ELSEVIER
ACADEMIC PRESS

Advances in
IMAGING and ELECTRON PHYSICS

Henning F. Harmuth
Beate Meffert

CALCULUS OF FINITE DIFFERENCES IN QUANTUM ELECTRODYNAMICS

Volume 129

ADVANCES IN IMAGING AND ELECTRON PHYSICS

VOLUME 129

CALCULUS OF FINITE DIFFERENCES IN QUANTUM ELECTRODYNAMICS

Advances in

Imaging and Electron Physics

Calculus of Finite Differences in Quantum Electrodynamics

HENNING F. HARMUTH
Retired, The Catholic University of America
Washington, DC, USA

BEATE MEFFERT
Humboldt-Universität
Berlin, Germany

VOLUME 129

ELSEVIER
ACADEMIC
PRESS

AMSTERDAM • BOSTON • HEIDELBERG • LONDON
NEW YORK • OXFORD • PARIS • SAN DIEGO
SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO
Academic Press is an imprint of Elsevier

This book is printed on acid-free paper. ♾

Academic Press
An Elsevier Imprint.
525 B Street, Suite 1900, San Diego, California 92101-4495, USA
http://www.academicpress.com

Academic Press
84 Theobald's Road, London WC1X 8RR, UK
http://www.academicpress.com

International Standard Book Number: 0-12-014771-8

PRINTED IN THE UNITED STATES OF AMERICA
03 04 05 06 07 08 9 8 7 6 5 4 3 2 1

CONTENTS

Equations are numbered consecutively within each of Sections 1.1 to 6.8. Reference to an equation in a different section is made by writing the number of the section in front of the number of the equation, e.g., Eq.(2.1-5) for Eq.(5) in Section 2.1.

Illustrations are numbered consecutively within each section, with the number of the section given first, e.g., Fig.1.1-1.

References are listed by the name of the author(s), the year of publication, and a lowercase Latin letter if more than one reference by the same author(s) is listed for that year.

3 Difference Equations for the Pure Radiation Field

4 Differential Equation for the Klein-Gordon Field

5 Difference Equation for the Klein-Gordon Field

6 Appendix

PREFACE

H. F. Harmuth is no newcomer to these Advances, for his early work on Maxwell's equations, which was highly controversial at the time, formed Supplements 14 (1981) and 15 (1984) to the series; an earlier study on sequency theory was published in 1977 as Supplement 9.

Some consequences of that work on Maxwell's equations have led H.F. Harmuth and B. Meffert to enquire how the discrete nature of many scientific phenomena could be incorporated in the theory. For this, the familiar differential equations must be replaced by difference equations; this in itself is a far from trivial task and it is not at all easy to draw general conclusions from difference equations even though all numerical work of course depends on them.

In this volume, the authors first introduce the basic differential equations involved and then examine two very important cases in detail. Two chapters are devoted to the pure radiation field (Maxwell's equations), one on the differential equations, the other on the corresponding difference equations. Two further chapters are devoted to the differential and difference equations for the Klein-Gordon field.

These studies raise important and fundamental questions concerning some of the basic ideas of physics: electromagnetic theory and quantum mechanics. They deserve careful study and reflection for although the authors do not attempt to provide the definitive answer to the questions, their work is undoubtedly a major step towards such an answer. I am delighted that this work will be presented to the scientific public in these Advances.

Peter Hawkes

This Page Intentionally Left Blank

FUTURE CONTRIBUTIONS

S. van Aert, A. den Dekker, A. van de Bos, and D. van Dyck (vol. 130)
Statistical experimental design for quantitative atomic-resolution transmission electron microscopy

G. Abbate
New developments in liquid-crystal-based photonic devices

S. Ando
Gradient operators and edge and corner detection

C. Beeli
Structure and microscopy of quasicrystals

G. Borgefors
Distance transforms

B. L. Breton, D. McMullan, and K. C. A. Smith, Eds (vol. 132)
Sir Charles Oatley and the scanning electron microscope

A. Bretto (vol. 131)
Hypergraphs and their use in image modelling

H. Delingette
Surface reconstruction based on simplex meshes

R. G. Forbes
Liquid metal ion sources

E. Förster and F. N. Chukhovsky
X-ray optics

A. Fox
The critical-voltage effect

L. Godo and V. Torra
Aggregration operators

A. Gölzhäuser
Recent advances in electron holography with point sources

A. M. Grigoryan and S. S. Agaian (vol. 130)
Transform-based image enchancement algorithms with performance measure

M. I. Herrera
The development of electron microscopy in Spain

D. Hitz
Recent progress on HF ECR ion sources

J. Hormigo and G. Cristobal (vol. 131)
Texture and the Wigner distribution

K. Ishizuka
Contrast transfer and crystal images

K. L. Jensen
Emission physics

G. Kögel
Positron microscopy

W. Krakow
Sideband imaging

N. Krueger (vol. 131)
The application of statistical and deterministic regularities in biological and artificial vision systems

B. Lahme
Karhunen-Loeve decomposition

B. Lencová
Modern developments in electron optical calculations

M. A. O'Keefe
Electron image simulation

N. Papamarkos and A. Kesidis
The inverse Hough transform

K. S. Pedersen, A. Lee, and M. Nielsen
The scale-space properties of natural images

M. Petrou (vol. 130)
Image registration

R. Piroddi and M. Petrou
Dealing with irregularly sampled data

M. Rainforth
Recent developments in the microscopy of ceramics, ferroelectric materials, and glass

E. Rau
Energy analysers for electron microscopes

H. Rauch
The wave-particle dualism

J. J. W. M. Rosink and N. van der Vaart (vol. 131)
HEC sources for the CRT

G. Schmahl
X-ray microscopy

S. Shirai
CRT gun design methods

N. Silvis-Cividjian and C.W. Hagen
Electron-beam–induced desposition

T. Soma
Focus-deflection systems and their applications

J.-L. Starck
The curvelet transform

I. Talmon
Study of complex fluids by transmission electron microscopy

M. Tonouchi
Terahertz radiation imaging

N. M. Towghi
I_p norm optimal filters

Y. Uchikawa
Electron gun optics

D. van Dyck
Very high resolution electron microscopy

K. Vaeth and G. Rajeswaran
Organic light-emitting arrays

C. D. Wright and E. W. Hill
Magnetic force microscopy-filtering for pattern recognition using wavelet transforms and neural networks

M. Yeadon
Instrumentation for surface studies

This Page Intentionally Left Blank

To the memory of Max Planck (1858 – 1947)
Founder of quantum physics and distinguished participant of the Morgenthau Plan, 1945–1948.

This Page Intentionally Left Blank

FOREWORD

It is about 20 years since it was found that Maxwell's equations generally cannot have solutions that satisfy the causality law. The problem was overcome by adding a term for magnetic dipole currents that are produced by rotating magnetic dipoles. Six books have been published on this topic since 1986. The most recent one, "Modified Maxwell Equations in Quantum Electrodynamics," shows that the problem of infinite zero-point energy and renormalization disappears if the corrected Maxwell equations are used.

Three more changes had to be made in addition to the modification of Maxwell's equations: a) Dipole currents – either electric or magnetic ones – can flow in vacuum even though monopole currents are prohibited by the conservation of charge; this is why an electric current can flow through a capacitor whose dielectric is vacuum. b) The quantization is applied to a wave that satisfies the causality law as well as the conservation law of energy, rather than to a field for which the meaning of these laws is not clear. c) Infinite space and time intervals were replaced by arbitrarily large but finite intervals.

The last point may be seen as nothing more than a physical explanation of the traditional box normalization. However, the elimination of infinite space and time intervals raises the question why infinitesimal space and time intervals should not be replaced by arbitrarily small but finite intervals. Since we can neither observe infinite nor infinitesimal intervals it would be inconsistent to eliminate the one but not the other.

The elimination of infinite intervals required no more than the replacement of the Fourier integral by a Fourier series with denumerable functions orthogonal in a finite interval and undefined outside that interval. If one wants to eliminate infinitesimal intervals one must replace the differential calculus by the calculus of finite differences, which is a much more demanding task. We will show for the pure radiation field and the Klein-Gordon field that one obtains significantly but not catastrophically different results with the calculus of finite differences. A particularly interesting result will be that a basic difference equation of quantum electrodynamics yields the same (energy) eigenvalues as the corresponding differential equation but quite different eigenfunctions.

Let us consider the replacement of differentials dx by arbitrarily small but finite differences Δx from a more general point of view. Elementary particles within the framework of the differential calculus must be treated as *point-like*, since by physical definition they should not have any spatial structure. The extension of such particles is of the order of dx. The mathematical method is chosen first and the physical features are matched to the mathematical requirements.

Assume now that the best available instruments for spatial measurements have a resolution Δx, which means we can distinguish something at the location x from something at the location $x + \Delta x$. A particle smaller than Δx cannot be resolved, which implies we cannot claim anything more about its spatial features than that it is smaller than Δx. The calculus of finite differences is a mathematical method that matches the physical situation. Since a finite interval Δx can be subdivided into nondenumerably many intervals dx we are a long way from defining elementary particles as point-like.

What we have said about elementary particles raises the question whether physics is a branch of mathematics or mathematics a tool for physics. This was a serious question for the ancient Greeks. Today we recognize mathematics as a science of the thinkable and physics as a science of the observable. The one can never be more than a tool or a source of inspiration for the other. Physics inspired the enormous development of the differential calculus in mathematics compared with the calculus of finite differences. Physics is used as a tool in mathematics whenever a few more digits of the numbers π or e are obtained with a computer. This book is an example how mathematics is used as a tool and a source of inspiration in physics.

The authors want to thank Humboldt-Universität in Berlin for providing computer and library services.

* * *

This is the fourteenth scientific book the lead-author wrote either alone or – after age 65 – with co-authors and it will probably be the last one. The following observations of a lifetime of scientific publishing may help young scientists.

Scientific advancement is universally based on the concept of many small steps or *incremental science*. This approach founders when one tries to introduce the causality law into electrodynamics in many small steps. Some new ideas cannot be advanced incrementally. They are extremely difficult to publish.

Perhaps the most important human idea ever was to climb out of the trees and to live on the ground. It was, no doubt, strongly opposed by the leading experts of tree-climbing who feared for their status. This principle has not changed. The peer review for scientific publications may or may not weed out publications that are below the level of the reviewing peers, but there is no such doubt about the elimination of anything above that level. Bulldozing through this barrier requires good health, long life, great tolerance for abuse, as well as a dedication to the advancement of knowledge.

It is much easier to publish books than journal articles since book publishers must publish to stay in business and books with new ideas are good for their reputation. This is not so for journals of scientific societies financed by membership dues. Dominant journals of large scientific societies are at the forefront of the battle against non-incremental science and the protection of the status of the leading experts of current activities. It would be practically impossible to publish non-incremental science if there were not some editors who understand the limitations of the peer review and make it their life's goal to overcome this barrier. They are among the few who will support ideas that contradict accepted ones. Finding such editors is difficult and time-consuming but essential.

From my own experience I rate the contribution of these editors as important as that of the authors. Science that is not published is no better than science that is not done. We have no way to estimate how much scientific progress is lost because authors were not able to overcome the barrier of the peer review. I want to use this opportunity to thank four editors who have helped me: The late Ladislaus L. Marton (Academic Press), the late Richard B. Schulz (IEEE Transactions on Electromagnetic Compatibility), Peter W. Hawkes (Academic Press), and Myron W. Evans (World Scientific Publishers).

Henning F. Harmuth

This Page Intentionally Left Blank

List of Frequently Used Symbols

$\mathbf{A}_\mathrm{e}$	As/m	electric vector potential
$\mathbf{A}_\mathrm{m}$	Vs/m	magnetic vector potential
$\mathbf{B}$	$\mathrm{Vs/m^2}$	magnetic flux density
c	m/s	299 792 458; velocity of light (definition)
$\mathbf{D}$	$\mathrm{As/m^2}$	electric flux density
d^2	-	$4[(2\pi\kappa)^2+\rho_2^2]$, Eq. (2.2-11)
d_Δ^2	-	see Eq. (3.2-31)
$\mathbf{E}$, E	V/m	electric field strength
e	As	electric charge
$\mathbf{g}_\mathrm{e}$	$\mathrm{A/m^2}$	electric current density
$\mathbf{g}_\mathrm{m}$	$\mathrm{V/m^2}$	magnetic current density
$\mathbf{H}$, H	A/m	magnetic field strength
h	Js	$6.626\,075\,5\times10^{-34}$, Planck's constant
$\hbar=h/2\pi$	Js	$1.054\,572\,7\times10^{-34}$
I_T	-	Eqs. (4.2-44), (5.2-32)
J	$\mathrm{Nms^2}$	mechanical moment of inertia, Eq.(1.4-26)
K	-	$c\tau N_\tau\lvert(\sigma Z-s/Z)\rvert/4\pi$, Eq.(2.2-12)
K_0	-	$(Nc\Delta t/4\pi)\lvert\sigma Z-s/Z\rvert$, Eq.(3.2-35)
$K_{\mathrm{N}/2}$	-	$(N/2)[1-(c\Delta t/2\pi)\lvert\sigma Z-s/Z\rvert$, Eq.(3.2-55)
m	kg	mass
m_0	kg	rest mass
m_mo	$\mathrm{Am^2}$	magnetic dipole moment, Eq.(1.4-26)
N	-	$T/\Delta t$, Eq.(3.1-1)
N_τ	-	T/τ, Eq.(2.1-79)
p	-	Eq.(1.4-12)
q	-	Eq.(1.4-12)
q_m	Vs	hypothetical magnetic charge
q_1, q_2	-	Eq.(2.2-29)
q_3, q_4	-	Eq.(2.2-34)
R	m	Fig.1.1-2
s	V/Am	magnetic conductivity
t	s	time variable
T	s	arbitrarily large but finite time interval
Δt	s	arbitrarily small but finite time interval
v	m/s	velocity
$Z=\mu/c$	V/A	376.730 314; wave impedance of empty space

Continued

α	-	$Ze^2/2h \approxeq 7.297\,535 \times 10^{-3}$, Eq.(1.3-37)
α_{e}	-	$ZecA_{\mathrm{e}}/m_0c^2$, Eq.1.3-37)
$\beta = v/c$	-	Eq.(1.4-17)
β_κ	-	Eq.(5.2-14)
γ_{e}	-	Eq.(1.4-18)
γ_κ	-	$[(2\pi\kappa/N_\tau)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2 - \lambda_1^2]^{1/2}$; Eq.(4.2-33)
$\epsilon = 1/Zc$	As/Vm	$1/\mu c^2$; permittivity
δ_ϵ	-	$\sigma\Delta t/\epsilon$, Eq.(1.2-13)
δ_μ	-	$s\Delta t/\mu$, Eq.(1.2-13)
$\tilde{\Delta}$	-	symbol for difference quotient: $\tilde{\Delta}F/\tilde{\Delta}x$
Δ	-	symbol for finite difference: $x + \Delta x$
ζ	-	$y/c\Delta t$, normalized distance
θ	-	$t/\tau,\ t/\Delta t$, normalized time
κ_0	-	Eq.(5.2-36)
λ_1	-	$ec\tau A_{\mathrm{m}0y}/\hbar$; Eq.(5.1-23)
λ_2	-	Eq. (5.1-23)
λ_3	-	$\phi_{\mathrm{e}0}/cA_{\mathrm{m}0y}$; Eq.(5.1-23)
λ_{a}	-	$\arcsin[0.5(d_\Delta^2 - \rho_1^2)^{1/2}]$, Eq.(3.2-60)
$\mu = Z/c$	Vs/Am	$4\pi \times 10^{-7}$; permeability
$\rho = R/ct$	-	Eq.(1.4-34)
ρ_{e}	As/m^3	electric charge density
ρ_{m}	Vs/m^3	magnetic charge density
ρ_{s}	-	Eqs.(2.1-49), (3.4-1)
ρ_1	-	$c\Delta t(\sigma Z - s/Z) = c^2\Delta t(\sigma\mu + s\epsilon)$, Eq.(3.1-1)
ρ_2^2	-	$(c\Delta t)^2\sigma s$, Eq.(3.1-1)
σ	A/Vm	electric conductivity, Eq.(1.4-19)
τ	s	Eqs.(1.4-12), (1.4-29), (1.4-34)
τ_{mp}	s	Eqs.(1.4-3), (1.4-29), (1.4-34)
τ_{p}	s	Eq.(1.4-7)
ϕ_{e}	V	electric scalar potential
ϕ_{m}	A	magnetic scalar potential
ω^2	-	$s\epsilon/\sigma\mu$

1 Introduction

1.1 Modified Maxwell Equations

Maxwell's equations dominated electrodynamics in the twentieth century like Newton's theory dominated mechanics in the eighteenth and nineteenth centuries. Relativistic mechanics was developed when Newton's mechanics failed at velocities close to the velocity of light. The discovery of this failure required an improvement of experimental techniques far beyond what was available at the time of Newton (1642–1727). No such advance in technology was required to find a basic problem of Maxwell's equations. Strictly theoretical studies led to the discovery that Maxwell's equations generally did not yield solutions that satisfied the causality law. This law was, of course, known at the time of Maxwell. Indeed it is by far the oldest, generally recognized physical law. The Greeks philosophized a great deal about it. The conservation laws of physics were recognized much later, none of them being accepted before 1800.

Two scientists working independently and using different approaches realized the problem of Maxwell's equations with the causality law at about the same time, which gave the result great credibility (Harmuth 1986a, b, c; Hillion[1] 1991; 1992a, b; 1993). Many attempts to derive causal solutions from Maxwell's equations were made from 1986 on, but did not and could not succeed.

Once the problem of Maxwell's equations with the causality law was recognized the question arose what modification would resolve the problem. The addition of a magnetic current density term in 1986 was a strictly pragmatic solution based on mathematics rather than physics. Several years passed before the physical meaning of such a term was understood (Harmuth 1991). It only permitted but did not demand the existence of magnetic monopoles. Rotating magnetic dipoles produce magnetic dipole currents just as rotating electric dipoles produce electric dipole currents, e.g., in a material like Barium-Titanate. Electric dipoles—either induced by an electric field strength or inherent—were always part of Maxwell's equations. They produce electric

[1]Hillion obtained his results earlier than suggested by the dates of his publications, but it was next to impossible to publish anything questioning Maxwell's equations, particularly in the journals of the leading physical societies. The editors Peter W.Hawkes of *Advances in Electronics and Electron Physics* (Academic Press) and the late Richard B.Schulz of *IEEE Transactions on Electromagnetic Compatibility* deserve the credit for upholding the freedom of scientific publishing.

dipole currents either by induced polarization or by orientation polarization.

A group of 15 scientists cooperating under the title "Alpha Foundation" arrived at the same modified Maxwell equations using a quite different approach (Anastasovski et al. 2001).

An unforeseen result of the use of the modified Maxwell equations was the elimination of the divergencies known as *infinite zero-point energy* in quantum field theory (Harmuth, Barrett, Meffert 2001); this problem had plagued quantum electrodynamics since 1930 and could previously be overcome only by renormalization, a method generally considered unsatisfactory.

Let us see how certain physical concepts enter a mathematical theory of a physical system or process. To this end we first spell out the causality law in the following form preferred if the transmission of information is important:

Every effect requires a sufficient cause that occurred a finite time earlier.

The words 'a finite time earlier' serve here two purposes. First, they mean that the effect comes after the cause, which introduces the universally observed distinguished direction of time. Second, the use of the term 'time' makes clear that we deal with a law of physics, not an axiom of mathematics, since the concept of time does not exist in pure mathematics. There are no space and time variables in pure mathematics but complex variables, real variables, random variables, prime variables, etc.

To see how physical laws or conditions can be imposed on a physical process that is described by a partial differential equation or a system of such equations in a coordinate system at rest we note that in this case one must find a function that satisfies three requirements:

1. The function satisfies the partial differential equation(s).
2. The function satisfies an initial condition that holds at a certain time t_0 for all values of the spatial variable(s).
3. The function satisfies a boundary condition that holds at all times t for certain values of the spatial variable(s).

The first condition is a purely mathematical requirement. The second and third condition permit the introduction of physical requirements. If we use initial and boundary conditions with finite energy and momentum we can expect results with finite energy and momentum. It has been shown with the help of group theory that in this case Maxwell's equations will conserve energy, linear momentum, angular momentum, and center of energy (Fushchich and Nikitin 1987, p. 98). Of course, if we use periodic, infinitely extended sinusoidal functions as solutions of Maxwell's equations we have infinite energy and the conservation law of energy becomes meaningless.

A solution that satisfies the causality law requires that the initial condition at the time $t = t_0$ is independent of the boundary condition at the times $t > t_0$. Without this requirement a cause at a time $t > t_0$ could have an effect at the earlier time $t = t_0$. Only the initial and boundary conditions are of interest but not what their processing by the partial differential equation does. The symmetries of the equations have no effect.

The causality law is of little interest for *steady state solutions* that describe typically the transmission of power and energy. The opposite is true for *signal solutions* that describe the transmission of information; the energy is of lesser interest as long as it is sufficient to make the signal detectable. We define a classical electromagnetic signal as an EM wave that is zero before a certain time and has finite energy. Signals are usually represented by field strengths, voltages or currents[2]. All produced or observed waves are of this type. This includes standing and captive waves since they must be excited before they can reach the steady state. Mathematically a signal is represented by a time function that is zero before a certain time and quadratically integrable[3]. If a signal is used as boundary condition we obtain *signal solutions*. They always satisfy both the causality law and the conservation law of energy.

There is a class of partial differential equations that does not permit independent initial and boundary conditions and thus do not have solutions that satisfy the causality law. Hillion (1991; 1992a, b; 1993) showed that Maxwell's equations belong to this class. The other proof for the failure of Maxwell's equations grew out of an attempt to develop a theory of the distortions and the propagation velocity of electromagnetic signals in seawater (Harmuth 1986a, b, c). It turned out that the electric field strength due to an electric excitation as boundary condition could be obtained but the associated magnetic field strength could not. This explains why it took so long to recognize the problem of Maxwell's equations with causality. Anyone satisfied with the electric field strength due to electric excitation never noticed that something was amiss.

We write the modified Maxwell equations in a coordinate system at rest using international units. The old-fashioned notation curl $\equiv \nabla\times$, div $\equiv \nabla\cdot$ and grad $\equiv \nabla$ is used since this notation was used when the lack of causal solutions was discovered; it may be that the notation curl, div and grad helps one grasp the physics underlying the mathematical manipulations:

$$\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{g}_\mathrm{e} \tag{1}$$

$$-\operatorname{curl}\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{g}_\mathrm{m} \tag{2}$$

$$\operatorname{div}\mathbf{D} = \rho_\mathrm{e} \tag{3}$$

$$\operatorname{div}\mathbf{B} = 0 \quad \text{or} \quad \operatorname{div}\mathbf{B} = \rho_\mathrm{m} \tag{4}$$

Here $\mathbf{E}$ and $\mathbf{H}$ stand for the electric and magnetic field strength, $\mathbf{D}$ and $\mathbf{B}$ for the electric and magnetic flux density, $\mathbf{g}_\mathrm{e}$ and $\mathbf{g}_\mathrm{m}$ for the electric and magnetic

[2]In the case of the Aharonov-Bohm effect the signal is represented by the magnetic vector potential (Aharonov-Bohm 1959, 1961, 1962, 1963).

[3]We generally think of signals as functions of time t at a fixed location x, but a signal could also be a function of location x at a fixed time t.

current density, ρ_e and ρ_m for the electric and a possible magnetic charge density. We note that $\mathbf{g}_m$ does not have to be zero if ρ_m is zero, since only a monopole current requires charge densities ρ_e or ρ_m while dipole and higher order multipole currents exist for $\rho_e = 0$ or $\rho_m = 0$.

The modified Maxwell equations differ definitely by the term $\mathbf{g}_m$ in Eq.(2) from the original Maxwell equations. The condition $\operatorname{div} \mathbf{B} = \rho_m$ in Eq.(4) is only permitted but not required. Hence, there is no requirement that magnetic charges must or must not exist.

Equations (1)–(4) are augmented by *constitutive equations* that connect $\mathbf{D}$ with $\mathbf{E}$, $\mathbf{B}$ with $\mathbf{H}$, $\mathbf{g}_e$ with $\mathbf{E}$ and $\mathbf{g}_m$ with $\mathbf{H}$. In the simplest case this connection is provided by scalar constants called permittivity ϵ, permeability μ, electric conductivity σ and magnetic conductivity s. The electric and magnetic conductivities may be *monopole current conductivities* as well as *dipole* or higher order *multipole current conductivities*:

$$\mathbf{D} = \epsilon \mathbf{E} \tag{5}$$
$$\mathbf{B} = \mu \mathbf{H} \tag{6}$$
$$\mathbf{g}_e = \sigma \mathbf{E} \tag{7}$$
$$\mathbf{g}_m = s \mathbf{H} \tag{8}$$

In more complicated cases ϵ, μ, σ and s may vary with location, time, and direction, which requires tensors for their representation or the equations may be replaced by partial differential equations.

The term $\mathbf{g}_m$ in Eq.(2) was originally added strictly for mathematical reasons, although with its physical implications in mind (Harmuth 1986a, b, c). From 1990 on it was understood that this term did not imply the existence of magnetic charges or monopoles. Rotating magnetic dipoles can cause magnetic dipole currents just as rotating electric dipoles in a material like Barium-Titanate can cause electric dipole currents. Hence, the term $\mathbf{g}_m$ in Eq.(2) is now based on the existence of rotating magnetic dipoles, which means on physics rather than mathematics.

The electric current density term $\mathbf{g}_e$ in Eq.(1) has always represented electric monopole currents carried by charges and dipole currents carried by induced or inherent dipoles; the difference between induced and inherent dipoles will be discussed presently. Maxwell called the dipole currents *polarization currents* since today's atomistic thinking did not exist in his time. Without a polarization or dipole current one cannot explain how an electric current can flow through the dielectric of a capacitor, which is an insulator for monopole currents.

For a brief discussion of dipole currents as well as induced and inherent dipoles consider Fig.1.1-1. On the left in Fig.1.1-1a we see a positive and a negative charge carrier between two metal plates, one with positive the other with negative voltage. The charge carriers move toward the plate with opposite

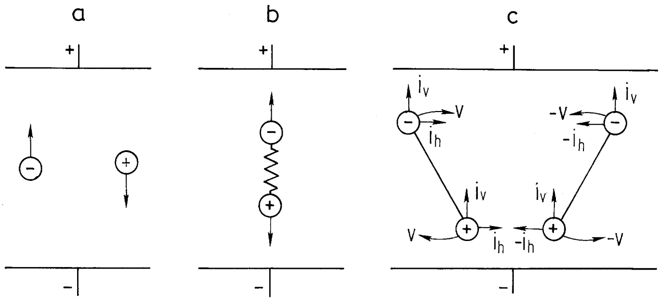

FIGURE 1.1-1. Current carried by independent positive and negative charges (a). Dipole current due to an induced dipole (b). Dipole current due to orientation polarization of inherent dipoles (c).

polarity. An electric monopole current is flowing as long as the charge carriers move.

An induced electric dipole is shown in Fig.1.1-1b. A neutral particle, such as a hydrogen atom, is not pulled in any direction by voltages at the metal plates. Instead, the positive nucleus moves toward the plate with negative voltage and the negative electron toward the plate with positive voltage. A restoring force, symbolized by a coil spring, will pull nucleus and electron together once the voltage at the plates is switched off. An electric dipole current is flowing as long as the two charge carriers are moving either apart or back together again. The lucidity of this simple model is lost if we say that the probability density function for the location of the electron loses its spherical symmetry and is deformed into the shape of an American football with the nucleus off-center in the elongated direction.

An induced dipole current can become a monopole current if the field strength between the metal plates exceeds the ionization field strength. One cannot tell initially whether an induced dipole current will become a monopole current or not, since this depends not only on the magnitude of the field strength but also on its duration. As a result a term in an equation representing an electric dipole current must be so that it can change to a monopole current. Vice versa, a term representing monopole currents must be so that it can change to a dipole current, since two particles with charges of opposite polarity may get close enough to become a neutral particle. The current density term $\mathbf{g}_e$ in Eq.(1) satisfies this requirement.

Most molecules, from H_2O to Barium-Titanate, are inherent dipoles while their atoms are inducible dipoles. Figure 1.1-1c shows two inherent electric dipoles represented by two electric charges with opposite polarity at the ends of rigid rods. A positive and a negative voltage applied to the metal plates will rotate these inherent dipoles to line up with the electric field strength. Dipole currents $2i_v$ in the direction of the field strength are carried by each rotating dipole. There are also dipole currents $2i_h$ perpendicular to the field

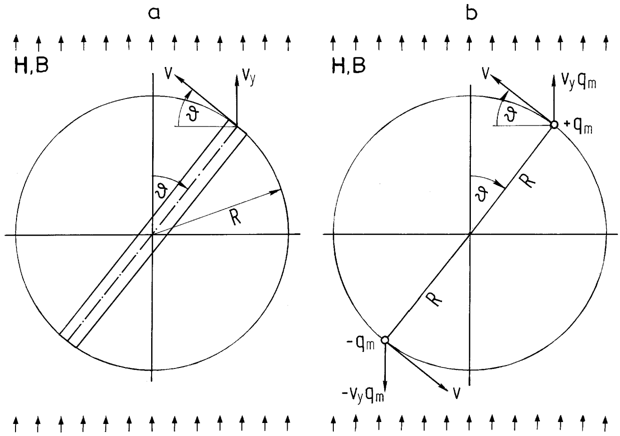

FIGURE 1.1-2. Ferromagnetic bar magnet in an homogeneous magnetic field (a) and its representation by a thin rod with hypothetical magnetic charges $\pm q_{\mathrm{m}}$ at its ends (b).

strength, but they compensate each other if there are counterrotating dipoles as shown. Only the currents in the direction of the field strength will remain observable if there are many dipoles with random orientation. Dipole currents due to *orientation polarization* exist for magnetic dipoles too. Such dipoles range from the hydrogen atom to the magnetic compass needle and the Earth.

Rotating magnetic dipoles behave very much like the rotating electric dipoles of Fig.1.1-1 if the electric quantities are replaced by magnetic quantities. This is made clear by Fig.1.1-2 which shows a ferromagnetic bar magnet on the left and its idealized representation by a thin rod with hypothetical magnetic charges on the right. There is no magnetic equivalent to the induced dipole of Fig.1.1-1b unless we find magnetic charges or monopoles. This explains why magnetic dipole currents are not part of Maxwell's equations. Before the arrival of atomistic thinking it was not possible to distinguish induced polarization due to induced dipoles and orientation polarization due to inherent dipoles. The concept of orientation polarization of inherent dipoles was not recognized and the lack of induced magnetic dipoles eliminated magnetic dipole currents.

In order to include magnetic dipole current densities one must use a term $\mathbf{g}_{\mathrm{m}}$ in Eq.(2). Equation (4) may be left unchanged in the form $\operatorname{div}\mathbf{B} = 0$ of Maxwell's equations for dipole currents, but the zero must be replaced by ρ_m if there are magnetic monopoles. The existence of magnetic monopoles has been a matter of contention for almost 70 years. We do not have an acceptable explanation for the quantization of electric charges without admitting magnetic monopoles, but we do not have a direct and convincing experimental proof for the existence of magnetic monopoles either. Fortunately, the choice $\operatorname{div}\mathbf{B} = 0$ or $\operatorname{div}\mathbf{B} = \rho_{\mathrm{m}}$ has no effect on the covariance of the modified Maxwell equations or the conservation of energy and momentum by these equations (Harmuth, Barrett, Meffert 2001, Sections 2.4 and 2.5).

1.2 Summary of Results in Classical Physics

Let us obtain the field strengths **E** and **H** of the modified Maxwell equations (1.1-1)–(1.1-8) for a planar TEM wave excited by an electric excitation function $E(0,t)$ at the plane $y=0$ and propagating in the direction y. We must write Eqs.(1.1-1)–(1.1-8) in Cartesian coordinates, make all derivates $\partial/\partial x$ and $\partial/\partial z$ zero to obtain the planar wave and choose $E_y=0$, $H_y=0$ to obtain a TEM wave[1]. With the substitutions

$$E = E_x = E_z, \quad H = H_x = -H_z \tag{1}$$

one obtains in the end the following two equations:

$$\partial E/\partial y + \mu\partial H/\partial t + sH = 0 \tag{2}$$
$$\partial H/\partial y + \epsilon\partial E/\partial t + \sigma E = 0 \tag{3}$$

The original Maxwell equations have $\mathbf{g}_{\mathrm{m}} = 0$ in Eq.(1.1-2) and $s=0$ in Eq.(1.1-8). Instead of Eqs.(2) and (3) one obtains:

$$\partial E/\partial y + \mu\partial H/\partial t = 0 \tag{4}$$
$$\partial H/\partial y + \epsilon\partial E/\partial t + \sigma E = 0 \tag{5}$$

The elimination of H from Eqs.(2) and (3) yields the following second order partial differential equation for E alone

$$\partial^2 E/\partial y^2 - \mu\epsilon\partial^2 E/\partial t^2 - (\mu\sigma + \epsilon s)\partial E/\partial t - s\sigma E = 0 \tag{6}$$

while the elimination of H from Eqs.(4) and (5) yields a different partial differential equation:

$$\partial^2 E/\partial y^2 - \mu\epsilon\partial^2 E/\partial t^2 - \mu\sigma\partial E/\partial t = 0 \tag{7}$$

The term $s\sigma E$ makes Eq.(6) significantly different from Eq.(7). Even if we solve Eq.(6) and *then* take the limit $s \to 0$ we may and actually do get a result different from the one provided by Eq.(7). We may say that Eqs.(1.1-1)–(1.1-8) have a singularity for $\mathbf{g}_{\mathrm{m}} = 0$, which prevents us from choosing $\mathbf{g}_{\mathrm{m}} = 0$ at the beginning rather than at the end of the calculation, or we may say the original and the modified Maxwell equations have a different group symmetry. The important fact is that Eqs.(6) and (7) are significantly different.

[1] Harmuth 1986a, b, c; Harmuth and Hussain 1994; Harmuth, Boules and Hussain 1999; Harmuth and Lukin 2000; Harmuth, Barrett and Meffert 2001.

Let us assume the electric field strength $E = E(y,t)$ has been derived from Eq.(6) for certain initial and boundary conditions. We may then obtain the associated magnetic field strength $H(y,t)$ from either Eq.(2) or (3):

$$H(y,t) = -\frac{1}{\mu} e^{-st/\mu} \int \frac{\partial E}{\partial y} e^{st/\mu} dt + H_{\mathrm{t}}(y) e^{-st/\mu} \tag{8}$$

$$H(y,t) = -\int \left(\epsilon \frac{\partial E}{\partial t} + \sigma E \right) dy + H_{\mathrm{y}}(t) \tag{9}$$

The two equations must yield the same value for $H(y,t)$. This requirement determines the integration constants $H_{\mathrm{t}}(y)$ and $H_{\mathrm{y}}(t)$.

If we obtain $E = E(y,t)$ for certain initial and boundary conditions from Eq.(7), derived from the original Maxwell equations, we may substitute it either into Eq.(4) or Eq.(5) to obtain the associated magnetic field strength $H(y,t)$:

$$H(y,t) = -\frac{1}{\mu} \int \frac{\partial E}{\partial y} dt + H_{\mathrm{t}}(y) \tag{10}$$

$$H(y,t) = -\int \left(\epsilon \frac{\partial E}{\partial t} + \sigma E \right) dy + H_{\mathrm{y}}(t) \tag{11}$$

Again, the requirement that both equations must yield the same field strength $H(y,t)$ should determine the integration constants.

We introduce the normalized time and space variables θ and ζ using a time interval Δt that is finite but has no other restrictions:

$$\theta = t/\Delta t, \quad \zeta = y/c\Delta t, \quad \Delta t > 0 \tag{12}$$

Equations (6), (8) and (9) become

$$\begin{gathered} \partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - (\delta_\epsilon + \delta_\mu)\partial E/\partial\theta - \delta_\epsilon\delta_\mu E = 0 \\ \delta_\epsilon + \delta_\mu = c^2\Delta t(\mu\sigma + \epsilon s), \quad \delta_\epsilon\delta_\mu = c^2(\Delta t)^2 s\sigma \\ \delta_\epsilon = \sigma\Delta t/\epsilon, \quad \delta_\mu = s\Delta t/\mu \end{gathered} \tag{13}$$

$$H(\zeta,\theta) = -\frac{1}{Z} e^{-\delta_\mu\theta} \int \frac{\partial E}{\partial\zeta} e^{\delta_\mu\theta} d\theta + H_\theta(\zeta) e^{-\delta_\mu\theta} \tag{14}$$

$$H(\zeta,\theta) = -\frac{1}{Z} \int \left(\frac{\partial E}{\partial\theta} + \delta_\epsilon E \right) d\zeta + H_\zeta(\theta) \tag{15}$$

while Eqs.(7), (10) and (11) assume the form

$$\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - \delta_\epsilon \partial E/\partial\theta = 0, \qquad \delta_\epsilon = \sigma\Delta t/\epsilon \tag{16}$$

$$H(\zeta,\theta) = -\frac{1}{Z}\int \frac{\partial E}{\partial\zeta} d\theta + H_\theta(\zeta) \tag{17}$$

$$H(\zeta,\theta) = -\frac{1}{Z}\int \left(\frac{\partial E}{\partial\theta} + \delta_\epsilon E\right) d\zeta + H_\zeta(\theta) \tag{18}$$

An electric force function with the time variation of a step function is introduced as boundary condition for the solution of Eq.(13):

$$\begin{aligned} E(0,\theta) = E_0 S(\theta) &= 0 \quad \text{for } \theta < 0 \\ &= E_0 \quad \text{for } \theta \geq 0 \end{aligned} \tag{19}$$

As initial condition we choose that the electric field strength shall be zero for all locations $\zeta > 0$ at $\theta = 0$:

$$E(\zeta,0) = 0 \quad \text{for } \zeta > 0 \tag{20}$$

For $\zeta = 0$ the value of $E(0,0)$ is already defined by the boundary condition of Eq.(19).

For the calculation we refer the reader to previous publications[2]. The choice $\Delta t = 2\epsilon/\sigma$ yields $\delta_\epsilon = 2$, $\delta_\mu = 2s\epsilon/\sigma\mu$ and the equations are greatly simplified. At the end of the calculation we make a further simplification by the trasition $s \to 0$, which implies that the magnetic dipole current $\mathbf{g}_\mathrm{m}$ in Eq.(1.1-2) approaches zero. We obtain the following electric field strength for Eq.(13):

$$\begin{aligned} E(\zeta,\theta) = E_0 \Bigg\{ 1 - \frac{2}{\pi} e^{-\theta} \\ \times \Bigg[\int_0^1 \left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}} \right) \frac{\sin\zeta\eta}{\eta}\, d\eta \\ + \int_1^\infty \left(\cos\left[(\eta^2-1)^{1/2}\theta\right] + \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}} \right) \frac{\sin\zeta\eta}{\eta}\, d\eta \Bigg] \Bigg\} \end{aligned} \tag{21}$$

Plots of $E(\zeta,\theta)/E_0$ are shown in Fig.1.2-1 for the locations $\zeta = 0, 1, 2, \ldots, 10$ in the time interval $0 \leq \theta \leq 60$ by the solid lines. The boundary condition of Eq.(19) is represented by the plot for $\zeta = 0$.

[2]Harmuth 1986c, Secs. 2.1, 2.4; Harmuth, Barrett, Meffert 2001, Secs. 1.3, 1.4, 6.1, 6.2.

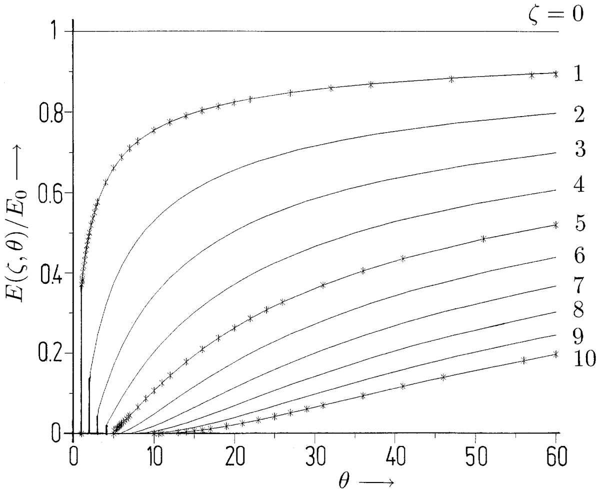

FIGURE 1.2-1. Electric field strengths $E(\zeta, \theta)/E_0$ according to Eq.(21) as function of the normalized time θ with the normalized distance ζ as parameter. The solid lines represent Eq.(21), the stars at $\zeta = 1$, 5, 10 the solutions of Eq.(16).

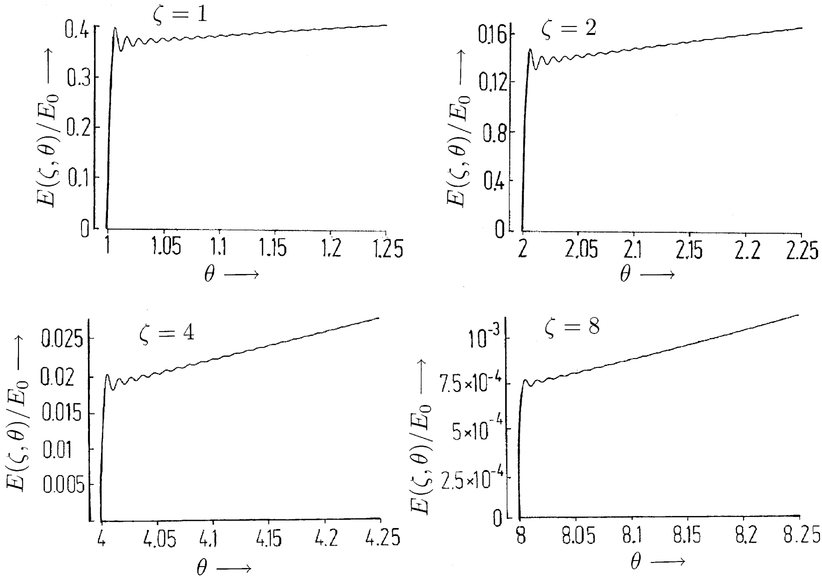

FIGURE 1.2-2. Plots of the electric field strength $E(\zeta, \theta)/E_0$ according to Eq.(21) in the vicinity of $\theta = \zeta$ with a large scale for θ; $\zeta = 1$, 2, 4, 8.

For $E(\zeta, \theta)/E_0$ close to the jumps at $\theta = \zeta$ the same plots are shown for $\zeta = 1$, 2, 4, 8 with much enlarged time scale in Fig.1.2-2. We recognize decaying oscillations for times θ slightly larger than ζ.

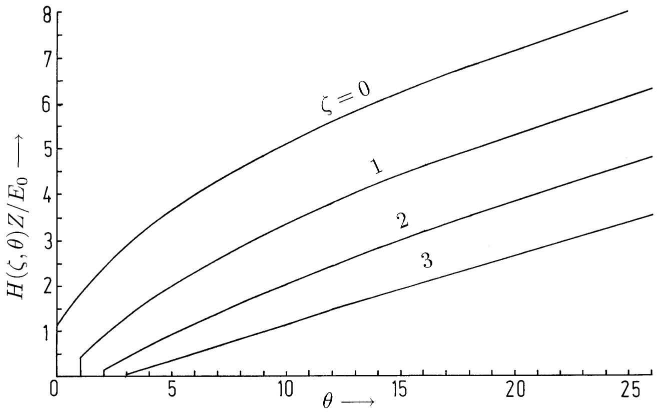

FIGURE 1.2-3. Normalized magnetic field strengths $H(\zeta,\theta)Z/E_0$ associated with the electric field strengths of Fig.1.2-1; $\theta = t/\Delta t = \sigma t/s\epsilon$; $\zeta = y/c\Delta t = yZ/s\sigma$, $s \to 0$.

The solution of Eq.(16) yields the points indicated by stars on the plots for $\zeta = 1$, 5, 10 in Fig.1.2-1. There appears to be perfect equality between the solutions of Eqs.(13) and (16). But this is not so in the vicinity of $\theta = \zeta$. The decaying oscillations of Fig.1.2-2 are not provided by Eq.(16).

We turn to the associated magnetic field strengths of Eqs.(14), (15) and (17), (18). Equations (15) and (18) are equal but Eqs.(14) and (17) are not. As a result, Eq.(14) converges but Eq.(17) does not. Since Eq.(18) converges we get a defined result for $H(\zeta,\theta)$ but Eq.(17) yields an undefined result. Since Eqs.(17) and (18) were derived from the original Maxwell equations we must conclude that these equations have no solution for the step function excitation of Eq.(19) while the modified Maxwell equations have a solution. This result has been generalized from the step excitation to the general excitation[3]

$$\begin{aligned} E(0,\theta) = E_0\theta^n S(\theta) &= 0 && \text{for } \theta < 0 \\ &= E_0\theta^n && \text{for } \theta \le 0, \quad n = 0,1,2,\ldots \end{aligned} \tag{22}$$

Let us emphasize that this result does not depend in any way on the method of solution of the differential equations (13) and (16). Only the derivation of the associated magnetic field strengths $H(\zeta,\theta)$ from the electric field strength $E(\zeta,\theta)$ in Eqs.(14), (15) and (17), (18) is used. Nothing can be gained by developing fancier methods for the solution of the differential equation (16).

Substitution of Eq.(21) differentiated with respect to ζ or θ into Eqs.(14) and (15) yields for $\Delta t = 2\epsilon/\sigma$, $\delta_\mu = 2s\epsilon/\sigma\mu$, and $s \to 0$ an equation for $H(\zeta,\theta)$ that is rather long[4]. Hence, we show in Fig.1.2-3 only plots for $\zeta = 0$, 1, 2, 3 in the time interval $0 \le \theta \le 25$. The plots for $\zeta = 1$, 4 are shown once more close

[3]Harmuth and Hussain 1994, Sec. 1.1; Harmuth, Barrett, Meffert 2001, Sec. 1.4.
[4]Harmuth 1986c, Sec. 2.4; Harmuth, Barrett, Meffert 2001 , Sec. 6.2.

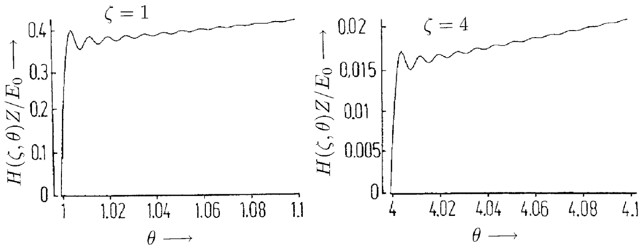

FIGURE 1.2-4. Magnetic field strengths as in Fig.1.2-3 for $\zeta = 1$, 4 but with a much larger time scale in the vicinity of $\theta = \zeta$.

to $\zeta = \theta$ with a much larger time scale in Fig.1.2-4. We see again the decaying oscillations observed in Fig.1.2-2 for the electric field strength $E(\zeta, \theta)$.

The step function excitation of Eq.(19) runs sometimes into problems of poor convergence if particles with mass are interacting with the electromagnetic wave. There is a second distinguished excitation function that rises gradually rather than with a discontinuity and yields better convergence. We call this excitation function the exponential ramp function:

$$\begin{aligned} E(0,\theta) = E_1 S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0 \\ &= E_1(1-e^{-\iota\theta}) && \text{for } \theta \geq 0 \end{aligned} \tag{23}$$

We obtain as solution[5] of the differential equation (13) for an initial condition $E(\zeta, 0) = 0$ and $\Delta t = 2\epsilon/\sigma$, $\delta_\epsilon = 2$, $\delta_\mu = 2s\epsilon/\sigma\mu$, $s \to 0$ the function

$$\begin{aligned} E(\zeta,\theta) = E_1 \Bigg[& \left(1 - e^{-2(1+\omega^2)\theta}\right) e^{-2\omega\zeta} \\ & - \frac{4}{\pi} e^{-\theta} \left(\int_0^1 \frac{\operatorname{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}} \frac{\sin\zeta\eta}{\eta} d\eta + \int_1^\infty \frac{\sin(\eta^2-1)^{1/2}\theta}{(\eta^2-1)^{1/2}} \frac{\sin\zeta\eta}{\eta} d\eta \right) \Bigg] \\ & \omega^2 = \epsilon s/\mu\sigma \end{aligned} \tag{24}$$

Plots of $E(\zeta, \theta)/E_1$ are shown in Fig.1.2-5 for various values of ζ and the time interval $0 \leq \theta \leq 10$. The associated magnetic field strengths are shown in Fig.1.2-6.

The excitation functions $E(0, \theta)$ in Eqs.(19) and (23) are extended to $\theta \to \infty$. This may readily be improved by subtracting an equal function with time delay Θ:

[5] Harmuth 1986c, Secs. 2.3, 2.9; Harmuth, Barrett, Meffert 2001, Secs. 1.5, 6.4, 6.5.

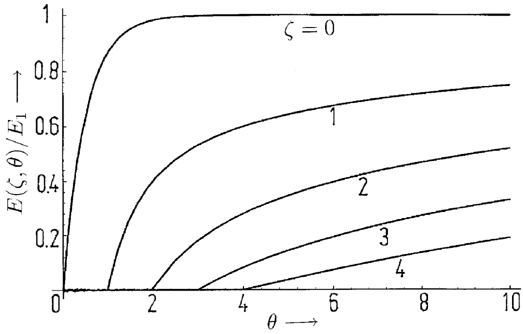

FIGURE 1.2-5. Electric field strengths $E(\zeta,\theta)/E_1$ due to electric exponential ramp function excitation according to Eq.(24) as function of the normalized time θ and with the normalized distance $\zeta = 0, 1, 2, 3, 4$ as parameter.

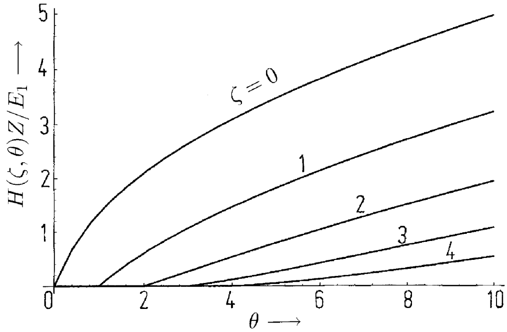

FIGURE 1.2-6. Associated magnetic field strengths $H(\zeta,\theta)Z/E_1$ for the electric field strengths of Fig.1.2-5 as functions of the normalized time θ and with the normalized distance $\zeta = 0$, 1, 2, 3, 4 as parameter.

$$E(0,\theta) = E_0[S(\theta) - S(\theta - \Theta)] \tag{25}$$

$$E(0,\theta) = E_1\left[S(\theta)\left(1 - e^{-\iota\theta}\right) - S(\theta - \Theta)\left(1 - e^{\iota(\theta-\Theta)}\right)\right] \tag{26}$$

The resulting electric field strengths $E(\zeta,\theta)$ are obtained from Eqs.(21) or (24) by subtracting the time-delayed function $E(\zeta,\theta - \Theta)$.

The function of Eq.(25) has a beginning at $\theta = 0$ and an end at $\theta = \Theta$. But Eq.(26) and the equations for $E(\zeta,\theta - \Theta)$ obtained from Eqs.(21) or (24) go from $\theta = 0$ or $\theta = \zeta$ to $\theta \to \infty$. This is of no more physical importance than the infinite time required by a capacitor to discharge through a resistor. However, it is the reason why we defined a signal as an electromagnetic wave that is zero before a finite time and has finite energy rather than a wave that is zero before and after two finite times.

Since Eqs.(21) and (24) would be twice as long if written for the excitation functions of Eqs.(25) and (26) we usually write them for the excitation functions of Eqs.(19) and (23) only. But one should keep in mind that Eqs.(25) and (26) can yield unexpected results[6].

1.3 Basic Relations for Quantum Mechanics

A number of basic relations derived from the modified Maxwell equations will be needed. They are listed here without derivation. References for their derivation are given.

The electric and magnetic field strength in Maxwell's equations are related to a vector potential $\mathbf{A}_{\mathrm{m}}$ and a scalar potential ϕ_{e}:

$$\mathbf{E} = -\frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} - \operatorname{grad} \phi_{\mathrm{e}} \tag{1}$$

$$\mathbf{H} = \frac{c}{Z} \operatorname{curl} \mathbf{A}_{\mathrm{m}} \tag{2}$$

For the modified Maxwell equations we have to add a vector potential $\mathbf{A}_{\mathrm{e}}$ and a scalar potential ϕ_{m}. Equations (1) and (2) are replaced by the following relations[1]:

$$\mathbf{E} = -Zc \operatorname{curl} \mathbf{A}_{\mathrm{e}} - \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} - \operatorname{grad} \phi_{\mathrm{e}} \tag{3}$$

$$\mathbf{H} = \frac{c}{Z} \operatorname{curl} \mathbf{A}_{\mathrm{m}} - \frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} - \operatorname{grad} \phi_{\mathrm{m}} \tag{4}$$

The vector potentials are not completely specified since Eqs.(3) and (4) only define $\operatorname{curl} \mathbf{A}_{\mathrm{e}}$ and $\operatorname{curl} \mathbf{A}_{\mathrm{m}}$. Two additional conditions can be chosen that we call the *extended Lorentz convention*:

$$\operatorname{div} \mathbf{A}_{\mathrm{m}} + \frac{1}{c^2} \frac{\partial \phi_{\mathrm{e}}}{\partial t} = 0 \tag{5}$$

$$\operatorname{div} \mathbf{A}_{\mathrm{e}} + \frac{1}{c^2} \frac{\partial \phi_{\mathrm{m}}}{\partial t} = 0 \tag{6}$$

The potentials of Eq.(3) and (4) then satisfy the following inhomogeneous partial differential equations:

$$\nabla^2 \mathbf{A}_{\mathrm{e}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\mathrm{e}}}{\partial t^2} \equiv \Box \mathbf{A}_{\mathrm{e}} = -\frac{1}{Zc} \mathbf{g}_{\mathrm{m}} \tag{7}$$

$$\nabla^2 \mathbf{A}_{\mathrm{m}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\mathrm{m}}}{\partial t^2} \equiv \Box \mathbf{A}_{\mathrm{m}} = -\frac{Z}{c} \mathbf{g}_{\mathrm{e}} \tag{8}$$

$$\nabla^2 \phi_{\mathrm{e}} - \frac{1}{c^2} \frac{\partial^2 \phi_{\mathrm{e}}}{\partial t^2} \equiv \Box \phi_{\mathrm{e}} = -Zc\rho_{\mathrm{e}} \tag{9}$$

$$\nabla^2 \phi_{\mathrm{m}} - \frac{1}{c^2} \frac{\partial^2 \phi_{\mathrm{m}}}{\partial t^2} \equiv \Box \phi_{\mathrm{m}} = -\frac{c}{Z} \rho_{\mathrm{m}} \tag{10}$$

---

[6]Harmuth, Barrett, Meffert 2001, Sec. 4.6.

[1]Harmuth and Hussain 1994, Sec. 1.7; Harmuth, Berrett, Meffert 2001, Sec. 1.6.

Particular solutions of these partial differential equations may be represented by integrals taken over the whole space. We note that the magnetic charge density $\rho_{\rm m}$ may be always zero, which implies $\phi_{\rm m} \equiv 0$:

$$\mathbf{A}_{\rm e}(x,y,z,t) = \frac{1}{4\pi Zc}\iiint \frac{\mathbf{g}_{\rm m}(\xi,\eta,\zeta,t-r/c)}{r}d\xi\, d\eta\, d\zeta \tag{11}$$

$$\mathbf{A}_{\rm m}(x,y,z,t) = \frac{Z}{4\pi c}\iiint \frac{\mathbf{g}_{\rm e}(\xi,\eta,\zeta,t-r/c)}{r}d\xi\, d\eta\, d\zeta \tag{12}$$

$$\phi_{\rm e}(x,y,z,t) = \frac{Zc}{4\pi}\iiint \frac{\rho_{\rm e}(\xi,\eta,\zeta,t-r/c)}{r}d\xi\, d\eta\, d\zeta \tag{13}$$

$$\phi_{\rm m}(x,y,z,t) = \frac{c}{4\pi Z}\iiint \frac{\rho_{\rm m}(\xi,\eta,\zeta,t-r/c)}{r}d\xi\, d\eta\, d\zeta \tag{14}$$

Here r is the distance between the coordinates ξ, η, ζ of the current and charge densities and the coordinates x, y, z of the potentials:

$$r = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{1/2} \tag{15}$$

If there is no magnetic charge $\rho_{\rm m}$, the scalar potential $\phi_{\rm m}$ drops out; if in addition there are no magnetic dipole current densities $\mathbf{g}_{\rm m}$, the vector potential $\mathbf{A}_{\rm e}$ drops out too. Equations (3) and (4) are then reduced to the conventional Eqs.(1) and (2). In the text before Eq.(1.2-22) we have pointed out that $\mathbf{H}(\zeta,\theta)$ does not have defined values but $\mathbf{E}(\zeta,\theta)$ has. This implies that $\mathbf{A}_{\rm m}$ in Eq.(2) is undefined but $\partial\mathbf{A}_{\rm m}/\partial t$ in Eq.(1) must be defined to yield defined values for $\mathbf{E}(\zeta,\theta)$. Hence, Eqs.(1) and (2) contain a contradiction and cannot be used[2].

We note that only Eqs.(1.1-1)–(1.1-4) are needed to derive Eqs.(3)–(14), the constitutive equations (1.1-5)–(1.1-8) are not used.

The Lagrange function and the Hamilton function shall be needed of a particle with mass m, charge e, and velocity $\mathbf{v}$ in an electromagnetic field. From the Lorentz equation of motion

$$\frac{\partial}{\partial t}(m\mathbf{v}) = e\mathbf{E} + \frac{Ze}{c}\mathbf{v}\times\mathbf{H} \tag{16}$$

one can derive for $v \ll c$ from the original Maxwell equations the Lagrange function[3] $\mathfrak{L}_{\rm M}$:

$$\mathfrak{L}_{\rm M} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + e(-\phi_{\rm e} + A_{{\rm m}x}\dot{x} + A_{{\rm m}y}\dot{y} + A_{{\rm m}z}\dot{z}) \tag{17}$$

The modified Maxwell equations yield a Lagrange function represented by a vector[4] $\boldsymbol{\mathfrak{L}}$:

[2] For a more detailed discussion of this contradiction see Harmuth, Barrett, Meffert 2001, Sec. 3.1.

[3] The subscript M refers to 'Maxwell'.

[4] The subscript c refers to 'correction'.

$$\boldsymbol{\mathfrak{L}} = \boldsymbol{\mathfrak{L}}_{\mathrm{M}} + \boldsymbol{\mathfrak{L}}_{\mathrm{c}} = (\mathfrak{L}_{\mathrm{M}} + \mathfrak{L}_{\mathrm{c}x})\mathbf{e}_x + (\mathfrak{L}_{\mathrm{M}} + \mathfrak{L}_{\mathrm{c}y})\mathbf{e}_y + (\mathfrak{L}_{\mathrm{M}} + \mathfrak{L}_{\mathrm{c}z})\mathbf{e}_z \tag{18}$$

The term $\mathfrak{L}_{\mathrm{M}}$ is the same as in Eq.(17) while $\mathfrak{L}_{\mathrm{c}x}$ is defined by

$$\begin{aligned}\mathfrak{L}_{\mathrm{c}x} = \frac{Ze}{c}\Bigg\{(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\dot{x} + \int\Bigg[-\left(\frac{\partial\phi_{\mathrm{m}}}{\partial z}\dot{y} - \frac{\partial\phi_{\mathrm{m}}}{\partial y}\dot{z}\right) + A_{\mathrm{e}z}\ddot{y} - A_{\mathrm{e}y}\ddot{z}\\ - c^2\left(\frac{\partial A_{\mathrm{e}z}}{\partial y} - \frac{\partial A_{\mathrm{e}y}}{\partial z}\right) + \left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\Bigg]dx\Bigg\}\end{aligned} \tag{19}$$

The terms $\mathfrak{L}_{\mathrm{c}y}$ and $\mathfrak{L}_{\mathrm{c}z}$ are obtained from $\mathfrak{L}_{\mathrm{c}x}$ by the cyclical replacements $x \to y \to z \to x$ and $x \to z \to y \to x$.

The derivatives $\dot{x}$, $\dot{y}$, $\dot{z}$ in Eq.(19) should and can be replaced by the components of the moment $\mathbf{p}$:

$$\mathbf{p} = p_x\mathbf{e}_x + p_y\mathbf{e}_y + p_z\mathbf{e}_z \tag{20}$$

$$p_x = \frac{\partial\mathfrak{L}_x}{\partial\dot{x}} = \frac{\partial(\mathfrak{L}_{\mathrm{m}} + \mathfrak{L}_{\mathrm{c}x})}{\partial\dot{x}} = m\dot{x} + eA_{\mathrm{m}x} + \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z}) \tag{21}$$

$$p_y = \frac{\partial\mathfrak{L}_y}{\partial\dot{y}} = m\dot{y} + eA_{\mathrm{m}y} + \frac{Ze}{c}(A_{\mathrm{e}x}\dot{z} - A_{\mathrm{e}z}\dot{x}) \tag{22}$$

$$p_z = \frac{\partial\mathfrak{L}_z}{\partial\dot{z}} = m\dot{z} + eA_{\mathrm{m}z} + \frac{Ze}{c}(A_{\mathrm{e}y}\dot{x} - A_{\mathrm{e}x}\dot{y}) \tag{23}$$

This is a major effort and we rewrite only the first component of $\mathfrak{L}_{\mathrm{c}x}$, denoted $\mathfrak{L}_{\mathrm{c}x1}$, in this form[5]:

$$\begin{aligned}\mathfrak{L}_{\mathrm{c}x1} = \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\dot{x} = \frac{Ze}{m^2c^2}\Bigg(A_{\mathrm{e}x}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z\\ + \frac{Ze}{mc}\Big\{A_{\mathrm{e}z}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})]_y - A_{\mathrm{e}y}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}]_z\Big\}\Bigg)\\ \times\Bigg[(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_x + \left(\frac{Ze}{mc}\right)^2 A_{\mathrm{e}x}\mathbf{A}_{\mathrm{e}}\cdot(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})\\ + \frac{Ze}{mc}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})]_x\Bigg]\left[1 + \left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\mathrm{e}}^2\right]^{-2}\end{aligned} \tag{24}$$

The Hamilton function $\boldsymbol{\mathfrak{H}}$ derived from the Lagrange function $\boldsymbol{\mathfrak{L}}$ of Eq.(18) is a vector too. If either the energy mc^2 is large compared with the energy due

[5]For the other components see Harmuth, Barrett, Meffert 2001, Sec. 3.2.

to the potential $\mathbf{A}_\mathrm{e}$ or the magnitude of the potential $\mathbf{A}_\mathrm{m}$ is large compared with the magnitude of $\mathbf{A}_\mathrm{e}$ we obtain the following simple equations:

$$\boldsymbol{\mathcal{H}} = \mathcal{H}_x \mathbf{e}_x + \mathcal{H}_y \mathbf{e}_y + \mathcal{H}_z \mathbf{e}_z \tag{25}$$

$$\mathcal{H}_x = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}x} \tag{26}$$

$$\mathcal{H}_y = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}y} \tag{27}$$

$$\mathcal{H}_z = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}z} \tag{28}$$

The terms $(1/2m)(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + e\phi_\mathrm{e}$ equal the conventional one derived from Maxwell's original equations.

If the simplifying assumptions made for the derivation of Eqs.(26)–(28) are not satisfied one obtains the following exact but much more complicated Hamilton function:

$$\boldsymbol{\mathcal{H}} = \frac{1}{2m}\bigg[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + \left(\frac{Ze}{mc}\right)^2 \Big\{2\big[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})\big]^2 + \big[\mathbf{A}_\mathrm{e}\times(\mathbf{p} - e\mathbf{A}_\mathrm{m})\big]^2\Big\}$$
$$+ \left(\frac{Ze}{mc}\right)^4 \mathbf{A}_\mathrm{e}^2\big[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})\big]^2\bigg]\left[1 + \left(\frac{Ze}{mc}\right)^2 \mathbf{A}_\mathrm{e}^2\right]^{-2} + e\phi_\mathrm{e} - \boldsymbol{\mathcal{L}}_\mathrm{c} \tag{29}$$

Terms multiplied by $(ZecA_\mathrm{e}/mc^2)^2$ or $(ZecA_\mathrm{e}/mc^2)^4$ have been added to the simplified terms of Eqs.(26)–(28).

Dropping the simplifying restriction $v \ll c$ we obtain more complicated expressions. In particular, the Hamilton function can be written with the help of series expansions only[6]. The relativistic generalization of the Lagrange function of Eq.(18) is:

$$\boldsymbol{\mathcal{L}} = -m_0c^2(1 - v^2/c^2)^{1/2} + e(-\phi_\mathrm{e} + \mathbf{A}_\mathrm{m}\cdot\mathbf{v}) + \boldsymbol{\mathcal{L}}_\mathrm{c} \tag{30}$$

In analogy to Eqs.(26)–(28) we write first an approximation for the three components of the Hamilton function that holds if the energy due to the potential $\mathbf{A}_\mathrm{e}$ is small compared with the energy $m_0c^2/(1 - v^2/c^2)^{1/2}$ and the magnitude of $\mathbf{A}_\mathrm{e}$ is small compared with the magnitude of $\mathbf{A}_\mathrm{m}$:

$$\mathcal{H}_x = c\big[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\big]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}x} \tag{31}$$

$$\mathcal{H}_y = c\big[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\big]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}y} \tag{32}$$

$$\mathcal{H}_z = c\big[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\big]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}z} \tag{33}$$

[6] Harmuth, Barrett, Meffert 2001, Sec. 3.3.

If we leave out the correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ we have the conventional relativistic Hamilton function for a charged particle in an electromagnetic field, written with three components rather than one. We call these equations the zero order approximation in $\alpha_{\mathrm{e}} = \alpha_{\mathrm{e}}(\mathbf{r},t) = ZecA_{\mathrm{e}}/m_0c^2$. Let us note that α_{e} is a diminsion-free normalization of the magnitude of the potential $\mathbf{A}_{\mathrm{e}}(\mathbf{r},t)$. A first order approximation in α_{e} is provided by the following equations:

$$\mathcal{H}_{\mathrm{x}} = c\big[(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\big]^{1/2}(1+\alpha_{\mathrm{e}}Q) + e\phi_{\mathrm{e}} - \mathcal{L}_{cx} \tag{34}$$

$$\mathcal{H}_{\mathrm{y}} = c\big[(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\big]^{1/2}(1+\alpha_{\mathrm{e}}Q) + e\phi_{\mathrm{e}} - \mathcal{L}_{cy} \tag{35}$$

$$\mathcal{H}_{\mathrm{y}} = c\big[(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\big]^{1/2}(1+\alpha_{\mathrm{e}}Q) + e\phi_{\mathrm{e}} - \mathcal{L}_{cy} \tag{36}$$

$$Q = \frac{1}{m_0^2c^2}\frac{(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2[\mathbf{A}_{\mathrm{e}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]^2}{[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{3/2}\,\mathbf{A}_{\mathrm{e}}^2\,(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2}$$

$$\alpha_{\mathrm{e}} = \alpha_{\mathrm{e}}(\mathbf{r},t) = \frac{ZecA_{\mathrm{e}}}{m_0c^2} = 2\frac{Ze^2}{2h}\frac{h}{m_0c}\frac{A_{\mathrm{e}}}{e} = 2\alpha\frac{\lambda_{\mathrm{C}}A_{\mathrm{e}}}{e}$$

$$\alpha = \frac{Ze^2}{2h} \approxeq 7.297\,535\times10^{-3} \text{ fine structure constant, } \lambda_{\mathrm{C}} = \frac{h}{m_0c}$$

$$\alpha_{\mathrm{e}} = 2.210\times10^5A_{\mathrm{e}} \text{ for electron, } \alpha_{\mathrm{e}} = 1.204\times10^2A_{\mathrm{e}} \text{ for proton} \tag{37}$$

The fine structure constant α is a universal constant of quantum physics. The factor $(h/c)A_{\mathrm{e}}/m_0e$ normalizes the magnitude A_{e} of the potential $\mathbf{A}_{\mathrm{e}}$ by the mass and charge of a particle interacting with the field; the factor h/c makes α_{e} dimension-free.

The correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ are defined by Eq.(19) and the text following it. The first term $\mathcal{L}_{cx1}$ of $\mathcal{L}_{cx}$ is shown by Eq.(24). In first order approximation in α_{e} it becomes:

$$\mathcal{L}_{cx1} = \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\dot{x} = \frac{\alpha_{\mathrm{e}}}{A_{\mathrm{e}}m_0}\frac{A_{\mathrm{e}z}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_z}{[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}} \times\frac{(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_x}{[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}} + O(\alpha_{\mathrm{e}}^2) \tag{38}$$

The good news about this series expansion in α_{e} is that it provides an unlimited number of topics for PhD theses. The bad news is the term Q of Eq.(37).

1.4 Dipole Currents

An electric charge density ρ_e moving with the velocity $\mathbf{v}$ produces a monopole current density $\mathbf{g}_e$:

$$\mathbf{g}_e = \rho_e \mathbf{v} \tag{1}$$

If the monopoles carrying the current have no mass, the current will follow any change of the electric field strength $\mathbf{E}$ instantly and we obtain Ohm's law

$$\mathbf{g}_e = \sigma \mathbf{E} \tag{2}$$

A mass m_0 of the monopoles will change Eq.(2) for $v \ll c$ to

$$\mathbf{g}_e + \tau_{mp} \frac{d\mathbf{g}_e}{dt} = \sigma \mathbf{E} \tag{3}$$

If $\mathbf{E}$ has the time variation of a step function, $\mathbf{E} = \mathbf{E}_0 S(t)$, we get the same time variation for $\mathbf{g}_e$ in Eq.(2) but $\mathbf{g}_e$ according to Eq.(3) becomes

$$\mathbf{g}_e(t) = \sigma \mathbf{E}_0 (1 - e^{-t/\tau_{mp}}) S(t) = \sigma \mathbf{E}_0 (1 - e^{-\theta/p}) S(\theta), \ \theta = t/\tau, \ p = \tau_{mp}/\tau \tag{4}$$

Plots of $\mathbf{g}_e(t)$ according to Eqs.(2) and (3) are shown in Fig.1.4-1.

In addition to the step function $\mathbf{E}_0 S(t)$ we use frequently the exponential ramp function

$$\mathbf{E}(t) = \mathbf{E}_0 (1 - e^{-t/\tau}) S(t) \tag{5}$$

The current density $\mathbf{g}_e$ according to Eq.(3) becomes in this case

$$\begin{aligned} \mathbf{g}_e &= \sigma \mathbf{E}_0 \left[1 - e^{-t/\tau_{mp}} + \frac{\tau}{\tau - \tau_{mp}} \left(e^{-t/\tau_{mp}} - e^{-t/\tau} \right) \right] && \text{for } \tau_{mp} \neq \tau \\ &= \sigma \mathbf{E}_0 \left[1 - \left(1 + \frac{t}{\tau_{mp}} e^{-t/\tau_{mp}} \right) \right], && \text{for } \tau_{mp} = \tau \end{aligned} \tag{6}$$

Plots of the exponential ramp function of Eq.(5) and representative current densities according to Eq.(6) are shown in Fig.1.4-2. The difference between the plot of the field strength $\mathbf{E}(t)/\mathbf{E}_0$ and the plots for the current densities is never as large as in Fig.1.4-1 for the step function.

Equation (3) is replaced for dipoles by the following equation[1]:

$$\mathbf{g}_e + \tau_{mp} \frac{d\mathbf{g}_e}{dt} + \frac{\tau_{mp}}{\tau_p^2} \int \mathbf{g}_e \, dt = \sigma_p \mathbf{E}, \quad \sigma_p = \frac{\rho_e e \tau_{mp}}{m_0} = \frac{N_0 e^2 \tau_{mp}}{m_0} \tag{7}$$

[1] Harmuth, Boules, Hussain 1999, Secs. 1.2–1.6; Harmuth, Barrett, Meffert 2001, Secs. 2.1–2.3.

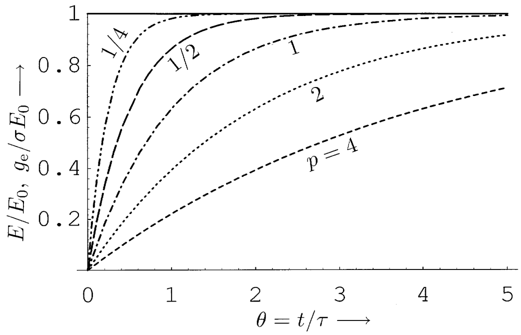

FIGURE 1.4-1. The step function excitation $\mathbf{E}/\mathbf{E}_0 = S(\theta)$ (solid line) and the lagging current densities $\mathbf{g}_e/\sigma\mathbf{E}_0$ according to Eq.(4) due to a finite mass of the charge or current carriers for $p = 1/4,\ 1/2,\ 1,\ 2,\ 4$.

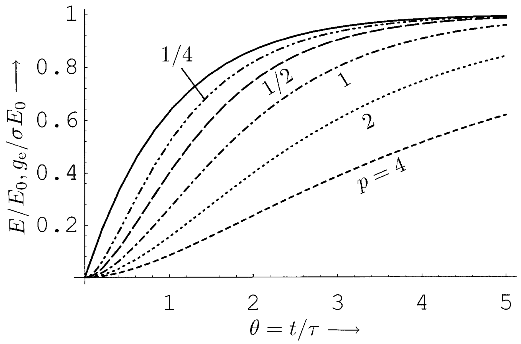

FIGURE 1.4-2. The exponential ramp function excitation according to Eq.(5) (solid line) and current densities $\mathbf{g}_e/\sigma\mathbf{E}_0$ according to Eq.(6) for $p = 1/4,\ 1/2,\ 1,\ 2,\ 4$.

The notation σ_p rather than σ is used to emphasize that it is the conductivity of dipole currents. Excitation of the dipole current $\mathbf{g}_e$ by a step function

$$\mathbf{E}(t) = \frac{2p}{q}\mathbf{E}_0 S(t) \tag{8}$$

with a factor $2p/q$ that will be explained presently yields the following current densities for various values of the parameter $p = \tau_{mp}/\tau_p$:

$$\mathbf{g}_e = 2\sigma_p\mathbf{E}_0\frac{1}{q^2}\theta e^{-\theta/q} \qquad \text{for } p = \frac{1}{2} \tag{9}$$

$$= 2\sigma_p\mathbf{E}_0\frac{p(e^{-\theta/\theta_1} - e^{-\theta/\theta_2})}{q\,(1-4p^2)^{1/2}} \qquad \text{for } p < \frac{1}{2} \tag{10}$$

$$= 2\sigma_p\mathbf{E}_0\frac{2pe^{-\theta/2pq}}{q\,(4p^2-1)^{1/2}}\sin\frac{(4p^2-1)^{1/2}\theta}{2pq} \qquad \text{for } p > \frac{1}{2} \tag{11}$$

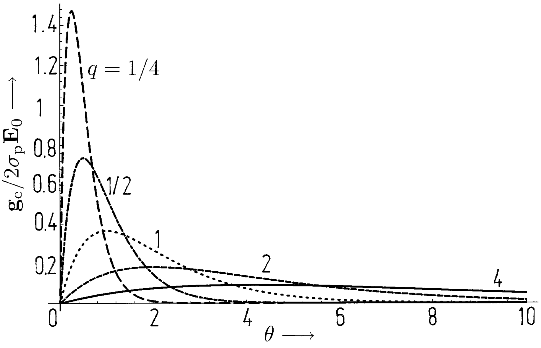

FIGURE 1.4-3. Time variation of dipole current densities according to Eq.(9) for $p = 1/2$ and $q = 1/4$, 1/2, 1, 2 4 in the interval $0 \leq \theta \leq 10$.

$$\theta_1 = q[1 + (1 - 4p^2)^{1/2}]/2p, \quad \theta_2 = q[1 - (1 - 4p^2)^{1/2}]/2p$$
$$\theta = t/\tau, \ q = \tau_\mathrm{p}/\tau, \ p = \tau_\mathrm{mp}/\tau \tag{12}$$

The use of the factor $2p/q$ in Eq.(8) is explained by the three integrals

$$\frac{1}{q^2}\int_0^\infty \theta e^{-\theta/q} d\theta = 1$$
$$\frac{p}{q\,(1-4p^2)^{1/2}}\int_0^\infty \left(e^{-\theta/\theta_1} - e^{-\theta/\theta_2}\right) d\theta = 1$$
$$\frac{2p}{q\,(4p^2-1)^{1/2}}\int_0^\infty e^{-\theta/2pq}\sin\frac{(4p^2-1)^{1/2}\theta}{2pq} d\theta = 1 \tag{13}$$

The same charge will pass through a certain cross section of the path for the current density during the time $0 < t < \infty$.

The time variations of the current densities $\mathbf{g}_\mathrm{e}$ according to Eqs.(9)–(11)are shown in Figs.1.4-3 to 1.4-5 for $p = 1/2$, 1/4, and 1. These are typical time variations of dipole currents while Figs.1.4-1 and 1.4-2 show typical time variations of monopole currents. We see, however, that for $q = 2$, 4 and small values of the time θ the dipole currents vary quite similarly to the monopole currents.

We replace the step function excitation of Eq.(8) by an exponential ramp function excitation in analogy to Eq.(5)

$$\mathbf{E} = \frac{q-2p}{q^2}\mathbf{E}_0(1 - e^{-t/\tau})S(t) = \frac{q-2p}{q^2}\mathbf{E}_0(1 - e^{-\theta/q})S(\theta)$$
$$\theta = t/\tau_\mathrm{p}, \ q = \tau/\tau_\mathrm{p} \tag{14}$$

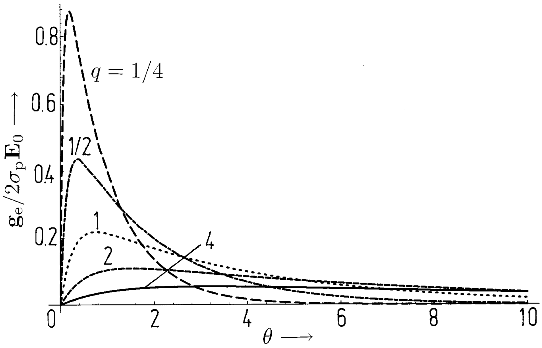

FIGURE 1.4-4. Time variation of dipole current densities according to Eq.(10) for $p = 1/4$ and $q = 1/4$, 1/2, 1, 2, 4 in the interval $0 \leq \theta \leq 10$.

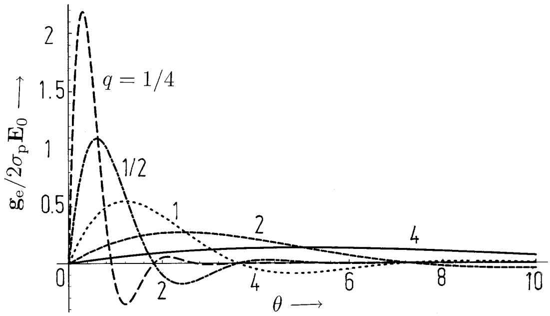

FIGURE 1.4-5. Time variation of dipole current densities according to Eq.(11) for $p = 1$ and $q = 1/4$, 1/2, 1, 2, 4 in the interval $0 \leq \theta \leq 10$.

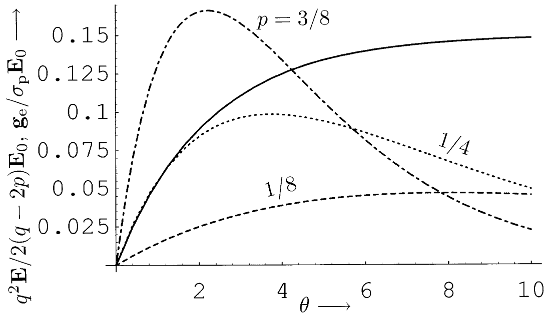

FIGURE 1.4-6. Electric exponential ramp function $q^2\mathbf{E}/2(q-2p)\mathbf{E}_0$ according to Eq.(14) for $p = 3/8$ (solid line) and dipole current densities $\mathbf{g}_e/\sigma_p\mathbf{E}_0$ according to Eq.(16) for $p = 3/8$, 1/4, 1/8.

where the factor $(q-2p)/q^2$ will be explained presently. Substitution of $\mathbf{E}$ into Eq.(7) yields a resonant solution for $\tau = \tau_1$ or $\tau = \tau_2$:

$$\frac{\tau_1}{\tau_\mathrm{p}} = \frac{1}{2p}\left[1+(1-4p^2)^{1/2}\right]$$
$$\frac{\tau_2}{\tau_\mathrm{p}} = \frac{1}{2p}\left[1-(1-4p^2)^{1/2}\right], \quad p = \frac{\tau_\mathrm{mp}}{\tau_\mathrm{p}} \tag{15}$$

We pursue only this resonant solution and restrict it further to values of $p < 1/2$. The current density according to Eq.(7) becomes:

$$\mathbf{g}_\mathrm{e} = \sigma_\mathrm{p}\mathbf{E}_0\frac{1}{q^2}\theta e^{-\theta/q}$$
$$q = \frac{1}{2p}\left[1+(1-4p^2)^{1/2}\right] \quad \text{for } \tau = \tau_1,\ \frac{1}{2p} < q < \frac{1}{p}$$
$$\frac{1}{q^2}\int_0^\infty \theta e^{-\theta/q}d\theta = 1 \tag{16}$$

Figure 1.4-6 shows plots of $q^2\mathbf{E}/2(q-2p)\mathbf{E}_0$ according to Eq.(14) for $p = 3/8$ and $\mathbf{g}_\mathrm{e}/\sigma_\mathrm{p}\mathbf{E}_0$ according to Eq.(16) for various values of p. A comparison with Fig.1.4-2 shows that the dipole current densities vary quite similarly to the monopole current densities for small times, particularly for small values of p. For larger times all the dipole current densities drop to zero while the monopole current densities in Fig.1.4-2 approach a constant value shown there as 1.

For the extension of our results to velocities $\mathbf{v}$ that are not restricted by the condition $v \ll c$ introduced with Eq.(3) we use the notation[2]

$$\beta = \frac{\mathbf{g}_\mathrm{e}}{\mathbf{g}_\mathrm{ec}} = \frac{g_\mathrm{e}}{g_\mathrm{ec}} = \frac{N_0ev}{N_0ec} = \frac{\rho_\mathrm{e}v}{\rho_\mathrm{e}c} = \frac{v}{c} \tag{17}$$

where $\mathbf{g}_\mathrm{ec}$ is a limiting current density of the charge ρ_e if its velocity v approaches the velocity c of light; N_0 is the density of the charge carrier with the charge e. Such a relativistic extension is evidently of interest only if the number of charge carriers or their density is limited. Otherwise a larger current density can be achieved by increasing the density of charge carriers rather than their velocity. Since the current density has always the same direction as the field strength $\mathbf{E}$ in Eq.(3) we may replace the vectors by their magnitudes. The relativistic generalization of Eq.(3) becomes:

[2]Harmuth, Barrett, Meffert 2001, Sec. 2.3; Harmuth, Boules, Hussain 1999, Sec. 1.5.

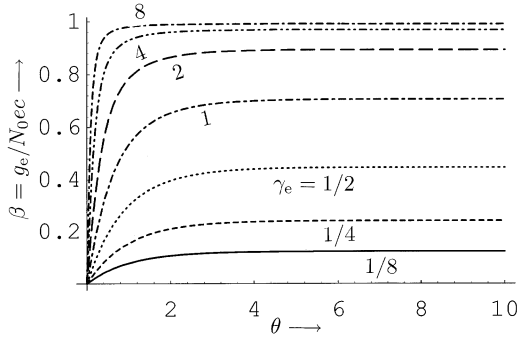

FIGURE 1.4-7. Plots of the normalized electric monopole current density $\beta = g_e/N_0ec$ for $E/E_0 = S(\theta)$, $pq = 1$, and $\gamma_0 = 1/8$, 1/4, 1/2, 1, 2, 4, 8 according to Eq.(18).

$$(1-\beta^2)^{-1/2}\left(\frac{1}{1-\beta^2}\frac{d\beta}{d\theta}+\frac{1}{pq}\beta\right)=\frac{\gamma_e E}{E_0}$$

$$\beta = v/c = N_0ev/N_0ec = \rho_e/\rho_{ec} = g_e/g_{ec},\ g_{ec} = N_0ec = \rho_e c$$

$$\gamma_e = \tau_{mp}eE_0/m_0c,\ q = \tau_p/\tau,\ p = \tau_{mp}/\tau_p,\ pq = \tau_{mp}/\tau,\ \theta = t/\tau \qquad (18)$$

To make the connection of Eq.(18) with Eq.(3) more recognizable we rewrite Eq.(18) for small values of β:

$$\frac{d\beta}{d\theta}+\frac{1}{pq}\beta=\frac{\gamma_e E}{E_0} \quad \text{or} \quad \tau_{mp}\frac{dg_e}{dt}+g_e=\sigma E$$

$$\sigma = \frac{N_0e^2\tau_{mp}}{m_0},\ g_e = \beta N_0ec = N_0ev = \rho_e v \qquad (19)$$

Although the square root in Eq.(18) makes its analytical solution impossible it is not difficult to produce plots of β as functions of time. Such plots are shown in Fig.1.4-7 for excitation by a step function $E/E_0 = S(\theta)$, $pq = 1$, and various values of γ_e. The normalized current density $\beta = g_e/g_{ec}$ never exceeds one.

We are used to thinking of electric dipole currents as the weak currents flowing in the capacitors of radio and television receivers, while monopole currents drive the motors, the water heaters, and the light bulbs. However, very large dipole currents occur in capacitive phase compensators used in electric power distribution grids to combat the phase shift between voltage and current due to inductive loading.

We turn to Eq.(7) for dipole current densities. Here we run into an important difference between monopole and dipole currents. If there is a fixed density N_0 of charge carriers in vacuum, the monopole current can be increased

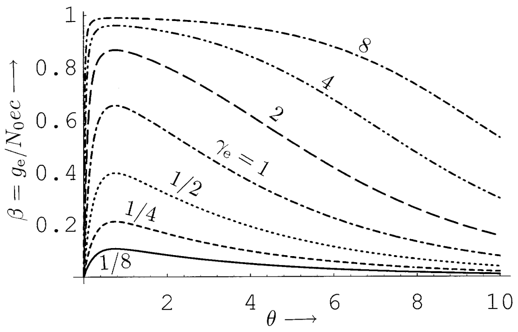

FIGURE 1.4-8. Plots of the normalized electric dipole current density $\beta(\theta)$ according to Eq.(21) for step function excitation, $E = E_0S(\theta)$, $q = 1$, $p = 1/4$, and $\gamma_e = 1/8$, 1/4, 1/2, 1, 2, 4 8.

only by increasing the velocity of the charge carriers. This is different for dipole currents. A fixed density of hydrogen atoms seems to call for an increase of the velocity of the electrons to increase the dipole current density. However, dipole currents can be produced in vacuum by the creation of electric dipoles without violating the conservation of charge. In this case a larger current density can be achieved by the creation of more dipoles rather than by an increase of the velocity of their positive and negative charge components. Only experimental work can decide whether the density of electric dipoles is ever limited.

Let us introduce the normalized time $\theta = t/\tau$ into Eq.(7) and differentiate with respect to θ. The vectors $\mathbf{g}_e$ and $\mathbf{E}$ are replaced by their magnitudes:

$$\frac{d^2g_e}{d\theta^2} + \frac{1}{pq}\frac{dg_e}{d\theta} + \frac{1}{q^2}g_e = \frac{\sigma_p}{pq}\frac{dE}{d\theta}$$
$$p = \tau_{mp}/\tau_p,\ q = \tau_p/\tau,\ pq = \tau_{mp}/\tau,\ \theta = t/\tau \tag{20}$$

Using the definitions of Eq.(18) we obtain the following relativistic form of Eq.(20):

$$(1-\beta^2)^{-1/2}\left[\frac{1}{1-\beta^2}\left(\frac{d^2\beta}{d\theta^2} + \frac{1}{pq}\frac{d\beta}{d\theta}\right) + \frac{3}{(1-\beta^2)^2}\beta\left(\frac{d\beta}{d\theta}\right)^2 + \frac{1}{q^2}\beta\right]$$
$$= \frac{\gamma_e}{pq}\frac{1}{E_0}\frac{dE}{d\theta} \tag{21}$$

For $\beta^2 \to 0$ and $\beta(d\beta/d\theta)^2 \to 0$ the nonrelativistic limit of this equation is obtained and Eq.(20) is regained:

$$\frac{d^2\beta}{d\theta^2} + \frac{1}{pq}\frac{d\beta}{d\theta} + \frac{1}{q^2}\beta = \frac{\gamma_e}{pq}\frac{1}{E_0}\frac{dE}{d\theta} = \frac{\sigma_p}{N_0ecpq}\frac{dE}{d\theta} \tag{22}$$

For the numerical evaluation of Eq.(21) we assume the current density is zero for $\theta = 0$, which yields $\beta(0) = 0$. A second initial condition is needed. We rewrite Eq.(21) for $\beta = 0$

$$\frac{d^2\beta}{d\theta^2} + \frac{1}{pq}\frac{d\beta}{d\theta} = \frac{\gamma_\mathrm{e}}{pq}\frac{1}{E_0}\frac{dE}{d\theta} \tag{23}$$

and integrate:

$$\frac{d\beta}{d\theta} + \frac{1}{pq}\beta = \frac{\gamma_\mathrm{e}}{pq}\frac{1}{E_0}E(\theta) \tag{24}$$

Since $\beta(0)$ is zero due to the first initial condition we get

$$\frac{d\beta(0)}{d\theta} = \frac{\gamma_e}{pq}\frac{1}{E_0}E(\theta), \quad \gamma_\mathrm{e} = \frac{\tau_\mathrm{mp}eE_0}{pq} \tag{25}$$

and a step function excitation $E(\theta) = E_0 S(\theta)$ yields γ_e/pq on the right side of Eq.(25) as the second initial condition. Figure 1.4-8 shows plots of $\beta = g_\mathrm{e}/N_0 ec$ for $q = 1$, $p = 1/4$, and various values of γ_e. The limitation of the current density at $\beta = 1$ is conspicuous.

For—hypothetical—induced magnetic dipoles we get essentially the same equations as for induced electric dipoles. More important are inherent magnetic dipoles as the one shown in Fig.1.1-2. Consider this dipole or ferromagnetic bar magnet of length $2R$ in a homogeneous magnetic field of strength $\mathbf{H}$ and flux density $\mathbf{B}$. We introduce the magnetic dipole moment m_mo with dimension[3] Am^2 and the mechanical moment of inertia J with dimension Nms^2 of the bar magnet. The equation of motion becomes:

$$J\frac{d^2\vartheta}{dt^2} = -m_\mathrm{mo}B\sin\vartheta \tag{26}$$

where ϑ is the angle between the field strength $\mathbf{H}$ and the bar magnet. The velocity of the end points of the bar has the value

$$v(t) = -R\frac{d\vartheta}{dt} \tag{27}$$

which suggests to introduce a velocity dependent attenuation term with the coefficient ξ_m into Eq.(26):

$$J\frac{d^2\vartheta}{dt} + \xi_\mathrm{m}R\frac{d\vartheta}{dt} + m_\mathrm{mo}B\sin\vartheta = 0 \tag{28}$$

[3]If we write $m_\mathrm{mo}\mathbf{B} = m_\mathrm{mo}\mu\mathbf{H}$, the term $m_\mathrm{mo}\mu$ has the dimension Vsm and the symmetry with the electric dipole moments eR [Asm] is maintained, if the electric charge $\pm e$ replaces $\pm q_\mathrm{m}$ in Fig.1.1-2. The product $(eR)\mathbf{E}$ [Asm×V/m] is then in complete analogy to the product $m_\mathrm{mo}\mu\mathbf{H}$ [Vsm×A/m].

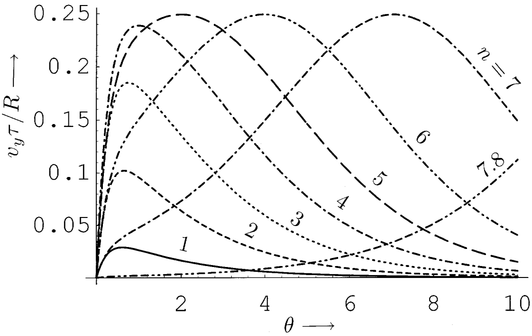

FIGURE 1.4-9. The function $v_y(\theta)\tau/R$ according to Eq.(31) for $p = 1/4$, $q = 1$, and $\vartheta_n = n\pi/8$ with n = 1, 2, 3, 4, 5, 6, 7, 7.8. The plot becomes zero for any finite value of θ for $n = 8$.

The term $\sin\vartheta$ makes an analytical solution of this differential equation generally impossible. Solutions for small angles $\vartheta \doteq \sin\vartheta$ were already studied by Gauss. The computer enables us to represent solutions by plots. We rewrite Eq.(28) in normalized form:

$$\frac{d^2\vartheta}{d\theta^2} + \frac{1}{pq}\frac{d\vartheta}{dt} + \frac{1}{q^2}\sin\vartheta = 0$$

$$\theta = \frac{t}{\tau},\ q = \frac{1}{\tau}\sqrt{\frac{J}{m_{\mathrm{mo}}B}} = \frac{\tau_{\mathrm{p}}}{\tau},\ p = \frac{J}{\xi_{\mathrm{m}}R\tau_{\mathrm{p}}} = \frac{\sqrt{Jm_{\mathrm{mo}}B}}{\xi_{\mathrm{m}}R} = \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}}$$

$$\tau_{\mathrm{p}} = \sqrt{J/m_{\mathrm{mo}}B},\ \tau_{\mathrm{mp}} = J/\xi_{\mathrm{m}}R,\ pq = \tau_{\mathrm{mp}}/\tau = J/\xi_{\mathrm{m}}R\tau \tag{29}$$

This differential equation can be solved numerically for the initial condition $\vartheta(0) = n\vartheta_0 = \vartheta_n$ and $d\vartheta(0)/d\theta = 0$. The numerical values obtained for $\vartheta(\theta)$ and $d\vartheta/d\theta$ may then be used to calculate the velocities $v(\theta)$ and $v_y(\theta)$ of Fig.1.1-2:

$$v(\theta) = -R\frac{d\vartheta}{dt} = -\frac{R}{\tau}\frac{d\vartheta}{d\theta} \tag{30}$$

$$v_y(\theta) = v(\theta)\sin\vartheta = -\frac{R}{\tau}\frac{d\vartheta}{d\theta}\sin\vartheta \tag{31}$$

Plots of $v_y(\theta)$ are shown in Fig.1.4-9 for $p = 1/2$, $q = 1$, and various values of ϑ_n.

In order to connect the velocity $v(t)$ with the current density $\mathbf{g}_{\mathrm{m}}(t)$ of a magnetic dipole current we replace the bar magnet in Fig.1.1-2a by a thin rod with fictitious magnetic charges $\pm q_{\mathrm{m}}$ at its ends as shown in Fig.1.1-2b. The magnetic dipole moment m_{mo} equals $2q_{\mathrm{m}}R$. The charge q_{m} must be connected with the magnetic dipole moment by the relation

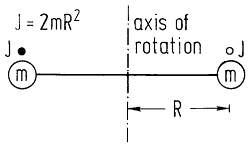

FIGURE 1.4-10. Dumbbell model of a rotating dipole with two masses m at the ends of a thin rod of length $2R$.

$$q_{\mathrm{m}}\,[\mathrm{Vs}] = \frac{\mu m_{\mathrm{mo}}}{2R} \left[\frac{\mathrm{Vs}}{\mathrm{Am}}\frac{\mathrm{Am}^2}{\mathrm{m}}\right] \tag{32}$$

where μ is the magnetic permeability, to obtain the dimension of q_{m}. The magnetic dipole current $2q_{\mathrm{m}}v_y(t) = \mu m_{\mathrm{mo}}v_y(t)/R$ is produced by such a bar magnet.

Just as in the case of the electric dipole we do not know whether a relativistic limitation exists for magnetic dipole currents since there is no conservation law for magnetic dipoles either. For a relativistic extension of Eq.(29) we must specify the moment of inertia J in more detail. We assume the bar magnet can be represented by a dumbbell shown in Fig.1.4-10 with the masses m at the end of a thin rod of length $2R$. Substitution of $J = 2mR^2$ into Eq.(28) yields:

$$2mR^2\frac{d^2\vartheta}{dt^2} + \xi_{\mathrm{m}}R\frac{d\vartheta}{dt} + m_{\mathrm{mo}}B\sin\vartheta = 0 \tag{33}$$

The relativistic generalization of this equation has the following form[4]:

$$\left[1-\rho^2\left(\frac{d\vartheta}{d\theta}\right)^2\right]^{-1/2}\left(\frac{1}{1-\rho^2\left(d\vartheta/d\theta\right)^2}\frac{d^2\vartheta}{d\theta^2} + \frac{1}{pq}\frac{d\vartheta}{d\theta}\right) + \frac{1}{q^2}\sin\vartheta = 0$$

$$\theta = \frac{t}{\tau},\ \rho = \frac{R}{c\tau},\ q = \frac{R}{\tau}\sqrt{\frac{2m_0}{m_{\mathrm{mo}}B}} = \frac{\tau_{\mathrm{p}}}{\tau},\ p = \frac{\sqrt{2m_0m_{\mathrm{mo}}B}}{\xi_{\mathrm{m}}} = \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}}$$

$$pq = \frac{\tau_{\mathrm{mp}}}{\tau} = \frac{2Rm_0}{\tau\xi_{\mathrm{m}}},\ \tau_{\mathrm{p}} = R\sqrt{\frac{2m_0}{m_{\mathrm{mo}}B}},\ \tau_{\mathrm{mp}} = \frac{2m_0R}{\xi_{\mathrm{m}}} \tag{34}$$

For $\rho^2(d\vartheta/d\theta)^2 \to 0$ one obtains from Eq.(34) the nonrelativistic limit of Eq.(29).

The initial conditions of Eq.(34) are $\vartheta(\theta) = n\vartheta_0 = \vartheta_n$ and $d\vartheta(0)/d\theta = 0$ just as for Eq.(29). The velocities $v(\theta)$ and $v_y(\theta)$ of Eqs.(30) and (31) become:

$$\frac{v(\theta)}{c} = -\frac{R}{c\tau}\frac{d\vartheta}{\theta} = -\rho\frac{d\vartheta}{d\theta} \tag{35}$$

$$\frac{v_y(\theta)}{c} = -\rho\frac{d\vartheta}{d\theta}\sin\vartheta \tag{36}$$

[4]Harmuth, Boules, Hussain 1999, Sec. 1.6; Harmuth, Barrett, Meffert 2001, Sec. 2.3.

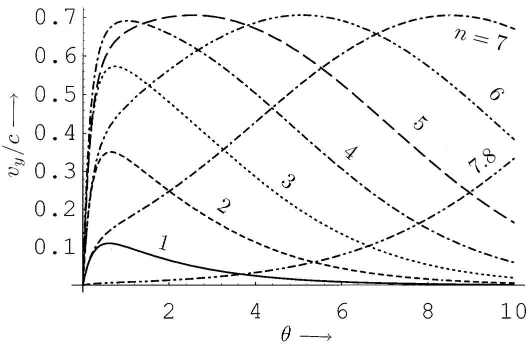

FIGURE 1.4-11. The normalized velocity $v_y(\theta)/c$ according to Eq.(36) for $p = 1/4$, $q = 1$, $\rho = 4$, and $\vartheta(0) = \vartheta_n = n\pi/8$ with $n = 1, 2, 3, 4, 5, 6, 7, 7.8$.

Plots of $v_y(\theta)/c$ are shown in Fig.1.4-11 for $p = 1/4$, $q = 1$ or $\tau = \tau_{\rm p}$, $\rho = 4$, and various values of $\vartheta(0) = \vartheta_n = n\pi/8$. The relativistic limitation $v/c < 1$ is not conspicuous but a comparison with Fig.1.4-9 shows how the peaks of the plots for $n = 5, 6, 7$ have been flattened.

1.5 Infinitesimal and Finite Differences for Space and Time

The discussion of finite or infinite divisibility of space and time has been going on for some 2500 years. Zeno of Elea (c. 490 – c. 430 B.C.) advanced the paradoxes of the race between Achilles and the turtle or the arrow which cannot fly that were supposed to show that infinite divisibility of space and time was not possible. A quote from Aristotle (384 – 322 B.C.) shows that infinite divisibility and thus the concept of the continuum was a matter of discussion before he wrote his *Physica*:

> Now a motion is thought to be one of those things which are continuous, and it is in the continuous that the infinite first appears; and for this reason, it often happens that those who define the continuous use the formula of the infinite, that is, they say that the continuous is that which is infinitely divisible [Apostle 1969, Book III (Γ), 1, § 2].

Aristotle argued the concept of the continuum for space, time and motion so convincingly[1] that it does not seem to have been challenged until Max Planck introduced quantum mechanics. Newton (1971) took this concept apparently so much for granted that he did not even mention it, even though he was very meticulous in listing and elaborating his assumptions. The differential calculus of Leibnitz and Newton made us carry the concept of infinite divisible one step further since we distinguish now between dividing a finite interval ΔX into denumerable or non-denumerable many intervals.

[1]Book V(E) 3, § 5 defines continuous, Book VI(Z) 1, 2 elaborates the concept of the continuum further; Zeno of Elea is refuted in Book VI(Z) 2 and 9 (Apostle 1969, Aristotle 1930).

A widely held view of a space-continuum is summed up in a quote by Weyl that emphasizes that this concept came from mathematics:

> From the essence of space remains in the hands of the mathematician, using such abstraction, only one truth: that it is a *three-dimensional continuum* (Weyl 1921; 1968, vol. II, p. 213).

The following two quotes give a good summary of our currently accepted thinking about space and time:

> So let us conclude that space has a definite real intrinsic structure in its metric, affinity, and topology. This means it has a shape and size in a way I have tried at length to make clear. It shows just how much space is a particular thing (Nehrlich 1976, p. 211).

> It is now generally taken for granted that public time is both infinitely divisible, or "dense" as the mathematician terms it, and continuous; that is, not only can we always consider any interval as made up of smaller ones, but we are entitled to apply even irrational numbers to the measurement of time ... Our concepts are not immune to revision; and in the case of time, we are already prepared, in some locations, to speak of it as though it were discrete. But to do so consistently would require a fairly radical revision of the concept. We should have to unthink as far back as Aristotle (Lucas 1973, pp. 29, 32, 33).

We avoid all questions of how spatial and time distances can be divided indefinitely since we never find a hint how such a division can be carried out experimentally or can be observed. Instead we replace the differentials dx, dt by arbitrarily small but finite differences Δx, Δt. They can always be equal or even smaller than the shortest observable distance. The difference between a finite distance of 10^{-100} m and an infinitesimal distance dx is not directly observable. The question arises whether such small values of Δx would not have to yield the same results as differentials dx. This question was answered by Hölder (1887) who showed that difference equations and differential equations define different classes of functions. In particular the Gamma function satisfies the algebraic difference equation

$$\Gamma(X+1) = X\Gamma(X), \quad X = x/\Delta x \tag{1}$$

but no algebraic differential equation.

The example of the Gamma function shows that difference equations can define continuous and differentiable functions. The use of finite differences Δx, Δt does not imply that only discrete functions defined at integer multiples of Δx and Δt can be obtained as solution of a partial difference equation.

Although this result may be of limited interest to the practical physicist it contributes to the philosophy of science. Our inability to make observations at x and $x + dx$ or t and $t + dt$ prevents any proof that there is a physical space-time continuum, but it also prevents a proof that there is NO physical space-time continuum. The question whether there is or is not a space-time continuum is no more answerable within the confines of a science based on observation than the question how many angels can dance on the point of a needle. We can use mathematics as a tool in physics, but we cannot use it as a source of concepts that are beyond observation.

Let us take one more step in the direction of philosophy of science and quote from Einstein and Infeld (1938, p. 311):

> The psychological subjective feeling of time enables us to order our impressions, to state that one event precedes another. But to connect every instant of time with a number, by the use of a clock, to regard time as an one-dimensional continuum, is already an invention. So also are the concepts of Euclidean and non-Euclidean geometry, and our space understood as a three-dimensional continuum.

When we use differential calculus and then permit this mathematical method to define physical space and space-time we make physics a special branch of mathematics. Since mathematics is a science of the thinkable while physics is a science of the observable, mathematics can never be more than a tool in physics or provide inspiration. The succesful solution of a physical problem by means of differential calculus only implies that the assumption of a mathematical continuum can yield results that correspond with physical observation, it does not imply the existence of a physical space-time continuum.

Let us see how this principle works for finite differences. We know from observation that we can only resolve finite space and time differences x and $x + \Delta x$ or t and $t + \Delta t$. If we look what mathematical tools are available that satisfy this requirement we find the calculus of finite differences. This calculus does not define any physical concept of space or space-time. It works for continuous functions, like the Gamma function, but does not suggest any particular topology of space or space-time. We could go one step further and require that x and t are integer multiples of Δx and Δt: $x = n\Delta x$, $t = m\Delta t$. If we did this we would introduce a cellular space or space-time into physics; we would repeat the mistake we made with differential calculus and the space-time continuum. There is no physical reason to do so.

The use of the calculus of finite differences reduces the concepts of space and space-time to coordinate systems and moving coordinate systems, which are obviously human constructs in line with the quote of Einstein and Infeld above. This is discussed in some detail in a book by Harmuth (1989). Long term the most important result of the use of the calculus of finite differences may be that it removes the mathematical concepts imposed on physics by the differential calculus. This would free our thinking to search for observations that can be associated with physical concepts of space and time, other than distances relative to a standard distance such as the meter and changes relative to a standard change, such as the one produced by a clock.

We shall forgo philosophy from here on and concentrate on more practical problems. The partial differential equations of the pure radiation field and the Klein-Gordon field will be replaced by difference equations. Solutions will be derived and represented by computer plots that show definite deviations of the results derived from the difference equations and the differential equations. The deviations will be small, but quantum electrodynamics is known to produce results corresponding very closely to observation. Large deviations would make one doubt the results of the difference theory. In Section 3.6 we will see that a basic difference equation of quantum electrodynamics yields the same (energy) eigenvalues as the corresponding differential equation but significantly different eigenfunctions.

2 Differential Equations for the Pure Radiation Field

2.1 Pure Radiation Field

Equations (1.3-7)–(1.3-10) define the potentials by means of the current and charge densities. We specify that there shall be neither electric nor magnetic charge densities ρ_e and ρ_m. According to Eqs.(1.3-13) and (1.3-14) we get:

$$\phi_\mathrm{e}(x,y,z,t) \equiv 0, \ \ \phi_\mathrm{m}(x,y,z,t) \equiv 0 \tag{1}$$

Only Eqs.(1.3-7) and (1.3-8) remain:

$$\nabla^2\mathbf{A}_\mathrm{e} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{e}}{\partial t^2} = -\frac{1}{Zc}\mathbf{g}_\mathrm{m} \tag{2}$$

$$\nabla^2\mathbf{A}_\mathrm{m} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{m}}{\partial t^2} = -\frac{Z}{c}\mathbf{g}_\mathrm{e} \tag{3}$$

Since we excluded charge densities the current densities $\mathbf{g}_\mathrm{m}$ and $\mathbf{g}_\mathrm{e}$ can refer to dipole or higher order multipole currents only. The absence of monopole currents and charges is a characteristic feature of a pure radiation field in vacuum. We derived Eq.(1.4-7) for electric dipole currents due to induced dipoles:

$$\mathbf{g}_\mathrm{e} + \tau_\mathrm{mp}\frac{d\mathbf{g}_\mathrm{e}}{dt} + \frac{\tau_\mathrm{mp}}{\tau_\mathrm{p}^2}\int \mathbf{g}_\mathrm{e}\,dt = \sigma_\mathrm{p}\mathbf{E} \tag{4}$$

We have shown in Figs.1.4-1 to 1.4-6 that for small values of $\theta = t/\tau$, small values of $p = \tau_\mathrm{mp}/\tau$, and large values of $q = \tau_\mathrm{p}/\tau$ one may approximate the dipole current of Eq.(4) by a monopole current defined in the simplest case by Ohm's law of Eq.(1.4-2):

$$\mathbf{g}_\mathrm{e} = \sigma\mathbf{E} \tag{5}$$

The situation is much worse if we want analytical results that apply to magnetic dipole currents. Equation (1.4-29) and equations derived from it can be used for numerical solutions, as we did in Figs.1.4-9 and 1.4-10, but the term $\sin\vartheta$

makes it useless for analytical solutions. In order to obtain some theoretical understanding we have little choice but to assume induced magnetic dipoles and look for results where the effect of magnetic dipoles is small compared with the effect of the electric dipoles. Exact results must be left to numerical evaluations. If we use an approximation of magnetic dipole current densities in analogy to Eq.(5)

$$\mathbf{g}_{\mathrm{m}} = s\mathbf{H} \tag{6}$$

we obtain from Eqs.(1.3-3) and (1.3-4) the following result:

$$\mathbf{g}_{\mathrm{e}} = \sigma\mathbf{E} = -\sigma\left(Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} + \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right) \tag{7}$$

$$\mathbf{g}_{\mathrm{m}} = s\mathbf{H} = s\left(\frac{c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}} - \frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right) \tag{8}$$

The current densities in Eqs.(2) and (3) can be eliminated. Two equations containing the vector potentials $\mathbf{A}_{\mathrm{m}}$ and $\mathbf{A}_{\mathrm{e}}$ only are obtained:

$$\nabla^2\mathbf{A}_{\mathrm{e}} - \frac{1}{c}\frac{\partial^2\mathbf{A}_{\mathrm{e}}}{\partial t^2} + \frac{s}{Zc}\left(\frac{c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}} - \frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right) = 0 \tag{9}$$

$$\nabla^2\mathbf{A}_{\mathrm{m}} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\mathrm{m}}}{\partial t^2} - \frac{Z\sigma}{c}\left(Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} + \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right) = 0 \tag{10}$$

The potentials $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$ are connected in these two equations even in the limit $s \to 0$, with which we may eliminate the induced magnetic dipoles at the end of the calculation. On the other hand, the assumption that the current densities $\mathbf{g}_{\mathrm{m}}$ and $\mathbf{g}_{\mathrm{e}}$ in Eqs.(1.3-7) and (1.3-8) are zero leads to $\sigma = s = 0$ in Eqs.(9) and (10), which eliminates any connection between $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$. This would lead to separate electric and magnetic theories with features such as independent electric and magnetic photons.

We want to derive solutions of Eqs.(7) and (8) for planar waves. To this end we write the vectors $\mathbf{A}$, $\mathrm{curl}\,\mathbf{A}$ and $\nabla^2\mathbf{A}$ in Cartesian coordinates using the unit vector $\mathbf{e}_{\mathrm{x}}$, $\mathbf{e}_{\mathrm{y}}$ and $\mathbf{e}_{\mathrm{z}}$:

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z \tag{11}$$

$$\mathrm{curl}\,\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z \tag{12}$$

$$\begin{aligned}\nabla^2\mathbf{A} = {}& \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\mathbf{e}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\mathbf{e}_y \\ & + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\mathbf{e}_z\end{aligned} \tag{13}$$

These relations permit us to write Eqs.(9) and (10) in Cartesian coordinates:

$$\frac{\partial^2 A_{\mathrm{e}x}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}x}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}x}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}z}}{\partial y}-\frac{\partial A_{\mathrm{m}y}}{\partial z}\right)-\frac{\partial A_{\mathrm{e}x}}{\partial t}\right]=0 \tag{14}$$

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}y}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z}-\frac{\partial A_{\mathrm{m}z}}{\partial x}\right)-\frac{\partial A_{\mathrm{e}y}}{\partial t}\right]=0 \tag{15}$$

$$\frac{\partial^2 A_{\mathrm{e}z}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}z}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}z}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}y}}{\partial x}-\frac{\partial A_{\mathrm{m}x}}{\partial y}\right)-\frac{\partial A_{\mathrm{e}z}}{\partial t}\right]=0 \tag{16}$$

$$\frac{\partial^2 A_{\mathrm{m}x}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}x}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}x}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{\mathrm{e}z}}{\partial y}-\frac{\partial A_{\mathrm{e}y}}{\partial z}\right)+\frac{\partial A_{\mathrm{m}x}}{\partial t}\right]=0 \tag{17}$$

$$\frac{\partial^2 A_{\mathrm{m}y}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}y}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{\mathrm{e}x}}{\partial z}-\frac{\partial A_{\mathrm{e}z}}{\partial x}\right)+\frac{\partial A_{\mathrm{m}y}}{\partial t}\right]=0 \tag{18}$$

$$\frac{\partial^2 A_{\mathrm{m}z}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}z}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}z}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}z}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{\mathrm{e}y}}{\partial x}-\frac{\partial A_{\mathrm{e}x}}{\partial y}\right)+\frac{\partial A_{\mathrm{m}z}}{\partial t}\right]=0 \tag{19}$$

These equations are simplified for a planar wave propagating in the direction of y. All derivatives with respect to x and z must be zero:

$$\frac{\partial A_{\mathrm{e}x}}{\partial x}=\frac{\partial A_{\mathrm{e}y}}{\partial x}=\frac{\partial A_{\mathrm{e}z}}{\partial x}=\frac{\partial A_{\mathrm{e}x}}{\partial z}=\frac{\partial A_{\mathrm{e}y}}{\partial z}=\frac{\partial A_{\mathrm{e}z}}{\partial z}=0 \tag{20}$$

$$\frac{\partial A_{\mathrm{m}x}}{\partial x}=\frac{\partial A_{\mathrm{m}y}}{\partial x}=\frac{\partial A_{\mathrm{m}z}}{\partial x}=\frac{\partial A_{\mathrm{m}x}}{\partial z}=\frac{\partial A_{\mathrm{m}y}}{\partial z}=\frac{\partial A_{\mathrm{m}z}}{\partial z}=0 \tag{21}$$

Equations (14)–(19) are reduced to the following form:

$$\frac{\partial^2 A_{\mathrm{e}x}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2} + \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{\mathrm{m}z}}{\partial y} - \frac{\partial A_{\mathrm{e}x}}{\partial t}\right) = 0 \tag{22}$$

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2} - \frac{s}{Zc}\frac{\partial A_{\mathrm{e}y}}{\partial t} = 0 \tag{23}$$

$$\frac{\partial^2 A_{\mathrm{e}z}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{\mathrm{m}x}}{\partial y} + \frac{\partial A_{\mathrm{e}z}}{\partial t}\right) = 0 \tag{24}$$

$$\frac{\partial^2 A_{\mathrm{m}x}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2} - \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{\mathrm{e}z}}{\partial y} + \frac{\partial A_{\mathrm{m}x}}{\partial t}\right) = 0 \tag{25}$$

$$\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2} - \frac{\sigma Z}{c}\frac{\partial A_{\mathrm{m}y}}{\partial t} = 0 \tag{26}$$

$$\frac{\partial^2 A_{\mathrm{m}z}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}z}}{\partial t^2} + \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{\mathrm{e}x}}{\partial y} - \frac{\partial A_{\mathrm{m}z}}{\partial t}\right) = 0 \tag{27}$$

A further simplification is achieved by specifying a transverse electromagnetic (TEM) wave with the components E_y and H_y equal to zero. The following relations for the potentials $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$ are obtained in this case from Eqs.(1.3-3) and (1.3-4) for $\phi_{\mathrm{e}} = \phi_{\mathrm{m}} = 0$:

$$E_y = -Zc\left(\frac{\partial A_{\mathrm{e}x}}{\partial z} - \frac{\partial A_{\mathrm{e}z}}{\partial x}\right) - \frac{\partial A_{\mathrm{m}y}}{\partial t} = 0 \tag{28}$$

$$H_y = \frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z} - \frac{\partial A_{\mathrm{m}z}}{\partial x}\right) - \frac{\partial A_{\mathrm{e}y}}{\partial t} = 0 \tag{29}$$

These two equations are simplified with the help of Eqs.(20) and (21):

$$\frac{\partial A_{\mathrm{m}y}}{\partial t} = 0, \quad \frac{\partial A_{\mathrm{e}y}}{\partial t} = 0 \tag{30}$$

Substitution into Eqs.(23) and (24) brings

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2} = 0 \tag{31}$$

$$\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2} = 0 \tag{32}$$

since the vanishing of the first partial derivatives of Eq.(30) does not mean that the second partial derivatives with respect to time must be zero.

The following substitution brings a further simplification; the subscript v alludes to 'variable':

$$A_{\mathrm{e}x} = A_{\mathrm{e}z} = A_{\mathrm{ev}}, \qquad A_{\mathrm{m}x} = -A_{\mathrm{m}z} = A_{\mathrm{mv}} \tag{33}$$

The four Eqs.(22) and (24) as well as (25) and (27) are reduced to two equations with the variables A_{ev} and A_{mv}:

$$\frac{\partial^2 A_{\mathrm{ev}}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{ev}}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{\mathrm{mv}}}{\partial y} + \frac{\partial A_{\mathrm{ev}}}{\partial t}\right) = 0 \tag{34}$$

$$\frac{\partial^2 A_{\mathrm{mv}}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{mv}}}{\partial t^2} - \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{\mathrm{ev}}}{\partial y} + \frac{\partial A_{\mathrm{mv}}}{\partial t}\right) = 0 \tag{35}$$

Instead of using the substitutions of Eq.(33) we may make the more general substitutions

$$\begin{aligned} A_{\mathrm{e}x} &= A_{\mathrm{e}} \cos\chi, \quad & A_{\mathrm{e}z} &= A_{\mathrm{e}} \sin\chi \\ A_{\mathrm{m}x} &= A_{\mathrm{m}} \sin\chi, \quad & A_{\mathrm{m}z} &= A_{\mathrm{m}} \cos\chi \end{aligned}$$

where χ is the *polarization angle*[1] measured from the positive x-axis to the vector $\mathbf{A}_{\mathrm{e}}$ or from the negative z-axis to the vector $\mathbf{A}_{\mathrm{m}}$ to obtain Eqs.(34) and (35). The physical meaning of A_{ev} and A_{mv} in Eqs.(34) and (35) is thus that of the magnitude of $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$. Since the polarization angle χ is constant, the time variation of $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$ is the same as that of their magnitudes A_{ev} and A_{mv}. Hence, we may write our equations for A_{e} and A_{m} rather than for $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$.

Circularly polarized waves can be obtained by replacing the constant polarization angle χ by a time-variable angle χ:

$$\begin{aligned} A_{\mathrm{e}x} &= A_{\mathrm{e}} \cos\omega t, \quad & A_{\mathrm{e}z} &= A_{\mathrm{e}} \sin\omega t \\ A_{\mathrm{m}x} &= A_{\mathrm{m}} \sin\omega t, \quad & A_{\mathrm{m}z} &= -A_{\mathrm{m}} \cos\omega t \end{aligned}$$

Substitution into Eqs.(22), (24) and (25), (27) yields again Eqs.(34) and (35). In this form one emphasizes functions with sinusoidal time variation. This restriction vanishes if one does not choose χ as a linear function of time, $\chi = \omega t$, but as a general function $\chi = f(t)$:

$$\begin{aligned} A_{\mathrm{e}x} &= A_{\mathrm{e}} \cos[f(t)], \quad & A_{\mathrm{e}z} &= A_{\mathrm{e}} \sin[f(t)] \\ A_{\mathrm{m}x} &= A_{\mathrm{m}} \sin[f(t)], \quad & A_{\mathrm{m}z} &= -A_{\mathrm{m}} \cos[f(t)] \end{aligned}$$

[1]Some authors distinguish between a polarization angle and a rotation angle. They would call χ a rotation angle.

The substitution of A_{ex}, A_{ez}, A_{mx}, A_{mz} into Eqs.(22), (24) and (25), (27) produces once more Eqs.(34) and (35).

The two wave equations (31) and (32) have the general d'Alembert solution for $y \geq 0$ and $t \geq 0$, where f_{e0}, f_{e1}, f_{m0}, and f_{m1} denote arbitrary functions:

$$A_{ey}(y,t) = A_{e0} f_{e}(y - ct), \qquad y \geq 0,\ t \geq 0 \tag{36}$$
$$A_{my}(y,t) = A_{m0} f_{m}(y - ct) \tag{37}$$

These solutions hold for excitation functions or boundary conditions $f_e(0,t)$ and $f_m(0,t)$ at the plane $y = 0$ for all times $t \geq 0$ as well as initial conditions $f_e(y,0)$ and $f_m(y,0)$ for $t = 0$ at all locations $y \geq 0$.

The variables A_{ev} and A_{mv} in Eqs.(34) and (35) can be separated with some effort. One differentiates Eq.(35) with respect to y, expresses $\partial A_{mv}/\partial y$ by Eq.(34), differentiates as often as needed with respect to t and y, and substitutes into the differentiated Eq.(35). Eventually one obtains the following two equations:

$$\frac{\partial^2 V_e}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 V_e}{\partial t^2} - \frac{1}{c}\left(\sigma Z + \frac{s}{Z}\right)\frac{\partial V_e}{\partial t} - \sigma s V_e = 0 \tag{38}$$
$$\frac{\partial^2 A_{ev}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ev}}{\partial t^2} = V_e(y,t) \tag{39}$$

The dimension of A_{ev} is As/m, which is a linear electric charge density, while the dimension of V_e is As/m^3, which is an electric charge density. We note the term $\sigma s V_e$. It becomes zero for either $s = 0$ or $\sigma = 0$ and one obtains a different differential equation that in turn yields different solutions. This is why it is important not to make σ or s zero before the end of the calculation. In essence, the current densities $\mathbf{g}_e$ and $\mathbf{g}_m$ cannot be ignored at the beginning of a calculation even if they turn out to be zero in the end. The term $(\sigma Z + s/Z)\partial V_m/\partial t$ is less sensitive. It does not become zero for $s = 0$ or $\sigma = 0$, only for $s = 0$ and $\sigma = 0$.

With the substitutions $A_{ev} \leftrightarrow A_{mv}$, $c/Z \leftrightarrow Zc$, $s \leftrightarrow \sigma$ one may transform Eq.(34) into Eq.(35) and vice versa. Equations (38) and (39) are then replaced by:

$$\frac{\partial^2 V_m}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 V_m}{\partial t^2} - \frac{1}{c}\left(\sigma Z + \frac{s}{Z}\right)\frac{\partial V_m}{\partial t} - \sigma s V_m = 0 \tag{40}$$
$$\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2} = V_m(y,t) \tag{41}$$

The dimension of A_{mv} is Vs/m, which is a hypothetical linear magnetic charge density, while the dimension of V_{m} is $\mathrm{Vs/m^3}$, which is a hypothetical magnetic charge density.

At this point we switch from the time and space variables t and y with dimensions s and m to normalized variables θ and ζ:

$$\theta = t/\tau, \quad \zeta = y/c\tau \tag{42}$$

Equations (38) and (39) are rewritten:

$$\frac{\partial^2 V_{\mathrm{e}}}{\partial \zeta^2} - \frac{\partial^2 V_{\mathrm{e}}}{\partial \theta^2} - \rho_1 \frac{\partial V_{\mathrm{e}}}{\partial \theta} - \rho_2^2 V_{\mathrm{e}} = 0$$

$$\rho_1 = c\tau(\sigma Z + s/Z) = c^2\tau(\sigma\mu + s\epsilon),\ \rho_2^2 = c^2\tau^2\sigma s,\ \rho_1^2 - 4\rho_2^2 \geq 0 \tag{43}$$

$$\frac{\partial^2 A_{\mathrm{ev}}}{\partial \zeta^2} - \frac{\partial^2 A_{\mathrm{ev}}}{\partial \theta^2} = c^2\tau^2 V_{\mathrm{e}}(\zeta, \theta) \tag{44}$$

Equation (44) is the inhomogeneous wave equation with one spatial variable. Its solution is known[2]:

$$A_{\mathrm{ev}}(\zeta, \theta) = -\frac{c^2\tau^2}{2} \int_0^{\theta} \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_{\mathrm{e}}(\zeta', \theta') d\zeta' \right) d\theta' \tag{45}$$

The variables ζ and θ of $V_{\mathrm{e}}(\zeta, \theta)$ in Eq.(44) have to be replaced by ζ' and θ' when Eq.(45) is used.

In analogy to Eqs.(43)–(45) one obtains for the variable $V_{\mathrm{m}}(y, t)$ of Eqs.(40) and (41) the following three equations:

$$\frac{\partial^2 V_{\mathrm{m}}}{\partial \zeta^2} - \frac{\partial^2 V_{\mathrm{m}}}{\partial \theta^2} - \rho_1 \frac{\partial V_{\mathrm{m}}}{\partial \theta} - \rho_2^2 V_{\mathrm{m}} = 0 \tag{46}$$

$$\frac{\partial^2 A_{\mathrm{mv}}}{\partial \zeta^2} - \frac{\partial^2 A_{\mathrm{mv}}}{\partial \theta^2} = c^2\tau^2 V_{\mathrm{m}}(\zeta, \theta) \tag{47}$$

$$A_{\mathrm{mv}}(\zeta, \theta) = -\frac{c^2\tau^2}{2} \int_0^{\theta} \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_{\mathrm{m}}(\zeta', \theta') d\zeta' \right) d\theta' \tag{48}$$

As before, the variables ζ and θ in Eqs.(46) and (47) must be replaced by ζ' and θ' when Eq.(48) is used.

[2] Smirnov 1961, vol. II, Cha. VII, § 1, Sec. 174, Eq. 95

One may obtain the component $A_{\rm mv}(\zeta,\theta)$ of an *associated potential* from either Eq.(34) or (35), if $A_{\rm ev}(\zeta,\theta)$ is found from Eq.(45). Consider first Eq.(34):

$$A_{\rm mv}(\zeta,\theta) = Z\rho_{\rm s}\int\left(\frac{\partial^2 A_{\rm ev}}{\partial\zeta^2} - \frac{\partial^2 A_{\rm ev}}{\partial\theta^2} - \frac{1}{\rho_{\rm s}}\frac{\partial A_{\rm ev}}{\partial\theta}\right)d\zeta$$

$$\rho_{\rm s} = \frac{Z}{sc\tau} = \frac{\mu}{s\tau} = \frac{1}{2\rho_2^2}\left[\rho_1 \pm (\rho_1^2 - 4\rho_2^2)^{1/2}\right] \tag{49}$$

Equation (35) yields a second expression for $A_{\rm mv}(\zeta,\theta)$ if we treat this equation as an inhomogeneous equation for $A_{\rm mv}$ with a known term $\partial A_{\rm ev}/\partial y$ or $\partial A_{\rm ev}/\partial\zeta$:

$$\frac{\partial^2 A_{\rm mv}}{\partial\zeta^2} - \frac{\partial^2 A_{\rm mv}}{\partial\theta^2} - \rho_\sigma\frac{\partial A_{\rm mv}}{\partial\theta} = Z\rho_\sigma\frac{\partial A_{\rm ev}}{\partial\zeta} \tag{50}$$

$$\rho_\sigma = Z\tau c\sigma = \frac{\sigma\tau}{\epsilon},\ \rho_{\rm s} = \frac{Z}{sc\tau} = \frac{\mu}{s\tau},\ \rho_{\rm s}\rho_\sigma = \frac{\sigma\mu}{s\epsilon} = \frac{1}{\omega^2} \tag{51}$$

It is easier to integrate Eq.(49) than Eq.(50), but we cannot ignore Eq.(50). Since Eqs.(47) and (48) must yield the same result for $A_{\rm mv}(\zeta,\theta)$ we generally need the solution of Eq.(50) to determine integration constants.

Alternately, if $A_{\rm mv}(\zeta,\theta)$ is found from Eq.(48) for certain boundary and initial conditions one may obtain an *associated potential* $A_{\rm ev}(\zeta,\theta)$ from either Eq.(34) or (35). First we get from Eq.(35):

$$A_{\rm ev}(\zeta,\theta) = \frac{1}{Z\rho_\sigma}\int\left(\frac{\partial^2 A_{\rm mv}}{\partial\zeta^2} - \frac{\partial^2 A_{\rm mv}}{\partial\theta^2} - \rho_\sigma\frac{\partial A_{\rm mv}}{\partial\theta}\right)d\zeta \tag{52}$$

The second expression for $A_{\rm ev}(\zeta,\theta)$ is obtained from Eq.(34) by treating it as an inhomogeneous equation for $A_{\rm ev}$ with a known term $\partial A_{\rm mv}/\partial y$ or $\partial A_{\rm mv}/\partial\zeta$:

$$\frac{\partial^2 A_{\rm ev}}{\partial\zeta^2} - \frac{\partial^2 A_{\rm ev}}{\partial\theta^2} - \frac{1}{\rho_{\rm s}}\frac{\partial A_{\rm ev}}{\partial\theta} = \frac{1}{Z\rho_{\rm s}}\frac{\partial A_{\rm mv}}{\partial\zeta} \tag{53}$$

Again, one must generally obtain $A_{\rm ev}(\zeta,\theta)$ from both Eq.(52) and (53) in order to determine the integration constants.

If we denote the solution of $A_{\rm ev}$ derived from Eq.(45) by $A_{\rm eve}$ and the associated solution obtained from $A_{\rm mv}$ via Eqs.(52) and (53) by $A_{\rm evm}$ we obtain the general solution of $A_{\rm ev}$ as the sum

$$A_{\rm ev}(\zeta,\theta) = A_{\rm eve}(\zeta,\theta) + A_{\rm evm}(\zeta,\theta) \tag{54}$$

Similarly, if we denote the solution derived for $A_{\rm mv}$ from Eq.(48) by $A_{\rm mvm}$ and the associated solution obtained from $A_{\rm ev}$ via Eqs.(49) and (50) by $A_{\rm mve}$ we obtain the general solution of $A_{\rm mv}$ as the sum

$$A_{\mathrm{mv}}(\zeta,\theta) = A_{\mathrm{mvm}}(\zeta,\theta) + A_{\mathrm{mve}}(\zeta,\theta) \tag{55}$$

This means we can choose initial and boundary conditions independently for A_{eve} and A_{mvm}, but the associated potentials A_{mve} and A_{evm} are always automatically excited with A_{eve} or A_{mvm}. We can never excite A_{ev} without exciting A_{mv} and vice versa.

So far we have followed closely previously published results[3]. From here on we shall deviate. For a first solution of Eq.(43) we assume as boundary condition at $\zeta = 0$ a step function:

$$\begin{aligned} V_{\mathrm{e}}(0,\theta) = V_{\mathrm{e0}}S(\theta) &= 0 \qquad \text{for } \theta < 0 \\ &= V_{\mathrm{e0}} \quad \text{for } \theta \geq 0 \end{aligned} \tag{56}$$

At large distances we want $V_{\mathrm{e}}(\zeta,\theta)$ to be finite. We write $\zeta \to \infty$ for a large distance but this notation will be made more specific later on:

$$V_{\mathrm{e}}(\infty,\theta) = \text{finite} \tag{57}$$

The boundary condition of Eq.(56) uses a step function that is not quadratically integrable. This may cause concern that an infinite energy is introduced but there is no such problem. The boundary condition of Eq.(56) excites an electromagnetic wave with finite energy. Of course, one may eliminate the concern about quadratical integrability by subtracting from $V_{\mathrm{e0}}S(\theta)$ a delayed step function $V_{\mathrm{e0}}S(\theta - \theta_1)$ and thus replacing the step function of Eq.(56) by a rectangular pulse[4].

Let us consider the initial condition(s). As initial condition at $\theta = 0$ we assume the relation

$$V_{\mathrm{e}}(\zeta,0) = 0 \tag{58}$$

but observe that this condition implies $V_{\mathrm{e}}(\zeta,\theta) = 0$ for $\theta < 0$ due to Eq.(56). Hence, the potential $\mathbf{A}_{\mathrm{e}}$ derived from A_{ev} will be zero for $\theta < 0$. We note that a function of time that describes a physical process subject to the causality law must be zero before a finite time. The term *causal function* is sometimes used by mathematicians for such functions.

If $V_{\mathrm{e}}(\zeta,0)$ is zero for all values $\zeta > 0$, its derivatives with respect to ζ must be zero too:

$$\partial^n V_{\mathrm{e}}(\zeta,0)/\partial\zeta^n = 0 \tag{59}$$

With the help of Eqs.(58) and (59) we obtain from Eq.(43) for $\theta = 0$:

[3]Harmuth, Barrett, Meffert 2001, Sec. 4.1
[4]Harmuth, Barrett, Meffert 2001, Sec. 4.6

$$\frac{\partial}{\partial\theta}\left(\frac{\partial V_{\rm e}(\zeta,\theta)}{\partial\theta} + \rho_1 V_{\rm e}(\zeta,\theta)\right) = 0 \tag{60}$$

This equation is satisfied by $V_{\rm e}(\zeta, 0) = 0$ of Eq.(58) and the additional condition:

$$\partial V_{\rm e}(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0 \tag{61}$$

We assume the general solution of Eq.(43) can be written as the sum of a steady state solution $F(\zeta)$ plus a deviation $w(\zeta,\theta)$ from it (Habermann 1987):

$$V_{\rm e}(\zeta,\theta) = V_{\rm e0}[F(\zeta) + w(\zeta,\theta)] \tag{62}$$

Substitution of $F(\zeta)$ into Eq.(43) yields a differential equation with the one variable ζ only:

$$\begin{gathered} d^2F/d\zeta^2 - \rho_2^2 F = 0 \\ F(\zeta) = A_{10}e^{-\rho_2\zeta} + A_{11}e^{\rho_2\zeta} \end{gathered} \tag{63}$$

The boundary condition of Eq.(57) demands $A_{11} = 0$. From Eq.(56) follows then $A_{11} = 1$:

$$F(\zeta) = e^{-\rho_2\zeta} \tag{64}$$

The introduction of $F(\zeta)$ transforms the boundary condition of Eq.(56) for $V_{\rm e}$ into a homogeneous boundary condition for w, which is the reason for using Eq.(62):

$$V_{\rm e}(0,\theta) = V_{\rm e0}[F(0) + w(0,\theta)] = V_{\rm e0} \qquad \text{for } \theta \geq 0 \tag{65}$$

$$w(0,\theta) = 0 \qquad \text{for } \theta \geq 0 \tag{66}$$

The boundary condition of Eq.(57) for great distances becomes

$$w(\infty,\theta) = \text{finite} \tag{67}$$

The initial conditions of Eqs.(58) and (61) yield:

$$F(\zeta) + w(\zeta,0) = 0 \qquad w(\zeta,0) = -e^{-\rho_2\zeta} \tag{68}$$

$$\partial w(\zeta,\theta)/\partial\theta = 0 \qquad \text{for } \theta = 0,\ \zeta > 0 \tag{69}$$

Substitution of Eq.(62) into Eq.(43) yields for $w(\zeta,\theta)$ the same equation as for $V_{\rm e}(\zeta,\theta)$:

$$\partial^2 w/\partial\zeta^2 - \partial^2 w/\partial\theta^2 - \rho_1 \partial w/\partial\theta - \rho_2^2 w = 0 \tag{70}$$

Particular solutions of this equation denoted $w_\kappa(\zeta,\theta)$ are obtained by means of Bernoulli's product method for the separation of variables:

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \tag{71}$$

$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{\psi}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\rho_1}{\psi}\frac{\partial\psi}{\partial\theta} + \rho_2^2 = -(2\pi\kappa/N_\tau)^2$$
$$N_\tau \geq 1,\ \kappa = 1,\ 2,\ldots, \tag{72}$$

We write $-(2\pi\kappa/N_\tau)^2$ rather than the usual $-(2\pi\kappa)^2$ as separation constant in order to obtain later on an orthogonality interval of length N_τ rather than 1. Two ordinary differential equations are obtained

$$d^2\phi/d\zeta^2 + (2\pi\kappa/N_\tau)^2\phi = 0 \tag{73}$$
$$d^2\psi/d\theta^2 + \rho_1 d\psi/d\theta + [(2\pi\kappa/N_\tau)^2 + \rho_2^2]\psi = 0 \tag{74}$$

with the solutions

$$\phi(\zeta) = A_{20}\sin\frac{2\pi\kappa\zeta}{N_\tau} + A_{21}\cos\frac{2\pi\kappa\zeta}{N_\tau} \tag{75}$$
$$\psi(\theta) = A_{30}\exp(\gamma_1\theta) + A_{31}\exp(\gamma_2\theta) \tag{76}$$

The coefficients γ_1 and γ_2 are the roots of the equation

$$\gamma^2 + \rho_1\gamma + [(2\pi\kappa/N_\tau)^2 + \rho_2^2] = 0$$
$$\gamma_1 = \frac{1}{2}[-\rho_1 + (\rho_1^2 - d^2)^{1/2}] \quad \text{for } d^2 < \rho_1^2$$
$$\gamma_2 = \frac{1}{2}[-\rho_1 - (\rho_1^2 - d^2)^{1/2}]$$
$$\gamma_1 = \frac{1}{2}[-\rho_1 + i(d^2 - \rho_1^2)^{1/2}] \quad \text{for } d^2 > \rho_1^2$$
$$\gamma_2 = \frac{1}{2}[-\rho_1 - i(d^2 - \rho_1^2)^{1/2}]$$
$$\rho_1 = c\tau(\sigma Z + s/Z) = c^2\tau(\sigma\mu + s\epsilon),\ d^2 = 4[(2\pi\kappa/N_\tau)^2 + \rho_2^2],\ \rho_2^2 = c^2\tau^2\sigma s \tag{77}$$

The boundary condition of Eq.(66) requires $A_{21} = 0$ in Eq.(75) and the particular solution $w_\kappa(\zeta,\theta)$ becomes:

$$w_\kappa(\zeta,\theta) = [A_1\exp(\gamma_1\theta) + A_2\exp(\gamma_2\theta)]\sin\frac{2\pi\kappa\zeta}{N_\tau} \tag{78}$$

The solution w_κ is usually generalized by making A_1 and A_2 functions of κ and integrating over all values of κ from zero to infinity. This would imply non-denumerably many oscillators or photons. It is usual in quantum field theory to reduce the non-denumerably many oscillators to denumerably many, using *box normalization* to accomplish this reduction. We shall follow the spirit of this reduction without recourse to box normalization.

To generalize w_κ of Eq.(78) one may make A_1 and A_2 functions of κ and take the sum of denumerably many values of $\kappa = 1, 2, \ldots$. The Fourier integral is replaced by the Fourier series. In the specific case of Eq.(78) we have only the function $\sin(2\pi\kappa\zeta/N_\tau)$. The constant term of the Fourier series and the terms multiplied by $\cos(2\pi\kappa\zeta/N_\tau)$—which is shown in Eq.(75)—have been eliminated by the boundary condition of Eq.(66). Hence, the solution w_κ of Eq.(78) is generalized by a Fourier sine integral or a Fourier sine series.

A Fourier series requires a finite interval for ζ in Eq.(78) which we must define. The creation of a finite interval is, of course, the goal of box normalization. This problem of having to define a finite interval does not occur if the Fourier integral is used for the generalization of Eq.(78) since the interval for the integral always runs from zero to infinity for both ζ and κ. We choose the finite interval for ζ to be

$$0 \le \zeta = y/c\tau \le T/\tau, \quad T/\tau = N_\tau \gg 1 \tag{79}$$

where the finite time interval T is arbitrarily large but finite. The boundary condition of Eq.(57) should be written

$$V_\mathrm{e}(T/\tau,\theta) = \text{finite}, \quad \zeta = T/\tau = N_\tau \gg 1 \tag{80}$$

but there is no great difference between the usual mathematical way of writing $\zeta \to \infty$ and the more physical way of writing $\zeta \to N_\tau \gg 1$.

It is common practice to continue a Fourier series outside its finite interval of definition periodically to $\pm\infty$. There is no need to do so. We may just as well claim that the series is continued as zero outside the interval of definition or simply avoid any claim about what is outside this interval. Inherently a finite value of T and cT is preferable in physics since one cannot make observations at infinite distances in space or time.

We generalize Eq.(78) by a sum of denumerably many terms of the variable κ and a finite interval for the variable ζ:

$$w(\zeta,\theta) = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\exp(\gamma_1\theta) + A_2(\kappa)\exp(\gamma_2\theta)]\sin\frac{2\pi\kappa\zeta}{N_\tau} \tag{81}$$

Since ζ is a continuous variable with non-denumerably many values but κ has only denumerably many values, the sum of Eq.(81) can represent $w(\zeta, \theta)$ only in the sense of a vanishing mean-square error.

The normalization constant τ may have any value. A logical choice would be to use for τ an arbitrarily small but finite difference Δt

$$\tau = \Delta t \tag{82}$$

but this will not work until we introduce difference equations in Chapter 3. The differentials dt, dy, $d\theta$ and $d\zeta$ will there be replaced by finite differences Δt, Δy, $\Delta\theta$ and $\Delta\zeta$. The variables t, y, θ and ζ remain continuous variables with an upper limit:

$$\begin{aligned} &0 \le t \le T, \qquad 0 \le y \le cT \\ &0 \le \theta \le N, \qquad 0 \le \zeta \le N, \qquad N = T/\Delta t \end{aligned} \tag{83}$$

In the following sections we shall work out the solution defined by Eq.(81). This is mainly for the purpose of comparison. There is an inconsistency if one solves the differential equation (43) and then replaces the differentials $d\zeta$, $d\theta$ by finite differences $\Delta\zeta$, $\Delta\theta$. One should replace Eq.(43) by a difference equation and see where this leads. The calculations from Eq.(1) to Eq.(43) are of a form that a mathematician should be able to rewrite from differential to difference form without reference to their physical significance.

The importance of distinguishing between differential and difference equations is due to the theorem of Hölder, cited in connection with Eq.(1.5-1). It states that differential and difference equations define different classes of functions.

The use of differentials in physics has been questioned for a long time. Zeno of Elea objected already to the notion of infinite divisibility of space and time. Pauli (1933) as well as Landau and Peierls (1931) pointed out that the assumption of arbitrarily accurate position and time measurements was probably unjustified in relativistic quantum mechanics, since the Compton effect limits the accuracy of the position measurement of a particle. However, Pauli concluded that there was no such limitation in nonrelativistic quantum mechanics. Today, as a result of the development of information theory, we must reject any assumption of measurements with unlimited accuracy, since they imply gathering and processing infinite information. In particular, non-denumerably infinite information is implied by 'differentially accurate' or *infinitesimally accurate* measurements. March (1948, 1951) postulated that the distance of two particles could only be measured with an error that was at least equal to an *elementary unit of length*. Such elementary units are difficult to reconcile with the special theory of relativity. No such difficulties arise if one assumes an arbitrarily small but finite distance between adjacent marks of a ruler. Such a distance does not have to be 10^{-18} m, it is still finite if it is 10^{-100} m. Many papers have been published on the use of finite differences in quantum physics. We list in chronological sequence Snyder (1947a, b), Flint (1948), Schild (1949), Hellund and Tanaka (1954), Hill (1955), Das (1960, 1966a, b, c), Yukawa (1966), Atkinson and Halpern (1967), Cole (1970, 1971, 1972a, b; 1973a, b), Hasebe (1972), Welch (1976), and Harmuth (1989).

We emphasize that the use of the finite differences Δt, Δy used from Chapter 3 on does *not* imply a quantized or cellular space-time. It strictly replaces differential equations by difference equations. As the Gamma function demonstrates a function defined by a difference equation can be continuous and differentiable except at certain poles.

2.2 Differential Solution for $w(\zeta,\theta)$

Equations (2.1-68) and (2.1-69) define initial conditions for $w(\zeta,\theta)$ and $\partial w(\zeta,\theta)/\partial\theta$ at $\theta = 0$. The derivative $\partial w(\zeta,\theta)/\partial\theta$ follows from Eq.(2.1-81):

$$\frac{\partial w}{\partial\theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1\exp(\gamma_1\theta) + A_2(\kappa)\gamma_2\exp(\gamma_2\theta)]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$0 \le t \le T, \quad 0 \le y \le cT$$
$$0 \le t/\tau = \theta \le T/\tau = N_\tau, \quad 0 \le y/c\tau = \zeta \le T/\tau = N_\tau \qquad (1)$$

With Eqs.(2.1-81) and (1) we may write the initial conditions of Eqs.(2.1-68) and (2.1-69) in the following form:

$$w(\zeta,0) = \sum_{\kappa=1}^{\infty}[A_1(\kappa) + A_2(\kappa)]\sin\frac{2\pi\kappa\zeta}{N_\tau} = -e^{-\rho_2\zeta} \qquad (2)$$

$$\frac{\partial w(\zeta,0)}{\partial\theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2]\sin\frac{2\pi\kappa\zeta}{N_\tau} = 0 \qquad (3)$$

In order to solve these two sets of equations for $A_1(\kappa)$ and $A_2(\kappa)$ we consider the Fourier series. Since only $\sin(2\pi\kappa\zeta/N_\tau)$ but not $\cos(2\pi\kappa\zeta/N_\tau)$ occurs in Eqs.(2) and (3) it is sufficient to use the Fourier sine series, which we write in the following form:

$$g_\mathrm{s}(\kappa) = \frac{2}{N_\tau}\int_0^{N_\tau} f_\mathrm{s}(\zeta)\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta, \quad f_\mathrm{s}(\zeta) = \sum_{\kappa=1}^{\infty} g_\mathrm{s}(\kappa)\sin\frac{2\pi\kappa\zeta}{N_\tau} \qquad (4)$$

We note that the variable κ/N_τ is used rather than κ. Figure 2.2-1 shows that this produces one cycle of the sinusoidal function $\sin 2\pi(\kappa/N_\tau)\zeta$ for $\kappa = 1$

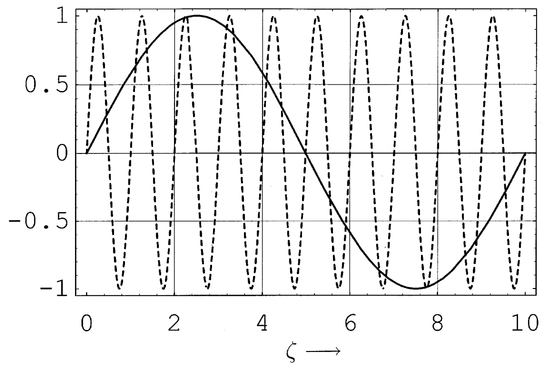

FIGURE 2.2-1. Sinusoidal function $\sin 2\pi(\kappa/N_\tau)\zeta$ with the lowest number $\kappa = 1$ of cycles in the interval $0 \leq \zeta \leq N_\tau = 10$ and with the highest number $\kappa = N_\tau = 10$ if $\tau = T/N_\tau$ is the smallest observable time resolution in the time interval of duration T.

in the interval $0 \leq \zeta \leq N_\tau = 10$ and $N_\tau = 10$ cycles for $\kappa = N_\tau = 10$. There is no inherent reason for κ not to assume values larger than N_τ. The number of cycles of $\sin 2\pi(\kappa/N_\tau)\zeta$ in the interval $0 \leq \zeta \leq N_\tau = 10$ can increase without bound as shown by the limit ∞ of the sum in Eq.(4).

The situation changes if we postulate that τ is the shortest time and $c\tau$ the shortest distance that can be resolved by the available instruments of observation. There is no need to claim a certain minimal value for τ. It may be arbitrarily small but must be finite. A value $\tau = 10^{-100}$ s will satisfy this condition. It is an uncontested principle that we cannot observe differential—or infinitesimal—space and time differences dx and dt, we can only calculate as if we were able to observe that accurately. The introduction of arbitrarily small but finite values of τ works the limited practical resolution into the differential theory using dx and dt. In Chapter 3 we will extend our results by means of the calculus of finite differences using arbitrarily small but finite differences Δx and Δt. We will obtain results that differ significantly—but not excessively—from the differential theory.

If we have an interval of length T and if τ is the smallest observable time we can distinguish $T/\tau = N_\tau$ sinusoidal functions and N_τ cosinusoidal functions. The value of κ in Eq.(4) runs then from $\kappa = 1$ to $\kappa = N_\tau$ rather than to infinity. In place of Eq.(4) we get:

$$g_s(\kappa) = \frac{2}{N_\tau}\int_0^{N_\tau} f_s(\zeta)\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta, \quad f_s(\zeta) = \sum_{\kappa=1}^{N_\tau} g_s(\kappa)\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$0 \leq \zeta \leq N_\tau, \; N_\tau = T/\tau \tag{5}$$

We identify $g_s(\kappa)$ first with $A_1(\kappa) + A_2(\kappa)$ of Eq.(2) and obtain

$$A_1(\kappa) + A_2(\kappa) = -\frac{2}{N_\tau}\int_0^{N_\tau} e^{-\rho_2\zeta}\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta \tag{6}$$

Then we identify $g_s(\kappa)$ with $A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2$ in Eq.(3) to obtain

$$A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 = 0 \tag{7}$$

The integral in Eq.(6) may be found in a table (Gradshtein and Ryzhik 1980, p. 196, 2663/1):

$$\int e^{px}\sin qx\,dx = \frac{e^{px}(p\sin qx - q\cos qx)}{p^2+q^2} \tag{8}$$

We obtain from Eqs.(6) and (8):

$$\begin{aligned}
A_1(\kappa) + A_2(\kappa) &= -\frac{1}{N_\tau}\frac{(4\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2} \\
&\doteq -\frac{2N_\tau\rho_2}{2\pi\kappa} = -\frac{2\rho_2}{2\pi\kappa/N_\tau} \qquad \text{for } N_\tau\rho_2 \ll 1 \\
&\doteq -\frac{1}{N_\tau}\frac{4\pi\kappa/N_\tau}{(2\pi\kappa/N_\tau)^2+\rho_2^2} \qquad \text{for } N_\tau\rho_2 \gg 1
\end{aligned} \tag{9}$$

The parameters γ_1 and γ_2 are defined in Eq.(2.1-77). The solution of Eqs.(7) and (9) for $A_1(\kappa)$ and $A_2(\kappa)$ yields:

$$\begin{aligned}
A_1(\kappa) &= -\frac{1}{N_\tau}\frac{(4\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\frac{\gamma_2}{\gamma_2-\gamma_1} \\
&= -\frac{1}{N_\tau}\frac{(2\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(1-\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right) \qquad \text{for } d^2 > \rho_1^2 \\
&= -\frac{1}{N_\tau}\frac{(2\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(1+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right) \qquad \text{for } \rho_1^2 > d^2
\end{aligned} \tag{10}$$

$$\begin{aligned}
A_2(\kappa) &= -\frac{1}{N_\tau}\frac{(4\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\frac{\gamma_1}{\gamma_1-\gamma_2} \\
&= -\frac{1}{N_\tau}\frac{(2\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(1+\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right) \qquad \text{for } d^2 > \rho_1^2 \\
&= -\frac{1}{N_\tau}\frac{(2\pi\kappa/N_\tau)(1-e^{-N_\tau\rho_2})}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(1-\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right) \qquad \text{for } \rho_1^2 > d^2
\end{aligned}$$

$$\rho_1 = c\tau(\sigma Z + s/Z),\ \rho_2^2 = (c\tau)^2\sigma s,\ d^2 = 4[(2\pi\kappa/N_\tau)^2+\rho_2^2] \tag{11}$$

Substitution of $A_1(\kappa)$, $A_2(\kappa)$ as well as of γ_1, γ_2 from Eq.(2.1-77) into Eq.(2.1-81) and observing the limit N_τ of the sum according to Eq.(5) yields the following equation for $w(\zeta,\theta)$:

$$
\begin{aligned}
w(\zeta,\theta) = -\frac{e^{-\rho_1\theta/2}(1-e^{-N_\tau\rho_2})}{N_\tau}\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\left(1+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right)\exp\frac{(\rho_1^2-d^2)^{1/2}\theta}{2} \\
+\left(1-\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right)\exp\frac{-(\rho_1^2-d^2)^{1/2}\theta}{2}\Bigg]\frac{2\pi\kappa/N_\tau}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N_\tau} \\
+\sum_{\kappa>K}^{N_\tau}\Bigg[\left(1-\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right)\exp\frac{i(d^2-\rho_1^2)^{1/2}\theta}{2} \\
+\left(1+\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right)\exp\frac{-i(d^2-\rho_1^2)^{1/2}\theta}{2}\Bigg]\frac{2\pi\kappa/N_\tau}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N_\tau}\Bigg\} \\
K = N_\tau(\rho_1^2-4\rho_2^2)^{1/2}/4\pi = c\tau N_\tau|(\sigma Z - s/Z)|/4\pi,\ \rho_1^2 \geq 4\rho_2^2 \\
d^2 = 4[(2\pi\kappa/N_\tau)^2+\rho_2^2] \qquad (12)
\end{aligned}
$$

Here $< K$ and $> K$ in the limits of the two sums mean the largest integer smaller than K or the smallest integer larger than K.

With the help of hyperbolic and trigonometric functions we may simplify the exponential terms in Eq.(12) and eliminate the imaginary terms:

$$
\begin{aligned}
w(\zeta,\theta) = -\frac{2}{N_\tau}e^{-\rho_1\theta/2}(1-e^{-N_\tau\rho_2})\Bigg[\sum_{\kappa=1}^{<K}\Bigg(\mathrm{ch}[(\rho_1^2-d^2)^{1/2}\theta/2] \\
+\frac{\rho_1\,\mathrm{sh}[(\rho_1^2-d^2)^{1/2}\theta/2]}{(\rho_1^2-d^2)^{1/2}}\Bigg)\frac{2\pi\kappa/N_\tau}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N_\tau} \\
+\sum_{\kappa>K}^{N_\tau}\Bigg(\cos[(d^2-\rho_1^2)^{1/2}\theta/2] \\
+\frac{\rho_1\sin[(d^2-\rho_1^2)^{1/2}\theta/2]}{(d^2-\rho_1^2)^{1/2}}\Bigg)\frac{2\pi\kappa/N_\tau}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N_\tau}\Bigg] \\
\rho_1 = c\tau(\sigma Z + s/Z),\ \rho_2^2 = c^2\tau^2\sigma s,\ d^2 = 4[(2\pi\kappa/N_\tau)^2+\rho_2^2] \\
K = N_\tau(\rho_1^2-4\rho_2^2)^{1/2}/4\pi = c\tau N_\tau|(\sigma Z - s/Z)|/4\pi \qquad (13)
\end{aligned}
$$

The function $V_{\mathrm{e}}(\zeta,\theta)$ of Eq.(2.1-45) is defined by $F(\zeta)$ of Eq.(2.1-64) and $w(\zeta,\theta)$ of Eq.(13). One obtains $A_{\mathrm{ev}}(\zeta,\theta)$ by the integrations of Eq.(2.1-45). The variables ζ, θ of $F(\zeta)$ and $w(\zeta,\theta)$ are replaced by ζ', θ' and the integration is carried out over ζ', θ'. We integrate first over ζ' and denote the result by $\partial A_{\mathrm{ev}}(\zeta,\theta')/\partial\theta'$:

$$\frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'} = -\frac{c^2\tau^2 V_{\mathrm{e}0}}{2}\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}[F(\zeta')+w(\zeta',\theta')]d\zeta' \tag{14}$$

Two integrals must be evaluated:

$$\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} F(\zeta)\,d\zeta = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-\rho_2\zeta'}d\zeta' = -\frac{1}{\rho_2}(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}) \tag{15}$$

$$\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \sin\frac{2\pi\kappa\zeta'}{N_\tau}\,d\zeta'$$
$$= \frac{\sin(2\pi\kappa\zeta/N_\tau)}{\pi\kappa/N_\tau}\left(\sin\frac{2\pi\kappa\theta}{N_\tau}\cos\frac{2\pi\kappa\theta'}{N_\tau} - \cos\frac{2\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\kappa\theta'}{N_\tau}\right) \tag{16}$$

We obtain for $\partial A_{\mathrm{ev}}(\zeta,\theta')/\partial\theta'$:

$$\begin{aligned}
\frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'} &= \frac{c^2\tau^2 V_{\mathrm{e}0}}{2}\bigg\{\frac{1}{\rho_2}(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}) \\
&+ \frac{4}{N_\tau}e^{-\rho_1\theta'/2}(1-e^{-N_\tau\rho_2})\bigg[\sum_{\kappa=1}^{<K}\left(\mathrm{ch}[(\rho_1^2-d^2)^{1/2}\theta'/2] + \frac{\rho_1\,\mathrm{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]}{(\rho_1^2-d^2)^{1/2}}\right) \\
&\times \frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(\sin\frac{2\pi k\theta}{N_\tau}\cos\frac{2\pi k\theta'}{N_\tau} - \cos\frac{2\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\kappa\theta'}{N_\tau}\right) \\
&+ \sum_{\kappa>K}^{N_\tau}\left(\cos[(d^2-\rho_1^2)^{1/2}\theta'/2] + \frac{\rho_1\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]}{(d^2-\rho_1^2)^{1/2}}\right) \\
&\times \frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\left(\sin\frac{2\pi\kappa\theta}{N_\tau}\cos\frac{2\pi\kappa\theta'}{N_\tau} - \cos\frac{2\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\kappa\theta'}{N_\tau}\right)\bigg]\bigg\}
\end{aligned} \tag{17}$$

The integration of Eq.(17) over θ' produces $A_{\mathrm{ev}}(\zeta,\theta)$:

$$A_{\mathrm{ev}}(\zeta,\theta) = \int\limits_0^\theta \frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'}d\theta' \tag{18}$$

The following integrals are obtained by the substitution of Eq.(17) into Eq.(18):

$$\frac{1}{\rho_2}\int_0^\theta \left(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}\right)d\theta' = \frac{2}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) \tag{19}$$

$$L_{11}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]\cos\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{20}$$

$$L_{12}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]\sin\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{21}$$

$$L_{13}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\operatorname{ch}[(\rho_1^2-d^2)^{1/2}\theta'/2]\cos\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{22}$$

$$L_{14}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\operatorname{ch}[(\rho_1^2-d^2)^{1/2}\theta'/2]\sin\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{23}$$

$$L_{15}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]\cos\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{24}$$

$$L_{16}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]\sin\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{25}$$

$$L_{17}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\cos[(d^2-\rho_1^2)^{1/2}\theta'/2]\cos\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{26}$$

$$L_{18}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\cos[(d^2-\rho_1^2)^{1/2}\theta'/2]\sin\frac{2\pi\kappa\theta'}{N_\tau}d\theta' \tag{27}$$

The integrals $L_{11}(\theta,\kappa)$ to $L_{18}(\theta,\kappa)$ are either tabulated or can readily be rewritten into a tabulated form. The function $A_{\mathrm{ev}}(\zeta,\theta)$ of Eq.(18) assumes the following form:

$$A_{\mathrm{ev}}(\zeta,\theta) = c^2\tau^2V_{e0}\Bigg(\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$$
$$+\frac{2}{N_\tau}(1-e^{-N_\tau\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\left(L_{13}(\theta,\kappa)+\frac{\rho_1L_{11}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\right)\sin\frac{2\pi\kappa\theta}{N_\tau}$$

$$
- \left(L_{14}(\theta,\kappa) + \frac{\rho_1 L_{12}(\theta,\kappa)}{(\rho_1^2 - d^2)^{1/2}} \right) \cos\frac{2\pi\kappa\theta}{N_\tau} \Bigg] \frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}
$$

$$
+ \sum_{\kappa>K}^{N_\tau} \Bigg[\left(L_{17}(\theta,\kappa) + \frac{\rho_1 L_{15}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2}} \right) \sin\frac{2\pi\kappa\theta}{N_\tau}
$$

$$
- \left(L_{18}(\theta,\kappa) + \frac{\rho_1 L_{16}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2}} \right) \cos\frac{2\pi\kappa\theta}{N_\tau} \Bigg] \frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2 + \rho_2^2} \Bigg\} \Bigg) \qquad (28)
$$

All terms of the two sums in Eq.(28) contain products that represent propagating sinusoidal waves:

$$
\sin\frac{2\pi\kappa\theta}{N_\tau} \sin\frac{2\pi\kappa\zeta}{N_\tau} = \frac{1}{2} \left(\cos\frac{2\pi\kappa(\zeta-\theta)}{N_\tau} - \cos\frac{2\pi\kappa(\zeta+\theta)}{N_\tau} \right)
$$

$$
\cos\frac{2\pi\kappa\theta}{N_\tau} \sin\frac{2\pi\kappa\zeta}{N_\tau} = \frac{1}{2} \left(\sin\frac{2\pi\kappa(\zeta-\theta)}{N_\tau} + \sin\frac{2\pi\kappa(\zeta+\theta)}{N_\tau} \right)
$$

The integrals $L_{15}(\zeta,\kappa)$ to $L_{18}(\zeta,\kappa)$ of Eqs.(24)–(27) can be written explicitly with the help of two new auxiliary variables $q_1 = q_1(\kappa)$ and $q_2 = q_2(\kappa)$:

$$
q_1 = \frac{1}{2}(d^2 - \rho_1^2)^{1/2} + \frac{2\pi\kappa}{N_\tau},\ q_2 = \frac{1}{2}(d^2 - \rho_1^2)^{1/2} - \frac{2\pi\kappa}{N_\tau}, \quad d^2 - \rho_1^2 > 0 \qquad (29)
$$

One obtains:

$$
\begin{aligned}
L_{15}(\theta,\kappa) &= L_{15\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2} L_{15\mathrm{B}}(\theta,\kappa) \\
&= \frac{1}{2} \Bigg[\frac{q_1}{(\rho_1/2)^2 + q_1^2} + \frac{q_2}{(\rho_1/2)^2 + q_2^2} - e^{-\rho_1\theta/2} \\
&\times \left(\frac{(\rho_1/2)\sin q_1\theta + q_1 \cos q_1\theta}{(\rho_1/2)^2 + q_1^2} + \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2 + q_2^2} \right) \Bigg]
\end{aligned} \qquad (30)
$$

$$
\begin{aligned}
L_{16}(\theta,\kappa) &= L_{16\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2} L_{16\mathrm{B}}(\theta,\kappa) \\
&= \frac{1}{2} \Bigg[-\frac{\rho_1}{2} \left(\frac{1}{(\rho_1/2)^2 + q_1^2} - \frac{1}{(\rho_1/2)^2 + q_2^2} \right) + e^{-\rho_1\theta/2} \\
&\times \left(\frac{(\rho_1/2)\cos q_1\theta - q_1 \sin q_1\theta}{(\rho_1/2)^2 + q_1^2} - \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2 + q_2^2} \right) \Bigg]
\end{aligned} \qquad (31)
$$

$$
\begin{aligned}
L_{17}(\theta,\kappa) &= L_{17\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2} L_{17\mathrm{B}}(\theta,\kappa) \\
&= \frac{1}{2} \Bigg[\frac{\rho_1}{2} \left(\frac{1}{(\rho_1/2)^2 + q_1^2} + \frac{1}{(\rho_1/2)^2 + q_2^2} \right) - e^{-\rho_1\theta/2} \\
&\times \left(\frac{(\rho_1/2)\cos q_1\theta - q_1 \sin q_1\theta}{(\rho_1/2)^2 + q_1^2} + \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2 + q_2^2} \right) \Bigg]
\end{aligned} \qquad (32)
$$

$$L_{18}(\theta,\kappa) = L_{18\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{18\mathrm{B}}(\theta,\kappa)$$
$$= \frac{1}{2}\left[\frac{q_1}{(\rho_1/2)^2+q_1^2} - \frac{q_2}{(\rho_1/2)^2+q_2^2} - e^{-\rho_1\theta/2}\right.$$
$$\left.\times\left(\frac{(\rho_1/2)\sin q_1\theta + q_1\cos q_1\theta}{(\rho_1/2)^2+q_1^2} - \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2+q_2^2}\right)\right] \quad (33)$$

For the explicit form of the integrals $L_{11}(\theta,\kappa)$ to $L_{14}(\theta,\kappa)$ of Eqs.(20)–(23) we introduce two more auxiliary variables $q_3 = q_3(\kappa)$ and $q_4 = q_4(\kappa)$:

$$q_3 = \frac{1}{2}[(\rho_1^2-d^2)^{1/2} - \rho_1],\; q_4 = \frac{1}{2}[(\rho_1^2-d^2)^{1/2} + \rho_1], \quad \rho_1^2 - d^2 > 0 \quad (34)$$

The integrals $L_{11}(\theta,\kappa)$ to $L_{14}(\theta,\kappa)$ may then be written explicitly in the following form:

$$L_{11}(\theta,\kappa) = L_{11\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{11\mathrm{B}}(\theta,\kappa)$$
$$= \frac{1}{2}\left[-\frac{q_3}{q_3^2+(2\pi\kappa/N_\tau)^2} - \frac{q_4}{q_4^2+(2\pi\kappa/N_\tau)^2} + e^{-\rho_1\theta/2}\right.$$
$$\times\left(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2][q_3\cos(2\pi\kappa\theta/N_\tau) + (2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)]}{q_3^2+(2\pi\kappa/N_\tau)^2}\right.$$
$$\left.\left.+\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2][q_4\cos(2\pi\kappa\theta/N_\tau) - (2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)]}{q_4^2+(2\pi\kappa/N_\tau)^2}\right)\right] \quad (35)$$

$$L_{12}(\theta,\kappa) = L_{12\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{12\mathrm{B}}(\theta,\kappa)$$
$$= \frac{1}{2}\left[\frac{2\pi\kappa/N_\tau}{q_3^2+(2\pi\kappa/N_\tau)^2} - \frac{2\pi\kappa/N_\tau}{q_4^2+(2\pi\kappa/N_\tau)^2} + e^{-\rho_1\theta/2}\right.$$
$$\times\left(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2][q_3\sin(2\pi\kappa\theta/N_\tau) - (2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)]}{q_3^2+(2\pi\kappa/N_\tau)^2}\right.$$
$$\left.\left.+\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2][q_4\sin(2\pi\kappa\theta/N_\tau) + (2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)]}{q_4^2+(2\pi\kappa/N_\tau)^2}\right)\right] \quad (36)$$

$$L_{13}(\theta,\kappa) = L_{13\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{13\mathrm{B}}(\theta,\kappa)$$
$$= \frac{1}{2}\left[-\frac{q_3}{q_3^2+(2\pi\kappa/N_\tau)^2} + \frac{q_4}{q_4^2+(2\pi\kappa/N_\tau)^2} + e^{-\rho_1\theta/2}\right.$$
$$\times\left(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2][q_3\cos(2\pi\kappa\theta/N_\tau) + (2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)]}{q_3^2+(2\pi\kappa/N_\tau)^2}\right.$$
$$\left.\left.-\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2][q_4\cos(2\pi\kappa\theta/N_\tau) - (2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)]}{q_4^2+(2\pi\kappa/N_\tau)^2}\right)\right] \quad (37)$$

$$
\begin{aligned}
L_{14}(\theta,\kappa) &= L_{14\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2} L_{14\mathrm{B}}(\theta,\kappa) \\
&= \frac{1}{2}\left[\frac{2\pi\kappa/N_\tau}{q_3^2 + (2\pi\kappa/N_\tau)^2} + \frac{2\pi\kappa/N_\tau}{q_4^2 + (2\pi\kappa/N_\tau)^2} + e^{-\rho_1\theta/2}\right. \\
&\times \left(\frac{\exp[(\rho_1^2 - d^2)^{1/2}\theta/2][q_3 \sin(2\pi\kappa\theta/N_\tau) - (2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)]}{q_3^2 + (2\pi\kappa/N_\tau)^2}\right. \\
&- \left.\left.\frac{\exp[-(\rho_1^2 - d^2)^{1/2}\theta/2][q_4 \sin(2\pi\kappa\theta/N_\tau) + (2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)]}{q_4^2 + (2\pi\kappa/N_\tau)^2}\right)\right]
\end{aligned} \tag{38}
$$

Equation (28) may be written in the following form:

$$
\begin{aligned}
A_{\mathrm{ev}}(\zeta,\theta) &= c^2\tau^2 V_{\mathrm{e}0}\left(\frac{1}{\rho_2^2} e^{-\rho_2\zeta}(1 - \operatorname{ch}\rho_2\theta)\right. \\
&+ \frac{2}{N_\tau}(1 - e^{-N_\tau\rho_2}) \sum_{\kappa=1}^{<K} \left\{\left(L_{13\mathrm{A}}(\kappa) + \frac{\rho_1 L_{11\mathrm{A}}(\kappa)}{(\rho_1^2 - d^2)^{1/2}}\right)\sin\frac{2\pi\kappa\theta}{N_\tau}\right. \\
&- \left(L_{14\mathrm{A}}(\kappa) + \frac{\rho_1 L_{12\mathrm{A}}(\kappa)}{(\rho_1^2 - d^2)^{1/2}}\right)\cos\frac{2\pi\kappa\theta}{N_\tau} \\
&+ e^{-\rho_1\theta/2}\left[\left(L_{13\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2 - d^2)^{1/2}}\right)\sin\frac{2\pi\kappa\theta}{N_\tau}\right. \\
&- \left.\left.\left(L_{14\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2 - d^2)^{1/2}}\right)\cos\frac{2\pi\kappa\theta}{N_\tau}\right]\right\}\frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2 + \rho_2^2} \\
&+ \frac{2}{N_\tau}(1 - e^{-N_\tau\rho_2}) \sum_{\kappa>K}^{N_\tau} \left\{\left(L_{17\mathrm{A}}(\kappa) + \frac{\rho_1 L_{15\mathrm{A}}(\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\sin\frac{2\pi\kappa\theta}{N_\tau}\right. \\
&- \left(L_{18\mathrm{A}}(\kappa) + \frac{\rho_1 L_{16\mathrm{A}}(\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\cos\frac{2\pi\kappa\theta}{N_\tau} \\
&+ e^{-\rho_1\theta/2}\left[\left(L_{17\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\sin\frac{2\pi\kappa\theta}{N_\tau}\right. \\
&- \left.\left.\left.\left(L_{18\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\cos\frac{2\pi\kappa\theta}{N_\tau}\right]\right\}\frac{\sin(2\pi\kappa\zeta/N_\tau)}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}\right)
\end{aligned} \tag{39}
$$

We recognize that the terms $L_{..\mathrm{A}}(\kappa)$ do not contain the variable θ and are connected with ζ and θ only via the products with $\sin(2\pi\kappa\theta/N_\tau)$, $\cos(2\pi\kappa\theta/N_\tau)$, and $\sin(2\pi\kappa\zeta/N_\tau)$. These terms have been quantized successfully (Harmuth, Barrett, Meffert 2001, Cha. 4). The terms $L_{..\mathrm{B}}(\theta,\kappa)$ contain θ and they are multiplied in addition by $e^{-\rho_1\theta/2}$. If we succeed in eliminating θ from $L_{..\mathrm{B}}(\theta,\kappa)$

and further eliminate $e^{-\rho_1\theta/2}$ we can use the previously developed quantization process. This elimination is possible by means of Fourier series expansions. The calculations are straight forward but lengthy. They will be found in Section 6.1. We recognize in Eq.(6.1-43) the last two lines of Eq.(39), and in Eq.(6.1-60) lines 4 and 5 of Eq.(39). The whole Eq.(39) is shown in Eq.(6.1-61), still with the factors $e^{-\rho_1\theta/2}$ and other features that need reworking. The required changes are carried out in the calculations that lead from Eq.(6.1-61) to (6.1-109) and its radically simplified form shown by Eq.(6.1-110). We copy this equation without the term $A_{\mathrm{e0}}(\zeta,\theta)$. If we did not ignore this term we would get terms $[\rho_2^{-2}\exp(-\rho_2\zeta)]^2$ and $\rho_2^{-2}\exp(-\rho_2\zeta)\sin(2\pi\kappa\zeta/N_\tau)$ in $(\partial A_{\mathrm{ev}}/d\zeta)^2$ in Eq.(2.3-16) in the following Section 2.3. The integrals of these terms over ζ would be small compared with those of terms not containing $\exp(-\rho_2\zeta)$:

$$\begin{aligned}
A_{\mathrm{ev}}(\zeta,\theta) &= c^2\tau^2 V_{\mathrm{e0}}\left(\sum_{\kappa=1}^{<K} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa>K}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)\\
&= c^2\tau^2 V_{\mathrm{e0}}\sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\\
C_{\mathrm{e}\kappa}(\theta) &= \frac{2}{N_\tau}\left[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right.\\
&\quad\left. + \sum_{\nu=1}^{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\right]
\end{aligned} \tag{40}$$

We write $A_{\mathrm{ev}}(\zeta,\theta)$ both as two sums from $\kappa=1$ to $\kappa<K$ and from $\kappa>K$ to N_τ as well as a single sum from $\kappa=1$ to N_τ. The single sum is shorter to write but the two sums emphasize that the function $C_{\mathrm{e}\kappa}(\theta)$ is calculated differently in the two intervals $1\le\kappa<K$ and $K<\kappa\le N_\tau$.

Let us turn to the potential $A_{\mathrm{mv}}(\zeta,\theta)$ that is associated with $A_{\mathrm{ev}}(\zeta,\theta)$ according to Eq.(2.1-49). Three integrals have to be evaluated. Here is the first one:

$$\begin{aligned}
A_{\mathrm{mv1}}(\zeta,\theta) &= Z\rho_{\mathrm{s}}\int\frac{\partial^2 A_{\mathrm{ev}}(\zeta,\theta)}{\partial\zeta^2}d\zeta = Z\rho_{\mathrm{s}}\frac{\partial A_{\mathrm{ev}}(\zeta,\theta)}{\partial\zeta}\\
&= c^2\tau^2 V_{\mathrm{e0}} Z\rho_{\mathrm{s}}\frac{\partial}{\partial\zeta}\sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\\
&= c^2\tau^2 V_{\mathrm{e0}} Z\rho_{\mathrm{s}}\sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa}{N_\tau}C_{\mathrm{e}\kappa}(\theta)\cos\frac{2\pi\kappa\zeta}{N_\tau}
\end{aligned} \tag{41}$$

The third integral in Eq.(2.1-49) yields:

$$
\begin{aligned}
A_{\mathrm{mv3}}(\zeta,\theta) &= -Z\int \frac{\partial A_{\mathrm{ev}}}{\partial\theta}\,d\zeta \\
&= c^2\tau^2 V_{\mathrm{e0}} Z \frac{\partial}{\partial\theta}\sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\frac{\cos(2\pi\kappa\zeta/N_\tau)}{2\pi\kappa/N_\tau} \\
&= c^2\tau^2 V_{\mathrm{e0}} Z \sum_{\kappa=1}^{N_\tau} \frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\,\frac{\cos(2\pi\kappa\zeta/N_\tau)}{2\pi\kappa/N_\tau}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta} &= \frac{2}{N_\tau}\left[\frac{2\pi\kappa}{N_\tau}\left(A_{\mathrm{es}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau} - A_{\mathrm{ec}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}\right)\right. \\
&\left. + \sum_{\nu=1}^{N_\tau}\frac{2\pi\nu}{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau} - B_{\mathrm{ec}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau}\right)\right] \qquad (42)
\end{aligned}
$$

The second integral in Eq.(2.1-49) calls for a second differentiation of $A_{\mathrm{mv3}}(\zeta,\theta)$ with respect to θ:

$$
\begin{aligned}
A_{\mathrm{mv2}}(\zeta,\theta) &= -Z\rho_{\mathrm{s}}\int \frac{\partial^2 A_{\mathrm{ev}}(\zeta,\theta)}{\partial\theta^2}\,d\zeta \\
&= c^2\tau^2 V_{\mathrm{e0}} Z\rho_{\mathrm{s}} \frac{\partial^2}{\partial\theta^2}\sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\frac{\cos(2\pi\kappa\zeta/N_\tau)}{2\pi\kappa/N_\tau} \\
&= c^2\tau^2 V_{\mathrm{e0}} Z\rho_{\mathrm{s}} \sum_{\kappa=1}^{N_\tau} \frac{\partial^2 C_{\mathrm{e}\kappa}(\theta)}{\partial\theta^2}\,\frac{\cos(2\pi\kappa\zeta/N_\tau)}{2\pi\kappa/N_\tau}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 C_{\mathrm{e}\kappa}(\theta)}{\partial\theta^2} &= -\frac{2}{N\tau}\left[\left(\frac{2\pi\kappa}{N_\tau}\right)^2\left(A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right)\right. \\
&\left. + \sum_{\nu=1}^{N_\tau}\left(\frac{2\pi\nu}{N_\tau}\right)^2\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\right] \qquad (43)
\end{aligned}
$$

The potential $A_{\mathrm{mv}}(\zeta,\theta) = A_{\mathrm{mve}}(\zeta,\theta)$ associated with $A_{\mathrm{ev}}(\zeta,\theta)$ of Eq.(40) is the sum of A_{mv1}, A_{mv2}, and A_{mv3}:

$$
\begin{aligned}
A_{\mathrm{mv}}(\zeta,\theta) &= A_{\mathrm{mv1}}(\zeta,\theta) + A_{\mathrm{mv2}}(\zeta,\theta) + A_{\mathrm{mv3}}(\zeta,\theta) \\
&= c^2\tau^2 V_{\mathrm{e0}} Z \sum_{\kappa=1}^{N_\tau} C_{\mathrm{m}\kappa}(\theta)\cos\frac{2\pi\kappa\zeta}{N_\tau}
\end{aligned}
$$

$$C_{\mathrm{m}\kappa}(\theta) = \frac{2\pi\kappa\rho_{\mathrm{s}}}{N_\tau} C_{\mathrm{e}\kappa}(\theta) + \frac{1}{2\pi\kappa/N_\tau}\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta} + \frac{\rho_{\mathrm{s}}}{2\pi\kappa/N_\tau}\frac{\partial^2 C_{\mathrm{e}\kappa}(\theta)}{\partial\theta^2}$$
$$= \frac{2}{N_\tau}\Bigg[- A_{\mathrm{ec}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{es}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$
$$+ \sum_{\nu=1}^{N_\tau}\left(C_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + C_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\Bigg]$$

$$C_{\mathrm{es}}(\kappa,\nu) = \frac{2\pi\kappa\rho_{\mathrm{s}}}{N_\tau}\left(1 - \frac{(2\pi\nu)^2}{(2\pi\kappa)^2}\right) B_{\mathrm{es}}(\kappa,\nu) - \frac{2\pi\nu}{2\pi\kappa} B_{\mathrm{ec}}(\kappa,\nu)$$
$$C_{\mathrm{ec}}(\kappa,\nu) = \frac{2\pi\kappa\rho_{\mathrm{s}}}{N_\tau}\left(1 - \frac{(2\pi\nu)^2}{(2\pi\kappa)^2}\right) B_{\mathrm{ec}}(\kappa,\nu) + \frac{2\pi\nu}{2\pi\kappa} B_{\mathrm{es}}(\kappa,\nu) \tag{44}$$

We emphasize that the coefficients $A_{\mathrm{ec}}(\kappa)$, $A_{\mathrm{es}}(\kappa)$, $B_{\mathrm{es}}(\kappa,\nu)$ and $B_{\mathrm{ec}}(\kappa,\nu)$ are calculated differently in the two intervals $1 \leq \kappa < K$ and $K < \kappa \leq N_\tau$.

2.3 Hamilton Function for Planar Wave

The use of the Fourier series expansion in Eq.(2.2-4) permits a largest time $t = T$ and a largest distance $y = cT$, where T is arbitrarily large but finite. In the directions x and z, which are perpendicular to the direction y of propagation, we have not specified any intervals. We chose them $-L/2 \leq x \leq L/2$, $-L/2 \leq z \leq L/2$. For $L/2 = cT$ no point on the y-axis can be reached during the time interval $0 \leq t \leq T$ from a point outside the area $-L/2 \leq x \leq L/2$, $-L/2 \leq z \leq L/2$ by a wave and thus cannot affect what is being observed along the y-axis. The energy U of the electric and magnetic field strength within the volume $0 \leq y \leq cT$, $-L/2 \leq x \leq L/2$, $-L/2 \leq z \leq L/2$ is defined by the integral[1]:

$$U = \frac{1}{2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\left[\int_0^{cT}\left(\frac{1}{Zc}E^2 + \frac{Z}{c}H^2\right)dy\right]dx\,dz \tag{1}$$
$$E^2 = \left(- Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} - \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2 \tag{2}$$
$$H^2 = \left(\frac{c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}} - \frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2 \tag{3}$$

Using Eqs.(2.1-33), (2.1-36), and (2.1-37) we obtain the following values for the components of $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$ in Eqs.(2) and (3):

[1] Harmuth, Barrett, Meffert 2001, Sec 4.3

$$\begin{aligned}
&A_{\mathrm{ex}}(\zeta,\theta)=A_{\mathrm{ev}}(\zeta,\theta) && A_{\mathrm{mx}}(\zeta,\theta)=A_{\mathrm{mv}}(\zeta,\theta)\\
&A_{\mathrm{ey}}(\zeta,\theta)=A_{\mathrm{e0}}f_{\mathrm{e}}(\zeta-\theta) && A_{\mathrm{my}}(\zeta,\theta)=A_{\mathrm{m0}}f_{\mathrm{m}}(\zeta-\theta)\\
&A_{\mathrm{ez}}(\zeta,\theta)=A_{\mathrm{ev}}(\zeta,\theta) && A_{\mathrm{mz}}(\zeta,\theta)=-A_{\mathrm{mv}}(\zeta,\theta)
\end{aligned}\tag{4}$$

The functions $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$ are defined by Eqs.(2.2-40) and (2.2-44), while $f_{\mathrm{e}}(\zeta-\theta)$ and $f_{\mathrm{m}}(\zeta-\theta)$ are arbitrary functions of $\zeta-\theta$.

The next task is to write the vector components on the right sides of Eqs.(2) and (3) in terms of $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$:

$$\left(-Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}}-\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2=Z^2c^2\,\mathrm{curl}^2\,\mathbf{A}_{\mathrm{e}}+2Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}}\cdot\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}+\left(\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2\tag{5}$$

$$\left(\frac{c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}}-\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2=\frac{c^2}{Z^2}\,\mathrm{curl}^2\,\mathbf{A}_{\mathrm{m}}-\frac{2c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}}\cdot\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}+\left(\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2\tag{6}$$

With the help of Eqs.(2.1-20), (2.1-21), (2.1-30), and (2.1-33) we obtain the following relations:

$$\theta=t/\tau,\ \zeta=y/c\tau,\ T/\tau=N_\tau$$

$$\mathrm{curl}^2\,\mathbf{A}_{\mathrm{e}}=2\left(\frac{\partial A_{\mathrm{ev}}}{\partial y}\right)^2=\frac{2}{c^2\tau^2}\left(\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\right)^2\tag{7}$$

$$\mathrm{curl}\,\mathbf{A}_{\mathrm{e}}\cdot\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}=2\frac{\partial A_{\mathrm{ev}}}{\partial y}\frac{\partial A_{\mathrm{mv}}}{\partial t}=\frac{2}{c\tau^2}\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\tag{8}$$

$$\left(\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2=2\left(\frac{\partial A_{\mathrm{mv}}}{\partial t}\right)^2=\frac{2}{\tau^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\tag{9}$$

$$\mathrm{curl}^2\,\mathbf{A}_{\mathrm{m}}=2\left(\frac{\partial A_{\mathrm{mv}}}{\partial y}\right)^2=\frac{2}{c^2\tau^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2\tag{10}$$

$$\mathrm{curl}\mathbf{A}_{\mathrm{m}}\cdot\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}=-2\frac{\partial A_{\mathrm{mv}}}{\partial y}\frac{\partial A_{\mathrm{ev}}}{\partial t}=-\frac{2}{c\tau^2}\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\tag{11}$$

$$\left(\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2=2\left(\frac{\partial A_{\mathrm{ev}}}{\partial t}\right)^2=\frac{2}{\tau^2}\left(\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\tag{12}$$

The squares E^2 and H^2 of the field strengths in Eqs.(2) and (3) assume the following form:

$$E^2 = \frac{2}{\tau^2}\left[Z^2\left(\frac{\partial A_{\rm ev}}{\partial\zeta}\right)^2 + 2Z\frac{\partial A_{\rm ev}}{\partial\zeta}\frac{\partial A_{\rm mv}}{\partial\theta} + \left(\frac{\partial A_{\rm mv}}{\partial\theta}\right)^2\right]$$
$$= \frac{2}{\tau^2}\left(Z\frac{\partial A_{\rm ev}}{\partial\zeta} + \frac{\partial A_{\rm mv}}{\partial\theta}\right)^2 \qquad (13)$$

$$H^2 = \frac{2}{Z^2\tau^2}\left[\left(\frac{\partial A_{\rm mv}}{\partial\zeta}\right)^2 + 2Z\frac{\partial A_{\rm mv}}{\partial\zeta}\frac{\partial A_{\rm ev}}{\partial\theta} + Z^2\left(\frac{\partial A_{\rm ev}}{\partial\theta}\right)^2\right]$$
$$= \frac{2}{Z^2\tau^2}\left(\frac{\partial A_{\rm mv}}{\partial\zeta} + Z\frac{\partial A_{\rm ev}}{\partial\theta}\right)^2 \qquad (14)$$

The energy U of Eq.(1) is rewritten into the following form:

$$U = \frac{c^2\tau}{Z}\int_{-L/2c\tau}^{L/2c\tau}\int_{-L/2c\tau}^{L/2c\tau}\left\{\int_0^{N_\tau}\left[\left(Z\frac{\partial A_{\rm ev}}{\partial\zeta} + \frac{\partial A_{\rm mv}}{\partial\theta}\right)^2 + \left(\frac{\partial A_{\rm mv}}{\partial\zeta} + Z\frac{\partial A_{\rm ev}}{\partial\theta}\right)^2\right]d\zeta\right\}d\left(\frac{x}{c\tau}\right)d\left(\frac{z}{c\tau}\right)$$

$$= \frac{c^2\tau}{Z}\left(\frac{L}{c\tau}\right)^2\int_0^{N_\tau}\left[Z^2\left(\frac{\partial A_{\rm ev}}{\partial\zeta}\right)^2 + Z^2\left(\frac{\partial A_{\rm ev}}{\partial\theta}\right)^2 + 2Z\left(\frac{\partial A_{\rm ev}}{\partial\zeta}\frac{\partial A_{\rm mv}}{\partial\theta} + \frac{\partial A_{\rm mv}}{\partial\zeta}\frac{\partial A_{\rm ev}}{\partial\theta}\right) + \left(\frac{\partial A_{\rm mv}}{\partial\zeta}\right)^2 + \left(\frac{\partial A_{\rm mv}}{\partial\theta}\right)^2\right]d\zeta \qquad (15)$$

For $A_{\rm ev}(\zeta,\theta)$ and $A_{\rm mv}(\zeta,\theta)$ we use Eqs.(2.2-40) and (2.2-44). Their substitution into Eq.(15) yields the following integrals:

$$\int_0^{N_\tau}\left(\frac{\partial A_{\rm ev}}{\partial\zeta}\right)^2 d\zeta = (c^2\tau^2V_{\rm e0})^2\int_0^{N_\tau}\left(\sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa}{N_\tau}C_{\rm e\kappa}(\theta)\cos\frac{2\pi\kappa\zeta}{N_\tau}\right)^2 d\zeta$$
$$= \frac{N_\tau}{2}(c^2\tau^2V_{\rm e0})^2\sum_{\kappa=1}^{N_\tau}\left(\frac{2\pi\kappa}{N_\tau}\right)^2 C_{\rm e\kappa}^2(\theta) \qquad (16)$$

$$\int_0^{N_\tau}\left(\frac{\partial A_{\rm ev}}{\partial\theta}\right)^2 d\zeta = (c^2\tau^2V_{\rm e0})^2\int_0^{N_\tau}\left(\sum_{\kappa=1}^{N_\tau}\frac{\partial C_{\rm e\kappa}(\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2 d\zeta$$
$$= \frac{N_\tau}{2}(c^2\tau^2V_{\rm e0})^2\sum_{\kappa=1}^{N_\tau}\left(\frac{\partial C_{\rm e\kappa}(\theta)}{\partial\theta}\right)^2 \qquad (17)$$

$$\int_0^{N_\tau} \frac{\partial A_{\text{ev}}}{\partial \zeta} \frac{\partial A_{\text{mv}}}{\partial \theta} d\zeta = (c^2\tau^2 V_{\text{e0}})^2 Z \int_0^{N_\tau} \left(\sum_{\kappa=1}^{N_\tau} \frac{2\pi\kappa}{N_\tau} C_{\text{e}\kappa}(\theta) \frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} \cos^2 \frac{2\pi\kappa\zeta}{N_\tau} \right) d\zeta$$
$$= \frac{N_\tau}{2} (c^2\tau^2 V_{\text{e0}})^2 Z \sum_{\kappa=1}^{N_\tau} \frac{2\pi\kappa}{N_\tau} C_{\text{e}\kappa}(\theta) \frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} \tag{18}$$

$$\int_0^{N_\tau} \frac{\partial A_{\text{mv}}}{\partial \zeta} \frac{\partial A_{\text{ev}}}{\partial \theta} d\zeta = (c^2\tau^2 V_{\text{e0}})^2 Z \int_0^{N_\tau} \left(\sum_{\kappa=1}^{N_\tau} -\frac{2\pi\kappa}{N_\tau} C_{\text{m}\kappa}(\theta) \frac{\partial C_{\text{e}\kappa}(\theta)}{\partial \theta} \sin^2 \frac{2\pi\kappa\zeta}{N_\tau} \right) d\zeta$$
$$= -\frac{N_\tau}{2} (c^2\tau^2 V_{\text{e0}})^2 Z \sum_{\kappa=1}^{N_\tau} \frac{2\pi\kappa}{N_\tau} C_{\text{m}\kappa}(\theta) \frac{\partial C_{\text{e}\kappa}(\theta)}{\partial \theta} \tag{19}$$

$$\int_0^{N_\tau} \left(\frac{\partial A_{\text{mv}}}{\partial \zeta} \right)^2 d\zeta = (c^2\tau^2 V_{\text{e0}} Z)^2 \int_0^{N_\tau} \left(\sum_{\kappa=1}^{N_\tau} -\frac{2\pi\kappa}{N_\tau} C_{\text{m}\kappa}(\theta) \sin \frac{2\pi\kappa\zeta}{N_\tau} \right)^2 d\zeta$$
$$= \frac{N_\tau}{2} (c^2\tau^2 V_{\text{e0}} Z)^2 \sum_{\kappa=1}^{N_\tau} \left(\frac{2\pi\kappa}{N_\tau} \right)^2 C_{\text{m}\kappa}^2(\theta) \tag{20}$$

$$\int_0^{N_\tau} \left(\frac{\partial A_{\text{mv}}}{\partial \theta} \right)^2 d\zeta = (c^2\tau^2 V_{\text{e0}} Z)^2 \int_0^{N_\tau} \left(\sum_{\kappa=1}^{N_\tau} \frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} \cos \frac{2\pi\kappa\zeta}{N_\tau} \right)^2 d\zeta$$
$$= \frac{N_\tau}{2} (c^2\tau^2 V_{\text{e0}} Z)^2 \sum_{\kappa=1}^{N_\tau} \left(\frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} \right)^2 \tag{21}$$

Substitution of Eqs.(16) to (21) into Eq.(15) yields the energy U of the planar wave as sum of the energy of the components $C_{\text{e}\kappa}(\theta)$ and $C_{\text{m}\kappa}(\theta)$:

$$U = \frac{1}{2} Z V_{\text{e0}}^2 L^2 T^3 c^4 \frac{1}{N_\tau^2} \sum_{\kappa=1}^{N_\tau} \left[\left(\frac{2\pi\kappa}{N_\tau} \right)^2 C_{\text{e}\kappa}^2(\theta) + \left(\frac{\partial C_{\text{e}\kappa}(\theta)}{\partial \theta} \right)^2 \right.$$
$$+ \frac{4\pi\kappa}{N_\tau} \left(C_{\text{e}\kappa}(\theta) \frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} - C_{\text{m}\kappa}(\theta) \frac{\partial C_{\text{e}\kappa}(\theta)}{\partial \theta} \right)$$
$$\left. + \left(\frac{2\pi\kappa}{N_\tau} \right)^2 C_{\text{m}\kappa}^2(\theta) + \left(\frac{\partial C_{\text{m}\kappa}(\theta)}{\partial \theta} \right)^2 \right] \tag{22}$$

No function of θ in Eq.(22) has a physical dimension. All dimensions are contained in the factor $ZV_{\mathrm{e}0}^2L^2T^3c^4$, which has the dimension VAs as required for an electromagnetic energy. The text following Eq.(2.1-39) gives the dimension of V_e as As/m^3; the constant $V_{\mathrm{e}0}$ has the same dimension according to Eq.(2.1-62).

Equation (22) can be rewritten into the form of a sum of two quadratic expressions:

$$U = \frac{1}{2}ZV_{\mathrm{e}0}^2L^2T^3c^4\frac{1}{N_\tau^2}\sum_{\kappa=1}^{N_\tau}\Bigg[\left(\frac{2\pi\kappa}{N_\tau}C_{\mathrm{e}\kappa}(\theta)+\frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta}\right)^2 + \left(\frac{2\pi\kappa}{N_\tau}C_{\mathrm{m}\kappa}(\theta)-\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\right)^2\Bigg] \tag{23}$$

In order to derive the Hamilton function for U we must work out the time variation of $C_{\mathrm{e}\kappa}(\theta)$ and $C_{\mathrm{m}\kappa}(\theta)$ explicitly. The functions $C_{\mathrm{e}\kappa}(\theta)$ and $C_{\mathrm{m}\kappa}(\theta)$ are shown in Eq.(2.2-40) and (2.2-44). Their substitution into the two terms with large parentheses in Eq.(23) yields:

$$\frac{2\pi\kappa}{N_\tau}C_{\mathrm{e}\kappa}(\theta)+\frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta} = \frac{2}{N_\tau}\sum_{\nu=1}^{N_\tau}\Bigg[\left(\frac{2\pi\kappa}{N_\tau}B_{\mathrm{es}}(\kappa,\nu)-\frac{2\pi\nu}{N_\tau}C_{\mathrm{ec}}(\kappa,\nu)\right)\sin\frac{2\pi\nu\theta}{N_\tau} + \left(\frac{2\pi\kappa}{N_\tau}B_{\mathrm{ec}}(\kappa,\nu)+\frac{2\pi\nu}{N_\tau}C_{\mathrm{es}}(\kappa,\nu)\right)\cos\frac{2\pi\nu\theta}{N_\tau}\Bigg] \tag{24}$$

$$\frac{2\pi\kappa}{N_\tau}C_{\mathrm{m}\kappa}(\theta)-\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta} = \frac{2}{N_\tau}\sum_{\nu=1}^{N_\tau}\Bigg[\left(\frac{2\pi\nu}{N_\tau}B_{\mathrm{ec}}(\kappa,\nu)+\frac{2\pi\kappa}{N_\tau}C_{\mathrm{es}}(\kappa,\nu)\right)\sin\frac{2\pi\nu\theta}{N_\tau} - \left(\frac{2\pi\nu}{N_\tau}B_{\mathrm{es}}(\kappa,\nu)-\frac{2\pi\kappa}{N_\tau}C_{\mathrm{ec}}(\kappa,\nu)\right)\cos\frac{2\pi\nu\theta}{N_\tau}\Bigg] \tag{25}$$

The sum of the squares of these two expressions yields a very long formula that must be broken into parts to become printable:

$$\left(\frac{2\pi\kappa}{N_\tau}C_{\mathrm{e}\kappa}(\theta)+\frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta}\right)^2 + \left(\frac{2\pi\kappa}{N_\tau}C_{\mathrm{m}\kappa}(\theta)-\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\right)^2 = U_{\mathrm{c}N}(\kappa)+U_{\mathrm{v}N}(\kappa,\theta) \tag{26}$$

The energy U of Eq.(23) consists of the sum of a constant part $U_{\mathrm{c}N}(\kappa)$ and a time-variable part $U_{\mathrm{v}N}(\kappa,\theta)$ with time-average zero:

$$U = \frac{1}{2} Z V_{e0}^2 L^2 T^3 c^4 \frac{1}{N_\tau^2} \sum_{\kappa=1}^{N_\tau} [U_{cN}(\kappa) + U_{vN}(\kappa, \theta)] \tag{27}$$

Considerable effort is required to work out $U_{cN}(\kappa)$ and $U_{vN}(\kappa, \theta)$ from Eqs.(24) to (26):

$$U_{cN}(\kappa) = \frac{2(2\pi\kappa)^2}{N_\tau^4} [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \tag{28}$$

We replace the variable $U_{cN}(\kappa)$ by a new variable $U_{c\kappa}$ which makes the results very similar to previously derived ones[2] except that N_τ is now finite rather than infinite and τ is finite rather than infinitesimal:

$$U_{c\kappa}(\kappa) = N_\tau^4 U_{cN}(\kappa) = 2(2\pi\kappa)^2 [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \tag{29}$$

The functions $U_{cs}^2(\kappa)$ and $U_{cc}^2(\kappa)$ may be written with the help of B_{ec}, B_{es}, C_{ec}, and C_{es} as follows:

$$U_{cs}^2(\kappa) = \sum_{\nu=1}^{N_\tau} \left[\left(B_{ec}(\kappa,\nu) + \frac{\nu}{\kappa} C_{es}(\kappa,\nu) \right)^2 + \left(B_{es}(\kappa,\nu) - \frac{\nu}{\kappa} C_{ec}(\kappa,\nu) \right)^2 \right] \tag{30}$$

$$U_{cc}^2(\kappa) = \sum_{\nu=1}^{N_\tau} \left[\left(\frac{\nu}{\kappa} B_{ec}(\kappa,\nu) + C_{es}(\kappa,\nu) \right)^2 + \left(\frac{\nu}{\kappa} B_{es}(\kappa,\nu) - C_{ec}(\kappa,\nu) \right)^2 \right] \tag{31}$$

In analogy to Eq.(29) we replace the variable $U_{cN}(\kappa, \theta)$ in Eq.(27) by a new variable $U_{v\kappa}(\kappa, \theta)$:

$$\begin{aligned} U_{v\kappa}(\kappa,\theta) = N_\tau^4 U_{vN}(\kappa,\theta) = \sum_{\nu=1}^{N_\tau} U_{v1}(\kappa,\nu) \cos\frac{4\pi\nu\theta}{N_\tau} \\ + \sum_{\substack{\nu=1 \\ \nu\neq\lambda}}^{N_\tau} \sum_{\lambda=1}^{N_\tau} \left(U_{v2}(\kappa,\nu,\lambda) \sin\frac{2\pi\nu\theta}{N_\tau} \sin\frac{2\pi\lambda\theta}{N_\tau} + U_{v3}(\kappa,\nu,\lambda) \cos\frac{2\pi\nu\theta}{N_\tau} \cos\frac{2\pi\lambda\theta}{N_\tau} \right) \\ + \sum_{\nu=1}^{N_\tau} \sum_{\lambda=1}^{N_\tau} U_{v4}(\kappa,\nu,\lambda) \sin\frac{2\pi\nu\theta}{N_\tau} \cos\frac{2\pi\nu\theta}{N_\tau} \end{aligned} \tag{32}$$

The four functions $U_{v1}(\kappa, \nu)$ to $U_{v4}(\kappa, \nu, \lambda)$ also may be written in terms of B_{ec}, B_{es}, C_{ec}, and C_{es}:

[2] Harmuth, Barrett, Meffert 2001, Eqs.4.3-28 to 4.3-46.

$$U_{\mathrm{v1}}(\kappa,\nu) = 2(2\pi\kappa)^2\Big(B_{\mathrm{ec}}^2(\kappa,\nu) - B_{\mathrm{es}}^2(\kappa,\nu) + C_{\mathrm{ec}}^2(\kappa,\nu) - C_{\mathrm{es}}^2(\kappa,\nu)$$
$$+\frac{\nu^2}{\kappa^2}[B_{\mathrm{es}}^2(\kappa,\nu) - B_{\mathrm{ec}}^2(\kappa,\nu) + C_{\mathrm{es}}^2(\kappa,\nu) - C_{\mathrm{ec}}^2(\kappa,\nu)]\Big) \quad (33)$$

$$U_{\mathrm{v2}}(\kappa,\nu,\lambda) = 4(2\pi\kappa)^2\Big(B_{\mathrm{es}}(\kappa,\nu)B_{\mathrm{es}}(\kappa,\lambda) + C_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda)$$
$$+\frac{\lambda}{\kappa}[B_{\mathrm{ec}}(\kappa,\lambda)C_{\mathrm{es}}(\kappa,\nu) - B_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)]$$
$$+\frac{\nu}{\kappa}[B_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda) - B_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)]$$
$$+\frac{\nu\lambda}{\kappa^2}[B_{\mathrm{ec}}(\kappa,\nu)B_{\mathrm{ec}}(\kappa,\lambda) + C_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)]\Big) \quad (34)$$

$$U_{\mathrm{v3}}(\kappa,\nu,\lambda) = 4(2\pi\kappa)^2\Big(B_{\mathrm{ec}}(\kappa,\nu)B_{\mathrm{ec}}(\kappa,\lambda) + C_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)$$
$$+\frac{\lambda}{\kappa}[B_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda) - B_{\mathrm{es}}(\kappa,\lambda)C_{\mathrm{ec}}(\kappa,\nu)]$$
$$+\frac{\nu}{\kappa}[B_{\mathrm{ec}}(\kappa,\lambda)C_{\mathrm{es}}(\kappa,\nu) - B_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)]$$
$$+\frac{\nu\lambda}{\kappa^2}[B_{\mathrm{es}}(\kappa,\nu)B_{\mathrm{es}}(\kappa,\lambda) + C_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda)]\Big) \quad (35)$$

$$U_{\mathrm{v4}}(\kappa,\nu,\lambda) = 4(2\pi\kappa)^2\Big(B_{\mathrm{ec}}(\kappa,\lambda)B_{\mathrm{es}}(\kappa,\nu) + C_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda)$$
$$+\frac{\lambda}{\kappa}[B_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda) - B_{\mathrm{es}}(\kappa,\lambda)C_{\mathrm{es}}(\kappa,\nu)]$$
$$+\frac{\nu}{\kappa}[B_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\lambda) - B_{\mathrm{ec}}(\kappa,\lambda)C_{\mathrm{ec}}(\kappa,\nu)]$$
$$-\frac{\nu\lambda}{\kappa^2}[B_{\mathrm{ec}}(\kappa,\nu)B_{\mathrm{es}}(\kappa,\lambda) + C_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\lambda)]\Big) \quad (36)$$

The term $U_{\mathrm{c}\kappa}(\kappa)$ in Eq.(29) does not depend on the time θ. It represents what we usually call the energy U of the wave in the volume L^2cT defined by Eq.(15). The term $U_{\mathrm{v}\kappa}(\kappa,\theta)$ of Eq.(32) varies with time but its time average is zero. If we write $U_{\mathrm{v}\kappa}(\kappa,t)/T$ instead of $U_{\mathrm{v}\kappa}(\kappa,\theta)$ we have a time variable power with average power or energy equal to zero. No widely accepted interpretation of this result exists yet, but it cannot be dismissed easily since it is a generally encountered result of the theory. For a physical interpretation consider two plates of a capacitor with vacuum between them. An electric field strength drives an electric dipole current through this vacuum. It could be that the dipoles are created by the field strength. But it is also possible that the dipoles are constantly created and annihilated even in the absence of a field strength. The random orientation of the dipoles would prevent any macroscopic effect. An applied electric field strength would orient the dipoles and produce a macroscopic effect in the form of a dipole current. From this point of view an electromagnetic wave propagating in vacuum has an energy due to its excitation, but a fluctuating power with time average zero is created by its interaction with the continuously created and annihilated electric and magnetic dipoles.

We denote the time-independent component of U in Eq.(27) by U_{Nc}:

$$\begin{aligned} U_{\mathrm{Nc}} &= \frac{1}{2} Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N_\tau^2} \sum_{\kappa=1}^{N_\tau} U_{\mathrm{cN}}(\kappa) \\ &= Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N_\tau^2} \sum_{\kappa=1}^{N_\tau} \left(\frac{2\pi\kappa}{N_\tau}\right)^2 [U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \end{aligned} \tag{37}$$

In analogy to Eq.(29) we introduce then a new variable U_{c} in order to eliminate N_τ^4 from Eq.(37):

$$\begin{aligned} U_{\mathrm{c}} &= N_\tau^4 U_{\mathrm{Nc}} \\ &= Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \sum_{\kappa=1}^{N_\tau} (2\pi\kappa)^2 [U_{\mathrm{cs}}(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \end{aligned} \tag{38}$$

The new function U_{c} equals the previously derived function U_{c} in Eq.(4.3-33) of the reference of footnote 2, except that the sum runs from $\kappa = 1$ to N_τ rather than to ∞. Hence, we may follow the procedure used there for the derivation of the Hamilton function and the quantization. First we normalize the energy U_{c} of Eq.(38):

$$\begin{aligned} \mathcal{H} &= U_{\mathrm{c}} / Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \\ &= \sum_{\kappa=1}^{N_\tau} \mathcal{H}_\kappa = \sum_{\kappa=1}^{N_\tau} (2\pi\kappa)^2 [U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \end{aligned} \tag{39}$$

It is shown in Section 6.1, Eq.(6.1-136), that $\mathcal{H}_\kappa$ decreases proportionate to $1/\kappa^2$ for $\kappa \gg 1$:

$$\mathcal{H}_\kappa = (2\pi\kappa)^2[U_{\rm cs}^2(\kappa) + U_{\rm cc}^2(\kappa)] \propto 1/(2\pi\kappa)^2 \quad \text{for } \kappa \gg 1 \tag{40}$$

This decrease implies that the sum of Eq.(39) is absolutely convergent if N_τ is permitted to go to infinity.

For the fluctuating power we see from Eqs.(6.1-143) and (6.1-144) that the slowest decreasing terms are $U_{\rm v1}(\kappa,\nu)$ and $U_{\rm v3}(\kappa,\nu,\lambda)$, which decrease proportionate to $1/(2\pi\kappa)^2$. Hence, $U_{\rm v\kappa}(\theta)$ of Eq.(32) decreases proportionate to $1/(2\pi\kappa)^2$

$$U_{\rm v\kappa}(\kappa,\theta) \propto 1/(2\pi\kappa)^2 \quad \text{for } \kappa \gg 1 \tag{41}$$

and the sum of Eq.(27) over κ would be absolutely convergent for any value of θ and unbounded values of N_τ.

The concept of absolute convergence is a carryover from the differential theory with infinite time or space intervals and infinitesimal resolution in time or space. In a theory with finite intervals T or cT and finite resolution τ or $c\tau$ the concept of convergence is of little meaning. In such a theory every term of a sum is either defined as well as finite and the sum exists or at least one term is infinite or undetermined and the sum does not exist. The computer plots in Section 2.5 will show immediately that everything is defined and finite. Of course, writing the program for the computer plots is as much trouble as analyzing convergence.

Having satisfied ourselves that everything is indeed finite for unbounded values of N_τ we may derive the Hamilton formulas in the usual way. We rewrite $\mathcal{H}_\kappa$ as follows:

$$\begin{aligned}\mathcal{H}_\kappa &= (2\pi\kappa)^2 \left\{ \left[U_{\rm cs}^2(\kappa) + U_{\rm cc}^2(\kappa)\right] \sin^2 2\pi\kappa\theta + \left[U_{\rm cs}^2(\kappa) + U_{\rm cc}^2(\kappa)\right] \cos^2 2\pi\kappa\theta \right\} \\ &= (2\pi\kappa)^2 \left[U_{\rm cs}(\kappa) + iU_{\rm cc}(\kappa)\right] (\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta) \\ &\qquad \times \left[U_{\rm cs}(\kappa) - iU_{\rm cc}(\kappa)\right] (\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\ &= -2\pi i\kappa p_\kappa(\theta) q_\kappa(\theta)\end{aligned} \tag{42}$$

For $p_\kappa(\theta)$ and $q_\kappa(\theta)$ we get:

$$\begin{aligned}p_\kappa(\theta) &= (2\pi i\kappa)^{1/2}[U_{\rm cs}(\kappa) + iU_{\rm cc}(\kappa)](\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta) \\ &= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa) - iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta}\end{aligned} \tag{43}$$

$$\dot{p}_\kappa = \frac{\partial p_\kappa(\theta)}{\partial\theta} = (2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa) - iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta} \tag{44}$$

$$\begin{aligned}q_\kappa(\theta) &= (2\pi i\kappa)^{1/2}[U_{\rm cs}(\kappa) - iU_{\rm cc}(\kappa)](\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\ &= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa) + iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta}\end{aligned} \tag{45}$$

$$\dot{q}_\kappa = \frac{\partial q_\kappa(\theta)}{\partial\theta} = -(2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa) + iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta} \tag{46}$$

The derivatives $\partial\mathcal{H}_\kappa/\partial q_\kappa$ and $\partial\mathcal{H}_\kappa/\partial p_\kappa$ equal:

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -2\pi i\kappa p_\kappa(\theta) = -(2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa) - iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta} \tag{47}$$

$$\frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = -2\pi i\kappa q_\kappa(\theta) = -(2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa) + iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta} \tag{48}$$

The comparison of Eqs.(47) and (48) with Eqs.(44) and (46) yields the proper relations for the components $\mathcal{H}_\kappa$ of the Hamilton function:

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -\dot{p}_\kappa, \quad \frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = \dot{q}_\kappa \tag{49}$$

Equation (42) may be rewritten as done previously[3] by means of the definitions

$$\begin{aligned} a_\kappa &= [U_{\rm cc}(\kappa) - iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta} \\ a_\kappa^* &= [U_{\rm cc}(\kappa) + iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta} \end{aligned} \tag{50}$$

to yield:

$$\mathcal{H} = -i\sum_{\kappa=1}^{N_\tau} 2\pi\kappa p_\kappa q_\kappa = \sum_{\kappa=1}^{N_\tau}(2\pi\kappa)^2 a_\kappa a_\kappa^* = \sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa}{T}\hbar\, b_\kappa b_\kappa^* = \sum_{\kappa=1}^{N_\tau}\mathcal{H}_\kappa$$

$$b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa^* \tag{51}$$

We shall want the relative frequency $r(\kappa)$ or the probability of the energy $U_{{\rm c}\kappa}(\kappa)$ of Eq.(29). It is defined by the fraction

$$r(\kappa) = \frac{U_{{\rm c}\kappa}(\kappa)}{S_{\rm c1}}, \quad S_{\rm c1} = \sum_{\kappa=1}^{N_\tau} U_{{\rm c}\kappa}(\kappa) \tag{52}$$

$$r(\kappa) \doteq \frac{U_{{\rm c}\kappa}(\kappa)}{S_{\rm cK}}, \quad S_{\rm cK} = \sum_{\kappa>K}^{N_\tau} U_{{\rm c}\kappa}(\kappa),\ K = N_\tau(\rho_1^2 - 4\rho_2^2)^{1/2}/4\pi \tag{53}$$

[3]Harmuth, Barrett, Meffert 2001, Eq.(4.4-44)

We note that the term probability rather than probability density must be used due to the finite value of terms with the variable κ, $1 \leq \kappa \leq N_\tau$ or $K \leq \kappa \leq N_\tau$.

2.4 Quantization of the Differential Solution

Equation (2.3-51) is written in a form that permits us to follow the standard ways of quantization. It works both for the Heisenberg and the Schrödinger approach. Consider the Heisenberg approach first. The complex terms b_κ and b_κ^* are replaced by operators b_κ^- and b_κ^+:

$$b_\kappa^* \to b_\kappa^+ = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right), \quad b_\kappa \to b_\kappa^- = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right) \tag{1}$$

Alternately one may interchange b_κ^* and b_κ:

$$b_\kappa \to b_\kappa^+ = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right), \quad b_\kappa^* \to b_\kappa^- = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right) \tag{2}$$

The two ways of quantization are a well known ambiguity of the theory (Becker 1963, 1964, vol. 2, § 52; Heitler 1954, p. 57). We use first Eq.(1) and obtain for $\mathcal{H}_\kappa$ of Eq.(2.3-51):

$$b_\kappa^- b_\kappa^+ = \frac{\mathcal{H}_\kappa T}{2\pi\kappa\hbar} \equiv \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{3}$$

Equation (1) applied to a function Φ yields:

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$
$$\frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{\partial^2}{\partial\zeta^2}\right)\Phi + \frac{1}{2}\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi \tag{4}$$

If we use Eq.(2) rather than Eq.(1) we obtain the following equations instead of Eqs.(3) and (4):

$$b_\kappa^+ b_\kappa^- = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{5}$$
$$\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{\partial}{\partial\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$
$$\frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{\partial^2}{\partial\zeta^2}\right)\Phi - \frac{1}{2}\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi \tag{6}$$

Becker (1964, vol. II, § 15) shows in some detail that the energy eigenvalues

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2, \ldots, N_\tau \tag{7}$$

are obtained from Eq.(4) by means of Heisenberg's matrix method while Eq.(6) yields

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n - \frac{1}{2}\right), \quad n = 0,\ 1,\ 2, \ldots, N_\tau \tag{8}$$

We turn to the Schrödinger approach. The products $b_\kappa^- b_\kappa^+$ or $b_\kappa^+ b_\kappa^-$ are applied to a function Φ:

$$(b_\kappa^- b_\kappa^+)\Phi = \frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi = \lambda_\kappa \Phi$$

$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{9}$$

We are going to use the Schrödinger approach here since we know how to rewrite a differential equation into a difference equation and we know how to solve the obtained difference equation. The substitution

$$\xi = \alpha\zeta \tag{10}$$

produces a standard form of the differential equation of the parabolic cylinder functions

$$\frac{\partial^2\Phi}{\partial\xi^2} + (2\lambda_\kappa - \xi^2)\Phi = 0 \tag{11}$$

The further substitution

$$\Phi = e^{-\xi^2/2}\chi(\xi) \tag{12}$$

leads to the differential equation of the Hermite polynomials[1] for χ:

$$\frac{\partial^2\chi}{\partial\xi^2} - 2\xi\frac{\partial\chi}{\partial\xi} + (2\lambda_\kappa - 1)\chi = 0 \tag{13}$$

We show in a few steps the solution of Eq.(13) since we will need it for comparison with the corresponding difference equation in Section 3.6. One starts with a power series

[1] Becker 1964, vol. II, § 15; Landau and Lifschitz 1966, vol. III, § 23; Abramowitz and Stegun 1964; Smirnov 1961, Part III,2, § 3/156.

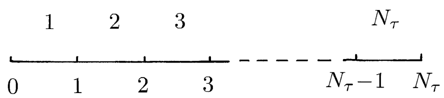

FIG.2.4-1. With $N_\tau + 1$ points one can define N_τ intervals.

$$\chi(\xi) = \sum_{j=0}^{\infty} a_j \xi^j \tag{14}$$

with the two choosable coefficients a_0 and a_1. Substitution of Eq.(14) into Eq.(13) brings

$$\begin{aligned} (j+2)(j+1)a_{j+2} - 2ja_j + (2\lambda_\kappa - 1)a_j &= 0 \\ a_{j+2} &= \frac{2j - (2\lambda_\kappa - 1)}{(j+2)(j+1)} a_j \end{aligned} \tag{15}$$

A polynomial solution with the highest power $j = n$ is obtained for

$$2j = 2n = 2\lambda_\kappa - 1, \quad \lambda_\kappa = n + \frac{1}{2}, \; n = 0,\ 1,\ 2, \ldots \tag{16}$$

If n is even one chooses $a_0 \neq 0$, $a_1 = 0$. For odd values of n one chooses $a_0 = 0$, $a_1 \neq 0$. The first three Hermite polynomials obtained from Eqs.(15) and (16) are:

$$H_0(\xi) = 1,\ H_1(\xi) = 2\xi,\ H_2(\xi) = 4\xi^2 - 2 \tag{17}$$

The energy eigenvalues $\mathsf{E}_\kappa = \mathsf{E}_{\kappa n}$ of Eq.(9) follow from Eq.(16):

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2, \ldots,\ N_\tau \tag{18}$$

The variable κ in Eq.(18) runs from 1 to N_τ as discussed in the text following Eq.(2.2-4). We introduced for ζ an arbitrarily large but finite interval $0 \leq \zeta \leq T/\tau = N_\tau$ in Eq.(2.2-4). The transformation $\xi = \alpha\zeta$ of Eq.(10) does not change the number N_τ of units of the intervals of ζ or ξ. Hence, there are N_τ unit intervals for the variable ξ and $N_\tau + 1$ points for ξ as explained by Fig.2.4-1. The $N_\tau + 1$ points permit us to define $N_\tau + 1$ orthogonal functions—parabolic cylinder functions in our case—and n in Eq.(18) can run from 0 to N_τ.

We obtain only one solution for the energy eigenvalues instead of the two solutions of Eqs.(7) and (8). This is a known problem that will not be discussed here.

The period number $\kappa = 1,\ 2,\ 3,\ \ldots,\ N_\tau$ assumes all positive integer values up to N_τ. The restriction to integers is due to the postulate that in an experimental science all distances in space or time must be finite, even though they can be arbitrarily large or small. Without the requirement for finite large or small distances, the period number κ would be a real, positive number. The number κ denotes the number of periods in the time interval T. If we replaced κ/T by a frequency f we would immediately face the problem that a frequency is defined strictly for an infinitely extended periodic sinusoidal function. If we use only a finite number of periods of such a function we get a *sinusoidal pulse* that may be represented by a superposition of non-denumerably many infinitely extended, periodic sinusoidal functions with frequencies in the whole interval $0 < f < \infty$. One would have to explain which of these many frequencies should be used, a problem not encountered with the period number κ. The restriction of κ to values $\kappa \leq N_\tau = T/\tau$ is due to the introduction of the finite resolution τ or $c\tau$ of any measurement. Waves with periods too short to be observable are excluded.

A second deviation from the conventional theory is caused by the increase of the energy $\mathsf{E}_{\kappa n}$ in Eq.(18) proportionately to κ. This is so even for $n = 0$ and we get the famous or rather infamous infinite zero-point energy that in the past could be overcome only by renormalization. We have pointed out in Eq.(2.3-39) that the component $\mathcal{H}_\kappa$ of the Hamilton function used in Eq.(9) decreases with κ like $1/\kappa^2$. If the energy of one photon increases proportionately to κ in Eq.(18), the number of photons must decrease like $1/\kappa^3$. In classical physics that would mean that there is no photon with more than a certain finite energy. In quantum physics we must modify this statement to say that the probability of observing a photon with infinite energy is zero. The introduction of the finite resolution τ or $c\tau$ yields a second limit on the possible values of κ, since κ cannot exceed the arbitrarily large but finite number N_τ.

So far we have emphasized that we deviate from the conventional theory by using (a) a modification of Maxwell's equations that generally permits solutions that satisfy the causality law, (b) arbitrarily large but finite distances in space or time, and (c) finite resolution in space and time. But there is a fourth difference. The solution $A_{\mathrm{ev}}(\zeta, \theta)$ of Eq.(2.2-28) of the inhomogeneous wave equation of Eq.(2.1-45) is represented by a sum of sinusoidal pulses $\sin(2\pi\kappa\theta/N_\tau)\sin(2\pi\kappa\zeta/N_\tau)$. The individual pulses as well as their sum have a finite energy. We replace this representation of a wave by a superposition of sinusoidal pulses by a superposition of differential operators representing photons. The energy of the photons must equal the energy of the individual pulses with period number κ as well as the energy of all the pulses or the wave. Hence, there is nothing surprising if no infinite energy is encountered. If there are $N_\tau(N_\tau + 1)$ finite energies $\mathsf{E}_{\kappa n}$ their sum must be finite too. The possibility of infinitely many finite energies yielding an infinite sum never arises. The conventional quantization refers to a field rather than to a superposition of pulses representing a wave. It is not obvious how the conservation law of energy applies to a field.

A comprehensive historical review of the infinities of quantum field theory is provided by Weinberg (1995, pp. 31–48). The renowned book by Berestezki, Lifschitz, and Pitajewski (1970, 1982; § 3. Photons, second paragraph) has the following to say about the infinite zero point energy of the conventional theory: *But already in this state each oscillator has the 'zero-point energy'* $2\pi f\hbar/2$, *which differs from zero. The summation over the infinitely many oscillators yields an infinite result. We meet here one of the 'divergencies' that the existing theory contains because it is not complete and not logically consistent.*

At another location in the same book we find the following statement (§ 1. Uncertainty Relations in the Relativistic Theory, second paragraph from the end): *The lack of complete logical consistency shows in this theory by the existence of divergent expressions when the mathematical methods are directly applied; however, there are unambiguous methods for the removal of these divergencies. Nevertheless, these methods have largely the character of semi-empirical recipes and our belief in the correctness of the result obtained in this way is based in the end on their excellent agreement with experiment, but not on the inner consistency and the logic lucidity of the basic principles of the theory.*

Becker writes the following (Becker 1964, vol. II, § 52): *The ground state represented by [n=0] corresponds to vacuum; it still contains zero-point vibrations, however, as in the case of the linear oscillator. Since we are dealing with an infinite number of oscillators the mean-square values of the field strengths* $\overline{\mathbf{E}^2}$, $\overline{\mathbf{H}^2}$, *must also be infinitely great. A completely satisfactory treatment of this anomaly does not yet exist* (footnote p. 311). *The anomaly is directly associated with divergent integrals. This divergence has for long been an insuperable difficulty of quantum theory; it has not yet been completely overcome, but has been ingeniously circumvented through the concept of the mass renormalization of the electron (Kramer, 1945)* (small print following Eq. 53.9).

2.5 Computer Plots for the Differential Theory

The energy $U_{c\kappa}(\kappa)$ is represented in Eq.(2.3-29) by a function of the period number κ. The function is defined for all integer values of κ in the interval $1 \leq \kappa \leq N_\tau$. We want to produce plots of $U_{c\kappa}(\kappa)$. This requires 34 core equations for the interval $1 \leq \kappa < K$ and equally many equations for the interval $K < \kappa \leq N_\tau$. The value of K will turn out to be 1.19366 so that the interval $1 \leq \kappa < K$ contains only the one value $\kappa = 1$. We shall ignore this one value and plot $U_{c\kappa}(\kappa)$ only in the interval $2 \leq \kappa \leq N_\tau$, which covers upward of 99% of all values of κ. In addition to the 34 core equations one needs additional equations for checking the program and plotting the final result. We explain in some detail how the computer program is written to facilitate checking by readers.

As a compromise between precision and computing time we choose $N_\tau = 100$. The further choice $\rho_1 = 1/4$ yields according to Eq.(2.1-43):

$$\sigma\tau(1 - s/\sigma Z^2) = \rho_1/cZ = 1/4 \times 3 \cdot 10^8 \times 377 = 2.2 \cdot 10^{-12}\ [\mathrm{As/Vm}] \tag{1}$$

The second choice $\rho_2 = 1/10 < \rho_1/2$ yields according to Eq.(2.1-43):

$$\tau\sqrt{\sigma s} = \rho_2/c = 1/10 \times 3 \cdot 10^8 = 3.3 \cdot 10^{-10}\ [\mathrm{s/m}] \tag{2}$$

For ρ_s we get from Eq.(2.1-49) two values:

$$\rho_{\mathrm{s}1} = 20,\ \rho_{\mathrm{s}2} = 5 \tag{3}$$

The choice $\rho_1 = 1/4$ and $\rho_2 = 1/10$ is made primarily to obtain simple numbers for ρ_s and to avoid very small or very large numbers that might increase the computing time. Whether the choice is physically reasonable cannot be discussed until we have some values for the electric and magnetic dipole conductivities σ and s.

Following the text of the last paragraph of Section 2.3 we are going to plot $r(\kappa) = U_{\mathrm{c}\kappa}(\kappa)/S_{\mathrm{c}K}$ according to Eq.(2.3-53). Table 2.5-1 gives step by step instructions for the writing of the computer program. First we write $d = d(\kappa)$ according to Eq.(6.1-1), then $q_1 = q_1(\kappa)$ according to Eq.(6.1-2), and so on until $r(\kappa)$ is written according to Eq.(2.3-53). These are our 34 core equations.

All entries in Table 2.5-1 except for $S_{\mathrm{c}K}$ are functions of κ or κ and ν. To check the program one must make in principle 33 plots for d to $r(\kappa)$, since $S_{\mathrm{c}K}$ is only one number. The computation of the 29 plots from d to $C_{\mathrm{ec}}(\kappa,\nu)$ requires seconds or fractions of seconds.

The equations with the one variable κ are plotted by the following instructions written in the programming language Mathematica:

$$\kappa = \texttt{k},\ \texttt{x} = 2\ \texttt{for}\ A_{57}(\kappa)\ \texttt{to}\ F_{68}(\kappa),\ \texttt{x} = N_\tau\ \texttt{otherwise}$$

```
f1 := If[k < 2, True, #]
p1 := Plot[f1, {k, 0, x}, PlotRange-> All]          (4)
```

The terms d, q_1, ... , $F_{68}(\kappa)$ in their computer language representation `d[k]`, `q1[k]`, ... , `f68[k]` must be substituted for #.

The equations with two variables κ, ν are displayed by a three-dimensional plot:

$$\kappa = \texttt{k},\ \nu = \texttt{nu},\ N_\tau = \texttt{n}$$

```
f1 := If[k < 2, True, #]
p1 := Plot3D[f1, {k, 1, n}, {nu, 1, n}, PlotRange-> All]          (5)
```

The terms $I_9(\kappa,\nu)$, $I_{10}(\kappa,\nu)$, ... , $C_{\mathrm{ec}}(\kappa,\nu)$ in their computer language representation `i9[k,nu]`, `i10[k,nu]`, ... `cec[k,nu]` must be substituted for #.

TABLE 2.5-1
THE 34 CORE EQUATIONS REQUIRED FOR A COMPUTER PROGRAM THAT PRODUCES A PLOT OF $r(\kappa)$ ACCORDING TO EQ.(2.3-53).

d	Eq.(6.1-1)	q_1	Eq.(6.1-2)	q_2	Eq.(6.1-3)
$A_{57}(\kappa)$	Eq.(6.1-16)	$B_{68}(\kappa)$	Eq.(6.1-31)	$C_{57}(\kappa)$	Eq.(6.1-18)
$C_{68}(\kappa)$	Eq.(6.1-32)	$D_{57}(\kappa)$	Eq.(6.1-19)	$D_{68}(\kappa)$	Eq.(6.1-33)
$E_{57}(\kappa)$	Eq.(6.1-39)	$E_{68}(\kappa)$	Eq.6.1-40)	$F_{57}(\kappa)$	Eq.(6.1-41)
$F_{68}(\kappa)$	Eq.(6.1-42)	$I_9(\kappa,\nu)$	Eq.(6.1-91)	$I_{10}(\kappa,\nu)$	Eq.(6.1-92)
$I_{11}(\kappa,\nu)$	Eq.(6.1-93)	$I_{12}(\kappa,\nu)$	Eq.(6.1-94)	$I_{13}(\kappa,\nu)$	Eq.(6.1-97)
$I_{14}(\kappa,\nu)$	Eq.(6.1-98)	$I_{15}(\kappa,\nu)$	Eq.(6.1-99)	$I_{16}(\kappa,\nu)$	Eq.(6.1-100)
$I_{17}(\kappa,\nu)$	Eq.(6.1-103)	$I_{18}(\kappa,\nu)$	Eq.(6.1-104)	$I_{19}(\kappa,\nu)$	Eq.(6.1-105)
$I_{20}(\kappa,\nu)$	Eq.(6.1-106)	$B_{\mathrm{es}}(\kappa,\nu)$	Eq.(6.1-107)	$B_{\mathrm{ec}}(\kappa,\nu)$	Eq.(6.1-107)
$C_{\mathrm{es}}(\kappa,\nu)$	Eq.(2.2-44)	$C_{\mathrm{ec}}(\kappa,\nu)$	Eq.(2.2-44)	$U^2_{\mathrm{cs}}(\kappa)$	Eq.(2.3-30)
$U^2_{\mathrm{cc}}(\kappa)$	Eq.(2.2-31)	$U_{\mathrm{c}\kappa}(\kappa)$	Eq.(2.3-29)	$S_{\mathrm{c}K}$	Eq.(2.3-53)
$r(\kappa)$	Eq.(2.3-53)				

The final plot of $r(\kappa)$ is produced by two more equations:

$$\begin{aligned}
&\kappa = \mathtt{k},\ r(\kappa) = \mathtt{r}[\mathtt{k}],\ K = \mathtt{k0},\ S_{\mathrm{c}K} = \mathtt{scK}\\
\mathtt{f01}[\mathtt{k_}] &:= \mathtt{If}[\mathtt{0} <= \mathtt{k} < \mathtt{k0},\ \mathtt{0.81},\ \mathtt{uck}[\mathtt{k}]/\mathtt{scK}]\\
\mathtt{t01} &:= \mathtt{Table}[\{\mathtt{k}, \mathtt{f01}[\mathtt{k}]\}, \{\mathtt{k}, \mathtt{0}, \mathtt{100}\}]\\
\mathtt{p01} &:= \mathtt{ListPlot}[\mathtt{t01}, \mathtt{Prolog}-> \mathtt{AbsolutePointSize}[5],\\
&\qquad \mathtt{PlotRange}-> \mathtt{All}, \mathtt{AxesOrigin}-> \{\mathtt{0},\mathtt{0}\}]
\end{aligned} \tag{6}$$

Figure 2.5-1 shows a plot of $r(\kappa)$ for $\rho_1 = 1/4$, $\rho_2 = 1/10$ $\rho_{\mathrm{s}} = 20$, and $K = 1.19366$ according to Eq.(2.3-53) in the interval $2 \leq \kappa \leq 100 = N_\tau$. Practically all energy is concentrated below $\kappa = 6$. The function drops so fast with increasing values of κ that it appears to be negligible compared with $r(2)$ almost everywhere. One cannot see whether $r(\kappa)$ drops proportionate to $1/\kappa^2$ as demanded by Eq.(2.3-40), but the finite number of values $\kappa = 2, 3, \ldots, N_\tau$ makes it possible to use a numerical plot as proof that $r(\kappa)$ is defined and finite for every value of κ. This would be, of course, not possible for denumerable or non-denumerable values of κ.

The range $2 \leq \kappa \leq 10$ of Fig.2.5-1 is shown expanded in Fig.2.5-2. The rapid drop of $r(\kappa)$ makes only the points for $\kappa = 2, 3, 4, 5$ visible above zero. The semilogarithmic plot of Fig.2.5-3 improves the presentation drastically. We note that the amplitude of $r(\kappa)$ drops essentially from 1 to 10^{-8}. The drop is not as fast as that of an exponential function $e^{-\kappa}$.

We had obtained the two values $\rho_{\mathrm{s}1} = 20$ and $\rho_{\mathrm{s}2} = 5$ in Eq.(3). Figures 2.5-1 to 2.5-3 hold for $\rho_{\mathrm{s}} = 20$. In Figs.2.5-4 to 2.5-6 we show the corresponding plots for $\rho_{\mathrm{s}} = 5$. The plots of Figs.2.5-4 to 2.5-6 look identical to those of Figs.2.5-1 to 2.5-3.

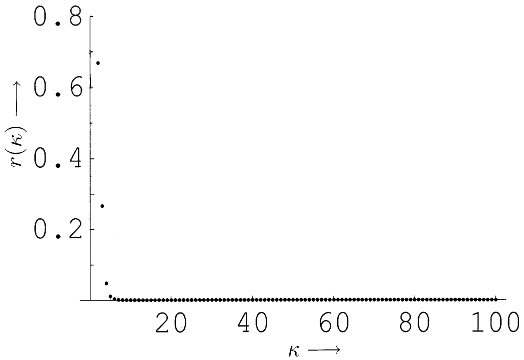

FIGURE 2.5-1. Point-plot of $r(\kappa)$ according to Eq.(2.3-53) for $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, $N_\tau = 100$, and $K = 1.19366$ for $\kappa = 2, 3, \dots, 100$; $S_{cK} = 4.78874 \times 10^{12}$.

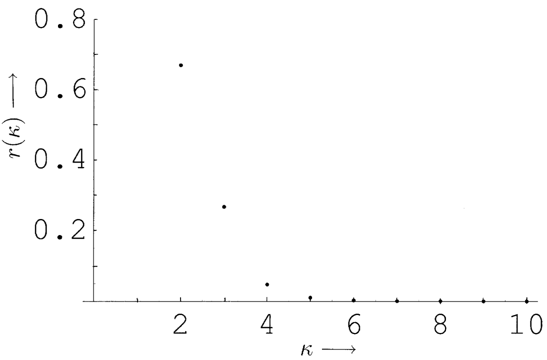

FIGURE 2.5-2. Point-plot of $r(\kappa)$ according to Eq.(2.3-53) for $N_\tau = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$ and $K = 1.19366$ for $\kappa = 2, 3, \dots, 10$; $S_{cK} = 4.78874 \times 10^{12}$.

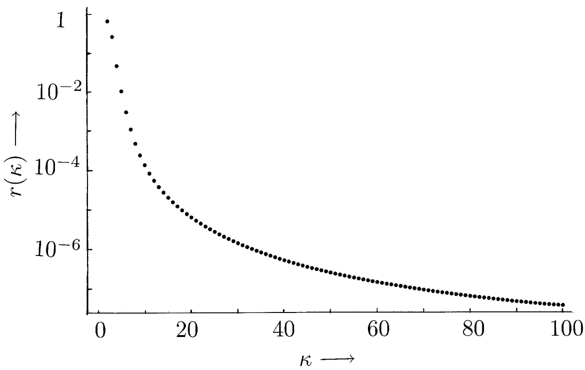

FIGURE 2.5-3. Semilogarithmic point-plot of $r(\kappa)$ according to Fig.2.5-1 or Eq.(2.3-53) for $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, $N_\tau = 100$ and $K = 1.19366$ for $\kappa = 2, 3, \dots, 100$; $S_{cK} = 4.78874 \times 10^{12}$.

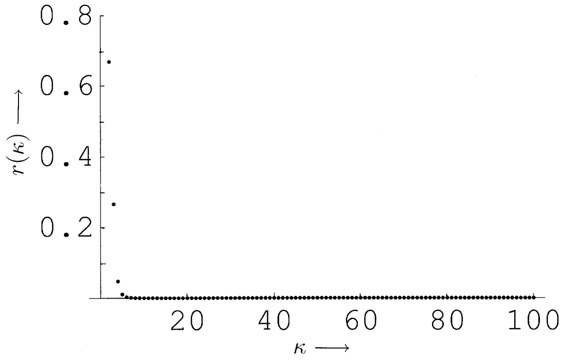

FIGURE 2.5-4. Point-plot of $r(\kappa)$ according to Eq.(2.3-53) for $N_\mathcal{T} = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$ and $K = 1.19366$ for $\kappa = 2, 3, \ldots, 100$; $S_{cK} = 2.99773 \times 10^{11}$.

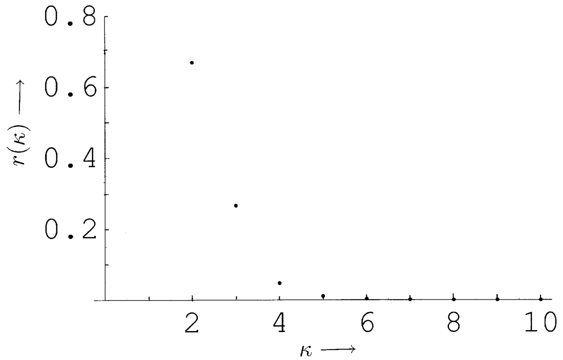

FIGURE 2.5-5. Point-plot of $r(\kappa)$ according to Eq.(2.3-53) for $N_\mathcal{T} = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$ and $K = 1.19366$ for $\kappa = 2, 3, \ldots, 10$; $S_{cK} = 2.99773 \times 10^{11}$.

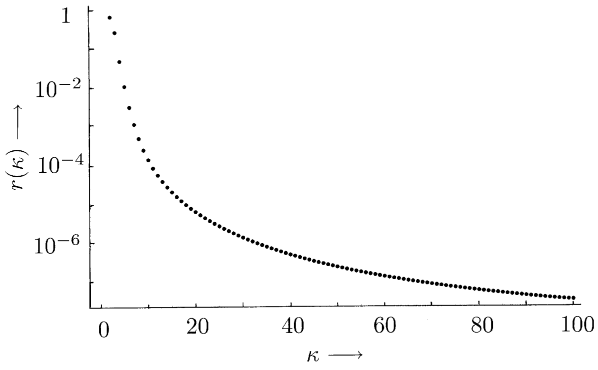

FIGURE 2.5-6. Semilogarithmic point-plot of $r(\kappa)$ according to Fig.2.5-4 or Eq.(2.3-53) for $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$, $N_\mathcal{T} = 100$ and $K = 1.19366$ for $\kappa = 2, 3, \ldots, 100$; $S_{cK} = 2.99773 \times 10^{11}$.

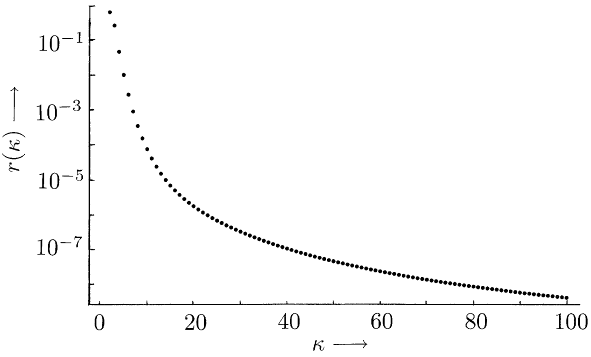

FIGURE 2.5-7. Semilogarithmic point-plot of $r(\kappa)$ according to Eq.(2.3-53) for $\rho_1 = 1/8$, $\rho_2 = 1/20$, $\rho_s = 40$, $N_\tau = 200$ and $K = 1.19366$ for $\kappa = 2, 3, \dots, 100$; $S_{cK} = 9.06075 \times 10^{15}$.

A further semilogarithmic plot for $N_\tau = 200$, $\rho_1 = 1/8$, $\rho_2 = 1/20$, $\rho_s = 40$, $K = 1.19366$ is shown in Fig.2.5-7. The different scale for $r(\kappa)$ shows a difference compared with Figs.2.5-3 and 2.5-6 that would not show up in a linear plot.

3 Difference Equations for the Pure Radiation Field

3.1 Basic Difference Equations

In order to derive a difference equation we rewrite the differential equation (2.1-43)

$$\frac{\partial^2 V_\mathrm{e}}{\partial\zeta^2} - \frac{\partial^2 V_\mathrm{e}}{\partial\theta^2} - \rho_1\frac{\partial V_\mathrm{e}}{\partial\theta} - \rho_2^2 V_\mathrm{e} = 0$$
$$\zeta = y/c\Delta t,\ \theta = t/\Delta t,\ 0 \le \theta \le N,\ 0 \le \zeta \le N,\ N = T/\Delta t$$
$$\rho_1 = c\Delta t(\sigma Z + s/Z) = c^2\Delta t(\sigma\mu + s\epsilon),\ \rho_2^2 = (c\Delta t)^2\sigma s \tag{1}$$

replacing τ by Δt. The symmetric difference quotient with respect to time is used to replace the first order differential[1]:

$$\frac{\partial V_\mathrm{e}(\zeta,\theta)}{\partial\theta} \to \frac{\tilde{\mathit{\Delta}} V_\mathrm{e}(\zeta,\theta)}{\tilde{\mathit{\Delta}}\theta} = \frac{V_\mathrm{e}(\zeta,\theta+\Delta\theta) - V_\mathrm{e}(\zeta,\theta-\Delta\theta)}{2\Delta\theta} \tag{2}$$

To simplify writing it is usual to choose $\Delta\theta = 1$:

$$\frac{\partial V_\mathrm{e}(\zeta,\theta)}{\partial\theta} \to \frac{1}{2}[V_\mathrm{e}(\zeta,\theta+1) - V_\mathrm{e}(\zeta,\theta-1)] \tag{3}$$

The choice $\Delta\theta = 1$ does not reduce the generality of the calculation since one may define a new variable $\theta' = \theta/\Delta\theta$ and then leave out the prime.

The second order difference quotient is practically always used in the symmetric form. The simplified notation $\tilde{\mathit{\Delta}}^2/(\tilde{\mathit{\Delta}}\theta)^2 = \tilde{\mathit{\Delta}}^2/\tilde{\mathit{\Delta}}\theta^2$ and $\tilde{\mathit{\Delta}}^2/(\tilde{\mathit{\Delta}}\zeta)^2 = \tilde{\mathit{\Delta}}^2/\tilde{\mathit{\Delta}}\zeta^2$ is introduced:

---

[1] See Harmuth 1989, Sec. 12.4 why the symmetric difference quotient is chosen over the right or left difference quotient. Note that an italic Delta with tilde $\tilde{\mathit{\Delta}}$ is used for operators like $\tilde{\mathit{\Delta}} V_\mathrm{e}(\zeta,\theta)/\tilde{\mathit{\Delta}}\theta$ but a roman Delta Δ for a difference $\theta + \Delta\theta$.

$$\begin{aligned}
\frac{\partial^2 V_\mathrm{e}}{\partial\theta^2} \to \frac{\tilde{\Delta}^2 V_\mathrm{e}}{(\tilde{\Delta}\theta)^2} = \frac{\tilde{\Delta}^2 V_\mathrm{e}}{\tilde{\Delta}\theta^2} &= \frac{V_\mathrm{e}(\zeta,\theta+\Delta\theta) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta,\theta-\Delta\theta)}{(\Delta\theta)^2} \\
&= V_\mathrm{e}(\zeta,\theta+1) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta,\theta-1) \quad \text{for } \Delta\theta = 1 \\
\frac{\partial^2 V_\mathrm{e}}{\partial\zeta^2} \to \frac{\tilde{\Delta}^2 V_\mathrm{e}}{(\tilde{\Delta}\zeta)^2} = \frac{\tilde{\Delta}^2 V_\mathrm{e}}{\tilde{\Delta}\zeta^2} &= \frac{V_\mathrm{e}(\zeta+\Delta\zeta,\theta) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta-\Delta\zeta,\theta)}{(\Delta\zeta)^2} \\
&= V_\mathrm{e}(\zeta+1,\theta) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta-1,\theta) \quad \text{for } \Delta\zeta = 1 \qquad (4)
\end{aligned}$$

Equation (1) assumes the following form as difference equation:

$$\begin{aligned}
&[V_\mathrm{e}(\zeta+1,\theta) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta-1,\theta)] \\
&\qquad - [V_\mathrm{e}(\zeta,\theta+1) - 2V_\mathrm{e}(\zeta,\theta) + V_\mathrm{e}(\zeta,\theta-1)] \\
&\qquad\quad - \frac{1}{2}\rho_1[V_\mathrm{e}(\zeta,\theta+1) - V_\mathrm{e}(\zeta,\theta-1)] - \rho_2^2 V_\mathrm{e}(\zeta,\theta) = 0 \qquad (5)
\end{aligned}$$

We look for a solution excited by a step function as boundary condition as in Eq.(2.1-56):

$$\begin{aligned}
V_\mathrm{e}(0,\theta) = V_{\mathrm{e}0}S(\theta) &= 0 \qquad \text{for } \theta < 0 \\
&= V_{\mathrm{e}0} \quad \text{for } \theta \geq 0 \qquad (6)
\end{aligned}$$

The boundary condition of Eq.(2.1-57) and the initial condition of Eqs.(2.1-58) become:

$$V_\mathrm{e}(N,\theta) = \text{finite} \quad \text{for } \zeta \to N \gg 1 \qquad (7)$$

$$V_\mathrm{e}(\zeta,0) = 0 \qquad (8)$$

Equations (2.1-59) to (2.1-61) show that in the differential case the initial condition $V_\mathrm{e}(\zeta,0) = 0$ leads to a second initial condition $\partial V_\mathrm{e}(\zeta,0)/\partial\theta = 0$. The derivation of the equivalent condition for the difference equation (5) for $\theta = 0$ proceeds as follows:

1. The three terms of the difference quotient of second order with respect to ζ in the first line of Eq.(5) are all equal to zero for $\theta = 0$ due to Eq.(8).
2. The difference quotient of second order with respect to θ in the second line does not exist for $\theta = 0$ since the functional value $V_\mathrm{e}(\zeta,-1)$ does not exist. A difference quotient of second order exists only for $\theta \geq 1$.
3. The last term $\rho_2^2 V_\mathrm{e}(\zeta,0)$ in the third line is zero due to Eq.(8).
4. What remains of Eq.(5) for $\theta = 0$ is the first order difference quotient

$$\frac{1}{2}[V_{\mathrm{e}}(\zeta,1)-V_{\mathrm{e}}(\zeta,-1)]=0$$

This difference quotient does not exist either but it can be replaced by the non-symmetric difference quotient

$$V_{\mathrm{e}}(\zeta,\theta+1)-V_{\mathrm{e}}(\zeta,\theta)=0 \quad \text{for } \theta=0 \tag{9}$$

Let us observe that the differential quotient $\partial V_{\mathrm{e}}(\zeta,\theta)/\partial\theta$ for $\theta = 0$ in Eq.(2.1-61) is derived from the non-symmetric difference quotient

$$\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}[V_{\mathrm{e}}(\zeta,\theta+\Delta\theta)-V_{\mathrm{e}}(\zeta,\theta)] \quad \text{for } \theta=0$$

while for $\theta > 0$ it may be derived from any one of the following three difference quotients:

$$\lim_{\Delta\theta\to 0}\frac{1}{2\Delta\theta}[V_{\mathrm{e}}(\zeta,\theta+\Delta\theta)-V_{\mathrm{e}}(\zeta,\theta-\Delta\theta)]$$
$$\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}[V_{\mathrm{e}}(\zeta,\theta+\Delta\theta)-V_{\mathrm{e}}(\zeta,\theta)]$$
$$\lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}[V_{\mathrm{e}}(\zeta,\theta)-V_{\mathrm{e}}(\zeta,\theta-\Delta\theta)]$$

The method of obtaining a general solution of a partial differential equation as the sum of a steady state solution $F(\zeta)$ plus a deviation $w(\zeta,\theta)$ used in Eq.(2.1-62) is extended to partial difference equations:

$$V_{\mathrm{e}}(\zeta,\theta)=V_{\mathrm{e}0}[F(\zeta)+w(\zeta,\theta)] \tag{10}$$

Substitution of $F(\zeta)$ into Eq.(5) yields:

$$[F(\zeta+1)-2F(\zeta)+F(\zeta-1)]-\rho_2^2F(\zeta)=0 \tag{11}$$

The usual method of solution of Eq.(11) is by means of the ansatz

$$F(\zeta)=A_1v^{\zeta}, \quad F(\zeta+1)=A_1v^{\zeta+1}, \quad F(\zeta-1)=A_1v^{\zeta-1} \tag{12}$$

which yields an equation for v:

$$v^2-(2+\rho_2^2)v+1=0$$
$$v_1=1+\frac{1}{2}\rho_2^2+\rho_2\left(1+\frac{1}{4}\rho_2^2\right)^{1/2}>1$$
$$v_2=1+\frac{1}{2}\rho_2^2-\rho_2\left(1+\frac{1}{4}\rho_2^2\right)^{1/2}<1 \tag{13}$$

The solution v_1 grows beyond all bounds for large values of ζ and is discarded. From Eq.(6) then follows $A_1 = 1$ and we obtain a result that approaches for $\rho_2 \ll 1$ that of Eq.(2.1-64):

$$F(\zeta) = v_2^\zeta = e^{\zeta \ln v_2} \doteq e^{-\rho_2 \zeta}$$

$$\ln v_2 = \ln \left[1 + \frac{1}{2}\rho_2^2 - \rho_2 \left(1 + \frac{1}{4}\rho_2^2 \right)^{1/2} \right] \doteq \ln(1-\rho_2) \doteq -\rho_2 \tag{14}$$

The substitution of $F(\zeta)$ into Eq.(10) transforms the boundary condition of Eq.(6) for V_e into a homogeneous boundary condition for w, which is the purpose of Eq.(10):

$$V_{\mathrm{e}0}[1 + w(0,\theta)] = V_{\mathrm{e}0} \quad \text{for } \theta \geq 0 \tag{15}$$

$$w(0,\theta) = 0 \qquad \text{for } \theta \geq 0 \tag{16}$$

The boundary condition of Eq.(7) becomes:

$$w(N,\theta) = \text{finite for } \zeta \to N \gg 1 \tag{17}$$

The initial conditions of Eqs.(8) and (9) yield:

$$F(\zeta) + w(\zeta,0) = 0, \qquad w(\zeta,0) = -v_2^\zeta \doteq -e^{-\rho_2\zeta} \tag{18}$$

$$w(\zeta,\theta+1) - w(\zeta,\theta) = 0 \quad \text{for } \theta = 0 \tag{19}$$

Substitution of $w(\zeta,\theta)$ into Eq.(5) yields the same equation with V_e replaced by w:

$$\begin{aligned}
&[w(\zeta+1,\theta) - 2w(\zeta,\theta) + w(\zeta-1,\theta)] \\
&\qquad - [w(\zeta,\theta+1) - 2w(\zeta,\theta) + w(\zeta,\theta-1)] \\
&\qquad\qquad - \frac{1}{2}\rho_1[w(\zeta,\theta+1) - w(\zeta,\theta-1)] - \rho_2^2 w(\zeta,\theta) = 0
\end{aligned} \tag{20}$$

Particular solutions $w_\kappa(\zeta,\theta)$ of this equation can be obtained by the extension of Bernoulli's product method for the separation of variables from differential to difference equations:

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \tag{21}$$

Substitution of Eq.(21) into Eq.(20) yields:

$$\begin{aligned}
&[\phi(\zeta+1)\psi(\theta) - 2\phi(\zeta)\psi(\theta) + \phi(\zeta-1)\psi(\theta)] \\
&\qquad - [\phi(\zeta)\psi(\theta+1) - 2\phi(\zeta)\psi(\theta) + \phi(\zeta)\psi(\theta-1)] \\
&\qquad - \frac{1}{2}\rho_1[\phi(\zeta)\psi(\theta+1) - \phi(\zeta)\psi(\theta-1)] - \rho_2^2\phi(\zeta)\psi(\theta) = 0
\end{aligned} \tag{22}$$

Multiplication with $1/\phi(\zeta)\psi(\theta)$ and separation of the variables yields in analogy to the procedure used for differential equations the following equation:

$$\begin{aligned}
&\frac{1}{\phi(\zeta)}[\phi(\zeta+1) - 2\phi(\zeta) + \phi(\zeta-1)] \\
&\quad = \frac{1}{\psi(\theta)}\left\{[\psi(\theta+1) - 2\psi(\theta) + \psi(\theta-1)] + \frac{1}{2}\rho_1[\psi(\theta+1) - \psi(\theta-1)]\right\} \\
&\qquad + \rho_2^2 = -(2\pi\rho_\kappa/N)^2
\end{aligned} \tag{23}$$

We have written a constant $-(2\pi\rho_\kappa/N)^2$ at the end of the equation since a function of ζ can be equal to a function of θ for any ζ and θ only if they are equal to a constant. The constant N will permit the use of an orthogonality interval of length N rather than 1 later on, in analogy to the use of N_τ in Eq.(2.1-72). Two ordinary difference equations are obtained from Eq.(23):

$$[\phi(\zeta+1) - 2\phi(\zeta) + \phi(\zeta-1)] + (2\pi\rho_\kappa/N)^2\phi(\zeta) = 0 \tag{24}$$

$$\begin{aligned}
&[\psi(\theta+1) - 2\psi(\theta) + \psi(\theta-1)] + \frac{1}{2}\rho_1[\psi(\theta+1) - \psi(\theta-1)] \\
&\qquad + [(2\pi\rho_\kappa/N)^2 + \rho_2^2]\psi(\theta) = 0
\end{aligned} \tag{25}$$

For the solution of Eq.(24) we proceed in analogy to Eq.(12) by making the following substitutions

$$\phi(\zeta) = A_2 v^\zeta,\ \phi(\zeta+1) = A_2 v^{\zeta+1},\ \phi(\zeta-1) = A_2 v^{\zeta-1} \tag{26}$$

and obtaining an equation for v:

$$v^2 - [2 - (2\pi\rho_\kappa/N)^2]v + 1 = 0 \tag{27}$$

We shall need the solution for $(2\pi\rho_\kappa/N)^2 \leq 4$ only:

for $(2\pi\rho_\kappa/N)^2 \leq 4$

$$v_3 = 1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \frac{2i\pi\rho_\kappa}{N}\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2} \tag{28}$$

$$v_4 = 1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 - \frac{2i\pi\rho_\kappa}{N}\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2} \tag{29}$$

For small values of $2\pi\rho_\kappa/N$ we may use the following approximations:

$$\text{for } \rho_\kappa/N \ll 1$$

$$v_3 \doteq 1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \frac{2i\pi\rho_\kappa}{N}\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{8}\right) \tag{30}$$

$$v_4 \doteq 1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 - \frac{2i\pi\rho_\kappa}{N}\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{8}\right) \tag{31}$$

Using the exact Eqs.(28) and (29) we observe the relation

$$\left[1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^2 + \left(\frac{2\pi\rho_\kappa}{N}\right)^2\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{4}\right) = 1 \tag{32}$$

which permits us to write v_3^ζ and v_4^ζ as follows:

$$v_3^\zeta = \left\{\left[1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^2 + \left(\frac{2\pi\rho_\kappa}{N}\right)^2\left(1 - \frac{(2\pi\rho_\kappa/N)^2}{4}\right)\right\}^{\zeta/2} e^{i\varphi_\kappa\zeta}$$

$$= e^{i\varphi_\kappa\zeta} = \cos\varphi_\kappa\zeta + i\sin\varphi_\kappa\zeta \tag{33}$$

$$v_4^\zeta = e^{-i\varphi_\kappa\zeta} = \cos\varphi_\kappa\zeta - i\sin\varphi_\kappa\zeta \tag{34}$$

$$\varphi_\kappa = \operatorname{arctg}\frac{(2\pi\rho_\kappa/N)[1 - (2\pi\rho_\kappa/N)^2/4]^{1/2}}{1 - (2\pi\rho_\kappa/N)^2/2} \tag{35}$$

The angle φ_κ in Eq.(35) will be of interest and we develop three approximations:

$$\frac{\varphi_\kappa}{\pi} = \frac{1}{\pi}\operatorname{arctg} x \doteq \frac{x}{\pi} = \frac{(2\rho_\kappa/N)[1 - (2\pi\rho_\kappa/N)^2/4]^{1/2}}{1 - (2\pi\rho_\kappa/N)^2/2} \tag{36}$$

$$\frac{\varphi_\kappa}{\pi} \doteq \frac{2\rho_\kappa}{N}\left[1 - \frac{(2\pi\rho_\kappa/N)^2}{8}\right]\left[1 + \frac{(2\pi\rho_\kappa/N)^2}{2}\right] \doteq \frac{2\rho_\kappa}{N}\left(1 + \frac{3(2\pi\rho_\kappa/N)^2}{8}\right) \tag{37}$$

$$\frac{\varphi_\kappa}{\pi} \doteq \frac{2\rho_\kappa}{N} \tag{38}$$

The real term $\cos\varphi_\kappa\zeta$ and the imaginary term $i\sin\varphi_\kappa\zeta$ in Eqs.(33) and (34) are both solutions. We obtain for $\phi(\zeta)$ the general solution

$$\phi(\zeta) = A_{20}\cos\varphi_\kappa\zeta + A_{21}\sin\varphi_\kappa\zeta \tag{39}$$

with φ_κ defined by Eq.(35). Since we will use Eq.(39) for a Fourier series we must choose $\varphi_\kappa = \varphi_\kappa(\rho_\kappa)$ so that we get an orthogonal system of sine and cosine functions with a maximum of N periods; the variable ζ varies from 0

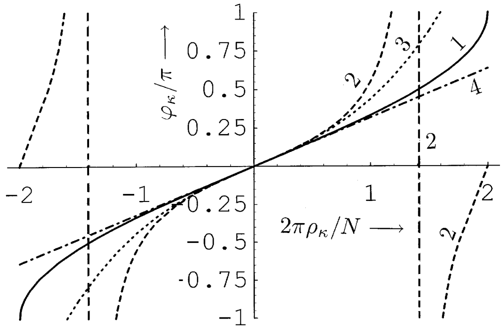

FIGURE 3.1-1. Plots of φ_κ/π according to Eq.(43) (solid line 1), Eq.(36) (dashed line 2), Eq.(37) (dotted line 3), and Eq.(38) (dashed-dotted line 4) in the interval $-2 \leq 2\pi\rho_\kappa/N \leq 2$ or $0 \leq (2\pi\rho_\kappa/N)^2 \leq 4$.

to $N = T/\Delta t$. For an orthogonal system of functions one must choose φ_κ as follows:

$$\begin{aligned} \varphi_\kappa N &= 0,\ \pm 1 \cdot 2\pi,\ \pm 2 \cdot 2\pi,\ \ldots,\ \pm\frac{N}{2} \cdot 2\pi \\ \varphi_\kappa &= 2\pi\frac{\kappa}{N}, \quad \kappa = 0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm N/2 \\ & 0 \leq \zeta \leq N = T/\Delta t \end{aligned} \tag{40}$$

We get $N+1$ values of φ_κ. There are N orthogonal sine and N orthogonal cosine functions. For $\kappa = 0$ we get $\sin\varphi_\kappa\zeta = 0$ and $\cos\varphi_\kappa\zeta = 1$, which is the constant in a Fourier series that is orthogonal to all sine and cosine functions with $\kappa \neq 0$. The function $\phi(\zeta) =$ constant is evidently a solution of Eq.(24) for $\varphi_\kappa = 2\pi\rho_\kappa/N = 0$.

In order to obtain the eigenvalues $(2\pi\rho_\kappa/N)^2$ associated with the angles φ_κ we must solve Eq.(35) for $(2\pi\rho_\kappa/N)^2$. Observing the relation

$$1 + \mathrm{tg}^2\,\varphi_\kappa = \frac{1}{\cos^2\varphi_\kappa}$$

we readily obtain two solutions:

$$\left(\frac{2\pi\rho_\kappa}{N}\right)^2 = 2(1 + \cos\varphi_\kappa) = 4\cos^2\frac{\varphi_\kappa}{2} \tag{41}$$

$$= 2(1 - \cos\varphi_\kappa) = 4\sin^2\frac{\varphi_\kappa}{2} \tag{42}$$

In order to see which solution to use we take the inverse of Eq.(42) and write φ_κ/π instead of φ_κ:

$$\frac{\varphi_\kappa}{\pi} = \frac{2}{\pi}\arcsin\frac{2\pi\rho_\kappa/N}{2} \tag{43}$$

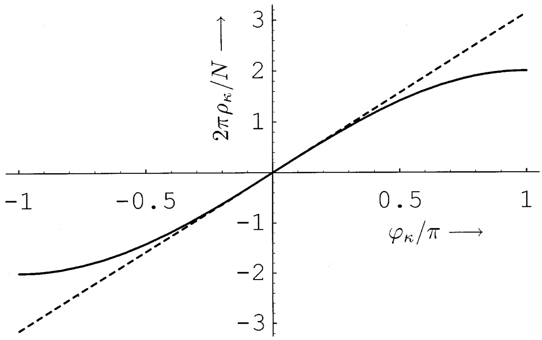

FIGURE 3.1-2. Exact plot of $2\pi\rho_\kappa/N$ according to Eq.(44) (solid line) and its approximation by φ_κ (dashed line) in the interval $-2 \leq \varphi_\kappa/\pi \leq 2$.

A plot of Eq.(43) is shown by the solid line in Fig.3.1-1. In essence this equation is the same as Eq.(35) but the multiple values of inverse trigonometric functions make Eq.(43) produce a simpler plot than Eq.(35). The approximations of Eqs.(36)–(38) are also shown in Fig.3.1-1. The simplest function provided by Eq.(38) yields the best approximation in the interval $-2 \leq 2\pi\rho_\kappa/N \leq 2$.

Since φ_κ is defined by Eq.(40) we are more interested in $2\pi\rho_\kappa/N$ as function of φ_κ than in φ_κ as function of $2\pi\rho_\kappa/N$. Using Eq.(42) and making a series expansion for small values of φ_κ yields:

$$\left(\frac{2\pi\rho_\kappa}{N}\right)^2 = 2(1-\cos\varphi_\kappa) \doteq 2(1-1+\varphi_\kappa^2/2\ \ldots) = \varphi_\kappa^2$$

$$\frac{2\pi\rho_\kappa}{N} = 2\sin\frac{\varphi_\kappa}{2} \doteq \varphi_\kappa \tag{44}$$

The same value is obtained for φ_κ/π as in Eq.(38). Hence, we shall use Eq.(44).

Plots of Eq.(44) for the exact values of $2\pi\rho_\kappa$ and their approximation by φ_κ are shown in Fig.3.1-2. For small values of φ_κ/π the exact plot and its approximation $2\pi\rho_\kappa/N = \varphi_\kappa$ match well. But for $\varphi_\kappa/\pi = 1$ we obtain $2\pi\rho_\kappa/N = 2\sin\pi/2 = 2$ for the exact plot and $2\pi\rho_\kappa/N = \pi$ for the approximation. Hence, there is a significant difference between the result obtained from the difference equation and the differential equation in Section 2.1. Whether this difference leads to an improvement of the theory remains to be shown.

3.2 Time Dependent Solution of $V_\mathrm{e}(\zeta,\theta)$

We turn to the time dependent difference equation of $\psi(\theta)$ shown by Eq.(3.1-25) and the value of $(2\pi\rho_\kappa/N)^2$, which is defined by Eqs.(3.1-41) and (3.1-42):

$$\psi(\theta+1)-2\psi(\theta)+\psi(\theta-1)+\frac{1}{2}\rho_1[\psi(\theta+1)-\psi(\theta-1)] \\ +[(2\pi\rho_\kappa/N)^2+\rho_2^2]\psi(\theta)=0$$

$$\left(\frac{2\pi\rho_\kappa}{N}\right)^2=4\sin^2\frac{\varphi_\kappa}{2},\ \varphi_\kappa=2\pi\frac{\kappa}{N},\ \kappa=0,\ \pm1,\ \dots,\ \pm\frac{N}{2},\ \rho_\kappa=\frac{N}{\pi}\sin\frac{\varphi_\kappa}{2} \tag{1}$$

Using the ansatz $\psi(\theta)=A_3v^\theta$ yields an equation for v:

$$v^2-\frac{2-(2\pi\rho_\kappa/N)^2-\rho_2^2}{1+\rho_1/2}v+\frac{1-\rho_1/2}{1+\rho_1/2}=0 \tag{2}$$

$$v=\frac{1}{1+\rho_1/2}\Bigg\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\rho_2^2 \\ \pm\frac{1}{2}\left[-2(2-\rho_2^2)\left(\frac{2\pi\rho_\kappa}{N}\right)^2-(4-\rho_2^2)\rho_2^2+\left(\frac{2\pi\rho_\kappa}{N}\right)^4+\rho_1^2\right]^{1/2}\Bigg\} \tag{3}$$

We neglect ρ_2^2 compared with 2 or 4. Furthermore, $1/(1+\rho_2/2)$ is replaced by $1-\rho_1/2$:

$$v=\left(1-\frac{\rho_1}{2}\right)\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\pm\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\} \tag{4}$$

The relation

$$-4\rho_2^2+\rho_1^2\geq 0 \tag{5}$$

follows from Eq.(3.1-1):

$$(c\Delta t)^2(-4\sigma s+\sigma^2Z^2+2\sigma s+s^2/Z^2)=(c\Delta t)^2(\sigma Z-s/Z)^2\geq 0 \tag{6}$$

Hence, we get from Eq.(4) a complex solution

$$\text{for } 4(2\pi\rho_\kappa/N)^2>(2\pi\rho_\kappa/N)^4-4\rho_2^2+\rho_1^2$$

$$v_{5,6}=\left(1-\frac{\rho_1}{2}\right)\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\pm\frac{2\pi i\rho_\kappa}{N}\left[1-\frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{4\rho_2^2-\rho_1^2}{4(2\pi\rho_\kappa/N)^2}\right]^{1/2}\right\} \tag{7}$$

and a real solution:

$$\text{for } 4(2\pi\rho_\kappa/N)^2 \le (2\pi\rho_\kappa/N)^4 - 4\rho_2^2 + \rho_1^2$$

$$v_{7,8} = \left(1-\frac{\rho_1}{2}\right)\left[1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 \pm \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\left(1-\frac{4}{(2\pi\rho_\kappa/N)^2}-\frac{4\rho_2^2-\rho_1^2}{(2\pi\rho_\kappa/N)^4}\right)^{1/2}\right] \quad (8)$$

The conditions that make either Eq.(7) or Eq.(8) apply are rewritten as an equation:

$$\begin{aligned}
&(2\pi\rho_\kappa/N)^4 - 4(2\pi\rho_\kappa/N)^2 - 4\rho_2^2 + \rho_1^2 = 0\\
&(2\pi\rho_\kappa/N)^2 \doteq 2 \pm 2(1+\rho_2^2-\rho_1^2/4)^{1/2} \doteq 4 - (-\rho_2^2+\rho_1^2/4) \quad \text{for } +\\
&\qquad \doteq -\rho_2^2 + \rho_1^2/4 \quad \text{for } -
\end{aligned} \quad (9)$$

Hence, if $(2\pi\rho_\kappa/N)^2$ is in the interval

$$-\rho_2^2 + \rho_1^2/4 \le (2\pi\rho_\kappa/N)^2 \le 4 - (-\rho_2^2 + \rho_1^2/4) \quad (10)$$

the complex solution of Eq.(7) will apply while Eq.(8) applies for smaller or larger values of $(2\pi\rho_\kappa/N)^2$. The solutions for $(2\pi\rho_\kappa/N)^2 > 4$ have been eliminated by Eq.(3.1-27) and the text following Eq.(3.1-44). But the case

$$0 \le (2\pi\rho_\kappa/N)^2 < -\rho_2^2 + \rho_1^2/4 \quad (11)$$

is important and will be investigated presently. First we turn to Eqs.(7) and (10).

We assign the positive sign of $2\pi i\rho_\kappa/N$ in Eq.(7) to v_5 and the negative sign to v_6. One may then write v_5 as follows for $\rho_1 \gg \rho_2^2,\ \rho_1^2$:

$$\text{for } -\rho_2^2 + \rho_1^2/4 < (2\pi\rho_\kappa/N)^2 < 4 - (-\rho_2^2 + \rho_1^2/4)$$

$$\begin{aligned}
v_5 = {}& \left(1-\frac{\rho_1}{2}\right)\left\{\left[1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^2\right.\\
&\left. + \left(\frac{2\pi\rho_\kappa}{N}\right)^2\left[1-\frac{(2\pi\rho_\kappa/N)^2}{4}+\frac{4\rho_2^2-\rho_1^2}{4(2\pi\rho_\kappa/N)^2}\right]\right\}^{1/2} e^{i\varphi_{\theta\kappa}}\\
\doteq {}& \left(1-\frac{\rho_1}{2}\right)e^{i\varphi_{\theta\kappa}} \doteq e^{-\rho_1/2}e^{i\varphi_{\theta\kappa}}
\end{aligned} \quad (12)$$

$$\operatorname{tg}\varphi_{\theta\kappa} = \frac{2\pi\rho_\kappa}{N}\left[1-\frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{4\rho_2^2-\rho_1^2}{4\,(2\pi\rho_\kappa/N)^2}\right]^{1/2}\left[1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^{-1} \quad (13)$$

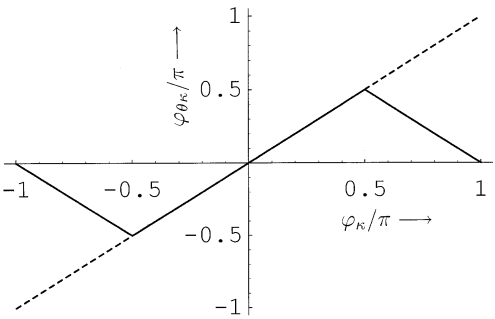

FIGURE 3.2-1. Plot of $\varphi_{\theta\kappa}$ according to Eq.(21) (solid line) and according to Eq.(22) (dashed line).

For v_6 one must replace $\exp(i\varphi_{\theta\kappa})$ by $\exp(-i\varphi_{\theta\kappa})$. We obtain for v_5^θ:

$$v_5^\theta \doteq e^{-\rho_1\theta/2} e^{i\varphi_{\theta\kappa}\theta} \tag{14}$$

Using the relations

$$\operatorname{tg}^2 \varphi_{\theta\kappa} = \frac{\sin^2 \varphi_{\theta\kappa}}{\cos^2 \varphi_{\theta\kappa}}, \quad \cos^2 \varphi_{\theta\kappa} = 1 - \sin^2 \varphi_{\theta\kappa}, \quad 1 \gg \rho_2^2,\ \rho_1^2$$

we obtain the formula

$$\sin^2 \varphi_{\theta\kappa} = \left(\frac{2\pi\rho_\kappa}{N}\right)^2 \left[1 - \frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right] + \rho_2^2 - \frac{1}{4}\rho_1^2$$

$$\varphi_{\theta\kappa} = \arcsin\left[4\left(1 - \sin^2\frac{\varphi_\kappa}{2}\right)\sin^2\frac{\varphi_\kappa}{2} + \rho_2^2 - \frac{1}{4}\rho_1^2\right]^{1/2}$$

$$\varphi_\kappa = 2\pi\kappa/N,\ \kappa = 0,\ \pm1,\ \pm2,\ \ldots,\ \pm N/2,\ 2\pi\rho_\kappa/N = 2\sin(\varphi_\kappa/2) \tag{15}$$

Let us emphasize that the terms $\rho_2^2 - \rho_1^2/4$ in this equation and the following Eqs.(16)–(18) should only be used when $4[1-\sin^2(\varphi_\kappa/2)]\sin^2(\varphi_\kappa/2)$ is close to zero, which means for φ_κ close to 0 or 2π. For small values of φ_κ we get the approximations

$$\varphi_{\theta\kappa} \doteq \left[4\left(1 - \sin^2\frac{\varphi_\kappa}{2}\right)\sin^2\frac{\varphi_\kappa}{2} + \rho_2^2 - \frac{1}{4}\rho_1^2\right]^{1/2}, \quad \varphi_\kappa \ll 1 \tag{16}$$

$$\doteq \left(4\sin^2\frac{\varphi_\kappa}{2} + \rho_2^2 - \frac{1}{4}\rho_1^2\right)^{1/2} \tag{17}$$

$$\doteq \left(\varphi_\kappa^2 + \rho_2^2 - \frac{1}{4}\rho_1^2\right)^{1/2} = \left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \rho_2^2 - \frac{1}{4}\rho_1^2\right]^{1/2} \tag{18}$$

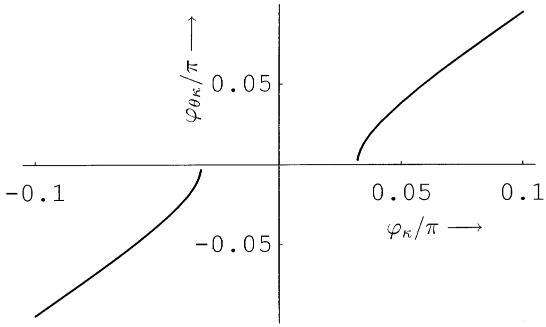

FIGURE 3.2-2. Plots according to Eq.(15) in the neighborhood of $\varphi_\kappa/\pi = 0$ for $\rho_2^2 - \rho_1^2/4 = -0.01$.

For a comparison with the results of the differential theory we return to Eq.(2.1-76) and write $\exp(\gamma_1\theta)$ in more detail:

$$e^{\gamma_1\theta} = e^{-\rho_1\theta/2} e^{i(d^2-\rho_1^2)^{1/2}\theta/2} = e^{-\rho_1\theta/2} e^{i\varphi_{\theta\kappa}\theta} \tag{19}$$

$$\varphi_{\theta\kappa} = \frac{1}{2}(d^2 - \rho_1^2)^{1/2} = \frac{1}{2}\left\{4\left[\left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \rho_2^2\right] - \rho_1^2\right\}^{1/2}$$
$$= \left[\left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \rho_2^2 - \frac{1}{4}\rho_1^2\right]^{1/2} \tag{20}$$

Equations (20) and (18) are equal for $N_\tau = N$ if one uses $\rho_\kappa = \kappa$ for the differential theory.

If φ_κ in Eq.(15) is in the order of $2\pi/N$ we cannot ignore ρ_2^2 and ρ_1^2, but for φ_κ larger than 0.01 we can. Hence, we plot $\varphi_{\theta\kappa}$ in Fig.3.2-1 according to the formula

$$\varphi_{\theta\kappa} = \arcsin\left[4\left(1 - \sin^2\frac{\pi}{2}\frac{\varphi_\kappa}{\pi}\right)\sin^2\frac{\pi}{2}\frac{\varphi_\kappa}{\pi}\right]^{1/2} \tag{21}$$

which turns out to consist of two triangles. The approximation according to Eq.(18)

$$\varphi_{\theta\kappa} \doteq \varphi_\kappa \quad \text{for } \varphi_\kappa^2 \gg \rho_2^2 - \rho_1^2/4 \tag{22}$$

is plotted too. Equation (22) also represents Eq.(20) of the differential theory for $\rho_\kappa = \kappa$ and $\rho_2^2 - \rho_1^2/4 \ll (2\pi\kappa)^2$. There is a perfect fit between Eqs.(21) and (22) in the interval $-\pi/2 \le \varphi_\kappa \le \pi/2$ but a significant deviation outside this interval.

If $\rho_2^2 - \rho_1^2/4$ is not zero we get in Fig.3.2-1 deviations near $\varphi_\kappa/\pi = 0$ and $\varphi_\kappa/\pi = \pm 1$. We note Eq.(5) that shows $\rho_2^2 - \rho_1^2/4$ is never larger than zero.

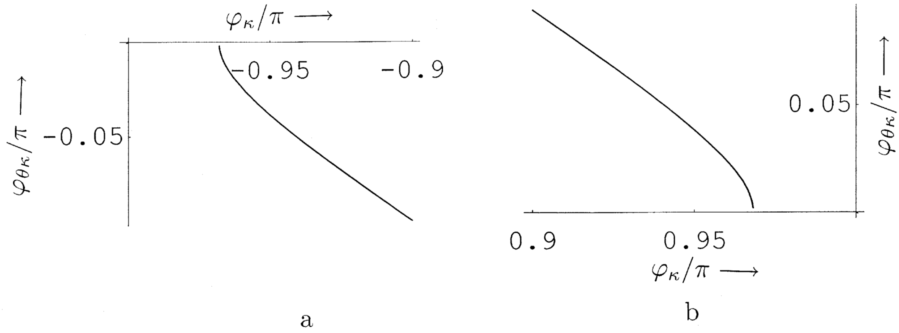

FIGURE 3.2-3. Plots according to Eq.(15) for $\rho_2^2 - \rho_1^2/4 = -0.01$ in the neighborhood of $\varphi_\kappa/\pi = -1$ (a) and $\varphi_\kappa/\pi = +1$ (b).

Figures 3.2-2 and 3.2-3 show plots of Eq.(15) for $\rho_2^2 - \rho_1^2/4 = -0.01$ and φ_κ close to 0 or ± 1.

We turn to Eqs.(11) and (8) to obtain the solutions v_7 and v_8 for the real values of the square root in Eq.(3) for v:

$$\text{for } 4(2\pi\rho_\kappa/N)^2 \leq (2\pi\rho_\kappa/N)^4 - 4\rho_2^2 + \rho_1^2$$

$$v_7 = \left(1 - \frac{\rho_1}{2}\right)\left\{1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4 - 4\left(\frac{2\pi\rho_\kappa}{N}\right)^2 - 4\rho_2^2 + \rho_1^2\right]^{1/2}\right\} \tag{23}$$

$$v_8 = \left(1 - \frac{\rho_1}{2}\right)\left\{1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 - \frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4 - 4\left(\frac{2\pi\rho_\kappa}{N}\right)^2 - 4\rho_2^2 + \rho_1^2\right]^{1/2}\right\} \tag{24}$$

Figures 3.2-4 and 3.2-5 show plots of $v_7/(1-\rho_1/2)$ and $v_8/(1-\rho_1/2)$ for various values of $-\rho_2^2 + \rho_1^2/4$ as function of φ_κ/π. The functions for $-\rho_2^2 + \rho_1^2/4 = 0.01$ fit into the interval $-(-\rho_2^2 + \rho_1^2/4)^{1/2} = -0.1/\pi = -0.0318 \leq \varphi_\kappa/\pi \leq +0.0318$ in Fig.3.2-2. All values of $v_7/(1 - \rho_1/2)$ are larger than 1 and all values of $v_8/(1 - \rho_1/2)$ are smaller than 1.

We are more interested whether v_7 and v_8 are larger or smaller than 1 rather than $v_7/(1-\rho_1/2)$ and $v_8/(1-\rho_1/2)$. Since the largest values of $v_7/(1-\rho_1/2)$ occur for $\varphi_\kappa/\pi = 0$ we investigate Eq.(23) for $\varphi_\kappa/\pi = 0$:

$$\begin{gathered} v_7 = (1 - \rho_1/2)[1 + (-\rho_2^2 + \rho_1^2/4)^{1/2}] \lesseqgtr 1 \\ 1 + (-\rho_2^2 + \rho_1^2/4)^{1/2} \lesseqgtr 1 + \rho_1/2 \end{gathered} \tag{25}$$

We may subtract 1 from both sides and compare the squares of the remaining terms:

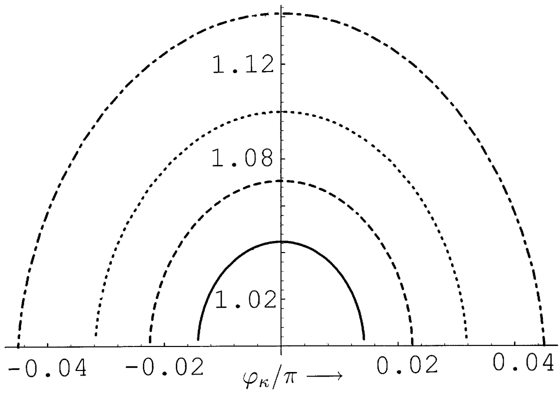

FIGURE 3.2-4. The function $v_7/(1-\rho_1/2)$ according to Eq.(23) for $-\rho_2^2+\rho_1^2/4=0.002$ (solid line), 0.005 (dashed line), 0.01 (dotted line), and 0.02 (dashed-dotted line) in the intervals $-(-\rho_2^2+\rho_1^2/4)^{1/2}/\pi \leq \varphi_\kappa \leq (-\rho_2^2+\rho_1^2/4)^{1/2}/\pi$.

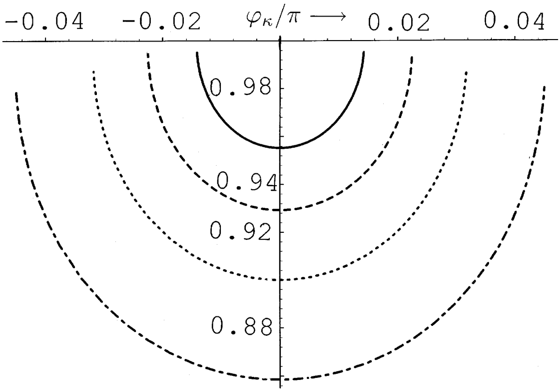

FIGURE 3.2-5. The function $v_8/(1-\rho_1/2)$ according to Eq.(24) for $-\rho_2^2+\rho_1^2/4=0.002$ (solid line), 0.005 (dashed line), 0.01 (dotted line), and 0.02 (dashed-dotted line) in the intervals $-(-\rho_2^2+\rho_1^2/4)^{1/2}/\pi \leq \varphi_\kappa \leq (-\rho_2^2+\rho_1^2/4)^{1/2}/\pi$.

$$-\rho_2^2+\rho_1^2/4<\rho_1^2/4 \tag{26}$$

Since $\rho_1^2/4$ is always larger than $\rho_1^2/4-\rho_2^2$ we conclude that v_7 is always smaller than 1. Hence, both solutions v_7 and v_8 can be used without violating the conservation law of energy. We get:

$$\text{for } 4(2\pi\rho_\kappa/N)^2 \leq (2\pi\rho_\kappa/N)^4-4\rho_2^2+\rho_1^2$$

$$v_7^\theta=e^{-\rho_1\theta/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}^\theta \tag{27}$$

$$v_8^\theta = e^{-\rho_1\theta/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}^\theta \quad (28)$$

In order to be able to write the general solution of $\psi(\theta)$ in a form that resembles Eq.(2.1-76) of the differential theory we may use the definitions

$$\gamma_5 = -\frac{1}{2}\rho_1 + i\varphi_{\theta\kappa} \quad (29)$$

$$\gamma_6 = -\frac{1}{2}\rho_1 - i\varphi_{\theta\kappa} \quad (30)$$

The function $\varphi_{\theta\kappa}$ of Eq.(15) can then be rewritten as follows:

$$\begin{aligned}
\varphi_{\theta\kappa} &= \arcsin\frac{1}{2}(d_\Delta^2-\rho_1^2)^{1/2}\\
d_\Delta^2 &= 4\left[4\left(1-\sin^2\frac{\varphi_\kappa}{2}\right)\sin^2\frac{\varphi_\kappa}{2}+\rho_2^2\right] \geq \rho_1^2\\
&\doteq 4\left[(2\pi\kappa/N)^2+\rho_2^2\right] \quad \text{for } \varphi_\kappa^2 = (2\pi\kappa/N)^2 \ll 1
\end{aligned} \quad (31)$$

Using this notation one obtains for γ_5 and γ_6 the following relations:

$$\begin{aligned}
\gamma_5 &= \frac{1}{2}\left[-\rho_1 + 2i\arcsin\frac{1}{2}\left(d_\Delta^2-\rho_1^2\right)^{1/2}\right] \qquad \text{for } d_\Delta^2 > \rho_1^2\\
&\doteq \frac{1}{2}\left[-\rho_1 + i\left(d_\Delta^2-\rho_1^2\right)^{1/2}\right],\ d_\Delta^2-\rho_1^2 \ll 1\\
\gamma_6 &= \frac{1}{2}\left[-\rho_1 - 2i\arcsin\frac{1}{2}\left(d_\Delta^2-\rho_1^2\right)^{1/2}\right]\\
&\doteq \frac{1}{2}\left[-\rho_1 - i\left(d_\Delta^2-\rho_1^2\right)^{1/2}\right],\ d_\Delta^2-\rho_1^2 \ll 1
\end{aligned}$$

$$\begin{aligned}
\rho_1 &= c\Delta t(\sigma Z + s/Z) = cT(\sigma Z+s/Z)/N\\
\rho_2^2 &= (c\Delta t)^2\sigma s = (cT/N)^2\sigma s
\end{aligned} \quad (32)$$

The relations for γ_5 and γ_6 should be compared with those for γ_1 and γ_2 in Eq.(2.1-77) for the differential theory.

From Eqs.(7) and (8) we obtain for the time-variable part $\psi(\theta)$ of our solution:

$$\begin{aligned}
\psi(\theta) &= A_5v_5^\theta + A_6v_6^\theta + A_7v_7^\theta + A_8v_8^\theta\\
A_5 = A_6 &= 0 \quad \text{for } 4(2\pi\rho_\kappa/N)^2 \leq (2\pi\rho_\kappa/N)^4 - \rho_2^2 + \rho_1^2/4\\
A_7 = A_8 &= 0 \quad \text{for } 4(2\pi\rho_\kappa/N)^2 > (2\pi\rho_\kappa/N)^4 - \rho_2^2 + \rho_1^2/4
\end{aligned} \quad (33)$$

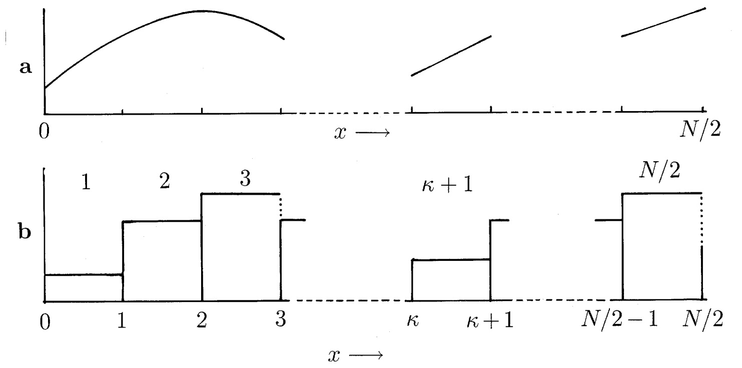

FIGURE 3.2-6. Continuous function $f(x)$ in the interval $0 \le x < N/2$ (a) and the corresponding step function with $\kappa = 0, 1, 2, \ldots, N/2-1$ having $N/2$ steps of width 1 (b).

The solution $w_\kappa(\zeta,\theta)$ according to Eq.(3.1-21) is the product of Eqs.(3.1-39) and (33). The boundary condition of Eq.(3.1-16) demands the coefficient $A_{20} = 0$ in Eq.(3.1-39):

$$\begin{aligned}
w_\kappa(\zeta,\theta) &= (A_5 v_5^\theta + A_6 v_6^\theta + A_7 v_7^\theta + A_8 v_8^\theta) \sin\frac{2\pi\kappa\zeta}{N} \\
&= e^{-\rho_1\theta/2}\Bigg[A_5 \exp(i\varphi_{\theta\kappa}\theta) + A_6 \exp(-i\varphi_{\theta\kappa}\theta) \\
&+A_7\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}^\theta \\
&+A_8\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}^\theta\Bigg] \\
&\qquad\times \sin\frac{2\pi\kappa\zeta}{N} \qquad (34)
\end{aligned}$$

The generalization of Eq.(34) in analogy to Eq.(2.1-81) is achieved by making A_5 to A_8 functions of κ and summing over the values of κ defined by Eqs.(3.1-40) or (15) and permitted by Eq.(33). To determine the summation limit for small values of κ we use the approximation $2\pi\rho_\kappa/N = \varphi_\kappa$ of Eq.(3.1-44):

$$\varphi_\kappa = 2\pi\rho_\kappa/N = 2\pi\kappa/N,\ \rho_\kappa = \kappa, \quad \rho_\mathrm{a} = \rho_1 N,\ \rho_\mathrm{b} = \rho_2 N$$

$$-\rho_2^2 + \rho_1^2/4 \le (2\pi\rho_\kappa/N)^2 = (2\pi\kappa/N)^2$$

$$\kappa \ge K_0 = \frac{1}{4\pi}\left(\rho_a^2 - 4\rho_b^2\right)^{1/2} = \frac{N}{4\pi}(\rho_1^2 - 4\rho_2^2)^{1/2} = \pm\frac{Nc\Delta t}{4\pi}\left|\sigma Z - \frac{s}{Z}\right| \quad (35)$$

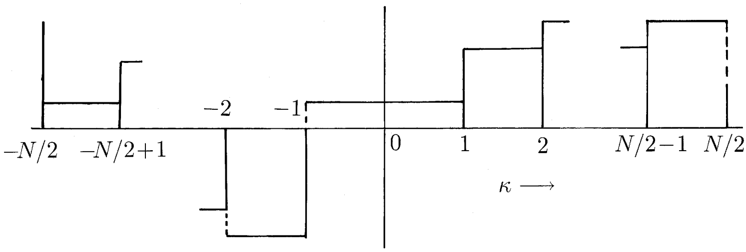

FIGURE 3.2-7. Symmetric generalization of Fig.3.2-6b from the interval $0 \le \kappa < N/2$ to the interval $-N/2 < \kappa < N/2$.

Hence, Eq.(34) has to be summed with a change of variable at $\kappa = \pm K_0$. We still have to decide what lower and upper limit should be used for the sum. If we used the Fourier integral rather than the sum in Eq.(2.1-81) we would calculate the area of a function $f(x)$ as shown in Fig.3.2-6a in the interval $0 \le x \le N/2$. A sum replaces this continuous function by a step function as shown in Fig.3.2-6b. The area of this step function is represented by the sum of the functional value $f(x)$ for $x = 0, 1, 2, \dots, \kappa, \dots N/2-1$ multiplied by the width $\Delta x = 1$ of the steps. Since the area of the step $\kappa+1$ is given by the step in the interval $\kappa \le x < \kappa+1$ we must sum κ from 0 to $N/2-1$ rather than from 0 to $N/2$. This appears to be a trivial distinction. However, we shall encounter results that are convergent in the half open interval $0 \le x < N/2$ or $\kappa = 0, 1, 2, \dots, N/2-1$ but not at $x = \kappa = N/2$. The generalization of Fig.3.2-6 from the interval $0 \le \kappa < N/2$ to the interval $-N/2 < \kappa < N/2$ is shown in Fig.3.2-7. This generalization makes $\kappa = 0$ count twice. The contribution of a particular interval is of little interest for large values of N as long as no convergence problem is introduced. Using Fig.3.2-7, Eq.(34) has to be summed from $\kappa = -N/2+1$ to $N/2-1$ with a change of variable at $\kappa = \pm K_0$, if K_0 is an integer. Non-integer values of K_0 and other refinements will be discussed later on in this section. Here we write

$$w(\zeta,\theta) = \sum_{\kappa=-N/2+1}^{N/2-1} w_\kappa(\zeta,\theta)$$

$$A_5 = A_6 = 0 \quad \text{for } -K_0 < \kappa < K_0$$

$$A_7 = A_8 = 0 \quad \text{for } -N/2 < \kappa < -K_0,\ K_0 < \kappa < N/2 \qquad (36)$$

The main difference between Eqs.(2.1-81) and (36) is that κ has no longer denumerably many values; instead it has an arbitrarily large but finite number of values that is smaller than N. Hence, the time interval T, the space interval cT, and κ can be arbitrarily large but finite while the time interval Δt and the space interval $c\Delta t$ can be arbitrarily small but finite.

For the determination of $A_5(\kappa)$ and $A_6(\kappa)$ we have the two initial conditions of Eqs.(3.1-18) and (3.1-19). We note that Eq.(3.1-18) equals Eq.(2.1-68) but the differential equation (2.1-69) is replaced by the difference equation (3.1-19). For Eq.(3.1-18) we obtain:

$$w(\zeta,0)=\sum_{\kappa=-N/2+1}^{N/2-1}[A_5(\kappa)+A_6(\kappa)+A_7(\kappa)+A_8(\kappa)]\sin\frac{2\pi\kappa\zeta}{N}=e^{-\rho_2\zeta} \quad (37)$$

Equation (3.1-19) yields:

$$\begin{aligned}w(\zeta,1)-w(\zeta,0)=&\sum_{\kappa=-N/2+1}^{N/2-1}\Bigg[A_5(\kappa)(e^{\gamma_5}-1)+A_6(\kappa)(e^{\gamma_6}-1)\\&+A_7(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\\&+A_8(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\Bigg]\\&\times\sin\frac{2\pi\kappa\zeta}{N}=0\end{aligned} \quad (38)$$

For small values of γ_5 and γ_6 we obtain the approximations

$$e^{\gamma_5}-1\doteq\gamma_5,\ e^{\gamma_6}-1\doteq\gamma_6 \quad (39)$$

and Eq.(38) assumes the form of Eq.(2.2-3) plus the additional terms with $A_7(\kappa)$ and $A_8(\kappa)$.

The Fourier series of Eq.(2.2-4) is rewritten in the following form for $\tau\to\Delta t$ and $T/\tau\to T/\Delta t=N$:

$$\begin{gathered}g_s(\kappa)=\frac{2}{N}\int_0^N f_s(\zeta)\sin\frac{2\pi\kappa\zeta}{N}\,d\zeta,\quad f_s(\zeta)=\sum_{\kappa=-N/2+1}^{N/2-1}g_s(\kappa)\sin\frac{2\pi\kappa\zeta}{N}\\0\le t\le T,\ 0\le y\le cT\\0\le t/\Delta t=\theta\le T/\Delta t=N,\ 0\le y/c\Delta t=\zeta\le N\end{gathered} \quad (40)$$

In analogy to Eqs.(2.2-6) and (2.2-7) we obtain from Eqs.(37) and (36):

$$A_5(\kappa)+A_6(\kappa)+A_7(\kappa)+A_8(\kappa)=-\frac{2}{N}\int_0^N e^{-\rho_2\zeta}\sin\frac{2\pi\kappa\zeta}{N}d\zeta \quad (41)$$

$$
\begin{aligned}
&A_5(\kappa)(e^{\gamma_5}-1)+A_6(\kappa)(e^{\gamma_6}-1)\\
&+A_7(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\\
&+A_8(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\\
&=0 \qquad (42)
\end{aligned}
$$

With the help of Eq.(2.2-8) we get:

$$
\begin{aligned}
-\frac{2}{N}\int_0^N e^{-\rho_2\zeta}\sin\frac{2\pi\kappa\zeta}{N}\,d\zeta &= -\frac{1}{N}\frac{(4\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2}, \quad N\rho_2=cT\sqrt{\sigma s}\\
&= -\frac{1}{N}\frac{4\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2} \quad \text{for } N\rho_2 \gg 1 \qquad (43)
\end{aligned}
$$

It seems that Eqs.(41) and (42) are two equations with four unknowns $A_5(\kappa)$ to $A_8(\kappa)$, but this is not so. According to Eq.(36) either $A_5(\kappa)$ and $A_6(\kappa)$ are zero or $A_7(\kappa)$ and $A_8(\kappa)$ are zero, depending on the value of κ. Hence, Eqs.(41) and (42) can be rewritten as two systems of two equations, each with two variables:

$$\text{for } -K_0<\kappa<K_0$$

$$A_7(\kappa)+A_8(\kappa)=-\frac{1}{N}\frac{4\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2} \qquad (44)$$

$$
\begin{aligned}
&A_7(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\\
&+A_8(\kappa)\left(e^{-\rho_1/2}\left\{1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^4-4\left(\frac{2\pi\rho_\kappa}{N}\right)^2-4\rho_2^2+\rho_1^2\right]^{1/2}\right\}-1\right)\\
&=0 \qquad (45)
\end{aligned}
$$

$$\text{for } -N/2<\kappa<-K_0,\ K_0<\kappa<N/2$$

$$A_5(\kappa)+A_6(\kappa)=-\frac{1}{N}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2} \qquad (46)$$

$$A_5(\kappa)(e^{\gamma_5}-1)+A_6(\kappa)(e^{\gamma_6}-1)=0 \qquad (47)$$

We solve first Eqs.(46) and (47). The exponents γ_5 and γ_6 are defined by Eq.(32):

$$
\begin{aligned}
A_5(\kappa) &= -\frac{1}{N}\frac{(4\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2}\,\frac{e^{\gamma_6}-1}{e^{\gamma_6}-e^{\gamma_5}} \qquad d_\Delta^2 > \rho_1^2 \\
&= -\frac{1}{N}\frac{(2\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2} \\
&\qquad \times\left(1-2i\frac{1-e^{-\rho_1/2}\cos\{\arcsin[(d_\Delta^2-\rho_1^2)^{1/2}/2]\}}{e^{-\rho_1/2}\,(d_\Delta^2-\rho_1^2)^{1/2}}\right) \\
&\doteq -\frac{1}{N}\frac{(2\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2}\left(1-\frac{i\rho_1}{(d_\Delta^2-\rho_1^2)^{1/2}}\right),\ |\gamma_5|,\ |\gamma_6|\ll 1 \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
A_6(\kappa) &= -\frac{1}{N}\frac{(4\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2}\,\frac{e^{\gamma_5}-1}{e^{\gamma_5}-e^{\gamma_6}} \qquad d_\Delta^2 > \rho_1^2 \\
&= -\frac{1}{N}\frac{(2\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2} \\
&\qquad \times\left(1+2i\frac{1-e^{-\rho_1/2}\cos\{\arcsin[(d_\Delta^2-\rho_1^2)^{1/2}/2]\}}{e^{-\rho_1/2}\,(d_\Delta^2-\rho_1^2)^{1/2}}\right) \\
&\doteq -\frac{1}{N}\frac{(2\pi\kappa/N)(1-e^{-N\rho_2})}{(2\pi\kappa/N)^2+\rho_2^2}\left(1+\frac{i\rho_1}{(d_\Delta^2-\rho_1^2)^{1/2}}\right),\ |\gamma_5|,\ |\gamma_6|\ll 1 \qquad (49)
\end{aligned}
$$

The solution of Eqs.(44) and (45) yields $A_7(\kappa)$ and $A_8(\kappa)$:

$$
\begin{aligned}
A_7(\kappa) &= \frac{1}{N}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2} \\
&\qquad \times\left(1-\frac{1-[2\sin(\pi\kappa/N)]^2-\rho_1/2}{\{[2\sin(\pi\kappa/N)]^4-4[2\sin(\pi\kappa/N)]^2-4\rho_2^2+\rho_1^2\}^{1/2}}\right) \\
&\doteq \frac{1}{N}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\left(1-\frac{1-(2\pi\kappa/N)^2-\rho_1/2}{[(2\pi\kappa/N)^4-4(2\pi\kappa/N)^2-4\rho_2^2+\rho_1^2]^{1/2}}\right) \qquad (50)
\end{aligned}
$$

$$
\begin{aligned}
A_8(\kappa) &= -\frac{1}{N}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2} \\
&\qquad \times\left(1+\frac{1-[2\sin(\pi\kappa/N)]^2-\rho_1/2}{\{[2\sin(\pi\kappa/N)]^4-4[2\sin(\pi\kappa/N)]^2-4\rho_2^2+\rho_1^2\}^{1/2}}\right) \\
&\doteq -\frac{1}{N}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\left(1+\frac{1-(2\pi\kappa/N)^2-\rho_1/2}{[(2\pi\kappa/N)^4-4(2\pi\kappa/N)^2-4\rho_2^2+\rho_1^2]^{1/2}}\right) \qquad (51)
\end{aligned}
$$

We must revisit the question for which values of κ one must use $A_5(\kappa)$, $A_6(\kappa)$ and for which $A_7(\kappa)$, $A_8(\kappa)$. From Eq.(9) or the definition of d_Δ^2 in Eq.(31) we obtain for $d_\Delta^2 - \rho_1^2 = 0$ the following equation:

$$\begin{aligned}
\sin^4\frac{\varphi_\kappa}{2} - \sin^2\frac{\varphi_\kappa}{2} - \frac{1}{4}\rho_2^2 + \frac{1}{16}d_\Delta^2 = 0 \\
\sin\frac{\varphi_\kappa}{2} = \sin\frac{\pi\kappa}{N} = \pm\left[\frac{1}{2} \pm \frac{1}{2}\left(1+\rho_2^2 - \frac{1}{4}d_\Delta^2\right)^{1/2}\right]^{1/2} \\
\kappa = K = \pm\frac{N}{\pi}\arcsin\left[\frac{1}{2} \pm \frac{1}{2}\left(1+\rho_2^2 - \frac{1}{4}\rho_1^2\right)^{1/2}\right]^{1/2} \quad \text{for } d_\Delta^2 = \rho_1^2
\end{aligned} \tag{52}$$

The positive sign $+N/\pi$ holds for $\kappa > 0$ or $\varphi_\kappa/\pi = 2\kappa/N$ according to Eq.(15) and Fig.3.2-1 while $-N/\pi$ holds for $\kappa < 0$. We may use the series expansion $\arcsin x \doteq x$ for the negative sign $-1/2$ in Eq.(52):

$$\begin{aligned}
K = K_0 &= \pm\frac{N}{\pi}\arcsin\left[\frac{1}{2} - \frac{1}{2}\left(1+\rho_2^2 - \frac{1}{4}\rho_1^2\right)^{1/2}\right]^{1/2} \\
&= \pm\frac{N}{4\pi}(\rho_1^2 - 4\rho_2^2)^{1/2} = \pm\frac{1}{4\pi}(\rho_a^2 - 4\rho_b^2)^{1/2} \\
&= \pm\frac{cT}{4\pi}\left|\sigma Z - \frac{s}{Z}\right| = \pm\frac{Nc\Delta t}{4\pi}\left|\sigma Z - \frac{s}{Z}\right|
\end{aligned} \tag{53}$$

These two values of K_0 correspond to the values in Eq.(2.2-12) and (35). But we get a second value for K by using the positive sign $+1/2$ in Eq.(52):

$$K = K_{N/2} = \pm\frac{N}{\pi}\arcsin\left[\frac{1}{2} + \frac{1}{2}\left(1+\rho_2^2 - \frac{1}{4}\rho_1^2\right)^{1/2}\right]^{1/2} \tag{54}$$

It is evident that K equals about $N/2$. The more detailed calculation with series expansions yields for even values of N:

$$\begin{aligned}
K_{N/2} &= \pm\left(\frac{N}{2} - \frac{1}{4\pi}(\rho_a^2 - 4\rho_b^2)^{1/2}\right) = \pm\left(\frac{N}{2} - K_0\right) \\
&= \pm\left(\frac{N}{2} - \frac{cT}{4\pi}\left|\sigma Z - \frac{s}{Z}\right|\right) = \pm\frac{N}{2}\left(1 - \frac{c\Delta t}{2\pi}\left|\sigma Z - \frac{s}{Z}\right|\right)
\end{aligned} \tag{55}$$

There is no equivalent in the differential theory. The deviation is due to the decrease of the plot shown with a solid line in Fig.3.2-1 for $\varphi_\kappa/\pi > 0.5$, which makes $\varphi_{\theta\kappa}/\pi$ zero for both $\varphi_\kappa/\pi = 0$ and $\varphi_\kappa/\pi = \pm 1$.

Using the limits $-K_{N/2}$ and $K_{N/2}$ we modify Eq.(36) as follows:

$$w(\zeta,\theta) = \sum_{\kappa=-N/2+1}^{N/2-1} w_\kappa(\zeta,\theta), \qquad N = \text{even}$$

$$\begin{aligned} &A_5 = A_6 = 0 \quad \text{for } -N/2 < \kappa < -K_{\mathrm{N}/2},\ -K_0 < \kappa < K_0,\ K_{\mathrm{N}/2} < \kappa < N/2 \\ &A_7 = A_8 = 0 \quad \text{for } -K_{\mathrm{N}/2} < \kappa < -K_0,\ K_0 < \kappa < K_{\mathrm{N}/2} \end{aligned} \tag{56}$$

The values of K_0 and $K_{\mathrm{N}/2}$ in Eqs.(53) and (55) would not deviate from 0 or $N/2$ in the case

$$\sigma Z - s/Z = 0 \quad \text{or} \quad \epsilon/\sigma - \mu/s = 0 \tag{57}$$

A corresponding relation

$$CR - LG = 0 \tag{58}$$

is known from the telegrapher's equation, where C, L, R, G are capacitance, inductance, resistance and (trans)conductance per unit length of a transmission line. A transmission line satisfying Eq.(58) is called *distortion-free*. In the design of transmission lines one can control the parameters C, R, L, G to satisfy Eq.(58), practically by increasing L. We have no such control over the wave impedance Z or the electric and magnetic dipole conductivities σ and s of vacuum.

We adopt the following notation for the sum of Eq.(56) to avoid repeated writing of the terms of the sum:

$$\begin{aligned} &\sum_{\kappa=-N/2+1}^{<-K_{\mathrm{N}/2}} + \sum_{\kappa>-K_{\mathrm{N}/2}}^{<-K_0} + \sum_{\kappa>-K_0}^{<K_0} + \sum_{\kappa>K_0}^{<K_{\mathrm{N}/2}} + \sum_{\kappa>K_{\mathrm{N}/2}}^{N/2-1} \\ &\qquad = \sum_{\substack{\kappa=-N/2+1\\ \kappa>-K_0\\ \kappa>K_{\mathrm{N}/2}}}^{\substack{N/2-1\\ <K_0\\ <-K_{\mathrm{N}/2}}} + \sum_{\substack{\kappa>-K_{\mathrm{N}/2}\\ \kappa>K_0}}^{\substack{<K_{\mathrm{N}/2}\\ <-K_0}} = \sum_{\substack{\kappa=-N/2+1\\ -K_0,K_{\mathrm{N}/2}}}^{\substack{N/2-1\\ -K_{\mathrm{N}/2},K_0}} + \sum_{\substack{\kappa>-K_{\mathrm{N}/2}\\ \kappa>K_0}}^{\substack{<K_{\mathrm{N}/2}\\ <-K_0}} \end{aligned} \tag{59}$$

The notation $<-K_0$, $<K_0$, $<-K_{\mathrm{N}/2}$, $<K_{\mathrm{N}/2}$ and $>-K_0$, $>K_0$, $>K_{\mathrm{N}/2}$, $>K_{\mathrm{N}/2}$ means the largest integer smaller than $-K_0$, K_0, $-K_{\mathrm{N}/2}$, $K_{\mathrm{N}/2}$ and the smallest integer larger than $-K_0$, K_0, $-K_{\mathrm{N}/2}$, $K_{\mathrm{N}/2}$.

Substitution of γ_5, γ_6 from Eq.(32), $w_\kappa(\zeta,\theta)$ from Eq.(34), as well as of $A_5(\kappa)$, $A_6(\kappa)$, $A_7(\kappa)$, and $A_8(\kappa)$ from Eqs.(48) to (51) into Eq.(36) yields the following equation for $w(\zeta,\theta)$:

$$w(\zeta,\theta) = \frac{e^{-\rho_1\theta/2}}{N}\Bigg\{ \sum_{\substack{\kappa=-N/2+1 \\ -K_0, K_{N/2}}}^{\substack{N/2-1 \\ -K_{N/2}, K_0}} \frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N}$$
$$\times\Bigg[\left(1-\frac{1-4[\sin(\pi\kappa/N)]^2-\rho_1/2}{h(\pi\kappa/N)}\right)\left(1-2[\sin(\pi\kappa/N)]^2+\frac{1}{2}h(\pi\kappa/N)\right)^\theta$$
$$-\left(1+\frac{1-4[\sin(\pi\kappa/N)]^2-\rho_1/2}{h(\pi\kappa/N)}\right)\left(1-2[\sin(\pi\kappa/N)]^2-\frac{1}{2}h(\pi\kappa/N)\right)^\theta\Bigg]$$
$$-2(1-e^{-N/\rho_2})\sum_{\substack{\kappa>-K_{N/2} \\ \kappa>K_0}}^{\substack{<K_{N/2} \\ <-K_0}} \frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N}$$
$$\times\left(\cos\lambda_a\theta+\frac{2(1-e^{-\rho_1/2}\cos\lambda_a)}{e^{-\rho_1/2}\,(d_\Delta^2-\rho_1^2)^{1/2}}\sin\lambda_a\theta\right)\Bigg\}$$
$$h(\pi\kappa/N) = \{16[\sin(\pi\kappa/N)]^4-16[\sin(\pi\kappa/N)]^2-4\rho_2^2+\rho_1^2\}^{1/2}$$
$$\lambda_a = \varphi_{\theta\kappa} = \arcsin\frac{1}{2}(d_\Delta^2-\rho_1^2)^{1/2}$$
$$d_\Delta^2 \text{ see Eq.(31)}, \quad \varphi_{\theta\kappa} \text{ see Eq.(15)} \tag{60}$$

This equation looks quite different from Eq.(2.2-13) but we may show the similarity by rewriting it. With the definitions

$$u(\pi\kappa/N) = \frac{1-4[\sin(\pi\kappa/N)]^2-\rho_1/2}{h(\pi\kappa/N)},\ v(\pi\kappa/N) = 1-2[\sin(\pi\kappa/N)]^2$$
$$\lambda_{A1} = \ln\left(v(\pi\kappa/N)+\frac{1}{2}h(\pi\kappa/N)\right),\ \lambda_{A2} = \ln\left(v(\pi\kappa/N)-\frac{1}{2}h(\pi\kappa/N)\right) \tag{61}$$

we may write the kernel of the first sum of Eq.(60) in the following form:

$$\left\{[1-u(\pi\kappa/N)]e^{\lambda_{A1}\theta}-[1+u(\pi\kappa/N)]e^{\lambda_{A2}\theta}\right\}\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N}$$
$$= \left[e^{\lambda_{A1}\theta}-e^{\lambda_{A2}\theta}-u(\pi\kappa/N)(e^{\lambda_{A1}\theta}+e^{\lambda_{A2}\theta})\right]\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2+\rho_2^2}\sin\frac{2\pi\kappa\zeta}{N} \tag{62}$$

For small values of $\pi\kappa/N$ we obtain:

$$
\begin{aligned}
v(\pi\kappa/N) &\doteq 1, \quad h(\pi\kappa/N) \ll 1 \\
v(\pi\kappa/N) + \frac{1}{2}h(\pi\kappa/N) &\doteq \frac{1}{v(\pi\kappa/N) - h(\pi\kappa/N)/2} \\
\lambda_{\mathrm{A2}} &\doteq -\lambda_{\mathrm{A1}}
\end{aligned} \tag{63}
$$

and the right side of Eq.(62) assumes the form:

$$
-2u(\pi\kappa/N)\left(\mathrm{ch}\,\lambda_{\mathrm{A1}}\theta - \frac{\mathrm{sh}\,\lambda_{\mathrm{A1}}\theta}{u(\pi\kappa/N)}\right)\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2 + \rho_2^2}\sin\frac{2\pi\kappa\zeta}{N} \tag{64}
$$

The kernel of the second sum in Eq.(60) may be rewritten with the following approximations

$$
\begin{aligned}
&\lambda_{\mathrm{a}} \doteq \frac{(d_\Delta^2 - \rho_1^2)^{1/2}}{2}, \quad e^{-\rho_1/2} \doteq 1 - \frac{\rho_1}{2} \\
&\frac{2(1 - e^{-\rho_1/2}\cos\lambda_{\mathrm{a}})}{e^{-\rho_1/2}\,(d_\Delta^2 - \rho_1^2)^{1/2}} \doteq \frac{\rho_1}{(d_\Delta^2 - \rho_1^2)^{1/2}}
\end{aligned} \tag{65}
$$

into the form

$$
\begin{aligned}
&-2e^{-\rho_1\theta/2}(1 - e^{-N\rho_2})\Bigg(\cos[(d_\Delta^2 - \rho_1^2)^{1/2}\theta/2] \\
&\qquad + \frac{\rho_1 \sin[(d_\Delta^2 - \rho_1^2)^{1/2}\theta/2]}{(d_\Delta^2 - \rho_1^2)^{1/2}}\Bigg)\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2 + \rho_2^2}\sin\frac{2\pi\kappa\zeta}{N}
\end{aligned} \tag{66}
$$

and Eq.(60) becomes for $N\rho_2 \gg 1$:

$$
\begin{aligned}
w(\zeta,\theta) = -\frac{2}{N}e^{-\rho_1\theta/2}\Bigg\{&\sum_{\substack{\kappa=-N/2+1 \\ -K_0, K_{\mathrm{N/2}}}}^{\substack{N/2-1 \\ -K_{\mathrm{N/2}}, K_0}} u(\pi\kappa/N)\left(\mathrm{ch}\,\lambda_{\mathrm{A1}}\theta - \frac{\mathrm{sh}\,\lambda_{\mathrm{A1}}\theta}{u(\pi\kappa/N)}\right) \\
&\qquad\times \frac{2\pi\kappa/N}{(2\pi\kappa/N)^2 + \rho_2^2}\sin\frac{2\pi\kappa\zeta}{N} \\
&+ \sum_{\substack{\kappa > -K_{\mathrm{N/2}} \\ \kappa > K_0}}^{\substack{< K_{\mathrm{N/2}} \\ < -K_0}} \Bigg(\cos[(d_\Delta^2 - \rho_1^2)^{1/2}\theta/2] \\
&\qquad + \frac{\rho_1 \sin[(d_\Delta^2 - \rho_1^2)^{1/2}\theta/2]}{(d_\Delta^2 - \rho_1^2)^{1/2}}\Bigg)\frac{2\pi\kappa/N}{(2\pi\kappa/N)^2 + \rho_2^2}\sin\frac{2\pi\kappa\zeta}{N}\Bigg\}
\end{aligned} \tag{67}
$$

The oscillating terms in Eq.(67) are quite similar to the oscillating terms of Eq.(2.2-13). The non-oscillating terms differ more, but a study of the limits of the two sums of Eq.(67) shows that $w(\zeta,\theta)$ consists mainly of oscillating terms.

3.3 Solution for $A_{\mathrm{ev}}(\zeta,\theta)$

In the differential theory we can obtain $A_{\mathrm{ev}}(\zeta,\theta)$ from $V_{\mathrm{e}}(\zeta,\theta)$ by solving the inhomogeneous wave equation defined by Eq.(2.1-44). Smirnov gave us the very simple and elegant solution of Eq.(2.1-45). Nothing comparable exists for the calculus of finite differences. Hence, we must develop the generalization of the required mathematical methods from the inhomogeneous *differential* wave equation to the inhomogeneous *difference* wave equation. This is done in Section 6.2.

We start with $V_{\mathrm{e}}(\zeta,\theta)$ of Eq.(3.1-10). The function $F(\zeta)$ is defined in Eq.(3.1-14) and $w(\zeta,\theta)$ is defined in Eq.(3.2-67):

$$V_e(\zeta,\theta) = V_{\mathrm{e}0}[F(\zeta) + w(\zeta,\theta)] \tag{1}$$

If we write Eq.(6.2-1) with the indices ev, e, and the constant $c^2\tau^2$ we obtain Eq.(6.2-23) in the form

$$\frac{\tilde{\Delta}^2 A_{\mathrm{ev}}}{\tilde{\Delta}\zeta^2} - \frac{\tilde{\Delta}^2 A_{\mathrm{ev}}}{\tilde{\Delta}\theta^2} = c^2\tau^2 V_{\mathrm{e}}(\zeta,\theta) \tag{2}$$

At this point the solution for finite differences $\Delta\zeta$, $\Delta\theta$ begins to deviate decisively from the solution for differentials $d\zeta$, $d\theta$. Let us compare the solution $V_{\mathrm{e}}(\zeta,\theta)$ of Eq.(2.1-62) for differentials with that of Eq.(1) for finite differences. For Eq.(2.1-62) the components $F(\zeta)$ and $w(\zeta,\theta)$ were derived in Eqs.(2.1-64) and (2.2-13). Their derivation is quite similar to the derivation of $F(\zeta)$ in Eq.(3.1-14) and of $w(\zeta,\theta)$ in Eq.(3.2-60). The next step is to obtain A_{ev} from Eq.(2.1-44) as done by Eq.(2.1-45) for the differential theory. This solution by Smirnov depends on several results of the differential theory that we cannot expect to derive here for finite differences. A less demanding solution of Eq.(2.1-44) than Eq.(2.1-45) is derived in Section 6.3. But it is still too demanding for an easy rewrite in terms of finite differences. The reason is that the variables ζ and θ in Eq.(6.3-1) are first replaced by the variables $\eta = \zeta + \theta$ and $\xi = \zeta - \theta$ from Eq.(6.3-2) on. The integration of Eq.(6.3-7) is done with ξ and η, but Eqs.(6.3-38) to (6.3-40) return to the original variables ζ and θ. No doubt this transformation of integration variables can be rewritten for the *summation* variables of Eq.(6.2-47), but this is a task for authors who want to advance the calculus of finite differences for its own sake. The calculus of finite differences is in a state of development that is well below the state of differential calculus and this fact will only be changed through applications in physics that are of comparable importance. We shall use an approach that requires a minimum of mathematical refinement for the calculus of finite differences. Equation (1) is rewritten for the variables $\eta = \zeta + \theta$ and $\xi = \zeta - \theta$, the whole calculation—including two summations—is done in terms of η and ξ, and only at the very end do we return from η, ξ to ζ, θ:

$$V_{\mathrm{e}}(\eta,\xi) = V_{\mathrm{e0}}[F(\eta,\xi) + w(\eta,\xi)] \tag{3}$$

We want to use Eq.(6.2-47). To do so we observe that $a = c\tau$ becomes $a = c\Delta t$ since τ was replaced by Δt in Eq.(3.1-1). Furthermore, we simplify Eq.(6.2-47) by leaving out the constants c_1 and c_2. This simplification will be explained presently:

$$\begin{aligned}\frac{\tilde{\Delta} A_{\mathrm{ev}}(\eta,\xi)}{\tilde{\Delta}\eta} &= \frac{1}{4}c^2(\Delta t)^2 \sum^{\xi} V_{\mathrm{e}}(\eta,\xi'+1)\Delta\xi' \\ &= \frac{1}{4}V_{\mathrm{e0}}c^2(\Delta t)^2 \sum^{\xi} [F(\eta,\xi'+1) + w(\eta,\xi'+1)]\Delta\xi' \end{aligned}\tag{4}$$

$$\begin{aligned}A_{\mathrm{ev}}(\eta,\xi) = \frac{1}{4}V_{\mathrm{e0}}(c\Delta t)^2 \sum^{\eta}\bigg(\sum^{\xi} &[F(\eta'+1,\xi'+1) \\ &+ w(\eta'+1,\xi'+1)]\Delta\xi'\bigg)\Delta\eta' \end{aligned}\tag{5}$$

$$\eta = \zeta + \theta, \quad \xi = \zeta - \theta, \quad \zeta = \frac{1}{2}(\eta+\xi), \quad \theta = \frac{1}{2}(\eta-\xi) \tag{6}$$

The functions $F(\zeta)$ of Eq.(3.1-14) and $w(\zeta,\theta)$ of Eq.(3.2-60) must be rewritten as functions $F(\eta,\xi)$ and $w(\eta,\xi)$:

$$F(\zeta) \doteq e^{-\rho_2\zeta} = F(\eta,\xi) = e^{-\rho_2\eta/2}e^{-\rho_2\xi/2} \tag{7}$$

The factor $1 - e^{-N\rho_2}$ in Eq.(3.2-60) is left out due to the relation $N\rho_2 = Tc/\sqrt{\sigma s} \gg 1$:

$$\begin{aligned} w(\zeta,\theta) = w(\eta,\xi) = &\frac{1}{N}e^{-\rho_1\eta/4}e^{\rho_1\xi/4} \\ \times\Bigg(&\sum_{\substack{\kappa=-N/2+1\\ -K_0,K_{\mathrm{N}/2}}}^{\substack{N/2-1\\ -K_{\mathrm{N}/2},K_0}} D_0(\kappa)[D_1(\kappa)G_1^{(\eta-\xi)/2}(\kappa) - D_2(\kappa)G_2^{(\eta-\xi)/2}(\kappa)]\sin\frac{\pi\kappa(\eta+\xi)}{N} \\ &- 2\sum_{\substack{\kappa>-K_{\mathrm{N}/2}\\ \kappa>K_0}}^{\substack{<K_{\mathrm{N}/2}\\ <-K_0}} \Big\{D_0(\kappa)\cos[\lambda_{\mathrm{a}}(\eta-\xi)/2] \\ &+ D_3(\kappa)\sin[\lambda_{\mathrm{a}}(\eta-\xi)/2]\Big\}\sin\frac{\pi\kappa(\eta+\xi)}{N}\Bigg) \end{aligned}\tag{8}$$

$$D_0(\kappa) = \frac{2\pi\kappa/N}{(2\pi\kappa/N)^2 + \rho_2^2} \tag{9}$$

$$D_1(\kappa) = 1 - \frac{1 - 4[\sin(\pi\kappa/N)]^2 - \rho_1/2}{h(\pi\kappa/N)} \tag{10}$$

$$G_1(\kappa) = 1 - 2[\sin(\pi\kappa/N)]^2 + \frac{1}{2}h(\pi\kappa/N) \tag{11}$$

$$D_2(\kappa) = 1 + \frac{1 - 4[\sin(\pi\kappa/N)]^2 - \rho_1/2}{h(\pi\kappa/N)} \tag{12}$$

$$G_2(\kappa) = 1 - 2[\sin(\pi\kappa/N)]^2 - \frac{1}{2}h(\pi\kappa/N) \tag{13}$$

$$h(\pi\kappa/N) = \{16[\sin(\pi\kappa/N)]^4 - 16[\sin(\pi\kappa/N)]^2 - 4\rho_2^2 + \rho_1^2\}^{1/2} \tag{14}$$

$$D_3(\kappa) = \frac{2(1 - e^{-\rho_1/2}\cos\lambda_{\text{a}})}{e^{-\rho_1/2}\left(d_\Delta^2 - \rho_1^2\right)^{1/2}} D_0(\kappa) \tag{15}$$

$$N,\ \rho_1,\ \rho_2^2 \text{ see Eq.(3.1-1)};\ d_\Delta^2 \text{ see Eq.(3.2-31)};\ \lambda_{\text{a}} \text{ see Eq.(3.2-60)}$$

$$K_0,\ K_{\text{N}/2} \text{ see Eqs.(3.2-53)–(3.2-55)},\ e^{-N\rho_2} \doteq 1$$

The function $w(\eta, \xi)$ must be written in a form that separates η and ξ:

$$\begin{aligned}
w(\eta,\zeta) = {} & \frac{1}{N} \sum_{\substack{\kappa=-N/2+1\\ -K_0, K_{\text{N}/2}}}^{\substack{N/2-1\\ -K_{\text{N}/2}, K_0}} e^{-\rho_1\eta/4} e^{\rho_1\xi/4} D_0(\kappa) \Big[D_1(\kappa) G_1^{\eta/2}(\kappa) G_1^{-\xi/2}(\kappa) \\
& - D_2(\kappa) G_2^{\eta/2}(\kappa) G_2^{-\xi/2}(\kappa) \Big] \left(\sin\frac{\pi\kappa\eta}{N} \cos\frac{\pi\kappa\xi}{N} + \cos\frac{\pi\kappa\eta}{N} \sin\frac{\pi\kappa\xi}{N} \right) \\
& - \frac{2}{N} \sum_{\substack{\kappa=-K_{\text{N}/2}\\ \kappa>K_0}}^{\substack{<K_{\text{N}/2}\\ <-K_0}} e^{-\rho_1\eta/4} e^{\rho_1\xi/4} \Bigg(D_0(\kappa) \Big\{ \sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \cos\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \\
& + \cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\xi\right] + \sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \\
& + \cos\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \Big\} \\
& - D_3(\kappa) \Big\{ \cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \cos\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \\
& + \sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\xi\right] - \cos\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \\
& + \sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_{\text{a}}}{2}\right)\eta\right] \sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_{\text{a}}}{2}\right)\xi\right] \Big\} \Bigg)
\end{aligned} \tag{16}$$

For the evaluation of the sum of $F(\eta, \xi' + 1)$ in Eq.(4) we need the sum of $\exp(-\rho_2\xi/2)$ in Eq.(7). We obtain it with the help of $e^{\gamma x}$ in Table 6.2-1. The constants c used in Table 6.2-1 are left out for two reasons. First, according to Eq.(2.3-15) only the derivatives of A_{ev} and A_{mv} are of interest and any summation constant will be eliminated. Second, if the summation of Eq.(4) yields a constant a it will become a function $a(\eta - c - 1)$ according to Table 6.3-1. The product $a(c+1)$ is a constant and may be ignored. The product $a\eta = a(\zeta + \theta)$ is a solution of the homogeneous difference wave equation according to Eq.(6.2-52). Since the general solution of the homogeneous equation has to be added to the solution of the inhomogeneous equation by Eq.(5) we may ignore $a\eta$ in the inhomogeneous solution. Hence, we get:

$$S_{00} = \mathop{\mathrm{S}}^{\xi} e^{-\rho_2(\xi'+1)/2} \Delta\xi' = -\frac{e^{-\rho_2\xi/2}}{\mathrm{sh}(\rho_2/2)} \tag{17}$$

For the summation of $w(\eta, \xi' + 1)$ in Eq.(4) we introduce the following notation for the terms $G_1(\kappa)$ and $G_2(\kappa)$ of the first sum of Eq.(16):

$$\lambda_{\mathrm{A1}} = \ln G_1(\kappa) = \ln\left(1 - 2[\sin(\pi\kappa/N)]^2 + \frac{1}{2}h(\pi\kappa/N)\right)$$
$$\lambda_{\mathrm{A2}} = \ln G_2(\kappa) = \ln\left(1 - 2[\sin(\pi\kappa/N)]^2 - \frac{1}{2}h(\pi\kappa/N)\right)$$
$$G_1(\kappa) = e^{\lambda_{\mathrm{A1}}},\ G_2(\kappa) = e^{\lambda_{\mathrm{A2}}},\ G_1^{-\xi/2} = e^{-\lambda_{\mathrm{A1}}\xi/2},\ G_2^{-\xi/2} = e^{-\lambda_{\mathrm{A2}}\xi/2} \tag{18}$$

The first sum in Eq.(16) requires the summation of the following four terms:

$$\exp[(\rho_1/2 - \lambda_{\mathrm{A1}})\xi/2]\cos(\pi\kappa\xi/N)$$
$$\exp[(\rho_1/2 - \lambda_{\mathrm{A1}})\xi/2]\sin(\pi\kappa\xi/N)$$
$$\exp[(\rho_1/2 - \lambda_{\mathrm{A2}})\xi/2]\cos(\pi\kappa\xi/N)$$
$$\exp[(\rho_1/2 - \lambda_{\mathrm{A2}})\xi/2]\sin(\pi\kappa\xi/N)$$

Such summations have also been worked out in Table 6.2-1. Here is a list of all summations needed for Eq.(16). Four are obtained from the first sum in Eq.(16):

$$S_{01} = \mathop{\mathrm{S}}^{\xi} e^{(\rho_1/2-\lambda_{\mathrm{A1}})(\xi'+1)/2} \cos\frac{\pi\kappa(\xi'+1)}{N}\Delta\xi'$$
$$= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A1}})\xi/2}}{[(\rho_1/2 - \lambda_{\mathrm{A1}})/2]^2 + (\pi\kappa/N)^2}\left(\lambda_{01}\cos\frac{\pi\kappa\xi}{N} + \gamma_{01}\sin\frac{\pi\kappa\xi}{N}\right) \tag{19}$$

$$S_{02} = \mathop{\large\S}^{\xi} e^{(\rho_1/2-\lambda_{\mathrm{A}1})(\xi'+1)/2}\sin\frac{\pi\kappa(\xi'+1)}{N}\Delta\xi'$$

$$= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A}1})\xi/2}}{[(\rho_1/2-\lambda_{\mathrm{A}1})/2]^2+(\pi\kappa/N)^2}\left(\lambda_{01}\sin\frac{\pi\kappa\xi}{N}-\gamma_{01}\cos\frac{\pi\kappa\xi}{N}\right) \tag{20}$$

$$\lambda_{01} = -\frac{2\{[(\rho_1/2-\lambda_{\mathrm{A}1})/2]^2+(\pi\kappa/N)^2\}\,\mathrm{sh}[(\rho_1/2-\lambda_{\mathrm{A}1})/2]\cos(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}1})} \tag{21}$$

$$\gamma_{01} = +\frac{2\{[(\rho_1/2-\lambda_{\mathrm{A}1})/2]^2+(\pi\kappa/N)^2\}\,\mathrm{ch}[(\rho_1/2-\lambda_{\mathrm{A}1})/2]\sin(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}1})} \tag{22}$$

$$S_{03} = \mathop{\large\S}^{\xi} e^{(\rho_1/2-\lambda_{\mathrm{A}2})(\xi'+1)/2}\cos\frac{\pi\kappa(\xi'+1)}{N}\Delta\xi'$$

$$= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A}2})\xi/2}}{[(\rho_1/2-\lambda_{\mathrm{A}2})/2]^2+(\pi\kappa/N)^2}\left(\lambda_{03}\cos\frac{\pi\kappa\xi}{N}+\gamma_{03}\sin\frac{\pi\kappa\xi}{N}\right) \tag{23}$$

$$S_{04} = \mathop{\large\S}^{\xi} e^{(\rho_1/2-\lambda_{\mathrm{A}2})(\xi'+1)/2}\sin\frac{\pi\kappa(\xi'+1)}{N}\Delta\xi'$$

$$= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A}2})\xi/2}}{[(\rho_1/2-\lambda_{\mathrm{A}2})/2]^2+(\pi\kappa/N)^2}\left(\lambda_{03}\sin\frac{\pi\kappa\xi}{N}-\gamma_{03}\cos\frac{\pi\kappa\xi}{N}\right) \tag{24}$$

$$\lambda_{03} = -\frac{2\{[(\rho_1/2-\lambda_{\mathrm{A}2})/2]^2+(\pi\kappa/N)^2\}\,\mathrm{sh}[(\rho_1/2-\lambda_{\mathrm{A}2})/2]\cos(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}2})} \tag{25}$$

$$\gamma_{03} = +\frac{2\{[(\rho_1/2-\lambda_{\mathrm{A}2})/2]^2+(\pi\kappa/N)^2\}\,\mathrm{ch}[(\rho_1/2-\lambda_{\mathrm{A}2})/2]\sin(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}2})} \tag{26}$$

The second sum in Eq.(16) requires four summations that all follow the pattern of S_{01} to S_{04}:

$$S_{05} = \mathop{\large\S}^{\xi} e^{\rho_1(\xi'+1)/4}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_{\mathrm{a}}}{2}\right)(\xi'+1)\right]\Delta\xi'$$

$$= \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_{\mathrm{a}}/2)^2}\left\{\lambda_{05}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_{\mathrm{a}}}{2}\right)\xi\right]\right.$$
$$\left.+\gamma_{05}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_{\mathrm{a}}}{2}\right)\xi\right]\right\} \tag{27}$$

$$S_{06} = \mathop{\mathbf{S}}^{\xi} e^{\rho_1(\xi'+1)/4} \sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_\mathrm{a}}{2}\right)(\xi'+1)\right]\Delta\xi'$$
$$= \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2 - (\pi\kappa/N - \lambda_\mathrm{a}/2)^2}\left\{\lambda_{05}\sin\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_\mathrm{a}}{2}\right)\xi\right] - \gamma_{05}\cos\left[\left(\frac{\pi\kappa}{N} - \frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\right\} \quad (28)$$

$$\lambda_{05} = -\frac{2[(\rho_1/4)^2 + (\pi\kappa/N - \lambda_\mathrm{a}/2)^2]\,\mathrm{sh}(\rho_1/4)\cos(\pi\kappa/N - \lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N - \lambda_\mathrm{a}) - \mathrm{ch}(\rho_1/2)} \quad (29)$$

$$\gamma_{05} = +\frac{2[(\rho_1/4)^2 + (\pi\kappa/N - \lambda_\mathrm{a}/2)^2]\,\mathrm{ch}(\rho_1/4)\sin(\pi\kappa/N - \lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N - \lambda_\mathrm{a}) - \mathrm{ch}(\rho_1/2)} \quad (30)$$

$$S_{07} = \mathop{\mathbf{S}}^{\xi} e^{\rho_1(\xi'+1)/4} \cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)(\xi'+1)\right]\Delta\xi'$$
$$= \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2 + (\pi\kappa/N + \lambda_\mathrm{a}/2)^2}\left\{\lambda_{07}\cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)\xi\right] + \gamma_{07}\sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\right\} \quad (31)$$

$$S_{08} = \mathop{\mathbf{S}}^{\xi} e^{\rho_1(\xi'+1)/4} \sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)(\xi'+1)\right]\Delta\xi'$$
$$= \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2 + (\pi\kappa/N + \lambda_\mathrm{a}/2)^2}\left\{\lambda_{07}\sin\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)\xi\right] - \gamma_{07}\cos\left[\left(\frac{\pi\kappa}{N} + \frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\right\} \quad (32)$$

$$\lambda_{07} = -\frac{2[(\rho_1/4)^2 + (\pi\kappa/N + \lambda_\mathrm{a}/2)^2]\,\mathrm{sh}(\rho_1/4)\cos(\pi\kappa/N + \lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N + \lambda_\mathrm{a}) - \mathrm{ch}(\rho_1/2)} \quad (33)$$

$$\gamma_{07} = +\frac{2[(\rho_1/4)^2 + (\pi\kappa/N + \lambda_\mathrm{a}/2)^2]\,\mathrm{ch}(\rho_1/4)\sin(\pi\kappa/N + \lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N + \lambda_\mathrm{a}) - \mathrm{ch}(\rho_1/2)} \quad (34)$$

Before we rewrite Eq.(4) with the help of the summations of Eqs.(19) to (34) we show some intermediate steps since few readers will be well versed with the calculus of finite differences. First we obtain from Eqs.(4), (7), and (17):

$$\mathop{\mathbf{S}}^{\xi} F(\eta,\xi'+1)\Delta\xi' = e^{-\rho_2\eta/2} \mathop{\mathbf{S}}^{\xi} e^{-\rho_2(\xi'+1)/2}\Delta\xi' = S_{00}e^{-\rho_2\eta/2} \tag{35}$$

Then we write the kernel of the first sum in Eq.(16) with the help of Eq.(19):

$$\begin{aligned}
&\frac{1}{N}e^{-\rho_1\eta/4}e^{\rho_1\xi/4}D_0(D_1G_1^{\eta/2}G_1^{-\xi/2} - D_2G_2^{\eta/2}G_2^{-\xi/2}) \\
&\qquad\qquad \times\left(\sin\frac{\pi\kappa\eta}{N}\cos\frac{\pi\kappa\xi}{N} + \cos\frac{\pi\kappa\eta}{N}\sin\frac{\pi\kappa\xi}{N}\right) \\
&= \frac{D_0}{N}\Bigg(D_1e^{-(\rho_1/2-\lambda_{\rm A1})\eta/2}\sin\frac{\pi\kappa\eta}{N}e^{(\rho_1/2-\lambda_{\rm A1})\xi/2}\cos\frac{\pi\kappa\xi}{N} \\
&\quad +D_1e^{-(\rho_1/2-\lambda_{\rm A1})\eta/2}\cos\frac{\pi\kappa\eta}{N}e^{(\rho_1/2-\lambda_{\rm A1})\xi/2}\sin\frac{\pi\kappa\xi}{N} \\
&\quad -D_2e^{-(\rho_1/2-\lambda_{\rm A2})\eta/2}\sin\frac{\pi\kappa\eta}{N}e^{(\rho_1/2-\lambda_{\rm A2})\xi/2}\cos\frac{\pi\kappa\xi}{N} \\
&\quad -D_2e^{-(\rho_1/2-\lambda_{\rm A2})\eta/2}\cos\frac{\pi\kappa\eta}{N}e^{(\rho_1/2-\lambda_{\rm A2})\xi/2}\sin\frac{\pi\kappa\xi}{N}\Bigg)
\end{aligned} \tag{36}$$

For the summation over ξ one must replace ξ by $\xi+1$. Doing so in the last four lines of Eq.(36) produces the kernels of the summations S_{01} to S_{04} in Eqs.(19) to (26). The kernel of the second sum of Eq.(16) can be rewritten in analogy to Eq.(36) but we get twice as many terms. We write only the first one:

$$-\frac{2}{N}e^{-\rho_1\eta/4}D_0\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\rm a}}{2}\right)\eta\right]e^{\rho_1\xi/4}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_{\rm a}}{2}\right)\xi\right]$$

The relation with S_{05} in Eq.(27) is evident.

After these preliminaries we may return Eq.(4) and rewrite it into the following form:

$$\begin{aligned}
\frac{\tilde{\Delta}A_{\rm ev}(\eta,\xi)}{\tilde{\Delta}\eta} &= \frac{1}{4}V_{\rm e0}(c\Delta t)^2\left(\mathop{\mathbf{S}}^{\xi}F(\eta,\xi'+1)\Delta\xi' + \mathop{\mathbf{S}}^{\xi}w(\eta,\xi'+1)\Delta\xi'\right) \\
&= \frac{1}{4}V_{\rm e0}(c\Delta t)^2\Bigg(S_{00}e^{-\rho_2\eta/2} \\
+\frac{1}{N}&\sum_{\substack{\kappa=-N/2\\ -K_0,K_{\rm N/2}}}^{\substack{N/2\\ -{\rm N}/2,K_0}} D_0\Bigg[D_1e^{-(\rho_1/2-\lambda_{\rm A1})\eta/2}\left(S_{01}\sin\frac{\pi\kappa\eta}{N}+S_{02}\cos\frac{\pi\kappa\eta}{N}\right)
\end{aligned}$$

$$- D_2 e^{-(\rho_1/2-\lambda_{A2})\eta/2}\left(S_{03}\sin\frac{\pi\kappa\eta}{N} + S_{04}\cos\frac{\pi\kappa\eta}{N}\right)\Bigg]$$

$$-\frac{2}{N}\sum_{\substack{\kappa>-K_{N/2}\\ \kappa>K_0}}^{\substack{<-K_{N/2}\\ <-K_0}}\Bigg\{(D_0S_{05}-D_3S_{06})e^{-\rho_1\eta/4}\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_a}{2}\right)\eta\right]$$

$$+(D_0S_{06}-D_3S_{05})e^{-\rho_1\eta/4}\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_a}{2}\right)\eta\right]$$

$$+(D_0S_{07}-D_3S_{08})e^{-\rho_1\eta/4}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_a}{2}\right)\eta\right]$$

$$+(D_0S_{08}+D_3S_{07})e^{-\rho_1\eta/4}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_a}{2}\right)\eta\right]\Bigg\}\Bigg) \qquad (37)$$

To obtain $A_{ev}(\eta,\xi)$ from $\Delta A_{ev}(\eta,\xi)/\Delta\eta$ we must sum each term in Eq.(37) that contains the variable η. This means in essence a repetition of Eqs.(17) and (19) to (34):

$$S_{10} = \mathop{\mathrm{S}}^{\eta} e^{-\rho_2(\eta'+1)/2}\Delta\eta' = -\frac{e^{-\rho_2\eta/2}}{\mathrm{sh}(\rho_2/2)} \qquad (38)$$

The first sum in Eq.(37) requires four summations S_{11} to S_{14}:

$$S_{11} = \mathop{\mathrm{S}}^{\eta} e^{-(\rho_1/4-\lambda_{A1}/2)(\eta'+1)}\sin\frac{\pi\kappa(\eta'+1)}{N}\Delta\eta'$$
$$= \frac{e^{-(\rho_1/4-\lambda_{A1}/2)\eta}}{(\rho_1/4-\lambda_{A1}/2)^2+(\pi\kappa/N)^2}\left(\lambda_{11}\sin\frac{\pi\kappa\eta}{N}-\gamma_{11}\cos\frac{\pi\kappa\eta}{N}\right) \qquad (39)$$

$$S_{12} = \mathop{\mathrm{S}}^{\eta} e^{-(\rho_1/4-\lambda_{A1}/2)(\eta'+1)}\cos\frac{\pi\kappa(\eta'+1)}{N}\Delta\eta'$$
$$= \frac{e^{-(\rho_1/4-\lambda_{A1}/2)\eta}}{(\rho_1/4-\lambda_{A1}/2)^2+(\pi\kappa/N)^2}\left(\lambda_{11}\cos\frac{\pi\kappa\eta}{N}+\gamma_{11}\sin\frac{\pi\kappa\eta}{N}\right) \qquad (40)$$

$$\lambda_{11} = +\frac{2[(\rho_1/4-\lambda_{A1}/2)^2+(\pi\kappa/N)^2]\,\mathrm{sh}(\rho_1/4-\lambda_{A1}/2)\cos(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{A1})} \qquad (41)$$

$$\gamma_{11} = +\frac{2[(\rho_1/4-\lambda_{A1}/2)^2+(\pi\kappa/N)^2]\,\mathrm{ch}(\rho_1/4-\lambda_{A1}/2)\sin(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{A1})} \qquad (42)$$

$$
\begin{aligned}
S_{13} &= \mathop{\mathfrak{S}}\limits^{\eta} e^{-(\rho_1/4-\lambda_{\mathrm{A}2}/2)(\eta'+1)} \sin\frac{\pi\kappa(\eta'+1)}{N}\Delta\eta \\
&= \frac{e^{-(\rho_1/4-\lambda_{\mathrm{A}2}/2)\eta}}{(\rho_1/4-\lambda_{\mathrm{A}2}/2)^2+(\pi\kappa/N)^2}\left(\lambda_{13}\sin\frac{\pi\kappa\eta}{N}-\gamma_{13}\cos\frac{\pi\kappa\eta}{N}\right)
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
S_{14} &= \mathop{\mathfrak{S}}\limits^{\eta} e^{-(\rho_1/4-\lambda_{\mathrm{A}2}/2)(\eta'+1)} \cos\frac{\pi\kappa(\eta'+1)}{N}\Delta\eta \\
&= \frac{e^{-(\rho_1/4-\lambda_{\mathrm{A}2}/2)\eta}}{(\rho_1/4-\lambda_{\mathrm{A}2}/2)^2+(\pi\kappa/N)^2}\left(\lambda_{13}\cos\frac{\pi\kappa\eta}{N}+\gamma_{13}\sin\frac{\pi\kappa\eta}{N}\right)
\end{aligned} \tag{44}
$$

$$
\lambda_{13} = +\frac{2[(\rho_1/4-\lambda_{\mathrm{A}2}/2)^2+(\pi\kappa/N)^2]\,\mathrm{sh}(\rho_1/4-\lambda_{\mathrm{A}2}/2)\cos(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}2})} \tag{45}
$$

$$
\gamma_{13} = +\frac{2[(\rho_1/4-\lambda_{\mathrm{A}2}/2)^2+(\pi\kappa/N)^2]\,\mathrm{ch}(\rho_1/4-\lambda_{\mathrm{A}2}/2)\sin(\pi\kappa/N)}{\cos(2\pi\kappa/N)-\mathrm{ch}(\rho_1/2-\lambda_{\mathrm{A}2})} \tag{46}
$$

The second sum in Eq.(37) requires four more summations:

$$
\begin{aligned}
S_{15} &= \mathop{\mathfrak{S}}\limits^{\eta} e^{-\rho_1(\eta'+1)/4}\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)(\eta'+1)\right]\Delta\eta' \\
&= \frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2+(\pi\kappa/N+\lambda_{\mathrm{a}}/2)^2}\left\{\lambda_{15}\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)\eta\right]\right. \\
&\qquad\left. -\gamma_{15}\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)\eta\right]\right\}
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
S_{16} &= \mathop{\mathfrak{S}}\limits^{\eta} e^{-\rho_1(\eta'+1)/4}\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)(\eta'+1)\right]\Delta\eta' \\
&= \frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2+(\pi\kappa/N+\lambda_{\mathrm{a}}/2)^2}\left\{\lambda_{15}\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)\eta\right]\right. \\
&\qquad\left. +\gamma_{15}\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_{\mathrm{a}}}{2}\right)\eta\right]\right\}
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\lambda_{15} &= +\frac{2[(\rho_1/4)^2+(\pi\kappa/N+\lambda_\mathrm{a}/2)^2]\,\mathrm{sh}(\rho_1/4)\cos(\pi\kappa/N+\lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N+\lambda_\mathrm{a})-\mathrm{ch}(\rho_1/2)}\\
&= -\lambda_{07} \qquad (49)\\
\gamma_{15} &= +\frac{2[(\rho_1/4)^2+(\pi\kappa/N+\lambda_\mathrm{a}/2)^2]\,\mathrm{ch}(\rho_1/4)\sin(\pi\kappa/N+\lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N+\lambda_\mathrm{a})-\mathrm{ch}(\rho_1/2)}\\
&= +\gamma_{07} \qquad (50)
\end{aligned}
$$

$$
\begin{aligned}
S_{17} &= \mathop{\mathrm{S}}^{\eta} e^{-\rho_1(\eta'+1)/4}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)(\eta'+1)\right]\Delta\eta'\\
&= \frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2}\left\{\lambda_{17}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right.\\
&\qquad\left.-\gamma_{17}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right\} \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
S_{18} &= \mathop{\mathrm{S}}^{\eta} e^{-\rho_1(\eta'+1)/4}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{N}\right)(\eta'+1)\right]\Delta\eta'\\
&= \frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2}\left\{\lambda_{17}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right.\\
&\qquad\left.+\gamma_{17}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right\} \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{17} &= +\frac{2[(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2]\,\mathrm{sh}(\rho_1/4)\cos(\pi\kappa/N-\lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N-\lambda_\mathrm{a})-\mathrm{ch}(\rho_1/2)}\\
&= -\lambda_{05} \qquad (53)\\
\gamma_{17} &= +\frac{2[(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2]\,\mathrm{ch}(\rho_1/4)\sin(\pi\kappa/N-\lambda_\mathrm{a}/2)}{\cos(2\pi\kappa/N-\lambda_\mathrm{a})-\mathrm{ch}(\rho_1/2)}\\
&= +\gamma_{05} \qquad (54)
\end{aligned}
$$

With the help of Eqs.(37) to (54) we may write $A_\mathrm{ev}(\eta,\xi)$ of Eq.(5) in the following form:

$$A_{\text{ev}}(\eta,\xi)=\frac{1}{4}V_{\text{e0}}(c\Delta t)^2\Bigg(S_{00}S_{10}$$

$$+\frac{1}{N}\sum_{\substack{\kappa=-N/2+1\\ -K_0,K_{\text{N}/2}}}^{\substack{N/2-1\\ -K_{\text{N}/2},K_0}} D_0(\kappa)\Big[D_1(\kappa)(S_{01}S_{11}+S_{02}S_{12})-D_2(\kappa)(S_{03}S_{13}+S_{04}S_{14})\Big]$$

$$-\frac{2}{N}\sum_{\substack{\kappa=-K_{\text{N}/2}\\ \kappa>K_0}}^{\substack{<K_{\text{N}/2}\\ <-K_0}} \Big\{[D_0(\kappa)S_{05}-D_3(\kappa)S_{06}]S_{15}+[D_0(\kappa)S_{06}-D_3(\kappa)S_{05}]S_{16}$$

$$+[D_0(\kappa)S_{07}-D_3(\kappa)S_{08}]S_{17}+[D_0(\kappa)S_{08}+D_3(\kappa)S_{07}]S_{18}\Big\}\Bigg) \quad (55)$$

We have succeeded in eliminating the summation signs of finite difference mathematics from Eq.(4). What remains to be done is to write Eq.(55) in terms of functions of η and ξ, and to substitute ζ and η for η and ξ. Furthermore, we want to write in analogy to Eq.(2.2-40) all functions of θ in terms of sinusoidal functions, which is possible by means of Fourier expansions. The calculations are carried out in Section 6.4 and the following representation of $A_{\text{ev}}(\eta,\xi)=A_{\text{ev}}(\zeta,\theta)$ is obtained in Eq.(6.4-83):

$$A_{\text{ev}}(\zeta,\theta)=(c\Delta t)^2V_{\text{e0}}\Bigg[\frac{e^{-\rho_2\zeta}}{4\,\text{sh}^2(\rho_2/2)}$$

$$+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\bigg(C_{\text{ec}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}+C_{\text{es}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}$$

$$+C_{\text{ec}}^{\text{a}}(\kappa,\theta)\cos\lambda_{\text{a}}\zeta+C_{\text{es}}^{\text{a}}(\kappa,\theta)\sin\lambda_{\text{a}}\zeta\bigg)\Bigg] \quad (56)$$

The coefficients $C_{\text{ec}}(\kappa,\theta)$ to $C_{\text{es}}^{\text{a}}(\kappa,\theta)$ are shown in Eqs.(6.4-84) and (6.4-85).

3.4 Magnetic Potential $A_{\text{mv}}(\zeta,\theta)$

In Section 3.1 we obtained a difference equation (3.1-5) for V_{e} by rewriting the differential equation (2.1-43) as a difference equation. After solving it for V_{e} we obtained the electric potential $A_{\text{ev}}(\eta,\xi)$ by means of Eq.(3.3-5), which is a finite difference substitute for Eq.(2.1-45). The electric potential was eventually brought into the form of Eq.(3.3-56).

To obtain a difference equation for the magnetic potential $A_{\text{mv}}(\zeta,\theta)$ we rewrite the differential equation (2.1-34) as a difference equation and treat the electric potential $A_{\text{ev}}(\zeta,\theta)$ as known according to Eq.(3.3-56).

Equation (2.1-34) is rewritten in normalized form and the terms are rearranged:

$$\frac{\partial A_{\mathrm{mv}}}{\partial \zeta} = Z\rho_{\mathrm{s}}\left(\frac{\partial^2 A_{\mathrm{ev}}}{\partial \zeta^2} - \frac{\partial^2 A_{\mathrm{ev}}}{\partial \theta^2} - \frac{1}{\hat{\rho}_{\mathrm{s}}}\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)$$
$$\theta = \frac{t}{\Delta t},\ \zeta = \frac{y}{c\Delta t},\ \rho_{\mathrm{s}} = \frac{Z}{sc\Delta t} = \frac{\mu}{s\Delta t} = \frac{1}{2\rho_2^2}[\rho_1 \pm (\rho_1^2 - 4\rho_2^2)^{1/2}] \qquad (1)$$

Using Eqs.(3.1-2) to (3.1-4) we obtain from Eq.(1) the following difference equation:

$$\begin{aligned}\frac{\tilde{\Delta} A_{\mathrm{mv}}(\zeta,\theta)}{\tilde{\Delta}\zeta} = Z\rho_{\mathrm{s}}\Big(&A_{\mathrm{ev}}(\zeta+1,\theta) - 2A_{\mathrm{ev}}(\zeta,\theta) + A_{\mathrm{ev}}(\zeta-1,\theta)\\ &- [A_{\mathrm{ev}}(\zeta,\theta+1) - 2A_{\mathrm{ev}}(\zeta,\theta) + A_{\mathrm{ev}}(\zeta,\theta-1)]\\ &- \frac{1}{2\rho_{\mathrm{s}}}[A_{\mathrm{ev}}(\zeta,\theta+1) - A_{\mathrm{ev}}(\zeta,\theta-1)]\Big) \qquad (2)\end{aligned}$$

Summation of $\Delta A_{\mathrm{mv}}(\zeta,\theta)/\Delta\zeta$ with respect to ζ produces $A_{\mathrm{mv}}(\zeta,\theta)$ just as the integration of $\partial A_{\mathrm{mv}}/\partial\zeta$ with respect to ζ in Eq.(2.1-49) produced the *differential magnetic potential*$A_{\mathrm{mv}}(\zeta,\theta)$. We obtain the first order difference quotient from the second order difference quotient by comparing Eqs.(3.1.2) and (3.1-4):

$$\begin{aligned}A_{\mathrm{mv}}(\zeta,\theta) =& A_{\mathrm{mv1}}(\zeta,\theta) + A_{\mathrm{mv2}}(\zeta,\theta) + A_{\mathrm{mv3}}(\zeta,\theta)\\ =& \frac{1}{2}Z\rho_{\mathrm{s}}[A_{\mathrm{ev}}(\zeta+1,\theta) - A_{\mathrm{ev}}(\zeta-1,\theta)]\\ &- Z\rho_{\mathrm{s}}\mathop{\large S}_{c}^{\zeta}[A_{\mathrm{ev}}(\zeta,\theta+1) - 2A_{\mathrm{ev}}(\zeta,\theta) + A_{\mathrm{ev}}(\zeta,\theta-1)]\Delta\zeta\\ &- \frac{Z}{2}\mathop{\large S}_{c}^{\zeta}[A_{\mathrm{ev}}(\zeta,\theta+1) - A_{\mathrm{ev}}(\zeta,\theta-1)]\Delta\zeta \qquad (3)\end{aligned}$$

In Eq.(2.1-49) we had ignored an integration constant since only derivatives of $A_{\mathrm{mv}}(\zeta,\theta)$ are used in Eq.(2.3-15). Equation (3) still contains the summation constant c but we simplify the calculation by leaving it out from here on.

Since $A_{\mathrm{mv}}(\zeta,\theta)$ will be integrated later on with respect to ζ we must write the first two terms $A_{\mathrm{ev}}(\zeta\pm1,\theta)$ in some detail using Eq.(3.3-56). This is done in Section 6.5. Equation (6.5-1) shows $A_{\mathrm{ev}}(\zeta\pm1,\theta)$ while Eq.(6.5-2) shows $A_{\mathrm{ev}}(\zeta+1,\theta) - A_{\mathrm{ev}}(\zeta-1,\theta)$.

The summations in Eq.(3) are obtained by evaluating the summation of $A_{\mathrm{ev}}(\zeta,\theta)$ and substituting $\theta\pm1$ for θ to obtain the summations of $A_{\mathrm{ev}}(\zeta,\theta+1)$

and $A_{\rm ev}(\zeta,\theta-1)$. According to Eq.(3.3-56) we need five summations that can all be obtained with the help of Table 6.2-1:

$$\mathop{\mathfrak{S}}^{\zeta} e^{-\rho_2\zeta}\Delta\zeta = -\frac{e^{-\rho_2\zeta}}{\operatorname{sh}\rho_2}, \quad \rho_2 > 0 \tag{4}$$

$$\mathop{\mathfrak{S}}^{\zeta} \cos\lambda_{\rm a}\zeta\Delta\zeta = \frac{\sin\lambda_{\rm a}\zeta}{\sin\lambda_{\rm a}}, \quad |\lambda_{\rm a}| < \pi \tag{5}$$

$$\mathop{\mathfrak{S}}^{\zeta} \sin\lambda_{\rm a}\zeta\Delta\zeta = -\frac{\cos\lambda_{\rm a}\zeta}{\sin\lambda_{\rm a}}, \quad |\lambda_{\rm a}| < \pi \tag{6}$$

$$\mathop{\mathfrak{S}}^{\zeta} \cos\frac{2\pi\kappa\zeta}{N}\Delta\zeta = \frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}, \quad |\kappa| < \frac{N}{2} \tag{7}$$

$$\mathop{\mathfrak{S}}^{\zeta} \sin\frac{2\pi\kappa\zeta}{N}\Delta\zeta = -\frac{\cos(2\pi\kappa\zeta/N}{\sin(2\pi\kappa/N)}, \quad |\kappa| < \frac{N}{2} \tag{8}$$

Substitution of Eqs.(4) to (8) into Eq.(3.3-56) yields the summation of $A_{\rm ev}(\zeta,\theta)$ with respect to ζ. We see from Eqs.(7) and (8) that the exclusion of the limits $\kappa = -N/2$ and $\kappa = +N/2$ in Eqs.(3.3-55) and (3.3-56) avoids a problem of divergence and that Eqs.(5), (6) may require further exclusions. More details may be found in the text before Eq.(3.2-36). We obtain the following formula from Eq.(3.3-56):

$$\begin{aligned}\mathop{\mathfrak{S}}^{\zeta} A_{\rm ev}(\zeta,\theta)\Delta\zeta = (c\Delta t)^2 V_{\rm e0}\Bigg[-\frac{e^{-\rho_2\zeta}}{4\operatorname{sh}^2(\rho_2/2)\operatorname{sh}\rho_2} + \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\ \Bigg(C_{\rm ec}(\kappa,\theta)\frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)} - C_{\rm es}(\kappa,\theta)\frac{\cos(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)} \\ + C^a_{\rm ec}(\kappa,\theta)\frac{\sin\lambda_{\rm a}\zeta}{\sin\lambda_{\rm a}} - C^a_{\rm es}(\kappa,\theta)\frac{\cos\lambda_{\rm a}\zeta}{\sin\lambda_{\rm a}}\Bigg)\Bigg]\end{aligned} \tag{9}$$

The summations of $A_{\rm ev}(\zeta,\theta+1)$ and $A_{\rm ev}(\zeta,\theta-1)$ in Eq.(3) are obtained from this formula by substituting $\theta \pm 1$ for θ. This is done in Section 6.5. Equation (6.5-14) shows $A_{\rm mv}(\zeta,\theta)$ in the following relatively compact form:

$$A_{\rm mv}(\zeta,\theta) = Z(c\Delta t)^2 V_{\rm e0}\Bigg[-\frac{\operatorname{sh}\rho_2\, e^{-\rho_2\zeta}}{4\operatorname{sh}^2(\rho_2/2)} + \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\bigg(C_{\rm mc}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C_{\rm ms}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N} + C^{\rm a}_{\rm mc}(\kappa,\theta)\cos\lambda_{\rm a}\zeta + C^{\rm a}_{\rm ms}(\kappa,\theta)\sin\lambda_{\rm a}\zeta\bigg)\Bigg] \tag{10}$$

The functions $C_{\rm mc}(\kappa,\theta)$ to $C^{\rm a}_{\rm ms}(\kappa,\theta)$ are listed in Eqs.(6.5-15) and (6.5-16).

3.5 Hamilton Function for Finite Differences

With $A_{\rm ev}(\zeta,\theta)$ of Eq.(3.3-56) and $A_{\rm mv}(\zeta,\theta)$ of Eq.(3.4-10) in a form comparable to $A_{\rm ev}(\zeta,\theta)$ and $A_{\rm mv}(\zeta,\theta)$ of Eqs.(2.2-40) and (2.2-44) derived by infinitesimal mathematics we may follow the procedure of Section 2.3 to obtain a Hamilton function. One question arises immediately: Should we use integrals as in Eq.(2.3-1) or should we use summations instead? The advantage of the integrals is the simpler calculation but the finite differences used by summations are better in the representation of physical observability. In the end one has no choice but to try both methods. If the difference between the results obtained by integration or by summation is significant, one knows that summation has to be used—if one wants to use the mathematics of finite differences at all. Unfortunately, the reverse is not true. An insignificant difference in one example does not prove there will never be another example with significant difference.

We choose integration. Since our functions $A_{\rm ev}(\zeta,\theta)$ and $A_{\rm mv}(\zeta,\theta)$ of Eqs.(3.3-56) and (3.4-10) are differentiable with respect to ζ and θ it is consistent to use differentiation as well as integration. Hence, we may use Eq.(2.3-1) as our starting point:

$$U = \frac{1}{2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\left[\int_0^{cT}\left(\frac{1}{Zc}E^2 + \frac{Z}{c}H^2\right)dy\right]dx\,dz \tag{1}$$

In Eqs.(2.3-7) to (2.3-12) we used $\tau = T/N_\tau$. Now we use $\tau = T/N = \Delta t$ and the normalizations $\theta = t/\Delta t$ and $\zeta = y/c\Delta t$ instead of $\theta = t/\tau$ and $\zeta = y/c\tau$:

$$\tau = \Delta t,\ \theta = t/\Delta t,\ \zeta = y/c\Delta t,\ T/\Delta t = N$$

$$\operatorname{curl}^2 \mathbf{A}_{\rm e} = 2\left(\frac{\partial A_{\rm ev}}{\partial y}\right)^2 = \frac{2}{(c\Delta t)^2}\left(\frac{\partial A_{\rm ev}}{\partial\zeta}\right)^2 \tag{2}$$

$$\operatorname{curl}\mathbf{A}_{\mathrm{e}}\cdot\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}=2\frac{\partial A_{\mathrm{ev}}}{\partial y}\frac{\partial A_{\mathrm{mv}}}{\partial t}=\frac{2}{c(\Delta t)^2}\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\tag{3}$$

$$\left(\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2=\left(\frac{\partial A_{\mathrm{mv}}}{\partial t}\right)^2=\frac{2}{(\Delta t)^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\tag{4}$$

$$\operatorname{curl}^2\mathbf{A}_{\mathrm{m}}=\left(\frac{\partial A_{\mathrm{mv}}}{\partial y}\right)^2=\frac{2}{(c\Delta t)^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2\tag{5}$$

$$\operatorname{curl}\mathbf{A}_{\mathrm{m}}\cdot\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}=-2\frac{\partial A_{\mathrm{mv}}}{\partial y}\frac{\partial A_{\mathrm{ev}}}{\partial t}=-\frac{2}{c(\Delta t)^2}\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\tag{6}$$

$$\left(\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2=2\left(\frac{\partial A_{\mathrm{ev}}}{\partial t}\right)^2=\frac{2}{(\Delta t)^2}\left(\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\tag{7}$$

The squares of the field strengths **E** and **H** in Eqs.(2.3-2) and (2.3-3) have the following values:

$$\begin{aligned}E^2&=\frac{2}{(\Delta t)^2}\left[Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\right)^2+2Z\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta}+\left(\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\right]\\&=\frac{2}{(\Delta t)^2}\left(Z\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}+\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\end{aligned}\tag{8}$$

$$\begin{aligned}H^2&=\frac{2}{Z^2(\Delta t)^2}\left[\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2+2Z\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta}+Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\right]\\&=\frac{2}{Z^2(\Delta t)^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}+Z\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\end{aligned}\tag{9}$$

The energy U of Eq.(1) is rewritten as follows:

$$\begin{aligned}U=\frac{c^2\Delta t}{Z}\int\limits_{-L/2c\Delta t}^{L/2c\Delta t}\int\limits_{-L/2c\Delta t}^{L/2c\Delta t}&\left\{\int\limits_0^N\left[\left(Z\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}+\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\right.\right.\\&\left.\left.+\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}+Z\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\right]d\zeta\right\}d\left(\frac{x}{c\Delta t}\right)d\left(\frac{z}{c\Delta t}\right)\\=\frac{c^2\Delta t}{Z}\left(\frac{L}{c\Delta t}\right)^2\int\limits_0^N&\left[Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\right)^2+Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2\right.\\&+2Z\left(\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta}+\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)\\&\left.+\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2+\left(\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2\right]d\zeta\end{aligned}$$

$$c^2\Delta t/Z=c^2T/ZN.\quad L/c\Delta t=LN/cT\tag{10}$$

In order to evaluate Eq.(10) we need $\partial A_{\rm ev}/\partial\zeta$, $\partial A_{\rm ev}/\partial\theta$, $\partial A_{\rm mv}/\partial\zeta$, and $\partial A_{\rm mv}/\partial\theta$ from Eqs.(3.3-56) and (3.4-10):

$$\begin{aligned}\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\zeta} &= (c\Delta t)^2 V_{\rm e0}\bigg[-\frac{\rho_2}{4\,{\rm sh}^2(\rho_2/2)}e^{-\rho_2\zeta}\\ &\quad+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\bigg(C_{\rm es}^{\zeta}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}+C_{\rm ec}^{\zeta}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\\ &\quad+C_{\rm es}^{{\rm a}\zeta}(\kappa,\theta)\cos\lambda_{\rm a}\zeta+C_{\rm ec}^{{\rm a}\zeta}(\kappa,\theta)\sin\lambda_{\rm a}\zeta\bigg)\bigg]\end{aligned}\tag{11}$$

$$\begin{aligned}C_{\rm es}^{\zeta}(\kappa,\theta) &= \frac{2\pi\kappa}{N}C_{\rm es}(\kappa,\theta)\\ C_{\rm ec}^{\zeta}(\kappa,\theta) &= -\frac{2\pi\kappa}{N}C_{\rm ec}(\kappa,\theta)\\ C_{\rm es}^{{\rm a}\zeta}(\kappa,\theta) &= \lambda_{\rm a}C_{\rm es}^{\rm a}(\kappa,\theta)\\ C_{\rm ec}^{{\rm a}\zeta}(\kappa,\theta) &= -\lambda_{\rm a}C_{\rm ec}^{\rm a}(\kappa,\theta)\end{aligned}\tag{12}$$

$$\begin{aligned}\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\theta} &= (c\Delta t)^2 V_{\rm e0}\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\bigg(\frac{\partial C_{\rm ec}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N}+\frac{\partial C_{\rm es}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\\ &\quad+\frac{\partial C_{\rm ec}^{\rm a}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\rm a}\zeta+\frac{\partial C_{\rm es}^{\rm a}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\rm a}\zeta\bigg)\end{aligned}\tag{13}$$

$$\begin{aligned}\frac{\partial A_{\rm mv}(\zeta,\theta)}{\partial\zeta} &= Z(c\Delta t)^2 V_{\rm e0}\bigg[-\frac{\rho_2}{4\,{\rm sh}^2(\rho_2/2)\,{\rm sh}\,\rho_2}e^{-\rho_2\zeta}\\ &\quad+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\bigg(C_{\rm ms}^{\zeta}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}+C_{\rm mc}^{\zeta}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\\ &\quad+C_{\rm ms}^{{\rm a}\zeta}(\kappa,\theta)\cos\lambda_{\rm a}\zeta+C_{\rm mc}^{{\rm a}\zeta}(\kappa,\theta)\sin\lambda_{\rm a}\zeta\bigg)\bigg]\end{aligned}\tag{14}$$

$$\begin{aligned}C_{\rm ms}^{\zeta}(\kappa,\theta) &= \frac{2\pi\kappa}{N}C_{\rm ms}(\kappa,\theta)\\ C_{\rm mc}^{\zeta}(\kappa,\theta) &= -\frac{2\pi\kappa}{N}C_{\rm mc}(\kappa,\theta)\\ C_{\rm ms}^{{\rm a}\zeta}(\kappa,\theta) &= \lambda_{\rm a}C_{\rm ms}^{\rm a}(\kappa,\theta)\\ C_{\rm mc}^{{\rm a}\zeta}(\kappa,\theta) &= -\lambda_{\rm a}C_{\rm mc}^{\rm a}(\kappa,\theta)\end{aligned}\tag{15}$$

$$\frac{\partial A_{\text{mv}}(\zeta,\theta)}{\partial\theta} = Z(c\Delta t)^2 V_{\text{e0}} \times \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \left(\frac{\partial C_{\text{mc}}(\kappa,\theta)}{\partial\theta} \cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\text{ms}}(\kappa,\theta)}{\partial\theta} \sin\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\text{mc}}^{\text{a}}(\kappa,\theta)}{\partial\theta} \cos\lambda_{\text{a}}\zeta + \frac{\partial C_{\text{ms}}^{\text{a}}(\kappa,\theta)}{\partial\theta} \sin\lambda_{\text{a}}\zeta \right) \quad (16)$$

The square of $\partial A_{\text{ev}}/\partial\zeta$ in Eq.(11) is required for the first term in Eq.(10). Since $\partial A_{\text{ev}}/\partial\zeta$ has 5 terms, its square has 15 terms. A simplification is needed. We ignore the terms $\exp(-\rho_2\zeta)$ in Eqs.(11) and (14), just as we did in Eqs.(2.2-40) and (2.2-44) of the differential theory[1].

$$\begin{aligned} U_1 = \frac{L^2 Z}{\Delta t}\int_0^N \left(\frac{\partial A_{\text{ev}}}{\partial\zeta}\right)^2 d\zeta = L^2 c^4 (\Delta t)^3 Z V_{\text{e0}}^2 \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \int_0^N \Big[C_{\text{es}}^{\zeta 2}(\kappa,\theta)\cos^2\frac{2\pi\kappa\zeta}{N} \\ + C_{\text{ec}}^{\zeta 2}(\kappa,\theta)\sin^2\frac{2\pi\kappa\zeta}{N} + C_{\text{es}}^{\text{a}\zeta 2}(\kappa,\theta)\cos^2\lambda_{\text{a}}\zeta + C_{\text{ec}}^{\text{a}\zeta 2}(\kappa,\theta)\sin^2\lambda_{\text{a}}\zeta \\ + 2C_{\text{es}}^{\text{a}\zeta}(\kappa,\theta)\cos\lambda_{\text{a}}\zeta\left(C_{\text{es}}^{\zeta}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C_{\text{ec}}^{\zeta}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\ + 2C_{\text{ec}}^{\text{a}\zeta}(\kappa,\theta)\sin\lambda_{\text{a}}\zeta\left(C_{\text{es}}^{\zeta}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C_{\text{ec}}^{\zeta}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\ + 2C_{\text{es}}^{\zeta}(\kappa,\theta)C_{\text{ec}}^{\zeta}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N} \\ + 2C_{\text{es}}^{\text{a}\zeta}(\kappa,\theta)C_{\text{ec}}^{\text{a}\zeta}(\kappa,\theta)\sin\lambda_{\text{a}}\zeta\cos\lambda_{\text{a}}\zeta\Big] d\zeta \quad (17) \end{aligned}$$

We have now 9 terms rather than 15, which is more manageable. The remaining 5 terms in Eq.(10) are written with the same approximation:

$$U_2 = \frac{L^2 Z}{\Delta t}\int_0^N \left(\frac{\partial A_{\text{ev}}}{\partial\theta}\right)^2 d\zeta = L^2 c^4 (\Delta t)^3 Z V_{\text{e0}}^2 \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \int_0^N \left[\left(\frac{\partial C_{\text{ec}}(\kappa,\theta)}{\partial\theta}\right)^2 \cos^2\frac{2\pi\kappa\zeta}{N} + \left(\frac{\partial C_{\text{es}}(\kappa,\theta)}{\partial\theta}\right)^2 \sin^2\frac{2\pi\kappa\zeta}{N}\right.$$

[1]From Eq.(17) to (36) we have to write some very long equations. On first reading one may want to read the text but glance over the equations.

$$
\begin{aligned}
&+ \left(\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\right)^2 \cos^2\lambda_{\mathrm{a}}\zeta + \left(\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\right)^2 \sin^2\lambda_{\mathrm{a}}\zeta \\
&+ 2\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ 2\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&\qquad + 2\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N} \\
&\qquad\qquad + 2\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\mathrm{a}}\zeta\cos\lambda_{\mathrm{a}}\zeta\Bigg]d\zeta \qquad (18)
\end{aligned}
$$

The following expressions for U_3 and U_4 are very long:

$$
\begin{aligned}
U_3 &= \frac{2L^2}{\Delta t}\int_0^N \frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta}d\zeta = 2L^2c^4(\Delta t)^3 ZV_{\mathrm{e0}}^2 \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
&\int_0^N \Bigg[C^{\zeta}_{\mathrm{es}}(\kappa,\theta)\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos^2\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{ec}}(\kappa,\theta)\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin^2\frac{2\pi\kappa\zeta}{N} \\
&\quad + C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos^2\lambda_{\mathrm{a}}\zeta + C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin^2\lambda_{\mathrm{a}}\zeta \\
&\quad + C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&\quad + C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&\quad + \frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{es}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{ec}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&\quad + \frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{es}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{ec}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&\quad + \left(C^{\zeta}_{\mathrm{es}}(\kappa,\theta)\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} + C^{\zeta}_{\mathrm{ec}}(\kappa,\theta)\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N} \\
&\quad + \left(C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} + C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)\cos\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}}\zeta\Bigg]d\zeta \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= \frac{2L^2}{\Delta t}\int_0^N \frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta}d\zeta = 2L^2c^4(\Delta t)^3 ZV_{\mathrm{e0}}^2 \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
&\int_0^N \Bigg[C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos^2\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin^2\frac{2\pi\kappa\zeta}{N}
\end{aligned}
$$

$$
\begin{aligned}
&+ C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos^2\lambda_{\mathrm{a}}\zeta + C^{\mathrm{a}\zeta}_{\mathrm{mc}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin^2\lambda_{\mathrm{a}}\zeta \\
&+ C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ C^{\mathrm{a}\zeta}_{\mathrm{mc}}\sin\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N} + \frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ \frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ \frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ \left(C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\right)\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N} \\
&+ \left(C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta} + C^{\mathrm{a}\zeta}_{\mathrm{mc}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\right)\cos\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}}\zeta\Bigg]d\zeta \qquad (20)
\end{aligned}
$$

The expressions for U_5 and U_6 are no longer than for U_1 and U_2:

$$
\begin{aligned}
U_5 &= \frac{L^2}{Z\Delta t}\int_0^N\left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2 d\zeta = L^2c^4(\Delta t)^3 ZV_{\mathrm{e0}}^2\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
&\int_0^N\Bigg[C^{\zeta 2}_{\mathrm{ms}}(\kappa,\theta)\cos^2\frac{2\pi\kappa\zeta}{N} + C^{\zeta 2}_{\mathrm{mc}}(\kappa,\theta)\sin^2\frac{2\pi\kappa\zeta}{N} \\
&+ C^{\mathrm{a}\zeta 2}_{\mathrm{ms}}(\kappa,\theta)\cos^2\lambda_{\mathrm{a}}\zeta + C^{\mathrm{a}\zeta 2}_{\mathrm{mc}}(\kappa,\theta)\sin^2\lambda_{\mathrm{a}}\zeta \\
&+ 2C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ 2C^{\mathrm{a}\zeta}_{\mathrm{mc}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}\zeta\left(C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N} + C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right) \\
&+ 2C^{\zeta}_{\mathrm{ms}}(\kappa,\theta)C^{\zeta}_{\mathrm{mc}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N} \\
&+ 2C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)C^{\mathrm{a}\zeta}_{\mathrm{mc}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}}\zeta\Bigg]d\zeta \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
U_6 &= \frac{L^2}{Z\Delta t}\int_0^N\left(\frac{\partial A_{\mathrm{mv}}}{\partial\theta}\right)^2 d\zeta = L^2c^4(\Delta t)^3 ZV_{\mathrm{e0}}^2\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
&\int_0^N\Bigg[\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)^2\cos^2\frac{2\pi\kappa\zeta}{N} + \left(\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\right)^2\sin^2\frac{2\pi\kappa\zeta}{N}
\end{aligned}
$$

$$+\left(\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)^2\cos^2\lambda_{\mathrm{a}}\zeta+\left(\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\right)^2\sin^2\lambda_{\mathrm{a}}\zeta$$
$$+2\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N}+\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right)$$
$$+2\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\lambda_{\mathrm{a}}\zeta\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N}+\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\sin\frac{2\pi\kappa\zeta}{N}\right)$$
$$+2\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa\zeta}{N}$$
$$\left.+2\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\cos\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}}\zeta\right]d\zeta \quad (22)$$

The following 10 integrals are required for Eqs.(17) to (22):

$$\int_0^N\sin^2\frac{2\pi\kappa\zeta}{N}d\zeta=\int_0^N\cos^2\frac{2\pi\kappa\zeta}{N}d\zeta=\frac{N}{2},\quad\int_0^N\sin\frac{2\pi\kappa\zeta}{N}\cos\frac{2\pi\kappa\zeta}{N}d\zeta=0 \quad (23)$$

$$\int_0^N\sin^2\lambda_{\mathrm{a}}\zeta\,d\zeta=\frac{N}{2}+\frac{\sin 2N\lambda_{\mathrm{a}}}{4\lambda_{\mathrm{a}}},\quad\int_0^N\cos^2\lambda_{\mathrm{a}}\zeta\,d\zeta=\frac{N}{2}-\frac{\sin 2N\lambda_{\mathrm{a}}}{4\lambda_{\mathrm{a}}}$$

$$\int_0^N\sin\lambda_{\mathrm{a}}\zeta\cos\lambda_{\mathrm{a}}\zeta\,d\zeta=\frac{1-\cos 2N\lambda_{\mathrm{a}}}{4\lambda_{\mathrm{a}}} \quad (24)$$

$$\int_0^N\sin\lambda_{\mathrm{a}}\zeta\sin\frac{2\pi\kappa\zeta}{N}d\zeta=\frac{N}{2}\left(\frac{\sin(N\lambda_{\mathrm{a}}-2\pi\kappa)}{N\lambda_{\mathrm{a}}-2\pi\kappa}-\frac{\sin(N\lambda_{\mathrm{a}}+2\pi\kappa)}{N\lambda_{\mathrm{a}}+2\pi\kappa}\right)$$

$$\int_0^N\sin\lambda_{\mathrm{a}}\zeta\cos\frac{2\pi\kappa\zeta}{N}d\zeta=\frac{N}{4}\left(\frac{\sin^2(N\lambda_{\mathrm{a}}/2-\pi\kappa)}{N\lambda_{\mathrm{a}}/2-\pi\kappa}+\frac{\sin^2(N\lambda_{\mathrm{a}}/2+\pi\kappa)}{N\lambda_{\mathrm{a}}/2+\pi\kappa}\right)$$

$$\int_0^N\cos\lambda_{\mathrm{a}}\zeta\sin\frac{2\pi\kappa\zeta}{N}d\zeta=-\frac{N}{4}\left(\frac{\sin^2(N\lambda_{\mathrm{a}}/2-\pi\kappa)}{N\lambda_{\mathrm{a}}/2-\pi\kappa}-\frac{\sin^2(N\lambda_{\mathrm{a}}/2+\pi\kappa)}{N\lambda_{\mathrm{a}}/2+\pi\kappa}\right)$$

$$\int_0^N\cos\lambda_{\mathrm{a}}\zeta\cos\frac{2\pi\kappa\zeta}{N}d\zeta=\frac{N}{2}\left(\frac{\sin(N\lambda_{\mathrm{a}}-2\pi\kappa)}{N\lambda_{\mathrm{a}}-2\pi\kappa}+\frac{\sin(N\lambda_{\mathrm{a}}+2\pi\kappa)}{N\lambda_{\mathrm{a}}+2\pi\kappa}\right) \quad (25)$$

The energy U of Eq.(10) is the sum of the energies U_1 to U_6 of Eqs.(17) to (22). Since the variable ζ has been eliminated we may drop the argument (κ,θ) in the following formula in order to shorten it.

$$
\begin{aligned}
U = \sum_{i=1}^{6} U_i = \frac{1}{2} Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N^2} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
\Bigg\{ \Bigg[C_{\mathrm{es}}^{\zeta 2} + C_{\mathrm{ec}}^{\zeta 2} + \left(\frac{\partial C_{\mathrm{ec}}}{\partial\theta}\right)^2 + \left(\frac{\partial C_{\mathrm{es}}}{\partial\theta}\right)^2 + C_{\mathrm{es}}^{\zeta} \frac{\partial C_{\mathrm{mc}}}{\partial\theta} + C_{\mathrm{ec}}^{\zeta} \frac{\partial C_{\mathrm{ms}}}{\partial\theta} \\
+ C_{\mathrm{ms}}^{\zeta} \frac{\partial C_{\mathrm{ec}}}{\partial\theta} + C_{\mathrm{mc}}^{\zeta} \frac{\partial C_{\mathrm{es}}}{\partial\theta} + C_{\mathrm{ms}}^{\zeta 2} + C_{\mathrm{mc}}^{\zeta 2} + \left(\frac{\partial C_{\mathrm{mc}}}{\partial\theta}\right)^2 + \left(\frac{\partial C_{\mathrm{ms}}}{\partial\theta}\right)^2 \Bigg] \\
+ \left(1 - \frac{\sin 2N\lambda_{\mathrm{a}}}{2N\lambda_{\mathrm{a}}}\right) \Bigg[C_{\mathrm{es}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta}\right)^2 + C_{\mathrm{es}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} \\
+ C_{\mathrm{ms}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta}\right)^2 \Bigg] \\
+ \left(1 + \frac{\sin 2N\lambda_{\mathrm{a}}}{2N\lambda_{\mathrm{a}}}\right) \Bigg[C_{\mathrm{ec}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta}\right)^2 + C_{\mathrm{ec}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} \\
+ C_{\mathrm{mc}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta}\right)^2 \Bigg] \\
+2\left(\frac{\sin(N\lambda_{\mathrm{a}} - 2\pi\kappa)}{N\lambda_{\mathrm{a}} - 2\pi\kappa} - \frac{\sin(N\lambda_{\mathrm{a}} + 2\pi\kappa)}{N\lambda_{\mathrm{a}} + 2\pi\kappa}\right) \Bigg[C_{\mathrm{es}}^{\mathrm{a}\zeta} C_{\mathrm{es}}^{\zeta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{es}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{mc}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{es}}^{\zeta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ms}}^{\zeta} \Bigg) + C_{\mathrm{ms}}^{\mathrm{a}\zeta} C_{\mathrm{ms}}^{\zeta} + \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{mc}}}{\partial\theta} \Bigg] \\
-\left(\frac{\sin^2(N\lambda_{\mathrm{a}}/2 - \pi\kappa)}{N\lambda_{\mathrm{a}}/2 - \pi\kappa} - \frac{\sin^2(N\lambda_{\mathrm{a}}/2 + \pi\kappa)}{N\lambda_{\mathrm{a}}/2 + \pi\kappa}\right) \Bigg[C_{\mathrm{es}}^{\mathrm{a}\zeta} C_{\mathrm{ec}}^{\zeta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{es}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ms}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ec}}^{\zeta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{mc}}^{\zeta} \Bigg) + C_{\mathrm{ms}}^{\mathrm{a}\zeta} C_{\mathrm{mc}}^{\zeta} + \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{ms}}}{\partial\theta} \Bigg] \\
-\left(\frac{\sin^2(N\lambda_{\mathrm{a}}/2 - \pi\kappa)}{N\lambda_{\mathrm{a}}/2 - \pi\kappa} + \frac{\sin^2(N\lambda_{\mathrm{a}}/2 + \pi\kappa)}{N\lambda_{\mathrm{a}}/2 + \pi\kappa}\right) \Bigg[C_{\mathrm{ec}}^{\mathrm{a}\zeta} C_{\mathrm{es}}^{\zeta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{ec}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{mc}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{es}}^{\zeta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ms}}^{\zeta} \Bigg) + C_{\mathrm{mc}}^{\mathrm{a}\zeta} C_{\mathrm{ms}}^{\zeta} + \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{mc}}}{\partial\theta} \Bigg] \\
+2\left(\frac{\sin(N\lambda_{\mathrm{a}} - 2\pi\kappa)}{N\lambda_{\mathrm{a}} - 2\pi\kappa} - \frac{\sin(N\lambda_{\mathrm{a}} + 2\pi\kappa)}{N\lambda_{\mathrm{a}} + 2\pi\kappa}\right) \Bigg[C_{\mathrm{ec}}^{\mathrm{a}\zeta} C_{\mathrm{ec}}^{\zeta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{ec}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ms}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ec}}^{\zeta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{mc}}^{\zeta} \Bigg) + C_{\mathrm{mc}}^{\mathrm{a}\zeta} C_{\mathrm{mc}}^{\zeta} + \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{ms}}}{\partial\theta} \Bigg] \\
+ \frac{1 - \cos 2N\lambda_{\mathrm{a}}}{N\lambda_{\mathrm{a}}} \Bigg[C_{\mathrm{es}}^{\mathrm{a}\zeta} C_{\mathrm{ec}}^{\mathrm{a}\zeta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{es}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{ec}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} \\
+ C_{\mathrm{ms}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta} \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} \Bigg) + C_{\mathrm{ms}}^{\mathrm{a}\zeta} C_{\mathrm{mc}}^{\mathrm{a}\zeta} + \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} \Bigg] \Bigg\} \qquad (26)
\end{aligned}
$$

In analogy to Section 2.3 from Eq.(2.3-23) on we must work out the time variation of the functions $C^{\cdot\cdot}_{\cdot\cdot} = C^{\cdot\cdot}_{\cdot\cdot}(\kappa,\theta)$ in Eq.(26). Consider first the terms with subscript es or ec in Eq.(26). We obtain with the help of Eqs.(12), (6.4-84), and (6.4-85):

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq 0 \text{ or } |\kappa|>K_{\mathrm{N/2}}$$

$$\begin{aligned}
C^{\zeta}_{\mathrm{ec}} &= C^{\zeta}_{\mathrm{ec}}(\kappa,\theta) = -(2\pi\kappa/N)C_{\mathrm{ec}}(\kappa,\theta)\\
&= \frac{2\pi\kappa}{N}\frac{1}{4N}\left[L^{\mathrm{c}}_{0}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{c}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{c}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C^{\zeta}_{\mathrm{es}} &= C^{\zeta}_{\mathrm{es}}(\kappa,\theta) = (2\pi\kappa/N)C_{\mathrm{es}}(\kappa,\theta)\\
&= \frac{2\pi\kappa}{N}\frac{1}{4N}\left[L^{\mathrm{s}}_{0}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C^{\mathrm{a}\zeta}_{\mathrm{ec}} &= C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta) = -\lambda_{\mathrm{a}}C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\nu) = 0\\
C^{\mathrm{a}\zeta}_{\mathrm{es}} &= C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta) = \lambda_{\mathrm{a}}C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta) = 0
\end{aligned}\tag{27}$$

$$\text{for } K_0<|\kappa|<K_{\mathrm{N/2}}$$

$$\begin{aligned}
C^{\zeta}_{\mathrm{ec}} &= C^{\zeta}_{\mathrm{ec}}(\kappa,\theta) = -(2\pi\kappa/N)C_{\mathrm{ec}}(\kappa,\theta)\\
&= -\frac{2\pi\kappa}{N}\frac{1}{2N}\left[L_{0\mathrm{c}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C^{\zeta}_{\mathrm{es}} &= C^{\zeta}_{\mathrm{es}}(\kappa,\theta) = (2\pi\kappa/N)C_{\mathrm{es}}(\kappa,\theta)\\
&= \frac{2\pi\kappa}{N}\frac{1}{2N}\left[L_{0\mathrm{s}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C^{\mathrm{a}\zeta}_{\mathrm{ec}} &= C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta) = -\lambda_{\mathrm{a}}C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)\\
&= -\frac{\lambda_{\mathrm{a}}}{2N}\left[L^{\mathrm{a}}_{0\mathrm{c}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{a}}_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C^{\mathrm{a}\zeta}_{\mathrm{es}} &= C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta) = \lambda_{\mathrm{a}}C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)\\
&= \frac{\lambda_{\mathrm{a}}}{2N}\left[L^{\mathrm{a}}_{0\mathrm{s}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{a}}_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]
\end{aligned}\tag{28}$$

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq 0 \text{ or } |\kappa|>K_{\mathrm{N/2}}$$

$$\frac{\partial C_{\mathrm{ec}}}{\partial\theta}=\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}=-\frac{1}{4N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L^{\mathrm{c}}_{\mathrm{s}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L^{\mathrm{c}}_{\mathrm{c}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)$$

$$\begin{aligned}
\frac{\partial C_{\rm es}}{\partial\theta} &= \frac{\partial C_{\rm es}(\kappa,\theta)}{\partial\theta} = \frac{1}{4N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\Big(L^{\rm s}_{{\rm s}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^{\rm s}_{{\rm c}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\Big)\\
\frac{\partial C^{\rm a}_{\rm ec}}{\partial\theta} &= \frac{\partial C^{\rm a}_{\rm ec}(\kappa,\theta)}{\partial\theta} = 0\\
\frac{\partial C^{\rm a}_{\rm es}}{\partial\theta} &= \frac{\partial C^{\rm a}_{\rm es}(\kappa,\theta)}{\partial\theta} = 0
\end{aligned} \tag{29}$$

for $K_0 < |\kappa| < K_{\rm N/2}$

$$\begin{aligned}
\frac{\partial C_{\rm ec}}{\partial\theta} &= \frac{\partial C_{\rm ec}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\Big(L_{\rm sc}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L_{\rm cc}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\Big)\\
\frac{\partial C_{\rm es}}{\partial\theta} &= \frac{\partial C_{\rm es}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\Big(L_{\rm ss}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L_{\rm cs}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\Big)\\
\frac{\partial C^{\rm a}_{\rm ec}}{\partial\theta} &= \frac{\partial C^{\rm a}_{\rm ec}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\Big(L^{\rm a}_{\rm sc}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^{\rm a}_{\rm cc}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\Big)\\
\frac{\partial C^{\rm a}_{\rm es}}{\partial\theta} &= \frac{\partial C^{\rm a}_{\rm es}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\Big(L^{\rm a}_{\rm ss}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^{\rm a}_{\rm cs}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\Big)
\end{aligned} \tag{30}$$

Next come the terms with subscript ms or mc in Eq.(26). Using Eqs.(15) and (6.5-17) to (6.5-22) we obtain the following relations:

for $-K_0 < \kappa < K_0,\ \kappa \neq 0$ or $|\kappa| > K_{\rm N/2}$

$$\begin{aligned}
C^{\zeta}_{\rm mc} &= C^{\zeta}_{\rm mc}(\kappa,\theta) = -(2\pi\kappa/N)C_{\rm mc}(\kappa,\theta)\\
&= -\frac{2\pi\kappa}{N}\frac{1}{4N}\Big[L_{00}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\Big(L_{01}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{02}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\Big)\Big]\\
C^{\zeta}_{\rm ms} &= C^{\zeta}_{\rm ms}(\kappa,\theta) = (2\pi\kappa/N)C_{\rm ms}(\kappa,\theta)\\
&= \frac{2\pi\kappa}{N}\frac{1}{4N}\Big[L_{03}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\Big(L_{04}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{05}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\Big)\Big]\\
C^{\rm a\zeta}_{\rm mc} &= C^{\rm a\zeta}_{\rm mc}(\kappa,\theta) = -\lambda_{\rm a}C^{\rm a}_{\rm mc}(\kappa,\theta) = 0\\
C^{\rm a\zeta}_{\rm ms} &= C^{\rm a\zeta}_{\rm ms}(\kappa,\theta) = \lambda_{\rm a}C^{\rm a}_{\rm ms}(\kappa,\theta) = 0
\end{aligned} \tag{31}$$

for $K_0 < |\kappa| < K_{\mathrm{N}/2}$

$$
\begin{aligned}
C_{\mathrm{mc}}^{\zeta} &= C_{\mathrm{mc}}^{\zeta}(\kappa,\theta) = -(2\pi\kappa/N)C_{\mathrm{mc}}(\kappa,\theta)\\
&= -\frac{2\pi\kappa}{N}\frac{1}{2N}\left[L_{00}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{01}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{02}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C_{\mathrm{ms}}^{\zeta} &= C_{\mathrm{ms}}^{\zeta}(\kappa,\theta) = (2\pi\kappa/N)C_{\mathrm{ms}}(\kappa,\theta)\\
&= \frac{2\pi\kappa}{N}\frac{1}{2N}\left[L_{03}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{04}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{05}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C_{\mathrm{mc}}^{\mathrm{a}\zeta} &= C_{\mathrm{mc}}^{\mathrm{a}\zeta}(\kappa,\theta) = -\lambda_{\mathrm{a}}C_{\mathrm{mc}}^{\mathrm{a}}(\kappa,\theta)\\
&= -\frac{\lambda_{\mathrm{a}}}{2N}\left[L_{00}^{\mathrm{a}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{01}^{\mathrm{a}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{02}^{\mathrm{a}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
C_{\mathrm{ms}}^{\mathrm{a}\zeta} &= C_{\mathrm{ms}}^{\mathrm{a}\zeta}(\kappa,\theta) = \lambda_{\mathrm{a}}C_{\mathrm{ms}}^{\mathrm{a}}(\kappa,\theta)\\
&= \frac{\lambda_{\mathrm{a}}}{2N}\left[L_{03}^{\mathrm{a}}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{04}^{\mathrm{a}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{05}^{\mathrm{a}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right] \qquad (32)
\end{aligned}
$$

for $-K_0 < \kappa < K_0,\ \kappa \neq 0$ or $|\kappa| > K_{\mathrm{N}/2}$

$$
\begin{aligned}
\frac{\partial C_{\mathrm{mc}}}{\partial\theta} &= \frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta} = \frac{1}{4N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L_{01}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L_{02}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
\frac{\partial C_{\mathrm{ms}}}{\partial\theta} &= \frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} = \frac{1}{4N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L_{04}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L_{05}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
\frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} &= \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}(\kappa,\theta)}{\partial\theta} = 0\\
\frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} &= \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}(\kappa,\theta)}{\partial\theta} = 0 \qquad (33)
\end{aligned}
$$

for $K_0 < |\kappa| < K_{\mathrm{N}/2}$

$$
\begin{aligned}
\frac{\partial C_{\mathrm{mc}}}{\partial\theta} &= \frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L_{01}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L_{02}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
\frac{\partial C_{\mathrm{ms}}}{\partial\theta} &= \frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} = \frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L_{04}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L_{05}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)
\end{aligned}
$$

$$\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}}{\partial\theta}=\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}=\frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L^{\mathrm{a}}_{01}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L^{\mathrm{a}}_{02}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)$$
$$\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}}{\partial\theta}=\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}=\frac{1}{2N}\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}\left(L^{\mathrm{a}}_{04}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}-L^{\mathrm{a}}_{05}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)$$
$$(34)$$

In order to bring Eq.(26) into a more lucid form we rewrite the first term $C^{\zeta 2}_{\mathrm{es}}=C^{\zeta 2}_{\mathrm{es}}(\kappa,\theta)$ with the help of Eq.(27) and (28):

for $-K_0<\kappa<K_0,\ \kappa\neq 0$ or $|\kappa|>K_{\mathrm{N/2}}$

$$\begin{aligned}
C^{\zeta 2}_{\mathrm{es}}(\kappa,\theta)=&\left(\frac{2\pi\kappa}{N}\right)^2\left(\frac{1}{4N}\right)^2\Bigg\{L^{\mathrm{s}2}_{0}(\kappa,0)+\frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{\mathrm{s}2}_{\mathrm{s}\kappa}(\kappa,\nu)+L^{\mathrm{s}2}_{\mathrm{c}\kappa}(\kappa,\nu)]\\
&-\sum_{\nu=1}^{N/2-1}[L^{\mathrm{s}2}_{\mathrm{s}\kappa}(\kappa,\nu)-L^{\mathrm{s}2}_{\mathrm{c}\kappa}(\kappa,\nu)]\cos\frac{4\pi\nu\theta}{N}\\
&+2L^{\mathrm{s}}_{0}(\kappa,0)\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\\
&+\sum_{\omega=1}^{N/2-1,\neq\nu}\sum_{\nu=1}^{N/2-1}\Bigg[L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\omega)\left(\cos\frac{2\pi(\nu-\omega)\theta}{N}-\cos\frac{2\pi(\nu+\omega)\theta}{N}\right)\\
&+2L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)L^{s}_{\mathrm{c}\kappa}(\kappa,\omega)\left(\sin\frac{2\pi(\nu-\omega)\theta}{N}+\sin\frac{2\pi(\nu+\omega)\theta}{N}\right)\\
&+L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\nu)L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\omega)\left(\cos\frac{2\pi(\nu-\omega)\theta}{N}+\cos\frac{2\pi(\nu+\omega)\theta}{N}\right)\Bigg]\Bigg\}\\
=&U_1(\kappa)+V_1(\kappa,\theta)
\end{aligned}\tag{35}$$

for $K_0<|\kappa|<K_{\mathrm{N/2}}$

$$\begin{aligned}
C^{\zeta 2}_{\mathrm{es}}(\kappa,\theta)=&\left(\frac{2\pi\kappa}{N}\right)^2\left(\frac{1}{2N}\right)^2\Bigg\{L^{2}_{0s}(\kappa,0)+\frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{2}_{\mathrm{ss}}(\kappa,\nu)+L^{2}_{\mathrm{cs}}(\kappa,\nu)]\\
&-\sum_{\nu=1}^{N/2-1}[L^{2}_{\mathrm{ss}}(\kappa,\nu)-L^{2}_{\mathrm{cs}}(\kappa,\nu)]\cos\frac{4\pi\nu\theta}{N}\\
&+2L_{0\mathrm{s}}(\kappa,0)\sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\omega=1}^{N/2-1,\neq\nu}\sum_{\nu=1}^{N/2-1}\left[L_{\rm ss}(\kappa,\nu)L_{\rm ss}(\kappa,\omega)\left(\cos\frac{2\pi(\nu-\omega)\theta}{N}-\cos\frac{2\pi(\nu+\omega)\theta}{N}\right)\right.\\
&+2L_{\rm ss}(\kappa,\nu)L_{\rm cs}(\kappa,\omega)\left(\sin\frac{2\pi(\nu-\omega)\theta}{N}+\sin\frac{2\pi(\nu+\omega)\theta}{N}\right)\\
&\left.\left.+L_{\rm cs}(\kappa,\nu)L_{\rm cs}(\kappa,\omega)\left(\cos\frac{2\pi(\nu-\omega)\theta}{N}+\cos\frac{2\pi(\nu+\omega)\theta}{N}\right)\right]\right\}\\
=&U_1(\kappa)+V_1(\kappa,\theta) \qquad (36)
\end{aligned}
$$

Let us return to Eq.(26). The energy U_{01} of the sum over the first term $C_{\rm es}^{\zeta 2}(\kappa,\theta)$ consists of the sum of the constant part $U_1(\kappa)$ and the time-variable part $V_1(\kappa,\theta)$:

$$
U_{01}=\frac{1}{2}ZV_{\rm e0}^2L^2T^3c^4\frac{1}{N^2}\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}[U_1(\kappa)+V_1(\kappa,\theta)] \qquad (37)
$$

The equations corresponding to Eqs.(35) and (36) for the terms $C_{\rm ec}^{\zeta 2}$, $(\partial C_{\rm ec}/\partial\theta)^2$, ..., $(\partial C_{\rm ms}/\partial\theta)^2$ in Eq.(26) are all listed in Section 6.6. They yield energies of the form $U_j(\kappa)+V_j(\kappa,\theta)$ as in Eqs.(35) and (36):

$$
U_{0j}=\frac{1}{2}ZV_{\rm e0}^2L^2T^3c^4\frac{1}{N^2}\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}[U_j(\kappa)+V_j(\kappa,\theta)]
$$
$$
j=1,\ 2,\ \ldots,\ 12 \qquad (38)
$$

There are 52 more terms $C_{\rm es}^{{\rm a}\zeta 2}$ to $(\partial C_{\rm mc}^{\rm a}/\partial\theta)(\partial C_{\rm ms}^{\rm a}/\partial\theta)$ in Eq.(26) that are multliplied by functions of $\lambda_{\rm a}$ and may be written in the form of Eq.(36) too. We show only the first example:

$$
\left(1-\frac{\sin 2N\lambda_{\rm a}}{2N\lambda_{\rm a}}\right)C_{\rm es}^{{\rm a}\zeta 2}=U_{13}(\kappa)+V_{13}(\kappa,\theta) \qquad (39)
$$

With this notation we may expand Eq.(38) from 12 to 64 terms:

$$
U_{0j}=\frac{1}{2}ZV_{\rm e0}^2L^2T^3c^4\frac{1}{N^2}\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}[U_j(\kappa)+V_j(\kappa,\theta)]
$$
$$
j=1,\ 2,\ \ldots,\ 64 \qquad (40)
$$

The total energy U of Eq.(26) may then be written in the following concentrated form:

$$U = U_{\mathrm{Nc}} + U_{\mathrm{Nv}}(\theta) = \frac{1}{2} Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N^2} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} [U_{\mathrm{cN}}(\kappa) + U_{\mathrm{vN}}(\kappa, \theta)]$$

$$U_{\mathrm{cN}}(\kappa) = \sum_{j=1}^{64} U_j(\kappa), \quad U_{\mathrm{vN}}(\kappa, \theta) = \sum_{j=1}^{64} V_j(\kappa, \theta) \tag{41}$$

The constant normalized energy $U_{\mathrm{cN}}(\kappa)$ is worked out in detail in Section 6.6. The normalized variable energy $U_{\mathrm{vN}}(\kappa, \theta)$ with time-average zero is not elaborated since it is not needed; in case of need one may obtain it in analogy to Eqs.(35) and (36). Following Eq.(2.3-28) we write $U_{\mathrm{cN}}(\kappa)$ in the form

$$U_{\mathrm{cN}}(\kappa) = N^{-4} U_{\mathrm{c}\kappa}(\kappa) = \frac{2(2\pi\kappa)^2}{N^4} [U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \tag{42}$$

without actually working out $U_{\mathrm{cs}}^2(\kappa)$ and $U_{\mathrm{cc}}^2(\kappa)$. The non-fluctuating part U_{Nc} of U in Eq.(41) becomes:

$$\begin{aligned} U_{\mathrm{Nc}} &= \frac{1}{2} Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N^2} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} U_{\mathrm{cN}}(\kappa) \\ &= Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N^2} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \left(\frac{2\pi\kappa}{N} \right)^2 [U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \end{aligned} \tag{43}$$

The normalized energy $U_{\mathrm{c}} = N^4 U_{\mathrm{Nc}}$ is the Hamilton function $\mathcal{H}$:

$$\begin{aligned} \mathcal{H} &= U_{\mathrm{c}} / Z V_{\mathrm{e0}}^2 L^2 T^3 c^4, \quad N = N_\tau \\ &= \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \mathcal{H}_\kappa = \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} (2\pi\kappa)^2 [U_{cs}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \end{aligned} \tag{44}$$

$$\mathcal{H}_\kappa = (2\pi\kappa)^2 [U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] = N^4 U_{\mathrm{cN}}(\kappa)/2 = U_{\mathrm{c}\kappa}(\kappa)/2 \tag{45}$$

Equations (44) and (45) are equal to Eq.(2.3-39). Our next task is to rewrite the Hamiltonian formalism of Eqs.(2.3-42) to (2.3-49) with finite differences rather than differentials. We start with Eq.(2.3-42), which remains unchanged. The following derivation is done in some detail due to its basic significance:

$$\begin{aligned} \mathcal{H}_\kappa &= (2\pi\kappa)^2 \left\{ \left[U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa) \right] \sin^2 2\pi\kappa\theta + \left[U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa \right] \cos^2 2\pi\kappa\theta \right\} \\ &= (2\pi\kappa)^2 \left[U_{\mathrm{cs}}(\kappa) + iU_{\mathrm{cc}}(\kappa) \right] (\sin 2\pi\kappa\theta - i \cos 2\pi\kappa\theta) \\ &\qquad \times \left[U_{\mathrm{cs}}(\kappa) - iU_{\mathrm{cc}}(\kappa) \right] (\sin 2\pi\kappa\theta + i \cos 2\pi\kappa\theta) \\ &= -2\pi i \kappa p_\kappa(\theta) q_\kappa(\theta) \end{aligned} \tag{46}$$

Using difference rather than differential operators we obtain for $p_\kappa(\theta)$ and $q_\kappa(\theta)$:

$$\begin{aligned}
p_\kappa(\theta) &= (2\pi i\kappa)^{1/2}\,[U_{\rm cs}(\kappa)+iU_{\rm cc}(\kappa)]\sin(2\pi\kappa\theta - i\cos 2\pi\kappa\theta)\\
&= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)-iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta} \qquad (47)\\
\dot p_\kappa &= \frac{\tilde\Delta p_\kappa(\theta)}{\tilde\Delta\theta} = \frac{p_\kappa(\theta+\Delta\theta)-p_\kappa(\theta-\Delta\theta)}{2\Delta\theta}\\
&= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)-iU_{\rm cs}(\kappa)]\frac{1}{2\Delta\theta}\left(e^{2\pi i\kappa(\theta+\Delta\theta)} - e^{2\pi i\kappa(\theta-\Delta\theta)}\right)\\
&= \frac{i\sin 2\pi\kappa\Delta\theta}{\Delta\theta}(2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)-iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta}\\
&\doteq (2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa)-iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta},\quad \Delta\theta\ll 1 \qquad (48)
\end{aligned}$$

$$\begin{aligned}
q_\kappa(\theta) &= (2\pi i\kappa)^{1/2}[U_{\rm cs}(\kappa)-iU_{\rm cc}(\kappa)](\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta)\\
&= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)+iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta} \qquad (49)\\
\dot q_\kappa &= \frac{\tilde\Delta q_\kappa(\theta)}{\tilde\Delta\theta} = \frac{q_\kappa(\theta+\Delta\theta)-q_\kappa(\theta-\Delta\theta)}{2\Delta\theta}\\
&= (2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)+iU_{\rm cs}(\kappa)]\frac{1}{2\Delta\theta}\left(e^{-2\pi i\kappa(\theta+\Delta\theta)} - e^{-2\pi i\kappa(\theta-\Delta\theta)}\right)\\
&= -\frac{i\sin 2\pi\kappa\Delta\theta}{\Delta\theta}(2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)+iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta}\\
&\doteq -(2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa)+iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta},\quad \Delta\theta\ll 1 \qquad (50)
\end{aligned}$$

The *finite derivatives* $\tilde\Delta\mathcal{H}_\kappa/\tilde\Delta q_\kappa$ and $\tilde\Delta\mathcal{H}_\kappa/\tilde\Delta p_\kappa$ equal:

$$\begin{aligned}
\frac{\tilde\Delta\mathcal{H}}{\tilde\Delta q_\kappa} &= \frac{\tilde\Delta[-2\pi i\kappa p_\kappa(\theta)q_\kappa(\theta)]}{\tilde\Delta q_\kappa(\theta)} = -2\pi i\kappa p_\kappa(\theta)\frac{\tilde\Delta q_\kappa(\theta)}{\tilde\Delta q_\kappa(\theta)}\\
&= -2\pi i p_\kappa(\theta)\frac{q_\kappa(\theta)+\Delta q_\kappa(\theta)-[q_\kappa(\theta)-\Delta q_\kappa(\theta)]}{2\Delta q_\kappa(\theta)}\\
&= -2\pi i\kappa p_\kappa(\theta) = -(2\pi i\kappa)^{1/2}[U_{\rm cc}(\kappa)-iU_{\rm cs}(\kappa)]e^{2\pi i\kappa\theta} \qquad (51)
\end{aligned}$$

$$\begin{aligned}
\frac{\tilde\Delta\mathcal{H}_\kappa}{\tilde\Delta p_\kappa} &= \frac{\tilde\Delta[-2\pi i\kappa p_\kappa(\theta)q_\kappa(\theta)]}{\tilde\Delta p_\kappa(\theta)} = -2\pi i\kappa q_\kappa(\theta)\frac{\tilde\Delta p_\kappa(\theta)}{\tilde\Delta p_\kappa(\theta)}\\
&= -2\pi i\kappa q_\kappa(\theta) = -(2\pi i\kappa)^{3/2}[U_{\rm cc}(\kappa)+iU_{\rm cs}(\kappa)]e^{-2\pi i\kappa\theta} \qquad (52)
\end{aligned}$$

The comparison of Eqs.(51) and (52) with Eqs.(48) and (50) yields the following relations for the components $\mathcal{H}_\kappa$ of the Hamilton function in terms of the calculus of finite differences:

$$\frac{\tilde{\Delta}\mathcal{H}_\kappa}{\tilde{\Delta}q_\kappa} = -\frac{\tilde{\Delta}p_\kappa}{\tilde{\Delta}\theta} = -\dot{p}_\kappa \tag{53}$$

$$\frac{\tilde{\Delta}\mathcal{H}_\kappa}{\tilde{\Delta}p_\kappa} = +\frac{\tilde{\Delta}q_\kappa}{\tilde{\Delta}\theta} = +\dot{q}_\kappa \tag{54}$$

The calculus of finite differences yields the same result for $\Delta\theta \ll 1$ as the differential calculus.

Equation (46) may be rewritten in analogy to Eqs.(2.3-50) and (2.3-51). Since these equations contain no differentials we obtain no change for Eq. (2.3-50) for the calculus of finite differences:

$$\begin{aligned} a_\kappa &= [U_{\mathrm{cc}}(\kappa) - iU_{\mathrm{cs}}(\kappa)]e^{2\pi i\kappa\theta} \\ a_\kappa^* &= [U_{\mathrm{cc}}(\kappa) + iU_{\mathrm{cs}}(\kappa)]e^{-2\pi i\kappa\theta} \end{aligned} \tag{55}$$

Equation (2.3-51) is only modified by the change of N_τ to N and the different summation limits:

$$\begin{aligned} \mathcal{H} &= \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \mathcal{H}_\kappa = -i \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} 2\pi\kappa p_\kappa q_\kappa = \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} (2\pi\kappa)^2 a_\kappa a_\kappa^* \\ &= \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \frac{2\pi\kappa}{T}\hbar b_\kappa b_\kappa^* \end{aligned}$$

$$b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa^* \tag{56}$$

3.6 Quantization of the Difference Solution

Equation (3.5-56) permits us to follow the standard way of quantization of Eq.(2.3-51) in Section 2.4 following the Schrödinger approach that led to Eq.(2.4-9). We start by writing difference operators for the differential operators of Eq.(2.4-1). An italic Delta with tilde $\tilde{\Delta}$ is used for operators and a roman Delta Δ for differences:

$$b_\kappa^* \to b_\kappa^+ = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{\tilde{\Delta}}{\tilde{\Delta}\zeta}\right), \quad b_\kappa \to b_\kappa^- = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{\tilde{\Delta}}{\tilde{\Delta}\zeta}\right) \tag{1}$$

The operator $\tilde{\Delta}/\tilde{\Delta}\zeta$ applied to a function $V_{\mathrm{e}}(\zeta,\theta)$ follows from Eq.(3.1-2)

$$\frac{\tilde{\Delta} V_{\mathrm{e}}(\zeta,\theta)}{\tilde{\Delta}\zeta} = \frac{V_{\mathrm{e}}(\zeta+\Delta\zeta,\theta) - V_{\mathrm{e}}(\zeta-\Delta\zeta,\theta)}{2\Delta\zeta} \tag{2}$$

and can be simplified like Eq.(3.1-3) by choosing $\Delta\zeta = 1$:

$$\frac{\tilde{\Delta} V_{\mathrm{e}}(\zeta,\theta)}{\tilde{\Delta}\zeta} = \frac{1}{2}[V_{\mathrm{e}}(\zeta+1,\theta) - V_{\mathrm{e}}(\zeta-1,\theta)], \quad \Delta\zeta = 1 \tag{3}$$

The second order difference quotient of $V_{\mathrm{e}}(\zeta,\theta)$ is defined by Eq.(3.1-4). The product $b_\kappa^- b_\kappa^+$ according to Eq.(1) becomes

$$b_\kappa^- b_\kappa^+ = \frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{\tilde{\Delta}^2}{\tilde{\Delta}\zeta^2}\right) \tag{4}$$

and yields an equation similar to Eq.(2.4-9):

$$\frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{\tilde{\Delta}^2}{\tilde{\Delta}\zeta^2}\right)\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi = \lambda_\kappa\Phi$$
$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{5}$$

The substitution

$$\xi = \alpha\zeta, \quad \zeta = \frac{1}{\alpha}\xi, \quad \frac{\tilde{\Delta}^2}{\tilde{\Delta}\zeta^2} = \alpha^2\frac{\tilde{\Delta}^2}{\tilde{\Delta}\xi^2} \tag{6}$$

produces a difference equation similar to the differential equation (2.4-11):

$$\frac{\tilde{\Delta}^2\Phi}{\tilde{\Delta}\xi^2} + (2\lambda_\kappa - \xi^2)\Phi = 0 \tag{7}$$

We can again use the substitution of Eq.(2.4-12):

$$\Phi = e^{-\xi^2/2}\chi(\xi) \tag{8}$$

The derivation of $\tilde{\Delta}^2\Phi/\tilde{\Delta}\xi^2$ for $\Delta\xi \ll 1$ is shown in detail:

$$
\begin{aligned}
\frac{\tilde{\Delta}^2\Phi}{\tilde{\Delta}\xi^2} &= \frac{1}{(\Delta\xi)^2}\big\{\exp[-(\xi+\Delta\xi)^2/2]\chi(\xi+\Delta\xi) - 2\exp(-\xi^2/2)\chi(\xi) \\
&\qquad + \exp[-(\xi-\Delta\xi)^2/2]\chi(\xi-\Delta\xi)\big\} \\
&= e^{-\xi^2/2}\frac{1}{(\Delta\xi)^2}\Big(\exp\big\{-[2\xi\Delta\xi+(\Delta\xi)^2]/2\big\}\chi(\xi+\Delta\xi) - 2\chi(\xi) \\
&\qquad + \exp\big\{-[-2\xi\Delta\xi+(\Delta\xi)^2]/2\big\}\chi(\xi-\Delta\xi)\Big) \\
&= e^{-\xi^2/2}\frac{1}{(\Delta\xi)^2}\bigg[\Big(1-\xi\Delta\xi-\frac{1}{2}(\Delta\xi)^2+\frac{1}{2}\xi^2(\Delta\xi)^2\Big)\chi(\xi+\Delta\xi) \\
&\qquad - 2\chi(\xi) + \Big(1+\xi\Delta\xi-\frac{1}{2}(\Delta\xi)^2+\frac{1}{2}\xi^2(\Delta\xi)^2\Big)\chi(\xi-\Delta\xi)\bigg] \\
&= e^{-\xi^2/2}\bigg(\frac{\chi(\xi+\Delta\xi)-2\chi(\xi)+\chi(\xi-\Delta\xi)}{(\Delta\xi)^2} - \xi\frac{\xi(\xi+\Delta\xi)-\chi(\xi-\Delta\xi)}{\Delta\xi} \\
&\qquad - \frac{1}{2}(1-\xi^2)[\chi(\xi+\Delta\xi)+\chi(\xi-\Delta\xi)]\bigg) \\
&= e^{-\xi^2/2}\bigg(\frac{\tilde{\Delta}^2\chi(\xi)}{\tilde{\Delta}\xi^2} - 2\xi\frac{\tilde{\Delta}\chi(\xi)}{\tilde{\Delta}\xi} + (\xi^2-1)\chi(\xi)\bigg) \qquad (9)
\end{aligned}
$$

Equation (7) becomes:

$$
\frac{\tilde{\Delta}^2\chi(\xi)}{\tilde{\Delta}\xi^2} - 2\xi\frac{\tilde{\Delta}\chi(\xi)}{\tilde{\Delta}\xi} + (2\lambda_\kappa - 1)\chi(\xi) = 0 \qquad (10)
$$

Written explicitly this equation assumes the form

$$
\begin{aligned}
\frac{\chi(\xi+\Delta\xi)-2\chi(\xi)+\chi(\xi-\Delta\xi)}{(\Delta\xi)^2} &- 2\xi\frac{\chi(\xi+\Delta\xi)-\chi(\xi-\Delta\xi)}{2\Delta\xi} \\
&+ (2\lambda_\kappa-1)\chi(\xi) = 0 \qquad (11)
\end{aligned}
$$

and the substitution

$$
x = \xi/\Delta\xi, \quad \Delta\xi = 1 \qquad (12)
$$

brings:

$$
\chi(x+1)-2\chi(x)+\chi(x-1)-x[\chi(x+1)-\chi(x-1)]+(2\lambda_\kappa-1)\chi(\xi)=0 \qquad (13)
$$

For the solution of a differential equation with variable coefficients, like Eq.(2.4-13), one starts with a power series (2.4-14) and develops a recursion formula (2.4-15) for the coefficients of the power series. A similar approach exists for difference equations but the power series is replaced by a factorial series (of the second kind). Assume that $\chi(x)$ can be represented by the following series:

$$\begin{aligned}\chi(x) &= b_0+b_1\frac{x-1}{1!}+b_2\frac{(x-1)(x-2)}{2!}+\dots+b_j\frac{(x-1)(x-2)\dots(x-j)}{j!}+\dots\\ &= b_0+b_1g_1(x)+b_2g_2(x)+\dots+b_jg_j(x)+\dots \qquad (14)\end{aligned}$$

Each term $g_j(x)$ must satisfy Eq.(13) independently. In order to obtain Eq.(13) for the particular solution $\chi(x)=g_j(x)$ we write $g_j(x)$, $g_j(x+1)$ and $g_j(x-1)$:

$$\begin{aligned}g_j(x) &= \frac{1}{j!}(x-1)(x-2)\dots(x-j)\\ &= \frac{1}{j!}(x-2)(x-3)\dots(x-j)[x-(j+1)+j]\\ &= \frac{1}{j!}\{(x-2)\dots[x-(j+1)]+j(x-2)\dots(x-j)\} \qquad (15)\end{aligned}$$

$$\begin{aligned}g_j(x+1) &= \frac{1}{j!}x(x-1)\dots[x-(j-1)] = \frac{1}{j!}(x-1)\dots[x-(j-1)](x-j+j)\\ &= \frac{1}{j!}\{(x-1)\dots(x-j)+j(x-1)\dots[x-(j-1)]\}\\ &= \frac{1}{j!}\{(x-2)\dots(x-j)[x-(j+1)+j]\\ &\qquad +j(x-2)\dots[x-(j-1)][x-j+(j-1)]\}\\ &= \frac{1}{j!}\{(x-2)\dots[x-(j+1)]+2j(x-2)\dots(x-j)\\ &\qquad +j(j-1)(x-2)\dots[x-(j-1)]\} \qquad (16)\end{aligned}$$

$$g_j(x-1) = \frac{1}{j!}(x-2)(x-3)\dots[x-(j+1)] \qquad (17)$$

Next we write $xg_j(x+1)$ and $xg_j(x-1)$ for the fourth and fifth term in Eq.(13):

$$\begin{aligned}xg_j(x+1) &= \frac{1}{j!}\{x(x-2)\dots[x-(j+1)]+2jx(x-2)\dots(x-j)\\ &\qquad +j(j-1)x(x-2)\dots[x-(j-1)]\}\\ &= \frac{1}{j!}\{(x-2)\dots[x-(j+1)][x-(j+2)+(j+2)]\\ &\qquad +2j(x-2)\dots(x-j)[(x-(j+1)+(j+1)]\\ &\qquad +j(j+1)(x-2)\dots[x-(j-1)](x-j+j)\}\end{aligned}$$

$$= \frac{1}{j!}\{(x-2)\dots[x-(j+2)] + (3j+2)(x-2)\dots[x-(j+1)] + 3j(j+1)(x-2)\dots(x-j) + j^2(j+1)(x-2)\dots[x-(j-1)]\} \quad (18)$$

$$\begin{aligned} xg_j(x-1) &= \frac{1}{j!}x(x-2)\dots[x-(j+1)] \\ &= \frac{1}{j!}(x-2)\dots[x-(j+1)][x-(j+2)+(j+2)] \\ &= \frac{1}{j!}\{(x-2)\dots[x-(j+2)]+(j+2)(x-2)\dots[x-(j+1)]\} \end{aligned} \quad (19)$$

From Eqs.(15)–(17) we get the second order difference quotient in Eq.(13) for $g_j(x)$ rather than $\chi(x)$:

$$g_j(x+1) - 2g_j(x) + g_j(x-1) = \frac{1}{(j-2)!}(x-2)\dots[x-(j-1)] \quad (20)$$

The first order difference quotient multiplied by x in Eq.(13) becomes with the help of Eqs.(18) and (19) for $g_j(x)$:

$$x[g_j(x+1) - g_j(x-1)] = \frac{1}{(j-1)!}\{2(x-2)\dots[x-(j+1)] + 3(j+1)(x-2)\dots(x-j) + j(j+1)(x-2)\dots[x-(j-1)]\} \quad (21)$$

The whole difference equation (13) written for $b_j g_j(x)$ becomes:

$$\begin{aligned} b_j\{g_j(x+1) &- 2g_j(x) + g_j(x-1) - x[g_j(x+1) - g_j(x-1)] + (2\lambda_\kappa - 1)g_j(x)\} \\ &= \frac{b_j}{j!}\{-j(j^2+1)(x-2)\dots[x-(j-1)] \\ &\quad + j[2\lambda_\kappa - 1 - 3(j+1)](x-2)\dots(x-j) \\ &\quad + (2\lambda_\kappa - 1 - 2j)(x-2)\dots[x-(j+1)]\} \end{aligned} \quad (22)$$

We still need to write this equation for $b_{j+1}g_{j+1}(x)$ and $b_{j+2}g_{j+2}(x)$. This requires replacement of j by $j+1$ and $j+2$ in Eq.(22):

$$\begin{aligned} b_{j+1}\{g_{j+1}(x+1) &- 2g_{j+1}(x) + g_{j+1}(x-1) - x[g_{j+1}(x+1) - g_{j+1}(x-1)] \\ &\quad + (2\lambda_\kappa - 1)g_{j+1}(x)\} \\ = \frac{b_{j+1}}{(j+1)!}\{&-(j+1)[(j+1)^2+1](x-2)\dots(x-j) \\ &+ (j+1)[2\lambda_\kappa - 1 - 3(j+2)](x-2)\dots[x-(j+1)] \\ &+ [2\lambda_\kappa - 1 - 2(j+1)](x-2)\dots[x-(j+2)]\} \end{aligned} \quad (23)$$

$$\begin{aligned}
b_{j+2}\{g_{j+2}(x+1) - 2g_{j+2}(x) + g_{j+2}(x-1) - x[g_{j+2}(x+1) - g_{j+2}(x-1)] \\
+ (2\lambda_\kappa - 1)g_{j+2}(x)\} \\
= \frac{b_{j+2}}{(j+2)!}\{ - (j+2)[(j+2)^2+1](x-2)\dots[x-(j+1)] \\
+ (j+2)[2\lambda_\kappa - 1 - 3(j+3)[(x-2)\dots[x-(j+2)] \\
+ [2\lambda_\kappa - 1 - 2(j+2)](x-2)\dots[x-(j+3)]\}
\end{aligned} \tag{24}$$

Consider line 4 in Eq.(22), line 4 in Eq.(23) and line 3 in Eq.(24). They all show terms multiplied by $(x-2)\dots[x-(j+1)]$. If the sum of these three lines vanishes, each term $g_j(x-1) = \hat{g}_j(x)$ in Eq.(14) will vanish and the factorial series of Eq.(14) will be a solution of the difference equation (13):

$$\begin{aligned}
\frac{1}{j!}(2\lambda_\kappa - 1 - 2j)b_j + \frac{j+1}{(j+1)!}[2\lambda_\kappa - 1 - 3(j+2)]b_{j+1} \\
-\frac{j+2}{(j+2)!}[(j+2)^2+1]b_{j+2} = 0
\end{aligned}$$

$$b_{j+2} = \frac{j+1}{(j+2)^2+1}\{(2\lambda_\kappa - 1 - 2j)b_j + [2\lambda_\kappa - 1 - 3(j+2)]b_{j+1}\} \tag{25}$$

Let us see whether there are polynomial solutions corresponding to the Hermite polynomials that satisfy Eq.(2.4-13). Let b_n for $2j = 2n = 2\lambda_\kappa - 1$ be the last coefficient of the polynomial that is unequal to zero:

$$b_n = \frac{n-1}{n^2+1}\{[2\lambda_\kappa - 1 - 2(n-2)]b_{n-2} + (2\lambda_\kappa - 1 + 3n)b_{n-1}\} \tag{26}$$

If we succeed in making b_{n+1} and b_{n+2} equal to zero the recursion formula of Eq.(25) will make b_{n+3}, b_{n+4}, ... zero too:

$$b_{n+1} = \frac{n}{(n+1)^2+1}\{[2\lambda_\kappa - 1 - 2(n-1)]b_{n-1} + [2\lambda_\kappa - 1 - 3(n+1)]b_n\} = 0 \tag{27}$$

$$b_{n+2} = \frac{n+1}{(n+2)^2+1}\{(2\lambda_\kappa - 1 - 2n)b_n + [2\lambda_\kappa - 1 - 3(n+2)]b_{n+1}\} = 0 \tag{28}$$

Equation (28) is satisfied for

$$2n = 2\lambda_\kappa - 1, \quad n = 0,\ 1,\ 2, \dots \tag{29}$$

since b_{n+1} is zero due to Eq.(27). To see whether Eq.(27) can be satisfied we observe that the recursion formula of Eq.(25) is a difference equation of second order. If we solve such an equation by the series of Eq.(14) we can choose two coefficients. First we choose b_0 without specifying its value to leave b_0 available as a normalization constant. The second choice is $b_{n+1} = 0$, which is Eq.(27). Let us show by example how this works. For a reason soon to be evident we write $b_{n,j}$ rather than b_j. First we choose $n = 0$ and obtain:

$$n = 0,\ b_{n,0} = b_{0,0},\ b_{n,1} = b_{0,1} = 0$$
$$b_{n,2} = b_{0,2} = \frac{1}{5}\{(2\lambda_\kappa - 1 - 0)b_{0,0} + (2\lambda_\kappa - 1 - 6)b_{0,1}\} = 0 \quad \text{for } 2\lambda_\kappa - 1 = 0 \tag{30}$$

This first example is not quite representative. We add a second and third example:

$$n = 1,\ b_{1,0},\ b_{1,2} = 0$$
$$b_{1,3} = \frac{2}{10}\{(2\lambda_k - 1 - 2)b_{1,1} + (2\lambda_\kappa - 1 - 9)b_{1,2}\} = 0 \quad \text{for } 2\lambda_\kappa - 1 = 2 \tag{31}$$

The missing constant $b_{1,1}$ follows from Eq.(25) for $j = 0$ and $n = 1$:

$$b_{1,2} = 0 = \frac{1}{10}\{(2 - 0)b_{1,0} + (2 - 6)b_{1,1}\}$$
$$b_{1,1} = \frac{1}{2}b_{1,0} \quad \text{for } 2\lambda_\kappa - 1 = 2 \tag{32}$$

The third example shows how the coefficients $b_{n,1}, b_{n,2}, \ldots, b_{n,n}$ are obtained in the general case:

$$n = 2,\ b_{2,0},\ b_{2,3} = 0$$
$$b_{2,4} = \frac{3}{17}\{(2\lambda_\kappa - 1 - 4)b_{2,2} + (2\lambda_\kappa - 1 - 12)b_{2,3}\} = 0 \quad \text{for } 2\lambda_\kappa - 1 = 4 \tag{33}$$

The two missing constants $b_{2,1}$ and $b_{2,2}$ follow from Eq.(25) for $j = 0$, $j = 1$ and $2\lambda_\kappa - 1 = 4$:

$$b_{2,2} = \frac{1}{5}\{(4-0)b_{2,0} + (4-6)b_{2,1}\},\quad b_{2,3} = 0 = \frac{2}{10}\{(4-2)b_{2,1} + (4-9)b_{2,2}\}$$
$$b_{2,1} = b_{2,0},\ b_{2,2} = \frac{2}{5}b_{2,0} \tag{34}$$

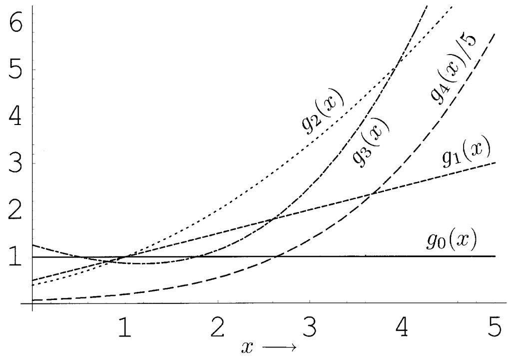

FIGURE 3.6-1. Plots of the functions $g_n(x)$ of Eq.(37). The constants $b_{n,0}$ are chosen equal to 1.

For $n = 3$ and 4 we only state the result:

$$\begin{aligned}
&n = 3,\ b_{3,0},\ b_{3,4} = 0 \\
&b_{3,1} = \frac{7}{5}b_{3,0},\ b_{3,2} = \frac{6}{5}b_{3,0},\ b_{3,3} = \frac{2}{5}b_{3,0}
\end{aligned} \tag{35}$$

$$\begin{aligned}
&n = 4;\ b_{4,0},\ b_{4,5} = 0, \\
&b_{4,1} = \frac{269}{154}b_{4,0},\ b_{4,2} = \frac{177}{77}b_{4,0},\ b_{4,3} = \frac{126}{77}b_{4,0},\ b_{4,4} = \frac{36}{77}b_{4,0}
\end{aligned} \tag{36}$$

The resulting polynomials are written with the help of Eq.(14):

$$\begin{aligned}
&n=0,\ g_0(x)/b_{0,0}=1 \\
&n=1,\ g_1(x)/b_{1,0}=1+\frac{1}{2}\frac{x-1}{1!} \\
&n=2,\ g_2(x)/b_{2,0}=1+\frac{x-1}{1!}+\frac{2}{5}\frac{(x-1)(x-2)}{2!} \\
&n=3,\ g_3(x)/b_{3,0}=1+\frac{7}{5}\frac{x-1}{1!}+\frac{6}{5}\frac{(x-1)(x-2)}{2!}+\frac{2}{5}\frac{(x-1)(x-2)(x-3)}{3!} \\
&n=4,\ g_4(x)/b_{4,0}=1+\frac{269}{154}\frac{x-1}{1!}+\frac{177}{77}\frac{(x-1)(x-2)}{2!}+\frac{126}{77}\frac{(x-1)(x-2)(x-3)}{3!} \\
&\qquad +\frac{36}{77}\frac{(x-1)(x-2)(x-3)(x-4)}{4!}
\end{aligned} \tag{37}$$

Figure 3.6-1 shows plots of the functions $g_n(x)$ of Eq.(37). The system of functions $\{\exp(-x^2/2)g_n(x)\}$ is not orthogonal. But the polynomials $g_n(x)$ are linearly independent. The solution of 10 linear equations with 10 variables will orthogonalize the first five functions of the system $\{\exp(-x^2/2)g_n(x)\}$. There is also the question of uniqueness for the solutions of Eq.(37). We can only hope that mathematicians will address these problems. The calculus of finite differences is very poorly developed due to the lack of a use that would motivate mathematicians to elaborate it. The success of the calculus of finite differences in quantum electrodynamics would provide such a motivation[1].

The substitution of Eq.(29) into Eq.(5) yields energy eigenvalues equal to those of the differential theory in Eq.(2.4-18):

$$\mathsf{E}_{\kappa} = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots,\ N \tag{38}$$

The comparison of the results obtained in this section with those of Section 2.4 shows the following:

1. The eigenvalues $\lambda_{\kappa} = n + 1/2$ are the same in Eqs.(2.4-16) and (29).
2. The recursion formula (25) has three terms while the recursion formula of Eq.(2.4-15) for the Hermite polynomials has only two terms.
3. The polynomials of Eq.(37) are significantly different from those of Eq. (2.4-17).

What we have learned is that a difference equation obtained by rewriting a differential equation can yield some results that are equal and others that are completely different. This is in line with Hölder's theorem published in the late 19th century (Hölder 1887).

3.7 Computer Plots for the Difference Theory

The variable $U_{\mathrm{cN}}(\kappa)$ in Eq.(3.5-41) represents energy as function of the period number κ. Its components $U_j(\kappa)$ are listed for $j = 1$ to 32 and $j = 49$ to 56 in Eqs.(6.6-11)–(6.6-15). The terms with other values of j have been eliminated by the approximation of Eq.(6.6-9). Our goal is to produce plots of $U_{\mathrm{cN}}(\kappa)$ according to Eq.(6.6-17) in the form of $r(\kappa)$ of Eq.(6.6-18) which is normalized to yield the probability fo $U_{\mathrm{cN}}(\kappa)$. We explain in some detail how the computer program is written since it requires more than 100 equations.

The parameters N, ρ_1 and ρ_2 have to be specified. As a compromise between precision and computing time we choose $N = 100$ for the beginning. The choice $\rho_1 = 1/4$ yields according to Eqs.(2.5-1) or (3.1-1) for $\tau = \Delta t$:

$$\sigma\Delta t(1 - s/\sigma Z) = \rho_1/cZ = 1/(4 \times 3 \cdot 10^8 \times 377) = 2.2 \cdot 10^{-12}\ [\mathrm{As/Vm}] \tag{1}$$

[1] Only seven books seem to have been published on the calculus of finite differences during the whole 20th century: Nörlund (1924, 1929), Milne-Thomson (1951), Gelfond (1958), Levy and Lessmann (1961), Smith (1982), Spiegel (1994). Originally one had to be able to read heavy mathematics in English, French, German and Russian, but Gelfond's book was translated into French and German which eliminated the Russian language requirement.

TABLE 3.7-1

THE 108 CORE EQUATIONS REQUIRED FOR A COMPUTER PROGRAM THAT PRODUCES A PLOT OF $U_{cN}(\kappa)$ ACCORDING TO EQ.(6.6-17) FOR THE INTERVAL $K_0 \leq |\kappa| \leq K_{N/2}$.

K_0	Eq.(3.2-53)	$K_{N/2}$	Eq.(3.2-55)	$d_\Delta(\kappa)$	Eq.(6.4-4)	$\lambda_a(\kappa)$	Eq(3.2-60)
$D_0(\kappa)$	Eq.(3.3-9)	$D_3(\kappa)$	Eq.(3.3-15)	$\lambda_{05}(\kappa)$	Eq.(3.3-29)	$\gamma_{05}(\kappa)$	Eq.(3.3-30)
$\lambda_{07}(\kappa)$	Eq.(3.3-33)	$\gamma_{07}(\kappa)$	Eq.(3.3-34)	$\lambda_{15}(\kappa)$	Eq.(3.3-49)	$\gamma_{15}(\kappa)$	Eq.(3.3-50)
$\lambda_{17}(\kappa)$	Eq.(3.3-53)	$\gamma_{17}(\kappa)$	Eq.(3.3-54)	$R_{15}(\kappa)$	Eq.(6.4-15)	$\Lambda_{50}(\kappa)$	Eq.(6.4-16)
$\Lambda_{51}(\kappa)$	Eq.(6.4-16)	$\Lambda_{52}(\kappa)$	Eq.(6.4-16)	$\Lambda_{53}(\kappa)$	Eq.(6.4-16)	$\Lambda_{70}(\kappa)$	Eq.(6.4-21)
$\Lambda_{71}(\kappa)$	Eq.(6.4-21)	$\Lambda_{72}(\kappa)$	Eq.(6.4-21)	$\Lambda_{73}(\kappa)$	Eq.(6.4-21)	$G_{01}(\kappa)$	Eq.(6.4-48)
$G_{02}(\kappa)$	Eq.(6.4-48)	$G_{03}(\kappa)$	Eq.(6.4-48)	$G_{04}(\kappa)$	Eq.(6.4-48)	$J_{11}(\kappa,\nu)$	Eq.(6.4-66)
$J_{12}(\kappa,\nu)$	Eq.(6.4-67)	$J_{13}(\kappa,\nu)$	Eq.(6.4-68)	$J_{14}(\kappa,\nu)$	Eq.(6.4-69)	$J_{15}(\kappa,\nu)$	Eq.(6.4-70)
$J_{16}(\kappa,\nu)$	Eq.(6.4-71)	$J_{17}(\kappa,\nu)$	Eq.(6.4-72)	$J_{18}(\kappa,\nu)$	Eq.(6.4-73)	$J_{19}(\kappa,\nu)$	Eq.(6.4-74)
$J_{20}(\kappa,\nu)$	Eq.(6-4-75)	$J_{21}(\kappa,\nu)$	Eq.(6.4-76)	$J_{22}(\kappa,\nu)$	Eq.(6.4-77)	$L^a_{0c}(\kappa)$	Eq.(6.4-79)
$L^a_{sc}(\kappa,\nu)$	Eq.(6.4-79)	$L^a_{cc}(\kappa,\nu)$	Eq.(6.4-79)	$L^a_{0s}(\kappa)$	Eq.(6.4-80)	$L^a_{ss}(\kappa,\nu)$	Eq.(6.4-80)
$L^a_{cs}(\kappa,\nu)$	Eq.(6.4-80)	$L_{0c}(\kappa)$	Eq.(6.4-81)	$L_{sc}(\kappa,\nu)$	Eq.(6.4-81)	$L_{cc}(\kappa,\nu)$	Eq.(6.4-81)
$L_{0s}(\kappa)$	Eq.(6.4-82)	$L_{ss}(\kappa,\nu)$	Eq.(6.4-82)	$L_{cs}(\kappa,\nu)$	Eq.(6.4-82)	$L^a_{00}(\kappa)$	Eq.(6.5-21)
$L^a_{01}(\kappa,\nu)$	Eq.(6.5-21)	$L^a_{02}(\kappa,\nu)$	Eq.(6.5-21)	$L^a_{03}(\kappa)$	Eq.(6.5-22)	$L^a_{04}(\kappa,\nu)$	Eq.(6.5-22)
$L^a_{05}(\kappa,\nu)$	Eq.(6.5-22)	$L_{00}(\kappa)$	Eq.(6.5-19)	$L_{01}(\kappa,\nu)$	Eq.(6.5-19)	$L_{02}(\kappa,\nu)$	Eq.(6.5-19)
$L_{03}(\kappa)$	Eq.(6.5-20)	$L_{04}(\kappa,\nu)$	Eq.(6.5-20)	$L_{05}(\kappa,\nu)$	Eq.(6.5-20)	$U_1(\kappa)$	Eq.(6.6-11)
$U_2(\kappa)$	Eq.(6.6-11)	$U_3(\kappa)$	Eq.(6.6-11)	$U_4(\kappa)$	Eq.(6.6-11)	$U_5(\kappa)$	Eq.(6.6-11)
$U_6(\kappa)$	Eq.(6.6-11)	$U_7(\kappa)$	Eq.(6.6-11)	$U_8(\kappa)$	Eq.(6.6-11)	$U_9(\kappa)$	Eq.(6.6-11)
$U_{10}(\kappa)$	Eq.(6.6-11)	$U_{11}(\kappa)$	Eq.(6.6-11)	$U_{12}(\kappa)$	Eq.(6.6-11)	$U_{13}(\kappa)$	Eq.(6.6-12)
$U_{14}(\kappa)$	Eq.(6.6-12)	$U_{15}(\kappa)$	Eq.(6.6-12)	$U_{16}(\kappa)$	Eq.(6.6-12)	$U_{17}(\kappa)$	Eq.(6.6-12)
$U_{18}(\kappa)$	Eq.(6.6-12)	$U_{19}(\kappa)$	Eq.(6.6-13)	$U_{20}(\kappa)$	Eq.(6.6-13)	$U_{21}(\kappa)$	Eq.(6.6-13)
$U_{22}(\kappa)$	Eq.(6.6-13)	$U_{23}(\kappa)$	Eq.(6.6-13)	$U_{24}(\kappa)$	Eq.(6.6-13)	$U_{25}(\kappa)$	Eq.(6.6-14)
$U_{26}(\kappa)$	Eq.(6.6-14)	$U_{27}(\kappa)$	Eq.(6.6-14)	$U_{28}(\kappa)$	Eq.(6.6-14)	$U_{29}(\kappa)$	Eq.(6.6-14)
$U_{30}(\kappa)$	Eq.(6.6-14)	$U_{31}(\kappa)$	Eq.(6.6-14)	$U_{32}(\kappa)$	Eq.(6.6-14)	$U_{49}(\kappa)$	Eq.(6.6-15)
$U_{50}(\kappa)$	Eq.(6.6-15)	$U_{51}(\kappa)$	Eq.(6.6-15)	$U_{52}(\kappa)$	Eq.(6.6-15)	$U_{53}(\kappa)$	Eq.(6.6-15)
$U_{54}(\kappa)$	Eq.(6.6-15)	$U_{55}(\kappa)$	Eq.(6.6-15)	$U_{56}(\kappa)$	Eq.(6.6-15)	$U_{cN}(\kappa)$	Eq.(6.6-17)
$U_{cN1}(\kappa)$	Eq.(6.6-17)	$U_{cN2}(\kappa)$	Eq.(6.6-17)	$r_1(\kappa)$	Eq.(6.6-18)	$r_2(\kappa)$	Eq.(6.6-18)

The further choice $\rho_2 = 1/10 < \rho_1/2$ yields according to Eqs.(2.5-2) or (3.1-1):

$$\Delta t\sqrt{\sigma s} = \rho_2/c = 1/(10 \times 3 \cdot 10^8) = 3.3 \cdot 10^{-10} \text{ [s/m]} \tag{2}$$

For ρ_s we get from Eqs.(2.1-49) or (3.4-1):

$$\rho_{s1} = 20, \;\; \rho_{s2} = 5 \tag{3}$$

Whether the choices of Eqs.(1)–(3) are physically reasonable or not cannot be discussed until we have some values for the electric and magnetic dipole conductivities σ and s. At this time the choice of $\rho_1 = 1/4$ and $\rho_2 = 1/10$ is primarily motivated by the desire to use numbers that reduce computing times.

We have used consistently the two intervals

$$K_0 < |\kappa| < K_{\mathrm{N}/2},\ \kappa \neq 0 \text{ or } |\kappa| > K_{\mathrm{N}/2}, \quad \kappa \lesseqgtr 0$$

For $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$ we obtain from Eqs.(3.2-53) and (3.2-55) the limits $K_0 = 1.19366$ and $K_{\mathrm{N}/2} = 48.80633$. Since $|\kappa|$ is in the interval $1 \leq |\kappa| \leq N/2 - 1 = 49$ the interval $-K_0 \leq |\kappa| \leq K_{\mathrm{N}/2}$ covers the fraction 47/49 or 96% of the total interval $1 \leq |\kappa| \leq 49$. This is the important interval and we compute the functions $U_j(\kappa)$ for this interval.

Table 3.7-1 gives step by step instructions for the writing of a computer program. First we write K_0 according to Eq.(3.2-53), then $K_{\mathrm{N}/2}$ according to Eq.(3.2-55), and so on until $U_{56}(\kappa)$ is written according to Eq.(6.6-15). Then comes the equation for $U_{\mathrm{cN}}(\kappa)$ according to Eq.(6.6-17) and $r(\kappa)$ of Eq.(6.6-18). These are our 108 core equations.

Except for the constants K_0, $K_{\mathrm{N}/2}$, S_{cK1}, S_{cK2} and S_{cK} all entries in Table 3.7-1 are functions of κ or κ and ν. To check the program we must in principle make 103 plots from $d_\Delta(\kappa)$ to $U_{\mathrm{cN}}(\kappa)$. Actually, four less are required due to the relations $\lambda_{15}(\kappa) = -\lambda_{07}(\kappa)$, $\gamma_{15}(\kappa) = -\gamma_{07}(\kappa)$, $\lambda_{17}(\kappa) = -\lambda_{05}(\kappa)$, and $\gamma_{17}(\kappa) = \gamma_{05}(\kappa)$.

The equations with the one variable κ are plotted by the following instructions shown in the programming language Mathematica:

$$\kappa = \mathtt{k},\ K_0 = \mathtt{k0},\ K_{N/2} = \mathtt{kN}$$

$$\mathtt{f1}{:=}\mathtt{Which}[-\mathtt{kN}<\mathtt{k}<-\mathtt{k0},\ \#[\mathtt{k}], -\mathtt{k0}<=\mathtt{k}<=\mathtt{k0}, \mathtt{True}, \mathtt{k0}<\mathtt{k}<\mathtt{kN},\ \#[\mathtt{k}]]$$

$$\mathtt{p1}{:=}\mathtt{Plot}[\mathtt{f1}, \{\mathtt{k}, -\mathtt{kN}, \mathtt{kN}\}, \mathtt{PlotRange}{-}{>}\ \mathtt{All}] \tag{4}$$

The terms d_Δ, λ_{a}, ... , U_{56} in their computer representation must be substituted for #.

The equations with two variables κ, ν are displayed by a three-dimensional plot:

$$\nu = \mathtt{nu},\ N = \mathtt{n}$$

$$\mathtt{f1}{:=}\mathtt{Which}[-\mathtt{kN}<\mathtt{k}<-\mathtt{k0},\ \#[\mathtt{k},\mathtt{nu}], -\mathtt{k0}<\mathtt{k}<\mathtt{k0}, \mathtt{True}, \mathtt{k0}<\mathtt{k}<\mathtt{kN},\ \#[\mathtt{k},\mathtt{nu}]]$$

$$\mathtt{p1}{:=}\mathtt{Plot3D}[\mathtt{f1}, \{\mathtt{k}, -\mathtt{n}/2+1, \mathtt{n}/2-1\}, \{\mathtt{nu}, 1, \mathtt{n}/2-1\}, \mathtt{PlotRange}{-}{>}\mathtt{All}] \tag{5}$$

Two equations produce the sums $U_{\mathrm{cN1}}(\kappa)$ and $U_{\mathrm{cN2}}(\kappa)$ representing the two sums of Eq.(6.6-17):

$$U_j(\kappa) = \mathtt{uj}[\mathtt{k}],\ U_{\mathrm{cN1}}(\kappa) = \mathtt{ucN1}[\mathtt{k}],\ U_{\mathrm{cN2}}(\kappa) = \mathtt{ucN2}[\mathtt{k}]$$

$$\begin{aligned}
\mathtt{ucN1}[\mathtt{k_}] :=& \mathtt{u1}[\mathtt{k}] + \mathtt{u2}[\mathtt{k}] + \cdots + \mathtt{u32}[\mathtt{k}] + \mathtt{u49}[\mathtt{k}] + \cdots + \mathtt{u56}[\mathtt{k}] \\
& \mathtt{u1}[-\mathtt{k}] + \mathtt{u2}[-\mathtt{k}] + \cdots + \mathtt{u32}[-\mathtt{k}] + \mathtt{u49}[-\mathtt{k}] + \cdots + \mathtt{u56}[-\mathtt{k}] \\
\mathtt{ucN2}[\mathtt{k_}] :=& \mathtt{u1}[\mathtt{k}] + \mathtt{u2}[\mathtt{k}] + \cdots + \mathtt{u24}[\mathtt{k}] \\
& + \mathtt{u1}[-\mathtt{k}] + \mathtt{u2}[-\mathtt{k}] + \cdots + \mathtt{u24}[-\mathtt{k}]
\end{aligned} \tag{6}$$

A plot of $r(\kappa) = U_{\mathrm{cN}}(\kappa)/S_{\mathrm{cK}}$ according to Eq.(6.6-18) is produced by two more equations:

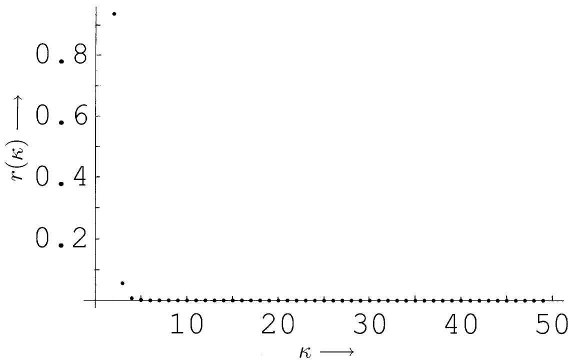

FIGURE 3.7-1. Point-plot of $r(\kappa)$ according to Eq.(6.6-18) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 48$; $S_{cK} = 59516.8$.

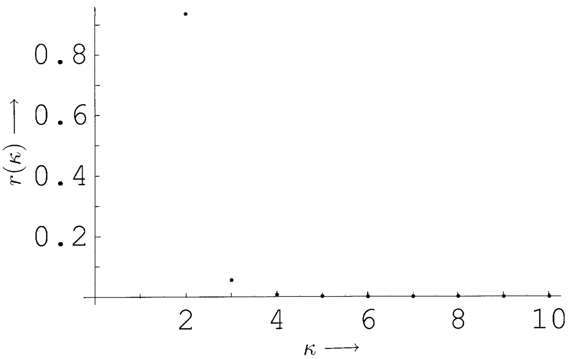

FIGURE 3.7-2. Point-plot of $r(\kappa)$ according to Eq.(6.6-18) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 10$; $S_{cK} = 59516.8$.

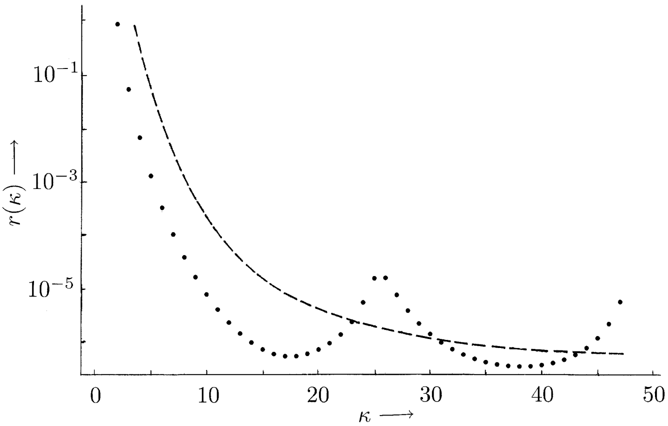

FIGURE 3.7-3. Semilogarithmic plot of $r(\kappa)$ according to Fig.3.7-1 and Eq.(6.6-18) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 50$; $S_{cK} = 59516.8$. The dashed line representsthe plot of Fig.2.5-3 for $\kappa = 2, 3, \ldots, 48$.

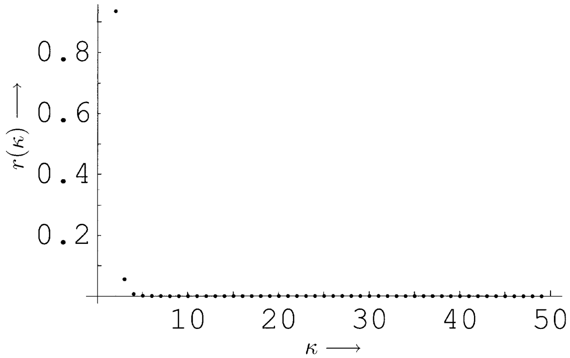

FIGURE 3.7-4. Point-plot of $r(\kappa)$ according to Eq.(6.6-18) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$, and $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 48$; $S_{cK} = 3732.64$.

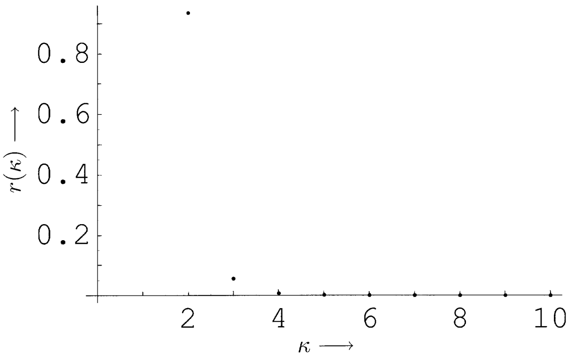

FIGURE 3.7-5. Point-plot of $r(\kappa)$ according to Eq.(6.6-18) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$, and $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 10$; $S_{cK} = 3732.64$.

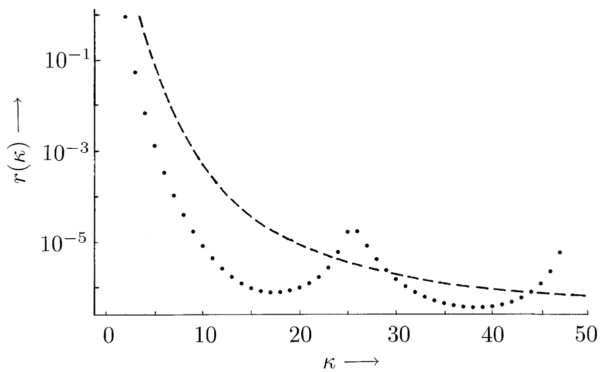

FIGURE 3.7-6. Semilogarithmic plot of $r(\kappa)$ according to Fig.3.7-4 and Eq.(6.6-18) for $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 5$, $N = 100$ and $K_0 = 1.19366$ for $\kappa = 2, 3, \ldots, 48$; $S_{cK} = 59516.8$. The dashed line represents the plot of Fig.2.5-6 for $\kappa = 2, 3, \ldots, 50$.

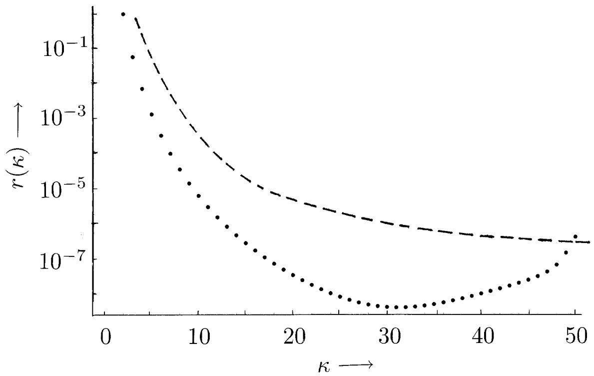

FIGURE 3.7-7. Semilogarithmic plot of $r(\kappa)$ according to Eq.(6.6-18) for $N = 200$, $\rho_1 = 1/8$, $\rho_2 = 1/20$, $\rho_s = 40$, $K_0 = 1.19366$ for $\kappa = 2, 3, \dots, 50$; $S_{cK} = 7.50154 \times 10^6$. The dashed line represents the plot of Fig.2.5-7 for $\kappa = 2, 3, \dots, 50$.

```
f01:=Which[0 <= k < k0,True,k0 <= k <= n/4,r1[k],n/4 < k <= kN,
                                        r2[k],kN < k <= n/2,True]
t01 :=Table[{k,f01[k]},{k,0,50}]
p12 :=ListPlot[t01,Prolog-> AbsolutePointSize[5],
                                        AxesOrigin-> {0,0}]      (7)
```

Figure 3.7-1 shows the plot for $r(\kappa)$ with $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$. There is a very fast drop close to $\kappa = 2$ and the function is essentially zero for $\kappa > 5$. This is quite similar to Fig.2.5-1.

The range $2 \leq \kappa \leq 10$ of Fig.3.7-1 is shown expanded in Fig.3.7-2 for the range $\kappa = 2, 3, \dots, 10$. The drop from $\kappa = 2$ to $\kappa = 3$ and 4 is much faster than in Fig.2.5-2. The semilogarithmic plot of Fig.3.7-3 gives a much better representation than Figs.3.7-1 and 3.7-2.

We had obtained the two values $\rho_{s1} = 20$ and $\rho_{s2} = 5$ in Eq.(3). Figures 3.7-1 to 3.7-3 hold for $\rho_s = 20$. In Figs.3.7-4 to 3.7-6 we show the corresponding plots for $\rho_s = 5$. The plots look identical to those in Figs.3.7-1 to 3.7-3.

The three semilogarithmic plots of Figs.3.7-3, 3.7-6 and 3.7-7 show that the dotted plots of the theory of finite differences drop significantly faster than the dashed plots of the differential theory for small values of κ. But the dotted plots show strange peaks at $\kappa = N/4$ and $\kappa = N/2$. We note that the largest value $r(2)$ in Fig.3.7-1 has the approximate value 0.92 while the peak at $\kappa = N/4 = 25$ in Fig.3.7-3 is somewhat larger than 10^{-5}, which is more than four orders of magnitude less than 0.92. Our computation is based on Eq.(6.6-9) which has 40 summands and is specifically called “shorter form” of Eq.(3.5-26) which has 64 summands. This simplification was done with the goal of keeping computation errors to less than the linewidth of a plot with *linear scale*. The plots of Figs.3.7-1 and 3.7-4 show that this goal was indeed achieved. The much more demanding semilogarithmic plots of Figs.3.7-3, 3.7-6 and 3.7-7 show the error introduced by using the approximation of Eq.(6.6-9) instead of Eq.(3.5-26). The 108 equations of Table 3.7-1 exceeded the capabilities of the program Mathematica 2.2 that was originally used and could only be processed by going to Mathematica 4.1. The use of Eq.(3.5-26) instead of Eq.(6.6-9) would increase the number of core equations in Table 3.7-1 to more than 175. We did not want to face this task.

Figures 3.7-8 to 3.7-12 on the following five pages display 40 plots of $U_1(\kappa)$ = `u1[k]` to $U_{56}(\kappa)$ = `u56[k]` of Table 3.7-1 to show the relative importance of these terms and also to help anyone who wants to write a program according to Table 3.7-1 with the tracking of errors in the program.

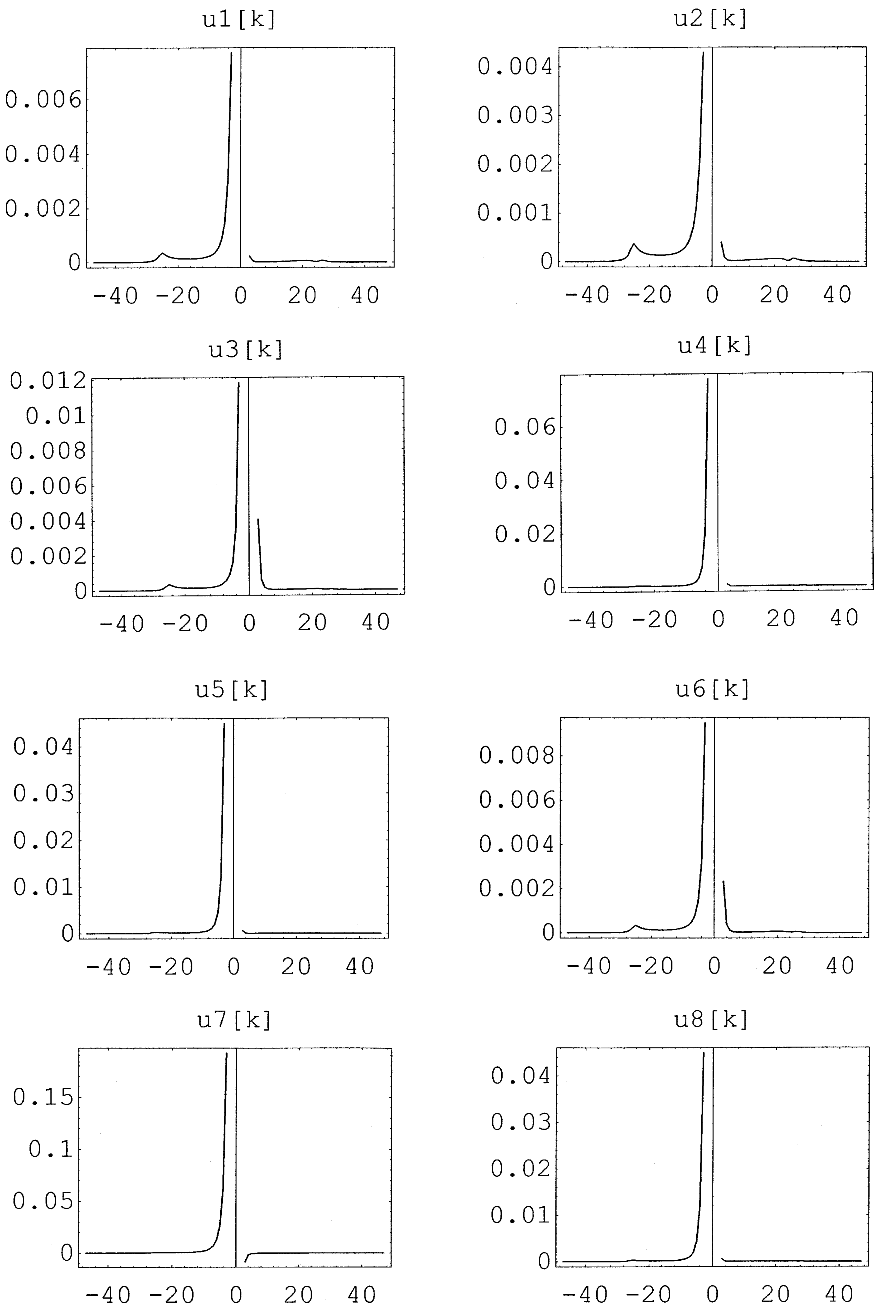

FIGURE 3.7-8. Plots of $U_1(\kappa) = \mathbf{u1[k]}$ to $U_8(\kappa) = \mathbf{u8[k]}$ according to Eq.(6.6-11) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ in the intervals $-48 \leq \kappa \leq -2$ and $2 \leq \kappa \leq 48$.

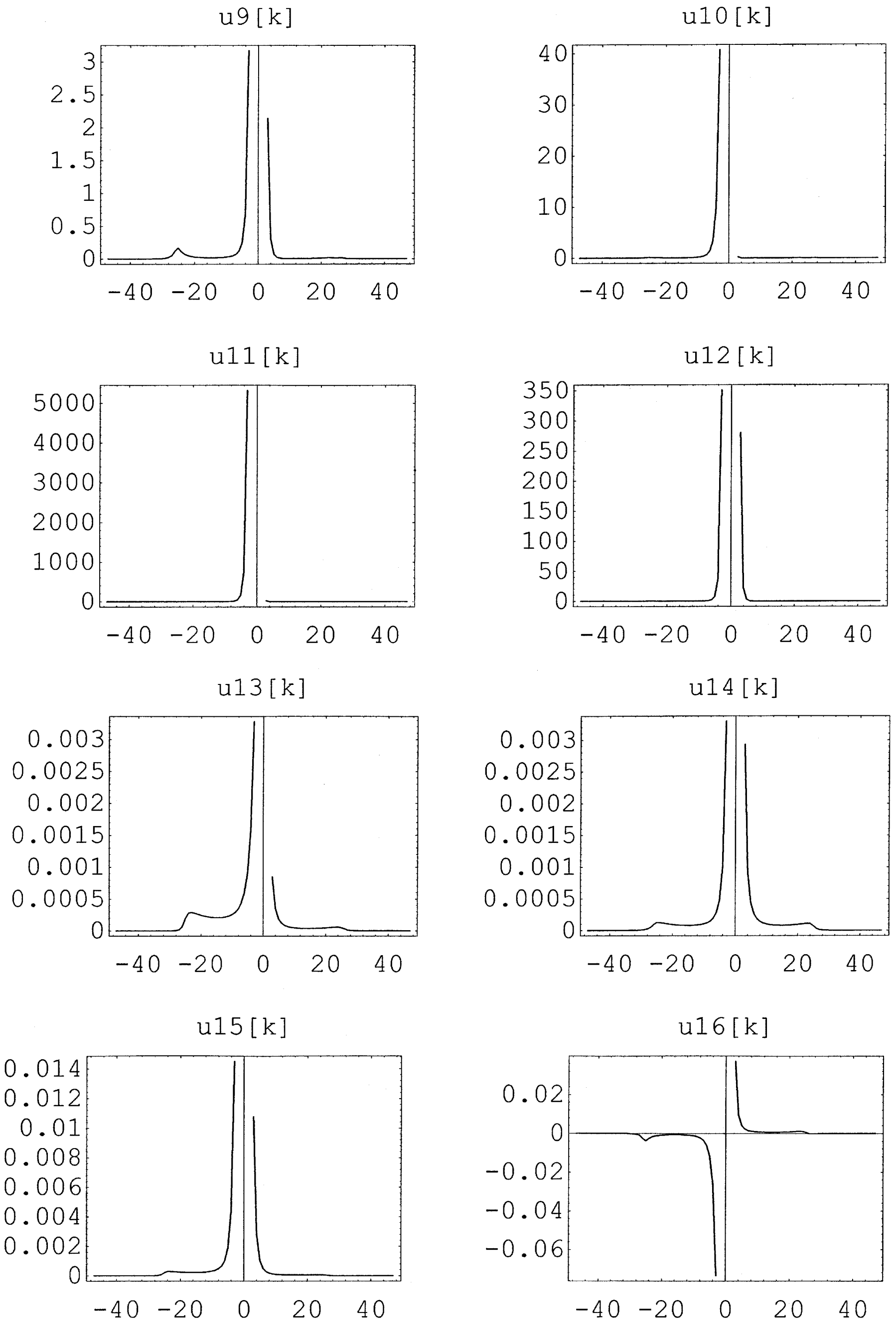

FIGURE 3.7-9. Plots of $U_9(\kappa)$ = **u9[k]** to $U_{16}(\kappa)$ = **u16[k]** according to Eqs.(6.6-11) and (6.6-12) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ in the intervals $-48 \leq \kappa \leq -2$ and $2 \leq \kappa \leq 48$.

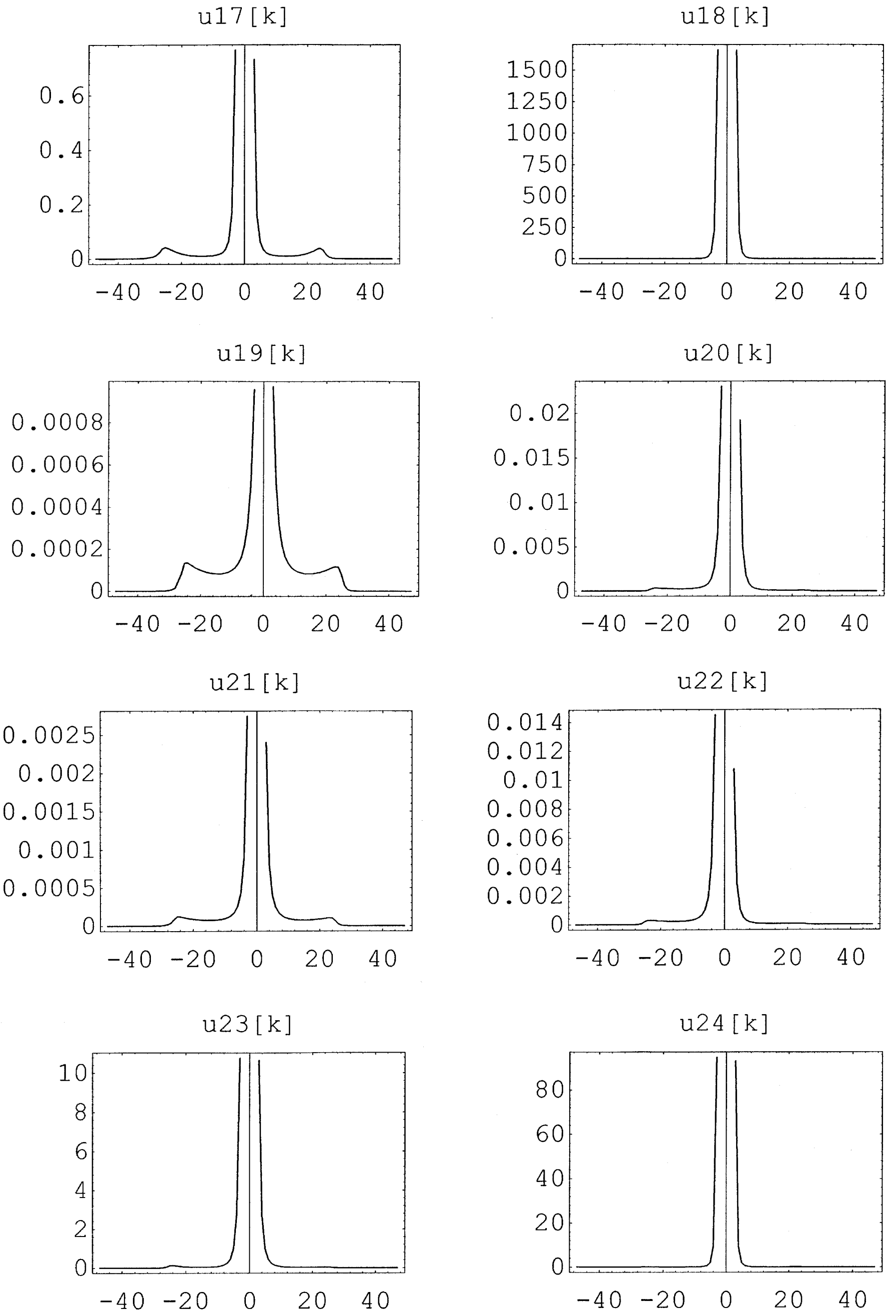

FIGURE 3.7-10. Plots of $U_{17}(\kappa) = \mathbf{u17[k]}$ to $U_{24}(\kappa) = \mathbf{u24[k]}$ according to Eqs.(6.6-12) and (6.6-13) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ in the intervals $-48 \leq \kappa \leq -2$ and $2 \leq \kappa \leq 48$.

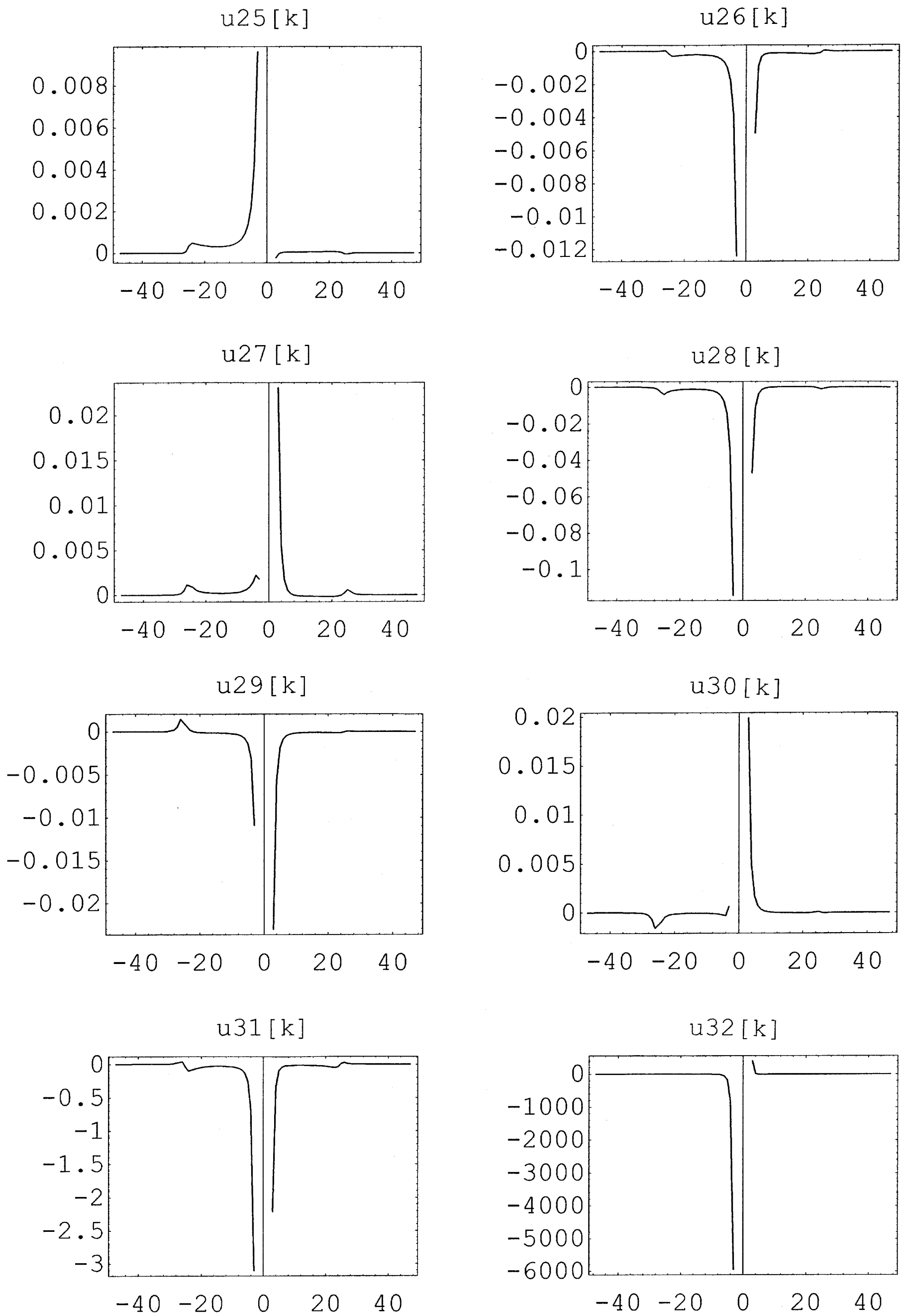

FIGURE 3.7-11. Plots of $U_{25}(\kappa)$ = **u25[k]** to $U_{32}(\kappa)$ = **u32[k]** according to Eq.(6.6-14) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ in the intervals $-48 \leq \kappa \leq -2$ and $2 \leq \kappa \leq 48$.

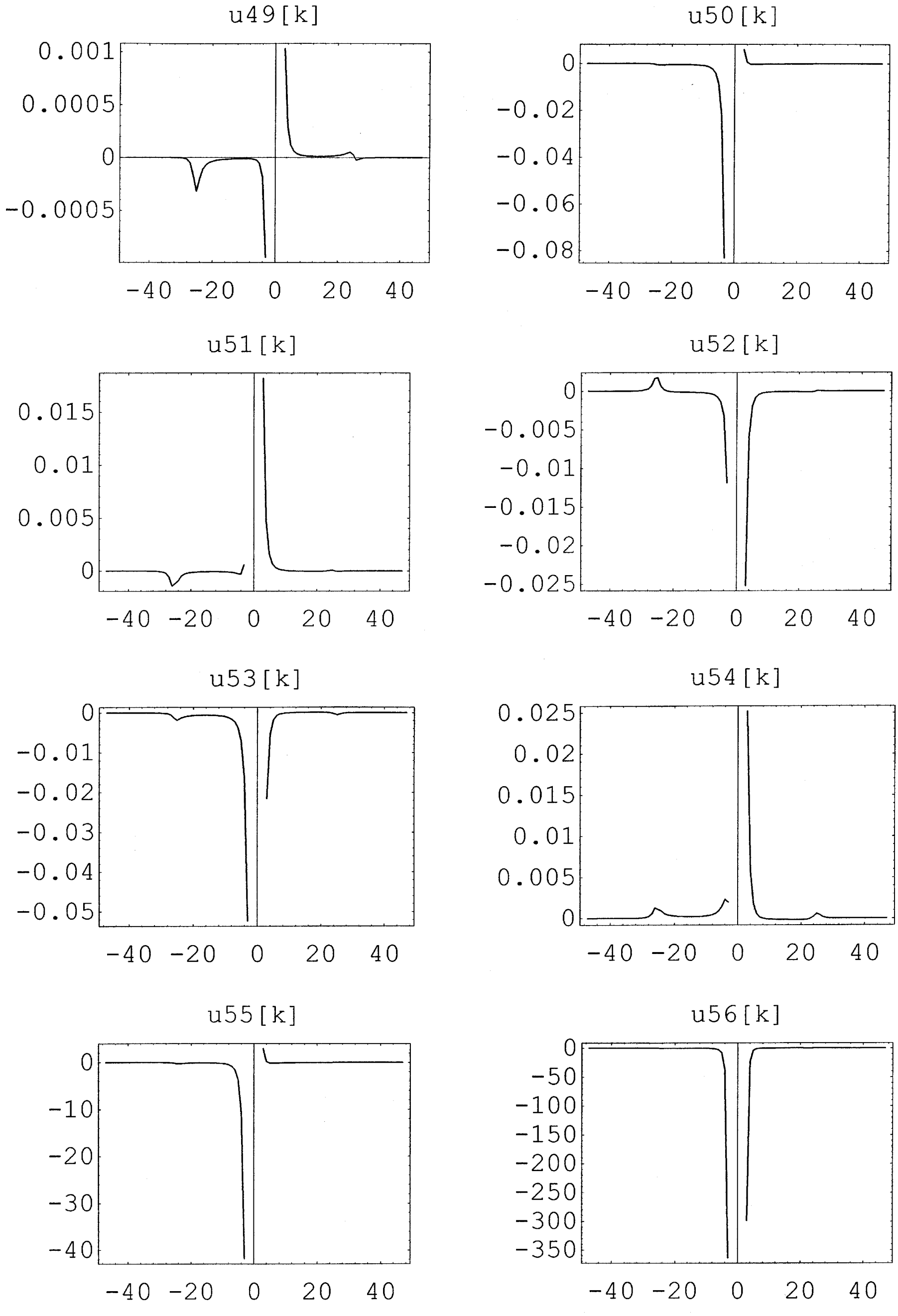

FIGURE 3.7-12. Plots of $U_{49}(\kappa) = \mathtt{u49[k]}$ to $U_{56}(\kappa) = \mathtt{u56[k]}$ according to Eq.(6.6-15) for $N = 100$, $\rho_1 = 1/4$, $\rho_2 = 1/10$, $\rho_s = 20$, and $K_0 = 1.19366$ in the intervals $-48 \leq \kappa \leq -2$ and $2 \leq \kappa \leq 48$.

4 Differential Equation for the Klein-Gordon Field

4.1 Klein-Gordon Equation With Magnetic Current Density

We begin with a summary of the derivation of the Klein-Gordon equation using Maxwell's equations with a term for magnetic (dipole) currents added. A detailed derivation may be found in Section 5.1 of a book by Harmuth, Barrett and Meffert (2001). From Eq.(18) on we shall deviate from the old text.

The usual Klein-Gordon equation without allowance for magnetic dipole currents can be derived from the following Hamilton function:

$$\mathcal{H} = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2} + e\phi_\mathrm{e} \tag{1}$$

This equation is rewritten into the following form:

$$(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_\mathrm{e})^2 = -m_0^2c^2$$

$$(p_x - eA_{\mathrm{m}x})^2 + (p_y - eA_{\mathrm{m}y})^2 + (p_z - eA_{\mathrm{m}z})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_\mathrm{e})^2 = -m_0^2c^2 \tag{2}$$

The following substitutions are made:

$$p_x \to \frac{\hbar}{i}\frac{\partial}{\partial x},\ p_y \to \frac{\hbar}{i}\frac{\partial}{\partial y},\ p_z \to \frac{\hbar}{i}\frac{\partial}{\partial z},\ \mathcal{H} \to -\frac{\hbar}{i}\frac{\partial}{\partial t} \tag{3}$$

Equation (2) assumes the form of the usual Klein-Gordon equation if these operators are substituted and applied to a function Ψ:

$$\left[\sum_{j=1}^{3}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 - \frac{1}{c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_\mathrm{e}\right)^2\right]\Psi = -m_0^2c^2\Psi \tag{4}$$

In order to allow for magnetic dipole currents we must generalize Eq.(1) by the introduction of a Hamilton function with three components $\mathcal{H}_x$, $\mathcal{H}_y$, $\mathcal{H}_z$. A first order approximation in α_e of this Hamilton function is given by Eqs.(1.3-34) to (1.3-36). We write the component $\mathcal{H}_x$ once more:

$$\mathcal{H}_x = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2}(1 + \alpha_\mathrm{e}Q) + e\phi_\mathrm{e} - \mathcal{L}_{\mathrm{c}x} \tag{5}$$

$$(\mathcal{H}_x - e\phi_\mathrm{e} + \mathcal{L}_{\mathrm{c}x})^2 = c^2[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2](1 + \alpha_\mathrm{e}Q)^2 \tag{6}$$

The terms α_e and Q are defined in Eq.(1.3-37), the term $\mathcal{L}_{\mathrm{c}x}$ in Eq.(1.3-19), and the potentials $\mathbf{A}_\mathrm{e}$, $\mathbf{A}_\mathrm{m}$, ϕ_e, ϕ_m in Eqs.(1.3-7) to (1.3-14).

Since Eqs.(5) and (6) hold only in first order of α_e we write $(1+\alpha_\mathrm{e}Q)^2 \doteq 1+2\alpha_\mathrm{e}Q$. Furthermore, we leave out the term $\mathcal{L}_{\mathrm{c}x}^2$ since it is multiplied according to Eq.(1.3-19) by

$$\left(\frac{Ze}{c}\right)^2 = \alpha_\mathrm{e}^2\left(\frac{m_0}{A_\mathrm{e}}\right)^2 \tag{7}$$

Equation (6) becomes in first order of α_e:

$$(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_\mathrm{e})^2 + \alpha_\mathrm{e}\bigg\{2[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]Q$$
$$- \frac{1}{c^2}\left[(\mathcal{H}_x - e\phi_\mathrm{e})\frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_\mathrm{e}} + \frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_\mathrm{e}}(\mathcal{H}_x - e\phi_\mathrm{e})\right]\bigg\} = -m_0^2c^2$$

$$\alpha_\mathrm{e} = \frac{ZecA_\mathrm{e}}{m_0c^2} = \frac{Ze}{c}\frac{A_\mathrm{e}}{m_0} = 2\alpha\frac{\lambda_\mathrm{C}A_\mathrm{e}}{e} \ll 1 \tag{8}$$

In order to obtain a solution as an expansion in powers of α_e we replace the function Ψ in Eq.(4) by the function Ψ_x:

$$\Psi_x = \Psi_{\mathrm{x}0} + \alpha_\mathrm{e}\Psi_{\mathrm{x}1} \tag{9}$$

Equation (8) assumes the following form when applied to the function Ψ_x:

$$\bigg\{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_\mathrm{e})^2 + \alpha_\mathrm{e}\bigg[2[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]Q$$
$$- \frac{1}{c^2}\left((\mathcal{H}_x - e\phi_\mathrm{e})\frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_\mathrm{e}} + \frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_\mathrm{e}}(\mathcal{H}_x - e\phi_\mathrm{e})\right)\bigg]\bigg\}(\Psi_{\mathrm{x}0} + \alpha_\mathrm{e}\Psi_{\mathrm{x}1})$$
$$= -m_0^2c^2(\Psi_{\mathrm{x}0} + \alpha_\mathrm{e}\Psi_{\mathrm{x}1}) \tag{10}$$

Since $\mathcal{L}_{\mathrm{c}x}$ in Eq.(1.3-19) is in essence multiplied by α_e according to Eq.(7) we may separate Eq.(10) into one equation of order $O(1)$ and a second one of order $O(\alpha_\mathrm{e})$:

$$\left((\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_\mathrm{e})^2 + m_0^2c^2\right)\Psi_{\mathrm{x}0} = 0 \tag{11}$$

$$\left((\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2-\frac{1}{c^2}(\mathcal{H}_x-e\phi_{\mathrm{e}})^2+m_0^2c^2\right)\Psi_{\mathrm{x}1}=-\Bigg[2[(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2+m_0^2c^2]Q \\ -\frac{1}{c^2}\left((\mathcal{H}_x-e\phi_{\mathrm{e}})\frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_{\mathrm{e}}}+\frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_{\mathrm{e}}}(\mathcal{H}_x-e\phi_{\mathrm{e}})\right)\Bigg]\Psi_{\mathrm{x}0} \quad (12)$$

Equation (11) is the usual Klein-Gordon equation of Eq.(4) while Eq.(12) is the same equation with an added inhomogeneous term.

The factor Q in Eq.(12) contains a term $[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{-3/2}$ according to Eq.(1.3-37). If we want to replace the momentum $\mathbf{p}$ by the differential operators of Eq.(3) we must explain what the resulting operators mean. One may multiply Eq.(12) with $[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{3/2}$, accumulate all terms multiplied with a square root on one side and square both sides of the equation to resolve the problem. This approach is made more complicated by the terms $\mathcal{L}_{\mathrm{c}x}$ as will soon be seen. A simplification is possible if we restrict the calculation to sufficiently small values of $(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2 \ll 1$ and use the following series expansion; the factor k is written instead of $-3/2$ since the same series expansion will be needed later on for $k=-1/2$:

$$[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^k \doteq 1+k(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2 \quad (13)$$

We still have to explain or eliminate the factor

$$Q_{02}=\frac{[\mathbf{A}_{\mathrm{e}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]^2}{\mathbf{A}_{\mathrm{e}}^2(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2} \quad (14)$$

of Q in Eq.(1.3-37). This is possible by replacing the vectors $\mathbf{A}_{\mathrm{e}}$ and $(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})$ in Eq.(14) by matrices of rank 3 whose components are vectors:

$$\mathbf{A}_{\mathrm{e}}=\begin{pmatrix} A_{\mathrm{e}x}\mathbf{e}_x & 0 & 0 \\ 0 & A_{\mathrm{e}y}\mathbf{e}_y & 0 \\ 0 & 0 & A_{\mathrm{e}z}\mathbf{e}_z \end{pmatrix} \quad (15)$$

$$\mathbf{p}-e\mathbf{A}_{\mathrm{m}}=\begin{pmatrix} (p_x-eA_{\mathrm{m}x})\mathbf{e}_x & 0 & 0 \\ 0 & (p_y-eA_{\mathrm{m}y})\mathbf{e}_y & 0 \\ 0 & 0 & (p_z-eA_{\mathrm{m}z})\mathbf{e}_z \end{pmatrix} \quad (16)$$

Substitution of Eqs.(15) and (16) into Eq.(14) yields $Q_{02}=1$.

With $Q_{02}=1$ the term Q of Eq.(1.3-37) is reduced to

$$Q=\frac{1}{m_0^2c^2}\frac{(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2}{[1+(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{3/2}} \quad (17)$$

Using Eq.(13) we may eliminate the denominator and bring Q into a form that permits the substitution of the operators of Eq.(3). A new problem arises.

Shall we write $(\mathbf{p}-e\mathbf{A}_{\rm m})^2$ as the first factor or shall we write $1+k(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2$ as the first factor? This is the perennial problem when one wants to replace commuting factors by non-commuting operators. There is no actual problem with Eq.(17) since $(\mathbf{p}-e\mathbf{A}_{\rm m})^2$ only has to commute with a constant or with itself. However, the problem will occur again in a non-trivial form. We must introduce an assumption or a postulate to make the replacement of two commuting factors by non-commuting factors unique. Any new theory requires a new assumption. If the assumption is successful it becomes eventually a law of nature. We shall replace a commuting product ab by

$$ab \to \frac{1}{2}(ab+ba) \tag{18}$$

to make the transition from a commuting product ab to a non-commuting one unique. Equation (18) has the features of symmetry and simplicity.

From Eqs.(13) and (17) we obtain a unique form for Q if $\mathbf{p}-e\mathbf{A}_{\rm m}$ is replaced by an operator:

$$Q \doteq (\mathbf{p}-e\mathbf{A}_{\rm m})^2\left(m_0^2c^2-\frac{3}{2}(\mathbf{p}-e\mathbf{A}_{\rm m})^2\right) \tag{19}$$

The potential $\mathbf{A}_{\rm e}$ that comes from the magnetic current density term $\mathbf{g}_{\rm m}$ in the modified Maxwell equations, Eq.(1.1-2), has disappeared from Eq.(19) due to the relation $Q_{02}=1$. But its influence has not disappeared because $\mathbf{p}-e\mathbf{A}_{\rm m}$ is now the matrix of Eq.(16) that was forced on us by $\mathbf{A}_{\rm e}$ in order to obtain a usable form of Q.

A further term needing explanation is $\mathcal{L}_{\mathrm{c}x}$ of Eqs.(12) and (1.3-19). We break it into five components. In order to avoid a long calculation we must refer to previously derived equations[1].

$$\begin{aligned}
\frac{1}{\alpha_{\rm e}}\mathcal{L}_{\mathrm{c}x1} &= \frac{m_0}{A_{\rm e}}(A_{{\rm e}x}\dot{y}-A_{{\rm e}y}\dot{z})\dot{x}\\
&= \frac{1}{A_{\rm e}m_0}\frac{[A_{{\rm e}z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y-A_{{\rm e}y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z](\mathbf{p}-e\mathbf{A}_{\rm m})_x}{1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2}+O(\alpha_{\rm e}^2)\\
&\doteq \frac{1}{2A_{\rm e}m_0}\Bigg[[A_{{\rm e}z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y-A_{{\rm e}y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z](\mathbf{p}-e\mathbf{A}_{\rm m})_x\\
&\qquad\times\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{m_0^2c^2}\right)+\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{m_0^2c^2}\right)\\
&\qquad\times[A_{{\rm e}z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y-A_{{\rm e}y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z](\mathbf{p}-e\mathbf{A}_{\rm m})_x\Bigg]
\end{aligned} \tag{20}$$

[1] Harmuth, Barrett, Meffert (2001), Eqs.(3.3-53) to (3.3-57).

$$\begin{aligned}
\frac{1}{\alpha_{\rm e}}\mathcal{L}_{{\rm c}x2} &= \frac{m_0}{A_{\rm e}}\int\left(\frac{\partial\phi_{\rm m}}{\partial y}\dot{z} - \frac{\partial\phi_{\rm m}}{\partial z}\dot{y}\right)dx \\
&= \frac{1}{A_{\rm e}}\int\left(\frac{\partial\phi_{\rm m}}{\partial y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z - \frac{\partial\phi_{\rm m}}{\partial z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y\right) \\
&\qquad\qquad \times\left(1+\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{m_0^2c^2}\right)^{-1/2}dx \\
&\doteq \frac{1}{2A_{\rm e}}\int\left[\left(\frac{\partial\phi_{\rm m}}{\partial y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z - \frac{\partial\phi_{\rm m}}{\partial z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y\right)\right. \\
&\quad \times\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right) + \left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right) \\
&\qquad\qquad \left.\times\left(\frac{\partial\phi_{\rm m}}{\partial y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z - \frac{\partial\phi_{\rm m}}{\partial z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y\right)\right]dx
\end{aligned} \tag{21}$$

$$\begin{aligned}
\frac{1}{\alpha_{\rm e}}\mathcal{L}_{{\rm c}x3} &= \frac{m_0}{A_{\rm e}}\int(A_{{\rm e}z}\ddot{y} - A_{{\rm e}y}\ddot{z})dx \\
&= \frac{1}{A_{\rm e}}\int\left[A_{{\rm e}z}\frac{\partial}{\partial t}\left(\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})_y}{[1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2]^{1/2}}\right)\right. \\
&\qquad\qquad \left. - A_{{\rm e}y}\frac{\partial}{\partial t}\left(\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})_z}{[1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2]^{1/2}}\right)\right]dx \\
&\doteq \frac{1}{2A_{\rm e}}\int\left\{A_{{\rm e}z}\frac{\partial}{\partial t}\left[\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right)(\mathbf{p}-e\mathbf{A}_{\rm m})_y\right.\right. \\
&\qquad\qquad \left. + (\mathbf{p}-e\mathbf{A}_{\rm m})_y\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right)\right] \\
&\quad - A_{{\rm e}z}\frac{\partial}{\partial t}\left[\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right)(\mathbf{p}-e\mathbf{A}_{\rm m})_z\right. \\
&\qquad\qquad \left.\left. + (\mathbf{p}-e\mathbf{A}_{\rm m})_z\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right)\right]\right\}dx
\end{aligned} \tag{22}$$

$$\frac{1}{\alpha_{\rm e}}\mathcal{L}_{{\rm c}x4} = \frac{m_0c^2}{A_{\rm e}}\int\left(\frac{\partial A_{{\rm e}y}}{\partial z} - \frac{\partial A_{{\rm e}z}}{\partial y}\right)dx \tag{23}$$

$$\begin{aligned}
\frac{1}{\alpha_{\rm e}}\mathcal{L}_{{\rm c}x5} &= \frac{m_0}{A_{\rm e}}\int\left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(A_{{\rm e}z}\dot{y} - A_{{\rm e}y}\dot{z})dx \\
&= \frac{1}{4A_{\rm e}m_0}\int\left\{\left[\left(1-\frac{(\mathbf{p}-e\mathbf{A}_{\rm m})^2}{2m_0^2c^2}\right)(\mathbf{p}-e\mathbf{A}_{\rm m})_y\right.\right.
\end{aligned}$$

$$+ (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y \left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2c^2}\right)\bigg] \frac{\partial}{\partial y}$$
$$+ \left[\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2c^2}\right)(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z\right.$$
$$\left. + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z \left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2c^2}\right)\right] \frac{\partial}{\partial z}\bigg\}$$
$$\times \left[\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2c^2}\right)[A_{\mathrm{e}z}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z]\right.$$
$$\left. + [A_{\mathrm{e}z}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z]\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2c^2}\right)\right] dx \quad (24)$$

We substitute the four operators of Eq.(3) into Eq.(11) but write matrices according to Eq.(16):

$$\left[\begin{pmatrix} \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)^2 & 0 \\ 0 & 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right)^2 \end{pmatrix}\right.$$
$$- \frac{1}{c^2}\begin{pmatrix} \left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\mathrm{e}}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\mathrm{e}}\right)^2 & 0 \\ 0 & 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\mathrm{e}}\right)^2 \end{pmatrix}$$
$$\left. + m_0^2c^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} \Psi_{\mathrm{x}0x} & 0 & 0 \\ 0 & \Psi_{\mathrm{x}0y} & 0 \\ 0 & 0 & \Psi_{\mathrm{x}0z} \end{pmatrix} = 0 \quad (25)$$

This is essentially three times Eq.(4) without the summation sign but with the index j retaining the values $j = 1, 2, 3$:

$$\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 - \frac{1}{c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\mathrm{e}}\right)^2 + m_0^2c^2\right]\Psi_{\mathrm{x}0x_j} = 0 \qquad (26)$$

Using the notation of Eq.(26) we may write Eq.(12) with the help of Eq.(19) for Q. There is no problem of commutability of Q with the factor in front of it in Eq.(12) and we get for that part of Eq.(12) the following result:

$$\begin{aligned} &2[(\mathbf{p}-e\mathbf{A}_{\rm m})^2+m_0^2c^2](\mathbf{p}-e\mathbf{A}_{\rm m})^2\left(m_0^2c^2-\frac{3}{2}(\mathbf{p}-e\mathbf{A}_{\rm m})^2\right)\\ &\qquad =2(\mathbf{p}-e\mathbf{A}_{\rm m})^2[m_0^2c^2+(\mathbf{p}-e\mathbf{A}_{\rm m})^2]\left(m_0^2c^2-\frac{3}{2}(\mathbf{p}-e\mathbf{A}_{\rm m})^2\right) \end{aligned} \tag{27}$$

Equation (12) becomes:

$$\begin{aligned} &\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2-\frac{1}{c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\rm e}\right)^2+m_0^2c^2\right]\Psi_{{\rm x}1x_j}\\ &\quad =-\left\{2\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2\left[m_0^2c^2+\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2\right]\right.\\ &\qquad\qquad \times\left[m_0^2c^2-\frac{3}{2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2\right]\\ &\quad +\frac{1}{c^2}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\rm e}\right)(\mathfrak{L}_{{\rm cx}1j}+\mathfrak{L}_{{\rm cx}2j}+\mathfrak{L}_{{\rm cx}3j}+\mathfrak{L}_{{\rm cx}4j}+\mathfrak{L}_{{\rm cx}5j})\right.\\ &\quad \left.\left.+(\mathfrak{L}_{{\rm cx}1j}+\mathfrak{L}_{{\rm cx}2j}+\mathfrak{L}_{{\rm cx}3j}+\mathfrak{L}_{{\rm cx}4j}+\mathfrak{L}_{{\rm cx}5j})\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\rm e}\right)\right]\right\}\Psi_{{\rm x}0x_j} \end{aligned} \tag{28}$$

The operators $\mathfrak{L}_{{\rm cx}1j}$ to $\mathfrak{L}_{{\rm cx}5j}$ follow from Eqs.(20)–(24) with the help of Eq.(3) and the substitution

$$\frac{1}{\alpha_{\rm e}}\mathcal{L}_{{\rm cxk}j}\to\mathfrak{L}_{{\rm cxk}j},\ k=1,\ 2,\ 3,\ 4,\ 5 \tag{29}$$

Note that $\mathcal{L}$ uses the font Euler Script medium while $\mathfrak{L}$ uses Euler Fraktur medium. The matrix $\boldsymbol{\mathfrak{L}}_{\rm cx1}$ has the terms $\mathfrak{L}_{{\rm cx}1j}$ along its main diagonal and zeroes everywhere else:

$$\begin{aligned} \mathfrak{L}_{{\rm cx}1j}&=\frac{1}{2A_{\rm e}m_0}\left\{\left[A_{{\rm e}z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}-eA_{{\rm m}y}\right)-A_{{\rm e}y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z}-eA_{{\rm m}z}\right)\right]\right.\\ &\qquad\times\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-eA_{{\rm m}x}\right)\left[1-\frac{1}{m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2\right]\\ &\quad+\left[1-\frac{1}{m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{{\rm m}x_j}\right)^2\right]\left[A_{{\rm e}z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}-eA_{{\rm m}y}\right)\right.\\ &\qquad\left.\left.-A_{{\rm e}y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z}-eA_{{\rm m}z}\right)\right]\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-eA_{{\rm m}x}\right)\right\}\\ &\qquad j=1,\ 2,\ 3;\quad x_1=x,\ x_2=y,\ x_3=z \end{aligned} \tag{30}$$

For clarification of the notation we observe that the terms

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2$$

are the terms of a matrix with rank 3 like the first matrix in Eq.(25) with the terms along the main diagonal varying according to $j = 1, 2, 3$. On the other hand, the terms

$$A_{\mathrm{e}z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right),\ A_{\mathrm{e}y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right),\ \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x}\right)$$

form matrices of rank 3 with equal values for all elements in the main diagonal like the second and third matrix in Eq.(25).

$$\begin{aligned}\mathfrak{L}_{\mathrm{cx}2j} = \frac{1}{2A_{\mathrm{e}}}\int\Bigg\{&\left[\frac{\partial\phi_m}{\partial y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right) - \frac{\partial\phi_{\mathrm{m}}}{\partial z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)\right]\\ \times&\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right] + \left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\\ &\times\left[\frac{\partial\phi_{\mathrm{m}}}{\partial y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right) - \frac{\partial\phi_{\mathrm{m}}}{\partial z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)\right]\Bigg\}dx\end{aligned} \tag{31}$$

$$\begin{aligned}\mathfrak{L}_{\mathrm{cx}3j} = \frac{1}{2A_{\mathrm{e}}}\int\Bigg(&A_{\mathrm{e}z}\frac{\partial}{\partial t}\left\{\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)\right.\\ &\left.+\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\right\}\\ &- A_{\mathrm{e}z}\frac{\partial}{\partial t}\left\{\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right)\right.\\ &\left.+\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right)\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\right\}\Bigg)dx\end{aligned} \tag{32}$$

$$\mathfrak{L}_{\mathrm{cx}4j} = \frac{m_0c^2}{A_{\mathrm{e}}}\int\left(\frac{\partial A_{\mathrm{e}y}}{\partial z} - \frac{\partial A_{\mathrm{e}z}}{\partial y}\right)dx \tag{33}$$

$$\begin{aligned}
\mathfrak{L}_{\mathrm{cx}5j} = \frac{1}{4A_{\mathrm{e}}m_0} \int \Bigg(\Bigg\{ & \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) \\
& + \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \Bigg\} \frac{\partial}{\partial y} \\
& + \Bigg\{ \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \right. \\
& + \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \Bigg\} \frac{\partial}{\partial z} \Bigg) \\
& \times \Bigg\{ \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \\
& \qquad \times \left[A_{\mathrm{e}x} \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) - A_{\mathrm{e}y} \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \right] \\
& + \left[A_{\mathrm{e}x} \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) - A_{\mathrm{e}y} \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \right] \\
& \qquad \times \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \Bigg\} dx \qquad (34)
\end{aligned}$$

Equations (30) to (34) make all terms in Eq.(28) defined for known values of the potentials $\mathbf{A}_{\mathrm{m}}$, $\mathbf{A}_{\mathrm{e}}$, ϕ_{m}, ϕ_{e} and the rest mass m_0 of a charged particle. A similar but not unique result has been derived previously (Harmuth, Barrett, Meffert 2001, Eqs.5.1-37 to 5.1-43). We have now a unique result *provided* we accept Eq.(18) as necessary. Only success of the derived equations can prove the necessity of Eq.(18). At this time it is perfectly possible that another relation than Eq.(18) yields unique and better results. The complexity of Eqs.(30) to (34) strongly suggests not to replace Eq.(18) by a more complex relation.

Equation (26) can be solved for certain initial and boundary conditions just like partial differential equations for the field strengths $\mathbf{E}$ and $\mathbf{H}$ or the potentials $\mathbf{A}_{\mathrm{m}}$, $\mathbf{A}_{\mathrm{e}}$, ϕ_{m}, ϕ_{e} derived from the modified Maxwell equations can be solved. We note that a vector with three components that are scalars is formally similar to a matrix of rank 3 with three components in the main diagonal that are vectors. If the solution is done by Fourier's method of standing waves we are led to a quantization procedure as in Section 2.4 for the pure radiation field.

The solution of Eq.(28) requires a particular solution of the inhomogeneous equation since the homogeneous equation is the same as in Eq.(26).

4.2 Step Function excitation

For the pure radiation field we developed Eqs.(2.1-38) and (2.1-39) as solutions for electric excitation while Eqs.(2.1-40) and (2.1-41) hold for magnetic excitation. The associated potentials were derived for the normalized variables ζ and θ in Eqs.(2.1-49) and (2.1-52). Here we have to develop solutions for Eqs.(4.1-26) and (4.1-28). To simplify the notation we write

$$x_j = y, \quad \Psi_{\mathrm{x}0x_j} = \Psi_{\mathrm{x}0y} = \Psi \tag{1}$$

The two terms in the squared parentheses in Eq.(4.1-26) are expanded:

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)^2 \Psi = -\hbar^2 \frac{\partial^2 \Psi}{\partial y^2} + 2i\hbar e A_{\mathrm{m}y}\frac{\partial \Psi}{\partial y} + \left(e^2 A_{\mathrm{m}y}^2 + i\hbar e \frac{\partial A_{\mathrm{m}y}}{\partial y}\right)\Psi \tag{2}$$

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\mathrm{e}}\right)^2 \Psi = -\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} - 2i\hbar e \phi_{\mathrm{e}}\frac{\partial \Psi}{\partial t} + \left(e^2 \phi_{\mathrm{e}}^2 - i\hbar e \frac{\partial \phi_{\mathrm{e}}}{\partial t}\right)\Psi \tag{3}$$

Substitution of these two equations into Eq.(4.1-26) yields:

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} - 2i\frac{e}{\hbar}\left(A_{\mathrm{m}y}\frac{\partial \Psi}{\partial y} + \frac{1}{c^2}\phi_{\mathrm{e}}\frac{\partial \Psi}{\partial t}\right) \\ - \frac{e^2}{\hbar^2}\left[A_{\mathrm{m}y}^2 - \frac{e^2}{c^2}\phi_{\mathrm{e}}^2 + i\frac{\hbar}{e}\left(\frac{\partial A_{\mathrm{m}y}}{\partial y} + \frac{1}{c^2}\frac{\partial \phi_{\mathrm{e}}}{\partial t}\right) + \frac{m_0^2 c^2}{e^2}\right]\Psi = 0\end{aligned} \tag{4}$$

The component $A_{\mathrm{m}y}$ of the vector potential and the scalar potential ϕ_{e} are generally functions of location and time. This means Eq.(4) is a partial differential equation of Ψ with variable coefficients. The differential equation (2.1-43) was rewritten in Eqs.(3.1-1) to (3.1-5) into a difference equation. We want to do the same eventually with Eq.(4). Equation (3.1-5) has constant coefficients but its solution was quite a challenge. The mathematics of difference calculus will need a good deal of further development before we can hope to solve a partial difference equation with variable coefficients corresponding to Eq.(4). At this time we have little choice but to assume that $A_{\mathrm{m}y}$ and ϕ_{e} can be represented by a series expansion. We have used the constant α_{e} of Eq.(1.3-37) for series expansions in Eqs.(1.3-34) to (1.3-36) and again in Eq.(4.1-5). Hence, we shall use it here too in order to make Eq.(4) an equation with constant coefficients:

$$\begin{aligned}\mathbf{A}_{\mathrm{m}} = \mathbf{A}_{\mathrm{m}0} + \alpha_{\mathrm{e}}\mathbf{A}_{\mathrm{m}1}(\mathbf{r}, t), \quad A_{\mathrm{m}x_j} = A_{\mathrm{m}0x_j} + \alpha_{\mathrm{e}} A_{\mathrm{m}1x_j}(x_j, t) \\ \phi_{\mathrm{e}} = \phi_{\mathrm{e}0} + \alpha_{\mathrm{e}}\phi_{\mathrm{e}1}(x_j, t), \quad x_j = x,\ y,\ z; \quad \Psi_{\mathrm{x}} = \Psi_{\mathrm{x}0} + \alpha_{\mathrm{e}}\Psi_{\mathrm{x}1}\end{aligned} \tag{5}$$

With the approximations of first order in α_{e}

$$(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})^2 \doteq (\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 - \alpha_{\mathrm{e}} e[\mathbf{A}_{\mathrm{m1}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}}) + (\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})\cdot\mathbf{A}_{\mathrm{m1}}] \quad (6)$$

$$(\mathcal{H}_x - e\phi_e)^2 \doteq (\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 - \alpha_{\mathrm{e}} e[\phi_{\mathrm{e1}}(\mathcal{H}_x - e\phi_{\mathrm{e0}}) + (\mathcal{H}_x - e\phi_{\mathrm{e0}})\phi_{\mathrm{e1}}] \quad (7)$$

we obtain the following equation instead of Eq.(4.1-10):

$$\begin{aligned}\Big\{(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 + \alpha_{\mathrm{e}}\Big[2[(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 + m_0^2c^2]Q \\ - \frac{1}{c^2}\Big((\mathcal{H}_x - e\phi_{\mathrm{e0}})\frac{\mathcal{L}_{cx}}{\alpha_{\mathrm{e}}} + \frac{\mathcal{L}_{cx}}{\alpha_{\mathrm{e}}}(\mathcal{H}_{\mathrm{x}} - e\phi_{\mathrm{e0}})\Big) - e[\mathbf{A}_{\mathrm{m1}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}}) + (\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})\cdot\mathbf{A}_{\mathrm{m1}}] \\ + \frac{e}{c^2}[\phi_{\mathrm{e1}}(\mathcal{H}_x - e\phi_{\mathrm{e0}}) + (\mathcal{H}_x - e\phi_{\mathrm{e0}})\phi_{\mathrm{e1}}]\Big]\Big\}(\Psi_{\mathrm{x0}} + \alpha_{\mathrm{e}}\Psi_{\mathrm{x1}}) \\ = -m_0^2c^2(\Psi_{\mathrm{x0}} + \alpha_{\mathrm{e}}\Psi_{\mathrm{x1}}) \quad (8)\end{aligned}$$

The separation of this equation into two equations of order $O(1)$ and $O(\alpha_{\mathrm{e}})$ as in the case of Eqs.(4.1-11) and (4.1-12) yields:

$$\Big((\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 + m_0^2c^2\Big)\Psi_{\mathrm{x0}} = 0 \quad (9)$$

$$\begin{aligned}\Big((\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 + m_0^2c^2\Big)\Psi_{\mathrm{x1}} = -\Big[2[(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})^2 + m_0^2c^2]Q \\ - \frac{1}{c^2}\Big((\mathcal{H}_x - e\phi_{\mathrm{e0}})\frac{\mathcal{L}_{\mathrm{cx}}}{\alpha_{\mathrm{e}}} + \frac{\mathcal{L}_{\mathrm{cx}}}{\alpha_{\mathrm{e}}}(\mathcal{H}_x - e\phi_{\mathrm{e0}})\Big) - e[\mathbf{A}_{\mathrm{m1}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m0}}) \\ + (\mathbf{p}-e\mathbf{A}_{\mathrm{m0}})\cdot\mathbf{A}_{\mathrm{m1}}] + \frac{e}{c^2}[\phi_{\mathrm{e1}}(\mathcal{H}_x - e\phi_{\mathrm{e0}}) + (\mathcal{H}_x - e\phi_{\mathrm{e0}})\phi_{\mathrm{e1}}]\Big]\Psi_{\mathrm{x0}} \quad (10)\end{aligned}$$

Equation (9) equals Eq.(4.1-11) if $\mathbf{A}_{\mathrm{m}}$ and ϕ_{e} are replaced by $\mathbf{A}_{\mathrm{m0}}$ and ϕ_{e0}. Equation (4) may be simplified with the approximations of Eq.(5):

$$\begin{aligned}\frac{\partial^2\Psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - 2i\frac{e}{\hbar}\Big(A_{\mathrm{m0}y}\frac{\partial\Psi}{\partial y} + \frac{1}{c^2}\phi_{\mathrm{e0}}\frac{\partial\Psi}{\partial t}\Big) \\ - \frac{e^2}{\hbar^2}\Big(A_{\mathrm{m0}y}^2 - \frac{1}{c^2}\phi_{\mathrm{e0}}^2 + \frac{m_0^2c^2}{e^2}\Big)\Psi = 0, \quad \Psi = \Psi_{\mathrm{x0}} \quad (11)\end{aligned}$$

From Eq.(10) we obtain a modification of Eq.(4.1-28):

$$\begin{aligned}
&\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)^2-\frac{1}{c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}0}\right)^2+m_0^2c^2\right]\Psi_{\mathrm{x}1x_j}\\
&\quad=-\left\{2\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)^2\left[m_0^2c^2+\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)^2\right]\right.\\
&\qquad\qquad\times\left[m_0^2c^2-\frac{3}{2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)^2\right]\\
&\quad+\frac{1}{c^2}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}0}\right)(\mathfrak{L}_{\mathrm{cx}1j}+\mathfrak{L}_{\mathrm{cx}2j}+\mathfrak{L}_{\mathrm{cx}3j}+\mathfrak{L}_{\mathrm{cx}4j}+\mathfrak{L}_{\mathrm{cx}5j})\right.\\
&\quad\left.+(\mathfrak{L}_{\mathrm{cx}1j}+\mathfrak{L}_{\mathrm{cx}2j}+\mathfrak{L}_{\mathrm{cx}3j}+\mathfrak{L}_{\mathrm{cx}4j}+\mathfrak{L}_{\mathrm{cx}5j})\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}0}\right)\right]\\
&\quad-e\left[A_{\mathrm{m}1x_j}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)+\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-eA_{\mathrm{m}0x_j}\right)A_{\mathrm{m}1x_j}\right]\\
&\qquad\left.-\frac{e}{c^2}\left[\phi_{\mathrm{e}1}\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}0}\right)+\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}0}\right)\phi_{\mathrm{e}1}\right]\right\}\Psi_{\mathrm{x}0x_j}
\end{aligned}\tag{12}$$

The homogeneous part of Eq.(12) is the same as that of Eq.(4.1-28) if $A_{\mathrm{m}x_j}$ and ϕ_{e} are replaced by $A_{\mathrm{m}0x_j}$ and $\phi_{\mathrm{e}0}$. Hence, this part is again represented by Eq.(11). The inhomogeneous part has four terms added to Eq.(4.1-28). We observe that $A_{\mathrm{m}\,.}$ must be replaced by $A_{\mathrm{m}0\,.}$ in Eqs.(4.1-30) to (4.1-34) for $\mathfrak{L}_{\mathrm{cx}1j}$ to $\mathfrak{L}_{\mathrm{cx}5j}$.

Equation (11) as well as the homogeneous part of Eq.(12) are now partial differential equations with constant coefficients. Only the inhomogeneous part of Eq.(12) contains variable components of the potentials $\mathbf{A}_{\mathrm{m}}$, $\mathbf{A}_{\mathrm{e}}$, ϕ_{m} and ϕ_{e}.

We make the transition from the non-normalized variables y, t to the variables normalized by τ as in Eq.(2.1-42):

$$\theta=t/\tau,\quad \zeta=y/c\tau \tag{13}$$

Substitution into Eq.(11) yields:

$$\begin{gathered}
\frac{\partial^2\Psi}{\partial\zeta^2}-\frac{\partial^2\Psi}{\partial\theta^2}-2i\lambda_1\left(\frac{\partial\Psi}{\partial\zeta}+\lambda_3\frac{\partial\Psi}{\partial\theta}\right)-\lambda_2^2\Psi=0\\
\lambda_1=\frac{ec\tau A_{\mathrm{m}0y}}{\hbar},\ \lambda_3=\frac{\phi_{\mathrm{e}0}}{cA_{\mathrm{m}0y}},\ N_\tau=\frac{T}{\tau}\\
\lambda_2^2=\lambda_1^2\left(1-\frac{\phi_{\mathrm{e}0}^2}{c^2A_{\mathrm{m}0y}^2}+\frac{m_0^2c^2}{e^2A_{\mathrm{m}0y}^2}\right)=\frac{\tau^2}{\hbar^2}(m_0^2c^4-e^2\phi_{\mathrm{e}0}^2+e^2c^2A_{\mathrm{m}0y}^2)\\
\lambda_2^2-\lambda_1^2=\frac{\tau^2}{\hbar^2}(m_0^2c^4-e^2\phi_{\mathrm{e}0}^2)>0\quad\text{for } m_0c^2>e\phi_{\mathrm{e}0}\\
\lambda_1^2\lambda_3^2+\lambda_2^2=\frac{\tau^2}{\hbar^2}(m_0^2c^4+e^2c^2A_{\mathrm{m}0y}^2),\ \lambda_1^2\lambda_3^2+\lambda_2^2-\lambda_1^2=\frac{\tau^2m_0^2c^4}{\hbar^2}
\end{gathered}\tag{14}$$

Equation (14) is similar to Eqs.(2.1-43) and (2.1-46). This suggests obtaining a solution of Eq.(14) by means of Fourier's method of standing waves that satisfies the causality law and the conservation law of energy. In analogy to Eq.(2.1-56) we try the ansatz

$$\begin{aligned}\Psi(0,\theta) = \Psi_0 S(\theta) &= 0 \quad \text{for } \theta < 0\\ &= \Psi_0 \text{ for } \theta \geq 0\end{aligned} \tag{15}$$

As in the case of Eq.(2.1-57) there is a problem with a boundary condition $\Psi(\infty,\theta)$ that has to be resolved later on. The comments following Eq.(2.1-57) apply again.

For the initial condition at the time $\theta = 0$ we follow Eq.(2.1-58) and observe the comments made there:

$$\Psi(\zeta,0) = 0 \tag{16}$$

If $\Psi(\zeta,0)$ is zero for all values $\zeta > 0$, all its derivatives with respect to ζ will be zero too:

$$\partial^n\Psi(\zeta,0)/\partial\zeta^n = 0 \tag{17}$$

Equation (14) yields with the help of Eqs.(16) and (17) for $\theta = 0$ another initial condition:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi(\zeta,\theta)}{\partial\theta} - 2i\lambda_1\lambda_3\Psi(\zeta,\theta)\right)_{\theta=0} = 0 \tag{18}$$

This equation is satisfied by Eq.(16) and the additional condition

$$\partial\Psi(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0 \tag{19}$$

We assume the general solution of Eq.(14) can be written as the sum of a steady state solution $F(\zeta)$ plus a deviation $w(\zeta,\theta)$ from it (Habermann 1987):

$$\Psi(\zeta,\theta) = \Psi_0[F(\zeta) + w(\zeta,\theta)] \tag{20}$$

Substitution of $F(\zeta)$ into Eq.(14) yields an ordinary differential equation with the variable ζ:

$$d^2F/d\zeta^2 - 2i\lambda_1 dF/d\zeta - \lambda_2^2 F = 0 \tag{21}$$

Its general solution is:

$$F(\zeta) = A_{10}\exp\{[(\lambda_2^2-\lambda_1^2)^{1/2} + i\lambda_1]\zeta\} + A_{11}\exp\{[-(\lambda_2^2-\lambda_1^2)^{1/2} + i\lambda_1]\zeta\} \tag{22}$$

According to Eq.(14) the difference $\lambda_2^2 - \lambda_1^2$ will be positive except for extremely large values of ϕ_{e0}^2. We restrict ourselves to the case $\lambda_2^2 - \lambda_1^2 > 0$. Furthermore, we want $F(0)$ to be 1. This can be achieved by choosing $A_{11} = 1$ and $A_{10} = 0$. Hence we obtain for $F(\zeta)$:

$$\begin{gathered}\lambda_2^2 - \lambda_1^2 > 0, \quad F(0) = 1, \quad A_{10} = 0, \quad A_{11} = 1 \\ F(\zeta) = \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta} \qquad (23)\end{gathered}$$

Substitution of Eq.(20) into the boundary condition of Eq.(15) yields the boundary condition for $w(0, \theta)$:

$$\begin{aligned}\Psi(0,\theta) = \Psi_0[F(0) + w(0,\theta)] = \Psi_0 \quad &\text{for } \theta \geq 0 \\ w(0,\theta) = 0 \quad &\text{for } \theta \geq 0 \qquad (24)\end{aligned}$$

We have achieved the homogeneous boundary condition $w(0, \theta) = 0$. The choice of A_{10} and A_{11} in Eq.(23) is justified. There are other choices for A_{10} and A_{11} that also yield $w(0, \theta) = 0$ but this would simply mean that both $F(\zeta)$ and $w(\zeta, \theta)$ in Eq.(20) are modified so that their sum remains unchanged. Only the sum is of interest.

Let us turn to the initial conditions of Eqs.(16) and (19). We obtain an equation for $w(\zeta, 0)$:

$$F(\zeta) + w(\zeta, 0) = 0, \qquad w(\zeta, 0) = -F(\zeta) \qquad (25)$$

$$\partial w(\zeta,\theta)/\partial\theta = 0 \qquad \text{for } \theta = 0,\ \zeta > 0 \qquad (26)$$

Substitution of Eq.(20) into Eq.(14) yields for $w(\zeta, \theta)$ the same equation as for Ψ, but we have now the homogeneous boundary condition of Eq.(24) for $w(\zeta, \theta)$:

$$\frac{\partial^2 w}{\partial\zeta^2} - \frac{\partial^2 w}{\partial\theta^2} - 2i\lambda_1\left(\frac{\partial w}{\partial\zeta} + \lambda_3\frac{\partial w}{\partial\theta}\right) - \lambda_2^2 w = 0 \qquad (27)$$

Separation of the variables yields particular solutions of this equation which we denote $w_\kappa(\zeta, \theta)$:

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \qquad (28)$$

$$\begin{gathered}\frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial\zeta^2} - 2i\lambda_1\frac{\partial\phi}{\partial\zeta}\right) = \frac{1}{\psi}\left(\frac{\partial^2\psi}{\partial\theta^2} + 2i\lambda_1\lambda_3\frac{\partial\psi}{\partial\theta}\right) + \lambda_2^2 = -(2\pi\kappa/N_\tau)^2 + \lambda_1^2 \\ N_\tau \geq 1,\ \kappa = 1,\ 2,\ \ldots \qquad (29)\end{gathered}$$

As in Eq.(2.1-72) we write $-(2\pi\kappa/N_\tau)^2$ rather than the usual $-(2\pi\kappa)^2$ as separation constant in order to obtain later on an orthogonality interval of length N_τ rather than 1. The reason for the additional constant λ_1^2 will be seen presently. Two ordinary differential equations are obtained:

$$d^2\phi/d\zeta^2 - 2i\lambda_1 d\phi/d\zeta + [(2\pi\kappa/N_\tau)^2 - \lambda_1^2]\phi = 0 \quad (30)$$

$$d^2\psi/d\theta^2 + 2i\lambda_1\lambda_3 d\psi/d\theta + \left[(2\pi\kappa/N_\tau)^2 - \lambda_1^2 + \lambda_2^2\right]\psi = 0 \quad (31)$$

The solutions are:

$$\phi(\zeta) = e^{i\lambda_1\zeta}(A_{20}e^{2\pi i\kappa\zeta/N_\tau} + A_{21}e^{-2\pi i\kappa\zeta/N_\tau}) \quad (32)$$

$$\begin{aligned}\psi(\theta) &= A_{30}e^{-i(\lambda_1\lambda_3+\gamma_\kappa)\theta} + A_{31}e^{-i(\lambda_1\lambda_3-\gamma_\kappa)\theta}\\ \gamma_\kappa &= \left[(2\pi\kappa/N_\tau)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2 - \lambda_1^2\right]^{1/2}\\ &= \left[(2\pi\kappa/N_\tau)^2 + m_0^2c^4\tau^2/\hbar^2\right]^{1/2} = \text{real}\end{aligned} \quad (33)$$

The boundary condition $w(0,\theta) = 0$ in Eq.(24) requires in Eq.(32) the relation

$$\begin{aligned}A_{21} &= -A_{20}\\ \phi(\zeta) &= 2iA_{20}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau)\end{aligned} \quad (34)$$

and the particular solution $w_\kappa(\zeta,\theta)$ assumes the following form:

$$w_\kappa(\zeta,\theta) = \left\{A_1\exp[-i(\lambda_1\lambda_3+\gamma_\kappa)\theta] + A_2\exp[-i(\lambda_1\lambda_3-\gamma_\kappa)\theta]\right\} \times e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \quad (35)$$

The usual way to generalize Eq.(35) is to make A_1 and A_2 functions of κ and integrate over all values of κ. We deviate and follow the text between Eqs.(2.1-78) and (2.1-79) in Section 2.1. Our finite time and space intervals are chosen to be $0 \le t \le T$ and $0 \le y \le cT$, where T is arbitrarily large but finite. The variables θ and ζ cover the intervals

$$0 \le \theta = t/\tau \le T/\tau, \qquad 0 \le \zeta = y/c\tau \le T/\tau, \qquad T/\tau = N_\tau \gg 1 \quad (36)$$

Instead of the Fourier sum of Eq.(2.1-81) we get the following sum if the upper summation limit $\kappa \to \infty$ is replaced by $\kappa = N_\tau$ according to Eq.(2.2-5):

$$w(\zeta,\theta) = \sum_{\kappa=1}^{N_\tau} \{A_1(\kappa)\exp[-i(\lambda_1\lambda_3+\gamma_\kappa)\theta] + A_2(\kappa)\exp[-i(\lambda_1\lambda_3-\gamma_\kappa)\theta]\} \times e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \quad (37)$$

The derivative of $w(\zeta,\theta)$ with respect to θ will be needed:

$$\frac{\partial w}{\partial\theta} = \sum_{\kappa=1}^{N_\tau} -i\Big\{A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa)\exp[-i(\lambda_1\lambda_3+\gamma_\kappa)\theta] + A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)\exp[-i(\lambda_1\lambda_3-\gamma_\kappa)\theta]\Big\}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \quad (38)$$

With the help of Eqs.(25) and (26) we get from Eqs.(37) and (38) two equations for the determination of $A_1(\kappa)$ and $A_2(\kappa)$:

$$w(\zeta,0) = \sum_{\kappa=1}^{N_\tau}[A_1(\kappa)+A_2(\kappa)]e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) = -F(\zeta) \quad (39)$$

$$\frac{\partial w(\zeta,0)}{\partial\theta} = \sum_{\kappa=1}^{N_\tau} -i[A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa) + A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)] \times e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) = 0 \quad (40)$$

In analogy to the solution of Eqs.(2.2-2) and (2.2-3) we use the Fourier series expansion of Eq.(2.2-4). The factors $-i$ and $e^{i\lambda_1\zeta}$ can be moved in front of the summation sign since they do not contain κ:

$$\sum_{\kappa=1}^{N_\tau}[A_1(\kappa)+A_2(\kappa)]\sin(2\pi\kappa\zeta/N_\tau) = -F(\zeta)e^{-i\lambda_1\zeta} = -\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] \quad (41)$$

$$\sum_{\kappa=1}^{N_\tau}[A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa) + A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)]\sin(2\pi\kappa\zeta/N_\tau) = 0 \quad (42)$$

Multiplication of Eqs.(41) and (42) with $\sin 2\pi j\zeta/N_\tau$ followed by integration over the orthogonality interval $0 < \zeta < N_\tau$ yields:

$$A_1(\kappa) + A_2(\kappa) = -\frac{2}{N_\tau}\int_0^{N_\tau} F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta = -I_{\mathrm{T}}(\kappa/N_\tau) \tag{43}$$

$$\begin{aligned} I_{\mathrm{T}}(\kappa/N_\tau) &= \frac{2}{N_\tau}\int_0^{N_\tau}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta \\ &= \frac{2}{N_\tau}\frac{(2\pi\kappa/N_\tau)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2} \end{aligned} \tag{44}$$

$$A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa) + A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa) = 0 \tag{45}$$

Equation (43) and (45) are solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa) = +I_{\mathrm{T}}(\kappa/N_\tau)\frac{\lambda_1\lambda_3-\gamma_\kappa}{2\gamma_\kappa} \tag{46}$$

$$A_2(\kappa) = -I_{\mathrm{T}}(\kappa/N_\tau)\frac{\lambda_1\lambda_3+\gamma_\kappa}{2\gamma_\kappa} \tag{47}$$

Substitution of $A_1(\kappa)$ and $A_2(\kappa)$ brings Eq.(37) into the following form:

$$\begin{aligned} w(\zeta,\theta) = \sum_{\kappa=1}^{N_\tau}\frac{I_{\mathrm{T}}(\kappa/N_\tau)}{2\gamma_\kappa}&\left[(\lambda_1\lambda_3-\gamma_\kappa)e^{-i\gamma_\kappa\theta}-(\lambda_1\lambda_3+\gamma_\kappa)e^{i\gamma_\kappa\theta}\right] \\ &\times e^{i\lambda_1(\zeta-\lambda_3\theta)}\sin\frac{2\pi\kappa\zeta}{N_\tau} \end{aligned} \tag{48}$$

4.3 Exponential Ramp Function Excitation

The step function excitation of Eq.(4.2-15) is replaced by an exponential ramp function excitation in order to produce a linear variation of the excitation at $\theta = 0$:

$$\begin{aligned} \Psi(0,\theta) = \Psi_1 S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0 \\ &= \Psi_1(1-e^{-\iota\theta}) && \text{for } \theta \geq 0 \end{aligned} \tag{1}$$

The initial conditions are the same as in Eqs.(4.2-16), (4.2-17) and (4.2-19):

$$\Psi(\zeta,0)=0 \tag{2}$$
$$\partial\Psi(\zeta,0)/\partial\zeta=0 \tag{3}$$
$$\partial\Psi(\zeta,\theta)/\partial\theta=0 \quad \text{for } \theta=0,\ \zeta\geq 0 \tag{4}$$

Instead of Eq.(4.2-20) we use the following ansatz to solve Eq.(4.2-14):

$$\Psi(\zeta,\theta)=\Psi_1[(1-e^{-\iota\theta})F(\zeta)+u(\zeta,\theta)] \tag{5}$$

Substitution of $\Psi_1(1-e^{-\iota\theta})F(\zeta)$ into Eq.(4.2-14) yields the following result:

$$(1-e^{-\iota\theta})\frac{\partial^2 F}{\partial\zeta^2}+\iota^2 e^{-\iota\theta}F-2i\lambda_1\left[(1-e^{-\iota\theta})\frac{\partial F}{\partial\zeta}+\lambda_3\iota e^{-\iota\theta}F\right] - \lambda^2(1-e^{-\iota\theta})F=0 \tag{6}$$

This equation will vanish if the terms with different functions of θ vanish separately. We obtain one ordinary differential equation and one algebraic equation:

$$d^2F/d\zeta^2-2i\lambda_1 dF/d\zeta-\lambda_2^2F=0 \tag{7}$$
$$\iota^2-2i\lambda_1\lambda_3\iota=0 \tag{8}$$

Equation (8) has a non-trivial solution:

$$\iota=2i\lambda_1\lambda_3 \tag{9}$$

Equation (7) equals Eq.(4.2-21). We use again the solution of Eq.(4.2-23):

$$F(\zeta)=\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta} \tag{10}$$

Equation (5) must satisfy the boundary condition of Eq.(1). Since Eq.(10) yields $F(0)=1$ we get:

$$\Psi_1[1-e^{-\iota\theta}+u(0,\theta)]=\Psi_1(1-e^{-\iota\theta}),\quad \theta\geq 0$$
$$u(0,\theta)=0 \tag{11}$$

We have again a homogeneous boundary condition for $u(0,\theta)$. The initial conditions of Eqs.(2) and (4) yield with $F(0)=1$:

$$\Psi(\zeta,0)=\Psi_1 u(\zeta,0)=0 \tag{12}$$
$$\iota e^{-\iota\theta}F(\zeta)+\partial u/\partial\theta=0,\quad \partial u/\partial\theta=-\iota F(\zeta)\quad \text{for } \theta=0,\ \zeta\geq 0 \tag{13}$$

The calculation of $u(\zeta,\theta)$ proceeds like that of $w(\zeta,\theta)$ in Section 4.2 until Eqs.(4.2-37) and (4.2-38) are reached. We replace $w(\zeta,\theta)$ by $u(\zeta,\theta)$ in these two equations:

$$u(\zeta,\theta)=\sum_{\kappa=1}^{N_\tau}\Big\{A_1(\kappa)\exp[-i(\lambda_1\lambda_3+\gamma_\kappa)\theta]+A_2(\kappa)\exp[-i(\lambda_1\lambda_3-\gamma_\kappa)\theta]\Big\}\times e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \quad (14)$$

$$\frac{\partial u(\zeta,\theta)}{\partial\theta}=\sum_{\kappa=1}^{N_\tau}-i\Big\{A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa)\exp[-i(\lambda_1\lambda_3+\gamma_\kappa)\theta]+A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)\exp[-i(\lambda_1\lambda_3-\gamma_\kappa)\theta]\Big\}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \quad (15)$$

The substitution of $u(\zeta,0)$ and $\partial u/\partial\theta$ of Eqs.(12) and (13) yields equations for the determination of $A_1(\kappa)$ and $A_2(\kappa)$. The factor $e^{i\lambda_1\zeta}$ in Eq.(15) is moved to the right side since it does not depend on κ:

$$u(\zeta,0)=\sum_{\kappa=1}^{N_\tau}[A_1(\kappa)+A_2(\kappa)]\sin(2\pi\kappa\zeta/N_\tau)=0 \quad (16)$$

$$\begin{aligned}\frac{\partial u(\zeta,\theta=0)}{\partial\theta}&=\sum_{\kappa=1}^{N_\tau}-i[A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa)+A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)]\sin(2\pi\kappa\zeta/N_\tau)\\&=-\iota F(\zeta)e^{-i\lambda_1\zeta}=-2i\lambda_1\lambda_3F(\zeta)e^{-i\lambda_1\zeta}\end{aligned} \quad (17)$$

Using once more the Fourier series expansion we obtain from Eqs.(16) and (17):

$$A_1(\kappa)+A_2(\kappa)=0 \quad (18)$$

$$\begin{aligned}A_1(\kappa)(\lambda_1\lambda_3+\gamma_\kappa)+A_2(\kappa)(\lambda_1\lambda_3-\gamma_\kappa)&=\frac{4\lambda_1\lambda_3}{N_\tau}\int_0^{N_\tau}F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta\\&=2\lambda_1\lambda_3I_{\mathrm{T}}(\kappa/N_\tau)\end{aligned} \quad (19)$$

Equations (18) and (19) are solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa)=-A_2(\kappa)=\frac{\lambda_1\lambda_3I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa} \quad (20)$$

Substitution of $A_1(\kappa)$ and $A_2(\kappa)$ brings Eq.(14) into the following form:

$$u(\zeta,\theta) = -2i\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3}{\gamma_\kappa}I_{\rm T}(\kappa/N_\tau)\sin\gamma_\kappa\theta e^{-i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau) \qquad (21)$$

Substitution of Eqs.(21) and (10) into Eq.(5) brings the result:

$$\Psi(\zeta,\theta) = \Psi_1\bigg((1-e^{-2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] - 2i\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3}{\gamma_\kappa}I_{\rm T}(\kappa/N_\tau)\sin\gamma_\kappa\theta e^{-i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N_\tau)\bigg) \qquad (22)$$

In order to see how $\Psi(\zeta,\theta)$ rises at $\theta=0$ we use the approximations

$$e^{-2i\lambda_1\lambda_3\theta} \doteq 1-2i\lambda_1\lambda_3\theta,\ e^{-i\lambda_1\lambda_3\theta} \doteq 1-i\lambda_1\lambda_3\theta,\ \sin\gamma_\kappa\theta \doteq \gamma_\kappa\theta \qquad (23)$$

and obtain in first order of θ:

$$\Psi(\zeta,\theta) \doteq 2i\Psi_1\lambda_1\lambda_3\theta e^{i\lambda_1\zeta}\bigg(\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] - \sum_{\kappa=1}^{N_\tau}I_{\rm T}(\kappa/N_\tau)\sin(2\pi\kappa\zeta/N_\tau)\bigg) \qquad (24)$$

Hence, $\Psi(\zeta,\theta)$ rises proportionate to θ from $\theta=0$. The imaginary unit i is of no consequence since the factor Ψ_1 can still be chosen.

4.4 Hamilton Function and Quantization

The Klein-Gordon equation defines a wave. Its energy density is given by the term T_{00} of the energy-impulse tensor[1]:

$$T_{00} = \frac{1}{c^2}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t} + \nabla\Psi^*\cdot\nabla\Psi + \frac{m_0^2c^2}{\hbar^2}\Psi^*\Psi \qquad (1)$$

The dimension of T_{00} is J/m^3 and the dimension of $\Psi^*\Psi$ must thus be J/m or VAs/m in electromagnetic units for the energy. In the case of Eq.(4.2-4) for a planar wave propagating in the direction y we have $\nabla=\partial/\partial y$.

The Fourier series expansion of Eq.(4.2-37) permits an arbitrarily large but finite time T and a corresponding spatial distance cT in the direction of y, using the intervals $0\le t\le T$ and $0\le y\le cT$. In the directions x and z

[1] Berestezki, Lifschitz, Pitajewski 1970, 1982; § 10, Eq.10.13.

we have not specified any intervals. We shall follow Eq.(2.3-1) and make them $-L/2 \le x \le L/2$, $-L/2 \le z \le L/2$, with $L = cT$. The Klein-Gordon wave has the energy U in this interval:

$$U = \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left[\int_0^{cT} \left(\frac{1}{c^2} \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial y} \frac{\partial \Psi}{\partial y} + \frac{m_0^2 c^2}{\hbar^2} \Psi^* \Psi \right) dy \right] dx\, dz \tag{2}$$

Since the dimension of $\Psi^*\Psi$ is VAs/m one obtains for U the dimension VAs. We turn to the normalized variables according to Eq.(4.2-13)

$$t \to t/\tau = \theta,\ y \to y/c\tau = \zeta,\ x \to x/c\tau,\ z \to z/c\tau \tag{3}$$

and rewrite U in the form of Eq.(2.3-15):

$$\begin{aligned} U &= c\tau \int_{-L/2c\tau}^{L/2c\tau} \int_{-L/2c\tau}^{L/2c\tau} \left[\int_0^{N_\tau} \left(\frac{\partial \Psi^*}{\partial \theta} \frac{\partial \Psi}{\partial \theta} + \frac{\partial \Psi^*}{\partial \zeta} \frac{\partial \Psi}{\partial \zeta} + \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi^* \Psi \right) d\zeta \right] \\ &\qquad \times d\left(\frac{x}{c\tau}\right) d\left(\frac{z}{c\tau}\right) \\ &= \frac{L^2}{c\tau} \int_0^{N_\tau} \left(\frac{\partial \Psi^*}{\partial \theta} \frac{\partial \Psi}{\partial \theta} + \frac{\partial \Psi^*}{\partial \zeta} \frac{\partial \Psi}{\partial \zeta} + \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi^* \Psi \right) d\zeta \end{aligned} \tag{4}$$

We use $\Psi(\zeta, \theta)$ of Eqs.(4.3-5) and (4.3-22) to produce the product $\Psi^*\Psi$. The constant Ψ_1^2 has the dimension VAs/m:

$$\begin{aligned} \Psi^*\Psi &= \Psi_1^2 \left[(1 - e^{2i\lambda_1\lambda_3\theta}) F^*(\zeta) + u^*(\zeta,\theta) \right] \left[(1 - e^{-2i\lambda_1\lambda_3\theta}) F(\zeta) + u(\zeta,\theta) \right] \\ &= 2\Psi_1^2 \Bigg[(1 - \cos 2\lambda_1\lambda_3\theta) \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta] \\ &\qquad - 2(1 - \cos 2\lambda_1\lambda_3\theta)\lambda_1\lambda_3 \sin \lambda_1\lambda_3\theta \\ &\qquad \times \sum_{\kappa=1}^{N_\tau} \frac{I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa} \sin \gamma_\kappa\theta \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta] \sin(2\pi\kappa\zeta/N_\tau) \\ &\qquad + 2\left(\sum_{\kappa=1}^{N_\tau} \frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa} \sin \gamma_\kappa\theta \sin \frac{2\pi\kappa\zeta}{N_\tau} \right)^2 \Bigg] \end{aligned}$$

$$\lambda_1, \lambda_2, \lambda_3 \text{ Eq.(4.2-14)},\ \ \gamma_\kappa \text{ Eq.(4.2-33)},\ \ I_{\mathrm{T}}(\kappa/N_\tau) \text{ Eq.(4.2-44)} \tag{5}$$

Differentiation of $\Psi(\zeta, \theta)$ of Eq.(4.3-22) with respect to θ or ζ yields:

$$\begin{aligned}\frac{\partial\Psi}{\partial\theta} &= \Psi_1\left(2i\lambda_1\lambda_3 e^{-2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\theta}\right)\\ \frac{\partial u}{\partial\theta} &= -2e^{i\lambda_1(\zeta-\lambda_3\theta)}\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_\mathrm{T}(\kappa/N_\tau)}{\gamma_\kappa}\\ &\qquad\times(\lambda_1\lambda_3\sin\gamma_\kappa\theta+i\gamma_\kappa\cos\gamma_\kappa\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\end{aligned}\tag{6}$$

$$\begin{aligned}\frac{\partial\Psi}{\partial\zeta} &= \Psi_1\bigg([-(\lambda_2^2-\lambda_1^2)^{1/2}+i\lambda_1](1-e^{-2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}\\ &\qquad\times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\zeta}\bigg)\\ \frac{\partial u}{\partial\zeta} &= 2e^{i\lambda_1(\zeta-\lambda_3\theta)}\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_\mathrm{T}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta\\ &\qquad\times\left(\lambda_1\sin\frac{2\pi\kappa\zeta}{N_\tau}-\frac{2\pi i\kappa}{N_\tau}\cos\frac{2\pi\kappa\zeta}{N_\tau}\right)\end{aligned}\tag{7}$$

The first term in Eq.(4) becomes:

$$\begin{aligned}\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} &= \Psi_1^2\bigg(-2i\lambda_1\lambda_3 e^{2i\lambda_1\lambda_3\theta}e^{-i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u^*}{\partial\theta}\bigg)\\ &\qquad\times\left(2i\lambda_1\lambda_3 e^{-2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\theta}\right)\\ &= 4\lambda_1^2\lambda_3^2\Psi_1^2\bigg[\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\\ &\qquad -2\sum_{\kappa=1}^{N_\tau}I_\mathrm{T}(\kappa/N_\tau)\bigg(\cos\gamma_\kappa\theta\cos\lambda_1\lambda_3\theta\\ &\qquad +\frac{1}{\gamma_\kappa}\sin\gamma_\kappa\theta\sin\lambda_1\lambda_3\theta\bigg)\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N_\tau}\\ &\qquad +\left(\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_\mathrm{T}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2\\ &\qquad +\left(\sum_{\kappa=1}^{N_\tau}I_\mathrm{T}(\kappa/N_\tau)\cos\gamma_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2\bigg]\end{aligned}\tag{8}$$

For the second term in Eq.(4) we obtain:

$$
\begin{aligned}
\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} &= \Psi_1^2\Bigg([-(\lambda_2^2-\lambda_1^2)^{1/2}-i\lambda_1](1-e^{2i\lambda_1\lambda_3\theta})e^{-i\lambda_1\zeta}\\
&\qquad\qquad \times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u^*}{\partial\zeta}\Bigg)\\
&\quad\times\Bigg([-(\lambda_2^2-\lambda_1^2)^{1/2}+i\lambda_1](1-e^{-2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}\\
&\qquad\qquad \times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\zeta}\Bigg)\\
&= 2\Psi_1^2\Bigg[\lambda_2^2(1-\cos 2\lambda_1\lambda_3\theta)\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\\
&\quad -4\sin\lambda_1\lambda_3\theta\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta\\
&\qquad \times\left(\lambda_1^2\sin\frac{2\pi\kappa\zeta}{N_\tau}-\frac{2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}}{N_\tau}\cos\frac{2\pi\kappa\zeta}{N_\tau}\right)\\
&\quad +2\left(\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1^2\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2\\
&\qquad +2\left(\sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}\sin\gamma_\kappa\theta\cos\frac{2\pi\kappa\zeta}{N_\tau}\right)^2\Bigg]
\end{aligned}
\tag{9}
$$

Equations (8), (9), and (5) have to be substituted into Eq.(4). The integration with respect to ζ is straight forward but lengthy. The calculations may be found in Section 6.7. The energy U of Eq.(4) is separated into a constant part U_{c} and a time-variable part $U_{\mathrm{v}}(\theta)$ that depends on sinusoidal functions of θ and has the time-average zero:

$$
U = U_{\mathrm{c}} + U_{\mathrm{v}}(\theta) \tag{10}
$$

For U_{c} we copy the three components of Eqs.(6.7-16), (6.7-20), and (6.7-12). The factor Ψ_1^2 has the dimension J/m=VAs/m:

$$
\begin{aligned}
U_{\mathrm{c}} &= U_{\mathrm{c}2}+U_{\mathrm{c}3}+U_{\mathrm{c}1}\\
&= \frac{L^2}{c\tau}\Psi_1^2 N_\tau\Bigg\{\left(2\lambda_1^2\lambda_3^2+\lambda_2^2+\frac{m_0^2c^2}{\hbar^2}\right)\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau}\\
&\quad +\lambda_1^2\lambda_3^2\sum_{\kappa=1}^{N_\tau}\frac{I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2}\left[1+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N_\tau}\right)^2+\frac{m_0^2c^4\tau^2}{\hbar^2}\right]\Bigg\}
\end{aligned}
$$

$$\doteq \frac{L^2}{c\tau}\Psi_1^2 N_\tau \lambda_1^2\lambda_3^2 \sum_{\kappa=1}^{N_\tau} \frac{I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \times \left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2c^4\tau^2}{\hbar^2}\right], \quad \text{for } N \gg 1$$

$$I_{\mathrm{T}}(\kappa/N_\tau) \text{ Eq.(4.2-44)}; \quad \lambda_1,\ \lambda_2,\ \lambda_3 \text{ Eq.(4.2-14)}; \quad \gamma_\kappa \text{ Eqs.(4.2-33)} \tag{11}$$

For the time-variable energy $U_{\mathrm{v}}(\theta)$ we obtain from Eqs.(6.7-17), (6.7-21) and (6.7-13) a very complicated expression that is greatly simplified for $N \gg 1$. We write it only for this case:

$$\begin{aligned} U_v(\theta) &= U_{\mathrm{v}2}(\theta) + U_{\mathrm{v}3}(\theta) + U_{\mathrm{v}1}(\theta) \\ &= \frac{L^2}{c\tau}\Psi_1^2 N_\tau \lambda_1^2\lambda_3^2 \sum_{\kappa=1}^{N_\tau} \frac{I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \\ &\times \left[\gamma_\kappa^2 - \lambda_1^2\lambda_3^2 - \lambda_1^2 - \left(\frac{2\pi\kappa}{N_\tau}\right)^2 - \frac{m_0^2c^4\tau^2}{\hbar^2}\right]\cos 2\gamma_\kappa\theta, \quad \text{for } N \gg 1 \end{aligned} \tag{12}$$

For the derivation of the Hamilton function $\mathcal{H}$ we need the constant energy U_{c} only since the average of the variable energy $U_{\mathrm{v}}(\theta)$ is zero. We normalize U_{c} in Eq.(11):

$$\frac{c\tau U_{\mathrm{c}}}{L^2\Psi_1^2 N_\tau} = \frac{cT\tau U_{\mathrm{c}}}{L^2\Psi_1^2 T N_\tau} = \frac{cTU_{\mathrm{c}}}{L^2\Psi_1^2}\frac{1}{N_\tau^2} = \frac{\mathcal{H}}{N_\tau^2} \tag{13}$$

$$\begin{aligned} \frac{cTU_{\mathrm{c}}}{L^2\Psi_1^2} &= \mathcal{H} = N_\tau^2\lambda_1^2\lambda_3^2 \sum_{\kappa=1}^{N_\tau} \frac{I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2}\left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2c^4\tau^2}{\hbar^2}\right] \\ \mathcal{H} &= \sum_{\kappa=1}^{N_\tau}\mathcal{H}_\kappa = \sum_{\kappa=1}^{N_\tau} d^2(\kappa) \\ d(\kappa) &= N_\tau\lambda_1\lambda_3\frac{I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2c^4\tau^2}{\hbar^2}\right]^{1/2} \end{aligned} \tag{14}$$

The component $\mathcal{H}_\kappa$ of the sum is rewritten as in Eq.(2.3-42):

$$\begin{aligned} \mathcal{H}_\kappa &= (2\pi\kappa)^2\frac{d(\kappa)}{2\pi\kappa}(\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta)\frac{d(\kappa)}{2\pi\kappa}(\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\ &= -2\pi i\kappa p_\kappa(\theta)q_\kappa(\theta) \end{aligned} \tag{15}$$

$$p_\kappa(\theta) = \sqrt{2\pi i\kappa}\,\frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta} \tag{16}$$

$$\dot{p}_\kappa = \frac{\partial p_\kappa}{\partial\theta} = (2\pi i\kappa)^{3/2}\frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta} = 2\pi i\kappa p_\kappa(\theta) \tag{17}$$

$$q_\kappa(\theta) = \sqrt{2\pi i\kappa}\,\frac{d(\kappa)}{2\pi\kappa}e^{-2\pi i\kappa\theta} \tag{18}$$

$$\dot{q}_\kappa = \frac{\partial q_\kappa}{\partial\theta} = -(2\pi i\kappa)^{3/2}\frac{d(\kappa)}{2\pi\kappa}e^{-2\pi i\kappa\theta} = -2\pi i\kappa q_\kappa(\theta) \tag{19}$$

The derivatives of $\mathcal{H}_\kappa$ with respect to q_κ and p_κ produce the proper relations for the components $\mathcal{H}_\kappa$ of the Hamilton function of Eq.(15):

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -2\pi i\kappa p_\kappa = -\dot{p}_\kappa \tag{20}$$

$$\frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = -2\pi i\kappa q_\kappa = +\dot{q}_\kappa \tag{21}$$

Equation (14) may be rewritten into the form of Eq.(2.3-51) by means of the following definitions that replace Eqs.(2.3-50)

$$a_\kappa = \frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta}, \quad a_\kappa^* = \frac{d(\kappa)}{2\pi\kappa}e^{-2\pi i\kappa\theta} \tag{22}$$

and we obtain:

$$\mathcal{H} = -i\sum_{\kappa=1}^{N_\tau}2\pi\kappa p_\kappa(\theta)q_\kappa(\theta) = \sum_{\kappa=1}^{N_\tau}(2\pi\kappa)^2 a_\kappa a_\kappa^* = \sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa}{T}\hbar b_\kappa b_\kappa^* = \sum_{\kappa=1}^{N_\tau}\mathcal{H}_\kappa \tag{23}$$

$$b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right) a_\kappa^*, \quad T = N_\tau\tau \tag{24}$$

For the quantization we follow Section 2.4 and the conventional procedure for quantization. Using the Schrödinger approach we obtain in analogy to Eqs.(2.4-9) and (2.4-18):

$$(b_\kappa^+ b_\kappa^-)\Phi = \frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = \frac{\mathcal{H}_\kappa T}{2\pi\kappa\hbar}\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi = \lambda_\kappa\Phi$$

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots,\ N_\tau \tag{25}$$

The upper limit N_τ for n was discussed in connection with Eq.(2.4-18). Again, there is no need for renormalization.

The energies $\mathsf{E}_{\kappa n}$ in Eqs.(2.4-18) and (25) are the same since the photons of a pure EM wave are the same as that of an EM wave interacting with bosons. What differs is the fraction of photons with a certain energy $U_{\mathrm{c}\kappa}(\kappa)$ in Section 2.3 or with energy $U_{\mathrm{c}\kappa}(\kappa)$ derived from Eq.(11):

$$U_{\mathrm{c}\kappa}(\kappa) = \frac{L^2\Psi_1^2}{cT} N_\tau^2 \lambda_1^2 \lambda_3^2 \frac{I_\mathrm{T}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2 c^4 \tau^2}{\hbar^2} \right] \quad (26)$$

The term $m_0^2 c^4/\hbar^2$ shows that one must get results different from those of the pure radiation field since a mass m_0 does not occur in the equations of a pure radiation field.

4.5 Plots for the Differential Theory

The energy $U_{\mathrm{c}\kappa}(\kappa)$ as function of the period number κ is defined by Eq. (4.4-26) with γ_κ from Eq.(4.2-33), $I_\mathrm{T}(\kappa/N_\tau)$ from Eq.(4.2-44) and λ_1^2, λ_2^2, λ_3^2 from Eq.(4.2-14):

$$\begin{gathered}
U_{\mathrm{c}\kappa}(\kappa) = \frac{L^2\Psi_1^2}{cT} N_\tau^2 \lambda_1^2 \lambda_3^2 \frac{I_\mathrm{T}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2 c^4 \tau^2}{\hbar^2} \right] \\
\gamma_\kappa = \left[(2\pi\kappa/N_\tau)^2 + m_0^2 c^4 \tau^2/\hbar^2 \right]^{1/2} \\
I_\mathrm{T}(\kappa/N_\tau) = \frac{2}{N_\tau} \frac{(2\pi\kappa/N_\tau)\{1 - \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]\}}{\lambda_2^2 - \lambda_1^2 + (2\pi\kappa/N_\tau)^2} \\
\lambda_2^2 - \lambda_1^2 = \tau^2(m_0^2 c^4 - e^2\phi_{\mathrm{e}0}^2)/\hbar^2,\ \lambda_1^2 = e^2 c^2 \tau^2 A_{\mathrm{m}0y}^2/\hbar^2 \\
\lambda_1^2\lambda_3^2 = e^2\tau^2\phi_{\mathrm{e}0}^2/\hbar^2,\ N_\tau = T/\tau
\end{gathered} \quad (1)$$

From Eq.(4.4-25) we get the energy of a photon with period number κ and a certain value of n:

$$\mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0\ \ldots\ N_\tau \quad (2)$$

The average value of $\mathsf{E}_{\kappa n}$ for all $N_\tau + 1$ values of n becomes:

$$\mathsf{E}_\kappa = \frac{2\pi\kappa\hbar}{T}\frac{1}{N_\tau + 1}\sum_{n=0}^{N_\tau}\left(n + \frac{1}{2}\right) = \frac{1}{2}(N_\tau + 1)\frac{2\pi\kappa\hbar}{T} \doteq \frac{1}{2}N_\tau\frac{2\pi\kappa\hbar}{T} \quad (3)$$

For a specific value of n the energy $U_{\mathrm{c}\kappa}(\kappa)$ requires the number $U_{\mathrm{c}\kappa}(\kappa)/\mathsf{E}_{\kappa n}$ of photons:

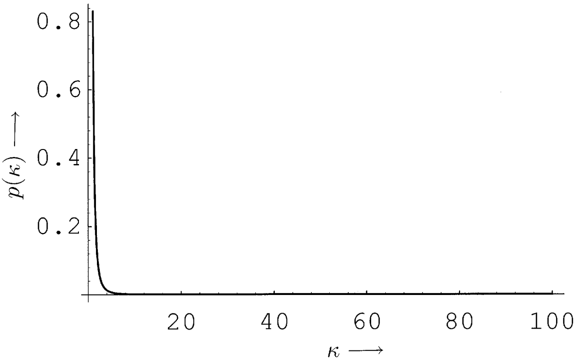

FIGURE 4.5-1. Plot of $p(\kappa)$ according to Eq.(14) for $N_\tau = 100$. The variable κ is treated as a continuous variable.

$$\frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{\kappa n}} = C_{rn} r(\kappa)$$
$$r(\kappa) = \frac{I_T^2(\kappa/N_\tau)}{\gamma_\kappa^2} \frac{1}{2\pi\kappa} \left[\gamma_\kappa^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2 + \left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \frac{m_0^2 c^4 \tau^2}{\hbar^2} \right]$$
$$C_{rn} = \frac{L^2 \Psi_1^2 N_\tau^2 \lambda_1^2 \lambda_3^2}{c\hbar} \left(n + \frac{1}{2} \right)^{-1} \tag{4}$$

If photons with various values of n are equally frequent we obtain the following relation instead of Eq.(4):

$$\frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{\kappa}} = C_r r(\kappa)$$
$$C_r = \frac{2L^2 \Psi_1^2 N_\tau \lambda_1^2 \lambda_3^2}{c\hbar} \tag{5}$$

The total number of photons is the sum over κ of Eqs.(4) or (5):

$$s_{rn} = \sum_{\kappa=1}^{N_\tau} \frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{\kappa n}} = C_{rn} \sum_{\kappa=1}^{N_\tau} r(\kappa) \tag{6}$$
$$s_r = \sum_{\kappa=1}^{N_\tau} \frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{\kappa}} = C_r \sum_{\kappa=1}^{N_\tau} r(\kappa) \tag{7}$$

The probability $p(\kappa)$ of a photon with period number κ is the same for Eqs.(4) and (5) since the constants C_{rn} and C_r drop out:

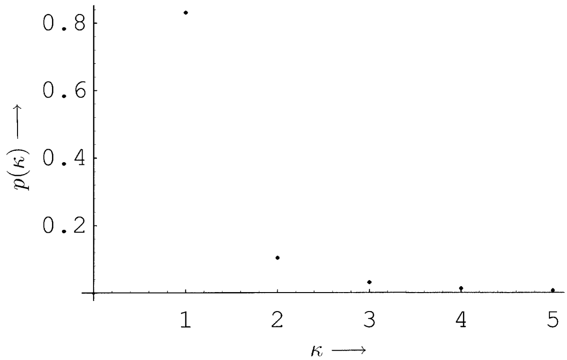

FIGURE 4.5-2. Plot of $p(\kappa)$ for $\kappa = 1, 2, 3, 4, 5$ according to Eq.(14) for $N_\tau = 100$.

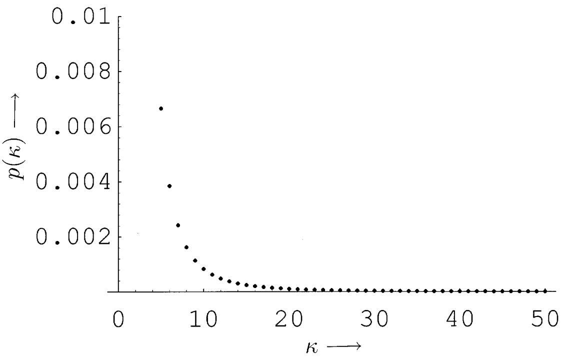

FIGURE 4.5-3. Plot of $p(\kappa)$ for $\kappa = 5, 6, \ldots, 50$ according to Eq.(14) for $N_\tau = 100$. The vertical scale is enlarged by almost a factor of 100 compared with Figs.4.5-1 and 4.5-2. The points for $\kappa = 1, 2, 3, 4$ are outside the plotting range.

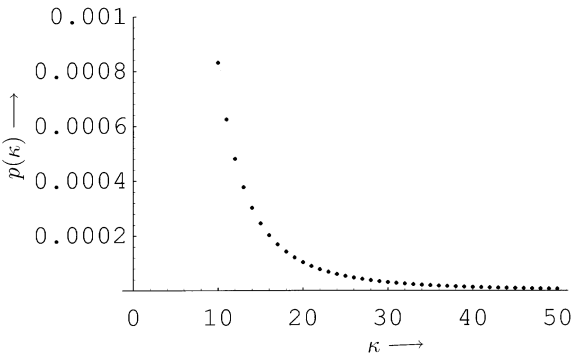

FIGURE 4.5-4. Plot of $p(\kappa)$ for $\kappa = 10, 11, \ldots, 50$ according to Eq.(14) for $N_\tau = 100$. The vertical scale is enlarged by a factor of 10 compared with Fig.4.5-3. The points for $\kappa = 1, 2, \ldots, 9$ are outside the plotting range.

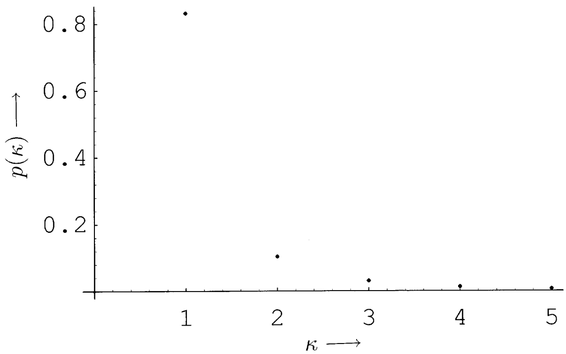

FIGURE 4.5-5. Plot of $p(\kappa)$ for $\kappa = 1, 2, 3, 4, 5$ according to Eq.(14) for $N_\tau = 50$. The plot differs from Fig.4.5-2 but the difference is not visible without magnification.

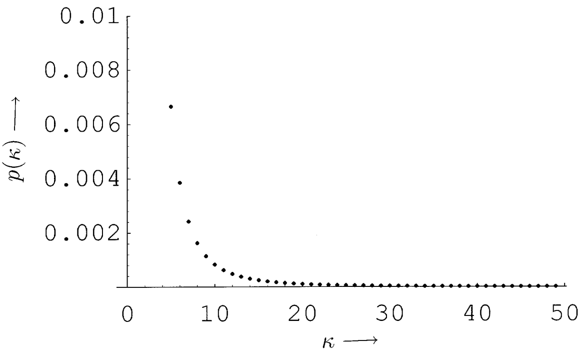

FIGURE 4.5-6. Plot of $p(\kappa)$ for $\kappa = 5, 6, \ldots, 49$ according to Eq.(14) for $N_\tau = 50$. The vertical scale is enlarged by almost a factor of 100 compared with Fig.4.5-5. The values of $p(\kappa)$ for $\kappa = 1, 2, 3, 4$ are outside the plotting range.

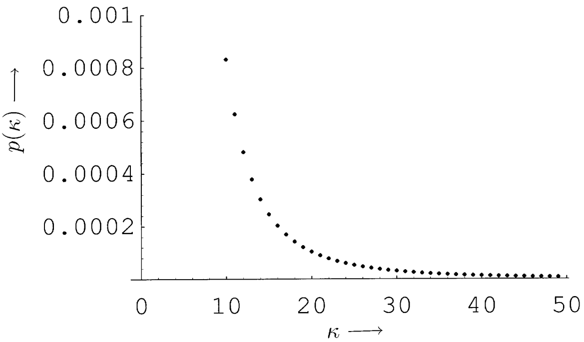

FIGURE 4.5-7. Plot of $p(\kappa)$ for $\kappa = 10, 11, \ldots, 49$ according to Eq.(14) for $N_\tau = 50$. The vertical scale is enlarged by a factor of 10 compared with Fig.4.5-6 and the points for $\kappa = 1, 2, \ldots, 9$ are outside the plotting range.

$$p(\kappa) = \frac{r(\kappa)}{\sum_{\kappa=1}^{N_\tau} r(\kappa)} \tag{8}$$

Various simplifications can be made for the computation of $r(\kappa)$ and $p(\kappa)$. We develop a few approximations for $\kappa \geq 1$:

$$\gamma_\kappa \doteq 2\pi\kappa/N \quad \text{for } \tau \ll \frac{2\pi\kappa}{N_\tau}\frac{\hbar}{m_0c^2} < \frac{h}{m_0c^2} \tag{9}$$

$$(\lambda_1^2 - \lambda_2^2)^{1/2}N_\tau = N_\tau\tau(m_0^2c^4 - e^2\phi_{\mathrm{e}0}^2)^{1/2}/\hbar \gg 1$$
$$\text{for } T \gg \frac{\hbar}{(m_0^2c^4 - e^2\phi_{\mathrm{e}0}^2)^{1/2}} \doteq \frac{1}{2\pi}\frac{h}{m_0c^2} \tag{10}$$

$$I_{\mathrm{T}}(\kappa/N_\tau) \doteq \frac{1}{\pi\kappa} \tag{11}$$

$$r(\kappa) \doteq \left(\frac{1}{\pi\kappa}\right)^3 \tag{12}$$

$$\sum_{\kappa=1}^{N_\tau}\left(\frac{1}{\pi\kappa}\right)^3 = \frac{1}{\pi^3}\sum_{\kappa=1}^{N_\tau}\frac{1}{\kappa^3} \doteq \frac{1.20205}{\pi^3}, \qquad N_\tau \gg 1 \tag{13}$$

$$p(\kappa) = \left(\frac{1}{\pi\kappa}\right)^3\frac{\pi^3}{1.20205} = \frac{1}{1.20205\kappa^3} \tag{14}$$

We note that Eq.(9) states that τ can be arbitrarily small but finite while Eq.(10) states that T can be arbitrarily large but finite. These are two assumptions introduced in Section 2.1 from Eqs.(2.1-79) to (2.1-83). The Compton period h/m_0c^2 equals 2.96241×10^{-23} s for the pions π^+ and π^-.

The results of Chapter 4 may now be represented by a number of plots. Figure 4.5-1 shows $p(\kappa)$ according to Eq.(14) for $N_\tau = 100$. This plot treats κ as a continuous variable.

A better representation is used in Fig.4.5-2 that shows $p(\kappa)$ only for $\kappa = 1$, 2, 3, 4, 5 but not for continuous values of κ in the same range.

Figure 4.5-3 extends the discrete representation of $p(\kappa)$ to the range $\kappa = 5$, 6, ... , 50. The vertical scale is enlarged by almost a factor 100 compared with Figs.4.5-1 and 4.5-2. This larger scale puts $p(1)$ to $p(4)$ outside the plotting range.

A further enlargement of the vertical scale by a factor 10 is shown in Fig.4.5-4. The probabilities $p(1)$ to $p(9)$ are now outside the plotting range. The need for the enlargements of the vertical scale in Figs.4.5-3 and 4.5-4 will be explained in Section 5.5.

Figures 4.5-5, 4.5-6 and 4.5-7 are almost a repetition of Figs.4.5-2, 4.5-3 and 4.5-4, but the parameter $N_\tau = 100$ has been replaced by $N_\tau = 50$ and only the probabilities $p(\kappa)$ for $\kappa = 1, 2, \dots, 49$ are plotted. There is a difference between Figs.4.5-2 to 4.5-4 and Figs.4.5-5 to 4.5-7, but it is too small to be recognizable without magnification. The main reason for showing the plots for $N_\tau = 50$ will be explained in Section 5.5. Here, we only note that the finer resolution of the plots with $N_\tau = 100$ compared with the plots for $N_\tau = 50$ has a minimal effect.

5 Difference Equation for the Klein-Gordon Field

5.1 Klein-Gordon Difference Equation

For the derivation of the Klein-Gordon difference equation we start from Eq.(4.1-2):

$$(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_{\mathrm{e}})^2 = -m_0^2c^2$$

$$(p_x - eA_{\mathrm{m}x})^2 + (p_y - eA_{\mathrm{m}y})^2 + (p_z - eA_{\mathrm{m}z})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_{\mathrm{e}})^2 = -m_0^2c^2 \quad (1)$$

Instead of the differential operators of Eq.(4.1-3) we use difference operators. Since the transition from first to second order difference operators is not as simple as in the case of differential operators we must define them both according to Eqs.(3.1-2) to (3.1-4):

$$p_{x_j}\Psi \to \frac{\hbar}{i}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}x_j} = \frac{\hbar}{i}\frac{\Psi(x_j + \Delta x_j) - \Psi(x_j - \Delta x_j)}{2\Delta x_j} \quad (2)$$

$$\mathcal{H}\Psi \to -\frac{\hbar}{i}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}t} = -\frac{\hbar}{i}\frac{\Psi(t + \Delta t) - \Psi(t - \Delta t)}{2\Delta t} \quad (3)$$

$$p_{x_j}^2\Psi \to -\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}x_j^2} = -\hbar^2\frac{\Psi(x_j + \Delta x_j) - 2\Psi(x_j) + \Psi(x_j - \Delta x_j)}{(\Delta x_j)^2} \quad (4)$$

$$\mathcal{H}^2\Psi \to -\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}t^2} = -\hbar^2\frac{\Psi(t + \Delta t) - 2\Psi(t) + \Psi(t - \Delta t)}{(\Delta t)^2} \quad (5)$$

For the normalized variables

$$\theta = t/\Delta t,\ \zeta_j = x_j/c\Delta t,\ N = T/\Delta t \quad (6)$$

we follow the simplified notation of Eqs.(3.1-3) and (3.1-4):

$$\frac{\hbar}{i}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}\zeta_j} = \frac{\hbar}{2i}[\Psi(\zeta_j+1) - \Psi(\zeta_j-1)] \tag{7}$$

$$-\frac{\hbar}{i}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}\theta} = -\frac{\hbar}{2i}[\Psi(\theta+1) - \Psi(\theta-1)] \tag{8}$$

$$-\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}\zeta_j^2} = -\hbar^2[\Psi(\zeta_j+1) - 2\Psi(\zeta_j) + \Psi(\zeta_j-1)] \tag{9}$$

$$-\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}\theta^2} = -\hbar^2[\Psi(\theta+1) - 2\Psi(\theta) + \Psi(\theta-1)] \tag{10}$$

These definitions suffice to rewrite all equations in Section 4.1 from Eq.(4.1-3) to Eq.(4.1-20). From Eq.(4.1-21) on we encounter integrations in addition to differentiations. To remain consistent we have no choice but to substitute summations for integrations as shown in Table 6.2-1:

$$\int \varphi(x)dx = \int^x \varphi(\nu)d\nu \to \mathop{S}^{x} \varphi(\nu+1)\Delta\nu \tag{11}$$

There are now enough substitutions to be able to rewrite all of Section 4.1 in terms of the calculus of finite differences. We may write the matrix equation (4.1-25) by substituting $\tilde{\Delta}/\tilde{\Delta}x$ for $\partial/\partial x$ to $\tilde{\Delta}/\tilde{\Delta}t$ for $\partial/\partial t$, but must observe that the squares $\tilde{\Delta}^2/\tilde{\Delta}x^2$ to $\tilde{\Delta}^2/\tilde{\Delta}t^2$ are replaced according to Eqs.(4) and (5). Instead of doing this rewriting we go directly to Section 4.2 and reduce the number of spatial variables to one as in Eq.(4.2-1):

$$x_j = y,\ \Psi_{\mathrm{x}0x_j} = \Psi_{\mathrm{x}0y} = \Psi \tag{12}$$

Next we rewrite Eq.(4.2-2) for finite differences Δy and Δt. To simplify the notation we do not write a variable that is not being changed, e.g, we write $\Psi(y+\Delta y) - \Psi(y-\Delta y)$ rather than $\Psi(y+\Delta y, t) - \Psi(y-\Delta y, t)$:

$$\begin{aligned}
&\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)^2\Psi \to \left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}y} - eA_{\mathrm{m}y}\right)^2\Psi \\
&\quad = -\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}y^2} + 2i\hbar eA_{\mathrm{m}y}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}y} + \left(e^2A_{\mathrm{m}y}^2 + i\hbar e\frac{\tilde{\Delta}A_{\mathrm{m}y}}{\tilde{\Delta}y}\right)\Psi \\
&\quad = -\hbar^2\frac{\Psi(y+\Delta y) - 2\Psi(y) + \Psi(y-\Delta y)}{(\Delta y)^2} + 2i\hbar eA_{\mathrm{m}y}\frac{\Psi(y+\Delta y) - \Psi(y-\Delta y)}{2\Delta y} \\
&\qquad + \left(e^2A_{\mathrm{m}y}^2 + i\hbar\frac{A_{\mathrm{m}y}(y+\Delta y) - A_{\mathrm{m}y}(y-\Delta y)}{2\Delta y}\right)\Psi
\end{aligned} \tag{13}$$

Equation (4.2-3) is also rewritten:

$$\begin{aligned}
\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+e\phi_{\mathrm{e}}\right)^2\Psi &\to \left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t}+e\phi_{\mathrm{e}}\right)^2\Psi\\
&= -\hbar^2\frac{\tilde{\Delta}^2\Psi}{\tilde{\Delta}t^2}-2i\hbar e\phi_{\mathrm{e}}\frac{\tilde{\Delta}\Psi}{\tilde{\Delta}t}+\left(e^2\phi_{\mathrm{e}}^2-i\hbar e\frac{\tilde{\Delta}\phi_{\mathrm{e}}}{\tilde{\Delta}t}\right)\Psi\\
&= -\hbar^2\frac{\Psi(t+\Delta t)-2\Psi(t)+\Psi(t-\Delta t)}{(\Delta t)^2}-2i\hbar e\phi_{\mathrm{e}}\frac{\Psi(t+\Delta t)-\Psi(t-\Delta t)}{2\Delta t}\\
&\qquad+\left(e^2\phi_{\mathrm{e}}^2-i\hbar e\frac{\phi_{\mathrm{e}}(t+\Delta t)-\phi_{\mathrm{e}}(t-\Delta t)}{2\Delta t}\right)\Psi \qquad (14)
\end{aligned}$$

Substitution of these two equations into Eq.(4.1-26) yields a difference equation instead of the differential equation (4.2-4). We write here both variables y and t of Ψ, $A_{\mathrm{m}y}$ and ϕ_{e}:

$$\begin{aligned}
&\frac{\Psi(y+\Delta y,t)-2\Psi(y,t)+\Psi(y-\Delta y,t)}{(\Delta y)^2}\\
&\qquad-\frac{1}{c^2}\frac{\Psi(y,t+\Delta t)-2\Psi(y,t)+\Psi(y,t-\Delta t)}{(\Delta t)^2}\\
&-2i\frac{e}{\hbar}\left(A_{\mathrm{m}y}(y,t)\frac{\Psi(y+\Delta y,t)-\Psi(y-\Delta y,t)}{2\Delta y}\right.\\
&\qquad\left.+\frac{1}{c^2}\phi_{\mathrm{e}}(y,t)\frac{\Psi(y,t+\Delta t)-\Psi(y,t-\Delta t)}{2\Delta t}\right)\\
&-\frac{e^2}{\hbar^2}\left[A_{\mathrm{m}y}^2(y,t)-\frac{1}{c^2}\phi_{\mathrm{e}}^2(y,t)+i\frac{\hbar}{e}\left(\frac{A_{\mathrm{m}y}(y+\Delta y,t)-A_{\mathrm{m}y}(y-\Delta y,t)}{2\Delta y}\right.\right.\\
&\qquad\left.\left.+\frac{1}{c^2}\frac{\phi_{\mathrm{e}}(y,t+\Delta t)-\phi_{\mathrm{e}}(y,t-\Delta t)}{2\Delta t}\right)+\frac{m_0^2c^2}{e^2}\right]\Psi(y,t)=0 \qquad (15)
\end{aligned}$$

Equation (15) is a partial difference equation of Ψ with variable coefficients since ϕ_{e} and $A_{\mathrm{m}y}$ are generally functions of location and time. A great deal of effort went into the solution of this and related difference equations. A number of interesting results were obtained but the calculations could never be carried through to the end because of mathematical difficulties (Harmuth 1977, 436-482; 1989, 269-328 [Russian edition], 230-296 [English edition]). We follow the text after Eq.(4.2-4) and restrict ourselves to potentials $A_{\mathrm{m}y}$ and ϕ_{e} that depend on location and time only in first or higher order of the dimension-free constant α_{e} of Eq.(1.3-37). We may again use Eqs.(4.2-5) to (4.2-10) since they are written with operators that may explicitly contain either differentials or finite differences. If we use again the representation of $\mathbf{A}_{\mathrm{m}}$ and ϕ_{e} as the sum of a constant and a variable term multiplied by α_{e} as in Eq.(4.2-5) we get:

$$\mathbf{A}_{\mathrm{m}} = \mathbf{A}_{\mathrm{m0}} + \alpha_{\mathrm{e}}\mathbf{A}_{\mathrm{m1}}(\mathbf{r}, t),\ A_{\mathrm{m}x_j} = A_{\mathrm{m0}x_j} + \alpha_{\mathrm{e}} A_{\mathrm{m1}x_j}(x_j, t)$$
$$\phi_{\mathrm{e}} = \phi_{\mathrm{e0}} + \alpha_{\mathrm{e}}\phi_{\mathrm{e1}}(x_j, t), \quad x_j = x,\ y,\ z; \quad \Psi_{\mathrm{x}} = \Psi_{\mathrm{x0}} + \alpha_{\mathrm{e}}\Psi_{\mathrm{x1}} \tag{16}$$

With the help of Eqs.(4.2-6) to (4.2-8) we obtain again Eqs.(4.2-9) and (4.2-10):

$$\left((\mathbf{p} - e\mathbf{A}_{\mathrm{m0}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 + m_0^2c^2\right)\Psi_{\mathrm{x0}} = 0 \tag{17}$$

$$\left((\mathbf{p} - e\mathbf{A}_{\mathrm{m0}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e0}})^2 + m_0^2c^2\right)\Psi_{\mathrm{x1}} = -\Bigg[2[(\mathbf{p} - e\mathbf{A}_{\mathrm{m0}})^2 + m_0^2c^2]Q$$
$$- \frac{1}{c^2}\left((\mathcal{H}_x - e\phi_{\mathrm{e0}})\frac{\mathcal{L}_{\mathrm{cx}}}{\alpha_{\mathrm{e}}} + \frac{\mathcal{L}_{\mathrm{cx}}}{\alpha_{\mathrm{e}}}(\mathcal{H}_x - e\phi_{\mathrm{e0}})\right) - e[\mathbf{A}_{\mathrm{m1}} \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m0}})$$
$$+ (\mathbf{p} - e\mathbf{A}_{\mathrm{m0}}) \cdot \mathbf{A}_{\mathrm{m1}}] + \frac{e}{c^2}[\phi_{\mathrm{e1}}(\mathcal{H}_x - e\phi_{\mathrm{e0}}) + (\mathcal{H}_x - e\phi_{\mathrm{e0}})\phi_{\mathrm{e1}}]\Bigg]\Psi_{\mathrm{x0}} \tag{18}$$

We only need to rewrite the differential equations (4.2-11) and (4.2-12) as difference equations to obtain Eqs.(17) and (18) expressed as difference equations:

$$\frac{\Psi(y + \Delta y, t) - 2\Psi(y, t) + \Psi(y - \Delta y, t)}{(\Delta y)^2}$$
$$- \frac{1}{c^2}\frac{\Psi(y, t + \Delta t) - 2\Psi(y, t) + \Psi(y, t - \Delta t)}{(\Delta t)^2}$$
$$- 2i\frac{e}{\hbar}\left(A_{\mathrm{m0}y}\frac{\Psi(y + \Delta y, t) - \Psi(y - \Delta y, t)}{2\Delta y}\right.$$
$$\left. + \frac{1}{c^2}\phi_{\mathrm{e0}}\frac{\Psi(y, t + \Delta t) - \Psi(y, t - \Delta t)}{2\Delta t}\right)$$
$$- \frac{e^2}{\hbar^2}\left(A_{\mathrm{m0}y}^2 - \frac{1}{c^2}\phi_{\mathrm{e0}}^2 + \frac{m_0^2c^2}{e^2}\right)\Psi(y, t) = 0, \quad \Psi = \Psi_{\mathrm{x0}} \tag{19}$$

The difference equation representing Eq.(18) is written in a more compact form corresponding to Eq.(4.2-12) by means of the definitions of Eqs.(3.1-2) and (3.1-4):

$$\begin{aligned}
&\left[\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right)^2 - \frac{1}{c^2}\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t} + e\phi_{\mathrm{e}0}\right)^2 + m_0^2c^2\right]\Psi_{\mathrm{x}1} \\
&\quad = -\left\{2\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right)^2\left[m_0^2c^2 + \left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right)^2\right]\right. \\
&\qquad\qquad \times\left[m_0^2c^2 - \frac{3}{2}\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right)^2\right] \\
&\quad + \frac{1}{c^2}\left[\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t} + e\phi_{\mathrm{e}0}\right)(\mathfrak{L}_{\mathrm{cx}1j} + \mathfrak{L}_{\mathrm{cx}2j} + \mathfrak{L}_{\mathrm{cx}3j} + \mathfrak{L}_{\mathrm{cx}4j} + \mathfrak{L}_{\mathrm{cx}5j})\right. \\
&\quad \left. + (\mathfrak{L}_{\mathrm{cx}1j} + \mathfrak{L}_{\mathrm{cx}2j} + \mathfrak{L}_{\mathrm{cx}3j} + \mathfrak{L}_{\mathrm{cx}4j} + \mathfrak{L}_{\mathrm{cx}5j})\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t} + e\phi_{\mathrm{e}0}\right)\right] \\
&\quad - e\left[A_{\mathrm{m}1x_j}\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right) + \left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}x_j} - eA_{\mathrm{m}0x_j}\right)A_{\mathrm{m}1x_j}\right] \\
&\qquad \left. - \frac{e}{c^2}\left[\phi_{\mathrm{e}1}\left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t} + e\phi_{\mathrm{e}0}\right) + \left(\frac{\hbar}{i}\frac{\tilde{\Delta}}{\tilde{\Delta}t} + e\phi_{\mathrm{e}0}\right)\phi_{\mathrm{e}1}\right]\right\}\Psi_{\mathrm{x}0} \qquad (20)
\end{aligned}$$

If we substitute Eqs.(13) and (14) in the first line of Eq.(20) and write $x_j = y$, $A_{\mathrm{m}0y}$ for $A_{\mathrm{m}y}$, $\phi_{\mathrm{e}0}$ for ϕ_{e}, and $\Psi_{\mathrm{x}1}$ for $\Psi = \Psi_{\mathrm{x}0}$ we obtain again Eq.(19). Hence, both the general solution of $\Psi_{\mathrm{x}0}$ and the homogeneous solution of $\Psi_{\mathrm{x}1}$ are represented by the same partial difference equation with constant coefficients. We need only a particular solution of the inhomogeneous equation with variable coefficients represented by Eq.(20).

We make the transition from the non-normalized variables y, t to the variables normalized by Δt rather than τ as in Eq.(4.2-13):

$$\theta = t/\Delta t,\ \zeta = y/c\Delta t \qquad (21)$$

This rewriting is most easily done by starting from Eq.(4.2-14), using Eqs.(3.1-3), (3.1-4) and the modification

$$\frac{\partial V_{\mathrm{e}}}{\partial\zeta} \to \frac{1}{2}[V_{\mathrm{e}}(\zeta+1,\theta) - V_{\mathrm{e}}(\zeta-1,\theta)] \qquad (22)$$

of Eq.(3.1-3). The following normalized partial difference equation is obtained:

$$\begin{aligned}
&[\Psi(\zeta{+}1,\theta) - 2\Psi(\zeta,\theta) + \Psi(\zeta{-}1,\theta)] - [\Psi(\zeta,\theta{+}1) - 2\Psi(\zeta,\theta) + \Psi(\zeta,\theta{-}1)] \\
&\quad - i\lambda_1\big\{[\Psi(\zeta{+}1,\theta) - \Psi(\zeta{-}1,\theta)] + \lambda_3[\Psi(\zeta,\theta{+}1) - \Psi(\zeta,\theta{-}1)]\big\} - \lambda_2^2\Psi(\zeta,\theta) = 0
\end{aligned}$$

$$\lambda_1 = \frac{ec\Delta t A_{\mathrm{m}0y}}{\hbar}, \quad \lambda_3 = \frac{\phi_{\mathrm{e}0}}{cA_{\mathrm{m}0y}}, \quad N = \frac{T}{\Delta t}$$
$$\lambda_2^2 = \lambda_1^2\left(1 - \frac{\phi_{\mathrm{e}0}^2}{c^2A_{\mathrm{m}0y}^2} + \frac{m_0^2c^2}{e^2A_{\mathrm{m}0y}^2}\right) = \frac{(\Delta t)^2}{\hbar^2}(m_0^2c^4 - e^2\phi_{\mathrm{e}0}^2 + e^2c^2A_{\mathrm{m}0y}^2)$$
$$\lambda_2^2 - \lambda_1^2 = (\Delta t/\hbar^2)^2(m_0^2c^4 - e^2\phi_{\mathrm{e}0}^2) > 0 \text{ for } m_0c^2 > e\phi_{\mathrm{e}0}$$
$$\lambda_1^2\lambda_3^2 + \lambda_2^2 = (\Delta t/\hbar)^2(m_0^2c^4 + e^2c^2A_{\mathrm{m}0y}^2), \ \lambda_1^2\lambda_3^2 + \lambda_2^2 - \lambda_1^2 = (m_0c^2\Delta t/\hbar)^2$$
$$\lambda_1^2\lambda_3^2 - \lambda_2^2 = -(\Delta t/\hbar)^2(m_0^2c^4 - 2e^2\phi_{\mathrm{e}0}^2 + e^2c^2A_{\mathrm{m}0y}^2) \tag{23}$$

We look for a solution excited by a step function as boundary condition as in Eqs.(4.2-15) or (3.1-6):

$$\begin{aligned}\Psi(0,\theta) = \Psi_0 S(\theta) &= 0 \quad \text{for } \theta < 0\\ &= \Psi_0 \quad \text{for } \theta \geq 0\end{aligned} \tag{24}$$

The boundary condition of Eq.(3.1-7) is not needed if N is finite. Any value of $\Psi(\zeta,\theta)$ must be finite, not only $\Psi(N,\theta)$. The initial condition of Eq.(3.1-8) is needed:

$$\Psi(\zeta,0) = 0 \tag{25}$$

In order to obtain a second initial condition we proceed according to the text following Eq.(3.1-8):

1. The three terms $\Psi(\zeta+1,\theta)$, $\Psi(\zeta,\theta)$, $\Psi(\zeta-1,\theta)$ of the difference quotient of second order with respect to ζ in Eq.(23) must be zero for $\theta = 0$ due to Eq.(25).
2. The difference quotient of second order with respect to θ in Eq.(23) does not exist for $\theta = 0$ since the functional value $\Psi(\zeta,-1)$ does not exist. A difference quotient of second order exists only for $\theta \geq 1$.
3. The terms $\Psi(\zeta+1,\theta)$ and $\Psi(\zeta-1,\theta)$ of the first order difference quotient with respect to ζ in Eq.(23) are zero for $\theta = 0$ due to Eq.(25).
4. The last term $\Psi(\zeta,\theta)$ in Eq.(23) is also zero for $\theta = 0$ because of Eq.(25).
5. What remains of Eq.(23) for $\theta = 0$ is the first order difference quotient with respect to θ:

$$\frac{1}{2}[\Psi(\zeta,+1) - \Psi(\zeta,-1)] = 0$$

This difference quotient does not exist either but it can be replaced by the non-symmetric difference quotient

$$\Psi(\zeta,\theta+1) - \Psi(\zeta,\theta) = 0 \quad \text{for } \theta = 0, \ \zeta \geq 0 \tag{26}$$

The two non-symmetric difference quotients of first order have been discussed in the text following Eq.(3.1-9).

We use again the ansatz of Eq.(3.1-10) to find a general solution from a steady state solution $F(\zeta)$ plus a deviation $w(\zeta,\theta)$ from $F(\zeta)$:

$$\Psi(\zeta,\theta) = \Psi_0[F(\zeta) + w(\zeta,\theta)] \tag{27}$$

Substitution of $F(\zeta)$ into Eq.(23) yields an ordinary difference equation with the variable ζ:

$$[F(\zeta+1) - 2F(\zeta) + F(\zeta-1)] - i\lambda_1[F(\zeta+1) - F(\zeta-1)] - \lambda_2^2 F(\zeta) = 0 \tag{28}$$

Following Eq.(3.1-12) we use again the ansatz

$$F(\zeta) = A_1 v^\zeta,\ F(\zeta+1) = A_1 v^{\zeta+1},\ F(\zeta-1) = A_1 v^{\zeta-1} \tag{29}$$

and obtain an equations for v:

$$\begin{aligned}(1-i\lambda_1)v^2 - (2+\lambda_2^2)v + 1 + i\lambda_1 &= 0\\ v_{1,2} &= \frac{1+i\lambda_1}{2(1+\lambda_1^2)}\left\{2+\lambda_2^2 \pm [4(\lambda_2^2-\lambda_1^2)+\lambda_2^4]^{1/2}\right\}\\ &\doteq (1+i\lambda_1)\left[1 \pm (\lambda_2^2-\lambda_1^2)^{1/2}\right], \quad \lambda_2^2,\, \lambda_1^2 \ll 1\end{aligned} \tag{30}$$

We get two complex solutions:

$$\begin{aligned}v_1 &= 1 - (\lambda_2^2-\lambda_1^2)^{1/2} + i\lambda_1\left[1-(\lambda_2^2-\lambda_1^2)^{1/2}\right]\\ &\doteq \exp\left[-(\lambda_2^2-\lambda_1^2)^{1/2}\right]e^{i\lambda_1}\end{aligned} \tag{31}$$

$$\begin{aligned}v_2 &= 1 + (\lambda_2^2-\lambda_1^2)^{1/2} + i\lambda_1\left[1+(\lambda_2^2-\lambda_1^2)^{1/2}\right]\\ &\doteq \exp\left[(\lambda_2^2-\lambda_1^2)^{1/2}\right]e^{i\lambda_1}\end{aligned} \tag{32}$$

Following Eqs.(4.2-22) and (4.2-23) we choose again the values of Eq. (4.2-23) and obtain:

$$\begin{aligned}F(\zeta) &= A_{10}v_1^\zeta + A_{20}v_2^\zeta\\ F(0) &= A_{10} + A_{11} = 1, \quad \lambda_2^2 - \lambda_1^2 > 0\\ F(\zeta) &= \exp\left[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta\right]e^{i\lambda_1\zeta}\end{aligned} \tag{33}$$

The substitution of $F(\zeta)$ into Eq.(27) transforms the boundary condition of Eq.(24) for Ψ into a homogeneous boundary condition for w, which is the purpose of Eq.(27):

$$\begin{aligned} \Psi_0[1+w(0,\theta)] &= \Psi_0 \quad \text{for } \theta \geq 0 \\ w(0,\theta) &= 0 \qquad \text{for } \theta \geq 0 \end{aligned} \tag{34}$$

The initial conditions of Eqs.(25) and (26) yield:

$$F(\zeta)+w(\zeta,0)=0, \quad w(\zeta,0)=-F(\zeta) \quad \text{for } \theta \geq 0 \tag{35}$$

$$w(\zeta,\theta+1)-w(\zeta,\theta)=0 \qquad \text{for } \theta = 0 \tag{36}$$

Substitution of $w(\zeta,\theta)$ into Eq.(23) yields the same equation with Ψ replaced by w:

$$\begin{aligned} &[w(\zeta+1,\theta)-2w(\zeta,\theta)+w(\zeta-1,\theta)]-[w(\zeta,\theta+1)-2w(\zeta,\theta)+w(\zeta,\theta-1)] \\ &-i\lambda_1\big\{[w(\zeta+1,\theta)-w(\zeta-1,\theta)]+\lambda_3[w(\zeta,\theta+1)-w(\zeta,\theta-1)]\big\}-\lambda_2^2 w(\zeta,\theta)=0 \end{aligned} \tag{37}$$

Particular solutions $w_\kappa(\zeta,\theta)$ of this equation can be obtained by the extension of Bernoulli's product method for the separation of variables from partial differential equations to partial difference equations:

$$w_\kappa(\zeta,\theta)=\phi(\zeta)\psi(\theta) \tag{38}$$

Substitution of $w_\kappa(\zeta,\theta)$ for $w(\zeta,\theta)$ in Eq.(37) yields:

$$\begin{aligned} &[\phi(\zeta+1)\psi(\theta)-2\phi(\zeta)\psi(\theta)+\phi(\zeta-1)\psi(\theta)] \\ &\qquad -[\phi(\zeta)\psi(\theta+1)-2\phi(\zeta)\psi(\theta)+\phi(\zeta)\psi(\theta-1)] \\ &-i\lambda_1\Big\{[\phi(\zeta+1)\psi(\theta)-\phi(\zeta-1)\psi(\theta]+\lambda_3[\phi(\zeta)\psi(\theta+1)-\phi(\zeta)\psi(\theta-1)]\Big\} \\ &\qquad -\lambda_2^2\phi(\theta)\psi(\theta)=0 \end{aligned} \tag{39}$$

Multiplication with $1/\phi(\zeta)\psi(\theta)$ and separation of the variables yields in analogy to the procedure used for partial differential equations the following equation:

$$\begin{aligned} &\frac{1}{\phi(\zeta)}\Big\{[\phi(\zeta+1)-2\phi(\zeta)+\phi(\zeta-1)]-i\lambda_1[\phi(\zeta+1)-\phi(\zeta-1)]\Big\} \\ &=\frac{1}{\psi(\theta)}\Big\{[\psi(\theta+1)-2\psi(\theta)+\psi(\theta-1)]+i\lambda_1\lambda_3[\psi(\theta+1)-\psi(\theta-1)]\Big\}+\lambda_2^2 \\ &=-(2\pi\rho_\kappa/N)^2 \end{aligned} \tag{40}$$

We have written a constant $-(2\pi\rho_\kappa/N)^2$ at the end of the equation since a function of ζ can be equal to a function of θ for any ζ and θ only if they are equal to a constant. The division by N will permit the use of an orthogonality interval of length N rather than 1 later on, in analogy to the use of N_τ in Eq.(4.2-29). We obtain two ordinary difference equations from Eq.(40):

$$[\phi(\zeta+1)-2\phi(\zeta)+\phi(\zeta-1)]-i\lambda_1[\phi(\zeta+1)-\phi(\zeta-1)] \\ +(2\pi\rho_\kappa/N)^2\phi(\zeta)=0 \quad (41)$$

$$[\psi(\theta+1)-2\psi(\theta)+\psi(\theta-1)]+i\lambda_1\lambda_3[\psi(\theta+1)-\psi(\theta-1)] \\ +[(2\pi\rho_\kappa/N)^2+\lambda_2^2]\psi(\theta)=0 \quad (42)$$

For the solution of Eq.(41) we proceed in analogy to the ansatz for $F(\zeta)$ in Eq.(29)

$$\phi(\zeta)=A_2v^\zeta,\ \phi(\zeta+1)=A_2v^{\zeta+1},\ \phi(\zeta-1)=A_2v^{\zeta-1} \quad (43)$$

and obtain an equation for v:

$$(1-i\lambda_1)v^2-[2-(2\pi\rho_\kappa/N)^2]v+1+i\lambda_1=0$$

$$v_{3,4}=\frac{1+i\lambda_1}{2(1+\lambda_1^2)}\left\{2-\left(\frac{2\pi\kappa\rho_\kappa}{N}\right)^2 \right. \\ \left. \pm 2i\,\frac{2\pi\rho_\kappa}{N}\left[1-\frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\frac{\lambda_1^2}{(2\pi\rho_\kappa/N)^2}\right]^{1/2}\right\} \quad (44)$$

This equation equals Eq.(3.1-27) for $\lambda_1=0$. We obtain the following solutions of Eq.(44) in first order of Δt. As in Eqs.(3.1-28) and (3.1-29) only the range $(2\pi\rho_\kappa/N)^2\le 4$ is of interest. The term containing λ_1^2 is of order $O(\Delta t)^2$ and thus of no consequence:

$$\text{for } (2\pi\rho_\kappa/N)^2<4+\frac{4\lambda_1^2}{(2\pi\rho_\kappa/N)^2}$$

$$v_3=1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2-\lambda_1\frac{2\pi\rho_\kappa}{N}\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2} \\ +i\left[\frac{2\pi\rho_\kappa}{N}\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2}+\lambda_1\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)\right]+O(\Delta t)^2 \quad (45)$$

$$v_4=1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_1\frac{2\pi\rho_\kappa}{N}\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2} \\ -i\left[\frac{2\pi\rho_\kappa}{N}\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)^{1/2}-\lambda_1\left(1-\frac{(2\pi\rho_\kappa/N)^2}{4}\right)\right]+O(\Delta t)^2 \quad (46)$$

With the help of Eqs.(44), (45) and (46), which ignore only terms of order $O(\Delta t)^2$, we may write v_3^ζ and v_4^ζ as follows:

$$\text{for } (2\pi\rho_\kappa/N)^2 < 4 + \frac{4\lambda_1^2}{(2\pi\rho_\kappa/N)^2}$$

$$v_{3,4} = (1+i\lambda_1)\left\{1 - \frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2 \pm i\frac{2\pi\rho_\kappa}{N}\left[1 - \frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^{1/2}\right\} \tag{47}$$

$$\begin{aligned} v_3^\zeta &= (1+i\lambda_1)^\zeta\left\{\left[1-\frac{1}{2}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]^2 + \left(\frac{2\pi\rho_\kappa}{N}\right)^2\left[1-\frac{1}{4}\left(\frac{2\pi\rho_\kappa}{N}\right)^2\right]\right\}^{\zeta/2} e^{i\varphi_\kappa\zeta} \\ &= (1+i\lambda_1)^\zeta\,\{\,1\,\}^{\zeta/2}\,e^{i\varphi_\kappa\zeta} \doteq (1+i\lambda_1\zeta)e^{i\varphi_\kappa\zeta} \doteq e^{i(\varphi_\kappa+\lambda_1)\zeta} \\ &= \cos(\varphi_\kappa+\lambda_1)\zeta + i\sin(\varphi_\kappa+\lambda_1)\zeta \end{aligned} \tag{48}$$

$$\begin{aligned} v_4^\zeta &\doteq (1+i\lambda_1)^\zeta e^{-i\varphi_\kappa\zeta} = e^{-i(\varphi_\kappa-\lambda_1)\zeta} \\ &\doteq \cos(\varphi_\kappa-\lambda_1)\zeta - i\sin(\varphi_\kappa-\lambda_1)\zeta \end{aligned} \tag{49}$$

$$\varphi_\kappa = \operatorname{arctg}\frac{(2\pi\rho_\kappa/N)[1-(2\pi\rho_\kappa/N)^2/4]^{1/2}}{1-(2\pi\rho_\kappa/N)^2/2}, \quad \lambda_1 = \frac{ecA_{\text{m}0y}\Delta t}{\hbar} \tag{50}$$

We write $\phi(\zeta)$ in complex form as the sum of Eqs.(48) and (49), with φ_κ and λ_1 defined by Eq.(50):

$$\phi(\zeta) = A_{30}v_3^\zeta + A_{31}v_4^\zeta = A_{30}e^{i(\varphi_\kappa+\lambda_1)\zeta} + A_{31}e^{-i(\varphi_\kappa-\lambda_1)\zeta} \tag{51}$$

The boundary condition $w(0,\theta) = 0$ in Eq.(34) requires in Eq.(51) the relation

$$\begin{aligned} A_{31} &= -A_{30} \\ \phi(\zeta) &= A_{30}\left(e^{i(\varphi_\kappa+\lambda_1)\zeta} - e^{-i(\varphi_\kappa-\lambda_1)\zeta}\right) = 2iA_{30}e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta \end{aligned} \tag{52}$$

Equation (3.1-39) contains only φ_κ but not λ_1. Equations (4.2-32) and (52) become equal if N_τ is replaced by N as will be seen presently.

Since Eqs.(51) and (52) will be used for a Fourier series we must choose $\varphi_\kappa = \varphi_\kappa(\rho_\kappa)$ so that we get an orthogonal system of sine and cosine functions in an interval $0 \le \zeta \le N = T/\Delta t$ with a maximum of N periods. One must choose φ_κ as follows:

$$\begin{gathered} \varphi_\kappa = 0,\ \pm1\cdot 2\pi,\ \pm2\cdot 2\pi,\ \ldots,\ \pm\frac{N}{2}\cdot 2\pi \\ \varphi_\kappa = 2\pi\kappa/N, \quad \kappa = 0,\ \pm1,\ \pm2,\ \ldots,\ \pm N/2 \\ 0 \le \zeta \le N = T/\Delta t \end{gathered} \tag{53}$$

There are $N+1$ values of φ_κ which yield N orthogonal sine functions and N orthogonal cosine functions. For $\kappa = 0$ one gets $\sin\varphi_\kappa\zeta = 0$ and $\cos\varphi_\kappa\zeta = 1$, which is the constant in a Fourier series. The function $\phi(\zeta) =$ constant is a solution of Eq.(41) for $\varphi_\kappa = 2\pi\rho_\kappa/N = 0$.

In order to obtain the eigenvalues $(2\pi\rho_\kappa/N)^2$ associated with the angles φ_κ we must solve Eq.(50) for $(2\pi\rho_\kappa/N)^2$. This was already done for Eq.(3.1-35). The results of Eqs.(3.1-41) to (3.1-44) as well as Figs.3.1-1 and 3.1-2 apply again. We only copy Eq.(3.1-44) in a modified form:

$$\begin{gathered}\rho_\kappa = \frac{N}{\pi}\sin\frac{\varphi_\kappa}{2} \doteq \frac{N}{2\pi}\varphi_\kappa \\ \varphi_\kappa = 2\pi\kappa/N, \quad \kappa = 0,\ \pm1,\ \pm2,\ \ldots,\ \pm N/2\end{gathered} \tag{54}$$

5.2 Time Dependent Solution of $\Psi(\zeta,\theta)$

We turn to the time dependent difference equation of $\psi(\theta)$ shown by Eq.(5.1-42) and the value of $(2\pi\rho_\kappa/N)^2$ defined by Eq.(5.1-54):

$$\begin{gathered}\psi(\theta+1) - 2\psi(\theta) + \psi(\theta-1) + i\lambda_1\lambda_3[\psi(\theta+1) - \psi(\theta-1)] \\ +[(2\pi\rho_\kappa/N)^2 + \lambda_2^2]\psi(\theta) = 0 \\ \varphi_\kappa = \frac{2\pi\kappa}{N},\ \kappa = 0,\ \pm1,\ \pm2,\ldots,\ \pm\frac{N}{2} \\ \left(\frac{2\pi\rho_\kappa}{N}\right)^2 = 4\sin^2\frac{\varphi_\kappa}{2},\ \rho_\kappa = \frac{N}{\pi}\sin\frac{\varphi_\kappa}{2}\end{gathered} \tag{1}$$

Substitution of

$$\psi(\theta) = v^\theta \tag{2}$$

into Eq.(1) yields the following equation for v that equals Eq.(3.2-2) for $i\lambda_1\lambda_3 = \rho_1/2$ and $\lambda_2^2 = \rho_2^2$:

$$(1 + i\lambda_1\lambda_3)v^2 - [2 - (2\pi\rho_\kappa/N)^2 - \lambda_2^2]v + 1 - i\lambda_1\lambda_3 = 0 \tag{3}$$

It has the solutions:

$$\begin{aligned}v_{5,6} = \frac{1 - i\lambda_1\lambda_3}{1 + \lambda_1^2\lambda_3^2}\Bigg(1 - \frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] \\ \pm\frac{1}{2}\left\{\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right]^2 - 4\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] - 4\lambda_1^2\lambda_3^2\right\}^{1/2}\Bigg)\end{aligned} \tag{4}$$

We do not make the simplifications that led from Eq.(3.2-3) to (3.2-4). The square root is imaginary if the condition

$$4[(2\pi\rho_\kappa/N)^2+\lambda_2^2]+4\lambda_1^2\lambda_3^2>[(2\pi\rho_\kappa/N)^2+\lambda_2^2]^2 \tag{5}$$

is satisfied. If the sign $>$ is replaced by $=$ we may work out the following quadratic equation

$$\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]^2-4\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]-4\lambda_1^2\lambda_3^2=0 \tag{6}$$

with the solutions

$$\begin{aligned}(2\pi\rho_\kappa/N)^2&=2-\lambda_2^2\pm2(1+\lambda_1^2\lambda_3^3)^{1/2}\doteq2-\lambda_2^2\pm(2+\lambda_1^2\lambda_3^2)\\&\doteq4-\lambda_2^2+\lambda_1^2\lambda_3^2\qquad\text{for }+\\&\doteq-\lambda_2^2-\lambda_1^2\lambda_3^2\qquad\text{for }-\end{aligned} \tag{7}$$

Hence, the square root in Eq.(4) is imaginary if the following two conditions, derived with the help of Eq.(5.1-23), are satisfied

$$\begin{aligned}4-\lambda_2^2+\lambda_1^2\lambda_3^2&>(2\pi\rho_\kappa/N)^2=4\sin^2(\varphi_\kappa/2)\\4-\left(\frac{\Delta t}{\hbar}\right)^2(m_0^2c^4-2e\phi_{\mathrm{e0}}^2+e^2c^2A_{\mathrm{m0}y}^2)&>(2\pi\rho_\kappa/N)^2\\\varphi_\kappa=\frac{2\pi\kappa}{N}&<2\arcsin\left(1-\frac{\lambda_2^2-\lambda_1^2\lambda_3^2}{4}\right)^{1/2}\\\kappa<\kappa_0=\frac{N}{\pi}&\arcsin\left(1-\frac{\lambda_2^2-\lambda_1^2\lambda_3^2}{4}\right)^{1/2}\end{aligned} \tag{8}$$

$$\begin{aligned}-\lambda_2^2-\lambda_1^2\lambda_3^2&<(2\pi\rho_\kappa/N)^2=4\sin^2(\varphi_\kappa/2)\\-\left(\frac{\Delta t}{\hbar}\right)^2(m_0^2c^4+e^2c^2A_{\mathrm{m0}y})&<(2\pi\rho_\kappa/N)^2\end{aligned} \tag{9}$$

The condition of Eq.(9) is always satisfied since the range of interest of $(2\pi\rho_\kappa/N)^2=4\sin^2(\varphi_\kappa/2)$ is defined by

$$-N/2\le\kappa\le N/2,\quad-1\le\varphi_\kappa/\pi\le+1,\quad0\le4\sin^2(\varphi_\kappa/2)\le4 \tag{10}$$

but the condition of Eq.(8) may or may not be satisfied. We must investigate both the real and the imaginary root of $v_{5,6}$. Consider the imaginary root first.

$$v_{5,6} = \frac{1 - i\lambda_1\lambda_3}{1 + \lambda_1^2\lambda_3^2}\Bigg(1 - \frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] \pm i\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right]^{1/2} \times \left\{1 - \frac{1}{4}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] + \frac{\lambda_1^2\lambda_3^2}{(2\pi\rho_\kappa/N)^2 + \lambda_2^2}\right\}^{1/2}\Bigg) \quad (11)$$

Using the relation

$$\left\{1 - \frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right]\right\}^2 + \left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] \times \left\{1 - \frac{1}{4}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2 + \lambda_2^2\right] + \frac{\lambda_1^2\lambda_3^2}{(2\pi\rho_\kappa/N)^2 + \lambda_2^2}\right\} = 1 + \lambda_1^2\lambda_3^2 \quad (12)$$

we may write

$$v_{5,6} = \frac{1 - i\lambda_1\lambda_3}{1 + \lambda_1^2\lambda_3^2}(1 + \lambda_1^2\lambda_3^2)e^{\pm i\beta_\kappa} = (1 - i\lambda_1\lambda_3)e^{\pm i\beta_\kappa} \doteq e^{-i\lambda_1\lambda_3}e^{\pm i\beta_\kappa} \quad (13)$$

$$\operatorname{tg}\beta_\kappa = \frac{[(2\pi\rho_\kappa/N)^2 + \lambda_2^2]^{1/2}\{1 - [(2\pi\rho_\kappa/N)^2 + \lambda_2^2]/4 + \lambda_1^2\lambda_3^2/[(2\pi\rho_\kappa/N)^2 + \lambda_2^2]\}^{1/2}}{1 - [(2\pi\rho_\kappa/N)^2 + \lambda_2^2]/2} \quad (14)$$

This equation should be compared with Eq.(3.2-13). We recognize that β_κ follows from Eq.(3.2-13) by the substitutions

$$(2\pi\rho_\kappa/N)^2 \to (2\pi\rho_\kappa/N)^2 + \lambda_2^2, \quad \rho_2^2 - \rho_1^2/4 \to \lambda_1^2\lambda_3^2 \quad (15)$$

The range of interest of $2\pi\rho_\kappa/N$ is defined by Eq.(10).

Equation (14) for β_κ can be simplified by substituting it into the following identity

$$1 + \operatorname{tg}^2\beta_\kappa = \frac{1}{\cos^2\beta_\kappa} \quad (16)$$

and solving for $1/\cos^2\beta_\kappa$. Substituting $2\pi\rho_\kappa/N$ from Eq.(1) and calculating for a while one obtains:

$$\frac{\beta_\kappa}{\pi} = \frac{1}{\pi}\arcsin\left(1 - \frac{[1 - \lambda_2^2/2 - 2\sin^2(\varphi_\kappa/2)]^2}{1 + \lambda_1^2\lambda_3^2}\right)^{1/2} \quad (17)$$

For small values of λ_1^2 and λ_2^2 we get again Eq.(3.2-21) written here in a modified form and with β_κ replacing $\varphi_{\kappa\theta}$:

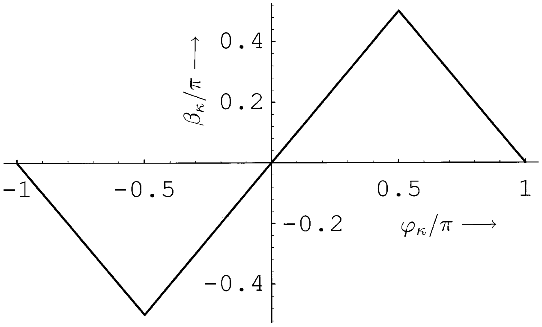

FIGURE 5.2-1. Plot of β_κ/π according to Eq.(18).

$$\frac{\beta_\kappa}{\pi} = \frac{1}{\pi}\arcsin\left[1 - \left(1 - 2\sin^2\frac{\pi}{2}\frac{\varphi_\kappa}{\pi}\right)^2\right]^{1/2} \tag{18}$$

Equation (18) is plotted in Fig.5.2-1. In this approximation the plot is equal to the plot shown with solid line in Fig.3.2-1.

The plot of Fig.5.2-1 assumes that $\lambda_2^2/2$ can be ignored compared with $2\sin(\varphi_\kappa/2)$. Hence, the plot needs to be analyzed in more detail in the neighborhood of $\varphi_\kappa/\pi = 0,\ 1,\ -1$. Consider φ_κ/π close to zero first. From Eq.(17) we get:

$$\begin{aligned}\frac{\beta_\kappa}{\pi} &= \frac{1}{\pi}\arcsin\left(1 - \frac{(1-\lambda_2^2/2)^2}{1+\lambda_1^2\lambda_3^2}\right)^2\\ &= 0.0225 \quad \text{for } \lambda_2^2 = 0.005,\ \lambda_1^2\lambda_3^2 = 0\end{aligned} \tag{19}$$

A plot of β_κ/π for $\lambda_2^2 = 0.005$, $\lambda_1^2\lambda_3^2 = 0$ is shown for the interval $-0.04 < \varphi_\kappa < 0.04$ according to Eq.(17) in Fig.5.2-2. This plot should be compared with the one of Fig.3.2-2. The important feature of Fig.5.2-2 is that β_κ/π does not become zero for $\varphi_\kappa/\pi = 0$ but assumes a value defined by Eq.(19):

$$\left|\frac{\beta_\kappa}{\pi}\right| \geq \frac{1}{\pi}\arcsin\left(1 - \frac{(1-\lambda_2^2)^2}{1+\lambda_1^2\lambda_3^2}\right)^{1/2} \doteq \frac{1}{\pi}(\lambda_2^2 + \lambda_1^2\lambda_3^2)^{1/2} \doteq \frac{\lambda_2}{\pi} \tag{20}$$

For the neighborhood of $\varphi_\kappa/\pi = 1$ we obtain from Eq.(8)

$$\begin{aligned}2\sin\frac{\varphi_\kappa}{2} &< (4 - \lambda_2^2 + \lambda_1^2\lambda_3^2)^{1/2} \doteq 2\left[1 - \frac{1}{8}(\lambda_2^2 - \lambda_1^2\lambda_3^2)\right]\\ \frac{\varphi_\kappa}{2} &< \arcsin 2\left[1 - \frac{1}{8}(\lambda_2^2 - \lambda_1^2\lambda_3^2)\right]\\ \frac{\varphi_\kappa}{\pi} &< \frac{2}{\pi}\arcsin 2\left[1 - \frac{1}{8}(\lambda_2^2 - \lambda_1^2\lambda_3^2)\right] = 0.9775\\ &\qquad \text{for } \lambda_2^2 = 0.005,\ \lambda_1^2\lambda_3^2 = 0\end{aligned} \tag{21}$$

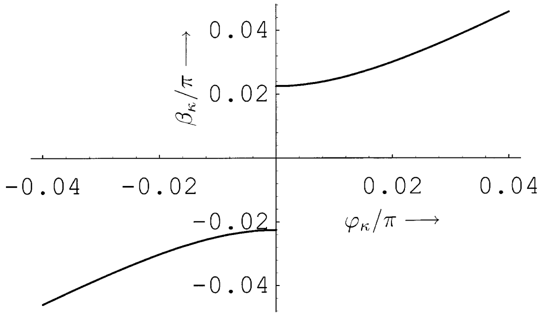

FIGURE 5.2-2. Plot of β_κ/π according to Eq.(17) in the neighborhood of $\varphi_\kappa/\pi = 0$ for $\lambda_2^2 = 0.005$, $\lambda_1^2\lambda_3^2 = 0$.

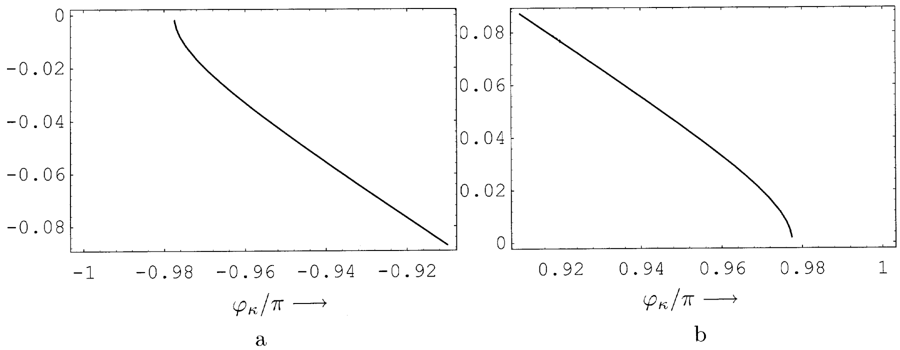

FIGURE 5.2-3. Plots of β_κ/π according to Eq.(17) in the neighborhood of $\varphi_\kappa/\pi = -1$ (a) and $\varphi_\kappa/\pi = +1$ (b) for $\lambda_2^2 = 0.005$, $\lambda_1^2\lambda_3^2 = 0$.

Plots of Eq.(17) for φ_κ/π close to ± 1 are shown in Figs.5.2-3a and b. They should be compared with those of Fig.3.2-3.

The function $v_{5,6}^\theta$ according to Eq.(13) may be written as follows if the conditions of Eqs.(8) and (9) are satisfied:

$$v_5^\theta = e^{-i(\lambda_1\lambda_3+\beta_\kappa)\theta} \tag{22}$$

$$v_6^\theta = e^{-i(\lambda_1\lambda_3-\beta_\kappa)\theta} \tag{23}$$

The solution of Eq.(1) becomes:

$$\psi(\theta) = A_{40}v_5^\theta + A_{41}v_6^\theta = A_{40}e^{-i(\lambda_1\lambda_3+\beta_\kappa)\theta} + A_{41}e^{-i(\lambda_1\lambda_3-\beta_\kappa)\theta} \tag{24}$$

The particular solution $w_\kappa(\zeta,\theta)$ of Eq.(5.1-38) assumes the following form with the help of Eqs.(5.1-52), (22) and (23):

$$w_\kappa(\zeta,\theta) = \Big\{A_1 \exp[-(\lambda_1\lambda_3+\beta_\kappa)\theta] + A_2 \exp[-i(\lambda_1\lambda_3-\beta_\kappa)\theta]\Big\} e^{i\lambda_1\zeta} \sin\varphi_\kappa\zeta \quad (25)$$

The usual way to generalize Eq.(25) is to make A_1 and A_2 functions of κ and integrate over all values of κ. As before we deviate and follow the text between Eqs.(2.1-78) and (2.1-79) in Section 2.1. Our finite time and space intervals are chosen to be $0 \leq t \leq T$ and $0 \leq y \leq cT$, where T is arbitrarily large but finite. The variables θ and ζ cover the intervals

$$0 \leq \theta = t/\Delta t \leq T/\Delta t,\ 0 \leq \zeta = y/c\Delta t \leq T/\Delta t,\quad T/\Delta t = N \gg 1 \quad (26)$$

Instead of the Fourier sum of Eq.(4.2-37) we obtain for $N_\tau \to N$ and the definition of κ_0 in Eq.(8) the following sum from Eq.(25), where $\kappa > -\kappa_0$ means the smallest integer larger than $-\kappa_0$ and $\kappa < \kappa_0$ the largest integer smaller than κ_0:

$$w(\zeta,\theta) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} \Big\{A_1(\kappa)\exp[-i(\lambda_1\lambda_3+\beta_\kappa)\theta] + A_2(\kappa)\exp[-i(\lambda_1\lambda_3-\beta_\kappa)\theta]\Big\} \times e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta \quad (27)$$

We note that the summation is symmetric over negative and positive values of κ just like in Eq.(3.2-60). The differential theory always yielded non-symmetric sums over positive values of κ, e.g., in Eqs.(2.1-81) and (4.2-37).

For the initial condition of Eq.(5.1-35) we obtain from Eq.(27) the following relation for $\theta = 0$:

$$w(\zeta,0) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} [A_1(\kappa)+A_2(\kappa)]e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta = -F(\zeta) \quad (28)$$

The second initial condition of Eq.(5.1-36) yields with $\theta = 0$:

$$w(\zeta,1) - w(\zeta,0) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} \Big[A_1(\kappa)\left(e^{-i(\lambda_1\lambda_3+\beta_\kappa)} - 1\right) + A_2(\kappa)\left(e^{-i(\lambda_1\lambda_3-\beta_\kappa)} - 1\right)\Big] e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta = 0 \quad (29)$$

We use again the Fourier series of Eq.(3.2-40) to obtain $A_1(\kappa)$ and $A_2(\kappa)$ from Eqs.(28) and (29). The factor $e^{i\lambda_1\zeta}$ in Eqs.(28) and (29) can be taken in front of the summation sign and cancelled:

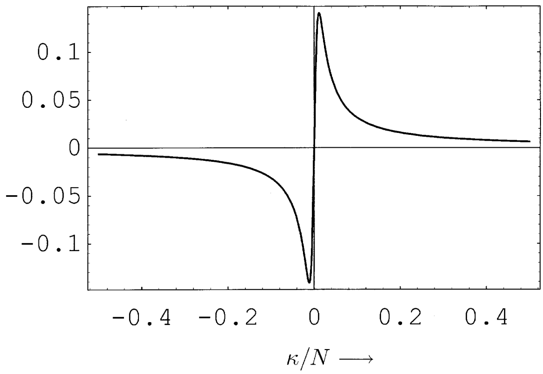

FIGURE 5.2-4. Plot of $I_\mathrm{T}(\kappa/N)$ according to Eq.(32) for $N = 100$, $\lambda_1^2 = 0$ and $\lambda_2^2 = 0.005$ in the interval $-0.5 \leq \kappa/N \leq 0.5$.

$$A_1(\kappa) + A_2(\kappa) = -\frac{2}{N}\int_0^N F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}d\zeta = -I_\mathrm{T} \tag{30}$$

$$A_1(\kappa)\left(e^{-i(\lambda_1\lambda_3+\beta_\kappa)} - 1\right) + A_2(\kappa)\left(e^{-i(\lambda_1\lambda_3-\beta_\kappa)} - 1\right) = 0 \tag{31}$$

Equation (5.1-33) defines the function $F(\zeta)$. We obtain for the integral I_T the following expression:

$$I_\mathrm{T}(\kappa/N) = \frac{2}{N}\int_0^N F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}d\zeta = \frac{2}{N}\int_0^N \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N}d\zeta$$
$$= \frac{2}{N}\frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2} \tag{32}$$

A plot of $I_\mathrm{T}(\kappa/N)$ for $N = 100$, $\lambda_1^2 = 0$ and $\lambda_2^2 = 0.005$ is shown in Fig.5.2-4 in the interval $-0.5 \leq \kappa/N \leq 0.5$. The same plot is shown expanded in the interval $-0.05 \leq \kappa/N \leq 0.05$ in Fig.5.2-5.

From Eqs.(30) and (31) one obtains the two complex variables $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa) = \frac{i}{2}I_\mathrm{T}(\kappa/N)\frac{e^{i\beta_\kappa} - e^{i\lambda_1\lambda_3}}{\sin\beta_\kappa}$$
$$= -\frac{1}{2}I_\mathrm{T}(\kappa/N)\frac{\sin\beta_\kappa - \sin\lambda_1\lambda_3 - i(\cos\beta_\kappa - \cos\lambda_1\lambda_3)}{\sin\beta_\kappa} \tag{33}$$

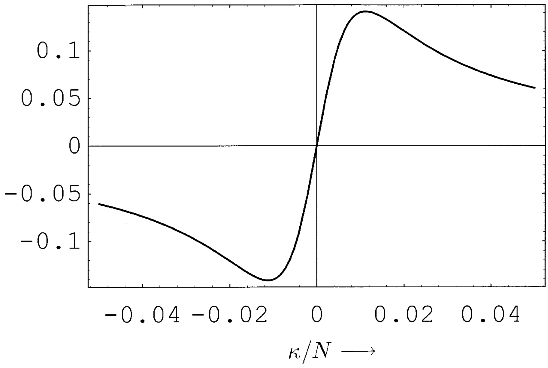

FIGURE 5.2-5. Plot of $I_{\rm T}(\kappa/N)$ according to Eq.(32) for $N=100$, $\lambda_1^2=0$ and $\lambda_2^2=0.005$ in the interval $-0.05 \leq \kappa/N \leq 0.05$.

$$\begin{aligned}A_2(\kappa) &= -\frac{i}{2}I_{\rm T}(\kappa/N)\frac{e^{-i\beta_\kappa}-e^{i\lambda_1\lambda_3}}{\sin\beta_\kappa}\\&= -\frac{1}{2}I_{\rm T}(\kappa/N)\frac{\sin\beta_\kappa+\sin\lambda_1\lambda_3+i(\cos\beta_\kappa-\cos\lambda_1\lambda_3)}{\sin\beta_\kappa}\end{aligned}\tag{34}$$

Substitution into Eq.(27) yields after summing the real and the imaginary parts:

$$\text{for } 0 \leq 4\sin^2(\varphi_\kappa/2) < 4-\lambda_2^2+\lambda_1^2\lambda_3^2$$

$$\begin{aligned}w(\zeta,\theta) = -\frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}&\frac{I_{\rm T}(\kappa/N)}{\sin\beta_\kappa}\\\times\big\{&[\sin\beta_\kappa+\cos\beta_\kappa-(\sin\lambda_1\lambda_3+\cos\lambda_1\lambda_3)]\sin[\lambda_1\zeta-(\lambda_1\lambda_3+\beta_\kappa)\theta]\\+&[\sin\beta_\kappa-\cos\beta_\kappa-(\sin\lambda_1\lambda_3-\cos\lambda_1\lambda_3)]\cos[\lambda_1\zeta-(\lambda_1\lambda_3+\beta_\kappa)\theta]\\+&[\sin\beta_\kappa-\cos\beta_\kappa+(\sin\lambda_1\lambda_3+\cos\lambda_1\lambda_3)]\sin[\lambda_1\zeta-(\lambda_1\lambda_3-\beta_\kappa)\theta]\\+&[\sin\beta_\kappa+\cos\beta_\kappa+(\sin\lambda_1\lambda_3-\cos\lambda_1\lambda_3)]\cos[\lambda_1\zeta-(\lambda_1\lambda_3-\beta_\kappa)\theta]\big\}\\&\times\sin\frac{2\pi\kappa\zeta}{N}\end{aligned}\tag{35}$$

We observe that the term $1/\sin\beta_\kappa$ does not become infinite at $\varphi_\kappa/\pi=0$ according to Fig.5.2-2. The situation is more critical close to $\pm\varphi_\kappa/\pi=\varphi_{\kappa 0}/\pi$. For $\kappa=\pm\kappa_0$ we would get $\sin\beta_\kappa=0$ but κ is an integer either larger than $-\kappa_0$ or smaller than $+\kappa_0$. If κ_0 happens to be an integer the square root in Eq.(4) would become zero for $|\kappa|=\kappa_0$. Hence, the largest value of $|\kappa|$ yielding an imaginary root would be $|\kappa|=\kappa_0-1$. Since all terms of the sum $w(\zeta,\theta)$ are finite and there is a finite number of close to $2\kappa_0+1$ terms the whole

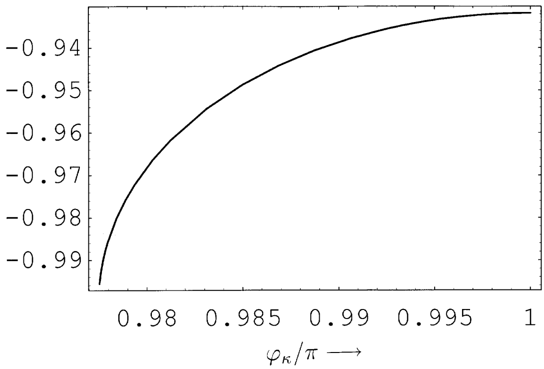

FIGURE 5.2-6. The function $V_7(\varphi_\kappa)$ of Eq.(37) for $\lambda_1^2 = 0$ and $\lambda_2^2 = 0.005$ in the interval $(2/\pi)\arcsin(1-\lambda_2^2+\lambda_1^2\lambda_3^2)^{1/2} < \varphi_\kappa/\pi \leq 1$.

sum $w(\zeta,\theta)$ must be finite for any value of ζ and θ. Note that the concept of convergence does not enter due to the finite value of N.

We turn to the case $4\sin^2(\varphi_\kappa/2) > 4-\lambda_2^2+\lambda_1^2\lambda_3^2$ of Eq.(8). A new solution is required to close the gaps

$$-N/2 \leq \kappa < -\kappa_0, \qquad \kappa_0 < \kappa \leq N/2$$
$$\kappa_0 = \frac{N}{\pi}\arcsin\left(1-\frac{\lambda_2^2-\lambda_1^2\lambda_3^2}{4}\right)^{1/2} \tag{36}$$

in Fig.5.2-3. We write $v_{7,8}$ in Eq.(4) if the square root is real and obtain the following two solutions:

$$v_7 = V_7(\varphi_\kappa)(1-i\lambda_1\lambda_3) \doteq V_7(\varphi_\kappa)e^{-i\lambda_1\lambda_3}, \quad \lambda_1^2\lambda_3^2 \ll 1$$
$$V_7(\varphi_\kappa) = 1-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]$$
$$+\frac{1}{2}\left\{\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]^2-4\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]-4\lambda_1^2\lambda_3^2\right\}^{1/2}$$
$$(2\pi\rho_\kappa/N)^2 = 4\sin^2(\varphi_\kappa/2) \tag{37}$$

$$v_8 = V_8(\varphi_\kappa)e^{-i\lambda_1\lambda_3}, \quad \lambda_1^2\lambda_3^2 \ll 1$$
$$V_8(\varphi_\kappa) = 1-\frac{1}{2}\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]$$
$$-\frac{1}{2}\left\{\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]^2-4\left[\left(\frac{2\pi\rho_\kappa}{N}\right)^2+\lambda_2^2\right]-4\lambda_1^2\lambda_3^2\right\}^{1/2} \tag{38}$$

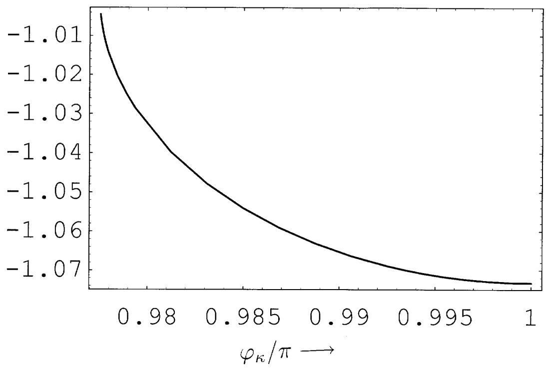

FIGURE 5.2-7. The function $V_8(\varphi_\kappa)$ of Eq.(38) for $\lambda_1^2 = 0$ and $\lambda_2^2 = 0.005$ in the interval $(2/\pi)\arcsin(1-\lambda_2^2+\lambda_1^2\lambda_3^2)^{1/2} < \varphi_\kappa/\pi \leq 1$.

Plots of $V_7(\varphi_\kappa)$ and $V_8(\varphi_\kappa)$ are shown in Figs.5.2-6 and 5.2-7 for $\lambda_1^2 = 0$ and $\lambda_2^2 = 0.005$. The negative values of both $V_7(\varphi_\kappa)$ and $V_8(\varphi_\kappa)$ suggest to use the notation

$$V_7(\varphi_\kappa) = |V_7(\varphi_\kappa)|e^{i\pi}, \quad v_7 = |V_7(\varphi_\kappa)|e^{i(\pi-\lambda_1\lambda_3)} \tag{39}$$
$$V_8(\varphi_\kappa) = |V_8(\varphi_\kappa)|e^{i\pi}, \quad v_8 = |V_8(\varphi_\kappa)|e^{i(\pi-\lambda_1\lambda_3)} \tag{40}$$

Hence, v_7^θ represents a damped oscillation and v_8^θ an amplified oscillation:

$$v_7^\theta = |V_7(\varphi_\kappa)|^\theta e^{i(\pi-\lambda_1\lambda_3)\theta} \tag{41}$$
$$v_8^\theta = |V_8(\varphi_\kappa)|^\theta e^{i(\pi-\lambda_1\lambda_3)\theta} \tag{42}$$

The solution of Eq.(2) for the interval $\kappa_0 < \kappa \leq N/2$ becomes

$$\psi(\theta) = A_7|V_7(\kappa)|^\theta e^{i(\pi-\lambda_1\lambda_3)\theta} + A_8|V_8(\kappa)|^\theta e^{i(\pi-\lambda_1\lambda_3)\theta}$$
$$V_7(\kappa) = V_7(\varphi_\kappa),\ V_8(\kappa) = V_8(\varphi_\kappa) \tag{43}$$

and the particular solution $w_\kappa(\zeta,\theta)$ of Eq.(5.1-38) assumes the following form with the help of Eqs.(5.1-52), (37) and (38):

$$w_\kappa(\zeta,\theta) = [A_7|V_7(\kappa)|^\theta + A_8|V_8(\kappa)|^\theta]e^{i(\pi-\lambda_1\lambda_3)\theta}e^{i\lambda_1\zeta}\sin(2\pi\kappa\zeta/N) \tag{44}$$

The generalization by means of a Fourier series yields:

$$w(\zeta,\theta)=\sum_{\substack{\kappa=-N/2\\>\kappa_0}}^{\substack{N/2\\<-\kappa_0}}[A_7(\kappa)|V_7(\kappa)|^\theta+A_8(\kappa)|V_8(\kappa)|^\theta]e^{i(\pi-\lambda_1\lambda_3)\theta}e^{i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N} \quad (45)$$

The first initial condition of Eq.(5.1-35) demands

$$w(\zeta,0)=\sum_{\substack{\kappa=-N/2\\>\kappa_0}}^{\substack{N/2\\<\kappa_0}}[A_7(\kappa)+A_8(\kappa)]e^{i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}=-F(\zeta) \quad (46)$$

and the second initial condition of Eq.(5.1-36) demands:

$$w(\zeta,1)-w(\zeta,0)=\sum_{\substack{\kappa=-N/2\\>\kappa_0}}^{\substack{N/2\\<-\kappa_0}}\Big\{[A_7(\kappa)|V_7(\kappa)|+A_8(\kappa)|V_8(\kappa)|]e^{i(\pi-\lambda_1\lambda_3)}$$
$$-[A_7(\kappa)+A_8(\kappa)]\Big\}e^{i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}=0 \quad (47)$$

Multiplication with $\sin(2\pi\kappa\zeta/N)$ and integration over ζ from 0 to N brings Eqs.(46) and (47) into the following form:

$$A_7(\kappa)+A_8(\kappa)=-\frac{2}{N}\int_0^N F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}d\zeta=-I_{\rm T}(\kappa/N) \quad (48)$$

$$\begin{aligned}A_7(\kappa)[|V_7(\kappa)|e^{i(\pi-\lambda_1\lambda_3)}-1]+A_8(\kappa)[|V_8(\kappa)|e^{i(\pi-\lambda_1\lambda_3)}-1]=0\\ A_7(\kappa)[V_7(\kappa)e^{-i\lambda_1\lambda_3}-1]+A_8(\kappa)[V_8(\kappa)e^{-i\lambda_1\lambda_3}-1]=0\end{aligned} \quad (49)$$

One obtains for $A_7(\kappa)$ and $A_8(\kappa)$ if $\lambda_1\lambda_3$ is small compared with 1:

$$A_7(\kappa)=+\frac{I_{\rm T}(\kappa/N)}{V_7(\kappa)-V_8(\kappa)}[V_8(\kappa)-e^{i\lambda_1\lambda_3}]\doteq+\frac{I_{\rm T}(\kappa/N)[V_8(\kappa)-1]}{V_7(\kappa)-V_8(\kappa)} \quad (50)$$

$$A_8(\kappa)=-\frac{I_{\rm T}(\kappa/N)}{V_7(\kappa)-V_8(\kappa)}[V_7(\kappa)-e^{i\lambda_1\lambda_3}]\doteq-\frac{I_{\rm T}(\kappa/N)[V_7(\kappa)-1]}{V_7(\kappa)-V_8(\kappa)} \quad (51)$$

Substitution of $A_7(\kappa)$ and $A_8(\kappa)$ into Eq.(45) yields a function with a largest term of magnitude $A_8(\kappa)|V_8(\kappa)|^N$ for $\theta=N$. This is of no concern since Eq.(36)

shows that small values of $\lambda_2^2 - \lambda_1^2\lambda_3^2$ produce $\kappa_0 \doteq N/2$ or $N/2-1 < \kappa_0 < N/2$. Small values of $\lambda_2^2 - \lambda_1^2\lambda_3^2$ can be achieved by choosing Δt sufficiently small according to Eq.(5.1-23). Hence, only one arbitrarily large term $|V_8(\kappa)|^N$ can occur in the sum of Eq.(45). We have discussed with the help of Fig.3.2-7 that the sum from $-N/2$ to $N/2$ only adds the areas from $-N/2+1$ to $N/2-1$. Hencxe, Eq.(45) is never needed and we may change the summation limits in Eq.(27) to $\kappa = -N/2+1$ and $\kappa = N/2-1$. The value $\kappa = 0$ is excluded.

5.3 Exponential Ramp Function as Boundary Condition

In Section 5.2 we used the step function excitation of Eq.(5.1-24). A less sudden excitation is desirable when dealing with a particle of finite mass m_0. We use again the exponential ramp function excitation of Eq.(4.3-1):

$$\begin{aligned}\Psi(0,\theta) = \Psi_1 S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0\\ &= \Psi_1(1-e^{-\iota\theta}) && \text{for } \theta \geq 0\end{aligned} \tag{1}$$

The initial conditions of Eqs.(5.1-25) and (5.1-26) are used again:

$$\Psi(\zeta,0) = 0 \tag{2}$$

$$\Psi(\zeta,\theta+1) - \Psi(\zeta,\theta) = 0 \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{3}$$

Equation (5.1-27) is modified according to Eq.(4.3-5) and the following ansatz is used:

$$\Psi(\zeta,\theta) = \Psi_1[(1-e^{-\iota\theta})F(\zeta) + u(\zeta,\theta)] \tag{4}$$

Substitution of $\Psi_1(1-e^{-\iota\theta})F(\zeta)$ into Eq.(5.1-23) produces the following relations:

$$\Psi(\zeta,\theta) = (1-e^{-\iota\theta})F(\zeta),\ \ \Psi(\zeta,\theta\pm1) = (1-e^{-\iota(\theta\pm1)})F(\zeta)$$
$$\Psi(\zeta\pm1,\theta) = (1-e^{-\iota\theta})F(\zeta\pm1)$$

$$\begin{aligned}&(1-e^{-\iota\theta})[F(\zeta+1) - 2F(\zeta) + F(\zeta-1)] + e^{-\iota\theta}(e^{-\iota} - 2 + e^{\iota})F(\zeta)\\ &\quad - i\lambda_1\Big\{(1-e^{-\iota\theta})[F(\zeta+1) - F(\zeta-1)] - \lambda_3 e^{-\iota\theta}(e^{-\iota} - e^{\iota})F(\zeta)\Big\}\\ &\quad\quad - \lambda_2^2(1-e^{-\iota\theta})F(\zeta) = 0\end{aligned} \tag{5}$$

A sufficiently general solution is obtained if we let the terms multiplied with $1-e^{-\iota\theta}$ and $e^{-\iota\theta}$ vanish separately:

$$F(\zeta+1) - 2F(\zeta) + F(\zeta-1) - i\lambda_1[F(\zeta+1) - F(\zeta-1)] - \lambda_2^2 F(\zeta) = 0 \quad (6)$$

$$e^{-\iota} - 2 + e^{\iota} + i\lambda_1\lambda_3(e^{-\iota} - e^{\iota}) = 0 \quad (7)$$

Equation (6) equals Eq.(5.1-28). We get again $F(\zeta)$ according to Eq.(5.1-33):

$$F(\zeta) = \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta} \quad (8)$$

Equation (7) yields a quadratic equation for e^{ι}

$$(1 - i\lambda_1\lambda_3)e^{2\iota} - 2e^{\iota} + 1 + i\lambda_1\lambda_3 = 0 \quad (9)$$

with the solution

$$e^{\iota} = \frac{(1 + i\lambda_1\lambda_3)(1 \pm i\lambda_1\lambda_3)}{1 + \lambda_1^2\lambda_3^2} \doteq 1 + i(\lambda_1\lambda_3 \pm \lambda_1\lambda_3) \quad (10)$$

As in the case of Eq.(4.3-8) we obtain a trivial solution $e^{\iota} = 1$, $\iota = 0$ and a non-trivial solution

$$e^{\iota} = 1 + 2i\lambda_1\lambda_3, \quad \iota \doteq 2i\lambda_1\lambda_3 \quad (11)$$

Equation (4) must satisfy the boundary condition of Eq.(1). Since Eq.(8) yields $F(0) = 1$ we get

$$\begin{aligned}\Psi_1[1 - e^{-\iota\theta} + u(0,\theta)] &= \Psi_1(1 - e^{-\iota\theta}), \quad \theta \geq 0\\ u(0,\theta) &= 0\end{aligned} \quad (12)$$

The boundary condition for $u(0,\theta)$ is once more homogeneous. The initial conditions of Eqs.(2) and (3) yield with $F(0) = 1$:

$$\Psi(\zeta, 0) = \Psi_1 u(\zeta, 0) = 0, \quad \zeta \geq 0 \quad (13)$$

$$\begin{aligned}&\Psi_1[(1 - e^{-\iota})F(\zeta) + u(\zeta,1)] - \Psi_1 u(\zeta,0) = 0\\ &u(\zeta,1) - u(\zeta,0) = -(1 - e^{-\iota})F(\zeta), \quad \zeta \geq 0\end{aligned} \quad (14)$$

The calculation of $u(\zeta,\theta)$ proceeds like the one of $w(\zeta,\theta)$ in Sections 5.1 and 5.2 from Eq.(5.1-37) on until Eq.(5.2-27) is reached. We replace $w(\zeta,\theta)$ by $u(\zeta,\theta)$ in this equation:

$$u(\zeta,\theta) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} \Big\{A_1(\kappa)\exp[-i(\lambda_1\lambda_3+\beta_\kappa)\theta] + A_2(\kappa)\exp[-i(\lambda_1\lambda_3-\beta_\kappa)\theta]\Big\} \times e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta \quad (15)$$

With the help of Eqs.(5.2-28) and (5.2-29) we obtain for the initial conditions of Eqs.(13) and (14) the following relations:

$$u(\zeta,0) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} [A_1(\kappa)+A_2(\kappa)]e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta = 0 \quad (16)$$

$$u(\zeta,1)-u(\zeta,0) = \sum_{\kappa>-\kappa_0}^{<\kappa_0} \Big[A_1(\kappa)\Big(e^{-i(\lambda_1\lambda_3+\beta_\kappa)}-1\Big)+A_2(\kappa)\Big(e^{-i(\lambda_1\lambda_3-\beta_\kappa)}-1\Big)\Big] \times e^{i\lambda_1\zeta}\sin\varphi_\kappa\zeta = -(1-e^{-2i\lambda_1\lambda_3})F(\zeta) \doteq -2i\lambda_1\lambda_3 F(\zeta) \quad (17)$$

Once more we use the Fourier series of Eq.(3.2-40) to obtain $A_1(\kappa)$ and $A_2(\kappa)$ from Eqs.(16) and (17). The factor $e^{i\lambda_1\zeta}$ can be taken in front of the summation sign. With $\varphi_\kappa = 2\pi\kappa/N$ we get:

$$A_1(\kappa)+A_2(\kappa) = 0 \quad (18)$$

$$A_1(\kappa)\Big(e^{-i(\lambda_1\lambda_3+\beta_\kappa)}-1\Big)+A_2(\kappa)\Big(e^{-i(\lambda_1\lambda_3-\beta_\kappa)}-1\Big) = -\frac{4i\lambda_1\lambda_3}{N}\int_0^N F(\zeta)e^{-i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N}d\zeta = -2i\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N) \quad (19)$$

The function $I_{\mathrm{T}} = I_{\mathrm{T}}(\kappa/N)$ is defined by Eq.(5.2-32). One obtains for $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa) = -A_2(\kappa) = \frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}e^{i\lambda_1\lambda_3} \quad (20)$$

Substitution of $A_1(\kappa)$ and $A_2(\kappa)$ into Eq.(15) yields:

$$u(\zeta,\theta) = -2i\lambda_1\lambda_3 e^{i\lambda_1\lambda_3}e^{i\lambda_1(\zeta-\lambda_3\theta)}\sum_{\kappa>-\kappa_0}^{<\kappa_0} I_{\mathrm{T}}(\kappa/N)\frac{\sin\beta_\kappa\theta}{\sin\beta_\kappa}\sin\frac{2\pi\kappa\zeta}{N} \quad (21)$$

Substitution of $F(\zeta)$ of Eq.(8) into Eq.(4) brings Eq.(4) into the following form:

$$\text{for } 0 \leq 4\sin^2(\varphi_\kappa/2) < 4 - \lambda_2^2 + \lambda_1^2\lambda_3^2$$

$$\begin{aligned}\Psi(\zeta,\theta) &= \Psi_1\big[(1-e^{-2i\lambda_1\lambda_3\theta})F(\zeta) + u(\zeta,\theta)\big]\\ &= \Psi_1\bigg[(1-e^{-2i\lambda_1\lambda_3\theta})\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta}\\ &\quad - 2i\lambda_1\lambda_3 e^{i\lambda_1\lambda_3}e^{i\lambda_1(\zeta-\lambda_3\theta)}\sum_{\kappa>-\kappa_0}^{<\kappa_0} I_\mathrm{T}(\kappa/N)\frac{\sin\beta_\kappa\theta}{\sin\beta_\kappa}\sin\frac{2\pi\kappa\zeta}{N}\bigg]\end{aligned}$$

$$\kappa_0 = \frac{N}{\pi}\arcsin\left(1-\frac{\lambda_2^2-\lambda_1^2\lambda_3^2}{4}\right)^{1/2} \tag{22}$$

To see how $\Psi(\zeta,\theta)$ rises at $\theta = 0$ we use the following approximations:

$$e^{-2i\lambda_1\lambda_3\theta} \doteq 1 - 2i\lambda_1\lambda_3\theta,\ e^{i\lambda_1(\zeta-\lambda_3\theta)} \doteq e^{i\lambda_1\zeta}(1-i\lambda_1\lambda_3\theta),\ \sin\beta_\kappa\theta \doteq \beta_\kappa\theta \tag{23}$$

and obtain in first order of θ:

$$\begin{aligned}\Psi(\zeta,\theta) \doteq 2i\lambda_1\lambda_3\theta\Psi\bigg(&\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta}\\ &- e^{i\lambda_1\lambda_3}e^{i\lambda_1\zeta}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_\mathrm{T}(\kappa/N)\beta_\kappa}{\sin\beta_\kappa}\sin\frac{2\pi\kappa\zeta}{N}\bigg)\end{aligned} \tag{24}$$

Hence, the real as well as the imaginary part of $\Psi(\zeta,\theta)$ rise proportionate to θ from $\theta = 0$.

For the gaps $-N/2 \leq \kappa < -\kappa_0$ and $\kappa_0 < \kappa \leq N/2$ we need again the real solutions $V_7(\varphi_\kappa)$ and $V_8(\varphi_\kappa)$ of Eqs.(5.2-37) and (5.2-38). Equation (5.2-41) for $w(\zeta,\theta)$ is obtained but $w(\zeta,\theta)$ is replaced by $u(\zeta,\theta)$. The initial conditions of Eqs.(5.2-46) and (5.2-47) must be modified according to Eqs.(16) and (17):

$$u(\zeta,0) = \sum_{\substack{\kappa=-N/2\\ >\kappa_0}}^{\substack{N/2\\ <-\kappa_0}} [A_7(\kappa)+A_8(\kappa)]e^{i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N} = 0 \tag{25}$$

$$\begin{aligned}u(\zeta,1)-u(\zeta,0) = \sum_{\substack{\kappa=-N/2\\ >\kappa_0}}^{\substack{N/2\\ <-\kappa_0}} &\{A_7(\kappa)[|V_7(\kappa)|-1] + A_8(\kappa)[|V_8(\kappa)|-1]\}\\ &\times e^{i\lambda_1\zeta}\sin\frac{2\pi\kappa\zeta}{N} = -(1-e^{-2i\lambda_1\lambda_3})F(\zeta)\end{aligned} \tag{26}$$

In analogy to Eqs.(5.2-48) and (5.2-49) we get:

$$A_7(\kappa) + A_8(\kappa) = 0 \tag{27}$$

$$\begin{aligned} A_7(\kappa)[|V_7(\kappa)| - 1] + A_8(\kappa)[|V_8(\kappa)| - 1] &= -\frac{2}{N}(1 - e^{-2i\lambda_1\lambda_3}) \\ &\times \int_0^N F(\zeta)e^{-i\lambda_1\zeta} \sin\frac{2\pi\kappa\zeta}{N} d\zeta = -(1 - e^{-2i\lambda_1\lambda_3}) I_{\mathrm{T}}(\kappa/N) \end{aligned} \tag{28}$$

If we use λ_1 from Eq.(5.1-23) we get $A_7(\kappa) = -A_8(\kappa) = O(\Delta t)$, which is much smaller than $A_7(\kappa)$ and $A_8(\kappa)$ of Eqs.(5.2-50) and (5.2-51). The comments following Eq.(5.2-51) apply again. Hence, we may replace the summation limits in Eqs.(15), (21), (22) and (24) by $\kappa = -N/2 + 1$, $\kappa = N/2 - 1$ and ignore $V_7(\kappa)$, $V_8(\kappa)$.

5.4 Hamilton Function for Difference Equation

Our goal is to derive the Hamilton function for Eq.(5.3-22) and quantize it in analogy to what was done in Section 4.3 with Eq.(4.3-22) of the differential theory. Equations (4.4-1) and (4.4-2) continue to apply. But in Eq.(4.4-3) we replace τ by Δt:

$$\begin{gathered} t \to t/\Delta t = \theta,\ y \to y/c\Delta t = \zeta,\ x \to x/c\Delta t,\ z \to z/c\Delta t \\ N_\tau \to N = T/\Delta t \end{gathered} \tag{1}$$

Equation (4.4-4) becomes:

$$\begin{aligned} U &= c\Delta t \int_{-L/2c\Delta t}^{L/2c\Delta t} \int_{-L/2c\Delta t}^{L/2c\Delta t} \left[\int_0^N \left(\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} + \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} + \frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\Psi^*\Psi \right) d\zeta \right] \\ &\qquad \times d\left(\frac{x}{c\Delta t}\right) d\left(\frac{z}{c\Delta t}\right) \\ &= \frac{L^2}{c\Delta t} \int_0^N \left(\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} + \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} + \frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\Psi^*\Psi \right) d\zeta \end{aligned} \tag{2}$$

Equation (4.4-5) is replaced by an equation derived from Eq.(5.3-22):

$$\begin{aligned}
\Psi^*\Psi &= \Psi_1^2[(1-e^{2i\lambda_1\lambda_3\theta)})F^*(\zeta)+u^*(\zeta,\theta)][(1-e^{-2i\lambda_1\lambda_3\theta)})F(\zeta)+u(\zeta,\theta)] \\
&= 4\Psi_1^2\bigg[\frac{1}{2}(1-\cos 2\lambda_1\lambda_3\theta)\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] \\
&\qquad -2\lambda_1\lambda_3\cos\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] \\
&\qquad\qquad \times\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N} \\
&\qquad\qquad +\lambda_1^2\lambda_3^2\bigg(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\bigg)^2\bigg]
\end{aligned} \tag{3}$$

This equation should be compared with Eq.(4.4-5).

Differentiation of $\Psi(\zeta,\theta)$ of the first line of Eq.(5.3-22) with respect to θ or ζ produces with the help of Eqs.(5.3-8) and (5.3-21) the following results:

$$\begin{aligned}
\frac{\partial\Psi}{\partial\theta} &= \Psi_1\bigg(2i\lambda_1\lambda_3e^{-2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\theta}\bigg) \\
\frac{\partial u}{\partial\theta} &= -2\lambda_1\lambda_3e^{-i\lambda_1\lambda_3(\theta-1)}e^{i\lambda_1\zeta} \\
&\qquad \times\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}(\lambda_1\lambda_3\sin\beta_\kappa\theta+i\beta_\kappa\cos\beta_\kappa\theta)\sin\frac{2\pi\kappa\zeta}{N}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{\partial\Psi}{\partial\zeta} &= \Psi_1\bigg((1-e^{-2i\lambda_1\lambda_3\theta})[-(\lambda_2^2-\lambda_1^2)^{1/2}+i\lambda_1] \\
&\qquad\qquad \times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]e^{i\lambda_1\zeta}+\frac{\partial u}{\partial\zeta}\bigg) \\
\frac{\partial u}{\partial\zeta} &= 2\lambda_1\lambda_3e^{-i\lambda_1\lambda_3(\theta-1)}e^{i\lambda_1\zeta}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta \\
&\qquad\qquad \times\bigg(\lambda_1\sin\frac{2\pi\kappa\zeta}{N}-\frac{2\pi i\kappa}{N}\cos\frac{2\pi\kappa\zeta}{N}\bigg)
\end{aligned} \tag{5}$$

The first term in Eq.(2) becomes:

$$\begin{aligned}
\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} &= \Psi_1^2\bigg(-2i\lambda_1\lambda_3e^{2i\lambda_1\lambda_3\theta}e^{-i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u^*}{\partial\theta}\bigg) \\
&\qquad \times\bigg(2i\lambda_1\lambda_3e^{-2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\theta}\bigg)
\end{aligned}$$

$$= 4\lambda_1^2\lambda_3^2\Psi_1^2\Bigg[\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]$$

$$-2\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}[\lambda_1\lambda_3\sin\lambda_1\lambda_3(\theta+1)\sin\beta_\kappa\theta$$

$$+\beta_\kappa\cos\lambda_1\lambda_3(\theta+1)\cos\beta_\kappa\theta]\sin\frac{2\pi\kappa\zeta}{N}$$

$$+\Bigg(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)\lambda_1\lambda_3}{\sin\beta_\kappa}\sin\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\Bigg)^2$$

$$+\Bigg(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)\beta_\kappa}{\sin\beta_\kappa}\cos\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\Bigg)^2\Bigg] \quad (6)$$

For the second term in Eq.(2) we get:

$$\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} = \Psi_1^2\Bigg((1-e^{2i\lambda_1\lambda_3\theta})e^{-i\lambda_1\zeta}[-(\lambda_2^2-\lambda_1^2)^{1/2}-i\lambda_1]$$

$$\times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u^*}{\partial\zeta}\Bigg)$$

$$\times\Bigg((1-e^{-2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}[-(\lambda_2^2-\lambda_1^2)^{1/2}+i\lambda_1]$$

$$\times\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]+\frac{\partial u}{\partial\zeta}\Bigg) \quad (7)$$

The evaluation of this equation is a challenge. In the end one obtains the following result:

$$\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} = \Psi_1^2\Bigg\{2\lambda_2^2(1-\cos2\lambda_1\lambda_3\theta)\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]$$

$$-8\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta$$

$$\times\Bigg[\cos\lambda_1\lambda_3\Bigg(\lambda_1^2\sin\frac{2\pi\kappa\zeta}{N}-\frac{2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}}{N}\cos\frac{2\pi\kappa\zeta}{N}\Bigg)$$

$$+\sin\lambda_1\lambda_3\Bigg(\lambda_1(\lambda_2^2-\lambda_1^2)^{1/2}\sin\frac{2\pi\kappa\zeta}{N}+\frac{2\pi\kappa\lambda_1}{N}\cos\frac{2\pi\kappa\zeta}{N}\Bigg)\Bigg]$$

$$+4\lambda_1^2\lambda_3^2\Bigg[\Bigg(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{\lambda_1 I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\Bigg)^2$$

$$+\Bigg(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\sin\beta_\kappa\theta\cos\frac{2\pi\kappa\zeta}{N}\Bigg)^2\Bigg]\Bigg\} \quad (8)$$

Equations (6), (8) and (3) have to be substituted into Eq.(2). The integration with respect to ζ is straightforward but lengthy. The reader will find the calculations in Section 6.8. The energy U of Eq.(2) is separated into a constant part U_c and a time variable part $U_\mathrm{v}(\theta)$ that depends on sinusoidal functions of θ and has the time-average zero:

$$U = U_\mathrm{c} + U_\mathrm{v}(\theta) \tag{9}$$

We copy for U_c the three components of Eqs.(6.8-10), (6.8-14) and (6.8-6):

$$\begin{aligned} U_\mathrm{c} &= U_{\mathrm{c}2} + U_{\mathrm{c}3} + U_{\mathrm{c}1} = \sum_{\kappa=-N/2+1}^{N/2-1} U_{\mathrm{c}\kappa}(\kappa) \\ &= \frac{L^2}{c\Delta t}\Psi_1^2 N\lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_\mathrm{T}(\kappa/N)}{\sin\beta_\kappa}\right)^2\left[\beta_\kappa^2+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2+\frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\right] \\ &\qquad \lambda_1,\ \lambda_2,\ \lambda_3\ \ \text{Eq.(5.1-23)},\quad I_\mathrm{T}(\kappa/N)\ \ \text{Eq.(5.2-32)},\quad \beta_\kappa\ \ \text{Eq.(5.2-17)} \end{aligned} \tag{10}$$

For the derivation of the Hamilton function $\mathcal{H}$ we need only the constant energy U_c since the average of the variable energy $U_\mathrm{v}(\theta)$ is zero. We normalize U_c in Eq.(10):

$$\frac{c\Delta t U_\mathrm{c}}{L^2\Psi_1^2 N} = \frac{cT\Delta t U_\mathrm{c}}{L^2\Psi_1^2 TN} = \frac{cTU_\mathrm{c}}{L^2\Psi_1^2N^2} = \frac{\mathcal{H}}{N^2} \tag{11}$$

$$\begin{aligned} &\frac{cTU_\mathrm{c}}{L^2\Psi_1^2} = \mathcal{H} = \sum_{\kappa>-\kappa_0}^{<\kappa_0}\mathcal{H}_\kappa = \sum_{\kappa>-\kappa_0}^{<\kappa_0} d^2(\kappa) \\ &d(\kappa) = \lambda_1\lambda_2 N\frac{I_\mathrm{T}(\kappa/N)}{\sin\beta_\kappa}\left[\beta_\kappa^2+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2+\frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\right]^{1/2} \end{aligned} \tag{12}$$

The component $\mathcal{H}_\kappa$ of the sum is rewritten as in Eq.(3.5-46):

$$\begin{aligned} \mathcal{H}_\kappa &= (2\pi\kappa)^2\frac{d(\kappa)}{2\pi\kappa}(\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta)\frac{d(\kappa)}{2\pi\kappa}(\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\ &= -2\pi i\kappa p_\kappa(\theta)q_\kappa(\theta) \end{aligned} \tag{13}$$

$$p_\kappa(\theta) = \sqrt{2\pi i\kappa}\frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta} \tag{14}$$

$$\dot{p}_\kappa = \frac{\tilde{\Delta}p_\kappa}{\tilde{\Delta}\theta} = (2\pi i\kappa)^{3/2}\frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta} = 2\pi i\kappa p_\kappa(\theta) \tag{15}$$

$$q_\kappa(\theta) = \sqrt{2\pi i\kappa}\frac{d(\kappa)}{2\pi\kappa}e^{-2\pi i\kappa(\theta)} \tag{16}$$

$$\dot{q}_\kappa = \frac{\tilde{\Delta}q_\kappa}{\tilde{\Delta}\theta} = -(2\pi i\kappa)^{3/2}\frac{d(\kappa)}{2\pi\kappa} = -2\pi i\kappa q(\theta) \tag{17}$$

As in Section 3.5 from Eq.(3.5-51) on we get again the proper relations for the finite derivatives $\tilde{\Delta}\mathcal{H}_\kappa/\tilde{\Delta}q_\kappa$ and $\tilde{\Delta}\mathcal{H}_\kappa/\tilde{\Delta}p_\kappa$:

$$\frac{\tilde{\Delta}\mathcal{H}_\kappa}{\tilde{\Delta}q_\kappa} = -\frac{\tilde{\Delta}p_\kappa}{\tilde{\Delta}\theta} = -\dot{p}_\kappa \tag{18}$$

$$\frac{\tilde{\Delta}\mathcal{H}_\kappa}{\tilde{\Delta}p_\kappa} = +\frac{\tilde{\Delta}q_\kappa}{\tilde{\Delta}\theta} = +\dot{q}_\kappa \tag{19}$$

Equation (13) may be rewritten into the form of Eq.(3.5-56) by means of the following definitions that replace Eqs.(3.5-55)

$$a_\kappa = \frac{d(\kappa)}{2\pi\kappa}e^{2\pi i\kappa\theta}, \quad a_\kappa^* = \frac{d(\kappa)}{2\pi\kappa}e^{-2\pi i\kappa\theta} \tag{20}$$

and we obtain

$$\mathcal{H} = -i\sum_{\kappa=0}^{N}2\pi\kappa p_\kappa q_\kappa = \sum_{\kappa=0}^{N}(2\pi\kappa)^2 a_\kappa a_\kappa^* = \sum_{\kappa=0}^{N}\frac{2\pi\kappa}{T}\hbar b_\kappa b_\kappa^* = \sum_{\kappa=0}^{N}\mathcal{H}_\kappa \tag{21}$$

$$b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa^*, \quad T = N\Delta t \tag{22}$$

For the quantization we follow the finite difference Schrödinger approach worked out in Section 3.6. We obtain in analogy to Eqs.(3.6-5) and (3.6-38):

$$\frac{1}{2}\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{\tilde{\Delta}^2}{\tilde{\Delta}\zeta^2}\right)\Phi = \frac{\mathcal{H}T}{2\pi\kappa\hbar}\Phi = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi = \lambda_\kappa\Phi$$

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots,\ N \tag{23}$$

The upper limit N_τ for n in Eq.(4.4-25) is replaced by N. There is no need for renormalization.

The energies $\mathsf{E}_{\kappa n}$ in Eq.(3.6-38) and (23) are equal since the photons of a pure EM wave are the same as of an EM wave interacting with bosons. What differs is the fraction of photons with a certain energy $U_{\mathrm{c}\kappa}(\kappa)$ according to Eq.(3.5-45) and one term of Eq.(10):

$$U_{\mathrm{c}\kappa}(\kappa)=\frac{L^2\Psi_1^2}{cT}N^2\lambda_1^2\lambda_3^2\,\frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}\left[\beta_\kappa^2+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2+\frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\right] \quad (24)$$

The term $m_0^2c^4/\hbar^2$ shows as at the end of Section 4.4 that one must get results different from those of the pure radiation field since a mass m_0 does not occur in the equation for a pure radiation field.

5.5 Plots for the Difference Theory

The energy $U_{\mathrm{c}\kappa}(\kappa)$ of the difference theory as function of the period number κ is defined by Eq.(5.4-24) with β_κ from Eq.(5.2-17), $I_{\mathrm{T}}(\kappa/N)$ from Eq.(5.2-32), λ_1, λ_2, λ_3 from Eq.(5.1-23) and κ_0 from Eq.(5.2-8):

$$\begin{aligned}
U_{\mathrm{c}\kappa}(\kappa) &= \frac{L^2\Psi_1^2N^2\lambda_1^2\lambda_3^2}{cT}\,\frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}\left[\beta_\kappa^2+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2+\left(\frac{m_0c^2\Delta t}{\hbar}\right)^2\right]\\
\beta_\kappa &= \arcsin\left(1-\frac{[1-\lambda_2^2/2-2\sin^2(\pi\kappa/N)]^2}{1+\lambda_1^2\lambda_3^2}\right)^{1/2}\\
\sin^2\beta_\kappa &= 1-\frac{[1-\lambda_2^2/2-2\sin^2(\pi\kappa/N)]^2}{1+\lambda_1^2\lambda_3^2}\\
I_{\mathrm{T}}(\kappa/N) &= \frac{2}{N}\,\frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2}\\
&\lambda_2^2-\lambda_1^2=(\Delta t)^2(m_0^2c^4-e^2\phi_{\mathrm{e}0}^2)/\hbar,\quad \lambda_1^2=e^2c^2(\Delta t)^2A_{\mathrm{m}0y}^2/\hbar^2\\
&\lambda_1^2\lambda_3^2=e^2(\Delta t)^2\phi_{\mathrm{e}0}^2/\hbar^2,\quad 0\le\kappa<\kappa_0=\frac{N}{\pi}\arcsin\left(1-\frac{\lambda_2^2-\lambda_1^2\lambda_3^2}{4}\right)^{1/2}\\
&\lambda_2^2-\lambda_1^2\lambda_3^2=(\Delta t)^2(m_0^2c^4-2e^2\phi_{\mathrm{e}0}^2+e^2c^2A_{\mathrm{m}0y}^2)/\hbar^2,\quad N=T/\Delta t
\end{aligned} \quad (1)$$

The energy of a photon with period number κ and a certain value of n is defined like in the differential theory by Eq.(5.4-23):

$$\mathsf{E}_{\kappa n}=\frac{2\pi\kappa\hbar}{T}\left(n+\frac{1}{2}\right),\quad n=0,\ 1,\ \ldots,\ N \quad (2)$$

The average value of $\mathsf{E}_{\kappa n}$ for all $N+1$ values of n is the same as in Eq.(4.5-3) but N_τ must be replaced by N:

$$\mathsf{E}_\kappa = \frac{2\pi\kappa\hbar}{T}\frac{1}{N+1}\sum_{\kappa=0}^{N}\left(n+\frac{1}{2}\right) = \frac{1}{2}(N+1)\frac{2\pi\kappa\hbar}{T} \doteq \frac{1}{2}N\frac{2\pi\kappa\hbar}{T} \tag{3}$$

For a specific value of n the energy $U_{c\kappa}(\kappa)$ requires the number $U_{c\kappa}(\kappa)/\mathsf{E}_{\kappa n}$of photons:

$$\begin{aligned}\frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{\kappa n}} &= D_{rn} r_\Delta(\kappa)\\ r_\Delta(\kappa) &= \frac{I_T^2(\kappa/N)}{\sin^2\beta_\kappa}\frac{1}{2\pi\kappa}\left[\beta_\kappa^2+\lambda_1^2\lambda_3^2+\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2+\frac{m_0^2c^4(\Delta t)^2}{\hbar^2}\right]\\ D_{rn} &= \frac{L^2\Psi_1^2N^2\lambda_1^2\lambda_3^2}{c\hbar}\left(n+\frac{1}{2}\right)^{-1}\end{aligned} \tag{4}$$

If photons with various values of n are equally frequent we obtain the following relation in place of Eq.(4):

$$\begin{aligned}\frac{U_{c\kappa}(\kappa}{\mathsf{E}_\kappa} &= D_r r_\Delta(\kappa)\\ D_r &= \frac{2L^2\Psi_1^2N\lambda_1^2\lambda_3^2}{c\hbar}\end{aligned} \tag{5}$$

The total number of photons equals the sum over κ in Eqs.(4) or (5):

$$s_{rn}^\Delta = \sum_{\kappa=0}^{<\kappa_0}\frac{U_{c\kappa}(\kappa)}{\mathsf{E}_{c\kappa}} = D_{rn}\sum_{\kappa=0}^{<\kappa_0} r_\Delta(\kappa) \tag{6}$$

$$s_r^\Delta = \sum_{\kappa=0}^{<\kappa_0}\frac{U_{c\kappa}(\kappa)}{\mathsf{E}_\kappa} = D_r\sum_{\kappa=0}^{<\kappa_0} r_\Delta(\kappa) \tag{7}$$

The probability $p_\Delta(\kappa)$ of a photon with period number κ is the same for Eqs.(4) and (5) since the constants D_{rn} and D_r drop out:

$$p_\Delta(\kappa) = \frac{r_\Delta(\kappa)}{\sum_{\kappa>0}^{<\kappa_0} r_\Delta(\kappa)} \tag{8}$$

For the computation of $r_\Delta(\kappa)$ and $p_\Delta(\kappa)$ one can make several simplifications in Eqs.(4) and (8). First we deal with the special case $\kappa = 0$. We see

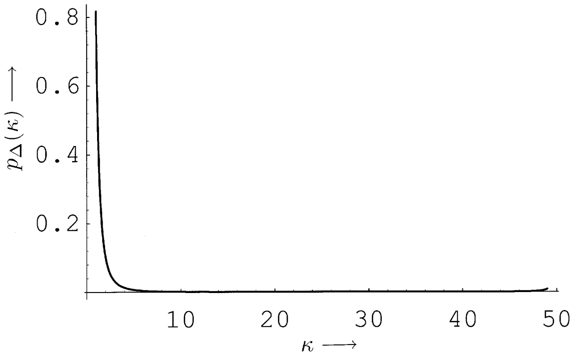

FIGURE 5.5-1. Plot of $p_\Delta(\kappa)$ according to Eq.(15) for $N = 100$ or $N/2-1 = 49$. The variable κ is treated as a continuous variable. Note the slight increase of the function at $\kappa = 49$.

from Fig.5.2-2 that β_κ and thus $\sin\beta_\kappa$ is not zero at $\varphi_\kappa = 2\pi\kappa/N = 0$ or $\kappa = 0$. The function $I_{\rm T}^2(\kappa/N)$ decreases like κ^2, the factor $1/2\pi\kappa$ increases like $1/\kappa$, and the terms in brackets become constant. Hence, $r_\Delta(0)$ and $p_\Delta(0)$ approach zero. For discrete values of κ we get $I_{\rm T}(\kappa/N)^2/2\pi\kappa = 0^2/0$ for $\kappa = 0$, which makes $r_\Delta(0)$ and $p_\Delta(0)$ undefined. We leave out the terms $\kappa = 0$ in Eqs.(6) to (8). This was alredy pointed out in the last sentence of Section 5.2. The following approximations hold for $1 \leq \kappa < \kappa_0$:

$$\kappa_0 = \frac{N}{\pi}\arcsin\left(1 - \frac{\lambda_2^2 - \lambda_1^2\lambda_3^2}{4}\right)^{1/2} < \frac{N}{\pi}\frac{\pi}{2} = \frac{N}{2}$$

$$1 \leq \kappa \leq N/2 - 1 < \kappa_0 \quad \text{for } \lambda_2^2 - \lambda_1^2\lambda_3^2 \ll 1 \tag{9}$$

$$\lambda_2^2 - \lambda_1^2\lambda_3^2 \ll 1 \quad \text{for } \Delta t \ll \hbar/m_0c^2 \tag{10}$$

$$(\lambda_1^2 - \lambda_2^2)^{1/2}N = N\Delta t(m_0^2c^4 - e^2\phi_{\rm e0}^2)^{1/2}/\hbar \gg 1, \quad \text{for } T \gg \hbar/m_0c^2 \tag{11}$$

$$r_\Delta(\kappa) = \left(\frac{2\pi\kappa/N}{\lambda_2^2 + (2\pi\kappa/N)^2}\right)^2 \frac{1}{\sin^2\beta_\kappa}\frac{1}{2\pi\kappa}\left[\beta_\kappa^2 + \left(\frac{2\pi\kappa}{N}\right)^2\right] \tag{12}$$

$$\lambda_2 = 2\pi q/N, \quad 0 < q < 1 \tag{13}$$

$$\sum_{\kappa=1}^{N/2-1} r_\Delta(\kappa) \doteq 98.743 \quad \text{for } N = 100,\ q = 0.1 \tag{14}$$

$$p_\Delta(\kappa) = r_\Delta(\kappa)/98.743 \tag{15}$$

Equation (10) states that Δt can be arbitrarily small but finite while Eq.(11) states that T can be arbitrarily large but finite. This is in line with our assumptions. We have the numerical value $h/m_0c^2 = 2.95241 \times 10^{-23}$ s for pions π^+ and π^-.

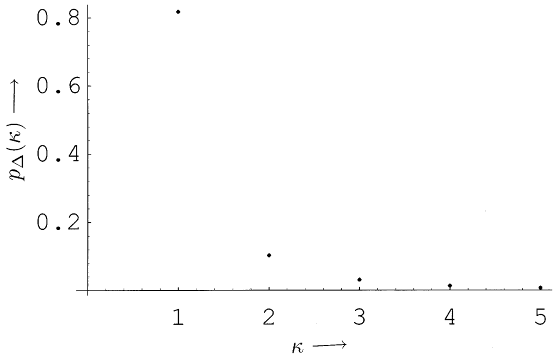

FIGURE 5.5-2. Plot of $p_\Delta(\kappa)$ for $\kappa = 5, 6, \ldots, 49$ according to Eq.(15) for $N = 100$.

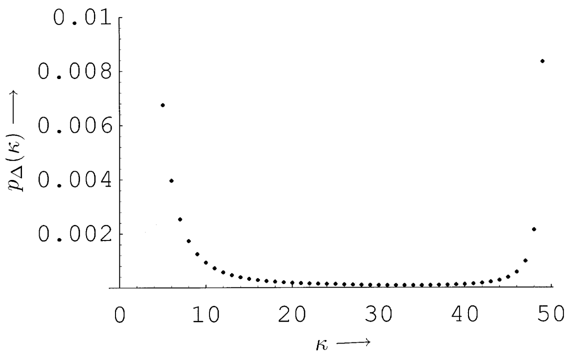

FIGURE 5.5-3. Plot of $p_\Delta(\kappa)$ for $\kappa = 5, 6, \ldots, 49$ according to Eq.(15) for $N = 100$. The vertical scale is enlarged by almost a factor of 100 compared with Figs.5.5-1 and 5.5-2. The points for $\kappa = 1, 2, 3, 4$ are outside the plotting range.

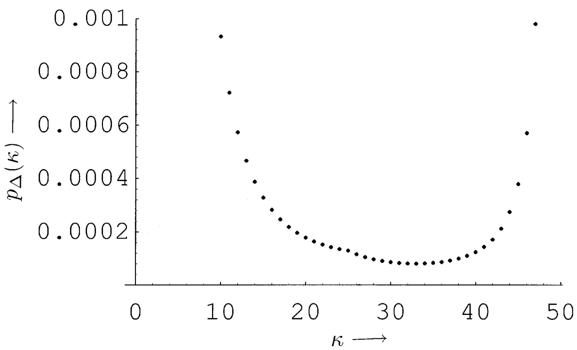

FIGURE 5.5-4. Plot of $p_\Delta(\kappa)$ for $\kappa = 10, 11, \ldots, 49$ according to Eq.(15) for $N = 100$. The vertical scale is enlarged by a factor of 10 compared with Fig.5.5-3. The points for $\kappa = 1, 2, \ldots, 9$ are outside the plotting range.

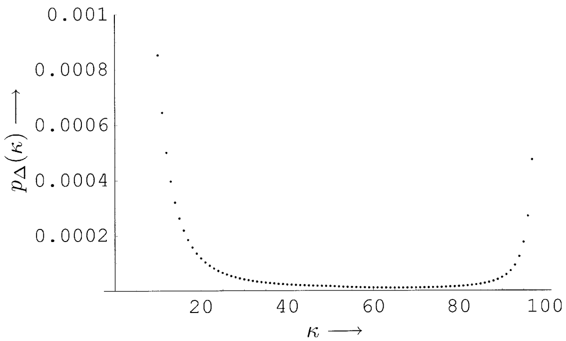

FIGURE 5.5-5. Plots of $p_\Delta(\kappa)$ for $\kappa = 10, 11, \ldots, 99$ according to Eq.(8) for $N = 200$, $\lambda_2 = 5 \times 10^{-5}$, $\lambda_1\lambda_3 = 0.1\lambda_2$ for the plot range $0 \leq p_\Delta(\kappa) \leq 0.001$.

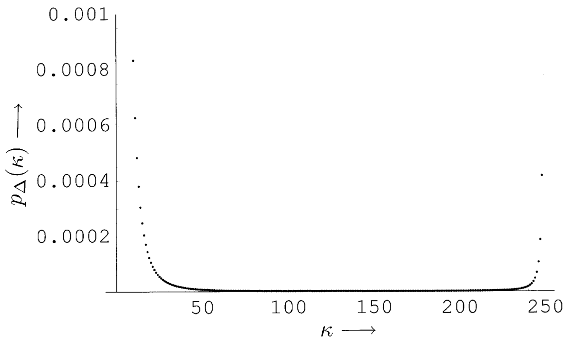

FIGURE 5.5-6. Plot of $p_\Delta(\kappa)$ for $\kappa = 10, 11, \ldots, 249$ according to Eq.(8) for $N = 500$, $\lambda_2 = 2 \times 10^{-5}$, $\lambda_1\lambda_3 = 0.1\lambda_2$ for the plot range $0 \leq p_\Delta(\kappa) \leq 0.001$.

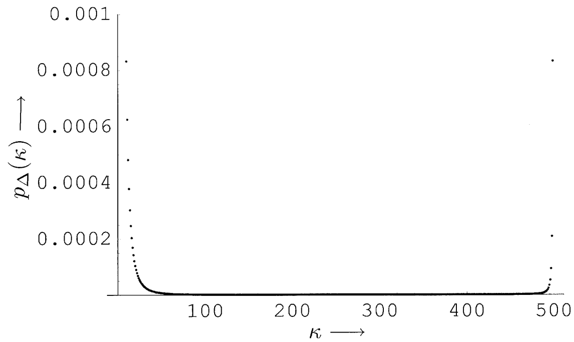

FIGURE 5.5-7. Plot of $p_\Delta(\kappa)$ for $\kappa = 10, 11, \ldots, 499$ according to Eq.(8) for $N = 1000$, $\lambda_2 = 10^{-5}$, $\lambda_1\lambda_3 = 0.1\lambda_2$ for the plot range $0 \leq p_\Delta(\kappa) \leq 0.001$.

Figure 5.5-1 shows $p_\Delta(\kappa)$ according to Eq.(15) for $N = 100$ or $N/2 - 1 = 49$ plotted as function of a continuous variable κ. We note the similarity to Fig.4.5-1; the only recognizable difference is the slight increase of the function in Fig.5.5-1 at the extreme right.

A discrete representation of $p_\Delta(\kappa)$ for $\kappa = 1, 2, 3, 4, 5$ is shown in Fig. 5.5-2. The extension of the discrete representation of $p_\Delta(\kappa)$ to the range $\kappa = 5, 6, \ldots, 49$ is shown in Fig.5.5-3. The vertical scale in Fig.5.5-3 is enlarged by almost a factor 100 compared with Figs.5.5-1 or 5.5-2. The larger scale puts $p_\Delta(1)$ to $p_\Delta(5)$ outside the plotting range.

A further enlargement of the vertical scale by a factor 10 is shown for $p_\Delta(\kappa)$ in Fig.5.5-4. The probabilities $p_\Delta(1)$ to $p_\Delta(9)$ are now outside the plotting range.

Let us compare the computer plots of Section 4.5 with the ones obtained here. In the difference theory the summation is symmetric, $-\kappa_0 < \kappa < \kappa_0$, as in Eq.(5.2-27), but the plotting is non-symmetric, $0 < \kappa < \kappa_0$. Hence, for $N = 100$ we sum 98 values from $\kappa = -N/2 + 1 > -\kappa_0$ to $\kappa = N/2 - 1 < \kappa_0$, but we plot 49 values from $\kappa = 1$ to $\kappa = N/2 - 1 < \kappa_0$. In the differential theory the summation is non-symmetric, $1 \leq \kappa \leq N_\tau$, as in Eq.(4.2-37) and the plotting is also non-symmetric, $1 \leq \kappa \leq N_\tau$. This difference turns out to be unimportant but in order to demonstrate this result we show plots for $N_\tau = 100$ in the range $1 \leq \kappa \leq 50$ in Figs.4.5-3 and 4.5-4, but plots for $N_\tau = 50$ in the range $1 \leq \kappa \leq 49$ in Figs.4.5-6 and 4.5-7; the upper limit 49 rather than 50 was chosen to increase the similarity with the plots of Figs.5.5-3 and 5.5-4.

The plots of Figs.4.5-1 and 5.5-1 are practically equal. Only a very keen observer will note the increase of $p_\Delta(\kappa)$ at $\kappa = 49$ in Fig.5.5-1 that does not exist close to $\kappa = 100$ in Fig.4.5-1. The enlarged vertical scale in Figs.4.5-3, 4.5-4 or 4.5-6, 4.5-7 and 5.5-3, 5.5-4 makes the deviation between the plots of the differential theory and the difference theory visible. We note that the largest value of $p(\kappa)$ in Figs.4.5-2 and 4.5-5 is about 0.8, just as in Fig.5.5-2 for $p_\Delta(\kappa)$. On the other hand, the smallest value of $p(\kappa)$ in Figs.4.5-4 and 4.5-7 is essentially zero while it is about $p_\Delta(33) \doteq 0.0001$ in Fig.5.5-4.

The question arises which features of Figs.5.5-2 to 5.5-4 are true results of the calculus of finite differences and which are only due to the choice $N = 100$. We show in Figs.5.5-5 to 5.5-7 plots like the one in Fig.5.5-4 but for larger values of N. We see that the center section of the plots becomes quite flat for increasing values of N. On the other hand, the increase of $p_\Delta(\kappa)$ in the range $40 \leq \kappa \leq 49$ in Fig.5.5-4 becomes an increase with about the same amplitudes but in the reduced range $490 \leq \kappa \leq 499$ in Fig.5.5-7. This appears more likely to be a true result of the use of finite differences.

To make this point more evident we show in Fig.5.5-8 the plot of Fig.5.5-7 but with the plot range reduced to 1/10. The idea is that an increase of N from 100 in Fig.5.5-4 to 1000 in Fig.5.5-8 should decrease the values of $p_\Delta(\kappa)$ by essentially a factor 1/10. Figure 5.5-8 shows a very flat center section and an increase of $p_\Delta(\kappa)$ restricted to about the interval $480 \leq \kappa \leq 499$.

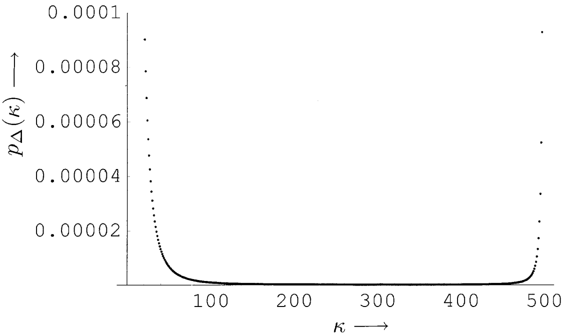

FIGURE 5.5-8. Plot of $p_\Delta(\kappa)$ for $\kappa = 19, 20, \ldots, 499$ according to Eq.(8) for $N = 1000$, $\lambda_2 = 10^{-5}$, $\lambda_1\lambda_3 = 0.1\lambda_2$ for the plot range $0 \leq p_\Delta(\kappa) \leq 0.0001$.

6 Appendix

6.1 Calculations for Section 2.2

The auxiliary variables d^2, q_1, q_2, q_3 and q_4 are used extensively. Here are some relations that simplify their use:

$$d^2 = 4[(2\pi\kappa/N_\tau)^2 + \rho_2^2] \tag{1}$$

$$q_1 = +\frac{1}{2}(d^2 - \rho_1^2)^{1/2} + 2\pi\kappa/N_\tau, \quad d^2 > \rho_1^2 \tag{2}$$

$$q_2 = +\frac{1}{2}(d^2 - \rho_1^2)^{1/2} - 2\pi\kappa/N_\tau, \quad d^2 > \rho_1^2 \tag{3}$$

$$q_3 = +\frac{1}{2}(\rho_1^2 - d^2)^{1/2} - \frac{1}{2}\rho_1, \quad d^2 < \rho_1^2 \tag{4}$$

$$q_4 = +\frac{1}{2}(\rho_1^2 - d^2)^{1/2} + \frac{1}{2}\rho_1, \quad d^2 < \rho_1^2 \tag{5}$$

$$q_2 = q_1 - 4\pi\kappa/N_\tau, \quad q_4 = q_3 + \rho_1 \tag{6}$$

$$q_1 = 2\pi\kappa/N_\tau + [(2\pi\kappa/N_\tau)^2 + \rho_2^2 - \rho_1^2/4]^{1/2} \tag{7}$$

$$q_2 = -2\pi\kappa/N_\tau + [(2\pi\kappa/N_\tau)^2 + \rho_2^2 - \rho_1^2/4)^{1/2} \tag{8}$$

$$(d^2 - \rho_1^2)^{1/2} = 2(q_1 - 2\pi\kappa/N_\tau) = 2(q_2 + 2\pi\kappa/N_\tau), \quad d^2 > \rho_1^2 \tag{9}$$

$$(\rho_1^2 - d^2)^{1/2} = 2q_3 + \rho_1 = 2q_4 - \rho_1, \quad d^2 < \rho_1^2 \tag{10}$$

$$q_1^2 + \left(\frac{\rho_1}{2}\right)^2 = 2\left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \rho_2^2 + 2\left(\frac{2\pi\kappa}{N_\tau}\right)^2\left(1 + \frac{\rho_2^2 - \rho_1^2/4}{(2\pi\kappa/N_\tau)^2}\right)^{1/2}, \ d^2 > \rho_1^2 \tag{11}$$

$$q_2^2 + \left(\frac{\rho_1}{2}\right)^2 = 2\left(\frac{2\pi\kappa}{N_\tau}\right)^2 + \rho_2^2 - 2\left(\frac{2\pi\kappa}{N_\tau}\right)^2\left(1 + \frac{\rho_2^2 - \rho_1^2/4}{(2\pi\kappa/N_\tau)^2}\right)^{1/2}, \ d^2 > \rho_1^2 \tag{12}$$

We start with the second sum holding for $\kappa > K$ in Eq.(2.2-28). Later we shall extend the investigation to the whole of Eq.(2.2-28). Equation (2.2-28) is shown once more in Eq.(2.2-39) with the terms $L_{15}(\theta,\kappa)$ to $L_{18}(\theta,\kappa)$ broken up into $L_{15\mathrm{A}}(\kappa)$, $L_{15\mathrm{B}}(\theta,\kappa)$, ... , $L_{18\mathrm{B}}(\theta,\kappa)$. We denote part of the first and second term of the second sum of Eq.(2.2-39) by $A_{\mathrm{es}}(\kappa)$ and $A_{\mathrm{ec}}(\kappa)$:

$$A_{\mathrm{es}}(\kappa) = \frac{1}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}\left(L_{17\mathrm{A}}(\kappa) + \frac{\rho_1 L_{15\mathrm{A}}(\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)$$
$$= \frac{\rho_1}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}\,\frac{1}{2(q_1 - 2\pi\kappa/N_\tau)}$$
$$\times\left(\frac{q_1 - \pi\kappa/N_\tau}{q_1^2 + (\rho_1/2)^2} + \frac{q_2 + \pi\kappa/N_\tau}{q_2^2 + (\rho_1/2)^2}\right) \qquad (13)$$

$$A_{\mathrm{ec}}(\kappa) = -\frac{1}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}\left(L_{18\mathrm{A}}(\kappa) + \frac{\rho_1 L_{16\mathrm{A}}(\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)$$
$$= -\frac{1}{(2\pi\kappa/N_\tau)^2 + \rho_2^2}\,\frac{1}{2(q_1 - 2\pi\kappa/N_\tau)}$$
$$\times\left(\frac{q_1(q_1 - 2\pi\kappa/N_\tau) - \rho_1^2/4}{q_1^2 + (\rho_1/2)^2} - \frac{q_2(q_2 + 2\pi\kappa/N_\tau) - \rho_1^2/4}{q_2^2 + (\rho_1/2)^2}\right)$$

$$\kappa > K = N_\tau(\rho_1^2 - 4\rho_2^2)^{1/2}/4\pi$$
$$= c\tau N_\tau|(\sigma Z - s/Z)|/4\pi \text{ if used for Eq.(2.2-28)} \qquad (14)$$

For the third term of the second sum in Eq.(2.2-39) we obtain with Eqs. (2.2-30)–(2.2-33):

$$\frac{(d^2-\rho_1^2)^{1/2}L_{17\mathrm{B}}(\theta,\kappa)+\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]} = -\frac{1}{2\,(2q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}$$
$$\times\left(\frac{[\rho_1(q_1-2\pi\kappa/N_\tau)+\rho_1 q_1]\cos q_1\theta - [2q_1(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2]\sin q_1\theta}{2\left[q_1^2+(\rho_1/2)^2\right]}\right.$$
$$\left.+\frac{[\rho_1(q_1-2\pi\kappa/N_\tau)+\rho_1 q_2]\cos q_2\theta - [2q_2(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2]\sin q_2\theta}{2\left[q_2^2+(\rho_1/2)^2\right]}\right)$$
$$= A_{57}(\kappa)\cos q_1\theta + B_{57}(\kappa)\sin q_1\theta + C_{57}(\kappa)\cos q_2\theta + D_{57}(\kappa)\sin q_2\theta \qquad (15)$$

$$A_{57}(\kappa) = -\frac{\rho_1(2q_1 - 2\pi\kappa/N_\tau)}{4(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]} \qquad (16)$$

$$B_{57}(\kappa) = +\frac{2q_1(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]} \qquad (17)$$

$$C_{57}(\kappa) = -\frac{\rho_1(q_1+q_2-2\pi\kappa/N_\tau)}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \tag{18}$$

$$D_{57}(\kappa) = +\frac{2q_2(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \tag{19}$$

The third term of the second sum in Eq.(2.2-39) may now be written as follows:

$$\begin{aligned}\left(L_{17\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)&\frac{\sin(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\\ &= A_{57}(\kappa)\cos q_1 \sin\frac{2\pi\kappa\theta}{N_\tau} + B_{57}(\kappa)\sin q_1 \sin\frac{2\pi\kappa\theta}{N_\tau}\\ &\quad + C_{57}(\kappa)\cos q_2\theta \sin\frac{2\pi\kappa\theta}{N_\tau} + D_{57}(\kappa)\sin q_2\theta \sin\frac{2\pi\kappa\theta}{N_\tau}\end{aligned} \tag{20}$$

Using the changed wavenumbers $2\pi\kappa/N_\tau \pm q_1$ and $2\pi\kappa/N_\tau \pm q_2$

$$2\pi\kappa/N_\tau - q_1 = \ \pi\kappa/N_\tau - (\pi^2\kappa^2/N_\tau^2+\rho_2^2-\rho_1^2/4)^{1/2} \tag{21}$$

$$2\pi\kappa/N_\tau + q_1 = 3\pi\kappa/N_\tau + (\pi^2\kappa^2/N_\tau^2+\rho_2^2-\rho_1^2/4)^{1/2} \tag{22}$$

$$2\pi\kappa/N_\tau - q_2 = 3\pi\kappa/N_\tau - (\pi^2\kappa^2/N_\tau^2+\rho_2^2-\rho_1^2/4)^{1/2} \tag{23}$$

$$2\pi\kappa/N_\tau + q_2 = \ \pi\kappa/N_\tau + (\pi^2\kappa^2/N_\tau^2+\rho_2^2-\rho_1^2/4)^{1/2} \tag{24}$$

we obtain for the products of sine and cosine functions in Eq.(20):

$$\sin(2\pi\kappa\theta/N_\tau)\cos q_1\theta = \frac{1}{2}[\sin(2\pi\kappa/N_\tau-q_1)\theta+\sin(2\pi\kappa/N_\tau+q_1)\theta] \tag{25}$$

$$\sin(2\pi\kappa\theta/N_\tau)\sin q_1\theta = \frac{1}{2}[\cos(2\pi\kappa/N_\tau-q_1)\theta-\cos(2\pi\kappa/N_\tau+q_1)\theta] \tag{26}$$

$$\sin(2\pi\kappa\theta/N_\tau)\cos q_2\theta = \frac{1}{2}[\sin(2\pi\kappa/N_\tau-q_2)\theta+\sin(2\pi\kappa/N_\tau+q_2)\theta] \tag{27}$$

$$\sin(2\pi\kappa\theta/N_\tau)\sin q_2\theta = \frac{1}{2}[\cos(2\pi\kappa/N_\tau-q_2)\theta-\cos(2\pi\kappa/N_\tau+q_2)\theta] \tag{28}$$

We turn to the fourth term of the second sum in Eq.(2.2-39). Instead of Eqs.(15)–(19) one obtains:

$$
\begin{aligned}
&-\frac{(d^2-\rho_1^2)^{1/2}L_{18\mathrm{B}}(\theta,\kappa)+\rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}=\frac{1}{2\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}\\
&\quad\times\Bigg(\frac{[\rho_1(q_1-2\pi\kappa/N_\tau)+\rho_1 q_1]\sin q_1\theta+[2q_1(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2]\cos q_1\theta}{2\left[q_1^2+(\rho_1/2)^2\right]}\\
&\quad-\frac{[\rho_1(q_1-2\pi\kappa/N_\tau)-\rho_1 q_2]\sin q_2\theta+[2q_2(q_1-2\pi\kappa/N_\tau)+\rho_1^2/2]\cos q_2\theta}{2\left[q_2^2+(\rho_1/2)^2\right]}\Bigg)\\
&\quad=A_{68}(\kappa)\sin q_1\theta+B_{68}(\kappa)\cos q_1\theta+C_{68}(\kappa)\sin q_2\theta+D_{68}(\kappa)\cos q_2\theta
\end{aligned}
\tag{29}
$$

$$A_{68}(\kappa)=+\frac{\rho_1(q_1-2\pi\kappa/N_\tau)+\rho_1 q_1}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]}=-A_{57}(\kappa) \tag{30}$$

$$B_{68}(\kappa)=+\frac{2q_1(q_1-2\pi\kappa/N_\tau)-\rho_1^2/2}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]}=+B_{57}(\kappa) \tag{31}$$

$$C_{68}(\kappa)=-\frac{\rho_1(q_1-2\pi\kappa/N_\tau)-\rho_1 q_2}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \tag{32}$$

$$D_{68}(\kappa)=-\frac{2q_2(q_1-2\pi\kappa/N_\tau)+\rho_1^2/2}{4\,(q_1-2\pi\kappa/N_\tau)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \tag{33}$$

The fourth term of the second sum in Eq.(2.2-39) may be written as follows:

$$
\begin{aligned}
&-\left(L_{18\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\cos(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}\\
&\qquad=A_{68}(\kappa)\sin q_1\theta\cos\frac{2\pi\kappa\theta}{N_\tau}+B_{68}(\kappa)\cos q_1\theta\cos\frac{2\pi\kappa\theta}{N_\tau}\\
&\qquad\quad+C_{68}(\kappa)\sin q_2\theta\cos\frac{2\pi\kappa\theta}{N_\tau}+D_{68}(\kappa)\cos q_2\theta\cos\frac{2\pi\kappa\theta}{N_\tau}
\end{aligned}
\tag{34}
$$

Using the changed wavenumbers of Eqs.(21)–(24) we obtain for the products of sine and cosine functions in Eq.(34):

$$\cos(2\pi\kappa\theta/N_\tau)\sin q_1\theta=\frac{1}{2}[-\sin(2\pi\kappa/N_\tau-q_1)\theta+\sin(2\pi\kappa/N_\tau+q_1)\theta] \tag{35}$$

$$\cos(2\pi\kappa\theta/N_\tau)\cos q_1\theta=\frac{1}{2}[+\cos(2\pi\kappa/N_\tau-q_1)\theta+\cos(2\pi\kappa/N_\tau+q_1)\theta] \tag{36}$$

$$\cos(2\pi\kappa\theta/N_\tau)\sin q_2\theta=\frac{1}{2}[-\sin(2\pi\kappa/N_\tau-q_2)\theta+\sin(2\pi\kappa/N_\tau+q_2)\theta] \tag{37}$$

$$\cos(2\pi\kappa\theta/N_\tau)\cos q_2\theta=\frac{1}{2}[+\cos(2\pi\kappa/N_\tau-q_2)\theta+\cos(2\pi\kappa/N_\tau+q_2)\theta] \tag{38}$$

We introduce four more auxiliary variables:

$$\begin{aligned} E_{57}(\kappa) &= \frac{1}{2}[C_{57}(\kappa) - C_{68}(\kappa)] \\ &= -\frac{\rho_1 q_1}{4\left(q_1 - 2\pi\kappa/N_\tau\right)\left[\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2\right]\left[q_2^2 + \left(\rho_1/2\right)^2\right]} \end{aligned} \quad (39)$$

$$\begin{aligned} E_{68}(\kappa) &= \frac{1}{2}[C_{57}(\kappa) + C_{68}(\kappa)] \\ &= -\frac{\rho_1(2q_1 - 2\pi\kappa/N_\tau)}{4\left(q_1 - 2\pi\kappa/N_\tau\right)\left[\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2\right]\left[q_2^2 + \left(\rho_1/2\right)^2\right]} \end{aligned} \quad (40)$$

$$\begin{aligned} F_{57}(\kappa) &= \frac{1}{2}[D_{57}(\kappa) + D_{68}(\kappa)] \\ &= -\frac{\rho_1^2/2}{4\left(q_1 - 2\pi\kappa/N_\tau\right)\left[\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2\right]\left[q_2^2 + \left(\rho_1/2\right)^2\right]} \end{aligned} \quad (41)$$

$$\begin{aligned} F_{68}(\kappa) &= -\frac{1}{2}[D_{57}(\kappa) - D_{68}(\kappa)] \\ &= -\frac{2q_2(q_1 - 2\pi\kappa/N_\tau)}{4\left(q_1 - 2\pi\kappa/N_\tau\right)\left[\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2\right]\left[q_2^2 + \left(\rho_1/2\right)^2\right]} \end{aligned} \quad (42)$$

The third and fourth terms of the second sum of Eq.(2.2-39) become:

$$\begin{aligned} &\left(L_{17\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\sin(2\pi\kappa\theta/N_\tau)}{\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2} \\ &\qquad - \left(L_{18\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\cos(2\pi\kappa\theta/N_\tau)}{\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2} \\ &= A_{57}(\kappa)\sin(2\pi\kappa/N_\tau - q_1)\theta + B_{68}(\kappa)\cos(2\pi\kappa/N_\tau - q_1)\theta \\ &\quad + E_{57}(\kappa)\sin(2\pi\kappa/N_\tau - q_2)\theta + E_{68}(\kappa)\sin(2\pi\kappa/N_\tau + q_2)\theta \\ &\quad + F_{57}(\kappa)\cos(2\pi\kappa/N_\tau - q_2)\theta + F_{68}(\kappa)\cos(2\pi\kappa/N_\tau + q_2)\theta \end{aligned} \quad (43)$$

We turn to the first sum in Eq.(2.2-28) that contains more complicated terms. With the help of Eqs.(2.2-35)–(2.2-38) we obtain:

$$\begin{aligned} A_{\mathrm{es}}(\kappa) &= \frac{(\rho_1^2-d^2)^{1/2}L_{13\mathrm{A}}(\kappa) + \rho_1 L_{11\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}\left[\left(2\pi\kappa/N_\tau\right)^2 + \rho_2^2\right]} \\ &= -\frac{q_3 q_4}{(2q_3+\rho_1)\left[\left(2\pi\kappa/N_\tau\right)^2+\rho_2^2\right]}\left(\frac{1}{\left(2\pi\kappa/N_\tau\right)^2+q_3^2} - \frac{1}{\left(2\pi\kappa/N_\tau\right)^2+q_4^2}\right) \end{aligned} \quad (44)$$

$$
\begin{aligned}
A_{\mathrm{ec}}(\kappa) &= -\frac{(\rho_1^2-d^2)^{1/2}L_{14\mathrm{A}}(\kappa)+\rho_1 L_{12\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]} \\
&= -\frac{2\pi\kappa/N_\tau}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}\left(\frac{q_4}{(2\pi\kappa/N_\tau)^2+q_3^2}+\frac{q_3}{(2\pi\kappa/N_\tau)^2+q_4^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\kappa < K &= N_\tau(\rho_1^2-4\rho_2^2)^{1/2}/4\pi \\
&= c\tau N_\tau|(\sigma Z - s/Z)|/4\pi \text{ if used for Eq.(3.2-28)}
\end{aligned}
\tag{45}
$$

The terms with the subscript B rather than A are more complicated:

$$
\begin{aligned}
&\frac{(\rho_1^2-d^2)^{1/2}L_{13\mathrm{B}}(\theta,\kappa)+\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]} = \frac{1}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]} \\
&\quad\times\Bigg(q_4 e^{(2q_3+\rho_1)\theta/2}\,\frac{q_3\cos(2\pi\kappa\theta/N_\tau)-(2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+q_3^2} \\
&\quad - q_3 e^{-(2q_3+\rho_1)\theta/2}\,\frac{q_4\cos(2\pi\kappa\theta/N_\tau)-(2\pi\kappa/N_\tau)\sin(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+q_4^2}\Bigg) \\
&= e^{(2q_3+\rho_1)\theta/2}\left(A_{13}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}+B_{13}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}\right) \\
&\qquad + e^{-(2q_3+\rho_1)\theta/2}\left(C_{13}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}+D_{13}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}\right)
\end{aligned}
\tag{46}
$$

$$
A_{13}(\kappa) = +\frac{q_3 q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_3^2\right]} \tag{47}
$$

$$
B_{13}(\kappa) = -\frac{(2\pi\kappa/N_\tau)q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_3^2\right]} \tag{48}
$$

$$
C_{13}(\kappa) = -\frac{q_3 q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_4^2\right]} \tag{49}
$$

$$
D_{13}(\kappa) = +\frac{(2\pi\kappa/N_\tau)q_3}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_4^2\right]} \tag{50}
$$

We still need the terms with subscripts 14B and 12B:

$$
\begin{aligned}
&-\frac{(\rho_1^2-d^2)^{1/2}L_{14\mathrm{B}}(\theta,\kappa)+\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}=-\frac{1}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]}\\
&\qquad\times\Bigg(q_4 e^{(2q_3+\rho_1)\theta/2}\ \frac{q_3\sin(2\pi\kappa\theta/N_\tau)-(2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+q_3^2}\\
&\qquad-q_3 e^{-(2q_3+\rho_1)\theta/2}\ \frac{q_4\sin(2\pi\kappa\theta/N_\tau)+(2\pi\kappa/N_\tau)\cos(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+q_4^2}\Bigg)\\
&=e^{(2q_3+\rho_1)\theta/2}\left(A_{24}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+B_{24}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right)\\
&\qquad\qquad+e^{-(2q_3+\rho_1)\theta/2}\left(C_{24}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+D_{24}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right)
\end{aligned}
\tag{51}
$$

$$
A_{24}(\kappa)=-\frac{q_3q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_3^2\right]} \tag{52}
$$

$$
B_{24}(\kappa)=+\frac{(2\pi\kappa/N_\tau)q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_3^2\right]} \tag{53}
$$

$$
C_{24}(\kappa)=+\frac{q_3q_4}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_4^2\right]} \tag{54}
$$

$$
D_{24}(\kappa)=+\frac{(2\pi\kappa/N_\tau)q_3}{(2q_3+\rho_1)\left[(2\pi\kappa/N_\tau)^2+\rho_2^2\right]\left[(2\pi\kappa/N_\tau)^2+q_4^2\right]} \tag{55}
$$

The comparison of Eqs.(52) to (55) with Eqs.(47) to (50) provides the following relations:

$$A_{24}(\kappa)=-A_{13}(\kappa) \tag{56}$$

$$B_{24}(\kappa)=-B_{13}(\kappa) \tag{57}$$

$$C_{24}(\kappa)=+C_{13}(\kappa) \tag{58}$$

$$D_{24}(\kappa)=+D_{13}(\kappa) \tag{59}$$

The sum of Eq.(46) multiplied by $\sin(2\pi\kappa\theta/N_\tau)$ and of Eq.(51) multiplied by $\cos(2\pi\kappa\theta/N_\tau)$ yields:

$$\left(L_{13\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\sin(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}$$
$$-\left(L_{14\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\cos(2\pi\kappa\theta/N_\tau)}{(2\pi\kappa/N_\tau)^2+\rho_2^2}$$
$$=-e^{(2q_3+\rho_1)\theta/2}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau}$$
$$+e^{-(2q_3+\rho_1)\theta/2}\left(C_{13}(\kappa)\sin\frac{4\pi\kappa\theta}{N_\tau}+D_{13}(\kappa)\right) \quad (60)$$

Substitution of Eqs.(13), (14), (43)–(45), and (60) into Eq.(2.2-28) yields:

$$A_{\mathrm{ev}}(\zeta,\theta)=c^2\tau^2V_{\mathrm{e0}}\Bigg(\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\mathrm{ch}\,\rho_2\theta)+\frac{2}{N_\tau}(1-e^{-N_\tau\rho_2})$$
$$\times\Bigg\{\sum_{\kappa=1}^{<K}\left(A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right)\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+e^{-\rho_1\theta/2}$$
$$\times\Bigg[\sum_{\kappa=1}^{<K}-e^{(2q_3+\rho_1)\theta/2}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau}$$
$$+\sum_{\kappa=1}^{<K}e^{-(2q_3+\rho_1)\theta/2}\left(C_{13}(\kappa)\sin\frac{4\pi\kappa\theta}{N_\tau}+D_{13}(\kappa)\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+\sum_{\kappa>K}^{N_\tau}\left(A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right)\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+e^{-\rho_1\theta/2}$$
$$\times\Bigg(\sum_{\kappa>K}^{N_\tau}[A_{57}(\kappa)\sin(2\pi\kappa/N_\tau-q_1)\theta+B_{68}(\kappa)\cos(2\pi\kappa/N_\tau-q_1)\theta]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+\sum_{\kappa>K}^{N_\tau}[E_{57}(\kappa)\sin(2\pi\kappa/N_\tau-q_2)\theta+F_{57}(\kappa)\cos(2\pi\kappa/N_\tau-q_2)\theta]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+\sum_{\kappa>K}^{N_\tau}[E_{68}(\kappa)\sin(2\pi\kappa/N_\tau+q_2)\theta+F_{68}(\kappa)\cos(2\pi\kappa/N_\tau+q_2)\theta]\sin\frac{2\pi\kappa\zeta}{N_\tau}\Bigg)\Bigg\}\Bigg)$$
$$\kappa=1,\ 2,\ \ldots,<K,\ >K,\ \ldots,N_\tau \quad (61)$$

The term $e^{(2q_3+\rho_1)\theta/2}$ may cause concern but it is multiplied with $e^{-\rho_1\theta/2}$. The exponent of the product equals:

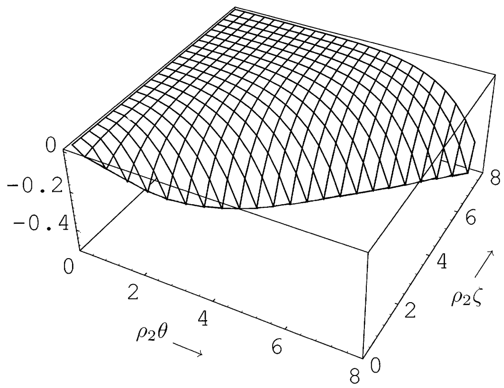

FIGURE 6.1-1. Three-dimensional plot of the function $\exp(-\rho_2\zeta)(1 - \operatorname{ch}\rho_2\theta)$ for $\zeta - \theta \geq 0$ in the interval $0 \leq \rho_2\zeta \leq 8$, $0 \leq \rho_2\theta \leq 8$.

$$\begin{aligned}[(2q_3 + \rho_1) - \rho_1]\theta/2 &= [(\rho_1^2 - d^2)^{1/2} - \rho_1]\theta/2 \\ &= \{[\rho_1^2 - 4(2\pi\kappa/N_\tau)^2 - 4\rho_2^2]^{1/2} - \rho_1\}\theta/2 \qquad (62)\end{aligned}$$

The relation

$$(\rho_1^2 - d^2)^{1/2} = [\rho_1^2 - 4(2\pi\kappa/N_\tau)^2 - 4\rho_2^2]^{1/2} > 0$$

implies

$$[\rho_1^2 - 4(2\pi\kappa/N_\tau)^2 - 4\rho_2^2] - \rho_1 < 0 \qquad (63)$$

and the terms in Eq.(61) multiplied with $e^{-\rho_1\theta/2}e^{(2q_3+\rho_1)\theta/2}$ become very small for large values of θ.

Consider the very first term in Eq.(61). To recognize what values it may assume we rewrite it as follows:

$$\begin{aligned}\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1 - \operatorname{ch}\rho_2\theta) &= \frac{1}{\rho_2^2}\left(e^{-\rho_2\zeta} - \frac{1}{2}e^{-\rho_2(\zeta-\theta)} - \frac{1}{2}e^{-\rho_2(\zeta+\theta)}\right) \\ &\doteq -\frac{1}{2\rho_2^2}e^{-\rho_2(\zeta-\theta)} \quad \text{for } \zeta,\ \theta \gg 1 \qquad (64)\end{aligned}$$

The constraint $\zeta - \theta \geq 0$ ensures that this term will vary only in the interval from 0 to $-1/2\rho_2^2$. Figure 6.1-1 shows this variation in detail.

Equation (61) contains functions like $e^{-\rho_1\theta/2}$ and $e^{(2q_3+\rho_1)\theta/2}$ that represent attenuation due to losses. Such terms have no meaning in quantum mechanics since photons are never attenuated. To eliminate these attenuation terms, as well as the phase shifted arguments $2\pi\kappa/N_\tau - q_1$ and $2\pi\kappa/N_\tau \pm q_2$ of some of the sine and cosine functions we resort to the Fourier series. Following Fig.2.2-1 and Eq.(2.2-5) we replace $\kappa\zeta$ by $\nu\theta$ and write as follows:

$$f_\kappa(\theta) = g_0 + \sum_{\nu=1}^{N_\tau}\left(g_{s\kappa}(\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + g_{c\kappa}(\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)$$

$$g_{s\kappa}(\nu) = \frac{2}{N_\tau}\int_0^{N_\tau} f_\kappa(\theta)\sin\frac{2\pi\nu\theta}{N_\tau}d\theta,\ \ g_{c\kappa}(\nu) = \frac{2}{N_\tau}\int_0^{N_\tau} f_\kappa(\theta)\cos\frac{2\pi\nu\theta}{N_\tau}d\theta$$

$$g_0 = \frac{1}{N_\tau}\int_0^{N_\tau} f_\kappa(\theta)\,d\theta, \quad 0 \le \theta = t/\tau \le T/\tau = N_\tau \tag{65}$$

We apply this series expansion to the second sum in Eq.(61):

$$f_{\kappa 1}(\theta) = e^{q_3\theta}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau} \tag{66}$$

$$g_{s\kappa 1}(\nu) = \frac{2}{N_\tau}\int_0^{N_\tau} e^{q_3\theta}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\nu\theta}{N_\tau}d\theta \tag{67}$$

$$g_{c\kappa 1}(\nu) = \frac{2}{N_\tau}\int_0^{N_\tau} e^{q_3\theta}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau}\cos\frac{2\pi\nu\theta}{N_\tau}d\theta \tag{68}$$

$$g_0 = 0 \tag{69}$$

The following integrals are required to evaluate $g_{s\kappa 1}(\nu)$ and $g_{c\kappa 1}(\nu)$:

$$I_1(\kappa,\nu) = 2\int_0^{N_\tau} e^{q_3\theta}\cos\frac{4\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\nu\theta}{N_\tau}d\theta = I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu) \tag{70}$$

$$I_2(\kappa,\nu) = 2\int_0^{N_\tau} e^{q_3\theta}\cos\frac{4\pi\kappa\theta}{N_\tau}\cos\frac{2\pi\nu\theta}{N_\tau}d\theta = I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu) \tag{71}$$

$$I_{31}(\kappa,\nu) = \int_0^{N_\tau} e^{q_3\theta}\sin\frac{2\pi(2\kappa+\nu)}{N_\tau}d\theta = +\frac{2\pi[(2\kappa+\nu)/N_\tau](1-e^{N_\tau q_3})}{q_3^2 + [2\pi(2\kappa+\nu)/N_\tau]^2} \tag{72}$$

$$I_{32}(\kappa,\nu) = \int_0^{N_\tau} e^{q_3\theta}\sin\frac{2\pi(2\kappa-\nu)}{N_\tau}d\theta = -\frac{2\pi[(2\kappa-\nu)/N_\tau](1-e^{N_\tau q_3})}{q_3^2 + [2\pi(2\kappa-\nu)/N_\tau]^2} \tag{73}$$

$$I_{41}(\kappa,\nu) = \int_0^{N_\tau} e^{q_3\theta} \cos\frac{2\pi(2\kappa+\nu)}{N_\tau} N_\tau d\theta = -\frac{q_3(1-e^{N_\tau q_3})}{q_3^2+[2\pi(2k+\nu)/N_\tau]^2} \tag{74}$$

$$I_{42}(\kappa,\nu) = \int_0^{N_\tau} e^{q_3\theta} \cos\frac{2\pi(2\kappa-\nu)}{N_\tau} d\theta = -\frac{q_3(1-e^{N_\tau q_3})}{q_3^2+[2\pi(2\kappa-\nu)/N_\tau]^2} \tag{75}$$

We obtain for $g_{s\kappa 1}(\nu)$, $g_{c\kappa 1}(\nu)$, and $f_{\kappa 1}(\theta)$:

$$g_{s\kappa 1}(\nu) = B_{13}(\kappa)[I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu)] \tag{76}$$

$$g_{c\kappa 1}(\nu) = B_{13}(\kappa)[I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu)] \tag{77}$$

$$\begin{aligned} f_{\kappa 1}(\theta) = \sum_{\iota=1}^{N_\tau} B_{13}(\kappa)\bigg([I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu)] \sin\frac{2\pi\nu\theta}{N_\tau} \\ + [I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu)] \cos\frac{2\pi\nu\theta}{N_\tau}\bigg) \end{aligned} \tag{78}$$

The second sum in Eq.(61) becomes:

$$\begin{aligned} &\sum_{\kappa=1}^{<K} -e^{q_3\theta} B_{13}(\kappa) \cos\frac{4\pi\kappa\theta}{N_\tau} \sin\frac{2\pi\kappa\zeta}{N_\tau} \\ &= -\sum_{\kappa=1}^{<K}\sum_{\nu=1}^{N_\tau} B_{13}(\kappa)\left(I_1(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + I_2(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\sin\frac{2\pi\kappa\zeta}{N_\tau} \end{aligned} \tag{79}$$

$$I_1(\kappa,\nu) = (1-e^{N_\tau q_3})\left(\frac{2\pi(2\kappa+\nu)/N_\tau}{q_3^2+[2\pi(2\kappa+\nu)/N_\tau]^2} - \frac{2\pi(2\kappa-\nu)/N_\tau}{q_3^2+[2\pi(2\kappa-\nu)/N_\tau]^2}\right) \tag{80}$$

$$I_2(\kappa,\nu) = -(1-e^{N_\tau q_3})\left(\frac{q_3}{q_3^2+[2\pi(2\kappa+\nu)/N_\tau]^2} + \frac{q_3}{q_3^2+[2\pi(2\kappa-\nu)/N_\tau]^2}\right) \tag{81}$$

We turn to the third sum in Eq.(61):

$$f_{\kappa 2}(\theta) = e^{-(q_3+\rho_1)\theta}\left(C_{13}(\kappa)\sin\frac{4\pi\kappa\theta}{N_\tau} + D_{13}(\kappa)\right) \tag{82}$$

Following the steps from Eq.(66) to (77) we get:

$$\sum_{\kappa=1}^{<K} e^{-(q_3+\rho_1)\theta}\left(C_{13}(\kappa)\sin\frac{4\pi\kappa\theta}{N_\tau}+D_{13}(\kappa)\right)\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$=\sum_{\kappa=1}^{<K}\sum_{\nu=1}^{N_\tau}\left[\left(C_{13}(\kappa)I_5(\kappa,\nu)+D_{13}(\kappa)I_6(\kappa,\nu)\right)\sin\frac{2\pi\nu\theta}{N_\tau}\right.$$
$$\left.+\left(C_{13}(\kappa)I_7(\kappa,\nu)+D_{13}(\kappa)I_8(\kappa,\nu)\right)\cos\frac{2\pi\nu\theta}{N_\tau}\right]\sin\frac{2\pi k\zeta}{N_\tau} \quad (83)$$

The four integrals $I_5(\kappa,\nu)$ to $I_8(\kappa,\nu)$ are defined as follows:

$$I_5(\kappa,\nu)=2\int_0^{N_\tau} e^{-(q_3+\rho_1)\theta}\sin\frac{4\pi\kappa\theta}{N_\tau}\sin\frac{2\pi\nu\theta}{N_\tau}d\theta=-(q_3+\rho_1)\left(1-e^{-N_\tau(q_3+\rho_1)}\right)$$
$$\times\left(\frac{1}{(q_3+\rho_1)^2+[2\pi(2\kappa+\nu)/N_\tau]^2}-\frac{1}{(q_3+\rho_1)^2+[2\pi(2\kappa-\nu)/N_\tau]^2}\right) \quad (84)$$

$$I_6(\kappa,\nu)=2\int_0^{N_\tau} e^{-(q_3+\rho_1)\theta}\sin\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$=2\left(1-e^{-N_\tau(q_3+\rho_1)}\right)\frac{2\pi\nu/N_\tau}{(q_3^2+\rho_1)^2+(2\pi\nu/N_\tau)^2} \quad (85)$$

$$I_7(\kappa,\nu)=2\int_0^{N_\tau} e^{-(q_3+\rho_1)\theta}\sin\frac{4\pi\kappa\theta}{N_\tau}\cos\frac{2\pi\nu\theta}{N_\tau}d\theta=\left(1-e^{-N_\tau(q_3+\rho_1)}\right)$$
$$\times\left(\frac{2\pi(2\kappa+\nu)/N_\tau}{(q_3+\rho_1)^2+[2\pi(2\kappa+\nu)/N_\tau]^2}+\frac{2\pi(2\kappa-\nu)/N_\tau}{(q_3+\rho_1)^2+[2\pi(2\kappa-\nu)/N_\tau]^2}\right) \quad (86)$$

$$I_8(\kappa,\nu)=2\int_0^{N_\tau} e^{-(q_3+\rho_1)\theta}\cos\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$=2\left(1-e^{-N_\tau(q_3+\rho_1)}\right)\frac{q_3+\rho_1}{(q_3+\rho_1)^2+(2\pi\nu/N_\tau)^2} \quad (87)$$

The first three sums in Eq.(61) may now be combined:

$$\sum_{\kappa=1}^{<K}\Bigg[A_{\rm es}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+A_{\rm ec}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$
$$-e^{q_3\theta}B_{13}(\kappa)\cos\frac{4\pi\kappa\theta}{N_\tau}$$
$$+e^{-(q_3+\rho_1)\theta}\left(C_{13}(\kappa)\sin\frac{4\pi\kappa\theta}{N_\tau}+D_{13}(\kappa)\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$=\sum_{\kappa=1}^{<K}\Bigg[A_{\rm es}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+A_{\rm ec}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$
$$+\sum_{\nu=1}^{N_\tau}\left(B_{\rm es}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau}+B_{\rm ec}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$B_{\rm es}(\kappa,\nu)=B_{13}(\kappa)I_1(\kappa,\nu)+C_{13}(\kappa)I_5(\kappa,\nu)+D_{13}(\kappa)I_6(\kappa,\nu)$$
$$B_{\rm ec}(\kappa,\nu)=B_{13}(\kappa)I_2(\kappa,\nu)+C_{13}(\kappa)I_7(\kappa,\nu)+D_{13}(\kappa)I_8(\kappa,\nu) \qquad (88)$$

We turn to the last three sums in Eq.(61). They are all multiplied by $e^{-\rho_1\theta/2}$ and they all have a shift $-q_1$ or $\pm q_2$ in the arguments of the sine and cosine functions. Following Eq.(82) we write:

$$f_{\kappa 3}(\theta)=e^{-\rho_1\theta/2}[A_{57}(\kappa)\sin(2\pi\kappa/N_\tau-q_1)\theta+B_{68}(\kappa)\cos(2\pi\kappa/N_\tau-q_1)\theta] \qquad (89)$$

Following the steps from Eq.(66) to (77) we get:

$$e^{-\rho_1\theta/2}\sum_{\kappa>K}^{N_\tau}\left[A_{57}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}-q_1\right)\theta+B_{68}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}-q_1\right)\theta\right]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$=\sum_{\kappa>K}^{N_\tau}\sum_{\nu=1}^{N_\tau}\Bigg([A_{57}(\kappa)I_9(\kappa,\nu)+B_{68}(\kappa)I_{10}(\kappa,\nu)]\sin\frac{2\pi\nu\theta}{N_\tau}$$
$$+[A_{57}(\kappa)I_{11}(\kappa,\nu)+B_{68}(\kappa)I_{12}(\kappa,\nu)]\cos\frac{2\pi\nu\theta}{N_\tau}\Bigg)\sin\frac{2\pi\kappa\zeta}{N_\tau} \qquad (90)$$

The integrals $I_9(\kappa,\nu)$ to $I_{12}(\kappa,\nu)$ are more complicated than the previous integrals $I_1(\kappa,\nu)$ to $I_8(\kappa,\nu)$:

$$I_9(\kappa,\nu) = 2\int\limits_0^{N_\tau} e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau} - q_1\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$= \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\cos q_1N_\tau + [2\pi(\kappa+\nu)/N_\tau - q_1]\sin q_1N_\tau\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa+\nu)/N_\tau - q_1]^2}$$
$$- \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\cos q_1N_\tau + [2\pi(\kappa-\nu)/N_\tau - q_1]\sin q_1N_\tau\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa-\nu)/N_\tau - q_1]^2} \quad (91)$$

$$I_{10}(\kappa,\nu) = 2\int\limits_0^{N_\tau} e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau} - q_1\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$= \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\sin q_1N_\tau - [2\pi(\kappa+\nu)/N_\tau - q_1]\cos q_1N_\tau\} + 2\pi(\kappa+\nu)/N_\tau - q_1}{(\rho_1/2)^2 + [2\pi(\kappa+\nu)/N_\tau - q_1]^2}$$
$$- \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\sin q_1N_\tau - [2\pi(\kappa-\nu)/N_\tau - q_1]\cos q_1N_\tau\} + 2\pi(\kappa-\nu)/N_\tau - q_1}{(\rho_1/2)^2 + [2\pi(\kappa-\nu)/N_\tau - q_1]^2} \quad (92)$$

$$I_{11}(\kappa,\nu) = 2\int\limits_0^{N_\tau} e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau} - q_1\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$= \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\sin q_1N_\tau - [2\pi(\kappa+\nu)/N_\tau - q_1]\cos q_1N_\tau\} + 2\pi(\kappa+\nu)/N_\tau - q_1}{(\rho_1/2)^2 + [2\pi(\kappa+\nu)/N_\tau - q_1]^2}$$
$$+ \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\sin q_1N_\tau - [2\pi(\kappa-\nu)/N_\tau - q_1]\cos q_1N_\tau\} + 2\pi(\kappa-\nu)/N_\tau - q_1}{(\rho_1/2)^2 + [2\pi(\kappa-\nu)/N_\tau - q_1]^2} \quad (93)$$

$$I_{12}(\kappa,\nu) = 2\int\limits_0^{N_\tau} e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau} - q_1\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta$$
$$= -\frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\cos q_1N_\tau + [2\pi(\kappa+\nu)/N_\tau - q_1]\sin q_1N_\tau\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa+\nu)/N_\tau - q_1]^2}$$
$$- \frac{e^{-N_\tau\rho_1/2}\{(\rho_1/2)\cos q_1N_\tau + [2\pi(\kappa-\nu)/N_\tau - q_1]\sin q_1N_\tau\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa-\nu)/N_\tau - q_1]^2} \quad (94)$$

The second sum from the end in Eq.(61) is written in analogy to Eq.(82) as follows:

$$f_{\kappa 4}(\theta) = e^{-\rho_1\theta/2}\left[E_{57}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau} - q_1\right)\theta + F_{57}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau} - q_2\right)\theta\right] \quad (95)$$

Again we follow the steps from Eq.(66) to (77) and obtain:

$$
\begin{aligned}
e^{-\rho_1\theta/2}\sum_{\kappa>K}^{N_\tau}\left[E_{57}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta+F_{57}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta\right]\sin\frac{2\pi\kappa\zeta}{N_\tau} \\
=\sum_{\kappa>K}^{N_\tau}\sum_{\nu=1}^{N_\tau}\Bigg([E_{57}(\kappa)I_{13}(\kappa,\nu)+F_{57}(\kappa)I_{14}(\kappa,\nu)]\sin\frac{2\pi\nu\theta}{N_\tau} \\
+[E_{57}(\kappa)I_{15}(\kappa,\nu)+F_{57}(\kappa)I_{16}(\kappa,\nu)]\cos\frac{2\pi\nu\theta}{N_\tau}\Bigg)\sin\frac{2\pi\kappa\zeta}{N_\tau} \qquad (96)
\end{aligned}
$$

The integrals $I_{13}(\kappa,\nu)$ to $I_{16}(\kappa,\nu)$ can be written with the help of the integrals $I_9(\kappa,\nu)$ to $I_{12}(\kappa,\nu)$:

$$
\begin{aligned}
I_{13}(\kappa,\nu) &= 2\int_0^{N_\tau} e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta \\
&= I_9(\kappa,\nu)\text{ with }q_1\text{ replaced by }q_2 \qquad (97) \\
I_{14}(\kappa,\nu) &= 2\int_0^{N_\tau} e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta \\
&= I_{10}(\kappa,\nu)\text{ with }q_1\text{ replaced by }q_2 \qquad (98) \\
I_{15}(\kappa,\nu) &= 2\int_0^{N_\tau} e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta \\
&= I_{11}(\kappa,\nu)\text{ with }q_1\text{ replaced by }q_2 \qquad (99) \\
I_{16}(\kappa,\nu) &= 2\int_0^{N_\tau} e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta \\
&= I_{12}(\kappa,\nu)\text{ with }q_1\text{ replaced by }q_2 \qquad (100)
\end{aligned}
$$

We turn to the last sum in Eq.(61). In analogy to Eq.(82) we write it in the following form:

$$
f_{\kappa 5}(\theta)=e^{-\rho_1\theta/2}\left[E_{68}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta+F_{68}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\right] \qquad (101)
$$

Following the steps from Eq.(66) to (77) a final time we obtain:

$$
\begin{aligned}
&e^{-\rho_1\theta/2}\sum_{\kappa>K}^{N_\tau}\left[E_{68}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta+F_{68}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\right]\sin\frac{2\pi\kappa\zeta}{N_\tau}\\
&\qquad=\sum_{\kappa>K}^{N_\tau}\sum_{\nu=1}^{N_\tau}\Bigg([E_{68}(\kappa)I_{17}(\kappa,\nu)+F_{68}(\kappa)I_{18}(\kappa,\nu)]\sin\frac{2\pi\nu\theta}{N_\tau}\\
&\qquad\quad+[E_{68}(\kappa)I_{19}(\kappa,\nu)+F_{68}(\kappa)I_{20}(\kappa,\nu)]\cos\frac{2\pi\nu\theta}{N_\tau}\Bigg)\sin\frac{2\pi\kappa\zeta}{N_\tau} \qquad (102)
\end{aligned}
$$

The integrals $I_{17}(k,\nu)$ to $I_{20}(k,\nu)$ can be written with the help of the integrals $I_9(k,\nu)$ to $I_{12}(k,\nu)$:

$$
\begin{aligned}
I_{17}(\kappa,\nu)&=2\int_0^{N_\tau}e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta\\
&=I_9(\kappa,\nu)\text{ with }q_1\text{ replaced by }-q_2 \qquad (103)\\
I_{18}(\kappa,\nu)&=2\int_0^{N_\tau}e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\,\sin\frac{2\pi\nu\theta}{N_\tau}d\theta\\
&=I_{10}(\kappa,\nu)\text{ with }q_1\text{ replaced by }-q_2 \qquad (104)\\
I_{19}(\kappa,\nu)&=2\int_0^{N_\tau}e^{-\rho_1\theta/2}\sin\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta\\
&=I_{11}(\kappa,\nu)\text{ with }q_1\text{ replaced by }-q_2 \qquad (105)\\
I_{20}(\kappa,\nu)&=2\int_0^{N_\tau}e^{-\rho_1\theta/2}\cos\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\,\cos\frac{2\pi\nu\theta}{N_\tau}d\theta\\
&=I_{12}(\kappa,\nu)\text{ with }q_1\text{ replaced by }-q_2 \qquad (106)
\end{aligned}
$$

The sums 4 to 7 of Eq.(61) may be combined:

$$
\begin{aligned}
&\sum_{\kappa>K}^{N_\tau}\Bigg\{A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau}+A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\\
&+e^{-\rho_1\theta/2}\Bigg[A_{57}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}-q_1\right)\theta+B_{68}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}-q_1\right)\theta\\
&\quad+E_{57}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta+F_{57}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}-q_2\right)\theta
\end{aligned}
$$

$$+ E_{68}(\kappa)\sin\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta + F_{68}(\kappa)\cos\left(\frac{2\pi\kappa}{N_\tau}+q_2\right)\theta\Bigg]\Bigg\}\sin\frac{2\pi\kappa\zeta}{N_\tau}$$

$$= \sum_{\kappa>K}^{N_\tau}\Bigg[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$

$$+\sum_{\nu=1}^{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$

$$B_{\mathrm{es}}(\kappa,\nu) = A_{57}(\kappa)I_9(\kappa,\nu) + B_{68}(\kappa)I_{10}(\kappa,\nu) + E_{57}(\kappa)I_{13}(\kappa,\nu) + F_{57}(\kappa)I_{14}(\kappa,\nu) + E_{68}(\kappa)I_{17}(\kappa,\nu) + F_{68}(\kappa)I_{18}(\kappa,\nu)$$

$$B_{\mathrm{ec}}(\kappa,\nu) = A_{57}(\kappa)I_{11}(\kappa,\nu) + B_{68}(\kappa)I_{12}(\kappa,\nu) + E_{57}(\kappa)I_{15}(\kappa,\nu) + F_{57}(\kappa)I_{16}(\kappa,\nu) + E_{68}(\kappa)I_{19}(\kappa,\nu) + F_{68}(\kappa)I_{20}(\kappa,\nu) \quad (107)$$

The very first term $e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$ in Eq.(61) can be represented by a product of Fourier series with ζ and θ as the transformed variables. We shall not do so now in order to preserve the compactness and clarity of the notation $e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$ compared with its Fourier representation. Equation (61) is written in the following form with the help of Eqs.(88) and (107):

$$A_{\mathrm{ev}}(\zeta,\theta) = c^2\tau^2V_{\mathrm{e}0}\Bigg(\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) + \frac{2}{N_\tau}(1-e^{-N_\tau\rho_2})$$
$$\times\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$
$$+\sum_{\nu=1}^{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}$$
$$+\sum_{\kappa>K}^{N_\tau}\Bigg[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}$$
$$+\sum_{\nu=1}^{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\Bigg]\sin\frac{2\pi\kappa\zeta}{N_\tau}\Bigg\}\Bigg) \quad (108)$$

A radical simplification of the writing of this equation is necessary to make it usable:

$$A_{\mathrm{ev}}(\zeta,\theta) = c^2\tau^2V_{\mathrm{e}0}\Bigg(A_{\mathrm{e}0}(\zeta,\theta) + \sum_{\kappa=1}^{<K}C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa>K}^{N_\tau}C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\Bigg)$$

$$
\begin{gathered}
= c^2\tau^2 V_{\mathrm{e0}}\left(A_{\mathrm{e0}}(\zeta,\theta) + \sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\right) \\
A_{\mathrm{e0}}(\zeta,\theta) = \frac{1}{\rho_2^2} e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) \\
C_{\mathrm{e}\kappa}(\theta) = \frac{2}{N_\tau}(1-e^{-N_\tau\rho_2})\left[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right. \\
\left. + \sum_{\nu=1}^{N_\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\right] \\
K = N_\tau(\rho_1^2 - 4\rho_2^2)^{1/2}/4\pi = c\tau N_\tau|(\sigma Z - s/Z)| \qquad (109)
\end{gathered}
$$

We may make two simplifications that hold generally by using the relation $N_\tau\rho_2 = cT\sqrt{\sigma s} \gg 1$ of Eq.(2.2-9). First the factor $1-\exp(-N_\tau\rho_2)$ is essentially equal to 1. Second, the absolute value of $A_{\mathrm{e0}}(\zeta,\theta)$ is less than $0.5/\rho_2^2$ according to Fig.6.1-1. Hence, we write:

$$
\begin{gathered}
A_{\mathrm{ev}}(\zeta,\theta) = c^2\tau^2 V_{\mathrm{e0}}\left(\sum_{\kappa=1}^{<K} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa>K}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau}\right) \\
= c^2\tau^2 V_{\mathrm{e0}}\sum_{\kappa=1}^{N_\tau} C_{\mathrm{e}\kappa}(\theta)\sin\frac{2\pi\kappa\zeta}{N_\tau} \\
C_{\mathrm{e}\kappa}(\theta) = \frac{2}{N_\tau}\left[A_{\mathrm{es}}(\kappa)\sin\frac{2\pi\kappa\theta}{N_\tau} + A_{\mathrm{ec}}(\kappa)\cos\frac{2\pi\kappa\theta}{N_\tau}\right. \\
\left. + \sum_{\nu=1}^{N\tau}\left(B_{\mathrm{es}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N_\tau} + B_{\mathrm{ec}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N_\tau}\right)\right] \qquad (110)
\end{gathered}
$$

We want approximations for large values of κ. Since the largest value of κ equals N_τ we could write $\kappa \to N_\tau$ rather than $\kappa \to \infty$. It seems better to generally use the notation $\kappa \ggg 1$, which means that κ becomes arbitrarily large but finite, since κ actually can equal N_τ but can only approach ∞. Those who are not satisfied with the approximations derived from here to the end of Section 6.1 are referred to Section 2.5 that shows computer plots of the energy density $U_{\mathrm{c}\kappa}(\kappa)$ and makes the approximation of $\mathcal{H}$ in Eq.(136) below unnecessary.

$$
\text{for } d^2 = 4[(2\pi\kappa/N_\tau)^2 + \rho_2^2] > \rho_1^2 \quad\text{and}\quad \kappa \ggg 1
$$

$$
d^2 \approx 4(2\pi\kappa/N_\tau)^2, \qquad \rho_1^2,\ \rho_2^2 \ll (2\pi)^2 \qquad (111)
$$

$$
q_1 \approx \frac{4\pi\kappa}{N_\tau} + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa/N_\tau}, \qquad q_2 \approx \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa/N_\tau} \qquad (112)
$$

$$(d^2 - \rho_1^2)^{1/2} \approx 4\pi\kappa/N_\tau \tag{113}$$

$$q_1^2 + \left(\frac{\rho_1}{2}\right)^2 \approx 4\left(\frac{2\pi\kappa}{N_\tau}\right)^2 + 2\rho_2^2 - \frac{1}{4}\rho_1^2 \tag{114}$$

$$q_2^2 + \left(\frac{\rho_1}{2}\right)^2 \approx \frac{1}{4}\rho_1^2 \tag{115}$$

$$q_1 - \frac{2\pi\kappa}{N_\tau} \approx \frac{2\pi\kappa}{N_\tau} + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa/N_\tau} \tag{116}$$

$$q_2 + \frac{2\pi\kappa}{N_\tau} \approx \frac{2\pi\kappa}{N_\tau} + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa/N_\tau} \tag{117}$$

Starting with $A_{\text{es}}(\kappa)$ of Eq.(13) we follow the listing and obtain for the limit $\kappa \ggg 1$ the following results:

$$A_{\text{es}}(\kappa) \approx \frac{1}{(2\pi\kappa/N_\tau)^2\rho_1} \qquad A_{\text{ec}}(\kappa) \approx \frac{1}{(2\pi\kappa/N_\tau)^3}\left(1 - \frac{\rho_2^2}{\rho_1^2}\right) \tag{118}$$

$$A_{57}(\kappa) \approx -\frac{3\rho_1}{16(2\pi\kappa/N_\tau)^4} \qquad B_{57}(\kappa) \approx \frac{1}{4(2\pi\kappa/N_\tau)^3} \tag{119}$$

$$C_{57}(\kappa) \approx -\frac{3}{(2\pi\kappa/N_\tau)^2\rho_1} \qquad D_{57}(\kappa) \approx -\frac{3}{4(2\pi\kappa/N_\tau)^3}\left(1 - \frac{4\rho_2^2}{3\rho_1^2}\right) \tag{120}$$

$$A_{68}(\kappa) \approx \frac{3\rho_1}{16(2\pi\kappa/N_\tau)^4} \approx -A_{57}(\kappa) \qquad B_{68}(\kappa) \approx \frac{1}{4(2\pi\kappa/N_\tau)^3} \approx B_{57}(\kappa) \tag{121}$$

$$C_{68}(\kappa) \approx -\frac{1}{(2\pi\kappa/N_\tau)^2\rho_1} \qquad D_{68}(\kappa) \approx -\frac{\rho_2^2 + \rho_1^2/4}{(2\pi\kappa/N_\tau)^3\rho_1^2} \tag{122}$$

$$E_{57}(\kappa) \approx -\frac{1}{(2\pi\kappa/N_\tau)^2\rho_1} \qquad E_{68}(\kappa) \approx -\frac{2}{(2\pi\kappa/N_\tau)^2\rho_1} \tag{123}$$

$$F_{57}(\kappa) \approx -\frac{1}{2(2\pi\kappa/N_\tau)^3} \qquad F_{68}(\kappa) \approx \frac{1}{4(2\pi\kappa/N_\tau)^3}\left(1 - 4\frac{\rho_2^2}{\rho_1^2}\right) \tag{124}$$

The integrals $I_1(\kappa,\nu)$ to $I_8(\kappa,\nu)$ are not needed for the limits $\kappa \ggg 1$, but the integrals $I_9(\kappa,\nu)$ to $I_{20}(\kappa,\nu$ are:

$$I_9(\kappa,\nu) \approx -\frac{4\rho_1\nu/N_\tau}{(2\pi\kappa/N_\tau)^3} \qquad I_{10}(\kappa,\nu) \approx -\frac{4\pi\nu/N_\tau}{(2\pi\kappa/N_\tau)^2} \tag{125}$$

$$I_{11}(\kappa,\nu) \approx -\frac{2}{2\pi\kappa/N_\tau} \qquad I_{12}(\kappa,\nu) \approx \frac{\rho_1}{(2\pi\kappa/N_\tau)^2} \tag{126}$$

$$I_{13}(\kappa,\nu) \approx 2\rho_1\frac{2\pi\nu/N_\tau}{(2\pi\kappa/N_\tau)^3} \qquad I_{14}(\kappa,\nu) \approx -\frac{4\pi\nu/N_\tau}{(2\pi\kappa/N_\tau)^2} \tag{127}$$

$$I_{15}(\kappa,\nu) \approx \frac{2}{2\pi\kappa/N_\tau} \qquad I_{16}(\kappa,\nu) \approx \frac{\rho_1}{2(2\pi\kappa/N_\tau)^2} \tag{128}$$

$$I_{17}(\kappa,\nu) \approx 2\rho_1 \frac{2\pi\nu/N_\tau}{(2\pi\kappa/N_\tau)^3} \qquad I_{18}(\kappa,\nu) \approx -\frac{2\pi\nu/N_\tau}{(2\pi\kappa/N_\tau)^2} \tag{129}$$

$$I_{19}(\kappa,\nu) \approx \frac{2}{2\pi\kappa/N_\tau} \qquad I_{20}(\kappa,\nu) \approx \frac{\rho_1}{2(2\pi\kappa/N_\tau)^2} \tag{130}$$

We may now produce the functions $B_{\text{es}}(\kappa,\nu)$ and $B_{\text{ec}}(\kappa,\nu)$ of Eq.(107). Since we do not want to evaluate the sums over ν we cannot write $\approx$ (approximately) but must write $\propto$ (proportionate):

$$B_{\text{es}}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^5} \qquad B_{\text{ec}}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^3} \tag{131}$$

The functions $C_{\text{es}}(\kappa,\nu)$ and $C_{\text{ec}}(\kappa,\nu)$ of Eq.(2.2-44) vary for $\kappa \ggg 1$ as follows:

$$C_{\text{es}}(\kappa,\nu) \approx \frac{2\pi\kappa\rho_{\text{s}}}{N_\tau} B_{\text{es}}(\kappa,\nu) - \frac{2\pi\nu}{2\pi\kappa} B_{\text{ec}}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^4} \tag{132}$$

$$C_{\text{ec}}(\kappa,\nu) \approx \frac{2\pi\kappa\rho_{\text{s}}}{N_\tau} B_{\text{ec}}(\kappa,\nu) + \frac{2\pi\nu}{2\pi\kappa} B_{\text{es}}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^2} \tag{133}$$

For $U_{\text{cs}}^2(\kappa)$ and $U_{\text{cc}}^2(\kappa)$ in Eqs.(2.3-30) and (2.3-31) we get:

$$U_{\text{cs}}^2(\kappa) \propto \left(B_{\text{ec}}(\kappa,\nu) + \frac{\nu}{\kappa} C_{\text{es}}(\kappa,\nu)\right)^2 + \left(B_{\text{es}}(\kappa,\nu) - \frac{\nu}{\kappa} C_{\text{ec}}(\kappa,\nu)\right)^2 \propto \frac{1}{(2\pi\kappa)^6} \tag{134}$$

$$U_{\text{cc}}^2(\kappa) \propto \left(\frac{\nu}{\kappa} B_{\text{ec}}(\kappa,\nu) + C_{\text{es}}(\kappa,\nu)\right)^2 + \left(\frac{\nu}{\kappa} B_{\text{es}}(\kappa,\nu) - C_{\text{ec}}(\kappa,\nu)\right)^2 \propto \frac{1}{(2\pi\kappa)^4} \tag{135}$$

The normalized energy of the component of the wave represented by the sinusoidal pulse with κ cycles in the interval $0 \leq y \leq cT$ or by all the photons with the period number κ varies for $\kappa \gg 1$ as follows:

$$\mathcal{H}_\kappa = (2\pi\kappa)^2 [U_{\text{cs}}^2(\kappa) + U_{\text{cc}}^2(\kappa)] \propto \frac{1}{(2\pi\kappa)^2} \qquad \text{for } \kappa \gg 1 \tag{136}$$

The following products are needed to determine the decrease of $U_{\text{v}\kappa}(\theta)$ of Eq.(2.3-32) for $\kappa \gg 1$:

$$(2\pi\kappa)^2 B_{\rm ec}^2(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^4} \qquad (2\pi\kappa)^2 B_{\rm es}^2(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^8} \tag{137}$$

$$(2\pi\kappa)^2 C_{\rm ec}^2(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^2} \qquad (2\pi\kappa)^2 C_{\rm es}^2(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^6} \tag{138}$$

$$2\pi\kappa B_{\rm ec}(\kappa,\nu) C_{\rm es}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^6} \qquad 2\pi\kappa B_{\rm es}(\kappa,\nu) C_{\rm ec}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^6} \tag{139}$$

$$2\pi\kappa B_{\rm ec}(\kappa,\nu) C_{\rm ec}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^4} \qquad 2\pi\kappa B_{\rm es}(\kappa,\nu) C_{\rm es}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^8} \tag{140}$$

$$(2\pi\kappa)^2 B_{\rm es}(\kappa,\nu) B_{\rm ec}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^6} \tag{141}$$

$$(2\pi\kappa)^2 C_{\rm es}(\kappa,\nu) C_{\rm ec}(\kappa,\lambda) \propto \frac{1}{(2\pi\kappa)^4} \tag{142}$$

One obtains for $U_{\rm v1}(\kappa,\nu)$ to $U_{\rm v4}(\kappa,\nu,\lambda)$ of Eqs.(2.3-32) to (2.3-36) the following relations for $\kappa \gg 1$:

$$U_{\rm v1}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^2} \qquad U_{\rm v2}(\kappa,\nu,\lambda) \propto \frac{1}{(2\pi\kappa)^6} \tag{143}$$

$$U_{\rm v3}(\kappa,\nu,\lambda) \propto \frac{1}{(2\pi\kappa)^2} \qquad U_{\rm v4}(\kappa,\nu,\lambda) \propto \frac{1}{(2\pi\kappa)^4} \tag{144}$$

6.2 Inhomogeneous Difference Wave Equation

The inhomogeneous differential wave equation (2.1-44) has Smirnov's simple and elegant solution of Eq.(2.1-45). No comparable solution has been derived for the corresponding difference equation. In order to find a solution we first develop a solution for the differential equation (2.1-44) that will serve as a guide. We write Eq.(2.1-44) in simplified form:

$$\frac{\partial^2 A}{\partial \zeta^2} - \frac{\partial^2 A}{\partial \theta^2} = a^2 V(\zeta,\theta), \quad A = A_{\rm ev},\ V = V_{\rm e},\ a^2 = c^2\tau^2 \tag{1}$$

Following d'Alembert we introduce new variables η and ζ

$$\begin{gathered}\eta = \zeta + \theta,\ \xi = \zeta - \theta,\ \zeta = \frac{1}{2}(\eta + \xi),\ \theta = \frac{1}{2}(\eta - \xi)\\ \partial\eta/\partial\zeta = 1,\ \partial\eta/\partial\theta = 1,\ \partial\xi/\partial\zeta = 1,\ \partial\xi/\partial\theta = -1\end{gathered} \tag{2}$$

and obtain:

$$\frac{\partial A}{\partial \zeta} = \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial \zeta} + \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial \zeta} = \frac{\partial A}{\partial \eta} + \frac{\partial A}{\partial \xi} \tag{3}$$

$$\frac{\partial A}{\partial \theta} = \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial \theta} + \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial \theta} = \frac{\partial A}{\partial \eta} - \frac{\partial A}{\partial \xi} \tag{4}$$

A second differentiation yields:

$$\begin{aligned}\frac{\partial^2 A}{\partial \zeta^2} &= \frac{\partial}{\partial \eta}\left(\frac{\partial A}{\partial \eta} + \frac{\partial A}{\partial \xi}\right)\frac{\partial \eta}{\partial \zeta} + \frac{\partial}{\partial \xi}\left(\frac{\partial A}{\partial \eta} + \frac{\partial A}{\partial \xi}\right)\frac{\partial \xi}{\partial \zeta} \\ &= \frac{\partial^2 A}{\partial \eta^2} + 2\frac{\partial^2 A}{\partial \eta \partial \xi} + \frac{\partial^2 A}{\partial \xi^2} \end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial^2 A}{\partial \theta^2} &= \frac{\partial}{\partial \eta}\left(\frac{\partial A}{\partial \eta} - \frac{\partial A}{\partial \xi}\right)\frac{\partial \eta}{\partial \theta} + \frac{\partial}{\partial \xi}\left(\frac{\partial A}{\partial \eta} - \frac{\partial A}{\partial \xi}\right)\frac{\partial \xi}{\partial \theta} \\ &= \frac{\partial^2 A}{\partial \eta^2} - 2\frac{\partial^2 A}{\partial \eta \partial \xi} + \frac{\partial^2 A}{\partial \xi^2} \end{aligned} \tag{6}$$

Substitution into Eq.(1)

$$4\frac{\partial^2 A}{\partial \eta \partial \xi} = a^2 V(\eta, \xi) \tag{7}$$

and integration with respect to η yields:

$$\frac{\partial A}{\partial \xi} = \frac{1}{4}a^2\left[\int V(\eta,\xi)d\eta + C_1(\xi)\right] \tag{8}$$

A further integration with respect to ξ provides a solution for Eq.(1):

$$A(\eta,\xi) = \frac{1}{4}a^2\int\left[\int V(\eta,\xi)d\eta + C_1(\xi)\right]d\xi + C_2 \tag{9}$$

The integration constants $C_1(\xi)$ and C_2 can still be chosen and a solution of the homogeneous Eq.(7) can be added.

We want to rewrite these calculations for finite differences. This requires some definitions of difference operators that go beyond Eqs.(3.1-2) to (3.1-4). A first order *symmetric* difference operator

$$\begin{aligned}\frac{\tilde{\Delta}V(\zeta)}{\tilde{\Delta}\zeta} &= \frac{V(\zeta+\Delta\zeta) - V(\zeta-\Delta\zeta)}{2\Delta\zeta} \\ &= \frac{1}{2}[V(\zeta+1) - V(\zeta-1)] \quad \text{for } \Delta\zeta = 1\end{aligned} \tag{10}$$

and a second order difference operator

$$\frac{\tilde{\Delta}^2 V(\zeta)}{\tilde{\Delta}\zeta^2} = \frac{\tilde{\Delta}}{\tilde{\Delta}\zeta}\left(\frac{\tilde{\Delta}V(\zeta)}{\tilde{\Delta}\zeta}\right) = \frac{V(\zeta+\Delta\zeta) - 2V(\zeta) + V(\zeta-\Delta\zeta)}{(\Delta\zeta)^2}$$
$$= V(\zeta+1) - 2V(\zeta) + V(\zeta-1) \quad \text{for } \Delta\zeta = 1 \tag{11}$$

are defined. The choice $\Delta\zeta = 1$ simplifies the relations without reducing the generality since we may define a new variable $\zeta' = \zeta/\Delta\zeta$ and then leave out the prime.

The second order difference quotient does not formally follow from using twice the first order symmetric difference quotient:

$$\frac{\tilde{\Delta}}{\tilde{\Delta}\zeta}\left(\frac{\tilde{\Delta}V(\zeta)}{\tilde{\Delta}\zeta}\right) \neq \frac{1}{2}\left\{\frac{1}{2}[V(\zeta+2) - V(\zeta)] - \frac{1}{2}[V(\zeta) - V(\zeta-2)]\right\}$$
$$= \frac{1}{4}[V(\zeta+2) - 2V(\zeta) + V(\zeta-2)] \tag{12}$$

One may get around this difficulty by defining a right and a left first order difference operator

$$\frac{\tilde{\Delta}_{\mathrm{r}} V(\zeta)}{\tilde{\Delta}\zeta} = V(\zeta+1) - V(\zeta)$$
$$\frac{\tilde{\Delta}_{\mathrm{l}} V(\zeta)}{\tilde{\Delta}\zeta} = V(\zeta) - V(\zeta-1) \tag{13}$$

and then the second order difference operator

$$\frac{\tilde{\Delta}_{\mathrm{r}}}{\tilde{\Delta}\zeta}\left(\frac{\tilde{\Delta}_{\mathrm{l}} V(\zeta)}{\tilde{\Delta}\zeta}\right) = \frac{\tilde{\Delta}_{\mathrm{l}}}{\tilde{\Delta}\zeta}\left(\frac{\tilde{\Delta}_{\mathrm{r}} V(\zeta)}{\tilde{\Delta}\zeta}\right)$$
$$= \frac{\tilde{\Delta}_{\mathrm{r}}}{\tilde{\Delta}\zeta}[V(\zeta) - V(\zeta-1)] = \frac{\tilde{\Delta}_{\mathrm{l}}}{\tilde{\Delta}\zeta}[V(\zeta+1) - V(\zeta)]$$
$$= [V(\zeta+1) - V(\zeta)] - [V(\zeta) - V(\zeta-1)]$$
$$= V(\zeta+1) - 2V(\zeta) + V(\zeta-1) \tag{14}$$

Mathematicians are usually satisfied with the right difference quotient. But its use in physics introduces an unsymmetry that is strictly due to mathematics and which may lead to divergencies avoided by the symmetric difference quotient[1]. Hence, we use the definitions of Eqs.(10) and (11) for the first and second order difference quotient.

[1] Harmuth 1989, Sec. 8.2

The formulas for the derivatives in Eq.(1) are replaced by the following formulas for finite differences:

$$\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\zeta} = \frac{1}{2}[\eta(\zeta+1) - \eta(\zeta-1)] = \frac{1}{2}[\zeta+1+\theta-(\zeta-1+\theta)] = 1 \tag{15}$$

$$\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\theta} = \frac{1}{2}[\eta(\theta+1) - \eta(\theta-1)] = \frac{1}{2}[\zeta+\theta+1-(\zeta+\theta-1)] = 1 \tag{16}$$

$$\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\zeta} = \frac{1}{2}[\xi(\zeta+1) - \xi(\zeta-1)] = \frac{1}{2}[\zeta+1-\theta-(\zeta-1-\theta)] = 1 \tag{17}$$

$$\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\theta} = \frac{1}{2}[\xi(\theta+1) - \xi(\theta-1)] = \frac{1}{2}\{\zeta-(\theta+1)-[\zeta-(\theta-1)]\} = -1 \tag{18}$$

Instead of Eqs.(3) and (4) we obtain:

$$\begin{aligned}\frac{\tilde{\Delta}A}{\tilde{\Delta}\zeta} &= \frac{1}{2}[A(\zeta+1) - A(\zeta-1)] = \frac{\tilde{\Delta}A}{\tilde{\Delta}\eta}\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\zeta} + \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\zeta} = \frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} + \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi} \\ &= \frac{1}{2}[A(\eta+1,\xi) - A(\eta-1,\xi)] + \frac{1}{2}[A(\eta,\xi+1) - A(\eta,\xi-1)]\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\tilde{\Delta}A}{\tilde{\Delta}\theta} &= \frac{1}{2}[A(\theta+1) - A(\theta-1)] = \frac{\tilde{\Delta}A}{\tilde{\Delta}\eta}\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\theta} + \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\theta} = \frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} - \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi} \\ &= \frac{1}{2}[A(\eta+1,\xi) - A(\eta-1,\xi)] - \frac{1}{2}[A(\eta,\xi+1) - A(\eta,\xi-1)]\end{aligned} \tag{20}$$

The second order difference quotient becomes:

$$\begin{aligned}\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\zeta^2} &= \frac{\tilde{\Delta}}{\tilde{\Delta}\eta}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} + \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right)\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\zeta} + \frac{\tilde{\Delta}}{\tilde{\Delta}\xi}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} + \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right)\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\zeta} \\ &= \frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta^2} + 2\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta\tilde{\Delta}\xi} + \frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\xi^2}\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\theta^2} &= \frac{\tilde{\Delta}}{\tilde{\Delta}\eta}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} - \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right)\frac{\tilde{\Delta}\eta}{\tilde{\Delta}\theta} + \frac{\tilde{\Delta}}{\tilde{\Delta}\xi}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\eta} - \frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right)\frac{\tilde{\Delta}\xi}{\tilde{\Delta}\theta} \\ &= \frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta^2} - 2\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta\tilde{\Delta}\xi} + \frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\xi^2}\end{aligned} \tag{22}$$

We write Eq.(1) as difference equation and substitute Eqs.(21), (22):

$$\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\zeta^2} - \frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\theta^2} = a^2 V(\zeta,\theta) \tag{23}$$

$$4\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta\tilde{\Delta}\xi} = a^2 V(\eta,\xi) \tag{24}$$

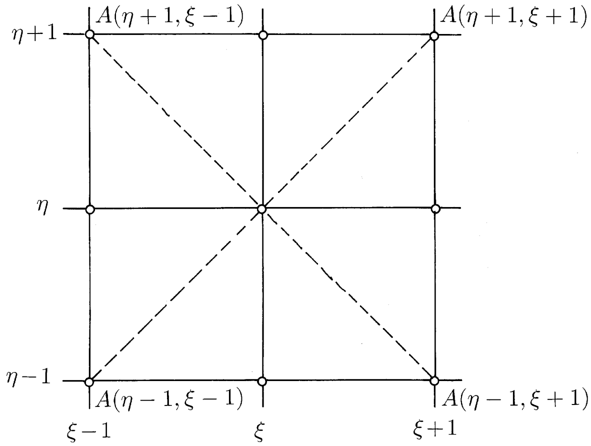

FIGURE 6.2-1. The difference of the averages of A taken at the points $\eta+1, \xi+1$ and $\eta-1, \xi-1$ as well as at $\eta+1, \xi-1$ and $\eta-1, \xi+1$ divided by 2 yields $\tilde{\Delta}^2 A/\tilde{\Delta}\eta\tilde{\Delta}\xi$.

This is the equivalent of Eq.(7) for finite differences $\Delta\eta$, $\Delta\xi$. Since Eq.(11) only defines the operator $\tilde{\Delta}^2/\tilde{\Delta}\zeta^2$ we still have to explain the operator $\tilde{\Delta}^2/\tilde{\Delta}\eta\tilde{\Delta}\zeta$:

$$
\begin{aligned}
\frac{\tilde{\Delta}^2 A}{\tilde{\Delta}\eta\tilde{\Delta}\xi} &= \frac{\tilde{\Delta}}{\tilde{\Delta}\eta}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right) = \frac{1}{2}\frac{\tilde{\Delta}}{\tilde{\Delta}\eta}[A(\eta,\xi+1) - A(\eta,\xi-1)] \\
&= \frac{1}{4}\{A(\eta+1,\xi+1) - A(\eta-1,\xi+1) \\
&\qquad - [A(\eta+1,\xi-1) - A(\eta-1,\xi-1)]\} \\
&= \frac{1}{2}\left\{\frac{1}{2}[A(\eta+1,\xi+1) + A(\eta-1,\xi-1)]\right. \\
&\qquad \left. - \frac{1}{2}[A(\eta+1,\xi-1) + A(\eta-1,\xi+1)]\right\}
\end{aligned}
\tag{25}
$$

In the last two lines of Eq.(25) we have the averages $0.5[A(\eta+1,\xi+1) + A(\eta-1,\xi-1)]$ and $0.5[A(\eta+1,\xi-1) + A(\eta-1,\xi+1)]$. Their first order difference yields $\tilde{\Delta}^2 A/\tilde{\Delta}\eta\tilde{\Delta}\xi$. Figure 6.2-1 shows the geometric meaning of this difference of two averages.

Equation (24) looks very similar to Eq.(7) and we will expect that a solution similar to Eq.(9) can be found. To obtain this solution we follow closely Nörlund and Milne-Thomson[2]. They use the notation

$$
\underset{\omega}{\tilde{\Delta}} u(x) = \frac{u(x+\omega) - u(x)}{\omega} = \varphi(x) \tag{26}
$$

[2] Nörlund 1924, Ch. 3; Milne-Thomson 1951, Ch. VIII

for an inhomogeneous difference equation of first order, where ω stands for $\Delta\zeta$ in Eq.(10) and the symmetric difference quotient $\tilde{\Delta}/\tilde{\Delta}\zeta$ is replaced by the right difference quotient $\tilde{\Delta}_{\mathrm{r}}/\tilde{\Delta}\zeta$ of Eq.(13). To connect Eq.(26) with our notation we can choose $\omega = 2$

$$\frac{1}{2}[u(x+2) - u(x)] = \varphi(x), \quad \omega = 2 \tag{27}$$

and substitute $x = x' - 1$:

$$\tilde{\Delta}u(x') = \frac{1}{2}[u(x'+1) - u(x'-1)] = \varphi(x'-1), \quad x' = x+1 \tag{28}$$

Consider the function

$$\begin{aligned} f(x) &= C_0 - \omega[\varphi(x) + \varphi(x+\omega) + \varphi(x+2\omega) + \ldots] \\ &= C_0 - \omega \sum_{s=0}^{\infty} \varphi(x + s\omega) \end{aligned} \tag{29}$$

and the shifted function $f(x+\omega)$:

$$f(x+\omega) = C_0 - \omega \sum_{s=0}^{\infty} [\varphi(x+\omega) + \varphi(x+2\omega) + \ldots] \tag{30}$$

Substitution of $f(x)$ for $u(x)$ in Eq.(26) satisfies that equation and we have found a *formal* solution of Eq.(26). The constant C_0 may be replaced by a definite integral

$$C_0 = \int_c^{\infty} \varphi(\nu)d\nu \tag{31}$$

and the *Hauptlösung* or *principal solution* of Eq.(26)—also called the *sum of the function* $\varphi(x)$—may be written in the form

$$F(x\,|\,\omega) = \int_c^{\infty} \varphi(\nu)d\nu - \omega \sum_{s=0}^{\infty} \varphi(x + s\omega) \tag{32}$$

The integral is introduced for the constant C_0 because a divergency of this integral may compensate a divergency of the sum, which a constant C_0 in Eq.(29) could not do. The principal solution of Eq.(26) is thus obtained by *summing* the function $\varphi(x)$. In analogy to the notation of integral calculus Nörlund introduced the notation

$$F(x\,|\,\omega) = \mathop{\large S}_{c}^{x} \varphi(\nu)\underset{\omega}{\Delta}\nu = \int_{c}^{\infty} \varphi(\nu)d\nu - \omega \sum_{s=0}^{\infty} \varphi(x+s\omega) \tag{33}$$

The function $F(x\,|\,\omega)$ is obtained by *summing* $\varphi(x)$ *from* c *to* x. The integral in Eq.(33) represents an 'integration' or 'summing' constant like the c in $\int_c^x f(x')dx'$ if it and the sum converge. In this case one may write a constant C_0 for the integral rather than evaluate Eq.(31).

Let us mention that Nörlund's definition of the sum of a function is more general than Eq.(32):

$$F(x\,|\,\omega) = \lim_{\mu\to 0}\left(\int_{c}^{\infty} \varphi(\nu)e^{-\mu\lambda(\nu)}d\nu - \omega\sum_{s=0}^{\infty}\varphi(x+s\omega)e^{-\mu\lambda(x+s\omega)}\right)$$

This more general definition is required here only to obtain the sum of the constant a in Table 6.2-1 (Milne-Thomson 1952, p. 203).

We rewrite $F(x\,|\,\omega)$ for the symmetric difference quotient on the left side of Eq,(28) with the substitutions $x = x' - 1$ and $\omega = 2$:

$$F(x'-1\,|\,2) = \mathop{\large S}_{c}^{x'-1} \varphi(\nu)\Delta\nu = \int_{c}^{\infty}\varphi(\nu)d\nu - 2\sum_{s=0}^{\infty}\varphi(x'-1+2s)$$

$$G(x) = F(x\,|\,2) = \mathop{\large S}_{c}^{x} \varphi(\nu)\Delta\nu = \int_{c}^{\infty}\varphi(\nu)d\nu - 2\sum_{s=0}^{\infty}\varphi(x+2s)$$

$$\underset{\omega}{\Delta}\nu = \Delta\nu \quad \text{for } \omega = 2 \tag{34}$$

As an example let $\varphi(x)$ in Eq.(26) be the exponential function e^{-x} and let ω as well as x be real and positive:

$$u(x) = F(x\,|\,\omega) = \mathop{\large S}_{c}^{x} e^{-\nu}\underset{\omega}{\Delta}\nu = \int_{c}^{\infty} e^{-\nu}d\nu - \omega\sum_{s=0}^{\infty} e^{-(x+s\omega)}$$

$$= e^{-c} - \frac{\omega e^{-x}}{1-e^{-\omega}} \tag{35}$$

In order to see what the symmetric difference quotient and the choice $\omega = 2$ do to this result we consider the difference equation

$$\tilde{\Delta}v(x') = \frac{1}{2}[v(x'+1) - v(x'-1)] = e^{-x'} = \varphi(x'), \quad u(x') = v(x')e$$

$$= \frac{1}{2}[u(x'+1) - u(x'-1)] = e^{-(x'-1)} = \varphi(x'-1) \tag{36}$$

The equation for $v(x')$ equals Eq.(28). We get with the help of Eq.(34) and $x' = x + 1$:

$$v(x') = e^{-1}u(x') = e^{-1}G(x') = \underset{c}{\overset{x'}{\mathbf{S}}} e^{-(\nu+1)}\Delta\nu = e^{-1}\left(\int_c^\infty e^{-\nu}d\nu - 2\sum_{s=0}^{\infty} e^{-(x'+2s)}\right)$$
$$= e^{-1}\left(e^{-c} - \frac{2e^{-x'}}{1-e^{-2}}\right) = e^{-(c+1)} - \frac{2}{e-e^{-1}}e^{-x'} = \frac{2}{e-e^{-1}}(C - e^{-x'}) \quad (37)$$

The integral $\int \exp(-x')dx'$ yields $C - e^{-x'}$. Equation (37) differs by a factor $2/(e - e^{-1}) \doteq 0.85092$ from this result.

It has been shown[3] that Eq.(35) not only holds for the exponential function $\varphi(x) = e^{-x}$ in Eq.(26) but generally for the exponential function $e^{\gamma x}$ in the complex plane as long as the condition $|\omega| < 2\pi/|\gamma|$ is satisfied:

$$\underset{c}{\overset{x}{\mathbf{S}}} e^{\gamma\nu} \underset{\omega}{\Delta}\nu = -\frac{e^{\gamma c}}{\gamma} - \frac{\omega e^{\gamma x}}{1 - e^{\gamma\omega}}, \quad |\omega| < \frac{2\pi}{|\gamma|}$$
$$\underset{c}{\overset{x'}{\mathbf{S}}} e^{\gamma(\nu+1)}\Delta\nu = -\frac{e^{\gamma(c+1)}}{\gamma} - \frac{2e^{\gamma x'}}{e^{-\gamma} - e^{\gamma}}, \quad |\gamma| < \pi \quad (38)$$

This relation makes it possible to solve Eq.(26) for $\varphi(x) = \sin\gamma x$:

$$\frac{1}{\omega}[u(x+\omega) - u(x)] = \sin\gamma x = \frac{1}{2i}\left(e^{i\gamma x} - e^{-i\gamma x}\right)$$
$$u(x) = \frac{1}{2i}\underset{c}{\overset{x}{\mathbf{S}}}\left(e^{i\gamma\nu} - e^{-i\gamma\nu}\right)\underset{\omega}{\Delta}\nu$$
$$= \frac{1}{2i}\left(-\frac{e^{i\gamma c}}{i\gamma} - \frac{\omega e^{i\gamma x}}{1 - e^{i\gamma\omega}} + \frac{e^{-i\gamma c}}{-i\gamma} + \frac{\omega e^{-i\gamma x}}{1 - e^{-i\gamma\omega}}\right)$$
$$= \frac{\cos\gamma c}{\gamma} - \frac{\omega}{2}\frac{\cos\gamma(x - \omega/2)}{\sin(\gamma\omega/2)}, \quad |\omega| < \frac{2\pi}{|\gamma|} \quad (39)$$

We need the corresponding result for Eq.(36). Intermediate steps are given to help with verification:

$$\tilde{\Delta}v(x') = \frac{1}{2}[v(x'+1) - v(x'-1)] = \sin\gamma x' = \frac{1}{2i}\left(e^{i\gamma x'} - e^{-i\gamma x'}\right) \quad (40)$$

[3]Nörlund 1924, p. 81; Milne-Thomson 1951, p. 231

$$\begin{aligned}
&\frac{1}{2}[v_1(x'+1)-v_1(x'-1)]=\frac{1}{2i}e^{i\gamma x'}, \quad u_1(x')=2ie^{-i\gamma}v_1(x')\\
&\frac{1}{2}[u_1(x'+1)-u_1(x'-1)]=e^{i\gamma(x'-1)}=\varphi(x'-1)
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\frac{1}{2}[v_2(x'+1)-v_2(x'-1)]=-\frac{1}{2i}e^{-i\gamma x'}, \quad u_2(x')=-2ie^{i\gamma}v_2(x')\\
&\frac{1}{2}[u_2(x'+1)-u_2(x'-1)]=e^{-i\gamma(x'-1)}=\varphi(x'-1)
\end{aligned} \tag{42}$$

$$\begin{aligned}
v(x')&=v_1(x')+v_2(x')=\frac{1}{2i}e^{i\gamma}u_1(x')-\frac{1}{2i}e^{-i\gamma}u_2(x')\\
&=\mathop{S}_{c}^{x'}\sin\gamma(\nu+1)\Delta\nu=\frac{1}{2i}e^{i\gamma}\mathop{S}_{c}^{x'}e^{i\gamma\nu}\Delta\nu-\frac{1}{2i}e^{-i\gamma}\mathop{S}_{c}^{x'}e^{-i\gamma\nu}\Delta\nu\\
&=\frac{1}{2i}\left(-\frac{e^{i\gamma(c+1)}}{i\gamma}-\frac{2e^{i\gamma(x'+1)}}{1-e^{2i\gamma}}+\frac{e^{-i\gamma(c+1)}}{-i\gamma}+\frac{2e^{-i\gamma(x'+1)}}{1-e^{-2i\gamma}}\right)\\
&=\frac{\cos[\gamma(c+1)]}{\gamma}-\frac{\cos\gamma x'}{\sin\gamma}, \quad |\gamma|<\pi
\end{aligned} \tag{43}$$

The integral $\int\sin\gamma x'\,dx'$ yields $-(\cos\gamma x')/\gamma+C$. Equation (43) has the factor $1/\sin\gamma$ rather than $1/\gamma$.

We shall need a few more results like Eqs.(37) and (43). Since there is no table of sums corresponding to the tables of integrals, except for the few examples provided by Nörlund, one must calculate these sums. Table 6.2-1 shows a collection of sums needed in this book. Except for the last two sums they are all due to Nörlund.

We have developed sufficient mathematical tools and return to Eq.(24). Let us use the notation

$$\frac{\tilde{\Delta}A(\eta,\xi)}{\tilde{\Delta}\eta}=v(\eta,\xi) \tag{44}$$

to obtain from Eq.(24) the relation

$$\frac{\tilde{\Delta}v(\eta,\xi)}{\tilde{\Delta}\xi}=\frac{1}{2}[v(\eta,\xi+1)-v(\eta,\xi-1)]=\frac{a^2}{4}V(\eta,\xi) \tag{45}$$

With the help of Eq.(33) we obtain a solution for $v(\eta,\xi)$:

TABLE 6.2-1

SUMS $v(x')=v(x)$ OF CERTAIN FUNCTIONS $\varphi(x')=\varphi(x)$ ACCORDING TO EQ.(36) FOR $\omega=2$. THE INTEGRALS OF $\varphi(x)$ ARE SHOWN FOR COMPARISON.

$\varphi(x)$	$v(x)=\mathop{\large S}_{c}^{x}\varphi(\nu+1)\Delta\nu$	$\int\varphi(x)dx$
a	$a(x-c-1)$	$ax+C$
e^{-x}	$-\dfrac{e^{-x}}{\operatorname{sh}1}+e^{-(c+1)}$	$-e^{-x}+C$
$e^{\gamma x}$	$\dfrac{e^{\gamma x}}{\operatorname{sh}\gamma}-\dfrac{e^{\gamma(c+1)}}{\gamma}$ γ complex, $\lvert\gamma\rvert<\pi$	$+\dfrac{e^{\gamma x}}{\gamma}+C$
$\sin\gamma x$	$-\dfrac{\cos\gamma x}{\sin\gamma}+\dfrac{\cos[\gamma(c+1)]}{\gamma},\ \lvert\gamma\rvert<\pi$	$-\dfrac{\cos\gamma x}{\gamma}+C$
$\cos\gamma x$	$\dfrac{\sin\gamma x}{\sin\gamma}+\dfrac{\sin[\gamma(c+1)]}{\gamma},\ \lvert\gamma\rvert<\pi$	$+\dfrac{\sin\gamma x}{\gamma}+C$
$e^{\lambda x}\sin\gamma x$	$\dfrac{e^{\lambda x}}{\lambda^2+\gamma^2}(\lambda_0\sin\gamma x-\gamma_0\cos\gamma x)+C_0$ $\lambda_0=-\dfrac{2(\lambda^2+\gamma^2)\operatorname{sh}\lambda\cos\gamma}{\cos2\gamma-\operatorname{ch}2\lambda}$ $\gamma_0=\dfrac{2(\lambda^2+\gamma^2)\operatorname{ch}\lambda\sin\gamma}{\cos2\gamma-\operatorname{ch}2\lambda}$ $C_0=-\dfrac{e^{\lambda(c+1)}}{\lambda^2+\gamma^2}\{\lambda\sin[\gamma(c+1)]$ $-\gamma\cos[\gamma(c+1)]\}$	$\dfrac{e^{\lambda x}}{\lambda^2+\gamma^2}(\lambda\sin\gamma x-\gamma\cos\gamma x)+C$
$e^{\lambda x}\cos\gamma x$	$\dfrac{e^{\lambda x}}{\lambda^2+\gamma^2}(\lambda_0\cos\gamma x+\gamma_0\sin\gamma x)+C_1$ λ_0 and γ_0 are shown above $C_1=-\dfrac{e^{\lambda(c+1)}}{\lambda^2+\gamma^2}\{\lambda\cos[\gamma(c+1)]$ $-\gamma\sin[\gamma(c+1)]\}$	$\dfrac{e^{\lambda x}}{\lambda^2+\gamma^2}(\lambda\cos\gamma x+\gamma\sin\gamma x)+C$

$$v(\eta,\xi)=\frac{a^2}{4}\mathop{\large S}_{c_1}^{\xi}V(\eta,\xi'+1)\Delta\xi' \tag{46}$$

Next we write

$$\frac{\tilde{\Delta}A(\eta,\xi)}{\tilde{\Delta}\eta}=\frac{1}{2}[A(\eta+1,\xi)-A(\eta-1,\xi)]=\frac{1}{2}[v(\eta+1,\xi)-v(\eta-1,\xi)]$$

$$= \frac{a^2}{4} \mathop{\large S}_{c_1}^{\xi} V(\eta, \xi' + 1)\Delta\xi'$$

$$A(\eta, \xi) = \frac{a^2}{4} \mathop{\large S}_{c_2}^{\eta} \left(\mathop{\large S}_{c_1}^{\xi} V(\eta' + 1, \xi' + 1)\Delta\xi' \right) \Delta\eta' \tag{47}$$

This is a particular solution of Eq.(24). To obtain the general solution we must add the general solution of the homogeneous equation

$$\frac{\tilde{\Delta}^2 A(\eta, \xi)}{\tilde{\Delta}\eta\tilde{\Delta}\xi} = 0 \tag{48}$$

We rewrite this equation

$$\frac{\tilde{\Delta}}{\tilde{\Delta}\eta}\left(\frac{\tilde{\Delta}A}{\tilde{\Delta}\xi}\right) = \frac{1}{2}\left(\frac{\tilde{\Delta}A(\eta+1)}{\tilde{\Delta}\xi} - \frac{\tilde{\Delta}A(\eta-1)}{\tilde{\Delta}\xi}\right) = 0 \tag{49}$$

This equation implies that $\tilde{\Delta}A/\tilde{\Delta}\xi$ does not vary with η and thus must be a function of ξ only:

$$\frac{\tilde{\Delta}A}{\tilde{\Delta}\xi} = f_0(\xi)$$

If we sum over ξ according to Eq.(34) we get

$$A = \mathop{\large S}_{c}^{\xi} f_0(\xi)\Delta\xi + f_2(\eta) \tag{50}$$

where $f_2(\eta)$ is an arbitrary function of η for which holds $\tilde{\Delta}f_2(\eta)/\tilde{\Delta}\xi = 0$. The first term on the right of Eq.(50) represents an arbitrary function of ξ plus a constant. We denote the arbitrary function with $f_1(\xi)$ and the constant with C:

$$A = f_1(\xi) + f_2(\eta) + C \tag{51}$$

Substitution of ξ and θ from Eq.(2) yields:

$$A(\zeta, \theta) = f_1(\zeta - \theta) + f_2(\zeta + \theta) + C \tag{52}$$

Hence, d'Alembert's general solution of the differential wave equation also applies to the difference wave equation. Let us find its solution for the usual two initial conditions. First we have for $\theta = 0$

$$A(\zeta, \theta)|_{\theta=0} = \varphi_1(\zeta) \tag{53}$$

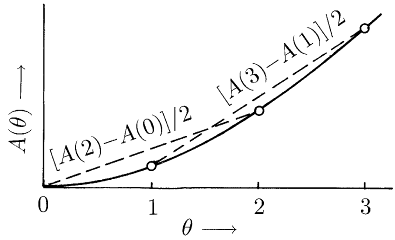

FIGURE 6.2-2. The average slope $[A(2)-A(0)]/2$ defined by the symmetric difference quotient of first order for $\theta = 1$ starts at $\theta = 0$.

The condition for the first derivative at $\theta = 0$ is replaced by

$$\frac{\tilde{\Delta} A}{\tilde{\Delta}\theta} = \frac{1}{2}[A(\zeta, \theta+1) - A(\zeta, \theta-1)]_{\theta=1} = \varphi_2(\zeta) \tag{54}$$

We have to take this condition for $\theta = 1$ rather than for $\theta = 0$ to avoid a value $\theta - 1$ smaller than 0. This is strictly a notational problem since the average slope $[A(2) - A(0)]/2$ at $\theta = 1$ begins at $\theta = 0$ according to Fig.6.2-2.

We take the symmetric first order difference quotient of Eq.(52) and use partial differenciation[4] according to Eq.(20):

$$\frac{\tilde{\Delta} A}{\tilde{\Delta}\theta} = \frac{\tilde{\Delta} f_1(\zeta-\theta)}{\tilde{\Delta}(\zeta-\theta)} \frac{\tilde{\Delta}(\zeta-\theta)}{\tilde{\Delta}\theta} + \frac{\tilde{\Delta} f_2(\zeta+\theta)}{\tilde{\Delta}(\zeta+\theta)} \frac{\tilde{\Delta}(\zeta+\theta)}{\tilde{\Delta}\theta} \tag{55}$$

With

$$\begin{aligned} \frac{\tilde{\Delta}(\zeta-\theta)}{\tilde{\Delta}\theta} &= \frac{1}{2}\{-(\theta+1) - [-(\theta-1)]\} = -1 \\ \frac{\tilde{\Delta}(\zeta+\theta)}{\tilde{\Delta}\theta} &= \frac{1}{2}[(\theta+1) - (\theta-1)] = +1 \end{aligned} \tag{56}$$

we obtain

$$\frac{\tilde{\Delta} A}{\tilde{\Delta}\theta} = -\frac{\tilde{\Delta} f_1(\zeta-\theta)}{\tilde{\Delta}(\zeta-\theta)} + \frac{\tilde{\Delta} f_2(\zeta+\theta)}{\tilde{\Delta}(\zeta+\theta} \tag{57}$$

Equations (52) and (53) yield

$$f_1(\zeta) + f_2(\zeta) + C = \varphi_1(\zeta) \quad \text{for } \theta = 0 \tag{58}$$

while Eqs.(54) and (57) produce:

[4]The word *differenciation* is used for finite differences, the word *differentiation* for infinitesimal differences.

$$-\frac{\tilde{\Delta} f_1(\zeta-\theta)}{\tilde{\Delta}(\zeta-\theta)}+\frac{\tilde{\Delta} f_2(\zeta+\theta)}{\tilde{\Delta}(\zeta+\theta)}=\varphi_2(\zeta) \quad \text{for } \theta=0 \tag{59}$$

Summation of Eq.(59) and multiplication with -1 brings:

$$f_1(\zeta)-f_2(\zeta)=-\mathop{\mathbb{S}}_{c}^{\zeta}\varphi_2(\zeta'+1)\Delta\zeta' \tag{60}$$

From Eqs.(58) and (60) we obtain $f_1(\zeta)$ and $f_2(\zeta)$:

$$f_1(\zeta)=\frac{1}{2}\varphi_1(\zeta)-\frac{1}{2}\mathop{\mathbb{S}}_{c}^{\zeta}\varphi_2(\zeta'+1)\Delta\zeta'-\frac{1}{2}C \tag{61}$$

$$f_2(\zeta)=\frac{1}{2}\varphi_1(\zeta)+\frac{1}{2}\mathop{\mathbb{S}}_{c}^{\zeta}\varphi_2(\zeta'+1)\Delta\zeta'-\frac{1}{2}C \tag{62}$$

Substitution of $f_1(\zeta)$ and $f_2(\zeta)$ into Eq.(52) brings:

$$A(\zeta,\theta)=\frac{1}{2}\varphi_1(\zeta-\theta)-\frac{1}{2}\mathop{\mathbb{S}}_{c}^{\zeta-\theta}\varphi_2(\zeta'+1)\Delta\zeta' +\frac{1}{2}\varphi_1(\zeta+\theta)+\frac{1}{2}\mathop{\mathbb{S}}_{c}^{\zeta+\theta}\varphi_2(\zeta'+1)\Delta\zeta' \tag{63}$$

Using a notation analogous to that used for integrals we may rewrite Eq.(63):

$$A(\zeta,\theta)=\frac{1}{2}[\varphi_1(\zeta-\theta)+\varphi_1(\zeta+\theta)]+\frac{1}{2}\mathop{\mathbb{S}}_{\zeta-\theta}^{\zeta+\theta}\varphi_2(\zeta'+1)\Delta\zeta' \tag{64}$$

This is the difference equivalent to the "classical solution" of the differential wave equation.

6.3 Differential Derivation of $A_{\mathrm{ev}}(\zeta,\theta)$

The derivation of $A_{\mathrm{ev}}(\zeta,\theta)$ in Eq.(2.2-40) used Eqs.(2.2-14) and (2.2-18) which come from the double integral of Eq.(2.1-45). This very elegant solution of the inhomogeneous wave equation (2.1-44) was derived by Smirnow (1961, Vol. II, VII, § 1, 175). The derivation of Smirnow's solution uses Poisson's formula (Smirnow, Vol. II, VII, § 1, 171). An extension of this formula from differential to difference mathematics does not seem to exist. We must either develop this extension or find a new solution of the inhomogeneous wave equation that is easier to extend to the mathematics of finite differences. The new solution is chosen here. Certain intermediate steps will be shown that one would usually leave out but these steps will help with the extension to finite differences.

We start with the inhomogeneous differential equation (2.1-44) for the potential $A_{\mathrm{ev}}(\zeta,\theta)$ but write it in simplified notation:

$$\frac{\partial^2 A}{\partial\zeta^2}-\frac{\partial^2 A}{\partial\theta^2}=a^2V(\zeta,\theta),\quad A=A_{\mathrm{ev}},\ V=V_{\mathrm{e}},\ a^2=c^2\tau^2 \tag{1}$$

Following the general solution of the wave equation by d'Alembert we introduce new variables η and ζ

$$\begin{aligned}&\eta=\zeta+\theta,\ \xi=\zeta-\theta,\ \zeta=\frac{1}{2}(\eta+\xi),\ \theta=\frac{1}{2}(\eta-\xi)\\&\partial\eta/\partial\zeta=1,\ \partial\eta/\partial\theta=1,\ \partial\xi/\partial\zeta=1,\ \partial\xi/\partial\theta=-1\end{aligned} \tag{2}$$

and obtain:

$$\frac{\partial A}{\partial\zeta}=\frac{\partial A}{\partial\eta}\frac{\partial\eta}{\partial\zeta}+\frac{\partial A}{\partial\xi}\frac{\partial\xi}{\partial\zeta}=\frac{\partial A}{\partial\eta}+\frac{\partial A}{\partial\xi} \tag{3}$$

$$\frac{\partial A}{\partial\theta}=\frac{\partial A}{\partial\eta}\frac{\partial\eta}{\partial\theta}+\frac{\partial A}{\partial\xi}\frac{\partial\xi}{\partial\theta}=\frac{\partial A}{\partial\eta}-\frac{\partial A}{\partial\xi} \tag{4}$$

A second differentiation yields:

$$\begin{aligned}\frac{\partial^2 A}{\partial\zeta^2}&=\frac{\partial}{\partial\eta}\left(\frac{\partial A}{\partial\eta}+\frac{\partial A}{\partial\xi}\right)\frac{\partial\eta}{\partial\zeta}+\frac{\partial}{\partial\xi}\left(\frac{\partial A}{\partial\eta}+\frac{\partial A}{\partial\xi}\right)\frac{\partial\xi}{\partial\zeta}\\&=\frac{\partial^2 A}{\partial\eta^2}+2\frac{\partial^2 A}{\partial\eta\partial\xi}+\frac{\partial^2 A}{\partial\xi^2}\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial^2 A}{\partial\theta^2}&=\frac{\partial}{\partial\eta}\left(\frac{\partial A}{\partial\eta}-\frac{\partial A}{\partial\xi}\right)\frac{\partial\eta}{\partial\theta}+\frac{\partial}{\partial\xi}\left(\frac{\partial A}{\partial\eta}-\frac{\partial A}{\partial\xi}\right)\frac{\partial\xi}{\partial\theta}\\&=\frac{\partial^2 A}{\partial\eta^2}-2\frac{\partial^2 A}{\partial\eta\partial\xi}+\frac{\partial^2 A}{\partial\xi^2}\end{aligned} \tag{6}$$

Substitution of Eqs.(5) and (6) into Eq.(1) yields:

$$\frac{\partial^2 A}{\partial\eta\partial\xi}=\frac{a^2}{4}V(\eta,\xi) \tag{7}$$

Integration with respect to η brings:

$$\frac{\partial A}{\partial \xi} = \frac{a^2}{4}\Big(\int^{\eta} V(\eta',\xi)d\eta' + F'_\eta(\xi)\Big) \tag{8}$$

A further integration with respect to ξ provides a general solution of Eq.(7):

$$\begin{aligned} A(\eta,\xi) &= \frac{a^2}{4}\Big[\int^{\xi}\Big(\int^{\eta} V(\eta',\xi')d\eta' + F'_\eta(\xi')\Big)d\xi' + C_\eta\Big] \\ &= \frac{a^2}{4}\Big[\int^{\xi}\Big(\int^{\eta} V(\eta',\xi')d\eta'\Big)d\xi' + F_\eta(\xi) + C_\eta\Big] \end{aligned} \tag{9}$$

The integration constants $F_\eta(\xi)$ and C_η must still be determined. To this end we substitute $\eta = \zeta + \theta$ and $\xi = \zeta - \theta$ in Eq.(9):

$$A(\zeta,\theta) = \frac{a^2}{4}\Big[\int^{\zeta-\theta}\Big(\int^{\zeta+\theta} V(\eta',\xi')d\eta'\Big)d\xi' + F_\eta(\zeta-\theta) + C_\eta\Big] \tag{10}$$

It is evident from Eq.(1) that the addition of a constant C_η to A leaves the equation unchanged. Since $A(\zeta,\theta)$ represents a potential we always use its derivatives either with respect to ζ or to θ and the constant C_η disappears. Hence, there is no need to determine C_η.

The function $F_\eta(\zeta-\theta)$ added to Eq.(1) also makes no difference since it is one half of d'Alembert's general solution of the homogeneous wave equation. If we want the other half we may integrate Eq.(7) first with respect to ξ and then with respect to η:

$$\frac{\partial A}{\partial \eta} = \frac{a^2}{4}\Big(\int^{\xi} V(\eta,\xi')d\xi' + F'_\xi(\eta)\Big) \tag{11}$$

$$\begin{aligned} A(\eta,\xi) &= \frac{a^2}{4}\Big[\int^{\eta}\Big(\int^{\xi} V(\eta',\xi')d\xi' + F'_\xi(\eta)\Big)d\eta' + C_\xi\Big] \\ &= \frac{a^2}{4}\Big[\int^{\eta}\Big(\int^{\xi} V(\eta',\xi')d\xi'\Big)d\eta' + F_\xi(\eta) + C_\xi\Big] \end{aligned} \tag{12}$$

The substitutions $\eta = \zeta + \theta$ and $\xi = \zeta - \theta$ bring:

$$A(\zeta,\theta) = \frac{a^2}{4}\Big[\int^{\zeta+\theta}\Big(\int^{\zeta-\theta} V(\eta',\xi')d\xi'\Big)d\eta' + F_\xi(\zeta+\theta) + C_\xi\Big] \tag{13}$$

A comparison with Eq.(10) shows that the double integral is the same since the sequence of integrations can be interchanged. Equation (13) contains the second half $F_\xi(\zeta+\theta)$ of d'Alembert's general solution of the homogeneous wave equation. Equation (10) describes waves that propagate in the direction of increasing values of the spatial variable ζ while Eq.(13) applies to waves propagating in the direction of decreasing values of ζ. The sum of Eqs.(10) and (13) is also a solution of Eq.(1) and permits solutions of the homogeneous equation of the form $F_\eta(\zeta-\theta)+F_\xi(\zeta+\theta)$:

$$A(\zeta,\theta)=\frac{a^2}{8}\left[2\int^{\zeta+\theta}\Big(\int^{\zeta-\theta}V(\eta',\xi')d\xi'\Big)d\eta'+F_\xi(\zeta+\theta)+F_\eta(\zeta-\theta)\right] \tag{14}$$

A constant $(C_\eta+C_\xi)a^2/4$ has been chosen to be zero in Eq.(14) in accordance with the text following Eq.(10).

We denote the double integral of Eq.(14) as a particular inhomogeneous solution $A_\mathrm{i}(\zeta,\theta)$ of Eq.(1)

$$A_\mathrm{i}(\zeta,\theta)=\frac{a^2}{4}\int^{\zeta+\theta}\Big(\int^{\zeta-\theta}V(\eta',\xi')d\xi'\Big)d\eta'=\frac{a^2}{4}\int^{\zeta+\theta}\Big(\int^{\zeta-\theta}V(\eta,\xi)d\xi\Big)d\eta \tag{15}$$

and

$$A_\mathrm{h}(\zeta,\theta)=\frac{a^2}{8}[F_\xi(\zeta+\theta)+F_\eta(\zeta-\theta)] \tag{16}$$

as the general homogeneous solution:

$$A(\zeta,\theta)=A_\mathrm{i}(\zeta,\theta)+A_\mathrm{h}(\zeta,\theta) \tag{17}$$

For the determination of the functions F_ξ and F_η we need two initial conditions. We choose them to be:

$$A(\zeta,0)=0,\quad \partial A(\zeta,\theta)/\partial\theta\Big|_{\theta=0}=\dot{A}(\zeta,\theta)\Big|_{\theta=0}=\dot{A}(\zeta,0)=0 \tag{18}$$

$$A_\mathrm{h}(\zeta,0)=-A_\mathrm{i}(\zeta,0) \tag{19}$$
$$\dot{A}_\mathrm{h}(\zeta,0)=-\dot{A}_\mathrm{i}(\zeta,0) \tag{20}$$

Equation (15) and (19) yield $A_\mathrm{h}(\zeta,0)$ as function of the inhomogeneous term $V(\eta,\xi)$:

$$A_\mathrm{h}(\zeta,0)=-A_\mathrm{i}(\zeta,0)=-\frac{a^2}{4}\int^{\zeta}\Big(\int^{\zeta}V(\eta,\xi)d\xi\Big)d\eta \tag{21}$$

In order to write $\dot{A}_{\rm h}$ of Eq.(20) as function of $V(\eta,\zeta)$ we must differentiate Eq.(16) with respect to θ. From Eq.(2) we obtain the differential relations

$$\partial\zeta/\partial\eta = 1/2, \quad \partial\theta/\partial\eta = 1/2, \quad \partial\zeta/\partial\xi = 1/2, \quad \partial\theta/\partial\xi = -1/2 \tag{22}$$

and Eq.(15) yields:

$$\frac{\partial A_{\rm i}}{\partial\xi} = \frac{\partial A_{\rm i}}{\partial\zeta}\frac{\partial\zeta}{\partial\xi} + \frac{\partial A_{\rm i}}{\partial\theta}\frac{\partial\theta}{\partial\xi} = \frac{1}{2}\left(\frac{\partial A_{\rm i}}{\partial\zeta} - \frac{\partial A_{\rm i}}{\partial\theta}\right)$$

$$\frac{\partial A_{\rm i}}{\partial\zeta} - \frac{\partial A_{\rm i}}{\partial\theta} = \frac{a^2}{2}\int^{\zeta+\theta} V(\eta,\zeta-\theta)d\eta \tag{23}$$

In order to be able to separate $\partial A_{\rm i}/\partial\zeta$ and $\partial A_{\rm i}/\partial\theta$ we differentiate Eq.(15) with respect to η:

$$\frac{\partial A_{\rm i}}{\partial\eta} = \frac{\partial A_{\rm i}}{\partial\zeta}\frac{\partial\zeta}{\partial\eta} + \frac{\partial A_{\rm i}}{\partial\theta}\frac{\partial\theta}{\partial\eta} = \frac{1}{2}\left(\frac{\partial A_{\rm i}}{\partial\zeta} + \frac{\partial A_{\rm i}}{\partial\theta}\right)$$

$$\frac{\partial A_{\rm i}}{\partial\zeta} + \frac{\partial A_{\rm i}}{\partial\theta} = \frac{a^2}{2}\int^{\zeta-\theta} V(\zeta+\theta,\xi)d\xi \tag{24}$$

Subtraction of Eq.(23) from Eq.(24) yields $\dot{A}_{\rm i}$:

$$\dot{A}_{\rm i} = \frac{\partial A_{\rm i}}{\partial\theta} = \frac{a^2}{4}\left(\int^{\zeta-\theta} V(\zeta+\theta,\xi)d\xi - \int^{\zeta+\theta} V(\eta,\zeta-\theta)d\eta\right) \tag{25}$$

In addition to Eq.(21) we have now a second initial condition for the homogeneous Eq.(16):

$$\dot{A}_{\rm h}(\zeta,0) = -\dot{A}_{\rm i}(\zeta,0) = -\frac{a^2}{4}\left(\int^{\zeta} V(\zeta,\xi)d\xi - \int^{\zeta} V(\eta,\zeta)d\eta\right) \tag{26}$$

Let us return to the homogeneous solution $A_{\rm h}(\zeta,\theta)$ of Eq.(16) and differentiate it with respect to θ:

$$\dot{A}_{\rm h}(\zeta,\theta) = \frac{a^2}{8}[F'_\xi(\zeta+\theta) - F'_\eta(\zeta-\theta)] \tag{27}$$

For $\theta = 0$ we use Eqs.(16) and (26) as initial conditions:

$$F_\xi(\zeta) + F_\eta(\zeta) = -\frac{8}{a^2}A_{\rm i}(\zeta,0) = -2\int^{\zeta}\left(\int^{\zeta} V(\eta,\xi)d\xi\right)d\eta \tag{28}$$

$$F'_\xi(\zeta) - F'_\eta(\zeta) = -\frac{8}{a^2}\dot{A}_{\rm i}(\zeta,0) = -2\left(\int^{\zeta} V(\zeta,\xi)d\xi - \int^{\zeta} V(\eta,\zeta)d\eta\right) \tag{29}$$

Equation (27) is integrated with respect to ζ:

$$F_\xi(\zeta) - F_\eta(\zeta) = -\frac{8}{a^2}\int^{\zeta} \dot{A}_{\rm i}(\zeta',0)d\zeta' + C_1 = \frac{8}{a^2}\int^{\zeta} \dot{A}_{\rm h}(\zeta',0)d\zeta' + C_1$$
$$= -2\int^{\zeta}\left(\int^{\zeta'} V(\zeta',\xi)d\xi - \int^{\zeta'} V(\eta,\zeta')d\eta\right)d\zeta' + C_1 \qquad (30)$$

The sum and difference of Eqs.(28) and (30) yield:

$$F_\xi(\zeta) = -\frac{4}{a^2}\left(A_{\rm i}(\zeta,0) + \int^{\zeta} \dot{A}_{\rm i}(\zeta',0)d\zeta' + C_1\right)$$
$$= -\left[\int^{\zeta}\left(\int^{\zeta} V(\eta,\xi)d\xi\right)d\eta\right.$$
$$\left. + \int^{\zeta}\left(\int^{\zeta'} V(\zeta',\xi)d\xi - \int^{\zeta'} V(\eta,\zeta')d\eta\right)d\zeta'\right] + \frac{C_1}{2} \qquad (31)$$

$$F_\eta(\zeta) = -\frac{4}{a^2}\left(A_{\rm i}(\zeta,0) - \int^{\zeta} \dot{A}_{\rm i}(\zeta',0)d\zeta' - C_1\right)$$
$$= -\left[\int^{\zeta}\left(\int^{\zeta} V(\eta,\xi)d\xi\right)d\eta\right.$$
$$\left. - \int^{\zeta}\left(\int^{\zeta'} V(\zeta',\xi)d\xi - \int^{\zeta'} V(\eta,\zeta')d\eta\right)d\zeta'\right] - \frac{C_1}{2} \qquad (32)$$

The homogeneous solution of Eq.(16) becomes:

$$\frac{a^2}{8}[F_\xi(\zeta+\theta)+F_\eta(\zeta-\theta)] = -\frac{a^2}{8}\left[\int^{\zeta+\theta}\left(\int^{\zeta+\theta} V(\eta,\xi)d\xi\right)d\eta + \int^{\zeta-\theta}\left(\int^{\zeta-\theta} V(\eta,\xi)d\xi\right)d\eta\right.$$
$$\left. + \int_{\zeta-\theta}^{\zeta+\theta}\left(\int^{\zeta'} V(\zeta',\xi)d\xi - \int^{\zeta'} V(\eta,\zeta')d\eta\right)d\zeta'\right] \qquad (33)$$

Equation (14) may be rewritten as follows:

$$A(\zeta,\theta)=\frac{a^2}{8}\Bigg[-\int\limits^{\zeta+\theta}\Bigg(\int\limits^{\zeta+\theta}V(\eta,\xi)d\xi-\int\limits^{\xi-\theta}V(\eta,\xi)d\xi\Bigg)d\eta$$
$$+\int\limits^{\zeta-\theta}\Bigg(\int\limits^{\zeta+\theta}V(\eta,\xi)d\xi-\int\limits^{\zeta-\theta}V(\eta,\xi)d\xi\Bigg)d\eta$$
$$-\int\limits_{\zeta-\theta}^{\zeta+\theta}\Bigg(\int\limits^{\zeta'}V(\zeta',\xi)d\xi-\int\limits^{\zeta'}V(\eta,\zeta')d\eta\Bigg)d\zeta'\Bigg] \qquad (34)$$

Substitution of $A=A_{\rm ev}$, $V=V_{\rm e}$, and $a^2=c^2\tau^2$ according to Eq.(1) brings Eq.(34) into the following form:

$$A_{\rm ev}(\zeta,\theta)=-\frac{c^2\tau^2}{8}\int\limits_{\zeta-\theta}^{\zeta+\theta}\Bigg[\Bigg(\int\limits_{\zeta-\theta}^{\zeta+\theta}V_{\rm e}(\eta,\xi)d\xi\Bigg)d\eta$$
$$+\Bigg(\int\limits^{\zeta'}V_{\rm e}(\zeta',\xi)d\xi-\int\limits^{\zeta'}V_{\rm e}(\eta,\zeta')d\eta\Bigg)d\zeta'\Bigg] \qquad (35)$$

Let us use the function $V_{\rm e}=V_{\rm e0}F(\zeta)=V_{\rm e0}e^{-\rho_2\zeta}$ of Eqs.(2.1-62) and (2.1-64). For simplification we choose $c^2\tau^2V_{\rm e0}=1$. We get from Eq.(2) for the last two terms:

$$\begin{aligned}\rho_2\zeta=\rho_2(\xi+\eta)/2&=\rho_2(\zeta'+\xi)/2\quad\text{for }\zeta'=\eta\\&=\rho_2(\zeta'+\eta)/2\quad\text{for }\zeta'=\xi\end{aligned}$$
$$\int\limits^{\zeta'}e^{-\rho_2(\zeta'+\xi)/2}d\xi-\int\limits^{\zeta'}e^{-\rho_2(\zeta'+\eta)/2}d\eta=0 \qquad (36)$$

The first term in Eq.(35) yields:

$$\begin{aligned}A_{\rm ev}(\zeta,\theta)&=-\frac{1}{8}\int\limits_{\zeta-\theta}^{\zeta+\theta}\Bigg(\int\limits_{\zeta-\theta}^{\zeta+\theta}e^{-\rho_2(\eta+\xi)/2}d\xi\Bigg)d\eta\\&=-\frac{2}{\rho_2^2}e^{-\rho_2\zeta}\,\mathrm{sh}^2(\rho_2\theta/2)\end{aligned} \qquad (37)$$

This worked quite well. However, if we use the function $w(\zeta,\theta)$ of Eqs.(2.1-62) and(2.2-13) we realize that the substitutions $\eta=\zeta+\theta$ and $\xi=\zeta-\theta$ lead to

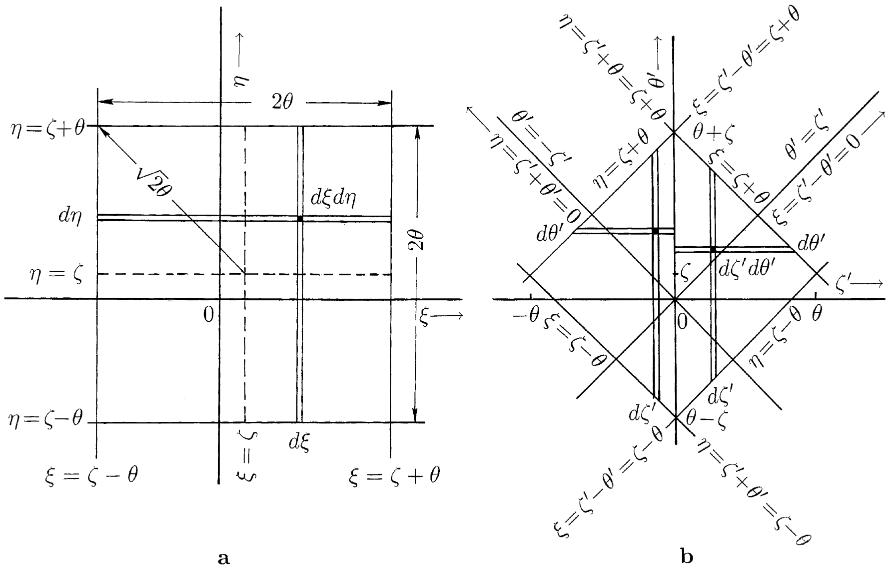

FIGURE 6.3-1. a) Integration limits for the variables ξ and η in the first integral of Eq.(35). b) Transformation of the integration limits from the coordinates ξ, η to the coordinates ζ', θ' in Eqs.(38) and (40).

more complicated integrals. This suggests to replace η and ξ in Eq.(35) by ζ and θ or rather by ζ' and θ'. For the first term we obtain with $\xi = \zeta' - \theta'$ and $\eta = \zeta' + \theta'$:

$$\int_{\zeta-\theta}^{\zeta+\theta} \left(\int_{\zeta-\theta}^{\zeta+\theta} V_{\mathrm{e}}(\eta, \xi) d\xi \right) d\eta = \int \left(\int V_{\mathrm{e}}(\zeta', \theta') D \, d\zeta' \right) d\theta'$$

$$= 2 \int \left(\int V_{\mathrm{e}}(\zeta', \theta') d\zeta' \right) d\theta' \qquad (38)$$

$$D = \begin{vmatrix} \partial\xi/\partial\zeta & \partial\xi/\partial\theta \\ \partial\eta/\partial\zeta & \partial\eta/\partial\theta \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad (39)$$

The interval $\zeta - \theta \le \xi \le \zeta + \theta$, $\zeta - \theta \le \eta \le \zeta + \theta$ is shown in Fig.6.3-1a. The function $V_{\mathrm{e}}(\eta, \xi)$ is defined for every pair of points η, ξ in this interval. Hence, the left side of Eq.(38) represents the volume of $V_{\mathrm{e}}(\eta, \xi)$ over the area defined by the integration limits. The transformation $\xi = \zeta' - \theta'$, $\eta = \zeta' + \theta'$ rotates the coordinate system ξ, η by $\pi/4$ relative to the coordinate system ζ', θ', as shown in Fig.6.3-1b. The volume of the function $V_{\mathrm{e}}(\eta, \xi)$ on the left in Eq.(38) becomes the volume of the function $V_{\mathrm{e}}(\zeta', \theta') D = 2V(\zeta', \theta')$ on the right. To obtain the same volume on both sides we must contract the coordinates ζ' and

θ' by a factor $1/\sqrt{2}$. The reduced lengths are used in Fig.6.3-1b. We obtain the following integration limits in Eq.(38) on the right:

$$
\begin{aligned}
&2\int\left(\int V_\mathrm{e}(\zeta',\theta')d\zeta'\right)d\theta' \\
&= 2\left[\int_{-\theta}^{0}\left(\int_{\zeta-(\theta+\theta')}^{\zeta+(\theta+\theta')} V_\mathrm{e}(\zeta',\theta')d\zeta'\right)d\theta' + \int_{0}^{\theta}\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_\mathrm{e}(\zeta',\theta')d\zeta'\right)d\theta'\right] \qquad (40)
\end{aligned}
$$

Consider the example of Eq.(37):

$$
\begin{aligned}
A_\mathrm{ev}(\zeta,\theta) &= -\frac{1}{8}\int_{\zeta-\theta}^{\zeta+\theta}\left(\int_{\zeta-\theta}^{\zeta+\theta} e^{-\rho_2(\eta+\xi)/2}d\xi\right)d\eta \\
&= -\frac{1}{4}\left[\int_{-\theta}^{0}\left(\int_{\zeta-(\theta+\theta')}^{\zeta+(\theta+\theta')} e^{-\rho_2\zeta'}d\zeta'\right)d\theta' + \int_{0}^{\theta}\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-\rho_2\zeta'}d\zeta'\right)d\theta'\right] \\
&= -\frac{2}{\rho_2^2}e^{-\rho_2\zeta}\,\mathrm{sh}^2(\rho_2\theta/2) \qquad (41)
\end{aligned}
$$

Alternately to Eqs.(37) and (41) we may use Eq.(2.1-45):

$$
A_\mathrm{ev}(\zeta,\theta) = -\frac{1}{2}\int_{0}^{\theta}\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-\rho_2\zeta'}d\zeta'\right)d\theta' = -\frac{2}{\rho_2^2}e^{-\rho_2\zeta}\,\mathrm{sh}^2(\rho_2\theta/2) \qquad (42)
$$

Equations (42) and (2.1-45) are much more desirable than Eqs.(37) and (41)—which require that the last two terms of Eq.(35) vanish as shown by Eq.(36)—but extending Eqs.(37) and (41) from differential calculus to the calculus of finite differences appears to be a great deal easier than the corresponding extensions of Eqs.(2.1-45) and (42).

6.4 Calculations for Section 3.3

We start with a collection of auxiliary variables that will be used extensively. From Eqs.(3.1-1), (3.2-15), (3.2-31), (3.2-61), and (3.2-60) we get:

$$N = T/\Delta t, \quad \rho_2^2 = (c\Delta t)^2\sigma s \tag{1}$$
$$\rho_1 = c\Delta t(\sigma Z + s/Z) = c^2\Delta t(\sigma\mu + s\epsilon) \tag{2}$$
$$\varphi_\kappa = 2\pi\kappa/N, \quad \kappa = 0,\ \pm1,\ \pm2, \dots,\ \pm N/2 \tag{3}$$
$$d_\Delta^2 = 4\left[4\left(1 - \sin^2\frac{\varphi_\kappa}{2}\right)\sin^2\frac{\varphi_\kappa}{2} + \rho_2^2\right] \tag{4}$$
$$\lambda_{\mathrm{A1}} = \ln\left(1 - 2[\sin(\pi\kappa/N)]^2 + \frac{1}{2}h(\pi\kappa/N)\right)$$
$$\lambda_{\mathrm{A2}} = \ln\left(1 - 2[\sin(\pi\kappa/N)]^2 - \frac{1}{2}h(\pi\kappa/N)\right)$$
$$\lambda_{\mathrm{a}} = \arcsin\frac{1}{2}(d_\Delta^2 - \rho_1^2)^{1/2} \tag{5}$$
$$\xi = \zeta - \theta,\ \eta = \zeta + \theta,\ \eta - \xi = 2\theta,\ \eta + \xi = 2\zeta \tag{6}$$
$$\lambda_{ik},\ \gamma_{ik},\ S_{ik},\ D_i(\kappa),\ h(\pi\kappa/N) \quad \text{see Section 3.3}$$

Equation (3.3-55) must be written so that the functions of η and ξ are clearly shown. We note that $D_0(\kappa)$ to $D_3(\kappa)$ are not functions of η or ξ, but the products $S_{00}S_{10}$ to $S_{08}S_{18}$ contain these variables. We postpone rewriting the product $S_{00}S_{10}$, since it is done in a way different from the rest, and start with $S_{01}S_{11}$. Equations (3.3-19), (3.3-39) and (3.3-20), (3.3-40) yield:

$$\begin{aligned}
S_{01}S_{11} = {} & \frac{e^{(\rho_1/2-\lambda_{\mathrm{A1}})\xi/2}}{[(\rho_1/2-\lambda_{\mathrm{A1}})/2]^2 + (\pi\kappa/N)^2}\,\frac{e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\eta/2}}{[(\rho_1/2-\lambda_{\mathrm{A1}})/2]^2 + (\pi\kappa/N)^2} \\
& \times\left(\gamma_{01}\lambda_{11}\sin\frac{\pi\kappa\xi}{N}\sin\frac{\pi\kappa\eta}{N} - \lambda_{01}\gamma_{11}\cos\frac{\pi\kappa\xi}{N}\cos\frac{\pi\kappa\eta}{N}\right. \\
& \left. - \gamma_{01}\gamma_{11}\sin\frac{\pi\kappa\xi}{N}\cos\frac{\pi\kappa\eta}{N} + \lambda_{01}\lambda_{11}\cos\frac{\pi\kappa\xi}{N}\sin\frac{\pi\kappa\eta}{N}\right) \\
= {} & R_{11}e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\theta}\left(\Lambda_{10}\cos\frac{2\pi\kappa\theta}{N} - \Lambda_{11}\cos\frac{2\pi\kappa\zeta}{N}\right. \\
& \left. - \Lambda_{12}\sin\frac{2\pi\kappa\theta}{N} + \Lambda_{13}\sin\frac{2\pi\kappa\zeta}{N}\right)
\end{aligned} \tag{7}$$

$$\begin{aligned}
S_{02}S_{12} = {} & \frac{e^{(\rho_1/2-\lambda_{\mathrm{A1}})\xi/2}}{[(\rho_1/2-\lambda_{\mathrm{A1}})/2]^2 + (\pi\kappa/N)^2}\,\frac{e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\eta/2}}{[(\rho_1/2-\lambda_{\mathrm{A1}})/2]^2 + (\pi\kappa/N)^2} \\
& \times\left(\lambda_{01}\gamma_{11}\sin\frac{\pi\kappa\xi}{N}\sin\frac{\pi\kappa\eta}{N} - \gamma_{01}\lambda_{11}\cos\frac{\pi\kappa\xi}{N}\cos\frac{\pi\kappa\eta}{N}\right. \\
& \left. + \lambda_{01}\lambda_{11}\sin\frac{\pi\kappa\xi}{N}\cos\frac{\pi\kappa\eta}{N} - \gamma_{01}\gamma_{11}\cos\frac{\pi\kappa\xi}{N}\sin\frac{\pi\kappa\eta}{N}\right)
\end{aligned}$$

$$= R_{11}e^{-(\rho_1/2-\lambda_{A1})\theta}\bigg(-\Lambda_{10}\cos\frac{2\pi\kappa\theta}{N} - \Lambda_{11}\cos\frac{2\pi\kappa\zeta}{N} + \Lambda_{12}\sin\frac{2\pi\kappa\theta}{N} + \Lambda_{13}\sin\frac{2\pi\kappa\zeta}{N}\bigg) \quad (8)$$

$$R_{11} = \frac{1}{\{[(\rho_1/2-\lambda_{A1})/2]^2+(\pi\kappa/N)^2\}^2} \quad (9)$$

$$\Lambda_{10} = \frac{1}{2}(\gamma_{01}\lambda_{11}-\lambda_{01}\gamma_{11}), \qquad \Lambda_{11} = \frac{1}{2}(\gamma_{01}\lambda_{11}+\lambda_{01}\gamma_{11})$$

$$\Lambda_{12} = \frac{1}{2}(\lambda_{01}\lambda_{11}+\gamma_{01}\gamma_{11}), \qquad \Lambda_{13} = \frac{1}{2}(\lambda_{01}\lambda_{11}-\gamma_{01}\gamma_{11}) \quad (10)$$

Furthermore, we get from Eqs.(3.3-23), (3.3-43) and (3.3-24), (3.3-44):

$$S_{03}S_{13} = R_{13}e^{-(\rho_1/2-\lambda_{A2})\theta}\bigg(\Lambda_{30}\cos\frac{2\pi\kappa\theta}{N} - \Lambda_{31}\cos\frac{2\pi\kappa\zeta}{N} - \Lambda_{32}\sin\frac{2\pi\kappa\theta}{N} + \Lambda_{33}\sin\frac{2\pi\kappa\zeta}{N}\bigg) \quad (11)$$

$$S_{04}S_{14} = R_{13}e^{-(\rho_1/2-\lambda_{A2})\theta}\bigg(-\Lambda_{30}\cos\frac{2\pi\kappa\theta}{N} - \Lambda_{31}\cos\frac{2\pi\kappa\zeta}{N} + \Lambda_{32}\sin\frac{2\pi\kappa\theta}{N} + \Lambda_{33}\sin\frac{2\pi\kappa\zeta}{N}\bigg) \quad (12)$$

$$R_{13} = \frac{1}{\{[(\rho_1/2-\lambda_{A2})/2]^2+(\pi\kappa/N)^2\}^2} \quad (13)$$

$$\Lambda_{30} = \frac{1}{2}(\gamma_{03}\lambda_{13}-\lambda_{03}\gamma_{13}), \qquad \Lambda_{31} = \frac{1}{2}(\gamma_{03}\lambda_{13}+\lambda_{03}\gamma_{13})$$

$$\Lambda_{32} = \frac{1}{2}(\lambda_{03}\lambda_{13}+\gamma_{03}\gamma_{13}), \qquad \Lambda_{33} = \frac{1}{2}(\lambda_{03}\lambda_{13}-\gamma_{03}\gamma_{13}) \quad (14)$$

The terms in the second sum of Eq.(3.3-55) are more complicated and there are eight of them rather than four. With the following five auxiliary variables

$$R_{15} = \frac{1}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_a/2)^2}\,\frac{1}{(\rho_1/4)^2+(\pi\kappa/N+\lambda_a/2)^2} \quad (15)$$

$$\Lambda_{50} = \frac{1}{2}(\gamma_{05}\lambda_{15}-\lambda_{05}\gamma_{15}), \qquad \Lambda_{51} = \frac{1}{2}(\gamma_{05}\lambda_{15}+\lambda_{05}\gamma_{15})$$

$$\Lambda_{52} = \frac{1}{2}(\gamma_{05}\gamma_{15}+\lambda_{05}\lambda_{15}), \qquad \Lambda_{53} = \frac{1}{2}(\gamma_{05}\gamma_{15}-\lambda_{05}\lambda_{15}) \quad (16)$$

we obtain expressions for $S_{05}S_{15}$, $S_{06}S_{15}$, $S_{05}S_{16}$, $S_{06}S_{16}$ with θ and ζ shown explicitly:

$$
\begin{aligned}
S_{05}S_{15} = {} & \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2}\,\frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2+(\pi\kappa/N-\lambda_\mathrm{a}/2)^2} \\
& \times\left\{\gamma_{05}\lambda_{15}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\,\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right. \\
& -\lambda_{05}\gamma_{15}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_\mathrm{a}}{2}\right)\eta\right] \\
& -\gamma_{05}\gamma_{15}\sin\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\cos\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_\mathrm{a}}{2}\right)\eta\right] \\
& \left.+\lambda_{05}\lambda_{15}\cos\left[\left(\frac{\pi\kappa}{N}-\frac{\lambda_\mathrm{a}}{2}\right)\xi\right]\,\sin\left[\left(\frac{\pi\kappa}{N}+\frac{\lambda_\mathrm{a}}{2}\right)\eta\right]\right\} \\
= {} & R_{15}e^{-\rho_1\theta/2}\left[\Lambda_{50}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta-\sin\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right)\right. \\
& -\Lambda_{51}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta-\sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right) \\
& +\Lambda_{52}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta+\cos\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right) \\
& \left.-\Lambda_{53}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta+\cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right)\right]
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
S_{06}S_{15} = R_{15}e^{-\rho_1\theta/2}\Bigg[& \Lambda_{52}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta-\sin\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right) \\
& +\Lambda_{53}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta-\sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right) \\
& -\Lambda_{50}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta+\cos\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right) \\
& -\Lambda_{51}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta+\cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right)\Bigg]
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
S_{05}S_{16} = R_{15}e^{-\rho_1\theta/2}\Bigg[& \Lambda_{52}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta-\sin\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right) \\
& -\Lambda_{53}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta-\sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right) \\
& -\Lambda_{50}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_\mathrm{a}\zeta+\cos\frac{2\pi\kappa\theta}{N}\sin\lambda_\mathrm{a}\zeta\right) \\
& +\Lambda_{51}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_\mathrm{a}\theta+\cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_\mathrm{a}\theta\right)\Bigg]
\end{aligned}
\tag{19}
$$

$$\begin{aligned}S_{06}S_{16} = -R_{15}e^{-\rho_1\theta/2}\bigg[\Lambda_{50}\bigg(&\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_{\mathrm{a}}\zeta - \sin\frac{2\pi\kappa\theta}{N}\sin\lambda_{\mathrm{a}}\zeta\bigg)\\ + \Lambda_{51}\bigg(&\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\mathrm{a}}\theta - \sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\mathrm{a}}\theta\bigg)\\ + \Lambda_{52}\bigg(&\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_{\mathrm{a}}\zeta + \cos\frac{2\pi\kappa\theta}{N}\sin\lambda_{\mathrm{a}}\zeta\bigg)\\ + \Lambda_{53}\bigg(&\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\mathrm{a}}\theta + \cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\mathrm{a}}\theta\bigg)\bigg]\end{aligned} \qquad (20)$$

For the products $S_{07}S_{17}$, $S_{08}S_{17}$, $S_{08}S_{18}$, $S_{07}S_{18}$ we may again use the auxiliary variable R_{15} of Eq.(15) but the auxiliary variables of Eq.(16) must be replaced:

$$\begin{aligned}\Lambda_{70} &= \frac{1}{2}(\gamma_{07}\lambda_{17} - \lambda_{07}\gamma_{17}), \quad \Lambda_{71} = \frac{1}{2}(\gamma_{07}\lambda_{17} + \lambda_{07}\gamma_{17})\\ \Lambda_{72} &= \frac{1}{2}(\gamma_{07}\gamma_{17} + \lambda_{07}\lambda_{17}), \quad \Lambda_{73} = \frac{1}{2}(\gamma_{07}\gamma_{17} - \lambda_{07}\lambda_{17})\end{aligned} \qquad (21)$$

The following four products are obtained from Eq.(3.3-55):

$$\begin{aligned}S_{07}S_{17} &= \frac{e^{\rho_1\xi/4}}{(\rho_1/4)^2 + (\pi\kappa/N + \lambda_{\mathrm{a}}/2)^2}\,\frac{e^{-\rho_1\eta/4}}{(\rho_1/4)^2 + (\pi\kappa/N - \lambda_{\mathrm{a}}/2)^2}\\ &\times\bigg\{\gamma_{07}\lambda_{17}\sin\bigg[\bigg(\frac{\pi\kappa}{N} + \frac{\lambda_{\mathrm{a}}}{2}\bigg)\xi\bigg]\sin\bigg[\bigg(\frac{\pi\kappa}{N} - \frac{\lambda_{\mathrm{a}}}{2}\bigg)\eta\bigg]\\ &\quad - \lambda_{07}\gamma_{17}\cos\bigg[\bigg(\frac{\pi\kappa}{N} + \frac{\lambda_{\mathrm{a}}}{2}\bigg)\xi\bigg]\cos\bigg[\bigg(\frac{\pi\kappa}{N} - \frac{\lambda_{\mathrm{a}}}{2}\bigg)\eta\bigg]\\ &\quad - \gamma_{07}\gamma_{17}\sin\bigg[\bigg(\frac{\pi\kappa}{N} + \frac{\lambda_{\mathrm{a}}}{2}\bigg)\xi\bigg]\cos\bigg[\bigg(\frac{\pi\kappa}{N} - \frac{\lambda_{\mathrm{a}}}{2}\bigg)\eta\bigg]\\ &\quad + \lambda_{07}\lambda_{17}\cos\bigg[\bigg(\frac{\pi\kappa}{N} + \frac{\lambda_{\mathrm{a}}}{2}\bigg)\xi\bigg]\sin\bigg[\bigg(\frac{\pi\kappa}{N} - \frac{\lambda_{\mathrm{a}}}{2}\bigg)\eta\bigg]\bigg\}\\ &= R_{15}e^{-\rho_1\theta/2}\bigg[\Lambda_{70}\bigg(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_{\mathrm{a}}\zeta + \sin\frac{2\pi\kappa\theta}{N}\sin\lambda_{\mathrm{a}}\zeta\bigg)\\ &\quad - \Lambda_{71}\bigg(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\mathrm{a}}\theta + \sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\mathrm{a}}\theta\bigg)\\ &\quad + \Lambda_{72}\bigg(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_{\mathrm{a}}\zeta - \cos\frac{2\pi\kappa\theta}{N}\sin\lambda_{\mathrm{a}}\zeta\bigg)\\ &\quad - \Lambda_{73}\bigg(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\mathrm{a}}\theta - \cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\mathrm{a}}\theta\bigg)\bigg]\end{aligned} \qquad (22)$$

$$S_{08}S_{17} = R_{15}e^{-\rho_1\theta/2}\Bigg[\Lambda_{72}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta + \sin\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$+ \Lambda_{73}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta + \sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)$$
$$- \Lambda_{70}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta - \cos\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$- \Lambda_{71}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta - \cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)\Bigg] \tag{23}$$

$$S_{07}S_{18} = R_{15}e^{-\rho_1\theta/2}\Bigg[\Lambda_{72}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta + \sin\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$- \Lambda_{73}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta + \sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)$$
$$- \Lambda_{70}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta - \cos\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$+ \Lambda_{71}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta - \cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)\Bigg] \tag{24}$$

$$S_{08}S_{18} = -R_{15}e^{-\rho_1\theta/2}\Bigg[\Lambda_{70}\left(\cos\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta + \sin\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$+ \Lambda_{71}\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta + \sin\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)$$
$$+ \lambda_{72}\left(\sin\frac{2\pi\kappa\theta}{N}\cos\lambda_{\rm a}\zeta - \cos\frac{2\pi\kappa\theta}{N}\sin\lambda_{\rm a}\zeta\right)$$
$$+ \Lambda_{73}\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\lambda_{\rm a}\theta - \cos\frac{2\pi\kappa\zeta}{N}\sin\lambda_{\rm a}\theta\right)\Bigg] \tag{25}$$

We turn to the term $S_{00}S_{10}$ in Eq.(3.3-55). Equations (3.3-17) and (3.3-38) yield:

$$S_{00}S_{10} = \frac{e^{-\rho_2\xi/2}}{\mathrm{sh}(\rho_1/2)}\,\frac{e^{-\rho_2\eta/2}}{\mathrm{sh}(\rho_2/2)} = \frac{e^{-\rho_2\zeta}}{\mathrm{sh}^2(\rho_2/2)} \tag{26}$$

The term for $\kappa = 0$ of the first sum in Eq.(3.3-55) might still have to be added to $S_{00}S_{10}$. We obtain from Eq.(3.2-15):

$$\varphi_\kappa = 2\pi\kappa/N = 0 \quad \text{for } \kappa = 0 \tag{27}$$

From Eq.(3.3-14) we get

$$h(0) = (-4\rho_2^2 + \rho_1^2)^{1/2}, \quad -4\rho_2^2 + \rho_1^2 \geq 0 \tag{28}$$

and Eq.(3.3-18) yields:

$$\begin{aligned} \lambda_{\mathrm{A}1} &= \ln[1 + (-4\rho_2^2 + \rho_1^2)^{1/2}] \doteq (-4\rho_2^2 + \rho_1^2)^{1/2} \\ \lambda_{\mathrm{A}2} &= \ln[1 - (-4\rho_2^2 + \rho_1^2)^{1/2}] \doteq -(-4\rho_2^2 + \rho_1^2)^{1/2} \end{aligned} \tag{29}$$

Further, we obtain from Eqs.(3.3-9), (3.3-10), (3.3-12), (3.3-19)–(3.3-26), and (3.3-39)–(3.3-46) for $\kappa = 0$:

$$D_0(0) = 0 \tag{30}$$

$$D_1(0) = 1 - \frac{1 - \rho_1/2}{(-4\rho_2^2 + \rho_1^2)^{1/2}} \tag{31}$$

$$D_2(0) = 1 + \frac{1 - \rho_1/2}{(-4\rho_2^2 + \rho_1^2)^{1/2}} \tag{32}$$

$$\begin{aligned} S_{01} &= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A}1})\xi/2}\lambda_{01}}{[(\rho_1/2 - \lambda_{\mathrm{A}1})/2]^2} \\ \lambda_{01} &= -\frac{2[(\rho_1/2 - \lambda_{\mathrm{A}1})/2]^2 \operatorname{sh}[(\rho_1/2 - \lambda_{\mathrm{A}1})/2]}{1 - \operatorname{ch}(\rho_1/2 - \lambda_{\mathrm{A}1})} \end{aligned} \tag{33}$$

$$S_{02} = -\frac{e^{(\rho_1/2-\lambda_{\mathrm{A}1})\xi/2}\gamma_{01}}{[(\rho_1/2 - \lambda_{\mathrm{A}1})/2]^2}, \quad \gamma_{01} = 0 \tag{34}$$

$$\begin{aligned} S_{03} &= \frac{e^{(\rho_1/2-\lambda_{\mathrm{A}2})\xi/2}\lambda_{03}}{[(\rho_1/2 - \lambda_{\mathrm{A}2})/2]^2} \\ \lambda_{03} &= -\frac{2[(\rho_1/2 - \lambda_{\mathrm{A}2})/2]^2 \operatorname{sh}[(\rho_1/2 - \lambda_{\mathrm{A}2})/2]}{1 - \operatorname{ch}(\rho_1/2 - \lambda_{\mathrm{A}2})} \end{aligned} \tag{35}$$

$$S_{04} = -\frac{e^{(\rho_1/2-\lambda_{\mathrm{A}2})\xi/2}\gamma_{03}}{[(\rho_1/2 - \lambda_{\mathrm{A}2})/2]^2}, \quad \gamma_{03} = 0 \tag{36}$$

$$S_{11} = -\frac{e^{-(\rho_1/4-\lambda_{\mathrm{A}1}/2)\eta}\gamma_{11}}{(\rho_1/4 - \lambda_{\mathrm{A}1}/2)^2}, \quad \gamma_{11} = 0 \tag{37}$$

$$\begin{aligned} S_{12} &= \frac{e^{-(\rho_1/4-\lambda_{\mathrm{A}1}/2)\eta}\lambda_{11}}{(\rho_1/4 - \lambda_{\mathrm{A}1}/2)^2} \\ \lambda_{11} &= \frac{2(\rho_1/4 - \lambda_{\mathrm{A}1}/2)^2 \operatorname{sh}(\rho_1/4 - \lambda_{\mathrm{A}1}/2)}{1 - \operatorname{ch}(\rho_1/2 - \lambda_{\mathrm{A}1})} \end{aligned} \tag{38}$$

$$S_{13} = -\frac{e^{-(\rho_1/4-\lambda_{\mathrm{A}2}/2)\eta}\gamma_{13}}{(\rho_1/4 - \lambda_{\mathrm{A}2}/2)^2}, \quad \gamma_{13} = 0 \tag{39}$$

$$S_{14} = \frac{e^{-(\rho_1/4-\lambda_{A2}/2)\eta}\lambda_{13}}{(\rho_1/4-\lambda_{A2}/2)^2}$$
$$\lambda_{13} = \frac{2(\rho_1/4-\lambda_{A2}/2)^2\,\mathrm{sh}(\rho_1/4-\lambda_{A2}/2)}{1-\mathrm{ch}(\rho_1/2-\lambda_{A2})} \tag{40}$$

We obtain for $\kappa = 0$:

$$S_{01}S_{11} = 0, \quad S_{02}S_{12} = 0, \quad S_{03}S_{13} = 0, \quad S_{04}S_{14} = 0 \tag{41}$$

With $D_1(0)$ and $D_2(0)$ finite in all cases except the distortion-free case mentioned after Eq.(3.2-58) we obtain zero for the term $\kappa = 0$ in the first sum of Eq.(3.3-55).

Using $S_{00}S_{10}$ of Eq.(26) wew may rewrite the first line of Eq.(3.3-55) into the following form

$$\frac{1}{4}V_{e0}(c\Delta t)^2 S_{00}S_{10} = \frac{1}{4}V_{e0}(c\Delta t)^2\frac{e^{-\rho_2\zeta}}{\mathrm{sh}^2(\rho_2/2)} \doteq V_{e0}(c\Delta t)^2\frac{1}{\rho_2^2}e^{-\rho_2\zeta}$$
$$\rho_2 = (c\Delta t)\sqrt{\sigma s} \ll 1 \tag{42}$$

which differs from the first line of Eq.(2.2-39):

$$L_1 = V_{e0}(c\tau)^2\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\mathrm{ch}\,\rho_2\theta) \tag{43}$$

One may add the general solution of the homogeneous difference wave equation of Eq.(6.2-52) to Eq.(42). If one chooses

$$f_1(\zeta-\theta) + f_2(\zeta+\theta) = -\frac{V_{e0}(c\Delta t)^2}{2\rho_2^2}\left(e^{-\rho_2(\zeta-\theta)} + e^{-\rho_2(\zeta+\theta)}\right)$$
$$= -V_{e0}(c\Delta t)^2\frac{1}{\rho_2^2}e^{-\rho_2}\,\mathrm{ch}\,\rho_2\theta \tag{44}$$

and adds Eq.(44) to Eq.(42) one obtains Eq.(43) for $\tau = \Delta t$. There is no incentive to do so. This term of the differential theory was ignored according to Eq.(6.1-64) and Fig.6.1-1 in order to save a Fourier series expansion of $1-\mathrm{ch}\,\rho_2\theta$. We are glad this term does not occur in Eq.(42) of the finite difference theory.

With the help of Eqs.(7)–(14) we may combine the terms of the first sum in Eq.(3.3-55):

$$D_0[D_1(S_{01}S_{11}+S_{02}S_{12}) - D_2(S_{03}S_{13}+S_{04}S_{14})]$$
$$= -e^{-\rho_1\theta/2}\Bigg(G_{11}e^{\lambda_{A1}\theta}\cos\frac{2\pi\kappa\zeta}{N} - G_{12}e^{\lambda_{A1}\theta}\sin\frac{2\pi\kappa\zeta}{N}$$
$$+G_{13}e^{\lambda_{A2}\theta}\cos\frac{2\pi\kappa\zeta}{N} - G_{14}e^{\lambda_{A2}\theta}\sin\frac{2\pi\kappa\zeta}{N}\Bigg) \tag{45}$$

$$
\begin{aligned}
G_{11}(\kappa) &= 2D_0D_1R_{11}\Lambda_{11} \qquad & G_{12}(\kappa) &= 2D_0D_1R_{11}\Lambda_{13} \\
G_{13}(\kappa) &= 2D_0D_2R_{13}\Lambda_{13} \qquad & G_{14}(\kappa) &= 2D_0D_2R_{13}\Lambda_{33}
\end{aligned} \tag{46}
$$

The second sum of Eq.(3.3-55) can be rewritten correspondingly with the help of Eqs.(17)–(25):

$$
\begin{aligned}
&(D_0S_{05} - D_3S_{06})S_{15} + (D_0S_{06} - D_3S_{05})S_{16} \\
&\qquad + (D_0S_{07} - D_3S_{08})S_{17} + (D_0S_{08} + D_3S_{07})S_{18} \\
&= -e^{-\rho_1\theta/2}\bigg[\bigg(G_{01}(\kappa)\cos\frac{2\pi\kappa\theta}{N} - G_{03}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\bigg)\cos\lambda_{\mathrm{a}}\zeta \\
&\qquad + \bigg(G_{03}(\kappa)\cos\frac{2\pi\kappa\theta}{N} + G_{01}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\bigg)\sin\lambda_{\mathrm{a}}\zeta \\
&\qquad + \bigg(G_{02}(\kappa)\cos\lambda_{\mathrm{a}}\theta - G_{04}(\kappa)\sin\lambda_{\mathrm{a}}\theta\bigg)\cos\frac{2\pi\kappa\zeta}{N} \\
&\qquad + \bigg(G_{04}(\kappa)\cos\lambda_{\mathrm{a}}\theta + G_{02}(\kappa)\sin\lambda_{\mathrm{a}}\theta\bigg)\sin\frac{2\pi\kappa\zeta}{N}\bigg]
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
G_{01}(\kappa) &= 2R_{15}D_3\Lambda_{52} \\
G_{02}(\kappa) &= 2R_{15}[D_0(\Lambda_{51} + \Lambda_{71}) + D_3\Lambda_{73}] \\
G_{03}(\kappa) &= 2R_{15}D_3\Lambda_{50} \\
G_{04}(\kappa) &= 2R_{15}[D_0(\Lambda_{53} + \Lambda_{73}) - D_3\Lambda_{71}]
\end{aligned} \tag{48}
$$

Using Eqs.(42), (45), and (47) we may write the potential $A_{\mathrm{ev}}(\eta,\xi) = A_{\mathrm{ev}}(\zeta,\theta)$ of Eq.(3.3-55) in the form of Eq.(6.1-61) of the differential theory:

$$
\begin{aligned}
A_{\mathrm{ev}}(\zeta,\theta) &= \frac{1}{4}(c\Delta t)^2V_{\mathrm{e0}}\bigg\{\frac{e^{-\rho_2\zeta}}{\mathrm{sh}^2(\rho_2/2)} \\
&\quad - \frac{1}{N}e^{-\rho_1\theta/2}\sum_{\substack{\kappa=-N/2+1\\-K_0,K_{\mathrm{N/2}}}}^{\substack{N/2-1\\-K_{\mathrm{N/2}},K_0,\neq 0}}\bigg[(G_{11}e^{\lambda_{\mathrm{A1}}\theta} + G_{13}e^{\lambda_{\mathrm{A2}}\theta})\cos\frac{2\pi\kappa\zeta}{N} \\
&\qquad - (G_{12}e^{\lambda_{\mathrm{A1}}\theta} + G_{14}e^{\lambda_{\mathrm{A2}}\theta})\sin\frac{2\pi\kappa\zeta}{N}\bigg]
\end{aligned}
$$

$$+\frac{2}{N}e^{-\rho_1\theta/2}\sum_{\substack{\kappa>-K_{\rm N/2}\\ \kappa>K_0}}^{\substack{<K_{\rm N/2}\\ <-K_0}}\Bigg[\left(G_{01}(\kappa)\cos\frac{2\pi\kappa\theta}{N}-G_{03}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\right)\cos\lambda_{\rm a}\zeta$$

$$+\left(G_{03}(\kappa)\cos\frac{2\pi\kappa\theta}{N}+G_{01}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\right)\sin\lambda_{\rm a}\zeta$$

$$+(G_{02}(\kappa)\cos\lambda_{\rm a}\theta-G_{04}(\kappa)\sin\lambda_{\rm a}\theta)\cos\frac{2\pi\kappa\zeta}{N}$$

$$+(G_{04}(\kappa)\cos\lambda_{\rm a}\theta+G_{02}(\kappa)\sin\lambda_{\rm a}\theta)\sin\frac{2\pi\kappa\zeta}{N}\Bigg]\Bigg\}\qquad(49)$$

Following the spirit of Section 6.1 starting after Eq.(6.1-64) we use Fourier series expansions to eliminate all terms in Eq.(49) with a time variation other than $\sin(2\pi\kappa\theta/N)$ or $\cos(2\pi\kappa\theta/N)$. But a change is required. The summation over κ in Eq.(6.1-61) ran from $\kappa=1$ to $\kappa=N_\tau$ and we chose the limits of the sum in Eq.(6.1-65) as $\nu=1$ and $\nu=N_\tau$. In Eq.(3.3-56) the summation is from $\kappa=-(N/2-1)$ to $\kappa=N/2-1$. This suggests to use the same limits for our Fourier series expansion. Since negative values $\nu=-1$ to $\nu=-(N/2-1)$ add nothing new to what is obtained from positive values $\nu=1$ to $\nu=N/2-1$ we restrict the range of summation to $1\leq\nu\leq N/2-1$. The exclusion of $\nu=N/2$ is important. With $\nu=N/2$ we get $\sin\pi\theta$ for $\sin(2\pi\nu\theta/N)$. Table 6.2-1 shows that the summation of $\sin\gamma\theta$ and $\cos\gamma\theta$ exists only for $|\gamma|<\pi$. Hence, the exclusion of $N/2$ and larger values is a must. We choose the following form of the Fourier series:

$$f_\kappa(\theta)=g_0+\sum_{\nu=1}^{N/2-1}[g_{{\rm s}\kappa}(\nu)\sin(2\pi\nu\theta/N)+g_{{\rm c}\kappa}(\nu)\cos(2\pi\nu\theta/N)]$$

$$g_{{\rm s}\kappa}(\nu)=\frac{2}{N}\int_0^N f_\kappa(\theta)\sin(2\pi\nu\theta/N)d\theta\quad g_{{\rm c}\kappa}(\nu)=\frac{2}{N}\int_0^N f_\kappa(\theta)\cos(2\pi\nu\theta/N)d\theta$$

$$g_0=\frac{1}{N}\int_0^N f_\kappa(\theta)d\theta,\qquad 0\leq\theta=t/\Delta t\leq T/\Delta t=N\qquad(50)$$

We apply this series expansion to the time-variable factors $\exp[(-\rho_1/2+\lambda_{\rm A1}\theta]$ and $\exp[(-\rho_1/2+\lambda_{\rm A2})\theta]$ in the first sum in Eq.(49):

$$f_{\kappa1}(\theta)=e^{-(\rho_1/2-\lambda_{\rm A1})\theta}=g_{01}+\sum_{\nu=1}^{N/2-1}[g_{{\rm s}\kappa1}(\nu)\sin(2\pi\nu\theta/N)$$
$$+g_{{\rm c}\kappa1}(\nu)\cos(2\pi\nu\theta/N)]\quad(51)$$

$$g_{\mathrm{s}\kappa 1}(\nu) = \frac{2}{N}\int_0^N e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\theta}\sin\frac{2\pi\nu\theta}{N}d\theta$$

$$= \frac{1}{N}\frac{4\pi\nu/N}{(\rho_1/2-\lambda_{\mathrm{A1}})^2+(2\pi\nu/N)^2}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A1}})N}) \tag{52}$$

$$g_{\mathrm{c}\kappa 1}(\nu) = \frac{2}{N}\int_0^N e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\theta}\cos\frac{2\pi\nu\theta}{N}d\theta$$

$$= \frac{1}{N}\frac{\rho_1-2\lambda_{\mathrm{A1}}}{(\rho_1/2-\lambda_{\mathrm{A1}})^2+(2\pi\nu/N)^2}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A1}})N}) \tag{53}$$

$$g_{01} = \frac{1}{N}\int_0^N e^{-(\rho_1/2-\lambda_{\mathrm{A1}})\theta}d\theta = \frac{1}{N}\frac{1}{\rho_1/2-\lambda_{\mathrm{A1}}}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A1}})N}) \tag{54}$$

$$f_{\kappa 2}(\theta) = e^{-(\rho_1/2-\lambda_{\mathrm{A2}})\theta} = g_{02} + \sum_{\nu=1}^{N/2-1}[g_{\mathrm{s}\kappa 2}(\nu)\sin(2\pi\nu\theta/N) + g_{\mathrm{c}\kappa 2}(\nu)\cos(2\pi\nu\theta/N)] \tag{55}$$

$$g_{\mathrm{s}\kappa 2}(\nu) = \frac{1}{N}\frac{4\pi\nu/N}{(\rho_1/2-\lambda_{\mathrm{A2}})^2+(2\pi\nu/N)^2}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A2}})N}) \tag{56}$$

$$g_{\mathrm{c}\kappa 2}(\nu) = \frac{1}{N}\frac{\rho_1-2\lambda_{\mathrm{A2}}}{(\rho_1/2-\lambda_{\mathrm{A2}})^2+(2\pi\nu/N)^2}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A2}})N}) \tag{57}$$

$$g_{02} = \frac{1}{N}\frac{1}{\rho_1/2-\lambda_{\mathrm{A2}}}(1-e^{-(\rho_1/2-\lambda_{\mathrm{A2}})N}) \tag{58}$$

The terms of the first sum in Eq.(49) are rewritten with the help of Eqs.(51)–(58):

$$-\frac{e^{-\rho_1\theta/2}}{N}\left[(G_{11}e^{\lambda_{\mathrm{A1}}\theta}+G_{13}e^{\lambda_{\mathrm{A2}}\theta})\cos\frac{2\pi\kappa\zeta}{N}-(G_{12}e^{\lambda_{\mathrm{A1}}\theta}+G_{14}e^{\lambda_{\mathrm{A2}}\theta})\right]\sin\frac{2\pi\kappa\zeta}{N}$$

$$= -\frac{1}{N}\left(L_0^{\mathrm{c}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[L_{\mathrm{s}\kappa}^{\mathrm{c}}(\kappa,\nu)\sin(2\pi\nu\theta/N) + L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu)\cos(2\pi\nu\theta/N)]\right)\cos\frac{2\pi\kappa\zeta}{N}$$

$$+\frac{1}{N}\left(L_0^{\mathrm{s}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu)\sin(2\pi\nu\theta/N) + L_{\mathrm{c}\kappa}^{\mathrm{s}}(\kappa,\nu)\cos(2\pi\nu\theta/N)]\right)\sin\frac{2\pi\kappa\zeta}{N} \tag{59}$$

$$\text{for } -K_0 < \kappa < K_0 \text{ or } |\kappa| > K_{\mathrm{N}/2}$$

$$\begin{aligned} L_0^{\mathrm{c}}(\kappa,0) &= G_{11}(\kappa)g_{01} + G_{13}(\kappa)g_{02} \\ L_{\mathrm{s}\kappa}^{\mathrm{c}}(\kappa,\nu) &= G_{11}(\kappa)g_{\mathrm{s}\kappa 1}(\nu) + G_{13}(\kappa)g_{\mathrm{s}\kappa 2}(\nu) \\ L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu) &= G_{11}(\kappa)g_{\mathrm{c}\kappa 1}(\nu) + G_{13}(\kappa)g_{\mathrm{c}\kappa 2}(\nu) \end{aligned} \tag{60}$$

$$\begin{aligned} L_0^{\mathrm{s}}(\kappa,0) &= G_{12}(\kappa)g_{01} + G_{14}(\kappa)g_{02} \\ L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu) &= G_{12}(\kappa)g_{\mathrm{s}\kappa 1}(\nu) + G_{14}(\kappa)g_{\mathrm{s}\kappa 2}(\nu) \\ L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu) &= G_{12}(\kappa)g_{\mathrm{c}\kappa 1}(\nu) + G_{14}(\kappa)g_{\mathrm{c}\kappa 2}(\nu) \end{aligned} \tag{61}$$

We turn to the second sum in Eq.(49). All terms of the sum are multiplied by $\exp(-\rho_1\theta/2)$. There are four different functions for which we need Fourier expansions:

$$\begin{aligned} &e^{-\rho_1\theta/2}\cos(2\pi\kappa\theta/N) \\ &\quad = J_{13}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[J_{11}(\kappa,\nu)\sin(2\pi\nu\theta/N) + J_{12}(\kappa,\nu)\cos(2\pi\nu\theta/N)] \end{aligned} \tag{62}$$

$$\begin{aligned} &e^{-\rho_1\theta/2}\sin(2\pi\kappa\theta/N) \\ &\quad = J_{16}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[J_{14}(\kappa,\nu)\sin(2\pi\nu\theta/N) + J_{15}(\kappa,\nu)\cos(2\pi\nu\theta/N)] \end{aligned} \tag{63}$$

$$\begin{aligned} &e^{-\rho_1\theta/2}\cos\lambda_{\mathrm{A}}\theta \\ &\quad = J_{19}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[J_{17}(\kappa,\nu)\sin(2\pi\nu\theta/N) + J_{18}(\kappa,\nu)\cos(2\pi\nu\theta/N)] \end{aligned} \tag{64}$$

$$\begin{aligned} &e^{-\rho_1\theta/2}\sin\lambda_{\mathrm{A}}\theta \\ &\quad = J_{22}(\kappa,0) + \sum_{\nu=1}^{N/2-1}[J_{20}(\kappa,\nu)\sin(2\pi\nu\theta/N) + J_{21}(\kappa,\nu)\cos(2\pi\nu\theta/N)] \end{aligned} \tag{65}$$

The following twelve integrals are obtained with the help of a table (Gradshtyn and Ryzhik 1980, 2.663/1, 2.663/3, 2.664/1):

$$J_{11}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos(2\pi\kappa\theta/N)\sin(2\pi\nu\theta/N)d\theta = \frac{1}{N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{2\pi(\kappa+\nu)/N}{(\rho_1/2)^2+[2\pi(\kappa+\nu)/N]^2} - \frac{2\pi(\kappa-\nu)/N}{(\rho_1/2)^2+[2\pi(\kappa-\nu)/N]^2}\right) \quad (66)$$

$$J_{12}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos(2\pi\kappa\theta/N)\cos(2\pi\nu\theta/N)d\theta = \frac{\rho_1}{2N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{1}{(\rho_1/2)^2+[2\pi(\kappa+\nu)/N]^2} + \frac{1}{(\rho_1/2)^2+[2\pi(\kappa-\nu)/N]^2}\right) \quad (67)$$

$$J_{13}(\kappa,\nu) = \frac{1}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos(2\pi\kappa\theta/N)d\theta = \frac{\rho_1}{2N}\,\frac{1-e^{-\rho_1 N/2}}{(\rho_1/2)^2+(2\pi\kappa/N)^2} \quad (68)$$

$$J_{14}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin(2\pi\kappa\theta/N)\sin(2\pi\nu\theta/N)d\theta = \frac{\rho_1}{2N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{1}{(\rho_1/2)^2+[2\pi(\kappa+\nu)/N]^2} - \frac{1}{(\rho_1/2)^2+[2\pi(\kappa-\nu)/N]^2}\right) \quad (69)$$

$$J_{15}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin(2\pi\kappa\theta/N)\cos(2\pi\nu\theta/N)d\theta = \frac{1}{N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{2\pi(\kappa+\nu)/N}{(\rho_1/2)^2+[2\pi(\kappa+\nu)/N]^2} + \frac{2\pi(\kappa-\nu)/N}{(\rho_1/2)^2+[2\pi(\kappa-\nu)/N]^2}\right) \quad (70)$$

$$J_{16}(\kappa,0) = \frac{1}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin(2\pi\kappa\theta/N)d\theta = \frac{1}{N}\,\frac{(2\pi\kappa/N)\left(1-e^{-\rho_1 N/2}\right)}{(\rho_1/2)^2+(2\pi\kappa/N)^2} \quad (71)$$

$$J_{17}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos\lambda_{\mathrm{a}}\theta\sin(2\pi\nu\theta/N)d\theta = \frac{1}{N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{\lambda_{\mathrm{a}}+2\pi\nu/N}{(\rho_1/2)^2+(\lambda_{\mathrm{a}}+2\pi\nu/N)^2} - \frac{\lambda_{\mathrm{a}}-2\pi\nu/N}{(\rho_1/2)^2+(\lambda_{\mathrm{a}}-2\pi\nu/N)^2}\right) \quad (72)$$

$$J_{18}(\kappa,\nu) = \frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos\lambda_{\mathrm{a}}\theta\cos(2\pi\nu\theta/N)d\theta = \frac{\rho_1}{2N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{1}{(\rho_1/2)^2+(\lambda_{\mathrm{a}}+2\pi\nu/N)^2} + \frac{1}{(\rho_1/2)^2+(\lambda_{\mathrm{a}}-2\pi\nu/N)^2}\right) \quad (73)$$

$$J_{19}(\kappa,0)=\frac{1}{N}\int\limits_0^N e^{-\rho_1\theta/2}\cos\lambda_{\rm a}\theta\,d\theta=\frac{\rho_1}{2N}\,\frac{1-e^{-\rho_1 N/2}}{(\rho_1/2)^2+\lambda_{\rm a}^2} \tag{74}$$

$$J_{20}(\kappa,\nu)=\frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin\lambda_{\rm a}\theta\sin(2\pi\nu\theta/N)d\theta=\frac{\rho_1}{2N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{1}{(\rho_1/2)^2+(\lambda_{\rm a}+2\pi\nu/N)^2}-\frac{1}{(\rho_1/2)^2+(\lambda_{\rm a}-2\pi\nu/N)^2}\right) \tag{75}$$

$$J_{21}(\kappa,\nu)=\frac{2}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin\lambda_{\rm a}\theta\cos(2\pi\nu\theta/N)d\theta=\frac{1}{N}\left(1-e^{-\rho_1 N/2}\right)$$
$$\times\left(\frac{\lambda_{\rm a}+2\pi\nu/N}{(\rho_1/2)^2+(\lambda_{\rm a}+2\pi\nu/N)^2}+\frac{\lambda_{\rm a}-2\pi\nu/N}{(\rho_1/2)^2+(\lambda_{\rm a}-2\pi\nu/N)^2}\right) \tag{76}$$

$$J_{22}(\kappa,0)=\frac{1}{N}\int\limits_0^N e^{-\rho_1\theta/2}\sin\lambda_{\rm a}\theta\,d\theta=\frac{1}{N}\left(1-e^{-\rho_1 N/2}\right)\frac{\lambda_{\rm a}}{(\rho_1/2)^2+\lambda_{\rm a}^2} \tag{77}$$

The terms in the second sum of Eq.(49) are rewritten with the help of Eqs.(62)–(77):

$$\frac{2}{N}\sum_{\substack{\kappa>-K_{\rm N/2}\\ \kappa>K_0}}^{\substack{<K_{\rm N/2}\\ <-K_0}} e^{-\rho_1\theta/2}\Bigg[\left(G_{01}(\kappa)\cos\frac{2\pi\kappa\theta}{N}-G_{03}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\right)\cos\lambda_{\rm a}\zeta$$
$$+\left(G_{03}(\kappa)\cos\frac{2\pi\kappa\theta}{N}+G_{01}(\kappa)\sin\frac{2\pi\kappa\theta}{N}\right)\sin\lambda_{\rm a}\zeta$$
$$+\left(G_{02}(\kappa)\cos\lambda_{\rm a}\theta-G_{04}(\kappa)\sin\lambda_{\rm a}\theta\right)\cos\frac{2\pi\kappa\zeta}{N}$$
$$+\left(G_{04}(\kappa)\cos\lambda_{\rm a}\theta+G_{02}(\kappa)\sin\lambda_{\rm a}\theta\right)\sin\frac{2\pi\kappa\zeta}{N}\Bigg]$$

$$=\frac{2}{N}\sum_{\substack{\kappa>-K_{\rm N/2}\\ \kappa>K_0}}^{\substack{<K_{\rm N/2}\\ <-K_0}}\Bigg[L^{\rm a}_{0\rm c}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\rm a}_{\rm sc}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\rm a}_{\rm cc}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\cos\lambda_{\rm a}\zeta$$
$$+L^{\rm a}_{0\rm s}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\rm a}_{\rm ss}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\rm a}_{\rm cs}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\sin\lambda_{\rm a}\zeta$$
$$+L_{0\rm c}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{\rm sc}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{\rm cc}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\cos\frac{2\pi\kappa\zeta}{N}$$

$$+L_{0s}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{ss}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{cs}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\sin\frac{2\pi\kappa\zeta}{N}\Bigg] \tag{78}$$

$$\text{for } K_0<|\kappa|<K_{\mathrm{N/2}}$$

$$\begin{aligned}
L^{\mathrm{a}}_{0\mathrm{c}}(\kappa,0)&=[G_{01}(\kappa)J_{13}(\kappa,0)+G_{03}(\kappa)J_{16}(\kappa,0)]\\
L^{\mathrm{a}}_{\mathrm{sc}}(\kappa,\nu)&=[G_{01}(\kappa)J_{11}(\kappa,\nu)+G_{03}(\kappa)J_{14}(\kappa,\nu)]\\
L^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\nu)&=[G_{01}(\kappa)J_{12}(\kappa,\nu)+G_{03}(\kappa)J_{15}(\kappa,\nu)]
\end{aligned} \tag{79}$$

$$\begin{aligned}
L^{\mathrm{a}}_{0\mathrm{s}}(\kappa,0)&=[G_{03}(\kappa)J_{13}(\kappa,0)-G_{01}(\kappa)J_{16}(\kappa,0)]\\
L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu)&=[G_{03}(\kappa)J_{11}(\kappa,\nu)-G_{01}(\kappa)J_{14}(\kappa,\nu)]\\
L^{\mathrm{a}}_{\mathrm{cs}}(\kappa,\nu)&=[G_{03}(\kappa)J_{12}(\kappa,\nu)-G_{01}(\kappa)J_{15}(\kappa,\nu)]
\end{aligned} \tag{80}$$

$$\begin{aligned}
L_{0\mathrm{c}}(\kappa,0)&=[G_{02}(\kappa)J_{19}(\kappa,0)+G_{04}(\kappa)J_{22}(\kappa,0)]\\
L_{\mathrm{sc}}(\kappa,\nu)&=[G_{02}(\kappa)J_{17}(\kappa,\nu)+G_{04}(\kappa)J_{20}(\kappa,\nu)]\\
L_{\mathrm{cc}}(\kappa,\nu)&=[G_{02}(\kappa)J_{18}(\kappa,\nu)+G_{04}(\kappa)J_{21}(\kappa,\nu)]
\end{aligned} \tag{81}$$

$$\begin{aligned}
L_{0\mathrm{s}}(\kappa,0)&=[G_{04}(\kappa)J_{19}(\kappa,0)-G_{02}(\kappa)J_{22}(\kappa,0)]\\
L_{\mathrm{ss}}(\kappa,\nu)&=[G_{04}(\kappa)J_{17}(\kappa,\nu)-G_{02}(\kappa)J_{20}(\kappa,\nu)]\\
L_{\mathrm{cs}}(\kappa,\nu)&=[G_{04}(\kappa)J_{18}(\kappa,\nu)-G_{02}(\kappa)J_{21}(\kappa,\nu)]
\end{aligned} \tag{82}$$

The sum of $\exp(-\rho_2\zeta)/\rho_2^2$, Eqs.(59) and (78) yields $A_{\mathrm{ev}}(\zeta,\theta)$ of Eq.(49). We write this sum in a compacted form that shows only the variable ζ explicitly:

$$\begin{aligned}
A_{\mathrm{ev}}(\zeta,\theta)&=(c\Delta t)^2V_{\mathrm{e}0}\left[\frac{e^{-\rho_2\zeta}}{4\,\mathrm{sh}^2(\rho_2/2)}\right.\\
&+\sum_{\kappa=-N/2+1}^{N/2-1,\neq0}\left(C_{\mathrm{ec}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}+C_{\mathrm{es}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right.\\
&\left.\left.+C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta+C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}\zeta\right)\right]
\end{aligned} \tag{83}$$

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq0 \text{ or } |\kappa|>K_{\mathrm{N/2}}$$

$$\begin{aligned}
C_{\mathrm{ec}}(\kappa,\theta)=-\frac{1}{4N}\Bigg[&L^{\mathrm{c}}_0(\kappa,0)\\
&+\sum_{\nu=1}^{N/2-1}\left(L^{\mathrm{c}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{c}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
C_{\mathrm{es}}(\kappa,\theta) &= \frac{1}{4N}\Bigg[L_0^{\mathrm{s}}(\kappa,0)\\
&\qquad + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{c}\kappa}^{\mathrm{s}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]\\
C_{\mathrm{ec}}^{\mathrm{a}}(\kappa,\theta) &= 0\\
C_{\mathrm{es}}^{\mathrm{a}}(\kappa,\theta) &= 0
\end{aligned}
\tag{84}
$$

for $K_0 < |\kappa| < K_{\mathrm{N}/2}$

$$
\begin{aligned}
C_{\mathrm{ec}}(\kappa,\theta) &= \frac{1}{2N}\Bigg[L_{0\mathrm{c}}(\kappa,0)\\
&\qquad + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]\\
C_{\mathrm{es}}(\kappa,\theta) &= \frac{1}{2N}\Bigg[L_{0\mathrm{s}}(\kappa,0)\\
&\qquad + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]\\
C_{\mathrm{ec}}^{\mathrm{a}}(\kappa,\theta) &= \frac{1}{2N}\Bigg[L_{0\mathrm{c}}^{\mathrm{a}}(\kappa,0)\\
&\qquad + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{sc}}^{\mathrm{a}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cc}}^{\mathrm{a}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]\\
C_{\mathrm{es}}^{\mathrm{a}}(\kappa,\theta) &= \frac{1}{2N}\Bigg[L_{0\mathrm{s}}^{\mathrm{a}}(\kappa,0)\\
&\qquad + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{ss}}^{\mathrm{a}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cs}}^{\mathrm{a}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\Bigg]
\end{aligned}
\tag{85}
$$

Equation (83) has the form of Eq.(6.1-109). But the time variable term $\operatorname{ch}\rho_2\theta$ of $A_{\mathrm{e0}}(\zeta,\theta)$ in Eq.(6.1-109) does no longer occur. Furthermore, the two spatial terms $\exp(-\rho_2\zeta)$ and $\sin 2\pi\kappa\zeta$ in Eq.(6.1-109) have been replaced by five spatial terms $\exp(-\rho_2\zeta)$, $\cos(2\pi\kappa\zeta/N)$, ... , $\sin\lambda_{\mathrm{a}}\zeta)$ in Eq.(83).

6.5 Calculations for Section 3.4

The first two terms $A_{\mathrm{ev}}(\zeta\pm 1,\theta)$ of Eq.(3.4-1) follow from Eq.(3.3-56) by the substitution of $\zeta\pm 1$ for ζ:

$$\begin{aligned}
A_{\mathrm{ev}}(\zeta\pm 1,\theta) &= (c\Delta t)^2 V_{\mathrm{e0}}\Bigg\{\frac{e^{-\rho_2(\zeta\pm 1)}}{4\,\mathrm{sh}^2(\rho_2/2)}\\
&+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\Bigg[\quad C_{\mathrm{ec}}(\kappa,\theta)\left(\cos\frac{2\pi\kappa\zeta}{N}\cos\frac{2\pi\kappa}{N}\mp\sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa}{N}\right)\\
&+C_{\mathrm{es}}(\kappa,\theta)\left(\sin\frac{2\pi\kappa\zeta}{N}\cos\frac{2\pi\kappa}{N}\mp\cos\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi\kappa}{N}\right)\\
&+C^a_{\mathrm{ec}}(\kappa,\theta)(\cos\lambda_{\mathrm{a}}\zeta\cos\lambda_{\mathrm{a}}\mp\sin\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}})\\
&+C^a_{\mathrm{es}}(\kappa,\theta)(\sin\lambda_{\mathrm{a}}\zeta\cos\lambda_{\mathrm{a}}\pm\cos\lambda_{\mathrm{a}}\zeta\sin\lambda_{\mathrm{a}})\Bigg]\Bigg\}
\end{aligned}\tag{1}$$

The difference $(Z\rho_{\mathrm{s}}/2)[A_{\mathrm{ev}}(\zeta+1,\theta)-A_{\mathrm{ev}}(\zeta-1,\theta)]$ becomes:

$$\begin{aligned}
A_{\mathrm{mv1}}(\zeta,\theta) &= \frac{1}{2}Z\rho_{\mathrm{s}}[A_{\mathrm{ev}}(\zeta+1,\theta)-A_{\mathrm{ev}}(\zeta-1,\theta)]\\
&= -Z\rho_{\mathrm{s}}(c\Delta t)^2 V_{\mathrm{e0}}\Bigg\{\frac{\mathrm{sh}\,\rho_2}{4\,\mathrm{sh}^2(\rho_2/2)}e^{-\rho_2\zeta}\\
&+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\Bigg[\quad \sin\frac{2\pi\kappa}{N}\left(C_{\mathrm{ec}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}-C_{\mathrm{es}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}\right)\\
&+\sin\lambda_{\mathrm{a}}[C^a_{\mathrm{ec}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}\zeta-C^a_{\mathrm{es}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta]\Bigg]\Bigg\}
\end{aligned}\tag{2}$$

We turn to Eq.(3.4-9) and substitute $\theta\pm 1$ for θ to obtain the summation of $A_{\mathrm{ev}}(\zeta,\theta\pm 1)$:

$$\begin{aligned}
\mathop{\mathbb{S}}^{\zeta} A_{\mathrm{ev}}(\zeta,\theta\pm 1)\Delta\zeta &= (c\Delta t)^2 V_{\mathrm{e0}}\Bigg\{-\frac{e^{-\rho_2\zeta}}{4\,\mathrm{sh}^2(\rho_2/2)\,\mathrm{sh}\,\rho_2}\\
&+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\Bigg[\quad C_{\mathrm{ec}}(\kappa,\theta\pm 1)\frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}-C_{\mathrm{es}}(\kappa,\theta\pm 1)\frac{\cos(2\pi\kappa\zeta/N}{\sin(2\pi\kappa/N)}\\
&+C^a_{\mathrm{ec}}(\kappa,\theta\pm 1)\frac{\sin\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}-C^a_{\mathrm{es}}(\kappa,\theta\pm 1)\frac{\cos\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}\Bigg]\Bigg\}
\end{aligned}\tag{3}$$

The term $A_{\mathrm{mv2}}(\zeta,\theta)$ in Eq.(3.4-3) becomes with the help of Eqs.(3.4-1) and (3):

$$A_{\mathrm{mv2}}(\zeta,\theta) = -Z\rho_{\mathrm{s}} \mathop{\mathrm{S}}^{\zeta} [A_{\mathrm{ev}}(\zeta,\theta+1) - 2A_{\mathrm{ev}}(\zeta,\theta) + A_{\mathrm{ev}}(\zeta,\theta-1)]\Delta\zeta$$

$$= -Z\rho_{\mathrm{s}}(c\Delta t)^2 V_{\mathrm{e0}} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}$$

$$\Bigg([C_{\mathrm{ec}}(\kappa,\theta+1) - 2C_{\mathrm{ec}}(\kappa,\theta) + C_{\mathrm{ec}}(\kappa,\theta-1)] \frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}$$

$$-[C_{\mathrm{es}}(\kappa,\theta+1) - 2C_{\mathrm{es}}(\kappa,\theta) + C_{\mathrm{es}}(\kappa,\theta-1)] \frac{\cos(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}$$

$$+[C_{\mathrm{ec}}^{a}(\kappa,\theta+1) - 2C_{\mathrm{ec}}^{a}(\kappa,\theta) + C_{\mathrm{ec}}^{a}(\kappa,\theta-1)] \frac{\sin\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}$$

$$-[C_{\mathrm{es}}^{a}(\kappa,\theta+1) - 2C_{\mathrm{es}}^{a}(\kappa,\theta) + C_{\mathrm{es}}^{a}(\kappa,\theta-1)] \frac{\cos\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}\Bigg) \tag{4}$$

For $A_{\mathrm{mv3}}(\zeta,\theta)$ in Eq.(3.4-3) we obtain:

$$A_{\mathrm{mv3}}(\zeta,\theta) = -\frac{1}{2} Z \mathop{\mathrm{S}}^{\zeta} [A_{\mathrm{ev}}(\zeta,\theta+1) - A_{\mathrm{ev}}(\zeta,\theta-1)]\Delta\zeta$$

$$= -\frac{1}{2} Z(c\Delta t)^2 V_{\mathrm{e0}} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}$$

$$\Bigg([C_{\mathrm{ec}}(\kappa,\theta+1) - C_{\mathrm{ec}}(\kappa,\theta-1)] \frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}$$

$$-[C_{\mathrm{es}}(\kappa,\theta+1) - C_{\mathrm{es}}(\kappa,\theta-1)] \frac{\cos(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}$$

$$+[C_{\mathrm{ec}}^{a}(\kappa,\theta+1) - C_{\mathrm{ec}}^{a}(\kappa,\theta-1)] \frac{\sin\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}$$

$$-[C_{\mathrm{es}}^{a}(\kappa,\theta+1) - C_{\mathrm{es}}^{a}(\kappa,\theta-1)] \frac{\cos\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}\Bigg) \tag{5}$$

The time functions $C_{\mathrm{ec}}(\kappa,\theta+1)$ to $C_{\mathrm{es}}^{\mathrm{a}}(\kappa,\theta-1)$ follow from Eqs.(6.4-84) and (6.4-85) by the substitution of $\theta \pm 1$ for θ:

$$\text{for } -K_0 < \kappa < K_0,\ \kappa \neq 0,\ \text{or } |\kappa| > K_{\mathrm{N/2}}$$

$$C_{\mathrm{ec}}(\kappa,\theta\pm 1) = -\frac{1}{4N}\Bigg[L_0^{\mathrm{c}}(\kappa,\theta)$$

$$+ \sum_{\nu=1}^{N/2-1} \left(L_{\mathrm{s}\kappa}^{\mathrm{c}}(\kappa,\nu) \sin\frac{2\pi\nu(\theta\pm 1)}{N} + L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu) \cos\frac{2\pi\nu(\theta\pm 1)}{N} \right)\Bigg]$$

$$= -\frac{1}{4N}\Bigg\{L_0^{\mathrm{c}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\Bigg[L_{\mathrm{s}\kappa}^{\mathrm{c}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \pm \cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right) + L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \mp \sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\}$$

$$C_{\mathrm{es}}(\kappa,\theta\pm 1) = \frac{1}{4N}\Bigg[L_0^{\mathrm{s}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu)\sin\frac{2\pi\nu(\theta\pm 1)}{N} + L_{\mathrm{c}\kappa}^{\mathrm{s}}(\kappa,\nu)\cos\frac{2\pi\nu(\theta\pm 1)}{N}\right)\Bigg]$$

$$= \frac{1}{4N}\Bigg\{L_0^{\mathrm{s}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\Bigg[L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \pm \cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right) + L_{\mathrm{c}\kappa}^{\mathrm{s}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \mp \sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\}$$

$$C_{\mathrm{ec}}^{\mathrm{a}}(\kappa,\theta\pm 1) = 0$$

$$C_{\mathrm{es}}^{\mathrm{a}}(\kappa,\theta\pm 1) = 0 \qquad (6)$$

for $K_0 < |\kappa| < K_{\mathrm{N/2}}$

$$C_{\mathrm{ec}}(\kappa,\theta\pm 1) = \frac{1}{2N}\Bigg[L_{0\mathrm{c}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu(\theta\pm 1)}{N} + L_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu(\theta\pm 1)}{N}\right)\Bigg]$$

$$= \frac{1}{2N}\Bigg\{L_{0\mathrm{c}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\Bigg[L_{\mathrm{sc}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \pm \cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right) + L_{\mathrm{cc}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N} \mp \sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\}$$

$$C_{\mathrm{es}}(\kappa,\theta\pm 1) = \frac{1}{2N}\Bigg[L_{0\mathrm{s}}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu(\theta\pm 1)}{N} + L_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu(\theta\pm 1)}{N}\right)\Bigg]$$

$$= \frac{1}{2N}\Bigg\{L_{0\mathrm{s}}(\kappa,0)$$

$$+\sum_{\nu=1}^{N/2-1}\Bigg[L_{\mathrm{ss}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\pm\cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)$$

$$+L_{\mathrm{cs}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\mp\sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\}$$

$$C^a_{\mathrm{ec}}(\kappa,\theta\pm1) = \frac{1}{2N}\Bigg[L^a_{0\mathrm{c}}(\kappa,0)$$

$$+\sum_{\nu=1}^{N/2-1}\left(L^a_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu(\theta\pm1)}{N}+L^a_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu(\theta\pm1)}{N}\right)\Bigg]$$

$$= \frac{1}{2N}\Bigg\{L^a_{0\mathrm{c}}(\kappa,0)$$

$$+\sum_{\nu=1}^{N/2-1}\Bigg[L^a_{\mathrm{sc}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\pm\cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)$$

$$+L^a_{\mathrm{cc}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\mp\sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\}$$

$$C^a_{\mathrm{es}}(\kappa,\theta\pm1) = \frac{1}{2N}\Bigg[L^a_{0\mathrm{s}}(\kappa,0)$$

$$+\sum_{\nu=1}^{N/2-1}\left(L^a_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu(\theta\pm1)}{N}+L^a_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu(\theta\pm1)}{N}\right)\Bigg]$$

$$= \frac{1}{2N}\Bigg\{L^a_{0\mathrm{s}}(\kappa,0)$$

$$+\sum_{\nu=1}^{N/2-1}\Bigg[L^a_{\mathrm{ss}}(\kappa,\nu)\left(\sin\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\pm\cos\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)$$

$$+L^a_{\mathrm{cs}}(\kappa,\nu)\left(\cos\frac{2\pi\nu\theta}{N}\cos\frac{2\pi\nu}{N}\mp\sin\frac{2\pi\nu\theta}{N}\sin\frac{2\pi\nu}{N}\right)\Bigg]\Bigg\} \quad (7)$$

Using Eqs.(6) and (7) we may simplify the four terms in brackets of Eq.(4). The variables C are replaced by new variables D:

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq0,\ \text{or } |\kappa|>K_{\mathrm{N/2}}$$

$$D_{\mathrm{ec}}(\kappa,\theta) = C_{\mathrm{ec}}(\kappa,\theta+1)-2C_{\mathrm{ec}}(\kappa,\theta)+C_{\mathrm{ec}}(\kappa,\theta-1)$$

$$= \frac{1}{N}\sum_{\nu=1}^{N/2-1}\sin^2\frac{\pi\nu}{N}\left(L^{\mathrm{c}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\mathrm{c}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)$$

$$
\begin{aligned}
D_{\mathrm{es}}(\kappa,\theta) &= -[C_{\mathrm{es}}(\kappa,\theta+1) - 2C_{\mathrm{es}}(\kappa,\theta) + C_{\mathrm{es}}(\kappa,\theta-1)] \\
&= \frac{1}{N}\sum_{\nu=1}^{N/2-1} \sin^2\frac{\pi\nu}{N}\left(L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right) \\
D^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta) &= 0 \\
D^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta) &= 0
\end{aligned} \tag{8}
$$

for $K_0 < |\kappa| < K_{\mathrm{N}/2}$

$$
\begin{aligned}
D_{\mathrm{ec}}(\kappa,\theta) &= C_{\mathrm{ec}}(\kappa,\theta+1) - 2C_{\mathrm{ec}}(\kappa,\theta) + C_{\mathrm{ec}}(\kappa,\theta-1) \\
&= -\frac{2}{N}\sum_{\nu=1}^{N/2-1} \sin^2\frac{\pi\nu}{N}\left(L_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right) \\
D_{\mathrm{es}}(\kappa,\theta) &= -[C_{\mathrm{es}}(\kappa,\theta+1) - 2C_{\mathrm{es}}(\kappa,\theta) + C_{\mathrm{es}}(\kappa,\theta-1)] \\
&= \frac{2}{N}\sum_{\nu=1}^{N/2-1} \sin^2\frac{\pi\nu}{N}\left(L_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right) \\
D^{a}_{\mathrm{ec}}(\kappa,\theta) &= C^{a}_{\mathrm{ec}}(\kappa,\theta+1) - 2C^{a}_{\mathrm{ec}}(\kappa,\theta) + C^{a}_{\mathrm{ec}}(\kappa,\theta-1) \\
&= -\frac{2}{N}\sum_{\nu=1}^{N/2-1} \sin^2\frac{\pi\nu}{N}\left(L^{a}_{\mathrm{sc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L^{a}_{\mathrm{cc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right) \\
D^{a}_{\mathrm{es}}(\kappa,\theta) &= -[C^{a}_{\mathrm{es}}(\kappa,\theta+1) - 2C^{a}_{\mathrm{es}}(\kappa,\theta) + C^{a}_{\mathrm{es}}(\kappa,\theta-1)] \\
&= \frac{2}{N}\sum_{\nu=1}^{N/2-1} \sin^2\frac{\pi\nu}{N}\left(L^{a}_{\mathrm{ss}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L^{a}_{\mathrm{cs}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)
\end{aligned} \tag{9}
$$

The terms of $A_{\mathrm{mv3}}(\zeta,\theta)$ in Eq.(5) may also be simplified significantly with the help of Eqs.(6) and (7) using new variables E instead of C:

for $-K_0 < \kappa < K_0,\ \kappa \neq 0,$ or $|\kappa| > K_{\mathrm{N}/2}$

$$
\begin{aligned}
E_{\mathrm{ec}}(\kappa,\theta) &= C_{\mathrm{ec}}(\kappa,\theta+1) - C_{\mathrm{ec}}(\kappa,\theta-1) \\
&= \frac{1}{2N}\sum_{\nu=1}^{N/2-1} \sin\frac{2\pi\nu}{N}\left(L^{\mathrm{c}}_{\mathrm{s}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^{\mathrm{c}}_{\mathrm{c}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right) \\
E_{\mathrm{es}}(\kappa,\theta) &= -[C_{\mathrm{es}}(\kappa,\theta+1) - C_{\mathrm{es}}(\kappa,\theta-1)] \\
&= -\frac{1}{2N}\sum_{\nu=1}^{N/2-1} \sin\frac{2\pi\nu}{N}\left(L^{\mathrm{s}}_{\mathrm{s}\kappa}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^{\mathrm{s}}_{\mathrm{c}\kappa}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right) \\
E^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\theta) &= 0 \\
E^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta) &= 0
\end{aligned} \tag{10}
$$

$$\text{for } K_0 < |\kappa| < K_{\mathrm{N}/2}$$

$$\begin{aligned}
E_{\mathrm{ec}}(\kappa,\theta) &= C_{\mathrm{ec}}(\kappa,\theta+1) - C_{\mathrm{ec}}(\kappa,\theta-1)\\
&= \frac{1}{N}\sum_{\nu=1}^{N/2-1}\sin\frac{2\pi\nu}{N}\left(L_{\mathrm{sc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L_{\mathrm{cc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
E_{\mathrm{es}}(\kappa,\theta) &= -[C_{\mathrm{es}}(\kappa,\theta+1) - C_{\mathrm{es}}(\kappa,\theta-1)]\\
&= -\frac{1}{N}\sum_{\nu=1}^{N/2-1}\sin\frac{2\pi\nu}{N}\left(L_{\mathrm{ss}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L_{\mathrm{cs}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
E^a_{\mathrm{ec}}(\kappa,\theta) &= C^a_{\mathrm{ec}}(\kappa,\theta+1) - C^a_{\mathrm{ec}}(\kappa,\theta-1)\\
&= \frac{1}{N}\sum_{\nu=1}^{N/2-1}\sin\frac{2\pi\nu}{N}\left(L^a_{\mathrm{sc}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^a_{\mathrm{cc}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right)\\
E^a_{\mathrm{es}}(\kappa,\theta) &= -[C^a_{\mathrm{es}}(\kappa,\theta+1) - C^a_{\mathrm{es}}(\kappa,\theta-1)]\\
&= -\frac{1}{N}\sum_{\nu=1}^{N/2-1}\sin\frac{2\pi\nu}{N}\left(L^a_{\mathrm{ss}}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N} - L^a_{\mathrm{cs}}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}\right) \qquad (11)
\end{aligned}$$

Equation (4) for the component $A_{\mathrm{mv2}}(\zeta,\theta)$ may be rewritten into the following form using the six equations of Eqs.(8) and (9):

$$\begin{aligned}
A_{\mathrm{mv2}}(\zeta,\theta) = -Z\rho_{\mathrm{s}}(c\Delta t)^2 V_{\mathrm{e0}} &\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\\
&\left(D_{\mathrm{ec}}(\kappa,\theta)\frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)} + D_{\mathrm{es}}(\kappa,\theta)\frac{\cos(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}\right.\\
&\left.+ D^a_{\mathrm{ec}}(\kappa,\theta)\frac{\sin\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}} + D^a_{\mathrm{es}}(\kappa,\theta)\frac{\cos\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}\right) \qquad (12)
\end{aligned}$$

The component $A_{\mathrm{mv3}}(\zeta,\theta)$ of Eq.(5) may be rewritten with the help of the six equations of Eqs.(10) and (11):

$$\begin{aligned}
A_{\mathrm{mv3}}(\zeta,\theta) = -\frac{1}{2}Z(c\Delta t)^2 V_{\mathrm{e0}} &\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0}\\
&\left(E_{\mathrm{ec}}(\kappa,\theta)\frac{\sin(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)} + E_{\mathrm{es}}(\kappa,\theta)\frac{\cos(2\pi\kappa\zeta/N)}{\sin(2\pi\kappa/N)}\right.\\
&\left.+ E^a_{\mathrm{ec}}(\kappa,\theta)\frac{\sin\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}} + E^a_{\mathrm{es}}(\kappa,\theta)\frac{\cos\lambda_{\mathrm{a}}\zeta}{\sin\lambda_{\mathrm{a}}}\right) \qquad (13)
\end{aligned}$$

We may combine $A_{\mathrm{mv1}}(\zeta,\theta)$ of Eq.(2), $A_{\mathrm{mv2}}(\zeta,\theta)$ of Eq.(12) and $A_{\mathrm{mv3}}(\zeta,\theta)$ of Eq.(13) according to Eq.(3.4-3):

$$\begin{aligned} A_{\mathrm{mv}}(\zeta,\theta) &= A_{\mathrm{mv1}}(\zeta,\theta)+A_{\mathrm{mv2}}(\zeta,\theta)+A_{\mathrm{mv3}}(\zeta,\theta) \\ &= Z(c\Delta t)^2V_{\mathrm{e0}}\Bigg[-\frac{\operatorname{sh}\rho_2}{4\operatorname{sh}^2(\rho_2/2)}e^{-\rho_2\zeta}+\sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\ &\qquad \left(C_{\mathrm{mc}}(\kappa,\theta)\cos\frac{2\pi\kappa\zeta}{N}+C_{\mathrm{ms}}(\kappa,\theta)\sin\frac{2\pi\kappa\zeta}{N}\right. \\ &\qquad \left.+C^a_{\mathrm{mc}}(\kappa,\theta)\cos\lambda_{\mathrm{a}}\zeta+C^a_{\mathrm{ms}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}\zeta\right)\Bigg] \end{aligned} \tag{14}$$

The functions $C_{\mathrm{mc}}(\kappa,\theta)$ to $C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)$ have the following values:

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq 0,\ \text{or } |\kappa|>K_{\mathrm{N}/2}$$

$$\begin{aligned} C_{\mathrm{mc}}(\kappa,\theta) &= +\rho_{\mathrm{s}}\left(C_{\mathrm{es}}(\kappa,\theta)\sin\frac{2\pi\kappa}{N}-\frac{D_{\mathrm{es}}(\kappa,\theta)}{\sin(2\pi\kappa/N)}\right)+\frac{E_{\mathrm{es}}(\kappa,\theta)}{2\sin(2\pi\kappa/N)} \\ C_{\mathrm{ms}}(\kappa,\theta) &= -\rho_{\mathrm{s}}\left(C_{\mathrm{ec}}(\kappa,\theta)\sin\frac{2\pi\kappa}{N}+\frac{D_{\mathrm{ec}}(\kappa,\theta)}{\sin(2\pi\kappa/N)}\right)-\frac{E_{\mathrm{ec}}(\kappa,\theta)}{2\sin(2\pi\kappa/N)} \\ C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta) &= 0 \\ C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta) &= 0 \end{aligned} \tag{15}$$

$$\text{for } K_0<|\kappa|<K_{\mathrm{N}/2}$$

$$\begin{aligned} C_{\mathrm{mc}}(\kappa,\theta) &= +\rho_{\mathrm{s}}\left(C_{\mathrm{es}}(\kappa,\theta)\sin\frac{2\pi\kappa}{N}-\frac{D_{\mathrm{es}}(\kappa,\theta)}{\sin(2\pi\kappa/N)}\right)+\frac{E_{\mathrm{es}}(\kappa,\theta)}{2\sin(2\pi\kappa/N)} \\ C_{\mathrm{ms}}(\kappa,\theta) &= -\rho_{\mathrm{s}}\left(C_{\mathrm{ec}}(\kappa,\theta)\sin\frac{2\pi\kappa}{N}+\frac{D_{\mathrm{ec}}(\kappa,\theta)}{\sin(2\pi\kappa/N)}\right)-\frac{E_{\mathrm{ec}}(\kappa,\theta)}{2\sin(2\pi\kappa/N)} \\ C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta) &= +\rho_{\mathrm{s}}\left(C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}-\frac{D^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\sin\lambda_{\mathrm{a}}}\right)+\frac{E^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{2\sin\lambda_{\mathrm{a}}} \\ C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta) &= -\rho_{\mathrm{s}}\left(C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)\sin\lambda_{\mathrm{a}}+\frac{D^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\sin\lambda_{\mathrm{a}}}\right)-\frac{E^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{2\sin\lambda_{\mathrm{a}}} \end{aligned} \tag{16}$$

Equations (15) and (16) will be needed in a form that shows the time variation explicitly. Using Eqs.(6.4-84), (8), and (10) we obtain from Eq.(15):

$$\text{for } -K_0<\kappa<K_0,\ \kappa\neq 0,\ \text{or } |\kappa|>K_{\mathrm{N}/2}$$

$$C_{\mathrm{mc}}(\kappa,\theta)=\frac{1}{4N}\left[L_{00}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L_{01}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L_{02}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]$$

$$
\begin{aligned}
L_{00}(\kappa,0) &= \rho_s \sin\frac{2\pi\kappa}{N} L_0^s(\kappa,0)\\
L_{01}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{s\kappa}^s(\kappa,\nu) + \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{c\kappa}^s(\kappa,\nu)\\
L_{02}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{c\kappa}^s(\kappa,\nu) - \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{s\kappa}^s(\kappa,\nu)
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
C_{ms}(\kappa,\theta) &= \frac{1}{4N}\left[L_{03}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{04}(\kappa,\nu)\sin\frac{2\pi\kappa\theta}{N} + L_{05}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
L_{03}(\kappa,0) &= \rho_s \sin\frac{2\pi\kappa}{N} L_0^c(\kappa,0)\\
L_{04}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{s\kappa}^c(\kappa,\nu) + \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{c\kappa}^c(\kappa,\nu)\\
L_{05}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{c\kappa}^c(\kappa,\nu) - \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{s\kappa}^c(\kappa,\nu)
\end{aligned}
\tag{18}
$$

We turn to Eq.(16). To show the time variation explicitly we use again Eq.(6.4-84) as well as Eqs.(9) and (11):

for $K_0 < |\kappa| < K_{N/2}$

$$
\begin{aligned}
C_{mc}(\kappa,\theta) &= \frac{1}{2N}\left[L_{00}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{01}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{02}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
L_{00}(\kappa,0) &= \rho_s \sin\frac{2\pi\kappa}{N} L_{0s}(\kappa,0)\\
L_{01}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{ss}(\kappa,\nu) + \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{cs}(\kappa,\nu)\\
L_{02}(\kappa,\nu) &= \rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{cs}(\kappa,\nu) - \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{ss}(\kappa,\nu)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
C_{ms}(\kappa,\theta) &= \frac{1}{2N}\left[L_{03}(\kappa,0) + \sum_{\nu=1}^{N/2-1}\left(L_{04}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N} + L_{05}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
L_{03}(\kappa,0) &= -\rho_s \sin\frac{2\pi\kappa}{N} L_{0c}(\kappa,0)\\
L_{04}(\kappa,\nu) &= -\left[\rho_s\left(\sin\frac{2\pi\kappa}{N} - \frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right) L_{sc}(\kappa,\nu) + \frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)} L_{cc}(\kappa,\nu)\right]
\end{aligned}
$$

TABLE 6.5-1

LOCATION OF THE DEFINITION OF VARIOUS CONSTANTS AND FUNCTIONS.

constant or function	equation
d_{Δ}^{2}	3.2-31
R_{11}, R_{13}, R_{15}	6.4-9, -13, -15
λ_{a}	3.2-60
λ_{A1}, λ_{A2}	3.2-61
ρ_1	3.1-1
ρ_2	3.1-1
ρ_{s}	2.1-49, 3.4-1
ρ_{σ}	2.1-51
φ_{κ}	3.2-15
$\varphi_{\theta\kappa}$	3.2-15, -16, -22
γ_{01}, γ_{03}, γ_{05}, γ_{07}	3.3-22, -26, -30, -34
γ_{11}, γ_{13}, γ_{15}, γ_{17}	3.3-42, -46, -50, -54
λ_{01}, λ_{03}, λ_{05}, λ_{07}	3.3-21, -25, -29, -33
λ_{11}, λ_{13}, λ_{15}, λ_{17}	3.3-41, -45, -49, -53
Λ_{10}, Λ_{11}, Λ_{12}, Λ_{13}	6.4-10
Λ_{30}, Λ_{31}, Λ_{32}, Λ_{33}	6.4-14
Λ_{50}, Λ_{51}, Λ_{52}, Λ_{53}	6.4-16
Λ_{70}, Λ_{71}, Λ_{72}, Λ_{73}	6.4-21
$C_{\mathrm{ec}}^{a}(\kappa,\theta)$, $C_{\mathrm{es}}^{a}(\kappa,\theta)$, $C_{\mathrm{ec}}(\kappa,\theta)$, $C_{\mathrm{es}}(\kappa,\theta)$	6.4-85
$C_{\mathrm{mc}}^{a}(\kappa,\theta)$, $C_{\mathrm{ms}}^{a}(\kappa,\theta)$, $C_{\mathrm{mc}}(\kappa,\theta)$, $C_{\mathrm{ms}}(\kappa,\theta)$	6.5-15 to -22
$D_0(\kappa)$, $D_1(\kappa)$, $D_2(\kappa)$, $D_3(\kappa)$	3.3-9, -10, -12, -15
$D_{\mathrm{ec}}(\kappa,\theta)$, $D_{\mathrm{es}}(\kappa,\theta)$	6.5-8
$D_{\mathrm{ec}}^{a}(\kappa,\theta)$, $D_{\mathrm{es}}^{a}(\kappa,\theta)$, $D_{\mathrm{ec}}(\kappa,\theta)$, $D_{\mathrm{es}}(\kappa,\theta)$	6.5-9
$E_{\mathrm{ec}}(\kappa,\theta)$, $E_{\mathrm{es}}^{\mathrm{a}}(\kappa,\theta)$	6.5-10
$E_{\mathrm{ec}}^{a}(\kappa,\theta)$, $E_{\mathrm{es}}^{a}(\kappa,\theta)$, $E_{\mathrm{ec}}(\kappa,\theta)$, $E_{\mathrm{es}}(\kappa,\theta)$	6.5-11
$G_{01}(\kappa)$, $G_{02}(\kappa)$, $G_{03}(\kappa)$, $G_{04}(\kappa)$	6.4-48
$G_{11}(\kappa)$, $G_{12}(\kappa)$, $G_{13}(\kappa)$, $G_{14}(\kappa)$	6.4-46
$L_{0}^{\mathrm{c}}(\kappa,0)$, $L_{\mathrm{s}\kappa}^{\mathrm{c}}(\kappa,\nu)$, $L_{\mathrm{c}\kappa}^{\mathrm{c}}(\kappa,\nu)$	6.4-60
$L_{0}^{\mathrm{s}}(\kappa,0)$, $L_{\mathrm{s}\kappa}^{\mathrm{s}}(\kappa,\nu)$, $L_{\mathrm{c}\kappa}^{\mathrm{s}}(\kappa,\nu)$	6.4-61
$L_{0\mathrm{c}}^{a}(\kappa,0), L_{\mathrm{sc}}^{a}(\kappa,\nu), L_{\mathrm{cc}}^{a}(\kappa,\nu), L_{0\mathrm{s}}^{a}(\kappa,0), L_{\mathrm{ss}}^{a}(\kappa,\nu), L_{\mathrm{cs}}^{a}(\kappa,\nu)$	6.4-79, -80
$L_{0\mathrm{c}}(\kappa,0), L_{\mathrm{sc}}(\kappa,\nu), L_{\mathrm{cc}}(\kappa,\nu), L_{0\mathrm{s}}(\kappa,0), L_{\mathrm{ss}}(\kappa,\nu)$, $L_{\mathrm{cs}}(\kappa,\nu)$	6.4-81, -82
$L_{00}^{\mathrm{a}}(\kappa,\theta), L_{01}^{\mathrm{a}}(\kappa,\theta), L_{02}^{\mathrm{a}}(\kappa,\theta), L_{03}^{\mathrm{a}}(\kappa,\theta), L_{04}^{\mathrm{a}}(\kappa,\theta), L_{05}^{\mathrm{a}}(\kappa,\theta)$	6.5-21, -22
$L_{00}(\kappa,\theta), L_{01}(\kappa,\theta), L_{02}(\kappa,\theta), L_{03}(\kappa,\theta), L_{04}(\kappa,\theta), L_{05}(\kappa,\theta)$	6.5-19, -20

$$L_{05}(\kappa,\nu)=-\left[\rho_{\rm s}\left(\sin\frac{2\pi\kappa}{N}-\frac{4\sin^2(\pi\nu/N)}{\sin(2\pi\kappa/N)}\right)L_{\rm cc}(\kappa,\nu)-\frac{\sin(2\pi\nu/N)}{\sin(2\pi\kappa/N)}L_{\rm sc}(\kappa,\nu)\right] \tag{20}$$

$$\begin{aligned}
C^{\rm a}_{\rm mc}(\kappa,\theta)&=\frac{1}{2N}\left[L^{\rm a}_{00}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\rm a}_{01}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\rm a}_{02}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
L^{\rm a}_{00}(\kappa,0)&=\rho_{\rm s}\sin\lambda_{\rm a}L^{\rm a}_{0\rm s}(\kappa,0)\\
L^{\rm a}_{01}(\kappa,\nu)&=\rho_{\rm s}\left(\sin\lambda_{\rm a}-\frac{4\sin^2(\pi\nu/N)}{\sin\lambda_{\rm a}}\right)L^{\rm a}_{\rm ss}(\kappa,\nu)+\frac{\sin(2\pi\nu/N)}{\sin\lambda_{\rm a}}L^{\rm a}_{\rm cs}(\kappa,\nu)\\
L^{\rm a}_{02}(\kappa,\nu)&=\rho_{\rm s}\left(\sin\lambda_{\rm a}-\frac{4\sin^2(\pi\nu/N)}{\sin\lambda_{\rm a}}\right)L^{\rm a}_{\rm cs}(\kappa,\nu)-\frac{\sin(2\pi\nu/N)}{\sin\lambda_{\rm a}}L^{\rm a}_{\rm ss}(\kappa,\nu)
\end{aligned} \tag{21}$$

$$\begin{aligned}
C^{\rm a}_{\rm ms}(\kappa,\theta)&=\frac{1}{2N}\left[L^{\rm a}_{03}(\kappa,0)+\sum_{\nu=1}^{N/2-1}\left(L^{\rm a}_{04}(\kappa,\nu)\sin\frac{2\pi\nu\theta}{N}+L^{\rm a}_{05}(\kappa,\nu)\cos\frac{2\pi\nu\theta}{N}\right)\right]\\
L^{\rm a}_{03}(\kappa,0)&=-\rho_{\rm s}\sin\lambda_{\rm a}L^{\rm a}_{0\rm c}(\kappa,0)\\
L^{\rm a}_{04}(\kappa,\nu)&=-\left[\rho_{\rm s}\left(\sin\lambda_{\rm a}-\frac{4\sin^2(\pi\nu/N)}{\sin\lambda_{\rm a}}\right)L^{\rm a}_{\rm sc}(\kappa,\nu)+\frac{\sin(2\pi\nu/N)}{\sin\lambda_{\rm a}}L^{\rm a}_{\rm cc}(\kappa,\nu)\right]\\
L^{\rm a}_{05}(\kappa,\nu)&=-\left[\rho_{\rm s}\left(\sin\lambda_{\rm a}-\frac{4\sin^2(\pi\nu/N)}{\sin\lambda_{\rm a}}\right)L^{\rm a}_{\rm cc}(\kappa,\nu)-\frac{\sin(2\pi\nu/N)}{\sin\lambda_{\rm a}}L^{\rm a}_{\rm sc}(\kappa,\nu)\right]
\end{aligned} \tag{22}$$

In order to help find all the constants and functions required for repeated substitutions to obtain $C_{\rm mc}(\kappa,\theta)$ to $C_{\rm ms}(\kappa,\theta)$ we provide a listing in Table 6.5-1.

6.6 Calculations for Section 3.5

According to Eq.(3.2-60) we have the relation $\lambda_{\rm a}=\varphi_{\theta\kappa}$. A good approximation of $\varphi_{\theta\kappa}$ is provided by Eq.(3.2-21). Figure 6.6-1 shows $\lambda_{\rm a}$ as function of κ for this approximation. We may represent $\lambda_{\rm a}$ by the following simpler functions:

$$\begin{aligned}
\lambda_{\rm a}&=2\pi\kappa/N && \text{for } -N/4\le\kappa\le N/4,\ \kappa\neq 0\\
&=\pi-2\pi\kappa/N && \text{for } \quad N/4<\kappa<N/2\\
&=-\pi-2\pi\kappa/N && \text{for } -N/2<\kappa<-N/4
\end{aligned} \tag{1}$$

We note that the points $\kappa=-N/2,\ 0,\ N/2$ are excluded in Eq.(1). The representation of Eq.(1) is applied to certain of the integrals of Eqs.(3.5-24) and (3.5-25) that are used in Eq.(3.5-26). The number N in the following formulas is even.

$$\begin{aligned} 1-\cos 2N\lambda_{\mathrm{a}} &= 1-\cos 4\pi\kappa = 0 && \text{for } -N/4 \le \kappa \le N/4 \\ &= 1-\cos 2\pi(N+2\kappa) = 0 && \text{for } \quad N/4 < \kappa < N/2 \\ &= 1-\cos 2\pi(N+2\kappa) = 0 && \text{for } -N/2 < \kappa < -N/4 \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{N}{2}\left(1-\frac{\sin 2N\lambda_{\mathrm{a}}}{2N\lambda_{\mathrm{a}}}\right) &= \frac{N}{2}\left(1-\frac{\sin 4\pi\kappa}{4\pi\kappa}\right) = \frac{N}{2} \\ &\qquad \text{for } -N/4 \le \kappa \le N/4,\ \kappa \ne 0 \\ &= \frac{N}{2}\left(1-\frac{\sin 2\pi(N-2\kappa)}{2\pi(N-2\kappa)}\right) = \frac{N}{2} \\ &\qquad \text{for } N/4 < \kappa < N/2,\ \kappa \ne N/2 \\ &= \frac{N}{2}\left(1-\frac{\sin 2\pi(N+2\kappa)}{2\pi(N+2\kappa)}\right) = \frac{N}{2} \\ &\qquad \text{for } -N/2 < \kappa < -N/4,\ \kappa \ne -N/2 \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{N}{2}\left(1+\frac{\sin 2N\lambda_{\mathrm{a}}}{2N\lambda_{\mathrm{a}}}\right) &= \frac{N}{2}\left(1+\frac{\sin 4\pi\kappa}{4\pi\kappa}\right) = \frac{N}{2} \\ &\qquad \text{for } -N/4 \le \kappa \le N/4,\ \kappa \ne 0 \\ &= \frac{N}{2}\left(1+\frac{\sin 2\pi(N-2\kappa)}{2\pi(N-2\kappa)}\right) = \frac{N}{2} \\ &\qquad \text{for } N/4 < \kappa < N/2,\ \kappa \ne N/2 \\ &= \frac{N}{2}\left(1+\frac{\sin 2\pi(N+2\kappa)}{2\pi(N+2\kappa)}\right) = \frac{N}{2} \\ &\qquad \text{for } -N/2 < \kappa < -N/4,\ \kappa \ne -N/2 \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\sin(N\lambda_{\mathrm{a}}-2\pi\kappa)}{N\lambda_{\mathrm{a}}-2\pi\kappa} &= \frac{\sin 2\pi(\kappa-\kappa)}{2\pi(\kappa-\kappa)} = 1 && \text{for } -N/4 \le \kappa \le N/4,\ \kappa \ne 0 \\ &= \frac{\sin \pi(N-4\kappa)}{\pi(N-4\kappa)} = 0 && \text{for } \quad N/4 < \kappa < N/2,\ \kappa \ne N/4 \\ &= \frac{\sin \pi(N+4\kappa)}{\pi(N+4\kappa)} = 0 && \text{for } -N/2 < \kappa < -N/4,\ \kappa \ne -N/4 \end{aligned} \tag{5}$$

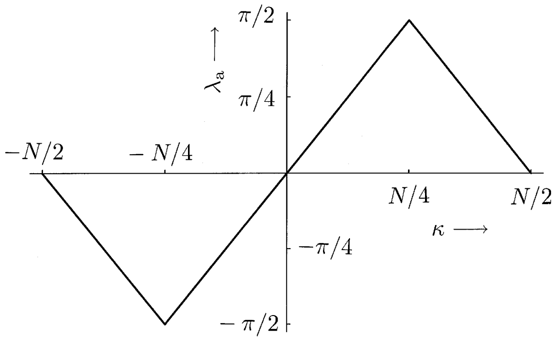

FIGURE 6.6-1. Plot of λ_a as function of κ using the approximation of Fig.3.2-1.

$$\begin{aligned}\frac{\sin(N\lambda_a + 2\pi\kappa)}{N\lambda_a - 2\pi\kappa} &= \frac{\sin 4\pi\kappa}{4\pi\kappa} = 0 \qquad \text{for } -N/4 \le \kappa \le N/4,\ \kappa \neq 0\\ &= \frac{\sin \pi N}{\pi N} = 0 \qquad \text{for } \quad N/4 < \kappa < N/2\\ &= \frac{\sin(-\pi N)}{-\pi N} = 0 \quad \text{for } -N/2 < \kappa < -N/4 \end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\sin^2(N\lambda_a/2 - \pi\kappa)}{N\lambda_a/2 - \pi\kappa} &= \frac{\sin^2 \pi(\kappa - \kappa)}{\pi(\kappa - \kappa)} = 0 \qquad \text{for } -N/4 \le \kappa \le N/2,\ \kappa \neq 0\\ &= \frac{\sin^2 \pi(N/2 - 2\kappa)}{\pi(N/2 - 2\kappa)} = 0 \quad \text{for } \quad N/4 < \kappa < N/2\\ &= \frac{\sin^2 \pi(N/2 + 2\kappa)}{\pi(N/2 + 2\kappa)} = 0 \quad \text{for } -N/2 < \kappa < -N/4 \end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\sin^2(N\lambda_a/2 + \pi\kappa)}{N\lambda_a/2 + \pi\kappa} &= \frac{\sin^2 2\pi\kappa}{2\pi\kappa} = 0 \qquad \text{for } -N/4 \le \kappa \le N/4\\ &= \frac{\sin^2 \pi N/2}{\pi N/2} = 0 \quad \text{for } \quad N/4 < \kappa < N/2\\ &= \frac{\sin^2 \pi N/2}{-\pi N/2} = 0 \quad \text{for } -N/2 < \kappa < -N/4 \end{aligned} \tag{8}$$

Using Eqs.(2) to (8) we reduce Eq.(3.5-26) to the following shorter form:

$$
\begin{aligned}
U = \sum_{i=1}^{6} U_i = \frac{1}{2} Z V_{\mathrm{e0}}^2 L^2 T^3 c^4 \frac{1}{N^2} \sum_{\kappa=-N/2+1}^{N/2-1,\neq 0} \\
\Bigg\{ \Bigg[C_{\mathrm{es}}^{\zeta 2} + C_{\mathrm{ec}}^{\zeta 2} + \left(\frac{\partial C_{\mathrm{ec}}}{\partial\theta}\right)^2 + \left(\frac{\partial C_{\mathrm{es}}}{\partial\theta}\right)^2 + C_{\mathrm{es}}^{\zeta}\frac{\partial C_{\mathrm{mc}}}{\partial\theta} + C_{\mathrm{ec}}^{\zeta}\frac{\partial C_{\mathrm{ms}}}{\partial\theta} \\
+ C_{\mathrm{ms}}^{\zeta}\frac{\partial C_{\mathrm{ec}}}{\partial\theta} + C_{\mathrm{mc}}^{\zeta}\frac{\partial C_{\mathrm{es}}}{\partial\theta} + C_{\mathrm{ms}}^{\zeta 2} + C_{\mathrm{mc}}^{\zeta 2} + \left(\frac{\partial C_{\mathrm{mc}}}{\partial\theta}\right)^2 + \left(\frac{\partial C_{\mathrm{ms}}}{\partial\theta}\right)^2 \Bigg] \\
+ \left[C_{\mathrm{es}}^{\mathrm{a}\zeta 2} + \left(\frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta}\right)^2 + C_{\mathrm{es}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta}\right)^2 \right] \\
+ \left[C_{\mathrm{ec}}^{\mathrm{a}\zeta 2} + \left(\frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta}\right)^2 + C_{\mathrm{ec}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta\,2} + \left(\frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta}\right)^2 \right] \\
+ 2S_1(\kappa)\Bigg[C_{\mathrm{es}}^{\mathrm{a}\zeta} C_{\mathrm{es}}^{\zeta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta}\frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{es}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{mc}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{es}}^{\zeta} + C_{\mathrm{ms}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{ec}}}{\partial\theta} + \frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ms}}^{\zeta}\Bigg) + C_{\mathrm{ms}}^{\mathrm{a}\zeta} C_{\mathrm{ms}}^{\zeta} + \frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}}{\partial\theta}\frac{\partial C_{\mathrm{mc}}}{\partial\theta}\Bigg] \\
+ 2S_1(\kappa)\Bigg[C_{\mathrm{ec}}^{\mathrm{a}\zeta} C_{\mathrm{ec}}^{\zeta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta}\frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{1}{2}\Bigg(C_{\mathrm{ec}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{ms}}}{\partial\theta} \\
+ \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{ec}}^{\zeta} + C_{\mathrm{mc}}^{\mathrm{a}\zeta}\frac{\partial C_{\mathrm{es}}}{\partial\theta} + \frac{\partial C_{\mathrm{es}}^{\mathrm{a}}}{\partial\theta} C_{\mathrm{mc}}^{\zeta}\Bigg) + C_{\mathrm{mc}}^{\mathrm{a}\zeta} C_{\mathrm{mc}}^{\zeta} + \frac{\partial C_{\mathrm{ms}}^{\mathrm{a}}}{\partial\theta}\frac{\partial C_{\mathrm{ms}}}{\partial\theta}\Bigg]\Bigg\} \qquad (9)
\end{aligned}
$$

Some of the brackets are retained in this equation to facilitate comparison with Eq.(3.5-26). The newly introduced function $S_1(\kappa)$ is defined as follows:

$$
\begin{aligned}
S_1(\kappa) &= 1 \quad \text{for } -N/4 \le \kappa \le N/4,\ \kappa \neq 0 \\
&= 0 \quad \text{for } -N/2 < \kappa < -N/4,\ N/4 < \kappa < N/2,\ \kappa = 0 \qquad (10)
\end{aligned}
$$

The functions C_{ec}^{ζ} to $\partial C_{\mathrm{ms}}^{\mathrm{a}}/\partial\theta$ are listed in Eqs.(3.5-27) to (3.5-34). Furthermore, the first square $C_{\mathrm{es}}^{\zeta\,2}$ in Eq.(9) has been worked out in Eqs.(3.5-35) and (3.5-36). There are 39 more such squares or products in Eq.(9) that still have to be elaborated. We simplify this task by writing only the constant part but not the time variable part - meaning the terms containing sine or cosine functions of θ. We have learned in Section 3.5 that the approximation of Fig.6.5-1 fails close to $\kappa = -N/2$, 0 and $N/2$. Hence the interval $K_0 < |\kappa| < K_{\mathrm{N/2}}$ for which Eq.(3.5-36) holds is of more interest than the interval for which Eq.(3.5-35) holds. It will be used for Eqs.(11)–(15):

$$\text{for } K_0 < |\kappa| < K_{\mathrm{N}/2}$$

$$C_{\mathrm{es}}^{\zeta\,2}(\kappa,\theta) = U_1(\kappa) + V_1(\kappa,\theta)$$

$$U_1(\kappa) = \left(\frac{2\pi\kappa}{N}\right)^2 \left(\frac{1}{2N}\right)^2 \left(L_{0\mathrm{s}}^2(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{\mathrm{ss}}^2(\kappa,\nu) + L_{\mathrm{cs}}^2(\kappa,\nu)]\right)$$

$$C_{\mathrm{ec}}^{\zeta 2}(\kappa,\theta) = U_2(\kappa) + V_2(\kappa,\theta)$$

$$U_2(\kappa) = \left(\frac{2\pi\kappa}{N}\right)^2 \left(\frac{1}{2N}\right)^2 \left(L_{0\mathrm{c}}^2(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{\mathrm{sc}}^2(\kappa,\nu) + L_{\mathrm{cc}}^2(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_3(\kappa) + V_3(\kappa,\theta)$$

$$U_3(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2 [L_{\mathrm{sc}}^2(\kappa,\nu) + L_{\mathrm{cc}}^2(\kappa,\nu)]$$

$$\left(\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_4(\kappa) + V_4(\kappa,\theta)$$

$$U_4(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2 [L_{\mathrm{ss}}^2(\kappa,\nu) + L_{\mathrm{cs}}^2(\kappa,\nu)]$$

$$C_{\mathrm{es}}^{\zeta}(\kappa,\theta)\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta} = U_5(\kappa) + V_5(\kappa,\theta)$$

$$U_5(\kappa) = \frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{\mathrm{cs}}(\kappa,\nu)L_{01}(\kappa,\nu) - L_{\mathrm{ss}}(\kappa,\nu)L_{02}(\kappa,\nu)]$$

$$C_{\mathrm{ec}}^{\zeta}(\kappa,\theta)\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} = U_6(\kappa) + V_6(\kappa,\theta)$$

$$U_6(\kappa) = \frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{\mathrm{sc}}(\kappa,\nu)L_{05}(\kappa,\nu) - L_{\mathrm{cc}}(\kappa,\nu)L_{04}(\kappa,\nu)]$$

$$C_{\mathrm{ms}}^{\zeta}(\kappa,\theta)\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta} = U_7(\kappa) + V_7(\kappa,\theta)$$

$$U_7(\kappa) = \frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{\mathrm{ss}}(\kappa,\nu)L_{05}(\kappa,\nu) - L_{\mathrm{cs}}(\kappa,\nu)L_{04}(\kappa,\nu)]$$

$$C_{\mathrm{mc}}^{\zeta}(\kappa,\theta)\frac{\partial C_{\mathrm{es}}(\kappa,\theta)}{\partial\theta} = U_8(\kappa) + V_8(\kappa,\theta)$$

$$U_8(\kappa)=\frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{\mathrm{cs}}(\kappa,\nu)L_{01}(\kappa,\nu)-L_{\mathrm{ss}}(\kappa,\nu)L_{02}(\kappa,\nu)]$$

$$C_{\mathrm{ms}}^{\zeta 2}(\kappa,\theta) = U_9(\kappa) + V_9(\kappa,\theta)$$

$$U_9(\kappa) = \left(\frac{2\pi\kappa}{N}\right)^2\left(\frac{1}{2N}\right)^2\left(L_{03}^2(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{04}^2(\kappa,\nu) + L_{05}^2(\kappa,\nu)]\right)$$

$$C_{\mathrm{mc}}^{\zeta 2}(\kappa,\theta) = U_{10}(\kappa) + V_{10}(\kappa,\nu)$$

$$U_{10}(\kappa) = \left(\frac{2\pi\kappa}{N}\right)^2\left(\frac{1}{2N}\right)^2\left(L_{00}^2(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{01}^2(\kappa,\nu) + L_{02}^2(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_{11}(\kappa) + V_{11}(\kappa,\theta)$$

$$U_{11}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{01}^2(\kappa,\nu) + L_{02}^2(\kappa,\nu)]$$

$$\left(\frac{\partial C_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_{12}(\kappa) + V_{12}(\kappa,\theta)$$

$$U_{12}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{04}^2(\kappa,\nu) + L_{05}^2(\kappa,\nu)] \tag{11}$$

$$C_{\mathrm{es}}^{\mathrm{a}\zeta\,2}(\kappa,\theta) = U_{13}(\kappa) + V_{13}(\kappa,\theta)$$

$$U_{13}(\kappa) = \lambda_{\mathrm{a}}^2\left(\frac{1}{2N}\right)^2\left(L_{0\mathrm{s}}^{\mathrm{a}\,2}(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{\mathrm{ss}}^{\mathrm{a}\,2}(\kappa,\nu) + L_{\mathrm{cs}}^{\mathrm{a}\,2}(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C_{\mathrm{ec}}^{\mathrm{a}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_{14}(\kappa) + V_{14}(\kappa,\theta)$$

$$U_{14}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{\mathrm{sc}}^{\mathrm{a}\,2}(\kappa,\nu) + L_{\mathrm{cc}}^{\mathrm{a}\,2}(\kappa,\nu)]$$

$$C_{\mathrm{es}}^{\mathrm{a}\zeta}(\kappa,\theta)\frac{\partial C_{\mathrm{mc}}^{\mathrm{a}}(\kappa,\theta)}{\partial\theta} = U_{15}(\kappa) + V_{15}(\kappa,\theta)$$

$$U_{15}(\kappa) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{01}^{\mathrm{a}}(\kappa,\nu)L_{\mathrm{cs}}^{\mathrm{a}}(\kappa,\nu)-L_{02}^{\mathrm{a}}(\kappa,\nu)L_{\mathrm{ss}}^{\mathrm{a}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta} = U_{16}(\kappa) + V_{16}(\kappa,\theta)$$

$$U_{16}(\kappa) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{05}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu) - L_{04}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta\,2}_{\mathrm{ms}}(\kappa,\theta) = U_{17}(\kappa) + V_{17}(\kappa,\theta)$$

$$U_{17}(\kappa) = \lambda_{\mathrm{a}}^2\left(\frac{1}{2N}\right)^2\left(L^{\mathrm{a}\,2}_{03}(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{\mathrm{a}\,2}_{04}(\kappa,\nu) + L^{\mathrm{a}\,2}_{05}(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_{18}(\kappa) + V_{18}(\kappa,\theta)$$

$$U_{18}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L^{\mathrm{a}\,2}_{01}(\kappa,\nu) + L^{\mathrm{a}\,2}_{02}(\kappa,\nu)] \tag{12}$$

$$C^{\mathrm{a}\zeta\,2}_{\mathrm{ec}}(\kappa,\theta) = U_{19}(\kappa) + V_{19}(\kappa,\theta)$$

$$U_{19}(\kappa) = \lambda_{\mathrm{a}}^2\left(\frac{1}{2N}\right)^2\left(L^{\mathrm{a}\,2}_{0\mathrm{c}}(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{\mathrm{a}\,2}_{\mathrm{sc}}(\kappa,\nu) + L^{\mathrm{a}\,2}_{\mathrm{cc}}(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta}\right)^2 = U_{20}(\kappa) + V_{20}(\kappa,\theta)$$

$$U_{20}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L^{\mathrm{a}\,2}_{\mathrm{ss}}(\kappa,\nu) + L^{\mathrm{a}\,2}_{\mathrm{cs}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta}_{\mathrm{ec}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\theta)}{\partial\theta} = U_{21}(\kappa) + V_{21}(\kappa,\theta)$$

$$U_{21}(\kappa) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\mathrm{a}}_{05}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{sc}}(\kappa,\nu) - L^{\mathrm{a}}_{04}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta}_{\mathrm{mc}}(\kappa,\theta)\frac{\partial C^{\mathrm{a}}_{\mathrm{es}}(\kappa,\theta)}{\partial\theta} = U_{22}(\kappa) + V_{22}(\kappa,\theta)$$

$$U_{22}(\kappa,\theta) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\mathrm{a}}_{01}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cs}}(\kappa,\nu) - L^{\mathrm{a}}_{02}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta\,2}_{\mathrm{mc}}(\kappa,\nu) = U_{23}(\kappa) + V_{23}(\kappa,\theta)$$

$$U_{23}(\kappa) = \lambda_{\mathrm{a}}^2\left(\frac{1}{2N}\right)^2\left(L^{\mathrm{a}\,2}_{00}(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{\mathrm{a}\,2}_{01}(\kappa,\nu) + L^{\mathrm{a}\,2}_{02}(\kappa,\nu)]\right)$$

$$\left(\frac{\partial C^{\mathrm{a}}_{\mathrm{ms}}(\kappa,\nu)}{\partial\theta}\right)^2 = U_{24}(\kappa) + V_{24}(\kappa,\nu)$$

$$U_{24}(\kappa) = \frac{1}{2}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1} \left(\frac{2\pi\nu}{N}\right)^2 [L^{\mathrm{a}\,2}_{04}(\kappa,\nu) + L^{\mathrm{a}\,2}_{05}(\kappa,\nu)] \tag{13}$$

The following terms are needed in the interval $-N/4 \le \kappa \le N/4$ only. We continue to determine them for the interval $K_0 < |\kappa| < K_{\mathrm{N/2}}$ but terms multiplied by $S_1(\kappa)$ in Eq.(9) are summed only from $-N/4$ to $N/4$ rather than from $-N/2+1$ to $N/2-1$; the value $\kappa = 0$ remains excluded:

for $K_0 < |\kappa| < K_{\mathrm{N/2}}$

$$2C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta)C^{\zeta}_{\mathrm{es}}(\kappa,\theta) = U_{25}(\kappa) + V_{25}(\kappa,\theta)$$

$$U_{25}(\kappa) = \frac{4\pi\kappa\lambda_{\mathrm{a}}}{N}\left(\frac{1}{2N}\right)^2 \Bigg(L_{0\mathrm{s}}(\kappa,0)L^{\mathrm{a}}_{0\mathrm{s}}(\kappa,0) + \frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{\mathrm{ss}}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu) + L_{\mathrm{cs}}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cs}}(\kappa,\nu)]\Bigg)$$

$$2\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta} = U_{26}(\kappa) + V_{26}(\kappa,\theta)$$

$$U_{26}(\kappa) = \left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2 [L_{\mathrm{sc}}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{sc}}(\kappa,\nu) + L_{\mathrm{cc}}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cc}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta}_{\mathrm{es}}(\kappa,\theta)\frac{\partial C_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta} = U_{27}(\kappa) + V_{27}(\kappa,\theta)$$

$$U_{27}(\kappa) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{01}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{cs}}(\kappa,\nu) - L_{02}(\kappa,\nu)L^{\mathrm{a}}_{\mathrm{ss}}(\kappa,\nu)]$$

$$\frac{\partial C^{\mathrm{a}}_{\mathrm{mc}}(\kappa,\theta)}{\partial\theta}C^{\zeta}_{\mathrm{es}}(\kappa,\theta) = U_{28}(\kappa) + V_{28}(\kappa,\theta)$$

$$U_{28}(\kappa) = \frac{2\pi\kappa}{N}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\mathrm{a}}_{01}(\kappa,\nu)L_{\mathrm{cs}}(\kappa,\nu) - L^{\mathrm{a}}_{02}(\kappa,\nu)L_{\mathrm{ss}}(\kappa,\nu)]$$

$$C^{\mathrm{a}\zeta}_{\mathrm{ms}}(\kappa,\theta)\frac{\partial C_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta} = U_{29}(\kappa) + V_{29}(\kappa,\theta)$$

$$U_{29}(\kappa) = \frac{\lambda_{\mathrm{a}}}{2}\left(\frac{1}{2N}\right)^2 \sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\mathrm{a}}_{05}(\kappa,\nu)L_{\mathrm{sc}}(\kappa,\nu) - L^{\mathrm{a}}_{04}(\kappa,\nu)L_{\mathrm{cc}}(\kappa,\nu)]$$

$$\frac{\partial C^{\mathrm{a}}_{\mathrm{ec}}(\kappa,\theta)}{\partial\theta}C^{\zeta}_{\mathrm{ms}}(\kappa,\theta) = U_{30}(\kappa) + V_{30}(\kappa,\theta)$$

$$U_{30}(\kappa)=\frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{05}(\kappa,\nu)L^{\rm a}_{\rm sc}(\kappa,\nu)-L_{04}(\kappa,\nu)L^{\rm a}_{\rm cc}(\kappa,\nu)]$$

$$2C^{\rm a\zeta}_{\rm ms}(\kappa,\theta)C^{\zeta}_{\rm ms}(\kappa,\theta)=U_{31}(\kappa)+V_{31}(\kappa,\theta)$$

$$U_{31}(\kappa)=\frac{4\pi\kappa\lambda_{\rm a}}{N}\left(\frac{1}{2N}\right)^2\Bigg(L_{03}(\kappa,0)L^{\rm a}_{03}(\kappa,0)+\frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{04}(\kappa,\nu)L^{\rm a}_{04}(\kappa,\nu)+L_{05}(\kappa,\nu)L^{\rm a}_{05}(\kappa,\nu)]\Bigg)$$

$$2\frac{\partial C^{\rm a}_{\rm mc}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\rm mc}(\kappa,\theta)}{\partial\theta}=U_{32}(\kappa)+V_{32}(\kappa,\theta)$$

$$U_{32}(\kappa)=\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{01}(\kappa,\nu)L^{\rm a}_{01}(\kappa,\nu)+L_{02}(\kappa,\nu)L^{\rm a}_{02}(\kappa,\nu)] \tag{14}$$

The terms $U_{33}(\kappa)$, $V_{33}(\kappa,\theta)$ to $U_{48}(\kappa)$, $V_{48}(\kappa,\theta)$ in lines 10 to 13 of Eq. (3.5-26) do not occur in the approximation of Eq.(9). Hence, we jump from $U_{32}(\kappa)$ in Eq.(14) to $U_{49}(\kappa)$:

for $K_0<|\kappa|<K_{\rm N/2}$

$$2C^{\rm a\zeta}_{\rm ec}(\kappa,\theta)C^{\zeta}_{\rm ec}(\kappa,\theta)=U_{49}(\kappa)+V_{49}(\kappa,\theta)$$

$$U_{49}(\kappa)=\frac{4\pi\kappa\lambda_{\rm a}}{N}\left(\frac{1}{2N}\right)^2\Bigg(L^{\rm a}_{0\rm c}(\kappa,0)L_{0\rm c}(\kappa,0)+\frac{1}{2}\sum_{\nu=1}^{N/2-1}[L^{\rm a}_{\rm sc}(\kappa,\nu)L_{\rm sc}(\kappa,\nu)+L^{\rm a}_{\rm cc}(\kappa,\nu)L_{\rm cc}(\kappa,\nu)]\Bigg)$$

$$2\frac{\partial C^{\rm a}_{\rm es}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\rm es}(\kappa,\theta)}{\partial\theta}=U_{50}(\kappa)+V_{50}(\kappa,\theta)$$

$$U_{50}(\kappa)=\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{\rm ss}(\kappa,\nu)L^{\rm a}_{\rm ss}(\kappa,\nu)+L_{\rm cs}(\kappa,\nu)L^{\rm a}_{\rm cs}(\kappa,\nu)]$$

$$C^{\rm a\zeta}_{\rm ec}(\kappa,\theta)\frac{\partial C_{\rm ms}(\kappa,\theta)}{\partial\theta}=U_{51}(\kappa)+V_{51}(\kappa,\theta)$$

$$U_{51}(\kappa)=\frac{\lambda_{\rm a}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{05}(\kappa,\nu)L^{\rm a}_{\rm sc}(\kappa,\nu)-L_{04}(\kappa,\nu)L^{\rm a}_{\rm cc}(\kappa,\nu)]$$

$$\frac{\partial C^{\rm a}_{\rm ms}(\kappa,\theta)}{\partial\theta}C^{\zeta}_{\rm ec}(\kappa,\theta)=U_{52}(\kappa)+V_{52}(\kappa,\theta)$$

$$U_{52}(\kappa)=\frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\rm a}_{05}(\kappa,\nu)L_{\rm sc}(\kappa,\nu)-L^{\rm a}_{04}(\kappa,\nu)L_{\rm cc}(\kappa,\nu)]$$

$$C^{{\rm a}\zeta}_{\rm mc}(\kappa,\theta)\frac{\partial C_{\rm es}(\kappa,\theta)}{\partial\theta}=U_{53}(\kappa)+V_{53}(\kappa,\theta)$$

$$U_{53}(\kappa)=\frac{\lambda_{\rm a}}{2}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L^{\rm a}_{01}(\kappa,\nu)L_{\rm cs}(\kappa,\nu)-L^{\rm a}_{02}(\kappa,\nu)L_{\rm ss}(\kappa,\nu)]$$

$$\frac{\partial C^{\rm a}_{\rm es}(\kappa,\theta)}{\partial\theta}C^{\zeta}_{\rm mc}(\kappa,\theta)=U_{54}(\kappa)+V_{54}(\kappa,\theta)$$

$$U_{54}(\kappa)=\frac{\pi\kappa}{N}\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\frac{2\pi\nu}{N}[L_{01}(\kappa,\nu)L^{\rm a}_{\rm cs}(\kappa,\nu)-L_{02}(\kappa,\nu)L^{\rm a}_{\rm ss}(\kappa,\nu)]$$

$$2C^{{\rm a}\zeta}_{\rm mc}(\kappa,\theta)C^{\rm a}_{\rm mc}(\kappa,\theta)=U_{55}(\kappa)+V_{55}(\kappa,\theta)$$

$$U_{55}(\kappa)=\frac{4\pi\kappa\lambda_{\rm a}}{N}\left(\frac{1}{2N}\right)^2\bigg(L_{00}(\kappa,0)L^{\rm a}_{00}(\kappa,0)+\frac{1}{2}\sum_{\nu=1}^{N/2-1}[L_{01}(\kappa,\nu)L^{\rm a}_{01}(\kappa,\nu)+L_{02}(\kappa,\nu)L^{\rm a}_{02}(\kappa,\nu)]\bigg)$$

$$2\frac{\partial C^{\rm a}_{\rm ms}(\kappa,\theta)}{\partial\theta}\frac{\partial C_{\rm ms}(\kappa,\theta)}{\partial\theta}=U_{56}(\kappa)+V_{56}(\kappa,\theta)$$

$$U_{56}(\kappa)=\left(\frac{1}{2N}\right)^2\sum_{\nu=1}^{N/2-1}\left(\frac{2\pi\nu}{N}\right)^2[L_{04}(\kappa,\nu)L^{\rm a}_{04}(\kappa,\nu)+L_{05}(\kappa,\nu)L^{\rm a}_{05}(\kappa,\nu)] \tag{15}$$

In analogy to Eq.(3.5-41) we may write:

$$\text{for } K_0\le|\kappa|\le N/4$$
$$U_{\rm cN}(\kappa)=\sum_{j=1}^{32}U_j(\kappa)+\sum_{j=49}^{56}U_j(\kappa),\quad U_{\rm vN}(\kappa,\theta)=\sum_{j=1}^{32}V_j(\kappa,\theta)+\sum_{j=49}^{56}V_j(\kappa,\theta)$$
$$\text{for } N/4<|\kappa|\le K_{\rm N/2}$$
$$U_{\rm cN}(\kappa)=\sum_{j=1}^{24}U_j(\kappa),\quad U_{\rm vN}(\kappa,\theta)=\sum_{j=1}^{24}V_j(\kappa,\theta) \tag{16}$$

Since we are usually not interested in distinguishing between positive and negative period numbers κ we may replace $U_j(\kappa)$ in Eq.(16) by $U_j(\kappa)+U_j(-\kappa)$

and write $U_{\mathrm{cN}}(\kappa)$ for the interval $0<\kappa\le N/2-1$ rather than $-N/2+1\le \kappa\le N/2-1$:

$$\text{for } K_0\le\kappa\le N/4$$
$$U_{\mathrm{cN}}(\kappa)=\sum_{j=1}^{32}[U_j(\kappa)+U_j(-\kappa)]+\sum_{j=49}^{56}[U_j(\kappa)+U_j(-\kappa)]=U_{\mathrm{cN1}}(\kappa)$$
$$\text{for } N/4<\kappa\le K_{\mathrm{N}/2}$$
$$U_{\mathrm{cN}}(\kappa)=\sum_{j=1}^{24}[U_j(\kappa)+U_j(-\kappa)]=U_{\mathrm{cN2}}(\kappa) \tag{17}$$

If we want to extend the range for κ in Eq.(16) to the interval $-N/2+1<\kappa<N/2-1$ we must write the 40 equations of Eq.(11) to (15) for the intervals $-K_0<\kappa<K_0$, $\kappa\neq 0$ and $K_{\mathrm{N}/2}<|\kappa|<N/2-1$.

We want the relative frequency $r(\kappa)$ or the probability of the energy $U_{\mathrm{cN}}(\kappa)$. If we ignore the intervals $-K_0<\kappa<K_0$, $\kappa\neq 0$ and $K_{\mathrm{N}/2}<|\kappa|<N/2-1$ we obtain in analogy to Eq.(2.3-53):

$$r(\kappa)=r_1(\kappa)+r_2(\kappa),\quad r_1(\kappa)=\frac{U_{\mathrm{cN1}}(\kappa)}{S_{\mathrm{cK}}},\quad r_2(\kappa)=\frac{U_{\mathrm{cN2}}(\kappa)}{S_{\mathrm{cK}}}$$
$$S_{\mathrm{cK}}=S_{\mathrm{cK1}}+S_{\mathrm{cK2}},\quad S_{\mathrm{cK1}}=\sum_{\kappa>K_0}^{\le N/4}U_{\mathrm{cN1}}(\kappa),\quad S_{\mathrm{cK2}}=\sum_{\kappa>N/4}^{<K_{\mathrm{N}/2}}U_{\mathrm{cN2}}(\kappa)$$
$$K_0=N(\rho_1^2-4\rho_2^2)^{1/2}/4\pi,\quad K_{\mathrm{N}/2}=(N/2-K_0) \tag{18}$$

Equation (18) is formally equal to Eq.(2.3-53) except for the upper limit $K_{\mathrm{N}/2}$ rather than $N/2$ of the sum.

6.7 Calculations for Section 4.4

The evaluation of Eq.(4.4-4) calls for the integration with respect to ζ of the expressions for $\Psi^*\Psi$, $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$ and $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$ in Eqs.(4.4-5), (4.4-8) and (4.4-9). A large number of simple steps is required. We write them in some detail since it is difficult to make so many steps without mistake even though each one is simple. First we rewrite the squared sums in these equations:

$$\left(\sum_{\kappa=1}^{N_\tau} \frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa} \sin\gamma_\kappa\theta \sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2 = \frac{1}{2}\sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2\lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2}(1-\cos 2\gamma_\kappa\theta)\sin^2\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa=1}^{N_\tau,\neq j}\sum_{j=1}^{N_\tau} \frac{\lambda_1^2\lambda_3^2 I_{\mathrm{T}}(\kappa/N_\tau) I_{\mathrm{T}}(j/N_\tau)}{\gamma_\kappa\gamma_j} \sin\gamma_\kappa\theta\sin\gamma_j\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\sin\frac{2\pi j\zeta}{N_\tau} \quad (1)$$

$$\left(\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}(\kappa/N_\tau)\cos\gamma_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\right)^2 = \frac{1}{2}\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)(1+\cos 2\gamma_\kappa\theta)\sin^2\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa=1}^{N_\tau,\neq j}\sum_{j=1}^{N_\tau} I_{\mathrm{T}}(\kappa/N_\tau) I_{\mathrm{T}}(j/N_\tau)\cos\gamma_\kappa\theta\cos\gamma_j\theta\sin\frac{2\pi\kappa\zeta}{N_\tau}\sin\frac{2\pi j\zeta}{N_\tau} \quad (2)$$

$$\left(\sum_{\kappa=1}^{N_\tau} \frac{2\pi\kappa\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}\sin\gamma_\kappa\theta\cos\frac{2\pi\kappa\zeta}{N_\tau}\right)^2 = \frac{1}{2}\sum_{\kappa=1}^{N_\tau}\left(\frac{2\pi\kappa\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}\right)^2(1-\cos 2\gamma_\kappa\theta)\cos^2\frac{2\pi\kappa\zeta}{N_\tau} + \sum_{\kappa=1}^{N_\tau,\neq j}\sum_{j=1}^{N_\tau}\left(\frac{2\pi\lambda_1\lambda_3}{N_\tau}\right)^2 \frac{\kappa j I_{\mathrm{T}}(\kappa/N_\tau) I_{\mathrm{T}}(j/N_\tau)}{\gamma_\kappa\gamma_j} \times \sin\gamma_\kappa\theta\sin\gamma_j\theta\cos\frac{2\pi\kappa\zeta}{N_\tau}\cos\frac{2\pi j\zeta}{N_\tau} \quad (3)$$

The following integrals will be needed:

$$\int_0^{N_\tau} \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta = N_\tau \frac{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau} \quad (4)$$

$$\int_0^{N_\tau} \exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\,d\zeta = N_\tau \frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau} \quad (5)$$

$$\int_0^{N_\tau} \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin(2\pi\kappa\zeta/N_\tau)\,d\zeta = N_\tau\,\frac{(2\pi\kappa/N_\tau)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau} \quad (6)$$

$$\int_0^{N_\tau} \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\cos(2\pi\kappa\zeta/N_\tau)\,d\zeta = N_\tau\,\frac{(\lambda_2^2-\lambda_1^2)^{1/2}\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau} \quad (7)$$

$$\begin{aligned}\int_0^{N_\tau} \sin(2\pi\kappa\zeta/N_\tau)\sin(2\pi j\zeta/N_\tau)\,d\zeta &= \frac{N_\tau}{2} && \text{for } \kappa=j\\ &= 0 && \text{for } \kappa\neq j \quad (8)\\ \int_0^{N_\tau} \cos(2\pi\kappa\zeta/N_\tau)\cos(2\pi j\zeta/N_\tau)\,d\zeta &= \frac{N_\tau}{2} && \text{for } \kappa=j\\ &= 0 && \text{for } \kappa\neq j \quad (9)\end{aligned}$$

According to Eq.(4.4-4) we need the integral of $\Psi^*\Psi$ with respect to ζ. We obtain from Eq.(4.4-5) with the help of Eq.(1):

$$\begin{aligned}\int_0^{N_\tau} \Psi^*\Psi\,d\zeta = 2\Psi_1^2\Bigg((1-\cos 2\lambda_1\lambda_3\theta)&\int_0^{N_\tau}\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta\\ &-2(1-\cos 2\lambda_1\lambda_3\theta)\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta\\ \times\sum_{\kappa=1}^{N_\tau}\frac{I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta&\int_0^{N_\tau}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta\\ +\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1^2\lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2}(1-\cos 2\gamma_\kappa\theta)&\int_0^{N}\sin^2\frac{2\pi\kappa\zeta}{N_\tau}d\zeta\end{aligned}$$

$$+ 2 \sum_{\kappa=1}^{N_\tau, \neq j} \sum_{j=1}^{N_\tau} \frac{\lambda_1^2 \lambda_3^2 I_{\mathrm{T}}(\kappa/N_\tau) I_{\mathrm{T}}(j/N_\tau)}{\gamma_\kappa \gamma_j}$$

$$\times \sin \gamma_\kappa \theta \sin \gamma_j \theta \int_0^{N_\tau} \sin \frac{2\pi\kappa\zeta}{N_\tau} \sin \frac{2\pi j \zeta}{N_\tau} d\zeta \Bigg) \quad (10)$$

The integrals are listed in Eqs.(5), (6) and (8):

$$\int_0^{N_\tau} \Psi^* \Psi \, d\zeta = \Psi_1^2 N_\tau \Bigg[2(1 - \cos 2\lambda_1\lambda_3\theta) \frac{1 - \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]}{2\left(\lambda_2^2 - \lambda_1^2\right)^{1/2} N_\tau}$$

$$- 4(1 - \cos 2\lambda_1\lambda_3\theta)\lambda_1\lambda_3 \sin \lambda_1\lambda_3\theta$$

$$\times \sum_{\kappa=1}^{N_\tau} \frac{I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa} \sin \gamma_\kappa \theta \frac{(2\pi\kappa/N_\tau)\{1 - \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]\}}{[\lambda_2^2 - \lambda_1^2 + (2\pi\kappa/N_\tau)^2] N_\tau}$$

$$+ \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2 \lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2} (1 - \cos 2\gamma_\kappa\theta) \Bigg) \quad (11)$$

We recognize that there are some terms in Eq.(11) that do not depend on θ while most vary according to sinusoidal functions of θ. We denote these constant terms with a subscript c:

$$U_{\mathrm{c1}} = \frac{L^2}{c\tau} \left(\int_0^{N_\tau} \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi^* \Psi \, d\zeta \right)_c = \frac{L^2}{c\tau} \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi_1^2 N_\tau$$

$$\times \left(\frac{1 - \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]}{\left(\lambda_2^2 - \lambda_1^2\right)^{1/2} N_\tau} + \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2 \lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \right)$$

$$\doteq \frac{L^2}{c\tau} \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi_1^2 N_\tau \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2 \lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \quad \text{for } N_\tau \gg 1,\ \lambda_1^2 \neq \lambda_2^2 \quad (12)$$

The remaining terms with sinusoidal functions of θ and time-average equal to zero are denoted with a subscript v:

$$U_{\mathrm{v1}}(\theta) = \frac{L^2}{c\tau} \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi_1^2 N_\tau \Bigg(-2 \frac{1 - \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]}{2\left(\lambda_2^2 - \lambda_1^2\right)^{1/2} N_\tau} \cos 2\lambda_1\lambda_3\theta$$

$$- 4(1 - \cos 2\lambda_1\lambda_3\theta)\lambda_1\lambda_3 \sin \lambda_1\lambda_3\theta$$

$$\times \sum_{\kappa=1}^{N_\tau} \frac{I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa} \frac{(2\pi\kappa/N_\tau)\{1 - \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]\}}{[\lambda_2^2 - \lambda_1^2 + (2\pi\kappa/N_\tau)^2] N_\tau} \sin \gamma_\kappa\theta$$

$$\left. - \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2\lambda_3^2 I_{\rm T}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \cos 2\gamma_\kappa\theta \right)$$

$$\doteq -\frac{L^2}{c\tau} \frac{m_0^2 c^4 \tau^2}{\hbar^2} \Psi_1^2 N_\tau \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2\lambda_3^2 I_{\rm T}^2(\kappa/N_\tau)}{\gamma_\kappa^2} \cos 2\gamma_\kappa\theta, \qquad N_\tau \gg 1 \qquad (13)$$

We turn to the integral over $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$ in Eq.(4.4-4). Equation (4.4-8) yields with the help of Eqs.(1) and (2) the following result:

$$\int_0^{N_\tau} \frac{\partial\Psi^*}{\partial\theta} \frac{\partial\Psi}{\partial\theta} d\zeta = 4\lambda_1^2\lambda_3^2\Psi_1^2 \left(\int_0^{N_\tau} \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta] d\zeta \right.$$

$$- 2 \sum_{\kappa=1}^{N_\tau} I_{\rm T}(\kappa/N_\tau) \left(\cos\gamma_\kappa\theta \cos\lambda_1\lambda_3\theta \right.$$

$$\left. + \frac{1}{\gamma_\kappa} \sin\gamma_\kappa\theta \sin\lambda_1\lambda_3\theta \right) \int_0^{N_\tau} \exp[-(\lambda_2^2 - \lambda_1^2)^{1/2}\zeta] \sin\frac{2\pi\kappa\zeta}{N_\tau} d\zeta$$

$$+ \frac{1}{2} \sum_{\kappa=1}^{N_\tau} \frac{\lambda_1^2\lambda_3^2 I_{\rm T}^2(\kappa/N_\tau)}{\gamma_\kappa^2} (1 - \cos 2\gamma_\kappa\theta) \int_0^{N_\tau} \sin^2\frac{2\pi\kappa\zeta}{N_\tau} d\zeta$$

$$+ \sum_{\kappa=1}^{N_\tau,\neq j} \sum_{j=1}^{N_\tau} \frac{\lambda_1^2\lambda_2^2 I_{\rm T}(\kappa/N_\tau) I_{\rm T}(j/N_\tau)}{\gamma_\kappa\gamma_j} \sin\gamma_\kappa\theta \sin\gamma_j\theta \int_0^{N_\tau} \sin\frac{2\pi\kappa\zeta}{N_\tau} \sin\frac{2\pi j\zeta}{N_\tau} d\zeta$$

$$+ \frac{1}{2} \sum_{\kappa=1}^{N_\tau} I_{\rm T}^2(\kappa/N_\tau)(1 + \cos 2\gamma_\kappa\theta \int_0^{N_\tau} \sin^2\frac{2\pi\kappa\zeta}{N_\tau} d\zeta$$

$$\left. + \sum_{\kappa=1}^{N_\tau,\neq j} \sum_{j=1}^{N_\tau} I_{\rm T}(\kappa/N_\tau) I_{\rm T}(j/N_\tau) \cos\gamma_\kappa\theta \cos\gamma_j\theta \int_0^{N_\tau} \sin\frac{2\pi\kappa\zeta}{N_\tau} \sin\frac{2\pi j\zeta}{N_\tau} d\zeta \right) \qquad (14)$$

We substitute the integrals from Eqs.(4), (5) and (8):

$$\int_0^{N_\tau} \frac{\partial\Psi^*}{\partial\theta} \frac{\partial\Psi}{\partial\theta} d\zeta = \Psi_1^2 N_\tau \lambda_1^2\lambda_3^2 \left(4 \frac{1 - \exp[-2(\lambda_2^2 - \lambda_1^2)^{1/2} N_\tau]}{2\,(\lambda_2^2 - \lambda_1^2)^{1/2}\, N_\tau} \right.$$

$$
\begin{aligned}
&-8\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}(\kappa/N_\tau)\left(\cos\gamma_\kappa\theta\cos\lambda_1\lambda_3\theta+\frac{1}{\gamma_\kappa}\sin\gamma_\kappa\theta\sin\lambda_1\lambda_3\theta\right)\\
&\qquad\times\frac{(2\pi\kappa/N_\tau)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau}\\
&+\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1^2\lambda_3^2 I_{\mathrm{T}}^2(\kappa/N_\tau)}{\gamma_\kappa^2}(1-\cos 2\gamma_\kappa\theta)+\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)(1+\cos 2\gamma_\kappa\theta)\Bigg) \qquad (15)
\end{aligned}
$$

As in the case of Eq.(11) there are terms that do not depend on θ and others that vary with sinusoidal functions of θ. We denote the constant terms again with a subscript c:

$$
\begin{aligned}
U_{\mathrm{c}2}&=\frac{L^2}{c\tau}\left(\int_0^{N_\tau}\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}d\zeta\right)_c=\frac{L^2}{c\tau}\Psi_1^2N_\tau\lambda_1^2\lambda_3^2\\
&\quad\times\left[4\,\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N_\tau}+\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)\left(\frac{\lambda_1^2\lambda_3^2}{\gamma_\kappa^2}+1\right)\right]\\
&\doteq\frac{L^2}{c\tau}\Psi_1^2N_\tau\lambda_1^2\lambda_3^2\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)\left(\frac{\lambda_1^2\lambda_3^2}{\gamma_\kappa^2}+1\right),\qquad N\gg 1 \qquad (16)
\end{aligned}
$$

The remaining terms with sinusoidal functions of θ and time-average zero are denoted with a subscript v:

$$
\begin{aligned}
U_{\mathrm{v}2}(\theta)&=\frac{L^2}{c\tau}\Psi_1^2N_\tau\lambda_1^2\lambda_3^2\\
&\quad\times\Bigg[-8\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}(\kappa/N_\tau)\frac{(2\pi\kappa/N_\tau)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau}\\
&\qquad\qquad\times\left(\cos\gamma_\kappa\theta\cos\lambda_1\lambda_3\theta+\frac{1}{\gamma_\kappa}\sin\gamma_\kappa\theta\sin\lambda_1\lambda_3\theta\right)\\
&\qquad\qquad-\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)\left(\frac{\lambda_1^2\lambda_3^2}{\gamma_\kappa^2}-1\right)\cos 2\gamma_\kappa\theta\Bigg]\\
&\doteq\frac{L^2}{c\tau}\Psi_1^2N_\tau\lambda_1^2\lambda_3^2\sum_{\kappa=1}^{N_\tau} I_{\mathrm{T}}^2(\kappa/N_\tau)\left(1-\frac{\lambda_1^2\lambda_3^2}{\gamma_\kappa^2}\right)\cos 2\gamma_\kappa\theta,\qquad N\gg 1 \qquad (17)
\end{aligned}
$$

The final integral over $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$ in Eq.(4.4-4) yields with the help of Eqs.(4.4-9), (1) and (3) the following expression:

$$\int_0^{N_\tau}\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta = 2\Psi_1^2\Bigg\{\lambda_2^2(1-\cos 2\lambda_1\lambda_3\theta)\int_0^{N_\tau}\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta$$

$$-4\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta\Bigg(\lambda_1^2\int_0^{N_\tau}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N_\tau}d\zeta$$

$$-\frac{2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}}{N_\tau}\int_0^{N_\tau}\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\cos\frac{2\pi\kappa\zeta}{N_\tau}d\zeta\Bigg)$$

$$+2\Bigg[\frac{1}{2}\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1^2\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2(1-\cos 2\gamma_\kappa\theta)\int_0^{N_\tau}\sin^2\frac{2\pi\kappa\zeta}{N_\tau}d\zeta$$

$$+\sum_{\kappa=1}^{N_\tau,\neq j}\sum_{j=1}^{N_\tau}\frac{\lambda_1^4\lambda_3^2 I_{\mathrm{T}}(\kappa/N_\tau)I_{\mathrm{T}}(j/N_\tau)}{\gamma_\kappa\gamma_j}\sin\gamma_\kappa\theta\sin\gamma_j\theta\int_0^{N_\tau}\sin\frac{2\pi\kappa\zeta}{N_\tau}\sin\frac{2\pi j\zeta}{N_\tau}d\zeta\Bigg]$$

$$+2\Bigg[\frac{1}{2}\sum_{\kappa=1}^{N_\tau}\left(\frac{2\pi\kappa\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}\right)^2(1-\cos 2\gamma_\kappa\theta)\int_0^{N_\tau}\cos^2\frac{2\pi\kappa\zeta}{N_\tau}d\zeta$$

$$+\sum_{\kappa=1}^{N_\tau,\neq j}\sum_{j=1}^{N_\tau}\frac{4\pi^2\kappa j\lambda_1^2\lambda_3^2 I_{\mathrm{T}}(\kappa/N_\tau)I_{\mathrm{T}}(j/N_\tau)}{N_\tau^2\gamma_\kappa\gamma_j}\sin\gamma_\kappa\theta\sin\gamma_j\theta$$

$$\times\int_0^{N_\tau}\cos\frac{2\pi\kappa\zeta}{N_\tau}\cos\frac{2\pi j\zeta}{N_\tau}d\zeta\Bigg]\Bigg\} \qquad (18)$$

Substitution of the integrals of Eqs.(5) to (9) brings:

$$\int_0^{N_\tau}\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta = \Psi_1^2 N_\tau\Bigg[2\lambda_2^2(1-\cos 2\lambda_1\lambda_3\theta)\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N_\tau}$$

$$-8\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{N_\tau}\frac{\lambda_1\lambda_3 I_{\mathrm{T}}(\kappa/N_\tau)}{\gamma_\kappa}\sin\gamma_\kappa\theta$$

$$\times\Bigg(\lambda_1^2\,\frac{(2\pi\kappa/N_\tau)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau}$$

$$-\frac{2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}}{N_\tau}\frac{(\lambda_2^2-\lambda_1^2)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N_\tau)^2]N_\tau}\Bigg)$$

$$+\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1^2\lambda_3 I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2(1-\cos 2\gamma_\kappa\theta)$$

$$+\sum_{\kappa=1}^{N_\tau}\left(\frac{2\pi\kappa\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}\right)^2(1-\cos 2\gamma\kappa\theta)\Bigg] \quad (19)$$

As before in Eqs.(11) and (15) there are terms that do not depend on θ while others have sinusoidal functions of θ. The constant terms are denoted by a subscript c:

$$U_{\rm c3} = \frac{L^2}{c\tau}\left(\int_0^{N_\tau}\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta\right)_c = \frac{L^2}{c\tau}\Psi_1^2 N_\tau\Bigg\{2\lambda_2^2\,\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N_\tau}$$

$$+\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N_\tau}\right)^2\right]\Bigg\}$$

$$\doteq \frac{L^2}{c\tau}\Psi_1^2 N_\tau\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N_\tau}\right)^2\right], \qquad N\gg 1 \quad (20)$$

The terms with sinusoidal functions of θ and time-average zero are denoted with a subscript v:

$$U_{\rm v3}(\theta) = \frac{L^2}{c\tau}\Psi_1^2 N_\tau\Bigg\{-2\lambda_2^2\,\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N_\tau}\cos 2\lambda_1\lambda_3\theta$$

$$-8\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{N_\tau}\frac{2\pi\kappa\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{N_\tau\gamma_\kappa}[\lambda_1^2-(\lambda_2^2-\lambda_1^2)^{1/2}]$$

$$\times\frac{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N_\tau]}{[\lambda_2^2-\lambda_1^2-(2\pi\kappa/N_\tau)^2]N_\tau}\sin\gamma_\kappa\theta$$

$$-\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N_\tau}\right)^2\right]\cos 2\gamma_\kappa\theta\Bigg\}$$

$$\doteq -\frac{L^2}{c\tau}\Psi_1^2 N_\tau\sum_{\kappa=1}^{N_\tau}\left(\frac{\lambda_1\lambda_3 I_{\rm T}(\kappa/N_\tau)}{\gamma_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N_\tau}\right)^2\right]\cos 2\gamma_\kappa\theta, \quad N\gg 1 \quad (21)$$

6.8 Calculations for Section 5.4

The evaluation of Eq.(5.4-2) requires the integration with respect to ζ of the expressions for $\Psi^*\Psi$, $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$ and $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$ in Eqs. (5.4-3), (5.4-6) and (5.4-8). Many simple steps are required. We write them in some detail to make it easier to check their correctness. We start with Eq.(5.4-3), go to Eq.(5.4-6) and end with Eq.(5.4-8):

$$\left(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\right)^2 = \frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}(1-\cos 2\beta_\kappa\theta)\sin^2\frac{2\pi\kappa\zeta}{N}$$
$$+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)}{\sin\beta_\kappa\sin\beta_j}\sin\beta_\kappa\theta\sin\beta_j\theta\sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi j\zeta}{N} \quad (1)$$

$$\left(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)\beta_\kappa}{\sin\beta_\kappa}\cos\beta_\kappa\theta\sin\frac{2\pi\kappa\zeta}{N}\right)^2 = \frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)\beta_\kappa}{\sin\beta_\kappa}\right)^2(1+\cos 2\beta_\kappa\theta)\sin^2\frac{2\pi\kappa\zeta}{N}$$
$$+\sum_{\kappa>\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)\beta_\kappa\beta_j}{\sin\beta_\kappa\sin\beta_j}\cos\beta_\kappa\theta\cos\beta_j\theta\sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi j\zeta}{N} \quad (2)$$

$$\left(\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\frac{2\pi\kappa}{N}\sin\beta_\kappa\theta\cos\frac{2\pi\kappa\zeta}{N}\right)^2 = \frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\cos^2\frac{2\pi\kappa\zeta}{N}$$
$$+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\frac{2\pi j I_{\mathrm{T}}(j/N)}{N\sin\beta_j}\sin\beta_\kappa\theta\sin\beta_j\theta\cos\frac{2\pi\kappa\zeta}{N}\cos\frac{2\pi j\zeta}{N} \quad (3)$$

According to Eq.(5.4-2) we need the integral of $\Psi^*\Psi$ with respect to ζ. We obtain it from Eq.(5.4-3) with the help of Eq.(1):

$$\int_0^N \Psi^*\Psi d\zeta = 2\Psi_1^2\Bigg((1-\cos 2\lambda_1\lambda_3\theta)\int_0^N \exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta$$

$$
\begin{aligned}
&- 4\lambda_1\lambda_3 \cos\lambda_1\lambda_3 \sin\lambda_1\lambda_3\theta \\
&\times \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa} \sin\beta_\kappa\theta \int_0^N \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta] \sin\frac{2\pi\kappa\zeta}{N} d\zeta \\
&+ \lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}(1-\cos 2\beta_\kappa\theta) \int_0^N \sin^2\frac{2\pi\kappa\zeta}{N} d\zeta \\
&+ 2\lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j} \sum_{j>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}(\kappa/N) I_{\mathrm{T}}(j/N)}{\sin\beta_\kappa \sin\beta_j} \sin\beta_\kappa\theta \sin\beta_j\theta \\
&\qquad \times \int_0^N \sin\frac{2\pi\kappa\zeta}{N} \sin\frac{2\pi j\zeta}{N} d\zeta \Bigg) \qquad (4)
\end{aligned}
$$

The integrals are evaluated in Eqs.(6.7-5), (6.7-6) and (6.7-9) if we substitute $\iota_\kappa = 2\pi\kappa/N$, $\iota_j = 2\pi j/N$ and $N = N_\tau$:

$$
\begin{aligned}
\int_0^N \Psi^*\Psi\, d\zeta = \Psi_1^2 N \Bigg[& 2(1-\cos 2\lambda_1\lambda_3\theta) \frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2} N} \\
& - 4\lambda_1\lambda_3 \cos\lambda_1\lambda_3 \sin\lambda_1\lambda_3\theta \\
& \times \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa} \sin\beta_\kappa\theta \frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2]N} \\
& + \lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}(1-\cos 2\beta_\kappa\theta) \Bigg] \qquad (5)
\end{aligned}
$$

Some terms in Eq.(5) do not depend on θ while most vary according to sinusoidal functions of θ. The constant terms are denoted with a subscript c:

$$
\begin{aligned}
U_{\mathrm{c}1} &= \frac{L^2}{c\Delta t} \left(\int_0^N \frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2} \Psi^*\Psi\, d\zeta \right)_c = \frac{L^2}{c\Delta t} \frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2} \Psi_1^2 N \\
&\quad \times \left(2\, \frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2} N} + \lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa} \right) \\
&\doteq \frac{L^2}{c\Delta t} \frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2} \Psi_1^2 N \lambda_1^2\lambda_3^2 \sum_{\kappa>-\kappa_0}^{<\kappa_0} \frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa} \quad \text{for } N \gg 1,\ \lambda_1^2 \neq \lambda_2^2 \qquad (6)
\end{aligned}
$$

The terms with sinusoidal functions of θ and time-average equal to zero in Eq.(5) are denoted with a subscript v. We obtain:

$$
\begin{aligned}
U_{\mathrm{v}1}(\theta) &= \frac{L^2}{c\Delta t}\left(\int_0^N \frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2}\Psi^*\Psi\, d\zeta\right)_v \\
&= \frac{L^2}{c\Delta t}\frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2}\Psi_1^2 N\Bigg[-2\cos 2\lambda_1\lambda_3\theta\,\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N} \\
&\qquad -4\lambda_1\lambda_3\cos\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta \\
&\qquad \times\frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2]N} \\
&\qquad -\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}\cos 2\beta_\kappa\theta\Bigg] \\
&\doteq -\frac{L^2}{c\Delta t}\frac{m_0^2 c^4 (\Delta t)^2}{\hbar^2}\Psi_1^2 N\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}^2(\kappa/N)}{\sin^2\beta_\kappa}\cos 2\beta_\kappa\theta \quad \text{for } N\gg 1 \qquad (7)
\end{aligned}
$$

We turn to the integral over $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$ in Eq.(5.4-2). Equation (5.4-6) yields with the help of Eqs.(1) and (2):

$$
\begin{aligned}
&\int_0^N \frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}d\zeta = 4\lambda_1^2\lambda_3^2\Psi_1^2\Bigg\{\int_0^N \exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta \\
&-2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}[\lambda_1\lambda_3\sin\lambda_1\lambda_3(\theta+1)\sin\beta_\kappa\theta+\beta_\kappa\cos\lambda_1\lambda_3(\theta+1)\cos\beta_\kappa\theta] \\
&\qquad \times\int_0^N \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N}d\zeta \\
&\qquad +\lambda_1^2\lambda_3^2\Bigg[\frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\int_0^N \sin^2\frac{2\pi\kappa\zeta}{N}d\zeta \\
&+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)}{\sin\beta_\kappa\sin\beta_j}\sin\beta_\kappa\theta\sin\beta_j\theta\int_0^N \sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi j\zeta}{N}d\zeta\Bigg] \\
&\qquad +\frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)\beta_\kappa}{\sin\beta_\kappa}\right)^2(1+\cos 2\beta_\kappa\theta)\int_0^N \sin^2\frac{2\pi\kappa\zeta}{N}d\zeta
\end{aligned}
$$

$$+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)\beta_\kappa\beta_j}{\sin\beta_\kappa\sin\beta_j}\cos\beta_\kappa\theta\cos\beta_j\theta\int_0^N\sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi j\zeta}{N}d\zeta\Bigg\} \tag{8}$$

Using Eqs.(6.7-5) to (6.7-9) brings:

$$\begin{aligned}\int_0^N\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}d\zeta &= \lambda_1^2\lambda_3^2\Psi_1^2N\Bigg[4\,\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N]}{2\,(\lambda_2^2-\lambda_1^2)^{1/2}\,N}\\
&-8\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}[\lambda_1\lambda_3\sin\lambda_1\lambda_3(\theta+1)\sin\beta_\kappa\theta+\beta_\kappa\cos\lambda_1\lambda_3(\theta+1)\cos\beta_\kappa\theta]\\
&\times\frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2]N}\\
&+\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\lambda_1\lambda_3I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos2\beta_\kappa\theta)+\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\beta_\kappa I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1+\cos2\beta_\kappa\theta)\Bigg]\\
&\doteq\lambda_1^2\lambda_3^2\Psi_1^2N\Bigg[\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\lambda_1\lambda_3I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos2\beta_\kappa\theta)\\
&+\sum_{\kappa>\kappa_0}^{<\kappa_0}\left(\frac{\beta_\kappa I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1+\cos2\beta_\kappa\theta)\Bigg],\quad N\gg1\end{aligned} \tag{9}$$

As in the case of Eq.(5) there are terms that do not depend on θ and others that vary with $\cos 2\beta_\kappa\theta$. We denote the constant terms again with a subscript c:

$$\begin{aligned}U_{\mathrm{c}2} &= \frac{L^2}{c\Delta t}\left(\int_0^N\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}d\zeta\right)_c\\
&\doteq\frac{L^2}{c\Delta t}\Psi_1^2N\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(\lambda_1^2\lambda_3^2+\beta_\kappa^2),\quad N\gg1\end{aligned} \tag{10}$$

The remaining terms with functions having sinusoidal time variation and time-average zero are denoted with a subscript v:

$$U_{\mathrm{v}2}(\theta)=\frac{L^2}{c\Delta t}\left(\int_0^N \frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}d\zeta\right)_v$$

$$\doteq\frac{L^2}{c\Delta t}\Psi_1^2N\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(\beta_\kappa^2-\lambda_1^2\lambda_3^2)\cos 2\beta_\kappa\theta,\ \ N\gg 1 \qquad (11)$$

The final integral over $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$ in Eq.(5.4-2) yields with the help of Eqs.(5.4-8), (1) and (3) the following expression:

$$\begin{aligned}
\int_0^N \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta &= 2\Psi_1^2\Bigg\{\lambda_2^2(1-\cos 2\lambda_1\lambda_3\theta)\int_0^N \exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]d\zeta\\
&\quad -4\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta\sum_{\kappa>-\kappa_0}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta\\
&\quad\times\Bigg([\lambda_1^2\cos\lambda_1\lambda_3+\lambda_1(\lambda_2^2-\lambda_1^2)^{1/2}\sin\lambda_1\lambda_3]\\
&\quad\times\int_0^N \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\sin\frac{2\pi\kappa\zeta}{N}d\zeta\\
&\quad-\frac{1}{N}[2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}\cos\lambda_1\lambda_3-2\pi\kappa\lambda_1\sin\lambda_1\lambda_3]\\
&\quad\times\int_0^N \exp[-(\lambda_2^2-\lambda_1^2)^{1/2}\zeta]\cos\frac{2\pi\kappa\zeta}{N}d\zeta\Bigg)\\
&\quad+2\lambda_1^2\lambda_3^2\Bigg[\frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\lambda_1 I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\int_0^N \sin^2\frac{2\pi\kappa\zeta}{N}d\zeta\\
&\quad+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{\lambda_1^2 I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)}{\sin\beta_\kappa\sin\beta_j}\sin\beta_\kappa\theta\sin\beta_j\theta\int_0^N \sin\frac{2\pi\kappa\zeta}{N}\sin\frac{2\pi j\zeta}{N}d\zeta\\
&\quad+\frac{1}{2}\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\int_0^N \cos^2\frac{2\pi\kappa\zeta}{N}d\zeta\\
&\quad+\sum_{\kappa>-\kappa_0}^{<\kappa_0,\neq j}\sum_{j>-\kappa_0}^{<\kappa_0}\frac{4\pi^2\kappa j I_{\mathrm{T}}(\kappa/N)I_{\mathrm{T}}(j/N)}{N^2\sin\beta_\kappa\sin\beta_j}\sin\beta_\kappa\theta\sin\beta_j\theta\int_0^N \cos\frac{2\pi\kappa\zeta}{N}\cos\frac{2\pi j\zeta}{N}d\zeta\Bigg]\Bigg\}
\end{aligned} \qquad (12)$$

The required integrals are listed in Eqs.(6.7-5) to (6.7-9):

$$
\begin{aligned}
\int_0^N \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta = \Psi_1^2 N\Bigg\{ & 2\lambda_2^2(1-\cos 2\lambda_1\lambda_3\theta)\frac{1-\exp[-2(\lambda_2^2-\lambda_1^2)^{1/2}N]}{2\left(\lambda_2^2-\lambda_1^2\right)^{1/2}N} \\
& - 8\lambda_1\lambda_3\sin\lambda_1\lambda_3\theta \sum_{\kappa>-\kappa}^{<\kappa_0}\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\sin\beta_\kappa\theta \\
& \times\Bigg([\lambda_1^2\cos\lambda_1\lambda_3+\lambda_1(\lambda_2^2-\lambda_1^2)^{1/2}\sin\lambda_1\lambda_3 \\
& \times\frac{(2\pi\kappa/N)\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2]N} \\
& -\frac{1}{N}[2\pi\kappa(\lambda_2^2-\lambda_1^2)^{1/2}\cos\lambda_1\lambda_3-2\pi\kappa\lambda_1\sin\lambda_1\lambda_3] \\
& \times\frac{(\lambda_2^2-\lambda_1^2)^{1/2}\{1-\exp[-(\lambda_2^2-\lambda_1^2)^{1/2}N]\}}{[\lambda_2^2-\lambda_1^2+(2\pi\kappa/N)^2]N} \\
& +\lambda_1^2\lambda_3^2\Bigg[\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\lambda_1 I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta) \\
& +\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\doteq \Psi_1^2 N\lambda_1^2\lambda_3^2\Bigg[& \sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{\lambda_1 I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta) \\
& +\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{2\pi\kappa I_{\mathrm{T}}(\kappa/N)}{N\sin\beta_\kappa}\right)^2(1-\cos 2\beta_\kappa\theta)\Bigg], \quad N\gg 1 \qquad (13)
\end{aligned}
$$

As before in Eqs.(5) and (9) there are terms that do not depend on θ while others have a sinusoidal function of θ with time-average zero. The constant terms are denoted with the subscript c:

$$
\begin{aligned}
U_{\mathrm{c}3} &= \frac{L^2}{c\Delta t}\left(\int_0^N \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta\right)_{\mathrm{c}} \\
&\doteq \frac{L^2}{c\Delta t}\Psi_1^2 N\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2\right],\ N\gg 1 \qquad (14)
\end{aligned}
$$

The terms with sinusoidal function of θ and time-average zero are denoted with a subscript v:

$$U_{\mathrm{v3}}(\theta)=\frac{L^2}{c\Delta t}\left(\int_0^N \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta\right)_v$$
$$\doteq -\frac{L^2}{c\Delta t}\Psi_1^2 N\lambda_1^2\lambda_3^2\sum_{\kappa>-\kappa_0}^{<\kappa_0}\left(\frac{I_{\mathrm{T}}(\kappa/N)}{\sin\beta_\kappa}\right)^2\left[\lambda_1^2+\left(\frac{2\pi\kappa}{N}\right)^2\right]\cos 2\beta_\kappa\theta,\ \ N\gg 1 \tag{15}$$

References and Bibliography

Abramowitz, M. and Stegun, I.A., eds. (1964). *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55. US Government Printing Office, Washington, DC.

Aharonov, Y. and Bohm, D. (1959). Significance of electromagnetic effects in the quantum theory. *Phys. Rev.*, vol. 115, no. 3, 485–491.

Aharonov, Y. and Bohm, D. (1961). Further considerations on electromagnetic potentials in the quantum theory. *Phys. Rev.*, vol. 123, no. 4, 1511–1524.

Aharonov, Y. and Bohm, D. (1962). Remarks on the possibility of quantum electrodynamics without potentials. *Phys. Rev.*, vol. 125, no. 6, 2192–2193.

Aharonov, Y. and Bohm, D. (1963). Further discussion of the role of electromagnetic potentials in the quantum theory. *Phys. Rev.*, vol. 130, no. 4, 1625–1632.

Anastasovski, P.K., Bearden, T.E., Ciubotariu, C., Coffey, W.T., Crowell, L.B., Evans, G.J., Evans, M.W., Flower, R., Jeffers, S., Labounsky, A., Lehnert, B., Mészaŕos, M., Molnár, P.R., Vigier, J.-P., and Roy, S. (2001). Empirical evidence for non-Abelian electrodynamics and theoretical development. *Annales Foundation Louis de Broglie*, vol. 26, no. 1, 1–23.

Apostle, H.G. (1969). *Aristotle's Physics, Translated with Commentaries and Glossary.* Indiana University Press, Bloomington.

Aristotle (1930). *The Works of Aristotle, vol.II, Physics.* R.R.Hardie and R.K.Gaye transl., Clarendon Press, Oxford.

Atkinson, D. and Halpern, M.B. (1967). Non-usual topologies on space–time and high-energy scattering, *J. Math. Phys.*, vol. 8, 373–387.

Becker, R. (1963). *Theorie der Elektrizität*, vol. 2 (revised by G.Leibfried and W.Breuig), 9th ed. Teubner, Stuttgart.

Becker, R. (1964). *Electromagnetic Fields and Interactions* (transl. by A.W.Knudsen of vol. 1, 16th ed. and by I.de Teissier of vol. 2, 8th ed. of *Theorie der Elektrizität*). Blaisdell, New York. Reprinted 1982 by Dover, New York.

Belfrage, C. (1954). *Seeds of Destruction. The Truth About the US-Occupation of Germany.* Cameron and Kahn, New York. Copyrighted by the Library of Congress but not listed in the online catalog www.loc.gov either under the name Belfrage, Cedric or the book title. Available at Deutsche Staatsbibliothek, Berlin, Signatur 8-29 MA 265 or at Bayrische Staatsbibliothek, München, Germany, Signatur BAY: L 1089.

Berestezki, W.B., Lifschitz, E.M. and Pitajewski, L.P. (1970). *Relativistische Quantentheorie*, vol. IVa of *Lehrbuch der Theoretischen Physik*, L.D.Landau and E.M.Lifschitz eds.; transl. from Russian. Akademie Verlag, Berlin.

Berestetskii, V.B., Lifshitz, E.M. and Pitajevskii, L.P. (1982). *Quantum Electrodynamics*, transl. from Russian. Pergamon Press, New York.

Blum, J.M. (1965). *From the Morgenthau Diaries, vol. 2: Years of Urgency, 1938–1941.* Houghton Mifflin Co., Boston.

Blum, J.M. (1967). *From the Morgenthau Diaries, vol 3: Years of War 1941–45.* Houghton Mifflin Co., Boston. German translation by U.Heinemann and I.Goldschmidt: *Deutschland ein Ackerland? (The Morgenthau Diaries).* Droste Verlag und Druckerei, Düsseldorf 1968.

Bub, J. (1999). *Interpreting the Quantum World.* Cambridge University Press, Cambridge.

Callender, C. and Hugget, N. (2000). *Physics Meets Philosophy at the Planck Scale.* Cambridge University Press, Cambridge.

Cao, T.Y. (1999). *Conceptual Foundations of Quantum Field Theory.* Cambridge University Press, Cambridge.

Chuang, I.L. and Nielsen, M.A. (2000). *Quantum Computation and Information.* Cambridge University Press, Cambridge.

Cole, E.A.B. (1970). Transition from continuous to a discrete space–time scheme. *Nuovo Cimento*, vol. 66A, 645–655.

Cole, E.A.B. (1971). Cellular space–time structures and the principle of covariance. *Nuovo Cimento*, vol. 18A, 445–458.

Cole, E.A.B. (1972a). The observer- dependence of cellular space-time structures. *Intern. J. Theor. Phys.*, 5, 3–14.

Cole, E.A.B. (1972b). The classification of displacements and rotations in a cellular space-time. *Intern. J. Theor. Phys.*, 5, 437–446.

Cole, E.A.B. (1973a). Perception and operation in the definition of observables. *Int. J. Theor. Phys.*, 8, 155–170.

Cole, E.A.B. (1973b). Transfromations between cellular space-time structures and the principle of covariance. *Nuovo Cimento*, 18A, 445–458.

Das, A. (1960). Cellular space–time and quantum field theory. *Nuovo Cimento*, vol. 18, 482–504.

Das, A. (1966a). Complex space–time and quantum theory of free fields. *J. Math. Phys.*, vol. 7, 52–60.

Das, A. (1966b). The quantized complex space-time and quantum theory of free fields. *J. Math. Phys.*, 7 ,52–60.

Das, A. (1966c). Complex space–time and geometrization of electromagnetism. *J. Math. Phys.*, vol. 7, 61–63.

Dickson, W.M. (1998). *Quantum Chance and Nonlocality.* Cambridge University Press, Cambridge.

Einstein, A. and Infeld, L. (1938). *The Evolution of Physics*, Simon and Schuster, New York.

Flint, H.T. (1948). The quantization of space and time. *Phys. Rev.*, vol. 74, 209–210.

Fushchich, W.I. and Nikitin, A.G. (1987). *Symmetries of Maxwell's Equation* (Transl. of Russian Original of 1983 by J.R.Schulenberger). Kluwer Academic/Reidel Publishing Co., Dordrecht.

Gelfond, A.O. (1958). *Differenzenrechnung* (transl. of Russian original, Moscow 1952). Deutscher Verlag der Wissenschaften, Berlin. French edition 1963.

Ghose, P. (1999). *Testing Quantum Mechanics on New Ground.* Cambridge University Press, Cambridge.

Gradshteyn, I.S. and Ryzhik, I.M. (1980). *Tables of Integrals, Series, and Products.* Academic Press, New York.

Greiner, B. (1995). *Die Morgenthau-Legende*, Hamburger Edition HIS Verlagsgesellschaft, Hamburg. References 133 books and 39 journal articles.

Habermann, R. (1987). *Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems.* 2nd ed., Prentice Hall, Englewood Cliffs, New Jersey.

Harman, P.M. (1998). *The Natural Philosophy of James Clerk Maxwell.* Cambridge University Press, Cambridge.

Harmuth, H.F. (1977). *Sequency Theory (Foundations and Applications).* Academic Press, New York.

Harmuth, H.F. (1986a). *Propagation of Nonsinusoidal Electromagnetic Waves.* Academic Press, New York.

Harmuth, H.F. (1986b, c). Correction of Maxwell's equations for signals I, II. *IEEE Trans. Electromagn. Compat.*, vol. EMC-28, 250–258, 259–266.

Harmuth, H.F. (1989). *Information Theory Applied to Space-Time Physics* (in Russian). MIR, Moscow. English edition by World Scientific Publishing Co., Singapore 1992.

Harmuth, H.F. (1991). Comments on "Riemann-Green function solution of transient electromagnetic plane waves in lossy media". *IEEE Trans. Electromagn. Compat.*, vol. EMC-33, 72–73.

Harmuth, H.F., Barrett, T.W. and Meffert, B. (2001). *Modified Maxwell Equations in Quantum Electrodynamics*. World Scientific Publishing Co., Singapore. Copies distributed to university libraries in Eastern Europe, Asia, North Africa, and Latin America contain one extra page.

Harmuth, H.F., Boules, R.N. and Hussain, M.G.M. (1999). *Electromagnetic Signals—Reflection, Focusing, Distortions, and Their Application*. Kluwer Academic/Plenum Publishers, New York.

Harmuth, H.F. and Hussain, M.G.M. (1994). *Propagation of Electromagnetic Signals*. World Scientific Publishing Co., Singapore.

Harmuth, H.F. and Lukin, K.A. (2000). *Interstellar Propagation of Electromagnetic Signals*. Kluwer Academic/Plenum Publishers, New York. The blank page 224 was censored. Uncensored copies are available at libraries of universities in Eastern Europe, Asia, North Africa, and Latin America.

Hasebe, K. (1972). Quantum space–time. *Prog. Theor. Phys.*, vol. 48, 1742–1750.

Heitler, W, (1954). *The Quantum Theory of Radiation*, 3rd ed. Oxford University Press, London.

Hellund, E.J. and Tanaka, K. (1954). Quantized space–time. *Phys. Rev.*, vol. 94, 192–195.

Hill, E.L. (1955). Relativistic theory of discrete momentum space and discrete space–time. *Phys. Rev.*, vol. 100, 1780–1783.

Hillion, P. (1991). Remarks on Harmuth's 'Correction of Maxwell's equations for signals I'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-33, 144.

Hillion, P. (1992a). Response to 'The magnetic conductivity and wave propagation'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-34, 376–377.

Hillion, P. (1992b). A further remark on Harmuth's problem. *IEEE Trans. Electromagn. Compat.*, vol. EMC-34, 377–378.

Hillion, P. (1993). Some comments on electromagnetic signals; in *Essays on the Formal Aspects of Electromagnetic Theory*. A.Lakhtakia ed., 127–137. World Scientific Publishing Co., Singapore.

Hölder, O. (1887). Über die Eigenschaft der Gammafunktion keiner algebraischen Differentialgleichung zu genügen. *Math. Annalen*, vol. 28, 1–13.

Icke, V. (1995). *The force of Symmetry*. Cambrdidge University Press, Cambridge.

Landau, L.D. and Lifschitz, E.M. (1966). *Lehrbuch der Theoretischen Physik* (German edition of Russian eoriginal by G.Heber). Akademie-Verlag, Berlin.

Landau, L. and Peierls, R. (1931). Erweiterung des Unbestimmtheitsprinzips für die relativistische Quantentheorie. *Z. Phys*, vol. 69, 56–69.

Leader, E. and Predazzi, E. (1996). *An Introduction to Gauge Theories and Modern Particle Physics*. Cambridge University Press, Cambridge.

Levy, H. and Lessman, F. (1961). *Finite Difference Equations*. McMillan, New York.

Lucas, J.R. (1973). *A treatise on Time and Space*. Methuen, London.

March, A. (1948). *Natur und Erkenntnis*. Springer-Verlag, Vienna.

March, A. (1951). *Quantum Mechanics of Particles and Wave Fields*. Wiley, New York.

Marsal, F. (1989). *Finite Differenzen und Elemente: numerische Lösung von Variationsproblemen und Partiellen Differentialgleichungen*. Springer-Verlag, Berlin.

Milne-Thomson, L.M. (1951). *The Calculus of Finite Differences*. MacMillan, London.

Morgenthau, H. (1945). *Germany Is Our Problem*. Harper & Brothers, New York and London.

Nehrlich, G. (1976). *The Shape of Space*. Cambridge University Press, London.

Newton, I. (1971). *Mathematical Principles*. A.Motte transl., F.Cajori ed., University of California Press, Berkeley.

Nörlund, N.E. (1924). *Vorlesungen über Differenzenrechnung*. Springer-Verlag, Berlin; reprinted Chelsea Publishing Co., New York 1954.

Nörlund, N. (1929). *Equations Linéaires aux Différences Finies*. Gauthier-Villars, Paris.

Ostrowski, A. (1919). Neuer Beweis des Hölderschen Satzes, daß die Gammafunktion keiner algebraischen Differentialgleichung genügt. *Math. Annalen*, vol. 79, 286–288.

Pauli, W. (1933). Die verallgemeinerten Prinzipien der Wellenmechanik, in *Handbuch der Physik*, 2nd ed., Geiger and Scheel eds., vol. 24, part 1. Reprinted in *Handbuch der Physik*, S. Flügge ed., vol. V, part I (1958), particularly p. 10. Springer-Verlag, Berlin.

Pearl, J. (2000). *Causality—Models, Reasoning, and Inference.* Cambridge University Press, Cambridge.

Richtmyer, R.D. (1957). *Difference Methods for Initial Value Problems.* Wiley Interscience, New York.

Schild, A. (1949). Discrete space–time and integral Lorentz transformations. *Canadian J. Math.*, vol. 1, 29–47.

Smirnov, W.I. (1961). *Lehrgang der höheren Mathematik*, 4th ed. (transl. of the 12th Russian edition). Deutscher Verlag der Wissenschaften, Berlin. English edition: *A Course in Higher Mathematics.* Pergamon Press, Oxford 1964.

Smith, G.D. (1982). *Theory and Problems of Calculus of Finite Differences and Difference Equations.* McGraw-Hill, New York.

Snyder, H.S. (1947a). Quantized space–time. *Phys. Rev.*, vol. 71, 38–41.

Snyder, H.S. (1947b). The electromagnetic field in quantized space-time. *Phys. Rev.*, vol. 72, 68–71.

Spiegel, M.R. (1994). *Theory and Problems of Calculus of Finite Differences and Difference Equations*; reprint of edition of 1971. McGraw-Hill (Schaum's Outline Series), New York. *Endliche Differenzen und Differenzengleichungen*; transl., McGraw-Hill, New York 1982. A basic introduction containing useful formulas.

Tsevelik, A.M. (1996). *Quantum Theory in Condensed Matter Physics.* Cambridge University Press, Cambridge.

Weyl, H. (1921). Das Raumproblem. *Jahresberichte der Deutschen Mathematikervereinigung*, vol. 30, 92–93.

Weyl, H. (1968). *Gesammelte Abhandlungen.* Springer-Verlag, Berlin.

Weinberg, S. (1995). *The Quantum Theory of Fields: I. Introduction.* Cambridge University Press, Cambridge.

Weinberg, S. (1996). *The Quantum Theory of Fields: II. Modern Applications.* Cambridge University Press, Cambridge.

Weinberg, S. (2000). *The Quantum Theory of Fields: III. Supersymmetry.* Cambridge University Press, Cambridge.

Welch, L.C. (1976). Quantum mechanics in a discrete space–time. *Nuovo Cimento*, vol. 31B, 279–288.

Yukawa, H. (1966). Atomistics and the divisibility of space and time. *Suppl. Prog. Theor. Phys.*, vol. 37–38, 512–523.

Index