent of $T = 40$, in units where $a_s = 1$.

four-dimensional background geometry [9; 11], and secondly to determine the effective action responsible for the observed large-scale features of this background geometry [12; 10]. Important information is contained in how the expectation values of the volume V_3 of spatial slices and the total time extent τ (the proper-time interval during which the spatial volumes $V_3 \gg 1$) of the observed universe behave as the total spacetime volume V_4 is varied. We find that to good approximation the spatially extended parts of the spacetimes for various four-volumes V_4 can be mapped onto each other by rescaling the spatial volumes and the proper times according to

$$V_3 \to V_3/V_4^{3/4}, \quad \tau \to \tau/V_4^{1/4}. \tag{18.9}$$

To quantify this we studied the so-called volume–volume correlator

$$\langle V_3(0)V_3(\delta)\rangle = \frac{1}{t^2}\sum_{j=1}^{t}\langle V_3(j)V_3(j+\delta)\rangle \tag{18.10}$$

for pairs of spatial slices an integer proper-time distance δ apart. Figure 18.3 shows the volume–volume correlator for five different spacetime volumes V_4, using the rescaling (18.9),[7] and exhibiting that it is almost perfect. An error estimate yields $d = 4 \pm 0.2$ for the large-scale dimension of the universe [10].

Another way of obtaining an effective dimension of the nonperturbative ground state, its so-called *spectral dimension* D_S, comes from studying a diffusion process

[7] In (18.10) we use discrete units such that successive spatial slices are separated by 1. For convenience we periodically identify $\mathcal{T}^{(3)}(T) = \mathcal{T}^{(3)}(0)$ and sum over all possible three-geometries $\mathcal{T}^{(3)}(0)$, rather than working with fixed boundary conditions. In this way (18.10) becomes a convenient translation-invariant measure of the spatial and temporal extensions of the universe (see [7] for a detailed discussion).

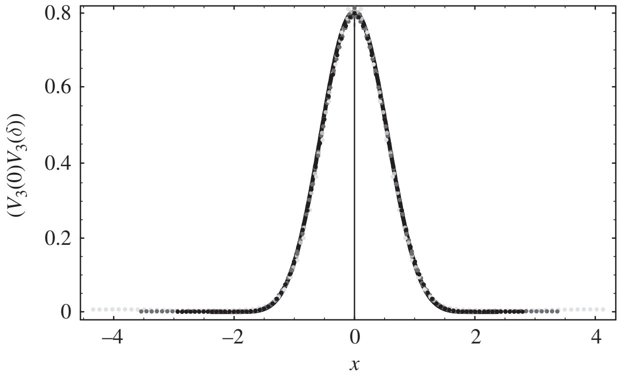

Fig. 18.3. The scaling of the volume–volume correlator, as function of the rescaled time variable $x = \delta/(N_4)^{1/4}$. Data points come from system sizes $N_4 =$ 22 500, 45 000, 91 000, 181 000 and 362 000 at $\kappa_0 = 2.2$, $\Delta = 0.6$ and $T = 80$.

on the underlying geometric ensemble. On a d-dimensional manifold with a fixed, smooth Riemannian metric $g_{ab}(\xi)$, the diffusion equation has the form

$$\frac{\partial}{\partial\sigma} K_g(\xi, \xi_0; \sigma) = \Delta_g K_g(\xi, \xi_0; \sigma), \tag{18.11}$$

where σ is a fictitious diffusion time, Δ_g the Laplace operator of the metric $g_{ab}(\xi)$ and $K_g(\xi, \xi_0; \sigma)$ the probability density of diffusion from point ξ_0 to point ξ in diffusion time σ. We will consider diffusion processes which initially are peaked at some point ξ_0, so that

$$K_g(\xi, \xi_0; \sigma = 0) = \frac{1}{\sqrt{\det g(\xi)}} \delta^d(\xi - \xi_0). \tag{18.12}$$

For the special case of a flat Euclidean metric, we have

$$K_g(\xi, \xi_0; \sigma) = \frac{\mathrm{e}^{-d_g^2(\xi,\xi_0)/4\sigma}}{(4\pi\sigma)^{d/2}}, \quad g_{ab}(\xi) = \delta_{ab}, \tag{18.13}$$

where d_g denotes the distance function associated with the metric g.

A quantity which is easier to measure in numerical simulations is the *average return probability* $P_g(\sigma)$, defined by

$$P_g(\sigma) := \frac{1}{V} \int d^d\xi \sqrt{\det g(\xi)}\; K_g(\xi, \xi; \sigma), \tag{18.14}$$

where V is the spacetime volume $V = \int d^d\xi \sqrt{\det g(\xi)}$. For an infinite flat space, we have $P_g(\sigma) = 1/(4\pi\sigma)^{d/2}$ and thus can extract the dimension d by taking the logarithmic derivative

$$-2\,\frac{d \log P_g(\sigma)}{d \log \sigma} = d, \tag{18.15}$$

independent of σ. For nonflat spaces and/or finite volume V, one can still use eq. (18.15) to extract the dimension, but there will be correction terms (see [10] for a detailed discussion).

In applying this set-up to four-dimensional Quantum Gravity in a path integral formulation, we are interested in measuring the expectation value of the average return probability $P_g(\sigma)$. Since $P_g(\sigma)$ defined according to (18.14) is invariant under reparametrizations, it makes sense to take its quantum average over all geometries of a given spacetime volume V_4,

$$P_{V_4}(\sigma) = \frac{1}{\tilde{Z}_{\mathrm{E}}(V_4)} \int \mathcal{D}[g_{ab}]\, e^{-\tilde{S}_{\mathrm{E}}(g_{ab})} \delta \left(\int d^4x \sqrt{\det g} - V_4 \right) P_g(\sigma), \quad (18.16)$$

where $\tilde{Z}_{\mathrm{E}}(V_4)$ is the Quantum Gravity partition function for spacetimes with constant four-volume V_4.

Our next task is to define a diffusion process on the class of metric spaces under consideration, the piecewise linear structures defined by the causal triangulations $\mathcal{T}$. We start from an initial probability distribution

$$K_{\mathcal{T}}(i, i_0; \sigma = 0) = \delta_{i,i_0}, \quad (18.17)$$

which vanishes everywhere except at a randomly chosen (4,1)-simplex i_0, and define the diffusion process by the evolution rule

$$K_{\mathcal{T}}(j, i_0; \sigma + 1) = \frac{1}{5} \sum_{k \to j} K_{\mathcal{T}}(k, i_0; \sigma), \quad (18.18)$$

where the diffusion time σ now advances in discrete integer steps. These equations are the simplicial analogues of (18.12) and (18.11), $k \to j$ denoting the five nearest neighbours of the four-simplex j. In this process, the total probability

$$\sum_j K_{\mathcal{T}}(j, i_0; \sigma) = 1 \quad (18.19)$$

is conserved. The probability of returning to the simplex i_0 is then defined as $P_{\mathcal{T}}(i_0; \sigma) = K_{\mathcal{T}}(i_0, i_0; \sigma)$ and its quantum average as

$$P_{N_4}(\sigma) = \frac{1}{\tilde{Z}_{\mathrm{E}}(N_4)} \sum_{\mathcal{T}_{N_4}} e^{-\tilde{S}_{\mathrm{E}}(\mathcal{T}_{N_4})} \frac{1}{N_4} \sum_{i_0 \in \mathcal{T}_{N_4}} K_{\mathcal{T}_{N_4}}(i_0, i_0; \sigma), \quad (18.20)$$

where $\mathcal{T}_{N_4}$ denotes a triangulation with N_4 four-simplices, and $\tilde{S}_{\mathrm{E}}(\mathcal{T}_{N_4})$ and $\tilde{Z}_{\mathrm{E}}(N_4)$ are the obvious simplicial analogues of the continuum quantities in eq. (18.16).

We can extract the value of the spectral dimension D_{S} by measuring the logarithmic derivative as in (18.15) above, that is,

$$D_{\mathrm{S}}(\sigma) = -2 \frac{d \log P_{N_4}(\sigma)}{d \log \sigma}, \quad (18.21)$$

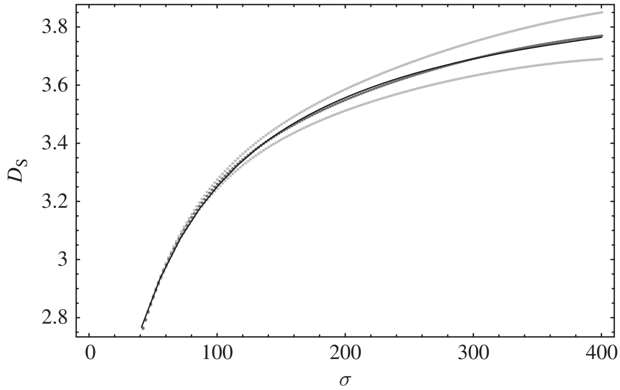

Fig. 18.4. The spectral dimension D_S of the universe as a function of the diffusion time σ, measured for $\kappa_0 = 2.2$, $\Delta = 0.6$ and $t = 80$, and a spacetime volume $N_4 = 181$k. The averaged measurements lie along the central curve, together with a superimposed best fit $D_S(\sigma) = 4.02 - 119/(54+\sigma)$ (thin black curve). The two outer curves represent error bars.

as long as the diffusion time is not much larger than N_4^{2/D_S}. The outcome of the measurements is presented in Fig. 18.4, with error bars included. (The two outer curves represent the envelopes to the tops and bottoms of the error bars.) The error grows linearly with σ, due to the presence of the $\log \sigma$ in (18.21).

The remarkable feature of the curve $D_S(\sigma)$ is its slow approach to the asymptotic value of $D_S(\sigma)$ for large σ. The new phenomenon we observe here is a *scale dependence of the spectral dimension*, which has emerged dynamically [11; 10].

As explained by [11], the best three-parameter fit which asymptotically approaches a constant is of the form

$$D_S(\sigma) = a - \frac{b}{\sigma + c} = 4.02 - \frac{119}{54 + \sigma}. \tag{18.22}$$

The constants a, b and c have been determined by using the data range $\sigma \in [40, 400]$ and the curve shape agrees well with the measurements, as can be seen from Fig. 18.4. Integrating (18.22) we obtain

$$P(\sigma) \sim \frac{1}{\sigma^{a/2}(1 + c/\sigma)^{b/2c}}, \tag{18.23}$$

from which we deduce the limiting cases

$$P(\sigma) \sim \begin{cases} \sigma^{-a/2} & \text{for large } \sigma, \\ \sigma^{-(a-b/c)/2} & \text{for small } \sigma. \end{cases} \tag{18.24}$$

Again we conclude that within measuring accuracy the large-scale dimension of spacetime in our model is four. We also note that the short-distance spectral dimension seems to be approximately $D_S = 2$, signalling a highly non-classical behaviour.

18.3.2 The effective action

Our next goal will be to understand the precise analytical form of the volume–volume correlator (18.10). To this end, let us consider the distribution of differences in the spatial volumes V_3 of successive spatial slices at proper times t and $t + \delta$, where δ is infinitesimal, i.e. $\delta = 1$ in lattice proper-time units. We have measured the probability distribution $P_{V_3}(z)$ of the variable

$$z = \frac{V_3(t+\delta) - V_3(t)}{V_3^{1/2}}, \qquad V_3 := V_3(t) + V_3(t+\delta) \tag{18.25}$$

for different values of V_3. As shown in Fig. 18.5 they fall on a common curve.[8] Furthermore, the distribution $P_{V_3}(z)$ is fitted very well by a Gaussian e^{-cz^2}, with a constant c independent of V_3. From estimating the entropy of spatial geometries, that is, the number of such configurations, one would expect corrections of the form V_3^{α}, with $0 \leq \alpha < 1$, to the exponent $c\,z^2$ in the distribution $P_{V_3}(z)$. Unfortunately it is impossible to measure these corrections directly in a reliable

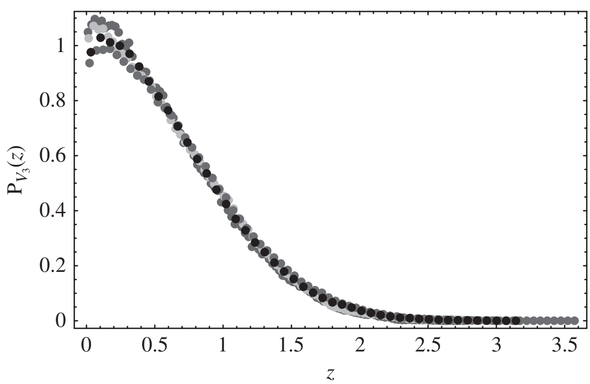

Fig. 18.5. Distribution $P_{V_3}(z)$ of volume differences of adjacent spatial slices, for three-volumes $V_3 = 10.000$, 20.000, 40.000 and 80.000 tetrahedra.

[8] Again we have applied finite-size scaling techniques, starting out with an arbitrary power V_3^{α} in the denominator in (18.25), and then determining $\alpha = 1/2$ from the principle of maximal overlap of the distributions for various V_3s.

way. We therefore make a general ansatz for the probability distribution for *large* $V_3(t)$ as

$$\exp\left[-\frac{c_1}{V_3(t)}\left(\frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\right)^2 - c_2 V_3^{\alpha}(t)\right], \tag{18.26}$$

where $0 \leq \alpha < 1$, and c_1 and c_2 are positive constants.

In this manner, we are led by "observation" to the effective action

$$S_{V_4}^{\mathrm{eff}} = \int_0^T \mathrm{d}t \, \left(\frac{c_1}{V_3(t)}\left(\frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\right)^2 + c_2 V_3^{\alpha}(t) - \lambda V_3(t)\right), \tag{18.27}$$

valid for large three-volume $V_3(t)$, where λ is a Lagrange multiplier to be determined such that

$$\int_0^T dt \; V_3(t) = V_4. \tag{18.28}$$

From general scaling of the above action it is clear that the only chance to obtain the observed scaling law, expressed in terms of the variable $t/V_4^{1/4}$, is by setting $\alpha = 1/3$. In addition, to reproduce the observed stalk for large times t the function $V_3^{1/3}$ has to be replaced by a function of V_3 whose derivative at 0 goes like V_3^{ν}, $\nu \geq 0$, for reasons that will become clear below. A simple modification, which keeps the large-V_3 behaviour intact, is given by

$$V_3^{1/3} \rightarrow (1 + V_3)^{1/3} - 1, \tag{18.29}$$

but the detailed form is not important. If we now introduce the (non-negative) *scale factor* $a(t)$ by

$$V_3(t) = a^3(t), \tag{18.30}$$

we can (after suitable rescaling of t and $a(t)$) write the effective action as

$$S_{V_4}^{\mathrm{eff}} = \frac{1}{G_N}\int_0^T \mathrm{d}t \, \left(a(t)\left(\frac{\mathrm{d}a(t)}{\mathrm{d}t}\right)^2 + a(t) - \lambda a^3(t)\right), \tag{18.31}$$

with the understanding that the linear term should be replaced using (18.30) and (18.29) for small $a(t)$. We emphasize again that we have been led to (18.31) entirely by "observation" and that one can view the small-$a(t)$ behaviour implied by (18.29) as a result of quantum fluctuations.

18.3.3 Minisuperspace

Let us now consider the simplest minisuperspace model for a closed universe in quantum cosmology, as for instance used by Hartle and Hawking in their semiclassical evaluation of the wave function of the universe [17]. In Euclidean signature and proper-time coordinates, the metrics are of the form

$$ds^2 = dt^2 + a^2(t)d\Omega_3^2, \tag{18.32}$$

where the scale factor $a(t)$ is the only dynamical variable and $d\Omega_3^2$ denotes the metric on the three-sphere. The corresponding Einstein–Hilbert action is

$$S^{\text{eff}} = \frac{1}{G_N}\int \mathrm{d}t \left(-a(t)\left(\frac{\mathrm{d}a(t)}{\mathrm{d}t}\right)^2 - a(t) + \lambda a^3(t)\right). \tag{18.33}$$

If no four-volume constraint is imposed, λ is the cosmological constant. If the four-volume is fixed to V_4, such that the discussion parallels the computer simulations reported above, λ should be viewed as a Lagrange multiplier enforcing a given size of the universe. In the latter case we obtain the same effective action as that extracted from the Monte Carlo simulations in (18.31), *up to an overall sign*, due to the infamous conformal divergence of the classical Einstein action evident in (18.33). From the point of view of the *classical* equations of motion this overall sign plays of course no role. Let us compare the two potentials relevant for the calculation of semiclassical Euclidean solutions associated with the actions (18.33) and (18.31). The "potential"[9] is

$$V(a) = -a + \lambda a^3, \tag{18.34}$$

and is shown in Fig. 18.6, without and with small-a modification, for the standard minisuperspace model and our effective model, respectively.

The quantum-induced difference for small a is important since the action (18.31) admits a classically stable solution $a(t) = 0$ which explains the "stalk" observed in the computer simulations (see Fig. 18.2). Moreover, it is appropriate to speak of a Euclidean "bounce" because $a = 0$ is a local maximum. If one therefore *naively* turns the potential upside down when rotating back to Lorentzian signature, the metastable state $a(t) = 0$ can tunnel to a state where $a(t) \sim V_4^{1/4}$, with a probability amplitude per unit time which is (the exponential of) the Euclidean action.

In order to understand how well the semiclassical action (18.31) can reproduce the Monte Carlo data, that is, the correlator (18.10) of Fig. 18.3, we have solved for the semiclassical bounce using (18.31), and presented the result as the black curve in Fig. 18.3. The agreement with the real data generated by the Monte Carlo simulations is clearly perfect.

[9] To obtain a standard potential – without changing "time" – one should first transform to a variable $x = a^{\frac{3}{2}}$ for which the kinetic term in the actions assumes the standard quadratic form. It is the resulting potential $\tilde{V}(x) = -x^{2/3} + \lambda x^2$ which in the case of (18.31) should be modified for small x such that $\tilde{V}'(0) = 0$.

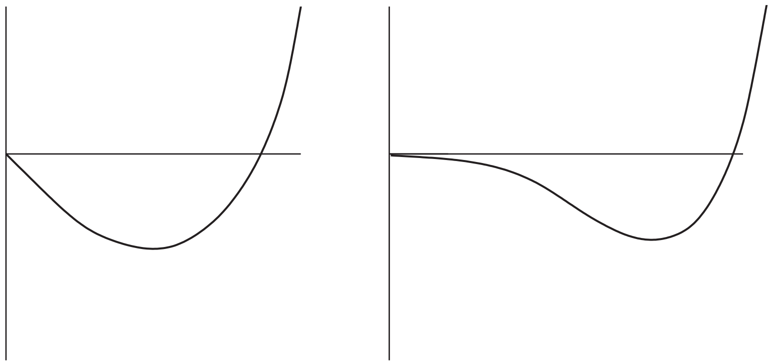

Fig. 18.6. The potential $V(a)$ of (18.34) underlying the standard minisuperspace dynamics (left) and the analogous potential in the effective action obtained from the full Quantum Gravity model, with small-a modification due to quantum fluctuations (right).

The picture emerging from the above for the effective dynamics of the scale factor resembles that of a universe created by tunneling from nothing (see, for example, [35; 36; 22; 31], although the presence of a preferred notion of time makes our situation closer to conventional quantum mechanics. In the set-up analysed here, there is apparently a state of vanishing spatial extension which can "tunnel" to a universe of finite linear extension of order $a \sim V_4^{1/4}$. Adopting such a tunneling interpretation, the action of the bounce is

$$S^{\text{eff}}_{V_4} \sim \frac{V_4^{1/2}}{G_N}, \tag{18.35}$$

and the associated probability per unit proper time for the tunneling is given by

$$P(V_4) \sim e^{-S^{\text{eff}}_{V_4}}. \tag{18.36}$$

18.4 Discussion

Causal dynamical triangulations (CDT) provide a regularized model of Quantum Gravity, which uses a class of piecewise linear geometries of Lorentzian signature (made from flat triangular building blocks) to define the regularized sum over geometries. The model is background-independent and has a diffeomorphism-invariant cut-off. For certain values of the bare gravitational and cosmological coupling constants we have found evidence that a continuum limit exists. The limit has been analysed by rotating the sum over geometries to Euclidean signature, made possible by our use of piecewise linear geometries. The geometries included in the sum thus originate from Lorentzian-signature spacetimes, a class different from (and smaller than) the class of geometries one would naturally include in a "native" Euclidean path integral. We have concentrated on computing a particular diffeomorphism-invariant quantity, the proper-time propagator, representing the sum over all geometries whose space-like boundaries are separated by a geodesic distance T. The sum over such geometries allows a simple and transparent implementation in terms of the above-mentioned building blocks.

In the Euclidean sector of the model, which can be probed by computer simulations we observe a four-dimensional macroscopic universe that can be viewed as a "bounce". When we integrate out (*after* having constructed the full path integral) all geometric degrees of freedom except for the global scale factor, the large-scale structure of the universe (the bounce) is described by the classical general-relativistic solution for a homogenous, isotropic universe with a cosmological constant on which (small) quantum fluctuations are superimposed. We find this result remarkable in view of the difficulties – prior to the introduction of causal dynamical triangulations – to obtain dynamically any kind of "quantum geometry" resembling a four-dimensional universe. In our construction, the restrictions imposed by causality before rotating to a Euclidean signature clearly have played a pivotal role.

A number of issues are being addressed currently to obtain a more complete understanding of the physical and geometric properties of the Quantum Gravity theory generated by CDT, and to verify in more detail that its classical limit is well defined. Among them are the following.

(i) A better understanding of the renormalization of the bare coupling constants in the continuum limit, with the currently favoured scenario being that of asymptotic safety [37]. There are very encouraging agreements between the results of CDT and those of a Euclidean renormalization group approach [23; 32; 27; 28; 29; 30]. See also [18; 19; 20; 21] for older, related work. In particular, both approaches obtain a scale-dependent spectral dimension which varies between four on large and two on short scales.

(ii) An identification and measurement of the "transverse" gravitational degrees of freedom, to complement the information extracted so far for the scale factor only. For background-independent and coordinate-free formulations like CDT we still lack a simple and robust prescription for how to extract information about the transverse degrees of freedom, a quantity analogous to the Wilson loop in non-Abelian gauge theories.

(iii) The inclusion of matter fields in the computer simulations. Of particular interest would be a scalar field, playing the role of an inflaton field. While it is straightforward to include a scalar field in the formalism, it is less obvious which observables one should measure, being confined to the Euclidean sector of the theory. Based on a well-defined CDT model for the nonperturbative quantum excitations of geometry and matter, moving the discussion of quantum cosmology and various types of inflation from handwaving arguments into the realm of quantitative analysis would be highly desirable and quite possibly already within reach.

Acknowledgments

All authors acknowledge support by ENRAGE (European Network on Random Geometry), a Marie Curie Research Training Network in the European Community's Sixth Framework Programme, network contract MRTN-CT-2004-005616. R. L. acknowledges support by the Netherlands Organisation for Scientific Research (NWO) under their VICI program. J. J. was partly supported by the Polish Ministry of Science and Information Society Technologies grant 1P03B04029(2005-2008).

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19

Quantum Regge calculus

R. WILLIAMS

19.1 Introduction

When Regge formulated the first discrete version of general relativity in 1961, one of his motivations was to set up a numerical scheme for solving Einstein's equations for general systems without a large amount of symmetry. The hope was that the formulation would also provide ways of representing complicated topologies and of visualising the resulting geometries. Regge calculus, as it has come to be known, has not only been used in large scale numerical calculations in classical general relativity but has also provided a basis for attempts at formulating a theory of Quantum Gravity.

The central idea in Regge calculus [59] is to consider spaces with curvature concentrated on codimension-two subspaces, rather than with continuously distributed curvature. This is achieved by constructing spaces from flat blocks glued together on matching faces. The standard example in two dimensions is a geodesic dome, where a network of flat triangles approximates part of a sphere. The curvature resides at the vertices, and the deficit angle, given by 2π minus the sum of the vertex angles of the triangles at that point, gives a measure of it. In general dimension n, flat n-simplices meet on flat $(n-1)$-dimensional faces and the curvature is concentrated on the $(n-2)$-dimensional subsimplices or hinges. The deficit angle at a hinge is given by 2π minus the sum of the dihedral angles of the simplices meeting at that hinge. The use of simplices is important because specification of their edge lengths determines their shapes exactly, and in Regge calculus the edge lengths are the fundamental variables, by analogy with the metric tensor in the continuum theory.

The analogue of the Einstein action

$$S = \frac{1}{2}\int R\sqrt{g}d^d x \tag{19.1}$$

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

is given by

$$S_{\mathrm{R}} = \sum_{hinges\ h} V_h \delta_h, \tag{19.2}$$

where V_h is the volume of the hinge h and δ_h is the deficit angle there. The principle of stationary action leads to the Regge calculus equivalent to Einstein's equations; the action S_{R} is varied with respect to edge lengths, giving

$$\sum_h \frac{\partial V_h}{\partial l_i} \delta_h = 0. \tag{19.3}$$

This is particularly simple because, as Regge showed, the variation of the deficit angles vanishes when summed over each simplex. In principle, Eq. (19.3) gives a complete set of equations, one for each edge, for determining the edge lengths and thus the simplicial geometry. In practice, the discrete analogues of the contracted Bianchi identities (see below) mean that the equations are not in general linearly independent, so there is freedom to specify some of the variables, as for the lapse and shift in the (3+1) version of continuum general relativity.

The Bianchi identities in Regge calculus were given a very simple topological interpretation by Regge [59] (see also [63]). It is simplest to see this interpretation in three dimensions, but the generalisation to higher dimensions is straightforward. (In two dimensions, both in Regge calculus and in continuum general relativity, there are no Bianchi identities.) In three-dimensional Regge calculus, a vector parallel-transported round an edge with non-zero deficit angle rotates. If it is parallel-transported along a path which does not enclose an edge, it does not rotate. Consider a number of edges meeting at a vertex. A path can be constructed which encircles each edge once but is topologically trivial: it can be deformed without crossing any edges into a path which obviously encloses no edges. (Try it with a loop of string and your fingers!) Consequently a vector parallel-transported along this path will *not* rotate. This means that there is a relation among the deficit angles at the edges: the product of the rotation matrices on each edge is the identity matrix. This is precisely the discrete Bianchi identity. Regge showed that in the limit of small deficit angles, it gives just the usual Bianchi identity of general relativity. The four-dimensional Bianchi identity in Regge calculus states that the product of the rotation matrices on all the triangles meeting along an edge is the identity matrix. The discrete Bianchi identities have been discussed further [57; 12; 67] and detailed forms given [29].

Closely connected with the Bianchi identities is the existence of diffeomorphisms. There are differing points of view on how to define diffeomorphisms in Regge calculus. One is that diffeomorphisms are transformations of the edge lengths which leave the *geometry* invariant. In this case, diffeomorphisms exist only in flat space and correspond to changes in the edge lengths as the vertices move around in that flat space [52; 42]. If the space is almost flat, then one can define approximate diffeomorphisms. The other view is that diffeomorphisms leave the *action* invariant and this gives rather more flexibility. It is easy to imagine changes in the edge lengths which could decrease the deficit angles in one region and compensatingly increase them in another region, producing no overall change in the action. The invariance could even be local in the sense that changes in the lengths of the edges meeting at one vertex could be made so that the action is unchanged.

In three dimensions, it is possible to construct transformations which are exact invariances of the action. This relies on the uniqueness of the embedding of the *star* of a vertex of a three-skeleton in a flat four-dimensional space. By a detailed counting argument [64], one can show that the number of degrees of freedom (i.e. the edge lengths in the star) is exactly equal to the number of coordinates for its embedding in four dimensions. Thus exact diffeomorphisms are defined locally at each point and consist of the three-parameter family of motions of the point (in the flat four-dimensional space defined by its star) which leave the action invariant. The corresponding argument does not go through in four dimensions because there is no unique embedding of a four-dimensional star in a higher dimensional flat space. Attempts have been made to find alternative definitions in four dimensions but none is completely satisfactory. Of course it is always possible to find approximate diffeomorphisms, in particular ones where the invariance holds to third order in the deficit angles [33].

A "gauge-fixed" version of Regge calculus was constructed by Römer and Zähringer [65] in which the simplices were all taken to be equilateral. This work was a forerunner to the scheme known as dynamical triangulations, in which all edge lengths are identical and the sum over histories involves the sum over triangulations [1] (see chapter 18 by Ambjørn *et al.*).

Another basic type of transformation in general relativity is a conformal transformation. One way to define this in Regge calculus [64] is to define a scalar function ϕ at each vertex. The procedure which guarantees that at least locally, the conformal transformations form a group, is to require that l_{xy}, the length of the edge joining vertices x and y, transforms into

$$l'_{xy} = \phi_x \phi_y l_{xy}. \tag{19.4}$$

However, the edge lengths are constrained: they must be such that the hypervolumes of all four-simplices are real. One can show that the product of two such conformal transformations, such that each separately preserves these constraints, is a transformation which will in general violate the constraints. Thus globally the group property is violated. Furthemore, no subset of the transformations forms a group. We conclude that it is only infinitesimal conformal transformations which are well-defined so far in Regge calculus.

A final point in this introductory section is the existence of a continuum limit. Cheeger, Müller and Schrader [14; 15] showed rigorously that the Regge action converges to the continuum action, in the sense of measures, provided that certain conditions on the flatness of the simplices are satisfied. From the opposite perspective, Friedberg and Lee [22] derived the Regge action from the continuum in a certain limit. Rather than considering the action, Barrett [4; 5] explored the relationship between the Regge variational equations and Einstein's equations, and set up a criterion [6] for solutions of the linearised Regge equations to converge to analytic solutions of the linearised Einstein equations.

Regge calculus has been used in many aspects of classical general relativity, but that is not our concern here. We now consider various ways in which it has been used in Quantum Gravity. Most use the sum over histories approach to calculate the partition function or transition amplitude, although of course it is also possible to use the canonical approach, as will be seen in the penultimate section.

19.2 The earliest quantum Regge calculus: the Ponzano–Regge model

The first application of Regge calculus to Quantum Gravity came about in a rather unexpected way. In a paper on the asymptotic behaviour of $6j$-symbols, Ponzano and Regge [58] formulated the following model. Triangulate a 3-manifold, and label each edge with a representation of $SU(2)$, $j_i = \{0, 1/2, 1, ...\}$. Assign a $6j$-symbol, $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ (a generalised Clebsch–Gordan coefficient, which relates bases of states when three angular momenta are added) to each tetrahedron. Form the state sum

$$Z = \sum_{j_i} \prod_i (2j_i + 1)(-1)^\chi \prod_{tetrahedra} \{6j\}, \tag{19.5}$$

where the χ in the phase factor is a function of the j_i. This sum is infinite in many cases, but it has some very interesting properties. In particular, the semi-classical limit exhibits a connection with Quantum Gravity. The edge lengths can be thought of as $\hbar(j_i + 1/2)$, and the limit is obtained by keeping these quantities finite while $\hbar$ tends to zero and j_i tends to infinity. Ponzano and Regge showed that, for large j_i, the asymptotic behaviour of the $6j$-symbol is

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_i j_i\theta_i + \pi/4\right), \tag{19.6}$$

where V is the volume of the corresponding tetrahedron and θ_i the exterior dihedral angle at edge i. In the sum over edge lengths, the large values dominate, so the sum over the j_i in the state sum can be replaced by an integral over the edge lengths, and the asymptotic formula used. By writing the cosine as a sum of exponentials, and interchanging the orders of summation over tetrahedra and over edges within a given tetrahedron in the expression for the state sum, we can show that it contains a term of the form

$$\int \prod_i dj_i(2j_i+1)\left(\prod_{tets\ k}\frac{1}{\sqrt{V_k}}\right)\exp\left(i\sum_{edges\ l} j_l\left(2\pi-\sum_{tets\ k\ni l}(\pi-\theta_l^k)\right)\right)$$

$$=\int \prod_i dj_i(2j_i+1)\left(\prod_{tets\ k}\frac{1}{\sqrt{V_k}}\right)\exp(iS_{\mathrm{R}}), \tag{19.7}$$

which looks precisely like a Feynman sum over histories with the Regge action in three dimensions, and with the other terms contributing to the measure.

This result was rather puzzling and there seemed to be no obvious way to generalise it to four dimensions, so it was virtually ignored for twenty years. Then, in the early 1990s, Turaev and Viro [68] wrote down a very similar expression which was made finite by the use of representations of the quantum group $Sl_q(2)$, rather than $SU(2)$. It was then realised that a regularised version of the Ponzano–Regge state sum provided a model of Quantum Gravity in three dimensions, with zero cosmological constant. These three-dimensional models then led on to the development of spin foam models which currently play an important role in the search for a theory of Quantum Gravity.

19.3 Quantum Regge calculus in four dimensions: analytic calculations

The complicated dependence on the edge lengths of the deficit angles in the Regge calculus action means that calculations have mainly involved either highly symmetric configurations or perturbation theory about a classical background. The earliest work was a comparison between the Regge propagator in the weak field limit and the continuum propagator [64]. This will be described in some detail as the formalism has been the basis for many calculations of this type.

Consider a lattice of four-dimensional unit hypercubes, with vertices labelled by coordinates (n_1, n_2, n_3, n_4), where each n_i is an integer. Each hypercube is divided into 24 4-simplices, by drawing in appropriate "forward-going" diagonals. The whole lattice can be generated by the translation of a set of edges based on the origin. We interpret the coordinates of vertices neighbouring the origin as binary numbers (so for example, (0, 1, 0, 0) is vertex 4, (1, 1, 1, 1) is vertex 15). The edges emanating from the origin in the positive direction join it to vertices 1, 2, 4 and 8 (coordinate edges), 3, 5, 6, 9, 10 and 12 (face diagonals), 7, 11, 13 and 14 (body diagonals) and 15 (hyperbody diagonal). Small perturbations are then made about the flat space edge lengths, so that

$$l_j^{(i)} = l_0^{(i)}(1 + \epsilon_j^{(i)}), \tag{19.8}$$

where the superscript i denotes the base point, the subscript j denotes the direction (1, 2,...,15) and $l_0^{(i)}$ is the unperturbed edge length (1, $\sqrt{2}$, $\sqrt{3}$ or 2). Thus for example, $\epsilon_{14}^{(1)}$ would be the perturbation in the length of the edge from vertex 1 in the 14-direction, i.e. to vertex 15. The ϵs are assumed to be small compared with 1.

The Regge action is evaluated for the hypercube based at the origin and then obtained for all others by translation. The lowest non-vanishing term in the total action is quadratic in the variations (the zeroth and first order terms vanish because the action is zero for flat space and also flat space is a stationary point of the action since it is a solution of the Regge analogue of the Einstein equations). It can be written symbolically as

$$S_{\mathrm{R}} = \sum \epsilon^{\dagger} \mathbf{M} \epsilon, \tag{19.9}$$

where ϵ is an infinite-dimensional column vector with 15 components per point and $\mathbf{M}$ is an infinite-dimensional sparse matrix. Since all the entries corresponding to fluctuations of the hyperbody diagonal are zero, these form a one-parameter family of zero eigenmodes. It can also be shown that physical translations of the vertices which leave the space flat form a four-parameter family of zero eigenmodes. These are the exact diffeomorphisms in this case.

The matrix $\mathbf{M}$ is then block diagonalised by Fourier transformation or expansion in periodic modes. This is achieved by setting

$$\epsilon_j^{(a,b,c,d)} = (\omega_1)^d (\omega_2)^c (\omega_4)^b (\omega_8)^a \epsilon_j^{(0)}, \tag{19.10}$$

where $\omega_k = e^{2\pi i/n_k}$, $k = 1, 2, 4, 8$. Acting on periodic modes $\mathbf{M}$ becomes a matrix with 15×15 dimensional blocks M_ω along the diagonal. This submatrix has the schematic form

$$M_\omega = \begin{pmatrix} A_{10} & B & 0 \\ B^{\dagger} & 18 I_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{19.11}$$

where A_{10} is a 10×10 dimensional matrix and B is a 4×10 dimensional one; their entries are functions of the ωs. Then M_ω itself is block diagonalised by a non-unitary but uni-modular similarity transformation, and the diagonal blocks are $Z = A_{10} - \frac{1}{18} B B^{\dagger}$, $18 I_4$ and 0. The 4×4 unit matrix block means that the fluctuations ϵ_j for $j = 7, 11, 13, 14$ have been decoupled; they are constrained to vanish by the equations of motion. Remarkably, the number of degrees of freedom per vertex has been reduced from 15 to the 10 that we expect in the continuum.

By working with trace-reversed metric fluctuations, it can be shown that after further transformation, the 10×10 matrix Z corresponds exactly to what is called $L_{\rm sym}$ in the continuum [69], where

$$L_{\rm sym} = L + \frac{1}{2} C_\mu^2, \tag{19.12}$$

with

$$L = -\frac{1}{2} \partial_\lambda h_{\alpha\beta} V_{\alpha\beta\mu\nu} \partial_\lambda h_{\mu\nu}, \tag{19.13}$$

where

$$V_{\alpha\beta\mu\nu} = \frac{1}{2} \delta_{\alpha\mu} \delta_{\beta\nu} - \frac{1}{4} \delta_{\alpha\beta} \delta_{\mu\nu} \tag{19.14}$$

and

$$C_\mu = \partial_\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h, \quad h = h_{\nu\nu}. \tag{19.15}$$

The C_μ term is a gauge-breaking term (see [69]). The long wave-length (or weak field) limit has been taken by expanding the ωs in powers of the momentum k, and in that limit, we have exact agreement of the propagators. A similar calculation was performed in the Lorentzian case [70] and an expression for the graviton propagator was also derived by Feinberg *et al.* [21].

More recent work has investigated Regge calculus in d dimensions, with d arbitrary and large [36]. The idea is to apply the methods of mean field theory to Regge calculus. This exploits the fact that in large dimensions each point is typically surrounded by many neighbours, whose action can then be either treated exactly, or included as some sort of local average. It is quite challenging to calculate volumes and deficit angles in arbitrary dimensions, even if the simplices are equilateral, so in this case, the perturbations were performed about an equilateral tessellation. The volumes and dihedral angles for a d-dimensional equilateral simplex are given by

$$V_d = \frac{1}{d!} \sqrt{\frac{d+1}{2^d}}, \tag{19.16}$$

and

$$\cos\theta_d = \frac{1}{d}. \tag{19.17}$$

The squared edge lengths are then perturbed according to

$$l_{ij}^2 = l_{ij}^{(0)\,2} + \delta\, l_{ij}^2. \tag{19.18}$$

We will set for convenience from now on $\delta\, l_{ij}^2 \;=\; \epsilon_{ij}$ and take $l_{ij}^{(0)} \;=\; 1$. Using the general formula for the volume of a d-simplex in terms of the determinant of a $(d+2) \times (d+2)$ matrix,

$$V_d \;=\; \frac{(-1)^{\frac{d+1}{2}}}{d!\,2^{d/2}} \begin{vmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & l_{12}^2 & \dots \\ 1 & l_{21}^2 & 0 & \dots \\ 1 & l_{31}^2 & l_{32}^2 & \dots \\ \dots & \dots & \dots & \dots \\ 1 & l_{d+1,1}^2 & l_{d+1,2}^2 & \dots \end{vmatrix}^{1/2}, \tag{19.19}$$

and for the dihedral angle in a d-dimensional simplex of volume V_d, between faces of volume V_{d-1} and V'_{d-1}

$$\sin\theta_d \;=\; \frac{d}{d-1}\,\frac{V_d\,V_{d-2}}{V_{d-1}\,V'_{d-1}}, \tag{19.20}$$

the contribution to the perturbed action from a single d-simplex can be evaluated.

The deficit angle when several simplices meet at a hinge is given by

$$\delta_d \;=\; 2\pi - \sum_{\text{simplices}} \theta_d \;=\; 2\pi - \sum_{\text{simplices}} \left\{ \arcsin\frac{\sqrt{d^2-1}}{d} + \cdots \right\}. \tag{19.21}$$

Since for large d, the arcsine term is approximately $\pi/2$, and the preferred tessellation is one in which the deficit angle is zero to lowest order, we shall use a tessellation in which four d-simplices meet at each $(d-2)$-dimensional hinge. The simplest example of this is the cross polytope [18].

Here we consider the surface of the cross polytope in $d+1$ dimensions, which is therefore an object of dimension d. It corresponds to a triangulated manifold with no boundary, homeomorphic to the sphere. It can be visualised as a set of $2d+2$ vertices arranged on a circle, with each vertex joined to every other vertex, except the one opposite it. The deficit angle is given to leading order by

$$\delta_d \;=\; 0 + \frac{4}{d} - \left(\epsilon_{d,d+1} + 3\,\text{terms} + \epsilon_{1,d}\,\epsilon_{1,d+1} + \cdots\right) + O(1/d^2, \epsilon/d, \epsilon^2/d). \tag{19.22}$$

When evaluated on such a manifold the lattice action becomes

$$\frac{\sqrt{d}\,2^{d/2}}{d!}\,2\left(\lambda - k\,d^3\right)\left[1 - \frac{1}{8}\sum \epsilon_{ij}^2 + \frac{1}{d}\left(\frac{1}{4}\sum \epsilon_{ij} + \frac{1}{8}\sum \epsilon_{ij}\,\epsilon_{ik}\right) + O(1/d^2)\right]. \tag{19.23}$$

Dropping the $1/d$ correction one obtains to leading order

$$\frac{\sqrt{d}\, 2^{d/2}}{d!}\, 2\, \left(\lambda - k\, d^3\right) \left(1 - \frac{1}{8} \sum \epsilon_{ij}^2 + \cdots \right). \tag{19.24}$$

The partition function can be formally computed via

$$Z = \int \prod_{i=1}^{N} d\epsilon_i\, e^{-\epsilon\, M\, \epsilon} = \frac{\pi^{N/2}}{\sqrt{\det M}}, \tag{19.25}$$

with $N = 2d(d+1)$. Convergence of the Gaussian integral then requires $kd^3 > \lambda$, and one has

$$\log Z = \frac{\sqrt{d}\, 2^{\frac{d}{2}+1}}{d!} \left(k\, d^3 - \lambda\right) - d\,(d+1)\, \log\left[\frac{\sqrt{d}\, 2^{\frac{d}{2}+1}}{d!} \left(k\, d^3 - \lambda\right)/8\pi\right]$$

with the first term arising from the constant term in the action, and the second term from the ϵ-field Gaussian integral. Therefore the general structure, to leading order in the weak field expansion at large d, is $\log Z = c_1(k\, d^3 - \lambda) - d(d+1)\log(kd^3 - \lambda) + c_2$ with c_1 and c_2 d-dependent constants, and therefore $\partial^2 \log Z/\partial k^2 \sim 1/(kd^3 - \lambda)^2$ with divergent curvature fluctuations in the vicinity of the critical point at $kd^3 = \lambda$.

If we apply the ideas of mean field theory, we need to keep the terms of order $1/d$ in Eq. (19.23). In the $\epsilon_{ij}\epsilon_{ik}$ term, we assume that the fluctuations are small and replace ϵ_{ik} by its average $\bar{\epsilon}$. Each ϵ_{ij} has $4d - 2$ neighbours (edges with one vertex in common with it); this has to be divided by 2 to avoid double counting in the sum, so the contribution is $(2d - 1)\,\bar{\epsilon}$. Then to lowest order in $1/d$, the action is proportional to

$$\left(\lambda - k\, d^3\right) \left[1 - \frac{1}{8} \sum \epsilon_{ij}^2 + \frac{1}{4}\, \bar{\epsilon} \sum \epsilon_{ij}\right]. \tag{19.26}$$

This gives rise to the same partition function as obtained earlier, and using it to calculate the average value of ϵ_{ij} gives $\bar{\epsilon}$, as required for consistency.

19.4 Regge calculus in quantum cosmology

In quantum cosmology, interest is focused on calculations of the wave function of the universe. According to the Hartle–Hawking prescription [41], the wave function for a given 3-geometry is obtained from a path integral over all 4-geometries which have the given 3-geometry as a boundary. To calculate such an object in all its glorious generality is impossible, but one can hope to capture the essential features by integrating over those 4-geometries which might, for whatever reason, dominate the sum over histories. This has led to the concept of *mini-superspace* models, involving the use of a single 4-geometry (or perhaps several). In the continuum theory, the calculation then becomes feasible if the chosen geometry depends only on a small number of parameters, but anything more complicated soon becomes extremely difficult. For this reason, Hartle [38] introduced the idea of summing over *simplicial* 4-geometries as an approximation tool in quantum cosmology. Although this is an obvious way of reducing the number of integration variables, there are still technical difficulties: the unboundedness of the Einstein action (which persists in the discrete Regge form) leads to convergence problems for the functional integral, and it is necessary to rotate the integration contour in the complex plane to give a convergent result [40; 54].

In principle, the sum over 4-geometries should include not only a sum over metrics but also a sum over manifolds with different topologies. One then runs into the problem of classifying manifolds in four and higher dimensions, which led Hartle [39] to suggest a sum over more general objects than manifolds, *unruly topologies*. Schleich and Witt [66] have explored the possibility of using conifolds, which differ from manifolds at only a finite number of points, and this has been investigated in some simple cases [10; 16; 17]. However, a sum over topologies is still very far from implementation.

A related problem is the calculation of the ground state wave function for linearised gravity. Hartle [37], using the path integral approach, and, before him, Kuchar [51], using the canonical approach, showed that for an asymptotically flat space with a flat boundary, the required wave function is given by

$$\Psi_0[h_{ij}^{TT}, t] \;=\; \mathcal{N}\ \exp\left\{-\frac{1}{4l_{\mathrm{P}}^2}\int d^3\mathbf{k}\ \omega_{\mathbf{k}}\ h_{ij}^{TT}(\mathbf{k})\ \bar{h}_{ij}^{TT}(\mathbf{k})\right\} \tag{19.27}$$

where $h_{ij}^{TT}(\mathbf{k})$ is a Fourier component of the transverse traceless part of the deviation of the three-metric from the flat three-metric in rectangular coordinates,

$$h_{ij}(\mathbf{x}, t) \;=\; {}^3g_{ij}(\mathbf{x}, t) - \delta_{ij}\ , \tag{19.28}$$

$\omega_{\mathbf{k}} = |\mathbf{k}|$, $\mathcal{N}$ is a normalization factor, and $l_{\mathrm{P}} = (16\pi G)^{1/2}$ is the Planck length in a system of units where $\hbar = c = 1$.

This calculation was repeated in Regge calculus [35] by performing the discrete functional integral over the interior metric perturbations for a lattice with boundary. As shown by Hartle and Sorkin [43], the deficit angle at a hinge on a boundary is given by π minus the sum of the dihedral angles, rather than 2π minus that sum. The formalism for the lattice described in Section 19.3 was used and the second variation of the action evaluated for a half-space bounded by $x_8 = 0$ say. Asymptotic flatness was assumed in the x_8-direction, and periodic boundary conditions imposed in the other directions.

At a typical interior vertex, the classical equations of motion were obtained for the variations $\epsilon_j^{(i)}$. New variables $f_j^{(i)}$ were introduced by

$$\epsilon_j^{(i)} = \hat{\epsilon}_j^{(i)} + f_j^{(i)}, \tag{19.29}$$

where the $\hat{\epsilon}_j^{(i)}$ satisfied the classical equations of motion. Use of these equations led to the elimination of the $\hat{\epsilon}_j^{(i)}$, leaving Gaussian integrals over the $f_j^{(i)}$, which contributed only to the normalisation.

The only remaining contributions to the action were those assigned to vertices on the boundary. Fourier transforms were taken in the directions with periodic boundary conditions. The fact that the scalar curvature is constrained to vanish on the boundary was used to eliminate many terms, and a careful identification of the boundary $\epsilon_j^{(i)}$s with the appropriate continuum h_{ij}s [64] led eventually to the Hartle–Kuchar expression.

19.5 Matter fields in Regge calculus and the measure

The work described so far has been for spaces devoid of matter, but clearly a theory of Quantum Gravity must include the coupling of gravity to all types of matter. On a lattice, it is conventional for a scalar field to be defined at the sites, and for a gauge field to be associated with edges, and this has been the standard method in Regge calculus (see for example, [31]). On the other hand, fermions need to be defined within the simplices, or rather on the sites of the dual lattice, with their coupling defined by way of the Lorentz transformation relating the frames in neighbouring simplices [61]. Following a suggestion of Fröhlich [23], Drummond [20] formulated a way of defining spinors on a Regge manifold, which could be modified to include the effect of torsion. It is not clear whether the method would overcome the problem of fermion doubling.

Since most of the quantum applications of Regge calculus involve the path integral approach, the definition of the measure is obviously very important. In his paper examining very basic questions in quantum Regge calculus, including matter fields as mentioned above, Fröhlich [23] discussed unitarity and reflection positivity, and also defined a measure on a sequence of incidence matrices and the volumes of their simplices. The dependence of the proposed measure on the cut-off would involve renormalization group techniques. The measure was also discussed by Cheeger, Müller and Schrader [14], Hartle [38] and Bander [3].

In spite of these early suggestions, there is still controversy over the form of the measure. It depends not only on the attitude to simplicial diffeomorphisms but also on the stage at which translation from the continuum to the discrete takes place. Hamber and Williams [33] argue that the local gauge invariance properties of the lattice action show that no Fadeev–Popov determinant is required in the gravitational measure, unless lattice perturbation theory is performed with a gauge-fixed action, such as the one arising in the lattice analogue of the conformal or harmonic gauges. In numerical simulations (see Section 19.6), a simple measure is frequently used:

$$\int \prod_{s} [V(S)]^{\sigma} \prod_{ij} dl_{ij}^2 \; \Theta(l_{ij}^2), \tag{19.30}$$

where the Θ-functions impose the triangle inequalities. The other terms are the lattice analogue of the DeWitt measure:

$$\int \prod_{x} (\sqrt{g(x)})^{\sigma} \prod_{\mu \geq \nu} dg_{\mu\nu}(x). \tag{19.31}$$

The continuum measure is derived from the DeWitt supermetric on the space of metrics, and the lattice version can be derived similarly from the simplicial supermetric [42], which is equivalent to the Lund–Regge metric [55]. In practice, rather than using powers of volumes, numerical simulations are often performed with the simple dl/l scale invariant measure or ldl which seems nearer to $dg_{\mu\nu}$. Beirl *et al.* [7] have shown that the choice between these two measure makes very little difference to their results.

The opposite view is that at least for weak field perturbation theory about flat space, it is necessary to divide through by the volume of the diffeomorphism group using the Fadeev–Popov determinant. Menotti and Peirano [56] insist on a non-local measure and have derived an expression for the functional measure in two-dimensional Regge gravity, starting from the DeWitt supermetric and giving exact expressions for the Fadeev–Popov determinant for both S^2 and $S^1 \times S^1$ topologies. However it is not clear how to extend their calculation to higher dimensions. In some circumstances it would not be necessary because, as pointed out by Hartle [38] the volume of the diffeomorphism group cancels in the evaluation of expectation values of operators.

19.6 Numerical simulations of discrete gravity using Regge calculus

The difficulties of analytic calculations in quantum Regge calculus, coupled with the need for a non-perturbative approach and also the availability of sophisticated techniques developed in lattice gauge theories, have combined to stimulate numerical work in Quantum Gravity, based on Regge calculus. One approach is to start with a Regge lattice for, say, flat space, and allow it to evolve using a Monte Carlo algorithm. Random fluctuations are made in the edge lengths and the new configuration is rejected if it increases the action, and accepted with a certain probability if it decreases the action. The system evolves to some equilibrium configuration, about which it makes quantum fluctuations, and expectation values of various operators can be calculated. It is also possible to study the phase diagram and search for phase transitions, the nature of which will determine the vital question of whether or not the theory has a continuum limit. Work in this area has been done mainly by three groups, Berg in Tallahassee, Hamber in Irvine and the Vienna group (of which Berg is sometimes part). For concreteness, we describe the methods of Hamber and collaborators (see [26; 30]). The basic lattice used is that of 4-dimensional hypercubes divided into 4-simplices, as described in Section 19.2. The form of the action is

$$S_{\mathrm{R}} = \sum_h \left(\lambda V_h - k\delta_h A_h + \frac{a\delta_h^2 A_h^2}{V_h} \right), \tag{19.32}$$

the lattice representation of the continuum expression

$$S = \int d^4x \sqrt{g} \left(\lambda - \frac{k}{2} R + \frac{a}{4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right), \tag{19.33}$$

where $k = 1/8\pi G$. The higher derivative term, quadratic in the curvature, was introduced by Hamber and Williams [30] to ensure that the action remained positive and so to avoid problems with the convergence of the functional integral. In practice, it was found that the coefficient of the higher derivative term could be taken to be arbitrarily small without any noticeable problems.

These types of simulations have been performed for the last twenty years and we shall now summarise some of the main results. Gross and Hamber [25] performed a two-dimensional simulation, keeping the total area constant, in order to compare their results with those of Knizhnik, Polyakov and Zamolodchikov using conformal field theory. There was good agreement for the torus and also subsequently for the sphere when an appropriate triangulation was used. The Haussdorf dimension for the model was found to be infinite. In simulations of the Ising model for both regular and random lattices, the Regge calculus results were found to be consistent with the flat space values, differing from what was obtained using dynamical triangulations. In four dimensions, with a lattice with topology T^4, the main result was that at strong coupling, the system developed a negative average-curvature. Finite size scaling and the renormalization group can be used to obtain phases and relations involving critical exponents. For example, on a 16^4 lattice, with $\lambda = 1$ and $a = 0$, Hamber [27; 28] obtained for $G < G_{\mathrm{c}}$ a weak coupling phase, with degenerate geometry behaving like a spiky branched polymer. For $G > G_{\mathrm{c}}$, there was a strong coupling phase, with smooth geometry at large scales and small negative average curvature. The phase transition occurred at $k_{\mathrm{c}} \approx 0.0636$, and the correlation length exponent ν, defined by

$$\xi \sim (k_{\mathrm{c}} - k)^{-\nu}, \tag{19.34}$$

where ξ is the correlation length, was approximately $1/3$. When scalar matter was included in the simulations, the effect on the critical exponents was small, but the results suggested that gravitational interactions could increase with distance [31]. In an investigation of the Newtonian potential in quantum Regge gravity, Hamber and Williams [32] computed correlations on the lattice between Wilson lines associated with two massive particles. In the smooth anti-de Sitter-like phase, the only region where a sensible lattice continuum limit could be constructed in the model, the shape and mass dependence of the attractive potential were studied close to the critical point in G. It was found that non-linear gravitational interactions gave rise to a Yukawa-like potential, with mass parameter decreasing towards the critical point where the average curvature vanished.

The other pioneer of these methods, Berg, did early simulations keeping the total volume constant [8]. His results indicated that an exponentially decreasing entropy factor in the measure might cure the problem of the unboundedness of the gravitaional action [9].

The group in Vienna has, over the years, explored many aspects of Regge lattice gravity. Recently, a Z_2 model, in which edge lengths could take just two discrete values, was compared with the standard Regge model with a continuous range of values for the edge lengths [11]. The results of the two models were similar. An extension of this [62] also included the model of Caselle *et al.* [13], where gravity is treated as a gauge theory, and the action involves the sine of the deficit angle. Evidence was found in all models of a continuous phase transition, and the results were compatible with the existence of massless spin-2 excitations. These types of comparison should be pursued as a means of investigating the very important question of the relationship between the universality classes of Regge calculus and dynamical triangulations.

More details and discussion of numerical work on quantum Regge calculus are given in the review by Loll [53].

19.7 Canonical quantum Regge calculus

By way of contrast, we mention finally some approaches to canonical Quantum Gravity using Regge calculus.

Immirzi set out to relate the canonical approach of loop quantum gravity to Regge calculus. He defined the Ashtekar variables for a Regge lattice, and introduced the Liouville form and Poisson brackets [44]. He found that it was impossible to quantise the model directly using complex variables, and leave the second class constraints to fix the metric of the quantum Hilbert space, because one cannot find a metric which makes the area variables hermitian [45].

In a long series of papers, Khatsymovsky has confronted many of the problems arising in setting up a canonical quantisation of Regge calculus [48]. Topics he has dealt with include tetrad variables [46], the constraint structure [47], matter fields [49] and the continuum limit [50].

A recent development is the application by Gambini and Pullin of their consistent discretisation [19] to Regge calculus. Their method is algebraic rather than geometric, and it seems to solve the problems of preservation of the constraints in numerical relativity and closure of the constraints in the quantum theory. Its application to Regge calculus [24] is valid in both the Euclidean and Lorentzian domains, and there is a natural elimination of spikes, which seemed to cause trouble in Regge calculus in the past [2] (but see also [34]). The method involves first formulating Regge calculus as a classical unconstrained canonical system, and then quantising it by implementing canonical transformations which give the discrete time evolution as a unitary quantum operator. The framework can incorporate topology change, in particular the evolution from a "no boundary" initial state.

19.8 Conclusions

Regge calculus was the first discretisation scheme in general relativity and the first form of simplicial Quantum Gravity (for more references, see the bibliography by Williams and Tuckey [71]). From it have developed a number of important and highly topical approaches to discrete Quantum Gravity, including Lorentzian dynamical triangulations and spin foam models [60]. It remains to be seen which approach will give rise to a fully satisfactory diffeomorphism-invariant and background-independent theory of Quantum Gravity. There are strong reasons for taking seriously Regge calculus and the other theories just mentioned, since they go to the basic level and seek to study the dynamical nature of quantum space-time.

Acknowledgements

This work has been supported in part by the UK Particle Physics and Astronomy Research Council. The author thanks Herbert Hamber for helpful comments on this chapter. She is particularly grateful to Girton College for its continuing support.

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20

Consistent discretizations as a road to Quantum Gravity

R. GAMBINI AND J. PULLIN

20.1 Consistent discretizations: the basic idea

There has long been the hope that lattice methods could be used as a non-perturbative approach to Quantum Gravity. This is in part based on the fact that lattice methods have been quite successful in the treatment of quantum chromodynamics. However, one needs to recall that one of the appeals of lattice methods in QCD is that they are gauge invariant regularization methods. In the gravitational context this is not the case. As soon as one discretizes space-time one breaks the invariance under diffeomorphisms, the symmetry of most gravitational theories of interest. As such, lattice methods in the gravitational context face unique challenges. For instance, in the path integral context, since the lattices break some of the symmetries of the theory, this may complicate the use of the Fadeev–Popov technique. In the canonical approach if one discretizes the constraints and equations of motion, the resulting discrete equations are inconsistent: they cannot be solved simultaneously. A related problem is that the discretized constraints fail to close a constraint algebra.

To address these problems we have proposed [16; 4] a different methodology for discretizing gravitational theories (or to use a different terminology "to put gravity on the lattice"). The methodology is related to a discretization technique that has existed for a while in the context of unconstrained theories called "variational integrators" [22]. In a nutshell, the technique consists in discretizing the action of the theory and working from it the discrete equations of motion. Automatically, the latter are generically guaranteed to be consistent. The resulting discrete theories have unique features that distinguish them from the continuum theories, although a satisfactory canonical formulation can be found for them [3]. The discrete theories do not have constraints associated with the space-time diffeomorphisms and as a consequence the quantities that in the continuum are the associated Lagrange multipliers (the lapse and the shift) become regular

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

variables of the discrete theories whose values are determined by the equations of motion. We call this approach in the context of constrained theories “consistent discretizations”.

The consistently discretized theories are both puzzling and attractive. On the one hand, it is puzzling that the Lagrange multipliers get fixed by the theory. Don’t the Lagrange multipliers represent the gauge freedom of general relativity? The answer is what is expected: the discretization breaks the freedom and solutions to the discrete theory that are different correspond, in the continuum limit, to the same solution of the continuum theory. Hence the discrete theory has more degrees of freedom. On the other hand, the lack of constraints make the consistently discretized theories extremely promising at the time of quantization. Most of the hard conceptual questions of Quantum Gravity are related to the presence of constraints in the theory. In comparison, the consistently discretized theories are free of these conceptual problems and can be straightforwardly quantized (to make matters even simpler, as all discrete theories, they have a finite number of degrees of freedom). In addition, they provide a framework to connect the path integral and canonical approaches to Quantum Gravity since the central element is a unitary evolution operator. In particular they may help reconcile the spin foam and canonical loop representation approaches. They also provide a natural canonical formulation for Regge calculus [20].

In this chapter we would like to briefly review the status of the consistent discretization approach, both in its application as a classical approximation to gravitational theories and as a tool for their quantization. Other brief reviews with different emphasis can be seen in [18; 19]. The organization of this chapter is as follows. In section 20.2 we consider the application of the technique to a simple, yet conceptually challenging mechanical model and discuss how features that one observes in the model are actually present in more realistic situations involving general relativity. In section 20.3 we outline various applications of the framework. In section 20.4 we discuss in detail the quantization of the discrete theories and in section 20.5 we outline how one can define the quantum continuum limit. We end with a summary and outlook.

20.2 Consistent discretizations

To introduce and illustrate the method in a simple – yet challenging – model we consider the model analyzed in detail by Rovelli [27] in the context of the problem of time in canonical Quantum Gravity: two harmonic oscillators with constant energy sum. We have already discussed this model in some detail in [19] but we would like to revisit it here to frame the discussion with a different emphasis.

The model has canonical coordinates q^1, q^2, p^1, p^2 with the standard Poisson brackets and a constraint given by,

$$C = \frac{1}{2}\left((p^1)^2 + (p^2)^2 + (q^1)^2 + (q^2)^2\right) - M = 0, \tag{20.1}$$

with M a constant. The model is challenging since no standard unconstrained Hamiltonian formulation can correspond to this dynamical system since the presymplectic space is compact and therefore cannot contain any $S \times R$ structure. Nevertheless, we will see that the consistent discretization approach does yield sensible results. This helps dispel certain myths about the consistent discretization scheme. Since it determines Lagrange multipliers, a lot of people tend to associate the scheme with some sort of "gauge fixing". For this model, however, a gauge fixing solution would be unsatisfactory, since it would only cover a portion of phase space. We will see that this is not the case in the consistent discretization scheme. We will also see that the evolution scheme is useful numerically in practice.

We start by writing a discrete Lagrangian for the model,

$$\begin{aligned} L(n, n+1) \quad = \quad & p_n^1\left(q_{n+1}^1 - q_n^1\right) + p_n^2\left(q_{n+1}^2 - q_n^2\right) \\ & -\frac{N_n}{2}\left((p_n^1)^2 + (p_n^2)^2 + (q_n^1)^2 + (q_n^2)^2 - 2M\right), \end{aligned} \tag{20.2}$$

and working out the canonical momenta for all the variables, i.e. P_q^1, P_q^2, P_p^1, P_p^2. The momenta of a variable at level n are obtained by differentiating $L(n, n+1)$ with respect to the variable at level $n+1$. One then eliminates the $p^{1,2}$ and the $P_p^{1,2}$ and is left with evolution equations for the canonical pairs,

$$q_{n+1}^1 = q_n^1 + N_n\left(P_{q,n}^1 - 2q_n^1\right) \tag{20.3}$$

$$q_{n+1}^2 = q_n^2 + N_n\left(P_{q,n}^2 - 2q_n^2\right) \tag{20.4}$$

$$P_{q,n+1}^1 = P_{q,n}^1 - N_n q_n^1 \tag{20.5}$$

$$P_{q,n+1}^2 = P_{q,n}^2 - N_n q_n^2. \tag{20.6}$$

The Lagrange multiplier gets determined by the solution(s) of a quadratic equation that is obtained by working out the momenta of the Lagrange multipliers,

$$\begin{aligned} & \left((q_n^1)^2 + (q_n^2)^2\right)(N_n)^2 - 2\left(P_{q,n}^1 q_n^1 + P_{q,n}^2 q_n^2\right) N_n \\ & + \left(P_{q,n}^1\right)^2 + \left(P_{q,n}^2\right)^2 + \left(q_n^1\right)^2 + \left(q_n^2\right)^2 - 2M = 0. \end{aligned} \tag{20.7}$$

The resulting evolution scheme when one eliminates the Lagrange multipliers using equation (20.7) constitutes a canonical transformation between instants n and $n+1$. This result may appear puzzling at first, a general discussion of how this can be framed in a Dirac-like approach for discrete theories can be seen in [3].

We would like to use this evolution scheme to follow numerically the trajectory of the system. For this, we need to give initial data. Notice that if one gives initial data that satisfy the constraint identically at level n, the quadratic equation for the lapse has a vanishing independent term and therefore the solution is that the lapse N vanishes (the non-vanishing root will be large and would imply a large time evolution step that puts us away from the continuum generically). To construct initial data one therefore considers a set for which the constraint vanishes and introduces a small perturbation on one (or more) of the variables. Then one will have evolution. Notice that one can make the perturbation as small as desired. The smaller the perturbation, the smaller the lapse and the closer the solution will be to the continuum.

For concreteness, we choose the following initial values for the variables, $M = 2$, $q_0^1 = 0$, $q_0^2 = (\sqrt{3} - \Delta)\sin(\frac{\pi}{4})$, $P_{q,0}^1 = 1$, $P_{q,0}^1 = (\sqrt{3} - \Delta)\cos(\frac{\pi}{4})$.

We choose the parameter Δ to be the perturbation, i.e. $\Delta = 0$ corresponds to an exact solution of the constraint, for which the observable $A = 1/2$ (see below for its definition). The evolution scheme can easily be implemented using a computer algebra program like Maple or Mathematica.

Before we show results of the evolution, we need to discuss in some detail how the method determines the lapse. As we mentioned it is obtained by solving the quadratic equation (20.7). This implies that for this model there will be two possible solutions and in some situations they could be negative or complex. One can choose either of the two solutions at each point during the evolution. This ambiguity can be seen as a remnant of the re-parameterization invariance of the continuum. It is natural numerically to choose one "branch" of the solution and keep with it. However, if one encounters that the roots become complex, we have observed that it is possible to backtrack to the previous point in the iteration, choose the alternate root to the one that had been used up to that point and continue with the evolution. A similar procedure could be followed when the lapse becomes negative. It should be noted that negative lapses are not a problem per se, it is just that the evolution will be retraced backwards. We have not attempted to correct such retracings, i.e. in the evolutions shown we have only "switched branches" whenever the lapse becomes complex. This occurs when the discriminant in the quadratic equation (20.7) changes sign.

We would like to argue that in some sense the discrete model "approximates" the continuum model well. This, however, turns out to be a challenging proposition in re-parameterization invariant theories. The first thing to try, to study the evolution of the quantities as a function of n is of course meaningless as a grounds to compare with the continuum. In the discrete theory we do not control the lapse, therefore plots of quantities as a function of n are meaningless. To try to get more meaningful information one would like to concentrate on "observables". In the continuum theory, these are quantities that have vanishing Poisson brackets with the constraints (also sometimes known as "perennials"). Knowing these quantities as functions of phase space allows one to know any type of dynamical physical behavior of the system. One can use them, for instance, to construct "evolving constants" [27]. The existence of perennials in the continuum theory is associated with symmetries of the theory. If such symmetries are not broken by the discretization process, then in the discrete theory one will have exact conserved quantities that correspond to the perennials of the continuum theory. The conserved quantities will be given by discretizations of the perennials of the continuum. It should be noted that in the continuum theory perennials as functions of phase space are defined up to the addition of multiples of the constraints. There are therefore infinitely many versions of a given perennial. When discretized these versions are inequivalent (since in the discrete theory the constraints of the continuum theory do not hold exactly) and only one of these versions will correspond to an exact conserved quantity of the discrete theory.

In this model there are two independent perennials in the continuum. One of them becomes straightforwardly upon discretization an exact conserved quantity of the discrete theory,

$$O_1 = p^1 q^2 - p^2 q^1. \tag{20.8}$$

Another perennial is given by

$$O_2 = (p^1)^2 - (p^2)^2 + (q^1)^2 - (q^2)^2. \tag{20.9}$$

This quantity is not an exact conserved quantity of the discrete model, it is conserved approximately, as we can see in figure 20.1. We at present do not know how to find an exact conserved quantity in the discrete theory that corresponds to a discretization of this perennial (plus terms proportional to the constraint). In the end, this will be the generic situation, since in more complicated models one will not know exact expressions either for the perennials of the continuum theory or the constants of motion of the discrete theory. Notice also that in the continuum, in order to recover physical information about the system, one generically needs the two perennials plus combinations involving the constraints. In the discrete theory these combinations will not be exactly preserved. Therefore even if we found exact conserved quantities for both perennials in the discrete theory, the extracted physics would still only be approximate, and the measure of the error will be given by how well the constraint of the continuum theory is satisfied in the discrete theory. It is in this sense that one can best say that the discrete theory "approximates the continuum theory well".

Figure 20.1 depicts the relative errors throughout evolution in the value of the second perennial we discussed. Interestingly, although in intermediate steps of the evolution the error grows, it decreases later.

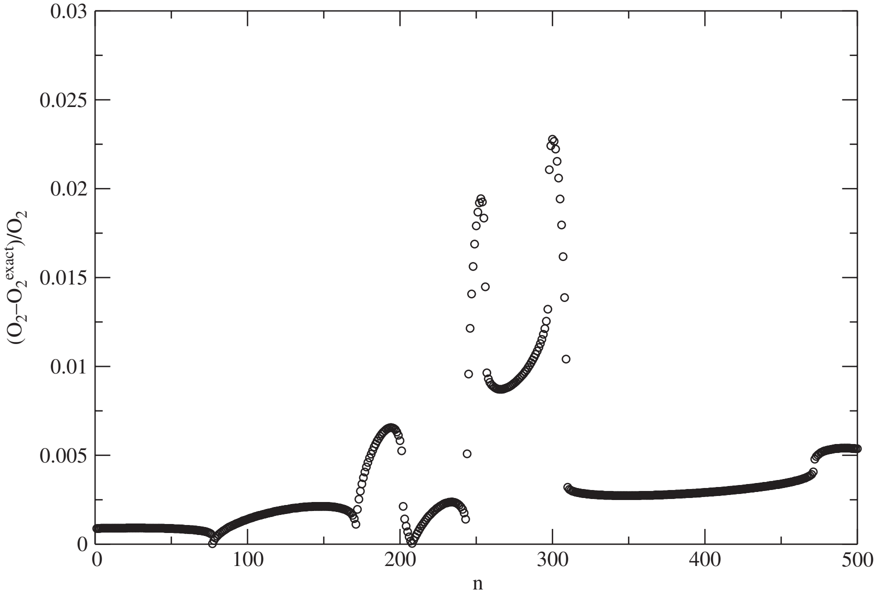

Fig. 20.1. The model has two "perennials". One of them is an exact conserved quantity of the discrete theory, so we do not present a plot for it. The second perennial (O_2) is approximately conserved. The figure shows the relative error in its computation in the discrete theory. It is worthwhile noticing that, unlike what is usual in free evolution schemes, errors do not accumulate, they may grow for a while but later they might diminish.

As we argued above, in the discrete theory quantities approximate those of the continuum with an error that is proportional to the value of the constraint. Therefore the value of the constraint is the real indicator of how accurately one is mirroring the continuum theory. It is a nice feature to have such an error indicator that is independent of the knowledge of the exact solution. Using this indicator one can, for instance, carry out convergence studies and show that the method does indeed converge for this model in a detailed way [19].

Figure 20.2 shows the trajectory in configuration space. As we see, the complete trajectory is covered by the discretized approach. This is important since many people tend to perceive the consistent discretization approach as "some sort of gauge fixing". This belief stems from the fact that when one gauge fixes a theory, the multipliers get determined. In spite of this superficial analogy, there are many things that are different from a gauge fixing. For instance, as we discussed before, the number of degrees of freedom changes (for more details see [17]). In addition to this, this example demonstrates another difference. If one indeed had gauge fixed this model, one would fail to cover the entire available configuration space, given its compact nature.

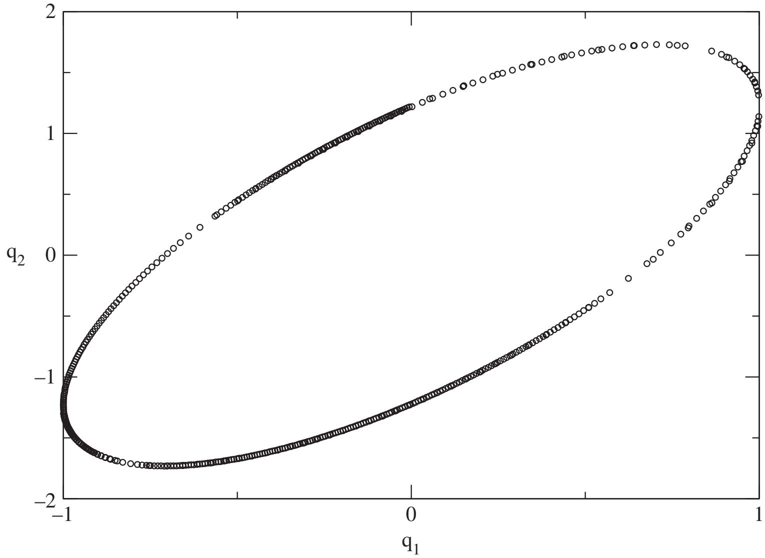

Fig. 20.2. The orbit in configuration space. As is readily seen, the consistent discrete approach covers the entire available configuration space. This clearly exhibits that the approach is not a "gauge fixing". Gauge fixed approaches cannot cover the entire configuration space due to its compact nature. The dynamical changes in the value of the lapse can be seen implicitly through the density of points in the various regions of the trajectory. Also apparent is that the trajectory is traced on more than one occasion in various regions. Deviation from the continuum trajectory is not noticeable in the scales of the plot.

To conclude this section, let us point out some hints that this model provides. To begin with, we see that the consistent discretization scheme successfully follows the classical continuum trajectory. One has control of how accurate things are by choosing the initial data. One can show that the approach converges using estimators of error that are independent of knowledge of exact solutions or other features generically not available. The solution of the equations for the Lagrange multipliers may develop branches, and one can use this to one's advantage in tackling problems where the topology of phase space is not simple.

What is the state of the art in terms of applying this approach as a classical numerical relativity tool? We have applied the method in homogeneous cosmologies and also in Gowdy cosmologies [8] where one has spatial dependence of the variables. All of the features we have seen in the model described in this section are present in the more complicated models, the only difference is computational complexity. How well does it compete with more traditional numerical relativity approaches? At the moment the method is too costly to compete well, since the evolution equations are implicit. But as traditional "free evolution" methods in numerical relativity keep on encountering problems of instabilities and constraint violations, and as computational power increases, the costliness of the consistent discretization approach may become less of a problem. A challenge to be overcome is that in situations of interest the problems have boundaries, and the approach has not yet been worked out in the presence of boundaries, although we are actively considering this point.

20.3 Applications

20.3.1 Classical relativity

As we argued before, our approach can be used to construct discrete theories that approximate general relativity. It is therefore suitable for doing numerical relativity. The main problem is that the resulting numerical schemes are implicit, and therefore very costly in situations of physical interest where there are no symmetries. Most of present numerical relativity is being pursued with explicit algorithms for that reason. In spite of this, our experience with the model analyzed by Rovelli and the Gowdy cosmologies indicates that our discretizations may have attractive features that are not present in more traditional discretization schemes. In particular the fact that errors do not seem to accumulate but rather grow and decrease in cycles as one evolves, could offer unique promises for long term evolutions like the ones desired in binary systems that emit gravitational waves. In addition to this, it has been shown [5] that our approach applied to linearized gravity yields a discretization that is "mimetic", that is, the constraints are automatically preserved without determining the Lagrange multipliers. This may suggest that at least at linearized level, our discretizations may perform better than others. In spite of these hints of a promise, there is a lot of terrain yet to cover before one could consider seriously using one of these schemes in problems of current interest. In particular, in numerical relativity the importance of having symmetric hyperbolic formulations has been increasingly recognized (see [26] for a review) and in particular of incorporating constraint preserving boundary conditions. Most symmetric hyperbolic formulations are constructed at the level of equations of motion and do not derive from an action principle. Therefore our discretization technique is not directly applicable. More work is clearly needed in this area.

Another area of recent progress [20] has been the application of these ideas to Regge calculus. In Regge calculus it had been observed that the canonical formulation was problematic. In particular it seemed to require that the Lagrange multipliers be fixed [7]. This is exactly the statement that we use as a starting point for our discrete construction. We have recently shown how one can construct an unconstrained version of canonical Regge calculus in which some of the lengths of the links are determined precisely mirroring what happens with the Lagrange multipliers in other theories. Although this is only a beginning, it suggests a novel technique to have a canonical formulation of Regge calculus that may have attractive implications quantum mechanically (for instance it contains a new prescription to define the path integral).

20.3.2 The problem of time

Since the discrete theory that one constructs through our procedure is constraint-free, it immediately circumvents most of the hard conceptual problems of canonical Quantum Gravity including the "problem of time". The issue is a bit more subtle than it initially appears. One indeed has a theory without constraints and a "genuine evolution", except that the latter is cast in terms of the discrete parameter n. This parameter cannot be accessed physically, it is not one of the variables one physically observes for the systems under study. This forces us to consider a "relational" formulation, in the same spirit as Page and Wootters [25] considered. The idea is to pick one of the physical variables and use it as a clock. One then asks relational questions, for instance "what is the conditional probability than one of the other variables takes a given value when the clock variable indicates a certain time". These questions can of course also be asked in continuum general relativity, but the detailed construction of the conditional probabilities is problematic, due to the difficulties of having a probabilistic interpretation of quantum states in canonical Quantum Gravity (see the discussion in [21]). In our approach, on the other hand, the conditional probabilities are well defined, since there are no constraints to generate problems with the probabilistic interpretation of states. For more details see [9].

20.3.3 Cosmological applications

We have applied the technique to cosmological models. The use of these discrete theories in cosmology has an attractive consequence. Since the lapse, and therefore the "lattice spacing in time", is determined by the equations of motion, generically one will avoid the singularity classically. Or to put it in a different way, one would have to "fine tune" the initial data to reach the singularity (unless one uses variables in which the singularity is on a boundary of phase space). Quantum mechanically, this implies that the singularity will be probabilistically suppressed. As the discrete theory tunnels through the singularity, there is a precise sense in which one can claim that the lattice spacing changes qualitatively. This could be used to argue that physical constants change when tunneling through a singularity since in lattice theories the "dressed" value of the coupling constants is related to the lattice spacing. Therefore this provides a concrete mechanism for Smolin's "The life of the cosmos" proposal [29; 30]. For more details see [15].

20.3.4 Fundamental decoherence, black hole information puzzle, limitations to quantum computing

Once one has solved the problem of time in the relational fashion discussed above, one notices that the resulting quantum theory fails to be unitary. This is reasonable. In our approach, when one quantizes, one would have a unitary evolution of the states as a function of the discrete parameter n. In the relational approach one picked some dynamical variable and called it time T. Suppose one chose a state in which this variable is highly peaked as a function of n. If one lets the system evolve, the variable will spread and at a later instant one would have a distribution of values of n that correspond to a given T (or vice versa). That means that if one started with a "pure" state, one ends with a mixed state. The underlying reason is that the physical clock T cannot remain in perfect lock-step with the evolution parameter n. A detailed discussion of the implications of this lack of unitarity is in [10; 11; 13]. Of course, this is not the first time that Quantum Gravity effects have been associated with loss of unitarity. However, unlike previous proposals (see [2]), the detailed evolution implied by the relational description we find conserves energy, which is a very desirable feature. One can give a bound on the smallness of the effect by taking into account what is the "best" clock one can construct from fundamental physical principles [23; 24]. The lack of unitarity makes the off diagonal elements of the density matrix go to zero exponentially. The exponent (for a system with two energy levels, for simplicity) is proportional to minus the Bohr frequency between the levels squared, to the Planck time to the $(4/3)$ power and to the time one waits for the state to lose coherence to the $(2/3)$ power (these results appear not even to be Galilean invariant, but this is not the case as discussed in detail in [12]. It is clear that the effect is negligible for most quantum systems. Chances of observing the effect in the lab (see for instance [28]) are at the moment remote, one would require a quantum system of macroscopic size. If one assumes energy differences of eV size, one would roughly need 10^{13} atoms. Bose–Einstein condensates at present can achieve states of this sort with perhaps hundreds of millions of atoms, but they do not involve energy differences of eVs per atom. Another important caveat of these types of discussions is that they have been carried out at a very naive level of Newtonian quantum mechanics. If one were to consider relativistic quantum field theory, one would have to have a "clock" variable per spatial point. This would imply that quantum states would lose coherence not only as time evolves, but also between points in space. Such effects could potentially have consequences that are much more amenable to experimental testing [28]. Once one accepts that quantum mechanics at a fundamental level contains loss of unitarity one may wish to reconsider the black hole information paradox. After all, the reason one has a paradox is that when a black hole evaporates, the final result is a mixed state, even if one built the black hole by collapsing a pure state. The question is: does this loss of unitarity occur faster or slower than the one we have found? If it is slower, then it will be unobservable. A priori one could expect that the effect we discussed should not be too important. We just argued in the previous paragraph that it is very small. However, black holes take a long time to evaporate. And as they evaporate their energy levels become more separated as the temperature increases. A detailed calculation shows that the order of magnitude of the off-diagonal elements of the density matrix at the time of complete evaporation would be approximately $M_{\mathrm{BH}}^{-2/3}$, with M_{BH} the black hole mass in Planck mass units [13]. For an astrophysical size black hole therefore the loss of unitarity is virtually complete and the paradox cannot be realized physically. What happens if one takes, say, a very small black hole? Can one reformulate the paradox in that case? The formulation we have is not precise enough to answer this question. We have only roughly estimated the magnitude of the decoherence just to give an order of magnitude estimate. Many aspects of the calculation are also questionable for small black holes, where true Quantum Gravity effects are also important. An interesting additional observation [14] is that the loss of quantum coherence we found can provide a fundamental limitation to how fast quantum computers can operate that is more stringent than other fundamental limits considered.

20.4 Constructing the quantum theory

As we argued above, the construction of the quantum theory starts by implementing the canonical transformation that gives the evolution in terms of the discrete parameter n as a unitary transformation. Before doing this one constructs the canonical theory that results from the elimination of the Lagrange multipliers. The resulting canonical theory generically has no constraints, and has evolution equations for its canonical variables. One picks a polarization, for instance $\Psi(q)$ where q is a set of configuration variables, and considers the unitary transformation as operating on the space of wavefunctions chosen. Since generically there are no constraints, one can pick as physical inner product the kinematical one and construct a Hilbert space of wavefunctions that are square integrable. If one is in the Schrödinger representation states evolve, so we label them as $\Psi_n(q)$ and the evolution is given by,

$$\Psi_{n+1}(q) = \int dq' U(q|q') \Psi_n(q'). \tag{20.10}$$

The transformation has to be such that it implements the evolution equations as operatorial relations acting on the space of wavefunctions in the Heisenberg representation, where

$$U(q|q') = \langle n+1, q'|n, q\rangle, \tag{20.11}$$

and where $|n+1, q'\rangle$ and $|n, q\rangle$ are the eigenvectors of the configuration operators $\hat{q}$ in the Heisenberg representation at levels $n+1$ and n respectively. The evolution equations take the form,

$$\langle n+1, q|\hat{q}_{n+1} - f(\hat{q}_n, \hat{p}_n)|n, q'\rangle \quad = \quad 0, \tag{20.12}$$

$$\langle n+1, q|\hat{p}_{n+1} - g(\hat{q}_n, \hat{p}_n)|n, q'\rangle \quad = \quad 0, \tag{20.13}$$

with f, g the quantum evolution equations, which are chosen to be self-adjoint in order for the transformation to be unitary. Explicit examples of this construction for cosmological models can be seen in [17]. If at the end of this process one has constructed a transformation that is truly unitary the quantization is complete in the discrete space and one has a well defined framework to rigorously compute the conditional probabilities that arise when one uses a relational time to describe the physical system. This is a major advantage over attempts to construct the relational picture with systems where one has constraints. There are some caveats to this construction that are worth pointing out. As we mentioned, our construction generically yields discrete theories that are constraint-free. To be more precise, the theories do not have the constraints associated with space-time diffeomorphisms. If the theory under consideration has other symmetries (for instance the Gauss law of Yang–Mills theory or gravity written in the new variable formulation), such symmetries may be preserved upon discretization (we worked this out explicitly for Yang–Mills and BF theory in [4]). The resulting discrete theory therefore will have some constraints. If this is the case, the above construction starts by considering as wavefunctions states that are gauge invariant and endowed with a Hilbert space structure given by a gauge invariant inner product. The resulting theory has true (free) Lagrange multipliers associated with the remaining constraints. The unitary transformation will depend on such parameters. An alternative is to work in a representation where the constraints are solved automatically (like the loop representation for the Gauss law). There one has no constraints left and the inner product is the kinematical one in the loop representation and the unitary transformation does not depend on free parameters. Other issues that may arise have to do with the fact that in many situations canonical transformation do not correspond quantum mechanically to unitary transformations. This problem has been discussed, for instance, by Anderson [1]. He noted that the only canonical transformations that can be implemented as unitary transformations are those that correspond to an isomorphism of a phase space into itself. This is important for the discrete theories in the following way. If one has a continuum constrained theory, its physical phase space is on the constraint surface. The discrete theories have a phase space that includes the constraint surface of the continuum theory. However, the discrete phase space variables cover only a subspace of the kinematical phase space of the continuum theory. There are inaccessible sectors that correspond to complex values of the Lagrange multipliers in the discrete theory. Therefore, in order that the canonical transformation of the discrete theory be an isomorphism, one may have to choose a physical Hilbert space for the discrete theory that is a subspace of the kinematical space instead of just taking it to be coincident. This has to be done carefully, since restricting the Hilbert space may imply that some physical quantities fail to be well defined in the physical Hilbert space. We have explored some of these issues in some quantum mechanical models that have a relational description. We have shown that one can successfully recover the traditional quantum mechanical results in a suitable continuum limit by carefully imposing a restriction on the kinematical Hilbert space, and that one can define variables that approximate any dynamical variable of the continuum theory in the continuum limit in the restricted Hilbert space (see [6]).

20.5 The quantum continuum limit

As we argued in the discussion of the model analyzed by Rovelli, a good measure of how close one is to the continuum theory in a given solution of the discrete theory is to evaluate the constraint of the continuum theory. Such constraint is only exactly satisfied in the continuum limit. An alternative way of presenting this is to consider the construction of a "Hamiltonian" such that exponentiated would yield the unitary evolution between n and $n+1$, $\hat{U} = \exp(i\hat{H})$ where $\hbar = 1$ and $\hat{H}$ has units of action. Such a Hamiltonian can only be constructed locally since in some points of the evolution the logarithm of the unitary transformation is not well defined. Such a Hamiltonian can be written as a formal expansion in terms of the constraint of the continuum theory (a way of seeing this is to notice that in the continuum limit this Hamiltonian has to vanish since it incorporates the timestep). If one chooses an initial state such that $\langle\hat{H}\rangle \ll 1$ the evolution will preserve this ($\hat{H}$ is an exact constant of the motion). This will continue until one reaches a point where $\hat{H}$ is not well defined. The evolution will continue, but it will not necessarily remain close to the continuum limit. In certain cosmological examples this point coincides with the point where the continuum theory has the singularity, for example [17]. Therefore a first condition on the quantum states in the continuum limit $\langle\hat{H}\rangle \ll 1$. A second condition is that the expectation values of the physical variables should not take values in the points where $\hat{H}$ is not well defined. A third condition is not to make measurements with "too much accuracy" on variables that do not commute with $\hat{H}$. This requirement stems from the fact that such measurements would introduce too much dispersion in $\hat{H}$ and one would violate the first requirement. In examples we have seen that this condition translates in not measuring q, p with sharper accuracy than that of the step of the evolution in the respective variable. This appears reasonable, a discrete theory should not allow the measurement of quantities with accuracies smaller than the discretization step. The variables that do not commute with $\hat{H}$ play a crucial role in the relational description since they are the variables that can be used as "clocks" as they are not preserved under evolution as constants of the motion.

20.6 Summary and outlook

One can construct discrete canonical theories that are constraint free and nevertheless approximate continuum constrained theories in a well defined sense. The framework has been tested at a classical level in a variety of models, including gravitational ones with infinitely many degrees of freedom. Further work is needed to make the framework computationally competitive in numerical relativity. In particular the use of better discretizations in time, including higher order ones, appears to be promising. Initial explorations we are carrying out in simple models indicate that one can achieve long-term stable and accurate evolutions using moderately large timesteps. This could be very attractive for numerical relativity if it turns out to be a generic property. Since the discrete theories are constraint free, they can be quantized without serious conceptual obstacles. In particular a relational time can be introduced in a well defined way and quantum states exhibit a non-unitary evolution that may have implications experimentally and conceptually (as in the black hole information puzzle). There is a reasonable proposal to construct the quantum continuum limit that has been tested in simple constrained models. The main challenge is to apply the framework at a quantum level in systems with field theoretic degrees of freedom. The fact that one has a well defined framework that is computationally intensive suggests that this is an avenue for conducting numerical Quantum Gravity.

This work was supported in part by grants NSF-PHY0244335, NASA-NAG5-13430 and funds from the Horace Hearne Jr. Laboratory for Theoretical Physics and CCT-LSU.

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21

The causal set approach to Quantum Gravity

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How can we reach a theory of Quantum Gravity? Many answers to this question are proposed in the different chapters of this book. A more specific set of questions might be: what demands should we put on our framework, so that it is best able to meet all the challenges involved in creating a theory of Quantum Gravity? What choices are most likely to give the correct theory, according to the clues we have from known physics? Are there any problems with our initial assumptions that may lead to trouble further down the road? The latter seems to be one of the most important strategic questions when beginning to formulate a candidate theory. For example, can a canonical approach overcome the multifaceted problem of time? And how far can a theory based on a fixed background spacetime be pushed? On the one hand, these questions may only be answered in the very attempt to formulate the theory. On the other, many such attempts have been made, and now that Quantum Gravity research has built up some history, perhaps it is time to plough some of the experience gained back into a new approach, laying the groundwork for our theory in such a way as to avoid well-known problems. The causal set program [1; 2; 3; 4; 5] represents such an attempt.

In this review, some answers to the above questions, as embodied by the causal set program, are set out and explained, and some of their consequences are given. As part of this, the results and open problems in the program are discussed. In section 21.1, reasons for hypothesising spacetime discreteness are reviewed. The definition of a causal set is given, along with the proposed correspondence principle between this structure and the effective continuum description of spacetime. Then some of the unique features of this discretisation scheme are discussed. In section 21.2, ideas for causal set dynamics are given. Next, a review is made of some phenomenological models based on this Quantum Gravity program, and successes and challenges in this line of work are summarised. Some results in this section touch on the issue of recovering locality for causal sets, something that is significant for

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

all the other subjects covered, and new results, which solve this problem in some situations for the first time, are mentioned.

Besides the present work, there are many other reviews available. One of the most recent is [5], while motivation and earlier work is covered in [2; 3; 4]. A philosophically oriented account of the conception of the causal set idea is given in [6], and there is a recent review which introduces some of the core concepts of causal set kinematics and dynamics [7]. Many of these articles, and other causal set resources, are most easily found at Rafael Sorkin's web site [8].

21.1 The causal set approach

This program is a development of "path-integral" or sum-over-histories (SOH) type approaches (for reasons to adopt this framework in Quantum Gravity, see [9; 10]). In such approaches, a space of histories is given, and an amplitude (or more generally a quantum measure), is assigned to sets of these histories, defining a quantum theory in analogy with Feynman's path integrals. A basic question, then, is what the space of histories should be for Quantum Gravity. Should they be the continuous Lorentzian manifolds of general relativity – or some discrete structure to which the manifold is only an approximation?

21.1.1 Arguments for spacetime discreteness

A number of clues from our present theories of physics point towards discreteness. The problematic infinities of general relativity and quantum field theory are caused by the lack of a short distance cut-off in degrees of freedom; although the renormalisation procedure ameliorates the problems in QFT, they return in naive attempts to quantise gravity (see [11] and references therein). Secondly, technical problems arise in the definition of a path-integral on a continuous history space that have never been fully resolved. On top of this, the history space of Lorentzian manifolds presents special problems of its own [12]. A discrete history space provides a well defined path integral, or rather a sum, that avoids these problems.

Perhaps the most persuasive argument comes from the finiteness of black hole entropy. With no short-distance cut-off, the so called "entanglement entropy" of quantum fields (the entropy obtained when field values inside a horizon are traced out) seems to be infinite (see [13; 14], and [15; 16] for some debate). If this entropy is indeed included in the black hole entropy, as many expect, a short distance cut-off of order the Planck scale must be introduced to allow agreement with the well-known semiclassical results. This, and similar analysis of the shape degrees of freedom of the black hole horizon [17; 18] lead to the conclusion that Planck scale discreteness is unavoidable, if the area–entropy relation for black holes is to arise from the statistical mechanics of a quantum theory.

Finally, suggestions of discreteness have come from various Quantum Gravity programs, like loop Quantum Gravity (see chapter 13 by Thiemann in this volume and section 21.1.6 of this chapter). Some of the most intriguing results come from so-called "analogue models" [19], where objects similar to black holes can be mocked up in condensed state matter systems. These analogies, as well as more direct arguments [20], suggest that the Einstein equation arises only as an equation of state, a thermodynamics of some more fundamental underlying theory. And the atomic discreteness of such systems provides a necessary cut-off to degrees of freedom at small scales. It is worth considering that atomic discreteness could never be found, for instance, by quantising some effective continuum theory describing a gas; it must be an independent hypothesis.

Quite apart from these more physical arguments, introducing discreteness can be of great utility. Conceptual problems, hidden under layers of technical complexity in continuum treatments, can sometimes be expressed more clearly in a discrete setting, and wrestled with more directly. This quality of discrete models has been of use in many Quantum Gravity programs. The successful definition of the "observables" in the "classical sequential growth" dynamics [21] (see section 21.2.1), an analogue of the problem of time in causal set theory, is an example of this.

21.1.2 What kind of discreteness?

Given these reasons for spacetime discreteness, in what way should we proceed? One might be disheartened by the sea of possibilities; how can we know, at this stage of knowledge, what the structure underlying the continuum manifold could be? However, the causal set offers a choice for the histories with a number of compelling and unique advantages.

The inspiration for the causal set idea comes from the remarkable amount of information stored in the causal order of spacetime. It has been proven that, given only this order information on the points, and volume information, it is possible to find the dimension, topology, differential structure, and metric of the original manifold [22; 23]. The points of a (weakly causal[1]) Lorentzian manifold, together with the causal relation on them, form a partially ordered set or *poset*, meaning that the set of points C and the order $\prec$ on them obey the following axioms.

(i) Transitivity: $(\forall x, y, z \in C)(x \prec y \prec z \implies x \prec z)$.
(ii) Irreflexivity: $(\forall x \in C)(x \not\prec x)$.

[1] A weakly causal Lorentzian manifold is one that contains no closed causal curves, otherwise called "causal loops".

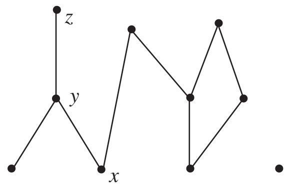

Fig. 21.1. A causal set. The figure shows an example of a Hasse diagram. In such a diagram, the elements of a causal set are represented by dots, and the relations not implied by transitivity are drawn in as lines (for instance, because $x \prec y$ and $y \prec z$, there is no need to draw a line from x to z, since that relation is implied by the other two). The element at the bottom of the line is to the past of the one at the top of the line.

If $x \prec y$ then we say "x is to the past of y", and if two points of the set C are unrelated by $\prec$ we say they are spacelike (in short, all the normal "causal" nomenclature is used for the partial order).

It is this partial order that we choose as fundamental. To achieve discreteness, the following axiom is introduced.

(iii) Local finiteness: $(\forall x, z \in C)\,(\mathbf{card}\,\{y \in C \mid x \prec y \prec z\} < \infty)$.

Here, **card** X is the cardinality of the set X. In other words, we have required that there only be a finite number of elements causally between any two elements in the structure (the term "element" replaces "point" in the discrete case). A locally finite partial order is called a causal set or *causet*, an example of which is illustrated in figure 21.1. Many researchers have independently been led to the same hypothesis [24; 25; 1]: that the causal set should be the structure that replaces the continuum manifold.

21.1.3 The continuum approximation

In all standard quantum theories, be they direct quantisations of a classical theory or discrete approximations, there is an approximate correspondence between at least some of the underlying histories and those of the limiting classical theory, needed in order to relate the quantum theory to known physics.[2] (Similarly, in the state vector formulation, there must be a correspondence between configurations at a particular time, each assigned a basis vector in the Hilbert space of the quantum theory, and the allowed configurations in the classical theory – something thought to be true even in loop Quantum Gravity [26]). The view taken in causal set theory

[2] In standard theories, this statement is dependent upon the choice of a Hilbert space basis, e.g. the position basis for a Shrödinger particle. The path integral can be expressed using other Hilbert space bases, but it is nevertheless always true that there exists some basis (usually a class of them) in which the above statement is true.

is that such a correspondence is necessary for any quantum theory, and so at least some of the histories in our Quantum Gravity SOH must be well approximated by Lorentzian manifolds. A full justification of this point is beyond the scope of this chapter, but support may be taken from other quantum theories, both standard and speculative, and from some of the seminal writings on quantum mechanics [27; 28]. Some explanation can be found in [29].

Here, the question is: when can a Lorentzian manifold $(\mathcal{M}, g)$ be said to be an approximation to a causet $\mathcal{C}$? Roughly, the order corresponds to the causal order of spacetime, while the volume of a region corresponds to the number of elements representing it.[3] It is interesting to note that the manifold and the metric on it have been unified into one structure, with counting replacing the volume measure; this is a realisation of Riemann's ideas on "discrete manifolds" [34] (see also the translated passages in [5]). But a more exact definition of the approximation is needed.

A causal set $\mathcal{C}$ whose elements are points in a spacetime $(\mathcal{M}, g)$, and whose order is the one induced on those points by the causal order of that spacetime, is said to be an *embedding* of $\mathcal{C}$ into $(\mathcal{M}, g)$.[4] Not all causal sets can be embedded into all manifolds. For example, the causal set in figure 21.2 cannot be embedded into 1+1D Minkowski space, but it *can* be embedded into 2+1D Minkowski space. There are analogues to this causal set for all higher dimensions [35], and surprisingly there are some causal sets that will not embed into Minkowski of *any* finite dimension. Thus, given a causal set, we gain some information about the manifolds into which it could be embedded. However, a manifold cannot be an approximation to any causal set that embeds into it; we could recover no volume information in

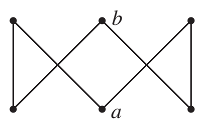

Fig. 21.2. A Hasse diagram of the "crown" causet. This causet cannot be embedded in 1+1D Minkowski space: if the above Hasse diagram is imagined as embedded into a 2D Minkowski spacetime diagram, the points at which elements a and b are embedded are not correctly related. In no such embedding can the embedded elements have the causal relations of the crown causet induced on them by the causal order of 1+1D Minkowski space. The causal set can however be embedded into 2+1D Minkowski space, where it resembles a three-pointed crown, hence its name.

[3] While this is the stance taken in what might be called the "causal set Quantum Gravity program", the causal set structure has also been useful elsewhere, although with different, or undefined, attitudes as to how it corresponds to the continuum. See for example [30; 31; 32; 33].

[4] Really an embedding of the isomorphism class of that causet (the "abstract causet"). The distinction between isomorphism classes and particular instances of causal sets is not crucial for the purposes of this chapter, and will be ignored.

this way, no discreteness scale is set, and there might not be enough embedded elements to "see" enough causal information. A further criterion is needed to ensure the necessary density of embedded elements.

So, to retrieve enough causal information, and to add the volume information, the concept of *sprinkling* is needed. A sprinkling is a random selection of points from a spacetime according to a Poisson process. The probability for sprinkling n elements into a region of volume V is

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}. \tag{21.1}$$

Here, ρ is a fundamental density assumed to be of Planckian order. Note that the probability depends on nothing but the volume of the region. The sprinkling defines an embedded causal set. The Lorentzian manifold $(\mathcal{M}, g)$ is said to approximate a causet $\mathcal{C}$ *if* $\mathcal{C}$ *could have come from sprinkling* $(\mathcal{M}, g)$ *with relatively high probability*.[5] In this case $\mathcal{C}$ is said to be *faithfully embeddable* in $\mathcal{M}$. On average, ρV elements are sprinkled into a region of volume V, and fluctuations in the number are typically of order $\sqrt{\rho V}$ (a standard result from the Poisson statistics), becoming insignificant for large V. This gives the promised link between volume and number of elements.

Can such a structure really contain enough information to provide a good manifold approximation? We do not want one causal set to be well-approximated by two spacetimes that are not similar on large scales. The conjecture that this cannot happen (sometimes called the "causal set haupvermutung", meaning "fundamental conjecture") is central to the program. It is proven in the limiting case where $\rho \to \infty$ [36], and there are arguments and examples to support it, but some steps remain to be taken for a general proof. One of the chief difficulties has been the lack of a notion of similarity between Lorentzian manifolds, or more properly, a distance measure on the space of such manifolds. Progress on this has now been made [37], raising hopes of a proof of the long-standing conjecture.

A further generalisation of this scheme may be necessary. Above, it was noted that certain small causal sets cannot be embedded into Minkowski space of any particular dimension. This means that, for $\mathcal{C}$ a large causal set that is faithfully embeddable into a region of n-dimensional Minkowski, by changing a small number of causal relations in $\mathcal{C}$ we can form a causet that no longer embeds. From our experience with quantum theories, we most likely will not want such "small

[5] The practical meaning of "relatively high probability" has so far been decided on a case-by-case basis. It is usually assumed that the random variable (function of the sprinkling) in question will not be wildly far from its mean in a faithfully embeddable causet. Beyond this, standard techniques involving χ^2 tests exist to test the distribution of sprinkled points for Poisson statistics.

fluctuations” to be physically significant, and so we may need a condition of “manifoldlikeness” that is more forgiving than faithful embedding. A possible method is given by *coarse-graining* [2]: removal of some points from the causal set $\mathcal{C}$ forming a new causal set $\mathcal{C}'$, before testing $\mathcal{C}'$ for faithful embeddability at the appropriate lower density of sprinkling ρ'. For example, this might reasonably be done at random with the same probability p for removal of each element, and $\rho' = \rho(1-p)$. This basically amounts to looking for a faithfully embeddable subset of a causal set, following a certain set of rules. Below, the criterion of faithful embeddability will be the one used, but it should be kept in mind that the causets being talked about could be coarse-grainings of some larger causet.

21.1.4 Reconstructing the continuum

The concept of faithful embedding gives the criterion for a manifold to approximate to a causet. But it is important to realise that the only use of sprinkling is to assign continuum approximations; the causal set itself is the fundamental structure. How then can this approximate discrete/continuum correspondence be used? That is, given a causal set that approximates a spacetime, how do we find an approximation to some particular property x of that spacetime? We need to find a property of the causal set itself, $x(\mathcal{C})$, that approximates the value of $x(\mathcal{M})$ with high probability for a sprinkling of a spacetime $\mathcal{M}$. Such estimators exist for dimension [38; 24; 39] timelike distance between points [40], and of course volumes. As another example, methods have been developed to retrieve topological information about spatial hypersurfaces in approximating Lorentzian manifolds, by reference only to the underlying causet [41].

A simple example of how such estimators work is given by one of the estimators of timelike distance. Firstly the volume of the interval causally between two elements can be easily estimated from the causal set (it is approximately proportional to the number of elements in that causally defined region). In Minkowski space, this volume is related to the distance between the points, in a way that depends on dimension. Therefore, given the dimension, this timelike distance can also be estimated. See [40] for a different distance measure conjectured to hold for curved spacetimes [24], and a way to identify approximations to timelike geodesics. As this other distance measure does not depend on the dimension, the two can be compared to give a dimension estimator.

Given a causal set $\mathcal{C}$ without an embedding (this is after all our fundamental structure) it would be of great utility to be able to say if it was faithfully embeddable into some spacetime or not – a criterion of “manifoldlikeness” – and if so to provide an embedding. The discrete-continuum correspondence given above does not directly answer this question; it would be highly impractical to carry out various sprinklings until we came up with a causet isomorphic to $\mathcal{C}$. Nevertheless, the measure of timelike distance and some simple geometry can be used, on computer, to attempt to find an embedding for $\mathcal{C}$, at least into small regions of Minkowski [42] (the idea has been implemented so far only in 2D). The success or failure of the attempted embedding gives a measure of manifoldlikeness for $\mathcal{C}$. Beyond this rough-and-ready computational scheme, several necessary conditions for manifoldlikeness are known (e.g. the matching of different dimension estimators, "self-similarity" [43], etc.), and it is hoped that a combination of these might yield a necessary and sufficient condition.

Given this discrete/continuum correspondence principle, some of the attractive features of the causal set structure can be noted. Firstly, there is no barrier to sprinkling into manifolds with spatial topology change, as long as it is degeneracy of the metric at a set of isolated points that enables topology change, and not the existence of closed timelike curves (one of these conditions must exist for topology change to occur, see e.g. [44] and references therein) – and in this discrete theory there is no problem with characterising the set of histories. For those who believe that topology change will be necessary in Quantum Gravity [10; 45], this is important. Secondly, the structure can represent manifolds of any dimension – no dimension is introduced at the kinematical level, as it is in Regge-type triangulations. In fact, scale dependent dimension and topology can be introduced with the help of coarse-graining, as explained in [38], giving an easy way to deal with notions of "spacetime foam". Also, it has been found necessary to incorporate some notion of causality at the fundamental level in other approaches to SOH Quantum Gravity, highlighting another advantage of using the causal set structure from the outset. But the property which really sets causal sets apart from other discrete structures is local Lorentz invariance.

21.1.5 Lorentz invariance and discreteness

For most discrete structures, local Lorentz invariance (LLI) is impossible to attain (see [46] for a brief explanation of why this is so). This can be a major problem if the locally Lorentz invariant spacetime we observe is to arise as an approximation to these structures. There is always the possibility that LLI does fail at higher energy scales, and discreteness of the Lorentz violating kind has been cited as a motivation when searching for such non-standard effects. As such studies progress, bounds on Lorentz violation from astrophysical observations are becoming ever more stringent [47; 48]. On top of this, Collins *et al.* [49; 50] argue that Lorentz symmetry breaking at the Planck scale would significantly affect the radiative corrections in the standard model, leading to results contrary to experiment unless additional fine tuning is introduced.

What does Lorentz invariance mean in this context? The answer should be guided by what is tested in the observations mentioned above. Let us begin with the statement for theories on a background Minkowski spacetime. Here, Lorentz invariance of a theory means that the dynamics should not distinguish a preferred Lorentz frame. Next, we want to say that the Minkowski space is only an approximation to some underlying discrete structure. In view of the statement of Lorentz invariance, we want to make sure that any dynamics on this approximating Minkowski is not forced to pick a preferred Lorentz frame because of the discreteness. This leads to the following: if the underlying structure, in and of itself, serves to pick out a preferred direction in the Minkowski space, then Lorentz invariance has been violated. This is the situation for lattice-like structures, and is arguably the most relevant statement for the current observational tests of Lorentz invariance.

In contrast, by this criterion, the causal set provides a locally Lorentz invariant discrete structure – the only one considered in any approach to Quantum Gravity. This property is achieved thanks to the random nature of the discrete/continuum correspondence principle given above.

As an analogy, consider a crystal, and a gas, as discrete systems of atoms whose behaviour can be given an approximate continuum treatment. The crystal has a regular underlying structure that breaks rotational symmetry, and this symmetry breaking can be observed macroscopically, by the existence of fracture planes and so on. The gas on the other hand has a *random* underlying structure, and the probability distribution of the molecules' positions at any time is rotationally invariant. There is no preferred direction in a gas that affects its behaviour in the effective continuum treatment. We could "cook up" a direction from the positions of the molecules – in any region containing two molecules we can of course draw a vector from one to the other. The point is that such "preferred directions" identifiable on microscopic scales have no effect on the bulk, continuum physics of the gas. Thus it is common to say that the behaviour of a gas is rotationally invariant.

The Lorentz invariance of the causal set is similar. As previously noted, in the Poisson process, the probability for sprinkling n elements into a region depends on no property of that region other than the volume. In Minkowski spacetime, to establish Lorentz invariance of the Poisson process rigorously we need only note the theorems proving the existence and uniqueness of the process with the distribution (21.1) for all measurable subsets of $\mathbb{R}^d$ and its invariance under all volume preserving linear maps (see e.g. [51]), which of course includes Lorentz transformations. In a curved spacetime, Lorentz invariance is to be understood to hold in the same sense that it holds in General Relativity: the equivalence of local Lorentz frames.

In some sense the situation is better than that for gases. In the case of a sprinkling of $\mathbb{R}^3$, a direction can be associated with a point in the sprinkling, in a way that commutes with rotations (i.e. finding the direction from the sprinkling and then rotating the direction gives the same result as first rotating the sprinkling and then finding the direction). An example is the map from the point in the sprinkling to the direction to its nearest neighbour. But owing to the non-compactness of the Lorentz group, there is no way to associate a preferred frame to a point in a sprinkling of Minkowski that commutes with Lorentz boosts [52]. In this sense each instance of the Poisson process, not just the distribution, is Lorentz invariant.

For causal sets that approximate to Minkowski space, the causal set does not pick out a preferred direction in the approximating manifold. We therefore expect that no alteration to the energy-momentum dispersion relation would be necessary for a wave moving on a causal set background. This property is explicit in at least one simple model [53].

Local Lorentz invariance of the causal set is one of the main things that distinguishes it among possible discretisations of Lorentzian manifolds. We now see that the daunting choice of the discrete structure to be used in Quantum Gravity is actually extremely limited, if the principle of local Lorentz invariance is to be upheld. Could other popular approaches to Quantum Gravity, based on graphs and triangulations, utilise sprinklings to incorporate Lorentz symmetry? The theorem mentioned above shows this to be impossible: no direction can be associated with a sprinkling of Minkowski in a way consistent with Lorentz invariance, and the same is true for a finite valancy graph or triangulation [52].

21.1.6 LLI and discreteness in other approaches

In the causal set approach, there is discreteness and LLI at the level of the individual histories of the theory. This seems to be the most obvious way to incorporate the symmetry, while ensuring that the foreseen problems with black hole entropy (and other infinities) are avoided. But is it necessary? Each approach to non-perturbative Quantum Gravity represents a different view on this, some of which can be found in the other chapters of this section. A brief "causal set perspective" on some of these ideas is given here.

In the broad category of "spin-foam approaches", the histories are also discrete, in the sense that they can be seen as collections of discrete pieces of data. But is there "enough discreteness" to evade the infinite black hole entropy arguments? This is a hard question to answer, not least because the approximate correspondence between these histories and Lorentzian manifolds has not been made explicit.[6] But if there is intended to be a correspondence between the area of

[6] However, the correspondence principle in the similar case of graphs corresponding to 3D space [26] could possibly be extended, somehow, to the 4D case.

a 2D surface and the number (and labels) of 2-surfaces in the spin-foam that "puncture" it (as is sometimes claimed), this suggests a kind of fundamental discreteness on such surfaces. It also suggests an upper bound on degrees of freedom per unit volume. But all this depends on the final form of the sum over triangulations in that approach, something not yet clarified.

The status of local Lorentz invariance in spin-foam models remains controversial. As stated above, the causal set, with the above sprinkling-based discrete-continuum correspondence, is the only known Lorentz invariant discrete structure, and spin-foams are not of this type. But the real debate is over whether this implies observable Lorentz violation (if spin-foams models really do imply an upper bound on degrees of freedom per unit volume). It is sometimes claimed that, although an individual spin-foam cannot be said to satisfy LLI, a quantum sum over many spin-foams may do (arguments from the closely related LQG program support this [54; 55]). An analogy is drawn with rotational invariance: in that case, the histories might only represent one component of the angular momentum of, say, an electron. In spite of this, the physics represented is in fact rotationally invariant.

However, in standard theories, at least the *macroscopic* properties we observe are properties of each history in some (nonempty) set, and we should expect the same for Quantum Gravity. Even these properties may fail to be present in the case of Lorentz transformations, if the histories are not Lorentz invariant in the sense that a causal set is. It is possible that further thought along these lines could lead to quantitative predictions of Lorentz violation from spin-foam models, giving an opportunity for observational support or falsification. A compromise between these views might be found in "doubly special relativity", in which Lorentz transformations are deformed. In this case observational tests are still possible.

In the loop Quantum Gravity program, the spectra of certain operators (e.g. the areas of 2D surfaces) are claimed to be discrete, although as yet the physical Hilbert space and operators have not been identified. Nevertheless, some arguments have been provided as to how the problems of spacetime singularities and black hole entropy might be solved in LQG. But without the physical observables, how this type of discreteness could circumvent the arguments mentioned in the introduction or in the previous paragraph, or even whether it would exist in a completed form of loop Quantum Gravity, is not clear as yet.

In dynamical triangulations, discreteness is used to solve the problems of defining the path integral, and coming to grips with technical issues in a manageable way, notably the Wick rotation. However, in this approach the discreteness is not considered fundamental and a continuum limit is sought. As the "causal dynamical triangulations" program is in the happy situation of possessing a working model, it would be of great interest for the debate on discreteness to see what becomes of black hole entropy (or more general forms of horizon entropy [16]) as the cut-off is removed. Will some previously unexpected effect keep the entropy finite, or are the arguments for a fundamental cut-off inescapable?

21.2 Causal set dynamics

Having introduced the causal set structure and discussed some of its special advantages, the most pressing question is how to construct a dynamics, with causal sets as the histories, that would be a satisfying theory of Quantum Gravity. The question is, perhaps unsurprisingly, a difficult one. In order to obtain the discrete, Lorentz invariant causal set we have thrown away much of the manifold structure that we are used to. For instance, states on spatial slices are not a natural notion for the causal set; only when "thickened" slices are considered can more pieces of approximate manifold information be recovered [41]. Therefore, as intended, the structure lends itself to sums over histories rather than the state vector formalism. But even the Feynman propagator crucially refers to states on spacelike hypersurfaces. This begs the question: what kind of dynamics should be used?

Generalised Quantum Mechanics [56], alternatively named Quantum Measure Theory [57], defines quantum processes as a generalisation of stochastic processes, allowing freedom from any reference to spatial slices. Most ideas for causal set dynamics are based on this framework.

21.2.1 Growth models

The most favoured approach to dynamics uses the simple structure and direct physical interpretation of the causal set to advantage. Given such a simple kinematical framework, and the dynamical framework of quantum measure theory, it is possible that physical principles could be used to constrain the dynamics until only a small class of theories remain (the ideal example being the derivation of GR from a small set of such principles). Particularly natural to the causal set are the concepts of general covariance and causality. As a warm-up for the quantal case, a set of stochastic processes on the space of past-finite causal sets has been developed, the so-called *classical sequential growth* (CSG) models [58; 59].

These models are based on the concept of randomly "growing" a causal set, following certain rules. From a single element, one new element is added in each of an infinite sequence of *transitions*. The new element is always added to the future of or spacelike to the existing elements. There are always several possibilities for how to add the new element, and a probability distribution is placed on these possibilities. These "transition probabilities" are then constrained by the chosen physical principles. A sequence of transitions can be thought of as a path through the partially ordered set of all finite causets, as illustrated in figure 21.3. The process generates

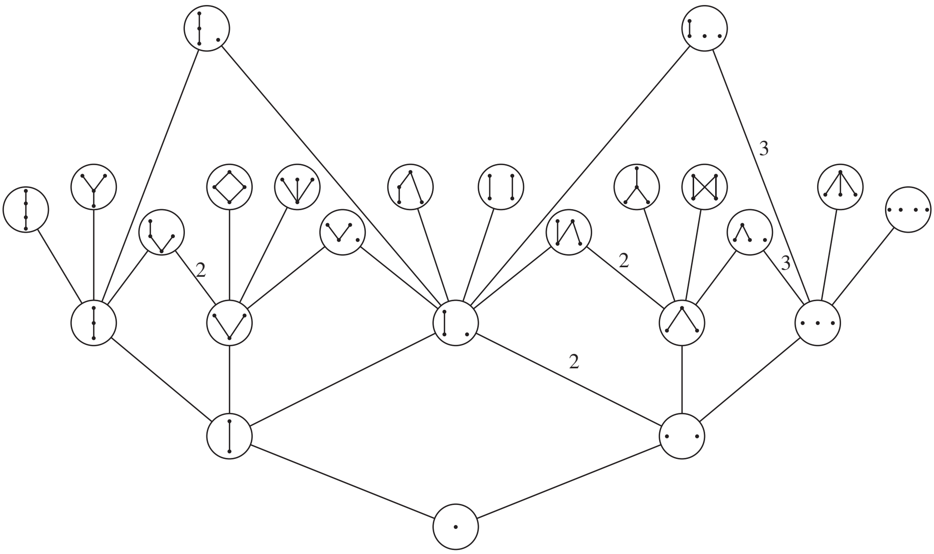

Fig. 21.3. An augmented Hasse diagram of "poscau", the partially ordered set of finite causets. The elements of this set are the finite causets. To the "future" of each causet are all the causets that can be generated from it by adding elements to the future of or spacelike to its elements (the numbers on some of the links represent the number of different ways the new element can be added, owing to automorphisms of the "parent" causet). Only causets of up to size 4 are shown here. An upwards path in poscau represents a sequence of transitions in a growth process. Each such path is given a probability by a CSG model. Because of the general covariance condition, the probabilities of paths ending at the same causet are the same. (Note that the apparent "left–right symmetry" of poscau does not survive above the 4-element causets.)

infinite-element causal sets. From the transition probabilities, a probability measure on the space of all infinite-element past-finite causal sets can be constructed.

The order of birth can be viewed as a labelling of the elements of the growing causet. A natural implementation of the principle of general covariance is that this labelling should not be physically significant. Another physical principle is introduced to ban superluminal influence, in a way appropriate to stochastic systems. With these constraints, the free parameters of the model are reduced to a series of real numbers.

The CSG models have made a useful testing ground for causal set dynamics, allowing some questions to be answered that would have relevance for quantum theories developed using the same method. For instance, the "typical" large causal set (i.e. the type that is most likely to be found from a uniform probability distribution over causal sets with some large number of elements) does not look like a manifold, but instead has a "flat" shape described more fully in [60]. It might be wondered what kind of a dynamics could overcome the great numbers of these

"Kleitman–Rothschild" causets – entropic effects might be expected to favour these typical causets. However, growth process models easily circumvent this worry. CSG models generically give low probabilities to these causets.

An important question is how to identify and characterise the physical questions that the theory can answer. In canonical theories this question can be phrased "what are the physical observables, and what do they mean?", and answering it is a central part of the problem of time. In some form it seems to afflict any indeterministic, generally covariant theory. This problem does not disappear in the CSG model, where it must be ensured that the "observables" (here, sets of histories which are assigned a probability, and are "covariant", meaning insensitive to the growth order labelling) can be characterised and given a physical interpretation. This was achieved for a generic class of CSG models in [21] and the results extended to the most general models in [61]. Most of the methods used will be directly applicable to any future "quantum sequential growth" model.

Another result with possible implications for the full quantum theory concerns the so-called "cosmic renormalisation" behaviour that the models exhibit [62; 63; 64]. Some generic models have a "bouncing cosmology" with many big-bang to big-crunch cycles. The large spatial extent of the universe in these models is not a result of fine-tuning, but simply a consequence of the extreme age of the universe, giving a mechanism for fixing parameters that may be useful in more realistic theories.

The CSG models have also been of use in developing tests of dynamically generated causal sets to look for manifoldlike behaviour [43], and computational techniques for causal sets. But it is important to note that the theories are not supposed to be a "classical limit" of a quantum dynamics; the situation is more analogous to the stochastic dynamics of Brownian motion, and its relationship with the quantum dynamics of the Schrödinger particle. The goal is to replace the probability measure used in the CSG model with a quantum measure, reworking the physical conditions to make sense in this case. Whether or not the CSG models can produce manifoldlike causal sets is not crucial for them to fulfil their role as a stepping-stone to the quantum case.

This ambitious approach to causal set dynamics has the advantage of simple, clean formalism and the prospect of going beyond what might be possible by attempting to approximate a continuum path integral. For instance, no dimension is specified anywhere in the founding principles of the theory, and so a successful "quantum sequential growth" model would give a real explanation for the 4D nature of large-scale spacetime from a small set of principles. However, challenges remain in the development of the full quantum theory. The generalisation of the relativistic causality principle to the quantum case has proved difficult [65]. It must also be ensured that there is as little freedom in the implementation of the fundamental principles as possible, lest we undermine the idea of directly proceeding from principles to dynamics. This question deserves a more thorough investigation even at the stochastic level. But many avenues for creating growth models lie open. Different physical principles could be used to constrain the dynamics, and a number of suggestions are currently under consideration.

21.2.2 Actions and amplitudes

Another approach to the dynamics is in closer analogy to that employed in other Quantum Gravity programs: assigning a complex amplitude to each history. It would be interesting to see how far this formalism can be pushed for causal set dynamics (its use was originally suggested in the early papers in the program [1; 2]). The first obstacle is the lack of an expression for the amplitude $\exp(i\,S(\mathcal{C}))$. We need to find an action for the causal set. The most obvious thing to do would be to find a function of the causal set that approximates the Einstein action for causal sets corresponding to 4D manifolds. This is another kinematical question like that of finding geodesic lengths, and dimension, discussed above. In the continuum, the Einstein action is the integral of a local quantity on the manifold, and so the causal set action should also be local, approximately. Indeed, the "natural" value of any such approximately local function should approximate the Einstein action, as argued in [1]. Then the kinematical task becomes the identification of approximately local causal set functions. The task of recovering locality has been a perennial theme of research in causal set theory, and has recently seen some exciting progress, leading to some possible expressions for the action, as discussed in section 21.3.2.

Then there is the question of what set of causal sets to sum over. Most satisfying would be to sum over *all* causal sets of a fixed number of elements (a "unimodular" sum over histories [66; 67; 68]). The action would have to be "slowly varying" in some appropriate sense near the classical solutions, and "quickly varying" elsewhere – here "elsewhere" means not only the causal sets corresponding to manifolds that are not solutions of GR, but also the (far greater number of) causal sets that do not correspond to *any* manifold. A less natural strategy would be to limit the history space to a subset of the full history space of causal sets, containing all causets faithfully embeddable into certain manifolds.

21.3 Causal set phenomenology

While progress is being made on the dynamics, a final theory is still not available. But it is still of use to ask the question "What are the consequences of the causal set hypothesis for phenomenology?". Does the use of Lorentz invariant, discrete histories suggest any measurable effects? Without the final dynamics any arguments will have to be heuristic; but, when it comes to phenomenology, advances are sometimes possible even before a full theory is defined [69]. Such considerations have led in many interesting directions. One prediction is that no violation of standard, undeformed LLI (as opposed to the deformed Lorentz invariance of "doubly special relativity") will be observed, as such an observation would undermine one of the major motivations for causal set theory. But this is a purely negative prediction, so it is useful to search for something more.

21.3.1 Predicting Λ

Perhaps the most significant phenomenological result for causal sets was the successful prediction of the cosmological constant from a heuristic argument. The argument is essentially a combination of unimodular ("volume-fixed") Quantum Gravity and the underlying random discreteness (see [2; 70; 71] for further details). From the classical theory, it can be seen that the spacetime volume V is conjugate to the cosmological constant Λ, in the sense that position and momentum are conjugate in particle dynamics. But in causal set theory, there is an intrinsic uncertainty in the volume of order $\pm\sqrt{V}$, where V is the past 4-volume of the universe in fundamental units. V cannot therefore be fixed on a sharp value. Plugging this uncertainty in V into the uncertainty relation, we can find the related "intrinsic" fluctuations in Λ:

$$\Delta\Lambda \sim \frac{1}{\Delta V} \sim \frac{1}{\sqrt{V}}, \tag{21.2}$$

using fundamental units. If we assume that the value of the cosmological constant is driven towards zero (taken as a natural assumption here), this equation tells us that it could not be exactly zero in our theory, but will have fluctuations of order 10^{-120} (again in Planckian units) in the present epoch. This prediction was subsequently verified by observation.

There are plenty of open questions surrounding this achievement. By this argument, the energy density in Λ is, *on average*, comparable to the matter and radiation energy density at all times. However, fluctuations in Λ are to be expected, and in [70] these fluctuations are modelled. The hope is that this will lead to more detailed predictions in cosmology. The path from theory to prediction in cosmology is typically a tortuous one, and the introduction of a varying cosmological "constant" breaks the assumptions used in standard cosmology. Much effort will be required to modify the standard predictions in the light of this idea, and then compare them to observation.

21.3.2 Swerving particles and almost local fields

How might particles and fields propagate on a causal set? Is it necessary, or at least natural in some sense, for them to behave in a non-standard way? This question has obvious phenomenological implications. The simple case of a point particle was discussed in [46]. A toy model was constructed of a point particle moving on a sprinkled causal set, replacing the continuum path by a set of timelike related elements. In the continuum we know that classical particles move on geodesics, and the velocity at any time is easily determined from the path up to that time. At the discrete level, however, the velocity of a particle has no exact value at any time, and cannot be accurately determined by looking at a short section of its discrete path. Assuming that the particle's dynamics at any time is only affected by its path within a certain proper time to the past (an assumption of approximate locality in time), it was found that the particle is subject to random (Lorentz invariant) acceleration – it "swerves" away from the geodesic. The assumption of Lorentz invariance leads to a generic diffusion law in velocity space with only one parameter, without direct reference to the particular microscopic toy model, in much the same way as the standard diffusion equation arises from many different microscopic processes.

The classical point particle picture, and the use of a fixed background causal set, are gross simplifications, and the strict form of approximate locality is by no means an absolute requirement. But the model is potentially testable, and that is the goal of studies of this heuristic, phenomenological type. Since any such model will predict acceleration of particles in space, the question is: do we see these particles? It is tempting to identify them with cosmic rays, the origin of which is currently a major problem in astrophysics [72]. While some features of the spectrum of cosmic rays can be reproduced by the simple model, the rate of diffusion needed to explain HECRs is incompatible with laboratory requirements. It is hoped that this could be corrected in a more sophisticated, quantum model.

For similar reasons, it would also be of interest to put fields on a fixed causal set background, and the easiest place to start is a scalar field. Early attempts to do this directly discretised Green's functions of the scalar field dynamics [73; 2]. This technique has recently been used [53] to show that it is possible for waves to travel on a causal set background without contradicting present observations. But, while useful for propagating a field from a source to a detector, the associated discretised d'Alembertian is unstable, and so evolving a field directly from some past configuration is not always possible. This is part of the problem of recovering approximate locality on a casual set background.

A new method, to be set out in [74], has some analytical backing and has survived the computational tests which previous ideas failed. The scheme can be seen as a "smearing" of a non-Lorentz invariant discretisation of the d'Alembertian (of the type normally used on lattices) over the whole Lorentz group. Significantly, a new "non-locality scale" must be introduced, above the Planck scale but macroscopically small, to allow for the non-locality of the causal set. The analysis of this discrete d'Alembertian has so far been carried out only in flat space, although it has been tentatively conjectured that the scheme will also be successful for sprinklings of curved spacetimes. Its discovery provides a way to define a classical dynamics of scalar fields on a fixed causal set background, giving a causal but non-local field theory, which may lead to hints on non-standard phenomenology. It would also be an interesting exercise to find a way to quantise the field, and look for similar results there.

One of the most intriguing uses is for causal set dynamics, as mentioned in section 21.2.2. How can this discretised d'Alembertian help us to find an action for causal sets? Consider the field $\Box\sigma(0, x)$, where $\sigma(x, y)$ is Synge's world function (i.e. half of the square of the geodesic distance between x and y) and 0 is some arbitrary origin of co-ordinates. It can be seen from some of the results in [75] that the d'Alembertian of *this* field at the origin gives the scalar curvature there:

$$R(0) = \Box\,\Box\,\sigma(0, x)\Big|_{x=0} . \tag{21.3}$$

The geodesic length between two timelike points in a causal set can be estimated (independently, it is conjectured, of curvature). Therefore, if we have a way of estimating the d'Alembertian of fields in curved space times, we also have a way of estimating the scalar curvature. If this method turns out to be correct, and the values found are stable and practically calculable, it will be of great significance for causal set dynamics.

These results are, hopefully, only the first handle on the problem of locality in causal sets, and consideration of what has been learnt may lead to the development of more techniques, as the reason for this success is more fully grasped. One goal would be would be to find an expression for the action which is combinatorially simple and compelling, and which gives sensible values for non-manifoldlike causal sets. Work on these topics has only just begun.

21.4 Conclusions

Discreteness provides a solution for many of the problems we confront in our attempts to construct a theory of Quantum Gravity. From the assumptions of discreteness and standard Lorentz invariance, we find that our choices of fundamental histories are extremely limited. Although this should not discourage other attempts to reconcile the two,[7] it has been argued here that, at present, the causal set is the only proposal that does so. The causal set program is an active and growing one. Many projects in progress have not been mentioned above. Attempts to identify the "atoms" that carry the black hole entropy have been made [77; 78], and this work is currently being extended to higher dimensions by Fay Dowker and Sara Marr. Further work on the question of "observables" has also been carried out [79]. Pros and cons of an amplitude-based dynamics are also being investigated. As well as this work, the basic causal set idea continues to inspire other approaches [80; 81]. Every statement of a result given here raises many more questions, only some of which are being pursued. This multiplicity of unanswered questions, the relatively small set of prerequisites needed to contribute to them, and the comparative, "strategic" perspective on Quantum Gravity that the approach offers, make causal sets an attractive field for both new and experienced researchers.

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Questions and answers

- **Q - J. Henson - to J. Ambjørn *et al*.**
  1. The CDT program borrows many techniques from lattice quantum field theory, and as there, some universality properties are presumably crucial here – many methods of discretisation should result in the same model in the continuum limit. But in this new type of model we cannot yet have the same level of confidence in this principle. Apart from the very encouraging results that you summarise above, which show that the model does have some desirable properties, what is known specifically about universality in this type of discretised Quantum Gravity model?
  2. Although the cut-off is diffeomorphism invariant in the sense that the discrete geometries only contain lengths and topological information, they are not in another important sense: the discretisation picks out a preferred foliation of spacetime, and one would expect that, were matter to be included, modes that were high frequency with respect to this foliation would be cut off. Then the hope would be, as in lattice QFT, that this discrete symmetry breaking has no significance in the continuum limit. What arguments are there for this, and can you envision a calculation that would verify it?
  - **A - J. Ambjørn *et al*.**:
    1. Not much is known about universality except the simple test of changing the coupling constants somewhat and observing the correlators can be mapped onto each other by rescaling of the time direction relative to the space direction.
    2. We simply do not know how the combined system of matter and geometry will behave. One can only hope that the time-foliation does not spoil the general properties of the matter system one would expect in GR. In flat spacetime we are of course allowed to consider an asymmetric lattice where the lattice spacing in the time direction is different from the lattice spacing in the spatial directions. It should not make any difference provided our action is adjusted correspondingly.

- **Q - C. Rovelli - to J. Ambjørn *et al*.**
  1. As far as I understand lattice gauge theory, a meaningful continuous theory is defined only if there is second order phase transition. This is because only in a second order phase transition the correlations functions of the discrete theory diverge in such a way that they give finite correlations functions in the continuum limit. If there is no second order phase transition, a continuum limit may still exist, but all the correlation functions in the continuum limit are trivial or diverge. As far as I understood the old dynamical triangulations program, this was, indeed, the main issue. That is, after having identified an interesting phase transition in the discrete model, the issue was to prove that it is second order. Now, I do not see this is the present approach. You focus on the transition between the crumpled and the smooth phase but you do not discuss if it is second order or not. Have you solved the problem? Circumvented it? Understood that it was a false problem?
  2. The sum over triangulations you study can be viewed as a Feynman sum over geometries, written in the time gauge, and weighted with the classical action. If this defines a consistent quantum theory, its classical limit is the field theory defined by the Einstein–Hilbert action for geometries in the time gauge. Well, this is not general relativity: one equation is missing. For the same reason that if you pose $A_0 = 0$ in the Maxwell action, you lose the equation $divE = 0$. The equation you loose is precisely the Hamiltonian constraint, which in a sense is where the core of the story is. It is well known, indeed, that to implement this key equation one has, so to speak, to integrate over all lapse functions, or all proper times. And, as far as I understand, you do not do that. If so, the theory you are studying is not general relativity, the theory that works so well empirically. What am I missing?
  - **A - J. Ambjørn *et al*.**:
    1. In the framework of statistical mechanics you refer to one imagining a critical surface where the correlation length is infinite. If you are not on the critical surface for some value of the coupling constants you have to fine-tune the coupling constants such that you approach that surface. An example is the fine-tuning of the temperature in magnetic systems to a second order phase transition between a magnetized phase and a phase where the magnetization is zero. The spin–spin correlation length will diverge and the long distance physics of the spin system can for a number of materials be described by a three-component ϕ^4 theory close to the Fisher–Wilson fixed point. The distance from the critical surface as related to the mass of the particle if we use field-theoretical language. For continuum theories with a mass gap, and this also includes theories like non-Abelian gauge theories, we will always have to stay a little away from the critical surface in a precise fine-tuned way such

that the lattice spacing times the correlation length measured in lattice units is constant (equal to the inverse physical mass). This is the way to recover the continuum limit of the lattice theory.

However, suppose we are already at the critical surface. As an explicit example consider a free massless scalar particle (in Euclidean spacetime). Put it on the lattice in the simplest possible way. The propagator is now

$$G(p) = \frac{1}{\sin^2(pa/2)} \to \frac{1}{a^2\ p^2} \quad \text{for } a \to 0.$$

Except for a prefactor, we have directly the continuum propagator when the lattice spacing $a \to 0$. No fine-tuning is needed. In other theories where massless particles can be put on the lattice in a natural way which does not generate a mass term, neither perturbatively nor non-perturbatively, we have the same situation. An example is four-dimensional lattice $U(1)$ theory. For the (lattice) coupling constant above the critical value, one has a confining lattice theory without a continuum limit, but for the coupling constant below the critical value one is automatically in the Coulomb phase where a trivial rescaling of the lattice spacing and fields leads to the continuum free field theory of the photon.

In CDT we seem to have the same situation: for some range of the bare gravitational coupling constant we obtain a lattice theory with no continuum limit. For another range of the gravitational coupling constant we obtain a continuum limit (to the extent one can trust the computer simulations) just by taking the lattice spacing to zero. If one wants to use the analogy with the $U(1)$ theory mentioned above, the interpretation would be that the graviton has been incorporated in a natural way which does not lead to a mass, so one is staying on the critical surface for a range of coupling constants. It is probably a good thing.

In the "old" Euclidean DT the situation was the following: for almost all values of the gravitational couplings constant the computer simulations showed just a lattice theory without any obvious continuum limit. Only near the phase transition between a pathologically crumpled phase and an equally pathologically "stretched" phase (where the geometry degenerated to so-called branched polymers) was there a chance to obtain something which was not a lattice artifact. Unfortunately the phase transition turned out to be a (weak) first order transition and the separation between the two phases would be sharp with increasing spacetime volume. Had it been a second order transition one could have hoped it would have been possible to define a continuum limit, in particular that there was a divergent correlation length associated with the transition. The situation in CDT is simpler and potentially more healthy.

2. There seems to be a misunderstanding here. No gauge fixing is performed since we pretend to sum over geometries. Whether we are covering the configuration space of geometries uniformly can be debated, but in this respect we appeal to universality, so if we at all find a critical point or critical coupling constant region, one can hope it is the right one. However, this has in the end to be settled by looking at the results we obtain.

Now the quantity we calculate is (as explained after eq. (18.4)) a special quantity where the two boundaries are separated a geodesic distance T. As remarked after (18.4) this is a diffeomorphism- invariant concept and there is no need to integrate over T. It is not a quantity usually considered in Quantum Gravity, which is maybe a pity, since it is much closer to the conventional idea of a "propagator" in field theory than what is usually considered. For a beautiful description of how it can be used to calculate more conventional amplitudes in the case of two-dimensional Quantum Gravity we refer to the article by Kawai *et al*. (*Nucl. Phys*. B474: 512–528, 1996).

- **Q - D. Oriti - to J. Ambjørn *et al*.**:

1. Regarding the problems you mention in defining and dealing with different topologies, I agree with you that in any naive definition of a sum over topologies, as for example in matrix models or group field theories, non-trivial topologies are likely to vastly dominate the sum, and that the sum itself is likely to be a divergent one. However, the above examples show that, a definition of the sum as a perturbative series expansion being given, it is a model dependent question whether this sum can be given a non-perturbative meaning or a physical interpretation that allows one to use it in spite of its divergence. In particular the example of the Borel summable modification of the Ponzano–Regge group field theory shows that this is not at all impossible at least, even though there is no physical understanding yet of the modification performed in that case to achieve summability. More generally, in models where extra data are present on top of the combinatorial ones, like indeed in group field theories, it is a possibility, admittedly not yet realised, that models can be constructed in which non- trivial topologies are suppressed or confined to the ultramicroscopic domain, even if present in the perturbative expansion of the partition function. Also, from the classical point of view, I don't understand what the motivation for escluding topology changing configurations would be, given that in a first order formalism geometries that are degenerate at isolated points are necessarily, or at least naturally, included in a path integral quantization, and this sort of degeneracy is enough to prevent causal pathologies in presence of topology change. Can you please give me your opinion on this?

2. The above applies also to your construction of causal dynamical triangulations, with the removal of baby universe configurations. Wouldn't it be more natural or satisfactorily to include such problematic configuration but having them confined to small (i.e. Planck size) volumes? Of course, this would require the presence of extra degrees of freedom on top of the combinatorial ones, for example volume information associated to each d-simplex (in d dimensions). Have you considered such possibility?
3. How would you modify your CDT construction to remove the gauge fixing correponding to the preferred foliation in T, assuming that it is indeed, as it better be, a gauge fixing? Is there already work going on in this direction?

– **A - J. Ambjørn *et al*.**:

1. Of course one could image a definition that suppresses topologies. The simplest mechanism is to leave them out by hand as we have suggested. Topology changes do not appear very natural in a metric formulation of Einstein's classical general relativity theory. That is one motivation for leaving them out. If one allows topology changes and then wants to suppress them, one has to have a physically motivated mechanism for doing it. Such a mechanism might exist, I am just not aware of one. The explicit example mentioned from 3d Ponzano–Regge group field theory is in my opinion well understood and explicitly non-physical. In fact it is in spirit very analogous to well studied examples in two-dimensional Quantum Gravity where one has been able to perform the summation over topologies and even obtain explicit analytical results. How does it work in 2d? You take 2d Euclidean Quantum Gravity, defined by some regularization, like dynamical triangulations, and you try to sum over all topologies. You discover that the sum is factorial divergent in the genus of the 2d manifold, which is not surprising. Most perturbative expansions are. No obvious way suggests itself for a summation of the series since the coefficients are all positive: it is not Borel summable. There is a physical reason for the coefficients being all positive: they are related to the counting of different geometries of a fixed topology. This number grows factorially with the genus of the topology. Now one could get the marvelous idea to modify this counting of positive numbers by introducing a new "geometric unit" apart from the triangles: the square (say), but with negative weight. At this point we have no real idea what we are doing, but let us be courageous and blindly proceed. It is worth emphasizing the picture in terms of the so-called matrix models which implement the explicit gluing of triangles, and the large-N expansion which gives the genus expansion. Starting with the gluing of the triangles we had an matrix model where the action was unbounded from below. Adding the squares produces an action which *is* bounded from below, and therefore well defined beyond perturbation theory, but the boundedness

of the action is directly linked to the fact that squares, viewed as geometric units, have negative weights. If we had included the squares with proper, understandable weight, the action would still, after the addition of the squares, have been unbounded from below. The bounded action allows us to define a non-perturbative sum over all genus. It is seen that the construction here is word by word the same as the one used in the 3d Ponzano–Regge group field theory referred to by Oriti. In the 2d case one can complete the analysis: it turns out that this contrived model *has* a decent interpretation: it represents a (2,5) minimal conformal field theory coupled to two-dimensional Quantum Gravity. The main point is that new non-geometric, non-unitary degrees of freedom have been introduced in the model and they totally dominate the high genus part of it. In this way one has tamed to topology, burying it in the dominant interactions of a non-unitary theory. It has (until now) proved impossible to repeat the same trick with unitary models couple to 2d Quantum Gravity for the simple reason that integrating out unitary matter always gives positive weight factors in Euclidean space. The situation in two dimensions is infinitely simpler than in three dimensions, not to speak about four dimensions.

To summarize: the suggestions for summation over topologies I have seen so far have in my opinion no chance to work. Of course this does not rule out that one day one will (1) understand that one should really sum over topologies and (2) understand how to do it.

2. Concerning the inclusion of baby universes or exclusion of baby universes, it is difficult to see the motivation for including them, but confining them to be of Planck size unless there is a natural mechanism which confines them to this size. Anyway, if they were included that way one could presumably just integrate them out again when one addresses physics at a slightly larger scale.

Actually one can address this question in a precise way in two-dimensional Quantum Gravity. As we have shown: if you start out with CDT and then allow baby universes (of all sizes), then you recover standard Euclidean two-dimensional Quantum Gravity (as described by dynamical triangulations, matrix models or Liouville field theory). Conversely, if you start out with Euclidean two-dimensional Quantum Gravity and chop away baby universes you obtain CDT. From the theory of Euclidean two-dimensional Quantum Gravity you know the precise distribution of baby universe volumes (it is governed by the so-called susceptibility (or entropy) exponent γ). The distribution is very strongly peaked at baby universes of cut-off scale, which one in this model would identify with the Planck-scale. So the model almost satisfies your requirement of having the baby universes confined to the Planck-scale simply by entropy. However, the rare larger baby universes are not unimportant, but it will take us too far to go into a discussion of the details. The point to emphasize in the present context is that the two-dimensional CDT model can be viewed as obtained from the "full" Euclidean model by integrating out the baby universes. Now the main motivation for introducing the CDT model in higher dimensional Quantum Gravity was the observation that allowing all geometries (of a fixed topology) led to the dominance of very degenerate geometries in higher than two dimensions, which in one way or another could be related to baby universes. We were therefore looking for a general principle to get rid of them.

- **Q - D. Oriti - to R. Williams**:
  Can you please clarify to me the rationale behind the search for the Regge calculus analogue of the diffeomorphism symmetry of continuum GR? I mean: Regge calculus being defined on simplicial complexes, as such there would possibly be no notion of diffeos at all, as diffeos are indeed maps between smooth manifolds by definition. This seems to me very different from the search for analogues of the Bianchi identities, that are a statement about the spacetime curvature and therefore admit an intrinsic definition on the simplicial complex, once the discrete analogue of the curvature has been defined on the complex. On the other hand it would seem to me that the search for diffeos in Regge calculus uses necessarily an embedding of the simplicial complex, and consequently of the geometric data assigned to it, in some continuum manifold, in which diffeos are indeed defined. This notion of diffeos would then not be "intrinsic" to the simplicial complex of Regge calculus alone, but it would require extra information about the embedding. Can you please clarify this procedure, if the above is correct, or point out where I am misunderstanding the situation? For example, is there a notion of an "analogue" of diffeos that is fully "intrinsic" to the simplicial complex, that somehow reduces to the usual continuum index continuum approximation notion in some limit?
  - **A - R. Williams**:
    The answer to the question depends on how you try to define the analogue of diffeomorphisms for a simplicial space. If they are transformations of the edge-lengths which leave the action invariant, there is no problem with embedding. However, if they are transformations of the edge-lengths which leave the geometry invariant (which seems closer to the continuum definition), then you are correct that there could be problems with embedding. Avoiding the notion of embedding, we see that we can really only define diffeomorphisms for flat space. Hartle has shown that one may define approximate diffeomorphisms in directions in which the action is approximately stationary. In the continuum limit, these diffeomorphisms become exact.

This is discussed very clearly in Section 5 of J. B. Hartle: "Simplicial Minisuperspace I: General Discussion", *J. Math. Phys*. 26 (1985) 804–814.

- **Q - D. Oriti - to R. Williams**:
Just for the sake of clarity, let me clarify my doubt a bit more. If I take a smooth manifold, I can define diffeos as smooth maps between points in the manifold, right? This definition does not need any notion of geometry, action, etc., I think. In a piecewise linear or simplicial space, is there an analogue notion of "diffeos", i.e. maps between points in the space, that does not require any extra information, like geometry or an embedding into the continuum, i.e. an "intrinsic" analogue of diffeos? Also, I am a bit puzzled, because I have always thought of edge lengths in Regge calculus as "spacetime distances", i.e. as the discrete analogue of integrals along geodesics of the line element (possibly, better as the sup or inf of such distances, according to whether the geodesic is timelike or spacelike). As such they would simply be invariant under diffeos in the continuum embedding, they would simply not transform at all under them. What is the interpretation of them that you are using and that is used in defining diffeos?
  - **A - R. M. Williams**:
  If you want an analogue of continuum diffeomorphisms as smooth transformations between points in the manifold (with no notion of preserving geometry or action), then one can define piecewise diffeomorphisms as one-to-one invertible maps of the simplicial space into itself, which are smooth on each simplex (e.g. relabelling vertices, or smooth diffeomorphisms of the interiors of simplices). For a general curved simplicial geometry, one expects diffeomorphisms in this sense to leave the edge lengths unchanged or change them only according to a trivial relabelling of the vertices (I am quoting Hartle here).

    As for the definition of edge lengths, it depends how one arrives at the simplicial complex. If it arises from the triangulation of a continuum manifold, then I would define the edge lengths by geodesic distances between vertices in the manifold. But if the complex is a "given", with no notion of an embedding, then the edge lengths are just "given" too and I do not see that one has a notion of invariant distance.
- **Q - D. Oriti - to R. Gambini and J. Pullin**:
What is the exact relation of your "consistent discretization" scheme with traditional Regge calculus? I understand from your work that your scheme allows for a definition of a canonical (Hamiltonian) formulation of Regge calculus, that had proven difficult to achieve in the usual formalism. But what are similarities and differences, advantages and disadvantages, with respect to the Lagrangian setting?

– **A - R. Gambini and J. Pullin**:
Indeed, our method of consistent discretizations using as starting point the Regge action yields a well defined canonical theory for Regge calculus. The formulation is equivalent to the original one classically (apart from some restrictions on the triangulations that are required to have a canonical formulation that has the same appearance at each point of the manifold). It should be pointed out that the formulation is canonical but not Hamiltonian, the evolution is given by a discrete canonical transformation instead of a continuous time evolution generated by a Hamiltonian. This is reasonable since Regge calculus discretizes both space and time. An interesting advantage of the Hamiltonian formulation is that since one naturally restricts the type of discretizations considered one eliminates the problem of “spikes" and other pathological structures that may develop in Regge calculus. The disadvantages include the fact that some of the edges that play the role of Lagrange multipliers get determined by the evolution equations through complicated equations that may yield undesired behaviors (like having complex solutions). In this context the only way of controlling the behavior of these variables is to choose judiciously the initial data. This type of difficulty has led to the construction of a special version of consistent discretizations called “uniform discretizations” where these problems are eliminated. It might be attractive to pursue Regge calculus with this new approach.

• **Q - L. Crane - to J. Henson**:
1. It seems one could equally well use a poset to approximate a Lorentzian manifold in any other dimension than 4. Is there an easy way to put conditions on a causet so that its dimension doesn’t vary from region to region?
2. In mathematics there are two very different notions of dimension, one topological and the other measure theoretic. The best known measure theoretical definition is Hausdorff dimension, which applies to fractals. Do you know of any approach to differentiating these for causets?

– **A - J. Henson**:
1. Yes, it is true that causets exist which correspond to manifolds of other dimensions. It is possible is estimate the dimension of the approximating manifold, given the causal set alone, and do as the question suggests. By different regions one might mean different intervals in the causal set, and the condition that these dimension estimators approximately match, and give the same value (most interestingly 4) in all regions, is a necessary condition for a causal set to be “manifoldlike”.
2. At the discrete level, the causal set does not retain the topological or metric structures of the continuum, which arise at an effective level. So, the question of comparison only makes sense for causal sets where a continuum

approximation exists. Nonetheless, you might imagine extending the definition of the continuum approximation to some fractals, by carrying out sprinklings of these structures, and then ask how the new dimension estimators compare to the more standard estimators of Hausdorff and Lebesque. It seems plausible that the causal set estimators we have would be more similar to the measure theoretic dimensions like the Hausdorff or Minkowski–Bouligand dimension (since the Lebesque measure is, I think, invariant under homeomorphisms, which seems at odds with scale-dependent dimension estimators), but this is not known (a question that is under investigation by David Meyer).

Part V

Effective models and Quantum Gravity phenomenology

22

Quantum Gravity phenomenology

G. AMELINO-CAMELIA

22.1 The "Quantum Gravity problem", as seen by a phenomenologist

The "Quantum Gravity problem" has been discussed for more than 70 years [1] assuming that no guidance could be obtained from experiments. But of course if there is to be a science of the Quantum Gravity problem it must be treated just like any other scientific problem, seeking desperately the guidance of experimental facts, and letting those facts take the lead in the development of new concepts. We must hope this works also for the Quantum Gravity problem, or else abandon it to the appetites of philosophers.

Unfortunately it is not unlikely that experiments might never give us any clear lead toward Quantum Gravity, especially if our intuition concerning the role of the tiny Planck length ($\sim 10^{-35}$ m) in setting the magnitude of the characteristic effects of the new theory turns out to be correct. But even if the new effects were really so small we could still try to uncover experimentally some manifestations of Quantum Gravity. This is hard, and there is no guarantee of success, but we must try.

Our estimate that the Quantum Gravity corrections should be very small in low-energy experiments is based on our experience with other similar situations; in fact, we expect that the Planck scale, since it is the energy scale where the current theories appear to break down, should also govern the magnitude of Quantum Gravity corrections to the analysis of processes involving particles with energies smaller than the Planck scale. For example, in processes involving two particles both with energy E the magnitude of the new effects should be set by some power of the ratio between E and the Planck scale E_{p} ($\sim 10^{28}$ eV). Since in all cases accessible to us experimentally E/E_{p} is extremely small, this is a key challenge for Quantum Gravity phenomenology. This is a challenge which, however, can be dealt with by relying on experience with other analogous situations in physics.[1]

[1] As I emphasized elsewhere [2; 3], ongoing studies of proton stability from the grand unification perspective and early 1900s studies of Brownian motion could be characterized by a very similar challenge.

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

For this phenomenology the key concern for a long time has not been the one of development but rather the one of articulating a basic claim of existence. The results of this effort allow me to provide, in this section, robust evidence that we can really do Quantum Gravity phenomenology.

Then Section 22.2 presents a (incomplete but representative) list of effects that should be considered as candidate Quantum Gravity effects, and in Section 22.3 I briefly describe the experiments and/or observations which are being analyzed as opportunities to provide related insight.

The rest of this chapter focuses on the most studied area of Quantum Gravity phenomenology, the one that concerns the possibility of Planck-scale departures from Poincaré (Lorentz) symmetry. It starts with a small aside (Section 22.4) on doubly special relativity, which I describe (as originally proposed) as a scenario for Planck-scale physics, rather than one or another choice of formalism. And I show that the doubly special relativity idea can be falsified, a rare example of a falsifiable proposal for Planck-scale physics.

Section 22.5 may be used as a compact point of entry to the literature on the phenomenology of Planck-scale departures from Poincaré (Lorentz) symmetry. I do not give detailed accounts but I try to touch on a representative subset of the ideas the community is pursuing, and in doing so I try to show as clearly as possible how important it is to rely on some carefully tailored, commonly adopted, test theories in mapping the progress of this phenomenology.

Some closing remarks are offered in Section 22.6.

22.1.1 Quantum Gravity phenomenology exists

Task number one for any phenomenology (usually an easy task but a challenging one here) is to show that effects of the type that could be expected from the relevant class of theories could be seen. The key source of pride for Quantum Gravity phenomenologists comes from the fact that, over the past few years, and over a time that indeed spanned only a handful of years, we managed to change the perception of Quantum Gravity research from the traditional "no help from experiments possible" to the present intuition, shared by most workers in the field, that these effects could be seen. We might need some luck actually to see them, but clearly it is not implausible. There is a legitimate phenomenology to be developed here.

Once task one is accomplished it is important to show that the type of observations that are doable not only provide opportunities to luckily stumble upon a manifestation of the new theory, but actually the data could be used to falsify candidate theories. This task two clearly requires much more of task one, both at the level of our understanding of the theories and for what concerns the quality of the data and their phenomenological analysis.

Tasks one and two really are preparatory work. The "fun" begins immediately after these first two tasks, when the relevant data are actually collected, possible departures from conventional theories are looked for, and the theories that could be falsified by those data are falsified.

22.1.2 Task one accomplished: some effects introduced genuinely at the Planck scale could be seen

Over the past few years several authors have shown in different ways and for different candidate Planck-scale effects that, in spite of the horrifying smallness of these effects, some classes of doable experiments and observations could see the effects. Just to make absolutely clear the fact that effects genuinely introduced at the Planck scale could be seen, let me exhibit one very clear illustrative example.

The Planck-scale effect I consider here is codified by the following energy-momentum (dispersion) relation

$$m^2 \simeq E^2 - \vec{p}^2 + \eta \vec{p}^2 \left(\frac{E^2}{E_p^2} \right), \tag{22.1}$$

where E_p denotes again the Planck scale and η is a phenomenological parameter. This is a good choice because convincing the reader that I am dealing with an effect introduced genuinely at the Planck scale is in this case effortless. It is in fact well known (see, e.g., Ref. [4]) that this type of E_p^{-2} correction to the dispersion relation can result from discretization of spacetime on a lattice with E_p^{-1} lattice spacing.[2]

If such a modified dispersion relation is part of a framework where the laws of energy-momentum conservation are unchanged one easily finds [5; 6; 7; 8] significant implications for the cosmic-ray spectrum. In fact, the "GZK cutoff", a key expected feature of the cosmic-ray spectrum, is essentially given by the threshold energy for cosmic-ray protons to produce pions in collisions with CMBR photons. In the evaluation of the threshold energy for $p+\gamma_{\mathrm{CMBR}} \rightarrow p+\pi$ the correction term $\eta \vec{p}^2 E^2/E_p^2$ of (22.1) can be very significant. Whereas the classical-spacetime prediction for the GZK cutoff is around 5.10^{19} eV, at those energies the Planck-scale

[2] The idea of a rigid lattice description of spacetime is not really one of the most advanced for Quantum Gravity research, but this consideration is irrelevant for task one: in order to get this phenomenology started we first must establish that the sensitivities we have are sufficient for effects as small as typically obtained from introducing structure at the Planck scale. The smallness of the effect in (22.1) is clearly representative of the type of magnitude that Quantum Gravity effects are expected to have, and the fact that it can also be obtained from a lattice with E_p^{-1} spacing confirms this point. It is at a later stage of the development of this phenomenology, much beyond task one, that we should become concerned with testing "plausible Quantum Gravity models" (whatever that means). Still it is noteworthy that, as discussed in some detail in Section 22.3, some modern Quantum Gravity-research ideas, such as the one of spacetime noncommutativity, appear to give rise to the same type of effect, and actually in some cases one is led to considering effects similar to (22.1) but with a weaker (and therefore more testable) Planck-scale correction, going like E_p^{-1} rather than E_p^{-2}.

correction to the threshold turns out [5; 6; 7; 8] to be of the order of $\eta E^4/(\epsilon E_{\rm p}^2)$, where ϵ is the typical CMBR-photon energy. For positive values of η, even somewhat smaller[3] than 1, this amounts to an observably large shift of the threshold energy, which should easily be seen (or excluded) once the relevant portion of the cosmic-ray spectrum becomes better known, with observatories such as the Pierre Auger Observatory.

Of course, the same effect is present and is even more significant if, instead of an $E_{\rm p}^{-2}$ correction, one introduces in the dispersion relation a correction of $E_{\rm p}^{-1}$ type.

22.1.3 Concerning task two

Task one is settled. Arguments such as the one offered in the previous subsection clearly show that this phenomenology has a right to existence. We do have at least a chance (perhaps slim, but this is not the point here) to see Planck-scale effects, and if we ever do see one such effect it will be wonderful. But a phenomenology should also be valuable when it does not find the effects it looks for, by setting limits on (and in some cases ruling out) corresponding theories. Have we proven that Quantum Gravity phenomenology can rule out Planck-scale theories?

Of course (also see later) the phenomenology will be based on some "test theories" and the parameters of the test theories will be increasingly constrained as data become available. But beyond the level of test theories there is the truly sought level of "theories", models which are not merely introduced (as is the case of test theories) as a language used in mapping the progress of experimental limits on some effects, but rather models which are originally motivated by some ideas for the solution of the Quantum Gravity problem. And in order to falsify such a theory we need to prove experimentally the absence of an effect which has been rigorously established to be a necessary consequence of the theory. But the theories used in Quantum Gravity research are so complex that we can rarely really establish that a given effect is necessarily present in the theory. What usually happens is that we find some "theoretical evidence" for the effect in a given Quantum Gravity theory and then we do the phenomenology of that effect using some test theories. The link from theory to effect is too weak to be used in reverse: we are usually not able to say that the absence of the effect really amounts to ruling out the theory.

Think, for example, of loop quantum gravity. Because of the "classical-limit problem" at present one is never really able to use that theory to provide a definite prediction for an effect to be looked for by experimentalists. And for string theory the situation might be worse, at least in the sense that one might not even be able to hope for better things for the future: at present it is not clear whether string

[3] Of course the Quantum Gravity intuition for η is $\eta \sim 1$.

theory is in principle able to make any definite predictions, since the formalism is so flexible, so capable of saying anything, that it is feared it will amount basically to saying nothing.

Usually in physics the demand that a theory be falsifiable is of course the first and most important requirement, but it is also usually not a tough one: any reasonable, however naive, concept of theory should give rise to a falsifiable theory. Two known causes of failure for falsifiability are the lack of logical consistency (so basically the candidate theory was not a theory after all, since the piece of mathematics introduced did not combine to produce a logically consistent overall structure) or the presence of unlimited flexibility, i.e. the scenario feared for string theory. Of course a theory that in principle is falsifiable but presents us with practically unsurmountable computational challenges (which might be the case of loop quantum gravity, if a satisfactory description of the classical limit does not become available) is for our purposes not falsifiable.

So concerning task two the situation does not look very healthy, but the problem resides on the theory side, not the phenomenology side. If they give us definite predictions we will do our best to honour them by killing their theories (which I still think should be the healthiest attitude to be adopted when doing phenomenology work). Presently, for most of the fashionable theories, no such honour can be given.

If indeed, at least for now, we cannot falsify loop quantum gravity and string theory, can we at least falsify some other theory used in Quantum Gravity research? I believe it is extremely important for Quantum Gravity phenomenology to find one such example. If we do find a first example then we can legitimately hope that the falsifiability of more and more theories will gradually be achieved. And because of the importance I give to this objective I have invested a lot of effort in the study of one of the formalisms used in Quantum Gravity research, that of the κ-Minkowski noncommutative spacetime. I do not necessarily "favour" this formalism, but I have the intuition that it should be falsifiable. This intuition must, however, still find full support in the analysis. The logical consistency of theories in κ-Minkowski has still not really been shown, at least not to the level desired by physicists, and we are presently unable to do many computations in this framework, which may be a manifestation of a serious unsurmountable challenge for computations; but at present it is still legitimate to hope (in my view rather reasonable to expect) that we will soon be able to do these computations. There have, for many years, been results on κ-Minkowski providing weak links (the usual weak links from theory to effects in Quantum Gravity research) to effects such as deformed dispersion relations, deformed energy-momentum-conservation laws, and deformed boost transformations. Some of these links have become gradually somewhat more robust. I expect that progress in this direction will accelerate thanks to the fact that we now know [9] that (a suitable adaptation of) the Noether theorem is applicable to theories in this noncommutative spacetime.

22.1.4 Neutrinos and task three

With task one completed and some promising partial results concerning task two, we certainly have enough encouragement to get started with actually developing test theories and looking for suitable observational/experimental contexts. Indeed this has been done for a few years now with great dedication by several research groups around the world. The presently available literature indicates that, besides the mentioned cosmic-ray opportunity, opportunities to see some corresponding candidate Planck-scale effects (and perhaps one day falsify theories) are found in several other contexts, including the study of gamma rays [10; 11; 12], studies of the neutral-kaon system [13; 14], and in modern interferometry [15; 16; 17]. For all of these possibilities there is at this point a quite sizeable literature, so I do not need to stress their importance here. I do find it appropriate, however, to spend a few words on a "new entry": planned neutrino observatories, such as ICECUBE, are likely to be very valuable. This had already been timidly suggested in a few earlier papers [18; 19; 20] and should now gain some momentum in light of the analysis reported in Ref. [21] (also see Refs. [22; 23]), which proposes a definite and apparently doable programme of studies.

A key reason for interest in these neutrino studies is the possibility of using them in combination with gamma-ray studies to seek evidence of a spin dependence of the way in which conjectured quantum properties of spacetime affect particle propagation. And, even assuming that there is no such spin dependence (so that gamma rays and neutrinos could serve exactly the same purposes), neutrinos might well be our best weapon for the study of certain candidate effects. This is because it is actually easier to detect high-energy neutrinos (at or above 10^{14} eV), rather than low-energy ones, whereas it is expected that high-energy gamma rays (starting at energies of a few TeV) will be absorbed by soft photons in the cosmic background. So neutrinos will effectively extend the energy range accessible to certain classes of studies, and energy is obviously a key factor for the sensitivity of Quantum Gravity phenomenology analyses.

22.2 Concerning Quantum Gravity effects and the status of Quantum Gravity theories

So far I have only built the case for the right to existence of Quantum Gravity phenomenology. I did mention some observations/experiments that may be used in this phenomenology, but without discussing in detail the type of effects that one could look for. It is actually not so obvious how to identify candidate Quantum Gravity effects. Analogous situations in other areas of physics are usually such that there are a few new theories which have started to earn our trust by successfully describing some otherwise unexplained data, and then often we let those theories guide us toward new effects that should be looked for. The theories we have for Quantum Gravity, in spite of all their truly remarkable mathematical beauty, and their extraordinary contribution to the investigation of the conceptual sides of the Quantum Gravity problem, cannot (yet) claim any success in the experimental realm. Moreover, even if we wanted to used them as guidance for experiments the complexity of these theories proves to be a forbidable obstruction. What we can do with these theories (and we must be content with it since we do not have many alternatives) is to look at their general structure and use this as a source of intuition for the proposal of a few candidate effects.

A similar type of path toward the identification of some candidate Quantum Gravity effects is the one based on the analysis of the general structure of the Quantum Gravity problem itself. It happens to be the case that, by looking at the type of presently-unanswered questions for which Quantum Gravity is being sought, one is automatically led to consider a few candidate effects.

Of course these ideas suggested from our perception of the structure of the Quantum Gravity problem and from our analysis of the general structure of some proposed Quantum Gravity theories could well turn out to be completely off the mark, but it still makes sense to investigate these ideas.

22.2.1 Planck-scale departures from classical spacetime symmetries

From the general structure of the Quantum Gravity problem, which clearly provides at least some encouragement to considering discretized (or otherwise "quantized") spacetimes, one finds encouragement for considering departures from classical spacetime symmetries. Consider, for example, the Minkowski limit, the one described by the classical Minkowski spacetime in current theories. There is a duality one-to-one relation between the classical Minkowski spacetime and the classical (Lie-) algebra of Poincaré symmetry. Poincaré transformations are smooth arbitrary-magnitude classical transformations and it is rather obvious that they should be put under scrutiny [24] if the classical description of spacetime is replaced by a quantized/discretized one.

One possibility that of course has been considered in detail is the one of some symmetry-breaking mechanism affecting Poincaré/Lorentz symmetry. An alternative, which I advocated a few years ago [25; 26], is the one of a "spacetime quantization" which deforms but does not break some classical spacetime symmetries.

22.2.2 Planck-scale departures from CPT symmetry

Perhaps the most intelligible evidence of a Planck-scale effect would be a violation of CPT symmetry. CPT symmetry is in fact protected by a theorem in our current (Minkowski-limit) theories, mainly as a result of locality and Poincaré symmetry. The fact that the structure of the Quantum Gravity problem invites us to consider spacetimes with some element of nonlocality and/or departures from Poincaré symmetry clearly opens a window of opportunity for Planck-scale violations of CPT symmetry.

22.2.3 Distance fuzziness and spacetime foam

The fact that the structure of the Quantum Gravity problem suggests that the classical description of spacetime should give way to a nonclassical one at scales of order of the Planck scale has been used extensively as a source of inspiration concerning the proper choice of formalism for the solution of the Quantum Gravity problem, but for a long time (decades) it had not inspired ideas relevant for phenomenology. The description that came closer to a physical intuition for the effects induced by spacetime nonclassicality is Wheeler's "spacetime foam", which however does not amount to a definition (at least not a scientific/operative definition). A few years ago I proposed a physical/operative definition of (at least one aspect of) spacetime fuzziness/foam, which makes direct reference to interferometry. According to this definition, the fuzziness/foaminess of a spacetime is established on the basis of an analysis of strain noise in interferometers set up in that spacetime. In achieving their remarkable accuracy modern interferometers must deal with several classical-physics strain noise sources (e.g. thermal and seismic effects induce fluctuations in the relative positions of the test masses). And, importantly, strain noise sources associated with effects due to ordinary Quantum Mechanics are also significant for modern interferometers (the combined minimization of *photon shot noise* and *radiation pressure noise* leads to a noise source which originates from ordinary Quantum Mechanics). The operative definition of fuzzy/foamy spacetime which I advocate characterizes the corresponding Quantum Gravity effects as an additional source of strain noise. A theory in which the concept of distance is fundamentally fuzzy in this operative sense would be such that the read-out of an interferometer would still be noisy (because of Quantum Gravity effects) even in the idealized limit in which all classical-physics and ordinary-quantum-mechanics noise sources are completely eliminated.

22.2.4 Decoherence

For approaches to the Quantum Gravity problem which assume that, in merging with General Relativity, Quantum Mechanics should revise one of the most popular effects is decoherence. This may be also motivated using heuristic arguments, based mainly on quantum field theory in curved spacetimes, which suggest that black holes radiate thermally, with an associated "information loss problem".

22.2.5 Planck-scale departures from the equivalence principle

Various perspectives on the Quantum Gravity problem appear to suggest departures from one or another (stronger or weaker) form of the equivalence principle. For brevity let me just summarize here my preferred argument, which is based on the observation that locality is a key ingredient of the present formulation of the equivalence principle. In fact, the equivalence principle ensures that (for the same initial conditions) two point particles would go on the same geodesic independently of their mass. But it is well established that this is not applicable to extended bodies, and presumably also not applicable to "delocalized point particles" (point particles whose position is affected by uncontrolled uncertainties). If spacetime structure is such to induce an irreducible limit on the localization of particles, it would seem natural to expect some departures from the equivalence principle.

22.2.6 Critical-dimension superstring theory

The most popular realization of string theory, with the adoption of supersymmetry and the choice of working in a "critical" number of spacetime dimensions, has given a very significant contribution to the conceptual aspects of the debate on Quantum Gravity, perhaps most notably the fact that, indeed thanks to research on string theory, we now know that Quantum Gravity might well be a perturbatively renormalizable theory (whereas this was once thought to be impossible). But for the prediction of physical effects string theory has not proven (yet?) to be rich. In spite of all the noteworthy mathematical structure that is needed for the analysis of string theory, from a wider perspective this is the approach that by construction assumes that the solution to the Quantum Gravity problem should bring about a rather limited amount of novelty. In particular, string theory is still introduced in a classical Minkowski spacetime and it is still a genuinely quantum-mechanical theory. None of the effects possibly due to spacetime quantization is therefore necessarily expected and all the departures-from-quantum-mechanics effects, like decoherence effects, are also not expected.

But on the other hand, as mentioned, string theory is turning out to be a remarkably flexible formalism and, therefore, while one can structure things in such a way that nothing interestingly new happens, one can also mould the formalism in such a way as to have some striking new effects,[4] and effects that fit within some intuitions concerning the Quantum Gravity problem. In particular there is a known scheme for having violations of the equivalence principle [27], and by providing a vacuum expectation value for a relevant antisymmetric tensor one can give rise [28] to departures from Poincaré symmetry (together with the emergence of an effective spacetime noncommutativity).

22.2.7 Loop quantum gravity

The only other approach with contributions to the conceptual debate on the Quantum Gravity problem of significance comparable to the ones of the string-theory approach is loop quantum gravity. In particular, it is thanks to work on loop quantum gravity that we now know that Quantum Gravity might fully preserve the diffeomorphism invariance of General Relativity (whereas this was once thought to be impossible). But loop quantum gravity, while excelling in the conceptual arena, has its difficulties in providing predictions to phenomenologists. While string theory may be perceived as frustratingly flexible, one might perhaps say that at the present stage of development loop quantum gravity appears not to have even the needed room to maneuver it down to the mundane arena of corrections to General Relativity and corrections to the Standard Model of particle physics. As a result of the much debated "classical-limit problem", in a certain sense loop quantum gravity provides a candidate description of everything but does not provide an explicit description of anything. One may attempt, however (and several groups have indeed attempted to do this), to infer from the general structure of the theory some ideas for candidate loop-quantum-gravity effects. In particular, several studies [12; 29] have argued that the type of discretization of spacetime observables usually attributed to loop quantum gravity could be responsible for Planck-scale departures from Lorentz symmetry. This hypothesis also finds encouragement [30] in light of the role apparently played by noncommutative geometry in the description of certain aspects of the theory.

Of course, as long as the "classical-limit problem" is not solved, the evidence of departures from Lorentz symmetry in (the Minkowski limit [31] of) loop quantum gravity must be considered weak, and any attempt to give a concrete formulation of these effects will have to rely at one point or another on heuristics. This remains a very valuable exercise for Quantum Gravity phenomenology, since it gives us ideas on effects that are worth looking for, but clearly at present phenomenologists are not given any chance of falsifying loop quantum gravity.

[4] One of the most noteworthy possibilities is the one of "large extra dimensions". This gives rise to a peculiar brand of Quantum Gravity phenomenology, which is not governed by the Planck scale. In this chapter I intend to focus on Planck-scale effects.

From the phenomenology perspective there is more than the Lorentz-symmetry issue at stake in the "classical-limit problem": it is not unlikely that structures relevant for CPT symmetry and the equivalence principle are also present, and loop quantum gravity could be a natural context where a physical intuition for spacetime foam could be developed.

22.2.8 Approaches based on noncommutative geometry

Noncommutative spacetimes so far have been considered has opportunities to look at specific aspects of the Quantum Gravity problem (whereas string theory and loop quantum gravity attempt to provide a full solution). It is perhaps fair to say that the most significant findings emerged in attempts to describe the Minkowski limit [31] of Quantum Gravity. One might say that these studies look at one half of the Quantum Gravity problem, the quantum-spacetime aspects. Because of the double role of the gravitational field, which in some ways is just like another field given in spacetime but also governs the structure of spacetime, in Quantum Gravity research one ends up considering two types of quantization: some sort of quantization of gravitational interactions and some sort of quantization of spacetime structure. At present one might say that only within the loop quantum gravity approach are we truly exploring both aspects of the problem. String theory, as long as it is formulated in a classical (background) spacetime, focuses in a sense on the quantization of the gravitational interaction, and sets aside the possible "quantization" of spacetime.[5] And the reverse is true of mainstream research on spacetime noncommutativity, which provides a way to quantize spacetime, but, at least for this early stage of development, does not provide a description of gravitational interactions.

The analysis of noncommutative deformations of Minkowski spacetime has provided some intuition for what could be the fate of (Minkowski-limit/Poincaré) symmetries at the Planck scale. And also valuable for the development of Quantum Gravity phenomenology is the fact that in some cases, such as the κ-Minkowski noncommutative spacetime, it is reasonable to hope that these studies will soon provide truly falsifiable predictions.

Unfortunately spacetime fuzziness, which is the primary motivation for most researchers to consider noncommutativity, frustratingly remains only vaguely characterized in current research on noncommutative spacetimes; certainly not characterized with the sharpness needed for phenomenology.

[5] As in noncommutative geometry, one hopes one day to obtain also the quantization of the interaction, by introducing a suitable noncommutative geometrodynamics, in approaches like string theory one may hope that the quantization of the interaction field may at advanced levels of analysis amount to spacetime quantization. Some string-theory results do encourage this hope [32; 33; 34; 35; 36] but the situation remains puzzling [37].

22.3 On the status of different areas of Quantum Gravity phenomenology

22.3.1 Planck-scale modifications of Poincaré symmetries

The most developed Quantum Gravity phenomenology research area is the one that considers the possibility of Planck-scale departures from Poincaré symmetry. I chose to treat this research area separately in a dedicated section (Section 22.5).

22.3.2 Planck-scale modifications of CPT symmetry and decoherence

The most studied opportunity to test CPT symmetry is provided by the neutral-kaon and the neutral-B systems [13; 14]. One finds that in these neutral-meson systems a Planck-scale departure from CPT symmetry could in principle be amplified. In particular, the neutral-kaon system hosts the peculiarly small mass difference between long-lived and the short-lived kaons $|M_L - M_S|/M_{L,S} \sim 7.10^{-15}$, and there are scenarios of Planck-scale CPT violation in the literature [13] in which the inverse of this small number amplifies a small (Planck-scale induced) CPT-violation effect. This in particular occurs in the most studied scenario for Planck-scale violations of CPT symmetry in the neutral-kaon system, in which the Planck-scale effects induce a difference between the terms on the diagonal of the $K^0, \bar{K}^0$ mass matrix. An analogous effect would be present in the neutral-B system but if the Planck-scale effect for the terms on the diagonal is momentum independent the best sensitivity is expected from studies of the neutral-kaon system. It is, however, not implausible [38] that the Planck-scale effects would introduce a correction to the diagonal terms of the neutral-meson mass matrix that depends on the momentum of the particle, and in this case, among the experiments currently done or planned, the best sensitivity would be obtained with the neutral-B system.

22.3.3 Distance fuzziness and spacetime foam

The phenomenology of distance fuzziness is being developed mainly in two directions: interferometry and observations of extragalactic sources.

In interferometry the debate [15; 16; 17] involves a variety of phenomenological models and different perspectives on what is the correct intuition that one should implement. It is perhaps best here to just give the simplest observation that can provide encouragement for these studies. As stressed in Subsection 22.2.3 in interferometry it is natural to look for Planck-scale contributions to the strain noise. And it is noteworthy that strain noise is natural described in terms of a function of frequency $\rho(\nu)$ (a tool for spectral analysis) that carries dimensions of Hz^{-1}. If one was to make a naive dimensional estimate of Planck scale effects one could simply pose $\rho \sim L_p/c$, which at first might seem not too encouraging since it leads to a very small estimate of ρ: $\rho \sim 10^{-44}\,\mathrm{Hz}^{-1}$. However, modern interferometers are achieving truly remarkable sensitivities, driven by their main objective of seeing classical gravity waves, and levels of ρ as small as $10^{-44}\,\mathrm{Hz}^{-1}$ are within their reach.

Another much discussed opportunity for constraining models of spacetime fuzziness is provided by the observation of extragalactic sources, such as distant quasars. Essentially it is argued [39; 40] that, given a wave description of the light observed from the source, spacetime fuzziness should introduce an uncertainty in the wave's phase that cumulates as the wave travels, and for sufficiently long propagation times this effect should scramble the wave front enough to prevent the observation of interferometric fringes. Also in this case plausible estimates suggest that, in spite of the smallness of the Planck-scale effects, thanks to the amplification provided by the long propagation times the sensitivity needed might soon be within our reach.

22.3.4 Decoherence

The development of test theories for decoherence is of course a challenging area of Quantum Gravity phenomenology, since the test theories must go beyond Quantum Mechanics. It is perhaps best if here I limit myself to directing the readers to the available dedicated reviews, such as Ref. [41]. Let me just mention that the neutral-kaon system, with its delicate balance of scales, besides taking center stage in the phenomenology of Planck-scale departures from CPT symmetry is also considered [13; 41] to be our best opportunity for laboratory studies of Planck-scale-induced decoherence.

22.3.5 Planck-scale departures from the equivalence principle

As mentioned, the Quantum Gravity problem also provides motivation to contemplate departures from the equivalence principle, and in some approaches (in particular in string theory) some structures suitable for describing departures from the equivalence principle are found. The phenomenology is very rich and in many ways goes well beyond the specific interests of Quantum Gravity research: the equivalence principle continues to be placed under careful scrutiny especially because of its central role in General Relativity. Interested readers could consider as points of entrance in the relevant literature the overall review in Ref. [42] and, more specifically for departures from the equivalence principle within the string theory approach, Ref. [27].

22.4 Aside on doubly special relativity: DSR as seen by the phenomenologist

In preparation for the next section, which focuses on the phenomenology of Planck-scale departures from Poincaré symmetry, I find it useful to provide here a short but self-contained introduction to "doubly special relativity" (DSR). This is a scenario that I proposed only a few years ago [25; 26], but is already rather extensively analyzed as an alternative to the standard scenario of Planck-scale effects that break Lorentz(/Poincaré) symmetry. As a result of this interest, at this point there are numerous attempts in the literature to review DSR research, so one might think this section could be unnecessary. However, DSR is becoming different things to different authors, and the differences are rather significant for the "phenomenologist perspective on Quantum Gravity" which I am here attempting to provide. The DSR proposal, which originally provided a physics scenario for the Planck scale, is now often identified with a certain (rather vaguely defined) mathematical framework, whether or not this mathematical framework turns out to be compatible with the DSR principles.

This recent mathematical twist of the DSR literature may well some day mature into a powerful tool for Quantum Gravity research, perhaps both at the conceptual level and for what concerns phenomenology, but at present it is certainly of no use for phenomenology (and even the conceptual side is only at an early stage of development). Instead for the thesis presented in this chapter my original DSR proposal is rather valuable since it provides a rare example of a physics idea that is powerful enough to make definite falsifiable predictions (even without any knowledge of the correct formalism that should implement it!). This is stressed in particular in the part of the next section devoted to photon stability.

22.4.1 Motivation

I introduced the doubly special relativity scenario as a sort of alternative perspective on the results on Planck-scale departures from Lorentz symmetry which had been reported in numerous articles [5; 6; 7; 8; 10; 11; 12; 29] between 1997 and 2000. These studies were advocating a Planck-scale modification of the energy-momentum dispersion relation, usually of the form $E^2 = p^2 + m^2 + \eta L_{\mathrm{p}}^n p^2 E^n + O(L_{\mathrm{p}}^{n+1} E^{n+3})$, on the basis of preliminary findings in the analysis of several formalisms in use for Planck-scale physics. The complexity of the formalisms is such that very little else was known about their physical consequences, but the evidence of a modification of the dispersion relation was becoming robust. In all of the relevant papers it was assumed that such modifications of the dispersion relation would amount to a breakup of Lorentz symmetry, with associated emergence of a preferred class of inertial observers (usually identified with the natural observer of the cosmic microwave background radiation).

I was intrigued by a striking analogy between these developments and the developments which led to the emergence of Special Relativity more than a century ago. In Galilei Relativity there is no observer-independent scale, and in fact the energy-momentum relation is written as $E = p^2/(2m)$. As experimental evidence in favour of Maxwell equations started to grow, the fact that those equations involve a fundamental velocity scale appeared to require the introduction of a preferred class of inertial observers. But in the end we figured out that the situation was not demanding the introduction of a preferred frame, but rather a modification of the laws of transformation between inertial observers. Einstein's Special Relativity introduced the first observer-independent relativistic scale (the velocity scale c), its dispersion relation takes the form $E^2 = c^2 p^2 + c^4 m^2$ (in which c plays a crucial role for what concerns dimensional analysis), and the presence of c in Maxwell's equations is now understood as a manifestation of the necessity to deform the Galilei transformations.

I argued in Refs. [25; 26] that it is not implausible that we might be presently confronted with an analogous scenario. Research in Quantum Gravity is increasingly providing reasons of interest in Planck-scale modifications of the dispersion relation, of the type mentioned above and, while it was customary to assume that this would amount to the introduction of a preferred class of inertial frames (a "Quantum Gravity ether"), the proper description of these new structures might require yet again a modification of the laws of transformation between inertial observers. The new transformation laws would have to be characterized by two scales (c and $L_{\rm p}$) rather than the single one (c) of ordinary Special Relativity.

22.4.2 Defining the DSR scenario

The "historical motivation" described above leads to a scenario for Planck-scale physics which is not intrinsically equipped with a mathematical formalism for its implementation, but still is rather well defined. With doubly special relativity one looks for a transition in the Relativity postulates, which should be largely analogous to the Galilei → Einstein transition. Just as it turned out to be necessary, in order to describe high-velocity particles, to set aside Galilei Relativity (with its lack of any characteristic invariant scale) and replace it with Special Relativity (characterized by the invariant velocity scale c), it is at least plausible that, in order to describe ultra-high-energy particles, we might have to set aside Special Relativity and replace it with a new relativity theory, a DSR, with two characteristic invariant scales, a new small-length/large-momentum scale in addition to the familiar velocity scale.

A theory will be compatible with the DSR principles if there is complete equivalence of inertial observers (Relativity Principle) and the laws of transformation between inertial observers are characterized by two scales, a high-velocity scale and a high-energy/short-length scale. Since in DSR one is proposing to modify the high-energy sector, it is safe to assume that the present operative characterization of the velocity scale c would be preserved: c is and should remain the speed of massless low-energy particles.[6] Only experimental data could guide us toward the operative description of the second invariant scale λ, although its size is naturally guessed to be somewhere in the neighborhood of the Planck length $L_{\rm p}$.

As a result of the "historical context" described in the preceding subsection most authors have explored the possibility that the second relativistic invariant be introduced through a modifications of the dispersion relation. This is a reasonable choice but it would be incorrect at present to identify (as is often done in the literature) the DSR proposal with the proposal of observer-independent modifications of the dispersion relation. For example, the dispersion relation might not be modified but there might instead be an observer-independent bound on the accuracy achievable in the measurement of distances.

In the search for a first example of formalism compatible with the DSR principles much work has been devoted to the study of κ-Minkowski. There are good reasons for this [25; 26; 31], but once again it would be incorrect to identify the DSR idea with κ-Minkowski. Of course we may one day stumble upon a very different formalism which is compatible with the DSR principles. And even within research on κ-Minkowski it must be noticed that the same mathematics can be used to obtain pictures which very clearly violate the DSR principles. For example, some authors introduce theories in κ-Minkowski in a way that leads to a law of conservation of energy-momentum based on a naive substitution of the usual sum rule with the "coproduct" sum rule, but this amounts [25; 26; 31] to breaking (rather than deforming) the Poincaré symmetries.

22.5 More on the phenomenology of departures from Poincaré symmetry

In this section I comment on some aspects of recent phenomenology work on departures from Poincaré symmetry, mostly as codified in modifications of the energy-momentum dispersion relation. I will start by stressing that the same modified dispersion relation can be introduced in very different test theories, leading to completely different physical predictions. But I also argue that, for most of the

[6] Note, however, the change of perspective imposed by the DSR idea: within Special Relativity c is the speed of all massless particles, but Special Relativity must be perceived as a low-energy theory (as viewed from the DSR perspective) and in taking Special Relativity as starting point for a high-energy deformation one is only bound to preserving c as the speed of massless low-energy particles.

ways in which a modified dispersion relation could manifest itself, we do have at least some hope of experimental study.

22.5.1 On the test theories with modified dispersion relation

The majority (see, e.g., Refs. [5; 6; 7; 8; 10; 11; 12; 29]) of studies concerning Planck-scale modifications of the dispersion relation adopt the phenomenological formula

$$m^2 \simeq E^2 - \vec{p}^2 + \eta \vec{p}^2 \left(\frac{E^n}{E_p^n} \right) + O\left(\frac{E^{n+3}}{E_{QG}^{n+1}} \right), \tag{22.2}$$

with real η (assumed to be of order $|\eta| \sim 1$) and integer n.

There is at this point a very large literature on the associated phenomenology, but I want to stress that actually some of the different phenomenological studies that compose this literature introduce this type of dispersion relation within different test theories. The limits obtained within different test theories are of course not to be compared. The same parametrization of the dispersion relation, if introduced within different test theories, actually gives rise to independent sets of parameters.

The potential richness of this phenomenology, for what concerns the development of test theories, mainly originates from the need to specify, in addition to the form of the dispersion relation, several other structural properties of the test theory.

It is necessary to state whether the theory is still "Hamiltonian", in the sense that the velocity along the x axis is obtained from the commutator with a Hamiltonian ($v \sim [x, H]$) and whether the Heisenberg commutator preserves its standard form ($[x, p] \sim \hbar$). This second concern is significant since some of the heuristic arguments that are used to motivate the presence of a modified dispersion relation at the Planck scale also suggest that the Heisenberg commutator should be correspondingly modified.

Then the test theory should formulate a law of energy-momentum conservation. For example, some types of spacetime noncommutativity which contributed to interest in modified dispersion relations appear to be such to require an accompanying modification of the law of energy-momentum conservation. And in particular a link between modification of the dispersion relation and associated modification of the law of energy-momentum conservation is required by the DSR principles (see below).

And one should keep clearly separate the test theories that intend to describe only kinematics and the ones that also adopt a scheme for Planck-scale dynamics. For example, in loop quantum gravity and some noncommutative spacetimes which provided motivation for considering modifications of the dispersion relation, while we might be close to having a correct picture of kinematics, it appears that we are still far from understanding Planck-scale corrections to dynamics.[7]

Elsewhere [43] I have tried to propose a handful of test theories that could provide a first level of language to handle this complexity. Here I shall be content with showing how, in different phenomenological studies based on modified dispersion relations, one ends up making assumptions about the points listed above.

22.5.2 Photon stability

It has recently been realized (see, e.g., Refs. [44; 45; 46]) that when Lorentz symmetry is broken at the Planck scale there can be significant implications for certain decay processes. At the qualitative level the most significant novelty would be the possibility for massless particles to decay. Let us consider, for example, a photon decay into an electron–positron pair: $\gamma \to e^+e^-$. And let us analyze this process using the dispersion relation (22.1), for $n = 1$, with the unmodified law of energy-momentum conservation. One easily finds a relation between the energy E_γ of the incoming photon, the opening angle θ between the outgoing electron–positron pair, and the energy E_+ of the outgoing positron, which, for the region of phase space with $m_e \ll E_\gamma \ll E_p$, takes the form $\cos(\theta) = (A + B)/A$, with $A = E_+(E_\gamma - E_+)$ and $B = m_e^2 - \eta E_\gamma E_+(E_\gamma - E_+)/E_p$ (m_e denotes of course the electron mass). The fact that for $\eta = 0$ this would require $\cos(\theta) > 1$ reflects the fact that if Lorentz symmetry is preserved the process $\gamma \to e^+e^-$ is kinematically forbidden. For $\eta < 0$ the process is still always forbidden, but for positive η and $E_\gamma \gg (m_e^2 E_p/|\eta|)^{1/3}$ one finds that $\cos(\theta) < 1$ in certain corresponding region of phase space.

The energy scale $(m_e^2 E_p)^{1/3} \sim 10^{13}$ eV is not too high for astrophysics. The fact that certain observations in astrophysics allow us to establish that photons of energies up to $\sim 10^{14}$ eV are not unstable (at least not noticeably unstable) could be used [44; 46] to set valuable limits on η.

This is quite a striking result, which however should be reported with caution: this is not a strategy to set direct limits on the parameters of the dispersion relation, since the analysis very explicitly requires us to specify also the form of the energy-momentum conservation law. Test theories that can make use of this phenomenological analysis must formulate at least both the dispersion relation and the law of energy-momentum conservation.

By changing the form of the law of energy-momentum conservation, for fixed form of the dispersion relation, one can indeed obtain very different results. This

[7] On the loop quantum gravity side this is linked once again with the "classical limit problem", while for the relevant noncommutative spacetime the concern originates from failures to produce consistent theories of quantum matter fields in those spacetimes.

is best illustrated contemplating the possibility that such a dispersion relation be introduced within a DSR framework. First of all let us notice that any theory compatible with the DSR principle will have stable massless particles, so that by looking for massless-particle decay one could falsify the DSR idea. A threshold-energy requirement for massless-particle decay (such as the $E_\gamma \gg (m_e^2 E_p/|\eta|)^{1/3}$ mentioned above) cannot of course be introduced as an observer-independent law, and is therefore incompatible with the DSR principles.

An analysis of the stability of massless particles that is compatible with the DSR principles can be obtained by combining the modification of the dispersion relation with an associated modification of the laws of energy-momentum conservation. The form of the new law of energy-momentum conservation can be derived from the requirement of being compatible both with the DSR principles and with the modification of the dispersion relation [25; 26], and in particular for the $a \rightarrow b + c$ case that I am considering one arrives at $E_\gamma \simeq E_+ + E_- - \lambda \vec{p}_+ \cdot \vec{p}_-$, $\vec{p}_\gamma \simeq \vec{p}_+ + \vec{p}_- - \lambda E_+ \vec{p}_- - \lambda E_- \vec{p}_+$. Using these in place of ordinary conservation of energy-momentum one ends up with a result for $\cos(\theta)$ which is still of the form $(A+B)/A$ but now with $A = 2E_+(E_\gamma - E_+) + \lambda E_\gamma E_+(E_\gamma - E_+)$ and $B = 2m_e^2$. Evidently this formula always gives $\cos(\theta) > 1$, consistently with the fact that $\gamma \rightarrow e^+e^-$ is forbidden in DSR.

22.5.3 Threshold anomalies

Another opportunity to investigate Planck-scale departures from Lorentz symmetry is provided by certain types of energy thresholds for particle-production processes that are relevant in astrophysics. This is a very powerful tool for Quantum Gravity phenomenology, and in fact at the beginning of this chapter I chose the evaluation of the threshold energy for $p + \gamma_{CMBR} \rightarrow p + \pi$ as a key example.

Numerous Quantum Gravity-phenomenology papers (see, e.g., Refs.[5; 6; 7; 8]) have been devoted to the study of Planck-scale-modified thresholds, so the interested readers can find an abundance of related materials. I should stress here that, for the purpose of the point I am trying to convey in this section, the study of threshold anomalies is not different from the study of the stability of massless particles: once again in the case in which the modified dispersion relation is combined with the unmodified law of energy-momentum conservation one finds a striking effect. But the size of this effect can change significantly if one also allows a modification of the law of energy-momentum conservation.

22.5.4 Time-of-travel analyses

A wavelength dependence of the speed of photons is obtained from a modified dispersion relation, if one assumes the velocity to be still described by $v = dE/dp$.

For the dispersion relation here considered one finds that at "intermediate energies" ($m < E \ll E_p$) the velocity law will take the form

$$v \simeq 1 - \frac{m^2}{2E^2} + \eta \frac{n+1}{2} \frac{E^n}{E_p^n} \, . \tag{22.3}$$

On the basis of this formula one would find that two simultaneously emitted photons should reach the detector at different times if they carry different energy. And this time-of-arrival-difference effect can be significant [10; 11] in the analysis of short-duration gamma-ray bursts that reach us from cosmological distances. For a gamma-ray burst it is not uncommon that the time travelled before reaching our Earth detectors be of order $T \sim 10^{17}$ s. Microbursts within a burst can have very short duration, as short as 10^{-3} s, and this means that the photons that compose such a microburst are all emitted at the same time, up to an uncertainty of 10^{-3} s. Some of the photons in these bursts have energies that extend at least up to the GeV range, and for two photons with energy difference of order $\Delta E \sim 1\,\text{GeV}$ a $\Delta E/E_p$ speed difference over a time of travel of 10^{17} s would lead to a difference in times of arrival of order $\Delta t \sim T \Delta \frac{E}{E_p} \sim 10^{-2}$ s, which is significant (the time-of-arrival differences would be larger than the time-of-emission differences within a microburst).

It is well established that the sensitivities achievable [47; 48] with the next generation of gamma-ray telescopes, such as GLAST [47; 48], could allow us to test very significantly (22.3) in the case $n = 1$, by possibly pushing the limit on $|\eta|$ far below 1. And, while probably beyond the reach of photon astrophysics, for the case $n = 2$ neutrino astronomy may lead to valuable insight [21; 22].

Notice, however, that in some test theories it may be incorrect to combine the limits obtained in this way with the limits obtained from the threshold analyses discussed above in these two different ways. As stressed above the threshold analyses require a test theory with certain forms of the law of energy-momentum conservation, whereas the law of energy-momentum conservation is not relevant for the time-of-travel studies. On the other hand a test theory using the time-of-travel analyses, in adopting $v = dE/dp$, should have a standard form of the Heisenberg commutator (so that $x \rightarrow \partial/\partial p$) and assign a standard role to the Hamiltonian (so that $v \sim [x, H]$).

22.5.5 Synchrotron radiation

As observed recently in Ref. [49], in the mechanism that leads to the production of synchrotron radiation a key role is played by the special-relativistic velocity law $v = dE/dp \simeq 1 - m^2/(2E^2)$. And an interesting observation is obtained by considering the velocity law (22.3) for the case $n = 1$. Assuming that all other aspects of the analysis of synchrotron radiation remain unmodified at the Planck scale, one is led [49] to the conclusion that, if $\eta < 0$, the energy/wavelength dependence of the Planck-scale term in (22.3) can affect the value of the cutoff energy for synchrotron radiation. This originates from the fact that according to (22.3), for $n = 1$ and $\eta < 0$, an electron cannot have a speed that exceeds the value $v_e^{max} \simeq 1 - (3/2)(|\eta| m_e/E_p)^{2/3}$, whereas in Special Relativity v_e can take values arbitrarily close to 1. This may be used to argue that for negative η the cutoff energy for synchrotron radiation should be lower than appears to be suggested by certain observations of the Crab nebula [49].

In making use of this striking observation it is, however, important to notice that synchrotron radiation is the result of the acceleration of the relevant electrons and therefore dynamics plays an implicit role in the derivation of the result [43]. From a field-theory perspective the process of synchrotron-radiation emission is described in terms of Compton scattering of the electrons with the virtual photons of the magnetic field, confirming the need to include a description of some aspects of dynamics and of energy-momentum conservation (for the vertices in the Compton-scattering analysis).

22.6 Closing remarks

The fact that Quantum Gravity phenomenology is at least worth trying is at this point rather widely acknowledged, and hopefully this chapter contributes to further spreading of knowledge. But unfortunately some misconceptions about Quantum Gravity phenomenology are still surprisingly common. In particular, it is often stated that the sensitivities achievable in Quantum Gravity phenomenology are inevitably not better than the ones needed for effects suppressed only linearly by the Planck length, but this ignores the few cases in which quadratic Planck-length sensitivity is within reach. I hope that a contribution to removing this misconception is given by the emphasis I placed on the analysis of the process $p + \gamma_{CMBR} \to p + \pi$ from the cosmic-ray perspective, with its associated quadratic Planck-length sensitivity. And Section 22.2 could act as an antidote for another common misconception: Quantum Gravity phenomenology is often identified with the study of Planck-scale departures from Lorentz symmetry, which ignores the numerous other types of candidate Planck-scale effects that this phenomenology is investigating.

Since we have robustly established some Planck-scale sensitivities and we even have encouraging progress toward falsifiability of some Planck-scale theories, it is now time to worry about adopting a correct methodology. We can stop worrying about proving the legitimacy of our efforts, and instead we had better start worrying about conducting these efforts in a correct manner. As stressed in Section 22.4, the adoption and proper use of some well tailored test theories should be very valuable in this respect.

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23

Quantum Gravity and precision tests

C. BURGESS

23.1 Introduction

Any of us who has used the Global Positioning System (GPS) in one of the gadgets of everyday life has also relied on the accuracy of the predictions of Einstein's theory of gravity, General Relativity (GR). GPS systems accurately provide your position relative to satellites positioned thousands of kilometres from the Earth, and their ability to do so requires being able to understand time and position measurements to better than 1 part in 10^{10}. Such an accuracy is comparable to the predicted relativistic effects for such measurements in the Earth's gravitational field, which are of order $GM_\oplus/R_\oplus c^2 \sim 10^{-10}$, where G is Newton's constant, $M_\oplus$ and $R_\oplus$ are the Earth's mass and mean radius, and c is the speed of light. GR also does well when compared with other precise measurements within the solar system, as well as in some extra-solar settings [1].

So we live in an age when engineers must know about General Relativity in order to understand why some their instruments work so accurately. And yet we also are often told there is a crisis in reconciling GR with quantum mechanics, with the size of quantum effects being said to be infinite (or – what is the same – to be unpredictable) for gravitating systems. But since precision agreement with experiment implies agreement within both theoretical and observational errors, and since uncomputable quantum corrections fall into the broad category of (large) theoretical error, how can uncontrolled quantum errors be consistent with the fantastic success of classical GR as a precision description of gravity?

This chapter aims to explain how this puzzle is resolved, by showing why quantum effects in fact *are* calculable within GR, at least for systems which are sufficiently weakly curved (in a sense explained quantitatively below). Since all of the extant measurements are performed within such weakly curved environments, quantum corrections to them can be computed and are predicted to be fantastically small. In this sense we quantitatively understand *why* the classical approximation to GR works so well within the solar system, and so why in practical situations

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

quantum corrections to gravity need not be included as an uncontrolled part of the budget of overall theoretical error.

More precisely, the belief that quantum effects are incalculable within GR arises because GR is what is called a non-renormalizable theory.

Non-renormalizability means that the short-wavelength divergences – which are ubiquitous within quantum field theory – cannot be absorbed into the definitions of a finite number of parameters (like masses and charges), as they are in renormalizable theories like Quantum Electrodynamics (QED) or the Standard Model (SM) of the strong and electroweak interactions. Although this does preclude making quantum predictions *of arbitrary accuracy*, it does not preclude making predictions to any finite order in an appropriate low-energy expansion, and this is what allows the predictivity on which precise comparison with experiment relies. In fact gravity is not at all special in this regard, as we know of other non-renormalizable theories which describe nature – such as the chiral perturbation theory which describes the low-energy interactions of pions and kaons, or the Fermi theory of the weak interactions, or a wide variety of condensed matter models. In many of these other systems quantum corrections are not only computable, they can be measured, with results which agree remarkably well with observations.

One thing this chapter is *not* intended to do is to argue that it is silly to think about the problems of Quantum Gravity, or that there are no interesting fundamental issues remaining to be addressed (such as many of those described elsewhere in these pages). What is intended is instead to identify more precisely where these more fundamental issues become important (at very short distances), and why they do not hopelessly pollute the detailed comparison of GR with observations. My presentation here follows that of my longer review of ref. [2], in which the arguments given here are provided in more detail.

23.2 Non-renormalizability and the low-energy approximation

Since the perceived difficulties with calculating quantum corrections in weak gravitational fields revolve around the problem of calculating with non-renormalizable theories, the first step is to describe the modern point of view as to how this should be done. It is convenient to do so first with a simpler toy model, before returning to GR in all of its complicated glory.

23.2.1 A toy model

Consider therefore the theory of a complex scalar field, ϕ, described by the Lagrangian density

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi) \,, \qquad (23.1)$$

with the following scalar potential

$$V = \frac{\lambda^2}{4}\,\left(\phi^*\phi - v^2\right)^2 . \tag{23.2}$$

This theory is renormalizable, so we can compute its quantum implications in some detail.

Since we return to it below, it is worth elaborating briefly on the criterion for renormalizability. To this end we follow standard practice and define the 'engineering' dimension of a coupling as p, where the coupling is written as (mass)^p in units where $\hbar = c = 1$ (which are used throughout).[1] For instance the coupling λ^2 which pre-multiplies $(\phi^*\phi)^2$ above is dimensionless in these units, and so has $p = 0$, while the coupling $\lambda^2 v^2$ pre-multiplying $\phi^*\phi$ has $p = 2$.

A theory is renormalizable if $p \geq 0$ for all of its couplings, and if for any given dimension all possible couplings have been included consistent with the symmetries of the theory. Both of these are clearly true for the Lagrangian of eqs. (23.1) and (23.2), since all possible terms are written consistent with $p \geq 0$ and the $U(1)$ symmetry $\phi \to e^{i\omega}\phi$.

23.2.1.1 Spectrum and scattering

We next analyze the spectrum and interactions, within the semiclassical approximation which applies in the limit $\lambda \ll 1$. In this case the field takes a nonzero expectation value, $\langle\phi\rangle = v$, in the vacuum. The particle spectrum about this vacuum consists of two weakly-interacting particle types, one of which – φ_0 – is massless and the other – φ_m – has mass $m = \lambda v$. These particles interact with one another through an interaction potential of the form

$$V = \frac{1}{2}\left[m\varphi_m + \frac{\lambda}{2\sqrt{2}}\left(\varphi_m^2 + \varphi_0^2\right)\right]^2 , \tag{23.3}$$

as may be seen by writing $\phi = v + (\varphi_m + i\varphi_0)/\sqrt{2}$. For instance, these interactions imply the following invariant scattering amplitude for the scattering process $\varphi_0(p)\varphi_0(q) \to \varphi_0(p')\varphi_0(q')$:

$$\begin{aligned}\mathcal{A} = &-\frac{3\lambda^2}{2} + \left(\frac{\lambda m}{\sqrt{2}}\right)^2\left[\frac{1}{(p+q)^2 + m^2 - i\epsilon}\right.\\ &\left.+\frac{1}{(p-p')^2 + m^2 - i\epsilon} + \frac{1}{(p-q')^2 + m^2 - i\epsilon}\right].\end{aligned} \tag{23.4}$$

This amplitude has an interesting property in the limit that the centre-of-mass scattering energy, E, is much smaller than the mass m. As may be explored by

[1] It is implicit in this statement that the relevant fields are canonically normalized, and so have dimensionless kinetic terms.

expanding $\mathcal{A}$ in powers of external four-momenta, in this limit the $O(\lambda^2)$ and $O(\lambda^2 E^2/m^2)$ terms both vanish, leaving a result $\mathcal{A} = O(\lambda^2 E^4/m^4)$. Clearly the massless particles interact more weakly than would be expected given a cursory inspection of the scalar potential, eq. (23.3).

The weakness of the scattering of φ_0 particles at low energy is a consequence of their being Nambu–Goldstone bosons [3; 4; 5; 6; 7] for the theory's $U(1)$ symmetry: $\phi \to e^{i\omega}\phi$. This can be seen more explicitly by changing variables to polar coordinates in field space, $\phi = \chi\, e^{i\theta}$, rather than the variables φ_0 and φ_m. In terms of θ and χ the action of the $U(1)$ symmetry is simply $\theta \to \theta + \omega$, and the model's Lagrangian becomes:

$$\mathcal{L} = -\partial_\mu \chi \partial^\mu \chi - \chi^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda^2}{4}(\chi^2 - v^2)^2 \,, \tag{23.5}$$

and semiclassical calculations can be performed as before by expanding in terms of canonically normalized fluctuations: $\chi = v + \hat{\varphi}_m/\sqrt{2}$ and $\theta = \hat{\varphi}_0/v\sqrt{2}$, revealing that $\hat{\varphi}_m$ describes the massive particle while $\hat{\varphi}_0$ describes the massless one. Because $\hat{\varphi}_0$ appears in $\mathcal{L}$ only explicitly differentiated (as it must because of the symmetry $\hat{\varphi}_0 \to \hat{\varphi}_0 + \omega v\sqrt{2}$), its scattering is suppressed by powers of E/m at low energies.

23.2.1.2 The low-energy effective theory

Properties such as this which arise (sometimes unexpectedly) when observables are expanded at low energies in powers of E/m are explored most easily by 'integrating out' the heavy particle to construct the *effective field theory* describing the low-energy dynamics of the massless particle alone. One way to do so in the case under consideration here would be to define 'light' degrees of freedom to be those modes (in momentum space) of $\hat{\varphi}_0$ which satisfy $p^2 < \Lambda^2$ (in Euclidean signature), for some cutoff Λ satisfying $E \ll \Lambda \ll m$. All other modes are, by definition, 'heavy'. Denoting the heavy and light modes schematically by h and ℓ, then the effective theory governing the light fields may be defined by

$$\exp\left[i \int d^4x \; \mathcal{L}_{\rm eff}(\ell, \Lambda) \right] = \int \mathcal{D}h_\Lambda \; \exp\left[i \int d^4x \; \mathcal{L}(\ell, h) \right] , \tag{23.6}$$

where the functional integral is performed over all of the heavy modes (including the large-momentum components of $\hat{\varphi}_0$).

$\mathcal{L}_{\rm eff}$ defined this way necessarily depends on Λ, but it does so in just the way required in order to have Λ cancel with the explicit Λs which cut off the loop integrals for the functional integration over the light fields, ℓ. All Λs must cancel in observables because Λ is just a bookmark which we use to organize the calculation. Because of this cancellation the detailed form of the regularization is largely immaterial and can be chosen for convenience of calculation.

For this reason it is actually preferable instead to define $\mathcal{L}_{\rm eff}$ using dimensional regularization rather than a cutoff. Paradoxically, this is possible even though one keeps both short- and long-wavelength modes of the light fields in the low-energy theory when dimensionally regularizing, which seems to contradict the spirit of what a low-energy effective theory is. In practice it is possible because the difference between the cutoff-regularized and dimensionally regularized low-energy theory can itself be parameterized by an appropriate choice for the effective couplings within the low-energy theory. This is the choice we shall make below when discussing quantum effects within the effective theory.

With this definition, physical observables at low energies are now computed by performing the remaining path integral over the light degrees of freedom only, weighted by the low-energy effective Lagrangian: $\exp\left[i\int d^4x\ \mathcal{L}_{\rm eff}(\ell)\right]$. The effects of virtual contributions of heavy states appear within this low-energy theory through the contributions of new effective interactions. When applied to the toy model to leading order in λ this leads to the following result for $\mathcal{L}_{\rm eff}$:

$$\mathcal{L}_{\rm eff} = v^2\left[-\partial_\mu\theta\,\partial^\mu\theta + \frac{1}{4m^2}(\partial_\mu\theta\,\partial^\mu\theta)^2 - \frac{1}{4m^4}\,(\partial_\mu\theta\,\partial^\mu\theta)^3 \right. \tag{23.7}$$
$$\left. +\frac{1}{4m^4}\,(\partial_\mu\theta\,\partial^\mu\theta)\partial_\lambda\partial^\lambda(\partial_\nu\theta\,\partial^\nu\theta) + \cdots\right],$$

where the ellipses in $\mathcal{L}$ represent terms which are suppressed by more than four inverse powers of m. The inverse powers of m which pre-multiply all of the interactions in this Lagrangian are a consequence of the virtual $\hat{\varphi}_m$ exchanges which are required in order to produce them within the full theory. The explicit numerical factors in each term are an artifact of leading order perturbation theory, and receive corrections order by order in λ. Computing 2-particle scattering using this effective theory gives a result for which the low-energy suppression by powers of E/m are explicit due to the derivative form of the interactions.

What is interesting about the Lagrangian, eq. (23.7), for the present purposes is that the successive effective couplings involve successively more powers of $1/m^2$. In particular, this keeps them from having non-negative engineering dimension and so makes the effective theory manifestly non-renormalizable. If someone were to hand us this theory we might therefore throw up our hands and conclude that we cannot predictively compute quantum corrections. However, in this case we know this theory simply expresses the low-energy limit of a full theory which *is* renormalizable, and so for which quantum corrections can be explicitly computed. Why can't these corrections also be expressed using the effective theory?

The answer is that they can, and this is by far the most efficient way to compute these corrections to observables in the low-energy limit where $E \ll m$. The key to computing these corrections is to systematically exploit the low-energy expansion in powers of E/m which underlies using the action, eq. (23.7) in the first place.

23.2.2 Computing loops

To explore quantum effects consider evaluating loop graphs using the toy-model effective Lagrangian, which we may write in the general form

$$\mathcal{L}_{\rm eff} = v^2 m^2 \sum_{id} \frac{c_{id}}{m^d}\, \mathcal{O}_{id}, \tag{23.8}$$

where the sum is over interactions, $\mathcal{O}_{id}$, involving i powers of the dimensionless field θ and d derivatives. The power of m pre-multiplying each term is chosen to ensure that the coefficient c_{id} is dimensionless, and we have seen that these coefficients are $O(1)$ at leading order in λ^2. To be completely explicit, in the case of the interaction $\mathcal{O} = (\partial_\mu\theta\, \partial^\mu\theta)^2$ we have $i = d = 4$ and we found earlier that $c_{44} = \frac{1}{4} + O(\lambda^2)$ for this term. Notice that Lorentz invariance requires d must be even, and the $U(1)$ symmetry implies every factor of θ is differentiated at least once, and so $d \geq i$. We may ignore all terms with $i = 1$ since these are linear in $\partial_\mu\theta$ and so must be a total derivative.[2] Furthermore, the only term with $i = 2$ is the kinetic term, which we take as the unperturbed Lagrangian, and so for the interactions we may restrict the sum to $i \geq 3$.

With these definitions it is straightforward to track the powers of v and m that interactions of the form (23.8) contribute to an L-loop contribution to a scattering amplitude, $\mathcal{A}(E)$, at centre-of-mass energy E. (The steps presented here closely follow the discussion of refs. [2; 7].) Imagine using this Lagrangian to compute a contribution to the scattering amplitude, $\mathcal{A}(E)$, coming from a Feynman graph involving $\mathcal{E}$ external lines; I internal lines and V_{ik} vertices. (The subscript i here counts the number of lines which converge at the vertex, while k counts the power of momentum which appears.) These constants are not all independent, since they are related by the identity $2I + E = \sum_{ik} i\, V_{ik}$. It is also convenient to trade the number of internal lines, I, for the number of loops, L, defined by $L = 1 + I - \sum_{ik} V_{ik}$.

We now use dimensional analysis to estimate the result of performing the integration over the internal momenta, using dimensional regularization to regulate the ultraviolet divergences. If all external momenta and energies are of order E then the size of a dimensionally regularized integral is given on dimensional grounds by the appropriate power of E, we find

[2] Terms like total derivatives, which do not contribute to the observables of interest, are called *redundant* and may be omitted when writing the effective Lagrangian.

$$\mathcal{A}(E) \sim v^2 m^2 \left(\frac{1}{v}\right)^{\mathcal{E}} \left(\frac{m}{4\pi v}\right)^{2L} \left(\frac{E}{m}\right)^{P}$$
$$\sim v^2 E^2 \left(\frac{1}{v}\right)^{\mathcal{E}} \left(\frac{E}{4\pi v}\right)^{2L} \prod_i \prod_{d>2} \left(\frac{E}{m}\right)^{(d-2)V_{id}}, \qquad (23.9)$$

where $P = 2 + 2L + \sum_{id}(d-2)V_{id}$. This is the main result, since it shows which graphs contribute to any order in E/m using a nonrenormalizable theory.[3]

To see how eqs. (23.9) are used, consider the first few powers of E in the toy model. For any $\mathcal{E}$ the leading contributions for small E come from *tree* graphs, i.e. those having $L = 0$. The tree graphs that dominate are those for which $\sum'_{id}(d-2)V_{id}$ takes the smallest possible value. For example, for 2-particle scattering $\mathcal{E} = 4$ and so precisely one tree graph is possible for which $\sum'_{id}(d-2)V_{id} = 2$, corresponding to $V_{44} = 1$ and all other $V_{id} = 0$. This identifies the single graph which dominates the 4-point function at low energies, and shows that the resulting leading energy dependence in this case is $\mathcal{A}(E) \sim E^4/(v^2 m^2)$, as was also found earlier in the full theory. The numerical coefficient can be obtained in terms of the effective couplings by more explicit evaluation of the appropriate Feynman graph.

The next-to-leading behaviour is also easily computed using the same arguments. Order E^6 contributions are achieved if and only if either: (i) $L = 1$ and $V_{i4} = 1$, with all others zero; or (ii) $L = 0$ and $\sum_i \left(4V_{i6} + 2V_{i4}\right) = 4$. Since there are no $d = 2$ interactions, no one-loop graphs having 4 external lines can be built using precisely one $d = 4$ vertex and so only tree graphs can contribute. Of these, the only two choices allowed by $\mathcal{E} = 4$ at order E^6 are therefore the choices: $V_{46} = 1$, or $V_{34} = 2$. Both of these contribute a result of order $\mathcal{A}(E) \sim E^6/(v^2 m^4)$.

Besides showing how to use the effective theory to compute to any order in E/m, eq. (23.9) also shows the domain of approximation of the effective-theory calculation. The validity of perturbation theory within the effective theory relies only on the assumptions $E \ll 4\pi v$ and $E \ll m$. In particular, it does *not* rely on the ratio $m/4\pi v = \lambda/4\pi$ being small, even though there is a factor of this order appearing for each loop. This factor does not count loops in the effective theory because it is partially cancelled by another factor, E/m, which also comes with every loop; $\lambda/4\pi$ *does* count loops within the full theory, of course. This calculation simply shows that the small-λ approximation is only relevant for predicting the values of the effective couplings, but are irrelevant to the problem of computing the energetics of scattering amplitudes given these couplings.

[3] It is here that the convenience of dimensional regularization is clear, since it avoids keeping track of powers of a cutoff like Λ, which drops out of the final answer for an observable in any case.

23.2.3 The effective Lagrangian logic

These calculations show how to calculate predictively – *including loops* – using a non-renormalizable effective theory.

Step I Choose the accuracy desired in the answer (e.g. a 1% accuracy might be desired).
Step II Determine how many powers of E/m are required in order to achieve the desired accuracy.
Step III Use a calculation like the one above to identify which effective couplings in $\mathcal{L}_{\text{eff}}$ can contribute to the observable of interest to the desired order in E/m. This always requires only a finite number (say: N) of terms in $\mathcal{L}_{\text{eff}}$ to any finite accuracy.

There are two alternative versions of the fourth and final step, depending on whether or not the underlying microscopic theory – like the ϕ theory in the toy model – is known.

Step IV-A If the underlying theory is known and calculable, then compute the required coefficients of the N required effective interactions to the accuracy required. Alternatively, use *Step IV-B*.
Step IV-B If the underlying theory is unknown, or is too complicated to permit the calculation of $\mathcal{L}_{\text{eff}}$, then leave the N required coefficients as free parameters. The procedure is nevertheless predictive if more than N observables can be identified whose predictions depend only on these parameters.

The effective Lagrangian is in this way seen to be predictive even though it is not renormalizable in the usual sense. Renormalizable theories are simply the special case of *Step IV-B* where one stops at zeroeth order in E/m, and so are the ones which dominate in the limit that the light and heavy scales are very widely separated. In fact, this is *why* renormalizable interactions are so important when describing Nature.

The success of the above approach is well-established in many areas outside of gravitational physics, with non-renormalizability being the signal that one is seeing the virtual effects due to some sort of heavier physics. Historically, one of earliest examples known was the non-renormalizable interactions of chiral perturbation theory which describe well the low-energy scattering of pions, kaons and nucleons. It is noteworthy that this success requires the inclusion of the loop corrections within this effective theory. The heavier physics in this case is the confining physics of the quarks and gluons from which these particles are built, and whose complicated dynamics has so far precluded calculating the effective couplings from first principles. The effective theory works so long as one restricts to centre-of-mass energies smaller than roughly 1 GeV.

The Fermi (or V–A) theory of the weak interactions is a similar example, which describes the effects of virtual W-boson exchange at energies well below the W-boson mass, $M_W = 80\,\text{GeV}$. This theory provides an efficient description of the low-energy experiments, with an effective coupling, $G_F/\sqrt{2} = g^2/8M_W^2$ which in this case is calculable in terms of the mass and coupling, g, of the W boson. In this case agreement with the precision of the measurements again requires the inclusion of loops within the effective theory.

23.3 Gravity as an effective theory

Given the previous discussion of the toy model, it is time to return to the real application of interest for this chapter: General Relativity. The goal is to be able to describe quantitatively quantum processes in GR, and to be able to compute the size of quantum corrections to the classical processes on which the tests of GR are founded.

Historically, the main obstacle to this program has been that GR is not renormalizable, as might be expected given that its coupling (Newton's constant), $G = (8\pi M_p^2)^{-1}$, has engineering dimension $(\text{mass})^{-2}$ in units where $\hbar = c = 1$. But we have seen that non-renormalizable theories can be predictive in much the same way as are renormalizable ones, provided that they are interpreted as being the low-energy limit of some more fundamental microscopic theory. For gravity, this more microscopic theory is as yet unknown, although these pages contain several proposals for what it might be. Happily, as we have seen for the toy model, their effective use at low energies does not require knowledge of whatever this microscopic theory might be. In this section the goal is to identify more thoroughly what the precise form of the low-energy theory really is for gravity, as well as to identify what the scales are above which the effective theory should not be applied.

23.3.1 The effective action

For GR the low-energy fields consist of the metric itself, $g_{\mu\nu}$. Furthermore, since we do not know what the underlying, more microscopic theory is, we cannot hope to compute the effective theory from first principles. Experience with the toy model of the previous section instead suggests we should construct the most general effective Lagrangian which is built from the metric and organize it into a derivative expansion, with the terms with the fewest derivatives being expected to dominate at low energies. Furthermore we must keep only those effective interactions which are consistent with the symmetries of the problem, which for gravity we can take to be general covariance.

These considerations lead us to expect that the Einstein–Hilbert action of GR should be considered to be just one term in an expansion of the action in terms of derivatives of the metric tensor. General covariance requires this to be written in terms of powers of the curvature tensor and its covariant derivatives,

$$-\frac{\mathcal{L}_{\rm eff}}{\sqrt{-g}} = \lambda + \frac{M_p^2}{2} R + a_1 R_{\mu\nu} R^{\mu\nu} + a_2 R^2 + a_3 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + a_4 \Box R + \frac{b_1}{m^2} R^3 + \frac{b_2}{m^2} R R_{\mu\nu} R^{\mu\nu} + \frac{b_3}{m^2} R_{\mu\nu} R^{\nu\lambda} R_\lambda{}^\mu + \cdots \tag{23.10}$$

where $R^\mu{}_{\nu\lambda\rho}$ is the metric's Riemann tensor, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ is its Ricci tensor, and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, each of which involves precisely two derivatives of the metric.

The first term in eq. (23.10) is the cosmological constant, which is dropped in what follows since observations imply λ is (for some reason) extremely small. Once this is done the leading term in the derivative expansion is the Einstein–Hilbert action whose coefficient, $M_{\rm p} \sim 10^{18}$ GeV, has dimensions of squared mass, whose value defines Newton's constant. This is followed by curvature-squared terms having dimensionless effective couplings, a_i, and curvature-cubed terms with couplings inversely proportional to a mass, b_i/m^2, (not all of which are written in eq. (23.10)). Although the numerical value of $M_{\rm p}$ is known, the mass scale m appearing in the curvature-cubed (and higher) terms is not. But since it appears in the denominator it is the lowest mass scale which has been integrated out which should be expected to dominate. For this reason m is unlikely to be $M_{\rm p}$, and one might reasonably use the electron mass, $m_{\rm e} = 5 \times 10^{-4}$ GeV, or neutrino masses, $m_\nu \gtrsim 10^{-11}$ GeV, when considering applications over the distances relevant in astrophysics.

Experience with the toy model shows that not all of the interactions in the Lagrangian (23.10) need contribute independently (or at all) to physical observables. For instance, for most applications we may drop total derivatives (like $\Box R$), as well as those terms which can be eliminated by performing judicious field redefinitions [2]. Since the existence of these terms does not affect the arguments about to be made, we do not bother to identify and drop these terms explicitly here.

23.3.2 Power counting

Of all of the terms in the effective action, only the Einstein–Hilbert term is familiar from applications of classical GR. Although we expect naively that this should dominate at low energies (since it involves the fewest derivatives), we now make this more precise by identifying which interactions contribute to which order in a low-energy expansion. We do so by considering the low-energy scattering of weak gravitational waves about flat space, and by repeating the power-counting exercise performed above for the toy model to keep track of how different effective couplings contribute. In this way we can see how the scales $M_{\rm p}$ and m enter into observables.

In order to perform this power counting we expand the above action around flat space, trading the full metric for a canonically normalized fluctuation: $g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}/M_{\rm p}$. For the present purposes what is important is that the expansion of the curvature tensor (and its Ricci contractions) produces terms involving all possible powers of $h_{\mu\nu}$, with each term involving precisely two derivatives. Proceeding as before gives an estimate for the leading energy-dependence of an L-loop contribution to the scattering amplitude, $\mathcal{A}$, which involves $\mathcal{E}$ external lines and V_{id} vertices involving d derivatives and i attached graviton lines. (The main difference from the previous section's analysis is the appearance here of interactions involving two derivatives, coming from the Einstein–Hilbert term.)

This leads to the estimate:

$$\mathcal{A}(E) \sim m^2 M_{\rm p}^2 \left(\frac{1}{M_{\rm p}}\right)^{\mathcal{E}} \left(\frac{m}{4\pi M_{\rm p}}\right)^{2L} \left(\frac{m^2}{M_{\rm p}^2}\right)^{Z} \left(\frac{E}{m}\right)^{P} \tag{23.11}$$

where $Z = \sum_{id}' V_{id}$ and $P = 2 + 2L + \sum_{id}'(d-2)V_{id}$. The prime on both of these sums indicates the omission of the case $d = 2$ from the sum over d. Grouping instead the terms involving powers of L and V_{ik}, eq. (23.11) becomes

$$\mathcal{A}_{\mathcal{E}}(E) \sim E^2 M_{\rm p}^2 \left(\frac{1}{M_{\rm p}}\right)^{\mathcal{E}} \left(\frac{E}{4\pi M_{\rm p}}\right)^{2L} \prod_i \prod_{d>2} \left[\frac{E^2}{M_{\rm p}^2}\left(\frac{E}{m}\right)^{(d-4)}\right]^{V_{id}}. \tag{23.12}$$

Notice that no negative powers of E appear here because d is even and because of the condition $d > 2$ in the product.

This last expression is the result we seek because it is what shows how to make systematic quantum predictions for graviton scattering. It does so by showing that the predictions of the full gravitational effective Lagrangian (involving all powers of curvatures) can be organized into powers of $E/M_{\rm p}$ and E/m, and so we can hope to make sensible predictions provided that both of these two quantities are small. Furthermore, all of the corrections involve powers of $(E/M_{\rm p})^2$ and/or $(E/m)^2$, implying that they may be expected to be *extremely* small for any applications for which $E \ll m$. For instance, notice that even if $E/m \sim 1$ then $(E/M_{\rm p})^2 \sim 10^{-42}$ if m is taken to be the electron mass. (Notice that factors of the larger parameter E/m do not arise until curvature-cubed interactions are important, and this first occurs at subleading order in $E/M_{\rm p}$.)

Furthermore, it shows in detail what we were in any case inclined to believe: that classical General Relativity governs the dominant low-energy dynamics of gravitational waves. This can be seen by asking which graphs are least suppressed by these small energy ratios, which turns out to be those for which $L = 0$ and $P = 2$. That is, arbitrary tree graphs constructed purely from the Einstein–Hilbert action – precisely the predictions of classical General Relativity. For instance, for 2-graviton scattering we have $\mathcal{E} = 4$, and so the above arguments predict the dominant energy-dependence to be $\mathcal{A}(E) \propto (E/M_p)^2 + \cdots$. This is borne out by explicit tree-level calculations [8] for graviton scattering, which give:

$$\mathcal{A}_{\text{tree}} = 8\pi i G \left(\frac{s^3}{tu} \right), \tag{23.13}$$

for an appropriate choice of graviton polarizations. Here $s = -(p_1 + p_2)^2$, $t = (p_1 - p_1')^2$ and $u = (p_1 - p_2')^2$ are the usual Lorentz-invariant Mandelstam variables built from the initial and final particle four momenta, all of which are proportional to E^2. This shows both that $\mathcal{A} \sim (E/M_p)^2$ to leading order, and that it is the physical, invariant centre-of-mass energy, E, which is the relevant energy for the power-counting analysis.

But the real beauty of a result like eq. (23.12) is that it also identifies which graphs give the subdominant corrections to classical GR. The leading such correction arises in one of two ways: either (i) $L = 1$ and $V_{id} = 0$ for any $d \neq 2$; or (ii) $L = 0$, $\sum_i V_{i4} = 1$, V_{i2} is arbitrary, and all other V_{id} vanish. That is, compute the one-loop corrections using only Einstein gravity; or instead work to tree level and include precisely one vertex from one of the curvature-squared interactions in addition to any number of interactions from the Einstein–Hilbert term. Both are suppressed compared to the leading term by a factor of $(E/M_p)^2$, and the one-loop contribution carries an additional factor of $(1/4\pi)^2$. This (plus logarithmic complications due to infrared divergences) are also borne out by explicit one-loop calculations [9; 10; 11]. Although the use of curvature-squared terms potentially introduces additional effective couplings into the results,[4] useful predictions can nonetheless be made provided more observables are examined than there are free parameters.

Although conceptually instructive, calculating graviton scattering is at this point a purely academic exercise, and is likely to remain so until gravitational waves are eventually detected and their properties are measured in detail. In practice it is of more pressing interest to obtain these power-counting estimates for observables which are of more direct interest for precision measurements of GR, such as within the solar system. It happens that the extension to these kinds of observables is often not straightforward (and in some cases has not yet been done in a completely systematic way), because they involve non-relativistic sources (like planets and stars).

[4] For graviton scattering in 4D with no matter no new couplings enter in this way because all of the curvature-squared interactions turn out to be redundant. By contrast, one new coupling turns out to arise describing a contact interaction when computing the sub-leading corrections to fields sourced by point masses.

Non-relativistic sources considerably complicate the above power-counting arguments because they introduce a new dimensionless small quantity, v^2/c^2, whose dependence is not properly captured by the simple dimensional arguments given above [12].

Nevertheless the leading corrections have been computed for some kinds of non-relativistic sources in asymptotically flat spacetimes [13; 14; 15; 16; 17]. These show that while relativistic corrections to the observables situated a distance r away from a gravitating mass M are of order GM/rc^2, the leading quantum corrections are suppressed by powers of the much smaller quantity $G\hbar/r^2c^3$. For instance, while on the surface of the Sun relativistic corrections are of order $GM_\odot/R_\odot c^2 \sim 10^{-6}$, quantum corrections are completely negligible, being of order $G\hbar/R_\odot^2 c^3 \sim 10^{-88}$. Clearly the classical approximation to GR is *extremely* good for solar-system applications.

Another important limitation to the discussion as given above is its restriction to perturbations about flat space. After all, quantum effects are also of interest for small fluctuations about other spacetimes. In particular, quantum fluctuations generated during a past epoch of cosmological inflationary expansion appear to provide a good description of the observed properties of the cosmic microwave background radiation. Similarly, phenomena like Hawking radiation rely on quantum effects near black holes, and the many foundational questions these raise have stimulated their extensive theoretical study, even though these studies may not lead in the near term to observational consequences. Both black holes and cosmology provide regimes for which detailed quantum gravitational predictions are of interest, but for which perturbations about flat space need not directly apply.

A proper power-counting of the size of quantum corrections is also possible for these kinds of spacetimes by examining perturbations about the relevant cosmological or black-hole geometry, although in these situations momentum-space techniques are often less useful. Position-space methods, like operator-product expansions, can then provide useful alternatives, although as of this writing comparatively few explicit power-counting calculations have been done using these. The interested reader is referred to the longer review, [2], for more discussion of this, as well as of related questions which arise concerning the use of effective field theories within time-dependent backgrounds and in the presence of event horizons.

23.4 Summary

General Relativity provides a detailed quantitative description of gravitational experiments in terms of a field theory which is not renormalizable. It is the purpose of the present chapter to underline the observation that gravity is not the only area of physics for which a non-renormalizable theory is found to provide a good description of experimental observations, and we should use this information to guide our understanding of what the limits to validity might be to its use. The lesson from other areas of physics is clear: the success of a non-renormalizable theory points to the existence of a new short-distance scale whose physics is partially relevant to the observations of interest. What makes this problematic for understanding the theory's quantum predictions is that it is often the case that we do often not understand what the relevant new physics is, and so its effects must be parameterized in terms of numerous unknown effective couplings. How can predictions be made in such a situation?

What makes predictions possible is the observation that only comparatively few of these unknown couplings are important at low energies (or long distances), and so only a finite number of them enter into predictions at any fixed level of accuracy. Predictions remain possible so long as more observables are computed than there are parameters, but explicit progress relies on being able to identify which of the parameters enter into predictions to any given degree of precision.

In the previous pages it is shown how this identification can be made for the comparatively simple case of graviton scattering in flat space, for which case the size of the contribution from any given effective coupling can be explicitly estimated. The central tool is a power-counting estimate which tracks the power of energy which enters into any given Feynman graph, and which duplicates for GR the similar estimates which are made in other areas of physics. The result shows how General Relativity emerges as the leading contribution to an effective theory of some more fundamental picture, with its classical contributions being shown to be the dominant ones, but with computable corrections which can be explicitly evaluated in a systematic expansion to any given order in a low-energy expansion. This shows how a theory's non-renormalizability need not preclude its use for making sensible quantum predictions, provided these are made only for low energies and long distances.

This kind of picture is satisfying because it emphasizes the similarity between many of the problems which are encountered in GR and in other areas of physics. It is also conceptually important because it provides control over the size of the theoretical errors which quantum effects would introduce into the classical calculations against which precision measurements of General Relativity are compared. These estimates show that the errors associated with ignoring quantum effects is negligible for the systems of practical interest.

There is a sense for which this success is mundane, in that it largely confirms our prejudices as to the expected size of quantum effects for macroscopic systems based purely on dimensional analysis performed by building dimensionless quantities out of the relevant parameters like G, $\hbar$, c, M and R. However, the power-counting result is much more powerful: it identifies which Feynman graphs contribute at any given power of energy, and so permits the detailed calculation of observables as part of a systematic low-energy expansion.

It is certainly true that the small size of quantum contributions in the solar system in no way reduces the fundamental mysteries described elsewhere in these pages that must be resolved in order to properly understand Quantum Gravity at fundamentally small distances. However, it is important to understand that these problems are associated with small distance scales and not with large ones, since this focusses the discussion as to what is possible and what is not when entertaining modifications to GR. In particular, although it shows that we are comparatively free to modify gravity at short distances without ruining our understanding of gravitational physics within the solar system, it also shows that we are not similarly protected from *long*-distance modifications to GR.

This observation is consistent with long experience, which shows that it is notoriously difficult to modify GR at long distances in a way which does not introduce unacceptable problems such as various sorts of instabilities to the vacuum. Such vacuum-stability problems are often simply ignored in some circles on the grounds that ‘Quantum Gravity’ is not yet understood, in the hope that once it is it will somehow also fix the stability issues. However, our ability to quantify the size of low-energy quantum effects in gravity shows that we need not await a more complete understanding of gravity at high energies in order to make accurate predictions at low energies. And since the vacuum is the lowest-energy state there is, we cannot expect unknown short-distance physics to be able to save us from long-distance sicknesses.

Calculability at low energies is a double-edged sword. It allows us to understand why precision comparison between GR and experiment is possible in the solar system, but it equally forces us to reject alternative theories which have low-energy problems (like instabilities) as being inadequate.

Acknowledgements

I thank Daniele Oriti for the kind invitation to contribute to this volume, and for his subsequent patience. My understanding of this topic was learned from Steven Weinberg, who had been making the points made here for decades before my arrival on the scene. My research on these and related topics is funded by the Natural Sciences and Engineering Research Council of Canada, as well as by funds from McMaster University and the Killam Foundation.

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24

Algebraic approach to Quantum Gravity II: noncommutative spacetime

S. MAJID

24.1 Introduction

In this chapter we present noncommutative geometry (NCG) not as a 'theory of everything' but as a bridge between any future, perhaps combinatorial, theory of Quantum Gravity and the classical continuum geometry that has to be obtained in some limit. We consider for the present that NCG is simply a more general notion of geometry that by its noncommutative nature should be the correct setting for the phenomenology and testing of first next-to-classical Quantum Gravity corrections. Beyond that, the mathematical constraints of NCG may give us constraints on the structure of Quantum Gravity itself in so far as this has to emerge in a natural way from the true theory.

Also in this chapter we focus on the role of quantum groups or Hopf algebras [10] as the most accessible tool of NCG, along the lines first introduced for Planck scale physics by the author in the 1980s [13; 14; 15; 16]. We provide a full introduction to our theory of 'bicrossproduct quantum groups', which is one of the two main classes of quantum group to come out of physics (the other class, the q-deformation quantum groups, came out of integrable systems rather than Quantum Gravity). The full machinery of noncommutative differential geometry such as gauge theory, bundles, quantum Riemannian manifolds, and spinors (at least in principle) has also been developed over the past two decades; these topics are deferred to a third article [11]. This should allow the present article to be read without prior knowledge of either NCG or quantum groups. The first article in the series is about the philosophical basis [17].

As is well-known, quantum groups are a generalised notion of symmetry. There is a theorem that *all* bicrossproduct quantum groups indeed have associated to them noncommutative spaces on which they canonically act. Thus the bicrossproduct Poincaré quantum group denoted $U(so_{3,1}){\triangleright\!\!\blacktriangleleft}\mathbb{C}[\mathbb{R}^3{>\!\!\!\triangleleft}\mathbb{R}]$ has associated to it the proposal [23]

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

$$[x_0, x_i] = \imath\lambda x_i \tag{24.1}$$

for a model of noncommutative 4D spacetime. Note that although (24.1) breaks usual Poincaré invariance, Special Relativity still holds in the form of the quantum group 'symmetry'. This is also the first noncommutative spacetime model with a genuine physical prediction [1], namely a variable speed of light (VSL). The NASA GLAST satellite launched in 2008 may among other things be able to test this prediction through a statistical analysis of gamma-ray bursts even in the worst case that we might expect for the parameter, namely $\lambda \sim 10^{-44}$ s (the Planck timescale). Note that the model should not be confused with an earlier κ-Poincaré group model [8] where the quantum group had quite different generators (for example the Lorentz generators did not close among themselves so the physical interpretation was fundamentally different) and where prior to [23] the spacetime on which it acts was assumed to be usual commutative Minkowski space (with nonsensical results). Similarly, the semidirect quantum group $U(su_2) \bowtie_\lambda \mathbb{C}[SU_2]$ of Euclidean motions (a special case of a bicrossproduct called a Drinfeld double) acts covariantly on

$$[x_i, x_j] = 2\imath\lambda\epsilon_{ij}{}^k x_k \tag{24.2}$$

as noncommutative space or Euclideanised 3D spacetime [2]. Indeed this algebra arises in a certain limit as an effective description of Euclideanized 3D Quantum Gravity as proposed in [2] and recently proven in [3]. It should not be confused with 'fuzzy spheres' as we do not quotient to a matrix algebra or use any (in our opinion ad hoc) matrix methods familiar in that context. One may also add a central x_0 to have a 4D spacetime [21]. Notice that these and other noncommutative spacetimes in the paper are geometrically flat, i.e. they are relevant to a weak gravity regime of Quantum Gravity. Instead the effects they encode are of curvature in momentum space or 'cogravity', a notion due to the author [9] as a potentially new and independent physical effect. Because of lack of space, we will focus mainly on (24.1) and its illustrative 2D version, for which we provide a full global treatment.

Of course, the algebraic machinery that we shall describe includes many more models of potential physical interest. The bicrossproduct family nevertheless remain the most interesting because they come from entirely classical (but nonlinear) data. This means that although they are excellent examples of NCG their structure can be described ultimately by classical nonlinear differential equations and classical pictures. The classical data are a local factorisation of some Lie group $X \approx GM$ and equivalent to solving a pair of 'matched pair' differential equations for an action of G on M and vice versa. In [13] these were introduced as toy models of Einstein's equations complete with 'event-horizon-like' singularities; in the present application where the bicrossproduct is viewed as a Poincaré quantum group the latter appear as limiting asymptotes in momentum space, which has been called a 'Planckian bound' on spatial momentum. This is a generic feature of all bicrossproduct models based on noncompact groups. Moreover, the classical group $X = SO_{4,1}$ in the model (24.1) acts on the momentum group $M = \mathbb{R}^3 {>\!\!\!\triangleleft} \mathbb{R}$ and using this action one can come up with an entirely classical picture equivalent to the model. The action of $G = SO_{3,1}$ is highly nonlinear and given by certain vector fields in [23]. We will demonstrate a new phenomenon for the model coming from this nonlinearity with explicit global formulae in the 2D case coming from $SO_{2,1}$.

Finally, a little knowledge can be a dangerous thing and certainly it is possible to claim any number of nonsensical 'predictions' based on an abuse of the mathematics. If one is arguing as a phenomenologist then this does not matter; it does not matter where a formula comes from, one can just posit it and see if it fits the data. However, for a theoretical prediction one must have an actual theory. For this one has to address the following.

- A somewhat complete mathematical framework within which to work (in our case this will be NCG).
- Is the proposal mathematically consistent?
- What are *all* the physical consequences (is it physically consistent?)

Typically in NCG if one modifies one thing then many other things have to be modified for mathematical consistency (e.g. the Poincaré quantum group does not act consistently on ordinary spacetime). There will be many such issues adopting (24.1) and after that is the interpretation of the mathematics physically consistent? If we suppose that a symbol p^0 in the mathematics is the energy then what else does this imply and is the whole interpretation consistent with other expectations? Or we can *suppose* that p^μ generators in the λ-Poincaré quantum group are the physically observed 4-momentum and from the deformed Casimir

$$||p||_\lambda^2 = \vec{p}^2 e^{\lambda p^0} - \frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1) \tag{24.3}$$

claim a VSL prediction but how to justify that? Our approach is to look at noncommutative plane waves (or quantum group Fourier theory) to at least begin to turn such a formula into a theoretical prediction [1]. The model (24.1) does then hold together fairly well for scalar or $U(1)$ fields. Spinors in the model remain problematic and more theoretical development would be needed before predictions involving neutrino oscillations or neutral kaon resonances etc. could have any meaning.

24.2 Basic framework of NCG

The framework that we use has the following elements.

- A spacetime coordinate algebra A, not necessarily commutative.
- Differential calculus done algebraically as a linear map $\mathrm{d} : A \to \Omega^1$ obeying some minimal axioms (here Ω^1 is a bimodule of '1-forms') .
- Symmetries done algebraically (e.g. as a quantum group)
- An *algebraic principle of equivalence:* all constructions are independent of any choice of generators of the algebras (the ability to change coordinates, cf. passive diffeomorphism invariance in usual geometry). This does not mean that we might not prefer to work in some gauge such as in special relativity.
- Insight into the new physics made possible by the particular framework. In our case it is that noncommutative spacetime corresponds to a very natural idea: curved momentum space or *cogravity*.

Taking these in turn, we briefly define a differential calculus. This is common to all approaches to NCG except that in the quantum groups approach one concentrates on Ω^1 in the first instance. Requiring it to be an A–A bimodule says that we can multiply '1-forms' by 'functions' from the left or the right and the two associate:

$$a((\mathrm{d}b)c) = (a\mathrm{d}b)c \quad \forall a, b, c \in A.$$

We also require that d obeys the Leibniz rule

$$\mathrm{d}(ab) = a\mathrm{d}b + (\mathrm{d}a)b$$

and that $\Omega^1 = \mathrm{span}\{a\mathrm{d}b\}$ which is more of a definition than a requirement (if not we would just make Ω^1 smaller). Finally there is an optional 'connectedness' condition that

$$\mathrm{d}a = 0 \Rightarrow a \propto 1.$$

These axioms are all more or less obvious and represent the minimum that any form of geometry would require. They are actually *weaker* than classical differential geometry even when the algebra A is commutative because we have not demanded anywhere that $[a, \mathrm{d}b] = 0$ for all a, b. Demanding that would imply that $\mathrm{d}[a, b] = 0$ for all a, b, which would violate the connectedness condition for any reasonably noncommutative algebra. Given Ω^1 there are some different schemes to extend this to an entire exterior algebra $\Omega = \oplus_n \Omega^n$ with $\mathrm{d}^2 = 0$, basically by some form of 'skew-symmetrised' tensor products of 1-forms.

As soon as one has a calculus one can start to do physics, such as gauge theory, at least at the level where a connection is a noncommutative (antihermitian) 1-form α. Gauge transformations are invertible (unitary) elements u of the noncommutative 'coordinate algebra' and the connection and curvature transform as

$$\alpha \to u^{-1}\alpha u + u^{-1}\mathrm{d}u$$

$$F(\alpha) = \mathrm{d}\alpha + \alpha \wedge \alpha \to u^{-1}F(\alpha)u.$$

Notice that the nonlinear term in F does not automatically vanish since we did not assume that functions and 1-forms commute. Hence we call this $U(1)$-Yang–Mills theory to distinguish it from the Maxwell theory where $F = \mathrm{d}\alpha$. The former detects noncommutative homotopy while the latter detects noncommutative de Rahm cohomology.

We do not actually need much from Hopf algebra theory other than the definitions and to be able to quote a couple of general results. A Hopf algebra or quantum group (we use the terms synonymously) means an algebra H with a unit which at the same time is a 'coalgebra with counit' in a compatible way. By a coalgebra, say over $\mathbb{C}$, we mean

$$\Delta : H \to H \otimes H, \quad \epsilon : H \to \mathbb{C}$$

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta, \quad (\mathrm{id} \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$$

(this is the same as the axioms of an algebra but with arrows reversed and Δ is called the 'coproduct', ϵ the 'counit'.) The compatibility with the algebra structure is that Δ, ϵ should be algebra homomorphisms. In addition for a true quantum group there should exist a map $S : H \to H$ called the 'antipode' such that

$$\cdot(\mathrm{id} \otimes S)\Delta = \cdot(S \otimes \mathrm{id})\Delta = 1\epsilon.$$

If H is a Hopf algebra then H^* is at least an algebra with 'convolution product' $(\phi\psi)(h) = (\phi \otimes \psi)(\Delta h)$ for all $\phi, \psi \in H^*$. For suitable notions of dual it is again a quantum group, the dual one. If H is a generalised symmetry algebra then H^* is like the coordinate algebra on a generalised group. The basic 'classical' example is when $H = U(\mathfrak{g})$ the enveloping algebra of a Lie algebra. This is a Hopf algebra with

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon x = 0, \quad Sx = -x, \quad \forall x \in \mathfrak{g}.$$

Its suitable dual is an algebra of coordinate functions $\mathbb{C}[G]$ on the associated Lie group. In the matrix Lie group case this is generated by matrix element coordinates $\Lambda^{\mu}{}_{\nu}$ with coproduct and counit

$$\Delta\Lambda^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\rho} \otimes \Lambda^{\rho}{}_{\nu}, \quad \epsilon\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}.$$

The antipode is given by matrix inversion. These two examples are all we need in most of the present chapter.

For a Hopf algebra H to act on an algebra A we require that the product map $A \otimes A \to A$ of the algebra is an intertwiner. The action of H on A extends to $A \otimes A$ via the coproduct, so we require

$$h{\triangleright}(ab) = \cdot((\Delta h){\triangleright}(a \otimes b)), \quad h{\triangleright}1 = \epsilon(h)1$$

where $h{\triangleright}a$ denotes the action of h on a and $\triangleright$ is similarly being used twice on the right hand side of the first expression. For a calculus on A to be covariant we require that H acts on Ω^1, and that d and the bimodule product maps are intertwiners. Part of the latter reads for example as

$$h{\triangleright}(a\mathrm{d}b) = \cdot(\mathrm{id} \otimes \mathrm{d})((\Delta h){\triangleright}(a \otimes b)).$$

Simply defining this as the action on Ω^1 and knowing that it is well-defined implies the rest. H always acts on H^* from both the left and the right by the coregular representation (e.g. the left action is $h{\triangleright}\phi = \phi((\ )h)$). In that case one can seek a calculus Ω^1 on H^* that is left and right covariant (bicovariant) [25]. This makes H^* into the coordinate algebra of a 'quantum Lie group'. Note that one can work entirely with H^* and never mention H provided one uses the broadly equivalent notion of a 'coaction' $\Delta_R : \Omega^1 \to \Omega^1 \otimes H^*$ instead of an action $\triangleright$ of H.

Similarly, an integral on an algebra A just means a linear map $\int : A \to \mathbb{C}$. It is said to be H-covariant if

$$\int (h{\triangleright}a) = \epsilon(h) \int a, \quad \forall a \in A,\ h \in H$$

with respect to a covariant action $\triangleright$ of H on A. For a quantum group $A = H^*$ say (see above) if an H-covariant integral exists it is unique, cf. the Haar measure on a group. Again one can define it entirely with respect to H^* if one uses the notion of a coaction.

The **principle of algebraic equivalence** is the analogue of the statement in usual geometry that all constructions are covariant under coordinate change. This should not be confused with the physical equivalence principle, it is valid even in Newtonian mechanics and just says that we are free to change variables for example from Cartesian to polar coordinates. This is what separates out the systematic framework of NCG from 'ad hoc' constructions. This also makes clear why from our point of view any argument for a physical prediction based on Casimirs in the Poincaré quantum group alone is completely empty. The reason is that most quantum groups including the bicrossproduct one for the spacetime (24.1) are *as algebras* isomorphic to the usual undeformed classical enveloping algebra. In other words there are new coordinates P^μ in which the quantum group is undeformed as an algebra and its Casimir is the usual $\vec{P}^2 - (P^0)^2$. In this case the so-called prediction is like mistakenly working in polar coordinates while thinking they were Cartesian coordinates and being excited by the form of the Laplacian. In fact in the P^μ

coordinates the coproduct of the quantum group also looks quite different but since the Casimir depends only on the algebra it does not see this. Where the coproduct shows up is in tensor product actions of the quantum group (see above) and in truth the classical dispersion relation is not fully characterised by being a Casimir but by further properties in relation to this. Equivalently, how do we justify that p^μ in (24.3) and not P^μ are the physical 4-momentum? The only way to know is to do experiments, and those experiments will likely involve objects such as plane waves that depend on the full quantum group structure not only the algebra. This means that early 'predictions' based only on the algebra were wishful speculations and not theoretical predictions.

On the topic of changing variables note that if x_i are generators of A then one might typically have $\mathrm{d}x_i$ forming a basis over A of Ω^1. In this case the conjugate partial derivatives are defined by

$$\mathrm{d}a = \sum_i (\partial^i a)\mathrm{d}x_i. \tag{24.4}$$

Notice that precisely when differentials do not commute with 1-forms, these ∂^i will not obey the usual Leibniz rule themselves. It is the coordinate-invariant object d which obeys the Leibniz rule. Bases of Ω^1 do not always exist and when they do they might not have the expected number, i.e. there might be additional auxiliary 1-forms beyond the classical basic 1-forms (see later). Moreover, under a change of coordinates we leave d unchanged and recompute the partial derivatives conjugate to the new basis. This is actually how it is done in classical differential geometry, only now we should do it in the noncommutative algebraic setting. The same remarks apply to the integral which will take a specific form when computed with one set of generators and another with a different set but with the same answer.

Finally, we promised one theorem and perhaps the most relevant is the quantum group Fourier transform [10, paperback edition.]. If H, H^* are a dual pair of Hopf algebras (for some suitable dual) with dual bases $\{e_a\}$ and $\{f^a\}$ respectively, we define

$$\mathcal{F} : H \to H^*, \quad \mathcal{F}(h) = \sum_a \int (e_a h) f^a, \quad \mathcal{F}^{-1}(\phi) = S^{-1} e_a \int f^a \phi$$

where we assume the antipode S is invertible (which is typical). This theory works nicely for finite-dimensional Hopf algebras but can also be applied at least formally to infinite-dimensional ones. Thus if $U(\mathfrak{g})$ and $\mathbb{C}[G]$ mentioned above are *suitably completed* one has at least formally

$$\mathcal{F} : \mathcal{C}[G] \to U(\mathfrak{g}), \quad \mathcal{F}^{-1} : U(\mathfrak{g}) \to \mathbb{C}[G].$$

The best approach here is actually to work with Hopf–von Neumann or C^*-algebra versions of these Hopf algebras. For example $\mathbb{C}[G]$ might become an algebra built from continuous functions on G with rapid decay at infinity in the noncompact case. The role of $U(\mathfrak{g})$ might become the group C^*-algebra which is a completion of the functions on G with convolution product. However, we do not need to make this too precise at least for the bicrossproduct model. Formally we take a basis $\{\delta_u\}$ of δ-functions on G (more precisely one should smear or approximate these). For dual basis we take the group elements $u \in U(\mathfrak{g})$ formally as exponential elements in the completed enveloping algebra. Then

$$\mathcal{F}(f) = \int_G \mathrm{d}u f(u)u \approx \int_{U\subset\mathbb{R}^n} \mathrm{d}^n k\, J(k) f(k) e^{\imath k^i e_i} \tag{24.5}$$

where e_i are a basis of $\mathfrak{g}$ so that the k^i are a local coordinate system for the group valid in some open domain U and $J(k)$ the Jacobian for this change of variables. There are subtleties particularly in the compact case (e.g. the case of $G = SU_2$ studied in detail in [4] as some kind of 'noncommutative sampling theory'). If G is a curved position space then the natural momenta e_i are noncommuting covariant derivates and in the highly symmetric case of a non-Abelian group manifold they generate noncommutative momemtum 'operators' $U(\mathfrak{g})$ instead of usual commutative coordinates. So actually physicists have been needing NCG – in momentum space – for about a century now, without knowing its framework. Indeed, Fourier transform is usually abandoned in any 'functional' form on a nonAbelian group (instead one works with the whole category of modules, $3j$ and $6j$-symbols, etc.) but quantum group methods allow us for the first time to revert to Fourier transform as a functional transform, just with noncommutative functions $U(\mathfrak{g})$. If this seems strange consider that the phase space of a particle on G is $T^*G = \mathfrak{g}^* \times G$ and has quantum algebra of observables $U(\mathfrak{g}){\triangleright\!\!\!<}\mathbb{C}[G]$ (in some form) – this is an example of Mackey quantisation. Here $U(\mathfrak{g})$ is contained in the algebra of obvervables as the quantisation of $\mathbb{C}[\mathfrak{g}^*]$. This explains the top line in Figure 24.1: gravity means noncommutative momentum space. Note that quantum mechanics itself is about cross relations between position and momentum as indicated for flat space in the bottom line of Figure 24.1. We work in units where its associated variable $\hbar = 1$.

On the other hand, now suppose that G is curved *momentum space* then the quantum group Fourier transform takes us equally well to a noncommutative enveloping algebra $U(\mathfrak{g})$ regarded as 'coordinate functions' on some noncommutative position space. This is the exact form of (24.1) and (24.2) where x_μ or x_i are the Lie algebra basis. So these noncommutative spacetimes are equivalent under quantum group Fourier transform to *classical* but curved momentum space. This is the middle line in Figure 24.1: noncommutativity in position space which should be interpreted as curvature in momentum space, i.e. the dual of gravity or **cogravity**. This is an

	Position	Momentum
Gravity	Curved $\sum_\mu x_\mu^2 = \frac{1}{\gamma^2}$	Noncommutative $[p_i, p_j] = 2\imath\gamma\epsilon_{ijk}p_k$
Cogravity	Noncommutative $[x_i, x_j] = 2\imath\lambda\epsilon_{ijk}x_k$	Curved $\sum_\mu p_\mu^2 = \frac{1}{\lambda^2}$
Quantum Mech.	$[x_i, p_j] = \imath\delta_{ij}$	

Fig. 24.1. Noncommutative spacetime means curvature in momentum space. The equations are for illustration.

independent physical effect and comes therefore with its own length scale which we denote λ. These ideas were introduced in this precise form by the author in the mid 1990s on the basis of the quantum group Fourier transform [9]. Other works on the quantum group Fourier transform in its various forms include [5; 6; 7]

24.3 Bicrossproduct quantum groups and matched pairs

We will give an explicit construction of the bicrossproduct quantum groups of interest, but let us start with a general theorem from the theory of Hopf algebras. The starting point is a theory of factorisation of a group X into subgroups M, G such that $X = MG$. It means every element of X can be uniquely expressed as a normal ordered product of elements in M, G. In this situation, define a left action $\triangleright$ of G on M and a right action $\triangleleft$ of M on G by the equation

$$us = (u\triangleright s)(u\triangleleft s), \quad \forall u \in G,\ s \in M. \tag{24.6}$$

These actions obey

$$\begin{aligned} u\triangleleft e &= u, \quad e\triangleright s = s, \quad u\triangleright e = e, \quad e\triangleleft s = e \\ (u\triangleleft s)\triangleleft t &= u\triangleleft(st), \quad u\triangleright(v\triangleright s) = (uv)\triangleright s \\ u\triangleright(st) &= (u\triangleright s)((u\triangleleft s)\triangleright t) \\ (uv)\triangleleft s &= (u\triangleleft(v\triangleright s))(v\triangleleft s) \end{aligned} \tag{24.7}$$

for all $u, v \in G$, $s, t \in M$. Here e denotes the relevant group unit element. A pair of groups equipped with such actions is said to be a ‘matched pair’ (M, G). One can then define a ‘double cross product group’ $M \bowtie G$ with product

$$(s, u).(t, v) = (s(u\triangleright t), (u\triangleleft t)v) \tag{24.8}$$

and with M, G as subgroups. Since it is built on the direct product space, the bigger group factorizes into these subgroups and in fact one recovers X in this way. These notions were known for finite groups since the 1910s but in a Lie group setting [12; 15] one has the similar notion of a ‘local factoristion’ $X \approx MG$ and a corresponding double cross sum $\mathfrak{m}\bowtie\mathfrak{g}$ of Lie algebras. Then the differential version of the equations (24.7) become a matter of a pair of coupled first order differential equations for families of vector fields α_ξ on M and β_ϕ on G labelled by $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{m}$ respectively. We write these vector fields in terms of Lie-algebra valued functions $A_\xi \in C^\infty(M,\mathfrak{m})$ and $B_\phi \in C^\infty(G,\mathfrak{g})$ according to left and right translation from the tangent space at the identity:

$$\alpha_\xi(s) = R_{s*}(A_\xi(s)), \quad \beta_\phi(u) = L_{u*}(B_\phi(u)). \tag{24.9}$$

In these terms the matched pair equations become

$$\begin{aligned} A_\xi(st) &= A_\xi(s) + \mathrm{Ad}_s(B_{\xi\triangleleft s}(t)), \quad A_\xi(e) = 0 \\ B_\phi(uv) &= \mathrm{Ad}_v^{-1}(A_{v\triangleright\phi}(u)) + B_\phi(v), \quad B_\phi(e) = 0 \end{aligned} \tag{24.10}$$

along with auxiliary data a pair of linear actions $\triangleright$ of G on $\mathfrak{m}$ and $\triangleleft$ of M on $\mathfrak{g}$ exponentiating Lie algebra actions $\triangleright, \triangleleft$ of $\mathfrak{g}, \mathfrak{m}$ respectively. Finally, (24.10) becomes a pair of differential equations if we let u, t be infinitesimal i.e. elements $\eta \in \mathfrak{g}$, $\psi \in \mathfrak{m}$ say of the Lie algebra. Then

$$\psi^{\mathrm{R}}(A_\xi)(s) = \mathrm{Ad}_s((\xi\triangleleft s)\triangleright\psi), \quad \eta^{\mathrm{L}}(B_\phi)(v) = \mathrm{Ad}_{v^{-1}}(\eta\triangleleft(v\triangleright\phi)) \tag{24.11}$$

where η^{L} is the left derivative on the Lie group G generated by η and ψ^{R} the right derivative on M generated by ψ. Note that this implies

$$\psi^{\mathrm{R}}(A_\xi)(e) = \xi\triangleright\psi, \quad \eta^{\mathrm{L}}(B_\phi)(e) = \eta\triangleleft\phi \tag{24.12}$$

which shows how the auxiliary data are determined. These nonlinear equations were proposed in [13] as a toy model of Einstein's equations and solved for $\mathbb{R}\bowtie\mathbb{R}$ where they were shown to have singularities and accumulation points not unlike a black-hole event horizon. Such accumulation points are a typical feature of (24.10) when both groups are noncompact. We have flipped conventions relative to [10] in order to have a left action of the Poincaré quantum group in our applications.

One has to solve these equations globally (taking account of any singularities) in order to have honest Hopf–von Neumann or Hopf C^*-algebra quantum groups; there are some interesting open problems there. However, for simply a Hopf algebra at an algebraic level one needs only the initial data (24.12) of the matched pair, namely the Lie algebra actions $\triangleright, \triangleleft$ corresponding to $\mathfrak{m} \bowtie \mathfrak{g}$. Clearly then $U(\mathfrak{m} \bowtie \mathfrak{g}) = U(\mathfrak{m}) \bowtie U(\mathfrak{g})$ as a Hopf algebra double cross product or factorisation of Hopf algebras [14]. We content ourselves with one theorem from this theory.

Theorem 1 *Let (H_1, H_2) be a matched pair of quantum groups with $H_1 \bowtie H_2$ the associated double cross product. Then (i) there is another quantum group denoted $H = H_2 \blacktriangleright\!\!\triangleleft H_1^*$ called the 'semidualisation' of the matched pair. (ii) This* *quantum group acts covariantly on $A = H_1$ from the left. (iii) Its dual is the other semidualisation $H^* = H_2^* \blacktriangleright\!\!\triangleleft H_1$ and coacts covariantly on H_1 from the right.*

Applying this theorem to $U(\mathfrak{m}) \bowtie U(\mathfrak{g})$ implies a bicrossproduct quantum group $U(\mathfrak{g}) \triangleright\!\!\blacktriangleleft U(\mathfrak{m})$ acting covariantly on $A = U(\mathfrak{m})$ from the left. Here it is assumed that $\mathbb{C}[M]$ is a suitable algebraic version of the coordinate algebra of functions on M dual to $U(\mathfrak{m})$. The bicrossproduct quantum group is generated by $U(\mathfrak{g})$ and the commutative algebra of functions on M, with cross relations and coproduct

$$[f, \xi] = \alpha_\xi(f) \tag{24.13}$$

$$\Delta\xi = \xi \otimes 1 + \Delta_L(\xi), \quad \Delta_{\mathrm{L}}(\xi) \in \mathbb{C}[M] \otimes \mathfrak{g}, \quad \Delta_{\mathrm{L}}(\xi)(s) = \xi \triangleleft s \tag{24.14}$$

where Δ_{L} is the left coaction induced by the auxiliary linear action $\triangleleft$ of M on $\mathfrak{g}$. Meanwhile, the coproduct on $f \in \mathbb{C}[M]$ is that of $\mathbb{C}[M]$ which appears as a subHopf algebra. This is how we shall construct the bicrossproduct Poincaré quantum group enveloping algebra. Its canonical action on $U(\mathfrak{m})$ from the theorem has $\xi \in \mathfrak{g}$ acting by the action $\triangleright$ on $\mathfrak{m}$ and $f \in \mathbb{C}[M]$ acting by $(\mathrm{id} \otimes f)\Delta$ using the coproduct of $U(\mathfrak{m})$.

Equally, there is a natural dual bicrossproduct as the Hopf algebra $\mathbb{C}[G] \blacktriangleright\!\!\triangleleft U(\mathfrak{m})$ coming from the same factorisation data. We denote by $a_\mu \in \mathfrak{m}$ the 'nonAbelian translation' generators of $U(\mathfrak{m})$ and by $\Lambda^\mu{}_\nu$ any mutually commutative classical coordinates of the 'Lorentz group' G (as they will be in our application). They obey

$$[a_\rho, \Lambda^\mu{}_\nu] = \beta_{a_\rho}(\Lambda^\mu{}_\nu), \quad \Delta a_\mu = 1 \otimes a_\mu + \Delta_{\mathrm{R}}(a_\mu) \tag{24.15}$$

where β is the other vector field in the matched pair and the coaction

$$\Delta_{\mathrm{R}}(a_\mu) = a_\nu \otimes \Lambda^\nu{}_\mu \in \mathfrak{m} \otimes \mathbb{C}[G], \quad \Delta_{\mathrm{R}}(\phi)(u) = u \triangleright \phi \tag{24.16}$$

is built similarly but now from $\triangleright$ in the matched pair data. *By definition* the $\Lambda^\mu{}_\nu$ are the coordinate functions appearing in Δ_{R} on the a_μ basis. The construction, like (24.13)–(24.14), is independent of any chosen generators but for Poincaré group coordinate functions one tends to use such notations. If we denote by x_μ the 'spacetime' generators of a second copy of $U(\mathfrak{m})$ then the coaction of $\mathbb{C}[G] \blacktriangleright\!\!\triangleleft U(\mathfrak{m})$ in Theorem 1 is

$$\Delta_{\mathrm{R}}^{\mathrm{Poinc}}(x_\mu) = 1 \otimes a_\mu + \Delta_R(x_\mu) = 1 \otimes a_\mu + x_\nu \otimes \Lambda^\nu{}_\mu. \tag{24.17}$$

In summary, the bicrossproduct theory constructs both the deformed Poincaré enveloping algebra and coordinate algebra at the same time and provides their canonical action and coaction respectively on another copy of $U(\mathfrak{m})$ as noncommutative spacetime.

24.3.1 Nonlinear factorisation in the 2D bicrossproduct model

Such models provide noncommutative spacetimes and Poincaré quantum groups in any dimension n based on a local factorisation of $SO_{n,1}$ or $SO_{n-1,2}$. The 4D model is known [23] but the 2D case has the same essential structure and we shall use this now to explore global and nonlinear issues, with full derivations.

The first remark in the 2D case is that for a convenient description of the global picture we work not with $SO_{2,1}$ exactly but its double cover $X = SL_2(\mathbb{R}) \to SO_{2,1}$. The map here at the Lie algebra level is

$$\tilde{a}_0 = \frac{\lambda}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \to \lambda\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \tilde{N} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$\tilde{a}_1 = \lambda\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \to \frac{\lambda}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

for xt, yt boosts and xy-rotations with $++-$ signature being generated by $-\imath\tilde{a}_0, \sqrt{2}\tilde{N}, \tilde{M} = -\imath\sqrt{2}(\lambda\tilde{N} - \tilde{a}_1)$ respectively. The $\tilde{a}_i$ close to the Lie algebra $[\tilde{a}_1, \tilde{a}_0] = \lambda\tilde{a}_1$ so generate a 2-dimensional nonAbelian Lie group $M = \mathbb{R}{\rtimes\!\!\!\triangleleft}\mathbb{R}$ along with $G = SO_{1,1} = \mathbb{R}$ generated by $\tilde{N}$. This gives a factorisation $SL_2(\mathbb{R}) \approx (\mathbb{R}{\rtimes\!\!\!\triangleleft}\mathbb{R}).SO_{1,1}$ as

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\mu & 0\\ \frac{ac-bd}{a\mu} & \frac{1}{a\mu}\end{pmatrix}\begin{pmatrix}\frac{1}{\mu} & \frac{b}{a\mu}\\ \frac{b}{a\mu} & \frac{1}{\mu}\end{pmatrix}; \quad \mu = \sqrt{1 - \frac{b^2}{a^2}}, \quad |b| < |a|.$$

This is valid in the domain shown which includes the identity in the group. It *cannot* be a completely global decomposition because topologically $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R}) = SO_{2,1}$ have a compact direction and so cannot be described globally by 3 unbounded parameters (there is a compact SO_2 direction generated by $\tilde{M}$). If one does not appreciate this and works with unbounded parameters one will at some stage encounter coordinate singularities, which is the origin of the Planckian bound for this model as well as other new effects (see below). From an alternative constructive point of view, as we solve the matched-pair equations for $(\mathbb{R}{\rtimes\!\!\!\triangleleft}\mathbb{R}){\bowtie}\mathbb{R}$ we must encounter a singularity due to the nonlinearity. Note that this nonAbelian factorisation and our construction of it cf. [23] is not the KAN decomposition into three subgroups.

In the factorisation we now change variables to

$$a\mu = e^{\frac{\lambda}{2}p^0}, \quad ac - bd = \lambda p^1 e^{\lambda p^0}, \quad \sinh\left(\frac{\theta}{2}\right) = \frac{b}{a\mu}$$

where we introduce p^0, p^1 as coordinates on the group M and θ as the coordinate of $SO_{1,1}$. Here λ is a fixed but arbitrary normalisation constant and we have $\theta/2$ because we are working with the double cover of $SO_{2,1}$. According to the group law of matrix multiplication, the p_i viewed abstractly as functions enjoy the coproduct

$$\Delta\begin{pmatrix} e^{\frac{\lambda}{2}p^0} & 0 \\ \lambda p^1 e^{\frac{\lambda}{2}p^0} & e^{-\frac{\lambda}{2}p^0}\end{pmatrix} = \begin{pmatrix} e^{\frac{\lambda}{2}p^0} & 0 \\ \lambda p^1 e^{\frac{\lambda}{2}p^0} & e^{-\frac{\lambda}{2}p^0}\end{pmatrix} \otimes \begin{pmatrix} e^{\frac{\lambda}{2}p^0} & 0 \\ \lambda p^1 e^{\frac{\lambda}{2}p^0} & e^{-\frac{\lambda}{2}p^0}\end{pmatrix}$$

where matrix multiplication is understood. Thus in summary we have

$$[p^0, p^1] = 0, \quad \Delta p^0 = p^0 \otimes 1 + 1 \otimes p^0, \quad \Delta p^1 = p^1 \otimes 1 + e^{-\lambda p^0} \otimes p^1 \tag{24.18}$$

$$S(p^0, p^1) = (-p^0, -e^{\lambda p^0} p^1) \tag{24.19}$$

as the Hopf algebra $\mathbb{C}[\mathbb{R}{\bowtie\!\!\!\triangleleft}\mathbb{R}]$ corresponding to our nonAbelian momentum group and its group inversion.

We now take group elements in the wrong order and refactorise:

$$\begin{pmatrix}\cosh(\frac{\theta}{2}) & \sinh(\frac{\theta}{2}) \\ \sinh(\frac{\theta}{2}) & \cosh(\frac{\theta}{2})\end{pmatrix}\begin{pmatrix} e^{\frac{\lambda}{2}p^0} & 0 \\ \lambda p^1 e^{\frac{\lambda}{2}p^0} & e^{-\frac{\lambda}{2}p^0}\end{pmatrix} = \begin{pmatrix}(C + S\lambda p^1)e^{\frac{\lambda}{2}p^0} & Se^{-\frac{\lambda}{2}p^0} \\ (S + C\lambda p^1)e^{\frac{\lambda}{2}p^0} & Ce^{-\frac{\lambda}{2}p^0}\end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{\lambda}{2}p^{0\prime}} & 0 \\ \lambda p^{1\prime} e^{\frac{\lambda}{2}p^{0\prime}} & e^{-\frac{\lambda}{2}p^{0\prime}}\end{pmatrix}\begin{pmatrix}\cosh(\frac{\theta'}{2}) & \sinh(\frac{\theta'}{2}) \\ \sinh(\frac{\theta'}{2}) & \cosh(\frac{\theta'}{2})\end{pmatrix}$$

where $S = \sinh(\theta/2)$, $C = \cosh(\theta/2)$, which gives according to (24.6):

$$p^{0\prime} = \theta\triangleright p^0 = p^0 + \frac{1}{\lambda}\ln\Big((C + S\lambda p^1)^2 - S^2 e^{-2\lambda p^0}\Big) \tag{24.20}$$

$$p^{1\prime} = \theta\triangleright p^1 = \frac{(C + S\lambda p^1)(S + C\lambda p^1) - SCe^{-2\lambda p^0}}{\lambda\left((C + S\lambda p^1)^2 - S^2 e^{-2\lambda p^0}\right)} \tag{24.21}$$

$$\theta' = \theta\triangleleft(p^0, p^1) = 2\mathrm{arcsinh}\left(\frac{Se^{-\lambda p^0}}{\sqrt{(C + S\lambda p^1)^2 - S^2 e^{-2\lambda p^0}}}\right) \tag{24.22}$$

where we have written formulae in the domain where $C + S\lambda p^1 > 0$. The refactorisation is possible (so the actions $\triangleright$, $\triangleleft$ are well-defined) only when

$$\Big(C + S(\lambda p^1 - e^{-\lambda p^0})\Big)\Big(C + S(\lambda p^1 + e^{-\lambda p^0})\Big) > 0. \tag{24.23}$$

This can be analysed in terms of the regions in Figure 24.2, which shows orbits under $\triangleright$ in (p^0, p^1) space. One can check from the expressions above that these orbits are lines of constant values of

$$||p||^2_\lambda = (p^1)^2 e^{\lambda p^0} - \frac{2}{\lambda^2}\big(\cosh(\lambda p^0) - 1\big) = \frac{e^{\lambda p^0}}{\lambda^2}\Big(\lambda^2 (p^1)^2 - (1 - e^{-\lambda p^0})^2\Big) \tag{24.24}$$

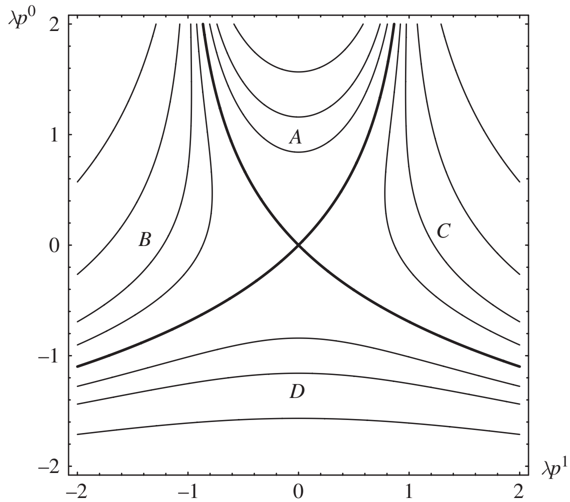

Fig. 24.2. Deformed orbits under the Lorentz group in the bicrossproduct model momentum group. Increasing θ moves anticlockwise along an orbit in regions A, D and clockwise in regions B, C.

which deforms the Minkowski norm in momentum space. It is also invariant under inversion in the curved momentum group and hence under the antipode S. Note that this has nothing to do with the Poincaré algebra which we have not constructed yet; it is part of the nonlinear geometry arising from the factorisation.

Theorem 2 *(i) The actions $\triangleright$, $\triangleleft$ are defined for all θ if and only if (p^0, p^1) lies in the upper mass shell (region A).*
(ii) For any other (p^0, p^1) there exists a finite boost θ_c that sends $p^0 \to -\infty$, after which $\triangleright$ breaks down.
(iii) For any θ there exists a critical curve not in region (A) such that approaching it sends $\theta \to \pm\infty$, after which $\triangleleft$ breaks down.

For the proof we use the shorthand $q \equiv e^{-\lambda p^0}$. We analyse the situation for the two cases $S > 0$ and $S < 0$; if $S = 0$ then the condition (24.23) always holds. Doing the first case, to lie in regions A, C means $\lambda p^1 + 1 - q \geq 0$. Hence

$$C + S(\lambda p^1 - q) = (C - S) + S(\lambda p^1 + 1 - q) > 0$$

which also implies that the other factor in (24.23) is also positive, so the condition holds. But conversely, strictly inside regions B, D mean that $q - \lambda p^1 > 1$ and $C + S(\lambda p^1 - q) = 0$ has a solution $\theta_c > 0$ according to $\coth(\frac{\theta_c}{2}) = q - \lambda p^1$. We also note that our assumption $C + S\lambda p^1 > 0$ holds here and for all smaller θ. As $\theta \to \theta_c$ from below, the denominator or argument of log in the actions (24.20)–(24.21) $\to 0$ and the transformed $p^{0\prime} \to -\infty$. If $S < 0$ then $\lambda p^1 + q - 1 \leq 0$ in regions A, B

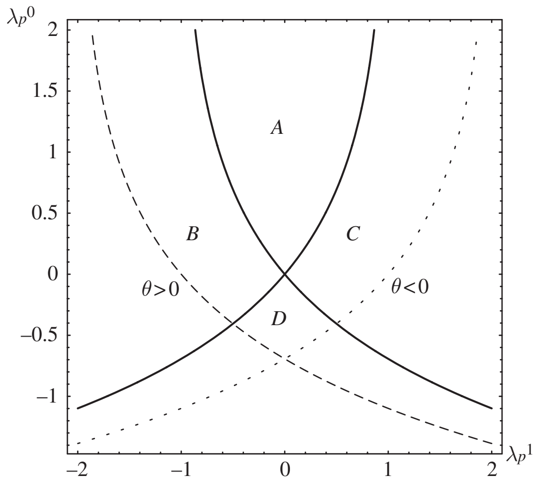

Fig. 24.3. Dashed (dotted) examples of critical curves for given θ. As (p^0, p^1) approaches from above its action sends $\theta \to \pm\infty$.

means that $C + S(\lambda p^1 - q) > 0$ and (24.23) holds as before. Conversely, to be strictly inside regions C, D means $\lambda p^1 + q > 1$ and hence $-\coth\frac{\theta_c}{2} = \lambda p^1 + q$ has a solution with $\theta_c < 0$, where the denominators or argument of log again$\to 0$ from above as $\theta \to \theta_c$ from above.

To give an example, consider a point in region D down from the origin, so $p^1 = 0$ and $p^0 < 0$. Then $e^{-\lambda p^{0\prime}} = e^{-\lambda p^0}/(1 - \sinh^2(\frac{\theta}{2})(e^{-2\lambda p^0} - 1))$ blows up as $|\theta| \to |\theta_c|$ from below, where

$$\theta_c = \pm 2\mathrm{arcsinh}\left(\frac{1}{\sqrt{e^{-2\lambda p^0} - 1}}\right) = \pm\ln\coth(-\frac{\lambda}{2}p^0).$$

Pushing the arguments the other way, for any value $\theta \neq 0$ we can clearly find a critical curve of constant $q - \lambda p^1$ from the domains B, D or of $q + \lambda p^1$ from C, D, according to the sign of θ, such that the same denominator factor, now in (24.22), vanishes as we approach the critical curve from the origin. This is shown in Figure 24.3. The physical meaning of this will be given later as infinite uncertainty when this happens.

In summary, the nonlinearity behind the matched pair equations and the resulting action and back reaction between momentum and $SO_{1,1}$ has several consequences. We see in Figure 24.2 that the $p^0 > 0$ mass shells are now cups with almost vertical walls, compressed into the vertical tube

$$|p^1| < \lambda^{-1}.$$

In other words, the spatial momentum is bounded above by the Planck momentum scale (if λ is the Planck time). Indeed, this is immediate from (24.24). Such singularities expressed in accumulation regions are a main discovery of the noncompact bicrossproduct theory visible already in the original examples [13]. They are a direct consequence of the nonlinearity but we also see their origin in the fact that the true group factorisation has a 'curled up' compact direction. Moreover, this much-noted feature of the model is only a small part of the story. We see that the fuller story is that any point outside this region is boosted to infinite negative p^0 by a finite boost with a similar story for θ and finite momentum as we saw in Figure 24.3. Indeed the actions $\triangleright$, $\triangleleft$ breakdown at such points as the factorisation itself breaks down. Note also that the group inversion which is the natural reversal under CPT symmetry takes us from the 'best' region A to the 'worst' region D, which is a **fundamental time-asymmetry or non-reversability** of the bicrossproduct model.

24.3.2 Bicrossproduct $U_\lambda(\mathrm{poinc}_{1,1})$ *quantum group*

Now, consider θ infinitesimal, i.e. we differentiate all expressions (24.20)–(24.21) by $\frac{\partial}{\partial\theta}|_0$ which is all we need for the algebraic part of the bicrossproduct Hopf algebra (the full operator algebra structure needs the full global data). Thus denoting N the Lie algebra generator conjugate to θ, we have from the above the vector field and actions:

$$\alpha_{\imath N} = \frac{\partial}{\partial\theta}|_0 = p^1\frac{\partial}{\partial p^0} + \frac{1}{2}\left(\frac{1-e^{-2\lambda p^0}}{\lambda} - \lambda(p^1)^2\right)\frac{\partial}{\partial p^1}$$

$$p^0\triangleleft N = -\imath\frac{\partial p^{0\prime}}{\partial\theta}|_0 = -\imath p^1,\quad p^1\triangleleft N = -\imath\frac{\partial p^{1\prime}}{\partial\theta}|_0 = -\frac{\imath}{2}\left(\frac{1-e^{-2\lambda p^0}}{\lambda} - \lambda(p^1)^2\right)$$

where the action $\triangleleft$ flips to the other way because θ is really a coordinate function on $SO_{1,1}$ now being evaluated against N. A right-handed cross product by this action gives the relations

$$[p^0, N] = -\imath p^1,\quad [p^1, N] = -\frac{\imath}{2}\left(\frac{1-e^{-2\lambda p^0}}{\lambda} - \lambda(p^1)^2\right).$$

Similarly differentiating the action (24.22) *on* θ at $\theta = 0$ gives the action of an element of $\mathbb{R}\bowtie\mathbb{R}$ on N, which we view equivalently as a coaction Δ_{L} of the coordinate algebra in algebraic terms, to find,

$$(p^0, p^1)\triangleright N = e^{-\lambda p^0}N \;\Rightarrow\; \Delta_{\mathrm{L}}(N) = e^{-\lambda p^0}\otimes N$$

which yields the coproduct and resulting antipode

$$\Delta N = N\otimes 1 + e^{-\lambda p^0}\otimes N,\quad SN = -e^{\lambda p^0}N$$

to complete the structure of $U_\lambda(\mathrm{poinc}_{1,1}) \equiv U(so_{1,1}) \triangleright\!\!\blacktriangleleft \mathbb{C}[\mathbb{R} \rtimes \mathbb{R}]$ along with (24.18). Note that as $\lambda \to 0$ we obtain the 2D Poincaré algebra with the usual additive coproduct of $U(\mathrm{poinc}_{1,1})$ as expected. Moreover, the deformed norm (24.24) is necessarily a constant of motion and hence killed by the vector $\tilde{N}$ (one may check this easily enough). Hence it is central (a Casimir) for the deformed algebra.

In the 4D case the factorisation $SO_{4,1} \approx (\mathbb{R}^3 \rtimes \mathbb{R}).SO_{3,1}$ leading to Poincaré quantum group $U(so_{3,1}) \triangleright\!\!\blacktriangleleft \mathbb{C}[\mathbb{R}^3 \rtimes \mathbb{R}]$ is too complicated to give explicitly but has similar global issues, likewise for $SO_{3,2}$. It was instead constructed in [23] by identifying the solution of the matched pair equations at the differentiated level as a result of finding the Hopf algebra itself (we have seen that only the differentials of the actions $\triangleright, \triangleleft$ enter into the Hopf algebra itself) and integrating these. The Hopf algebra now has commuting translation generators p^μ, rotations M_i and boosts N_i with cf. [23] but in opposite conventions for the coproduct:

$$[p^\mu, p^\nu] = 0, \quad [M_i, M_j] = \imath \epsilon_{ij}{}^k M_k, \quad [N_i, N_j] = -\imath \epsilon_{ij}{}^k M_k$$

$$[M_i, N_j] = \imath \epsilon_{ij}{}^k N_k, \quad [p^0, M_i] = 0, \quad [p^i, M_j] = \imath \epsilon^i{}_{jk} p^k, \quad [p^0, N_i] = -\imath p_i,$$

as usual, and the modified relations and coproduct

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j \left(\frac{1 - e^{-2\lambda p^0}}{\lambda} + \lambda \vec{p}^2 \right) + \imath \lambda p^i p_j,$$

$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda \epsilon_{ij}{}^k p^j \otimes M_k,$$

$$\Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

and the usual additive coproducts on p^0, M_i. The deformed Minkowski norm now has the same form as stated in (24.3) with the same picture as in Figure 24.2 except that now the horizontal axis is any one of the p_i (there is a suppressed rotational symmetry among them). As before, for the same fundamental reasons of nonlinearity of the matched pair equations (24.10), we have a Planckian bound $|\vec{p}| < \lambda^{-1}$ for particles on the $p^0 > 0$ mass-shell but we also have finite boosts sending off-shell or retarded momenta off to infinitely negative 'energy'.

We have skipped over the 3D case, which is of a similar form but without as many rotations and boosts. It was the first example in the general family to be found, by the author in [9; 12; 16; 21] as the bicrossproduct $U(so_3) \blacktriangleright\!\!\triangleleft \mathbb{C}[\mathbb{R}^2 \rtimes \mathbb{R}]$ (initially in a Hopf–von Neumann algebra setting), from the factorisation $SO_{3,1} = (\mathbb{R}^2 \rtimes \mathbb{R}).SO_3$. We have similarly

$$SO_{m,n} \approx (\mathbb{R}^{m+n-2} \rtimes \mathbb{R}).SO_{m,n-1}$$

and it has been conjectured that the resulting bicrossproducts are all (nontrivially) isomorphic to certain contractions of the q-deformation quantum groups $U_q(so_{m,n})$. In the 4D case the contraction of $U_q(so_{3,2})$ was found first [8] with the bicrossproduct of form found later in [23]. Note that the physical interpretation of the generators coming from contractions is completely different from the bicrossproduct one.

24.3.3 Bicrossproduct $\mathbb{C}_\lambda$[Poinc] quantum group

We now apply the same matched pair factorisation data (24.20)–(24.22) but now to construct the dual Hopf algebra. We start with $\mathbb{C}[SO_{1,1}]$ naturally described by generators $s = \sinh(\theta)$ and $c = \cosh(\theta)$ with relations $c^2 - s^2 = 1$ (which form the matrix $\Lambda^\mu{}_\nu$) and matrix coproduct

$$\Delta\begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} c & s \\ s & c \end{pmatrix} \otimes \begin{pmatrix} c & s \\ s & c \end{pmatrix}, \quad S\begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix}.$$

To see how this arises in our theory, recall that we worked with $S = \sinh(\frac{\theta}{2})$ and $C = \cosh(\frac{\theta}{2})$ which (similarly) describe the double cover of $SO_{1,1}$ in coordinate form. We differentiate (24.22) written in terms of S by $\frac{\partial}{\partial p^\mu}|_{p^\mu=0}$ to obtain the vector fields β and infinitesimal left action of the Lie algebra $[a_0, a_1] = \imath\lambda a_1$ on functions of θ:

$$\beta_{\imath a_0} = \frac{\partial}{\partial p^0}|_0 = -2\lambda CS\frac{\partial}{\partial\theta}, \quad \beta_{\imath a_1} = \frac{\partial}{\partial p^1}|_0 = -2\lambda S^2\frac{\partial}{\partial\theta}$$

$$a_0{\triangleright}S = -\imath\frac{\partial}{\partial p^0}|_0 \sinh(\frac{\theta'}{2}) = \imath\lambda SC^2, \quad a_1{\triangleright}S = -\imath\frac{\partial}{\partial p^1}|_0 \sinh(\frac{\theta'}{2}) = \imath\lambda CS^2.$$

Note also that $\sinh(\theta) = 2CS$ and $\cosh(\theta) = C^2 + S^2$. Hence from (24.15) we find the relations

$$[a_0, \begin{pmatrix} c \\ s \end{pmatrix}] = \imath\lambda s\begin{pmatrix} s \\ c \end{pmatrix}, \quad [a_1, \begin{pmatrix} c \\ s \end{pmatrix}] = \imath\lambda(c-1)\begin{pmatrix} s \\ c \end{pmatrix}$$

of the bicrossproduct $\mathbb{C}_\lambda[\mathrm{Poinc}_{1,1}] \equiv \mathbb{C}[SO_{1,1}]{\blacktriangleright\!\!\triangleleft} U(\mathbb{R}{>\!\!\triangleleft}\mathbb{R})$. Finally, differentiate (24.20)–(24.21) to have the coaction Δ_R of $\mathbb{C}[SO_{1,1}]$ on the a_μ:

$$\frac{\partial p^{0\prime}}{\partial p^0}|_0 = C^2 + S^2 = \frac{\partial p^{1\prime}}{\partial p^1}|_0, \quad \frac{\partial p^{0\prime}}{\partial p^1}|_0 = 2CS = \frac{\partial p^{1\prime}}{\partial p^0}|_0$$

$$\Rightarrow \quad \Delta_R(a_0, a_1) = (a_0, a_1) \otimes \begin{pmatrix} C & S \\ S & C \end{pmatrix}^2 = (a_0, a_1) \otimes \begin{pmatrix} c & s \\ s & c \end{pmatrix}$$

which along with the antipode completes the Hopf algebra structure constructed from (24.15)–(24.17). One can similarly describe $\mathbb{C}_\lambda[\mathrm{Poinc}_{3,1}] =$

$\mathbb{C}[SO_{3,1}]\blacktriangleright\!\!\triangleleft U(\mathbb{R}^3{>\!\!\triangleleft}\mathbb{R})$ in such a form, fitting in with a classification of Poincaré coordinate quantum groups in a certain ansatz in [24].

24.4 Noncommutative spacetime, plane waves and calculus

Until now we have given a quite technical construction of certain Poincaré Hopf algebras and spoken of 'mass-shells' and 'energy', etc., but such appellations are meaningless until we consider the spacetime on which the algebra acts. Expressions such as (24.24) depend only on the algebra and can look however one wants depending on the arbitrary choice of generators named p^μ. By contrast, the pair consisting of the quantum group *and* the spacetime on which it acts together have features independent of any choice of generators and this is where the actual physics lies as explained in Section 24.2. We turn to this now.

In the bicrossproduct models we know from Theorem 1 that there is a canonical choice for this and it is noncommutative. Thus Poincaré quantum groups in the form (24.13)–(24.17) act (coact) on $U(\mathfrak{m})$ and we recall that we denote the generators of this copy by x_μ, which for the family above have the relations (24.1). We focus on the 4D case where $i = 1, 2, 3$. The 3D case of these relations is the Lie algebra $\mathfrak{m}$ in [15].

The first thing to do here is to explain the choice of momentum space coordinates in the previous section in terms of potentially physical quantities on this noncommutative spacetime, namely the noncommutative plane waves. The choice of momentum coordinates is arbitrary and as we change them the plane waves will look different. For our choice,

$$\psi_{\vec{p},p^0} = e^{\imath\vec{p}\cdot x}e^{\imath p^0 x_0}, \quad \psi_{\vec{p},p^0}\psi_{\vec{p}',p^{0\prime}} = \psi_{\vec{p}+e^{-\lambda p^0}\vec{p}',p^0+p^{0\prime}}$$

which shows the classical but nonAbelian group law of the Lie group $\mathbb{R}^3{>\!\!\triangleleft}\mathbb{R}$ as read off from the product of plane waves. It has exactly the same form as the coproduct (24.18) before. Moreover, the quantum group Fourier transform reduces to the usual one but normal-ordered,

$$\mathcal{F}(f) = \int_{\mathbb{R}^4} \mathrm{d}^4p\, f(p)e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p^0 x_0}$$

and turns quantum differential operators on the noncommutative spacetime into multiplication operators. Put another way, the properly defined quantum differential operators will be diagonal on the noncommutative plane waves, as a general feature of all such models.

To complete the picture we need these quantum differentials, in order to describe the action of the λ-Poincaré generators on the noncommutative spacetime as differential operators. It is this action that physically specifies its role as a Poincaré group to allow predictions. In the present model we have a natural differential calculus Ω^1 with basis $\mathrm{d}x_\mu$ and

$$(\mathrm{d}x_j)x_\mu = x_\mu \mathrm{d}x_j, \quad (\mathrm{d}x_0)x_\mu - x_\mu \mathrm{d}x_0 = \imath\lambda \mathrm{d}x_\mu$$

which leads to the partial derivatives

$$\partial^i \psi =: \frac{\partial}{\partial x_i}\psi(\vec{x}, x_0) := \imath p^i \triangleright \psi \tag{24.25}$$

$$\partial^0 \psi =: \frac{\psi(\vec{x}, x_0 + \imath\lambda) - \psi(\vec{x}, x_0)}{\imath\lambda} := \frac{\imath}{\lambda}(1 - e^{-\lambda p^0})\triangleright\psi \tag{24.26}$$

for normal ordered polynomial functions ψ or in terms of the action of the momentum operators p^μ. These ∂^μ do respect our implicit $*$-structure (unitarity) on the noncommutative spacetime but in a Hopf algebra sense which is not the usual sense since the action of the antipode S is not just $-p^\mu$. This is fixed by adjusted derivatives $L^{-\frac{1}{2}}\partial^\mu$ where

$$L\psi =: \psi(x, x_0 + \imath\lambda) := e^{-\lambda p^0}\triangleright\psi.$$

In this case the natural 4D Laplacian is $L^{-1}((\partial^0)^2 - \sum_i (\partial^i)^2)$, which by (24.25)–(24.26) acts on plane waves as (24.3), thereby giving meaning to the latter as describing the physical mass-shell.

Finally, for the analysis of an experiment we assume the identification of noncommutive waves in the above normal ordered form with classical ones that a detector might register. In that case one may argue [1] that the speed for such waves can be computed as $|\frac{\partial p^0}{\partial p^i}| = e^{\lambda p^0}$ in units where 1 is the usual speed of light. So the prediction is that the speed of light depends on energy. What is remarkable is that even if $\lambda \sim 10^{-44}$ s (the Planck time scale), this prediction could in principle be tested, for example using γ-ray bursts. These are known in some cases to travel cosmological distances before arriving here, and have a spread of energies of 0.1–100 MeV. According to the above, the relative time delay Δ_T on travelling distance L for energies p^0, $p^0 + \Delta_{p^0}$ is

$$\Delta_T \sim \lambda\Delta_{p^0}\frac{L}{c} \sim 10^{-44}\,\mathrm{s} \times 100\,\mathrm{MeV} \times 10^{10}\,\mathrm{y} \sim 1\ \mathrm{ms},$$

which is in principle observable by statistical analysis of a large number of bursts correlated with distance (determined for example by using the Hubble telescope to lock in on the host galaxy of each burst). Although the above is only one of a class of predictions, it is striking that even Planck scale effects are now in principle within experimental reach.

24.5 Physical interpretation

We have given the bicrossproduct model to the point of first predictions. However, there are still many issues for this and all other models. The key problem is that in using NCG to model physics one still has to relate the mathematical objects to actual physics. That there is a fundamental issue here is evident in the following two questions.

(i) How could we see a noncommutative plane wave? How would we precisely measure any particular coordinates p^μ, etc., labelling our plane waves? Without answering this, one has no prediction.
(ii) How would we physically detect the order of 'addition' in the nonAbelian momentum group law? For example, if we smash together two waves of nonAbelian momentum p, p', which way round do we form the composite?

24.5.1 Prequantum states and quantum change of frames

The correct way to address the first issue according to current understanding is to treat the noncommutative algebra as an operator algebra, construct representations or 'states' of this 'prequantum system' and consider that what would be observed macroscopically are expectation values $\langle x_\mu \rangle$, $\langle \psi_p(x) \rangle$, etc., in this state. Typically there exist 'minimum uncertainty' coherent states where the x_μ appear localised as much as possible around $\langle x_\mu \rangle$ and the plane waves expectations in such coherent states have a specific signature that could be looked for, or conversely other states could be viewed as a superposition of these. For the model (24.2) see [2; 21]. In general the deeper theory of Quantum Gravity has to provide these states and their behaviour *in addition* to the noncommutative spacetime and Poincaré algebra. Here λ is treated as mathematically analogous to Planck's constant but is not Planck's constant (we work in units where $\hbar = 1$), which is why we call this 'prequantum' theory not quantum mechanics. It is something more fundamental.

Actually Quantum Gravity has to provide much more than this. It has to provide a representation of and hence expectation values for the entire coordinate algebra $\mathbb{C}_\lambda$[Poinc]. Only given such a state would a quantum Poincaré transformation become an actual numerical transformation (as needed for example to pass to a rest frame) of the form

$$\langle x_\mu \rangle \to \langle x_\nu \rangle \langle \Lambda^\nu{}_\mu \rangle + \langle a_\mu \rangle + O(\lambda)$$

where (say) the a_μ are the quantum group coordinates in the translation sector and $\Lambda^\mu{}_\nu$ are those in the Lorentz sector. In general one may not have such a decomposition, but even if one does, if one makes two such transformations, one will have in general that

$$\langle a_\mu a_\nu \rangle = \langle a_\mu \rangle \langle a_\nu \rangle + O(\lambda), \quad \langle \Lambda^\mu{}_\nu a_\rho \rangle = \langle \Lambda^\mu{}_\nu \rangle \langle a_\rho \rangle + O(\lambda) \tag{24.27}$$

$$\langle \Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \rangle = \langle \Lambda^\mu{}_\nu \rangle \langle \Lambda^\alpha{}_\beta \rangle + O(\lambda) \tag{24.28}$$

reflecting that the quantum Poincaré coordinates do not commute in NCG; they are not given by actual numbers. NonAbelianness of the momentum group appears here in the first of (24.27) which says that physical states provided by Quantum Gravity will not have classical numerical values for all the momentum coordinate operators a_μ simultaneously. This should not be confused with angular momentum (for example) where the enveloping algebra generators cannot be simultaneous diagonalised but where the coordinate algebra can be (actual classical values of angular momentum). Our situation is dual to that. We similarly cannot measure $\Lambda^\mu{}_\nu$ and a_ρ simultaneously due to the second of (24.27) when the commutation relations are nontrivial.

In the bicrossproduct model the $\Lambda^\mu{}_\nu$ mutually commute (the Lorentz coordinates are not deformed) so (24.28) does not need any $O(\lambda)$ corrections. States in this sector can be given by actual points in $SO_{3,1}$ or numerical angles. Meanwhile, the second of (24.27) has corrections due to (24.15) given by the vector fields β or in a global Hopf–von Neumann algebra setting by the global action $\triangleright$ as in (24.22), which we have seen blows up as in Figure 24.3. This implies some form of 'infinite nocommutativity' or 'infinite uncertainty' for certain states. Thus, while we have perfectly good Hopf algebras, they only see the differentiated data of the matched pair and miss the singular global picture. This enters when we try to represent them as operator algebras in actual states.

In summary, a quantum Poincaré transformation makes sense algebraically but to realise it numerically one needs expectation values or representations of the generators of $\mathbb{C}_\lambda$[Poinc] (this is not to be confused with representations of U_λ(poinc) which have their usual meaning as particle states). The lesson is that we need both in Quantum Gravity.

24.5.2 The •-product, classicalisation and effective actions

An alternative approach to operator 'prequantum' methods as above is to view the noncommutative spacetime algebra as a deformation on the same vector space as classically but with a new product •. This comes with an identification ϕ of vector spaces, which we call the 'classicalisation map', and which defines the modified product by

$$f \bullet g = \phi(\phi^{-1}(f)\phi^{-1}(g))$$

for classical functions f, g. We can add to this the **working hypothesis** that noncommutative variables are to be observed by applying ϕ and observing the classical image. This brings with it a wealth of questions about why one should make such a postulate or what kind of supposition it makes about the experimental set up. In fact specifying ϕ is essentially equivalent to saying what one believes the noncommutative plane waves $\psi_p(x)$ look like, the implicit assumption being that these are to coincide under ϕ with their classical counterparts $e^{\imath p^\mu X_\mu}$ where we use X^μ for the classical spacetime coordinates. In that case

$$e^{\imath p\cdot X} \bullet e^{\imath p'\cdot X} = \phi(\psi_p \psi_{p'}) = \phi(\psi_{pp'}) = e^{\imath (pp')\cdot X} \tag{24.29}$$

where pp' denotes the (possibly nonAbelian) momentum group composition law in the chosen coordinate system.

Thus for the bicrossproduct Minkowski spacetime the quantum plane waves that we used are equivalent to

$$\phi(: f(x) :) = f(X), \quad \text{e.g.} \quad \phi(\psi_p(x)) = e^{\imath p^\mu X^\mu}$$

for any classical expression $f(X)$ and where : : means putting all the x_i to the left of all the x_0 as explained in [23]. In experimental terms it means that experimental kit should (somehow) measure first x_0 and then x_i, the order mattering in view of the noncommutation relations. The bullet product implied here on classical functions is then

$$f \bullet g = \cdot \left(e^{\imath\lambda \frac{\partial}{\partial X_0} \otimes X_i \frac{\partial}{\partial X_i}} (f \otimes g) \right) = f(\vec{X}, X_0 + \imath\lambda \deg(g)) g(\vec{X}, X_0) \tag{24.30}$$

for classical functions f, g, where $\deg(g)$ is the total degree in the X_i in the case where g is homogeneous. Here one applies the operator shown and then multiplies the results using the classical product of functions on Minkowski space to give this result. This operator is a 2-cocycle in any Hopf algebra containing $\frac{\partial}{\partial X_0}$, $X_i \frac{\partial}{\partial X_i}$ which means it also fits into a 'twist functor approach to quantisation' [18; 22] leading to a different NCG on the same algebra than the one from the bicrossproduct picture. We will not be able to cover the twist functor approach here due to lack of space but other twist functor models include the Moyal product or θ-spacetime (*aka* the Heisenberg algebra) $[x_\mu, x_\nu] = \imath\theta_{\mu\nu}$.

Next, the classicalisation map allows one to write an NCG action like

$$\mathcal{L} = \int \mathrm{d}f \wedge \star \mathrm{d}f + m^2 f^2 + \mu f^3, \tag{24.31}$$

etc., where f is an element of the quantum spacetime algebra and we assume we are given a covariant $\int$ and a Hodge $\star$-operator, in terms of ordinary fields $\phi = \phi(f)$ with action

$$\mathcal{L} = \int_{\mathbb{R}^4} \mathrm{d}^4 X \, \frac{\partial}{\partial X_\mu}\phi \bullet \frac{\partial}{\partial X^\mu}\phi + m^2 \phi \bullet \phi + \mu \phi \bullet \phi \bullet \phi, \tag{24.32}$$

etc., using classical integration and calculus, but with the $\bullet$ product in place of the usual product of functions. This assumes that $\int = \int d^4X\phi(\,)$ and that the quantum differentials become classical through ϕ as is the case for the simplest NCG models (including θ-spacetimes and the 3D quantum double model (24.2)). In the case of the bicrossproduct spacetime model the quantum integration is indeed defined by the normal ordering ϕ and we have seen (24.25)–(24.26) that spatial quantum differentials indeed relate to the classical ones, but the ∂^0 direction relates under ϕ to a finite difference in the imaginary time direction. Hence a noncommutative action will *not* have a usual $\bullet$ form (24.32) but will involve finite differences for ∂^0. One also has the problem that the quantum calculus and hence the NCG action is not necessarily λ-Poincaré covariant (even though the spacetime itself is), there is an anomaly for the Poincaré group at the differential level. One can replace the calculus by a 5D covariant one but then one has to interpret this extra direction. We expect it (see below) to relate to the renormalization-group flow in the QFT on the spacetime. Again the physics of these issues remains fully to be explored at the time of writing.

24.6 Other noncommutative spacetime models

The 4D bicrossproduct model is the simplest noncommutative spacetime model that could be a deformation of our own world with its correct signature. There are less developed models and we outline them here.

We start with (24.2) for which $U_\lambda(\mathrm{poinc}_{2,1}) = U(so_{2,1})\rhd\!\!\!<\mathbb{C}[SO_{2,1}]$ as a special case of a bicrossproduct where the back-reaction β is trivial. Here $X = SO_{2,1\,\mathrm{Ad}}\!\bowtie\!\!\lhd SO_{2,1}$ and from the general theory we know that it acts on $U(so_{2,1})$ as a 3D noncommutative spacetime. Its Euclideanised version $U(su_2)$ is the algebra (24.2) proposed for 3D Quantum Gravity in [2]. For the plane waves, we use the canonical form

$$\psi_{\vec{k}} = e^{\imath k\cdot x}, \quad |\vec{k}| < \frac{\pi}{\lambda}$$

in terms of the local 'logarithmic' coordinates as in Section 24.2. The composition law for plane waves is the SU_2 product in these coordinates (given by the CBH formula) and we have a quantum Fourier transform (24.5) with $e_i = x_i$ in the present application. We also have [2]:

$$dx_i = \lambda\sigma_i, \quad x_i\Theta - \Theta x_i = \imath\frac{\lambda^2}{\mu}dx_i,$$

$$(dx_i)x_j - x_j dx_i = \imath\lambda\epsilon_{ij}{}^k dx_k + \imath\mu\delta_{ij}\Theta,$$

where Θ is the 2×2 identity matrix which, together with the Pauli matrices σ_i completes the basis of left-invariant 1-forms. The 1-form Θ provides a natural time direction, even though there is no time coordinate, and the new parameter $\mu \neq 0$ appears as the freedom to change its normalisation. The partial derivatives ∂^i are defined by

$$\mathrm{d}\psi(x) = (\partial^i \psi)\mathrm{d}x_i + (\partial^0\psi)\Theta$$

and act diagonally on plane waves as

$$\partial^i = \imath \frac{k^i}{\lambda|\vec{k}|}\sin(\lambda|\vec{k}|), \quad \partial^0 = \imath\frac{\mu}{\lambda}(\cos(\lambda|\vec{k}|) - 1) = \imath\frac{\mu}{2}\vec{\partial}^2 + O(\lambda^2).$$

Finally, there is a classicalisation map [4]

$$\phi(\psi_{\vec{k}}(x)) = e^{\imath p^\mu x_\mu}, \quad p^0 = \cos(\lambda|\vec{k}|), \quad p^i = \frac{\sin(\lambda|\vec{k}|)}{\lambda|\vec{k}|}k^i.$$

One can also label the noncommutative plane waves directly by p^μ as we did for the model (24.1). The map ϕ reproduces (24.2) by its $\bullet$ product and commutes with ∂_i (but not ∂^0), which means that actions such as (24.32) proposed in [3] as an effective theory for 3D Quantum Gravity essentially coincide with the NCG effective actions such as (24.31) as in [2]. Here $\int = \sum_{j\in\mathbb{N}}(j+1)\mathrm{Tr}_j$ is the sum of traces in the spin $j/2$ representation. The noncommutative action has an extra term involving ∂^0, which can be suppressed only by assuming that the 4D Hodge $*$-operator is degenerate. Moreover, the map ϕ sees only the integer spin information in the model which is not the full NCG, see [4].

Note that μ cannot be taken to be zero due to an anomaly for translation invariance of the DGA. This anomaly forces an extra dimension much as we saw for (24.1) before. The physical meaning of this extra direction ∂^0 from the point of view of Euclidanized 3D Quantum Gravity is as a renormalization group flow direction associated to blocking of the spins in the Ponzano–Regge model [4]. Alternatively, one can imagine this noncommuative spacetime arising in other nonrelativistic limits of a 4D theory, with the extra 'time' direction x_0 adjoined by [21]

$$\Theta = \mathrm{d}x_0, \quad [x_0, x_i] = 0, \quad [x_0, \mathrm{d}x_i] = \imath\frac{\lambda^2}{\mu}\mathrm{d}x_i, \quad [x_0, \Theta] = \imath\frac{\lambda^2}{\mu}\Theta$$

and new partial derivatives ∂^μ on the extended algebra. Then the 'stationary' condition in the new theory is $\mathrm{d}\psi = O(\mathrm{d}x_i)$ or $\partial^0\psi = 0$, i.e.

$$\psi(\vec{x}, x_0 + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1+\lambda^2\vec{\partial}^2}\right)\psi(\vec{x}, x_0) \tag{24.33}$$

which in the $\lambda \to 0$ limit becomes the Schroedinger equation for a particle of mass $m = 1/\mu$. Plane wave solutions exist in the form

$$e^{\imath k^\mu x_\mu}, \quad k^0 = -\frac{1}{m\lambda^2}\ln\cos(\lambda|\vec{k}|), \quad |\vec{k}| < \frac{\pi}{2\lambda}$$

showing the Planckian bound.

Another major noncommutative spacetime, more or less fully explored by the author in the 1990s using braided methods is $C_q[\mathbb{R}^{3,1}]$ or 'q-Minkowski space'. It has a matrix of generators, relations, $*$-structure and braided coproduct

$$\beta\alpha = q^2\alpha\beta, \quad \gamma\alpha = q^{-2}\alpha\gamma, \quad \delta\alpha = \alpha\delta,$$

$$\beta\gamma = \gamma\beta + (1-q^{-2})\alpha(\delta-\alpha),$$

$$\delta\beta = \beta\delta + (1-q^{-2})\alpha\beta,\ \gamma\delta = \delta\gamma + (1-q^{-2})\gamma\alpha,$$

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^* = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}, \quad \underline{\Delta}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\underline{\otimes}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

and is also denoted $B_q[M_2]$ as the algebra of braided 2×2 Hermitian matrices [19]. If we quotient by the braided determinant relation $\alpha\delta - q^2\gamma\beta = 1$ we have the unit hyperboloid in $\mathbb{C}_q[R^{3,1}]$ which is the braided group $B_q[SU_2]$ as obtained canonically from $\mathbb{C}_q[SU_2]$ by a process called 'transmutation'. Interestingly, the braided group is self-dual, $B_q[SU_2] \approx BU_q(su_2) = U_q(su_2)$ as an algebra, provided q is generic; this is a purely quantum phenomenon. It means that q-Minkowski space has two limits, one is classical Minkowski space and the other after scaling and then taking the limit, is the enveloping algebra of $su_2 \times u(1)$. There is also an additive braided coproduct $\underline{\Delta}\alpha = \alpha\otimes 1 + 1\otimes\alpha$, etc., which corresponds to the usual (flat) additive structure of $\mathbb{R}^{3,1}$. Finally, from braided group theory there is a 'bosonisation' construction $U_q(\widetilde{\mathrm{poinc}_{3,1}}) = U_q(\widetilde{so_{3,1}}) \ltimes \mathbb{C}_q[\mathbb{R}^{3,1}]$ which acts covariantly on $\mathbb{C}_q[\mathbb{R}^{3,1}]$ as q-Poincaré quantum group with dilation [20]. Once again there is an anomaly which requires an extra generator, here a dilation indicated by $\widetilde{\ }$. It has been proposed that q-deformed models relate to Quantum Gravity with cosmological constant.

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25

Doubly special relativity

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25.1 Introduction: what is DSR?

The definition of doubly special relativity (DSR) [2; 3; 4] (see [11] for review) is deceptively simple. Recall that Special Relativity is based on two postulates: the relativity principle for inertial observers and the existence of a single observer independent scale associated with the velocity of light. In this DSR replaces the second postulate by assuming the existence of *two* observer-independent scales: the old one of velocity plus the scale of mass (or of momentum, or of energy). That's it.

Adding a new postulate has consequences, however. The most immediate one is the question: what does the second observer-independent scale mean physically? Before trying to answer this question, let us recall the concept of an observer-independent scale. It can be easily understood, when contrasted with the notion of dimensionful coupling constant, like the Planck constant $\hbar$ or the gravitational constant G. What is their status in relativity? Do they transform under Lorentz transformation? Well, naively, one would think that they should because they are given by dimensional quantities. But of course they do not. The point is that there is a special operational definition of these quantities. Namely each observer, synchronized with all the other observers, by means of the standard Einstein synchronization procedure, measures their values in an identical quasi-static experiment in her own reference frame (like the Cavendish experiment). Then the relativity principle ensures that the numerical value of such a constant will turn out to be the same in all experiments (the observers could check the validity of the relativity principle by comparing values they obtained in their experiments). With an observer-independent scale the situation is drastically different. Like the speed of light it cannot be measured in quasi-static experiments; all the observers now measure a quantity associated with a single object (in Special Relativity, all the observers could find out what the speed of light is just by looking at the same single photon).

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

Now DSR postulates the presence of the second observer-independent scale. What is the physical object that carries this scale, like the photon carrying the scale of velocity of light? We do not know. One can speculate that black hole remnants will do so, but to understand them we need, presumably, the complete theory of Quantum Gravity. Fortunately, there is another way one can think of the observer-independent scale. If such a scale is present in the theory, and since, as explained above, it is operationally defined in terms of experiments, in which one physical object is observed by many distinct observers, who all measure the same value of the scale, it follows that the scale must appear as a parameter in the transformation rules, relating observers to each other. For example, the velocity of light is present as a parameter in Lorentz transformations. If we have a theory of spacetime with two observer-independent scales, both should appear in the transformations. As an example one can contemplate the following form of infinitesimal action of Lorentz generators, rotations M_i and boosts N_i satisfying the standard Lorentz algebra, on momenta (so called DSR1), with the scale of mass κ

$$[M_i, P_j] = \epsilon_{ijk} P_k, \quad [M_i, P_0] = 0$$

$$\left[N_i, P_j\right] = \delta_{ij} \left(\frac{1}{2} \left(1 - e^{-2P_0/\kappa}\right) + \frac{\mathbf{P}^2}{2\kappa} \right) - \frac{1}{\kappa} P_i P_j,$$

$$[N_i, P_0] = P_i. \tag{25.1}$$

This algebra is a part of κ-Poincaré quantum algebra, see [14]. One can also imagine a situation in which the scale κ appears not in the rotational, but in the translational sector of the modified, deformed Poincaré group.

One may think of the second scale also in terms of synchronization of observers. Recall that the velocity of light scale is indispensable in Special Relativity because it provides the only meaningful way of synchronizing different observers. However, this holds for spacetime measurements (lengths and time intervals) only. To define momenta and energy, one must relate them to velocities. On the other hand, using the momentum scale, one could, presumably, make both the spacetime and momentum space synchronization, independently, and perhaps could even describe the phase space as a single entity. Thus it seems that in DSR the primary concept would be the phase space not the configuration one.

In the limit when the second scale is very large (or very small depending on how the theory is constructed) the new theory should reduce to the old one; for example, when the second scale κ of DSR goes to infinity, DSR should reduce to Special Relativity. Putting it another way we can think of DSR as some sort of deformation of SR. Following this understanding some researchers would translate the acronym

DSR to deformed Special Relativity. But of course, deformation requires a deformation scale, so even semantically both terms are just equivalent, just stressing different aspects of DSR. Note that in addition to the modified, deformed algebra of spacetime symmetries, like the one in eq. (25.1), the theory is to be equipped with an additional structure(s), so as to make sure that its algebra cannot be reduced to the standard algebra of spacetime symmetries of Special Relativity, by rearrangement of generators. Only in such a case DSR will be physically different from Special Relativity.

In the framework of DSR we want to understand if there are any modifications to the standard particle kinematics as described by Special Relativity, at very high energies, of order of Planck scale. The motivation is both phenomenological and theoretical. First there are indications from observations of cosmic rays carrying energy higher than the GZK cutoff that the standard special relativistic kinematics might be not an appropriate description of particle scatterings at energies of order of 10^{20} eV (in the laboratory frame). Similar phenomenon, the violation of the corresponding cutoff predicted by the standard special relativistic kinematics for ultra-high energy photons seems also to be observed. It should be noted, however, that in both these cases we do not really control yet all the relevant astrophysical details of the processes involved (for example in the case of cosmic rays we do not really know what are the sources, though it it is hard to believe that they are not at the cosmological distances.) The extended discussion of these issues can be found, for example, in [1]. If violation of the GZK cutoff is confirmed, and if indeed the sources are at the cosmological distances, this will presumably indicate deviation from Lorentz kinematics. One of the major goals of DSR is to work out robust theoretical predictions concerning the magnitude of such effects. I will briefly discuss the "DSR phenomenology" below.

25.2 Gravity as the origin of DSR

The idea of DSR arose from the desire to describe possible deviations from the standard Lorentz kinematics on the one hand and, contrary to the Lorentz breaking schemes, to preserve the most sacred principle of physics – the Relativity Principle. Originally the view was that one may be forced by phenomenological data to replace Special Relativity by DSR, and then, on the basis of the latter one should construct its curved space extension, "doubly general relativity". Then it was realized that, in fact, the situation is likely to be quite opposite: DSR might be *the* correct flat space limit of gravity coupled to particles (see [5] and [8]). We are thus facing the fundamental theoretical question: is Special Relativity indeed, as it is believed, the correct limit of (Quantum) Gravity in the case when spacetime is flat? From the perspective of gravity, flat Minkowski spacetime is some particular configuration of the gravitational field, and as such is to be described by the theory of gravity. It corresponds to configurations of the gravitational field in which this field vanishes. However, equations governing the gravitational field are differential equations and thus describe the solutions only locally. In the case of Minkowski space particle kinematics we have to deal not only with a (flat) gravitational field but also with particles themselves. The particles are, of course, the sources of the gravitational field and even in the flat space limit the trace of the particles' back reaction on spacetime might remain in the form of some global information, even if locally, away from the locations of the particle, the spacetime is flat. Of course we know that in general relativity the energy-momentum of matter curves spacetime, and the strength of this effect is proportional to gravitational coupling (Newton's constant.) Thus we are interested in the situation in which the transition from general relativity to Special Relativity corresponds to smooth switching off the couplings. In principle two situations are possible (in four dimensions):

(i) weak gravity, semiclassical limit of Quantum Gravity

$$G,\ \hbar \to 0, \quad \sqrt{\frac{\hbar}{G}} = \kappa \ \text{ remains finite;} \tag{25.2}$$

(ii) weak gravity, small cosmological constant limit of Quantum Gravity

$$\Lambda \to 0, \quad \kappa \ \text{ remains finite.} \tag{25.3}$$

The idea is therefore to devise a controllable transition from the full (Quantum) Gravity coupled to point particles to the regime, in which all local degrees of freedom of gravity are switched off. Then it is expected that locally, away from the particles' worldlines, gravity will take the form of Minkowski (for $\Lambda = 0$) or (anti) de Sitter space, depending on the sign of Λ. Thus it is expected that DSR arises as a limit of general relativity coupled to point particles in the topological field theory limit. To be more explicit, consider the formulation of gravity as the constrained topological field theory, proposed in [10]:

$$S = \int \left(B_{IJ} \wedge F^{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL5} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} \right). \tag{25.4}$$

Here F^{IJ} is the curvature of $SO(4,1)$ connection A^{IJ}, and B_{IJ} is a two-form valued in the algebra $SO(4,1)$. The dimensionless parameters α and β are related to gravitational and cosmological constants, and the Immirzi parameter. The α term breaks the symmetry, and for $\alpha \neq 0$ this theory is equivalent to general relativity.

On the other hand there are various limits in which this theory becomes a topological one. For example, for $\alpha \to 0$ all the local degrees of freedom of gravity disappear, and only the topological ones remain. One hopes that, after coupling this theory to point particles, one derives DSR in an appropriate, hopefully natural, limit. This hope is based on experience with the 2+1 dimensional case, which I will now discuss.

25.3 Gravity in 2+1 dimensions as DSR theory

It is well known that gravity in 2+1 does not possess local degrees of freedom and is described by a topological field theory. Even in the presence of point particles with mass and spin the 2+1 dimensional spacetime is locally flat. Thus 2+1 gravity is a perfect testing ground for DSR idea. There is also a simple argument that it is not just a toy model, but can tell us something about the full 3+1 dimensional case. It goes as follows.

As argued above, what we are interested in is the flat space limit of gravity (perhaps also the semiclassical one in the quantum case.) Now consider the situation when we have 3+1 gravity coupled to a planar configuration of particles. When the local degrees of freedom of gravity are switched off this configuration has translational symmetry along the direction perpendicular to the plane. But now we can make a dimensional reduction and describe the system equivalently with the help of 2+1 gravity coupled to the particles. The symmetry algebra in 2+1 dimensions must therefore be a subalgebra of the full 3+1 dimensional one. Thus if we find that the former is not the 2+1 Poincaré algebra but some modification of it, the latter must be some appropriate modification of the 3+1 dimensional Poincaré algebra. Thus if DSR is relevant in 2+1 dimensions, it is likely that it is going to be relevant in 3+1 dimensions as well.

Let us consider the analog of situation (ii) listed in the previous section. We start therefore with the 2+1 gravity with a positive cosmological constant. Then it is quite well established (see for example [15]) that the excitations of 3d Quantum Gravity with a cosmological constant transform under representations of the quantum deformed de Sitter algebra $SO_q(3,1)$, with $z = \ln q$ behaving in the limit of small $\Lambda\hbar^2/\kappa^2$ as $z \approx \sqrt{\Lambda}\hbar/\kappa$, where κ is equal to inverse 2+1 dimensional gravitational constant, and has dimension of mass.

I will not discuss at this point the notion of quantum deformed algebras (Hopf algebras) in much detail. It suffices to say that quantum algebras consist of several structures, the most important for our current purposes would be the universal enveloping algebra, which could be understood as an algebra of brackets among generators, which are equal to some analytic functions of them. Thus the quantum algebra is a generalization of a Lie algebra, and it is worth observing that the former reduces to the latter in an appropriate limit. The other structures of Hopf algebras, like co-product and antipode, are also relevant in the context of DSR, and I will introduce them in the next section.

In the case of quantum algebra $SO_q(3, 1)$ the algebraic part looks as follows (the parameter z used below is related to q by $z = \ln q$)

$$\begin{aligned}
[M_{2,3}, M_{1,3}] &= \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3})\\
[M_{2,3}, M_{1,2}] &= M_{1,3}\\
[M_{2,3}, M_{0,3}] &= M_{0,2}\\
[M_{2,3}, M_{0,2}] &= \frac{1}{z}\sinh(zM_{0,3})\cosh(zM_{1,2})\\
[M_{1,3}, M_{1,2}] &= -M_{2,3}\\
[M_{1,3}, M_{0,3}] &= M_{0,1}\\
[M_{1,3}, M_{0,1}] &= \frac{1}{z}\sinh(zM_{0,3})\cosh(zM_{1,2})\\
[M_{1,2}, M_{0,2}] &= -M_{0,1}\\
[M_{1,2}, M_{0,1}] &= M_{0,2}\\
[M_{0,3}, M_{0,2}] &= M_{2,3}\\
[M_{0,3}, M_{0,1}] &= M_{1,3}\\
[M_{0,2}, M_{0,1}] &= \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3}).
\end{aligned} \tag{25.5}$$

Observe that on the right hand sides we do not have linear functions generators, as in the Lie algebra case, but some (analytic) functions of them. However, we still assume that the brackets are antisymmetric and, it is easy to show, that Jacobi identity holds. Note that in the limit $z \to 0$ the algebra (25.5) becomes the standard algebra $SO(3, 1)$, and this is the reason for using the term $SO_q(3, 1)$.

The $SO(3, 1)$ Lie algebra is the 2+1 dimensional de Sitter algebra and it is well known how to obtain the 2+1 dimensional Poincaré algebra from it. First of all one has to single out the energy and momentum generators of the right physical dimension (note that the generators $M_{\mu\nu}$ of (25.5) are dimensionless): one identifies three-momenta $P_\mu \equiv (E, P_i)$ ($\mu = 1, 2, 3$, $i = 1, 2$) as appropriately rescaled generators $M_{0,\mu}$ and then one takes the Inömü–Wigner contraction limit. In the quantum algebra case, the contraction is a bit more tricky, as one has to convince oneself that after the contraction the structure one obtains is still a quantum algebra. Such contractions have been discussed in [13].

Let us try to contract the algebra (25.5). To this aim, since momenta are dimensionful, while the generators M in (25.5) are dimensionless, we must first rescale some of the generators by an appropriate scale, provided by a combination of dimensionful constants present in the definition of the parameter z:

$$\begin{aligned} E &= \sqrt{\Lambda}\hbar\, M_{0,3} \\ P_i &= \sqrt{\Lambda}\hbar\, M_{0,i} \\ M &= M_{1,2} \\ N_i &= M_{i,3}. \end{aligned} \tag{25.6}$$

Taking into account the relation $z \approx \sqrt{\Lambda}\hbar/\kappa$, which holds for small Λ, from

$$[M_{2,3}, M_{1,3}] = \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3})$$

we find

$$[N_2, N_1] = \frac{\kappa}{\hbar\sqrt{\Lambda}}\sinh(\hbar\sqrt{\Lambda}/\kappa M)\cosh(E/\kappa). \tag{25.7}$$

Similarly from

$$[M_{0,2}, M_{0,1}] = \frac{1}{z}\sinh(zM_{1,2})\cosh(zM_{0,3})$$

we get

$$[P_2, P_1] = \sqrt{\Lambda}\hbar\kappa\ \sinh(\sqrt{\Lambda}\hbar/\kappa M)\cosh(E/\kappa). \tag{25.8}$$

Similar substitutions can be made in other commutators of (25.5). Now going to the contraction limit $\Lambda \to 0$, while keeping κ constant, we obtain the following algebra

$$\begin{aligned} &[N_i, N_j] = -M\epsilon_{ij}\ \cosh(E/\kappa) \\ &[M, N_i] = \epsilon_{ij}N^j \\ &[N_i, E] = P_i \\ &[N_i, P_j] = \delta_{ij}\,\kappa\ \sinh(E/\kappa) \\ &[M, P_i] = \epsilon_{ij}P^j \\ &[E, P_i] = 0 \\ &[P_2, P_1] = 0. \end{aligned} \tag{25.9}$$

This algebra is called three dimensional κ-Poincaré algebra (in the standard basis.)

Let us pause for a moment here to make a couple of comments. First of all, one easily sees that in the limit $\kappa \to \infty$ from κ-Poincaré algebra (25.9) one gets standard Poincaré algebra. Second, we see that in this algebra both the Lorentz and translation sectors are deformed. However, in the case of quantum algebras one is free to change the basis of generators in an arbitrary, analytic way (contrary to the case of Lie algebras, where only linear transformations of generators are allowed). It turns out that there exists such a change of the basis that the Lorentz part of the algebra becomes classical (i.e. undeformed.) This basis is called the bicrossproduct one, and the doubly special relativity model (both in three and four dimensions) based on such an algebra is called DSR1. In this basis, the 2+1 dimensional κ-Poincaré algebra appears as follows:

$$\begin{aligned}
&[N_i, N_j] = -\epsilon_{ij}\, M \\
&[M, N_i] = \epsilon_{ij} N^j \\
&[N_i, E] = P_i \\
&[N_i, P_j] = \delta_{ij}\, \frac{\kappa}{2}\left(1 - e^{-2E/\kappa} + \frac{\vec{P}^{\,2}}{\kappa^2}\right) - \frac{1}{\kappa} P_i P_j \\
&[M, P_i] = \epsilon_{ij} P^j \\
&[E, P_i] = 0 \\
&[P_1, P_2] = 0\,.
\end{aligned} \tag{25.10}$$

The algebra (25.10) is nothing but the 2+1 dimensional analog of the algebra (25.1) we started our discussion with. Thus we conclude that, in the case of 2+1 dimensional Quantum Gravity on de Sitter space, in the flat space, i.e. vanishing cosmological constant limit, the standard Poincaré algebra is replaced by (quantum) κ-Poincaré algebra.

It is noteworthy that in the remarkable paper by Freidel and Livine [9] κ-Poincaré algebra has been also found by direct quantization of 2+1 gravity without a cosmological constant, coupled to point particles, in the weak gravitational constant limit. Even though the structures obtained by them and those one gets from contraction are very similar, their relation remains to be understood.

Let me summarize. In 2+1 gravity (in the limit of a vanishing cosmological constant) the scale κ arises naturally. It can be also shown that instead of the standard Poincaré symmetry we have to deal with the deformed algebra, with deformation scale κ.

There is one interesting and important consequence of the emergence of κ-Poincaré algebra (25.10). As in the standard case this algebra can be interpreted both as the algebra of spacetime symmetries and gauge algebra of gravity *and* as the algebra of charges associated with a particle (energy momentum and spin.) It is easy to observe that this algebra can be interpreted as an algebra of Lorentz symmetries of momenta if the momentum space is the de Sitter space of curvature κ. It can be shown that one can extend this algebra to the full phase space algebra of a point particle, by adding four (non-commutative) coordinates (see [12]). The resulting spacetime of the particle becomes the so-called κ-Minkowski spacetime with the non-commutative structure

$$[x_0, x_i] = -\frac{1}{\kappa}\, x_i . \tag{25.11}$$

On κ-Minkowski spacetime one can build field theory, which in turn could be used to discuss phenomenological issues, mentioned in the Introduction. In the next section I will show how, in a framework of such a theory, one discovers the full power of quantum κ-Poincaré algebra.

25.4 Four dimensional field theory with curved momentum space

As I said above, κ-Poincaré algebra can be understood as an algebra of Lorentz symmetries of momenta, for the space of momenta being the curved de Sitter space, of radius κ. Let us therefore try to built the scalar field theory on such a space (see also [7]). Usually field theory is constructed on spacetime, and then, by Fourier transform, is turned to the momentum space picture. Nothing, however, prevents us from constructing field theory directly on the momentum space, flat or curved. Let us see how this can be done.

Let the space of momenta be de Sitter space of radius κ:

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2. \tag{25.12}$$

To find contact with κ-Poincaré algebra we introduce the coordinates on this space as follows

$$\begin{aligned} \eta_0 &= -\kappa \,\sinh \frac{P_0}{\kappa} - \frac{\vec{P}^{\,2}}{2\kappa}\, e^{\frac{P_0}{\kappa}} \\ \eta_i &= -P_i\, e^{\frac{P_0}{\kappa}} \\ \eta_4 &= \kappa \,\cosh \frac{P_0}{\kappa} - \frac{\vec{P}^{\,2}}{2\kappa}\, e^{\frac{P_0}{\kappa}} . \end{aligned} \tag{25.13}$$

Then one can easily check that the commutators of P_μ with generators of Lorentz subgroup, $SO(3, 1)$ of the full symmetry group $SO(4, 1)$ of (25.12) form exactly the κ-Poincaré algebra (25.1).

In the standard case of flat momentum space, the action for free massive scalar field has the form

$$S_0 = \int d^4P\, \mathcal{M}_0(P)\, \Phi(P)\, \Phi(-P) \tag{25.14}$$

with $\mathcal{M}_0(P) = P^2 - m^2$ being the mass shell condition. In the case of de Sitter space of momenta we should replace $\mathcal{M}_0(P)$ with some generalized mass shell condition and also modify somehow $\Phi(-P)$, because "$-P$" does not make sense on curved space.

It is clear what should replace $\mathcal{M}_0(P)$. It should be just the Casimir of the algebra (25.1). As a result of the presence of the scale κ, contrary to the Special

Relativistic case, there is an ambiguity here. However, since the Lorentz generators can be identified with the generators of the $SO(4,1)$ algebra of symmetries of the quadratic form (25.12), operating in the $\eta_0 - \eta_3$ sector, and leaving η_4 invariant, it is natural to choose the mass shell condition to be just (rescaled) η_4, to wit

$$m^2 = \kappa\, \eta_4 - \kappa^2$$

so that

$$\mathcal{M}_\kappa(P) = (2\kappa \sinh P_0/2\kappa)^2 - \mathbf{P}^2\, e^{P_0/\kappa} - m^2. \tag{25.15}$$

Equation (25.15) is the famous dispersion relation of DSR1. Notice that it implies that the momentum is bounded from above by κ, while the energy is unbounded.

Let us now turn to the "$-P$" issue. To see what is to replace it in the theory with curved momentum space let us trace the origin of it. In Special Relativity the space of momenta is flat, and equipped with the standard group of motions. The space of momenta has a distinguished point, corresponding to zero momentum. An element of translation group $g(P)$ moves this point to a point of coordinates P. This defines coordinates on the energy momentum space. Now we *define* the point with coordinates $S(P)$ to be the one obtained from the origin by the action of the element $g^{-1}(P)$. Since the group of translations on flat space is an Abelian group with addition, $S(P) = -P$.

Now, since in the case of interest the space of momenta is de Sitter space, which is a maximally symmetric space, we can repeat exactly the same procedure. The result, however, is not trivial now, to wit

$$S(P_0) = -P_0, \quad S(P_i) = -e^{P_0/\kappa}\, P_i. \tag{25.16}$$

Actually one can check that the S operator in this case is nothing but the antipode of κ-Poincaré quantum algebra. Thus we can write down the action for the scalar field on curved momentum space as

$$S_\kappa = \int d^4P\, \mathcal{M}_\kappa(P)\, \Phi(P)\, \Phi(S(P)). \tag{25.17}$$

De Sitter space of momenta has the ten dimensional group of symmetries, which can be decomposed to six "rotations" and four remaining symmetries, forming the deformed κ-Poincaré symmetry (25.1). We expect therefore that the action (25.17) should, if properly constructed, be invariant under the action of this group. We will find that this is indeed the case; however, the story will take an unexpected turn here: the action will turned out to be invariant under the action of the *quantum group*.

Let us consider the four parameter subgroup of symmetries that, in the standard case, would correspond to spacetime translation. It is easy to see that, in the standard case, the translation in spacetime fields is in the one-to-one correspondence with the phase transformations of the momentum space ones. This suggests that the ten parameter group of Poincaré symmetries in spacetime translates into a six parameter Lorentz group plus four independent phase transformations in the momentum space, being representations of the same algebra.

Using this insight let us turn to the case in hand. Consider first the infinitesimal phase transformation in energy direction[1] (to simplify the notation I put $\kappa = 1$)

$$\delta_0\Phi(P_0, \mathbf{P}) = i\epsilon\, P_0\Phi(P_0, \mathbf{P}), \tag{25.18}$$

where ϵ is an infinitesimal parameter. It follows that

$$\delta_0\Phi(S(P_0), S(\mathbf{P})) = i\epsilon\, S(P_0)\,\Phi(S(P_0), S(\mathbf{P})) = -i\epsilon\, P_0\,\Phi(S(P_0), S(\mathbf{P})) \tag{25.19}$$

and using Leibniz rule we easily see that the action is indeed invariant. Let us now consider the phase transformation in the momentum direction. Assume that in this case

$$\delta_i\Phi(P_0, \mathbf{P}) = i\epsilon\, P_i\,\Phi(P_0, \mathbf{P}). \tag{25.20}$$

But then

$$\delta_i\Phi(S(P_0), S(\mathbf{P})) = i\epsilon\, S(P_i)\,\Phi(S(P_0), S(\mathbf{P})) = -i\epsilon\, e^{P_0}\, P_i\,\Phi(S(P_0), S(\mathbf{P})) \tag{25.21}$$

and the action is not invariant, if we apply the Leibniz rule.

The way out of this problem is to replace the Leibniz rule by the co-product one. To this end we take

$$\begin{aligned}\delta_i\,\{\Phi(P_0, \mathbf{P})\Phi(S(P_0), S(\mathbf{P}))\} \equiv{}& \delta_i\,\{\Phi(P_0, \mathbf{P})\}\,\Phi(S(P_0), S(\mathbf{P})) \\ &+ \left\{e^{-P_0}\Phi(P_0, \mathbf{P})\right\}\,\delta_i\,\{\Phi(S(P_0), S(\mathbf{P}))\} = 0,\end{aligned}$$

i.e. we generalize the Leibniz rule by multiplying $\Phi(P_0, \mathbf{P})$ in the second term by e^{-P_0}. Note that this definition is consistent with the fact that the fields are commuting, because

$$\delta_i\,(\Phi(S(P_0), S(\mathbf{P}))\Phi(P_0, \mathbf{P})) =$$

$$\left(i\epsilon\, S(P_i) + i\epsilon\, e^{-S(P_0)}\, P_i\right)\Phi(S(P_0), S(\mathbf{P}))\Phi(P_0, \mathbf{P}) = 0.$$

We see therefore that, in order to make the action invariant with respect to infinitesimal phase transformations, one must generalize the standard Leibniz rule to the non-symmetric co-product one.

The rule of how an algebra acts on the (tensor) product of objects is called the co-product, and is denoted by Δ. If the Leibniz rule holds the co-product is trivial $\Delta\delta = \delta \otimes 1 + 1 \otimes \delta$. Quantum groups can be characterized by the fact that the

[1] Note that since the function $\mathcal{M}$ is real, $\delta_0\mathcal{M}_\kappa = \delta_i\mathcal{M}_\kappa = 0$.

Leibniz rule is generalized to a non-trivial coproduct rule. We discovered that in the case of κ-Poincaré algebra it takes the form

$$\Delta\delta_0 = \delta_0 \otimes 1 + 1 \otimes \delta_0, \quad \Delta\delta_i = \delta_i \otimes 1 + e^{-P_0} \otimes \delta_i. \tag{25.22}$$

One can check that, similarly, the co-product for the rotational part of the symmetry algebra is also non-trivial. The presence of a non-trivial co-product in the algebraic structure of DSR theory has, presumably, far reaching consequences for particle kinematics. I will return to this point below.

25.5 DSR phenomenology

DSR emerged initially from the Quantum Gravity phenomenology investigations, as a phenomenological theory, capable of describing possible future observations disagreeing with predictions of Special Relativity. Two of these effects, the possible energy dependence of the speed of light, which could be observed by GLAST satellite, and the previously mentioned possible violation of the GZK cutoff, which could be confirmed by the Pierre Auger Observatory, have been quite extensively discussed in the literature. Let me now briefly describe what would be the status of these (possible) effects vis à vis the approach of DSR I have analyzed above[2].

The prediction of the energy dependence of the speed of light is based on the rather naive observation that since in (some formulations of) DSR the dispersion relation is being deformed, the formula for velocity $v = \partial E/\partial p$ gives, as a rule, a result which differs from that of Special Relativity. It turns out, however, that this conclusion may not stand if the effects of non-commutative spacetime are taken into account.

In the classical theory, the noncommutativity is replaced by the nontrivial structure of the phase space of the particle and, as in the standard case, one calculates the three velocity of the particle as the ratio of $\dot{x} = \{x, H\}$ and $\dot{t} = \{t, H\}$: $v = \dot{x}/\dot{t}$. Then it can generally be proved that the effect of this nontrivial phase space structure cancels neatly the effect of the modified dispersion relation (see [6] for details.) Thus, in the framework of this formulation of DSR, the speed of massless particles is always 1, though there are deviations from the standard Special Relativistic formulas in the case of massive particles. However, the leading order corrections are here of order of m/κ, presumably beyond the reach of any feasible experiment.

Similarly one can argue that deviations from the GZK cutoff should be negligibly small in any natural DSR theory. The reasoning goes as follows (a similar argument

[2] It should be stressed that DSR was originally proposed as an idea, not a formally formulated theory, and therefore it may well happen that the particular realization of this idea described above could be replaced by another one in the future.

can be found in [4]). Consider *experimental* measurement of the threshold energy for the reaction $p + \gamma = p + \pi^0$, which is one of the relevant ones in the ultra high energy cosmic rays case, but details are not relevant here. To measure this energy we take the proton initially at rest and bombard it with more and more energetic photons. At some point, when the photon energy is of order of $E^0_{\text{th}} = 145$ MeV, the pion is being produced. Note that the threshold energy is just E^0_{th}, exactly as predicted by Special Relativity, and the corrections of DSR (if any) are much smaller than the experimental error bars ΔE^0_{th}. Thus whichever kinematics is the real one we have the robust result for the value of the threshold energy.

Now there comes the major point. Since DSR respects the Relativity Principle by definition, we are allowed to boost the photon energy down to the CMB energy (this cannot be done in the Lorentz breaking schemes, where the velocity of the observer with respect to the ether matters), and to calculate the value of the corresponding rapidity parameter. Now we boost the proton with the same value of rapidity, using the DSR transformation rules, and check the modified threshold. Unfortunately, the leading order correction to the standard Special Relativistic transformation rule would be of the form $\sim \alpha E_{\text{proton}}/\kappa$, where E_{proton} is the energy of the proton after boost, and α is the numerical parameter fixed in any particular formulation of DSR. It is natural to expect that α should be of order 1, so that in order to have sizeable effect we need κ of order of 10^{19} eV, quite far from the expected Planck scale.[3] One may contemplate the idea that since the proton is presumably, from the perspective of the Planck scale physics, a very complex composite system, we do not have to deal here with "fundamental" κ, but with some effective one instead, but then this particular value should be explained (it is curious to note in this context that, as observed in [4], 10^{19} eV is of the order of the geometric mean of the Planck energy and the proton rest mass.) However, the conclusion for now seems inevitably to be that, with the present formulation of DSR, the explanation of possible violation of GZK cutoff offered by this theory is, at least, rather unnatural.

25.6 DSR – facts and prospects

Let me summarize. Above I stressed two facts that seem to be essential features of DSR theory.

First, (Quantum) Gravity in 2+1 dimensions coupled to point particles is just a DSR theory. Since the former is rather well understood, it is a perfect playground for trying to understand better the physics of the latter. In 3+1 dimensions the situation is much less clear. Presumably, DSR emerges in an appropriate limit of

[3] Note that in this reasoning we do not have to refer to any particular DSR kinematics, the form of energy-momentum conservation, etc. The only input here is the Relativity Principle.

(Quantum) Gravity, coupled to point particles, when the dynamical degrees of freedom of gravitational field are switched off, and only the topological ones remain. However, it is not known exactly what this limit would be, and how to perform the limiting procedure in the full dynamical theory. There is an important insight, coming from an algebraic consideration, though. In 3+1 dimension one can do almost exactly the same procedure as the one I presented for the 2+1 case above. It suffices to replace the $SO_q(3, 1)$ group with $SO_q(4, 1)$. It happens, however, that in the course of the limiting procedure one has to further rescale the generators corresponding to energy and momentum. The possible rescalings are parametrized by the real, positive parameter r: for $r > 1$ the contraction does not exist, for $0 < r < 1$ as the result of contraction one gets the standard Poincaré algebra, and only for one particular value $r = 1$ one finds κ-Poincaré algebra. This result is not understood yet and, if DSR is indeed a limit of gravity, gravity must tell us why one has to choose this particular contraction.

Second, as I explained above there is a direct interplay between the non-trivial co-product and the fact that momentum space is curved. In addition, curved momentum space naturally implies non-commutative spacetime. While the relation between these three properties of DSR theory has been well established, it still requires further investigations.

The presence of the non-trivial co-product in DSR theory has its direct consequences for particle kinematics. Namely, the co-product can be understood as a rule of momentum composition. This fact has been again well established in the 2+1 dimensional case. However, the 3+1 situation requires further investigation. The main problem is that the co-product composition rule is not symmetric: the total momentum of the system (particle$_1$ + particle$_2$) is not equal, in general, to that of the total momentum of the (particle$_2$ + particle$_1$) one. This can be easily understood in 2+1 dimensions if one thinks of particles in terms of their worldlines, and where the theory takes care of the worldlines' braiding. In 3+1 dimensions the situation is far from clear, though. Perhaps a solution could be replacing holonomies that characterize particles in 2+1 dimensions by surfaces surrounding particles in 3+1 dimensions. If this is true, presumably the theory of gerbes will play a role in DSR (and gravity coupled with particles, for that matter.)

Related to this is the problem of "spectators". If the co-product rule is indeed correct, any particle would feel the non-local influence of other particles of the universe. This means in particular, that the LSZ theorem of quantum field theory, which requires the existence of free asymptotic states, presumably does not hold in DSR, and thus all the basic properties of QFT will have to be reconsidered.

Arguably one of the most urgent problems of DSR is the question "what is the momentum?". Indeed, as I mentioned above, in the κ-Poincaré case one has the freedom to redefine momentum and energy by any function of them and the κ

scale, restricted only by the condition that in the limit $\kappa \to \infty$ they all reduce to the standard momenta of Special Relativity. In particular some of them might be bounded from above, and some not. For example in DSR1 momentum is bounded from above and energy is not, in another model, called DSR2 both energy and momentum are bounded, and there are models in which neither is. Thus the question arises as to which one of them is physical? Which momentum and energy do we measure in our detectors?

There is a natural answer to this question. Namely, the physical momentum is the charge that couples to gravity. Indeed if DSR is an emergent theory, being the limit of gravity, the starting point should be, presumably, gravity coupled to particles' Poincaré charges in the canonical way.

To conclude: there seem to be important and deep interrelations between developments in Quantum Gravity and our understanding of DSR. Proper control over semiclassical Quantum Gravity would provide an insight into the physical meaning and relevance of DSR. And vice versa, DSR, being a possible description of ultra high energetic particle behavior, will perhaps become a workable model of Quantum Gravity phenomenology, to be confronted with future experiments.

Acknowledgement

This work is partially supported by the KBN grant 1 P03B 01828.

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26

From quantum reference frames to deformed special relativity

F. GIRELLI

26.1 Introduction

Quantum Gravity (QG) theory was thought for a long time to be just a mathematical theory since it was relevant only at extreme energies: for example right after the Big Bang or very close to the singularity in a black hole. It is hard to probe any of the physics happening there for obvious reasons. This situation changed recently: the possible existence of extra dimensions lowers the QG typical energy scale and so could make it possible to see QG effects in the new particle accelerators (see for example [1] and references therein). Even without extra dimensions, it was proposed that some extreme astrophysical situations might provide ways to probe the quantum (more exactly the semiclassical) structure of spacetime, see for example Amelino-Camelia's contribution to this book (chapter 22).

In the context of loop quantum gravity (LQG) and spinfoams, different models exist and it is unclear if they are equivalent or not. If one were able to construct a semiclassical limit for those, one would be able to make predictions for the different models and wait for the forthcoming experiments to falsify some of them. QG would be about to become true physics!

More explicitly one should calculate the full partition function

$$\mathcal{S} = \int d\phi_{\mathrm{M}} dg e^{i\int \mathcal{L}_{\mathrm{M}}(\phi_{\mathrm{M}},g) + \mathcal{L}_{\mathrm{GR}}(g)},$$

where the ϕ_{M} represent all the matter and interactions fields other than gravitational which are encoded in the metric g, and $\mathcal{L}_{\mathrm{M}}(\phi_{\mathrm{M}}, g)$, $\mathcal{L}_{\mathrm{GR}}(g)$ are respectively the Lagrangian for matter and gravity. To make valuable predictions for the next experiments we would like to integrate out all the QG degrees of freedom around the flat metric η (assuming that the cosmological constant is zero) to obtain an effective action for matter encoding the QG physics:

$$\mathcal{S} = \int d\phi_{\mathrm{M}} e^{i\int \tilde{\mathcal{L}}_{\mathrm{M}}(\phi)}.$$

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

The new Lagrangian $\tilde{\mathcal{L}}_{\mathrm{M}}(\phi_{\mathrm{M}})$ describes in an effective way the QG fluctuations. A natural consequence is then an important modification of matter dynamics and spacetime concepts. For example, field evolution in this context might not be unitary since we have integrated out some degrees of freedom. In the case of a simple classical relativistic particle, we would expect the dynamics to be described by a modified mass shell condition.

The explicit calculation can currently be done explicitly only in a three dimensional spacetime [2]. In this case a non-commutative spacetime emerges, as well as a modified notion of multiparticles states.

Unfortunately, deriving this semiclassical limit is still a challenge in the four dimensional case. Instead of trying to derive it by brute force, one can try to cook up a theory describing the semiclassical spacetime. We intend to put at the kinematical level the QG fluctuations, to have an effective notion of flat semiclassical spacetime. A modification of the Poincaré symmetries is then present. To my knowledge, deformed (or doubly) special relativity (DSR) is one of the best candidates to describe this setting. There is a number of heuristic arguments to show how DSR can be derived from a 4d QG theory [3; 4], but not yet any solid mathematical argument as in 3d [5]. Under the name of DSR actually go many different approaches (Snyder's approach [6], modified measurement [7; 8], quantum groups approach [9]), which are not clearly equivalent. They all have common features: in general a deformation of the Poincaré symmetries, a non-commutative spacetime and a modification of the multiparticles states. There are two ways to understand the apparent freedom in the choice of DSR type: either there is only one physical deformation together with one set of physical phase space coordinates (this is what happens in the 3d case), or all the different DSR structures can be unified in one general new structure. This question needs to be clarified at least in the simplest example, the "free" particle, before getting to quantum field theory and so on.

Clearly, there are now two jobs to be done: on one hand to check that DSR is really the semiclassical limit of some QG theory; on the other, to understand its physics and be able to make predictions to confront it with the forthcoming experiments.

Here I would like to argue that understanding the physics of DSR can be related to understanding the QG physics. Indeed General Relativity is a constrained theory, which means that observables must be relational, and in particular constructed in terms of physical reference frames [12]. When moving to the QG regime, one should talk about quantum reference frames (QRF), therefore quantum coordinates and so on. Moving on to the semiclassical limit, one should still feel the funny QG physics. From this perspective DSR will arise as a modified measurement theory, owing to the modification of the notion of reference frame still bearing some quantum/gravitational features.

In section 26.2 I will quickly recall the construction of observables in QG, but also ask a number of questions that should be answered to my mind, to understand the QG physics. It is hard of course to do so in a QG theory like LQG, so I will illustrate the possible answers using a little toy model consisting in a universe of spin $\frac{1}{2}$ (qubits).

In section 26.3 I want to describe what kind of flat semiclassical spacetime we can expect to recover. For this I will start by recalling how a modification of the measurement theory can be seen as implementing a deformation of the symmetries. A strong analogy holds with the toy model, an analogy that can be seen as another heuristic argument indicating that DSR is the right QG semiclassical limit. The deformation is usually done in the momentum space, that is the cotangent space. I will argue then that the geometry (that is, the tangent bundle picture) associated to this flat semiclassical spacetime can be a described by a Finsler geometry [10]. Since each type of deformed reference frame will correspond to a deformation of the symmetries, it is natural to ask if there is a global structure that allows one to unify the different constructions. I will show that indeed these different choices of reference frames just correspond to different choices of gaugefixings (or choice of observers) in an extended phase space [11]. This allows us also to specify in an unambiguous way the symplectic form and the physical spacetime coordinates. I will conclude with some comments on the multiparticles states.

26.2 Physics of Quantum Gravity: quantum reference frame

The symmetry group of General Relativity is the diffeomorphisms group. Invariance under this group means that the physics should not depend on the choice of coordinates. The coordinates x_μ are parameters, they should not have any physical meaning. To understand that was an essential step in the GR construction. It also led to a long-standing misunderstanding. Indeed when doing physics it is natural to use coordinates systems: there exists a reference frame (clock, rulers) that allows us to measure a spacetime position, and so provide physical coordinates. The confusion arose since it seems that a coordinate system must be at the same time physical and not physical. As so often, the answer to this paradox lies in its formulation: the measured coordinates do not have the same status as the coordinates met in the GR mathematical definition.

To define the physical coordinates, we must use some degrees of freedom [12]: the reference frame (that often can be confused with the measurement apparatus) is made of matter (clock and rulers) or gravitational degrees of freedom. This is a general feature: any physical quantity that is the outcome of some measurement quantifies the relation between two systems (the reference frame or apparatus and the system under study).

The discussion can be made more precise when addressed in the Hamiltonian formalism. GR is an example of a constrained theory: there is a set of first class constraints that encode the diffeomorphism symmetry.[1] Observable quantities are functions on phase space that commute with the constraints. It is pretty hard to construct a general complete set of observables. However, taking advantage of the fact that physics should be relational allowed us to construct a large set of such observables [13].

To simplify the analysis let us consider the relativistic free particle: in this case, we have time reparametrization invariance, encoded in the mass shell constraint $H = p^2 - m^2 = 0$. It is easy to construct the algebra of observables: it is given by the Poincaré algebra $\{J_{\mu\nu},\ p_\mu\}$. This set of observables does not include the important notion of position. To define this concept, we need to introduce the following Rovelli terminology of a *partial observable*[2] $b(\tau)$ as a clock. A natural observable[3] is then the value of another partial observable $a(\tau)$ when $b(\tau)$ is T. If separately a and b are not observable, since not commuting with the mass shell constraint, the quantity

$$a(b^{-1}(T)) = \int d\tau a(\tau)\dot{b}(\tau)\delta\left(b(\tau) - T\right) \tag{26.1}$$

is clearly time reparametrization invariant, and therefore observable. For example, if we take b to be x_0, and a to be x_i, we obtain the trajectories of the relativistic particle in terms of the time x_0:

$$x_\mu(T) = x_\mu + \frac{p_\mu}{p_0}(x_0 - T).$$

Notice that of course this observable can be constructed from the Poincaré algebra [14].

In the language of constrained mechanics, $b(\tau) = T$ is a gauge fixing, or a second class constraint. From the physics point of view, the degree of freedom b is the reference frame. Obviously there is the issue of the invertibility of b. In general the choice of clock might not lead to a function which is invertible everywhere. This means that the clock ceased to be a good clock. This issue has to be studied in a case per case analysis.

Introducing second class constraints means that we can reduce the phase space to obtain the physical phase space. The reduced symplectic form is called the Dirac bracket, and is not in general identical to the canonical Poisson bracket. This leads to complications when one wants to quantize such system.

[1] There might be more constraints according to the choice of variables. For example, if using the pair (tetrad, connection), there is also the Gauss law.

[2] Let be f a function on phase space, not commuting with the first class constraint H, then we define $f(\tau) = e^{\tau\{H,.\}}f$.

[3] That is the outcome of a measurement or, according to Rovelli, a complete observable [12].

The general theory of quantization of a constrained system has been set up by Dirac [15]. Let us deal only with a set of first class constraints $\{C_i\}$. We first quantize the algebra of partial observables $\mathcal{A}_k$ and then construct the kinematical Hilbert space $\mathcal{H}_k$ carrying the representation of $\mathcal{A}_k$. We quantize then the constraints C_i:

$$\mathcal{A}_k \to \hat{\mathcal{A}}_k,\ C_i \to \hat{C}_i.$$

In general it might be difficult to construct $\mathcal{H}_k$ but also to quantize the constraints C_i, since they might be non-polynomial functions of partial observables. This is precisely what happened in the first try to quantize GR [16].

The great accomplishment of LQG was to describe GR in terms of variables that allowed one to construct the Hilbert space and quantize the constraints.

From the kinematical Hilbert space, we construct physical Hilbert $\mathcal{H}_{\text{phys}}$ space, which is in the kernel of the constraints $\hat{C}_i$. For this, it is useful to introduce the projector[4]

$$P : \mathcal{H}_k \to \mathcal{H}_{\text{phys}} \equiv \{P|\psi\rangle_k\} \text{ with } P \sim \int d\lambda e^{i\lambda \hat{C}} \sim \delta(\hat{C}).$$

The physical quantum observables are also obtained upon projection

$$\hat{A} \in \hat{\mathcal{A}}_k \to P\hat{A}P \in \hat{\mathcal{A}}_{\text{phys}}.$$

In order to have some physical quantities, we can construct the relational observables analog to (26.1):

$$P\delta(\hat{B} - b)\hat{A}\delta(\hat{B} - b)P, \text{ with } \hat{A}, \hat{B} \in \hat{\mathcal{A}}_k,$$

where $\delta(\hat{B} - b)$ denotes the projection of $\hat{B}$ on the eigenspace with eigenvalue b. The degrees of freedom $\hat{B}$ can be identified as a quantum reference frame (QRF): the physics of QG should be understood in terms of QRF. This new type of physics is extremely rich and interesting to explore. Since in the context of QG it is a bit hard to explore this, we can look for some toy models to mimic this structure.

In fact interestingly the notion of QRF has already been introduced in quantum information theory (QIT), in concrete models that can also be experimentally tested! Let us choose, for example,[5] a quantum universe made of N qubits $\vec{\sigma}^i$,[6] which is globally invariant under $SO(3)$ rotations [17].

[4] It will be a projector if the constraints have zero in their discrete spectrum. If this is not the case, it is not a projector; we need to use distributions. This a mathematical subtlety important in QG, but not relevant to the current discussion.

[5] There are many other little toy models where one can play around to mimic gravitational effects. For example, one can look at *analog gravity* models [20], or constrained harmonic oscillators [21]. Of course all these models are finite dimensional, which (over?)simplifies drastically the analysis.

[6] I use the notational shortcut $\vec{\sigma}^i \equiv \mathbf{1} \otimes \ldots \otimes \vec{\sigma} \otimes \mathbf{1} \otimes \ldots$, the Pauli matrices being at the ith position.

The first key questions to answer to understand the QG physics are the following.

Can we construct a complete set of complete observables?
As I just recalled, this is a hard question in the QG context [13; 18]. In simple cases, like the qubits universe, this can be done exactly. It is not difficult to realize that the qubits universe can be seen as an intertwinner: we have a bunch of spin representations that should be invariant under rotations. The algebra of complete observables associated to this model has been determined in [19].

It is more interesting, however, to concentrate on specific observables, namely the analog of the coordinates.◇

Is there a noncommutativity naturally appearing?
Since coordinates are a key tool in physics, one needs to understand what is a quantum coordinate. This would allow also to relate a quantum geometry with a noncommutative geometry. This latter is usually described by a modification of the symplectic form, on the configuration space. That is, we have for example

$$[x_\mu, x_\nu] = \theta_{\mu\nu} + \epsilon^{\alpha}_{\mu\nu} x_\alpha + \cdots .$$

This interesting approach has been described in Majid's contribution to this book (see chapter 24). Unfortunately there is not yet any link between the LQG program and this program, though they should be definitely related.[7] In the toy model, we can easily construct some observables: the relative angles are clearly quantities invariant under global rotations. They allow us to construct the notion of coordinates: take two (non-intersecting) sets of qubits $\vec{J}_a$, which define the reference frame (the third vector is $\vec{J}_3 = \vec{J}_1 \wedge \vec{J}_2$). The quantities

$$\tilde{\sigma}^i_a = \vec{\sigma}^i . \vec{J}_a, \text{ with } a = 1, 2, 3, \tag{26.2}$$

define the (quantum) relative coordinates between $\vec{\sigma}^i$ and the reference frame $\{\vec{J}_a\}$. It is not difficult to be convinced that

$$[\tilde{\sigma}^i_a, \tilde{\sigma}^i_b] \neq \epsilon^c_{ab} \tilde{\sigma}_c. \tag{26.3}$$

This shows that the symplectic form has been modified, so that this would correspond to a noncommutative geometry.◇

What is a measurement?
The question of measurement is a difficult question in Quantum Mechanics (QM). Since QG comes from the canonical quantization scheme applied to General

[7] A first step has been done in [19]

Relativity, it is not clear how QG could help to solve this problem. Different interpretations to QM favor better understanding of the measurement procedure (though in general not solving it). In particular, treating QM as a theory about information (QIT) allows us to describe nicely what is a measurement in the presence of a quantum reference frame. This has been analyzed by [22] in the qubits universe. Let me recall the construction quickly in the case of the measurement of a qubit with respect to another qubit. Since we have the tensor product of two spins $\frac{1}{2}$, following the Schur lemma,[8] it is natural to decompose any measurement E_λ with outcome λ along the projectors $\Pi_{0,1}$ in the basis $0 \oplus 1 \sim \frac{1}{2} \otimes \frac{1}{2}$. The projectors $\Pi_{0,1}$ are observable, that is invariant under global rotations:

$$E_\lambda = a_{\lambda,1}\Pi_1 + a_{\lambda,0}\Pi_0,$$

where the coefficients $a_{\lambda,i}$ satisfy the necessary conditions to make an E_λ a projector operator valued measurement (POVM) [23]. To be in the eigenspace of one of the projectors $\Pi_{\frac{1}{2}\pm\frac{1}{2}}$ tells us if the spins are aligned or anti-aligned.

The idea is now to use the Bayes theorem, which from a prior distribution of knowledge describes how to update it. One starts with a prior distribution $p(\alpha)$ on α. Upon obtaining the outcome λ, we can update our knowledge from the prior distribution $p(\alpha)$ to $p_\lambda(\alpha) = p(\alpha|\lambda)$:

$$p(\alpha|\lambda) = \mathrm{Tr}(E_\lambda \rho_\alpha)\frac{p(\alpha)}{p(\lambda)}, \tag{26.4}$$

with ρ_α a physical state that is rotationally invariant, and $p(\lambda) = \int \mathrm{Tr}\,(E_\lambda \rho_\alpha)\, p(\alpha) d\alpha$.◇

Is our quantum reference frame robust?

In the classical case, a reference frame can happen to be not a good reference frame globally. This is related to the problem of invertibility of the partial observable as argued in the previous section. More physically a clock can for example decay, lose its precision, owing to various interactions with its environment. In the quantum case we can have some similar situations. For example, by making many consecutive measurements the QRF will get blurred since in general the QRF gets entangled with the system. Once again this has been analyzed in the context of QIT [24; 25]. For example after one measurement, forgetting about the outcome of the previous measurement, one has the new QRF state

$$\rho_{\mathrm{RF}}^{(1)} = \mathrm{Tr}_S \sum_{a=0,1} \Pi_a \rho_{\mathrm{RF}} \otimes \rho_S \Pi_a.$$

[8] Since we want to make a physical measurement, that is $R E_\lambda R^{-1} = E_\lambda$, for any global rotation R.

By doing a sequence of measurements, the QRF will not evolve in general unitarly, which precisely means that it decoheres. This robustness is of course interesting to explore to have some idea on which kind of reference frame can survive in the (semi)classical limit.◇

What kind of symmetries do we have?

Once one has constructed the physical observables, one gets rid of the symmetries encoded in the constraints. But one needs to be able to see what transformations relate the different choices of reference frame. Clearly, in the context of the qubits universe, one can choose a reference frame $\vec{J}_a$, or another one $\vec{J}'_a$; the two of them are related by some rotations, $R^b{}_a \vec{J}_b = \vec{J}'_a$. This means in particular that the observable $\vec{\tilde{\sigma}}$ transforms linearly under change of reference frame $\vec{\tilde{\sigma}}' = R \cdot \vec{\tilde{\sigma}}$. When the QRF starts to degrade after some measurements, it doesn't depend anymore in a linear way on the initial RF state: changing of reference frame implies that a rotation acts in a non-linear way on the $\tilde{\sigma}$. Indeed, we have clearly

$$R\rho^{(1)}_{\mathrm{RF}} R^{-1} \neq \mathrm{Tr}_S \sum_{a=0,1} \Pi_a R \rho_{\mathrm{RF}} R^{-1} \otimes \rho_S \Pi_a,$$

where R is the rotation. The symmetry gets deformed (or non-linearly realized) because of the degradation of the reference frame.◇

Is the notion of multiparticles states affected?

In the context of QG, one should be able first to define a particle or the notion of fields, which is not easy to do. In the context of a relational physics we can expect the multiparticles states to be modified. More precisely the tensor product structure can be modified. Indeed since we look at degrees of freedom encoded in relations, two physical degrees of freedom defined in terms of the same reference frame do share the reference frame degrees of freedom, possibly spoiling the usual multiparticles structure. The qubits universe allows us to illustrate this. A two qubits state can be constructed operationally: we take two spins σ^k and $\sigma^{k'}$ and consider the relational observable

$$\tilde{\sigma}^{\mathrm{tot}}_a = \left(\vec{\sigma}^k \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\sigma}^{k'}\right) \cdot \vec{J}_a = \tilde{\sigma}^k_a \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{\sigma}^{k'}_a.$$

The two particles state structure is modified since

$$[\tilde{\sigma}^k_a \otimes \mathbb{1}, \mathbb{1} \otimes \tilde{\sigma}^{k'}_b] \neq 0.$$

Note also that the multiparticles states are usually seen as tensor product of representations of the symmetry group. If the symmetry is moreover non-linearly realized as we argued above, this might imply further complications. ◇

What is the semiclassical limit?
To be able to define the semiclassical limit in the context of LQG is the big question. In particular the notion of flat semiclassical spacetime is a key notion to understand to make predictions to the forthcoming experiments. The natural flat semiclassical limit should be a theory of Special Relativity modified in order to account for some quantum gravitational fluctuations. In 3d the semiclassical limit is given by the deformed special relativity (DSR) theory. There are good hints now that in 4d, DSR is also the QG semiclassical limit [3; 4]. We expect to have some non-trivial physics happening owing to the modification of the notion of reference frame, the notion of measurements etc. These modifications should be traced back to an effective description of some gravitational or quantum features.

In the qubits universe,[9] the semiclassical limit is just given by taking the QRF semiclassical as well as the system. After measurements there is still a kick back of the system, due to quantum effects, on the reference frame making the physics non- trivial: deformation of the symmetry, modification of the multiparticles states. It is only in a very large limit that these effects disappear. $\diamond$

26.3 Semiclassical spacetimes

In the semiclassical limit one has $\hbar \to 0$. In 4d, since the Planck scales L_{P} and M_{P} are proportional to $\hbar$, they both go to zero. Since we are interested in studying the QG fluctuations around a flat spacetime, we can also take the limit $G \to 0$. Since the Planck mass is a ratio $M_{\mathrm{P}}{}^2 \sim \frac{\hbar}{G}$, to have the limit well defined it is important to specify how G goes to zero with respect to $\hbar$. For example we can take $G \sim \hbar \to 0$, so that M_{P} is fixed: this flat semiclassical limit is therefore described by the Planck mass. In this regime gravitational effects are comparable to the quantum effects, this is the DSR regime. M_{P} can be associated to a 3d momentum, to a rest mass, or energy. This regime is then effectively encoded in a modified Casimir, that is a modified dispersion relation (MDR) taking into account M_{P}. The starting point of the QG phenomenology is therefore the general MDR

$$E^2 = m^2 + p^2 + F(p, \mu, M_{\mathrm{P}}), \tag{26.5}$$

where F is a function of dimension mass two, μ is a possible set of extra mass parameters (like Higgs mass), and $p = |\vec{p}|$. This MDR can be also interpreted as a manifestation of Lorentz invariance violation (LIV). Using the effective field theory framework, some strong constraints have been set on the first terms when compared to data (e.g. coming from the Crab nebula) [27]. From the DSR point of view, it is natural to expect the deformation of the symmetries, to accommodate

[9] Semiclassical analysis have been done with other constrained toy models [26].

M_P as a maximum mass, to be first done in momentum space, and then try to reconstruct from there the flat semiclassical spacetime. I am going to recall first how the MDR can be associated to a modified measurement, as encountered in the previous section. I will then describe the new geometry associated to this effective spacetime. I will show then how the different deformations can be unified in one common scheme, as gaugefixings.

26.3.1 Modified measurement

In order to explain the physics of (26.5), Liberati *et al*. proposed a modified notion of measurements [7; 8]. Let us note π_μ, the momentum intrinsic to the particle. To do a measurement we need to introduce a reference frame $e^\mu{}_\alpha$, the tetrad [28]. The μ indices are spacetime indices and transform as tensor indices. Note that this is very similar to the reference frame introduced in the toy model $\vec{J}_a \sim J^i{}_a$. The outcome of measurement are scalars p_α, obtained upon projection of π on the reference frame e:

$$p_\alpha = \pi_\mu e^\mu{}_\alpha. \tag{26.6}$$

In the Minkowski case, the tetrad is trivial so that $e^\mu{}_\alpha \sim \delta^\mu{}_\alpha$, this just means that π and p coincide.

If one considers another reference frame $\tilde{e}^\mu_\alpha = \Lambda^\beta{}_\alpha e^\mu{}_\beta$, which is related to the first by a Lorentz transformation, the new outcome of the measurement is then $p'_\alpha = \pi_\mu \bar{e}^\mu{}_\alpha = \Lambda^\beta{}_\alpha e^\mu{}_\beta \pi_\mu = \Lambda^\beta{}_\alpha p_\beta$. We have naturally a linear realization of the Lorentz symmetries.

As we have seen in section 26.2, it might happen that the measurement in the quantum context mixes in an intricate way RF and system. We can then expect that in QG a similar situation could occur: an effective treatment of the (quantum) gravitational fluctuations can also generate such a non-trivial mixing [7; 8]. For example, naively, the tetrad could also encompass the gravitational field generated by the (quantum) particle (which is usually neglected) and so be dependent on the particle momentum.

The outcome of the measurement p_α is therefore a non-linear function U of the intrinsic momentum π:

$$p_\alpha = U_\alpha(\pi) \sim \pi_\mu e^\mu{}_\alpha(\pi).$$

Upon change of reference frame under Lorentz transformation, p_α will clearly transform non-linearly. It can be explicitly written as [29]:

$$\tilde{\Lambda}^\beta_\alpha p_\beta = U_\alpha \left(\Lambda^\beta{}_\alpha U^{-1}_\beta(p)\right).$$

In this sense, we do not have a symmetry breaking but a deformation of the symmetry more exactly a non-linear realization. The intrinsic momentum provides the undeformed mass shell condition $\pi_\mu \eta^{\mu\nu} \pi_\nu = m^2$, which allows us to construct the modified Casimir

$$m^2 = U_\alpha^{-1}(p)\eta^{\alpha\beta}U_\beta^{-1}(p) = p_\alpha p_\beta e_\mu{}^\alpha(p)\eta^{\mu\nu}e_\nu{}^\beta(p) = E^2 - p^2 - F(p, \mu, M_{\mathrm{P}}).$$

This is a similar construction for the qubits universe, as in section 26.2. Up to now, the construction was made only with momentum, that is forms. We need to construct the notion of spacetime and geometry associated to this new notion of tetrad.

26.3.2 Spacetimes reconstruction

26.3.2.1 Finsler geometry

Since we are working with momentum, it means that we are in the Hamiltonian formalism, that is the cotangent bundle. To completely specify the physics, we need to introduce the configuration space x^μ, that is the physical spacetime coordinates, but also the symplectic form relating x to p. Since from our approach we have no indication about the configuration space we can take the physical coordinates x to be canonically conjugated to p.[10]

The cotangent space is now endowed with a very non-trivial metric structure given by the momentum dependent tetrad $e^\mu{}_\alpha(p)$ which is clearly not issued from a (pseudo-)Riemannian structure.[11]

To understand the new geometry involved, it is natural to perform a Legendre transform to express the particle action in the Lagrangian formalism, or in the tangent bundle [10].

We start therefore with the DSR particle action which is encoded in the constraint associated to the MDR:

$$\mathcal{S} = \int dx^\mu p_\mu - \lambda(E^2 - m^2 - p^2 - F(p, \mu, M_{\mathrm{P}})),$$

where λ is the Lagrange multiplier. The key feature of this action is its time reparametrization, encoded in the constraint. The Hamilton equations specify

[10] In fact, following different approaches, it happens often that spacetime is non-commutative and that x is not canonically related to p. However, using the Darboux theorem, we can always introduce locally some phase space coordinates (y^μ, P_μ) such that $\{y^\mu, P_\nu\} = \delta^\mu_\nu$. Note that we can always also do a non-linear transformation on momentum space such that the MDR (26.5) just become the usual dispersion relation $P^2 = m^2$. The symplectic form will then be in general modified and non-trivial and so will be the multiparticles states. In this sense it is wrong to say that DSR is just like Special Relativity in some non-linear coordinates. Physics in the two regimes is very different.

[11] That is a not scalar product on the space of forms.

the Legendre transform, which is in general hard to invert, if not by perturbations in M_{P}:

$$\frac{dx^\mu}{ds} = \dot{x}^\mu = \lambda\{x^\mu, E^2 - m^2 - p^2 - F(p, \mu, M_{\mathrm{P}})\}, \quad \frac{dp_\mu}{ds} = 0.$$

The Lagrangian $\mathcal{L}$ then obtained will be in general a non-bilinear function $\mathcal{F}(\dot{x})$ of $\dot{x}^\mu$,

$$\mathcal{S} = \int \mathcal{F}(\dot{x}) ds.$$

The key feature is that it is still time reparametrization invariant so that upon rescaling of the vector $\dot{x} \to a\dot{x}$, we have $\mathcal{F}(a\dot{x}) = |a|\mathcal{F}(\dot{x})$.[12] This means that $\mathcal{F}$ can be identified with a norm (pseudo-norm if the kernel of $\mathcal{F}$ is not trivial). The particle lives then in a space the metric of which is given by

$$g_{\mu\nu}(\dot{x}) = \frac{1}{2}\frac{\partial \mathcal{F}^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu}.$$

This is a Finsler metric [30] and is the natural generalization of Riemannian metrics: the latter arises from a norm which is a bilinear form on the tangent space, whereas a Finsler metric arises from general norms

$$\mathcal{F}_{\mathrm{riem}}(x, \dot{x}) = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu \qquad \mathcal{F}_{\mathrm{fins}}(x, \dot{x}) = g_{\mu\nu}(x, \dot{x})\dot{x}^\mu \dot{x}^\nu.$$

All the geometrical objects (curvature, Killing vectors) arising in Riemannian geometry have been generalized by mathematicians to the Finsler case, though often with some ambiguities. In particular the notion of tetrad becomes here clearly vector dependent, as proposed in the previous section. What is left now is to explore this new concept of geometry, and to try to understand how these mathematical structures can provide a better understanding of the semiclassical spacetimes, but also to possible new experimental tests.

A key feature of this approach is to keep the usual notion of tangent bundle, as a vector bundle. Another possible interpretation of the MDR is to say that momentum space is curved, so that we lose the vector bundle structure for the tangent bundle. This is the standard interpretation of DSR.

26.3.2.2 Extended phase space

The choice of symplectic structure and therefore the choice of physical configuration coordinates was pretty arbitrary in the previous section. It is natural to ask if one can have some canonical way to derive the full (non-trivial) phase space. For this it would be convenient to construct a linear momentum in some space, define the canonical conjugated configuration coordinates and inverse the map to recover

[12] The homogeneity might be true only for $a > 0$ in which case the MDR is not invariant under time inversion.

the physical coordinates. In fact by adding two extra dimensions to phase space, we can do more, that is, see all the different deformations as different gaugefixings or different choices of non-equivalent observers.

Since we have an extra fundamental mass parameter M_{P} in the game we can rewrite the MDR (26.5) as

$$E^2 - p^2 - M(p)^2 = -M_{\mathrm{P}}{}^2, \text{ with } M(p)^2 = m^2 + F(p, \mu, M_{\mathrm{P}}) - M_{\mathrm{P}}{}^2,$$

such that it looks like a five dimensional mass shell condition

$$P_\mu P^\mu - P_4^2 = -M_{\mathrm{P}}{}^2. \tag{26.7}$$

$P_4^2 = M(p)^2$ can be interpreted as having a variable mass. This is something natural from the GR point of view. For example an extended object has a varying mass in a curved spacetime [28]. Since a quantum particle cannot be localized, curvature might introduce some slight variations to its mass. Note that now, momentum space is identified with the de Sitter space, so that we are out of the usual geometrical scheme, the cotangent bundle is not a vector fiber bundle anymore. This will have direct consequences on the addition of momenta as we shall see in the next section.

From the QG point of view, one can expects the Newton constant G to be renormalized to encompass quantum corrections [31]. Instead of considering fluctuations in G and a fixed mass, we can describe the theory in an effective way as a fixed G, with a fluctuating mass: $G(x)\, m \;\rightarrow\; G\, m(x)$. In fact all this is related to the choice of units. The notion of a variable mass in terms of units has already been studied in detail by Bekenstein [32]. The Planck units system $(M_{\mathrm{P}},\ L_{\mathrm{P}},\ T_{\mathrm{P}})$, is independent of any particle data. All the different fundamental constants can be expressed in terms of these quantities, and in these units are fixed. Now consider a particle with a variable mass, that is expressed in the Planck units we have $m = \chi M_{\mathrm{P}}$. If one moves to the particle unit, for example the Compton unit $(M_{\mathrm{C}} = m,\ L_{\mathrm{C}} = \frac{\hbar}{mc},\ T_{\mathrm{C}} = cL_{\mathrm{C}}^{-1})$, we do the scale transformation $L_{\mathrm{P}} \rightarrow\ L_{\mathrm{P}}\chi = L_{\mathrm{C}}$ that can be chosen to keep the speed of light c fixed as well as $\hbar$. G becomes however, variable $G \rightarrow \chi^2 G$: in the particle units we have a fixed mass but a variable G. Since the mass becomes a variable (in Planck units) encoding the QG fluctuations, it is natural to extend the configuration space to include it as a true variable. This goes naturally as encoding G as a new universal constant [33], since it allows us to transform a mass into a length. We consider now our extended phase space as given by a configuration space $(y^A = y_\mu, y_4 = \frac{G}{c^2}x_4)$, where x_4 has mass dimension, and the momentum space given by P_A.

A DSR particle will be described by the action

$$\mathcal{S}_{5\mathrm{d}} = \int dy^A P_A - \lambda_1(P_A P^A + M_{\mathrm{P}}{}^2) - \lambda_2(P_4 - \mathcal{M}),$$

where λ_i are Lagrange multipliers implementing the two first class constraints, $\mathcal{M}$ is a constant that will specify the mass m. In order to recover a 4d particle, we need to introduce a gauge fixing C that will allow us to reduce the ten dimensional phase space $\left(y^A, P_A\right)$ to an eight dimensional one $\left(x^\mu, \mathcal{P}_\mu\right)$ together with the constraint $H = P_A P^A + {M_\mathrm{P}}^2$ [11]. The symplectic form on the eight dimensional phase space is not arbitrary anymore but given by the the Dirac bracket

$$\{\phi, \psi\}_D = \{\phi, \psi\} - \{\phi, C\}\frac{1}{\{H, C\}}\{H, \psi\} + \{\phi, H\}\frac{1}{\{H, C\}}\{C, \psi\},$$

where ψ, ϕ, are functions on phase space and H is the constraint, such that $\{C, H\} \neq 0$. The reduced phase space coordinates are determined such that they commute with both the constraint and the gauge fixing. Note also that $\mathcal{P}$ can be interpreted as a coordinates system on the de Sitter space defined by the 5d mass shell condition $dS \sim \{P_A,\ P_A P^A = -\,{M_\mathrm{P}}^2\}$.

As a first example, we can introduce the gauge fixing $C = y^A \pi_A - T$. It is then easy to see that a choice of physical coordinates is just given by the Snyder coordinates

$$\mathcal{P}_\mu \equiv M_\mathrm{P}\frac{P_\mu}{P_4}, \qquad x_\mu = \frac{\hbar}{M_\mathrm{P}} J_{\mu 4} \equiv \frac{\hbar}{M_\mathrm{P}}(y_\mu P_4 - y_4 P_\mu). \tag{26.8}$$

The symplectic form is the Snyder symplectic form

$$\{x_\mu, x_\nu\} = \left(\frac{\hbar}{M_\mathrm{P}}\right)^2 J_{\mu\nu}, \quad \{x_\mu, \mathcal{P}_\nu\} = \hbar\left(\eta_{\mu\nu} - \frac{\mathcal{P}_\mu\mathcal{P}_\nu}{{M_\mathrm{P}}^2}\right). \tag{26.9}$$

The physical mass m can be determined solely from the constants in the action namely, $\mathcal{M},\ M_\mathrm{P}$, which both have dimension mass.

$$\mathcal{P}^2 = m^2 = \ {M_\mathrm{P}}^2\frac{\mathcal{M}^2 - {M_\mathrm{P}}^2}{\mathcal{M}^2}.$$

The rest mass m is bounded by M_P since we need $\mathcal{M}^2 - M_\mathrm{P}^2 \geq 0$.

A different gauge fixing $C = \frac{y_0 - y_4}{P_0 - P_4} - T$, provides the bicrossproduct basis [9]. The physical phase space variables are

$$\begin{aligned} \mathcal{P}_0 &\equiv M_\mathrm{P} \ln\frac{P_4 - P_0}{M_P}, \quad \mathcal{P}_i \equiv M_\mathrm{P}\frac{P_i}{P_0 - P_4}, \\ x_0 &\equiv \frac{\hbar}{M_\mathrm{P}} J_{40}, \quad x_i \equiv \frac{\hbar}{M_\mathrm{P}}(J_{i0} - J_{i4}), \end{aligned} \tag{26.10}$$

on the domain $P_4 - P_0 > 0$. These variables encode the so called κ-Minkowski symplectic structure on the 4d phase space,

$$\{x_0, \mathcal{P}_0\} = 1, \quad \{x_i, \mathcal{P}_j\} = -\delta_{ij}, \tag{26.11}$$

$$\{x_0, x_i\} = +\frac{1}{M_\mathrm{P}} x_i, \quad \{x_0, \mathcal{P}_i\} = -\frac{1}{M_\mathrm{P}}\mathcal{P}_i, \tag{26.12}$$

with all other brackets vanishing. The 4d Hamiltonian constraint $\mathcal{H}_{4d} = P_4 - \mathcal{M}$ can be re-written as:

$$\mathcal{H}_{4d} = \frac{1}{2M_P}\mathcal{H} + M_P - \mathcal{M}, \quad \text{with } \mathcal{H} = (2M_P \sinh \frac{\mathcal{P}_0}{2M_P})^2 - \vec{\mathcal{P}}^2 e^{\frac{\mathcal{P}_0}{M_P}}. \quad (26.13)$$

Then $\mathcal{H}_{4d} = 0$ reduces to the κ-Poincaré mass-shell condition $\mathcal{H} = m^2$ for the rest mass $m^2 = 2M_P(\mathcal{M} - M_P)$, where we restricted $\mathcal{M} \geq M_P$.

Finally this last gaugefixing $C = y_4 - T$ provides the usual 4d relativistic particle. In this case the physical phase space variables are just

$$\mathcal{P}_\mu \equiv P_\mu, \quad x_\mu \equiv y_\mu.$$

The reduced symplectic form is the canonical one

$$\{x_\mu, x_\nu\} = 0 = \{\mathcal{P}_\mu, \mathcal{P}_\nu\} \quad \{x_\mu, \mathcal{P}_\nu\} = \eta_{\mu\nu}. \quad (26.14)$$

Finally the mass is just given by $m^2 = -\, {M_P}^2 + \mathcal{M}^2$, where we restricted $\mathcal{M}^2 \geq {M_P}^2$.

In conclusion, by extending phase space with two extra coordinates related to the mass, the main DSR types as well as the usual relativistic particle, can be seen as different *inequivalent* gaugefixings. This approach should be compared to the passage from galilean physics to relativistic physics: space is unified to time and the Galilean physics arise as a specific gaugefixing $x_0 = t$ (together with the limit $c \to 0$). Following this philosophy it seems therefore that the 5d picture should be the correct underlying picture since now mass is unified to spacetime, and the different Special Relativity types arise as different gauge fixings.

To relate the 5d approach to the reference frame approach one can take two different points of view: either the intrinsic momentum π is given by P (then $\mathcal{P} \equiv p$), in which case, we have in fact a 5d intrinsic momentum space, or the physical momentum p is just given by P, in which case we are really living in a 5d space. At this stage we cannot clearly prefer one case over the other, further work is needed.

26.3.3 Multiparticles states

Following the little toy model in section 26.2, it appears that the notion of multiparticles could be modified in the QG semiclassical limit. In DSR, there is no modification of the tensor product. However, since we are dealing with a non-linear realization of symmetries, one can expect to have a modification of the meaning of two particles considered as one particle. Moreover, since there is also an ambiguity on which momenta is physical $\mathcal{P}$, or P, there is an ambiguity on which addition is the physical one.

We can define different types of addition using the mathematical structures at hand. In the first case, since $\mathcal{P}$ lives on the de Sitter space $dS \sim SO(4,1)/SO(3,1)$, we can use the coset structure to define the addition just as in Special Relativity where one uses the coset structure of the hyperboloid $H \sim SO(3,1)/SO(3)$ to define the speeds addition. This definition has some peculiar drawbacks: the addition is in general non-commutative but also non-associative, properties which are clearly due to the coset structure. For example in the Snyder case, a coset element is given by $e^{i\mathcal{P}^\mu J_{4\mu}} \sim e^{i\mathcal{P}^\mu x_\mu}$ and the addition is constructed from

$$e^{i\mathcal{P}_1^\mu J_{4\mu}} e^{i\mathcal{P}_2^\mu J_{4\mu}} = \Lambda(\mathcal{P}_1, \mathcal{P}_2) e^{i(\mathcal{P}_1 \oplus \mathcal{P}_2)^\mu J_{4\mu}},$$

where Λ is a Lorentz transformation, encoding a Lorentz precession. The addition is clearly non-commutative, non-associative. The bicrossproduct case corresponds to the parameterization of the coset $e^{i\mathcal{P}^0 J_{40}} e^{i\mathcal{P}^i \tilde{J}_{4i}}$, with $\tilde{J}_{4i} = J_{0i} - J_{4i}$, that gives a (non-Abelian) group structure to the coset:

$$e^{i\mathcal{P}_1^0 J_{40}} e^{i\mathcal{P}_1^i \tilde{J}_{4i}} e^{i\mathcal{P}_2^0 J_{40}} e^{i\mathcal{P}_2^i \tilde{J}_{4i}} = e^{i(\mathcal{P}_1^0 \oplus \mathcal{P}_2^0) J_{40}} e^{i(\mathcal{P}_1^i \oplus \mathcal{P}_2^i) \tilde{J}_{4i}}.$$

The addition is then non-commutative but associative, a natural feature since this construction arises using quantum groups.

This construction has, however, a further physical draw back: $\mathcal{P}$ lives on dS and is bounded by the Planck mass (either the rest mass in the Snyder case or the 3d momentum in the bicrossproduct case). The sum of momenta being defined on the de Sitter space is then still bounded by the Planck mass: there can be no object with rest mass or 3d momentum bigger than the Planck mass. This is of course a contradiction with everyday experience, therefore this addition seems to be ill defined. This problem has been called the soccer ball problem by Amelino-Camelia. A possible way out is to consider interacting particles or fields as suggested by Freidel [35].

Another way out is to argue that the physical momentum to add is the 5d momentum P [34]. It is easy to add since it is a linear momentum, carrying the linear representation of the 5d Poincaré group $ISO(4,1)$. In this case the sum is trivially

$$P_{\text{tot}} = P_1 + P_2,$$

and the new representation of $ISO(4,1)$ is given by a new parameter κ, which can be for example $\kappa = 2M_{\text{P}}$. In this way we have a rescaling of the radius of the de Sitter space and therefore of the maximum mass as we would have expected. In this way we escape the soccer ball problem. This argument can be also extended to the case where P actually represents the intrinsic momentum, so that $\mathcal{P}$ is the actual physical momentum. Indeed the P addition induces a non-linear addition on $\mathcal{P}$, commutative and associative, free of the soccer ball problem. For this we use the inspiration following the non-linear realization: we transform the $\mathcal{P}_i$ back to the linear momenta P_i, add them and then transform them back, taking into account the change of representation or de Sitter radius:

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = U_\kappa \left(U_{M\mathrm{P}}^{-1}(\mathcal{P}_1) + U_{M\mathrm{P}}^{-1}(\mathcal{P}_2) \right),$$

where, to emphasize that $\mathcal{P}$ lives on the de Sitter space of radius ρ, I used the notation $P = U_\rho^{-1}(\mathcal{P})$.

26.4 Conclusion

Since, in a few years, we expect some data on possible QG effects, it is urgent to understand the semiclassical limit of QG. In particular, one needs to understand the QG physics that is supposed to mix both quantum mechanical and gravitational effects. One key feature for understanding physics in this context is the notion of reference frame. As I recalled using a toy model, the notion of a Quantum Reference Frame leads to interesting physics: the notion of quantum coordinates, possibly a non-linear realization of the symmetries and a modification of the multi-particles states. These features are expected to appear also in the QG semiclassical limit. DSR naturally incorporates these features as a modified measurement procedure and can be seen as the effective description of a flat semiclassical spacetime. From the geometric point of view, DSR could be seen as a generalization of the Riemannian geometry, where the metric is not given in terms of a scalar product anymore. It involves therefore in a non-trivial way the full tangent bundle structure: the notion of symmetry, curvature and so on have to be understood once again. There is contact now with a large mathematical theory that is left to explore from the physics perspective, promising new exciting developments.

As I argued as well, there are many different types of DSR due mainly to the freedom in reconstructing spacetime. Most of them can be unified under a common framework. With this respect, DSR could be compared to Maxwell's electromagnetism theory when Lorentz introduced his symmetries at the end of the nineteenth century. All the theoretical ingredients were there but it was not until Einstein came up with some new physical principles (axiomatic) and some operational guidance that the theory was fully understood. At this time DSR still lacks these fundamental principles to be definitely understood. This is clear, for example, when we see that we have no clue to decide which momentum is physical and how it should add. At this stage, according to me, an axiomatic derivation of DSR is necessary before going to any quantum field theory: the modified notion of reference frame should definitely matter and provide guidance to these new physical principles.

To conclude, the quest to understand the notion of semiclassical spacetime allows us to relate to deep mathematical theories like Finsler geometry or quantum groups, but also pushes for fundamental thinking about spacetime right at the intersection between Special Relativity, General Relativity and Quantum Mechanics, the icing on the cake being the forthcoming experiments hopefully falsifying or confirming the different approaches: a lot of excitement is coming up!

Acknowledgements

I want to thank the Perimeter Institute for hospitality, where part of this work was achieved. The content of this contribution has been shaped by discussions/work with E. Livine, D. Oriti, and D. Poulin. I want also to thank L. Freidel, C. Rovelli and L. Smolin for inspiring discussions.

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27

Lorentz invariance violation and its role in Quantum Gravity phenomenology

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27.1 Introduction

Although there is enormous uncertainty about the nature of Quantum Gravity (QG), one thing is quite certain: the commonly used ideas of space and time should break down at or before the Planck length is reached. For example, elementary scattering processes with a Planck-sized center-of-mass energy create large enough quantum fluctuations in the gravitational field that space-time can no longer be treated as a classical continuum. It is then natural to question the exactness of the Lorentz invariance (LI) that is pervasive in all more macroscopic theories. Exact LI requires that an object can be arbitrarily boosted. Since the corresponding Lorentz contractions involve arbitrarily small distances, there is an obvious tension with the expected breakdown of classical space-time at the Planck length. Indeed, quite general arguments are made that lead to violations of LI within the two most popular approaches towards QG: string theory [27; 28] and loop quantum gravity [31; 2; 3]

This has given added impetus to the established line of research dedicated to the investigation of ways in which fundamental symmetries, like LI or CPT, could be broken [41; 42; 43; 44; 45]. It was realized that extremely precise tests could be made with a sensitivity appropriate to certain order of magnitude estimates of violations of LI [7].

The sensitivity of the tests arises because there is a universal maximum speed when LI holds, and even small modifications to the standard dispersion relation relating energy and 3-momentum give highly magnified observable effects on the propagation of ultra-relativistic particles. One possible modification is

$$E^2 = P^2 + m^2 + \frac{\xi}{M_{\mathrm{Pl}}} E^3. \tag{27.1}$$

Here E and p are a particle's energy and momentum in some preferred frame, m is its mass, while ξ is a dimensionless parameter arising from the details of the

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

QG effects on the particular particle type. Note that ξ could depend on the particle species and its polarization. The dispersion relation can be written in a covariant fashion:

$$P^\mu P_\mu = m^2 + \frac{\xi}{M_{\rm Pl}}(P^\mu W_\mu)^3, \tag{27.2}$$

where P^μ is the particle's 4-momentum, and W^μ is the 4-velocity of the preferred frame. Amelino-Camelia *et al.* [7] noted that photons ($m = 0$) with different energies would then travel with different velocities. For a gamma ray burst originating at a distance D from us, the difference in time of arrival of different energy components would be $\Delta t = \xi D \Delta E / M_{\rm Pl}$. If the parameter ξ were of order 1 and $D \sim 100\,\text{Mpc}$, then for $\Delta E \sim 100\,\text{MeV}$, we would have $\Delta t \sim 10^{-2}\,\text{s}$, making it close to measurable in gamma ray bursts.

A second possible modification is that the parameter normally called the speed of light, c, is different for different kinds of particle. This is implemented by a non-universal particle-dependent coefficient of P^2 in Eq. (27.1). The differences in the maximum speeds of propagation also gives sensitive tests: vacuum Cerenkov radiation etc. [19].

There are in fact two lines of inquiry associated with modified dispersion relations. One is the initial approach, where the equivalence of all reference frames fails, essentially with the existence of a preferred frame. A second popular approach preserves the postulate of the equivalence of all frames, but tries to find modifications of the standard Lorentz or Poincaré symmetries. The most popular version, with the name of doubly special relativity (DSR), replaces the standard Poincaré algebra by a non-linear structure [6; 52; 48; 51]. Another line of argument examines a deformed algebra formed by combining the Poincaré algebra with coordinate operators one [71; 17; 16]. Related to these are field theories on non-commutative space-time [15; 9; 24; 69]; they give a particular kind of LIV at short distances that fits into the general field theoretic framework we will discuss.

In this chapter we will concentrate on the first issue, actual violations of LI. Regarding DSR and its relatives, we refer the reader to the other contributions in this volume and to critiques by Schützhold & Unruh [62; 63], by Rembieliński & Smoliński [59], and by Sudarsky [67]. A problem that concerns us is that the proposed symmetry algebras all contain as a subalgebra the standard Poincaré algebra, and thus they contain operators for 4-momentum that obey the standard properties. The DSR approach uses a modified 4-momentum that has non-linear functions of what we regard as the standard momentum operators. This of course raises the issue of which are the operators directly related to observations. In the discussion section 27.9, we will summarize a proposal by Liberati, Sonego and Visser [49] who propose that it is the measurement process that picks out the modified 4-momentum operators as the measurable quantities.

We will also touch on an aspect with important connections to the general field of QG: the problem of a physical regularization and construction of Quantum Field Theories (QFT).

27.2 Phenomenological models

Methodical phenomenological explorations can best be quantified relative to a definite theoretical context. In our case, of Lorentz invariance violation (LIV) at accessible energies, the context should minimally incorporate known microscopic physics, including Quantum Mechanics and Special Relativity (in order to consider small deviations therefrom). This leads to the use of a conventional interacting Quantum Field Theory but with the inclusion of Lorentz violating terms in the Lagrangian.

One proposal is the Standard Model Extension (SME) of Colladay & Kostelecký [20] and Coleman & Glashow [19]. This incorporates within the Standard Model of particle physics all the possible renormalizable Lorentz violating terms, while preserving $SU(3) \times SU(2) \times U(1)$ gauge symmetry and the standard field content. For example, the terms in the free part of the Lagrangian density for a free fermion field ψ are:

$$\begin{aligned} \mathcal{L}_{\text{free}} = i\bar{\psi}(\gamma_\mu + c_{\mu\nu}\gamma^\nu + d_{\mu\nu}\gamma_5\gamma^\nu + e_\mu + i f_\mu\gamma_5 + \tfrac{1}{2}g_{\mu\nu\rho}\sigma^{\nu\rho})\partial^\mu\psi \\ - \bar{\psi}(m + a_\nu\gamma^\nu + b_\nu\gamma_5\gamma^\nu + \tfrac{1}{2}H_{\nu\rho}\sigma^{\nu\rho})\psi. \end{aligned} \tag{27.3}$$

Here the quantities a_μ, b_μ, $c_{\mu\nu}$, $d_{\mu\nu}$, e_μ, f_μ, $g_{\mu\nu\rho}$ and $H_{\mu\nu}$ are numerical quantities covariantly characterizing LIV, and can be thought of as arising from the VEV of otherwise dynamical gravitational fields. The interacting theory is then obtained in the same way as usual, with $SU(3) \times SU(2) \times U(1)$ gauge fields and a Higgs field. The expected renormalizability was shown by Kostelecký and Mewes [46] and Kostelecký *et al.* [47].

A second approach, as used by Myers and Pospelov [54] is to take the LIV terms as higher dimension non-renormalizable operators. This is a natural proposal if one supposes that LIV is produced at the Planck scale with power suppressed effects at low energy; it gives modified dispersion relations at tree approximation. For example, there are dimension-5 terms with $1/M_{\text{Pl}}$ suppression in the free part of the Lagrangian, such as

$$\frac{1}{M_{\text{Pl}}} W^\mu W^\nu W^\rho \bar{\psi}(\xi_f + \xi_{5f}\gamma_5)\gamma_\mu\partial_\nu\partial_\rho\psi, \tag{27.4}$$

where W^μ specifies a preferred frame. Similar terms can be written for scalar fields and gauge fields. Dimensionless parameters ξ in these terms specify the degree of LIV in each sector.

Each of the proposed Lagrangians can be regarded as defining an effective low-energy theory. Such a theory systematically provides an approximation, valid at low energies, to a more exact microscopic theory.

In Sections 27.4 and 27.5, we will analyze the applicability of LIV effective theories. But first, we will make some simple model calculations, to illustrate generic features of the relation between microscopic LIV and low-energy properties of a QFT.

27.3 Model calculation

The central issue is associated with the UV divergences of conventional QFT. Even if the actual divergences are removed because of the short-distance properties of a true microscopic theory, we know that QFT gives a good approximation to the true physics up to energies of at least a few hundred GeV. So at best the UV divergences are replaced by large finite values which still leave observable low-energy physics potentially highly sensitive to short-distance phenomena.

Of course, UV divergences are normally removed by renormalization, i.e. by adjustment of the parameters of the Lagrangian. The observable effects of short-distance physics now appear indirectly, not only in the values of the renormalized parameters, but also in the presence in the Lagrangian of all terms necessary for renormalizability.

The interesting and generic consequences in the presence of Lorentz violation we now illustrate in a simple Yukawa theory of a scalar field and a Dirac field. Before UV regularization the theory is defined by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - M_0)\psi + g_0\phi\bar{\psi}\psi. \tag{27.5}$$

We make the theory finite by introducing a cutoff on spatial momenta (in a preferred frame defined by a 4-velocity W^μ). We use a conventional real-time formalism, so that the cutoff theory is within the framework of regular quantum theory in 3 space dimensions. The cutoff is implemented as a modification of the free propagators:

$$\frac{i}{\gamma^\mu p_\mu - m_0 + i\epsilon} \to \frac{if(|\mathbf{p}|/\Lambda)}{\gamma^\mu p_\mu - m_0 + \Delta(|\mathbf{p}|/\lambda) + \mathbf{i}\epsilon}, \tag{27.6}$$

$$\frac{i}{p^2 - M_0^2 + i\epsilon} \to \frac{i\tilde{f}(|\mathbf{p}|/\Lambda)}{p^2 - M_0^2 + \tilde{\Delta}(|\mathbf{p}|/\lambda) + \mathbf{i}\epsilon}. \tag{27.7}$$

Here, the functions $f(|\mathbf{p}|/\Lambda)$ and $\tilde{f}(|\mathbf{p}|/\Lambda)$ go to 1 as $|\mathbf{p}|/\Lambda \to \mathbf{0}$, to reproduce normal low-energy behavior, and they go to zero as $|\mathbf{p}|/\Lambda \to \infty$, to provide UV finiteness. The functions Δ and $\tilde{\Delta}$ are inspired by concrete proposals for modified dispersion relations, and they should go to zero when $|\mathbf{p}|/\Lambda \to \mathbf{0}$. But in our calculations we will set Δ and $\tilde{\Delta}$ to exactly zero. We will assume Λ to be of the order of the Planck scale.

Corrections to the propagation of the scalar field are governed by its self-energy[1] $\Pi(p)$, which we evaluate to one-loop order. We investigate the value when p^μ and the physical mass m are much less than the cutoff Λ. Without the cutoff, the graph is quadratically divergent, so that differentiating three times with respect to p gives a convergent integral (i.e. one for which the limit $\Lambda \to \infty$ exists). Therefore we write

$$\Pi(p) = A + p^2 B + p^\mu p^\nu W_\mu W_\nu \tilde{\xi} + \Pi^{(\mathrm{LI})}(p^2) + \mathcal{O}(p^4/\Lambda^2), \tag{27.8}$$

in a covariant formalism with $p^2 = p^\mu p^\nu \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the space-time metric. The would-be divergences at $\Lambda = \infty$ are contained in the first three terms, quadratic in p, so that we can take the limit $\Lambda \to \infty$ in the fourth term $\Pi^{(\mathrm{LI})}(p^2)$, which is therefore Lorentz invariant. The fifth term is Lorentz violating but power-suppressed. The coefficients A and B correspond to the usual Lorentz-invariant mass and wave function renormalization, and the only unsuppressed Lorentz violation is in the third term. Its coefficient $\tilde{\xi}$ is finite and independent of Λ, and explicit calculation [22] gives:

$$\tilde{\xi} = \frac{g^2}{6\pi^2}\left[1 + 2\int_0^\infty dx\, x f'(x)^2\right]. \tag{27.9}$$

Although the exact value depends on the details of the function f, it is bounded below by $g^2/6\pi^2$. Lorentz violation is therefore of the order of the square of the coupling, rather than power-suppressed. The LIV term in (27.8) behaves like a renormalization of the metric tensor and hence of the particle's limiting velocity. The renormalization depends on the field and the size of the coupling, so that we expect different fields in the Standard Model to have limiting velocities differing by $\sim 10^{-2}$. The rough expected size depends only on UV power counting and Standard Model couplings.

The expected size is in extreme contrast to the measured limits. To avoid this, either Lorentz-violation parameters in the microscopic theory are extremely fine-tuned, or there is a mechanism that automatically removes low-energy LIV even though it is present microscopically. More exact calculations would use

[1] In perturbation theory, the sum over one-particle-irreducible two-point graphs.

renormalization group methods. But we know from the running of Standard-Model couplings, that this can produce changes of one order of magnitude, not twenty.

We could also perform the same calculation in conventional renormalization theory. We would use a Lorentz-invariant UV regulator followed by renormalization and removal of the regulator. The results would be of the same form, except that that coefficients A and B would change in value and ξ would be zero. If we regard our theory with the spatial-momentum cutoff as an analog of a true Lorentz-violating microscopic theory, we deduce that it agrees with conventional Yukawa theory with suitable values of its parameters provided only that an explicitly Lorentz violating term proportional to $(W \cdot \partial\phi)^2$ is added to its Lagrangian.

27.4 Effective long-distance theories

Normally, the details of physical phenomena on very small distance scales do not directly manifest themselves in physics on much larger scales. For example, a meteorologist treats the atmosphere as a continuous fluid on scales of meters to many kilometers, without needing to know that the atmosphere is not a continuum but is made up of molecules.

In a classical field theory or the tree approximation of a QFT, the transition from a discrete approximation to a continuum is a simple matter of replacing discrete derivatives by true derivatives, without change of parameters. But in QFT, the situation is much less trivial, and is formalized in the concept of a "long-distance effective theory". This provides an approximation to a more exact microscopic theory, and the errors are a power of l/D, where l is the intrinsic distance scale associated with the microscopic theory, while D is the much larger distance scale of the macroscopic phenomena under consideration.

The effective field theory approach has become particularly important because of the repeated discovery of particles corresponding to fields with ever higher mass. To the extent that gravity is ignored so that we can stay within the framework of QFT, the relation between effective theories appropriate for different scales has become extremely well understood (e.g. [60]). The basic theorems build from the decoupling theorem of Appelquist and Carazzone [8] (see also [74]).

Both the ideas of an effective field theory and the complications when the microscopic theory is Lorentz violating were illustrated by our calculation in the previous section. For phenomena at low energies relative to some large intrinsic scale Λ of a complete theory, we have agreement, up to power-suppressed corrections, of the following.

(i) Calculations in the exact microscopic theory. This theory, as concerns Quantum Gravity, is not yet known.
(ii) Calculations in a renormalized low-energy continuum field theory whose Lagrangian contains only renormalizable terms, i.e., of dimension four or less, possibly supplemented by power-suppressed higher-dimension non-renormalizable terms.

A basic intuition is obtained by the use of Wilsonian methods, where the most microscopic degrees of freedom are integrated out. At the one-loop level, these give unsuppressed contributions to low-energy phenomena of a form equivalent to vertices in a renormalizable Lagrangian, as with the first three terms in Eq. (27.8). This and its generalizations to all orders of QFT show that a renormalized effective QFT gives the dominant low-energy effects of the microscopic theory. A renormalizable low-energy effective theory is self-contained and self-consistent: it contains no direct hints that it is an approximation to a better theory. In constructing candidate approximate theories of physics, we now treat renormalizability not as an independent postulate but as a theorem.

In our model calculations, the theory with a cutoff stands in for the true microscopic theory. Our calculations and their generalizations show that the low energy effective theory is an ordinary renormalizable QFT but with a LIV Lagrangian, just like the Standard Model Extension.

Higher-power corrections in p/Λ can be allowed for by including higher-dimension non-renormalizable terms in the Lagrangian of the effective theory, as in Eq. (27.4). Loop corrections derived from the non-renormalizable terms involve a series of counterterm operators in the Lagrangian with ever higher dimension. But these also correspond to a suppression by more inverse powers of Λ, so it is consistent to truncate the series. The natural sizes of the coefficients in the Lagrangian are set in the Wilsonian fashion by integrals in the effective theory with cutoffs of order the intrinsic scale Λ of the full theory.

However, the phenomenological use of non-renormalizable terms does imply a definite upper limit on the energies where it is appropriate to use them. A classic case is the four-fermion form of weak interactions, where the limit is a few hundred GeV. The form of the interaction gave enough hints to enable construction of the full Standard Model. The four-fermion interaction (with some additions) now arises as the low-energy limit of processes with exchange of W and Z bosons.

An issue very important to the treatment of LIV and Quantum Gravity is that, normally, the terms in the Lagrangian a low-energy effective theory must be all those consistent with the unbroken symmetries of the microscopic theory. If some of the terms are observed to be absent, that gives strong implications about the microscopic theory. A good example is given by QCD. At short distances, weak interactions lead to violations of electromagnetic strength of symmetries such as parity. But at energies of a few GeV, it is measured that these symmetries are much more exact; that is why the weak interactions are called weak. As Weinberg [72] showed, a generic unified theory would not give this weak parity violation. He then observed that if the strong-interaction group commutes with the weak-interaction group, then the unobserved symmetry violation can be removed by a redefinition of the fields. This leads essentially uniquely to QCD as the strong-interaction part of the Standard Model.

In one respect, the situation with gravity is different from the usual kinds of effective field theory. Low energy gravitational physics is described by a non-renormalizable Lagrangian but is not power suppressed. The reasons are that the graviton has zero mass and that macroscopic *classical* gravitational fields occur, with coherent addition of the sources. The standard power-law suppression of gravitation occurs for quantum interactions of small numbers of elementary particles. Unsuppressed gravitational phenomena involve macroscopic classical fields, which need not be treated by quantum theoretic methods.

Modulo this qualification, we get the standard result that the total (leading-power) effect of the microscopic (Plank-scale) physics on GeV-scale physics is in determining the values of the renormalized parameters of the theory, and in changing them from the values obtained from the naive classically motivated considerations. This accounts for the folklore that macroscopic manifestations of Planck-scale physics are to be found only in power-suppressed phenomena.

However, for our purposes, the folklore is wrong because it ignores the price of the low-energy effective theory: that its Lagrangian must contain *all* renormalizable terms consistent with the symmetries of the *micro*scopic theory. If Lorentz symmetry is violated by Planck-scale physics, then we are inexorably led not to the Lorentz-invariant Standard Model, but to its Lorentz-violating extension. Observe that because logarithmic divergences are momentum-independent they are not associated with Lorentz violation. It is the self-energy (and related graphs) with higher divergences that are associated with Lorentz violation. Note that the true microscopic theory might well be UV finite. The UV divergences concern the ordinary continuum limit for the low-energy effective theory; their existence is a diagnostic for the presence of unsuppressed contributions at low energy.

27.5 Difficulties with the phenomenological models

The expected sizes of the Lorentz-violating parameters in the models summarized in Section 27.2 raise some serious difficulties, which we now discuss. We assume that on appropriate distance scales, presumably comparable to the Planck length, there is considerable Lorentz violation. This is the kind associated with space-time granularity, and leads in classical theory or tree approximation to modified dispersion relations like (27.1).

In the case of the SME, which contains only renormalizable terms, the natural size of the LIV parameters is then that of a one-loop Standard-Model correction. Although this appears to have been recognized by Kostelecký and Potting [45], the point is quite obscured in that paper. The conflict with data means either that there is also very small Lorentz violation at the Planck scale or that Quantum Gravity contains a mechanism for automatically restoring macroscopic Lorentz invariance. In either case, it is unjustified to adhere to the naive expectation that Lorentz violation is expected to be suppressed by a power of energy divided by M_{Pl}, as in (27.1).

The scheme of Myers and Pospelov [54] at first appears more natural. The renormalizable part of their effective low-energy Lagrangian is the usual Lorentz-invariant one, to which is added a 5-dimensional operator suppressed by $1/M_{\mathrm{Pl}}$ coefficient.

But as noted by these authors, consistent use of the effective theory requires that radiative corrections are needed; insertion of a dimension-5 operator in a self-energy generically leads to large Lorentz violation from the same power counting as in our model calculation. In general it even gives dimension-3 operators enhanced with a factor of M_{Pl}. They found that they could avoid these problems by postulating a certain antisymmetry structure for the tensor coefficient in the dimension-5 operator.

This is still not sufficient. Consistent use of the theory also requires iteration of the physical effects that give the dimension-5 operators, and hence, within the effective theory, *multiple* insertions of these operators. As shown by Perez and Sudarsky [57], this leads back to the LIV dimension-4 operators that one was trying to avoid.

The overall result is simply a set of particular cases of the general rule that the terms in the renormalizable part of the Lagrangian are all those not prohibited by symmetries of the microscopic theory. Lorentz symmetry is, by the initial hypothesis of all this work, not among the symmetries. Starting with Lorentz-violating modifications of dispersion relations that by themselves are only large at Planck-scale energies, bringing in virtual loop corrections in QFT generates integrals over all momenta up to the Planck scale, complete with the hypothesized Lorentz violation. This is a direct consequence of known properties of relativistic QFT, of which the Standard Model is only one example, and must be obeyed by any theory of Quantum Gravity that reproduces known Standard Model physics in the Standard Model's domain of validity. Extreme fine tuning of the parameters of the microscopic theory could be used to evade the conclusion, but this is generally considered highly inappropriate for a fundamental microscopic theory of physics.

Thus a very important requirement of a theory of QG is that it should ensure the absence of the macroscopic manifestation of effects of any presumed Lorentz-violating microscopic structure of space-time. This feature should be robust, without requiring any fine tuning. Note that such overriding general considerations have played a critical role in the discovery of key physical theories in the twentieth century, from relativity to QCD. As to experimental data, it can be seen in retrospect that only a relatively very small set of experimental data was essential in determining the course of these developments.

27.6 Direct searches

We now give a short account of some of the methods that have yielded the most important bounds on Lorentz violation. These experimental results are important independently of our critiques of their theoretical motivations. For a very complete summary of the situation we refer the reader to the recent review by Mattingly [53].

In the introduction, we have already mentioned the idea of Amelino-Camelia *et al.* [7] to search for energy-dependent differences in the times of arrival of gamma rays from gamma bursts. Actual bounds ($\xi < 10^2$) have been obtained this way recently [29].

Another interesting source of information relies on the expected parity-violating nature of some of the natural proposals for LIV effects in the propagation of photons [31; 54]. This would lead to differences in the propagation velocity for photons with different helicities. It was observed that the effects would lead to a depolarization of linearly polarized radiation as it propagates towards the Earth. Therefore the observation of linearly polarized radiation from distant sources could be used to set important bounds on such effects. For instance, [34] found a bound of the order 10^{-4} for the parameter ξ for the photon.

Another type of bound can be obtained by noting that is quite unlikely that the Earth would be at rest in the preferred rest frame associated with the sought-for LIV. Thus in an Earth-bound laboratory Lorentz-violation could appear as violation of the isotropy of the laws of physics. Using the prescription for the expected effects on fermions which arise in the loop quantum gravity scenarios [2; 3], one arrives at an effective SME description. Measurements rely on the extreme sensitivity of the Hughes driver type of test of the isotropy of physics using nuclear magnetic clocks [18; 11]. The bounds obtained this way are of the order 10^{-5} and 10^{-9} on parameters that were originally expected to be of order unity. Then one obtains very stringent bounds on the parameters characterizing the state of the quantum geometry [65]. Similar constraints can be placed on the effects that arise in the string theory scenarios [44; 66].

A further source of severe constraints uses the possibility that different particle species have different values of their limiting velocity, as in the SME. Tests are made by examining the resulting changes in thresholds and decay properties of common particles. Coleman and Glashow [19] obtained a dimensionless bound of 10^{-23} on this kind of Lorentz violation. Other related arguments connected to the existence of a bound to the propagation velocity of particles for modified dispersion relations have been used by Jacobson *et al.* [36; 37; 38]. These authors noted that the 100 MeV synchrotron radiation from the Crab nebula requires extremely high energy electrons. They combined the upper bound on the frequency of synchrotron radiation for electrons with a given velocity in a given magnetic field with the fact that there would be an upper bound for any electron's velocity if ξ for the electron had a particular sign. In fact the analysis, carried out within the Myers and Pospelov framework, indicates that at least for one of the electron's helicities a corresponding ξ parameter, if it had a particular sign, could not have a magnitude larger than about 10^{-7}.

Finally there is the reported detection of cosmic rays with energies beyond the GZK cutoff. We recall that these ultrahigh energy cosmic rays are thought to be protons whose interaction with the photons of the cosmic microwave background would prevent them from traveling more than about 50 Mpc, while the likely sources are located much further away. This anomaly is often presented as candidate observational evidence for LIV [13; 26; 70; 1; 12; 10]. Our own feeling is that the list of unexplored alternative explanations of this anomaly, even if one needs to go beyond established physics, is much too broad at this time, and thus its interpretation as a signature of a LIV – given the difficulties we discussed here – is at best premature. Fortunately the Auger Experiment will become fully operational soon and its results should help clarify the situation.

27.7 Evading the naturalness argument within QFT

Several proposals have been made to evade the naturalness problem for Lorentz violation.

One argument relies essentially on the possibility that a fiducial symmetry would protect Lorentz symmetry. Jain and Ralston [40] and Nibbelink and Pospelov [55] argue that supersymmetry could be such a symmetry. At the one-loop level this indeed works: contributions to self-energy graphs with particles and their superpartners have the same couplings but opposite signs. This cancellation is very reminiscent of the one for the cosmological constant in the same theories. However, the authors note that, as the Lorentz algebra is a subalgebra of the supersymmetry algebra, invoking the latter to protect the former is not entirely consistent [35]. They then observe that they would actually need only the translation subalgebra of the Poincaré algebra to be unbroken. However, it is hard to envision a situation in which a granular space-time would have the full translation group as a full continuous symmetry. Moreover as is well known, even if it is there at some level, supersymmetry must be broken at low energies. Then it is difficult to understand how it could protect the low energy phenomena from the LIV we have been discussing, while allowing at the same time for violations to be observable at higher energy scales that are closer to that energy regime where supersymmetry is presumably unbroken. In fact in a recent work Bolokov, Nibbelink and Pospelov [14] noted that the most supersymmetry seems to do is to decrease the severity of the required fine tuning. It seems that in the case of noncommutative field theories not even exact supersymmetry would prevent large violations of Lorentz invariance.

Liberati *et al.* [50] treat a condensed matter model of two component Bose–Einstein condensate as a model system. LI is associated with monometricity in the propagation of the two types of quasi-particles. In this type of study one says that there is monometricity if the various independently propagating modes do so in the same "effective metric" that results from the condensed matter background. The authors show that LI can, under certain conditions, be violated at high energies while being preserved at low energies. This is achieved by fine tuning a certain parameter in the model (the interaction with an external laser source) to ensure monometricity in the hydrodynamical limit. The fine tuning is in agreement with our general results.

The conclusion the authors reach in those studies is in agreement with well known expectations: that an emergent symmetry could give protection for the Lorentz Invariance. In their case the monometricity appears to be protected by an emergent SO(2) symmetry, in the sense that once imposed at the hydrodynamical level it is only residually broken beyond that limit. For us the issues would be then: What physical mechanism is that which ensures monometricity at the hydrodynamical level? What is its analog in the space-time/particle-physics arena? Finally, what are the hopes that this type of mechanism would succeed in ensuring monometricity for a very different type of propagating modes, such as gauge fields as compared with standard fermion matter fields?

As Liberati, Sonego and Visser [49] discuss in another paper, which we will summarize in the discussion in Section 27.9, it is possible that more fundamental issues come into play, perhaps concerned with measurement in a theory with a dynamical space-time. These issues would of course make even the principles of the derivation of an EFT quite different than in normal QFTs. But they would also remove the rationale for simple estimates for the sizes of higher dimension Lorentz-violating operators in an EFT.

Another proposal was made by Alfaro [4; 5] for a way to generate naturally small Lorentz violations. His general idea is to generate LIV in the integration measure for Feynman graphs. The proposal involves two concrete schemes. One uses a Lorentz violating cutoff that contains a parameter which when set to zero recovers a Lorentz invariant situation; the scheme thus has a parametrizably small LIV. The second scheme involves a Lorentz-violating dimensional regularization scheme, where the standard Minkowski metric $\eta_{\mu\nu}$ is replaced by $g_{\mu\nu} = \eta_{\mu\nu} + \alpha\epsilon W_\mu W_\nu$, where $\epsilon = n-4$ is the small parameter in the dimensional regularization scheme.

In the first scheme the regularization of a one-loop integral is to modify it by multiplying the integrand by

$$R(k) = \frac{-\Lambda^2}{k^2 - \Lambda^2 + ak_0^2 + i\epsilon}, \tag{27.10}$$

where $a = 0$ is the Lorentz-invariant case. This suffers from a routing dependence and is therefore not well-defined, certainly not as a complete theory. Furthermore, in the Lorentz-invariant case $a = 0$, the regulator factor has a pole at $k^2 = \Lambda^2$. This is very similar to Pauli–Villars regularization, which gives negative metric states and therefore the regulated theory cannot be considered a normal quantum theory. This scheme therefore does not address the actual situation we are concerned with in Quantum Gravity.

The second scheme uses dimensional regularization and modifies the metric in a way that depends on the $\epsilon = 0$ pole in the integral being calculated. This graph-dependent modification of the metric does not correspond to any normal definition of a QFT, and no rationale is given.

27.8 Cutoffs in QFT and the physical regularization problem

Our results also have important implications for the definition of QFT. Given the well-known complications of renormalization, it is sensible to try defining a QFT as the limit of an ordinary quantum mechanical theory defined on a lattice of points in real space. One could also make time a discrete variable, but this is unnecessary. Continuum field theory is defined by taking the limit of zero lattice spacing, with appropriate renormalization of the bare parameters of the theory. However, if the cutoff theory is defined on an ordinary spatial lattice, boost invariance is completely broken by the rest frame of the lattice. Therefore all the issues discussed in this paper apply to the construction of the renormalized continuum limit, and fine-tuning is needed to get Lorentz invariance. This is acceptable for a mathematical definition of a QFT, but not in a theory that has a claim on being a fundamental theory.

Normal methods of calculation avoid the problem, but in none of them is the regulated theory a normal quantum mechanical model. For instance, the functional integral, as used in lattice gauge theories, is defined in Euclidean space-time. The regulated theory on a lattice is a purely Euclidean construct. Discrete symmetries under exchange of coordinate axes are enough to restrict counterterms to those that give SO(4) invariance in the continuum limit. Continuum QFT in Minkowski space is obtained by analytic continuation of the time variable, and the compact SO(4) symmetry group of the Euclidean functional integral corresponds to the non-compact Lorentz group in real space-time.

On the other hand, a Pauli–Villars regulator can preserve LI in the regulated theory, but only at the expense of negative metric particles. That is, the regulated theory is not a normal quantum mechanical model.

Finally, dimensional regularization does preserve LI and many other symmetries. In this method, space is treated as having a non-integer dimension. Technically, space is made infinite dimensional, and this allows nonstandard definitions to be made of the integrals used in Feynman graphs so that they behave as if space has an arbitrary complex dimension [21]. However, it is not even known how to formulate Quantum Field Theories non-perturbatively within this framework.

Therefore we pose the problem of whether there exists a physical regularization of QFT in which LI is preserved naturally. A physical regularization means that the regulated theory is a normal quantum theory whose existence can be taken as assured.

One proposal of this kind was made by Evens *et al.* [30] and it uses a nonlocal regularization. However, Jain and Joglekar [39] argue that the scheme violates causality and thus is physically unacceptable.

So one is left with a spatial lattice, or some variant, as the only obvious physical regulator of a QFT.

The need to treat gravity quantum mechanically provides the known limits to the physical applicability of the concepts and methods of QFT. Therefore the observed Lorentz invariance of real phenomena indicates that a proper theory of Quantum Gravity will provide a naturally Lorentz invariant physical regulator of QFT. So perhaps a discovery of a better method of defining a QFT in Minkowski space might lead to important clues for a theory of QG.

27.9 Discussion

It is well-known that a nontrivial space-time structure is expected at the Planck scale, and this could easily lead to Lorentz-violating phenomena. The simplest considerations suggest that the observable Lorentz violation is suppressed by at least one power of particle energy divided by the Planck energy; this small expectation has led to an ingenious set of sensitive measurements, with so far null results.

However, an examination of field theoretic loop corrections shows that the expectation is incorrect, in general. Standard theorems in Quantum Field Theory show that the low-energy effects of Planck-scale phenomena can be summarized in an effective low-energy QFT whose Lagrangian contains all renormalizable terms compatible with the symmetries of the microscopic theory and the appropriate low-energy field content; this is the Standard Model Extension. If there is Lorentz violation in the fundamental theory, then in the effective theory, the Lorentz violating parameters are, as we have shown, of the size of normal one-loop corrections in the Standard Model, in violent contradiction with data. Without some special mechanism, extreme fine tuning is needed.

It is already known [68; 73] that there are fine-tuning problems with the Standard Model, involving at least the cosmological constant, mass hierarchies and the Higgs mass term. These, of course, suggest to many physicists that the Standard Model is not the ultimate microscopic theory, but is a low-energy approximation to some more exact theory where fine-tuning is not needed. Our results show that Lorentz invariance should be added to the list of fine tuning problems that should be solved by a good theory that includes Quantum Gravity, or alternatively by a new theory that supersedes currently known ideas. We thus suggest that a search for a physically meaningful, Minkowskian space-time, Poincaré and gauge-invariant regulator for the Standard Model could be intimately connected with the search for a theory of QG and with its possible phenomenological manifestations. The lack of a physical regularization for QFT besides the lattice makes the non-naturalness of Lorentz invariance a particularly important problem even when gravity is left out of the discussion.

We conclude by mentioning some intriguing ideas.

Some ideas regarding how a discrete nature of space-time can be made consistent with Lorentz invariance are explored by Rovelli and Speziale [61] and by Dowker, Henson and Sorkin [25]. In particular, Dowker *et al*. show that by using a random lattice or causal set methods one can evade the problem that regular spatial lattices prevent a physical realization of Lorentz contraction.

There are also considerations of other possible types of manifestations of QG. For instance there are proposals regarding nonstandard couplings to the Weyl tensor [23], fundamental quantum decoherence [32; 33], and QG induced collapse of the wave function [56; 58].

Finally, there are proposals invoking fundamental modifications of the Lorentz or Poincaré structures. This is the subject of doubly special relativity (DSR) which we discussed briefly in our introduction, Section 27.1, together with some critiques of the physical significance of DSR.

An interesting idea, with more general applicability, is the proposal by Liberati, Sonego and Visser [49] for resolving the problem in DSR that the measurable momentum operators differ from the operators, also present in DSR, that obey the standard commutation relations with the Lorentz generators. They suggest that the modifications of the momentum operators are a non-trivial effect of quantum mechanical measurement when Quantum Gravity effects are important. To our mind, this impinges on an important foundational problem in QFT and QG as compared with elementary Quantum Mechanics, including the issue of the relation between an effective field theory and an underlying theory in which space-time is genuinely dynamical.

In simple quantum mechanical theories of systems like the Schrödinger equation for a single atom, measurement involves an external apparatus. But with an interacting QFT, the theory is sufficiently broad in scope that it describes both the system being measured and the experimental apparatus measuring it. If the Standard Model is valid, it accurately governs all strong, electromagnetic and weak interactions, and therefore it includes particle detectors as well as particle collisions. An interacting QFT has a claim on being a theory of everything (in a certain universe-wide domain) in a way that a few-body Schrödinger equation does not. Measurement theory surely has a different status in QFT. This point is exemplified by the analysis by Sorkin [64]. This should apply even more so when Quantum Gravity is included. A localized measurement of a sufficiently elementary particle of sufficiently super-Planck energy could have a substantial effect on the local space-time metric and thus on the meaning of the energy being measured.

The emergence of the field known as QG phenomenology is certainly a welcome development for a discipline long considered as essentially removed from the empirical realm. However, one should avail oneself of all the other established knowledge in physics, in particular, the extensive development both at the theoretical and experimental level of QFT. Ignoring the lessons it provides, and the range of its successful phenomenology is not a legitimate option, unless one has a good substitute for it. The unity of physics demands that we work to advance in our knowledge by seeking to *expand* the range covered by our theories, therefore we should view with strong skepticism, and even with alarm any attempt to extrapolate in one direction – based essentially on speculation – at the price of having to cede established ground in any other.

Acknowledgments

This work was supported in part by the U.S. Department of Energy under grant number DE-FG02-90ER-40577 and by DGAPA-UNAM IN108103 and CONACyT 43914-F grants (México). We would like to thank J. Banavar, Y. Chen, C. Chryssomalakos, L. Frankfurt, J. Jain, and M. Strikman for useful discussions.

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28

Generic predictions of quantum theories of gravity

L. SMOLIN

28.1 Introduction

How does a proposal for unification go from an interesting body of mathematical results to a plausible explanation of natural phenomena? While evidence of mathematical consistency is ultimately important, what is often decisive is that a proposed unification leads to predictions of phenomena that are both new and generic. By generic I mean that the new phenomena are general consequences of the proposed unification and thus hold for a wide range of parameters as well as for generic initial conditions. The proposal becomes an explanation when some of those new generic phenonena are observed.

Generic consequences of unification often involve processes in which the things unified transform into each other. For example, electromagnetic waves are a generic consequence of unifying electricity and magnetism, weak vector bosons are a generic consequence of unifying the weak and electromagnetic interactions, and light bending is a generic consequence of the equivalence principle which unifies gravity and inertia.

Looking at history, we see that the reasons why proposals for unification succeed or fail often have to do with their generic consequences. In successful cases the consequences do not conflict with previous experiments but are easily confirmed when looked for in new experiments. These are cases in which we come to *celebrate the unification.* In bad cases the consequences generically disagree with experiment. Some of these cases still survive for some time because the theory has parameters that can be tuned to hide the consequences of the unification. But these then succumb to lack of predictability which follows from the same flexibility that allows the generic consequences to be hidden.

It is often the case that heuristic arguments are sufficient to uncover generic consequences of new theories even before precise predictions can be made. It was understood that QED would lead to a Lamb shift before there were

Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. Daniele Oriti. Published by Cambridge University Press.

precise predictions by Feynman and Schwinger. Einstein was able to predict that a theory based on the equivalence principle would lead to light bending before GR had been precisely formulated. Thus, uncovering generic consequences gives both experimentalists and theorists something to focus their attention on.

Moreover, physicists often have not needed to solve a theory exactly, or rigorously prove its consistency, to work out generic consequences and extract precise predictions that could be tested experimentally. This was certainly the case with both GR and QED. It is then incumbent on us to look at generic consequences of different proposed unifications of quantum theory, spacetime and particle physics and try to determine if they are cases in which there is a chance to celebrate, rather than hide, their consequences.

In this contribution I will attempt to do this for a large class of Quantum Gravity theories. These are theories which are *background independent* in that classical fields, such as a background metric, play no role in their formulation. To make the discussion concrete I will be interested in a large class of theories which I call *causal spin network theories* [1]. These include the different versions of loop Quantum Gravity [2; 3; 4] and spin foam models [5]. They include also a large class of theories describable in the general mathematical and conceptual language of LQG that have not, however, been derived from the quantization of any classical theory. These theories have been much studied in the 20 years since Ashtekar wrote down his reformulation of General Relativity as a gauge theory [6]. There remain significant open problems; nevertheless, I hope to convince the reader that we know enough about these theories to argue for several generic consequences.

My intention here is to explain the basic physical reasons for these generic consequences. Consequently, the discussion will be heuristic and I will often sketch arguments that are made fully elsewhere [2; 3; 4; 7; 8].

In the next section I will list the main postulates of causal spin network theories. Following that, I will discuss seven generic consequences.

(i) Discreteness of quantum geometry and ultraviolet finiteness.
(ii) Elimination of spacetime singularities.
(iii) Entropy of black hole and cosmological horizons.
(iv) Positive cosmological constant spacetimes are hot.
(v) Deformed special relativity.
(vi) The emergence of matter from quantum geometry.
(vii) Disordered locality.

The first four are well established. The next is the subject of recent progress and the last two are new.

28.2 Assumptions of background independent theories

Four generic assumptions define a class of background independent Quantum Gravity theories that have been the subject of most study.

- **Quantum Mechanics** We assume the basic postulates of Quantum Mechanics.
- **Partial background independence** The theory is formulated without reference to any fixed spacetime metric or other classical fields. There may however be some fixed structures including dimension, topology and boundary conditions. General Relativity is a partly background independent theory. There is an argument, to be found in [9], that any quantum theory of gravity must be so.[1]
- **Discreteness** The Hilbert space $\mathcal{H}$ has a countable basis given by discrete or combinatorial structures. The dynamics is generated by moves local in the topology of these structures. These define the events of the theory. The dynamics is specified by giving the amplitudes for the possible events.
- **Causality** The histories of the theory have causal structure, in the sense that the events define a partially ordered, or causal set.

There are a number of such theories, which depend on different choices for the combinatorial structure used to model quantum geometry. These include dynamical triangulations [10], causal set models [11], quantum causal history models [12] and consistent discretization models [13]. Important things have been learned from each of them. Here I will discuss the following class of theories, which I call *causal spin network theories* [1].

(i) The Hilbert space has a countable basis indexed by all embeddings, up to topology, of a class of graphs Γ in a fixed topological manifold Σ.

(ii) The graphs may be labeled. If so, the labeling is determined by a choice of a Lie algebra or quantum group $\mathcal{A}$. The edges of Γ are labeled with irreducible representations j of $\mathcal{A}$ and the nodes are labeled with invariants in the product of the incident representations. Labeled graphs are called *spin networks* (see Figure 28.1).
In the nicest examples $\mathcal{A}$ is a compact Lie algebra, or its quantum deformation at a root of unit, so that the labels form a discrete set.

(iii) There are a small number of local moves, for example those in Figures 28.2 and 28.3. The amplitude of a local move is a function of the labels involved. There are three basic kinds of moves. Expansion moves when a node is blown up to a symplex, for example a triangle, contraction moves, which are the reverse and exchange moves whereby two neighboring nodes exchange connections to other nodes.

(iv) A history is made of a sequence of local moves, which take the state from an initial spin network state to a final one. The moves have a partial order structure defined by domains of influence [1].

[1] One may also try to make theories that are more fully background independent, but they will not be discussed here.

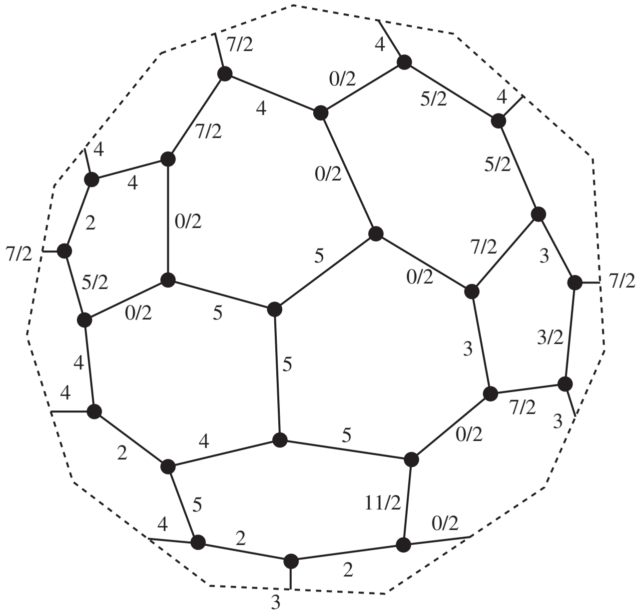

Fig. 28.1. An example of a spin network (from J. Baez).

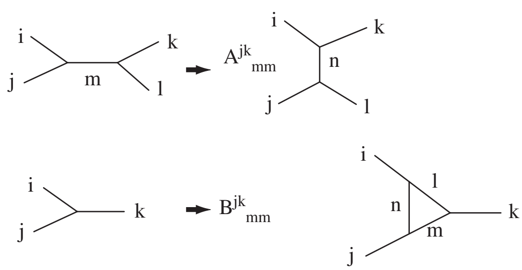

Fig. 28.2. The basic local moves on trivalent graphs

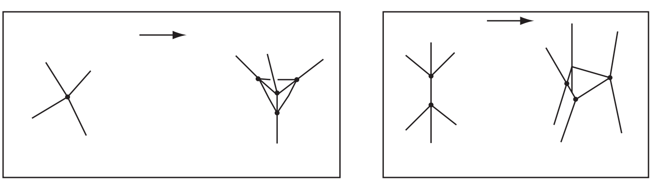

Fig. 28.3. The basic local moves on four-valent graphs

We call the set of graphs, embedded in Σ up to topology, the fundamental configuration space, $\mathcal{S}_{\Sigma,\mathcal{A}}$. In the quantum theory each labeled graph embedding corresponds to an element of an orthonormal basis of the Hilbert space $\mathcal{H}_{\Sigma,\mathcal{A}}$.

Some theories of this kind can be derived from classical theories which are diffeomorphism invariant gauge theories. It is the great discovery of Ashtekar that

General Relativity is a theory of this type [6]. This is true for any dimension and it is also true for any version of supergravity or coupling to any matter.

The classical configuration space, $\mathcal{C}$, is then the space of connections valued in $\mathcal{A}$, on the spatial manifold Σ, modulo gauge transformations. The conjugate electric field turns out to be related to the metric. There is also a diffeomorphism invariant configuration space, $\mathcal{C}^{\text{diffeo}}$ consisting of the orbits of $\mathcal{C}$ under $\mathit{Diff}(\Sigma)$.

The relationship to the previous definitions is based on the following two principles [4].

- **Gauge-graph duality** The Hilbert space $\mathcal{H}_{\Sigma,\mathcal{A}}$ is the quantization of the classical configuration space $\mathcal{C}^{\text{diffeo}}$ just defined.
- **Constrained or perturbed topological field theory** Gravitational theories, including General Relativity, and supergravity can be expressed simply in terms of constraining or perturbing topological field theories.

We discuss each in turn. To realize the graph-gauge duality, we express the theory, not in terms of the connection, A, but in terms of the holonomy,

$$U[\gamma, A] = Pe^{\int_\gamma A}. \tag{28.1}$$

Then, $T[\gamma] = TrU[\gamma, A]$ is called the Wilson loop observable. The conjugate operator is the electric flux through a surface S,

$$E(S, f) = \int_S E_i f_i. \tag{28.2}$$

This depends also on a Lie algebra valued function on S, given by f_i. These satisfy a closed Poisson algebra,

$$\{U[\gamma, A], E(S, f)\} = l_{\text{Pl}}^2 \, \mathit{Int}[\gamma, S] \, U[\gamma_S, A] f, \tag{28.3}$$

where $\mathit{Int}[\gamma, S]$ is the intersection number of the surface and loop and γ_S is the loop beginning and ending at the point it intersects the surface.

Wilson loops can be extended to spin networks in the following way: to each edge of a spin network Γ, write the holonomy in the representation indicated by the label on the edge; then tie these up with the invariants on the nodes to get a gauge invariant functional of Γ and the connection called $T[\Gamma, A]$.

There are several key features of the quantization in terms of these variables.

- The Fock space plays no role at all, as that depends on a background metric.
- Instead, there is a uniqueness theorem [14; 15] that tells us that there is a unique representation of the algebra (28.3) such that (i) the Wilson loop operators create normalizable states and (ii) it carries a unitary representation of the diffeomorphism group of Σ. We call this $\mathcal{H}^{\text{kin}}$ for kinematical Hilbert space.

  Any consequence of this unique representation is then a generic consequence of a large class of Quantum Gravity theories.

In $\mathcal{H}^{\text{kin}}$ there is no operator that represents A_a, all that is represented are the Wilson loops. Similarly there is no representation of infinitesimal diffeomorphisms, only finite ones.

The graph-gauge field duality is represented explicitly by a functional transform,

$$\Psi[\Gamma] = \int d\mu_{\text{AL}}(A) T[\Gamma, A] \Psi[A] \tag{28.4}$$

where $d\mu_{\text{AL}}$ stands for the rigorously defined, *Ashtekar–Lewandowski measure* [7; 8]. There is a basis element for every distinct embedding of a spin network, so $\mathcal{H}^{\text{kin}}$ is not separable. This is remedied by going to the subspace of diffeomorphism invariant states.

- The diffeomorphism invariant Hilbert space, $\mathcal{H}^{\text{diffeo}}$ is constructed by moding out by the action of $Diff(\Sigma)$ in the dual of $\mathcal{H}^{\text{kin}}$. There is a basis element for each (piecewise smooth) diffeomorphism class of graphs, so it is separable. As it is constructed from a unique kinematical space by a unique operation, $\mathcal{H}^{\text{diffeo}}$ is also unique.

Thus, we arrive uniquely at the kinematical structure of an evolving spin network theory, because[2] $\mathcal{H}^{\text{diffeo}} = \mathcal{H}_{\Sigma,\mathcal{A}}$.

All known classical gravity theories such as GR and supergravity in any dimension are diffeomorphism invariant gauge theories. Hence they all provide examples of causal spin network theories.

But it's even better than this, because the dynamics turns out to act simply on the spin network states, through local moves of the kind descrived above. This is a consequence of the second principle, which is that the dynamics of all known classical relativistic gravitational theories are arrived at by perturbing around [16] or constraining topological field theories [17; 18].

In 4 dimensions one route to this is through the Plebanski action [19; 20; 21; 22]. Pick $G = SU(2)$ and consider the action

$$S^{BF} = \int \left(B_i \wedge F^i - \frac{\Lambda}{2} B^i \wedge B_i \right) \tag{28.5}$$

where B_i is a two form valued in the Lie algebra of $SU(2)$ and F^i is the $SU(2)$ field strength. This has no local degrees of freedom as the field equations are

$$F^i = \Lambda B^i; \quad D \wedge B^{IJ} = 0. \tag{28.6}$$

Now consider the following action, which differs from it by just a constraint.

$$S^{BF} = \int \left(B_i \wedge F^i - \frac{\Lambda}{2} B^i \wedge B_i + \Phi_{ij} B^i \wedge B^j \right) \tag{28.7}$$

It is not hard to see that this is an action for General Relativity [47].

[2] Because equivalence of graph embeddings under piecewise smooth embeddings is equivalent to topological equivalence.

Starting with the action in this form, one can write a path integral representation of the dynamics of the spin network embeddings [2; 3; 5; 17; 18]. Details are discussed elsewhere in this volume, the result is to give amplitudes to a set of local moves.

28.3 Well studied generic consequences

Let us begin with some well studied generic consequences of the class of theories we have just described.

28.3.1 Discreteness of quantum geometry and ultraviolet finiteness

It is well understood that such theories are generically ultraviolet finite. The demonstrations of finiteness depend only on the assumptions that lead to the unique $\mathcal{H}^{\text{diffeo}}$ and they are now confirmed by rigorous results [7; 8]. But the reason these theories are discrete and finite can also be understood intuitively. The key point is that Wilson loop operators create normalizable states. This means that they realize precisely an old conjecture about quantum non-Abelian gauge theories which is that the electric flux is quantized so the operators that measure total electric flux through surfaces have discrete spectra. This used to be called the dual superconductor hypothesis. This is relevant for Quantum Gravity because *the uniqueness theorem tells us that the Hilbert space of any quantum theory of gravity describes a dual superconductor: the graphs are then the states of quantized electric flux.*

In the connection to gravitational theories the total electric flux through a surface translates to the area of the surface. Hence the areas of all surfaces are quantized, and there is a smallest non-zero area eigenvalue. This turns out to extend to other geometrical observables including volumes, angles, and lengths.

This discreteness of quantum geometry in turn implies that the theory is ultraviolet finite. The theory has no states in which areas, volumes or lengths smaller than Planck scale are meaningfully defined. There are consequently no modes with wavelength smaller than the Planck length. It has also been shown that for a class of theories the path integral is ultraviolet finite [5].

It can be asked whether the volume or area operators are physical observables, so that their discreteness is a physical prediction. The answer is yes. To show this one may first gauge fix the time coordinate, to give the theory in a version where there is a Hamiltonian evolution operator. Then one can construct the diffeomorphism invariant operator representing the volume of the gauge fixed spatial slices [23]. In cases where one fixes a spatial boundary, the area of the boundary is also a physically meaningful operator. One can also define diffeomorphism invariant area operators by using physical degrees of freedom to pick out the surfaces to be measured [24]. In several cases, such physically defined geometrical observables have been shown to have the same discrete spectra as their kinematical counterparts [24]. Hence, the discreteness is a true generic consequence of these theories.

28.3.2 Elimination of spacetime singularities

There has long been the expectation that spacetime singularities would be eliminated in quantum theories of gravity. In the context of LQG-like models this has been investigated so far in the context of a series of models [25; 26; 27; 28; 29; 30; 31; 32]. These have a reduced number of quantum degrees of freedom corresponding to approximating the spacetime near a spacetime or black hole singularity by its homogeneous degrees of freedom. In the cosmological case the number of degrees of freedom is finite [25; 26; 27; 28; 29; 30], while in the black hole case the theory is a $1 + 1$ dimensional field theory, which is the symmetry of the interior of a Schwarzchild black hole[31; 32].

All results so far confirm the expectation that the spacelike singularities are removed and replaced by bounces [25; 26; 27; 28; 29; 30; 31; 32]. Time continues to the future of where the singularity would have been and the region to the future is expanding.

These models are different from the old fashion quantum cosmological models, based on $\mathcal{L}^2(R^+)$. The key features by which they differ parallel the features mentioned above of the full diffeomorphism invariant quantum field theories. For example, as in the full field theory, there is no operator corresponding to A_a^i, rather the connection degrees of freedom (which are the variables conjugate to the spatial metric) are represented by the exponential (28.1). The elimination of singularities can be directly tied to the features of this new quantization. Thus, there is reason to expect that the same discreteness will apply to the full diffeomorphism invariant quantum field theory. Work aimed at resolving this question is in progress.

28.3.3 Entropy of black hole and cosmological horizons

Generic LQG theories have a universal mechanism for describing states on certain kinds of boundaries, which includes black hole and cosmological horizons. In the presence of a boundary, we have to add a boundary term to get a good variational principle. The details are described in [33] the key point is that through the connection to topological quantum field theory the boundary theories end up described in terms of a $2 + 1$ dimensional topological field theory, which is Chern–Simons theory. This follows from the fact that the deSitter or AdS spacetime represent solutions to the pure topological field theory, this implies that the topological field theory should dominate on the boundary of an asymptotically dS or AdS spacetime [33; 47]. It turns out that the same conditions hold on horizons [34; 35; 36; 37; 38]. In all these cases, the boundary term is of the form

$$S^{\text{boundary}} = \frac{k}{4\pi} \int Y^{\text{CS}}(A), \tag{28.8}$$

where A is the pull back of a connection one form to the boundary. The connection with Chern–Simons theory is a direct consequence of the relationship of General Relativity to topological field theory and hence is generic.

Chern–Simons theory is used to describe anyons in $2+1$ dimensional condensed matter physics. The states are labeled by punctures on the two dimensional sphere which is the spatial cross-section of the horizon. The punctures are points where the graphs attach to the boundary, and serve also as quanta of area on the boundary. As a result of the boundary conditions that identify the surface as a horizon, the connection is constrained to be flat everywhere except at the punctures. The physics on a horizon is then identical to that of a system of anyons, with the area being proportional to the total charge carried by the anyons.

Physicists know how to count the states of such $2+1$ dimensional theories. Not surprisingly, the entropy ends up proportional to the area. Getting the constant of proportionality right requires fixing a constant, the Immirzi constant.[3] Once that is done all results, for all black hole and cosmological horizons, agree with Hawking's prediction, to leading order [39; 40] (see also Figure 28.4). Past leading order there are corrections to the black hole entropy and thermal spectrum which are

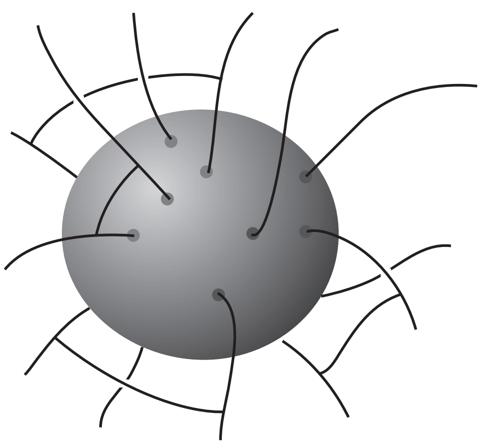

Fig. 28.4. A black hole in LQG.

[3] A heuristic argument that fixes the value of the constant in terms of a correspondence with the quasi normal mode spectrum was given by Dreyer [39]. When the states of the horizon are correctly counted, one gets the same value [40; 41].

Quantum Gravity effects [42; 43]. These corrections introduce a fine structure into the Hawking radiation, which is discussed in [44].

28.3.4 Heat and the cosmological constant

There turns out to be a natural role for the cosmological constant, which is that it parameterizes a quantum deformation of the algebra $\mathcal{A}$. For the case of $3+1$ dimensions, this leads to $\mathcal{A}$ being $SL_q(2)$ with $q = e^{\frac{2\pi\imath}{k+2}}$ where the level k is given by [33; 45; 46; 47]

$$k = \frac{6\pi}{G\Lambda}. \tag{28.9}$$

The quantum deformation of the symmetry algebra has a simple physical meaning, at least for $\Lambda > 0$. The ground state should be de Sitter spacetime, which has a horizon with an area

$$A = \frac{12\pi}{\Lambda}. \tag{28.10}$$

By the Bekenstein bound there should be a finite number of degrees of freedom observable on the horizon, given by

$$N = \frac{A}{4G\hbar} = \frac{3\pi}{G\hbar\Lambda}. \tag{28.11}$$

This relationship has been called the N-bound and has been conjectured by Banks and Fishler to be fundamental [48]. If an observer rotates they see the horizon rotate around them, hence these degrees of freedom should fall into a single irreducible representation. But if the Bekenstein bound is a real limit, there should not be any irreducible representation with more than N states in it. This is precisely true if the rotational symmetry is quantum deformed by (28.9). Thus the N-bound is a consequence of the quantum deformation of the symmetry induced by the cosmological constant [47].

A consequence of the quantum deformation of the label set is that the graphs are framed, so edges are represented by ribbons or tubes [33; 45; 46; 49; 50].

A classic result of quantum field theory in curved spacetime is that QFTs on the background of de Sitter spacetime are thermal, with a temperature

$$T = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}. \tag{28.12}$$

It turns out that one can extend this to Quantum Gravity at the non-perturbative level using a simple argument based on the few facts we have already mentioned. The key is that de Sitter spacetime corresponds to the solution of the topological field theory (28.6.). In terms of the configuration and momenta variables of the Ashtekar formulation, which are the $SU(2)_L$ connection A_a^i and its conjugate momenta $\tilde{E}_i^a \approx e \wedge e$ this becomes

$$F_{ab}^i + \frac{\Lambda}{3}\epsilon_{abc}\tilde{E}_i^c = 0. \tag{28.13}$$

One can solve this with a Hamilton–Jacobi function on configuration space [51; 47], which is a function $S(A^i)$ such that $\tilde{E}_i^a = \frac{\delta S(A^i)}{\delta A_a^i}$. This leads to the equation

$$F_{ab}^i + \frac{\Lambda}{3}\epsilon_{abc}\frac{\delta S(A^i)}{\delta A_a^i} = 0. \tag{28.14}$$

There is also the Gauss's law constraint which requires that

$$\mathcal{D}_a\tilde{E}_i^a = \mathcal{D}_a\frac{\delta S(A^i)}{\delta A_a^i} = 0. \tag{28.15}$$

These have a unique solution

$$S(A^i) = -\frac{k}{4\pi}\int Y_{\text{CS}}(A^i) \tag{28.16}$$

where $Y(A^i)$ is the Chern–Simons invariant. Thus we can consider the Chern–Simons invariant to be a time functional on the Euclidean configuration space.

If we choose $\Sigma = S^3$ we find that there is a periodicity due to the property that under large gauge transformations with winding number n

$$\int Y_{\text{CS}}(A^i) \to \int Y_{\text{CS}}(A^i) + 8\pi^2 n \tag{28.17}$$

This means that the Euclidean configuration space is a cylinder which further implies that *all correlation functions, for any fields in the theory* are periodic in an imaginary time variable given by the Chern–Simons functional $S(A^i)$. But by the KMS theorem, this means that the theory is at a finite temperature. If one works out the periodicity one finds precisely the temperature (28.12).

This applies to the full Quantum Gravity theory because it means that any quantum state on the full configuration space of the theory will be periodic in imaginary time. Thus, with very little effort we greatly extend the significance of the de Sitter temperature. This is an example of the power of seeing General Relativity in terms of connection variables and it is also an example of the importance of topological field theory to the physics of Quantum Gravity.

28.4 The problem of the emergence of classical spacetime

We have just seen that LQG gets several things about gravitational physics right, including the entropy of horizons and the temperature of de Sitter spacetime. There are a number of other results that tell us that LQG and related theories have real physics that we know in them. One thing that was done early in the development of the theory was to investigate classes of semiclassical states and show that their excitations, in the long wavelength limit were massless, spin two particles, i.e. gravitons [52]. It was further shown that when the theory was coupled to matter fields, one could recover the matter QFT on a classical background by expanding around semiclassical states [51].

This is encouraging, but we should ask more. We want to show that these results follow from expanding around the true ground state of the theory. As the fundamental Hilbert space is described in combinatorial and algebraic terms, the key issue is that classical spacetime is not fundamental, it must be an emergent, approximate description, analogous to thermodynamics. This was a problem that took some time to develop the tools to address, but in the past year or so there have been four separate developments that represent progress.

(i) Rovelli and collaborators have computed the graviton propagator in spin foam models [53]. They work in the Euclidean theory and fix a boundary, which is a four sphere, large in Planck units. They compute the amplitude for a graviton to travel from one point on the boundary to another, through the interior, which they treat by a particular form of the spin foam path integral. They get the right answer in the long wavelength limit. This shows that the theory has gravitons and reproduces Newton's gravitational force law.

(ii) Freidel and Livine have computed the spin foam path integral for $2+1$ gravity coupled to matter [54]. They derive an effective field theory for the matter, by which they show that the full effect of Quantum Gravity in this case is to deform the symmetry of flat spacetime from the Poincaré group to a quantum group called κ-Poincaré [62]. I will discuss the meaning of this below.

(iii) Ambjorn, Jurkiewicz and Loll have constructed a simple discrete and background independent model of spacetime, which implements discreteness and causal structure, called the causal dynamical triangulations model [10]. They find that it has a continuum limit which defines a theory which has a large universe limit. They can measure the dimension of spacetime by several means and it is to within error $3+1$.

(iv) Krebs and Markpoulou have proposed new criteria for the emergence of classical spacetime in terms of quantum information theory [55]. They address the low energy physics by asking whether there are local excitations that remain coherent in spite of the fact that they are continually in interaction with the quantum fluctuations in the geometry. The answer is that excitations will remain coherent when they are protected by emergent symmetries. The idea is then to analyze the low energy physics in terms of the symmetries that control the low energy coherent quantum states rather than in terms of emergent classical geometry. To address this problem it was shown that one can apply the technology of noiseless subsystems, or NS, from quantum information theory [56; 57; 58; 59]. In this framework subsystems which propagate coherently are identified by their transforming under emergent symmetries that commute with the interactions of the subsystem with an environment. In this way they protect the subsystems from decoherence. In the application of this idea to Quantum Gravity proposed in [55], the environment is the quantum fluctuations of geometry and the emergent particle states are to be identified as noiseless subsystems [60].

28.5 Possible new generic consequences

Given that there is progress on this key issue, we can go on to discuss three more generic consequences which might be associated with the low energy behavior of quantum theories of gravity.

28.5.1 Deformed Special Relativity

A new physical theory should not just reproduce the old physics, it should lead to new predictions for doable experiments. The problem of the classical limit is important not just to show that General Relativity is reproduced, but to go beyond that and derive observable Quantum Gravity effects. It turns out that such effects are observable in Quantum Gravity, from experiments that probe the symmetry of spacetime.

A big difference between a background independent and background dependent theory is that only in the former is the symmetry of the ground state a prediction of the theory. In a theory based on a fixed background, the background, and hence its symmetry, are inputs. But a background independent theory must predict the symmetry of the background.

There are generally three possibilities for the outcome.

(1) Unbroken Poincaré invariance.
(2) Broken Poincaré invariance, so there is a preferred frame [61].
(3) Deformed Poincaré invariance or, as it is sometimes known, Deformed or Double Special Relativity (DSR) [62].

There is a general argument why the third outcome is to be expected from a background independent theory, so long as it has a classical limit. As the theory has no background structure it is unlikely to have a low energy limit with a preferred frame of reference. This is even more unlikely if the dynamics is instituted by a Hamiltonian constraint, which is essentially the statement that there is no preferred frame of reference. Thus, we would expect the symmetry of the ground state to be Poincaré invariance. But at the same time, there is as we have described above, a discreteness scale, which is expected to be the minimal length at which a continuous geometry makes sense. This conflicts with the Lorentz transformations, according to which there cannot be a minimal length.

The resolution of this apparent paradox is that the symmetry can be DSR, which is a deformation of Poincaré invariance that preserves two invariant scales, a velocity and a length.

There are then two questions. Are there consistent interacting quantum theories with DSR symmetry? And if so, is DSR a generic prediction of background independent Quantum Gravity theories?

The results mentioned above by Freidel and Livine show that DSR is the correct description for Quantum Gravity, coupled to matter in $2 + 1$ dimensional worlds [54]. This answers the first question positively. What about $3 + 1$?

There are heuristic calculations that indicate that LQG in $3 + 1$ dimensions has a semiclassical approximation characterized by DSR [63]. But there is as yet no rigorous proof of this. One reason to expect a DSR theory is to notice that the symmetry group of the ground state of the theory with a non-zero cosmological constant is, by (28.9), the quantum deformation of the de Sitter or anti-de Sitter algebra. The contraction of this is, under plausible assumptions for the scaling of the energy and momentum generators, no longer the Poincaré algebra, it is the κ-Poincaré algebra that characterizes DSR theories [64].

The three possibilities are distinguished by different experiments in progress. We expect that a DSR theory will show itself by (a) the presence of a GZK threshold and similar thresholds for TeV photons but (b) a first order in l_{Pl} and parity even increase of the speed of light with energy [63]. This is in contrast to the implications of breaking Lorentz invariance, which are a parity odd energy dependent shift in the speed of light and a possible shift in the GZK threshold.

28.5.2 Emergent matter

In this and the next section I would like to describe two new possible generic consequences that have only recently been studied.

We are used to thinking that causal spin-network theories are theories of the quantum gravitational field alone. The problem of unification with fermions and the other forces is then postponed. This turns out to be wrong. In fact, it has recently been realized that many causal spin network theories have emergent local degrees of freedom that can be interpreted as elementary particles [65; 66]. That this is a feature of loop Quantum Gravity and similar theories might have been realized long ago, but it was only recently understood due to the application of the noiseless subsystem methods of Kribs and Markopoulou [55]. The reason is that there are emergent quantum numbers which measure knotting and braiding of the embeddings of the graphs [66]. These are preserved under some forms of the local moves: no matter how many local moves are applied there are features of the braiding of edges which are conserved.

These emergent conserved quantum numbers label local structures like braiding, which then can be seen to label noiseless subsystems of the quantum geometry.

One very interesting example of this is in theories with non-zero cosmological constants, in which case the relevant graphs are framed, and are represented by ribbons embedded in Σ (see Figure 28.5). The simplest of these conserved local states, preserved under the rules shown in turn out to correspond, with one additional assumption, to the first generation of quarks and leptons of the standard model [67; 66]. Some of these are shown in Figure 28.6

We see from this that causal spin network theories including loop Quantum Gravity are also unified theories, in which matter degrees of freedom are automatically included. It is also very interesting that the classification of these emergent matter degrees of freedom appears to depend only weakly on the properties of the theory, and so are generic over large classes of theories.

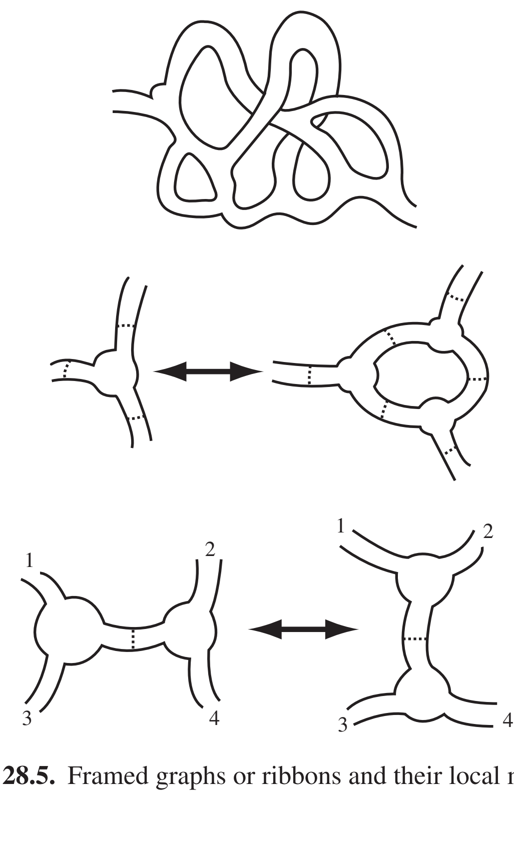

Fig. 28.5. Framed graphs or ribbons and their local moves.

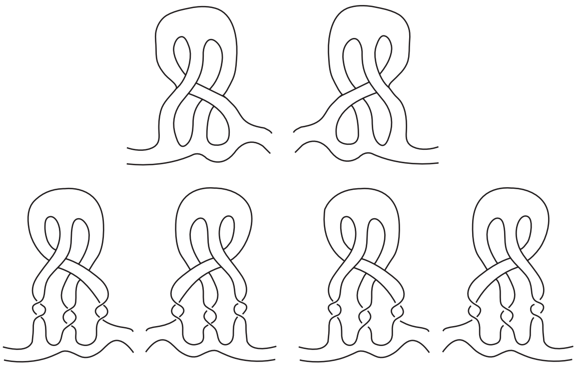

Fig. 28.6. Some braid state preserved under local moves. Under the correspondence proposed in [67; 66]. The first set corresponds to neutrinos, the second to electrons.

28.5.3 Disordered locality

Each spin network state, Γ has a *microscopic locality* given by the connections in the graph. Let us suppose that a semiclassical state exists

$$|\Psi\rangle = \sum_{\Gamma} a_\Gamma |\Gamma\rangle \tag{28.18}$$

corresponding to a classical spatial metric q_{ij}. That metric defines a notion of *macroscopic locality*. The correspondence may be defined by measurements of coarse grained geometrical observables, such as volumes and areas. We may also require that excitations of $|\Psi\rangle$, corresponding to graviton or matter degrees of freedom, propagate as if they were on the background metric q_{ij}.

But, as argued by Markopoulou, in [12], it may still not be the case that there is a complete correspondence between the macrolocality defined by q_{ij} and the microlocality defined by some or all of the graphs Γ whose states have significant amplitude in (28.18).

Consider, for example, the case of a "weave state", which is a random lattice constructed to approximate a flat background metric q^0_{ij} on a torus T^3. This consists of a graph Γ_0 embedded in the torus such that only nodes of order Planck distance apart in q^0_{ab} are connected. The spins and labels on nodes are chosen so that measurements of areas and volumes in the state $|\Gamma_0\rangle$ coincide with the metric q^0_{ij}. Let the total volume be $V = N l_{\rm Pl}^3$ for some very large N. We can then, for example, choose Γ_0 to be four valent with N nodes and $2N$ edges. Such a $|\Gamma_0\rangle$ is a state corresponding to the metric q^0_{ij} in which microlocality and macrolocality coincide.

But now let us add to the graph Γ_0, a new link connecting two randomly chosen nodes of Γ_0. It is not hard to see that we can adjust the labels on the edges and nodes so that no large areas or volumes are changed. In fact, we can do this M times, at least so long as $M \ll 2N$, without changing any large areas or volumes. Each of these M new links connects two randomly chosen nodes of Γ_0, making a new graph Γ'. The corresponding state $|\Gamma' >$ is still a semiclassical state for the metric q^0_{ab} and will reproduce it when sufficiently coarse grained observables are measured. But it provides an example of Markopoulou's observation that micro and macro notions of locality need not coincide even in the low energy limit [68]. We may call this phenomena, *disordered locality.*

At first sight it seems as if disordered locality would kill the theory, because there would be macroscopic violations of locality in the low energy limit. But it turns out that this need not the case, if the disagreement between micro and macro locality is rare enough. For example, suppose that the probability that a node has a non-local edge, $p = \frac{M}{2N}$ is on the order of 10^{-100}. This would still mean there are on the order of 10^{80} random non-local edges within the Hubble volume (see Figure 28.7). Could we do any measurements to tell that the quantum geometry of our universe was based on Γ' rather than Γ?

It would be very unlikely that any two nodes within the earth are connected by one of the non-local edges, so it would be very hard to directly detect non-locality. Moreover, since the defects were at the Planck scale, the amplitude for low energy quanta to jump across a non-local link would be suppressed by $l_{\text{Pl}}^2 E^2$. So we are unlikely to see fermions appearing and disappearing across the links. Moreover since the whole universe is in thermal equilibrium at the same temperature the transfers of energy through the non-local links would also be hard to observe. Studies have been done of the thermodynamics of spin systems on networks with disordered locality and the main effect for small p is to raise the Curie temperature by an amount of order p without strongly affecting the correlation functions [69].

Would dynamics suppress such non-local links? The answer is that the dynamics cannot. The reason is that the dynamics is micro-local, and hence defined by the connectivity of the graph Γ'. The local moves that generate the dynamics cannot

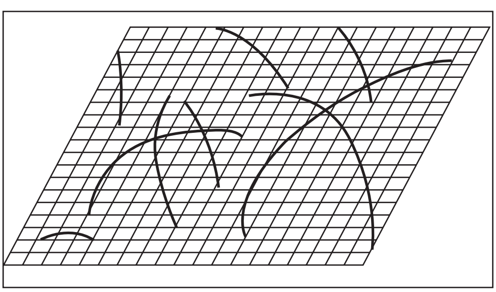

Fig. 28.7. A lattice with disordered locality from a contamination of non-local links.

remove non-local edges connecting two nodes far away in Γ^0 and hence q^0_{ab}. This was shown for stochastic evolution of graphs in [71], but there is no reason to believe the results will be different for quantum evolution.

Where then would the effects of disordered locality show up? The following are speculative suggestions which are presently under investigation.

- At cosmological scales there may be new effects coming from the fact that the shortest distance between points will go through the non-local edges. Could this have something to do with the dark energy and dark matter problems [72]?
- If electric flux is trapped in a non-local edge its ends look like charged particles. This provides a quantum mechanical version of Wheeler's old hypothesis that matter comes from charged wormholes.
- Suppose we have a subsystem, large enough to contain the ends of many non-local links but small enough that almost all of these connect it to the rest of the universe. Even at zero temperature the subsystem is subject to a random disorder coming from its connections to the rest of the universe through the non-local links. There are results that indicate that this could be the origin of quantum phenomena [70].
- Non-local links could connect regions of the universe to others beyond the horizon. This could provide a solution to the horizon problem without inflation. Could it also lead to the generation of a scale invariant spectrum of fluctuations? This is discussed next [73].

28.5.4 Disordered locality and the CMB spectrum

Here is a simple estimate that shows that effects of disordered locality could be responsible for the power spectrum observed in the CMB [73]. Assume that there is a random (and hence scale invariant) distribution of pairs of points in the universe that are connected by a non-local link. We call these pairs x_i and y_i, for $i = 1, ..., N_{\text{NL}}$. For practical purposes these pairs can be considered to be identified, as they are the equivalent of a Planck distance apart. We can estimate the contribution these points make to the two point correlation function for energy fluctuations, as

$$D(x, y)_{\text{NL}} =< \frac{\delta\rho}{\rho}(x)\,\frac{\delta\rho}{\rho}(y) >= l_{\text{Pl}}^2 T^2 \sigma_U^2 \sum_{i=1}^{N} \delta^3(x, x_i)\delta^3(y, y_i). \tag{28.19}$$

The factor $l_{\text{Pl}}^2 T^2$ is due to the cross-section of a Planck scale edge being roughly the Planck area. The factor σ_U^2 is the local (because the points connected by a non-local link are identified) fluctuation in energy

$$\sigma_U^2 = \frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} = \frac{T}{\rho V} = \frac{1}{VT^3}, \tag{28.20}$$

where T is the temperature and V is the volume of space within the horizon. The power spectrum is related to the Fourier transform

$$D(k)_{\mathrm{NL}} = \int_V d^3x \int_V d^3y D(x, y)_{\mathrm{NL}} e^{\imath k\cdot(x-y)}. \tag{28.21}$$

Since the connected pairs are distributed randomly, we find the correct scale invariant spectrum of fluctuations,

$$D(k)_{\mathrm{NL}} = \frac{A}{Vk^3} \,. \tag{28.22}$$

This should hold outside the horizon at decoupling, when there are no other long ranged correlations possible. The amplitude is given by

$$A = 2\pi^2 l_{\mathrm{Pl}}^2 T^2 N_{\mathrm{NL}} \sigma_U^2. \tag{28.23}$$

If we evaluate σ_U^2 at decoupling we find around 10^{-90}. This tells us that we get the correct amplitude of 10^{-10} with an $N_{\mathrm{NL}} \approx 10^{124}$. This gives us a $p \sim 10^{-56}$ which from the above discussion is well within observable limits. This is very rough, but it shows that distributed locality can comfortably do the job inflation does of solving the horizon problem in a way that leads to a scale invariant distribution of fluctuations outside the horizon, of the observed amplitude.

28.6 Conclusions

To summarize, the causal spin network theories, including loop Quantum Gravity and spin foam models, do a number of things that are expected of any sensible quantum theory of spacetime. They are finite, they predict that quantum geometry is discrete, they remove spacelike singularities and explain the entropy of black hole and cosmological horizons as well as the temperature of de Sitter spacetime. If one adds to this that there is progress understanding whether and how classical spacetime emerges from the quantum geometry, we see that these continue to show promise as plausible models of Quantum Gravity. While there is certainly still much to do, the last years have given us a well defined foundation to build on.

But theories triumph not because they do what is expected, but because of the surprises they lead to. A good theory must predict new phenomena, which are then observed. In the case of causal spin network theories we see several unexpected consequences which all have implications for experiment and observations. These are as follows.

- The symmetry of the ground state is DSR, leading to an energy dependent, parity even, speed of light.
- There is evidence that *LQG* predicts that spacelike singularities bounce. This opens up the possibility of tuning the parameters that govern low energy physics through a dynamical mechanism like cosmological natural selection (CNS) [74; 75].

- These theories have emergent local degrees of freedom, hence they automatically unify geometry and matter.
- Disordered locality has consequences for cosmological observations because even at small levels that make it unobservable in local experiments it dominates in the early universe and at cosmological scales. A rough estimate of such effects shows that this mechanism has a possibility to naturally solve the horizon problem while predicting the correct spectrum of fluctuations of the CMB.

So which kind of theories are LQG and other causal spin network theories? Are they the good kind of unification that leads to consequences we celebrate or the embarrassing kind that lead to consequences that must be hidden. The discovery that these theories generically predict emergent particle states certainly leaves them vulnerable to quick falsification. While there is preliminary evidence that a large class of theories can reproduce some features of the standard model, there is a lot that these theories have to get right so as not to disagree with observation.

Disordered locality certainly offers other possibilities for falsification. If the deviations from locality are small, disordered locality gives rise to new mechanisms for solving hard problems like the horizon problem and dark energy. This means they lead to falsifiable predictions, for there is only one parameter, p which controls these effects. But what if the deviations from locality are not small? One possibility is the proposal of Markopoulou, who argues in [12] that the macroscopic causal structure will be defined by the interactions of the coherent excitations which are the elementary particles. As described there, the test of this program is then whether the Einstein equations are reproduced.

Finally, the expectation the the low energy limit is DSR has to be counted as fortunate, as this experiment is sensitive enough to test the implied predictions that are expected in the next few years.

Thus, there appears to be a good possibility to use these generic consequences to test whether the correct unification of spacetime and quantum theory is in terms of a causal spin network theory. In the next few years we may hope to sharpen up the arguments described here to detailed predictions that may be confirmed or falsified in upcoming experiments.

Acknowledgements

I would like to thank the many people who contributed the ideas and results described here, for many discussions over the years. The proposal that locality is disordered in background independent theories, and the observation that these theories have emergent particle states, are due to Fotini Markopoulou. I would also like to thank M. Ansari, S. Bilson-Thompson, O. Dreyer, H. Finkel, T. Konopka,

J. Magueijo, S. Majid, J. Moffat, M. Paczuski, I. Premont-Schwarz and Y. Wan for collaborations and discussions which were very helpful for exploring these new ideas.

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Questions and answers

- **Q - L. Crane - to C. Burgess**:
  Can any of the approximate calculations you describe be used to make any predictions concerning the long distance interferometry tests which are being considered for QG?
  - **A - C. Burgess**:
    In principle yes, although the prediction is generically that the quantum effects to be expected are negligibly small. (Of course the details will depend on the precise tests which are of interest.) Although this is disappointing if the goal is to detect these quantum effects, it is what justifies the classical analyses of these tests which are usually performed.
- **Q - D. Oriti - to C. Burgess**:
  Assuming one takes your suggested point of view on gravity as an effective field theory, and is also re-assured by your explanation of how we can use it satisfactorily to make predictions at low energy, what if he/she wants to go further, i.e. what if he/she wants to find the fundamental (ultra-)microscopic theory of spacetime from which GR emerges at low energy? What can the effective field theory point of view teach us about the properties of the fundamental theory, if it exists? If spacetime and gravity emerge from the unknown microscopic theory (that therefore does not use our familiar notions of space and time) in the same way as hydrodynamic concepts and field theories emerge in many condensed matter systems from the underlying quantum (field) theories of "atoms", how much and what exactly can we deduce about the quantum (field) theory of fundamental "space atoms" from the effective theory (GR) we know (e.g. symmetries, type of degrees of freedom)?
  - **A - C. Burgess**:
    Unfortunately, this is the hard part! Based on experience with other interactions, the properties of the effective theory can point you to the energy scales at which the more fundamental theory becomes important, but it does not

say much about what this theory must be. But if you do have a candidate for what this fundamental theory is, the effective theory is among the most efficient ways to identify its observational consequences (and so to compare between different candidates for the fundamental theory). For instance, calculating the effective theory which is appropriate requires first identifying what the low-energy degrees of freedom are and what are their approximate low-energy symmetries. Then computing the coefficients of the relevant effective theory efficiently identifies what combination of the properties of the underlying theory are relevant in low-energy observables, and so can be accessed experimentally.

For gravity, the process of identifying the relevant low-energy theory is fairly well developed for the case where the candidate fundamental theory is string theory, with the result being supergravity theories in various dimensions. The comparison of string theory with its competitors in their implications for observations would be much easier if the implications of the alternative theories in weakly-curved spaces were similarly expressed.

- **Q - D. Sudarsky - to S. Majid**:

  1. Regarding eqs. (24.1) and (24.2): what are we to make of their meaning? If X_i has anything to do with the coordinates X that we use to parameterize space-time (in a given frame, and having chosen an origin for them), it would follow (using the interpretation you suggest in Section 24.5.1) that one can not measure position and time simultaneously except if we are considering located at the origin of coordinates (i.e. the uncertainty relation is $\Delta X_0 \; Delta X_i \leq 1\kappa\langle X_i\rangle$). Even if the X are not precisely the space-time positions that we measure, but have anything to do with them, it seems clear that the precision limitations to coincident measurements of space and time would increase with the distance to some origin. In fact in eq. (1.27) the quantities of order λ are also of order $\langle X\rangle$. So where in the universe is this special point?

  2. If on the other hand, these quantities above have nothing to do with the space-time coordinates we might measure, why do we talk about non-commutative space-time?

  3. You say that the model in Section 24.5.1 has been "taken to the point of first predictions", but then you acknowledge that without answering your questions about the physical (i.e. measurement related) meaning of the momentum coordinates, and the physical meaning of the order of addition in momentum addition law, you can have no predictions at all! Can you explain this apparent contradiction?

  – **A - S. Majid**:

    1. Indeed eq. (24.1) is in a specific frame of reference as is the conclusion that the uncertainty in that frame gets worse further out from the origin in that

frame. Just as a frame of reference may have limited validity due to global geometry, here even if spacetime is flat, its noncommutativity accumulates the uncertainty the further out one goes from the origin of that frame. Is it a problem? Only if some other observer with some other origin does not reach the same conclusion. The other observer would have transformed coordinates defined via eq. (24.17) which describes a quantum Poincaré transformation, in particular a shift is allowed. The new variables x'_μ defined by the RHS of eq. (24.17) obey the relations eq. (24.1) but are shifted by a_μ from the original. The only thing, which I explain in Section 24.5.1 is that the transformation parameters such as a_μ are themselves operators (its a quantum group not a classical group) so the new variables are not simply related to the old ones by a numerical matrix. In short, there is clearly no classical Poincaré invariance of eq. (24.1) but there is a quantum one. If one takes expectation values one then has real numbers but the expectation values do not then transform under a usual Poincaré transformation as the questioner perhaps assumes. Just because the uncertainty relations are not usual-Poincaré invariant does not mean an origin is being singled out in the universe. Rather to actually relate a new observer's expectations to the old one, one has to know the expectation value of the a_μ and face also that they need not commute with the x_μ. In short, a quantum frame transformation is itself "fuzzy" which is not surprising since the different observers' own locations should be fuzzy. To be sure one has approximated Poincaré invariant to $O(\lambda)$ but the equations such as eq. (24.1) are themselves at that level (both sides are zero if $\lambda = 0$ and we have usual commuting x_μ). My goal in Section 24.5.1 is indeed to get physicists thinking properly about quantum frame rotations as a theory of Quantum Gravity has to address their expectation values too. However, I don't see any inconsistency.

2. The x_μ are operators whose expectation values, we suppose, are the physically observed macroscopic spacetime coordinates at which a particle might be approximately located. A theory of Quantum Gravity has to provide the states on which these expectations are computed so the noncommutative algebra is not the whole of the observed physics. It's a joint effort between the (proposed) noncommutative geometry and the effective quantum state in which the operators are observed.

3. There is no contradiction. The "first predictions" I refer to are order of magnitude computations for a time-or-arrival experiment that can be done without solving all problems of interpretation of momentum and their addition. Addition of momenta would be more relevant in the many particle theory. For a single photon modelled as a noncommutative plane wave, one does not need to have solved the many particle theory. One does still need some sort of

insight into what a single plane wave is and how it could be measured and this is what we did for the time of flight experiment in ref. [1] using a normal ordering prescription, as explained in Section 24.5.2. I agree that some such justification was needed to have any valid prediction and that this is a problem that has plagued and still plagues much of the literature on this model.

Also, a general point made in the article is that noncommutative spacetime is most likely an effective description of some limit of a deeper Quantum Gravity theory. In an effective description one isolates the relevant quantities and their approximate behaviour without necessarily understanding the whole of the full theory. There is more than one way that one might do this and its an area that definitely needs more attention. Section 1.5 aims to bring out some of the issues here.

- **Q - D. Sudarsky - to J. Kowalski-Glikman**:
  1. In the second paragraph below eq. (25.1) you state that one could think of scales in terms of synchronization. That “in SR the velocity of light is indispensable for synchronization, as it provides the only meaningful way of synchronizing different observers”. I do not see why. Consider two inertial observers A and B who want to synchronize their clocks, first of all they must find out if they are at rest relative to each other. To do this A sends a proton (no photon) with a given energy and asks B to return another proton with the same energy as the one he received. Then A compares the energy of the proton he receives with the one that he sends, if they are the same A and B are at relative rest. To synchronize the clocks A tells B to set his clock to zero at the time it receives the above mentioned proton, while A sets his clock to zero at midtime between the moments he sends the proton and he receives a proton back. Note that there are no photons involved. So do you stand by your claim?
  2. Referring to that same paragraph: In the above we see that one can use things that travel to synchronize clocks, and photons are certainly useful in this way, precisely because they travel, but how can one talk of using a scale – related to what physical aspect of nature – to synchronize anything? In fact what is the meaning of momentum space synchronization? What is being synchronized?
  3. Is the modification of SR the only option to explain the GZK anomaly (if it is confirmed), or are there are other alternatives?
  4) You have acknowledged in Section 25.6 that there are serious problems interpreting the formalism of DSR, we do not know what to make of the order dependence of the addition law for momenta, we do not know what is the quantity we must identify with the measured momentum, we have the spectator problem, etc., etc. The question is: how can we consider doing phenomenology, using a formalism that we do not know how to interpret?

– **A - J. Kowalski-Glikman**:
1. You are certainly right that one could use any objects: photons, protons, or potatoes to synchronize two identical clocks placed at two distinct points, at rest with respect to each other. Yet it would be extremely odd to do that by means of anything but light in view of the Einstein postulate: "Clocks can be adjusted in such a way that the propagation velocity of every light ray in vacuum – measured by means of these clocks – becomes everywhere equal to a universal constant c, provided that the coordinate system is not accelerated." Such clocks provide Einstein synchronization.
2. I do not know exactly, but a general idea is that in momentum space, instead of clocks and rulers you will have a device measuring energy and momentum. If I have an observer independent fundamental scale of energy, carried by an object, which I call planckion, it would be convenient to synchronize the energy meters in such a way that "the energy of every planckion – measured by means of these meters – becomes everywhere equal to a universal constant κ, provided that the coordinate system is not accelerated."
3. If the GZK anomaly indeed is there (which means that we see $10^2 1$ eV protons, whose source is at the cosmological distance, and all the astrophysical data used to calculate the mean free path of such protons are correct) then I do not see any other explanation.
4. We obviously cannot do phenomenology if we do not understand it. However we already have some generic understanding of DSR formalism which leads to at least two robust predictions: there is no energy dependence of the speed of light, and, as I argued in my contribution, it is extremely unlikely that there are any sizable DSR corrections to GZK threshold.

• **Q - L. Crane:**
I think your explanation of the origin of the deformation of Lorentz transformations is very interesting. But wouldn't it then depend on the size and distance of the system and the state of motion of the observer?

– **A - F. Girelli**:
The deformation can be read out from the dispersion relation encoding the particle dynamics. This dispersion relation can be particle dependent, that is the extra terms encoding the deformation could depend on the helicity, spin, intrinsic properties of the particle. In this sense the deformation would be really particle dependent. Then the deformation depends also on the factor M_{P}, the Planck scale. This parameter is a priori universal. However, I argued that for many particles one should allow a rescaling of the maximum mass, in order to avoid the soccer ball problem, that is, the emergence problem of macroscopic objects. Indeed the maximal mass as a Scharwschild mass should rescale linearly in terms of its typical length. If we agree on that, if we

consider a composite object, the deformation will then depend on the typical size of the object or roughly on the number of particles making the object. This option should be, however, improved in the context of field theory since we can have virtual particles that would then spoil this simple interpretation. The deformation inducing the non-linear realization is really dependent on the system and not on the observer, this is why this is really a deformation of the usual relativity principle. In this sense the status of DSR is the same as Special Relativity regarding the state of motion of the observer.

DSR is a (a priori effective) theory supposed to describe flat semi classical spacetime, so that we encode approximately, effectively, some quantum and gravitational features in the kinematics. This is really a zero order approximation, where both quantum and gravitational effects are small but not negligible, modifying the symmetry. For example as I argued shortly in the article, the notion of consecutive measurements can implement a non-trivial dependence of the reference frame on the system, this irrespective of the distance between them. This is related to entanglement and is a purely quantum feature. Gravitational effects can also generate this deformation in a way independent to the particle distance: typically one can expect the gravitational fluctuations to be expressed in terms of the fundamental physical scale present there, provided by the particle: its Compton or de Broglie lengths. For example in the paper *Phys. Rev*. D74:085017 (2006), gr-qc/0607024, Aloisio *et al*. looked at a particle, together with some stochastic fluctuations of the gravitational field. The typical scale of these fluctuations being expressed as a function of the physical scale present there is the particle de Broglie length. It then implied naturally a deformation of the symmetries as well as a nonlinear dispersion relation.

In any case, I feel that still at this stage, a better understanding of DSR is needed. In particular to really understand what is the fundamental meaning of the deformed relativity principle, together with a better understanding of the DSR operational aspects are for me still open issues that deserve further (deep!) thinking.

- **Q - D. Oriti - to L. Smolin**:

I have one comment and one question. The comment is the following: it seems to me that the quantum discreteness of geometry and the ultraviolet finiteness that you discuss are a bit less generic than one would hope. In fact, the discreteness of geometric operators in the canonical formulation, as well as the uniqueness results that you mention for the same formulation, depend very much on the choice of a compact symmetry group G for labelling states and observables. This choice, although certainly well-motivated and rather convenient, is not the only possible one, and in fact there exist, for example, spin foam models where this choice is not made and one uses the full non-compact Lorentz group instead, in which case the spectra of some geometric observables are continuous and not bounded from below (e.g. no minimal spacelike areas or lengths exist), and no uniqueness result is, unfortunately, available to us. Some of these models remain ultraviolet finite despite this, as you correctly mention, but this seems to be a result of very specific models (more precisely, of a very specific choice of quantum amplitudes for the geometric configurations one sums over in the spin foam setting) and not a generic feature of this class of theories. I fully agree, of course, that the class of models you discuss remains truly "discrete" in the sense that it bases its description of spacetime geometry on discrete and combinatorial structures (graphs and their histories) and local discrete evolution moves. The question is the following. In the model of emergent matter that you present, where matter degrees of freedom are encoded in the braiding of the framed graphs on which the theory is based, where does the mass of such matter come from? Do you expect that this could be defined in terms of something like the holonomy "around these braids", when one endowes the graphs with geometric data, e.g. a connection field or group elements, as in the coupling of particles in topological field theories and 3d Quantum Gravity? If so, would you imagine a sort of coherent (noiseless) propagation of such "holonomy + braiding" degrees of freedom to encode the conservation of mass, or do you envisage a sort of "variable mass" field theory description for the dynamics for these matter degrees of freedom, in the continuum approximation?

– **A - L. Smolin**:

  Regarding your first comment, this of course depends on whether we take the view that the theory is derived by quantization of GR or invented. If we take the first view then my view is that the canonical theory is more fundamental for sorting out the quantum kinematics. The canonical theory leads to labels in SU(2) which is compact and thus implies the discreteness of area and volume. At the very least the canonical theory and the path integral theory should be related so that the path integral gives amplitudes for evolution or defines a projection operator for states in the canonical theory. It is unfortunately the case that none of the spin foam models which have so far been well developed do this, although I am told there is work in progress which remedies this. In the original papers of Reisenberger and Reisenberger and Rovelli as well as in the first paper of Markopoulou the spin foam amplitudes are defined in terms of evolution of states in the canonical theory. This to me is the preferred way as it is well defined and does not lead to ambiguities in choices of representations or whether one sums over triangulations or not. When the path integral is defined from the canonical theory all faces in the spin foam are spacelike and all should be labeled from finite dimensional reps of SU(2).

As a result, while I admire the beautiful work that friends and colleagues have done with spin foam models with representations of the Lorentz or even Poincaré groups I do not believe that ultimately this will be the choice that corresponds to nature.

One might of course, take the other view, which is that the spin foam model is to be invented independently of any quantization from a classical theory. I am sympathetic to this as quantum physics must be prior logically to classical physics, but in this case also I have two arguments against using the representations of the Lorentz or Poincaré group in a spin foam model.

The first argument starts with the observation that Lorentz and Poincaré must in the quantum theory be considered global symmetries. Someone might claim that they are local symmetries, but the equivalence principle is limited in quantum theory because the wavelength of a state is a limit to how closely you can probe geometry. When the curvature is large, the equivalence principle must break down, and thus it cannot be assumed in formulating the path integral, which will be dominated by histories with large curvatures. Thus, you cannot assume the equivalence principle for the quantum theory and as a consequence I dont think you can regard local symmetries derived from the equivalence principle as fundamental. On the other hand, global symmetries are not fundamental in General Relativity – because the generic solution has no symmetries at all and there are – as Kuchar showed – no symmetries on the configuration space of GR. Any appearance of a global symmetry in GR is either imposed by boundary conditions or a symmetry only of a particular solution.

Thus, the Lorentz and Poincaré groups are not fundamental to GR, they are instead symmetries only of a solution of the theory. Hence I cannot believe that a spin foam model using labels from Lorentz or Poincaré reps can be fundamental.

My second argument is that I believe that physics at the smallest possible scale should be simple and involve only finite calculations. I cannot believe that the universe must do an infinite amount of computation in a Planck time in each Planck volume just to figure out what happens next. I would thus propose that the computation required in the smallest unit of time in the smallest possible volume of space must be elementary and must require only a minimal number of bits of information and a minimal number of steps. My own bet would then be that at the Planck scale the graphs which label quantum geometry are purely combinatorial, in which case there are no representation labels at all.

You could push me by arguing that this is quantum theory and a minimal process should involve a small number of q-bits and not classical bits. This would allow small finite dimensional vector spaces, which is what is involved in the representation theory of SU(2). Indeed, q-bits are elementary reps of SU(2). So I could imagine being pushed to go far enough to believe in one or a few q-bits per Planck volume, evolving in a way that requires one q-gate per Planck time. But this does not allow the representation theory of non-compact groups.

As for your question, the answer to it is actually pretty straightforward, one has to compute the propagator for such states, under the evolution given by the local moves. The mass matrix is then the inverse propagator at zero momentum. To derive the propagator there are three steps. (1) Show that the braids do propagate on spin networks by local moves. (This is shown for the three valent moves in a paper in preparation by Jonathan Hackett and for the four valent case in another paper we have in preparation with Wan.) (2) Show that if the spin network has an approximate translation symmetry there are noise free subsystems spanned by identical braids in different positions, so that momentum is an approximate conserved quantity. (This is done in principle as it is a consequence of the Kribs and Markopoulous paper.) (3) In a given spin foam model, which gives amplitudes to the local moves, one then computes the propagator analogously to how the graviton propagator was recently computed.

I can also report that the extension of the results to the 4-valent case has been accomplished, thanks mainly to some insights of Yidun Wan and is now being written up. This is relevant for the Barrett–Crane and similar spin foam models. We show that the braid preon states both propagate and interact with each other in the 4-valent case.

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