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Supersymmetric Mechanics – Vol. 2

The Attractor Mechanism and Space Time Singularities

Springer

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Preface

This is the second volume in a series of books on the general theme of supersymmetric mechanics which are based on lectures and discussions held in 2005 and 2006 at the INFN – Laboratori Nazionali di Frascati. The first volume is published as Lecture Notes in Physics **698**, *Supersymmetric Mechanics – Vol. 1: Noncommutativity and Matrix Models*, 2006 (ISBN: 3-540-33313-4).

The present one is an expanded version of the series of lectures *"Attractor Mechanism, Black Holes, Fluxes and Supersymmetry"* given by S. Ferrara at the *SSM05 – Winter School on Modern Trends in Supersymmetric Mechanics*, held at the Laboratori Nazionali di Frascati, 7–12 March, 2005. Such lectures were aimed to give a pedagogical introduction at the nonexpert level to the attractor mechanism in space-time singularities. In such a framework, supersymmetry seems to be related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the thermodynamics of black holes. The attractor mechanism determines the long-range behavior of the flows in such (dissipative) systems, characterized by the following phenomenon: when approaching the fixed points, properly named "attractors," the orbits of the dynamical evolution lose all memory of their initial conditions, although the overall dynamics remains completely deterministic. After a qualitative overview, explicit examples realizing the attractor mechanism are treated at some length; they include relevant cases of asymptotically flat, maximal and nonmaximal, extended supergravities in four and five dimensions. Finally, we shortly overview a number of recent advances along various directions of research on the attractor mechanism.

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Contents

1

Black Holes and Supergravity

These lectures deal with black holes (BHs) in different space–time (s-t) dimensions and their relation to supersymmetry (SUSY). On the same footing of monopoles, massless point-particles, charged massive particles, and so on, BHs are indeed in the spectrum of the general theories that are supposed to unify gravity with elementary particle interactions, namely superstring theory, and its generalization, called M-theory.

In general relativity (GR) a BH is nothing but a singular metric satisfying the Einstein equations. The simplest and oldest example is given by the four-dimensional (4-d) Schwarzchild (Schw.) BH metric

$$ds^2_{Schw.}(M) = \left(1 - \frac{r_g(M)}{r}\right) c^2\, dt^2 - \left(1 - \frac{r_g(M)}{r}\right)^{-1} dr^2 - r^2 d\Omega\ , \quad (1.1)$$

where $d\Omega$ is the 2-d square angular differential and $r_g(M) \equiv \frac{2G_0 M}{c^2}$ is the Schwarzchild radius of the BH (c and G_0 are the light speed in vacuum and the 4-d gravitational Newton constant, respectively; unless otherwise indicated, in the following we will choose a suitable system of units, putting $c = \hbar = G_0 = 1$).

Therefore, M being the mass of the BH, (1.1) describes a one-parameter family of static, spherically symmetric, asymptotically flat uncharged singular metrics in $d = 4$ s-t dimensions.

The metric functions diverge at two points, $r = r_g$ and $r = 0$. The first one is just a "coordinate singularity," because actually the Riemann–Christoffel (RC) curvature tensor is well-behaved there. The surface at $r = r_g$ is called event horizon (EH) of the BH. The EH is a quite particular submanifold of the 4-d Schw. background, because it is a null hypersurface, i.e., a codimension-1 surface locally tangent to the light-cone structure. Otherwise speaking, the normal four-vector n_μ to such an hypersurface is lightlike. By denoting with dx^μ the set of tangent directions to the EH, n_μ is the covariant one-tensor satisfying

$$n_\mu dx^\mu = 0, \quad 0 = n_\mu n^\mu = g^{\mu\nu} n_\mu n_\nu\ . \quad (1.2)$$

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Thus, n^{μ} is both normal and tangent to the EH, and it represents the direction along which the local light-cone structure, described by the (local) constraint $g_{\mu\nu}(x)\, dx^{\mu}\, dx^{\nu} = 0$, is tangent to the EH. From a physical perspective, the tangency between the EH and the local light-cone (and the fact that spatial sections of the EH may be shown to be compact) characterizes the EH as the boundary submanifold, topologically separating the outer part of the BH, where light can escape to infinity, from the "inner" part, where no escape is allowed.

The singular behavior of the Schw. BH is fully encoded in the limit $r \longrightarrow 0^{+}$, in which the RC tensor diverges.

The observability of such an s-t singularity may be avoided by formulating the so-called cosmic censorship principle (CCP), for which every point of the s-t continuum having a singular RC tensor should be "covered" by a surface, named event horizon, having the property of being an asymptotical locus for the dynamics of particle probes falling toward the singularity, and preventing any information going from the singularity to the rest of the universe through the horizon. This means that the region inside the EH (the "internal part" of the BH) is not in the backward light-cone of future timelike infinity.[1] In other words, the CCP forbids the existence of "naked" singularities, i.e., of directly physically detectable points of s-t with singular curvature. From this point of view, BHs are simply solutions of Einstein field equations that exhibit an EH.

The simplest way to see this in the Schw. case is to consider the radial geodesic dynamics of a pointlike massless probe falling into the BH; in the reference frame of a distant observer, such a massless probe will travel from a radius r_0 to a radius r (both bigger than r_g) in a time given by the following formula:

$$\begin{aligned}\Delta t(r) &= \int_{r_0}^{r} \frac{dt}{dr} dr = \int_{r_0}^{r} \sqrt{\frac{g_{rr}}{g_{tt}}}\, dr = \int_{r_0}^{r} \frac{dr}{1 - \frac{r_g}{r}} \\ &= (r_0 - r) + r_g \ln\left(\frac{r_0 - r_g}{r - r_g}\right) \longrightarrow \infty \text{ for } r \longrightarrow r_g^{+}. \qquad (1.3)\end{aligned}$$

Such a mathematical diverging behavior may be consistently physically interpreted in the following way. A distant observer will see the massless probe reaching the EH in an infinite time: the physically detectable dynamics of infalling physical entities will be asymptotically converging to the EH, which covers the real s-t singularity located at $r = 0$.

Two important quantities related to the EH are its area A_H and the surface gravity κ_s. A_H is simply the area of the two-sphere S^2 defined by the EH. The surface gravity κ_s, which is constant on the horizon, is related to

[1] It is worth pointing out that many of the classical features of BH dynamics should be modified by quantum effects, starting from the famous Hawking radiation process. However, such issues are outside the scope of this work, and therefore they will be omitted here

the force (measured at spatial infinity) that holds a unit test mass in place, or equivalently to the redshifted acceleration of a particle staying "still" on the horizon. More formally, κ_s may be defined as the coefficient relating the Riemann-covariant directional derivative of the horizon normal four-vector n^μ along itself to n^μ:

$$n^\nu \nabla_\nu n^\mu = \kappa_s n^\mu \ . \tag{1.4}$$

Let us now ask the following question: *may SUSY be incorporated in such a framework?*

As it is well known, GR may be made supersymmetric by adding a spin $s = \frac{3}{2}$ Rarita–Schwinger (RS) field, namely the gravitino, to the field content of the considered GR theory. The result will be the $\mathcal{N}=1$ supergravity (SUGRA) theory. It is then clear that setting the gravitino field to zero, the Schw. BH is still a singular solution of $\mathcal{N}=1$, $d=4$ SUGRA, because it is nothing but the bosonic sector of such a theory. Nevertheless, it breaks SUSY: indeed, no fermionic Killing symmetries are preserved by the Schw. BH metric background. Otherwise speaking,

$$\delta_{\varepsilon(x)} \Psi_\mu \big|_{Schw.BH} = 0 \tag{1.5}$$

has no solutions, with $\varepsilon(x)$ being the fermionic local SUSY transformation parameter, and Ψ_μ denoting the gravitino RS field.

On the other hand, in general (Riemann-)flat metric backgrounds preserve SUSY. For instance, 4-d Minkowski space preserves four supersymmetries, because in such a space there exist four constant spinors, which are actually the components of a 4-d Majorana spinor, thus allowing one to include fermionic Killing symmetries in the isometries of the considered manifold.

Summarizing, while 4-d Minkowski space preserves four supersymmetries corresponding to constant spinors, the Schw. BH background metric does not have any fermionic isometry, and therefore it breaks all SUSY degrees of freedom (d.o.f.s). Of course, due to the asymptotically Minkowskian nature of the Schw. singular metric, such SUSY d.o.f.s are restored in the limit $r \to \infty$. This feature will characterize all singular spherically symmetric, static, asymptotically Minkowskian solutions to SUGRA field equations, which we will consider in the following.

As it is well known, other (partially) SUSY-preserving BH metric solutions exist; the first ones were found long ago, in the classical Maxwell–Einstein theory. The simplest example is given by the 4-d Reissner–Nördstrom (RN) BH metric

$$\begin{aligned} ds^2_{RN}\left(M, q^2\right) = & \left(1 - \frac{r_g\left(M\right)}{r} + \frac{q^2}{r^2}\right) dt^2 \\ & - \left(1 - \frac{r_g\left(M\right)}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega, \end{aligned} \tag{1.6}$$

which reduces to Schw. BH metric when the total electric charge q of the BH vanishes. Therefore (1.6) describes a two-parameter family of spherically symmetric, static, asymptotically flat, electrically charged singular metrics in $d = 4$. In this case, beside the real s-t singularity at $r = 0$, there are two distinct "coordinate-singular" surfaces, at

$$r_{\pm} = M \pm \sqrt{M^2 - q^2} \; . \tag{1.7}$$

The outer one, placed at r_+, is called "Cauchy horizon," while the one at r_- is the proper EH. It should be reminded that Schw. and RN BHs belong to the large family of spherically symmetric, static, asymptotically flat 4-d singular metric backgrounds of Maxwell–Einstein theory. They may be obtained from the Kerr–Newman solution (describing a spherically symmetric, rotating, charged BH, and therefore parameterized by the triplet (M, q, J), with J denoting the total angular momentum) by putting $q = 0 = J$ and $J = 0$, respectively.

As it is clear from (1.7), the reality of the radii crucially depends on the ratio between the mass and the total electric charge of the RN BH. Indeed, $M^2 < q^2$ implies that no BH horizons exist at all, and therefore that the s-t RN singularity is "naked," thus being physically detectable, because the asymptotical dynamics of infalling particles would end in the singularity itself.

In order to prevent this from happening, it may explicitly be proven that the CCP is, in general, equivalent to the constraint

$$M^2 \geqslant q^2 \; . \tag{1.8}$$

Such a condition is stunningly similar to the Bogomol'ny–Prasad–Sommerfeld (BPS) bound for the stability of monopole solutions in spontaneously broken gauge theories, formulated in a suitable system of units.

When the BPS-like condition arising from the CCP is "saturated," i.e., when

$$M^2 = q^2 \; , \tag{1.9}$$

the EH and the Cauchy horizon coincide; the resulting RN BH is said to become "extremal"[2] (or "extreme"), acquiring an extra feature of $\frac{1}{2}$-BPS SUSY-enhancement. Indeed, it may be rigorously shown that an extremal RN BH preserves four supersymmetries out of the eight related to the asymptotical $\mathcal{N}= 2$ Minkowski background.[3] The appearance of the BPS-saturated bound

[2] In Sect. 4 we will give a general, equivalent characterization of extreme (and nonextreme) BHs, pointing out that extreme RN BHs are only a particular subset of the class of 4-d static, spherically symmetric, asymptotically flat extreme BHs

[3] A generalization to electrically and magnetically charged static BHs yields a BPS-like saturated bound of the kind

$$M^2 = q^2 + m^2,$$

allowing one to interpret the considered s-t singularity as a Schwinger dyonic massive particle with electric charge q and magnetic charge m (related by the Dirac–Schwinger quantization relation).

(1.9) should not be a surprise, because actually the extremal RN BH metric background is a soliton stationary solution of field equations in $\mathcal{N}=2$, $d=4$ Maxwell–Einstein supergravity theory (MESGT).

For a generic RN BH, the surface gravity reads

$$\kappa_s = \frac{1}{2}\frac{r_+ - r_-}{r_+^2} = \frac{\sqrt{M^2 - q^2}}{r_+^2} \, . \tag{1.10}$$

It is worth noticing that in the case of a Schw. BH ($q^2 = 0$, $r_+ = r_g(M)$), the usual expression for the surface gravity of a massive star is recovered:

$$\kappa_s = \frac{1}{4M} \, . \tag{1.11}$$

But the most interesting consequence of (1.10) is that the saturation (1.9) of the BPS bound implies the vanishing of the surface gravity. Actually, the extreme RN BH is just a particular example of 4-d static, spherically symmetric and asymptotically flat extreme BHs, which, within such fundamental structural features, may be characterized as the most general $(U(1))^n$-charged class of singular Riemann backgrounds with vanishing surface gravity (with $n \in \mathbb{N}$).

As it is well known, the $\mathcal{N}=2$, $d=4$ MESGT may be obtained from the classical, non-SUSY, 4-d Maxwell–Einstein theory (whose field content is given by the Riemann metric $g_{\mu\nu}$ and the Maxwell vector potential A_μ) just by adding two $s=\frac{3}{2}$ RS gravitino fields $\Psi^A_{\mu a}(x)$ ($A = 1, 2$ is the SUSY index, while μ and a are the Lorentz vector and spinor indices, respectively). Notice that in such an approach to supersymmetrization, no extra bosonic fields are introduced; consequently, all non-SUSY solutions of Maxwell–Einstein theory (including RN BH) are also solutions of $\mathcal{N}=2$ MESGT, provided that fermions are set to zero.[4]

For generic values of the couple of parameters (M, q), the RN BH does not have a regular horizon geometry, nor it preserves any of the eight supersymmetries of the local maximal $\mathcal{N}=2$, $d=4$ SUSY algebra. The necessary

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This is the first example of electric–magnetic duality, due to the $U(1)$-invariance of the classical Maxwell equations, corresponding to $SL(2,\mathbb{R})$-duality rotational covariance on the Abelian field strength F and its Hodge dual ${}^*F$. In the presence of n electric and n magnetic charges, the electric–magnetic duality group is enlarged to $Sp(2n,\mathbb{R})$ [1, 2]. As it will be seen later, the existence of dyons is strictly related to the number of s-t dimensions being considered.

In what follows we will not explicitly consider magnetic charges, but such a fact will not touch the core and the generality of the whole treatment

4 Such an argument is very powerful and versatile; for instance, it may be applied to disentangle some symmetry structures of ordinary pure QCD. In fact, such a theory (containing only gluons) may be supersymmetrized just by adding some $s=\frac{3}{2}$ fermionic fields; such an additive procedure makes nothing but explicit some hidden SUSY properties of the starting theory. For instance, this has been used in literature in the calculation of tree-level gluonic amplitude in pure QCD

condition to obtain a minimal regularity of the geometric structure in proximity of the horizon(s) is expressed by the CCP BPS-like constraint (1.8).

The eight supersymmetries related to the asymptotical maximally SUSY Minkowski background in $\mathcal{N}=2$, $d=4$ MESGT simply come from the existence of two Majorana spinors, each with four real components. Moreover, in the same way the positive energy theorem can be proved in GR with the use of SUSY, in $\mathcal{N}=2$, $d=4$ MESGT it is possible to prove the CCP by using the local SUSY algebra. Roughly speaking, we may obtain the condition $M^2 \geqslant q^2$ from the requirement of positivity of the operators appearing in the right-hand sides (r.h.s.'s) of the anticommutator of two supercharges in the RN BH metric background. The saturation of the CCP BPS-like bound (1.8) makes the RN BH "extremal," and allows one to obtain four independent solutions to the spinor Killing equations

$$\delta_{\varepsilon(x)} \Psi_\mu^A \big|_{\text{extreme RN BH}} = 0\,. \tag{1.12}$$

Thus, BPS-saturated RN BHs can be actually described in terms of massive charged particles, corresponding to $\left(M, q^2\right)$-parameterized, pointlike representations of the $\mathcal{N}=2$, $d=4$ SUSY algebra. BPS-saturation implies nothing but the extreme RN BH solution to preserve one half of the supersymmetries related to 4-d asymptotical Minkowski background.

Another fundamental feature of the $\mathcal{N}=2$ ($d=4$) extreme RN BHs is the restoration of maximal SUSY at the EH.

Denoting with $r_H \equiv r_+ = r_-$ the radius of the EH, for an arbitrary value $r > r_H$ of the radius the spherically symmetric solutions of $\mathcal{N}=2$, $d=4$ MESGT represented by extreme RN BHs preserve only one half of the eight supersymmetries related to their asymptotical limit, i.e., to the 4-d Minkowski space, and therefore to the associated $\mathcal{N}=2$, $d=4$ superPoincaré algebra. Going toward the EH, i.e., performing the limit $r \to r_H^+$, one gets a restoration of the previously lost four additional supersymmetries, reobtaining a maximally symmetric $\mathcal{N}=2$ metric background, namely the 4-d Bertotti–Robinson (BR) $AdS_2 \times S^2$ BH metric[5] [3]– [5].

It is instructive to explicitly show that the "near-horizon" limit of the extreme RN BH metric in $d=4$ is the BR metric $AdS_2 \times S^2$. First of all, let us BPS-saturate the 4-d RN BH metric given by (1.6), by simply putting $M^2 = q^2$:

$$\begin{aligned} ds^2_{RN,extreme}\left(M\right) &\equiv ds^2_{RN}\left(M, q^2\right)\big|_{q^2=M^2} \\ &= \left(1-\frac{r_g\left(M\right)}{2r}\right)^2 dt^2 - \left(1-\frac{r_g\left(M\right)}{2r}\right)^{-2} dr^2 - r^2 d\Omega. \end{aligned} \tag{1.13}$$

[5] Actually, the BR metric provides the first example of the celebrated Maldacena's AdS/CFT conjecture, namely the AdS_2/CFT1 case. Indeed, the dynamics of superstring theories in the bulk of AdS_2 may be associated with a supersymmetric conformal field theory on the 1-d boundary of such a space, i.e., with the superconformal (quantum) mechanics (see, e.g., [6] and [72])

Equation (1.13) describes a one-parameter family of static, spherically symmetric, asymptotically flat, charged singular metrics in $d = 4$. The metric functions diverge at two points, namely at $r = 0$ (real s-t singularity) and at $r_H \equiv r_g(M)/2$ (EH), where $r_g(M) \equiv \frac{2G_0 M}{c^2}$ is the Schwarzchild radius. It is worth noting that the charged nature of the extreme RN BH decreases the radial coordinate of the EH, which is now at one half of the value related to the corresponding uncharged Schw. BH with the same mass.

Redefining $r_H \equiv r'_g \equiv r_g(M)/2$, and dropping the prime and the notation of the dependence on M, we get

$$\begin{aligned} ds^2_{RN,extreme}(M) &= \left(1 - \frac{r_g}{r}\right)^2 dt^2 - \left(1 - \frac{r_g}{r}\right)^{-2} dr^2 - r^2 d\Omega \\ &= \frac{1}{r^2}(r - r_g)^2 dt^2 - r^2 (r - r_g)^{-2} dr^2 - r^2 d\Omega. \end{aligned} \tag{1.14}$$

By performing the limit $r \to r_g^+$ and considering only the leading order, we therefore obtain

$$lim_{r \to r_g^+}\left[ds^2_{RN,extreme}(M)\right] = \frac{1}{r_g^2}(r - r_g)^2 dt^2 - r_g^2 (r - r_g)^{-2} dr^2 - r_g^2 d\Omega\ . \tag{1.15}$$

The mass of the spherically symmetric BR geometry is related to the area $A_H = 4\pi r_g^2$ of its EH by the simple relation

$$M^2_{BR} = \frac{A_H}{4\pi} = r_g^2\ . \tag{1.16}$$

By substituting such a relation in (1.15), we get

$$\begin{aligned} &lim_{r \to r_g^+}\left[ds^2_{RN,extreme}(M)\right] \\ &= \frac{1}{M^2_{BR}}(r - r_g)^2 dt^2 - M^2_{BR}(r - r_g)^{-2} dr^2 - M^2_{BR} d\Omega. \end{aligned} \tag{1.17}$$

Now, by performing the change of radial variable

$$r' \equiv r - r_g \tag{1.18}$$

and dropping out the prime once again, we get the following expression:

$$lim_{r \to 0^+}\left[\left. ds^2_{RN,extreme}(M)\right|_{r^{(\prime)} \equiv r - r_g}\right] = \frac{r^2}{M^2_{BR}} dt^2 - \frac{M^2_{BR}}{r^2}\left(dr^2 + r^2 d\Omega\right)\ . \tag{1.19}$$

It is easy to recognize that this is nothing but the BR metric $AdS_2 \times S^2$, with opposite scalar curvatures for AdS_2 and S^2. Indeed, the metric given by (1.19) belongs to the general class of static 4-d black hole metrics of the kind

$$ds^2 = e^{2U(\underline{x})} dt^2 - e^{-2U(\underline{x})} d\underline{x}^2\ , \tag{1.20}$$

with $U(\underline{x})$ satisfying the 3-d D'Alembert equation

$$\Delta e^{-U(\underline{x})} = 0 \ . \tag{1.21}$$

In particular, the spherically symmetric BR metric corresponds to the choice

$$e^{-2U(\underline{x})} = \frac{A_H}{4\pi |\underline{x}|^2} = \frac{M_{BR}^2}{r^2} \ , \tag{1.22}$$

which consequently relates $U(\underline{x})$ to the Newtonian gravitational potential (see Subsects. 4.1 and 4.2).

Notice that the change of radial coordinate specified by (1.18) encodes the very relationship between the extremal RN BH and the BR metric background: indeed (1.18) yields that the real s-t singularity of the BR geometry is on the EH of the extreme RN BH, which, as previously observed, is at one half of the gravitational radius of the Schw. BH of the same mass. Consequently, the BR geometry may be seen as the "near-horizon" asymptotical metric structure of the extreme RN BH[6]; the r.h.s. of (1.19) should always be considered for small values of the radius (i.e., for $r \to 0^+$), physically corresponding to the proximity to the EH of the extreme RN BH.

The BR metric $AdS_2 \times S^2$ yielded by (1.19) corresponds to the direct product of two spaces of constant (and opposite) Riemann–Christoffel scalar curvature. Consequently, it is R-flat, and it may also be shown that it is conformally flat, i.e., that all components of the related Weyl tensor vanish. Such a peculiar feature may be made manifest by choosing a suitable system of coordinates, called "conformal coordinates," defined as follows:

$$\rho \equiv \frac{M_{BR}^2}{r} \Leftrightarrow |\underline{y}| \equiv \frac{M_{BR}^2}{|\underline{x}|} \ . \tag{1.23}$$

By exploiting such a change of coordinates, we finally get

$$\begin{aligned} lim_{\rho\to\infty}\left[ds^2_{RN,extreme}(M)\Big|_{\rho\equiv\frac{M_{BR}^2}{r}}\right] &= \frac{M_{BR}^2}{\rho^2}dt^2 - \frac{M_{BR}^2}{\rho^2}\left(d\rho^2 - \rho^2 d\Omega\right) \\ &= \frac{M_{BR}^2}{|\underline{y}|^2}\left(dt^2 - d\underline{y}^2\right) , \end{aligned} \tag{1.24}$$

which is manifestly conformally flat, as it can be also seen by explicitly checking that the Weyl tensor vanishes:

$$C_{\mu\nu\lambda\delta} = 0 \ . \tag{1.25}$$

Notice that the conformal coordinates make the conformal flatness of the BR metric manifest by giving a stereographic treatment of the singularity, because they map the real s-t singularity at $r = 0$ to the point at the infinity $\rho \to \infty$.

[6] In Sublsects. 4.1 and 4.2 we will see that such a result may be extended to a generic (4-d, static, spherically symmetric, and asymptotically flat) extreme BH

The phenomenon of the doubling of the SUSY near the EH was discovered for the first time in MESGT in [9] (see [10] for an introductory report and further References). As we will see later, it is related to the appearance of a covariantly constant on-shell superfield of $\mathcal{N}=2$ ($d=4$) SUGRA [11]. In the presence of a dilaton such a mechanism was studied in [12]. In the context of exact 4-d BHs, string theory and conformal theories on the worldsheet, the BR metric has been studied in [13]. Finally, the idea of extremal, singular p-branes metric configurations interpolating between maximally symmetric asymptotical backgrounds has been developed in [14].

Therefore, for what concerns the SUSY-preserving features of the considered extreme RN BHs, there is a strong similarity between the asymptotical ($r \to \infty$) and near-horizon ($r \to r_H^+$) limits. They share the property of corresponding to maximally SUSY metric backgrounds in four dimensions, thus preserving eight different supersymmetries, but they also deeply differ on the algebraic side. The asymptotical 4-d. Minkowski flat background is associated with the $\mathcal{N}=2$, $d=4$ superPoincaré algebra (rigid SUSY asymptotical algebra). Instead, the horizon geometry has an $AdS_2 \times S^2$ structure of direct product of two spaces with nonvanishing, constant (and opposite) curvature, and it is associated with another 4-d maximal $\mathcal{N}=2$ SUSY algebra, namely to $\mathfrak{psu}(1,1\,|2)$.

$\mathfrak{psu}(1,1\,|2)$ is an interesting example of superalgebra containing not Poincaré nor semisimple groups, but (direct products of) simple groups as maximal bosonic subalgebra (m.b.s.). Indeed, in this case the m.b.s. is $\mathfrak{so}(1,2) \oplus \mathfrak{su}(2)$, with related maximal spin bosonic subalgebra (m.s.b.s.) $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$. This perfectly matches the corresponding bosonic isometry group of the BR metric, which is nothing but the direct product of a 2-d hyperboloid and a two-sphere

$$AdS_2 \times S^2 = \frac{SO(1,2)}{SO(1,1)} \times \frac{SO(3)}{SO(2)} \,. \tag{1.26}$$

Summarizing, it may be shown that the $\mathcal{N}=2$, $d=4$ extreme RN BH is a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N}=2$, $d=4$ MESGT. It interpolates between two maximally supersymmetric metric backgrounds, namely Minkowski for $r \to \infty$ and BR for $r \to r_H^+$, related to two different 4-d $\mathcal{N}=2$ superalgebras, i.e., respectively to the rigid $\mathcal{N}=2$, $d=4$ SUSY algebra given by the superPoincaré algebra and to the $\mathfrak{psu}(1,1\,|2)$ superalgebra.[7] See Fig. 1.1 for a graphical synthesis.

[7] $\mathcal{N}=2$, $d=4$ superPoincarè and $\mathfrak{psu}(1,1\,|2)$ are the only superalgebras compatible with the constraint of asymptotically flat metric background in the considered case.

The situation drastically changes when one removes such a constraint (i.e., when generic, asymptotically Riemann geometries are considered). For example, asymptotical maximally symmetric metric configurations could be considered; among the Riemann manifolds with nonzero constant Riemann–Christoffel intrinsic scalar curvature, one of the most studied in such a framework is the anti de Sitter (AdS) space. When endowing the AdS background with some local SUSY, one obtains a particular case of the so-called “gauged” SUGRAs

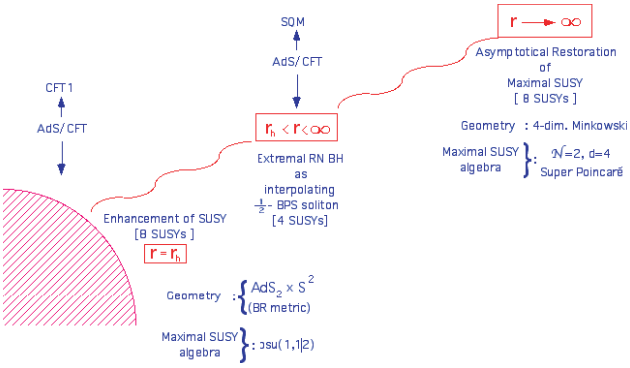

Fig. 1.1. The $d = 4$ extreme RN BH as a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N} = 2$, $d = 4$ MESGT. It interpolates between two maximally supersymmetric metric backgrounds, namely Minkowski (related to the rigid $\mathcal{N} = 2$, $d = 4$ superPoincaré algebra) for $r \to \infty$ and Bertotti–Robinson (related to the $\mathfrak{psu}(1,1\,|2)$ superalgebra) for $r \to r_H^+$. SQM stands for supersymmetric (but not superconformal) quantum mechanics, related by ADS/CFT correspondence to the interpolating regime of the considered RN extremal BH

There exists an interesting connection with the statistical mechanics of dynamical systems, which will be amply treated in the following sections; here we anticipate that the radius r_H of the EH of the extreme RN BH may be considered as an "attractor" for the evolution dynamics of the (scalar fields of the) physical system being considered, corresponding to the restoration of maximal SUSY.

Generalizations of the previous treatment to the case of p-d objects in d s-t dimensions are also possible. Nevertheless, as we will discuss later, it may be shown that for $d \geqslant 6$ it is not possible to have regular (generalized) Horizon geometries, and the calculations of the entropy of the considered (possibly extended) s-t singularities always give vanishing (or unphysical constant) results. The aforementioned case of the extreme RN BH is a particular example of $p = 0$-d brane in $d = 4$ s-t dimensions, and, as shown by Gibbons and Townsend in [14]

In general, a p-d extreme black brane in d s-t dimensions is an extended p-d object saturating a suitable generalization of the BPS bound (1.9), for which the $(p+1)$-d. generalization of EH may be introduced, together with a dimensionally extended version of the CCP. Also notice that in this case the real s-t singularity extends over a p-d. (hyper)volume in s-t. The near-horizon asymptotical geometry of a p-d. black brane is given by the (p,d)-generalization of BR metric, namely by the direct product

$$AdS_{p+2} \times S^{d-p-2}\,. \tag{1.27}$$

In general, the request of asymptotically Minkowski d-d s-t geometry in presence of a p-brane implies the consistence condition [15]

$$p < d - 3 \, . \tag{1.28}$$

Moreover, in d s-t dimensions an electric p-brane has a $(d-p-4)$-brane as magnetic dual. In the particular case in which the dimensions of an electric brane and of its magnetic dual coincide, namely when the couple (p, d) satisfies the condition

$$\frac{d}{2} = p + 2 \, , \tag{1.29}$$

the considered p-brane can be dyonic; i.e., it may have both electric and magnetic charges, respectively denoted with e and m. Finally, when the p satisfying the dyonic condition (1.29) is odd, the related p-brane may be self- (or anti-self-)dual, i.e., with $e = \pm m$, depending on the projective (or antiprojective) nature of the Hodge $*$-operator

$$(*)^2 = \pm \mathbb{I} \, . \tag{1.30}$$

Therefore, in $d = 4$ the only possible choice is $p = 0$, corresponding to the extreme BHs. Moreover, the couple $(p, d) = (0, 4)$ satisfies the dyonic condition (1.29), but p is not odd. Consequently, in $d = 4$ the 0-brane may be dyonic, but not self- (or anti-self-) dual. In other words, the extreme BH, such as the extreme RN one, may have simultaneously electric and magnetic charges, but they will not be related by the simple relation $e = \pm m$.

For $d = 5$ the condition (1.28) yields $p = 0, 1$ as allowed values. The relation (1.29) is never satisfied, and therefore 5-d dyons do not exist.

1. $p = 0$ corresponds to the Tangherlini extreme BH [21, 22]; its near-horizon geometry corresponds to $AdS_3 \times S^2$, admitting two Killing spinors. Moreover, by AdS/CFT it corresponds to completely solvable superconformal field theory (SCFT2) on the 2-d Minkowski manifold corresponding to the boundary of AdS_3.
2. $p = 1$ corresponds to a "black-string" in five dimensions, which is the magnetic dual of the Tangherlini extreme BH. It has an $AdS_2 \times S^3$ near-horizon geometry and, by application of the AdS/CFT correspondence, it yields a completely solvable superconformal quantum mechanics (SCFT1).

The most famous realization of Maldacena's AdS/CFT correspondence [193] (for a comprehensive review, see e.g. [194]) is given by the 10-d manifold $AdS_5 \times S^5$. By the previous reasonings, this may correspond to the near-horizon geometry of a three-black brane in a 10-d s-t. It is worth noticing that, by the previous analysis, in $d = 10$ the asymptotical flatness implies $0 \leqslant p \leqslant 6$, and the dyonic condition (1.29) holds true for the odd value $p = 3$. Therefore, a three-black brane in $d = 10$ may be dyonic, with $e = \pm m$, depending on the projectivity (or antiprojectivity) of the 10-d Hodge $*$-operator.

Actually, $AdS_5 \times S^5$ describes the 'near-horizon geometry of a D3-brane in $\mathcal{N}= 2$, $d = 10$ Type IIB SUGRA.[8] In such a context, the flat asymptotical

[8] We do not consider Type IIA SUGRA simply because it does not admit D3-black branes as solutions. In general, the p-d black-brane solutions have p even in IIA and p odd in IIB theories

($r \to \infty$) geometry is the 10-d Minkowski space with the associated maximally symmetric $\mathcal{N}=2$, $d=10$ rigid superPoincaré algebra (32 supersymmetries, related to the existence of two Majorana–Weyl spinors, each with 16 real components). On the other side, also $AdS_5 \times S^5$ is maximally supersymmetric, being related to the $\mathfrak{psu}(2,2\,|4)$ superalgebra[9] (with 32 real fermionic generators).

$\mathfrak{psu}(2,2\,|4)$ is another example of superalgebra containing not Poincaré nor semisimple groups, but (direct products of) simple groups as m.b.s.; indeed, in this case the m.b.s. and m.s.b.s. are respectively $\mathfrak{so}(4,2) \oplus \mathfrak{so}(6)$ and $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$, and there is a perfect matching with the corresponding bosonic isometry group of $AdS_5 \times S^5$, which is nothing but the direct product of a 5-d hyperboloid and a five-sphere

$$AdS_5 \times S^5 = \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} \,. \tag{1.31}$$

Notice that the isometry group $SO(4,2)$ of AdS_5 is nothing but the conformal group in four dimensions, i.e., the symmetry group of the $\mathcal{N}=4$ super Yang–Mills (SYM) gauge theory on the 4-d Minkowski space corresponding to the boundary of the 5-d hyperboloid AdS_5. Thus, the conformally invariant 4-d $\mathcal{N}=4$ SYM gauge theory stands to the embedding of a D3-black brane in a 10-d (asymptotically flat) s-t, as the superconformal quantum mechanics ($SC\,(Q)\,M = CFT1$) stands to an extreme BH, eventually of the extremal RN type treated above, in 4-d (asymptotically flat) s-t.

Such cases are different realizations of the AdS/CFT enlightenment, conjecturing a close (holographic) duality between gravity theories (superstrings and their low-energy limit given by SUGRA theories) in the bulk of AdS manifolds and strongly coupled, conformally invariant gauge theories on the flat Minkowskian boundaries of such spaces.

Thus, as shown in Fig. 1.2, the considered asymptotically flat D3-black bran is a soliton solution of $\mathcal{N}=2$, $d=10$ Type IIB SUGRA, which interpolates between two maximally supersymmetric metric backgrounds, namely Minkowski at $r \to \infty$ (by construction) and $AdS_5 \times S^5$ (which may be seen as a higher dimensional generalization of BR metric) in the near-horizon limit. It corresponds to a consistent $\frac{1}{2}$-BPS solution, therefore preserving 16 supersymmetries out of the 32 related to the maximally SUSY backgrounds.

[9] The considered Lie superalgebras $\mathfrak{psu}(1,1\,|2)$ and $\mathfrak{psu}(2,2\,|4)$ belong to the so-called unitary series of superalgebras $\mathfrak{psu}\,(n_1,n_2|m)$, admitting $\mathfrak{su}\,(n_1,n_2) \oplus \mathfrak{su}\,(m) \oplus (1-\delta_{n_1+n_2,m})\,\mathfrak{u}(1)$ as m.s.b.s.

In general, Lie SUSY algebras admit a classification similar to their nonsupersymmetric counterparts (see, e.g., [16]– [20]). For instance, besides the exceptional cases, another infinite series of Lie superalgebras is the orthosymplectic one, namely $\mathfrak{osp}\,(n_1,n_2|m)$, admitting $\mathfrak{so}(n_1,n_2) \oplus \mathfrak{sp}(2m)$ as m.s.b.s.

In general, the fermionic generators are in the bifundamental representation of the corresponding superalgebra, e.g., in (n_1+n_2,m)-repr. for both $\mathfrak{psu}\,(n_1,n_2|m)$ and $\mathfrak{osp}\,(n_1,n_2|m)$

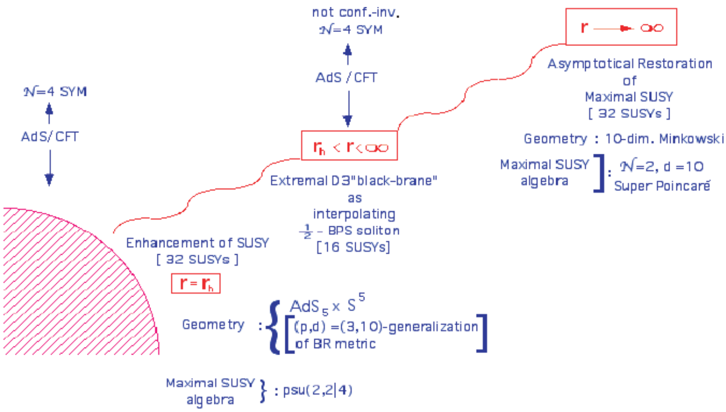

Fig. 1.2. The asymptotically flat D3-black brane as a $\frac{1}{2}$-BPS SUSY-preserving soliton solution in $\mathcal{N}=2$, $d=10$ Type IIB SUGRA. It interpolates between two maximally supersymmetric metric backgrounds, namely 10-d Minkowski (related to the rigid $\mathcal{N}=2$, $d=10$ superPoincaré algebra) for $r \longrightarrow \infty$ and $AdS_5 \times S^5$ (related to the $\mathfrak{psu}(2,2\,|4)$ superalgebra) for $r \longrightarrow r_H^+$

It is worth noticing that such a $\frac{1}{2}$-BPS solution can still be interpreted in terms of a $\mathcal{N}=4$ SYM gauge theory, but the conformal invariance is lost (or better, spontaneously broken) for a generic value of[10] $r_H < r < \infty$. This is due to the fact that for $r_H < r < \infty$ the $\mathcal{N}=4$ SYM gauge theory "living" on the boundary may be approximately described in terms of a Born-Infeld action, containing higher order derivative terms which (spontaneously) break conformal invariance.[11] The conformal invariance of the 4-d $\mathcal{N}=4$ SYM gauge theory defined on the boundary manifold is restored only in the near-horizon limit, i.e., when $r \longrightarrow r_H^+$, and therefore when the bulk tends to a direct product structure $AdS_5 \times S^5$. The restoration of the maximal supersymmetry of the metric background at the (generalized) EH (from 16 to 32 preserved supersymmetries) yields an enhancement of the symmetry features exhibited by the (holographically) related "boundary" (strongly coupled) $\mathcal{N}=4$ SYM gauge theory, which correspondingly becomes conformally invariant.

Concluding, in d-d $\mathcal{N}$-extended SUGRAs there exist stable (i.e., BPS-saturated), static, spherically symmetric, asymptotically flat $p\,(< d-3)$-d solitonic metric background solutions. They interpolate between two maximally supersymmetric backgrounds, namely the d-d flat Minkowski space in the limit $r \longrightarrow \infty$, and the d-d generalized BR geometry. This latter is obtained in the near-horizon limit $r \longrightarrow r_H^+$, and it may be expressed by the direct product

[10] r_H now stands for (the set of parameters specifying) the suitable generalization of the EH in the case of spatially extended s-t singularities embedded in higher dimensions

[11] Such a mechanism may actually be understood also in terms of non-linear realizations (see [195, 196] and references therein).

of a constant, (strictly) negative-curvature space (the $(p+2)$-d hyperboloid, or anti de Sitter space $AdS_{p+2} = \frac{SO(p+1,2)}{SO(p+1,1)}$) and of a constant, (strictly) positive-curvature space (the $(d-p-2)$-d. sphere $S^{d-p-2} = \frac{SO(d-p-1)}{SO(d-p-2)}$).

Depending on the number of (real) supersymmetries preserved by the maximal backgrounds (and therefore depending on d and $\mathcal{N}$), the interpolating solitonic solutions may have different BPS SUSY-preserving features. Despite being extremal (i.e., saturating-a suitable generalization of-the BPS-like bound (1.8)), they may also be non-BPS; i.e., they may also not preserve any of the supersymmetries of the two regimes considered above. For example, in 4-d $\mathcal{N} = 8$-extended SUGRA (having 32 real fermionic generators) we may have $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS, $\frac{1}{8}$-BPS, and non-BPS stable (i.e., BPS-saturated) singular solitonic metric backgrounds, with 16, 8, 4, and 0 supersymmetries preserved out of 32, respectively.

As we will see in Sect. 6, it is possible to classify the BPS-preservation features of such solutions in an invariant way, using the lowest order invariants and the orbits of the U-duality symmetry groups of the starting SUGRA theory.

In the cases which we will overview in Sect. 6, such groups are Lie noncompact exceptional groups of various ranks. They correspond to the isometry groups of the manifold of the nonlinear sigma model related to the relevant set of scalar fields. Such a manifold is nothing but a (particular type of) moduli space of the considered SUGRA theory. As it will hopefully be clearer later, the process of restoration of maximal SUSY in the near-horizon dynamics of the considered system is deeply related to the so-called attractor mechanism in the moduli space.

2

Attractors and Entropy

There exists an impressive coincidence between the laws of thermodynamics and the laws of BH mechanics. As it is well known, the first law of thermodynamics reads

$$\delta E = T\delta S - p\delta V \ , \tag{2.1}$$

and expresses the total variation of the energy E as equal to the temperature T times the variation of the entropy S, plus other work terms, such as a term proportional (through the pressure p) to the change of the volume V of the considered system. The corresponding formula for BHs is [23]

$$\delta M = \frac{\kappa_s}{4\pi}\frac{\delta A_H}{4} + \phi\delta q + \omega\delta J \ . \tag{2.2}$$

It states that the variation of the mass M of the BH is related to the variation of the EH area A_H, with two kinds of additional terms: a work term proportional (through the rotational angular frequency ω) to the variation of the total angular momentum J, and another term proportional (through the electric/magnetic potential ϕ evaluated at the horizon) to the variation of the charge q. Hawking [24–26] has shown that $\frac{\kappa_s}{4\pi}$ can be interpreted precisely as the temperature of the BH:

$$T_{BH} = \frac{\kappa_s}{4\pi} \ . \tag{2.3}$$

Therefore, by comparing (2.1) and (2.2), one obtains the famous Bekenstein–Hawking entropy–area (BHEA) formula, relating the entropy S of a s-t singularity with the area A_H of its EH (that should be always there, if one forbids the existence of "naked" singularities by advocating the CCP):

$$S = \frac{A_H}{4} \ . \tag{2.4}$$

In (2.2) and (2.4) the various quantities have been defined in Planck units, namely they have been made dimensionless by multiplication with an appropriate power of Newton's constant G_0 (recall, we set $\hbar = c = G_0 = 1$). By

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recalling that such a constant appears in the Einstein–Hilbert Lagrangian density

$$\mathcal{L}_{EH} = \frac{1}{16\pi G_0}\sqrt{|g|}R\ , \tag{2.5}$$

it is clear that the chosen normalization makes all quantities appearing in the first law of BH mechanics independent of the scale of the metric.

In the case of extreme BHs in SUGRA theories, formula (2.4) may be macroscopically determined by using the U-duality symmetries of string theories encoded in the SUGRA low-energy actions.[1] More specifically, the classical Einstein–Maxwell theory may be naturally embedded into $\mathcal{N}=2$ MESGT, leading to extensions involving a number of Abelian gauge fields and a related variety of massless scalar moduli fields. The BH mass M will, in general, depend on the values taken by the moduli at the spatial infinity, and therefore additional terms on the r.h.s. of (2.2) will appear.

For Schw. BHs the only relevant parameter is clearly the mass M, and, beside (1.11), we get the relation[2]

$$A_H = 16\pi M^2 = 4\pi r^2_{H,Schw.}\ , \tag{2.6}$$

where $r_{H,Schw.} \equiv r_g\left(M\right) \equiv 2M$. By differentiation, (2.6) is consistent with (2.2) constrained by $\left(\delta\right)q = 0 = \left(\delta\right)J$.

For the RN BH, the situation is more involved, due to the previously performed classification based on the ratio between M and q. As previously pointed out, for extreme RN BHs (i.e., with $M = |q|$), the surface gravity vanishes; the other relevant relations read

$$A_H = 4\pi M^2 = 4\pi r^2_{H,extreme\ RN}, \quad \phi = \sqrt{\frac{4\pi}{A_H}}q = \frac{q}{r_{H,extreme\ RN}}\ , \tag{2.7}$$

where $r_{H,extreme\ RN} \equiv M = \frac{r_{H,Schw.}}{2}$. As it has to be, by differentiating, we obtain consistence with (2.2) constrained in the subspace of static, extreme RN BHs (i.e., with $\delta J = 0$ and $\delta M = \delta q$). Since in this case $\kappa_s = 0$, and therefore the extreme RN BHs, as all extreme BHs, have $T_{BH} = 0$, by the "BH counterpart" of the third law of thermodynamics one would expect that the entropy vanishes. Clearly, this is not the case, because (2.7) yields that the area of the horizon remains finite for zero surface gravity (and thus, by (2.3), for $T_{BH} = 0$), and the BHEA (2.4) still holds, yielding[3]

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[1] We recall here the work of Cvetic and coworkers [197–200], where the BH entropy was computed making use of invariants. For an exhaustive review on BHs in string theory, see e.g. [201].

Moreover, it should be here mentioned the noteworthy, extremely symmetric case of the so-called *stu* BHs, whose *triality symmetry* has been investigated in a number of works (see e.g. [202–205]).

[2] In all spherically symmetric 4-d BHs $A_H = 4\pi r^2_H$, where r_H is the radius of the EH of the BH

[3] See Subsect. 4.2 for further insights

$$S_{BH} = \pi M^2 = \pi r^2_{H,extreme\ RN} \ . \tag{2.8}$$

These lectures deal with a general dynamical principle, named "attractor mechanism" (AM), which governs the dynamics inside the moduli space, and therefore allows one to determine the BH entropy through the special role that the moduli of the theory have in (generalized) BR geometries. In such a framework, SUSY is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system.[4] When the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) systems is the following: in approaching the fixed points, properly named "attractors," the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of extreme BHs in $\mathcal{N}= 2$, $d = 4$ and 5 MESGTs coupled with matter multiplets (namely, Abelian vector multiplets and hypermultiplets) [27, 28]. The corresponding dynamical system to be considered in this case is the one related to the radial evolution of the configurations of the relevant set of scalar fields of such theories (in this case, as it will be explained later, only the scalars from the vector multiplets have to be taken into account for the dynamics in the "near-horizon" limit).

Otherwise speaking, we have to consider the behavior of the moduli fields of the theory as they approach the core of the s-t singularity. When reaching the proximity of the EH, they dynamically run into fixed points, getting some fixed values which are only function (of the ratios) of the electric and magnetic charges of the configuration of Abelian Maxwell vector potentials being considered.

The inverse distance to the horizon is the fundamental evolution parameter in the dynamics toward the fixed points represented by the "attractor configurations" of the moduli. Such "near-horizon" configurations of the moduli, which "attracts" the dynamical evolutive flows in the moduli space, are completely independent of the initial data of such an evolution, i.e., on the asymptotical ($r \to \infty$) configurations of the moduli. Therefore, for what concerns the dynamics of the moduli, the system completely loses memory of its initial data, because the dynamical evolution will be "attracted" by some

[4] We recall that a point x_{fix} where the phase velocity $v\left(x_{fix}\right)$ vanishes is called a fixed point, and it gives a representation of the considered dynamical system in its equilibrium state,

$$v\left(x_{fix}\right) = 0 \ .$$

The fixed point is said to be an *attractor* of some motion $x\left(t\right)$ if

$$\lim_{t\to\infty} x(t) = x_{fix} \ .$$

fixed configuration points, exclusively depending on the electric and magnetic charges of the Maxwell vector field content of the theory.

Thus, there is a substantial (and irreversible) loss of physical information in the motion of moduli configurations toward the EH of the extreme BHs, which therefore may be considered as dissipative dynamical systems from an information theory perspective (for recent developments along this line, see, e.g., [29]).

Now, it should be reminded that there exists an interesting phenomenon in the physics of BHs, described by the so-called no-hair theorem: there is a limited number of parameters describing (geo)metric structures and physical fields far away from the s-t singularity represented by the BH, i.e., in the $r \to \infty$ limit. In other words, the spatial asymptotical configurations of BH metric are finitely determined.

In the framework of SUGRA theories extreme BHs may be interpreted as BPS-saturated interpolating metric singularities in the low-energy effective limit of higher dimensional superstring or M theory. Their asymptotically relevant parameters include the mass, the (conserved, quantized) electrical and magnetic charges (defined by integrating the fluxes of related field strengths over two-spheres at the infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields.

From what shortly mentioned above, we may generalize and strengthen the no-hair theorem for extreme BHs in SUGRA theories, stating that such BHs lose all their "scalar hair" near the EH.[5] This means that the extreme BH metric solutions, in the "near-horizon" limit in which they approach the BR metric, are characterized only by those discrete (quantized) parameters which correspond to the conserved charges associated with the gauge symmetries of the theory, but not by the continuously varying asymptotical values of the (dynamically relevant set of) scalar fields.

Thence, it appears evident that our ability to make (microscopic) sense of the entropy of a BPS-saturated BH in SUGRA is deeply based on the AM.

Indeed, by such a general dynamical principle, starting from unconstrained, continuously varying scalar field configurations, in the "near-horizon" limit $r \to r_H^+$ we obtain some discrete, "attractor" field configurations, completely independent of the initial data of the evolution, but instead totally determined by the conserved charges of the system.

The change of the nature (continuous→discrete, quantized) of the scalar field configurations approaching the EH allows one to consistently define the concept of entropy of an extreme s-t singularity, at least in a microscopic approach. Indeed, being the moduli some continuous parameters which can be freely specified in the asymptotical Minkowskian metric background of the theory, in general one could think that the entropy might depend on such values. Such a dependence on unconstrained values of the moduli would

[5] As it will be shown in Subsect. 4.2, such a phenomenon holds, under certain conditions, also in generic, non (necessarily)-supersymmetric frameworks

presumably lead to a possible violation of the second law of thermodynamics. Indeed, due to the functional moduli-dependence exhibited by the entropy, it might be possible to quasi-statically decrease it by performing infinitesimal transformations in the moduli space. Thanks to the AM, the entropy actually depends only on the values of the moduli at the EH of the BH, and such "attractor configurations" of the moduli turn out to be insensitive to the asymptotical continuous moduli configurations. Therefore, the BH entropy ends being a function purely of the (quantized) conserved charges of the system.

At this point, one could (and should) ask the following question: *how the initial-data-independent "attractor" moduli configurations are fixed?*

A priori, one would expect that the answer would be completely model-dependent, i.e., that such fixed, quantized values of the "near-horizon" moduli configurations would (strictly) depend on the features of the dynamical dissipative system given by the evolution of the (dynamically relevant set of) scalar fields in the moduli space. In other words, one would expect that such an answer would (heavily) rely on the (geo)metrical structure of the moduli space of the considered SUGRA theory.

But actually this is *not* the story. Indeed, at least in supersymmetric frameworks, the AM characterizes the "attractors" as stable fixed points corresponding to the critical points of the absolute value of the "central charge function" Z in the moduli space. This is a universal, model-independent feature of the "attractors."[6] The area A_H of the EH is proportional to the square of such an absolute value, computed at the point where it is extremized in the moduli space [30].

Let us denote with $\{\varphi\}$ a configuration of the relevant set of scalar fields of the considered SUGRA theory. $\{\varphi\}$ will correspond to a point in the moduli space M and, in general, it will depend on the continuously varying, unconstrained initial configuration $\{\varphi_\infty\}$, i.e., on the initial point of the dynamical flow in M corresponding to the radial evolution of the moduli (which is the only relevant in the considered class of static, spherically symmetric SUGRA solutions):

$$\{\varphi\} = \{\varphi(\varphi_\infty)\} \ . \tag{2.9}$$

The AM states that the "near-horizon" asymptotical moduli configurations $\{\varphi_H\} \equiv \lim_{r \to r_H^+} \{\varphi\}$ will be *independent* of $\{\varphi\}$. Moreover, at least at the quantum level, it will be discrete, since it exclusively depends on the

[6] Strictly speaking, this holds only for supersymmetric extreme BH attractors, i.e., for attractor configurations which preserve $\frac{1}{2}$ of the original supersymmetries of the $\mathcal{N}=2$, $d=4$ MESGT being considered.

But nonsupersymmetric extreme BH attractors may exist, too. Such a class of attractor configurations, which has been recently pointed out to be "discretely disjoint" from the class of supersymmetric attractors (at least in the one-modulus case, see [72]), is defined as the class of critical points of a suitably defined "BH effective potential" function V_{BH}, which are not also critical points of $|Z|$. For a detailed teatment, see Sect. 4, and in particular the Subsubsects. 4.4.1 and 4.4.2

(quantized) asymptotical values of the electric charges $\{q\}$ and magnetic charges $\{p\}$ of the system

$$AM : \begin{cases} \{\varphi_H\} \neq \{\varphi_H(\varphi_\infty)\}, \\ \\ \{\varphi_H\} = \{\varphi_H(p,q)\}. \end{cases} \tag{2.10}$$

Such a functional dependence on the charges may be determined by solving the general, model-independent "attractor" or "extremal" equations (AEs)

$$\left.\frac{\partial |Z(\varphi; p, q)|}{\partial \varphi}\right|_{\varphi=\varphi_H(q,p)} = 0 , \tag{2.11}$$

where Z is the "central charge" function of the SUSY algebra in $\mathcal{N} = 2$ SUGRAs, or the highest absolute-valued eigenvalue of the complex antisymmetric central charge matrix in $\mathcal{N} >$ two-extended SUGRAs (see Sect. 6 for explanations).

Equation (2.11) has the following meaning. The (charge-dependences of the) "near-horizon" moduli configurations $\{\varphi_H\}$ are such that, when substituted in the function $Z(q,p,\varphi)$, they give an extremum value of Z with respect to (w.r.t.) its functional dependence on $\{\varphi\}$. Otherwise speaking, the "near-horizon" value (independent of $\{\varphi_\infty\}$)

$$Z_H(q,p) \equiv Z(q,p,\varphi_\infty = \varphi_H(q,p)) \tag{2.12}$$

is an extremum value in the functional dependence of Z on $\{\varphi\}$ at given BH charges (p,q).

Remark: It is worth noticing that usually such an extremum is assumed to be a (local, not necessarily global) minimum, as it can be explicitly verified in some models.

However, for the time being it is not possible to exclude situations with different extrema (such as local or global maxima, flex or cusp points), or also cases in which (2.11) does *not* admit solutions.

By the way, due to positive definiteness of the potential in SUSY theories, for sure a minimum will exist, but a priori nothing forbids the existence of an entire, discrete or continuous family of minima. If this happens, the horizon geometry of a p-d "black brane" in a d-d s-t will still be given by the (p,d)-generalization of BR metric, namely by the direct product $AdS_{p+2} \times S^{d-p-2}$, but such a limit geometry will now be realized by *each one* of the "near-horizon" moduli configurations belonging to the considered family.

Also, given the set of moduli $\{\varphi^i\}_{i\in I}$, it could happen that a subset J of the discrete index range I exists, such that

$$\nexists \lim_{r\to r_H^+} \varphi^j, \quad \forall j \in J \subseteq I, \tag{$*$}$$

namely, that a certain subset of the moduli does not admit a "near-horizon" limit.

Consequently, in order to preserve the core of the AM in such a particular case, a priori a number of possible assumptions may be made:

1. actually $Z = Z(q, p, \{\varphi^k\}_{k\in K})$, where K is the complementary set of J with respect to I; *or*
2. AEs should be slightly generalized as
$$\left.\frac{\partial \left|Z\left(\varphi ;p,q\right)\right|}{\partial \varphi^{k}}\right|_{\varphi^{k}=\varphi_{H}^{k}\left(p,q\right)}=0,\quad \forall k\in K\ ,$$
meaning that the "near-horizon" extremization of the central charge function happens only w.r.t. the moduli well defined at the EH. Thus, in the limit $r \to r_H^+$ the central charge function, extremized w.r.t. its functional dependence on $\left\{\varphi^k\right\}_{k\in K}$, might still possibly depend on the subset of unconstrained, continuously varying asymptotical configurations of moduli $\left\{\varphi^j_\infty\right\}_{j\in J}$:
$$Z_H\left(\left\{\varphi^j_\infty\right\}_{j\in J};p,q\right) \equiv Z\left(\left\{\varphi^k\right\}_{k\in K}=\left\{\varphi^k_H\left(p,q\right)\right\}_{k\in K},\left\{\varphi^j_\infty\right\}_{j\in J};p,q\right)\ .$$
Such a possibility, however, should be disregarded, because, in general, it should lead to a violation of the second principle of thermodynamics in the BH physics; *or*
3. in general, $(*)$ corresponds to a vanishing horizon value of the central charge function
$$Z_H\left(p,q\right) \equiv Z\left(\varphi_\infty = \varphi_H\left(p,q\right);p,q\right) = 0\ ,$$
and therefore the BHEA (and Arnowitt–Deser–Misner [ADM] mass – see a bit further below in the main text) formulae become inconsistent and inapplicable, leading to a nonregular horizon geometry. As we will see later, this happens for all nonminimal BPS SUSY-preserving extremal solutions in $\mathcal{N} > 2$-extended, $d = 4, 5$-d SUGRAs, and also in $\mathcal{N} \geqslant 2$-extended, $d \geqslant 6$-d SUGRAs (where the BHEA formula may also give unphysical, constant nonzero results).

In the present pedagogical treatment we will implicitly assume, for simplicity's sake, that the AEs admit, at least in relation to the minimal BPS SUSY-preserving extremal backgrounds, (at least) one regular solution, corresponding to a purely charge-dependent "near-horizon" moduli configuration.

Finally, it should be mentioned that for an arbitrary geometry of the moduli space the form of the relevant central charge function $Z\left(\varphi;p,q\right)$ may also be very complicated. For instance, this is what happens for the $\mathcal{N} = 2$, $d = 4$ SUGRA obtained by the compactification of $\mathcal{N} = 2$, $d = 10$ type IIB SUGRA on Calabi–Yau threefolds.

Nevertheless, despite this fact, the extremization procedure expressed by the AEs allows one to consistently compute the entropy of the corresponding extremal singular metric backgrounds following a model-independent, universal procedure.

As far as we know, no existence and/or uniqueness theorems have been proved for (2.11), even though substantial progress has been made in the study of the topological properties of the moduli spaces as "attractor varieties" (see, e.g., [31], [32], and [33]).

A simple example illustrating the AM at work may be given by the $\mathcal{N} = 2$, $d = 4$ dilatonic BH of the heterotic string theory. In this case the BPS-saturation condition fixes the so-called ADM mass of the BH to be equal to the absolute value of the central charge function, which in turn will be a function of the electric charge q and magnetic charge p of the BH, and of the asymptotical value ϕ_∞ of the dilaton

$$\begin{aligned}&M_{ADM}\left(q,p,\phi_\infty\right)=\left|Z\left(q,p,\phi_\infty\right)\right|=\tfrac{1}{2}\left(e^{-\phi_\infty}\left|p\right|+e^{\phi_\infty}\left|q\right|\right),\\&\phi_\infty\in\mathbb{R},\quad\left(q,p\right)\in\mathbb{Z}^2\ \text{(in suitable units)}.\end{aligned}\tag{2.13}$$

The general theory based on the AM, when applied to the present case, gives the following "four-step recipe" to obtain the entropy of the dilatonic BH:

1. Write down the extremization condition for the absolute value of the central charge function depending on the dilatonic function $g\left(\phi\right)\equiv e^{\phi}$, at fixed values of the charges $\left(p,q\right)$

$$\frac{\partial\left|Z\left(\phi\left(g\right);p,q\right)\right|}{\partial g}=\frac{1}{2}\frac{\partial}{\partial g}\left(\frac{1}{g}\left|p\right|+g\left|q\right|\right)=-\frac{1}{g^2}\left|p\right|+\left|q\right|=0\,.\tag{2.14}$$

2. Solve such a condition, obtaining the fixed value of the dilatonic function

$$\frac{\partial\left|Z\left(\phi\left(g\right);p,q\right)\right|}{\partial g}=0\Leftrightarrow g=g_H\left(p,q\right)=\left|\frac{p}{q}\right|^{\frac{1}{2}},\tag{2.15}$$

   and therefore of the dilatonic moduli at the EH,

$$\phi_H\left(g\right)\equiv\phi\left(g_H\left(p,q\right)\right)=ln\left[g_H\left(p,q\right)\right]=\frac{1}{2}ln\left|\frac{p}{q}\right|.\tag{2.16}$$

   An example of the evolution of the moduli-dependent dilatonic function $g^{-2}\left(\phi\right)\equiv e^{-2\phi}$ toward a purely charge-dependent value at the EH of the $\mathcal{N}=2$, $d=4$ dilatonic BH is shown in Fig. 2.1.
3. Insert such a fixed value into the expression of the central charge function, by putting $\phi\left(g\right)=\phi_H\left(g\right)$. In such a way, one gets the fixed value $\left|Z_H\left(p,q\right)\right|$ of the absolute value of the central charge function at the EH of the dilatonic BH; clearly, due to the saturation of the BPS bound, it equals the value of the ADM mass of the EH, too (see (2.13))

$$\begin{aligned}M_{ADM,H}\left(p,q\right)&=M_{ADM}\left(\phi\left(g\right)=\phi_H\left(g\right)=\frac{1}{2}ln\left|\frac{p}{q}\right|;p,q\right)\\&=\left|Z_H\left(p,q\right)\right|=\left|Z\left(\phi\left(g\right)=\phi_H\left(g\right)=\frac{1}{2}ln\left|\frac{p}{q}\right|;p,q\right)\right|\\&=\left|pq\right|^{\frac{1}{2}}\,.\end{aligned}\tag{2.17}$$

4. Use the BHEA formula to get the (semiclassical, leading-order) entropy of the $\mathcal{N}=2$, $d=4$ dilatonic BH:

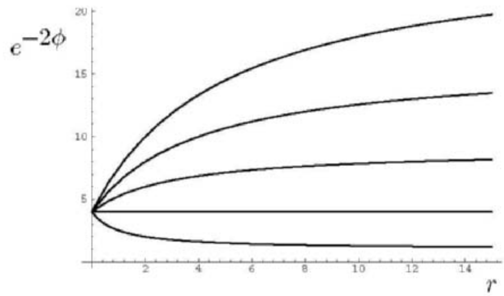

Fig. 2.1. Realization of the attractor mechanism in the $\mathcal{N}=2$, $d=4$ extremal $\frac{1}{2}$-BPS dilatonic BH. Independently of the set of initial (asymptotical $r \to \infty$) moduli configurations (corresponding to the initial data of the dynamical flow inside the moduli space), the "near-horizon" ($r \to 0^+$, with r denoting the radial distance from the EH) evolution of the moduli-dependent dilatonic function $g^{-2}(\phi) \equiv e^{-2\phi}$ converges toward a fixed "attractor" value, which is purely dependent on the (ratio of the) quantized conserved charges of the BH. Such a purely charge-dependent phenomenon of "attraction" of the moduli field configurations encodes the intrinsic loss of information in the (equilibrium) thermodynamics of the extremal dilatonic BH

$$S_{BH} = \frac{A_{Horizon}}{4} = \pi M^2_{ADM,H}(p,q) = \pi \left|Z_H(p,q)\right|^2 = \pi \left|pq\right| \ , \quad (2.18)$$

where we used the definition of the ADM mass at the EH of the BH:

$$M^2_{ADM,H} \equiv \frac{A_{Horizon}}{4\pi} \ . \quad (2.19)$$

Notice that the BH entropy given by (2.18) is purely charge-dependent, and it may be checked that it coincides with the result obtained by completely different (model-dependent, microscopic) methods.

In the $d=4$ (5)-d $\mathcal{N}=2$ SUGRAs coupled to n_V Abelian vector multiplets (named $\mathcal{N}=2$ n_V-fold MESGTs), the extremization of the central charge function Z through (2.11) may be made "coordinate-free" in the moduli space M_{n_V}, by using the fact that such a n_V-d complex manifold has actually a (real) special Kähler metric structure. The geometric properties of the moduli space and the overall symplectic structure of such $\mathcal{N}=2$ SUGRAs will be considered in the next section.

The final result of the AM in such theories is the macroscopic, model-independent derivation of BHEA formula, yielding

$$S_{BH} = \frac{A}{4} = \pi \left|Z_H(p,q)\right|^2 \quad (2.20)$$

and

$$S_{BH} = \frac{A}{4} \sim \left|Z_H(p,q)\right|^{\frac{3}{2}} \ , \quad (2.21)$$

in $d=4$ and $d=5$, respectively.

Recently, many applications of the above ideas have been worked out, especially in the case of string theory compactified on 3-d Calabi–Yau manifolds. Also, by using some properly formulated D-brane techniques, the topological entropy formula of BH has been obtained, by counting the related microstates in string theory. The results of such a procedure, whenever obtainable, are in agreement with the model-independent determination of the entropy which uses the attractor mechanism. The four-step algorithm given by (2.14)–(2.18) is just one of the possible realizations of such a model-independent approach to the equilibrium thermodynamics of BHs.

It should be also mentioned that several properties of the fixed "attractor" moduli configurations have been investigated. In particular, it has been shown that the attractor mechanism is also relevant in the discussion of the BH thermodynamics out of the extremality (i.e., when the BPS-like bound (1.8) is not saturated).

In the remaining part of these introductory lectures we will see how the AM works in two relevant contexts, namely in the so-called $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGTs, and then in the $\mathcal{N} = 8$, $d = 4$, and 5 maximal SUGRAs endowed with the exceptional Lie groups $E_{7(7)}$ and $E_{6(6)}$ as noncompact U-symmetries, respectively.

3

Attractor Mechanism in $\mathcal{N}=2$, $d=4$ Maxwell–Einstein Supergravity

The multiplet content of a completely general $\mathcal{N}=2$, $d=4$ supergravity (SUGRA) theory is the following (see, e.g., [34] and [35]):

1. the *gravitational multiplet*

$$\left(V_{\mu}^{a}, \psi^{A}, \psi_{A}, A^{0}\right) \ , \tag{3.1}$$

described by the Vielbein one-form V^a $(a=0,1,2,3)$ (together with the spin-connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A ($A=1,2$, with the upper and lower indices respectively denoting right and left chirality, i.e., $\gamma_5\psi_A = -\gamma_5\psi^A = 1$), and the graviphoton one-form A^0;

2. n_V *vector supermultiplets*

$$\left(A^{I}, \lambda^{iA}, \overline{\lambda}_{A}^{\bar{i}}, z^{i}\right) \ , \tag{3.2}$$

each containing a gauge boson one-form A^I $(I=1,\ldots,n_V)$, a doublet of gauginos (zero-form spinors) $\lambda^{iA}, \overline{\lambda}_{A}^{\bar{i}}$, and a complex scalar field (zero-form) z^i $(i=1,\ldots,n_V)$. The scalar fields z^i can be regarded as arbitrary coordinates on a complex manifold $M_{n_V}(dim_{\mathbb{C}}M_{n_V}=n_V)$, which is actually a special Kähler manifold;

3. n_H *hypermultiplets*

$$\left(\zeta_{\alpha}, \zeta^{\alpha}, q^{u}\right) \ , \tag{3.3}$$

each formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha, \zeta^\alpha$ $(\alpha=1,\ldots,2n_H)$, and four real scalar fields q^u $(u=1,\ldots,4n_H)$, which can be considered as arbitrary coordinates of a quaternionic manifold $\mathcal{Q}_{n_H}$ $(dim_{\mathbb{R}}\mathcal{Q}_{n_H}=4n_H)$.

In this section we will sketchy report the formulation of the $\mathcal{N}=2$, $d=4$ SUGRA coupled to n_V Abelian vector multiplets in the presence of electric and magnetic charges, i.e., of the so-called $\mathcal{N}=2$, $d=4$ n_V-fold MESGT.

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We will then show how the attraction mechanism explicitly works, in relation to the special Kähler geometry of the manifold M_{n_V} of the scalars z^i's from the Abelian vector supermultiplets, finally specializing the AE (2.11) for such a framework.[1]

3.1 Special Kähler–Hodge Geometry and Symplectic Structure of Moduli Space

Let us start by considering the moduli space M_{n_V} of the $\mathcal{N}=2$, $d=4$ n_V-fold MESGT; it is a complex n_V-d manifold having the n_V scalar complex

[1] Here we will not deal with the n_H hypermultiplets. Indeed, in the $\mathcal{N}=2$, $d=4$ n_V-fold MESGT the symplectic special Kähler geometry is completely determined by the n_V complex scalar fields coming from the considered n_V Abelian vector supermultiplets.

Such a fact may be understood by looking at the transformation properties of the Fermi fields: the hyperinos $\zeta_\alpha, \zeta^\alpha$'s transform independently of the vector fields, whereas the gauginos' SUSY transformations depend on the Maxwell vector fields.

Consequently, the contribution of the hypermultiplets may be dynamically decoupled from the rest of the physical system. Thus, it is also completely independent from the evolution dynamics of the complex scalars z^i's coming from the vector multiplets (i.e., from the evolution flow in the moduli space M_{n_V}).

Disregarding for simplicity's sake the fermionic and gauging terms, the SUSY transformations of hyperinos (see (3.2.1) further below) read

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB}\mathbb{C}_{\alpha\beta} \ . \qquad (**)$$

$(**)$ does not constrain the asymptotical configurations of the quaternionic scalars of the hypermultiplets, which therefore may continuously vary in the manifold $\mathcal{Q}_{n_H}$ of the related quaternionic nonlinear sigma model.

In the gauged $\mathcal{N}$-extended SUGRA (generally corresponding to asymptotically nonflat backgrounds), and consequently also in the $\mathcal{N}=2$, $d=4$, (n_V, n_H)-fold-gauged MESGT, the situation is much more complicated.

Of course, the geometry of the scalar sigma models remains the same, since it is completely fixed by the internal metric structure of the kinetic terms of the scalars. For a generic value of $(n_V, n_H) \in \mathbb{N}^2$, it is given by the direct product

$$M_{n_V} \times \mathcal{Q}_{n_H}$$

of the special Kähler–Hodge manifold of the complex scalars from the Maxwell vector supermultiplets and of the quaternionic manifold of the scalar fields from the hypermultiplets.

But, differently from the "ungauged," asymptotically flat case, which will be treated in the following pages, some interaction terms between the above two different sets of scalars will arise in the bosonic part of the gauged SUGRA Lagrangian. Such terms are generated by the Killing vectors coming from the introduction of covariant derivatives w.r.t. the gauging of (some of) the isometries of $\mathcal{Q}_{n_H}$, and they do *not* allow one to dynamically decouple the hypermultiplets any more.

fields z^i $(i = 1, \ldots, n_V)$ as local coordinates; such fields come from the vector multiplets coupling to $\mathcal{N} = 2$, $d = 4$ SUGRA.

The key feature is that M_{n_V} is a Kähler–Hodge manifold with special Kähler structure, namely a n_V-d special Kähler–Hodge manifold with symplectic structure.[2]

First, M_{n_V} is a Kähler manifold, i.e., a complex Hermitian manifold with the metric

$$G_{i\bar{j}}(z, \bar{z}) \equiv \bar{\partial}_{\bar{j}} \partial_i K(z, \bar{z}) \ , \tag{3.1.1}$$

where $K(z, \bar{z})$ is the so-called (real) "Kähler potential" scalar function. The Hermiticity of the metric directly follows from the reality of K (and from the fact that such a function is assumed to satisfy the Schwarz lemma about partial derivatives on M_{n_V})

$$\overline{G_{i\bar{j}}} = \overline{\bar{\partial}_{\bar{j}} \partial_i K} = \partial_j \bar{\partial}_{\bar{i}} K = \bar{\partial}_{\bar{i}} \partial_j K = G_{j\bar{i}} \ . \tag{3.1.2}$$

Second, since M_{n_V} is a special Kähler manifold, its Riemann–Christoffel curvature tensor satisfies the so-called special Kähler geometry (SKG) constraints

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{j}} G_{k\bar{l}} + G_{i\bar{l}} G_{k\bar{j}} - C_{ikp} \overline{C}_{\bar{j}\bar{l}\bar{p}} G^{p\bar{p}} \ , \tag{3.1.3}$$

where C_{ijk} is the rank-3, completely symmetric, Kähler-covariantly holomorphic tensor of SKG

$$\begin{cases} C_{ijk} = C_{(ijk)} \ ; \\ \\ \overline{D}_{\bar{l}} C_{ijk} = 0 \ . \end{cases} \tag{3.1.4}$$

It is also immediate to show that [56]

$$D_{[l} C_{i]jk} = 0 \ , \tag{3.1.5}$$

where square brackets denote antisymmetrization w.r.t. the enclosed indices. Indeed, the (differential) Bianchi identities for the Riemann–Christoffel tensor read

$$D_{[l} R_{i]\bar{k}j\bar{p}} = 0 \ ; \tag{3.1.6}$$

by using the SKG constraints (3.1.3) and recalling the Kähler-covariant holomorphicity of C_{ijk} ($\overline{D}_{\bar{l}} C_{ijk} = 0$) and the validity of the metric postulate in M_{n_V} ($D_k G_{i\bar{j}} = 0$), one immediately gets

$$\left(D_{[l} C_{i]jn} \right) \overline{C}_{\bar{k}\bar{p}\bar{n}} G^{n\bar{n}} = 0 \ , \tag{3.1.7}$$

and (3.1.5) follows from the observation that (3.1.7) holds for any (nonvanishing) $\overline{C}_{\bar{k}\bar{p}}{}^{n} = \overline{C}_{\bar{k}\bar{p}\bar{n}} G^{n\bar{n}}$.

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[2] For recent advances in local and rigid special Kähler geometry, the reader is addressed e.g. to [206–208].

Since in a (commutative) Kähler manifold the completely covariant Riemann–Christoffel tensor $R_{i\bar{j}l\overline{m}}$ may be rewritten as[3]

$$R_{i\bar{j}k\bar{l}} = -G^{m\overline{n}}\left(\overline{\partial}_{\bar{l}}\overline{\partial}_{\bar{j}}\partial_m K\right)\partial_i\overline{\partial}_{\overline{n}}\partial_k K + \overline{\partial}_{\bar{l}}\partial_i\overline{\partial}_{\bar{j}}\partial_k K \ , \tag{3.1.8}$$

the SKG constraints (3.1.3) may be reformulated as follows:

$$\begin{aligned} &-G^{m\overline{n}}\left(\overline{\partial}_{\bar{l}}\overline{\partial}_{\bar{j}}\partial_m K\right)\partial_i\overline{\partial}_{\overline{n}}\partial_k K + \overline{\partial}_{\bar{l}}\partial_i\overline{\partial}_{\bar{j}}\partial_k K \\ &= \left(\overline{\partial}_{\bar{j}}\partial_i K\right)\overline{\partial}_{\bar{l}}\partial_k K + \left(\overline{\partial}_{\bar{l}}\partial_i K\right)\overline{\partial}_{\bar{j}}\partial_k K - C_{ikp}\overline{C}_{\bar{j}\bar{l}\overline{p}}G^{p\overline{p}}; \end{aligned} \tag{3.1.9}$$

$$\Updownarrow$$

$$\begin{aligned} &G^{m\overline{n}}\left[\left(\overline{\partial}_{\bar{l}}\overline{\partial}_{\bar{j}}\partial_m K\right)\left(\partial_i\overline{\partial}_{\overline{n}}\partial_k K\right) - C_{ikm}\overline{C}_{\bar{j}\bar{l}\overline{n}}\right] \\ &= \overline{\partial}_{\bar{l}}\partial_i\overline{\partial}_{\bar{j}}\partial_k K - \left(\overline{\partial}_{\bar{j}}\partial_i K\right)\overline{\partial}_{\bar{l}}\partial_k K - \left(\overline{\partial}_{\bar{l}}\partial_i K\right)\overline{\partial}_{\bar{j}}\partial_k K; \end{aligned} \tag{3.1.10}$$

$$\Updownarrow$$

$$\begin{aligned} &C^{\overline{n}}{}_{km}\overline{C}_{\overline{n}\bar{j}\bar{l}} \\ &= \left(\overline{\partial}_{\bar{j}}\partial_i K\right)\overline{\partial}_{\bar{l}}\partial_k K + \left(\overline{\partial}_{\bar{l}}\partial_i K\right)\overline{\partial}_{\bar{j}}\partial_k K \\ &-\overline{\partial}_{\bar{l}}\partial_i\overline{\partial}_{\bar{j}}\partial_k K + G^{m\overline{n}}\left(\overline{\partial}_{\bar{l}}\overline{\partial}_{\bar{j}}\partial_m K\right)\left(\partial_i\overline{\partial}_{\overline{n}}\partial_k K\right) , \end{aligned} \tag{3.1.11}$$

where, as usual, the contravariant and covariant metric tensors are related by the orthonormality condition

$$G^{i\bar{j}}G_{l\bar{j}} = G^{i\bar{j}}\overline{\partial}_{\bar{j}}\partial_l K = \delta^i_l \ . \tag{3.1.12}$$

Third, since M_{n_V} is a Kähler–Hodge manifold, it admits a $U(1)$ line (Hodge) bundle $\Im$, whose first Chern class coincides with the Kähler class of M_{n_V}

$$c_1\left[\Im\right] = \mathcal{K}\left[M_{n_V}\right] \ . \tag{3.1.13}$$

Such a property allows one to locally write the $U(1)$ connection Q as

[3] It should also be recalled that in a Kählerian manifold the (completely covariant) Riemann-Christoffel tensor, as well as its contractions $R_{i\bar{j}}$ and R and the (completely covariant) Weyl conformal curvature tensor $C_{i\bar{j}k\bar{l}}$, are all real [37]

$$\overline{R_{i\bar{j}k\bar{l}}} = R_{\bar{i}j\bar{k}l}; \qquad \overline{R_{i\bar{j}}} = R_{\bar{i}j};$$

$$\overline{R} = R; \qquad \overline{C_{i\bar{j}k\bar{l}}} = C_{\bar{i}j\bar{k}l}.$$

$$Q = -\frac{i}{2}\left[(\partial_i K)\, dz^i - \left(\overline{\partial}_{\overline{i}} K\right) d\overline{z}^{\overline{i}}\right] . \tag{3.1.14}$$

Let us now consider a Kähler transformation

$$K(z, \overline{z}) \to K(z, \overline{z}) + f(z) + \overline{f}(\overline{z}) , \tag{3.1.15}$$

where f is an arbitrary holomorphic function. Clearly, due to definition (3.1.1), such a transformation does not affect the Kähler metric structure, and thus it actually expresses an intrinsic gauge metric degree of freedom of the considered manifold. Otherwise speaking, the metric properties of the manifold will not change if one chooses $K(z, \overline{z})$ or a Kähler potential transformed according to Eq. (3.1.15).

Consequently, beside the usual Hermitian covariance, one will have to take it into account when writing down the Kähler-covariant derivatives of any tensor quantity. In a general (commutative) Kähler geometry, a generic vector $\mathcal{V}^i$ which under (3.1.15) transforms as

$$\mathcal{V}^i(z, \overline{z}) \to \exp\left\{-\frac{1}{2}\left[p f(z) + \overline{p}\overline{f}(\overline{z})\right]\right\} \mathcal{V}^i(z, \overline{z}) , \quad (p, \overline{p}) \in \mathbb{R}^2 , \tag{3.1.16}$$

is said to have Kähler weights[4] $(p, \overline{p})$. Its Kähler-covariant derivatives are defined as follows:

$$\begin{cases} D_j \mathcal{V}^i(z, \overline{z}) = \partial_j \mathcal{V}^i(z, \overline{z}) + \Gamma_{jk}{}^i(z, \overline{z})\, \mathcal{V}^k(z, \overline{z}) + \frac{p}{2}\left(\partial_j K(z, \overline{z})\right) \mathcal{V}^i(z, \overline{z}) ; \\ \overline{D}_{\overline{j}} \mathcal{V}^i(z, \overline{z}) = \overline{\partial}_{\overline{j}} \mathcal{V}^i(z, \overline{z}) + \frac{\overline{p}}{2}\left(\overline{\partial}_{\overline{j}} K(z, \overline{z})\right) \mathcal{V}^i(z, \overline{z}) , \end{cases} \tag{3.1.17}$$

where $\Gamma_{jk}{}^i(z, \overline{z})$ denotes the symmetric connection given by the Christoffel symbols of the second kind of the Kähler metric

$$\begin{aligned} \Gamma_{jk}{}^i(z, \overline{z}) &\equiv \left\{{}^{\,i}_{jk}\right\}(z, \overline{z}) = G^{i\overline{l}}(z, \overline{z})\, \partial_j G_{k\overline{l}}(z, \overline{z}) \\ &= G^{i\overline{l}}(z, \overline{z})\, \partial_j \overline{\partial}_{\overline{l}} \partial_k K(z, \overline{z}) = \Gamma_{(jk)}{}^i(z, \overline{z}) . \end{aligned} \tag{3.1.18}$$

The Kähler transformation property (3.1.16) may be rewritten as follows:

$$\mathcal{V}^i(z, \overline{z}) \to \exp\left\{-\frac{1}{2}(p + \overline{p})\, Re\left(f(z)\right)\right\} \exp\left\{-\frac{i}{2}(p - \overline{p})\, Im\left(f(z)\right)\right\} \mathcal{V}^i(z, \overline{z}) \; ; \tag{3.1.19}$$

it is then immediate to realize that a generic Kähler transformation may always be decomposed in a $U(1)$ phase transformation (singled out by $p = -\overline{p}$)

[4] The Kähler weights are real. Notice that $\overline{p}$ is not the complex conjugate of the holomorphic Kähler weight p, but it rather simply stands for the antiholomorphic Kähler weight.

and in a proper Kähler transformation (singled out by $p=\overline{p}$). Due to the reality of the Kähler weights, the complex conjugation of (3.1.16) yields

$$\overline{\mathcal{V}}^{\overline{i}}\left(z,\overline{z}\right)\rightarrow \exp\left\{-\frac{1}{2}\left[\overline{p}f(z)+p\overline{f}(\overline{z})\right]\right\}\overline{\mathcal{V}}^{\overline{i}}\left(z,\overline{z}\right)\ , \tag{3.1.20}$$

and thus one gets that the complex conjugation simply exchanges the Kähler weights: if $\mathcal{V}^i$ has Kähler weights $(p,\overline{p})$, then $\overline{\mathcal{V}}^{\overline{i}}$ has Kähler weights $(\overline{p},p)$.

Since we are considering a $U(1)$ line bundle $\Im$ over the moduli space M_{n_V}, only the quantities with Kähler weights constrained by $p=-\overline{p}$ will properly belong to the related $U(1)$ ring. Clearly, real or (anti)holomorphic quantities will not belong to such a $U(1)$ ring, unless they are Kähler gauge-invariant, i.e., they have $(p,\overline{p})=(0,0)$. An example of tensor belonging to the $U(1)$ ring is the completely symmetric, Kähler-covariantly holomorphic rank-3 tensor $C_{ijk}\left(z,\overline{z}\right)$, having Kähler weights $(2,-2)$; as a consequence of the general formulae (3.1.17), its Kähler-covariant derivatives read

$$\begin{cases} D_l C_{ijk}\left(z,\overline{z}\right) \\ = \partial_l C_{ijk}\left(z,\overline{z}\right) - \mathit{\Gamma}_{li}^{\ \ m}\left(z,\overline{z}\right) C_{mjk}\left(z,\overline{z}\right) - \mathit{\Gamma}_{lj}^{\ \ m}\left(z,\overline{z}\right) C_{imk}\left(z,\overline{z}\right) \\ -\mathit{\Gamma}_{lk}^{\ \ m}\left(z,\overline{z}\right) C_{ijm}\left(z,\overline{z}\right) + \left(\partial_l K\left(z,\overline{z}\right)\right) C_{ijk}\left(z,\overline{z}\right); \\ \\ \overline{D}_{\overline{l}} C_{ijk}\left(z,\overline{z}\right) = \overline{\partial}_{\overline{l}} C_{ijk}\left(z,\overline{z}\right) - \left(\overline{\partial}_{\overline{l}} K\left(z,\overline{z}\right)\right) C_{ijk}\left(z,\overline{z}\right) = 0. \end{cases} \tag{3.1.21}$$

Therefore, the integrability condition (3.1.5) may be rewritten as follows:

$$\partial_{[l}C_{i]jk} - \mathit{\Gamma}_{[li]}^{\ \ \ m}C_{mjk} - \mathit{\Gamma}_{[l|j}^{\ \ \ m}C_{|i]mk} - \mathit{\Gamma}_{[l|k}^{\ \ \ m}C_{|i]jm} + \left(\partial_{[l}K\right)C_{i]jk} = 0\ . \tag{3.1.22}$$

A more intrinsic characterization of M_{n_V}, which makes its $Sp\left(2n_V+2\right)$-covariance manifest, is the following one.

Let us start by defining the (Kähler-covariantly holomorphic with Kähler weights $(1,-1)$) symplectic sections of the Hodge bundle $\Im$ on M_{n_V} ($\mathit{\Lambda}=0,1,\ldots,n_V$)

$$V\left(z,\overline{z}\right)\equiv\begin{pmatrix} L^{\mathit{\Lambda}}\left(z,\overline{z}\right) \\ \\ M_{\mathit{\Lambda}}\left(z,\overline{z}\right)\end{pmatrix},\ \text{with}\ \overline{D}_{\overline{i}}V=\left(\overline{\partial}_{\overline{i}}-\frac{1}{2}\overline{\partial}_{\overline{i}}K\right)V=0\ . \tag{3.1.23}$$

Notice that such sections may be arranged in a $Sp\left(2n_V+2\right)$-covariant vector V. By defining a scalar product in the related representation space using the $(2n_V+2)$-d symplectic metric

$$\epsilon \equiv \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.1.24}$$

($\mathbb{I}$ stands for the $(n_V + 1)$-d identity), the symplectic sections may be normalized as follows:

$$\langle V, \overline{V} \rangle \equiv V^T \epsilon \overline{V} = \overline{L}^\Lambda M_\Lambda - \overline{M}_\Lambda L^\Lambda \equiv -i \; . \tag{3.1.25}$$

Therefore, it is natural to introduce the $(2n_V + 2)$-d vector of the holomorphic Kähler-covariant derivatives of the sections[5] of $\Im$:

$$U_i \equiv D_i V = \left(\partial_i + \frac{1}{2} \partial_i K \right) \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \equiv \begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix} \; ; \tag{3.1.26}$$

consequently, the Kähler-covariant holomorphicity of V implies $U_{\overline{i}} \equiv \overline{D}_{\overline{i}} V = 0 = \overline{U}_i \equiv D_i \overline{V}$. It may be then shown that in SKG

$$D_i U_j = i C_{ijk} G^{k\overline{k}} \overline{U}_{\overline{k}} \; , \tag{3.1.27}$$
$$D_i \overline{U}_{\overline{j}} = G_{i\overline{j}} \overline{V} \; ; \tag{3.1.28}$$

here C_{ijk} may be defined to be the $(2, -2)$-Kähler-weighted section of $(T^*)^3 \otimes \Im^2$, totally symmetric in its indices and Kähler-covariantly holomorphic.[6] In [36] it was shown that (3.1.23)–(3.1.28) (with the properties (3.1.4) and the constraints (3.1.3), or equivalently the integrability condition (3.1.5) for C_{ijk} derivatives always coincide with the ordinary, flat derivatives.

It is worth mentioning that, while (3.1.27) is typical of SKG, (3.1.28) holds in contexts more general than SKG. To clarify such a point, let us derive it, by considering, without any loss of generality, the section L^Λ. As stated above, this is a Kähler-covariantly holomorphic symplectic section with Kähler weights $(1, -1)$; thus, it holds that

---

[5] In general, f_i^Λ and $h_{i\Lambda}$ are functions defined in M_{n_V}, with a local index i and a global index Λ. As the n-bein allows one to transform local Poincaré-covariant indices in global diffeomorphism-covariant indices (and viceversa), similarly such quantities allow one to switch between global $Sp\,(2n_V + 2)$-covariant indices and local indices in the Kähler–Hodge manifold M_{n_V} associated with the nonlinear σ-model of the complex scalars coming from the n_V considered vector multiplets.

It is also worth noticing that, in the particular cases in which such a manifold is a symmetric space of the kind G/H (as it happens for all $\mathcal{N} \geqslant 3$, $d = 4$ SUGRAs, and in particular for the maximal $\mathcal{N} = 8$, $d = 4$ SUGRA with noncompact $E_{7(7)}$ symmetry: see Subsect. 6.2), the functions f_i^Λ and $h_{i\Lambda}$ are nothing but the representative cosets of such a space.

[6] In an alternative defining approach, (3.1.23), (3.1.26), (3.1.27), and (3.1.28) may be also considered as the fundamental differential constraints defining the local special Kähler geometry of M_{n_V}. Indeed, it may be shown that they yield the SKG constraints (3.1.3) (see, e.g., [38]). For a thorough analysis of the various approaches to the definitions of (global and local) SKG, see, e.g., [40] and [41].

$$
\begin{aligned}
D_i\overline{D}_{\overline{j}}\overline{L}^{\Lambda} &= \left(\partial_i - \frac{1}{2}\left(\partial_i K\right)\right)\left(\overline{\partial}_{\overline{j}} + \frac{1}{2}\left(\overline{\partial}_{\overline{j}}K\right)\right)\overline{L}^{\Lambda} \\
&= \partial_i\overline{\partial}_{\overline{j}}\overline{L}^{\Lambda} + \frac{1}{2}\left(\partial_i\overline{\partial}_{\overline{j}}K\right)\overline{L}^{\Lambda} + \frac{1}{2}\left(\overline{\partial}_{\overline{j}}K\right)\partial_i\overline{L}^{\Lambda} \\
&\quad -\frac{1}{2}\left(\partial_i K\right)\overline{\partial}_{\overline{j}}\overline{L}^{\Lambda} - \frac{1}{4}\left(\partial_i K\right)\left(\overline{\partial}_{\overline{j}}K\right)\overline{L}^{\Lambda};
\end{aligned}
\tag{3.1.29}
$$

now, by recalling (3.1.1) and using the fact that the Kähler-covariant holomorphicity of L^{Λ} implies

$$
\partial_i\overline{L}^{\Lambda} = \frac{1}{2}\left(\partial_i K\right)\overline{L}^{\Lambda}\ , \tag{3.1.30}
$$

one gets

$$
D_i\overline{D}_{\overline{j}}\overline{L}^{\Lambda} = \partial_i\overline{\partial}_{\overline{j}}\overline{L}^{\Lambda} + \frac{1}{2}G_{i\overline{j}}\overline{L}^{\Lambda} - \frac{1}{2}\left(\partial_i K\right)\overline{\partial}_{\overline{j}}\overline{L}^{\Lambda}\ ; \tag{3.1.31}
$$

since $\overline{L}^{\Lambda}$ satisfies the Schwarz lemma on (flat) partial derivatives in M_{n_V}, by reusing (3.1.30), this implies

$$
D_i\overline{D}_{\overline{j}}\overline{L}^{\Lambda} = G_{i\overline{j}}\overline{L}^{\Lambda}\ . \tag{3.1.32}
$$

By repeating the same procedure for $\overline{M}_{\Lambda}$, one obtains the result (3.1.28), which therefore relies only on the Kähler-covariant holomorphicity of the vector V with Kähler weights $(1,-1)$ (and, rigorously, on the commutation of flat partial derivatives acting on K and V).

The SG constraints (3.1.3) (or (3.1.9)–(3.1.10)) may be solved by formulating the following fundamental Ansätze:

$$
M_{\Lambda}\left(z,\overline{z}\right) = \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)L^{\Sigma}\left(z,\overline{z}\right), \tag{3.1.33}
$$

$$
h_{i\Lambda}\left(z,\overline{z}\right) = \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)f_i^{\Sigma}\left(z,\overline{z}\right), \tag{3.1.34}
$$

where $\mathcal{N}_{\Lambda\Sigma}$ is a complex symmetric matrix. Such Ansätze are the fundamental relations on which the symplectic special Kähler geometry of the $\mathcal{N}=2$, $d=4$ n_V-fold MESGT is founded. They express the $Sp(2n_V+2)$ symmetry acting on the special Kähler geometry of the moduli space M_{n_V}.

By conjugating (3.1.33), the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and the conditions of normalization of sections given by (3.1.25) imply

$$
\begin{gathered}
-i \equiv \left\langle V,\overline{V}\right\rangle = \overline{L}^{\Lambda}M_{\Lambda} - \overline{M}_{\Lambda}L^{\Lambda} \\
= \overline{L}^{\Lambda}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^{\Sigma}L^{\Lambda} \\
= \left(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}\right)L^{\Lambda}\overline{L}^{\Sigma} = 2iIm\left(\mathcal{N}_{\Lambda\Sigma}\right)L^{\Lambda}\overline{L}^{\Sigma}; \\
\Updownarrow \\
Im\left(\mathcal{N}_{\Lambda\Sigma}\right)L^{\Lambda}\overline{L}^{\Sigma} = -\frac{1}{2}.
\end{gathered}
\tag{3.1.35}
$$

Thence, by using (3.1.25), (3.1.27), (3.1.28), (3.1.33) and (3.1.34), it may be explicitly calculated that

$$\begin{cases} I.\ \langle V, U_{\bar{i}} \rangle = 0 \Leftrightarrow \left\langle \overline{V}, \overline{U}_i \right\rangle = 0; \\ \\ II.\ G_{i\bar{j}} = -i \left\langle U_i, \overline{U}_{\bar{j}} \right\rangle; \\ \\ III.\ C_{ijk} = \langle D_i U_j, U_k \rangle . \end{cases} \tag{3.1.36}$$

Notice that the first result, namely $\langle V, U_{\bar{i}} \rangle = 0$, is trivial because $U_{\bar{i}} \equiv \overline{D}_{\bar{i}} V = 0$ by construction.

Moreover, it can also be proved that

$$\langle V, U_i \rangle = 0 \Longleftrightarrow \left\langle \overline{V}, \overline{U}_{\bar{i}} \right\rangle = 0 . \tag{3.1.37}$$

Indeed, by exploiting the distributivity of the Kähler-covariant derivative w.r.t. the symplectic scalar product $\langle \cdot, \cdot \rangle$ and the Kähler-covariant holomorphicity of V, and using (3.1.27), one gets

$$D_i \left\langle \overline{V}, U_j \right\rangle = \left\langle D_i \overline{V}, U_j \right\rangle + \left\langle \overline{V}, D_i U_j \right\rangle = i C_{ijk} G^{k\bar{k}} \left\langle \overline{V}, \overline{U}_{\bar{k}} \right\rangle . \tag{3.1.38}$$

Now, by also recalling the normalization (3.1.25), it holds that

$$0 = D_i \left\langle V, \overline{V} \right\rangle = \left\langle D_i V, \overline{V} \right\rangle + \left\langle V, D_i \overline{V} \right\rangle = \left\langle U_i, \overline{V} \right\rangle = -\left\langle \overline{V}, U_i \right\rangle . \tag{3.1.39}$$

By substituting such a result back into (3.1.38), one gets

$$\begin{gathered} C_{ijk}(z, \bar{z})\, G^{k\bar{k}}(z, \bar{z}) \left\langle \overline{V}(z, \bar{z}), \overline{U}_{\bar{k}}(z, \bar{z}) \right\rangle = 0,\ \forall (z, \bar{z}) \in M_{n_V} \\ \Updownarrow \\ \left\langle \overline{V}, \overline{U}_{\bar{i}} \right\rangle = 0 \Longleftrightarrow \langle V, U_i \rangle = 0 , \end{gathered} \tag{3.1.40}$$

q.e.d.

Moreover, it is straightforward to calculate

$$\begin{aligned} \left\langle V, \overline{U}_{\bar{i}} \right\rangle &\equiv V^T \epsilon \overline{U}_{\bar{i}} = -L^\Lambda \overline{h}_{\bar{i}\Lambda} + M_\Lambda \overline{f}_{\bar{i}}^\Lambda \\ &= -L^\Lambda \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma + \mathcal{N}_{\Lambda\Sigma} L^\Sigma f_i^\Lambda = 0 , \end{aligned} \tag{3.1.41}$$

where in the second line we used the Ansätze (3.1.33) and (3.1.34) and the symmetry of $\mathcal{N}_{\Lambda\Sigma}$. Summarizing, in the SKG framework the vector V is symplectically orthogonal to all its Kähler-covariant derivatives

$$\begin{cases} \langle V, U_i \rangle = 0 ; \\ \\ \langle V, U_{\bar{i}} \rangle = 0 ; \\ \\ \left\langle V, \overline{U}_i \right\rangle = 0 ; \\ \\ \left\langle V, \overline{U}_{\bar{i}} \right\rangle = 0 . \end{cases} \tag{3.1.42}$$

Notice that (3.1.27) and the last relation of (3.1.42) yield

$$\langle V, D_i U_j \rangle = \langle V, D_i D_j V \rangle = i C_{ijk} G^{k\overline{k}} \left\langle V, \overline{D}_{\overline{k}} \overline{V} \right\rangle = 0\,. \tag{3.1.43}$$

By applying the Kähler-covariant holomorphic derivative to $\left\langle V, \overline{U}_{\overline{i}} \right\rangle = 0$ and using (3.1.25) and (3.1.28), it is immediate to prove the result II of (3.1.36); indeed

$$\begin{aligned} 0 &= D_j \left\langle V, \overline{U}_{\overline{i}} \right\rangle = D_j \left\langle V, \overline{D}_{\overline{i}} \overline{V} \right\rangle = \left\langle D_j V, \overline{D}_{\overline{i}} \overline{V} \right\rangle + \left\langle V, D_j \overline{D}_{\overline{i}} \overline{V} \right\rangle \\ &= \left\langle D_j V, \overline{D}_{\overline{i}} \overline{V} \right\rangle + G_{j\overline{i}} \left\langle V, \overline{V} \right\rangle \Longleftrightarrow \left\langle D_j V, \overline{D}_{\overline{i}} \overline{V} \right\rangle = i G_{j\overline{i}}. \end{aligned} \tag{3.1.44}$$

Now, by complex conjugating the Ansatz (3.1.33) and considering the Ansatz (3.1.34), one gets

$$\begin{cases} \overline{M}_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \overline{L}^\Sigma, \\ h_{i\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma. \end{cases} \tag{3.1.45}$$

It then appears natural to define some square matrices with $(n_V+1)^2$ complex entries, corresponding to completing the $(n_V+1)\times n_V$ complex matrices f_i^Λ and $h_{i\Lambda}$ to a square form as follows:

$$f_I^\Lambda \equiv \left(f_i^\Lambda, \overline{L}^\Lambda \right), \quad h_{I\Lambda} \equiv \left(h_{i\Lambda}, \overline{M}_\Lambda \right)\ ; \tag{3.1.46}$$

consequently, $\overline{\mathcal{N}}_{\Lambda\Sigma}$ may be written as

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{I\Lambda} \left(f^{-1} \right)_\Sigma^I\ . \tag{3.1.47}$$

It is clear that (3.1.47), through the definitions given by (3.1.46), is completely equivalent to the set of Ansätze (3.1.33) and (3.1.34):

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z}) = h_{I\Lambda}(z,\overline{z}) \left(f^{-1} \right)_\Sigma^I (z,\overline{z}) \overset{(3.1.46)}{\Longleftrightarrow} \begin{cases} M_\Lambda(z,\overline{z}) = \mathcal{N}_{\Lambda\Sigma}(z,\overline{z})\, L^\Sigma(z,\overline{z}), \\ h_{i\Lambda}(z,\overline{z}) = \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z})\, f_i^\Sigma(z,\overline{z}). \end{cases} \tag{3.1.48}$$

Moreover, by Kähler-covariantly differentiating (3.1.33) and using (3.1.34) and (3.1.23), we may obtain the following results:

$$\left(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma} \right) f_i^\Sigma = - \left(D_i \mathcal{N}_{\Lambda\Sigma} \right) L^\Sigma\ . \tag{3.1.49}$$

$$\left(\overline{D}_{\overline{i}} \mathcal{N}_{\Lambda\Sigma} \right) L^\Sigma = 0\ . \tag{3.1.50}$$

Clearly, the very definition of $\mathcal{N}_{\Lambda\Sigma}$ by (3.1.33) implies that such a matrix has vanishing Kähler weights, because M_Λ and L^Λ are components of the same $(2n_V+2)$-tet in the vector representation of the symplectic group $Sp(2n_V+2)$.

If $\mathcal{N}_{\Lambda\Sigma}$ were not Kähler gauge-invariant, it would violate the symplectic inner structure of the special Kähler–Hodge geometry of M_{n_V} (such a feature of $\mathcal{N}_{\Lambda\Sigma}$ is clear also by looking at (3.1.35), by simply noticing that a quantity and its complex conjugate have always opposite Kähler weights: see (3.1.16) and (3.1.20)).

Therefore, (3.1.49) and (3.1.50) may actually be rewritten as follows:

$$\begin{gathered}\left(\mathcal{N}_{\Lambda\Sigma}-\overline{\mathcal{N}}_{\Lambda\Sigma}\right)f_i^{\Sigma}=-\left(\partial_i\mathcal{N}_{\Lambda\Sigma}\right)L^{\Sigma}\\ \Updownarrow\\ 2i\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma}f_i^{\Sigma}=-\left(\partial_i\mathcal{N}_{\Lambda\Sigma}\right)L^{\Sigma};\end{gathered}\tag{3.1.51}$$

$$\left(\overline{\partial}_{\overline{i}}\mathcal{N}_{\Lambda\Sigma}\right)L^{\Sigma}=0\,.\tag{3.1.52}$$

It is worth mentioning that (3.1.52) does not imply the holomorphicity of $\mathcal{N}_{\Lambda\Sigma}$, as it will be clear further below.

Now, due to the Kähler-covariant holomorphicity of the sections L^{Λ}'s and M_{Λ}'s of the Hodge bundle $\Im$ over M_{n_V}, we may define some symplectic-indexed holomorphic functions $X^{\Lambda}(z)$ and $F_{\Lambda}(z)$ in the moduli space M_{n_V} by using the related Kähler potential

$$\begin{aligned}L^{\Lambda}(z,\overline{z})&\equiv\exp\left(\tfrac{1}{2}K(z,\overline{z})\right)X^{\Lambda}(z);\\ M_{\Lambda}(z,\overline{z})&\equiv\exp\left(\tfrac{1}{2}K(z,\overline{z})\right)F_{\Lambda}(z);\end{aligned}\tag{3.1.53}$$

we may then arrange them in the holomorphic $(2n_V+2)$-d symplectic vector

$$\Phi(z)\equiv\begin{pmatrix}X^{\Lambda}(z)\\ \\ F_{\Lambda}(z)\end{pmatrix}=\exp\left(-\frac{1}{2}K(z,\overline{z})\right)V(z,\overline{z})\ .\tag{3.1.54}$$

It is also easy to realize that (3.1.53) define nothing but sections of an holomorphic line bundle over M_{n_V}, and all previous formulae may be rewritten in terms of such sections. First of all, we may obtain a simple symplectic-invariant expression of the Kähler potential in the moduli space by recalling the normalization of the sections given by (3.1.25)

$$\begin{gathered}-i=\left\langle V,\overline{V}\right\rangle=\exp\left(K(z,\overline{z})\right)\left\langle\Phi(z),\overline{\Phi}(\overline{z})\right\rangle\\ \Updownarrow\\ K(z,\overline{z})=-ln\left[i\left\langle\Phi(z),\overline{\Phi}(\overline{z})\right\rangle\right]\equiv-ln\left[i\Phi^T(z)\,\epsilon\overline{\Phi}(\overline{z})\right]\end{gathered}\tag{3.1.55}$$

$$
\begin{aligned}
&= -ln \left[i \left(X^{\Lambda}(z)\,,\, F_{\Lambda}(z) \right) \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \overline{X}^{\Lambda}(\overline{z}) \\ \\ \overline{F}_{\Lambda}(\overline{z}) \end{pmatrix} \right] \\
&= -ln \left[i \left(X^{\Lambda}(z)\,,\, F_{\Lambda}(z) \right) \begin{pmatrix} -\overline{F}_{\Lambda}(\overline{z}) \\ \\ \overline{X}^{\Lambda}(\overline{z}) \end{pmatrix} \right] \\
&= -ln \left\{ i \left[\overline{X}^{\Lambda}(\overline{z})\, F_{\Lambda}(z) - X^{\Lambda}(z)\, \overline{F}_{\Lambda}(\overline{z}) \right] \right\}
\end{aligned}
\tag{3.1.56}
$$

$$\Updownarrow$$

$$
\exp\left[-K(z,\overline{z})\right] = i \left[\overline{X}^{\Lambda}(\overline{z})\, F_{\Lambda}(z) - X^{\Lambda}(z)\, \overline{F}_{\Lambda}(\overline{z}) \right] . \tag{3.1.57}
$$

(3.1.54) trivially yields

$$
\begin{gathered}
|V(z,\overline{z})\rangle = \exp\left(\tfrac{1}{2}K(z,\overline{z})\right) \begin{pmatrix} X^{\Lambda}(z) \\ \\ F_{\Lambda}(z) \end{pmatrix} \\
\Downarrow \\
|U_i(z,\overline{z})\rangle = \exp\left(\tfrac{1}{2}K(z,\overline{z})\right) \begin{pmatrix} (\partial_i K)\, X^{\Lambda}(z) + \partial_i X^{\Lambda}(z) \\ \\ (\partial_i K)\, F_{\Lambda}(z) + \partial_i F_{\Lambda}(z) \end{pmatrix} ,
\end{gathered}
\tag{3.1.58}
$$

and therefore, using (3.1.40), we get that

$$
\langle V, U_i \rangle = 0 \Leftrightarrow X^{\Lambda}(z)\, \partial_i F_{\Lambda}(z) - \left(\partial_i X^{\Lambda}(z)\right) F_{\Lambda}(z) = 0, \quad \forall i = 1, \ldots, n_V \, . \tag{3.1.59}
$$

Notice that the symplectic holomorphic vector $\varPhi$ has Kähler weights $(2,0)$, i.e., under a Kähler gauge transformation (3.1.15) it transforms as $\varPhi(z) \rightarrow \varPhi(z)\, e^{-f(z)}$. This is clearly due to the fact that the symplectic holomorphic sections $X^{\Lambda}(z)$ and $F_{\Lambda}(z)$ have Kähler weights $(2,0)$, and therefore under a Kähler gauge transformation (3.1.15) they respectively transform as

$$
\begin{cases}
X^{\Lambda}(z) \longrightarrow e^{-f(z)} X^{\Lambda}(z)\, ; \\
\\
F_{\Lambda}(z) \longrightarrow e^{-f(z)} F_{\Lambda}(z)\, .
\end{cases}
\tag{3.1.60}
$$

Thus, $X^{\Lambda}(z)$ and $F_{\Lambda}(z)$ may be considered as symplectic sections of the holomorphic line bundle over M_{n_V}. Due to its Kähler transformation properties (3.1.60), the set $\left\{X^{\Lambda}\right\}_{\Lambda=0,1,\ldots,n_V}$ may be regarded, at least locally, as a set of homogeneous coordinates in the Kähler–Hodge manifold M_{n_V}, provided that the $n_V \times n_V$ holomorphic matrix of change between the Kähler gauge-invariant sets of coordinates $\left\{z^i\right\}_{i=1,\ldots,n_V}$ and

$$\{t^a(z)\}_{a=1,\ldots,n_V} \equiv \left\{ \frac{X^a(z)}{X^0(z)} \right\}_{a=1,\ldots,n_V} , \tag{3.1.61}$$

namely

$$e_i^a(z) \equiv \partial_i \left(\frac{X^a(z)}{X^0(z)} \right) , \tag{3.1.62}$$

is invertible.

If, as we suppose, this is the case, then

$$F_\Lambda(z) = F_\Lambda(z(X)) = F_\Lambda(X) . \tag{3.1.63}$$

By using relation (3.1.59) and the homogeneity of degree 1 of $F_\Lambda(X)$

$$X^\Sigma \partial_\Sigma F_\Lambda = F_\Lambda \tag{3.1.64}$$

(where $\partial_\Sigma \equiv \partial/\partial X^\Sigma$); it is thus possible to state that a symplectic coordinate frame $\left\{X^\Lambda\right\}$ always exists such that $F_\Lambda(X)$ may be written in terms of a scalar potential F, holomorphic and homogeneous of degree 2 in the X^Λ's:

$$\begin{aligned} &F_\Lambda(X) = \partial_\Lambda F(X) , \\ &X^\Sigma \partial_\Sigma F = 2F . \end{aligned} \tag{3.1.65}$$

The function $F(X) = F(X(z))$ is the (holomorphic) prepotential of vector multiplet couplings [39, 42–44] in the considered $\mathcal{N} = 2$ $(d = 4)$ MESGT. Due to the additivity of the Kähler weights, by definition the prepotential F has Kähler weights $(4, 0)$.

From definition (3.1.61) it follows that the t^a's are Kähler gauge-invariant coordinates, i.e., they have Kähler weights $(0, 0)$. It is also possible to choose a particular set of homogeneous coordinates in M_{n_V}, named "special coordinates" [36, 45–48], corresponding to the position

$$e_i^a(z) \equiv \partial_i \left(\frac{X^a(z)}{X^0(z)} \right) = \delta_i^a , \tag{3.1.66}$$

i.e., to

$$\begin{cases} X^0 = 1; \\ X^i = t^i = z^i . \end{cases} \Longrightarrow f_i^\Lambda \equiv D_i L^\Lambda = e^{\frac{1}{2}K} D_i X^\Lambda = e^{\frac{1}{2}K}\left(\delta_i^\Lambda + (\partial_i K) X^\Lambda\right) . \tag{3.1.67}$$

By such considerations, it is then clear that the coordinates z^i's and $\overline{z}^{\overline{i}}$'s and the related partial differential operators ∂_i's and $\overline{\partial}_{\overline{i}}$'s have vanishing Kähler weights.

By using definitions (3.1.53), the lower boundedness of the Kähler potential allows one to rewrite (3.1.52) as follows:

$$\left(\overline{\partial}_{\overline{i}}\mathcal{N}_{\Lambda\Sigma}\right)X^{\Sigma}=0\ . \tag{3.1.68}$$

By considering the set of local homogeneous coordinates $\{t^{a}(z)\}_{a=1,\dots,n_V}$ previously defined, the above result may be recast in the following form:

$$\begin{gathered}\left[\frac{\partial}{\partial\overline{z}^{\overline{i}}}\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)\right]X^{\Sigma}\left(z\right)=0\\ \Updownarrow\\ \overline{\partial}_{\overline{i}}\left[\overline{\left(\frac{X^{a}}{X^{0}}\right)}\left(\overline{z}\right)\right]\left[\frac{\partial}{\partial\overline{\left(\frac{X^{a}}{X^{0}}\right)}}\mathcal{N}_{\Lambda\Sigma}\left(X,\overline{X}\right)\right]X^{\Sigma}=0\\ \Updownarrow\\ \overline{e}_{\overline{i}}^{\overline{a}}\left(\overline{z}\right)\left[\frac{\partial}{\partial\overline{\left(\frac{X^{a}}{X^{0}}\right)}}\mathcal{N}_{\Lambda\Sigma}\left(X,\overline{X}\right)\right]X^{\Sigma}=0\ ,\end{gathered} \tag{3.1.69}$$

where in the last line we introduced $\overline{e}_{\overline{i}}^{\overline{a}}\left(\overline{z}\right)\equiv\overline{e_{i}^{a}\left(z\right)}$. By specializing (3.1.69) to "special coordinates," we may rewrite it as

$$\frac{\partial}{\partial\overline{X}^{\overline{i}}}\mathcal{N}_{\Lambda0}\left(X,\overline{X}\right)+\left[\frac{\partial}{\partial\overline{X}^{\overline{i}}}\mathcal{N}_{\Lambda j}\left(X,\overline{X}\right)\right]X^{j}=0\ . \tag{3.1.70}$$

Thus, provided that the matrix e_{i}^{a} exists (and it is invertible), due to the generally nontrivial dependence of $\mathcal{N}_{\Lambda\Sigma}$ on the (eventually "special") homogeneous coordinates of M_{n_V}, it is clear that

$$\left(\overline{\partial}_{\overline{i}}\mathcal{N}_{\Lambda\Sigma}\right)L^{\Sigma}=0\nRightarrow\overline{\partial}_{\overline{i}}\mathcal{N}_{\Lambda\Sigma}=0\ , \tag{3.1.71}$$

as previously announced.

At this point, in order to investigate more in depth the differential properties of the complex symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$, let us consider the nontrivial orthogonal relation given by (3.1.40), and let us use the Ansätze (3.1.33) and (3.1.34)

$$\begin{gathered}0=\langle V,U_{i}\rangle\equiv V^{T}\epsilon U_{i};\\ \Updownarrow\\ M_{\Lambda}f_{i}^{\Lambda}-L^{\Lambda}h_{i\Lambda}=2i\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma}L^{\Lambda}f_{i}^{\Sigma}=0.\end{gathered} \tag{3.1.72}$$

Thence, (3.1.51) and (3.1.72) imply (for lower bounded Kähler potential)

$$\left(\partial_{i}\mathcal{N}_{\Lambda\Sigma}\right)L^{\Lambda}L^{\Sigma}=0\Leftrightarrow\left(\partial_{i}\mathcal{N}_{\Lambda\Sigma}\right)X^{\Lambda}X^{\Sigma}=0\ . \tag{3.1.73}$$

It is interesting to notice that, despite the symmetry of $\partial_{i}\mathcal{N}_{\Lambda\Sigma}$ and $X^{\Lambda}X^{\Sigma}$ in the symplectic indices, the dependence of $\mathcal{N}_{\Lambda\Sigma}$ on the X's is such as to make the product $\left(\partial_{i}\mathcal{N}_{\Lambda\Sigma}\right)X^{\Lambda}X^{\Sigma}$ vanish. Thus, the differential properties of $\mathcal{N}_{\Lambda\Sigma}$ may be summarized as follows:

$$\begin{cases} \left(\overline{\partial}_{\bar{i}}\mathcal{N}_{\Lambda\Sigma}\right) X^{\Sigma} = 0 \; ; \\ \\ \left(\partial_i \mathcal{N}_{\Lambda\Sigma}\right) X^{\Lambda} X^{\Sigma} = 0 \; . \end{cases} \tag{3.1.74}$$

It should also be noticed that under coordinate transformations the holomorphic symplectic vector of sections $\Phi(z)$ transforms as

$$\widetilde{\Phi}(z) = e^{-f_{\mathcal{S}}(z)}\mathcal{S}(z)\Phi(z) \;, \tag{3.1.75}$$

where the holomorphic $(2n_V + 2) \times (2n_V + 2)$ matrix $\mathcal{S}(z)$ has a symplectic real structure, i.e., it is an element of $Sp(2n_V+2,\mathbb{R})$, preserving the $(2n_V + 2)$-d symplectic metric defined in (3.1.24); the $\mathcal{S}$-dependent factor $e^{-f_{\mathcal{S}}(z)}$ corresponds to a (an holomorphic) Kähler transformation. We may naturally divide $\mathcal{S}(z)$ in $(n_V + 1)$-d subblocks

$$\mathcal{S}(z) = \begin{pmatrix} A(z) & B(z) \\ \\ C(z) & D(z) \end{pmatrix} . \tag{3.1.76}$$

The symplecticity condition $\mathcal{S}^T(z)\epsilon\mathcal{S}(z) = \epsilon$ then implies the following relations among the subblocks:

$$\begin{cases} A^T D - C^T B = \mathbb{I}, \\ \\ A^T C - C^T A = B^T D - D^T B = 0. \end{cases} \tag{3.1.77}$$

Now, by differentiating both sides of the degree 2 homogeneity property of $F(X)$ (with Kähler weights $(4,0)$)

$$F(X) = \frac{1}{2} X^{\Lambda} F_{\Lambda} \;, \tag{3.1.78}$$

we trivially reobtain that F_{Λ} (having Kähler weights $(2,0)$) is homogeneous of degree 1 in the X^{Λ}'s (see (3.1.64))

$$F_{\Sigma} = X^{\Lambda} F_{\Lambda\Sigma} \;, \tag{3.1.79}$$

where we defined the Kähler gauge-invariant rank-2 symmetric tensor $F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^{\Lambda}\partial X^{\Sigma}}$, denoted with $\mathcal{F}(z)$ in symplectic matrix notation.[7] By iterating the differentiation, we get

$$X^{\Lambda} F_{\Lambda\Sigma\Xi} = 0 \;, \tag{3.1.80}$$

[7] Attention should be paid to carefully distinguish between:
1) the quantities F, F_{Λ}, $F_{\Lambda\Sigma} \equiv \frac{\partial^2 F}{\partial X^{\Lambda}\partial X^{\Sigma}} \equiv \mathcal{F}$
and
2) the quantities $\mathcal{F}^{-\Lambda}$, $\mathcal{F}^{+\Lambda}$, $\mathcal{F}^{\Lambda}$ and ${}^*\mathcal{F}^{\Lambda}$, which are related to the Abelian vector field strengths in the $\mathcal{N} = 2$, $d = 4$ n_V-fold MESGT; they will be introduced in Subsect. 3.2.

simply meaning that $F_{\Lambda\Sigma}$ is homogeneous of degree 0 in the coordinates X^{Λ}'s.

By recalling definitions (3.1.26) and (3.1.53), the Kähler covariant derivatives of $F_{\Lambda\Sigma}$ read

$$\begin{aligned} D_i F_{\Lambda\Sigma} &= \partial_i F_{\Lambda\Sigma} = \frac{\partial F_{\Lambda\Sigma}(z)}{\partial z^i} = D_i X^{\Xi}(z) \frac{\partial F_{\Lambda\Sigma}(z)}{\partial X^{\Xi}} \\ &= e^{-\frac{1}{2}K(z,\overline{z})} D_i L^{\Xi}(z,\overline{z}) \frac{\partial F_{\Lambda\Sigma}(z)}{\partial X^{\Xi}} = e^{-\frac{1}{2}K(z,\overline{z})} f_i^{\Xi}(z,\overline{z}) F_{\Lambda\Sigma\Xi}(z). \end{aligned} \tag{3.1.81}$$

Consequently, by using such a result we may write

$$\begin{aligned} h_{i\Lambda} &\equiv D_i M_{\Lambda} = e^{\frac{1}{2}K} D_i F_{\Lambda} = e^{\frac{1}{2}K} D_i \left(X^{\Sigma} F_{\Lambda\Sigma}\right) \\ &= e^{\frac{1}{2}K}\left(D_i X^{\Sigma}\right) F_{\Lambda\Sigma} + e^{\frac{1}{2}K} X^{\Sigma} D_i F_{\Lambda\Sigma} = \left(D_i L^{\Sigma}\right) F_{\Lambda\Sigma} + f_i^{\Xi} X^{\Sigma} F_{\Lambda\Sigma\Xi} \\ &= F_{\Lambda\Sigma} f_i^{\Sigma}. \end{aligned} \tag{3.1.82}$$

By recalling the Ansatz (3.1.34), we thus obtain that the two following formulae are equivalent:

$$h_{i\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^{\Sigma}; \tag{3.1.83}$$

$$h_{i\Lambda} = F_{\Lambda\Sigma} f_i^{\Sigma}. \tag{3.1.84}$$

Nevertheless, it is worth pointing out that, whereas (3.1.83) always holds, (3.1.84) is meaningful only in the cases in which the prepotential F may be defined.[8]

Now, by using definition (3.1.26) and (3.1.81) and (3.1.82), from the third of (3.1.36) we get

$$\begin{aligned} C_{ijk} &= \langle D_i U_j, U_k \rangle \equiv \left(D_i U_j\right)^T \epsilon U_k \\ &= \left(D_i D_j L^{\Lambda},\, D_i D_j M_{\Lambda}\right) \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} D_k L^{\Lambda} \\ D_k M_{\Lambda} \end{pmatrix} \\ &= \left(D_i D_j L^{\Lambda},\, D_i D_j M_{\Lambda}\right) \begin{pmatrix} -D_k M_{\Lambda} \\ D_k L^{\Lambda} \end{pmatrix} \\ &= -\left(D_i D_j L^{\Lambda}\right) D_k M_{\Lambda} + \left(D_i D_j M_{\Lambda}\right) D_k L^{\Lambda} \\ &= -h_{k\Lambda} D_i f_j^{\Lambda} + f_k^{\Lambda} D_i h_{j\Lambda} = -\overline{\mathcal{N}}_{\Lambda\Sigma} f_k^{\Sigma} D_i f_j^{\Lambda} + f_k^{\Lambda} D_i \left(\overline{\mathcal{N}}_{\Lambda\Sigma} f_j^{\Sigma}\right) \\ &= -\overline{\mathcal{N}}_{\Lambda\Sigma} f_k^{\Sigma} D_i f_j^{\Lambda} + f_k^{\Lambda}\left(D_i \overline{\mathcal{N}}_{\Lambda\Sigma}\right) f_j^{\Sigma} + f_k^{\Lambda} \overline{\mathcal{N}}_{\Lambda\Sigma} D_i f_j^{\Sigma} \\ &= f_k^{\Lambda}\left(\partial_i \overline{\mathcal{N}}_{\Lambda\Sigma}\right) f_j^{\Sigma} = f_k^{\Lambda}\left(\partial_i F_{\Lambda\Sigma}\right) f_j^{\Sigma} \\ &= e^{-\frac{1}{2}K} f_k^{\Lambda} f_i^{\Xi} f_j^{\Sigma} F_{\Lambda\Sigma\Xi} = e^{-\frac{1}{2}K} f_i^{\Lambda} f_j^{\Sigma} f_k^{\Xi} F_{\Lambda\Sigma\Xi}, \end{aligned} \tag{3.1.85}$$

[8] For a discussion of some relevant cases in which F does not exist (such as the low energy effective action of $\mathcal{N}=2$ heterotic string theory), see, e.g., [38].

where we also used the Kähler gauge-invariance of the complex matrix $\mathcal{N}_{\Lambda\Sigma}$ and the symmetry of the tensor C_{ijk}. The symplectic-invariant and Kähler-covariant expression (3.1.85) for C_{ijk} may be further elaborated (at the price of losing the manifest Kähler covariance) by expliciting the Kähler-covariant derivative encoded in f_i^Λ and using (3.1.80):

$$\begin{aligned}
C_{ijk} &= e^{-\frac{1}{2}K} f_i^\Lambda f_j^\Sigma f_k^\Xi F_{\Lambda\Sigma\Xi} = e^{-\frac{1}{2}K} \left(D_i L^\Lambda\right)\left(D_j L^\Sigma\right)\left(D_k L^\Xi\right) F_{\Lambda\Sigma\Xi} \\
&= e^K \left(D_i X^\Lambda\right)\left(D_j X^\Sigma\right)\left(D_k X^\Xi\right) F_{\Lambda\Sigma\Xi} \\
&= e^K \left[\partial_i X^\Lambda + (\partial_i K) X^\Lambda\right]\left[\partial_j X^\Sigma + (\partial_j K) X^\Sigma\right]\left[\partial_k X^\Xi + (\partial_k K) X^\Xi\right] F_{\Lambda\Sigma\Xi} \\
&= e^K \left(\partial_i X^\Lambda\right)\left(\partial_j X^\Sigma\right)\left(\partial_k X^\Xi\right) F_{\Lambda\Sigma\Xi}. \qquad (3.1.86)
\end{aligned}$$

By further specializing such a result in the symplectic frame (3.1.67) of "special coordinates," for which $f_i^\Lambda = e^{\frac{1}{2}K}\delta_i^\Lambda$, one finally gets (see also (3.1.61))

$$C_{ijk} = e^K \delta_i^\Lambda \delta_j^\Sigma \delta_k^\Xi F_{\Lambda\Sigma\Xi}(t) = e^K F_{ijk}(t) = e^K \partial_i \partial_j \partial_k F(t), \qquad (3.1.87)$$

which is symplectic-invariant, but manifestly Kähler-noncovariant.

It is easy to see that, in the case in which the prepotential F exists, the symplectic-orthogonality relation (3.1.40) between the $Sp(2n_V+2)$-covariant vectors V and U_i reduces to nothing but an integrability condition in the "special coordinates" (3.1.67) of M_{n_V}. In order to show this, let us firstly explicit relation (3.1.40), by writing

$$\begin{aligned}
0 &= \langle V, U_i \rangle \equiv V^T \epsilon U_i = \left(L^\Lambda, M_\Lambda\right) \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} D_i L^\Lambda \\ D_i M_\Lambda \end{pmatrix} \\
&= \left(L^\Lambda, M_\Lambda\right) \begin{pmatrix} -D_i M_\Lambda \\ D_i L^\Lambda \end{pmatrix} = M_\Lambda D_i L^\Lambda - L^\Lambda D_i M_\Lambda \\
&= -e^K \left(X^\Lambda D_i F_\Lambda - F_\Lambda D_i X^\Lambda\right) \\
&= -e^K \left\{X^\Lambda \left[\partial_i F_\Lambda + (\partial_i K) F_\Lambda\right] - F_\Lambda \left[\partial_i X^\Lambda + (\partial_i K) X^\Lambda\right]\right\} \\
&= -e^K \left(X^\Lambda \partial_i F_\Lambda - F_\Lambda \partial_i X^\Lambda\right) = e^K \left[\partial_i \left(X^\Lambda F_\Lambda\right) - 2F_\Lambda \partial_i X^\Lambda\right] \\
&= e^K \left[\partial_i \left(X^\Lambda F_\Lambda\right) - 2\partial_i F\right] = e^K \partial_i \left(X^\Lambda F_\Lambda - 2F\right). \qquad (3.1.88)
\end{aligned}$$

If we now specify the result (3.1.88) to the "special coordinates" (3.1.67) and recall the property (3.1.80) of homogeneity of degree 0 of the function $F_{\Lambda\Sigma}$, we get

$$\begin{gathered}
\langle V, U_i \rangle = 0 \Longleftrightarrow \partial_i F - X^j \partial_i \partial_j F = 0; \\
\Downarrow \\
\partial_k \partial_i F - \partial_i \partial_k F - X^j \partial_k \partial_i \partial_j F = 0; \\
\Updownarrow \\
\partial_k \partial_i F(t) = \partial_i \partial_k F(t), \ \forall (i,k) \in \{1, \ldots, n_V\}^2, \qquad (3.1.89)
\end{gathered}$$

which is satisfied iff the function $F(t)$ satisfies the Schwarz lemma on partial derivatives, i.e., if it is integrable in the "special coordinates" of M_{n_V}. Thus, we may say that, whenever the prepotential F exists, its generalized, symplectic-invariant integrability condition in the moduli space M_{n_V} is given by the orthogonality relation $\langle V, U_i \rangle = 0$.

Now, by recalling (3.1.75)–(3.1.77) and using the relation (3.1.78), we may rewrite the transformation law of X^Λ and $F(X)$ under symplectic transformations (disregarding the Kähler transformation factors) respectively as follows:

$$\widetilde{X}(z) = \left[A(z) + B(z)\mathcal{F}(z)\right] X(z); \tag{3.1.90}$$

$$\begin{aligned}\widetilde{F}\left(\widetilde{X}\right) &= \widetilde{F}\left((A + B\mathcal{F})X\right) = \tfrac{1}{2}\widetilde{X}^\Lambda \widetilde{F}_\Lambda \\ &= \left[F(X) + X^\Lambda \left(C^T B\right)_\Lambda^{\ \Sigma} F_\Sigma + \tfrac{1}{2} X^\Lambda \left(C^T A\right)_{\Lambda\Sigma} X^\Sigma + \tfrac{1}{2} F_\Lambda \left(D^T B\right)^{\Lambda\Sigma} F_\Sigma\right].\end{aligned} \tag{3.1.91}$$

Analogously, by using the Ansatz (3.1.33), the transformation property (3.1.75) yields the following transformation law for the matrix $\mathcal{N}$:

$$\widetilde{\mathcal{N}}_{\Lambda\Sigma}\left(\widetilde{X}, \widetilde{F}\right) = \left(C + D\mathcal{N}(X,F)\right)\left(A + B\mathcal{N}(X,F)\right)^{-1}. \tag{3.1.92}$$

Equation (3.1.90) shows that the transformation $X \to \widetilde{X}$ can eventually be singular, thus implying the nonexistence of the prepotential $F(X)$, depending on the choice of the symplectic gauge [38,49]. On the other hand, some physically interesting cases, such as the $\mathcal{N}=2 \longrightarrow \mathcal{N}=0$ SUSY breaking [50], correspond to situations in which $F(X)$ does not exist. Therefore, the tensor calculus constructions of the $\mathcal{N}=2$ theories actually turn out to be not completely general, because they use special coordinates from the very beginning, and they are essentially founded on the existence of the prepotential $F(X)$.

By considering the low-energy $\mathcal{N}=2$, $d=4$ MESGT Lagrangian density, we may observe that $Im\left(\mathcal{N}_{\Lambda\Sigma}\right)$ and $Re\left(\mathcal{N}_{\Lambda\Sigma}\right)$ are respectively related to the kinetic and topological terms $\mathcal{F}^2$ and $\mathcal{F}\widetilde{\mathcal{F}}$ of the (field strenghts of the) Maxwell vector fields; for this reason, usually the matrix $\mathcal{N}_{\Lambda\Sigma}$ is referred to as the "kinetic matrix" of the $\mathcal{N}=2$, $d=4$ MESGT.

Furthermore, from the Ansatz (3.1.34) and the second result of (3.1.36) we obtain an interesting relation, relating the metric $G_{i\bar{j}}$ of the Kähler–Hodge manifold M_{n_V} to the symplectic vector functions f_i^Λ and $h_{i\Lambda}$ through $Im\left(\mathcal{N}_{\Lambda\Sigma}\right)$

$$\begin{aligned}G_{i\bar{j}} &= -i\left\langle U_i, \overline{U}_{\bar{j}}\right\rangle \equiv -iU_i^T \epsilon \overline{U}_{\bar{j}} \\ &= -i\left[-\left(D_i L^\Lambda\right)\overline{D}_{\bar{j}}\overline{M}_\Lambda + \left(D_i M_\Lambda\right)\overline{D}_{\bar{j}}\overline{L}^\Lambda\right] \\ &= -i\left[-\left(D_i L^\Lambda\right)\mathcal{N}_{\Lambda\Sigma}\overline{D}_{\bar{j}}\overline{L}^\Sigma + \overline{\mathcal{N}}_{\Lambda\Sigma}\left(D_i L^\Sigma\right)\overline{D}_{\bar{j}}\overline{L}^\Lambda\right] \\ &= -2Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\left(D_i L^\Lambda\right)\overline{D}_{\bar{j}}\overline{L}^\Sigma = -2Im\left(\mathcal{N}_{\Lambda\Sigma}\right) f_i^\Lambda \overline{f}_{\bar{j}}^\Sigma.\end{aligned} \tag{3.1.93}$$

Whenever the prepotential F may be defined, by using (3.1.84) it may be analogously obtained that

$$G_{i\bar{j}} = 2Im\,(F_{\Lambda\Sigma})\, f_i^{\Lambda}\overline{f}_{\bar{j}}^{\Sigma} \ . \tag{3.1.94}$$

As previously noticed, the function f_i^{Λ}, endowed with a local index in the SKG of M_{n_V} and with a global index in $Sp(2n_V+2)$ (symplectic symmetry), plays a key role in intertwining such two different levels of symmetry, revealing the inner special Kähler–Hodge symplectic structure of the $\mathcal{N}=2$, $d=4$ MESGT.

In regular SKG, the Kähler metric $G_{i\bar{j}}$ is (strictly) positive definite in all M_{n_V}. By using (3.1.93), this implies the (strictly) negative definiteness of the real $(n_V+1)\times(n_V+1)$ matrix $Im\,(\mathcal{N}_{\Lambda\Sigma})$; a shorthand notation for such a condition, encoding the regularity of the SKG of M_{n_V}, reads

$$Im\,(\mathcal{N}_{\Lambda\Sigma}) < 0 \ , \tag{3.1.95}$$

which also follows from the position of such term in the low-energy $\mathcal{N}=2$ $(d=4)$ MESGT Lagrangian density.

At this point, whenever the Jacobian matrix $e_i^a\,(z)$ exists and it is invertible, a number of useful formulae may be obtained, relating the two main symplectic matrices introduced so far, namely the "kinetic" one $(\mathcal{N}_{\Lambda\Sigma})$ and the one given by the double symplectic derivatives of the prepotential ($F_{\Lambda\Sigma}$, also denoted with $\mathcal{F}$). The main result is[9]

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} - 2i\overline{T}_{\Lambda}\overline{T}_{\Sigma}\,(L_I m\,(\mathcal{F})\,L) \ , \tag{3.1.96}$$

where the symplectic vector T_{Λ} is defined as follows:

$$T_{\Lambda} \equiv -i\frac{\left(Im\,(\mathcal{F})\,\overline{L}\right)_{\Lambda}}{\overline{L}Im\,(\mathcal{F})\,\overline{L}} = 2i\,(Im\,(\mathcal{N})\,L)_{\Lambda} \ , \tag{3.1.97}$$

---

[9] Provided that the holomorphic prepotential $F\,(X)$ satisfies the Schwarz lemma in the moduli space, the symmetry of the $(n_V+1)\times(n_V+1)$ complex matrix $\mathcal{N}_{\Lambda\Sigma}$ is evident from (3.1.96), which anyway holds true only whenever the holomorphic Jacobian matrix $e_i^a\,(z)$, defined by (3.1.62), exists and it is invertible.

In general, the fundamental Ansätze, expressed by (3.1.33) and (3.1.34) and formulated in order to solve the so-called special geometry constraints given by (3.1.3), does not imply the symmetry of $\mathcal{N}_{\Lambda\Sigma}$. Therefore, assuming such a property, which is then largely used, would seem to imply some loss of generality.

Actually, also in the particular cases in which it is not possible to define a local system of homogeneous coordinates in the moduli space (i.e., when the matrix $e_i^a\,(z)$ does not exist or it is not invertible), it may be shown that (3.1.33) and (3.1.34) are always solved by a symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$.

Thus,

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Sigma\Lambda}$$

does not yield any loss of generality in the study of the symplectic special Kähler structure (of the moduli space) of the $\mathcal{N}=2$ $(d=4)$ n_V-fold MESGT.

and the following relations hold:

$$\begin{cases} L Im\left(\mathcal{F}\right)\overline{L} = -\frac{1}{2}; \\ \\ T_{\Lambda}\overline{L}^{\Lambda} = -i; \\ \\ 4\overline{L} Im\left(\mathcal{F}\right)\overline{L} = \left(L Im\left(\mathcal{N}\right)L\right)^{-1}. \end{cases} \tag{3.1.98}$$

Now, instead of saturating the symplectic indices of the product $f_i^{\Lambda}\overline{f}_{\overline{j}}^{\Sigma}$, as made in (3.1.93) and (3.1.94), we may instead saturate the Kähler ones, and the obvious choice is to use $G^{i\overline{j}}$; by doing this, we introduce the symplectic tensor

$$U^{\Lambda\Sigma} \equiv G^{i\overline{j}} f_i^{\Lambda}\overline{f}_{\overline{j}}^{\Sigma} = -\frac{1}{2}\left(\left(Im\left(\mathcal{N}\right)\right)^{-1}\right)^{\Lambda\Sigma} - \overline{L}^{\Lambda}L^{\Sigma}, \tag{3.1.99}$$

where in the last passage we used (3.1.36) and (3.1.93). Notice that in our notation $\mathcal{N}^{\Lambda\Sigma}$ is nothing but the inverse of $\mathcal{N}_{\Lambda\Sigma}$

$$\mathcal{N}^{\Lambda\Sigma}\mathcal{N}_{\Sigma\Delta} \equiv \delta_{\Delta}^{\Lambda}, \tag{3.1.100}$$

and moreover it holds that

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Sigma\Lambda} \Longrightarrow \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Sigma} = \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Lambda}. \tag{3.1.101}$$

By considering (3.1.93), (3.1.94), and (3.1.99), we finally get

$$\begin{aligned} U^{\Lambda\Sigma} &= \frac{1}{2}\left(\left(Im\left(\mathcal{F}\right)\right)^{-1}\right)^{\Lambda\Sigma} + L^{\Lambda}\overline{L}^{\Sigma} \\ &\equiv \mathcal{T}_I^{\Lambda}G^{IJ}\overline{\mathcal{T}}_J^{\Sigma}. \end{aligned} \tag{3.1.102}$$

In the second line we defined the (n_V+1)-d square matrix

$$\mathcal{T}_I^{\Lambda} \equiv \left(\mathcal{T}_i^{\Lambda}, \mathcal{T}_0^{\Lambda} \equiv L^{\Lambda}\right) \tag{3.1.103}$$

and, similarly to what previously done for the f's and h's, we extended the Kähler metric to a (n_V+1)-d block form

$$G^{IJ} = \begin{pmatrix} G^{i\overline{j}} & 0 \\ \\ 0 & -1 \end{pmatrix}. \tag{3.1.104}$$

Because of (3.1.93), (3.1.94,) and (3.1.95), we obtain that $Im\left(\mathcal{F}\right)$ is an (n_V+1)-d square symplectic matrix, with n_V positive and one negative eigenvalues. $U^{\Lambda\Sigma}$ is an (n_V+1)-d square symplectic matrix, too, but instead it has rank n_V because, as it may be explicitly shown, it annihilates the vector[10] T_{Λ} and its conjugate $\overline{T}_{\Lambda}$

[10] In the next subsection we will see that T_{Λ} is nothing but the graviphoton projector in the $\mathcal{N}=2$, $d=4$ MESGT.

$$T_\Lambda U^{\Lambda\Sigma} = U^{\Lambda\Sigma}\overline{T}_\Sigma = 0 \ . \tag{3.1.105}$$

From (3.1.102) we can further compute

$$\left[det\left(2Im\left(\mathcal{F}\right)\right)\right]^{-1} = det\left(U^{\Lambda\Sigma} - L^\Lambda\overline{L}^\Sigma\right) \ , \tag{3.1.106}$$

and using the (n_V+1)-d square matrices $\mathcal{T}_I^\Lambda$ and G^{IJ}, we obtain

$$det\left(2Im\left(\mathcal{F}\right)\right) = -det\left(G_{i\overline{j}}\right)\left|det\left(\mathcal{T}\right)\right|^{-2} \ . \tag{3.1.107}$$

By means of simple properties of the determinants, such relations yield the following result:

$$det\left(\mathcal{T}\right) = \exp\left[\left(n_V+1\right)\frac{1}{2}K\right]\left(det\left(e\right)\right)\left(X^0\right)^{n_V+1} \tag{3.1.108}$$

$$\Downarrow$$

$$\left|det\left(\mathcal{T}\right)\right|^2 = \exp\left[\left(n_V+1\right)K\right]\left|det\left(e\right)\right|^2\left(X^0\overline{X}^0\right)^{n_V+1} \ , \tag{3.1.109}$$

where $det\left(e\right)$ is the determinant of the (n_V+1)-d square matrix defined in (3.1.62), i.e., it is nothing but the Jacobian of the change of basis of coordinates

$$\left\{z^i\right\}_{i=1,\dots,n_V} \longleftrightarrow \left\{t^a\left(z\right)\right\}_{a=1,\dots,n_V} \equiv \left\{\frac{X^a\left(z\right)}{X^0\left(z\right)}\right\}_{a=1,\dots,n_V} \tag{3.1.110}$$

in the Kähler–Hodge moduli space M_{n_V} of the considered theory.

It then follows that [40,41,43,44]

$$\partial_i\overline{\partial}_{\overline{j}}ln\left(det\left(Im\left(\mathcal{F}\right)\right)\right) = \partial_i\overline{\partial}_{\overline{j}}ln\left(det\left(G_{k\overline{l}}\right)\right) - \left(n_V+1\right)G_{i\overline{j}} \ . \tag{3.1.111}$$

At this point, by using the SG constraints (3.1.3) satisfied by the RC tensor in the moduli space, we may compute the corresponding Ricci tensor as follows:

$$R_{i\overline{j}} \equiv R_{i\overline{j}l\overline{l}}G^{l\overline{l}} = \left(n_V+1\right)G_{i\overline{j}} - C_{ilp}\overline{C}_{\overline{j}\overline{l}\overline{p}}G^{l\overline{l}}G^{p\overline{p}} \ . \tag{3.1.112}$$

By using (3.1.111), and recalling that on a Kähler manifold the Ricci tensor can always be written as [37]

$$R_{i\overline{j}} = \partial_i\overline{\partial}_{\overline{j}}ln\left(det\left(G_{k\overline{l}}\right)\right) \ , \tag{3.1.113}$$

we finally get

$$\partial_i\overline{\partial}_{\overline{j}}ln\left(det\left(Im\mathcal{F}\right)\right) = -C_{ilp}\overline{C}_{\overline{j}\overline{l}\overline{p}}G^{l\overline{l}}G^{p\overline{p}} \ . \tag{3.1.114}$$

Notice that such a result generally characterizes every special Kähler–Hodge symplectic manifold which admits local coordinates defined by means of the (ratios of the) sections of the related holomorphic line bundle, i.e., for which the (n_V+1)-d square matrix defined in (3.1.62) exists and it is invertible.

3.2 Electric–Magnetic Duality, Central Charge, and Attractor Mechanism

In this subsection we will briefly report how, in $\mathcal{N}=2$, $d=4$ SUGRA coupled with n_V Abelian vector multiplets (and n_H hypermultiplets), the phenomenon of the doubling of preserved supersymmetries (and therefore of the restoration of maximal SUSY) occurs near the EH of the $\frac{1}{2}$-BPS stable soliton metric solution, whose simplest example is represented by the previously considered extremal RN BH. Furthermore, we will see the AM at work in the dynamical evolution of the relevant set of scalars, namely in the on-shell dynamics of the manifold M_{n_V}.

For simplicity's sake, let us set the fermionic bilinears and the $\mathcal{N}=2$ generalization of the Fayet–Iliopoulos term to zero (i.e., let us disregard the fermionic contributions and the presence of supersymmetric gaugings). We may then write the local SUSY transformations for the gravitino, for the gaugino, and for the hyperino in a manifestly symplectic covariant way as follows (see [34] and [35], where the complete, general SUGRA transformations may be found, too):

$$\begin{cases} \delta\psi_{A\mu} = \mathcal{D}_\mu \varepsilon_A + \epsilon_{AB}\varepsilon^B T^-_{\mu\nu}\gamma^\nu; \\ \delta\lambda^{iA} = i\varepsilon^A\gamma^\mu\partial_\mu z^i + \epsilon^{AB}\varepsilon_B \mathcal{F}^{i-}_{\mu\nu}\gamma^{\mu\nu}; \\ \delta\zeta_\alpha = i\epsilon_{AB}\varepsilon^A \mathcal{U}^{B\beta}_u \gamma^\mu \mathbb{C}_{\alpha\beta}\partial_\mu q^u, \end{cases} \tag{3.2.1}$$

where we recall once again that λ^{iA}, $\psi_{A\mu}$, and ζ_α respectively are the chiral gaugino, gravitino, and hyperino fields. Moreover, ε_A and ε^A respectively denote the chiral and antichiral local SUSY parameters, and ϵ^{AB} is the $SO(2)$ Ricci tensor

$$\epsilon^{AB} = -\epsilon^{BA}, \epsilon^2 = -\mathbb{I}\ , \tag{3.2.2}$$

namely the 2-d contravariant counterpart of the symplectic metric defined by (3.1.24). The moduli-dependent, symplectic-invariant quantities $T^-_{\mu\nu}$ and $\mathcal{F}^{i-}_{\mu\nu}$ respectively are the (imaginary self-dual) graviphoton and vector field strenghts. For what concerns the gravitino field $\psi_{A\mu}$, apart from being a spinor-valued one-form on s-t, it behaves as a section of the bundle $\Im \otimes \mathcal{SU}$, where $\mathcal{SU}$ is an $SU(2)$ principal bundle on the quaternionic scalar manifold M_{n_H} ($dim_{\mathbb{R}} M_{n_H} = 4n_H$) related to the considered n_H hypermultiplets. Consequently, the derivative $\mathcal{D}_\mu$ appearing in the first line of the transformations (3.2.1) is the covariant derivative w.r.t. the (ungauged) $U(1)$-bundle $\Im$ on M_{n_V} (with connection Q given by (3.1.14)) and the (ungauged) $SU(2)$ principal bundle $\mathcal{SU}$ on M_{n_H} (with connection ω^x, where x is an $SU(2)$ index). In the formalism of forms, we thus have

$$\mathcal{D}\varepsilon_A = d\varepsilon_A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \varepsilon_A + \frac{i}{2}Q \wedge \varepsilon_A + \frac{i}{2}\omega^x\left(\sigma_x\right)^B_A \wedge \varepsilon_B\ , \tag{3.2.3}$$

where d is the flat s-t differential, ω^{ab} is the spin-connection, and σ_x denotes the vector of Pauli matrices. Finally, $\mathcal{U}_u^{B\beta}$ and $\mathbb{C}_{\alpha\beta}$ respectively stand for the quaternionic Vielbein one-form and the $Sp(2n_H)$-invariant flat metric

$$\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}, \mathbb{C}^2 = -\mathbb{I} \, . \tag{3.2.4}$$

In order to describe the restoration of the maximal SUSY of the metric background of the $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT, i.e., the doubling of the number of preserved supersymmetries with respect to the four ones preserved by the $\frac{1}{2}$-BPS stable solitonic solution represented by the extremal (eventually RN) BH, we have to find solutions with unbroken $\mathcal{N} = 2$, $d = 4$ local SUSY.

In the present case, the relevant spinor Killing conditions to be solved are those given by (3.2.1), with the r.h.s.'s set to zero, i.e.,

$$\delta\psi_{A\mu} = \delta\lambda^{iA} = \delta\zeta_\alpha = 0 \, . \tag{3.2.5}$$

I. The first solution to (3.2.5) is the one corresponding to the standard flat vacuum, which is the asymptotical limit ($r \to \infty$) of the spherically symmetric, static extremal RN BH metric background. The corresponding unbroken, maximal $\mathcal{N} = 2$, $d = 4$ SUSY algebra is the $\mathcal{N} = 2$, $d = 4$ superPoincaré one (*asymptotical rigid* $\mathcal{N} = 2$ *SUSY*).
Concerning the field content of the theory in such a case, the 4-d metric is the flat, Minkowski $\eta_{\mu\nu}$, there are no vector fields, and all complex scalar fields in the considered n_V Abelian vector supermultiplets, as well as the quaternionic scalars in the n_H hypermultiplets, take arbitrary constant values

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu}, \\ T^-_{\mu\nu} = 0 = \mathcal{F}^{i-}_{\mu\nu}, \\ \partial_\mu z^i = 0 \Leftrightarrow z^i = z^i_\infty \in \mathbb{C}, \\ \partial_\mu q^u = 0 \Leftrightarrow q^u = q^u_\infty \in \mathbb{H}. \end{cases} \tag{3.2.6}$$

z^i_∞ is an unconstrained scalar field configuration in the n_V-d Kähler–Hodge complex moduli space M_{n_V} of the $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT. The positions (3.2.6) do provide solutions for the unbroken $\mathcal{N} = 2$, $d = 4$ SUGRA Killing spinor equations with constant, unconstrained values of the SUSY parameter ε_A, which therefore makes the local SUSY structure "rigid," i.e., global.
Thus, the unbroken SUSY manifests itself in the fact that each nonvanishing scalar field is the first component of a covariantly constant $\mathcal{N} = 2$ superfield for the vector and/or the hypermultiplet, but the supergravity superfield vanishes.

II. The second solution to (3.2.5) is much more sophisticated; as we will see by solving the related consistency conditions, it corresponds to the 4-d

BR metric, which is the "near-horizon" limit ($r \rightarrow r_H^+$) of the spherically symmetric, static extremal RN BH metric background.

First, it is possible to solve the Killing conditions for the gaugino and the hyperino just by using a suitable part of the previous Ansätze (3.2.6), namely

$$\begin{cases} \mathcal{F}^{i-}_{\mu\nu} = 0, \\ \\ \partial_\mu z^i = 0 \Leftrightarrow z^i = z^i_\infty \in \mathbb{C}, \\ \\ \partial_\mu q^u = 0 \Leftrightarrow q^u = q^u_\infty \in \mathbb{H}. \end{cases} \tag{3.2.7}$$

Second, we observe that the Killing equation for the gravitino

$$\delta\psi_{A\mu} = \mathcal{D}_\mu \varepsilon_A + \epsilon_{AB} T^-_{\mu\nu} \gamma^\nu \varepsilon^B = 0 \tag{3.2.8}$$

is not gauge-invariant. Consequently, without loss of generality we may consider variation of the gravitino field strength in a particular, suitable way, as shown in [11] and [12].

For what concerns the s-t metric, we may consider the geometry of the background with vanishing Riemann–Christoffel intrinsic scalar curvature R, vanishing Weyl tensor $C_{\mu\nu\lambda\delta}$ and covariantly constant graviphoton field strength $T^-_{\mu\nu}$

$$\begin{cases} R = 0, \\ \\ C_{\mu\nu\lambda\delta} = 0, \\ \\ \mathcal{D}_\lambda \left(T^-_{\mu\nu}\right) = 0. \end{cases} \tag{3.2.9}$$

While the first solution had a vanishing supergravity superfield, it may be shown that such a configuration corresponds to a covariantly constant superfield of $\mathcal{N}=2$, $d=4$, n_V-fold MESGT $W_{\alpha\beta}(x,\theta)$, whose first component is given by a two-component graviphoton field strength $T_{\alpha\beta}$.

The phenomenon of the doubling of preserved supersymmetries near the EH of the extremal RN BH may be qualitatively explained as follows.

It may be shown that the algebraic condition for the choice of broken versus unbroken $\mathcal{N}=2$, $d=4$ local SUSY is given in terms of a combination of the Weyl tensor and of the Riemann-covariant derivative of the graviphoton field strength. However, by the set (3.2.9) of Ansätze on the structure of the "near-horizon" metric background, both the Weyl tensor $C_{\mu\nu\lambda\delta}$ and the Riemann-covariant derivative of the graviphoton field strength $\mathcal{D}_\lambda \left(T^-_{\mu\nu}\right)$ separately vanish in proximity of the EH. Thus, all supersymmetries are restored in this limit, and one gets a covariantly constant superfield of $\mathcal{N}=2$, $d=4$, n_V-fold MESGT $W_{\alpha\beta}(x,\theta)$.

Considering a generic configuration of such a theory, in which the supergravity multiplet interacts with n_V Abelian vector supermultiplets and n_H hypermultiplets, we obtain that, beside the $\mathcal{N}=2$ supergravity superfield

$W_{\alpha\beta}(x,\theta)$, we also have covariantly constant $\mathcal{N}=2$ superfields, whose first component is given, similarly to what happened for the first solution, by the scalars of the corresponding multiplets.

However, whereas the flat vacuum given by the first solution admitted any value of the scalars, in the present case the nontrivial geometry of the metric background (which will then reveal to be the 4-d BR metric[11]), defined by the positions (3.2.9), imposes two consistency conditions for this second solution, namely

1. The Riemann–Christoffel tensor must match the product of two graviphoton field strengths
$$R_{\alpha\beta\alpha'\beta'} = T_{\alpha\beta}T_{\alpha'\beta'}. \tag{3.2.10}$$
2. The vector field strength must vanish (as given by the first position of Ansätze (3.2.7), too)
$$\mathcal{F}^{i-}_{\mu\nu} = 0 \,. \tag{3.2.11}$$

Later on, we will analyze the consistency conditions (3.2.10) and (3.2.11) more in depth. Now we move to deal with some noteworthy symplectic features of the special geometry of the n_V-d Kähler–Hodge complex moduli space M_{n_V} of such a theory. The additional symplectic structure allows one to introduce a central extension operator (and the related Kähler-covariant derivative) by purely geometric reasonings and in a completely symplectic-invariant way.

Considering the low-energy effective action of the $\mathcal{N}=2$, $d=4$ MESGT, the Kähler metric of M_{n_V} appears in the kinetic term of the complex scalars coming from the considered n_V Maxwell vector multiplets; it reads

$$G_{i\bar{j}}\partial_\mu z^i \partial_\nu \overline{z}^{\bar{j}} g^{\mu\nu}\sqrt{-g} \,. \tag{3.2.12}$$

As previously mentioned, the symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$ appears in the vector part of the action, which reads (setting the fermionic contributions to zero)

$$-\,2Im\left(\mathcal{F}^{-\Lambda}_{\mu\nu}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma\mu\nu}\right) = -2Im\left(\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{G}^{-\mu\nu}_{\Lambda}\right) \,, \tag{3.2.13}$$

where $\mathcal{F}^{-\Lambda}_{\mu\nu}$ is the complex, imaginary self-dual Maxwell field strength (see below). Instead, in general $\mathcal{G}^{-}_{\Lambda}$ is the Legendre transform of $\mathcal{F}^{-\Lambda}$

$$\mathcal{G}^{-\mu\nu}_{\Lambda} \equiv \frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-\Lambda}_{\mu\nu}} \,. \tag{3.2.14}$$

As yielded by (3.2.13), in the symplectic structure of the $\mathcal{N}=2$, $d=4$ n_V-fold MESGT, the above functional derivative may be equivalently reexpressed as the following linear combination:

[11] Notice indeed that the conditions of vanishing Weyl tensor and zero (overall) scalar curvature, respectively, expressed by the second and first position of the Ansätze (3.2.9), are compatible with the properties of the BR metric (see Sect. 1).

$$\mathcal{G}_{\Lambda}^{-} \equiv \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} \, . \tag{3.2.15}$$

$\mathcal{F}^{-\Lambda}$ is clearly moduli-independent

$$\partial_i \mathcal{F}^{-\Lambda} = 0 = \overline{\partial}_{\bar{i}}\mathcal{F}^{-\Lambda} \, . \tag{3.2.16}$$

Through the functional derivative of $\mathcal{L}$ given by (3.2.14), instead $\mathcal{G}_{\Lambda}^{-}$ depends on the moduli purely through the matrix $\overline{\mathcal{N}}_{\Lambda\Sigma}$ which however, as previously pointed out, has vanishing Kähler weights, because otherwise the Kähler structure of M_{n_V} would clash with the $Sp(2n_V+2)$-covariance of electric–magnetic duality of the theory. In general, the differential properties of $\mathcal{G}_{\Lambda}^{-}$ are the following:

$$\begin{cases} D_i\mathcal{G}_{\Lambda}^{-} = \partial_i\mathcal{G}_{\Lambda}^{-} = \left(\partial_i\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} = \left(D_i\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} \neq 0, \\ \\ \overline{D}_{\bar{i}}\mathcal{G}_{\Lambda}^{-} = \overline{\partial}_{\bar{i}}\mathcal{G}_{\Lambda}^{-} = \left(\overline{\partial}_{\bar{i}}\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} = \left(\overline{D}_{\bar{i}}\overline{\mathcal{N}}_{\Lambda\Sigma}\right)\mathcal{F}^{-\Sigma} \neq 0 \, . \end{cases} \tag{3.2.17}$$

The superscript "−" in $\mathcal{F}^{-\Lambda}$ and $\mathcal{G}_{\Lambda}^{-}$ denotes the (imaginary) self-duality of such complex symplectic vectors. In order to clarify such a point, let us now briefly address the issue of the general structure of an Abelian theory of vectors endowed with Hodge duality (for more details, see, e.g., [1,2], and [34]).

In general, in the considered context we may introduce a formal operator $\mathcal{H}$ that maps an Abelian field strength into its Hodge dual

$$\left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\mu\nu} \equiv \left({}^{*}\mathcal{F}^{\Lambda}\right)_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda\rho\sigma} = \frac{1}{2}g^{\rho\lambda}g^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\lambda\tau} \, , \tag{3.2.18}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the 4-d completely antisymmetric Ricci-Levi-Civita tensor. It is immediate to check that such an operator is antiprojective

$$\begin{aligned} \left(\mathcal{H}^2\mathcal{F}^{\Lambda}\right)_{\mu\nu} &= \left({}^{**}\mathcal{F}^{\Lambda}\right)_{\mu\nu} = \frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\mu\nu\gamma\delta}\left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\alpha\beta} \\ &= \frac{1}{4}g^{\alpha\gamma}g^{\beta\delta}g^{\lambda\sigma}g^{\rho\tau}\epsilon_{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\sigma\tau}\mathcal{F}^{\Lambda}_{\lambda\rho} \\ &= -\frac{1}{2}\left(\mathcal{F}^{\Lambda}_{\mu\nu} - \mathcal{F}^{\Lambda}_{\nu\mu}\right) = -\mathcal{F}^{\Lambda}_{\mu\nu}, \end{aligned} \tag{3.2.19}$$

where we used the result

$$g^{\alpha\gamma}g^{\beta\delta}g^{\lambda\sigma}g^{\rho\tau}\epsilon_{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\sigma\tau} = \epsilon_{\mu\nu}{}^{\alpha\beta}\epsilon_{\alpha\beta}{}^{\lambda\rho} = -2\left(\delta^{\lambda}_{\mu}\delta^{\rho}_{\nu} - \delta^{\rho}_{\mu}\delta^{\lambda}_{\nu}\right) \, . \tag{3.2.20}$$

Thus, since $\mathcal{H}^2 = -\mathbb{I}$, its eigenvalues are $\pm i$, and out of the real Abelian field strengths $\mathcal{F}^{\Lambda}_{\mu\nu}$ we can introduce imaginary anti-self-dual and imaginary self-dual complex combinations, respectively, as follows:

$$\mathcal{F}^{\pm\Lambda}_{\mu\nu} \equiv \mathcal{F}^{\Lambda}_{\mu\nu} \pm i\left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\mu\nu} = \mathcal{F}^{\Lambda}_{\mu\nu} \pm \frac{i}{2}g^{\rho\lambda}g^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda}_{\lambda\tau} \, , \tag{3.2.21}$$

such that

$$\begin{aligned}\left(\mathcal{H}\mathcal{F}^{\pm\Lambda}\right)_{\mu\nu} &= \left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\mu\nu} \pm i\left(\mathcal{H}^2\mathcal{F}^{\Lambda}\right)_{\mu\nu} \\ &= \mp i\mathcal{F}^{\Lambda}_{\mu\nu} + \left(\mathcal{H}\mathcal{F}^{\Lambda}\right)_{\mu\nu} = \mp i\mathcal{F}^{\pm\Lambda}_{\mu\nu}. \end{aligned} \tag{3.2.22}$$

Notice also that

$$\overline{\mathcal{F}^{\pm\Lambda}_{\mu\nu}} = \mathcal{F}^{\mp\Lambda}_{\mu\nu}\,. \tag{3.2.23}$$

In $\mathcal{N}=2$, $d=4$ n_V-fold MESGT the symplectic symmetry underlying the geometric structure of the equations of motion becomes elegantly manifest by considering the following four different kinds of vectors:

1. The $(2n_V+2)\times 1$, $Sp\,(2n_V+2)$-covariant complex symplectic vector of imaginary self-dual Abelian field strengths

$$\mathcal{Z}^- \equiv \begin{pmatrix} \mathcal{F}^{-\Lambda} \\ \\ \mathcal{G}^-_{\Lambda} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}^{\Lambda} - i\mathcal{H}\mathcal{F}^{\Lambda} \\ \\ \mathcal{G}_{\Lambda} - i\mathcal{H}\mathcal{G}_{\Lambda} \end{pmatrix}\;; \tag{3.2.24}$$

   recall that $Sp\,(2n_V+2,\mathbb{R})$ is the generalized electric–magnetic duality symmetry group, i.e., the U-duality symmetry group,[12] in the $\mathcal{N}=2$, $d=4$ n_V-fold MESGT.
2. By complex-conjugating $\mathcal{Z}^-$, we get the $(2n_V+2)\times 1$, $Sp\,(2n_V+2)$-covariant complex symplectic vector of imaginary anti-self-dual Abelian field strengths

$$\mathcal{Z}^+ \equiv \overline{(\mathcal{Z}^-)} = \begin{pmatrix} \overline{\mathcal{F}^{-\Lambda}} \\ \\ \overline{\mathcal{G}^-_{\Lambda}} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{+\Lambda} \\ \\ \mathcal{G}^+_{\Lambda} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}^{\Lambda} + i\mathcal{H}\mathcal{F}^{\Lambda} \\ \\ \mathcal{G}_{\Lambda} + i\mathcal{H}\mathcal{G}_{\Lambda} \end{pmatrix}\,. \tag{3.2.25}$$

   By definition, the real and imaginary parts of $\mathcal{Z}^-$ and its complex conjugate $\mathcal{Z}^+$ are the real Abelian field strengths of the theory and their Hodge-duals, respectively, reading

[12] More correctly, it should be said that $Sp\,(2n_V+2,\mathbb{R})$ is the “classical supergravity limit” of the U-duality group of the corresponding quantum theory, i.e., of the “discrete” version $Sp\,(2n_V+2,\mathbb{Z})$.

Indeed, the quantization of the conserved charges (related to the Abelian gauge-invariance exhibited by the MESGT) leads to the “discretization” of the numeric field of definition of the group classifying the electric–magnetic transformations. In the case at hand, this yields

$$Sp\,(2n_V+2,\mathbb{R}) \to Sp\,(2n_V+2,\mathbb{Z})\,.$$

The classical formulation of the theories is recovered in the (semiclassical) limit of large values of the integer quantized charges.

For simplicity’s sake and with a slight abuse of language, in the following treatment we will simply talk about “discrete” and “continuous” versions of the same U-group.

3.

$$\mathcal{Z} \equiv Re\left(\mathcal{Z}^{-}\right) = \begin{pmatrix} \mathcal{F}^{\Lambda} \\ \\ \mathcal{G}_{\Lambda} \end{pmatrix} . \tag{3.2.26}$$

4.

$$^{*}\mathcal{Z} \equiv \mathcal{H}\mathcal{Z} = \mathcal{H}\left[\frac{1}{2}\left(\mathcal{Z}^{-}+\mathcal{Z}^{+}\right)\right] = \frac{i}{2}\left(\mathcal{Z}^{-}-\mathcal{Z}^{+}\right) = -Im\left(\mathcal{Z}^{-}\right) = \begin{pmatrix} \mathcal{H}\mathcal{F}^{\Lambda} \\ \\ \mathcal{H}\mathcal{G}_{\Lambda} \end{pmatrix} . \tag{3.2.27}$$

Thus, we may summarize (3.2.24)–(3.2.27) as follows:

$$\mathcal{Z}^{\pm} = \begin{pmatrix} \mathcal{F}^{\pm\Lambda} \\ \\ \mathcal{G}^{\pm}_{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^{\Lambda} \pm i\mathcal{H}\mathcal{F}^{\Lambda} \\ \\ \mathcal{G}_{\Lambda} \pm i\mathcal{H}\mathcal{G}_{\Lambda} \end{pmatrix} , \tag{3.2.28}$$

with

$$\overline{\mathcal{Z}^{\pm}} = \mathcal{Z}^{\mp}, \quad \mathcal{H}\mathcal{Z}^{\pm} = \mp i\mathcal{Z}^{\pm} . \tag{3.2.29}$$

Since $\mathcal{Z}^{-}$, $\mathcal{Z}^{+}$, $\mathcal{Z}$, and $^{*}\mathcal{Z}$ are all $Sp\left(2n_V+2,\mathbb{R}\right)$-covariant vectors, it is clear that

$$\left[\mathcal{C}, Sp\left(2n_V+2,\mathbb{R}\right)\right] = 0 = \left[\mathcal{H}, Sp\left(2n_V+2,\mathbb{R}\right)\right] , \tag{3.2.30}$$

where $\mathcal{C}$ is the complex conjugation operator, $\mathcal{H}$ stands for the Hodge duality operator, and "$Sp\left(2n_V+2,\mathbb{R}\right)$" denotes the covariance w.r.t. the action of such a group. Otherwise speaking, the complex coniugation and/or the Hodge Abelian dualization do not have any effect on the symplectic covariance.

Using the summarizing relations (3.2.28), it is therefore possible to decompose (3.2.15) into the real and imaginary parts

$$\begin{aligned} \mathcal{G}^{-}_{\Lambda} &\equiv \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} = \left[Re\left(\mathcal{N}_{\Lambda\Sigma}\right) - iIm\left(\mathcal{N}_{\Lambda\Sigma}\right)\right]\left(\mathcal{F}^{\Sigma} - i\,{}^{*}\mathcal{F}^{\Sigma}\right) \\ &= \left[Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^{\Sigma} - Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\ {}^{*}\mathcal{F}^{\Sigma}\right] - i\left[Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\ {}^{*}\mathcal{F}^{\Sigma} + Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^{\Sigma}\right] , \end{aligned} \tag{3.2.31}$$

implying, for instance

$$\mathcal{G}_{\Lambda} \equiv Re\left(\mathcal{G}^{-}_{\Lambda}\right) = Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^{\Sigma} - Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\ {}^{*}\mathcal{F}^{\Sigma} . \tag{3.2.32}$$

Thus, in a source-free theory we may write, in the differential form language

$$d\left[Re\left(\mathcal{Z}^{-}\right)\right] = 0 . \tag{3.2.33}$$

Instead, in the presence of electric and magnetic sources with nonvanishing fluxes, we obtain the following "space-dressing" of the components of $Re\left(\mathcal{Z}^{-}\right)$:

$$\begin{aligned} \int_{S^2_\infty} \mathcal{F}^{\Lambda} &\equiv n^{\Lambda}_{m}, \\ \int_{S^2_\infty} \mathcal{G}_{\Lambda} &\equiv n^{e}_{\Lambda}, \end{aligned} \tag{3.2.34}$$

where the integration is performed in the physical space, and S^2_∞ is the 2-sphere at the infinity.

The integration of $\mathcal{F}^\Lambda$ and its Legendre transform $\mathcal{G}_\Lambda$ performed in (3.2.34) may respectively be considered as the definition, in a suitable system of units, of the asymptotical values of the magnetic and electric charges characterizing the charge configuration of the $n_V + 1$ Maxwell vector fields of the theory (indeed we get a vector potential from the gravity multiplet plus another one for each considered vector multiplet).

Clearly, the quantization of such conserved charges (related to the $(U(1))^{n_V+1}$ gauge invariance of the $\mathcal{N} = 2$, $d = 4$ n_V-fold MESGT) implies a discrete range for the quantities on the r.h.s.'s of (3.2.34), and therefore a "discretization" of the symplectic covariance. Consequently, since for a fixed Λ $\left(n^\Lambda_m, n^e_\Lambda\right) \in \mathbb{Z}^2$, it is clear that the "dressings" (3.2.34) will be covariant only under $Sp\,(2n_V + 2, \mathbb{Z})$, which is the "discrete" counterpart of the symplectic symmetry group $Sp\,(2n_V + 2, \mathbb{R})$.

Therefore, the defining equations (3.2.34) allow one to introduce the $(2n_V + 2)$-d symplectic vector of the electric and magnetic charges of the system as the (asymptotical) "space- dressing" of $Re\,(\mathcal{Z}^-)$.

$$n \equiv \int_{S^2_\infty} Re\left(\mathcal{Z}^-\right) = \begin{pmatrix} n^\Lambda_m \\ \\ n^e_\Lambda \end{pmatrix} . \tag{3.2.35}$$

Once again, due to the quantization of the electric and magnetic charges, such a vector is actually $Sp\,(2n_V + 2, \mathbb{Z})$-covariant.

Particular attention should be paid to the issue of moduli dependence. As it is clear from (3.2.16) and (3.2.17), $\mathcal{Z}^-$ is composed of a moduli-independent term $\mathcal{F}^{-\Lambda}$ and a moduli-dependent Kähler-scalar $\mathcal{G}^-_\Lambda$. Of course, the same holds for its real part $Re\,(\mathcal{Z}^-)$. The subtle, key point is that n, which, as defined in (3.2.35), is nothing but the (asymptotical) "space–dressing" of $Re\,(\mathcal{Z}^-)$, is completely moduli-independent

$$\partial_i n = \partial_i \left(\int_{S^2_\infty} Re\left(\mathcal{Z}^-\right) \right) = 0 = \overline{\partial}_{\bar{i}} \left(\int_{S^2_\infty} Re\left(\mathcal{Z}^-\right) \right) = \overline{\partial}_{\bar{i}} n \,. \tag{3.2.36}$$

In particular

1. $$\int_{S^2_\infty} \mathcal{F}^\Lambda \equiv n^\Lambda_m \tag{3.2.37}$$

   defines (in suitable units) the magnetic charges of the system; we have moduli independence both at the "predressing" and "postdressing" stages.
2. By recalling (3.2.14) and (3.2.32) we obtain that

$$\int_{S^2_\infty} \mathcal{G}_\Lambda = \int_{S^2_\infty} Re\left(\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-\Lambda}}\right) \\ = \int_{S^2_\infty} \left[Re\left(\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)\right)\mathcal{F}^\Sigma - Im\left(\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)\right)\ {}^*\mathcal{F}^\Sigma\right] \equiv n^e_\Lambda \tag{3.2.38}$$

defines (in suitable units) the electric charges of the system; while at the "predressing" stage there is nontrivial moduli-dependence through $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$, the (asymptotical) "space-dressing" of $\mathcal{G}_\Lambda$ is such that in the "postdressing" stage there is no moduli dependence.

By using the previously introduced symplectic-invariant scalar product, we can now define two symplectic-invariant combinations of the symplectic field strength vector $\mathcal{Z}^-$.

The first one is

$$\begin{aligned} T^- \equiv -\left\langle \mathcal{Z}^-, V\right\rangle &= M_\Lambda \mathcal{F}^{-\Lambda} - L^\Lambda \mathcal{G}^-_\Lambda \\ &= \mathcal{N}_{\Lambda\Sigma} L^\Sigma \mathcal{F}^{-\Lambda} - L^\Lambda \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma} \\ &= 2i\left(Im\left(\mathcal{N}\right)L\right)_\Lambda \mathcal{F}^{-\Lambda} \\ &= T_\Lambda \mathcal{F}^{-\Lambda}, \end{aligned} \tag{3.2.39}$$

where use of the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and of (3.1.33), (3.1.97), and (3.2.15) has been made. T_Λ may be considered the symplectic vector counterpart of the graviphoton field strength $T^-_{\mu\nu}$ (or, more rigorously, the graviphoton projector).

In general, since the U-duality group $Sp\left(2n_V+2,\mathbb{R}\right)$ is defined over the real numbers, a complex symplectic invariant will yield two distinct real symplectic invariants, given by its real and imaginary parts, or by (linear) combination of them. In such a "decomposition" the symplectic invariance is mantained simply due to the saturation of symplectic, uppercase Greek indices. Further below, we will see that the two fundamental SKG Ansätze (3.1.33) and (3.1.34) will always determine the vanishing of one of the two real symplectic invariants obtained by some kind of "decomposition" of a complex $Sp\left(2n_V+2,\mathbb{R}\right)$-invariant quantity.

Let us start by considering the complex $Sp\left(2n_V+2\right)$-invariant T^- defined in (3.2.39). By using (3.2.26), we obtain

$$T^- \equiv -\left\langle \mathcal{Z}^-, V\right\rangle = -2\left\langle Re\left(\mathcal{Z}^-\right), V\right\rangle + \left\langle \mathcal{Z}^+, V\right\rangle\ . \tag{3.2.40}$$

Moreover, (3.1.33), (3.2.29), and (3.2.15) yield

$$\begin{aligned} \left\langle \mathcal{Z}^+, V\right\rangle &= L^\Lambda \mathcal{G}^+_\Lambda - M_\Lambda \mathcal{F}^{+\Lambda} = \mathcal{N}_{\Lambda\Sigma} L^\Lambda \mathcal{F}^{+\Sigma} - M_\Lambda \mathcal{F}^{+\Lambda} \\ &= M_\Lambda \mathcal{F}^{+\Lambda} - M_\Lambda \mathcal{F}^{+\Lambda} = 0. \end{aligned} \tag{3.2.41}$$

Thus, the first complex $Sp\left(2n_V+2\right)$-invariant may be written as

$$T^{-}\equiv-\left\langle\mathcal{Z}^{-},V\right\rangle=-2\left\langle Re\left(\mathcal{Z}^{-}\right),V\right\rangle=2M_{\Lambda}\mathcal{F}^{\Lambda}-2L^{\Lambda}\mathcal{G}_{\Lambda}\,. \tag{3.2.42}$$

On the other hand, the second complex symplectic-invariant combination which may be considered reads (recall that $\overline{D}_{\overline{j}}$ denotes the antiholomorphic Kähler-covariant derivative in the moduli space)

$$\mathcal{F}^{-i}\equiv-G^{i\overline{j}}\left\langle\mathcal{Z}^{-},\overline{D}_{\overline{j}}\overline{V}\right\rangle=G^{i\overline{j}}\left[\left(\overline{D}_{\overline{j}}\overline{M}_{\Lambda}\right)\mathcal{F}^{-\Lambda}-\left(\overline{D}_{\overline{j}}\overline{L}^{\Lambda}\right)\mathcal{G}_{\Lambda}^{-}\right]\,. \tag{3.2.43}$$

By complex-conjugating it, we get

$$\begin{aligned}\mathcal{F}^{+\overline{j}}\equiv\overline{\mathcal{F}^{-j}}&=-\overline{G^{j\overline{i}}\left\langle\mathcal{Z}^{-},\overline{D}_{\overline{i}}\overline{V}\right\rangle}=-G^{i\overline{j}}\left\langle\mathcal{Z}^{+},D_iV\right\rangle\\&=G^{i\overline{j}}\left[\left(D_iM_{\Lambda}\right)\mathcal{F}^{+\Lambda}-\left(D_iL^{\Lambda}\right)\mathcal{G}_{\Lambda}^{+}\right].\end{aligned} \tag{3.2.44}$$

By using (3.2.26), we obtain

$$\mathcal{F}^{+\overline{j}}=-G^{i\overline{j}}\left\langle\mathcal{Z}^{+},D_iV\right\rangle=-2G^{i\overline{j}}\left\langle Re\left(\mathcal{Z}^{-}\right),D_iV\right\rangle+G^{i\overline{j}}\left\langle\mathcal{Z}^{-},D_iV\right\rangle. \tag{3.2.45}$$

As before, (3.1.33), (3.2.29), and (3.2.15) yield

$$\begin{aligned}\left\langle\mathcal{Z}^{-},D_iV\right\rangle&=\left(D_iL^{\Lambda}\right)\mathcal{G}_{\Lambda}^{-}-\left(D_iM_{\Lambda}\right)\mathcal{F}^{-\Lambda}\\&=\overline{\mathcal{N}}_{\Lambda\Sigma}\left(D_iL^{\Lambda}\right)\mathcal{F}^{-\Sigma}-\overline{\mathcal{N}}_{\Lambda\Sigma}\left(D_iL^{\Sigma}\right)\mathcal{F}^{-\Lambda}=0.\end{aligned} \tag{3.2.46}$$

Thus, the second complex $Sp\left(2n_V+2\right)$-invariant may be written as

$$\begin{aligned}\mathcal{F}^{+\overline{j}}&=-G^{i\overline{j}}\left\langle\mathcal{Z}^{+},D_iV\right\rangle=-2G^{i\overline{j}}\left\langle Re\left(\mathcal{Z}^{-}\right),D_iV\right\rangle\\&=2G^{i\overline{j}}\left[\left(D_iM_{\Lambda}\right)\mathcal{F}^{\Lambda}-\left(D_iL^{\Lambda}\right)\mathcal{G}_{\Lambda}\right].\end{aligned} \tag{3.2.47}$$

By complex-conjugating (3.2.41) and (3.2.46), we may summarize the obtained symplectic-orthogonality relations as follows:

$$\begin{aligned}&\left\langle\mathcal{Z}^{-},\overline{V}\right\rangle=0\Leftrightarrow\left\langle\mathcal{Z}^{+},V\right\rangle=0,\\&\left\langle\mathcal{Z}^{-},D_iV\right\rangle=0\Leftrightarrow\left\langle\mathcal{Z}^{+},\overline{D}_{\overline{i}}\overline{V}\right\rangle=0\,.\end{aligned} \tag{3.2.48}$$

Let us now consider the "space-dressing" of $-\frac{1}{2}T^{-}$ in the case of staticity and spherical symmetry of the moduli configurations (which therefore will at most be radially dependent $z^i=z^i\left(r\right)$). (3.2.37), (3.2.38), and (3.2.42) yield

$$\begin{aligned}-\frac{1}{2}\int_{S^2_\infty}T^{-}&=\int_{S^2_\infty}L^{\Lambda}\left(z\left(r\right),\overline{z}\left(r\right)\right)\mathcal{G}_{\Lambda}-\int_{S^2_\infty}M_{\Lambda}\left(z\left(r\right),\overline{z}\left(r\right)\right)\mathcal{F}^{\Lambda}\\&=L^{\Lambda}_{\infty}\int_{S^2_\infty}\mathcal{G}_{\Lambda}-M_{\Lambda,\infty}\int_{S^2_\infty}\mathcal{F}^{\Lambda}\\&=L^{\Lambda}_{\infty}n^e_{\Lambda}-M_{\Lambda,\infty}n^{\Lambda}_m\\&\equiv Z\left(z_\infty,\overline{z}_\infty;n_m,n^e\right),\end{aligned} \tag{3.2.49}$$

where z_∞, L^Λ_∞, and $M_{\Lambda,\infty}$ respectively stand for the asymptotical values of the moduli and of the symplectic sections[13]

$$z^i_\infty \equiv lim_{r\to\infty} z^i(r); \tag{3.2.50}$$

$$L^\Lambda_\infty \equiv L^\Lambda\left(z_\infty,\overline{z}_\infty\right) = lim_{r\to\infty} L^\Lambda\left(z(r),\overline{z}(r)\right); \tag{3.2.51}$$

$$M_{\Lambda,\infty} \equiv M_\Lambda\left(z_\infty,\overline{z}_\infty\right) = lim_{r\to\infty} M_\Lambda\left(z(r),\overline{z}(r)\right). \tag{3.2.52}$$

Rigorously, the Z defined by (3.2.49) should be denoted[14] by Z_∞: the central charge of the asymptotical SUSY algebra is the asymptotical value of the so-called central charge function

$$Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right) \equiv L^\Lambda\left(z\left(r\right),\overline{z}\left(r\right)\right)n^e_\Lambda - M_\Lambda\left(z\left(r\right),\overline{z}\left(r\right)\right)n^\Lambda_m\,. \tag{3.2.53}$$

In the considered static and spherically symmetric case,[15] (3.2.49) and (3.2.53) yield

$$Z_\infty\left(z_\infty,\overline{z}_\infty;n_m,n^e\right) = lim_{r\to\infty} Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)\,. \tag{3.2.54}$$

From the above definitions, it follows that both the central charge Z_∞ and the central charge function $Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$ are symplectic invariant and they have the same Kähler weights as the symplectic, Kähler-covariantly holomorphic sections L^Λ and M_Λ, namely $(1,-1)$. Here and in the following treatment, unless otherwise noted, we will formulate the hypotheses of staticity and spherical symmetry.

---

[13] Equations (3.2.51) and (3.2.52) clearly yield the assumption that the asymptotical limit $r\to\infty$ is "smooth" for the symplectic sections $L^\Lambda\left(z(r),\overline{z}(r)\right)$ and $M_\Lambda\left(z(r),\overline{z}(r)\right)$.

[14] As for the symplectic sections $L^\Lambda\left(z(r),\overline{z}(r)\right)$ and $M_\Lambda\left(z(r),\overline{z}(r)\right)$, the asymptotical limit $r\to\infty$ is assumed to be "smooth" also for $Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$.

In what follows, we will mainly deal with "central charge" function $Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$. The distinction from the central charge Z_∞ of the asymptotical SUSY algebra will usually be clear from the context, thus we will sometimes omit the subscript "∞".

[15] When considering the most general case in which the hypotheses of spherical symmetry and staticity are both removed, at least the central charge Z_∞ may still be defined as follows:

$$\begin{aligned} Z_\infty &\equiv -\frac{1}{2}\int_{S^2_\infty} T^- \\ &= \int_{S^2_\infty} L^\Lambda\left(z\left(t,r,\theta,\varphi\right),\overline{z}\left(t,r,\theta,\varphi\right)\right)\mathcal{G}_\Lambda - \int_{S^2_\infty} M_\Lambda\left(z\left(t,r,\theta,\varphi\right),\overline{z}\left(t,r,\theta,\varphi\right)\right)\mathcal{F}^\Lambda, \end{aligned}$$

where (r,θ,φ) denotes the usual spherical spatial coordinates. Clearly, in this case Z_∞ will generally be a nontrivial function of the time t and of the asymptotical configurations $\left(z_\infty,\overline{z}_\infty\right)$ of the moduli.

Let us now "space-dress" $-\frac{1}{2}\mathcal{F}^{+\bar{j}}G_{i\bar{j}}$; by recalling (3.2.47), we obtain

$$-\frac{1}{2}\int_{S^2_\infty}\mathcal{F}^{+\bar{j}}G_{i\bar{j}}=\int_{S^2_\infty}G_{i\bar{j}}G^{l\bar{j}}\left[\left(D_lL^\Lambda\right)\mathcal{G}_\Lambda-\left(D_lM_\Lambda\right)\mathcal{F}^\Lambda\right]$$
$$=\int_{S^2_\infty}\left[\left(D_iL^\Lambda\right)\mathcal{G}_\Lambda-\left(D_iM_\Lambda\right)\mathcal{F}^\Lambda\right]. \qquad (3.2.55)$$

Here the following subtlety arises. For what concerns the first term, by using (3.2.32) we get

$$\int_{S^2_\infty}\left(D_iL^\Lambda\right)\mathcal{G}_\Lambda=\int_{S^2_\infty}\left[D_i\left(L^\Lambda\mathcal{G}_\Lambda\right)-L^\Lambda D_i\mathcal{G}_\Lambda\right]$$
$$=\int_{S^2_\infty}\left\{D_i\left(L^\Lambda\mathcal{G}_\Lambda\right)-L^\Lambda D_i\left[Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^\Sigma\right.\right.$$
$$\left.\left.-Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\ {}^*\mathcal{F}^\Sigma\right]\right\}$$
$$=D_{i,\infty}\int_{S^2_\infty}L^\Lambda\mathcal{G}_\Lambda-\int_{S^2_\infty}L^\Lambda\left[\partial_i\left(Re\left(\mathcal{N}_{\Lambda\Sigma}\right)\right)\mathcal{F}^\Sigma\right.$$
$$\left.-\partial_i\left(Im\left(\mathcal{N}_{\Lambda\Sigma}\right)\right)\ {}^*\mathcal{F}^\Sigma\right], \qquad (3.2.56)$$

where $D_{i,\infty}$ denotes the Kähler-covariant derivative w.r.t. the asymptotical configurations of the moduli defined by (3.2.50). Therefore the holomorphic Kähler-covariant derivative cannot be moved outside the "space-dressing" integral, because $\mathcal{G}_\Lambda$ is a moduli-dependent Kähler-scalar.

Nevertheless, it should be recalled that the asymptotical "dressing" of such a Kähler-scalar, namely the electric charge (see (3.2.34) and (3.2.38)), is by definition moduli-independent. Therefore it holds that

$$\int_{S^2_\infty}\left(D_iL^\Lambda\right)\mathcal{G}_\Lambda=\left(D_iL^\Lambda\right)_\infty\int_{S^2_\infty}\mathcal{G}_\Lambda=D_{i,\infty}\left[L^\Lambda_\infty\left(\int_{S^2_\infty}\mathcal{G}_\Lambda\right)\right]. \qquad (3.2.57)$$

For what concerns the second term, no problems arise, because $\mathcal{F}^\Lambda$ is moduli-independent, and therefore we may move D_i outside the spatial integral over S^2_∞ after collecting the term $D_i\left(M_\Lambda\mathcal{F}^\Lambda\right)$ inside it

$$\int_{S^2_\infty}\left(D_iM_\Lambda\right)\mathcal{F}^\Lambda=\int_{S^2_\infty}D_i\left(M_\Lambda\mathcal{F}^\Lambda\right)$$
$$=D_{i,\infty}\int_{S^2_\infty}M_\Lambda\mathcal{F}^\Lambda=D_{i,\infty}\left(M_{\Lambda,\infty}\int_{S^2_\infty}\mathcal{F}^\Lambda\right). \qquad (3.2.58)$$

Thus, by collecting (3.2.57) and (3.2.58), we finally get

$$
\begin{aligned}
-\frac{1}{2}\int_{S^2_\infty}\mathcal{F}^{+\overline{j}}G_{i\overline{j}} &= \int_{S^2_\infty}\left[\left(D_iL^\Lambda\right)\mathcal{G}_\Lambda-\left(D_iM_\Lambda\right)\mathcal{F}^\Lambda\right] \\
&= D_{i,\infty}\left[L^\Lambda_\infty\left(\int_{S^2_\infty}\mathcal{G}_\Lambda\right)\right]-D_{i,\infty}\left(M_{\Lambda,\infty}\int_{S^2_\infty}\mathcal{F}^\Lambda\right) \\
&= D_{i,\infty}\left[L^\Lambda_\infty\int_{S^2_\infty}\mathcal{G}_\Lambda-M_{\Lambda,\infty}\int_{S^2_\infty}\mathcal{F}^\Lambda\right] \\
&= D_{i,\infty}Z_\infty\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)\equiv Z_{i,\infty}\left(z_\infty,\overline{z}_\infty;n_m,n^e\right),
\end{aligned}
\tag{3.2.59}
$$

where in the last line we recalled the definition of the central charge Z_∞ given by (3.2.49). Once again, the quantity[16] $Z_{i,\infty}\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)$, defined by (3.2.59) may be seen as the asymptotical limit[17] of the function

$$
Z_i\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)\equiv\left(D_iZ\right)\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)\ . \tag{3.2.60}
$$

Such a function is the Kähler-covariant derivative (w.r.t. the r-dependent moduli) of the central charge function defined by (3.2.53). Thus, in the considered static and spherically symmetric case, (3.2.59) and (3.2.60) yield

$$
Z_{i,\infty}\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)=lim_{r\to\infty}\left(D_iZ\right)\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)\ . \tag{3.2.61}
$$

Summarizing, in the assumed hypotheses of staticity and spherical symmetry, (3.2.49) and (3.2.59) respectively are the definitions of the central charge of the asymptotical $\mathcal{N}=2$, $d=4$ SUSY algebra and of the Kähler-covariant derivative of such a central extension operator. Such equations are nothing but the asymptotical limit of the definitions of the central charge function $Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$ and of the function $\left(D_iZ\right)\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$, respectively, given by (3.2.53) and (3.2.60).

A number of equivalent expressions for the central charge function and the related Kähler-covariant derivative[18] may be easily obtained. First of all, we may rewrite (3.2.53) by recalling (3.1.33) and (3.1.53):

[16] In what follows, we will mainly consider the function $\left(D_iZ\right)\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$. The distiction from $Z_{i,\infty}\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)$ will usually be clear from the context, thus we will sometimes omit the subscript "∞".

[17] As for the central charge function $Z\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$ and for the symplectic sections $L^\Lambda\left(z(r),\overline{z}(r)\right)$ and $M_\Lambda\left(z(r),\overline{z}(r)\right)$, the asymptotical limit $r\to\infty$ is assumed to be "smooth" also for the functions $Z_i\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)$.

[18] Since we will always be dealing with functions in the r-dependent moduli space, in the following treatment we will omit to say "function".

$$
\begin{aligned}
Z\left(z,\overline{z};n_m,n^e\right) &= L^{\Lambda}\left(z,\overline{z}\right)n^e_{\Lambda} - M_{\Lambda}\left(z,\overline{z}\right)n^{\Lambda}_m \\
&= L^{\Lambda}\left(z,\overline{z}\right)n^e_{\Lambda} - \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)L^{\Sigma}\left(z,\overline{z}\right)n^{\Lambda}_m \\
&= \left(n^e_{\Sigma} - \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^{\Lambda}_m\right)L^{\Sigma}\left(z,\overline{z}\right) \\
&= \left[\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\right]\left[X^{\Lambda}\left(z\right)n^e_{\Lambda} - F_{\Lambda}\left(z\right)n^{\Lambda}_m\right] \\
&= \left[\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\right]\left(n^e_{\Sigma} - \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^{\Lambda}_m\right)X^{\Sigma}\left(z\right).
\end{aligned}
\tag{3.2.62}
$$

Equations (3.1.23) and (3.2.53) directly yield the Kähler-covariant holomorphicity of the central charge $Z\left(z,\overline{z};n_m,n^e\right)$ of $\mathcal{N}=2$, $d=4$, n_V-fold MESGT

$$
\overline{D}_{\overline{i}}Z\left(z,\overline{z};n_m,n^e\right)=0 \Leftrightarrow D_i\overline{Z}\left(z,\overline{z};n_m,n^e\right)=0\ . \tag{3.2.63}
$$

By definition, a Kähler-covariantly holomorphic scalar function f with anti-holomorphic Kähler weight -1 satisfies

$$
\begin{gathered}
\overline{D}_{\overline{i}}f\left(z,\overline{z}\right)=\left(\overline{\partial}_{\overline{i}}-\tfrac{1}{2}\overline{\partial}_{\overline{i}}K\left(z,\overline{z}\right)\right)f\left(z,\overline{z}\right)=0 \\
\Updownarrow \\
f\left(z,\overline{z}\right)=\left[\exp\left(\tfrac{1}{2}K\left(z,\overline{z}\right)\right)\right]g\left(z\right),\quad \overline{\partial}_{\overline{i}}g\left(z\right)=0.
\end{gathered}
\tag{3.2.64}
$$

By considering $f\left(z,\overline{z}\right)=Z\left(z,\overline{z};n_m,n^e\right)$, clearly (3.2.62) implies

$$
g\left(z;n_m,n^e\right)=\left(n^e_{\Sigma}-\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^{\Lambda}_m\right)X^{\Sigma}\left(z\right). \tag{3.2.65}
$$

Such a function, despite the presence of $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$, is holomorphic due to the differential property of $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$ expressed by (3.1.68); indeed

$$
\begin{aligned}
\overline{\partial}_{\overline{i}}g\left(z;n_m,n^e\right) &= \overline{\partial}_{\overline{i}}\left[\left(n^e_{\Sigma}-\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^{\Lambda}_m\right)X^{\Sigma}\left(z\right)\right] \\
&= -n^{\Lambda}_m\left(\overline{\partial}_{\overline{i}}\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)\right)X^{\Sigma}\left(z\right)=0.
\end{aligned}
\tag{3.2.66}
$$

Now we apply the holomorphic Kähler-covariant derivative to the central charge; by using (3.1.26), (3.1.34), (3.2.36) and (3.2.62), we obtain

$$
\begin{aligned}
Z_i\left(z,\overline{z};n_m,n^e\right) \equiv D_i Z\left(z,\overline{z};n_m,n^e\right) &= D_i\left[L^\Lambda\left(z,\overline{z}\right)n^e_\Lambda - M_\Lambda\left(z,\overline{z}\right)n^\Lambda_m\right]\\
&= \left(D_i L^\Lambda\left(z,\overline{z}\right)\right)n^e_\Lambda - \left(D_i M_\Lambda\left(z,\overline{z}\right)\right)n^\Lambda_m\\
&= \left(D_i L^\Lambda\left(z,\overline{z}\right)\right)n^e_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)\left(D_i L^\Sigma\left(z,\overline{z}\right)\right)n^\Lambda_m\\
&= \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)D_i L^\Sigma\left(z,\overline{z}\right)\\
&= \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)f^\Sigma_i\left(z,\overline{z}\right)\\
&= \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)\left(\partial_i + \tfrac{1}{2}\partial_i K\right)\left[\exp\left(\tfrac{1}{2}K\left(z,\overline{z}\right)\right)X^\Sigma(z)\right]\\
&= \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)\left(\partial_i K\right)\left[\exp\left(\tfrac{1}{2}K\left(z,\overline{z}\right)\right)X^\Sigma\left(z\right)\right]\\
&+ \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)\exp\left(\tfrac{1}{2}K\left(z,\overline{z}\right)\right)\left(\partial_i X^\Sigma\left(z\right)\right)\\
&= \left(\partial_i K\right)Z\left(n_m,n^e,z,\overline{z}\right)\big|_{\mathcal{N}_{\Lambda\Sigma}\to\overline{\mathcal{N}}_{\Lambda\Sigma}}\\
&+ \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)\exp\left(\tfrac{1}{2}K\left(z,\overline{z}\right)\right)\left(\partial_i X^\Sigma\left(z\right)\right).
\end{aligned}
\tag{3.2.67}
$$

Attention should be paid to the complex conjugation of the Kähler-covariant derivative of the central charge. Indeed

$$
Z_{\overline{i}} \equiv \overline{D}_{\overline{i}}Z = 0 \neq \overline{Z}_{\overline{i}} = \overline{\left(D_i Z\right)} = \overline{D}_{\overline{i}}\overline{Z}\,. \tag{3.2.68}
$$

On the other hand, $\overline{Z}_{\overline{i}} = 0 \Leftrightarrow Z_i = 0$.

By using the second relation of (3.1.98), expression (3.2.39) of the symplectic-invariant quantity T^- may be rewritten as follows:

$$
T^- = 2i\left(Im\left(\mathcal{N}\right)L\right)_\Lambda \mathcal{F}^{-\Lambda} = T_\Lambda \mathcal{F}^{-\Lambda} = i\overline{L}^\Lambda T_\Sigma \mathcal{F}^{-\Sigma} T_\Lambda. \tag{3.2.69}
$$

On the other hand, by using the Ansatz (3.1.34) and (3.1.26) and (3.2.15), expression (3.2.43) of the other symplectic-invariant quantity $\mathcal{F}^{-i}$ yields

$$
\mathcal{F}^{-i} = G^{i\overline{j}}\left[\left(\overline{D}_{\overline{j}}\overline{M}_\Lambda\right)\mathcal{F}^{-\Lambda} - \left(\overline{D}_{\overline{j}}\overline{L}^\Lambda\right)\mathcal{G}^-_\Lambda\right] = 2iG^{i\overline{j}}\left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma}\overline{f}^\Lambda_{\overline{j}}\mathcal{F}^{-\Sigma}. \tag{3.2.70}
$$

Now, we can introduce $\widehat{\mathcal{F}}^{-\Lambda}$ as the component of the imaginary self-dual Maxwell field strength $\mathcal{F}^{-\Lambda}$ orthogonal to the graviphoton projector T_Λ:

$$
\widehat{\mathcal{F}}^{-\Lambda}T_\Lambda \equiv 0\,. \tag{3.2.71}
$$

By putting

$$\widehat{\mathcal{F}}^{-\Lambda} \equiv \mathcal{F}^{-\Lambda} + \breve{\mathcal{F}}^{-\Lambda} , \tag{3.2.72}$$

(3.2.69) yields

$$\breve{\mathcal{F}}^{-\Lambda} = -i\overline{L}^{\Lambda} T_{\Sigma} \mathcal{F}^{-\Sigma} , \tag{3.2.73}$$

and therefore

$$\begin{aligned} \widehat{\mathcal{F}}^{-\Lambda} &= \mathcal{F}^{-\Lambda} - i\overline{L}^{\Lambda} T_{\Sigma} \mathcal{F}^{-\Sigma} \\ &= \left(\delta^{\Lambda}_{\Sigma} - i\overline{L}^{\Lambda} T_{\Sigma}\right) \mathcal{F}^{-\Sigma}. \end{aligned} \tag{3.2.74}$$

Let us now apply the antiholomorphic Kähler-covariant derivative to (3.1.35); by using (3.1.23), (3.1.26), and (3.1.50), we get

$$\overline{D}_{\bar{i}} \left(Im \left(\mathcal{N}_{\Lambda\Sigma}\right) \overline{L}^{\Lambda} L^{\Sigma} \right) = 0 \Leftrightarrow Im\mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{i}} L^{\Sigma} = 0 \, . \tag{3.2.75}$$

Notice that such a result cannot be obtained by complex conjugating (3.1.72). By adding (3.1.72) to (3.2.75), one gets

$$\left(Im\mathcal{N}_{\Lambda\Sigma}\right) L^{\Lambda} \left(Ref^{\Sigma}_{i}\right) = 0 \Leftrightarrow \begin{cases} \left(Im\mathcal{N}_{\Lambda\Sigma}\right) \left(ReL^{\Lambda}\right) \left(Ref^{\Sigma}_{i}\right) = 0; \\ \\ \left(Im\mathcal{N}_{\Lambda\Sigma}\right) \left(ImL^{\Lambda}\right) \left(Ref^{\Sigma}_{i}\right) = 0. \end{cases} \tag{3.2.76}$$

On the other hand, by subtracting (3.1.72) from (3.2.75), one instead obtains

$$\left(Im\mathcal{N}_{\Lambda\Sigma}\right) L^{\Lambda} \left(Imf^{\Sigma}_{i}\right) = 0 \Leftrightarrow \begin{cases} \left(Im\mathcal{N}_{\Lambda\Sigma}\right) \left(ReL^{\Lambda}\right) \left(Imf^{\Sigma}_{i}\right) = 0; \\ \\ \left(Im\mathcal{N}_{\Lambda\Sigma}\right) \left(ImL^{\Lambda}\right) \left(Imf^{\Sigma}_{i}\right) = 0. \end{cases} \tag{3.2.77}$$

Now, due to (3.1.72), in (3.2.70) we may substitute $\mathcal{F}^{-\Lambda}$ with $\widehat{\mathcal{F}}^{-\Lambda}$ given by (3.2.74), because the extra term is zero

$$\begin{aligned} Im\mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{j}} \widehat{\mathcal{F}}^{-\Sigma} &= Im\mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{j}} \mathcal{F}^{-\Sigma} - iIm\mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{j}} \overline{L}^{\Sigma} T_{\Delta} \mathcal{F}^{-\Delta} \\ &= Im\mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{j}} \mathcal{F}^{-\Sigma}. \end{aligned} \tag{3.2.78}$$

Consequently, it holds that

$$\mathcal{F}^{-i} = 2iG^{i\bar{j}} \left(Im\left(\mathcal{N}\right)\right)_{\Lambda\Sigma} \overline{f}^{\Lambda}_{\bar{j}} \widehat{\mathcal{F}}^{-\Sigma} , \tag{3.2.79}$$

and the symplectic-invariant quantity $\mathcal{F}^{-i}$ is orthogonal to the graviphoton projector T_{Λ}, too.

This result allows us to interpret (3.2.59) as the geometrization of the fluxes of those Maxwell field strengths which are orthogonal to the graviphoton projector T_{Λ}.

It is also worth noticing that actually, by the previous construction, the "charge operators" (Z, Z_i) are in correspondence with the integer conserved charges $\left(n_m^{\Lambda}, n_{\Lambda}^{e}\right)$, but they refer to the eigenstates of the vector supermultiplets, and therefore, in general, they exhibit a nontrivial functional dependence on the moduli.

In a generic point of the n_V-d Kähler–Hodge complex moduli space of the $\mathcal{N}=2$, $d=4$, n_V-fold MESGT there exist, in general, two independent $Sp\left(2n_V+2\right)$-invariants homogeneous of degree two in the (quantized) electric and magnetic charges of the system. Such invariants may be expressed in a model-independent way as follows [51] :

$$\begin{aligned} I_1\left(z,\overline{z};n_m,n^e\right) &\equiv |Z|^2\left(z,\overline{z};n_m,n^e\right) \\ &\qquad +G^{i\overline{j}}\left(z,\overline{z}\right)Z_i\left(z,\overline{z};n_m,n^e\right)\overline{Z}_{\overline{j}}\left(z,\overline{z};n_m,n^e\right), \\ I_2\left(z,\overline{z};n_m,n^e\right) &\equiv |Z|^2\left(z,\overline{z};n_m,n^e\right) \\ &\qquad -G^{i\overline{j}}\left(z,\overline{z}\right)Z_i\left(z,\overline{z};n_m,n^e\right)\overline{Z}_{\overline{j}}\left(z,\overline{z};n_m,n^e\right) . \end{aligned} \tag{3.2.80}$$

At this point it is useful to introduce the real symplectic $(2n_V+2)$-d square matrix

$$\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)\equiv\mathcal{R}^T\left(Re\left(\mathcal{N}\right)\right)\mathcal{D}\left(Im\left(\mathcal{N}\right)\right)\mathcal{R}\left(Re\left(\mathcal{N}\right)\right), \tag{3.2.81}$$

where

$$\mathcal{R}\left(Re\left(\mathcal{N}\right)\right)\equiv\begin{pmatrix}\mathbb{I} & 0\\ -Re\left(\mathcal{N}\right) & \mathbb{I}\end{pmatrix},\quad \mathcal{D}\left(Im\left(\mathcal{N}\right)\right)\equiv\begin{pmatrix}Im\left(\mathcal{N}\right) & 0\\ 0 & \left(Im\left(\mathcal{N}\right)\right)^{-1}\end{pmatrix}; \tag{3.2.82}$$

consequently

$$\begin{aligned} &\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right) \\ &=\begin{pmatrix}Im\left(\mathcal{N}\right)+Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1}Re\left(\mathcal{N}\right) & -Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1}\\ -\left(Im\left(\mathcal{N}\right)\right)^{-1}Re\left(\mathcal{N}\right) & \left(Im\left(\mathcal{N}\right)\right)^{-1}\end{pmatrix}. \end{aligned} \tag{3.2.83}$$

Notice that

$$\left[Re\left(\mathcal{N}\right),\left(Im\left(\mathcal{N}\right)\right)^{-1}\right]\neq 0 \tag{3.2.84}$$

but, since $\mathcal{N}_{\Lambda\Sigma}=\mathcal{N}_{(\Lambda\Sigma)}$ and

$$\left[Re\left(\mathcal{N}\right)\left(Im\left(\mathcal{N}\right)\right)^{-1}\right]^{T}=\left(Im\left(\mathcal{N}\right)\right)^{-1}Re\left(\mathcal{N}\right)\ , \tag{3.2.85}$$

the real matrix $\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)$ is symmetric.

By using (3.1.96)–(3.1.98), (3.2.35), and (3.2.83) and by recalling the definition of the (n_V+1)-d complex symmetric square matrix $F_{\Lambda\Sigma}\equiv\frac{\partial^2 F}{\partial X^\Lambda\partial X^\Sigma}$, denoted with $\mathcal{F}(z)$ in matrix notation, we can rewrite the two symplectic-invariants of degree two as follows:

$$\begin{aligned}I_1\left(z,\overline{z};n_m,n^e\right)&=-\frac{1}{2}n^T\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)n\\&=-\frac{1}{2}\left(n^e_\Lambda-\overline{\mathcal{N}}_{\Lambda\Sigma}n^\Sigma_m\right)\left(\left(Im\left(\mathcal{N}\right)\right)^{-1}\right)^{\Lambda\Delta}\left(n^e_\Delta-\mathcal{N}_{\Delta\Gamma}n^\Gamma_m\right),\end{aligned} \tag{3.2.86}$$

$$\begin{aligned}I_2\left(z,\overline{z};n_m,n^e\right)&=-\frac{1}{2}n^T\mathcal{M}\left(Re\left(\mathcal{F}\right),Im\left(\mathcal{F}\right)\right)n\\&=-\frac{1}{2}\left(n^e_\Lambda-\overline{F}_{\Lambda\Sigma}n^\Sigma_m\right)\left(\left(Im\left(\mathcal{F}\right)\right)^{-1}\right)^{\Lambda\Delta}\left(n^e_\Delta-F_{\Delta\Gamma}n^\Gamma_m\right);\end{aligned} \tag{3.2.87}$$

as it is evident, (3.2.86) and (3.2.87) are simply related by the matrix interchange $\mathcal{N}\leftrightarrow\mathcal{F}$.

By considering (3.2.81)–(3.2.83), it is easy to realize that $Re\left(\mathcal{N}\right)$ and $Re\left(\mathcal{F}\right)$ do not play any role in expressions (3.2.86) and (3.2.87), because the matrix function $\mathcal{R}$ can be included in the symplectic vector n by a simple redefinition. Indeed, defining $n_{\mathcal{R}(\mathcal{K})}\equiv\mathcal{R}\left(\mathcal{K}\right)n$ (where $\mathcal{K}=\mathcal{N},\mathcal{F}$ in this case), one immediately gets

$$\begin{aligned}n^T\mathcal{M}\left(Re\left(\mathcal{K}\right),Im\left(\mathcal{K}\right)\right)n&=n^T\mathcal{R}^T\left(Re\left(\mathcal{K}\right)\right)D\left(Im\left(\mathcal{K}\right)\right)\mathcal{R}\left(Re\left(\mathcal{K}\right)\right)n\\&=n^T_{\mathcal{R}(\mathcal{K})}\mathcal{D}\left(Im\left(\mathcal{K}\right)\right)n_{\mathcal{R}(\mathcal{K})}.\end{aligned} \tag{3.2.88}$$

Therefore, by looking at the signatures of the quadratic forms appearing on the r.h.s.'s of (3.2.86) and (3.2.87), we get that $n^T_{\mathcal{R}(\mathcal{K})}\mathcal{D}\left(Im\left(\mathcal{K}\right)\right)n_{\mathcal{R}(\mathcal{K})}$ is manifestly a quadratic form with negative signature for $\mathcal{K}=\mathcal{N}$, and with n_V positive and one negative eigenvalues for $\mathcal{K}=\mathcal{F}$. Summarizing, (3.2.86) and (3.2.87) reflect the fact that $Im\left(\mathcal{N}\right)$ is negative definite and that, as previously mentioned, $Im\left(\mathcal{F}\right)$ has an $(n_V,1)$ signature (i.e., has n_V positive and one negative eigenvalues).

We will now explicitly derive some important identities of the SKG of M_{n_V}, which generalize the calculations of Ferrara and Kallosh in [30]. Further below, we will see that, when evaluated at some particular points in M_{n_V}, the obtained identities will yield the so-called *non(-BPS)-SUSY extreme BH attractor equations*, recently rediscovered by Kallosh [52] (and explicitly checked in some examples in [53]), but which had actually already been written in a slightly different fashion in [54].

Let us start by considering $\overline{D}_{\bar{i}}\overline{Z}$; by recalling definition (3.2.53), we may write

$$\overline{D}_{\bar{i}}\overline{Z} = n_{\Lambda}^{e}\overline{D}_{\bar{i}}\overline{L}^{\Lambda} - n_{m}^{\Lambda}\overline{D}_{\bar{i}}\overline{M}_{\Lambda}\ ; \tag{3.2.89}$$

by using the Ansatz (3.1.34) we thus get

$$\overline{D}_{\bar{i}}\overline{Z} = n_{\Lambda}^{e}\overline{D}_{\bar{i}}\overline{L}^{\Lambda} - n_{m}^{\Lambda}\mathcal{N}_{\Lambda\Delta}\overline{D}_{\bar{i}}\overline{L}^{\Delta}\ . \tag{3.2.90}$$

The contraction of both sides with $G^{i\bar{i}}D_iL^{\Sigma}$ then yields

$$G^{i\bar{i}}\left(D_iL^{\Sigma}\right)\overline{D}_{\bar{i}}\overline{Z} = n_{\Lambda}^{e}G^{i\bar{i}}\left(D_iL^{\Sigma}\right)\overline{D}_{\bar{i}}\overline{L}^{\Lambda} - n_{m}^{\Lambda}\mathcal{N}_{\Lambda\Delta}G^{i\bar{i}}\left(D_iL^{\Sigma}\right)\overline{D}_{\bar{i}}\overline{L}^{\Delta}; \tag{3.2.91}$$

now, by using (3.1.26) and (3.1.99) and the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse $\mathcal{N}^{\Lambda\Sigma}$ (see (3.1.101)), such an expression may be further elaborated as

$$\begin{aligned}
&G^{i\bar{i}}\left(D_iL^{\Sigma}\right)\overline{D}_{\bar{i}}\overline{Z}\\
&= n_{\Lambda}^{e}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Lambda} - \overline{L}^{\Sigma}L^{\Lambda}\right] - n_{m}^{\Lambda}\mathcal{N}_{\Lambda\Delta}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Delta} - \overline{L}^{\Sigma}L^{\Delta}\right]\\
&= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Lambda}n_{\Lambda}^{e} - \overline{L}^{\Sigma}L^{\Lambda}n_{\Lambda}^{e}\\
&+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Delta}\left[\left(Re\mathcal{N}\right)_{\Delta\Lambda} + i\left(Im\mathcal{N}\right)_{\Delta\Lambda}\right]n_{m}^{\Lambda} + \overline{L}^{\Sigma}\mathcal{N}_{\Lambda\Delta}L^{\Delta}n_{m}^{\Lambda}\\
&= -\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Lambda}n_{\Lambda}^{e} - \overline{L}^{\Sigma}\left(L^{\Lambda}n_{\Lambda}^{e} - M_{\Lambda}n_{m}^{\Lambda}\right)\\
&+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Delta}\left(Re\mathcal{N}\right)_{\Delta\Lambda}n_{m}^{\Lambda} + \tfrac{i}{2}n_{m}^{\Sigma}\\
&= \tfrac{i}{2}n_{m}^{\Sigma} - \overline{L}^{\Sigma}Z + \tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Delta}\left(Re\mathcal{N}\right)_{\Delta\Lambda}n_{m}^{\Lambda} - \tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Sigma\Lambda}n_{\Lambda}^{e},
\end{aligned} \tag{3.2.92}$$

where in the last two lines we used the Ansatz (3.1.33) and definition (3.2.53). Now, by subtracting to expression (3.2.92) its complex conjugate, one gets

$$n_{m}^{\Lambda} = 2Re\left[i\overline{Z}L^{\Lambda} + iG^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{L}^{\Lambda}\right)D_iZ\right] = -2Im\left[\overline{Z}L^{\Lambda} + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{L}^{\Lambda}\right)D_iZ\right]. \tag{3.2.93}$$

On the other hand, the contraction of both sides of (3.2.90) with $G^{i\bar{i}}D_iM_{\Sigma}$ yields

$$G^{i\bar{i}}\left(D_iM_\Sigma\right)\overline{D}_{\bar{i}}\overline{Z}=n^e_\Lambda G^{i\bar{i}}\left(D_iM_\Sigma\right)\overline{D}_{\bar{i}}\overline{L}^\Lambda-n^\Lambda_m\mathcal{N}_{\Lambda\Delta}G^{i\bar{i}}\left(D_iM_\Sigma\right)\overline{D}_{\bar{i}}\overline{L}^\Delta$$
$$=n^e_\Lambda G^{i\bar{i}}\overline{\mathcal{N}}_{\Sigma\Delta}\left(D_iL^\Delta\right)\overline{D}_{\bar{i}}\overline{L}^\Lambda-n^\Lambda_m\mathcal{N}_{\Lambda\Delta}G^{i\bar{i}}\overline{\mathcal{N}}_{\Sigma\Xi}\left(D_iL^\Xi\right)\overline{D}_{\bar{i}}\overline{L}^\Delta, \tag{3.2.94}$$

where in the last line we used the Ansatz (3.1.34). Once again, by using (3.1.99) and the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ and its inverse $\mathcal{N}^{\Lambda\Sigma}$, the above expression may be further elaborated as follows:

$$G^{i\bar{i}}\left(D_iM_\Sigma\right)\overline{D}_{\bar{i}}\overline{Z}$$
$$=n^e_\Lambda\overline{\mathcal{N}}_{\Sigma\Delta}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}-\overline{L}^\Delta L^\Lambda\right]$$
$$-n^\Lambda_m\mathcal{N}_{\Lambda\Delta}\overline{\mathcal{N}}_{\Sigma\Xi}\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}-\overline{L}^\Xi L^\Delta\right]$$
$$=\left[-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}-\overline{L}^\Delta L^\Lambda\right]\left[\left(Re\mathcal{N}\right)_{\Sigma\Delta}-i\left(Im\mathcal{N}\right)_{\Sigma\Delta}\right]n^e_\Lambda$$
$$+\left[\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}+\overline{L}^\Xi L^\Delta\right]\left[\left(Re\mathcal{N}\right)_{\Sigma\Xi}-i\left(Im\mathcal{N}\right)_{\Sigma\Xi}\right]\mathcal{N}_{\Lambda\Delta}n^\Lambda_m$$
$$=-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{L}^\Delta L^\Lambda\overline{\mathcal{N}}_{\Sigma\Delta}n^e_\Lambda$$
$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\mathcal{N}_{\Lambda\Delta}n^\Lambda_m-\tfrac{i}{2}\mathcal{N}_{\Lambda\Sigma}n^\Lambda_m+\overline{L}^\Xi L^\Delta\overline{\mathcal{N}}_{\Sigma\Xi}\mathcal{N}_{\Lambda\Delta}n^\Lambda_m$$
$$=-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{M}_\Sigma L^\Lambda n^e_\Lambda$$
$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\left[\left(Re\mathcal{N}\right)_{\Lambda\Delta}+i\left(Im\mathcal{N}\right)_{\Lambda\Delta}\right]n^\Lambda_m$$
$$-\tfrac{i}{2}\mathcal{N}_{\Lambda\Sigma}n^\Lambda_m+\overline{M}_\Sigma\mathcal{M}_\Lambda n^\Lambda_m$$
$$=-\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Lambda}\left(Re\mathcal{N}\right)_{\Sigma\Delta}n^e_\Lambda+\tfrac{i}{2}n^e_\Sigma-\overline{M}_\Sigma Z$$
$$+\tfrac{1}{2}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Delta}\left(Re\mathcal{N}\right)_{\Sigma\Xi}\left(Re\mathcal{N}\right)_{\Lambda\Delta}n^\Lambda_m+\tfrac{i}{2}\left(Re\mathcal{N}\right)_{\Sigma\Lambda}n^\Lambda_m$$
$$-\tfrac{i}{2}\left[\left(Re\mathcal{N}\right)_{\Lambda\Sigma}+i\left(Im\mathcal{N}\right)_{\Lambda\Sigma}\right]n^\Lambda_m$$

$$
\begin{aligned}
&= -\tfrac{1}{2}\left((Im\mathcal{N})^{-1}\right)^{\Delta\Lambda}(Re\mathcal{N})_{\Sigma\Delta}\, n^e_\Lambda + \tfrac{i}{2} n^e_\Sigma - \overline{M}_\Sigma Z \\
&+\tfrac{1}{2}\left((Im\mathcal{N})^{-1}\right)^{\Xi\Delta}(Re\mathcal{N})_{\Sigma\Xi}\,(Re\mathcal{N})_{\Lambda\Delta}\, n^\Lambda_m + \tfrac{1}{2}\,(Im\mathcal{N})_{\Lambda\Sigma}\, n^\Lambda_m,
\end{aligned}
\tag{3.2.95}
$$

where in the last lines we used the Ansatz (3.1.33) and definition (3.2.53). Thence, by subtracting to expression (3.2.95) its complex conjugate, one gets

$$
n^e_\Lambda = 2Re\left[i\overline{Z}M_\Lambda + iG^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{M}_\Lambda\right)D_iZ\right] = -2Im\left[\overline{Z}M_\Lambda + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{M}_\Lambda\right)D_iZ\right].
\tag{3.2.96}
$$

By expressing the identities (3.2.93) and (3.2.96) in a vector $Sp\,(2n_V+2)$-covariant notation, one finally gets

$$
\begin{aligned}
\begin{pmatrix} n^\Lambda_m \\ \\ n^e_\Lambda \end{pmatrix} &= 2Re\left[i\overline{Z}\begin{pmatrix} L^\Lambda \\ \\ M_\Lambda \end{pmatrix} + iG^{i\bar{i}}\begin{pmatrix} \left(\overline{D}_{\bar{i}}\overline{L}^\Lambda\right) \\ \\ \left(\overline{D}_{\bar{i}}\overline{M}_\Lambda\right) \end{pmatrix} D_iZ\right] \\
&= -2Im\left[\overline{Z}\begin{pmatrix} L^\Lambda \\ \\ M_\Lambda \end{pmatrix} + G^{i\bar{i}}\begin{pmatrix} \left(\overline{D}_{\bar{i}}\overline{L}^\Lambda\right) \\ \\ \left(\overline{D}_{\bar{i}}\overline{M}_\Lambda\right) \end{pmatrix} D_iZ\right],
\end{aligned}
\tag{3.2.97}
$$

or in compact form

$$
n = 2Re\left[i\overline{Z}V + iG^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{V}\right)D_iZ\right] = -2Im\left[\overline{Z}V + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{V}\right)D_iZ\right],
\tag{3.2.98}
$$

where we recalled definitions (3.1.23) and (3.2.35) of the $(2n_V+2)\times 1$ vectors V and n, respectively. It is worth pointing out that the vector identity (3.2.98) has been obtained only by using the properties of the SKG of M_{n_V}. Such an identity expresses nothing but a change of basis in the lattice Γ of BH charge configurations, between the real basis $\left(n^\Lambda_m, n^e_\Lambda\right)_{\Lambda=0,1,\dots,n_V}$ and the complex basis $(Z, D_iZ)_{i=1,\dots,n_V}$. Such a change of basis also introduces a nontrivial dependence on the moduli $(z^i,\overline{z}^{\bar{i}})$, since the complex charges $(Z,D_iZ)_{i=1,\dots,n_V}$ refer to the supermultiplet eigenstates, and they thus are moduli-dependent. The relations yielded by the identity (3.2.98) are $2n_V+2$ real ones, but they have been obtained by starting from an expression for $\overline{D}_{\bar{i}}\overline{Z}$, corresponding to n_V complex, and therefore $2n_V$ real, degrees of freedom. The two redundant real degrees of freedom are encoded in the homogeneity (of degree 1) of the identity (3.2.98) under complex rescalings of the symplectic BH charge vector n; indeed, by recalling definition (3.2.53) it is immediate to check that the r.h.s. of identity (3.2.98) acquires an overall factor λ under a global rescaling of n of the kind

$$n \longrightarrow \lambda n, \quad \lambda \in \mathbb{C} . \tag{3.2.99}$$

The summation of expressions (3.2.92) and (3.2.95) with their complex conjugates respectively yields

$$\begin{aligned} &\left((Im\mathcal{N})^{-1}\right)^{\Delta\Lambda} (Re\mathcal{N})_{\Delta\Sigma} n_m^{\Sigma} - \left((Im\mathcal{N})^{-1}\right)^{\Lambda\Sigma} n_{\Sigma}^{e} \\ &= 2Re\left[\overline{Z}L^{\Lambda} + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{L}^{\Lambda}\right) D_i Z\right]; \end{aligned} \tag{3.2.100}$$

$$\begin{aligned} &\left[Im\mathcal{N}_{\Lambda\Sigma} + \left((Im\mathcal{N})^{-1}\right)^{\Xi\Delta} (Re\mathcal{N})_{\Lambda\Xi} (Re\mathcal{N})_{\Sigma\Delta}\right] n_m^{\Sigma} \\ &- \left((Im\mathcal{N})^{-1}\right)^{\Delta\Sigma} (Re\mathcal{N})_{\Lambda\Delta} n_{\Sigma}^{e} \\ &= 2Re\left[\overline{Z}M_{\Lambda} + G^{i\bar{i}}\left(\overline{D}_{\bar{i}}\overline{M}_{\Lambda}\right) D_i Z\right]. \end{aligned} \tag{3.2.101}$$

In order to elaborate a shorthand notation for the obtained SKG identities (3.2.93), (3.2.96) and (3.2.100), (3.2.101), let us now reconsider the starting expressions (3.2.92) and (3.2.95), respectively, reading

$$\begin{aligned} &\left[\delta_{\Sigma}^{\Lambda} - i\left((Im\mathcal{N})^{-1}\right)^{\Lambda\Delta} (Re\mathcal{N})_{\Delta\Sigma}\right] n_m^{\Sigma} + i\left((Im\mathcal{N})^{-1}\right)^{\Lambda\Sigma} n_{\Sigma}^{e} \\ &= -2i\overline{L}^{\Lambda} Z - 2iG^{i\bar{i}}\left(D_i L^{\Lambda}\right) \overline{D}_{\bar{i}}\overline{Z}; \end{aligned} \tag{3.2.102}$$

$$\begin{aligned} &-i\left[\left((Im\mathcal{N})^{-1}\right)^{\Xi\Delta} (Re\mathcal{N})_{\Lambda\Xi} (Re\mathcal{N})_{\Sigma\Delta} + (Im\mathcal{N})_{\Lambda\Sigma}\right] n_m^{\Sigma} \\ &+ \left[\delta_{\Lambda}^{\Sigma} + i\left((Im\mathcal{N})^{-1}\right)^{\Delta\Sigma} (Re\mathcal{N})_{\Lambda\Delta}\right] n_{\Sigma}^{e} \\ &= -2i\overline{M}_{\Lambda} Z - 2iG^{i\bar{i}}\left(D_i M_{\Lambda}\right) \overline{D}_{\bar{i}}\overline{Z}. \end{aligned} \tag{3.2.103}$$

By recalling definitions (3.1.23), (3.1.24), and (3.2.35) and (3.2.83), the identities (3.2.102) and (3.2.103) may be synthesized in vector notation as follows:

$$n - i\epsilon\mathcal{M}(\mathcal{N})\, n = -2i\overline{V}Z - 2iG^{i\bar{i}}\left(D_i V\right) \overline{D}_{\bar{i}}\overline{Z}, \tag{3.2.104}$$

where $\mathcal{M}(\mathcal{N})$ denotes the $(2n_V+2) \times (2n_V+2)$ real matrix $\mathcal{M}(Re(\mathcal{N}), Im(\mathcal{N}))$ given by (3.2.83). By using the symplectic-orthogonality relations given by (3.1.25), II of (3.1.36), (3.1.43) and first and fourth of (3.1.42), the SKG identity (3.2.104) yields the following relations:

$$\begin{cases} \langle V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle = -2Z; \\ \\ \left\langle \overline{V}, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\right\rangle = 0; \\ \\ \langle D_i V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle = 0; \\ \\ \left\langle \overline{D}_{\overline{i}} V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\right\rangle = -2\overline{D}_{\overline{i}}\overline{Z}. \end{cases} \tag{3.2.105}$$

The real part of the general, fundamental SKG vector identity (3.2.104) yields

$$\begin{aligned} n &= -2Re\left[i\overline{V}Z + iG^{i\overline{i}}\left(D_i V\right)\overline{D}_{\overline{i}}\overline{Z}\right] = 2Re\left[iV\overline{Z} + iG^{i\overline{i}}\left(\overline{D}_{\overline{i}}\overline{V}\right)D_i Z\right] \\ &= 2Im\left[\overline{V}Z + G^{i\overline{i}}\left(D_i V\right)\overline{D}_{\overline{i}}\overline{Z}\right] = -2Im\left[V\overline{Z} + G^{i\overline{i}}\left(\overline{D}_{\overline{i}}\overline{V}\right)D_i Z\right], \end{aligned} \tag{3.2.106}$$

which is nothing but the SKG vector identity (3.2.98), which in turn summarizes the identities (3.2.93) and (3.2.96). On the other hand, the imaginary part of (3.2.104) yields

$$\begin{aligned} \epsilon\mathcal{M}(\mathcal{N})\, n &= 2Im\left[i\overline{V}Z + iG^{i\overline{i}}\left(D_i V\right)\overline{D}_{\overline{i}}\overline{Z}\right] \\ &= -2Im\left[-iV\overline{Z} - iG^{i\overline{i}}\left(\overline{D}_{\overline{i}}\overline{V}\right)D_i Z\right] \\ &= 2Re\left[\overline{V}Z + G^{i\overline{i}}\left(D_i V\right)\overline{D}_{\overline{i}}\overline{Z}\right] \\ &= 2Re\left[V\overline{Z} + G^{i\overline{i}}\left(\overline{D}_{\overline{i}}\overline{V}\right)D_i Z\right], \end{aligned} \tag{3.2.107}$$

and it summarizes the identities (3.2.100) and (3.2.101). Notice that the imaginary and real parts of the SKG identity (3.2.104) are linearly related by the $(2n_V+2)\times(2n_V+2)$ real matrix

$$\begin{aligned} &\epsilon\mathcal{M}(\mathcal{N}) \\ &= \begin{pmatrix} (Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -(Im(\mathcal{N}))^{-1} \\ \\ Im(\mathcal{N}) + Re(\mathcal{N})(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -Re(\mathcal{N})(Im(\mathcal{N}))^{-1} \end{pmatrix}. \end{aligned} \tag{3.2.108}$$

By transporting such a relation to the r.h.s.'s of the identities (3.2.106) and (3.2.107), one obtains

$$Re\left[\overline{V}Z + G^{i\bar{i}}\left(D_iV\right)\overline{D}_{\bar{i}}\overline{Z}\right] = \epsilon\mathcal{M}\left(\mathcal{N}\right)Im\left[\overline{V}Z + G^{i\bar{i}}\left(D_iV\right)\overline{D}_{\bar{i}}\overline{Z}\right]: \tag{3.2.109}$$

the real and imaginary parts of the symplectic-invariant quantity $\overline{V}Z + G^{i\bar{i}}\left(D_iV\right)\overline{D}_{\bar{i}}\overline{Z}$ are simply related through a "symplectic rotation" given by the matrix $\epsilon\mathcal{M}\left(\mathcal{N}\right)$, explicited in (3.2.108). Clearly, all this is consistent with the previously performed counting of the real degrees of freedom, since there are only $2n_V$ real independent relations.

In Sect. 4 we will see that the algebraic attractor equations, both for the $\frac{1}{2}$-*BPS-SUSY extreme BH "attractor(s)"* and for the *non(-BPS)-SUSY extreme BH "attractor(s)"*, are given by nothing but the evaluation of the SKG identity (3.2.106) at some peculiar points in the moduli space M_{n_V}, namely at the critical points of a suitably defined *"BH effective potential"* function $V_{BH}\left(z,\overline{z};n_m,n^e\right)$.

At this point, we may come back and reconsider the consistency conditions (3.2.10) and (3.2.11) for the second solution of the unbroken $\mathcal{N}=2$, $d=4$, n_V-fold MESGT Killing spinor equation (3.2.5).

In particular, condition (3.2.11) expresses the vanishing of the Abelian vector field strengths of the vector supermultiplets. It may be shown that it is nothing but an extremum condition for the radial dependence of the moduli of the theory; i.e., we may equivalently reformulate condition (3.2.11) as follows ($\forall i=1,\ldots,n_V$ understood throughout):

$$\frac{d}{dr}z^i\left(r\right)=0\ , \tag{3.2.110}$$

where r is the radial distance from the surface of the EH. It should be recalled that the radial dependence is the only relevant in this framework, due to the spherical symmetry of the (geo)metric structures involved. Let us remind also that the moduli of the considered $\mathcal{N}=2$, $d=4$, n_V-fold MESGT are the n_V complex scalar fields coming from the n_V Abelian vector supermultiplets coupled to the supergravity one.

Notice that (3.2.110), even though not resembling the previously considered AEs, is the first case in which some extremizing equation arises in the dynamics of extremal supersymmetric BHs.

By using the whole formal-geometrical machinery reported above, it may be proved that (3.2.110) implies the vanishing of the holomorphic Kähler-covariant derivative of the central charge:

$$Z_i \equiv D_iZ = \left(\partial_i + \frac{1}{2}\partial_iK\right)Z\left(z,\overline{z};n_m,n^e\right)=0\ . \tag{3.2.111}$$

As explained in Sect. 2, the fixed values of the moduli at the EH of the extremal RN BH will be obtained by solving (3.2.111), provided that such equations do have (at least one) solution, i.e., provided that the n_V-d Kähler–Hodge complex moduli space M_{n_V} of the $\mathcal{N}=2$, $d=4$, n_V-fold MESGT may be characterized as an "attractor variety" with at least one "attractor" point [31–33]. When existing, such "attractor" solutions will be independent of the asymptotical values of the moduli, i.e., on the initial data of their dynamical evolution flow inside the moduli space, and instead will depend only on the conserved, quantized electric and magnetic charges of the considered system.

Thus, (3.2.111) should be more precisely specified at the attractor points

$$\begin{aligned} Z_i\big|_{(z,\overline{z})=(z_H,\overline{z}_H)} &\equiv (D_i Z)_{(z,\overline{z})=(z_H,\overline{z}_H)} \\ &= \left[\left(\partial_i + \tfrac{1}{2}\partial_i K\right) Z\left(z,\overline{z};n_m,n^e\right)\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0\ , \end{aligned} \tag{3.2.112}$$

where $(z_H,\overline{z}_H) = \left(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right)\right)$ determines the position of the attractor point in M_{n_V}. As already pointed out, such a point is independent of the set of continuously varying, unconstrained initial (asymptotical $r\to\infty$) data $(z_\infty,\overline{z}_\infty) \equiv lim_{r\to\infty}\left(z(r),\overline{z}(r)\right) \in M_{n_V}$, but instead depends only on the set of quantized electric and magnetic charges $(n_m,n^e)\in\Gamma$ of the system. Consequently, $\left(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right)\right)$ generally corresponds to a discrete set of quantized attractor fixed points.

Therefore, beside being always a Kähler-covariantly holomorphic function (see (3.2.63)), in correspondence with the attractor point(s) the central charge becomes a Kähler-covariantly antiholomorphic function, too. Otherwise speaking, the set of attractor point(s) in M_{n_V} could be characterized as follows:

$$M_{n_V} \ni \left\{\left(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right)\right)\right\} : \begin{cases} \left(D_i Z\right)\left(z_H,\overline{z}_H;n_m,n^e\right) = 0, \\ \left(\overline{D}_{\overline{i}} Z\right)\left(z_H,\overline{z}_H;n_m,n^e\right) = 0. \end{cases} \tag{3.2.113}$$

Such a set of Kähler-covariant differential conditions may be seen as the realization of the attractor mechanism in the moduli space, or equivalently as the Kähler-covariant extremization of the central extension operator of the considered superalgebra. The AM selects the configurations of the moduli at the EH as the ones that make the central charge Kähler-covariantly antiholomorphic. Indeed, we will show that for nonvanishing Z (3.2.113) is the Kähler-covariant form of the general, model-independent "attractor" or "extremal" equation (2.11), the so-called $\frac{1}{2}$-*BPS extreme BH attractor equation.*

Before doing this, let us briefly comment on the Kähler weights of the central charge Z.

As previously mentioned, from its very definition (3.2.53) it follows that Z is a Kähler-scalar function in the moduli space M_{n_V} with Kähler weights $(1,-1)$. Therefore, as largely used above, its Kähler-covariant derivatives read

$$\begin{cases} D_i Z = \left(\partial_i + \frac{1}{2}\partial_i K\right) Z, \\ \overline{D}_{\overline{i}} Z = \left(\overline{\partial}_{\overline{i}} - \frac{1}{2}\overline{\partial}_{\overline{i}} K\right) Z. \end{cases} \tag{3.2.114}$$

As, in general, it follows from (3.1.16) and (3.1.20), the complex conjugation acts as a parity on the Kähler weights. Thus, $\overline{Z}$ is a Kähler-scalar function in M_{n_V} with Kähler weights $(-1,1)$, and its Kähler-covariant derivatives read

$$\begin{cases} D_i\overline{Z} = \left(\partial_i - \frac{1}{2}\partial_i K\right)\overline{Z} = \overline{\overline{D}_{\overline{i}} Z}, \\ \overline{D}_{\overline{i}}\overline{Z} = \left(\overline{\partial}_{\overline{i}} + \frac{1}{2}\overline{\partial}_{\overline{i}} K\right)\overline{Z} = \overline{D_i Z}\ . \end{cases} \tag{3.2.115}$$

Since the Kähler weights are additive under multiplication, it is clear that the square absolute value of Z, i.e., $|Z|^2 \equiv Z\overline{Z}$, is a Kähler gauge-invariant quantity, i.e., it has Kähler weights $(0,0)$. Consequently, the Kähler-covariant derivatives of such a Kähler-scalar trivially correspond to the ordinary, flat ones; this can be seen also by explicitly calculating that the terms of Kähler connections $\partial_i K$ cancel each other:

$$\begin{aligned} D_i\left(|Z|^2\right) &= D_i\left(Z\overline{Z}\right) = \left(D_i Z\right)\overline{Z} + Z\left(D_i\overline{Z}\right) \\ &= \left[\left(\partial_i + \frac{1}{2}\partial_i K\right)Z\right]\overline{Z} + Z\left[\left(\partial_i - \frac{1}{2}\partial_i K\right)\overline{Z}\right] \\ &= \partial_i\left(Z\overline{Z}\right) = \partial_i\left(|Z|^2\right) = 2\,|Z|\,\partial_i\,|Z|\,. \end{aligned} \tag{3.2.116}$$

Let us now calculate[19]

$$\begin{aligned} \partial_i\,|Z| = \partial_i\sqrt{Z\overline{Z}} &= \frac{1}{2\,|Z|}\left[\left(\partial_i Z\right)\overline{Z} + Z\left(\partial_i\overline{Z}\right)\right] \\ &= \frac{1}{2\,|Z|}\left[\left(\partial_i Z\right)\overline{Z} + \frac{1}{2}\left(\partial_i K\right)Z\overline{Z}\right] = \frac{\overline{Z}}{2\,|Z|}D_i Z, \end{aligned} \tag{3.2.117}$$

where in the second line we used the Kähler-covariant antiholomorphicity of $\overline{Z}$ expressed by (3.2.63), recalling (3.2.114).
Thus

$$\partial_i\,|Z| = 0 \Leftrightarrow D_i Z = 0\ . \tag{3.2.118}$$

This means that, when considering a Kähler-covariant holomorphic Z, its Kähler-covariant extremization is equivalent to the ordinary extremization of its absolute value. Thus, we may complete (3.2.113), obtaining (2.11), i.e., the general form of the $\frac{1}{2}$-*BPS extreme BH attractor equation*

[19] Throughout these lectures we will, in general, assume the nonvanishing of the central charge

$$|Z| \neq 0 \Leftrightarrow Z \neq 0 \Leftrightarrow \overline{Z} \neq 0.$$

$$
\begin{gathered}
Z_i|_{(z,\overline{z})=(z_H,\overline{z}_H)} \equiv (D_i Z)_{(z,\overline{z})=(z_H,\overline{z}_H)} \\
= \left[\left(\partial_i + \tfrac{1}{2}\partial_i K\right) Z\left(z,\overline{z};n_m,n^e\right)\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0 \\
\Updownarrow \\
\left[\partial_i \left|Z\left(z,\overline{z};n_m,n^e\right)\right|\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0 \\
\Updownarrow \\
\left[\overline{\partial}_{\overline{i}} \left|Z\left(z,\overline{z};n_m,n^e\right)\right|\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0.
\end{gathered}
\tag{3.2.119}
$$

Thus, in a generic supergravity theory (having a Kähler moduli space) with a nonvanishing and Kähler-covariantly holomorphic central charge Z, we explicitly showed that the Kähler-covariant extremization of such a function (expressed by (3.2.112)) is equivalent to the ordinary, flat extremization of its absolute value (given by (3.2.119)).

Now, we can specialize the general form (3.2.119) of the $\frac{1}{2}$-*BPS extreme BH attractor equation* to $\mathcal{N}=2$, $d=4$, n_V-fold MESGT. Such a theory has a complex moduli space M_{n_V} endowed with SKG, and the explicit form of the central charge is given by (3.2.62). Thus, by also using (3.2.67), we get a more explicit (model-dependent) form of the Kähler-covariant extremization of Z at the EH

$$
\begin{gathered}
(D_i Z)_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0, \\
\Updownarrow \\
\left[\left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z})\, n^\Lambda_m\right) f^\Sigma_i(z,\overline{z})\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0, \\
\Updownarrow \\
\left[\begin{array}{l}
\left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z})\, n^\Lambda_m\right)(\partial_i K)\left[\exp\left(\tfrac{1}{2}K(z,\overline{z})\right) X^\Sigma(z)\right] + \\
+ \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z})\, n^\Lambda_m\right)\exp\left(\tfrac{1}{2}K(z,\overline{z})\right)\left(\partial_i X^\Sigma(z)\right)
\end{array}\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0, \\
\Updownarrow \\
\left[\begin{array}{l}
(\partial_i K)\, Z\left(n_m,n^e,z,\overline{z}\right)|_{\mathcal{N}_{\Lambda\Sigma}\to\overline{\mathcal{N}}_{\Lambda\Sigma}} + \\
+ \left(n^e_\Sigma - \overline{\mathcal{N}}_{\Lambda\Sigma}(z,\overline{z})\, n^\Lambda_m\right)\exp\left(\tfrac{1}{2}K(z,\overline{z})\right)\left(\partial_i X^\Sigma(z)\right)
\end{array}\right]_{(z,\overline{z})=(z_H,\overline{z}_H)} = 0.
\end{gathered}
\tag{3.2.120}
$$

By using (3.2.119) and recalling (3.2.53), we finally get

$$
\begin{aligned}
&\left[\begin{array}{l}
\left.\left(\partial_i K\right) Z\left(n_m, n^e, z, \overline{z}\right)\right|_{\mathcal{N}_{\Lambda\Sigma}\to\overline{\mathcal{N}}_{\Lambda\Sigma}}+ \\
+\left(n^e_\Sigma-\overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right) n^\Lambda_m\right)\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\left(\partial_i X^\Sigma\left(z\right)\right)
\end{array}\right]_{(z,\overline{z})=(z_H,\overline{z}_H)}=0 \\
&\Updownarrow \\
&\left[\partial_i\left|\left(n^e_\Sigma-\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right) n^\Lambda_m\right) L^\Sigma\left(z,\overline{z}\right)\right|\right]_{(z,\overline{z})=(z_H,\overline{z}_H)}=0 \\
&\Updownarrow \\
&\left\{\partial_i\left|\left[\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\right]\left(n^e_\Sigma-\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right) n^\Lambda_m\right) X^\Sigma\left(z\right)\right|\right\}_{(z,\overline{z})=(z_H,\overline{z}_H)}=0 \\
&\Updownarrow \\
&\left\{\overline{\partial}_{\overline{i}}\left|\left[\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\right]\left(n^e_\Sigma-\overline{\mathcal{N}}_{\Lambda\Sigma}\left(z,\overline{z}\right) n^\Lambda_m\right) \overline{X}^\Sigma\left(\overline{z}\right)\right|\right\}_{(z,\overline{z})=(z_H,\overline{z}_H)}=0,
\end{aligned}
\tag{3.2.121}
$$

where in the last three lines the flat derivatives may be substituted by the Kähler-covariant ones, due to the vanishing of the Kähler weights of the absolute value of the central charge Z. (3.2) and (3.2) are the $\frac{1}{2}$-*BPS extreme BH attractor equation* of $\mathcal{N}=2$, $d=4$, n_V-fold MESGT.

Now, it should be recalled that in ($\mathcal{N}=2$) supersymmetric theories the saturation of the BPS bound fixes the ADM mass of the BH to be equal to the absolute value of the central charge

$$
M_{ADM}\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)=\left|Z\right|\left(z_\infty,\overline{z}_\infty;n_m,n^e\right)\ . \tag{3.2.122}
$$

By admitting an extension of such a saturated bound to the r-dependent moduli space M_{n_V}, one gets[20]

$$
M_{ADM}\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)=\left|Z\right|\left(z\left(r\right),\overline{z}\left(r\right);n_m,n^e\right)\ , \tag{3.2.123}
$$

such that (3.2.122) is the asymptotical limit $r\to\infty$ of (3.2.123). Thus, in the considered case of $\frac{1}{2}$-BPS extremal BHs we may directly translate the previous results in terms of the ADM mass function, obtaining

$$
\left[\partial_i M_{ADM}\left(z,\overline{z};n_m,n^e\right)\right]_{(z,\overline{z})=(z_H,\overline{z}_H)}=0 \tag{3.2.124}
$$

$$
\Updownarrow
$$

$$
\left[\overline{\partial}_{\overline{i}} M_{ADM}\left(z,\overline{z};n_m,n^e\right)\right]_{(z,\overline{z})=(z_H,\overline{z}_H)}=0. \tag{3.2.125}
$$

Moreover, at the EH it holds that

$$
\begin{aligned}
&M_{ADM}\left(z=z_H\left(n_m,n^e\right),\overline{z}=\overline{z}_H\left(n_m,n^e\right);n_m,n^e\right) \\
&=M_{ADM,H}\left(n_m,n^e\right)=M_{BR}\left(n_m,n^e\right)\ ,
\end{aligned}
\tag{3.2.126}
$$

[20] Otherwise speaking, we move to consider the "ADM mass" function in M_{n_V}; moreover, we assume the limits $r\to\infty$ and $r\to r_H^+$ to be "smooth" also for such a function.

where we recalled that the "near-horizon" geometry is described by the BR metric. Consequently, the extremum value of the ADM mass function of the BPS solutions at the EH is equal to the mass of the BR geometry.

Thus, we may conclude that the AM for $\frac{1}{2}$-BPS extremal BHs, encoded in the condition of Kähler-covariant antiholomorphicity of the central charge (see (3.2.113)), also implies the extremization of the ADM mass function w.r.t. its dependence on z and $\overline{z}$.

More in particular, by considering (3.2.62), we get the explicit expression of the ADM mass function of the $\frac{1}{2}$-BPS extremal BHs in the framework of the $\mathcal{N}=2$, $d=4$, n_V-fold MESGT

$$\begin{aligned} M_{ADM}\left(z,\overline{z};n_m,n^e\right) &= \left|\left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)L^\Sigma\left(z,\overline{z}\right)\right| \\ &= \left[\exp\left(\frac{1}{2}K\left(z,\overline{z}\right)\right)\right]\left|\left(n^e_\Sigma - \mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)n^\Lambda_m\right)X^\Sigma\left(z\right)\right| . \end{aligned} \tag{3.2.127}$$

An example of the extremization[21] of the absolute value of the central charge function Z of the local SUSY algebra (or equivalently for BPS extremal BHs, of the ADM mass function) in the Kähler–Hodge moduli space M_{n_V} of the $\mathcal{N}=2$, $d=4$, n_V-fold MESGT is shown in Fig. 3.1.

Moreover, at the attractor point(s) corresponding to the radius $r=r_H$, the two independent $Sp\left(2n_V+2\right)$-invariants I_1 and I_2 homogeneous of degree two in the (quantized) electric and magnetic charges (defined in (3.2.80) and then explicited in (3.2.86) and (3.2.87)) coincide one with the other, "degenerating" in one unique value[22]

$$\begin{aligned} & I_H\left(n_m,n^e\right) \equiv I_1(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right);n_m,n^e) \\ & = I_2(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right);n_m,n^e) \\ & = \left|Z(z_H\left(n_m,n^e\right),\overline{z}_H\left(n_m,n^e\right);n_m,n^e)\right|^2 \equiv \left|Z_h(n_m,n^e)\right|^2 \\ & = M^2_{ADM,H}\left(n_m,n^e\right) = M^2_{BR}\left(n_m,n^e\right); \\ & \Updownarrow \end{aligned} \tag{3.2.128}$$

[21] In Subsect. 4.4 it will be shown that in $\mathcal{N}=2$, $d=4$, n_V-fold MESGT with strictly positive definite metric of the moduli space and with a single continuous branch of the function $|Z|\left(z,\overline{z};n_m,n^e\right)$, at most only one extremum point exists, and it is a minimum. Clearly, the situation completely changes if the hypotheses of strictly positive definiteness of the metric and/or single continuous branch for $|Z|$ are removed. See, e.g., [55].

[22] As done above, we assume that the limit $r\to r^+_H$ is "smooth" for all considered functions in the moduli space.

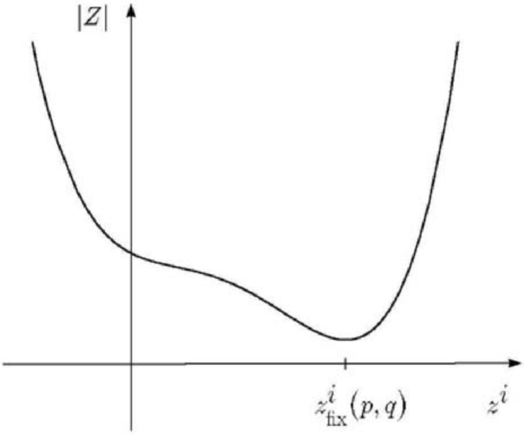

Fig. 3.1. Minimization of the absolute value of the "central charge" function $|Z|(z, \overline{z}; n_m, n^e)$ of the local SUSY algebra in the (holomorphic part of the) Kähler–Hodge complex moduli space M_{n_V} of the $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT. In the picture $z^i_{FIX}(p,q)$ stands for $z_H(n_m, n^e)$, i.e., for the "attractor," purely charge-dependent value of the moduli at the EH of the considered $\frac{1}{2}$-BPS extremal (eventually RN) BH. The attractor mechanism fixes the extrema of the central charge to correspond to the discrete "fixed" points of the "attractor variety" [31–33] M_{n_V}. Of course, the moduli-dependence of the central charge is shown at a fixed charge configuration of the system, i.e., for a fixed $(2n_V + 2)$-d symplectic-covariant vector n defined in (3.2.35)

$$
\begin{aligned}
& n^T \mathcal{M} \left(\mathcal{N} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) \right) n \\
& = n^T \mathcal{M} \left(\mathcal{F} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) \right) n \\
& = \left[n^e_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) n^\Sigma_m \right] \\
& \left(\left(Im \left(\left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) \right) \right)^{-1} \right)^{\Lambda\Delta} \cdot \\
& \cdot \left[n^e_\Delta - \mathcal{N}_{\Delta\Gamma} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) n^\Gamma_m \right] \\
& = \left[n^e_\Lambda - \overline{F}_{\Lambda\Sigma} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) n^\Sigma_m \right] \\
& \left(\left(Im \left(\mathcal{F} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) \right) \right)^{-1} \right)^{\Lambda\Delta} \cdot \\
& \cdot \left[n^e_\Delta - F_{\Delta\Gamma} \left(z_H \left(n_m, n^e \right), \overline{z}_H \left(n_m, n^e \right) \right) n^\Gamma_m \right] \\
& = M^2_{ADM,H} \left(n_m, n^e \right) = M^2_{BR} \left(n_m, n^e \right) = |Z|^2_H \left(n_m, n^e \right).
\end{aligned}
\tag{3.2.129}
$$

$|Z|_H(n_m, n^e)$ is the purely charge-dependent extremized value of the absolute value of the central charge function of the local $\mathcal{N}=2$, $d=4$ SUSY algebra, reached at the EH of the BPS extremal (RN) BH.

Now, by recalling the relation between the Horizon area and the BR mass

$$M_{BR}^2(n_m, n^e) = \frac{A_H}{4\pi} \tag{3.2.130}$$

and by using the BHEA formula, we may relate the entropy of the extremal BPS (RN) BH to the area of its EH, and therefore to its ADM mass function, whose near-horizon limit coincides with the BR mass.

Thus, the final result is the expression of the entropy of the extremal BPS (RN) BH in terms of the extremized (minimized) square absolute value of the central charge function of the local $\mathcal{N}=2$, $d=4$ SUSY algebra, reached in correspondence with the discrete attractor moduli configuration(s) at the EH

$$\begin{aligned} S_{BH} &= \frac{A_H}{4} = \pi M_{BR}^2(n_m, n^e) \\ &= \pi M_{ADM}^2\left(z_H(n_m, n^e), \overline{z}_H(n_m, n^e); n_m, n^e\right) \\ &= \pi M_{ADM,H}^2((n_m, n^e)) \\ &= \pi |Z|_H^2(n_m, n^e). \end{aligned} \tag{3.2.131}$$

As mentioned above, a key feature of the $d=4$ and 5, $\mathcal{N}=2$ SUGRAs coupled to n_V Abelian vector supermultiplets is the fact that the extremization of the central charge function Z through the AEs may be made "coordinate-free" in the moduli space M_{n_V}, by using the fact that such a n_V-d complex manifold is endowed with a special Kähler metric structure, on which we reported above for the $d=4$ case.

Clearly, the U-duality-invariant, i.e., symplectic-invariant, (re)formulation of the BHEA in the case of $d=4$ and 5, $\mathcal{N}=2$ MESGTs has various advantages, coming from its manifest symmetry.

Finally, one can also check the first consistency condition (3.2.10) for unbroken $\mathcal{N}=2$ SUSY at the EH; such a relation relates the Riemann-Christoffel tensor of the metric background to the graviphoton field strength. By using the definition of the central charge function, and by evaluating it at the attractor fixed point(s), it is possible to show that one obtains nothing but the BPS-saturation condition for the BR metric, expressing the validity of the cosmic censorship principle, and consequently yielding the existence of an EH with a regular geometry covering the inner s-t singularity

$$M_{BR}^2(n_m, n^e) = |Z|_H^2(n_m, n^e). \tag{3.2.132}$$

Whence, by recalling (3.2.130), one reobtains the main result given by (3.2.131).

4

Black Holes and Critical Points in Moduli Space

As we have seen, the $d = 4$, $\mathcal{N} = 2$ ungauged SUGRAs have two types of geometries: the s-t geometry and the moduli space geometry. In this section, mainly following the seminal paper [55] of Ferrara, Gibbons, and Kallosh, (see also [61]) we will consider the fundamental interplay between these two geometries, especially in relation with the attractor mechanism.

4.1 Black Holes and Constrained Geodesic Motion

Let us start by considering the 4-d Lagrangian density of a system of real scalars and abelian gauge fields coupled to gravity, [55]

$$
\begin{aligned}
\mathcal{L}_4 = &-\frac{R}{2} + \frac{1}{2}G_{ab}\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu} - \frac{1}{4}\mu_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho} \\
&-\frac{1}{4}\nu_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\,{}^*\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho},
\end{aligned}
\tag{4.1.1}
$$

with s-t lower Greek indices running $0, 1, \ldots, 3$, moduli lower Latin indices running $1, \ldots, m_\phi$, and symplectic capital Greek indices running $1, \ldots, n_V + 1$. $g^{\mu\nu}(x)$ and $G_{ab}(\phi)$ are the 4-d s-t metric and the m_ϕ-d moduli space metric, respectively. $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ respectively are the real, strictly[1] positive definite, moduli-dependent matrices of dilatonic and axionic couplings of the abelian gauge fields (they may be considered symmetric without loss of generality). Finally, ${}^*\mathcal{F}^{\Sigma}_{\lambda\rho}$ denotes the usual Hodge $*$-dual (see Eq. (3.2.18)).

[1] We may disregard the possibility to have vanishing eigenvalues for the matrices $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$. Indeed, such zero modes would correspond to abelian gauge fields with vanishing kinetic term, which can be thus omitted from the considered Lagrangian density (4.1.1). Consequently, since the matrices $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ are real, symmetric, and without zero modes, they are always invertible by an orthogonal transformation. By the way, as it will be evident by looking at (4.1.21), only $\mu_{\Lambda\Sigma}$ needs to be invertible in order for V_{BH} to be consistently defined

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We restrict our attention to static (i.e., time independent) metric backgrounds, described by the metric Ansatz[2] (remind that, unless otherwise indicated, we put $c = \hbar = G_0 = 1$ and $i, j = 1, 2, 3$)

$$ds^2 = e^{2U(\underline{x})} dt^2 - e^{-2U(\underline{x})} \gamma_{ij} \left(\underline{x}\right) dx^i, dx^j. \tag{4.1.2}$$

Such an Ansatz is a generalization (with nonnecessarily Euclidean spatial sections) of the previously considered 4-d BH metric given by (1.20). The assumption of staticity allows one to get a 3-d effective Lagrangian density, from which the field equations may be derived:

$$\mathcal{L}_3 = \frac{R\left[\gamma_{ij}\right]}{2} - \frac{1}{2}\gamma^{ij}\partial_i \widehat{\phi}^{\widehat{a}} \partial_j \widehat{\phi}^{\widehat{b}} \widehat{G}_{\widehat{a}\widehat{b}} \,, \tag{4.1.3}$$

where $R\left[\gamma_{ij}\right]$ denotes the intrinsic scalar curvature related to the 3-d spatial metric $\gamma_{ij}\left(\underline{x}\right)$. Moreover, the "hatted" scalar fields include, beside the scalar fields ϕ^a of the 4d theory, also the function $U\left(\underline{x}\right)$ defining the s-t metric and the electrostatic ψ^Λ and magnetostatic χ_Λ potentials related to the $U(1)$ gauge fields:

$$\widehat{\phi}^{\widehat{a}} \equiv \left(U, \phi^a, \psi^\Lambda, \chi_\Lambda\right), \tag{4.1.4}$$

with the "hatted" indices $\widehat{a}$ ranging in a set of cardinality $m_\phi + 2n_V + 3$.

In other words, in the passage from the 4d theory to the related effective 3d theory, it is convenient to enlarge the scalar manifold $\mathcal{M}_\phi$ as follows:

$$\left(\mathcal{M}_\phi, \left\{\phi^a\right\}, G_{ab}\left(\phi\right)\right) \longrightarrow \left(\mathcal{M}_{\widehat{\phi}}, \left\{\widehat{\phi}^{\widehat{a}}\right\}, \widehat{G}_{\widehat{a}\widehat{b}}\left(\widehat{\phi}\right)\right), \tag{4.1.5}$$

where it should be noted that the $U(1)^{n_V+1}$ gauge invariance implies that $\widehat{G}_{\widehat{a}\widehat{b}}$ is independent of the electromagnetic potentials

$$\widehat{G}_{\widehat{a}\widehat{b}}\left(\widehat{\phi}\right) = \widehat{G}_{\widehat{a}\widehat{b}}\left(U, \phi\right). \tag{4.1.6}$$

We further increase the symmetry of the considered s-t metric background, by formulating the hypothesis of spherical symmetry corresponding to the Ansatz [55]

$$\gamma_{ij}\left(\underline{x}\right) dx^i, dx^j = \frac{\mathbf{c}^4 d\tau^2}{sinh^4\left(\mathbf{c}\tau\right)} + \frac{\mathbf{c}^2}{sinh^2\left(\mathbf{c}\tau\right)}\left(d\theta^2 + sin^2\theta \, d\varphi^2\right), \tag{4.1.7}$$

where

$$\tau \equiv \frac{1}{r_H - r}. \tag{4.1.8}$$

Therefore, since $r \in [r_H, +\infty)$, it follows that τ runs from $-\infty$ (BH EH) to 0^- (spatial infinity). Moreover,

---

[2] It has been shown by Tod [81] that in $\mathcal{N} = 2$ supergravity theories the general form of static metrics admitting supersymmetries is given by (4.1.2)

$$\mathbf{c}^2 \equiv \frac{\kappa_s A_H}{8\pi} = 2 S_{BH} T_{BH}, \tag{4.1.9}$$

where in the last passage we recalled (2.3) and (2.4) (S_{BH} and T_{BH} respectively denote the entropy and the temperature of the BH).

Summarizing, we are considering the following 4-d static, spherically symmetric BH metrics:

$$ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\frac{\mathbf{c}^4 d\tau^2}{sinh^4(\mathbf{c}\tau)} + \frac{\mathbf{c}^2}{sinh^2(\mathbf{c}\tau)}\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right], \tag{4.1.10}$$

where τ is the 1-d effective evolution parameter defined in (4.1.8), and we introduced $U^{'}(\tau) = U(r)$ and dropped the prime out. By further using the spherical symmetry (i.e., the (θ, φ)-independence) of the BH metric (4.1.10), one obtains a 1-d, τ-dependent effective theory.

It can be shown that the 1-d effective Lagrangian from which the radial equations of motion may be derived has the purely geodesic form [57]

$$\mathcal{L}_1 = \widehat{G}_{\widehat{a}\widehat{b}}(U,\phi)\frac{d\widehat{\phi^{a}}(\tau)}{d\tau}\frac{d\widehat{\phi^{b}}(\tau)}{d\tau} \tag{4.1.11}$$

constrained by the condition

$$\widehat{G}_{\widehat{a}\widehat{b}}(U,\phi)\frac{d\widehat{\phi^{a}}(\tau)}{d\tau}\frac{d\widehat{\phi^{b}}(\tau)}{d\tau} = \mathbf{c}^2, \tag{4.1.12}$$

which characterizes τ as a "generalized proper time" for the enlarged scalar manifold $\mathcal{M}_{\widehat{\phi}}$.

Consequently, by assuming the s-t symmetries expressed by (4.1.2) and (4.1.7), the dynamics related to the starting 4-d Lagrangian (4.1.1) may be shown to reduce to a geodesic, constrained dynamics described by (4.1.11) and (4.1.12).

In order to further explicit $\mathcal{L}_1$, we may formulate the following "block-diagonal" Ansatz[3] for $\widehat{G}_{\widehat{a}\widehat{b}}$

$$\widehat{G}_{\widehat{a}\widehat{b}}(U,\phi) = \begin{pmatrix} 1 & & & \\ & \frac{1}{2}G_{ab}(\phi) & & \\ & & \widehat{G}_{\Lambda\Sigma}(U,\phi) & \\ & & & \widehat{G}^{\Lambda\Sigma}(U,\phi) \end{pmatrix}, \tag{4.1.13}$$

[3] A particular(ly simple) formulation of the "block-diagonal" Ansatz (4.1.13) reads

$$\widehat{G}_{\widehat{a}\widehat{b}}(U,\phi) = \begin{pmatrix} 1 & & \\ & \frac{1}{2}G_{ab}(\phi) & \\ & & \epsilon_{n_V+1} \end{pmatrix},$$

where ϵ_{n_V+1} is the $(2n_V+2)$-d symplectic metric given by (3.1.24).
The factor $\frac{1}{2}$ in front of $G_{ab}(\phi)$ is introduced for later convenience

where as usual

$$\widehat{G}^{\Lambda\Sigma}(U,\phi)\,\widehat{G}_{\Sigma\Xi}(U,\phi)=\delta^{\Lambda}_{\Xi},\quad \forall U,\phi, \tag{4.1.14}$$

and the unwritten components vanish. Therefore, $\mathcal{L}_1$ reads

$$\begin{aligned}\mathcal{L}_1 = &\left(\frac{dU(\tau)}{d\tau}\right)^2+\frac{1}{2}G_{ab}(\phi)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau}\\ &+\widehat{G}_{\Lambda\Sigma}(U,\phi)\frac{d\psi^{\Lambda}(\tau)}{d\tau}\frac{d\psi^{\Sigma}(\tau)}{d\tau}+\widehat{G}^{\Lambda\Sigma}(U,\phi)\frac{d\chi_{\Lambda}(\tau)}{d\tau}\frac{d\chi_{\Sigma}(\tau)}{d\tau}.\end{aligned} \tag{4.1.15}$$

Now, since $\widehat{G}_{\hat{a}\hat{b}}$ is independent of ψ^{Λ} and χ_{Λ}, we obtain that

$$\frac{dp^{\Lambda}}{d\tau}=0,\qquad \frac{dq_{\Lambda}}{d\tau}=0, \tag{4.1.16}$$

where

$$\begin{cases} p^{\Lambda}\equiv\frac{1}{2}\frac{\delta\mathcal{L}_1}{\delta\left(\frac{d\chi_{\Lambda}}{d\tau}\right)}=\widehat{G}^{\Lambda\Sigma}\frac{d\chi_{\Sigma}}{d\tau};\\[2ex] q_{\Lambda}\equiv\frac{1}{2}\frac{\delta\mathcal{L}_1}{\delta\left(\frac{d\psi^{\Lambda}}{d\tau}\right)}=\widehat{G}_{\Lambda\Sigma}\frac{d\psi^{\Sigma}}{d\tau}, \end{cases} \tag{4.1.17}$$

are identified with the magnetic and electric charges of the BH, respectively ($p^{\Lambda}\equiv n^{\Lambda}_{m}, q_{\Lambda}\equiv n^{e}_{\Lambda}$). Thus, by using definitions (4.1.17), (4.1.15) can be further elaborated as

$$\mathcal{L}_1=\left(\frac{dU(\tau)}{d\tau}\right)^2+\frac{1}{2}G_{ab}(\phi)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau}+q_{\Lambda}\frac{d\psi^{\Lambda}(\tau)}{d\tau}+p^{\Lambda}\frac{d\chi_{\Lambda}(\tau)}{d\tau}. \tag{4.1.18}$$

Now, it can be shown [57–59] and [60]; see also [61] that

$$q_{\Lambda}\frac{d\psi^{\Lambda}(\tau)}{d\tau}+p^{\Lambda}\frac{d\chi_{\Lambda}(\tau)}{d\tau}=e^{2U}V_{BH}(\phi;p,q), \tag{4.1.19}$$

where $V_{BH}(\phi;p,q)$ is the so-called BH effective potential, i.e., a particular, positive function of the scalars ϕ's and of the BH charges, constructed from the (strictly) positive definite couplings $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ as follows:

$$V_{BH}(\phi,p;q)\equiv\frac{1}{2}\left(p^{\Lambda},q_{\Lambda}\right)\mathbf{M}(\phi)\begin{pmatrix}p^{\Sigma}\\ q_{\Sigma}\end{pmatrix}, \tag{4.1.20}$$

where the $(2n_V+2)\times(2n_V+2)$, ϕ-dependent matrix $\mathbf{M}(\phi)$ is defined as

$$\mathbf{M}(\phi) \equiv \begin{pmatrix} \mu_{\Lambda\Sigma}(\phi) + \nu_{\Lambda\Delta}(\phi)\left(\mu^{-1}(\phi)\right)^{\Delta\Xi}\nu_{\Xi\Sigma}(\phi) & \nu_{\Lambda\Xi}(\phi)\left(\mu^{-1}(\phi)\right)^{\Xi\Sigma} \\ \\ \left(\mu^{-1}(\phi)\right)^{\Lambda\Xi}\nu_{\Xi\Sigma}(\phi) & \left(\mu^{-1}(\phi)\right)^{\Lambda\Sigma} \end{pmatrix}. \tag{4.1.21}$$

The reality, symmetry, and (strict) positive definiteness[4] of $\mu_{\Lambda\Sigma}(\phi)$ and $\nu_{\Lambda\Sigma}(\phi)$ imply the reality, symmetry, and (strict) positive definiteness of the matrix $\mathbf{M}(\phi)$, and consequently the positivity of $V_{BH}(\phi, p; q)$ in all $\mathcal{M}_\phi \times \Gamma$.

By substituting (4.1.19) in (4.1.18), we can finally write the 1-d effective Lagrangian density as

$$\begin{aligned} &\mathcal{L}_1\left[U(\tau), \phi(\tau); p, q\right] \\ &= \left(\tfrac{dU(\tau)}{d\tau}\right)^2 + \tfrac{1}{2} G_{ab}(\phi(\tau)) \tfrac{d\phi^a(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau} + e^{2U(\tau)} V_{BH}(\phi(\tau); p, q). \end{aligned} \tag{4.1.22}$$

Analogously, it may be shown that the constraint (4.1.12) is equivalent to

$$\begin{aligned} &\left(\tfrac{dU(\tau)}{d\tau}\right)^2 + \tfrac{1}{2} G_{ab}(\phi(\tau)) \tfrac{d\phi^a(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau} - e^{2U(\tau)} V_{BH}(\phi(\tau); p, q) \\ &= \mathbf{c}^2 = 2 S_{BH} T_{BH}. \end{aligned} \tag{4.1.23}$$

The equation of motion of U reads

$$\begin{gathered} \frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U(\tau), \phi(\tau); p, q\right]}{\delta\left(\frac{dU(\tau)}{d\tau}\right)} = \frac{\delta\mathcal{L}_1\left[U(\tau), \phi(\tau); p, q\right]}{\delta U(\tau)} \\ \Updownarrow \\ \frac{d^2U(\tau)}{d\tau^2} = e^{2U(\tau)} V_{BH}(\phi(\tau); p, q). \end{gathered} \tag{4.1.24}$$

For what concerns the equations of motion of the real scalars ϕ^a, from (4.1.22) we get

$$\begin{gathered} \frac{\delta\mathcal{L}_1\left[U(\tau), \phi(\tau); p, q\right]}{\delta\left(\frac{d\phi^a(\tau)}{d\tau}\right)} = G_{ab}(\phi(\tau))\frac{d\phi^b(\tau)}{d\tau}; \\ \Downarrow \end{gathered}$$

[4] It is worth pointing out once again that, in order for $\mathbf{M}(\phi)$ to be well defined, at least $\mu_{\Lambda\Sigma}(\phi)$ must be strictly positive definite on the whole moduli space $\mathcal{M}_\phi$

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\left(\frac{d\phi^a(\tau)}{d\tau}\right)}=\frac{\partial G_{ab}\left(\phi\right)}{\partial\phi^c}\frac{d\phi^c\left(\tau\right)}{d\tau}\frac{d\phi^b\left(\tau\right)}{d\tau}+G_{ab}\left(\phi\left(\tau\right)\right)\frac{d^2\phi^b\left(\tau\right)}{d\tau^2}; \tag{4.1.25}$$

$$\frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\phi^a\left(\tau\right)}=\frac{1}{2}\frac{\partial G_{cb}\left(\phi\right)}{\partial\phi^a}\frac{d\phi^c\left(\tau\right)}{d\tau}\frac{d\phi^b\left(\tau\right)}{d\tau}+e^{2U(\tau)}\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}. \tag{4.1.26}$$

Thus

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\left(\frac{d\phi^a(\tau)}{d\tau}\right)}=\frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\phi^a\left(\tau\right)} \tag{4.1.27}$$

$$\Updownarrow$$

$$\begin{aligned}&G_{ab}\left(\phi\left(\tau\right)\right)\tfrac{d^2\phi^b(\tau)}{d\tau^2}+\tfrac{\partial G_{ab}(\phi)}{\partial\phi^c}\tfrac{d\phi^c(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau}\\&-\tfrac{1}{2}\tfrac{\partial G_{cb}(\phi)}{\partial\phi^a}\tfrac{d\phi^c(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau}\\&=e^{2U(\tau)}\tfrac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a}.\end{aligned} \tag{4.1.28}$$

By contracting both sides with $G^{ac}\left(\phi\right)$ and recalling that $G^{ac}\left(\phi\right)G_{ab}\left(\phi\right)=\delta^c_b$, one gets

$$\begin{aligned}&\tfrac{d^2\phi^c(\tau)}{d\tau^2}+G^{ac}\left(\phi\left(\tau\right)\right)\tfrac{\partial G_{ab}(\phi)}{\partial\phi^c}\tfrac{d\phi^c(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau}\\&-\tfrac{1}{2}G^{ac}\left(\phi\left(\tau\right)\right)\tfrac{\partial G_{cb}(\phi)}{\partial\phi^a}\tfrac{d\phi^c(\tau)}{d\tau}\tfrac{d\phi^b(\tau)}{d\tau}\\&=e^{2U(\tau)}G^{ac}\left(\phi\left(\tau\right)\right)\tfrac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a}.\end{aligned} \tag{4.1.29}$$

Now, by assuming the geometry of the real parametrization $\mathcal{M}_\phi$ of the moduli space to be a (regular[5]) Riemannian one, the Riemann-covariant derivative of a scalar ϕ^a reads

$$D_b\phi^a=\partial_b\phi^a+\mathit{\Gamma}_{bd}{}^{a}\left(\phi\right)\phi^d=\delta^a_b+\mathit{\Gamma}_{bd}{}^{a}\left(\phi\right)\phi^d, \tag{4.1.30}$$

where the connection is given by the Christoffel symbols of the second kind of the metric $G_{ab}\left(\phi\right)$ (the round brackets denote symmetrization w.r.t. the enclosed indices):

$$\begin{aligned}\mathit{\Gamma}_{bd}{}^{a}\left(\phi\right)&=\left\{\begin{smallmatrix}a\\bd\end{smallmatrix}\right\}\left(\phi\right)=\left\{\begin{smallmatrix}a\\(bd)\end{smallmatrix}\right\}\left(\phi\right)\\&=\frac{1}{2}G^{ac}\left(\phi\right)\left[\partial_bG_{cd}\left(\phi\right)+\partial_dG_{cb}\left(\phi\right)-\partial_cG_{bd}\left(\phi\right)\right].\end{aligned} \tag{4.1.31}$$

[5] We call *regular* a geometry endowed with a strictly positive definite metric

Thence, the Riemann-covariant differential of a scalar ϕ^a is

$$D\phi^a = (D_b\phi^a)\,d\phi^b = \delta^a_b + {\Gamma_{bd}}^a(\phi)\,\phi^d d\phi^b; \tag{4.1.32}$$
$$\Downarrow$$
$$\frac{D\phi^a(\tau)}{d\tau} = (D_b\phi^a(\tau))\,\frac{d\phi^b(\tau)}{d\tau} = \frac{d\phi^a(\tau)}{d\tau} + {\Gamma_{bd}}^a(\phi(\tau))\,\phi^d(\tau)\,\frac{d\phi^b(\tau)}{d\tau}. \tag{4.1.33}$$

Consequently, one gets

$$\begin{aligned}\frac{D\phi^a(\tau)}{d\tau^2} &= \frac{D\left(\frac{d\phi^a(\tau)}{d\tau}\right)}{d\tau} = \left[D_b\frac{d\phi^a(\tau)}{d\tau}\right]\frac{d\phi^b(\tau)}{d\tau}\\ &= \frac{d^2\phi^a(\tau)}{d\tau^2} + {\Gamma_{bd}}^a(\phi(\tau))\,\frac{d\phi^b(\tau)}{d\tau}\frac{d\phi^d(\tau)}{d\tau}\\ &= \frac{d^2\phi^a(\tau)}{d\tau^2} + \frac{1}{2}G^{ac}(\phi(\tau))\left[\partial_b G_{cd}(\phi(\tau)) + \partial_d G_{cb}(\phi(\tau))\right.\\ &\quad \left.-\partial_c G_{bd}(\phi(\tau))\right]\frac{d\phi^b(\tau)}{d\tau}\frac{d\phi^d(\tau)}{d\tau}.\end{aligned} \tag{4.1.34}$$

By using the result

$$G^{ca}(\partial_d G_{ab})\,\frac{d\phi^b}{d\tau}\frac{d\phi^d}{d\tau} = \frac{1}{2}G^{ca}(\partial_d G_{ab} + \partial_b G_{ad})\,\frac{d\phi^b}{d\tau}\frac{d\phi^d}{d\tau}, \tag{4.1.35}$$

one finally obtains

$$\begin{aligned}\frac{D\phi^c(\tau)}{d\tau^2} &= \frac{d^2\phi^c(\tau)}{d\tau^2} + G^{ca}(\phi(\tau))\left[\partial_d G_{ab}(\phi(\tau))\right]\frac{d\phi^b(\tau)}{d\tau}\frac{d\phi^d(\tau)}{d\tau}\\ &\quad -\frac{1}{2}G^{ac}(\phi(\tau))\left[\partial_a G_{bd}(\phi(\tau))\right]\frac{d\phi^b(\tau)}{d\tau}\frac{d\phi^d(\tau)}{d\tau},\end{aligned} \tag{4.1.36}$$

and (4.1) may be rewritten as

$$\frac{D\phi^c(\tau)}{d\tau^2} = e^{2U(\tau)}G^{bc}(\phi(\tau))\,\frac{\partial V_{BH}(\phi;p,q)}{\partial\phi^b} = e^{2U(\tau)}D^c V_{BH}(\phi;p,q), \tag{4.1.37}$$

where we used the scalar nature of V_{BH} in $\mathcal{M}_\phi$ in order to write $G^{bc}\partial_b V_{BH} = G^{bc}D_b V_{BH} = D^c V_{BH}$. By recontracting with G_{ac}, (4.1.37) yields

$$G_{ac}(\phi(\tau))\,\frac{D\phi^c(\tau)}{d\tau^2} = e^{2U(\tau)}\frac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a}. \tag{4.1.38}$$

Since we assume the metric postulate (i.e., the covariant constancy of the metric tensor) to hold in the Riemann geometry of $\mathcal{M}_\phi$, we may use

$$G_{ac}D\phi^c = D\left(G_{ac}\phi^c\right) = D\phi_a, \tag{4.1.39}$$

in order to get the final expression for the equations of motion of the real scalars ϕ^a's:

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\left(\frac{d\phi^a(\tau)}{d\tau}\right)} = \frac{\delta\mathcal{L}_1\left[U(\tau),\phi\left(\tau\right);p,q\right]}{\delta\phi^a\left(\tau\right)}$$
$$\Updownarrow$$
$$\frac{D\phi_a(\tau)}{d\tau^2} = e^{2U(\tau)}\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}, \tag{4.1.40}$$

where D denotes the Riemann-covariant differential in the real, Riemann parametrization $\mathcal{M}_\phi$ of the moduli space.

The boundary conditions for $U(\tau)$ and $\phi^a\left(\tau\right)$ respectively read

$$U(0) = 0, \quad lim_{\tau\rightarrow-\infty}U(\tau) = \mathbf{c}\tau; \tag{4.1.41}$$

$$\phi^a\left(0\right) = \phi^a_\infty, \quad lim_{\tau\rightarrow-\infty}\frac{d\phi^a\left(\tau\right)}{d\tau} = O\left(e^{\mathbf{c}\tau}\right), \quad \forall a\in\left\{1,\ldots,m_\phi\right\}. \tag{4.1.42}$$

The above "near-horizon" (i.e., $\tau\rightarrow-\infty$) boundary conditions for U and $\frac{d\phi^a(\tau)}{d\tau}$ hold, in general, only for $\mathbf{c}\neq0$. In the case of vanishing $\mathbf{c}$ (corresponding to extreme BHs), they will be substituted by the conditions (4.2.5) and (4.2.6) below.

The BH mass may be defined as

$$M_{BH}\equiv lim_{\tau\rightarrow0^-}\frac{dU\left(\tau\right)}{d\tau}; \tag{4.1.43}$$

therefore, the spatial asymptotical expression of the potential U reads

$$lim_{\tau\rightarrow0^-}U\left(\tau\right) = M_{BH}\tau + U_0. \tag{4.1.44}$$

The boundary, asymptotical condition $U(0) = 0$ fixes $U_0 = 0$.

Let us now instead consider the Mac-Laurin expansion of the real scalars $\phi^a\left(\tau\right)$'s, i.e., the spatial asymptotical expansion of the moduli of the spherically symmetric (nonnecessarily supersymmetric) theory being considered:

$$\phi^a\left(\tau\right) = \textstyle\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n\phi^a(\tau)}{d\tau^n}\right|_{\tau\rightarrow0^-}\tau^n$$

$$= \phi_\infty^a + \left.\frac{d\phi^a(\tau)}{d\tau}\right|_{\tau\to 0^-} \tau + \frac{1}{2}\left.\frac{d^2\phi^a(\tau)}{d\tau^2}\right|_{\tau\to 0^-} \tau^2 + O\left(\tau^3\right); \qquad (4.1.45)$$

$$\Downarrow$$

$$\frac{d\phi^a(\tau)}{d\tau} = \left.\frac{d\phi^a(\tau)}{d\tau}\right|_{\tau\to 0^-} + \left.\frac{d^2\phi^a(\tau)}{d\tau^2}\right|_{\tau\to 0^-} \tau + O\left(\tau^2\right); \qquad (4.1.46)$$

$$\Downarrow$$

$$lim_{\tau\to 0^-}\frac{d\phi^a(\tau)}{d\tau} = \left.\frac{d\phi^a(\tau)}{d\tau}\right|_{\tau\to 0^-} \equiv \Sigma^a. \qquad (4.1.47)$$

Σ^a is defined as the "scalar charge," asymptotically associated with the scalar field ϕ^a; it is given by the coefficient of $\frac{1}{2}\tau^2$ in the third term of the Mac-Laurin expansion of $\phi^a(\tau)$.

Now, by considering the spatial asymptotical limit of the constraint (4.1.23) and using (4.1.41), (4.1.43), and (4.1.47), under suitable assumptions of "regularity" and smoothness of the concerned functions (allowing one to factorize the $lim_{\tau\to 0^-}$ operation) one gets

$$\begin{aligned}
&lim_{\tau\to 0^-}\left[\left(\frac{dU(\tau)}{d\tau}\right)^2 + \frac{1}{2}G_{ab}\left(\phi(\tau)\right)\frac{d\phi^a(\tau)}{d\tau}\frac{d\phi^b(\tau)}{d\tau} - e^{2U(\tau)}V_{BH}\left(\phi(\tau);p,q\right)\right]\\
&= \mathbf{c}^2 = 2S_{BH}T_{BH};
\end{aligned}$$

$$\Updownarrow$$

$$M_{BH}^2 + \frac{1}{2}G_{ab}\left(\phi_\infty\right)\Sigma^a\Sigma^b - V_{BH}\left(\phi_\infty;p,q\right) = \mathbf{c}^2 = 2S_{BH}T_{BH} \geqslant 0. \qquad (4.1.48)$$

This is the so-called "antigravity" bound for 4-d, $\mathbf{c}^2$-parameterized BHs [58, 59]. Its physical meaning is the following: the attractive forces of BH gravity (M_{BH}^2) and of the mutual interactions of the scalar fields ($G_{ab}\left(\phi_\infty\right)\Sigma^a\Sigma^b$) are never exceeded by the repulsive self-force due to the Abelian charged vectors ($V_{BH}\left(\phi_\infty;p,q\right)$). As a consequence, the resulting total self-force exerted on the BH is always attractive. In particular, it vanishes for extreme BHs ($\mathbf{c}^2 = 0$), thus determining a static, presumably neutral, equilibrium. Scherk named such a phenomenon "antigravity" [62, 63].

In general, we may rewrite the antigravity bound (4.1.48) as follows:

$$M_{BH}^2 + \frac{1}{2}G_{ab}\left(\phi_\infty\right)\Sigma^a\Sigma^b \geqslant V_{BH}\left(\phi_\infty;p,q\right), \qquad (4.1.49)$$

where the bound is saturated only for extreme BHs, which are then exactly "antigravitating."

Since $V_{BH}\left(\phi_\infty;p,q\right)$ is a symplectic-invariant, positive definite generalization of the quantities q^2 and q^2+p^2 appearing in the r.h.s.'s of (1.8) and (1.9) – and in the related Footnote 3 of Sect. 1 – when considering the BPS bound and its saturation by extremal (RN) BHs, it is clear that the antigravity bound (4.1.48)–(4.1.49) generalizes the simple expression $M_{BH}^2 \geqslant q^2+p^2$

to the case of “scalar hairs” and to a (not necessarily supersymmetric) system described by the Lagrangian density (4.1.1). While the BPS bound was derived in [64] in a supersymmetric framework by requiring duality invariance, the antigravity bound (4.1.48)–(4.1.49) holds also without such ingredients.

The general formalism described above, which allows one to treat 4-d static, spherically symmetric, $\mathbf{c}^2$-parameterized BHs with “scalar hairs” coupled to Abelian vector fields, essentially relies on the metric $G_{ab}(\phi)$ of the moduli space $\mathcal{M}_\phi$ and on the “effective BH potential” function $V_{BH}(\phi; p, q)$.

To a certain extent, the presented geodesic formulation is the most symmetrical one, in which the hatted fields $\widehat{\phi}$ comprise the real scalars ϕ^a, as well as the electro-magnetic potentials $\psi^\Lambda, \chi_\Lambda$, and the Newtonian gravitational potential U. The enlargement of the scalar manifold is related to the performed dimensional reduction procedure ($d = 4 \to d = 1$), which allows one to put U, ϕ^a and $\psi^\Lambda, \chi_\Lambda$ all on the same footing.

Physically, by exploiting the $(U(1))^{n_V+1}$ gauge invariance of $\widehat{G}_{\widehat{a}\widehat{b}}$, it is more convenient to eliminate the potentials $\psi^\Lambda, \chi_\Lambda$ by introducing their canonically conjugate variables q_Λ, p^Λ, corresponding to the BH electric and magnetic charges. Such a procedure allows one to define a BH effective potential function $V_{BH}(\phi; p, q)$, whereas the real scalars ϕ^a’s and the Newtonian potential U remain on the same footing, and they are described by a simple dynamical model (4.1.22) in the (U, ϕ)-space, with a potential $V_{BH}(\phi; p, q)$, and constrained and $\mathbf{c}^2$-parameterized by (4.1.23).

4.2 Extreme Black Holes and Attractor Mechanism without SUSY

Extreme (or extremal) BHs are obtained from the previous treatment by setting $\mathbf{c}^2 = 0$ (“$\mathbf{c}^2$-extremization”). Consequently, from Ansatz (4.1.7) we get that the extreme 3-d spatial metric reads

$$\gamma_{ij}(\underline{x})\, dx^i dx^j = \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + sin^2\theta d\varphi^2\right), \tag{4.2.1}$$

and therefore, from (4.1.10) the extreme BH metric reads

$$ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right]. \tag{4.2.2}$$

By using definition (4.1.8), such a static and spherically symmetric 4-d BH metric may be rewritten as

$$\begin{aligned} ds^2 &= e^{2U(r-r_H)}dt^2 - e^{-2U(r-r_H)}\left[dr^2 + (r - r_H)^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right] \\ &= e^{2U(r)}dt^2 - e^{-2U(r)}\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right] \\ &= e^{2U(r)}dt^2 - e^{-2U(r)}d\underline{x}^2; \end{aligned} \tag{4.2.3}$$

in the first line we introduced $U(\tau) = U'(r - r_H)$ and dropped the prime out, whereas in the second line we redefined $r' \equiv r - r_H$ and dropped the prime out once again. $d\underline{x}^2$ denotes the Euclidean 3-d infinitesimal metric element. Therefore, the $\mathbf{c}^2$-extremization of metric (4.1.10) given by (4.2.3) is the spherically symmetric version of the static BH metric (1.20). The asymptotical boundary condition

$$U(\tau = 0) = 0 \Longleftrightarrow \quad lim_{r\to\infty} U(r) = 0 \tag{4.2.4}$$

yields the asymptotical flatness of the extreme BH metric (4.2.3).

Now, the extreme BH metric (4.2.2) and (4.2.3) is assumed to be constrained by the so-called finite horizon area condition, reading

$$lim_{\tau\to-\infty}\, e^{-2U(\tau)} = \frac{A_H}{4\pi}\tau^2, \tag{4.2.5}$$

where clearly $A_H \neq 0$ (otherwise a 'naked' singularity arises out). An additional requirement, corresponding to the "near-horizon" boundary condition on $\frac{d\phi^a}{d\tau}$ for extreme BHs, is the following:

$$lim_{\tau\to-\infty} G_{ab}\left(\phi\left(\tau\right)\right) \frac{d\phi^a\left(\tau\right)}{d\tau}\frac{d\phi^b\left(\tau\right)}{d\tau} e^{2U(\tau)}\tau^4 < \infty. \tag{4.2.6}$$

By using once again definition (4.1.8) and putting $r' \equiv r - r_H$ with prime dropped out, we see that condition (4.2.5) constrains the "near-horizon" form of $e^{-2U(\tau)}$ to be nothing but the corresponding expression of the BR metric (see (1.22)). Correspondingly, condition (4.2.5) implies that in the limit $\tau \to -\infty$ the function $e^{-2U(\tau)}$ satisfies the D'Alembert equation (whose general, not necessarily spherically symmetric, form is given by (1.21)).

Thus, by putting $\mathbf{c}^2 = 0$ in the 4-d, static, spherically symmetric BH metric described by the Ansatz (4.1.10), one gets the $\mathbf{c}^2$-extremization of such a metric, given by the asymptotically flat expressions (4.2.2) and (4.2.3). Furthermore, the imposed "finite horizon area condition" (4.2.5) determines, in suitable coordinates, the decomposition of the near-horizon limit of the metrics (4.2.2) and (4.2.3) in the direct product $AdS_2 \times S^2$ (BR metric). Indeed, (4.2.3) and (4.2.5) yield

$$\begin{aligned}
&lim_{\tau\to-\infty} ds^2\\
&= lim_{\tau\to-\infty}\left\{e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\tfrac{d\tau^2}{\tau^4} + \tfrac{1}{\tau^2}\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right]\right\}\\
&= \tfrac{4\pi}{A_H}\tfrac{dt^2}{\tau^2} - \tfrac{A_H}{4\pi}\left[\tfrac{d\tau^2}{\tau^2} + \left(d\theta^2 + sin^2\theta d\varphi^2\right)\right]\\
&\overset{r^{(\prime)}\equiv r-r_H}{=} \tfrac{4\pi r^2}{A_H}dt^2 - \tfrac{A_H}{4\pi r^2}\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right]\\
&= \tfrac{r^2}{M_{BR}^2}dt^2 - \tfrac{M_{BR}^2}{r^2}\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right] = ds^2_{BR},
\end{aligned} \tag{4.2.7}$$

where in the last line we used (1.16) and the results of Sect. 1.

By considering (1.14), one gets that the extremal 4-d RN BH is a particular case of the extreme BH metric (4.2.3), with ($c = 1 = G_0$)

$$e^{2U(r)} = \frac{r^2}{(r+M)^2}, \tag{4.2.8}$$

clearly satisfying, by (1.16) and the identification $M_{BR} = M_{BH} \equiv M$, the constraint (4.2.5). Thus, the previously treated extremal 4-d Reissner-Nördstrom BH is just a particular case of static, spherically symmetric and asymptotically flat 4-d extreme BH metric background, interpolating between the two maximally supersymmetric[6] backgrounds M_4 (4-d Minkowski at spatial infinity) and $AdS_2 \times S^2$ (4-d BR at the horizon).

By defining $lim_{\tau \to -\infty} \phi^a(\tau) \equiv \phi^a_H$, the near-horizon boundary conditions (4.2.5) and (4.2.6) for extremal BHs may be rewritten as[7]

$$\begin{aligned} &lim_{\tau \to -\infty} G_{ab}(\phi(\tau)) \tfrac{d\phi^a(\tau)}{d\tau} \tfrac{d\phi^b(\tau)}{d\tau} e^{2U(\tau)} \tau^4 \\ &= G_{ab}(\phi_H) \, lim_{\tau \to -\infty} \tfrac{d\phi^a(\tau)}{d\tau} \tfrac{d\phi^b(\tau)}{d\tau} \tfrac{4\pi}{A_H} \tau^2 \equiv 2X^2 < \infty, \quad X \in \mathbb{R}. \end{aligned} \tag{4.2.9}$$

Since A_H does not vanish, this is equivalent to

$$lim_{\tau \to -\infty} G_{ab}(\phi(\tau)) \frac{d\phi^a(\tau)}{d\tau} \frac{d\phi^b(\tau)}{d\tau} = \frac{A_H}{2\pi} \frac{X^2}{\tau^2} < \infty. \tag{4.2.10}$$

Due to the assumed regularity of the Riemann geometry of $\mathcal{M}_\phi$, the metric G_{ab}, and consequently the quadratic form $G_{ab} \frac{d\phi^a}{d\tau} \frac{d\phi^b}{d\tau}$, is strictly positive definite, and (4.2.10) means that the near-horizon limit $\tau \to -\infty$ of such a quadratic form is finite and still in the regular region of $\mathcal{M}_\phi$, which appears to be a reasonable regularity assumption.

Now, condition (4.2.5) implies

$$lim_{\tau \to -\infty} \left(\frac{dU(\tau)}{d\tau} \right)^2 \frac{1}{\tau^2}; \tag{4.2.11}$$

consequently, by taking the near-horizon limit of the constraint (4.1.23) in the extreme case ($\mathbf{c}^2 = 0$) and using (4.2.5), (4.2.10), and (4.2.11), one obtains

[6] If any SUSY is present at all, of course. Indeed, it should be clearly pointed out that in the treatment of Subsects. 4.1 and 4.2 no SUSY is, in general, a priori assumed to exist

[7] As pointed out before (4.1.48), here and in the following treatment we assume suitable properties of regularity and smoothness of the involved functions, in order to perform the relevant operations with the near-horizon and space-asymptotical limits $\tau \to -\infty$ and $\tau \to 0^-$

$$\begin{aligned} & lim_{\tau \to -\infty} \left(\tfrac{dU(\tau)}{d\tau} \right)^2 + \tfrac{1}{2} G_{ab} (\phi) \tfrac{d\phi^a(\tau)}{d\tau} \tfrac{d\phi^b(\tau)}{d\tau} - e^{2U} V_{BH} (\phi; p, q) \\ & = \tfrac{1}{\tau^2} + \tfrac{A_H}{4\pi} \tfrac{X^2}{\tau^2} - \tfrac{4\pi}{A_H} \tfrac{1}{\tau^2} V_{BH} (\phi_H; p, q) = 0; \end{aligned} \tag{4.2.12}$$

$$\Updownarrow$$

$$1 + \frac{A_H}{4\pi} X^2 - \frac{4\pi}{A_H} V_{BH} (\phi_H; p, q) = 0; \tag{4.2.13}$$

$$\Updownarrow$$

$$\frac{4\pi}{A_H} V_{BH} (\phi_H; p, q) = 1 + \frac{A_H}{4\pi} X^2 \geqslant 1; \tag{4.2.14}$$

$$\Updownarrow$$

$$4\pi V_{BH} (\phi_H; p, q) \geqslant A_H. \tag{4.2.15}$$

Now, in order to proceed further, it is convenient to introduce the variable

$$\omega \equiv -ln (-\tau) ; \tag{4.2.16}$$

by doing this, the factorized, BR nature of the near-horizon extreme BH metric (4.2.7) becomes completely manifest:

$$\begin{aligned} lim_{\tau \to -\infty} ds^2 &= \frac{4\pi}{A_H} e^{2\omega} dt^2 - \frac{A_H}{4\pi} d\omega^2 - \frac{A_H}{4\pi} \left(d\theta^2 + sin^2\theta d\varphi^2 \right) \\ &= ds^2_{AdS_2} - ds^2_{S^2} = ds^2_{BR}. \end{aligned} \tag{4.2.17}$$

By introducing $U' (\omega) \equiv U (\tau = -e^{-\omega})$, $\phi'^a (\omega) \equiv \phi^a (\tau = -e^{-\omega})$ and $G'_{ab} \equiv G_{ab} (\phi' (\omega))$ and dropping the primes out as usual, we may rewrite the conditions (4.2.5) and (4.2.6), respectively, as follows:

$$lim_{\omega \to -\infty} e^{-2U(\omega)} = \frac{A_H}{4\pi} e^{-2\omega}; \tag{4.2.18}$$

$$lim_{\omega \to -\infty} G_{ab} (\phi (\omega)) \frac{d\phi^a (\omega)}{d\omega} \frac{d\phi^b (\omega)}{d\omega} e^{2U(\omega)} e^{-2\omega} < \infty. \tag{4.2.19}$$

The merging of (4.2.18) and (4.2.19) yields

$$\begin{aligned} & lim_{\omega \to -\infty} G_{ab} (\phi (\omega)) \tfrac{d\phi^a(\omega)}{d\omega} \tfrac{d\phi^b(\omega)}{d\omega} e^{2U(\omega)} e^{-2\omega} \\ & = \tfrac{4\pi}{A_H} G_{ab} (\phi_H) \, lim_{\omega \to -\infty} \tfrac{d\phi^a(\omega)}{d\omega} \tfrac{d\phi^b(\omega)}{d\omega} \equiv 2X^2 < \infty \end{aligned} \tag{4.2.20}$$

$$\Updownarrow$$

$$G_{ab} (\phi_H) \, lim_{\omega \to -\infty} \frac{d\phi^a (\omega)}{d\omega} \frac{d\phi^b (\omega)}{d\omega} = \frac{A_H}{2\pi} X^2 < \infty. \tag{4.2.21}$$

Now, independently of the signature and thus of the definiteness of $G_{ab}(\phi_H)$, we have two possibilities[8]:

1. $A_H X^2 \neq 0$.
   Since, in order to avoid naked singularities, we assume $A_H \neq 0$, this happens iff $X^2 \neq 0$ and implies

$$\left.\begin{array}{c} lim_{\omega\to-\infty}\frac{d\phi^a(\omega)}{d\omega} = \widetilde{\phi}^a \neq 0, \\ \Updownarrow \\ lim_{\omega\to-\infty}\phi^a(\omega) = \widetilde{\phi}^a\omega + \widetilde{\phi}^a_0, \\ \Downarrow \\ lim_{\omega\to-\infty}|\phi^a(\omega)| = \infty, \end{array}\right\} \text{for } a \in \mathcal{A} \subseteq \{1,\dots,m_\phi\}, \mathcal{A} \neq \emptyset. \tag{4.2.22}$$

   Therefore, by assuming that *all* the real scalar fields ϕ^a's do not diverge at the BH horizon, i.e., that

$$lim_{\omega\to-\infty}|\phi^a(\omega)| < \infty,\ \forall a \in \{1,\dots,m_\phi\}, \tag{4.2.23}$$

   such a case must be disregarded.
2. $A_H X^2 = 0 \overset{A_H\neq 0}{\Longleftrightarrow} X^2 = 0$.
   We have two subcases: the first subcase corresponds to (4.2.22) holding true for at least two a's, and thus, by formulating the hypothesis (4.2.23), it must be disregarded. The second subcase, which is the only one compatible with (4.2.23), corresponds to

$$lim_{\omega\to-\infty}\frac{d\phi^a(\omega)}{d\omega} = 0,\ \forall a \in \{1,\dots,m_\phi\}. \tag{4.2.24}$$

By substituting $X^2 = 0$ in (4.2.14), one gets

$$V_{BH}(\phi_H; p, q) = \frac{A_H}{4\pi}. \tag{4.2.25}$$

Thus, at the leading order (i.e., when the BHEA law holds true), one obtains that, for a given BH charge configuration, the BH entropy is proportional to the value of the BH effective potential function at the horizon:

$$S_{BH}(\phi_H; p, q) = \frac{A_H}{4} = \pi V_{BH}(\phi_H; p, q). \tag{4.2.26}$$

[8] The totally degenerate case of all eigenvalues of $G_{ab}(\phi_H)$ vanishing is not considered

Thus, we have shown that for the considered class of static, spherically symmetric and asymptotically flat 4-d extreme BHs ($\mathbf{c}^2 = 0$), from the requirements (4.2.5) and (4.2.6) and from the assumption (4.2.23) of nondiverging real scalars ϕ^a's at the horizon, it follows the fundamental relation (4.2.25) between the area A_H of the horizon and the horizon value $V_{BH}(\phi_H; p, q)$ of the BH effective potential function.

Such a relation holds also for the so-called double-extreme BHs, which are extreme BHs ($\mathbf{c}^2 = 0$) with constant moduli

$$\frac{d\phi^a(r)}{dr} = 0 \Longleftrightarrow \phi^a = \phi_0^a, \quad \forall r \in [r_H, +\infty), \quad \forall a \in \{1, \ldots, m_\phi\}. \tag{4.2.27}$$

Therefore, in such a case

$$G_{ab}(\phi(\tau)) \frac{d\phi^a(\tau)}{d\tau} \frac{d\phi^b(\tau)}{d\tau} = 0, \ \forall \tau \in (-\infty, 0], \tag{4.2.28}$$

and the constraint (4.1.23) becomes

$$\left(\frac{dU(\tau)}{d\tau}\right)^2 - e^{2U(\tau)} V_{BH}(\phi(\tau); p, q) = 0, \quad \forall \tau \in (-\infty, 0]. \tag{4.2.29}$$

Since, by definition (4.1.47), the moduli of a double-extreme BH all have vanishing scalar charges, the spatial asymptotical limit ($\tau \to 0^-$) of such an expression, namely the antigravity bound (4.1.48) and (4.1.49) for double-extreme BHs, reads

$$M_{BH}^2 = V_{BH}(\phi_\infty; p, q) = V_{BH}(\phi_H; p, q), \tag{4.2.30}$$

where in the last passage we used (4.2.27). Thus, by using (4.2.25) and (4.2.26), the following relation, holding for double-extreme BHs, is obtained:

$$S_{BH}(\phi_H; p, q) = \frac{A_H}{4} = \pi V_{BH}(\phi_H; p, q) = \pi M_{BH}^2. \tag{4.2.31}$$

By recalling (4.2.5) and (4.2.11), the near-horizon limit of (4.2.29) yields

$$\begin{aligned} 0 &= lim_{\tau \to -\infty} \left(\frac{dU(\tau)}{d\tau}\right)^2 - e^{2U(\tau)} V_{BH}(\phi(\tau); p, q) \\ &= \frac{1}{\tau^2} - \frac{4\pi}{A_H} \frac{1}{\tau^2} V_{BH}(\phi_H; p, q), \end{aligned} \tag{4.2.32}$$

implying the results (4.2.25) and (4.2.26). Thus, we have also shown that the horizon area of extreme BHs coincides with the horizon area of the double-extreme BHs with the same BH charge configuration (p^Λ, q_Λ). The same holds for the leading order BH entropy, which, by the BHEA law, is nothing but one quarter of the horizon area.

The universal features of the near-horizon limit of the extreme and double-extreme BHs expressed by (4.2.25) and (4.2.26) were firstly obtained in [27, 28], and [65] as a consequence of SUSY, i.e., by considering extreme BHs as $\frac{1}{2}$-BPS supersymmetric solutions in $\mathcal{N}=2$, $d=4$ SUGRA. Instead, it is worth pointing out once again that, in the presented treatment the universal properties of the horizon area of extreme BHs are deduced only from some requirements of "minimal regularity" of the moduli space geometry and of the moduli near the horizon, namely from the conditions (4.2.5), (4.2.6), and (4.2.23).

Let us now rewrite (4.2.24) in terms of the variable τ. By recalling the definition (4.2.16), one gets

$$\frac{d\phi^a(\omega)}{d\omega}=-\tau\frac{d\phi^a(\tau)}{d\tau}\Leftrightarrow\frac{d\phi^a(\tau)}{d\tau}=e^{\omega}\frac{d\phi^a(\omega)}{d\omega},\tag{4.2.33}$$

and therefore

$$lim_{\omega\rightarrow-\infty}\frac{d\phi^a(\omega)}{d\omega}=0\Longleftrightarrow lim_{\tau\rightarrow-\infty}\tau\frac{d\phi^a(\tau)}{d\tau}=0,\ \forall a\in\{1,\ldots,m_\phi\}\,.\tag{4.2.34}$$

In other words, the request of nondiverging scalars in the near-horizon limit implies that, when approaching the BH horizon (i.e., in the limit $r\rightarrow r_H^+\Leftrightarrow\tau\rightarrow-\infty$), $\frac{d\phi^a(\tau)}{d\tau}$ vanishes faster than $\frac{1}{\tau}=r_H-r$.

By further derivating (4.2.33) w.r.t. τ and using the definition (4.2.16), one obtains

$$\begin{aligned}\frac{d^2\phi^a(\tau)}{d\tau^2}&=\frac{d}{d\tau}\left(\frac{d\phi^a(\omega)}{d\omega}\frac{d\omega(\tau)}{d\tau}\right)=\frac{d}{d\tau}\left(-\frac{1}{\tau}\frac{d\phi^a(\omega)}{d\omega}\right)\\&=\frac{1}{\tau^2}\frac{d\phi^a(\omega)}{d\omega}-\frac{1}{\tau}\frac{d^2\phi^a(\omega)}{d\omega^2}\frac{d\omega(\tau)}{d\tau}\\&=e^{2\omega}\left[\frac{d\phi^a(\omega)}{d\omega}+\frac{d^2\phi^a(\omega)}{d\omega^2}\right].\end{aligned}\tag{4.2.35}$$

Thus, by using (4.2.33) and (4.2.35), the equations of motion (4.1) for ϕ^c's may be rewritten as

$$\begin{aligned}&\frac{d^2\phi^c(\omega)}{d\omega^2}+\frac{d\phi^c(\omega)}{d\omega}+G^{ac}\left(\phi\left(\omega\right)\right)\left[\frac{\partial G_{ab}(\phi)}{\partial\phi^c}-\frac{1}{2}\frac{\partial G_{cb}(\phi)}{\partial\phi^a}\right]\frac{d\phi^c(\omega)}{d\omega}\frac{d\phi^b(\omega)}{d\omega}\\&=e^{-2\omega}e^{2U(\omega)}G^{ac}\left(\phi\left(\omega\right)\right)\frac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a}.\end{aligned}\tag{4.2.36}$$

Such equations hold for any $\omega\in(-\infty,0]$ and for any value of $\mathbf{c}^2\in\mathbb{R}^+$.

Condition (4.2.5) for extreme BHs may be rewritten in terms of the variable ω as

$$lim_{\omega\to-\infty}e^{-2U(\omega)}=\frac{A_H}{4\pi}e^{-2\omega}. \tag{4.2.37}$$

By using such a condition and recalling (4.2.24), one finally gets that the near-horizon limit of equations of motion (4.2) for real scalars ϕ^c's in the case of extreme BHs ($\mathbf{c}^2=0$) reads

$$\begin{gathered}lim_{\omega\to-\infty}\left\{\frac{d^2\phi^c(\omega)}{d\omega^2}+\frac{d\phi^c(\omega)}{d\omega}+G^{ac}\left(\phi\left(\omega\right)\right)\left[\frac{\partial G_{ab}(\phi)}{\partial\phi^c}-\frac{1}{2}\frac{\partial G_{cb}(\phi)}{\partial\phi^a}\right]\frac{d\phi^c(\omega)}{d\omega}\frac{d\phi^b(\omega)}{d\omega}\right\}\\ =lim_{\omega\to-\infty}e^{-2\omega}e^{2U(\omega)}G^{ac}\left(\phi\left(\omega\right)\right)\frac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a};\end{gathered} \tag{4.2.38}$$

$$\Updownarrow$$

$$lim_{\omega\to-\infty}\frac{d^2\phi^c\left(\omega\right)}{d\omega^2}=\frac{4\pi}{A_H}G^{ac}\left(\phi_H\right)lim_{\omega\to-\infty}\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}; \tag{4.2.39}$$

$$\Updownarrow$$

$$lim_{\omega\to-\infty}\frac{d^2\phi^c\left(\omega\right)}{d\omega^2}=\frac{4\pi}{A_H}G^{ac}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}\right|_{\phi=\phi_H}. \tag{4.2.40}$$

By inverting (4.2.35) and taking its near-horizon limit, we get

$$lim_{\omega\to-\infty}\frac{d^2\phi^c\left(\omega\right)}{d\omega^2}=lim_{\tau\to-\infty}\tau^2\frac{d^2\phi^c\left(\tau\right)}{d\tau^2}, \tag{4.2.41}$$

and thus (4.2.38) may equivalently be rewritten as

$$lim_{\tau\to-\infty}\frac{d^2\phi^c\left(\tau\right)}{d\tau^2}=\frac{4\pi}{A_H}G^{ac}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}\right|_{\phi=\phi_H}\frac{1}{\tau^2}. \tag{4.2.42}$$

This is the near-horizon limit of equations of motion (4.1) or (4.2) for real scalars ϕ^c's in the case of extreme BHs ($\mathbf{c}^2=0$), in terms of the variable $\tau\equiv\left(r_H-r\right)^{-1}$. By using (4.2.25), one gets the equivalent expression

$$\begin{aligned}lim_{\tau\to-\infty}\frac{d^2\phi^c\left(\tau\right)}{d\tau^2}&=G^{ac}\left(\phi_H\right)\left[\frac{1}{V_{BH}\left(\phi;p,q\right)}\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}\right]_{\phi=\phi_H}\frac{1}{\tau^2}\\&=\frac{1}{\tau^2}G^{ac}\left(\phi_H\right)\left.\frac{\partial ln\left[V_{BH}\left(\phi;p,q\right)\right]}{\partial\phi^a}\right|_{\phi=\phi_H}.\end{aligned} \tag{4.2.43}$$

The solution of the ordinary second-order differential equations (4.2.42) and (4.2.43) reads as follows:

$$\begin{aligned} lim_{\tau\to-\infty}\phi^{a}\left(\tau\right) &= \frac{4\pi}{A_H}G^{ab}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^{b}}\right|_{\phi=\phi_H} ln\left(-\tau\right)+\varphi^{a}\tau+\varsigma^{a} \\ &= G^{ab}\left(\phi_H\right)\left.\frac{\partial ln\left[V_{BH}\left(\phi;p,q\right)\right]}{\partial\phi^{b}}\right|_{\phi=\phi_H} ln\left(-\tau\right)+\varphi^{a}\tau+\varsigma^{a}. \end{aligned} \tag{4.2.44}$$

Such a solution implies

$$lim_{\tau\to-\infty}\frac{d\phi^{a}\left(\tau\right)}{d\tau}=\frac{4\pi}{A_H}G^{ab}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^{b}}\right|_{\phi=\phi_H}\frac{1}{\tau}+\varphi^{a}; \tag{4.2.45}$$

this has to be consistent with the condition (4.2.34) of finiteness of the near-horizon scalar fields. Consequently, (4.2.34) and (4.2.45) yield ($A_H\neq 0$)

$$lim_{\tau\to-\infty}\frac{4\pi}{A_H}G^{ab}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^{b}}\right|_{\phi=\phi_H}+\varphi^{a}\tau=0,\ \forall a\in\{1,\ldots,m_\phi\} \tag{4.2.46}$$

$$\Updownarrow$$

$$\forall a\in\{1,\ldots,m_\phi\}\begin{cases} I) & \varphi^{a}=0; \\ II) & G^{ab}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^{b}}\right|_{\phi=\phi_H}=0. \end{cases} \tag{4.2.47}$$

By substituting the results given by (4.2.47) back in (4.2.44), one finally gets

$$lim_{\tau\to-\infty}\phi^{a}\left(\tau\right)=\varsigma^{a}\equiv\phi^{a}_{H}. \tag{4.2.48}$$

Let us analyze the condition II of (4.2.47): since $V_{BH}\left(\phi;p,q\right)$ is a scalar in $\mathcal{M}_\phi$, it holds that

$$\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^{b}}=\partial_b V_{BH}\left(\phi;p,q\right)=D_b V_{BH}\left(\phi;p,q\right); \tag{4.2.49}$$

$$\Downarrow$$

$$G^{ab}\partial_b V_{BH}\left(\phi;p,q\right)=G^{ab}D_b V_{BH}\left(\phi;p,q\right)=D^{a}V_{BH}\left(\phi;p,q\right)=\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi_a}. \tag{4.2.50}$$

Moreover, in the assumption that the horizon configuration(s) of the scalar fields ϕ^a's belong to the "regular" zone of the Riemann moduli space $\mathcal{M}_\phi$ (i.e., by assuming $G^{ab}\left(\phi_H\right)$ – strictly – positive definite and therefore with – strictly – positive determinant and maximal rank m_ϕ), it is clear that

$$G^{ab}\left(\phi_H\right)\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^b}\right|_{\phi=\phi_H}=0$$
$$\Updownarrow$$
$$\left.\frac{\partial V_{BH}\left(\phi;p,q\right)}{\partial\phi^a}\right|_{\phi=\phi_H}=0,\quad\forall a\in\{1,\ldots,m_\phi\}\,. \tag{4.2.51}$$

Equations (4.2.48) and (4.2.51) are the essence of the attractor mechanism in the moduli space of extreme BHs: (4.2.48) states that the horizon configuration(s) of the considered m_ϕ moduli are "regular" (they remain finite, do not blow up in proximity of the EH of the BH), whereas (4.2.51) characterizes such configuration(s) as critical point(s) of the BH effective potential $V_{BH}\left(\phi;p,q\right)$ seen as a function, for an arbitrary but fixed BH charge configuration $\left(p^\Lambda,q_\Lambda\right)$, of the m_ϕ real scalars ϕ^a's parametrizing $\mathcal{M}_\phi$.

Equation (4.2.51) may be considered as the fundamental, differential formulation of the attractor equations which, as we will see later, may be reformulated in a simpler, equivalent way as a set of algebraic equations in the context of the (regular) special Kähler geometry of the vector supermultiplets' moduli space of $\mathcal{N}=2,d=4$ Maxwell–Einstein SUGRA. Equation (4.2.51) yields that the horizon configuration(s) of the scalars are dynamical "attractors" in the radial evolution of such fields: indeed, they do not depend on the initial data ϕ^a_∞'s (spatial asymptotical configurations), but they instead exclusively depend on the specified BH charge configuration:

$$lim_{\tau\to-\infty}\phi^a\left(\tau\right)=\varsigma^a\equiv\phi^a_H=\phi^a_H\left(p,q\right). \tag{4.2.52}$$

In particular, for double-extreme 4-d BHs, the values of the constant moduli may be determined as the ones corresponding to the critical points of $V_{BH}\left(\phi;p,q\right)$ in $\mathcal{M}_\phi$; in this case the merging of (4.2.27) and (4.2.51) yields

$$\begin{gathered}\tfrac{d\phi^a(r)}{dr}=0\Longleftrightarrow\phi^a=\phi^a_0=\phi^a_H=\phi^a_\infty:\left.\tfrac{\partial V_{BH}(\phi;p,q)}{\partial\phi^a}\right|_{\phi=\phi_0}=0,\\ \forall r\in\left[r_H,+\infty\right),\quad\forall a\in\{1,\ldots,m_\phi\}\,.\end{gathered} \tag{4.2.53}$$

By substituting back the exclusively charge-dependent attractor horizon configuration(s) (4.2.52) in (4.2.25) and (4.2.26), one gets that the horizon value of V_{BH}, and thus (by the BHEA law) the value of the leading order BH entropy, only depend on the BH charges, and not on the initial data ϕ^a_∞'s of the radial evolution (from spatial infinity $r\to\infty\Leftrightarrow\tau\to0^-$ to the BH EH $r\to r_H^+\Leftrightarrow\tau\to-\infty$) of the moduli, as it must be for physical consistence of the BH entropy:

$$V_{BH}\left(\phi_H;p,q\right)=V_{BH}\left(\phi_H\left(p,q\right);p,q\right)=V_{BH}\left(p,q\right)=\frac{A_H}{4\pi}; \tag{4.2.54}$$

$$S_{BH}=\frac{A_H}{4}=\pi V_{BH}\left(\phi_H\left(p,q\right);p,q\right)=\pi V_{BH}\left(p,q\right)=S_{BH}\left(p,q\right). \tag{4.2.55}$$

Summarizing, (4.2.25), (4.2.26), (4.2.48), and (4.2.51) express the attractor mechanism working in the moduli space of the considered class of 4-d extreme (and double-extreme) BHs. It is worth stressing once again that in order to obtain such results, *no SUSY was used, thus they hold true also, in general, nonsupersymmetric frameworks.* Beside the working hypotheses of

- staticity;
- spherical symmetry;
- asymptotical flatness;
- extremality ($\mathbf{c}^2 = 0$) and
- Riemann "regular" geometry (i.e., positive definiteness of the metric tensor) of the m_ϕ-d real manifold $\mathcal{M}_\phi$,

  only some "minimal regularity" requirements on the moduli space geometry and on the moduli near the horizon were imposed, namely
- condition (4.2.5) ("finiteness of BH horizon area");
- condition (4.2.6) ("regularity of horizon scalar manifold geometry") and
- condition (4.2.23) ("regularity and finiteness of the horizon configurations of the moduli").

Thence, results (4.2.25), (4.2.26), (4.2.48), and (4.2.51) were obtained by studying the classical equations of motion of the scalars ϕ^a's, given by the relevant Euler-Lagrange equations of the 1-d effective Lagrangian $\mathcal{L}_1 [U, \phi; p, q]$ given by (4.1.22), in turn resulting from a s-t dimensional reduction from the initial 4-d Lagrangian density $\mathcal{L}_4$ given by (4.1.1), which determines the most general set of theories to which the performed treatment applies.

In general, the attractors in the moduli space of the considered 4-d extreme BHs may always be characterized, for a given BH charge configuration, as the critical points of the function $V_{BH} (\phi; p, q)$ in $\mathcal{M}_\phi$. Such points are identified with the horizon, "attracting" configurations of the scalar fields of the theory, corresponding to a BR vacuum state, which in the presence of SUSY may be seen as a maximally supersymmetric 4-d metric background. Extreme BH backgrounds correspond to dynamical trajectories in $\mathcal{M}_\phi$, starting from a point $\{\phi^a_\infty\}$ of initial data at spatial infinity ($r \to \infty \Leftrightarrow \tau \to 0^-$) and ending at a critical, attractor point $\{\phi^a_H\}$ at the BH EH ($r \to r^+_H \Leftrightarrow \tau \to -\infty$).

Attractor mechanism essentially consists in the loss of memory of the scalars at the horizon about the initial data of their radial evolution: the critical attractor point $\{\phi^a_H\}$ is completely independent of the position of the initial point $\{\phi^a_\infty\}$, but instead only depends on the specified BH charge configuration. The particular case of 4-d double-extreme BHs, characterized by "frozen," constant moduli, corresponds to "degenerate," pointlike trajectories in $\mathcal{M}_\phi$, because no radial evolution (the only possible evolution in the formulated hypotheses of staticity and spherical symmetry) takes place.

Recently, nonsupersymmetric 4-d BH attractors have been investigated also in relation to nonasymptotically flat metric backgrounds, such as 4-d asymptotically $(A)dS$ ones (see, e.g., [66]).

Finding explicit trajectories $\{\phi^a(\tau)\}$ in $\mathcal{M}_\phi$ amounts to explicitly solving the second-order differential equations of motion (4.1.40) of the moduli ϕ^a's, and it is, in general, quite difficult, even though some explicit solutions, such as the previously considered RN 4-d extreme BH, are known (for axion-dilatonic BHs, see, e.g., [67–69], and [70]; for not necessarily supersymmetric extreme BHs, see, e.g., [66] and [71]).

However, in Subsects. 4.3 ande 4.4 we will see that in the context of $\mathcal{N}=2, d=4$ SUGRA coupled to Abelian vector supermultiplets, by using the properties of the (regular) special Kähler geometry of the vector supermultiplets' moduli space, such a problem may considerably be simplified. In particular, we will see that in such a context different typologies of attractors may exist, depending on how much (and if) SUSY is preserved at such points.

Concluding this subsection, we may say that, by formulating reasonable "regularity" requirements, the attractor mechanism works in the moduli space of the 4-d extreme BHs for a wide set of theories, which do not necessarily have to be supersymmetric; SUSY turns out to be a sufficient, but nonnecessary, condition for attractor dynamics in the moduli space. The issue of non (necessarily)-supersymmetric BH attractors was firstly treated by Ferrara, Gibbons, and Kallosh in [55], whereas the attractor mechanism was obtained as a consequence of SUSY, i.e., by considering extreme BHs as $\frac{1}{2}$-BPS supersymmetric solutions in $\mathcal{N}=2$, $d=4$ SUGRA, in [27, 28] and [65]. Recently, the non (necessarily)-supersymmetric BH attractors have been rediscovered and investigated by a number of authors; we refer the reader to Sect. 9 for a glance at the latest developments and related bibliography. Also in relation to such recent advances, it would be interesting trying to remove some (if not all) of the assumptions made above, and see if still some kind of attractor mechanism works. For instance, it would be intriguing to extend all reasonings and results presented above to the case of 4-d nonextreme BHs ($\mathbf{c}^2 \neq 0$); some recent results seem to point out that in such a case no attractor dynamics in the moduli space exists at all [66].

4.3 Extreme Black Holes and Special Kähler Geometry

We now reconsider the previously introduced n_V-fold $\mathcal{N}=2$, $d=4$ Maxwell–Einstein supergravity theory (MESGT), i.e., a $\mathcal{N}=2$, $d=4$ supergravity theory in which the gravity multiplet is coupled to n_V Abelian vector supermultiplets, and therefore the overall gauge group is $(U(1))^{n_V+1}$. We will see how the (regular) special Kähler geometry (SKG) of the moduli space of such a theory allows one to simplify the investigation of the critical points of the function V_{BH}. In this and in the next subsection we will refer to and complete the treatment presented in Sect. 3. We will denote the BH charges as follows: $n^e_\Lambda \equiv q_\Lambda$, $n^\Lambda_m \equiv p^\Lambda$.

Let us start by switching to a complex parametrization of the moduli space: in order to do this, we assume m_ϕ to be even, i.e., $m_\phi = 2n_\phi$, $n_\phi \in \mathbb{N}$.

Therefore, by complexifying the $2n_\phi$-d real Riemann manifold $\mathcal{M}_\phi$ (with local coordinates $\{\phi^a\}$, $a = 1, \ldots, m_\phi$), we obtain a n_ϕ-d complex Hermitian manifold $\mathcal{M}_{z,\overline{z}}$ with local coordinates $\{z^i, \overline{z}^{\overline{i}}\}$ $(i, \overline{i} = 1, \ldots, n_\phi)$ [72]:

$$G_{ab}(\phi)\, d\phi^a d\phi^b = 2G_{i\overline{j}}(z,\overline{z})\, dz^i d\overline{z}^{\overline{j}}, \quad \overline{G_{i\overline{j}}} = G_{j\overline{i}}. \tag{4.3.1}$$

In particular, as it pertains to the framework of n_V-fold $\mathcal{N} = 2$, $d = 4$ MESGT, we assume that such an Hermitian geometry is a Kählerian one, regular (i.e., with the metric tensor strictly positive definite everywhere), and of the special type; namely, we assume that

$$G_{ab}(\phi)\, d\phi^a d\phi^b = 2\frac{\partial^2 K(z,\overline{z})}{\partial \overline{z}^{\overline{j}} \partial z^i} dz^i d\overline{z}^{\overline{j}}, \ K(z,\overline{z}) = \overline{K(z,\overline{z})}; \tag{4.3.2}$$

$$G_{i\overline{j}}(z,\overline{z}) \text{ strictly positive definite } \forall (z,\overline{z}) \in \mathcal{M}_{z,\overline{z}}; \tag{4.3.3}$$

$$R_{i\overline{j}l\overline{m}} = G_{i\overline{j}} G_{l\overline{m}} + G_{i\overline{m}} G_{l\overline{j}} - C_{ilp} \overline{C}_{\overline{j}\overline{m}\overline{p}} G^{p\overline{p}}, \tag{4.3.4}$$

where the real function $K(z,\overline{z})$ (satisfying the Schwarz lemma in $\mathcal{M}_{z,\overline{z}}$) is called Kähler potential, $R_{i\overline{j}l\overline{m}}$ is the Kähler Riemann–Christoffel curvature tensor and C_{ilm} is the rank 3, completely symmetric, Kähler-covariantly holomorphic tensor of SKG (with Kähler weights $(2,2)$).

Now, in order to study the BH effective potential function $V_{BH}(z,\overline{z};p,q)$ in (regular) SKG, we need to identify it with a symplectic-invariant, Kähler gauge-invariant, real positive function in such a geometric context. The natural and immediate choice is given by the first invariant $I_1(z,\overline{z};p,q)$ of the SKG, defined as [51]

$$I_1(z,\overline{z};p,q) \equiv |Z|^2(z,\overline{z};p,q) + G^{i\overline{i}}(z,\overline{z})(D_i Z)(z,\overline{z};p,q)\left(\overline{D}_{\overline{i}}\overline{Z}\right)(z,\overline{z};p,q), \tag{4.3.5}$$

where $Z(z,\overline{z};p,q)$ is the central charge function of n_V-fold $\mathcal{N} = 2$, $d = 4$ MESGT; let us also recall that (3.1.53) and (3.2.53) yield

$$Z(z,\overline{z};p,q) = L^\Lambda(z,\overline{z})\, q_\Lambda - M_\Lambda(z,\overline{z})\, p^\Lambda = e^{\frac{1}{2}K(z,\overline{z})}\left[X^\Lambda(z)\, q_\Lambda - F_\Lambda(z)\, p^\Lambda\right]. \tag{4.3.6}$$

By recalling (3.2.86), I_1 may also be defined as

$$I_1(z,\overline{z};p,q) \equiv -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right) \mathcal{M}\left(Re(\mathcal{N}), Im(\mathcal{N})\right) \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix}, \tag{4.3.7}$$

with $\mathcal{M}(Re(\mathcal{N}), Im(\mathcal{N}))$ defined by (3.2.81)–(3.2.83) to be the real $(2n_V+2) \times (2n_V+2)$, $(z,\overline{z})$-dependent symmetric matrix

$$\mathcal{M}\left(Re\left(\mathcal{N}\left(z,\overline{z}\right)\right),Im\left(\mathcal{N}\left(z,\overline{z}\right)\right)\right)\equiv$$

$$\equiv\left(\begin{array}{cc} \begin{array}{l} Im\left(\mathcal{N}\right)_{\Lambda\Sigma}+ \\ +Re\left(\mathcal{N}\right)_{\Lambda\Delta}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Delta\Xi}Re\left(\mathcal{N}\right)_{\Xi\Sigma} \end{array} & -Re\left(\mathcal{N}\right)_{\Lambda\Xi}\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Xi\Sigma} \\ & \\ -\left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Xi}Re\left(\mathcal{N}\right)_{\Xi\Sigma} & \left(\left(Im\mathcal{N}\right)^{-1}\right)^{\Lambda\Sigma} \end{array}\right). \tag{4.3.8}$$

Consequently, by performing the fundamental identification

$$V_{BH}\left(z,\overline{z};p,q\right)=I_1\left(z,\overline{z};p,q\right), \tag{4.3.9}$$

the comparison of (4.1.20) and (4.1.21) with (4.3.7)–(4.3) yields

$$\left.\begin{array}{r} Re\left(\mathcal{N}\left(z,\overline{z}\right)\right)_{\Lambda\Sigma}=-\nu_{\Lambda\Sigma}\left(z,\overline{z}\right) \\ \\ Im\left(\mathcal{N}\left(z,\overline{z}\right)\right)_{\Lambda\Sigma}=-\mu_{\Lambda\Sigma}\left(z,\overline{z}\right) \end{array}\right\}\Longrightarrow\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)=-\nu_{\Lambda\Sigma}\left(z,\overline{z}\right)-i\mu_{\Lambda\Sigma}\left(z,\overline{z}\right). \tag{4.3.10}$$

The reality, symmetry, and (strict) positive definiteness of the matrices $\mu_{\Lambda\Sigma}\left(z,\overline{z}\right)$ and $\nu_{\Lambda\Sigma}\left(z,\overline{z}\right)$ imply the reality, symmetry, and (strict) negative definiteness of the matrix $\mathcal{N}_{\Lambda\Sigma}\left(z,\overline{z}\right)$, and thence of its real and imaginary parts separately (concerning its imaginary part, this was already noted in (3.1.95)). Consequently, the matrix $\mathcal{M}\left(Re\left(\mathcal{N}\right),Im\left(\mathcal{N}\right)\right)$ is (strictly) negative definite, and (4.3.7) yields that $I_1\left(z,\overline{z};p,q\right)$ (and thus, by the identification (4.3.9), the BH effective potential function $V_{BH}\left(z,\overline{z};p,q\right)$) is (real and) positive in all $\mathcal{M}_{z,\overline{z}}\times\Gamma$. The (strict) negative definiteness of the quadratic form of BH charges appearing in the r.h.s. of (4.3.7) implies that I_1 and V_{BH} vanish iff the fluxes of the n_V+1 Abelian vector field strengths all vanish:

$$\begin{gathered} I_1\left(z,\overline{z};p,q\right)=0=V_{BH}\left(z,\overline{z};p,q\right) \\ \Updownarrow \\ p^{\Lambda}=0=q_{\Lambda},\ \forall\Lambda=0,1,\ldots,n_V. \end{gathered} \tag{4.3.11}$$

By using (4.3.1) ($G_{i\overline{j}}=\overline{\partial}_{\overline{j}}\partial_i K$ understood throughout) and (4.3.10), we may rewrite the 4-d Lagrangian density (4.1.1) as follows:

$$
\begin{aligned}
&\mathcal{L}_4 = -\tfrac{R}{2} + G_{i\overline{j}}\partial_\mu z^i \partial_\nu \overline{z}^{\overline{j}} g^{\mu\nu} \\
&+\tfrac{1}{2}\left(Im\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho} + \tfrac{1}{2}\left(Re\mathcal{N}_{\Lambda\Sigma}\right)\mathcal{F}^{\Lambda}_{\mu\nu}\ {}^{*}\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho}.
\end{aligned}
\tag{4.3.12}
$$

Now $\mathcal{L}_4$ denotes the purely bosonic part of the Lagrangian density of n_V-fold $\mathcal{N}=2$, $d=4$ MESGT, with $i,\overline{i} \in \{1,\dots,n_V\}$ and $\Lambda,\Sigma \in \{0,1,\dots,n_V\}$.

Let us now consider the infinitesimal Kählerian metric interval in $\mathcal{M}_{z,\overline{z}}$; by using (4.3.2) we get

$$
\left|\frac{dz}{d\tau}\right|^2 = G_{i\overline{j}}\frac{dz^i}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau} = \left(\overline{\partial}_{\overline{j}}\partial_i K\right)\frac{dz^i}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau} = \frac{1}{2}G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau}.
\tag{4.3.13}
$$

Thus, by recalling (4.3.5) and (4.3.9), (4.1.22) and (4.1.23) may be respectively rewritten as

$$
\begin{aligned}
&\mathcal{L}_1\left[U,z,\overline{z};p,q\right] = \left(\tfrac{dU(\tau)}{d\tau}\right)^2 + \left|\tfrac{dz(\tau)}{d\tau}\right|^2 + \\
&+e^{2U(\tau)}\begin{bmatrix} \left|Z\right|^2(z,\overline{z};p,q) + G^{i\overline{i}}(z,\overline{z})\cdot \\ \cdot\left(D_iZ\right)(z,\overline{z};p,q)\left(\overline{D}_{\overline{i}}\overline{Z}\right)(z,\overline{z};p,q)\end{bmatrix}.
\end{aligned}
\tag{4.3.14}
$$

$$
\begin{aligned}
&2S_{BH}T_{BH} = \mathbf{c}^2 = \left(\tfrac{dU(\tau)}{d\tau}\right)^2 + \left|\tfrac{dz(\tau)}{d\tau}\right|^2 + \\
&-e^{2U(\tau)}\begin{bmatrix} \left|Z\right|^2(z,\overline{z};p,q) + G^{i\overline{i}}(z,\overline{z})\cdot \\ \cdot\left(D_iZ\right)(z,\overline{z};p,q)\left(\overline{D}_{\overline{i}}\overline{Z}\right)(z,\overline{z};p,q)\end{bmatrix}.
\end{aligned}
\tag{4.3.15}
$$

Let us write down the Euler-Lagrange equations of $\mathcal{L}_1$, which correspond to the equations of motion for $U(\tau)$ and $z^i(\tau)$. For what concerns $U(\tau)$, we obtain the "complexified" version of (4.1.24), i.e., the form of (4.1.24) related to the moduli space $\mathcal{M}_{z,\overline{z}}$

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta\left(\frac{dU(\tau)}{d\tau}\right)}=\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta U(\tau)}$$

$$\Updownarrow$$

$$\frac{d^2U(\tau)}{d\tau^2}=e^{2U(\tau)}V_{BH}\left(z,\overline{z};p,q\right)=e^{2U(\tau)}I_1\left(z,\overline{z};p,q\right)=$$

$$=e^{2U(\tau)}\left[\begin{array}{l}\left|Z\right|^2\left(z,\overline{z};p,q\right)+G^{i\overline{i}}\left(z,\overline{z}\right)\cdot\\ \\ \cdot\left(D_iZ\right)\left(z,\overline{z};p,q\right)\left(\overline{D}_{\overline{i}}\overline{Z}\right)\left(z,\overline{z};p,q\right)\end{array}\right].\tag{4.3.16}$$

The equations of motion for the complex moduli $z^i(\tau)$ read as follows:

$$\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta\left(\frac{dz^i(\tau)}{d\tau}\right)}=G_{i\overline{j}}\frac{d\overline{z}^{\overline{j}}}{d\tau}\tag{4.3.17}$$

$$\Downarrow$$

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta\left(\frac{dz^i(\tau)}{d\tau}\right)}=\frac{d}{d\tau}\left(G_{i\overline{j}}\frac{d\overline{z}^{\overline{j}}}{d\tau}\right)=$$

$$=\left(\partial_kG_{i\overline{j}}\right)\frac{dz^k}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau}+\left(\overline{\partial}_{\overline{k}}G_{i\overline{j}}\right)\frac{d\overline{z}^{\overline{k}}}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau}+G_{i\overline{j}}\frac{d^2\overline{z}^{\overline{j}}}{d\tau^2};\tag{4.3.18}$$

$$\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta z^i(\tau)}=$$

$$=2e^{2U}\left|Z\right|\partial_i\left|Z\right|+e^{2U}\partial_i\left[G^{k\overline{k}}\left(D_kZ\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\right]+\left(\partial_iG_{j\overline{k}}\right)\frac{dz^j}{d\tau}\frac{d\overline{z}^{\overline{k}}}{d\tau}=$$

$$=e^{2U}\partial_iV_{BH}+\left(\partial_iG_{j\overline{k}}\right)\frac{dz^j}{d\tau}\frac{d\overline{z}^{\overline{k}}}{d\tau}.\tag{4.3.19}$$

Thence

$$\frac{d}{d\tau}\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta\left(\frac{dz^i(\tau)}{d\tau}\right)}=\frac{\delta\mathcal{L}_1\left[U,z,\overline{z};p,q\right]}{\delta z^i(\tau)}$$

$$\Updownarrow$$

$$\begin{aligned}&G_{i\overline{j}}\frac{d^2\overline{z}^{\overline{j}}}{d\tau^2}+\left(\partial_k G_{i\overline{j}}\right)\frac{dz^k}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau}\\&+\left(\overline{\partial}_{\overline{k}}G_{i\overline{j}}\right)\frac{d\overline{z}^{\overline{k}}}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau}=e^{2U}\partial_i V_{BH}+\left(\partial_i G_{j\overline{k}}\right)\frac{dz^j}{d\tau}\frac{d\overline{z}^{\overline{k}}}{d\tau};\end{aligned} \tag{4.3.20}$$

by contracting this last expression with $G^{i\overline{k}}$, one obtains

$$\begin{aligned}&\frac{d^2\overline{z}^{\overline{k}}}{d\tau^2}+G^{k\overline{k}}\left[\left(\partial_i G_{k\overline{j}}\right)\frac{dz^i}{d\tau}+\left(\overline{\partial}_{\overline{i}}G_{k\overline{j}}\right)\frac{d\overline{z}^{\overline{i}}}{d\tau}-\left(\partial_k G_{i\overline{j}}\right)\frac{dz^i}{d\tau}\right]\frac{d\overline{z}^{\overline{j}}}{d\tau}\\&=e^{2U}G^{i\overline{k}}\partial_i V_{BH}.\end{aligned} \tag{4.3.21}$$

In a general Hermitian (commutative) geometric framework, such an expression may be rewritten as

$$\frac{\overline{\mathcal{D}}\overline{z}^{\overline{k}}}{d\tau^2}+G^{k\overline{k}}\left[\partial_i G_{k\overline{j}}-\partial_k G_{i\overline{j}}\right]\frac{dz^i}{d\tau}\frac{d\overline{z}^{\overline{j}}}{d\tau}=e^{2U}G^{i\overline{k}}\partial_i V_{BH}=e^{2U}\mathcal{D}^{\overline{k}}V_{BH} \tag{4.3.22}$$

or, by complex conjugating

$$\frac{\mathcal{D}z^k}{d\tau^2}+G^{k\overline{k}}\left[\overline{\partial}_{\overline{i}}G_{j\overline{k}}-\overline{\partial}_{\overline{k}}G_{j\overline{i}}\right]\frac{dz^j}{d\tau}\frac{d\overline{z}^{\overline{i}}}{d\tau}=e^{2U}\overline{\mathcal{D}}^k V_{BH}, \tag{4.3.23}$$

where $\mathcal{D}$ and $\mathcal{D}^{\overline{k}}$ respectively denote the Hermitian-covariant differential and the Hermitian-contravariant derivative defined by the Hermitian connection $\Theta_{jk}{}^i\equiv G^{i\overline{i}}\partial_j G_{k\overline{i}}$. As it is evident, we used the scalar nature of V_{BH} in order to write $G^{i\overline{k}}\partial_i V_{BH}=G^{i\overline{k}}\mathcal{D}_i V_{BH}=\mathcal{D}^{\overline{k}}V_{BH}$.

When specializing to (commutative) Kähler geometry, as it pertains to the present treatment, we get that $G_{i\overline{j}}=\overline{\partial}_{\overline{j}}\partial_i K$, and consequently, since the Kähler potential is assumed to satisfy the Schwarz lemma for the partial derivatives, (4.3.23) becomes

$$\frac{Dz^k}{d\tau^2}=e^{2U}\overline{D}^k V_{BH} \tag{4.3.24}$$

or equivalently, by specifying the τ- and (p,q)-dependence and recalling the identification (4.3.9)

$$\frac{Dz^k(\tau)}{d\tau^2} = e^{2U(\tau)}\overline{D}^k \left[\begin{array}{l} |Z|^2 (z(\tau), \overline{z}(\tau); p, q) + G^{i\bar{i}} (z(\tau), \overline{z}(\tau)) \cdot \\ \cdot (D_i Z)(z(\tau), \overline{z}(\tau); p, q) \left(\overline{D}_{\bar{i}} \overline{Z}\right)(z(\tau), \overline{z}(\tau); p, q) \end{array} \right]. \tag{4.3.25}$$

D and $\overline{D}^k$ now respectively denote the Kähler-covariant differential and the (complex conjugate of the) Kähler-contravariant derivative defined by the Kählerian connection

$$\mathcal{K}_{jk}{}^{i} = \mathcal{K}_{(jk)}{}^{i} \equiv \Gamma_{jk}{}^{i} = \left\{ {}^{\,i}_{jk} \right\} = G^{i\bar{i}} \partial_j G_{k\bar{i}} = G^{i\bar{i}} \partial_j \overline{\partial}_{\bar{i}} \partial_k K, \tag{4.3.26}$$

such that (4.3.24) may be expanded as

$$\frac{dz^k}{d\tau^2} + \Gamma_{ij}{}^{k} \frac{dz^i}{d\tau}\frac{dz^j}{d\tau} = \frac{dz^k}{d\tau^2} + G^{k\bar{k}} \partial_i \overline{\partial}_{\bar{k}} \partial_j K \frac{dz^i}{d\tau}\frac{dz^j}{d\tau} = e^{2U}\overline{D}^k V_{BH}. \tag{4.3.27}$$

(4.3.25) and (4.3.27) are the complexified version of (4.1.37), i.e., the form of (4.1.37) related to the moduli space $\mathcal{M}_{z,\overline{z}}$.

By using the properties of the SKG of $\mathcal{M}_{z,\overline{z}}$, the effective 1-d Lagrangian $\mathcal{L}_1$ given by (4.3.14) may be rewritten in a particular form, which is convenient in order to solve the equations of motion for the $n_\phi \equiv n_V$ complex scalars $z^i(\tau)$ in the extreme case ($\mathbf{c}^2 = 0$). To show this, let us elaborate the following quantities:

$$\left(\frac{dU}{d\tau} \pm e^U |Z| \right)^2 = \left(\frac{dU}{d\tau} \right)^2 + e^{2U} |Z|^2 \pm 2e^U \frac{dU}{d\tau} |Z|; \tag{4.3.28}$$

$$\begin{aligned} \frac{d}{d\tau}\left(e^U |Z|\right) &= \frac{dU}{d\tau} e^U |Z| + e^U \frac{d|Z|}{d\tau} \\ &= \frac{dU}{d\tau} e^U |Z| + e^U \frac{\partial |Z|}{\partial z^i}\frac{dz^i}{d\tau} + e^U \frac{\partial |Z|}{\partial \overline{z}^{\bar{i}}}\frac{d\overline{z}^{\bar{i}}}{d\tau}; \end{aligned} \tag{4.3.29}$$

$$
\begin{aligned}
&\left|\tfrac{dz^i}{d\tau} \pm \tfrac{Z}{|Z|}e^U G^{i\overline{k}}\overline{D}_{\overline{k}}\overline{Z}\right|^2 \equiv G_{i\overline{j}}\left(\tfrac{dz^i}{d\tau} \pm \tfrac{Z}{|Z|}e^U G^{i\overline{k}}\overline{D}_{\overline{k}}\overline{Z}\right)\left(\tfrac{d\overline{z}^{\overline{j}}}{d\tau} \pm \tfrac{\overline{Z}}{|Z|}e^U G^{k\overline{j}} D_k Z\right)\\
&= G_{i\overline{j}}\tfrac{dz^i}{d\tau}\tfrac{d\overline{z}^{\overline{j}}}{d\tau} \pm \tfrac{\overline{Z}}{|Z|}e^U G_{i\overline{j}}G^{k\overline{j}}\left(D_k Z\right)\tfrac{dz^i}{d\tau} \pm \tfrac{Z}{|Z|}e^U G_{i\overline{j}}G^{i\overline{k}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\tfrac{d\overline{z}^{\overline{j}}}{d\tau}\\
&+e^{2U}G_{i\overline{j}}G^{i\overline{k}}G^{k\overline{j}}\left(D_k Z\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\\
&= \left|\tfrac{dz}{d\tau}\right|^2 \pm \tfrac{\overline{Z}}{|Z|}e^U\left(D_i Z\right)\tfrac{dz^i}{d\tau} \pm \tfrac{Z}{|Z|}e^U\left(\overline{D}_{\overline{i}}\overline{Z}\right)\tfrac{d\overline{z}^{\overline{i}}}{d\tau} + e^{2U}G_{i\overline{j}}\left(\overline{D}^i\overline{Z}\right)\left(D^{\overline{j}}Z\right)\\
&= \left|\tfrac{dz}{d\tau}\right|^2 \pm 2e^U\left(\partial_i|Z|\right)\tfrac{dz^i}{d\tau} \pm 2e^U\left(\overline{\partial}_{\overline{i}}|Z|\right)\tfrac{d\overline{z}^{\overline{i}}}{d\tau} + e^{2U}G_{i\overline{j}}\left(\overline{D}^i\overline{Z}\right)\left(D^{\overline{j}}Z\right),
\end{aligned}
\tag{4.3.30}
$$

where in the last line we used (3.2.117). By using such results, it is immediate to check that an equivalent form for $\mathcal{L}_1$ reads

$$
\begin{aligned}
&\mathcal{L}_1 = \left(\tfrac{dU}{d\tau} \pm e^U|Z|\right)^2\\
&+G_{i\overline{j}}\left(\tfrac{dz^i}{d\tau} \pm \tfrac{Z}{|Z|}e^U G^{i\overline{k}}\overline{D}_{\overline{k}}\overline{Z}\right)\left(\tfrac{d\overline{z}^{\overline{j}}}{d\tau} \pm \tfrac{\overline{Z}}{|Z|}e^U G^{k\overline{j}} D_k Z\right)\\
&\mp 2\tfrac{d}{d\tau}\left(e^U|Z|\right).
\end{aligned}
\tag{4.3.31}
$$

Let us now consider 4-d extreme BHs; from (4.3.14) and (4.3.15) with $\mathbf{c}^2 = 0$ we get

$$
\mathcal{L}_{1,\mathbf{c}^2=0} = 2\left(\frac{dU}{d\tau}\right)^2 + 2\left|\frac{dz}{d\tau}\right|^2 = 2e^{2U}\left[|Z|^2 + G^{i\overline{i}}\left(D_i Z\right)\left(\overline{D}_{\overline{i}}\overline{Z}\right)\right].
\tag{4.3.32}
$$

The equations of motion for $U(\tau)$ and $z^i(\tau)$ obtained from $\mathcal{L}_{1,\mathbf{c}^2=0}$ may be solved by postulating the following first-order Ansätze,[9] which are suggested by rewriting (4.3.31) of the 1-d effective Lagrangian density (4.3.14):

$$
\frac{dU}{d\tau} = \mp e^U|Z|; \tag{4.3.33}
$$

$$
\frac{dz^i}{d\tau} = \mp\frac{Z}{|Z|}e^U G^{i\overline{k}}\overline{D}_{\overline{k}}\overline{Z}. \tag{4.3.34}
$$

Such first-order Ansätze solve the second-order differential equations of motion (4.3) and (4.3.25)–(4.3.27) for $U(\tau)$ and $z^i(\tau)$ in the extreme case $\mathbf{c}^2 = 0$. Indeed, (4.3.31), (4.3.33), and (4.3.34) yield

[9] Let us notice that (4.3.33) and (4.3.34) were derived in [27] and [28] by using SUSY; instead, the derivation reported here, firstly given in [55], does not rely on SUSY at all

$$\mathcal{L}_1 = \frac{d^2U}{d\tau^2}, \tag{4.3.35}$$

whereas (4.3.33), (4.3.34), and (3.2.117) imply

$$\begin{aligned}
\frac{d^2U}{d\tau^2} &= \frac{d}{d\tau}\left(\mp e^U |Z|\right) = \mp \frac{dU}{d\tau} e^U |Z| \mp e^U \frac{d|Z|}{d\tau} \\
&= e^{2U} |Z|^2 \mp e^U \left(\partial_i |Z|\right) \frac{dz^i}{d\tau} \mp e^U \left(\overline{\partial}_{\overline{i}} |Z|\right) \frac{d\overline{z}^{\overline{i}}}{d\tau} \\
&= e^{2U} |Z|^2 + \left(\partial_i |Z|\right) e^{2U} \frac{Z}{|Z|} G^{i\overline{k}} \overline{D}_{\overline{k}} \overline{Z} + \left(\overline{\partial}_{\overline{i}} |Z|\right) e^{2U} \frac{\overline{Z}}{|Z|} G^{k\overline{i}} D_k Z \\
&= e^{2U} |Z|^2 + e^{2U} G^{i\overline{k}} D_i Z \overline{D}_{\overline{k}} \overline{Z} = e^{2U} \left[|Z|^2 + G^{i\overline{i}} \left(D_i Z\right) \left(\overline{D}_{\overline{i}} \overline{Z}\right)\right].
\end{aligned} \tag{4.3.36}$$

Thus, (4.3.35) and (4.3.36) yield

$$\mathcal{L}_1 = 2e^{2U} \left[|Z|^2 + G^{i\overline{i}} \left(D_i Z\right) \left(\overline{D}_{\overline{i}} \overline{Z}\right)\right] = \mathcal{L}_{1,\mathbf{c}^2=0}. \tag{4.3.37}$$

Let us now analyze the two opposite regimes (the near-horizon limit $\tau \to -\infty$ and the asymptotical limit $\tau \to 0^-$) of the Ansätze (4.3.33) and (4.3.34), which solve the dynamical equations (4.3) and (4.3.25)–(4.3.27) for $U(\tau)$ and $z^i(\tau)$ in the extreme case $\mathbf{c}^2 = 0$.

For what concerns the asymptotical limit $r \to \infty \Leftrightarrow \tau \to 0^-$, the Ansatz (4.3.33), by using (4.1.41) and (4.1.43), yields

$$\begin{gathered}
lim_{\tau \to 0^-} \frac{dU(\tau)}{d\tau} = \mp lim_{\tau \to 0^-} e^{U(\tau)} \left|Z\left(z(\tau), \overline{z}(\tau); p, q\right)\right| \\
\Updownarrow \\
M_{BH} = \mp \left|Z\left(z_\infty, \overline{z}_\infty; p, q\right)\right|;
\end{gathered} \tag{4.3.38}$$

in order to get a positive BH mass, we necessarily have to disregard the unphysical "−" solution. From now on, we will consider only the Ansätze (4.3.33) and (4.3.34) with the "+" on their r.h.s.'s. Notice that (4.3.38) is nothing but the saturated BPS bound (3.2.122); this means that the first-order Ansatz (4.3.33) solves the equation of motion (4.3) for $U(\tau)$ in the extreme ($\mathbf{c}^2 = 0$), BPS case.

On the other hand, the Ansatz (4.3.34), by using (4.1.41) and (4.1.47), yields

$$lim_{\tau\rightarrow 0^-}\frac{dz^i(\tau)}{d\tau}$$

$$= lim_{\tau\rightarrow 0^-}\frac{Z(z(\tau),\overline{z}(\tau);p,q)}{|Z(z(\tau),\overline{z}(\tau);p,q)|}e^{U(\tau)}G^{i\overline{k}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\overline{D}_{\overline{k}}\overline{Z}\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right); \tag{4.3.39}$$

$$\Updownarrow$$

$$\begin{aligned}\Sigma^i &= G^{i\overline{k}}\left(z_\infty,\overline{z}_\infty\right)\frac{Z(z_\infty,\overline{z}_\infty;p,q)}{|Z(z_\infty,\overline{z}_\infty;p,q)|}\;\overline{D}_{\overline{k}}\overline{Z}\left(z,\overline{z};p,q\right)\Big|_{(z,\overline{z})=(z_\infty,\overline{z}_\infty)}\\ &= \frac{Z(z_\infty,\overline{z}_\infty;p,q)}{|Z(z_\infty,\overline{z}_\infty;p,q)|}\;\overline{D}^i\overline{Z}\left(z,\overline{z};p,q\right)\Big|_{(z,\overline{z})=(z_\infty,\overline{z}_\infty)};\end{aligned} \tag{4.3.40}$$

$$\Updownarrow$$

$$\overline{\Sigma}^{\overline{i}} = \frac{\overline{Z}\left(z_\infty,\overline{z}_\infty;p,q\right)}{\left|Z\left(z_\infty,\overline{z}_\infty;p,q\right)\right|}\;D^{\overline{i}}Z\left(z,\overline{z};p,q\right)\Big|_{(z,\overline{z})=(z_\infty,\overline{z}_\infty)}. \tag{4.3.41}$$

For what concerns the near-horizon limit $r\rightarrow r_H^+\Leftrightarrow\tau\rightarrow-\infty$, the condition (4.2.5) implies

$$lim_{\tau\rightarrow-\infty}\frac{dU(\tau)}{d\tau} = -\frac{1}{\tau}; \tag{4.3.42}$$

thus, the Ansatz (4.3.33), by recalling condition (4.2.5), yields

$$lim_{\tau\rightarrow-\infty}\left|Z\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)\right| = lim_{\tau\rightarrow-\infty}e^{-U(\tau)}\frac{dU(\tau)}{d\tau}$$

$$\Updownarrow$$

$$\left|Z\left(z_H,\overline{z}_H;p,q\right)\right| = \left(\frac{A_H}{4\pi}\right)^{1/2}. \tag{4.3.43}$$

On the other hand, the Ansatz (4.3.34) yields

$$ZG^{i\overline{k}}\overline{D}_{\overline{k}}\overline{Z} = e^{-U}\left|Z\right|\frac{dz^i}{d\tau}, \tag{4.3.44}$$

and consequently, by using (4.2.5) and (4.3.43), one gets

$$lim_{\tau\rightarrow-\infty}Z\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)G^{i\overline{k}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\overline{D}_{\overline{k}}\overline{Z}\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)$$

$$= lim_{\tau\rightarrow-\infty}e^{-U(\tau)}\left|Z\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)\right|\frac{dz^i(\tau)}{d\tau}$$

$$\Updownarrow$$

$$\begin{aligned}&Z\left(z_H,\overline{z}_H;p,q\right)G^{i\overline{k}}\left(z_H,\overline{z}_H\right)\overline{D}_{\overline{k}}\overline{Z}\left(z_H,\overline{z}_H;p,q\right)\\ &= -\frac{A_H}{4\pi}lim_{\tau\rightarrow-\infty}\tau\frac{dz^i(\tau)}{d\tau} = 0,\end{aligned} \tag{4.3.45}$$

where in the last passage (4.2.34), implying that for $\tau \to -\infty$, $\frac{dz^i(\tau)}{d\tau}$ vanishes faster than $\frac{1}{\tau}$, was used. In general, we assume that $A_H \neq 0$ (no naked singularities) and $Z \neq 0$ everywhere in $\mathcal{M}_{z,\overline{z}}$, and in particular at the horizon, critical attractor point(s) (for any BH charge configuration being considered). Thus, in regular SKG (4.3.45) yields

$$
\begin{gathered}
G^{i\overline{k}}\left(z_H,\overline{z}_H\right)\overline{D}_{\overline{k}}\overline{Z}\left(z_H,\overline{z}_H;p,q\right)=\overline{D}^i\overline{Z}\left(z_H,\overline{z}_H;p,q\right)=0\\
\Updownarrow\\
D^{\overline{i}}Z\left(z_H,\overline{z}_H;p,q\right)=0\\
\Downarrow\\
\overline{D}_{\overline{i}}\overline{Z}\left(z_H,\overline{z}_H;p,q\right)=G_{i\overline{i}}\left(z_H,\overline{z}_H\right)\overline{D}^i\overline{Z}\left(z_H,\overline{z}_H;p,q\right)=0\\
\Updownarrow\\
D_iZ\left(z_H,\overline{z}_H;p,q\right)=0.
\end{gathered}
\tag{4.3.46}
$$

Equations (4.3.46) are nothing but the so-called $\frac{1}{2}$-*BPS extreme BH attractor equations* (see the end of Subsect. 3.2, and, e.g., (3.2.113)).

Finally, let us consider the asymptotical limit of the constraint (4.3.15), by separately calculating the asymptotical limit of its terms. By using equations (4.1.41), (4.3.41), (4.3.44), and (4.1.47), one gets

$$
\begin{aligned}
2S_{BH}T_{BH}=\mathbf{c}^2=lim_{\tau\to 0^-}&\left\{\left(\frac{dU(\tau)}{d\tau}\right)^2+\left|\frac{dz(\tau)}{d\tau}\right|^2\right.\\
&\left.-e^{2U(\tau)}\left[\begin{array}{l}|Z|^2\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)+G^{i\overline{i}}\left(z\left(\tau\right),\overline{z}\left(\tau\right)\right)\cdot\\ \cdot\left(D_iZ\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)\left(\overline{D}_{\overline{i}}\overline{Z}\right)\left(z\left(\tau\right),\overline{z}\left(\tau\right);p,q\right)\end{array}\right]\right\}
\end{aligned}
\tag{4.3.47}
$$

$$\Updownarrow$$

$$
\begin{aligned}
&M_{BH}^2+G_{i\overline{i}}\left(z_\infty,\overline{z}_\infty\right)\Sigma^i\overline{\Sigma}^{\overline{i}}-\left|Z\left(z_\infty,\overline{z}_\infty;p,q\right)\right|^2-G_{i\overline{i}}\left(z_\infty,\overline{z}_\infty\right)\Sigma^i\overline{\Sigma}^{\overline{i}}\\
&=\mathbf{c}^2=2S_{BH}T_{BH}
\end{aligned}
\tag{4.3.48}
$$

$$\Updownarrow$$

$$
M_{BH}^2=\left|Z\left(z_\infty,\overline{z}_\infty;p,q\right)\right|^2+\mathbf{c}^2\geqslant\left|Z\left(z_\infty,\overline{z}_\infty;p,q\right)\right|^2;
\tag{4.3.49}
$$

note that the second and fourth terms in the r.h.s. of (4.3.15) cancel in the considered limit. Therefore the contribution of the scalar charges vanishes at spatial infinity, where the constraint (4.3.15) becomes a BPS-like bound, saturated only for $\mathbf{c}^2 = 0$, i.e., only for extreme, BPS, static, spherically symmetric, and asymptotically flat BHs in $\mathcal{N} = 2$, $d = 4$ n_V-fold MESGT:

$$M_{BH}^2 = |Z(z_\infty, \overline{z}_\infty; p, q)|^2 . \qquad (4.3.50)$$

For such a particular class of BHs, we have seen that equations of motion (4.3) and (4.3.25)–(4.3.27) for $U(\tau)$ and $z^i(\tau)$ may be solved by the first-order Ansätze (4.3.33) and (4.3.34).

4.4 Critical Points of Black Hole Effective Potential in Special Kähler Geometry

We will now study the critical points of the BH-effective potential function $V_{BH}(z, \overline{z}; p, q)$ in the (regular) special Kähler geometry (SKG) of the vector supermultiplets' moduli space $\mathcal{M}_{z,\overline{z}}$ of the n_V-fold $\mathcal{N} = 2$, $d = 4$ MESGT. As previously pointed out, such critical points are attractors in the dynamical system describing the radial evolution of the moduli from $r \to \infty$ to $r \to r_H^+$. In order to perform such an analysis, we need to recall a few results from SKG[10]; beside the Kähler-covariant holomorphicity of Z, i.e., (see (3.2.63))

$$\overline{D}_{\overline{i}} Z = 0 \Leftrightarrow D_i \overline{Z} = 0, \qquad (4.4.1)$$

we will largely use (3.1.27) and (3.1.28) which, by definition (3.2.53), yield

$$D_i D_j Z = i C_{ijk} G^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z}; \qquad (4.4.2)$$

$$D_i \overline{D}_{\overline{j}} \overline{Z} = G_{i\overline{j}} \overline{Z} \Leftrightarrow \overline{D}_{\overline{i}} D_j Z = G_{j\overline{i}} Z. \qquad (4.4.3)$$

Let us start from the fundamental identification (4.3.9)

$$\begin{aligned} V_{BH}(z, \overline{z}; p, q) &= I_1(z, \overline{z}; p, q) \\ &\equiv |Z|^2 (z, \overline{z}; p, q) + G^{i\overline{i}}(z, \overline{z}) (D_i Z)(z, \overline{z}; p, q) \left(\overline{D}_{\overline{i}} \overline{Z}\right)(z, \overline{z}; p, q) . \end{aligned} \qquad (4.4.4)$$

Thence, by recalling that V_{BH} and $|Z(z, \overline{z}; p, q)|$ are Kähler-gauge invariant scalars in $\mathcal{M}_{z,\overline{z}}$, by using (3.2.117), (4.4.2), and (4.4.3) we can calculate (also remind that in $\mathcal{M}_{z,\overline{z}}$ the metric postulate holds)

[10] Beside Sect. 3, see, e.g., [36,44–47,56,73–75], and [76] for further insights on SKG and moduli space geometries of $\mathcal{N} = 2$ SUGRA more in general

$$
\begin{aligned}
D_i V_{BH} = \partial_i V_{BH} &= \partial_i \left[|Z|^2 + G^{j\overline{k}} \left(D_j Z \right) \left(\overline{D}_{\overline{k}} \overline{Z} \right) \right] \\
&= 2 |Z| \partial_i |Z| + D_i \left[G^{j\overline{k}} \left(D_j Z \right) \left(\overline{D}_{\overline{k}} \overline{Z} \right) \right] \\
&= \overline{Z} D_i Z + G^{j\overline{k}} \left(D_i D_j Z \right) \left(\overline{D}_{\overline{k}} \overline{Z} \right) + G^{j\overline{k}} \left(D_j Z \right) \left(D_i \overline{D}_{\overline{k}} \overline{Z} \right) \\
&= \overline{Z} D_i Z + i C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \left(\overline{D}_{\overline{k}} \overline{Z} \right) + G^{j\overline{k}} \left(D_j Z \right) G_{i\overline{k}} \overline{Z} \\
&= 2 \overline{Z} D_i Z + i C_{ijk} G^{j\overline{m}} G^{k\overline{k}} \left(\overline{D}_{\overline{m}} \overline{Z} \right) \left(\overline{D}_{\overline{k}} \overline{Z} \right). \qquad (4.4.5)
\end{aligned}
$$

Therefore, we get that the critical points of $|Z|$ are critical points also for V_{BH}; indeed, by assuming that $Z \neq 0$ (everywhere in $\mathcal{M}_{z,\overline{z}}$, and in particular at the horizon, critical attractor points) and using (3.2.117), (4.4.5) yields

$$
\partial_i |Z| = 0 \Leftrightarrow D_i Z = 0 \Longrightarrow \partial_i V_{BH} = 0. \qquad (4.4.6)
$$

It should be stressed that the opposite, in general, is not true:

$$
\partial_i V_{BH} = 0 \nRightarrow \partial_i |Z| = 0 \Leftrightarrow D_i Z = 0. \qquad (4.4.7)
$$

Thus, in the framework of the n_V-fold $\mathcal{N} = 2$, $d = 4$ MESGT with (regular) SKG of $\mathcal{M}_{z,\overline{z}}$, the horizon, attractor points for the considered extreme BH, i.e., the critical points of the BH effective potential function $V_{BH}(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$, may be divided in two disjoint classes:

1. The attractors which are critical points also of the absolute value of the central charge $|Z|(z, \overline{z}; p, q)$ (the so-called $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractors, treated in Subsubsect. 4.4.1)
   and
2. Those that are *not* critical points of $|Z|(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$ (the so-called *non-(BPS-)SUSY* extreme BH attractors, treated in Subsect. 4.4.2).

Clearly, such a distinction (and the whole treatment given below) is parametrically dependent on the BH charge configuration, i.e., it is parameterized by the $Sp(2n_V + 2)$-covariant vector $\left(p^\Lambda, q_\Lambda\right)$, with the group $Sp(2n_V + 2)$ defined on $\mathbb{R}$ at classical level and on $\mathbb{Z}$ when the charge quantization is taken into account.

4.4.1 Supersymmetric Attractors

Let us start by considering the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractors, i.e., the points $(z_{susy}(p, q), \overline{z}_{susy}(p, q))$ in $\mathcal{M}_{z,\overline{z}}$ defined by

$$\forall i \in \{1, \ldots, n_V\} : \begin{cases} (\partial_i |Z|)_{(z_{susy}, \overline{z}_{susy})} = 0 \Leftrightarrow (D_i Z)_{(z_{susy}, \overline{z}_{susy})} = 0; \\ \Updownarrow \\ \left\{ \left[\partial_i + \frac{1}{2} \partial_i K(z, \overline{z}) \right] Z(z, \overline{z}; p, q) \right\}_{(z_{susy}, \overline{z}_{susy})} = 0; \\ \Downarrow \\ (\partial_i V_{BH})_{(z_{susy}, \overline{z}_{susy})} = 0. \end{cases} \tag{4.4.1.1}$$

In order to eventually characterize such points as maxima or minima of the function $|Z|(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$, we have at least to calculate the Kähler-covariant second derivatives of $|Z|$, and then evaluate them at $(z_{susy}, \overline{z}_{susy})$. By using (4.4.1), (3.2.117), (4.4.2), and (4.4.3), we obtain

$$\begin{aligned} D_i D_j |Z| &= D_i \partial_j |Z| = D_i \left(\frac{\overline{Z}}{2|Z|} D_j Z \right) \\ &= \frac{1}{2|Z|} \left(D_i \overline{Z} \right) D_j Z - \frac{\overline{Z}}{2|Z|^2} (D_i |Z|) D_j Z + \frac{\overline{Z}}{2|Z|} D_i D_j Z \\ &= -\frac{\overline{Z}^2}{4|Z|^3} (D_i Z) D_j Z + i \frac{\overline{Z}}{2|Z|} C_{ijk} G^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z} \\ &= i \frac{\overline{Z}}{2|Z|} \left[i \frac{\overline{Z}}{2|Z|^2} (D_i Z) D_j Z + C_{ijk} G^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z} \right]; \end{aligned} \tag{4.4.1.2}$$

$$\begin{aligned} \overline{D}_{\overline{i}} D_j |Z| &= \overline{D}_{\overline{i}} \partial_j |Z| = \overline{D}_{\overline{i}} \left(\frac{\overline{Z}}{2|Z|} D_j Z \right) \\ &= \frac{1}{2|Z|} \left(\overline{D}_{\overline{i}} \overline{Z} \right) D_j Z - \frac{\overline{Z}}{2|Z|^2} \left(\overline{D}_{\overline{i}} |Z| \right) D_j Z + \frac{\overline{Z}}{2|Z|} \overline{D}_{\overline{i}} D_j Z \\ &= \frac{1}{4|Z|} \left(\overline{D}_{\overline{i}} \overline{Z} \right) D_j Z + \frac{1}{2} |Z| G_{j\overline{i}}. \end{aligned} \tag{4.4.1.3}$$

On the other hand, by recalling (4.3.26) and the general properties of Hermitian and Kählerian manifolds [37], one gets

$$\begin{aligned} D_i D_j |Z| &= D_i \partial_j |Z| = \partial_i \partial_j |Z| - \Gamma_{ij}{}^k \partial_k |Z| \\ &= \partial_i \partial_j |Z| - G^{k\overline{i}} \left(\partial_i \overline{\partial}_{\overline{i}} \partial_j K \right) \partial_k |Z|; \end{aligned} \tag{4.4.1.4}$$

$$\overline{D}_{\overline{i}} D_j |Z| = \overline{D}_{\overline{i}} \partial_j |Z| = \overline{\partial}_{\overline{i}} \partial_j |Z|. \tag{4.4.1.5}$$

Since the Kähler potential K and the central charge $|Z|$ are both assumed to satisfy the Schwarz lemma in $\mathcal{M}_{z,\overline{z}}$, such equations respectively yield

$$D_i D_j |Z| = D_j D_i |Z|; \tag{4.4.1.6}$$

$$\overline{D}_{\overline{i}} D_j |Z| = D_j \overline{D}_{\overline{i}} |Z|, \tag{4.4.1.7}$$

as it can be checked by looking at the explicit expressions (4.4.1.2) and (4.4.1.3). Consequently, by evaluating at the point(s) $(z_{susy}, \overline{z}_{susy})$ in $\mathcal{M}_{z,\overline{z}}$ defined by (4.4.1.1), (4.4.1.2)–(4.4.1.5) yield

$$(D_i D_j |Z|)_{(z_{susy},\overline{z}_{susy})} = (\partial_i \partial_j |Z|)_{(z_{susy},\overline{z}_{susy})} = 0; \tag{4.4.1.8}$$

$$\begin{aligned}\left(\overline{D}_{\overline{i}} D_j |Z|\right)_{(z_{susy},\overline{z}_{susy})} &= \left(\overline{\partial}_{\overline{i}} \partial_j |Z|\right)_{(z_{susy},\overline{z}_{susy})} \\ &= \frac{1}{2} |Z| (z_{susy}, \overline{z}_{susy}; p, q)\, G_{j\overline{i}} (z_{susy}, \overline{z}_{susy}).\end{aligned} \tag{4.4.1.9}$$

It is now possible to introduce the $2n_V \times 2n_V$ complex Hessian matrix $H^{|Z|}_{\hat{i}\hat{j}}$ of the function $|Z|(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$, as follows:

$$\begin{aligned} H^{|Z|}_{\hat{i}\hat{j}} (z, \overline{z}; p, q) &= \begin{pmatrix} H^{|Z|}_{ij} & H^{|Z|}_{i\overline{j}} \\ H^{|Z|}_{j\overline{i}} & H^{|Z|}_{\overline{i}\overline{j}} \end{pmatrix} \\ &\equiv \begin{pmatrix} D_i D_j |Z| & D_i \overline{D}_{\overline{j}} |Z| \\ D_j \overline{D}_{\overline{i}} |Z| & \overline{D}_{\overline{i}} \overline{D}_{\overline{j}} |Z| \end{pmatrix} = \begin{pmatrix} D_i D_j |Z| & \overline{\overline{D}_{\overline{i}} D_j |Z|} \\ \overline{D}_{\overline{i}} D_j |Z| & \overline{D_i D_j |Z|} \end{pmatrix} \\ &= \begin{pmatrix} i\frac{\overline{Z}}{2|Z|} \left[\begin{array}{c} i\frac{\overline{Z}}{2|Z|^2} (D_i Z) D_j Z \\ + C_{ijk} G^{k\overline{k}} \overline{D}_{\overline{k}} \overline{Z} \end{array} \right] & \frac{1}{4|Z|} (D_i Z) \overline{D}_{\overline{j}} \overline{Z} + \frac{1}{2} |Z| G_{i\overline{j}} \\ \frac{1}{4|Z|} \left(\overline{D}_{\overline{i}} \overline{Z}\right) D_j Z + \frac{1}{2} |Z| G_{j\overline{i}} & -i\frac{Z}{2|Z|} \left[\begin{array}{c} -i\frac{Z}{2|Z|^2} \left(\overline{D}_{\overline{i}} \overline{Z}\right) \overline{D}_{\overline{j}} \overline{Z} \\ + \overline{C}_{\overline{i}\overline{j}\overline{k}} G^{k\overline{k}} D_k Z \end{array} \right] \end{pmatrix}, \end{aligned} \tag{4.4.1.10}$$

where the hatted indices $\hat{\imath}$ and $\hat{\jmath}$ may be holomorphic or anti-holomorphic $(n_\phi = n_V)$:

$$\hat{\imath}, \hat{\jmath} \in \left\{ \begin{array}{ccc} \text{Holomorphic } i\text{-indices} & & \text{Anti-holomorphic } \bar{\imath}\text{-indices} \\ 1, \ldots, n_\phi & , & n_\phi + 1, \ldots, m_\phi \end{array} \right\}. \tag{4.4.1.11}$$

Thus, by evaluating at the point(s) $\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right)\right)$ in $\mathcal{M}_{z,\overline{z}}$ defined by (4.4.1.1), the Hessian becomes

$$\begin{aligned} & H^{|Z|}_{\hat{\imath}\hat{\jmath}}\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right); p,q\right) \\ & = \tfrac{1}{2}\left|Z\right|\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right); p,q\right) \\ & \cdot \begin{pmatrix} 0 & G_{i\bar{\jmath}}\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right)\right) \\ G_{j\bar{\imath}}\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right)\right) & 0 \end{pmatrix}; \end{aligned} \tag{4.4.1.12}$$

since $\overline{G_{i\bar{\jmath}}} = G_{j\bar{\imath}}$, we obtain

$$\left(H^{|Z|}_{\hat{\imath}\hat{\jmath}}\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right); p,q\right)\right)^{\dagger} = H^{|Z|}_{\hat{\imath}\hat{\jmath}}\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right); p,q\right). \tag{4.4.1.13}$$

Equation (4.4.1.13) means that the $2n_V \times 2n_V$ complex Hessian matrix $H^{|Z|}_{\hat{\imath}\hat{\jmath}}$ evaluated at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s) $\left(z_{susy}, \overline{z}_{susy}\right)$ in $\mathcal{M}_{z,\overline{z}}$ is Hermitian for any BH charge configuration. Consequently, $H^{|Z|}_{\hat{\imath}\hat{\jmath}}\left(z_{susy}, \overline{z}_{susy}; p,q\right)$ is always diagonalizable by a unitary transformation, and it has $2n_V$ real eigenvalues; from (4.4.1.12) and well-known theorems of mathematical analysis, it then follows that for an arbitrary but fixed BH charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right) \in \Gamma$

$$\begin{gathered} G_{i\bar{\jmath}}\left(z_{susy}, \overline{z}_{susy}\right) \text{ strictly positive (negative) definite} \\ \Updownarrow \\ \left(z_{susy}, \overline{z}_{susy}\right) \textit{ at least local} \text{ miminum (maximum) of } \left|Z\right|\left(z, \overline{z}; p,q\right) \text{ in } \mathcal{M}_{z,\overline{z}}. \end{gathered} \tag{4.4.1.14}$$

Since we assume that the SKG of $\mathcal{M}_{z,\overline{z}}$ is regular, namely, that the metric $G_{i\bar{\jmath}}$ is strictly positive definite everywhere, we obtain at least a local minimum of $|Z|$ at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s). However, if we go beyond the regular regime of SKG, $G_{i\bar{\jmath}}$ may be singular (i.e., not invertible) and/or without a well-defined definiteness (i.e., with some positive as well as negative eigenvalues); in such a case, (4.4.1) yields that the eventually existing (at least local) maxima of $|Z|$ are reached out of the regular SKG of $\mathcal{M}_{z,\overline{z}}$. In general, going beyond the regular regime of SKG, some "phase transitions" may happen in $\mathcal{M}_{z,\overline{z}}$, corresponding to a breakdown of the 1-d effective Lagrangian picture[11] of 4-d (extreme) BHs presented in Subsects. 4.1–4.3, unless new massless states appear [55].

Moreover, by recalling (4.4.4) and using the very definition (4.4.1.1), the value of the function V_{BH} at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s) reads

$$V_{BH}\left(z_{susy},\overline{z}_{susy};p,q\right)=\left|Z\right|^{2}\left(z_{susy},\overline{z}_{susy};p,q\right), \tag{4.4.1.15}$$

implying that the (semiclassical, leading order) entropy at such $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor(s) is

$$S_{BH,susy}=\pi\left|Z\right|^{2}\left(z_{susy},\overline{z}_{susy};p,q\right). \tag{4.4.1.16}$$

Now, in order to establish if the points $\left(z_{susy},\overline{z}_{susy}\right)$ are eventually maxima or minima of $V_{BH}\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$, we have at least to calculate the Kähler-covariant second derivatives of V_{BH}, and then evaluate them at $\left(z_{susy},\overline{z}_{susy}\right)$. By using (3.1.4), (4.4.1), (4.4.2), (4.4.3), and (4.4.5) and exploiting the validity of the metric postulate in $\mathcal{M}_{z,\overline{z}}$, we obtain

$$\begin{aligned}
&D_iD_jV_{BH}\\
&=D_i\left[2\overline{Z}D_jZ+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\right]\\
&=2\left(\overline{D}_i\overline{Z}\right)D_jZ+2\overline{Z}D_iD_jZ+i\left(D_iC_{jkl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\\
&+iC_{jkl}\left(D_iG^{k\overline{m}}\right)G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)+iC_{jkl}G^{k\overline{m}}\left(D_iG^{l\overline{l}}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\\
&+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(D_i\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(D_i\overline{D}_{\overline{l}}\overline{Z}\right)\\
&=2iC_{ijk}G^{k\overline{k}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{Z}+i\left(D_iC_{jkl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\\
&+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}G_{i\overline{m}}\overline{Z}\overline{D}_{\overline{l}}\overline{Z}+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)G_{i\overline{l}}\overline{Z}\\
&=2iC_{ijk}G^{k\overline{k}}\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{Z}+i\left(D_iC_{jkl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{l}}\overline{Z}\right)\\
&+iC_{jkl}G^{k\overline{m}}G^{l\overline{l}}\overline{Z}\left(G_{i\overline{m}}\overline{D}_{\overline{l}}\overline{Z}+G_{i\overline{l}}\overline{D}_{\overline{m}}\overline{Z}\right)
\end{aligned}$$

[11] Such a 1-d effective framework should be understood as being obtained by integrating all massive states of the theory out

$$
\begin{aligned}
&= 2i\left[2C_{ijk}G^{k\bar{k}}\overline{Z}\overline{D}_{\bar{k}}\overline{Z} + \tfrac{1}{2}\left(D_iC_{jkl}\right)G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\bar{l}}\overline{Z}\right]\\
&= 2i\left[2C_{(ij)k}G^{k\bar{k}}\overline{Z}\overline{D}_{\bar{k}}\overline{Z} + \tfrac{1}{2}\left(D_{(i}C_{j)kl}\right)G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\bar{l}}\overline{Z}\right];
\end{aligned}
\tag{4.4.1.17}
$$

$$
\begin{aligned}
&\overline{D}_{\bar{i}}D_jV_{BH}\\
&= \overline{D}_{\bar{i}}\left[2\overline{Z}D_jZ + iC_{jkl}G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\right]\\
&= 2\left(\overline{D}_{\bar{i}}\overline{Z}\right)D_jZ + 2\overline{Z}\overline{D}_{\bar{i}}D_jZ + i\left(\overline{D}_{\bar{i}}C_{jkl}\right)G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\\
&+iC_{jkl}\left(\overline{D}_{\bar{i}}G^{k\overline{m}}\right)G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right) + iC_{jkl}G^{k\overline{m}}\left(\overline{D}_{\bar{i}}G^{l\bar{l}}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\\
&+iC_{jkl}G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\bar{i}}\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right) + iC_{jkl}G^{k\overline{m}}G^{l\bar{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{i}}\overline{D}_{\bar{l}}\overline{Z}\right)\\
&= 2\left(\overline{D}_{\bar{i}}\overline{Z}\right)D_jZ + 2G_{j\bar{i}}\left|Z\right|^2 + C_{jkl}G^{k\overline{m}}G^{l\bar{l}}\overline{C}_{\bar{i}\overline{m}\bar{k}}G^{n\bar{k}}\left(D_nZ\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\\
&+C_{jkl}G^{k\overline{m}}G^{l\bar{l}}\overline{C}_{\bar{i}\bar{l}\bar{k}}G^{n\bar{k}}\left(D_nZ\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\\
&= 2\left(\overline{D}_{\bar{i}}\overline{Z}\right)D_jZ + 2G_{j\bar{i}}\left|Z\right|^2 + C_{jkl}G^{k\overline{m}}G^{l\bar{l}}G^{n\bar{k}}(D_nZ)\left(\overline{C}_{\bar{i}\overline{m}\bar{k}}\overline{D}_{\bar{l}}\overline{Z}+\overline{C}_{\bar{i}\bar{l}\bar{k}}\overline{D}_{\overline{m}}\overline{Z}\right)\\
&= 2\left[\left(\overline{D}_{\bar{i}}\overline{Z}\right)D_jZ + G_{j\bar{i}}\left|Z\right|^2 + C_{jkl}\overline{C}_{\bar{i}\overline{m}\bar{k}}G^{k\overline{m}}G^{l\bar{l}}G^{n\bar{k}}\left(D_nZ\right)\overline{D}_{\bar{l}}\overline{Z}\right],
\end{aligned}
\tag{4.4.1.18}
$$

where in the last lines of both equations we used the symmetry of the rank-3 tensor C_{jkl}: $C_{jkl} = C_{(jkl)}$, and in the last line of (4.4.1.17) also the symmetry of the Kähler-covariant derivative of such a tensor ($D_{[i}C_{j]kl} = 0$, see (3.1.5)).

By using (3.1.3), (3.1.8), and (3.1.9)–(3.1.11), expression (4.4.1.18) can be further elaborated as follows:

$$
\begin{aligned}
&\overline{D}_{\bar{i}} D_j V_{BH} \\
&= 2\left[\left(\overline{D}_{\bar{i}}\overline{Z}\right) D_j Z + G_{j\bar{i}} |Z|^2 + C_{jkl}\overline{C}_{\bar{i}\bar{m}\bar{k}} G^{k\bar{m}} G^{l\bar{l}} G^{n\bar{k}} \left(D_n Z\right) \overline{D}_{\bar{l}}\overline{Z}\right] \\
&= 2\left[\left(\overline{D}_{\bar{i}}\overline{Z}\right) D_j Z + G_{j\bar{i}} |Z|^2 + \left(\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}} G_{j\bar{i}} - G^{l\bar{l}} G^{n\bar{k}} R_{j\bar{i}l\bar{k}}\right) \left(D_n Z\right) \overline{D}_{\bar{l}}\overline{Z}\right] \\
&= 2\left[G_{j\bar{i}} |Z|^2 + \left(2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}} G_{j\bar{i}} - G^{l\bar{l}} G^{n\bar{k}} R_{j\bar{i}l\bar{k}}\right) \left(D_n Z\right) \overline{D}_{\bar{l}}\overline{Z}\right]
\end{aligned}
$$

$$
= 2\left\{
\begin{array}{l}
\left(\overline{\partial}_{\bar{i}}\partial_j K\right) |Z|^2 \\[2ex]
+\left[
\begin{array}{l}
2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}} \left(\overline{\partial}_{\bar{i}}\partial_j K\right) \\[2ex]
+G^{l\bar{l}} G^{n\bar{k}} G^{m\bar{m}} \left(\overline{\partial}_{\bar{k}}\overline{\partial}_{\bar{i}}\partial_m K\right) \partial_j \overline{\partial}_{\bar{m}}\partial_l K - G^{l\bar{l}} G^{n\bar{k}} \overline{\partial}_{\bar{k}}\partial_l \overline{\partial}_{\bar{i}}\partial_j K
\end{array}
\right] \cdot \\[2ex]
\cdot \left[\left(\partial_n + \frac{1}{2}\partial_n K\right) Z\right] \left[\left(\overline{\partial}_{\bar{l}} + \frac{1}{2}\overline{\partial}_{\bar{l}} K\right) \overline{Z}\right].
\end{array}
\right\}. \tag{4.4.1.19}
$$

On the other hand, by recalling (4.3.26) and the general properties of Hermitian and Kählerian manifolds [37], one gets

$$
\begin{aligned}
D_i D_j V_{BH} &= D_i \partial_j V_{BH} \\
&= \partial_i \partial_j V_{BH} + \Gamma_{ij}^{\ \ k} \partial_k V_{BH} = \partial_i \partial_j V_{BH} + G^{k\bar{i}} \partial_i \overline{\partial}_{\bar{i}} \partial_j K \partial_k V_{BH};
\end{aligned} \tag{4.4.1.20}
$$

$$
\overline{D}_{\bar{i}} D_j V_{BH} = \overline{D}_{\bar{i}} \partial_j V_{BH} = \overline{\partial}_{\bar{i}} \partial_j V_{BH}. \tag{4.4.1.21}
$$

Since the Kähler potential K and BH effective potential V_{BH} are both assumed to satisfy the Schwarz lemma in $\mathcal{M}_{z,\overline{z}}$, such equations respectively yield

$$
D_i D_j V_{BH} = D_j D_i V_{BH}; \tag{4.4.1.22}
$$

$$
\overline{D}_{\bar{i}} D_j V_{BH} = D_j \overline{D}_{\bar{i}} V_{BH}, \tag{4.4.1.23}
$$

as it can be checked by looking at expressions (4.4.1.17), (4.4.1.18), and (4.4.1.19).

For completeness, since $\overline{D}_{\bar{k}}\overline{Z} = \left(\overline{\partial}_{\bar{k}} + \frac{1}{2}\overline{\partial}_{\bar{k}} K\right) \overline{Z}$ and C_{jkl} is a rank-3 completely symmetric, Kähler-covariantly holomorphic tensor with Kähler weights $(2, -2)$ for which then (see (3.1.21))

$$
D_i C_{jkl} = \partial_i C_{jkl} + \left(\partial_i K\right) C_{jkl} - \Gamma_{ij}^{\ \ m} C_{mkl} - \Gamma_{ik}^{\ \ m} C_{jml} - \Gamma_{il}^{\ \ m} C_{jkm}. \tag{4.4.1.24}
$$

Equation (4.4.1.17) may be further elaborated as follows (by also recalling (4.3.26)):

$$
\begin{aligned}
&D_i D_j V_{BH} \\
&= 4iC_{ijk}G^{k\overline{k}}\overline{Z}\overline{D}_{\overline{k}}\overline{Z} + i\left(D_{(i}C_{j)kl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{l}}\overline{Z} \\
&= 4iC_{ijk}G^{k\overline{k}}\overline{Z}\left[\left(\overline{\partial}_{\overline{k}} + \tfrac{1}{2}\overline{\partial}_{\overline{k}}K\right)\overline{Z}\right] \\
&+i\left[\partial_{(i}C_{j)kl} + \left(\partial_{(i}K\right)C_{j)kl} - \mathit{\Gamma}_{(ij)}{}^{m}C_{mkl} - \mathit{\Gamma}_{(i|k}{}^{m}C_{|j)ml} - \mathit{\Gamma}_{(i|l}{}^{m}C_{|j)km}\right]\cdot \\
&\cdot G^{k\overline{m}}G^{l\overline{l}}\left[\left(\overline{\partial}_{\overline{m}} + \tfrac{1}{2}\overline{\partial}_{\overline{m}}K\right)\overline{Z}\right]\left[\left(\overline{\partial}_{\overline{l}} + \tfrac{1}{2}\overline{\partial}_{\overline{l}}K\right)\overline{Z}\right] \\
&= iG^{k\overline{m}}\left[\left(\overline{\partial}_{\overline{m}} + \tfrac{1}{2}\overline{\partial}_{\overline{m}}K\right)\overline{Z}\right]\cdot \\
&\cdot\left\{\begin{array}{l} 4\overline{Z}C_{ijk} + G^{l\overline{l}}\left[\left(\overline{\partial}_{\overline{l}} + \tfrac{1}{2}\overline{\partial}_{\overline{l}}K\right)\overline{Z}\right]\cdot \\ \cdot\left[\partial_{(i}C_{j)kl} + \left(\partial_{(i}K\right)C_{j)kl} - G^{m\overline{k}}C_{mkl}\overline{\partial}_{\overline{k}}\partial_i\partial_j K - 2G^{m\overline{k}}C_{ml(j|}\overline{\partial}_{\overline{k}}\partial_{|i)}\partial_k K\right] \end{array}\right\}.
\end{aligned}
\tag{4.4.1.25}
$$

Now, by evaluating at the point(s) $\left(z_{susy}\left(p,q\right), \overline{z}_{susy}\left(p,q\right)\right)$ in $\mathcal{M}_{z,\overline{z}}$ defined by (4.4.1.1), (4.4.1.17), (4.4.1.25), (4.4.1.18), and (4.4.1.19) yield

$$
\left(D_i D_j V_{BH}\right)_{\left(z_{susy},\overline{z}_{susy}\right)} = \left(\partial_i\partial_j V_{BH}\right)_{\left(z_{susy},\overline{z}_{susy}\right)} = 0; \tag{4.4.1.26}
$$

$$
\begin{aligned}
\left(\overline{D}_{\overline{i}}D_j V_{BH}\right)_{\left(z_{susy},\overline{z}_{susy}\right)} &= \left(\overline{\partial}_{\overline{i}}\partial_j V_{BH}\right)_{\left(z_{susy},\overline{z}_{susy}\right)} = \\
&= 2\left|Z\right|^2\left(z_{susy},\overline{z}_{susy};p,q\right)G_{j\overline{i}}\left(z_{susy},\overline{z}_{susy}\right).
\end{aligned}
\tag{4.4.1.27}
$$

As previously done for the function $|Z|$, it is now possible to introduce the $2n_V \times 2n_V$ complex Hessian matrix $H_{\hat{i}\hat{j}}^{V_{BH}}$ of the function $V_{BH}\left(z,\overline{z};p,q\right)$ in $\mathcal{M}_{z,\overline{z}}$, as follows:

$$H_{\hat{i}\hat{j}}^{V_{BH}}(z,\overline{z};p,q)=\begin{pmatrix} H_{ij}^{V_{BH}} & H_{i\overline{j}}^{V_{BH}} \\ \\ H_{j\overline{i}}^{V_{BH}} & H_{\overline{i}\overline{j}}^{V_{BH}} \end{pmatrix}$$

$$\equiv\begin{pmatrix} D_iD_jV_{BH} & D_i\overline{D}_{\overline{j}}V_{BH} \\ \\ D_j\overline{D}_{\overline{i}}V_{BH} & \overline{D}_{\overline{i}}\overline{D}_{\overline{j}}V_{BH} \end{pmatrix}=\begin{pmatrix} D_iD_jV_{BH} & \overline{\overline{D}_{\overline{i}}D_jV_{BH}} \\ \\ \overline{D}_{\overline{i}}D_jV_{BH} & \overline{D_iD_jV_{BH}} \end{pmatrix}$$

$$=2\begin{pmatrix} i\begin{bmatrix} 2C_{ijk}G^{k\overline{k}}\overline{Z}\overline{D}_{\overline{k}}\overline{Z}+ \\ +\frac{1}{2}D_{(i}C_{j)kl}\cdot \\ \cdot G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{l}}\overline{Z} \end{bmatrix} & \begin{bmatrix} \left(D_iZ\right)\overline{D}_{\overline{j}}\overline{Z}+G_{i\overline{j}}\left|Z\right|^2 \\ +\overline{C}_{\overline{j}\overline{k}\overline{l}}C_{imk}G^{m\overline{k}}G^{l\overline{l}}G^{k\overline{n}}\cdot \\ \cdot\left(\overline{D}_{\overline{n}}\overline{Z}\right)D_lZ \end{bmatrix} \\ \\ \begin{bmatrix} \left(\overline{D}_{\overline{i}}\overline{Z}\right)D_jZ+G_{j\overline{i}}\left|Z\right|^2 \\ +C_{jkl}\overline{C}_{\overline{i}\overline{m}\overline{k}}G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{k}}\cdot \\ \cdot\left(D_nZ\right)\overline{D}_{\overline{l}}\overline{Z} \end{bmatrix} & -i\begin{bmatrix} 2\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{k\overline{k}}ZD_kZ+ \\ +\frac{1}{2}\overline{D}_{(\overline{i}}\overline{C}_{\overline{j})\overline{k}\overline{l}} \\ \cdot G^{m\overline{k}}G^{l\overline{l}}\left(D_mZ\right)D_lZ \end{bmatrix} \end{pmatrix}. \tag{4.4.1.28}$$

Thus, by evaluating at the point(s) $\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right)$ in $\mathcal{M}_{z,\overline{z}}$ defined by (4.4.1.1) and recalling (4.4.1.15), the Hessian becomes

$$H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)$$

$$=2\left|Z\right|^2\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)$$

$$
\cdot \begin{pmatrix} 0 & G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \\ G_{j\bar{i}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) & 0 \end{pmatrix}
$$

$$
= 2V_{BH}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)
$$

$$
\cdot \begin{pmatrix} 0 & G_{i\bar{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \\ G_{j\bar{i}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) & 0 \end{pmatrix}. \tag{4.4.1.29}
$$

Since $\overline{G_{i\bar{j}}} = G_{j\bar{i}}$, also in this case we obtain

$$
\left(H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right)\right)^{\dagger} = H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right);p,q\right), \tag{4.4.1.30}
$$

i.e., the $2n_V \times 2n_V$ complex Hessian matrix $H^{V_{BH}}_{\hat{i}\hat{j}}$ evaluated at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor, point(s) $(z_{susy}, \overline{z}_{susy})$ in $\mathcal{M}_{z,\overline{z}}$ is Hermitian for any BH charge configuration. Thus, $H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right)$ is always diagonalizable by a unitary transformation, and it has $2n_V$ real eigenvalues; from (4.4.1.29) it then follows that, for an arbitrary but fixed BH charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right) \in \Gamma$,

$$
\begin{gathered}
G_{i\bar{j}}\left(z_{susy}, \overline{z}_{susy}\right) \text{ strictly positive (negative) definite} \\
\Updownarrow \\
\left(z_{susy}, \overline{z}_{susy}\right) \textit{ at least local } \text{miminum (maximum) of } V_{BH}\left(z, \overline{z}; p, q\right) \text{ in } \mathcal{M}_{z,\overline{z}}.
\end{gathered} \tag{4.4.1.31}
$$

Such a result also follows from the comparison of $H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right)$ (given by (4.4.1.12)) with $H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right)$ (given by (4.4.1.29)), yielding

$$
\begin{aligned}
H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right) &= 4\left|Z\right|\left(z_{susy}, \overline{z}_{susy}; p, q\right) H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right) \\
&= 4(V_{BH}\left(z_{susy}, \overline{z}_{susy}; p, q\right))^{1/2} H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy}, \overline{z}_{susy}; p, q\right),
\end{aligned} \tag{4.4.1.32}
$$

where in the last line we recalled (4.4.1.15).

As mentioned above, since we assume that the SKG of $\mathcal{M}_{z,\overline{z}}$ is regular, we obtain at least a local minimum of V_{BH} at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s). However, different situations may arise if we go beyond the regular regime of SKG; in such a case, (4.4.1) yields that the eventually existing (at least local) maxima of V_{BH} are reached out of the regular SKG of $\mathcal{M}_{z,\overline{z}}$.

Summarizing, in the context of regular SKG of $\mathcal{M}_{z,\overline{z}}$, all $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor points, defined by the differential (4.4.1.1) ($\forall i = 1, \ldots, n_V$),

$$\left(\partial_i \left|Z\right|\right)\left(z_{susy}, \overline{z}_{susy}; p, q\right) = 0 \tag{4.4.1.33}$$

$$\Updownarrow$$

$$\begin{aligned} &\left(D_i Z\right)\left(z_{susy}, \overline{z}_{susy}; p, q\right) \\ &= \left[\left(\partial_i Z\right)\left(z, \overline{z}; p, q\right) + \tfrac{1}{2}\left(\partial_i K\right)\left(z, \overline{z}\right) Z\left(z, \overline{z}; p, q\right)\right]_{\left(z_{susy}, \overline{z}_{susy}\right)} = 0 \end{aligned} \tag{4.4.1.34}$$

$$\Downarrow$$

$$\left(\partial_i V_{BH}\right)\left(z_{susy}, \overline{z}_{susy}; p, q\right) = 0, \tag{4.4.1.35}$$

are (at least local) minima[12] of both the real, positive functions $V_{BH}\left(z, \overline{z}; p, q\right)$ and $\left|Z\right|\left(z, \overline{z}; p, q\right)$, for the arbitrary but fixed BH charge configuration being considered.

However, if one considers only one (p, q)-parameterized continuous branch of $V_{BH}\left(z, \overline{z}; p, q\right)$ and $\left|Z\right|\left(z, \overline{z}; p, q\right)$ in $\mathcal{M}_{z,\overline{z}}$, then just one critical point

$$\left(z_{susy}\left(p, q\right), \overline{z}_{susy}\left(p, q\right)\right) = lim_{r \to r_H^+}\left(z\left(r\right), \overline{z}\left(r\right)\right) \tag{4.4.1.36}$$

exists as solution of the set of n_V complex differential equations (4.4.1.33)–(4.4.1.34), and it is a global minimum for the (p, q)-parameterized continuous branch of $V_{BH}\left(z, \overline{z}; p, q\right)$ and $\left|Z\right|\left(z, \overline{z}; p, q\right)$ in $\mathcal{M}_{z,\overline{z}}$.

Clearly, the situation changes if, for the considered $Sp\left(2n_V + 2\right)$-covariant BH charge configuration $\left(p^\Lambda, q_\Lambda\right) \in \Gamma$, more than one continuous branch of $V_{BH}\left(z, \overline{z}; p, q\right)$ and $\left|Z\right|\left(z, \overline{z}; p, q\right)$ may exist in $\mathcal{M}_{z,\overline{z}}$, or also if one considers not only the continuous branch(es) of V_{BH} and/or $\left|Z\right|$. In such cases, one would obtain that a variety of critical points may exist, corresponding to (at least local) minima of V_{BH} and $\left|Z\right|$ in $\mathcal{M}_{z,\overline{z}}$, in $1:1$ correspondence with possibly existing disconnected continuous branches of such functions, or in (not necessarily $1:1$) correspondence with eventually existing disconnected, noncontinuous branches of V_{BH} and $\left|Z\right|$.

[12] Since the real functions $\left|Z\right|$ and V_{BH} are (strictly) positive in $\mathcal{M}_{z,\overline{z}}$, their stable critical points are (at least local) minima. Because the attractor points are generally defined as stable critical points, it follows that in the considered framework of regular SKG of $\mathcal{M}_{z,\overline{z}}$ all critical points of $\left|Z\right|$ and V_{BH} are actually attractors.

In general, this does not continue to hold when the assumption of regularity of SKG is removed

Furthermore, by going beyond the regular SKG of $\mathcal{M}_{z,\overline{z}}$, and thus by admitting changes of definiteness of the Kählerian metric $G_{i\overline{j}}$, one would obtain various possible cases[13]:

1.

$$G_{i\overline{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \text{ strictly positive definite} \tag{4.4.1.37}$$

$$\Updownarrow$$

$$\left.\begin{array}{l} H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy},\overline{z}_{susy};p,q\right) \\ \\ H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy},\overline{z}_{susy};p,q\right) \end{array}\right\} \text{strictly positive definite} \tag{4.4.1.38}$$

$$\Updownarrow$$

$$\begin{array}{l} \left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) = lim_{r\rightarrow r_H^+}\left(z\left(r\right),\overline{z}\left(r\right)\right) \\ \text{(at least local) } \textit{minimum} \text{ for both } V_{BH} \text{ and } |Z| \text{ in } \mathcal{M}_{z,\overline{z}} \\ \text{(for the considered } \left(p^{\Lambda},q_{\Lambda}\right)\in\Gamma) \\ [\textit{proper } \tfrac{1}{2}\textit{-BPS supersymmetric extreme BH attractor}]; \end{array} \tag{4.4.1.39}$$

2.

$$G_{i\overline{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \text{ strictly negative definite} \tag{4.4.1.40}$$

$$\Updownarrow$$

$$\left.\begin{array}{l} H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{susy},\overline{z}_{susy};p,q\right) \\ \\ H^{|Z|}_{\hat{i}\hat{j}}\left(z_{susy},\overline{z}_{susy};p,q\right) \end{array}\right\} \text{strictly negative definite} \tag{4.4.1.41}$$

$$\Updownarrow$$

$$\begin{array}{l} \left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) = lim_{r\rightarrow r_H^+}\left(z\left(r\right),\overline{z}\left(r\right)\right) \\ \text{(at least local) } \textit{maximum} \text{ for both } V_{BH} \text{ and } |Z| \text{ in } \mathcal{M}_{z,\overline{z}} \\ \text{(for the considered } \left(p^{\Lambda},q_{\Lambda}\right)\in\Gamma); \end{array} \tag{4.4.1.42}$$

---

[13] Notice that the not strict positive (negative) definiteness of $G_{i\overline{j}}$ at the critical points $\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right)$ is only a necessary, but not necessarily a sufficient, condition for them to be (at least local) minima (maxima) for the functions V_{BH} and $|Z|$ in $\mathcal{M}_{z,\overline{z}}$.

Indeed, when the positive (negative) definiteness of $G_{i\overline{j}}$ is not strict, explicit counterexamples may be considered in which $\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right)$ is a saddle point for V_{BH} and $|Z|$. Thus, when the definiteness of $G_{i\overline{j}}$ is not strict, in order to discriminate between the different possibilities a more detailed investigation is needed, for instance consisting in the study of the function V_{BH} and/or $|Z|$ in a neighborhood of the considered critical point(s) $\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right)$

3.

$$\begin{array}{l} G_{i\overline{j}}\left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) \\ \text{neither positive nor negative definite} \\ \text{(i.e., it has some positive, some negative,} \\ \text{and possibly some vanishing, eigenvalues)} \end{array} \tag{4.4.1.43}$$

$$\Updownarrow$$

$$\left.\begin{array}{l} H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{susy},\overline{z}_{susy};p,q\right) \\ \\ H_{\hat{i}\hat{j}}^{|Z|}\left(z_{susy},\overline{z}_{susy};p,q\right) \end{array}\right\} \text{neither positive nor negative definite} \tag{4.4.1.44}$$

$$\Updownarrow$$

$$\begin{array}{l} \left(z_{susy}\left(p,q\right),\overline{z}_{susy}\left(p,q\right)\right) = lim_{r\rightarrow r_H^+}\left(z\left(r\right),\overline{z}\left(r\right)\right) \\ \textit{saddle point} \text{ for both } V_{BH} \text{ and } |Z| \text{ in } \mathcal{M}_{z,\overline{z}} \\ \text{(for the considered } \left(p^{\Lambda},q_{\Lambda}\right)\in\Gamma\text{).} \end{array} \tag{4.4.1.45}$$

Thus, when going beyond the regular SKG of the vector supermultiplets' moduli space $\mathcal{M}_{z,\overline{z}}$, one gets a much richer casistics, for example consisting in the possibility to have different maxima and minima, together with saddle points, also for only one (p,q)-parameterized continuous branch of the functions $V_{BH}\left(z,\overline{z};p,q\right)$ and $|Z|\left(z,\overline{z};p,q\right)$. In such a nonregular geometric framework, also disconnected and/or noncontinuous branch(es) of V_{BH} and $|Z|$ might be considered.

In [77] Kallosh et al. performed a detailed analysis of the issue of the uniqueness of the critical points of both V_{BH} and $|Z|$, not necessarily relying on the regularity of the Kähler geometry. They worked in the framework of $\mathcal{N}=2$, $d=5$ MESGT, whose moduli space is endowed with a "very special" (or "real special") Kähler geometry. An analogous approach in the corresponding 4-d framework of SKG of the vector supermultiplets' moduli space $\mathcal{M}_{z,\overline{z}}$ of n_V-fold $\mathcal{N}=2$, $d=4$ MESGT was sketchily outlined in [55].

4.4.2 Nonsupersymmetric Attractors

Let us now consider the case of the nonsupersymmetric, non-BPS (*NON-(BPS-)SUSY*) extreme BH attractors. They are stable critical points of $V_{BH}\left(z,\overline{z};p,q\right)$, but *not* of $|Z|\left(z,\overline{z};p,q\right)$, in $\mathcal{M}_{z,\overline{z}}$. In the considered context of n_V-fold $\mathcal{N}=2$, $d=4$ MESGT, their existence has been firstly pointed out in [55]; recently, they have been rediscovered and studied in a number of papers, also in not necessarily supersymmetric frameworks [61,66,71,72,78–80].

The elements of such a particular class of critical points of $V_{BH}(z,\overline{z};p,q)$ will be denoted as $(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))$. They are horizon, attractor vector supermultiplets' scalar configurations which do *not* preserve any supersymmetric degree of freedom out of the ones of the underlying n_V-fold $\mathcal{N}=2$, $d=4$ MESGT. By recalling (4.4.5), they are defined by the following set of differential conditions (also remind that we assume $Z(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q);p,q)\neq 0$):

$$\begin{cases}\begin{cases}(D_iV_{BH})(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q);p,q)\\ =\left[2\overline{Z}D_iZ+iC_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\right]_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))}=0,\\ \forall i\in\{1,\dots,n_V\};\end{cases}\\ \\ \begin{cases}(D_iZ)(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q);p,q)\neq 0\\ \Updownarrow\\ (\partial_i|Z|)(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q);p,q)\neq 0,\\ i\in\mathcal{I}\subseteq\{1,\dots,n_V\},\mathcal{I}\neq\emptyset,\end{cases}\end{cases}\tag{4.4.2.1}$$

which may be fully explicited as follows:

$$\begin{cases}\begin{cases}\left\{\begin{array}{l}2\overline{Z}(z,\overline{z};p,q)\left[\left(\partial_i+\frac{1}{2}\partial_iK(z,\overline{z})\right)Z(z,\overline{z};p,q)\right]\\ +iC_{ijk}(z,\overline{z})G^{j\overline{m}}(z,\overline{z})G^{k\overline{k}}(z,\overline{z})\\ \cdot\left[\left(\overline{\partial}_{\overline{m}}+\frac{1}{2}\overline{\partial}_{\overline{m}}K(z,\overline{z})\right)\overline{Z}(z,\overline{z};p,q)\right]\\ \cdot\left[\left(\overline{\partial}_{\overline{k}}+\frac{1}{2}\overline{\partial}_{\overline{k}}K(z,\overline{z})\right)\overline{Z}(z,\overline{z};p,q)\right]\end{array}\right\}_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))}=0,\\ \forall i\in\{1,\dots,n_V\};\end{cases}\\ \\ \begin{cases}\left\{\left(\partial_i+\frac{1}{2}\partial_iK(z,\overline{z})\right)Z(z,\overline{z};p,q)\right\}_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))}\neq 0;\\ \Updownarrow\\ \left\{\partial_i\left(|Z|(z,\overline{z};p,q)\right)\right\}_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))}\neq 0;\\ i\in\mathcal{I}\subseteq\{1,\dots,n_V\},\mathcal{I}\neq\emptyset.\end{cases}\end{cases}\tag{4.4.2.2}$$

Thus, by inserting the explicit expressions of $K(z,\overline{z})$, $C_{ijk}(z,\overline{z})$, and $Z(z,\overline{z};p,q)$ as input,[14] the set of differential conditions (4.4.2.1) and (4.4.2.2) should give, as output, the purely charge-dependent *NON-(BPS-)SUSY* extreme BH attractors

$$\left(z_{non-susy}(p,q), \overline{z}_{non-susy}(p,q)\right) = lim_{r \to r_H^+} \left(z(r), \overline{z}(r)\right). \tag{4.4.2.3}$$

Let us now reconsider the condition of criticality for $V_{BH}(z,\overline{z};p,q)$ in $\mathcal{M}_{z,\overline{z}}$. From (4.4.5) it reads ($\forall i \in \{1,\ldots,n_V\}$)

$$2\overline{Z}D_iZ = -iC_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right); \tag{4.4.2.4}$$

$$\Updownarrow$$

$$D_iZ = -\frac{i}{2}\frac{Z}{|Z|^2}C_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right); \tag{4.4.2.5}$$

$$\Updownarrow$$

$$\overline{D}_{\overline{i}}\overline{Z} = \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{ijk}}G^{m\overline{j}}G^{k\overline{k}}\left(D_mZ\right)\left(D_kZ\right). \tag{4.4.2.6}$$

By using (4.4.2.4)–(4.4.2.6), the evaluation of (4.4.1.17) at the critical points of $V_{BH}(z,\overline{z};p,q)$ in $\mathcal{M}_{z,\overline{z}}$ yields

$$D_iD_jV_{BH}\big|_{\partial_rV_{BH}=0,\ \forall r\in\{1,\ldots,n_V\}}$$

$$= 2i\left[\begin{array}{l} 2C_{ijk}G^{k\overline{k}}\overline{Z}\overline{D}_{\overline{k}}\overline{Z} \\ \\ +\frac{1}{2}\left(D_{(i}C_{j)kl}\right)G^{k\overline{m}}G^{l\overline{l}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{l}}\overline{Z} \end{array}\right]_{\partial_rV_{BH}=0,\ \forall r\in\{1,\ldots,n_V\}}$$

$$= 2i\left[\begin{array}{l} iG^{k\overline{k}}C_{ijk}\overline{C}_{\overline{klm}}\frac{\overline{Z}^2}{|Z|^2}G^{l\overline{l}}G^{m\overline{m}}\left(D_lZ\right)D_mZ \\ \\ -\frac{1}{8}\frac{\overline{Z}^2}{|Z|^4}\left(D_{(i}C_{j)kl}\right)\overline{C}_{\overline{mnp}}\overline{C}_{\overline{lqr}} \\ \\ \cdot G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}G^{r\overline{r}}\left(D_nZ\right)\left(D_pZ\right)\left(D_qZ\right)D_rZ \end{array}\right]_{\partial_rV_{BH}=0,\ \forall r\in\{1,\ldots,n_V\}}$$

[14] It is worth recalling once again that, beside $G_{i\overline{j}} = \partial_{\overline{j}}\partial_iK$, the Kähler potential K also determines the contravariant metric by the orthonormality condition

$$G^{i\overline{j}}\partial_{\overline{j}}\partial_lK = \delta^i_l.$$

$$= 2i \left\{ \frac{\overline{Z}^2}{|Z|^2} (D_n Z)(D_p Z) \cdot \left[\begin{array}{l} i\left(\delta_i^n \delta_j^p + \delta_i^p \delta_j^n - G^{n\overline{n}} G^{p\overline{p}} R_{i\overline{n}j\overline{p}}\right) \\ -\frac{1}{8|Z|^2} \left(D_{(i} C_{j)kl}\right) \overline{C}_{\overline{mnp}} \overline{C}_{\overline{lqr}} \\ \cdot G^{k\overline{m}} G^{l\overline{l}} G^{n\overline{n}} G^{p\overline{p}} G^{q\overline{q}} G^{r\overline{r}} (D_q Z) D_r Z \end{array} \right] \right\}_{\partial_r V_{BH}=0,\ \forall r \in \{1,\dots,n_V\}}$$

$$= 2i \left\{ \frac{\overline{Z}^2}{|Z|^2} (D_n Z)(D_p Z) \cdot \left[\begin{array}{l} i\left(\delta_i^n \delta_j^p + \delta_i^p \delta_j^n - G^{n\overline{n}} G^{p\overline{p}} R_{i\overline{n}j\overline{p}}\right) \\ +\frac{1}{8|Z|^2} \left[D_{(i|}\left(G^{l\overline{l}} G^{n\overline{n}} G^{p\overline{p}} G^{q\overline{q}} G^{r\overline{r}} R_{|j)\overline{n}l\overline{p}} \overline{C}_{\overline{lqr}}\right)\right] (D_q Z) D_r Z \end{array} \right] \right\}_{\partial_r V_{BH}=0,\ \forall r \in \{1,\dots,n_V\}} . \tag{4.4.2.7}$$

In the last line of (4.4.2.7) we used the result

$$\begin{aligned} &(D_i C_{jkl}) \overline{C}_{\overline{mnp}} \overline{C}_{\overline{lqr}} G^{k\overline{m}} G^{l\overline{l}} G^{n\overline{n}} G^{p\overline{p}} G^{q\overline{q}} G^{r\overline{r}} \\ &= -D_{(i|}\left(G^{l\overline{l}} G^{n\overline{n}} G^{p\overline{p}} G^{q\overline{q}} G^{r\overline{r}} R_{|j)\overline{n}l\overline{p}} \overline{C}_{\overline{lqr}}\right), \end{aligned} \tag{4.4.2.8}$$

following from the metric postulate in $\mathcal{M}_{z,\overline{z}}$, from the SKG constraints (3.1.3) and from the Bianchi identities (3.1.6) for the Riemann–Christoffel tensor.

On the other hand, by means of the criticality conditions (4.4.2.4)–(4.4.2.6) of V_{BH} (holding for nonvanishing Z, as we assumed) and recalling (4.4.1.19), we can express $\overline{D}_{\overline{i}} D_j V_{BH}$ at the critical points by using either holomorphic or anti-holomorphic Kähler-covariant derivatives of Z, respectively as it follows:

$$\left.\overline{D}_{\overline{i}} D_j V_{BH}\right|_{\partial_r V_{BH}=0,\ \forall r \in \{1,\dots,n_V\}} = 2 \left[\begin{array}{l} G_{j\overline{i}} |Z|^2 \\ + \left(2\delta_j^n \delta_{\overline{i}}^{\overline{l}} + G^{n\overline{l}} G_{j\overline{i}} - G^{l\overline{l}} G^{n\overline{k}} R_{j\overline{i}l\overline{k}}\right) (D_n Z) \overline{D}_{\overline{l}} \overline{Z} \end{array} \right]_{\partial_r V_{BH}=0,\ \forall r \in \{1,\dots,n_V\}}$$

$$
= 2 \left[\begin{array}{l} G_{j\bar{i}} |Z|^2 \\ \\ + \frac{i}{2} \frac{\overline{Z}}{|Z|^2} \left(2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}} G_{j\bar{i}} - G^{l\bar{l}} G^{n\bar{k}} R_{j\bar{i}l\bar{k}} \right) \\ \\ \cdot \overline{C}_{\bar{l}\overline{mp}} G^{m\overline{m}} G^{p\overline{p}} \left(D_n Z \right) \left(D_m Z \right) D_p Z \end{array} \right]_{\partial_r V_{BH}=0,\ \forall r \in \{1,\ldots,n_V\}}
$$

$$
= 2 \left[\begin{array}{l} G_{j\bar{i}} |Z|^2 \\ \\ - \frac{i}{2} \frac{Z}{|Z|^2} \left(2\delta_j^n \delta_{\bar{i}}^{\bar{l}} + G^{n\bar{l}} G_{j\bar{i}} - G^{l\bar{l}} G^{n\bar{k}} R_{j\bar{i}l\bar{k}} \right) \\ \\ \cdot C_{nrs} G^{r\overline{r}} G^{s\overline{s}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \left(\overline{D}_{\overline{r}} \overline{Z} \right) \overline{D}_{\overline{s}} \overline{Z} \end{array} \right]_{\partial_r V_{BH}=0,\ \forall r \in \{1,\ldots,n_V\}} . \tag{4.4.2.9}
$$

Notice that expressions (4.4.2.7) and (4.4.2.9) are manifestly symmetric, as, in general, it holds true for (4.4.1.17), (4.4.1.18), (4.4.1.19), and (4.4.1.25).

In general, (4.4.2.7) and (4.4.2.9) hold for every critical point of the function $V_{BH}(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$. In the case of $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s) (which, by definition (4.4.1.1) are also critical points of the function $|Z|(z, \overline{z}; p, q)$), such equations reduce to the much simpler expressions (4.4.1.26) and (4.4.1.27), respectively. Thus, since we already treated the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractors in Subsect. 4.4.1, we will here understand (4.4.2.7) and (4.4.2.9) in their nontrivial form in n_V-fold $\mathcal{N} = 2$, $d = 4$ MESGT, i.e., evaluated at the *NON-(BPS-)SUSY* extreme BH attractor(s) which, by definitions (4.4.2.1) and (4.4.2.2), are critical points of V_{BH}, but *not* of $|Z|$.

Thus, by evaluating the Hessian $H_{\hat{i}\hat{j}}^{V_{BH}}(z, \overline{z}; p, q)$ at the point(s) $(z_{non-susy}(p, q), \overline{z}_{non-susy}(p, q))$ in $\mathcal{M}_{z,\overline{z}}$ defined by the differential conditions (4.4.2.1) and (4.4.2.2), we get

$$H^{V_{BH}}_{\hat{i}\hat{j}}\left(z_{non-susy}\left(p,q\right),\overline{z}_{non-susy}\left(p,q\right);p,q\right)=$$

$$=\begin{pmatrix}
\begin{array}{l} 2i\frac{\overline{Z}^2}{|Z|^2}\left(D_nZ\right)\left(D_pZ\right)\cdot \\ \cdot\left[\begin{array}{l} i\left(\begin{array}{l}\delta_i^n\delta_j^p+\delta_i^p\delta_j^n+\\ -G^{n\overline{n}}G^{p\overline{p}}R_{i\overline{n}j\overline{p}}\end{array}\right)+\\ -\frac{1}{8|Z|^2}\left(D_{(i}C_{j)kl}\right)\overline{C}_{\overline{mnp}}\overline{C}_{\overline{lqr}}\cdot\\ \cdot G^{k\overline{m}}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}\cdot\\ \cdot G^{q\overline{q}}G^{r\overline{r}}\left(D_qZ\right)D_rZ\end{array}\right]\end{array}
&
2\left[\begin{array}{l} G_{i\overline{j}}\left|Z\right|^2+\\ -\frac{i}{2}\frac{Z}{|Z|^2}\cdot\\ \cdot\left(\begin{array}{l}2\delta_{\overline{j}}^{\overline{n}}\delta_i^l+G^{l\overline{n}}G_{i\overline{j}}+\\ -G^{l\overline{l}}G^{k\overline{n}}R_{i\overline{j}k\overline{l}}\end{array}\right)\cdot\\ \cdot C_{lmp}G^{m\overline{m}}G^{p\overline{p}}\cdot\\ \cdot\left(\overline{D}_{\overline{n}}\overline{Z}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{p}}\overline{Z}\end{array}\right]
\\ \\
2\left[\begin{array}{l} G_{j\overline{i}}\left|Z\right|^2+\\ +\frac{i}{2}\frac{\overline{Z}}{|Z|^2}\cdot\\ \cdot\left(\begin{array}{l}2\delta_j^n\delta_{\overline{i}}^{\overline{l}}+G^{n\overline{l}}G_{j\overline{i}}+\\ -G^{l\overline{l}}G^{n\overline{k}}R_{j\overline{i}l\overline{k}}\end{array}\right)\cdot\\ \cdot\overline{C}_{\overline{lmp}}G^{m\overline{m}}G^{p\overline{p}}\cdot\\ \cdot\left(D_nZ\right)\left(D_mZ\right)D_pZ\end{array}\right]
&
\begin{array}{l} -2i\frac{Z^2}{|Z|^2}\left(\overline{D}_{\overline{n}}\overline{Z}\right)\left(\overline{D}_{\overline{p}}\overline{Z}\right)\cdot\\ \cdot\left[\begin{array}{l} -i\left(\begin{array}{l}\delta_{\overline{i}}^{\overline{n}}\delta_{\overline{j}}^{\overline{p}}+\delta_{\overline{i}}^{\overline{p}}\delta_{\overline{j}}^{\overline{n}}+\\ -G^{n\overline{n}}G^{p\overline{p}}R_{n\overline{i}p\overline{j}}\end{array}\right)+\\ -\frac{1}{8|Z|^2}\left(\overline{D}_{(\overline{i}}\overline{C}_{\overline{j})\overline{kl}}\right)C_{mnp}C_{lqr}\cdot\\ \cdot G^{m\overline{k}}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}\cdot\\ \cdot G^{q\overline{q}}G^{r\overline{r}}\left(\overline{D}_{\overline{q}}\overline{Z}\right)\overline{D}_{\overline{r}}\overline{Z}\end{array}\right]\end{array}
\end{pmatrix}_{non-susy}= \tag{4.4.2.10}$$

$$
= \left(\begin{array}{cc}
\begin{array}{l}
2i\frac{\overline{Z}^2}{|Z|^2}\left(D_nZ\right)\left(D_pZ\right)\cdot \\
\cdot\left[\begin{array}{l}
i\left(\begin{array}{l}\delta_i^n\delta_j^p+\delta_i^p\delta_j^n+ \\ -G^{n\overline{n}}G^{p\overline{p}}R_{i\overline{n}j\overline{p}}\end{array}\right)+ \\
+\frac{1}{8|Z|^2}\cdot \\
\cdot\left[D_{(i|}\left(\begin{array}{l}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}\cdot \\ \cdot G^{r\overline{r}}R_{|j)\overline{n}l\overline{p}}\overline{C}_{\overline{l}\overline{q}\overline{r}}\end{array}\right)\right]\cdot \\
\cdot\left(D_qZ\right)D_rZ
\end{array}\right]
\end{array}
&
2\left[\begin{array}{l}
G_{i\overline{j}}|Z|^2+ \\
+\frac{i}{2}\frac{\overline{Z}}{|Z|^2}\cdot \\
\cdot\left(\begin{array}{l}2\delta_{\overline{j}}^{\overline{n}}\delta_i^l+G^{l\overline{n}}G_{i\overline{j}}+ \\ -G^{l\overline{l}}G^{k\overline{n}}R_{i\overline{j}k\overline{l}}\end{array}\right)\cdot \\
\cdot\overline{C}_{\overline{n}\overline{r}\overline{s}}G^{r\overline{r}}G^{s\overline{s}}\cdot \\
\cdot\left(D_lZ\right)\left(D_rZ\right)D_sZ
\end{array}\right]
\\
2\left[\begin{array}{l}
G_{j\overline{i}}|Z|^2+ \\
-\frac{i}{2}\frac{Z}{|Z|^2}\cdot \\
\cdot\left(\begin{array}{l}2\delta_j^n\delta_{\overline{i}}^{\overline{l}}+G^{n\overline{l}}G_{j\overline{i}}+ \\ -G^{l\overline{l}}G^{n\overline{k}}R_{j\overline{i}l\overline{k}}\end{array}\right)\cdot \\
\cdot C_{nrs}G^{r\overline{r}}G^{s\overline{s}}\cdot \\
\cdot\left(\overline{D}_{\overline{l}}\overline{Z}\right)\left(\overline{D}_{\overline{r}}\overline{Z}\right)\overline{D}_{\overline{s}}\overline{Z}
\end{array}\right]
&
\begin{array}{l}
-2i\frac{Z^2}{|Z|^2}\left(\overline{D}_{\overline{n}}\overline{Z}\right)\left(\overline{D}_{\overline{p}}\overline{Z}\right)\cdot \\
\cdot\left[\begin{array}{l}
-i\left(\begin{array}{l}\delta_{\overline{i}}^{\overline{n}}\delta_{\overline{j}}^{\overline{p}}+\delta_{\overline{i}}^{\overline{p}}\delta_{\overline{j}}^{\overline{n}}+ \\ -G^{n\overline{n}}G^{p\overline{p}}R_{n\overline{i}p\overline{j}}\end{array}\right)+ \\
+\frac{1}{8|Z|^2}\cdot \\
\cdot\left[\overline{D}_{(\overline{i}|}\left(\begin{array}{l}G^{l\overline{l}}G^{n\overline{n}}G^{p\overline{p}}G^{q\overline{q}}\cdot \\ \cdot G^{r\overline{r}}R_{n|\overline{j})p\overline{l}}C_{lqr}\end{array}\right)\right]\cdot \\
\cdot\left(\overline{D}_{\overline{q}}\overline{Z}\right)\overline{D}_{\overline{r}}\overline{Z}
\end{array}\right]
\end{array}
\end{array} \right)_{non-susy}, \tag{4.4.2.11}
$$

where the subscript "$non-susy$" means that everything inside the matrix is evaluated at the point(s) $\left(z_{non-susy}(p,q), \overline{z}_{non-susy}(p,q)\right)$ in $\mathcal{M}_{z,\overline{z}}$ defined by the differential conditions (4.4.2.1) and (4.4.2.2).

It is worth pointing out that at the critical points of V_{BH} the Kähler-covariant Hessian of V_{BH} coincides with the "flat," ordinary Hessian, defined through ordinary derivatives:

$$
\begin{aligned}
&H_{\hat{i}\hat{j}}^{V_{BH}}\left(z,\overline{z};p,q\right)\Big|_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}}\\
&=\left(\begin{array}{cc} H_{ij}^{V_{BH}} & H_{i\overline{j}}^{V_{BH}} \\ \\ H_{j\overline{i}}^{V_{BH}} & H_{\overline{i}\overline{j}}^{V_{BH}} \end{array}\right)_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}}\\
&\equiv\left(\begin{array}{cc} D_iD_jV_{BH} & D_i\overline{D}_{\overline{j}}V_{BH} \\ \\ D_j\overline{D}_{\overline{i}}V_{BH} & \overline{D}_{\overline{i}}\overline{D}_{\overline{j}}V_{BH} \end{array}\right)_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}}\\
&=\left(\begin{array}{cc} \partial_i\partial_jV_{BH} & \partial_i\overline{\partial}_{\overline{j}}V_{BH} \\ \\ \partial_j\overline{\partial}_{\overline{i}}V_{BH} & \overline{\partial}_{\overline{i}}\overline{\partial}_{\overline{j}}V_{BH} \end{array}\right)_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}}.
\end{aligned}
\tag{4.4.2.12}
$$

This is clearly due to the fact that the (regular) special Kähler moduli space $\mathcal{M}_{z,\overline{z}}$ is linearly connected (see (4.4.1.20) and (4.4.1.21)).

Now, by knowing the explicit expressions of the functions $Z\left(z,\overline{z};p,q\right)$, $K\left(z,\overline{z}\right)$, and $C_{ijk}\left(z,\overline{z}\right)$, and by solving the differential conditions (4.4.2.1) and (4.4.2.2), one should explicitly calculate the Hessian $H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{non-susy}\left(p,q\right),\right.$ $\left.\overline{z}_{non-susy}\left(p,q\right);p,q\right)$ given by (4.4.2.10) and (4.4.2.11) and study, case by case (if more than one solution exists to (4.4.2.1) and (4.4.2.2)), the definiteness of such an Hessian, i.e., the sign of its eigenvalues.

By denoting $H_{\hat{i}\hat{j}}^{V_{BH}}\left(z_{non-susy}\left(p,q\right),\overline{z}_{non-susy}\left(p,q\right);p,q\right)\equiv H_{\hat{i}\hat{j},non-susy}^{V_{BH}}$ $\left(p,q\right)$, it is immediate to check that the Hessian $H_{\hat{i}\hat{j},non-susy}^{V_{BH}}\left(p,q\right)$ is not Hermitian:

$$
\begin{aligned}
\left(H_{\hat{i}\hat{j},non-susy}^{V_{BH}}\left(p,q\right)\right)^{\dagger}&=\left(\begin{array}{cc} H_{\overline{i}\overline{j},non-susy}^{V_{BH}}\left(p,q\right) & H_{i\overline{j},non-susy}^{V_{BH}}\left(p,q\right) \\ \\ H_{j\overline{i},non-susy}^{V_{BH}}\left(p,q\right) & H_{ij,non-susy}^{V_{BH}}\left(p,q\right) \end{array}\right)\\
&\neq\left(\begin{array}{cc} H_{ij,non-susy}^{V_{BH}}\left(p,q\right) & H_{i\overline{j},non-susy}^{V_{BH}}\left(p,q\right) \\ \\ H_{j\overline{i},non-susy}^{V_{BH}}\left(p,q\right) & H_{\overline{i}\overline{j},non-susy}^{V_{BH}}\left(p,q\right) \end{array}\right)\\
&=H_{\hat{i}\hat{j},non-susy}^{V_{BH}}\left(p,q\right).
\end{aligned}
\tag{4.4.2.13}
$$

Such a nonhermiticity is, in general, due to the diagonal terms of the above block-diagonal arrangement, i.e., essentially to the $n_V \times n_V$ matrix $H^{V_{BH}}_{ij,non-susy}(p,q)$, which is symmetric but, in general, not real, and therefore not Hermitian (see also [72]).

Let us now evaluate the BH effective potential function V_{BH} at its critical points. By using (4.4.4), (4.4.5), and (4.4.2.4)–(4.4.2.6), one gets that the (semiclassical, leading order) BH entropy reads

$$
\begin{aligned}
S_{BH} &= \pi\, V_{BH}\big|_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}} \\
&= \pi \left\{ |Z|^2 + G^{i\bar{i}} \left(D_i Z\right) \overline{D}_{\bar{i}}\overline{Z} \right\}_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}} \\
&= \pi \left\{ \begin{array}{l} |Z|^2 \\ +\frac{1}{4}\frac{G^{i\bar{i}}}{|Z|^2} C_{ijk}\overline{C}_{\bar{i}\bar{j}\bar{n}} G^{j\overline{m}} G^{k\bar{k}} G^{m\bar{j}} G^{n\bar{n}} \left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(D_m Z\right) D_n Z \end{array} \right\}_{\substack{\partial_r V_{BH}=0,\\ \forall r\in\{1,\dots,n_V\}}} \\
&= \pi \left\{ \begin{array}{l} |Z|^2 \\ +\frac{1}{4|Z|^2} C^{\bar{i}\overline{m}\bar{k}}\overline{C}_{\bar{i}}{}^{mn} \left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(D_m Z\right) D_n Z \end{array} \right\}_{\partial_r V_{BH}=0,\ \forall r\in\{1,\dots,n_V\}} .
\end{aligned}
\tag{4.4.2.14}
$$

Thus, the (semiclassical, leading order) BH entropy at the *NON-(BPS-)SUSY* extreme BH attractor(s) is

$$
\begin{aligned}
S_{BH,non-susy} &\equiv S_{BH}\left(z_{non-susy}(p,q), \overline{z}_{non-susy}(p,q); p, q\right) \\
&= \pi \left\{ |Z|^2 + \frac{1}{4|Z|^2} C^{\bar{i}\overline{m}\bar{k}}\overline{C}_{\bar{i}}{}^{mn} \left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(D_m Z\right) D_n Z \right\}_{non-susy} \\
&= \pi \left\{ |Z|^2 + \frac{1}{4|Z|^2} \left| C_{ijk} G^{j\overline{m}} G^{k\bar{k}} \left(\overline{D}_{\overline{m}}\overline{Z}\right) \overline{D}_{\bar{k}}\overline{Z} \right|^2 \right\}_{non-susy} ,
\end{aligned}
\tag{4.4.2.15}
$$

where the subscript "*non* − *susy*" in the r.h.s. has the same meaning as above.

$\left| C_{ijk} G^{j\overline{m}} G^{k\bar{k}} \left(\overline{D}_{\overline{m}}\overline{Z}\right) \overline{D}_{\bar{k}}\overline{Z} \right|^2$ is the square norm of the complex, Kähler gauge-invariant covariant vector $C_{ijk} G^{j\overline{m}} G^{k\bar{k}} \left(\overline{D}_{\overline{m}}\overline{Z}\right) \overline{D}_{\bar{k}}\overline{Z}$ in $\mathcal{M}_{z,\overline{z}}$. Since we assume the SKG of $\mathcal{M}_{z,\overline{z}}$ to be regular, i.e., that the metric tensor $G_{i\overline{j}}$ is strictly positive definite in all $\mathcal{M}_{z,\overline{z}}$, it holds true that

$$\left|C_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{k}}\overline{Z}\right|^2 \equiv C^{\overline{i}\overline{m}\overline{k}}\overline{C}_{\overline{i}}{}^{mn}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(D_m Z\right)D_n Z \geqslant 0, \tag{4.4.2.16}$$

vanishing iff

$$C_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\overline{D}_{\overline{k}}\overline{Z} = 0, \forall i \in \{1,\ldots,n_V\}. \tag{4.4.2.17}$$

Notice that condition (4.4.2.17) is trivially satisfied at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s) defined by the differential conditions (4.4.1.1). However, it might happen also that, depending on the BH charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right) \in \Gamma$ and on the explicit expressions of C_{ijk}, K and Z, condition (4.4.2.17) is satisfied at some particular *NON-(BPS-)SUSY* extreme BH attractor(s).

Thus, by recalling (4.4.1.16) one gets that the BH entropy $S_{BH,non-susy}$ at the *NON-(BPS-)SUSY* extreme BH attractor(s) is larger than the entropy $S_{BH,susy}$ at the $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractor point(s) (having the same $|Z|^2$). In other words, by assuming

$$\begin{aligned}&|Z|^2\left(z_{susy}(p,q), \overline{z}_{susy}(p,q); p,q\right)\\ &= |Z|^2\left(z_{non-susy}(p,q), \overline{z}_{non-susy}(p,q); p,q\right) \equiv |Z|^{\prime 2}_{cr}(p,q),\end{aligned} \tag{4.4.2.18}$$

it holds that

$$\begin{aligned}&\Delta(p,q) \equiv S_{BH,non-susy} - S_{BH,susy}\\ &= \frac{\pi}{4}\left[\frac{1}{|Z|^2}C^{\overline{i}\overline{m}\overline{k}}\overline{C}_{\overline{i}}{}^{mn}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\left(D_m Z\right)D_n Z\right]_{non-susy} \geqslant 0.\end{aligned} \tag{4.4.2.19}$$

The above expressions can be further elaborated by using the SKG constraints expressed by (3.1.3).

Consequently, at the *NON-(BPS-)SUSY* extreme BH attractor(s) it holds that

$$
\begin{aligned}
&\left[G^{i\bar{i}}\left(D_iZ\right)\overline{D}_{\bar{i}}\overline{Z}\right]_{non-susy}\\
&=\left[\frac{1}{4}\frac{G^{i\bar{i}}}{|Z|^2}C_{ilm}\overline{C}_{\bar{i}\overline{n}\overline{p}}G^{l\bar{l}}G^{m\overline{m}}G^{n\overline{n}}G^{p\overline{p}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(D_nZ\right)D_pZ\right]_{non-susy}\\
&=\left[\begin{array}{l}\frac{1}{4|Z|^2}\left(G_{l\overline{n}}G_{m\overline{p}}+G_{l\overline{p}}G_{m\overline{n}}-R_{l\overline{n}m\overline{p}}\right)\\ \cdot G^{l\bar{l}}G^{m\overline{m}}G^{n\overline{n}}G^{p\overline{p}}\left(\overline{D}_{\bar{l}}\overline{Z}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(D_nZ\right)D_pZ\end{array}\right]_{non-susy}\\
&=\left[\frac{1}{4|Z|^2}\left(G^{n\bar{l}}G^{p\overline{m}}+G^{n\overline{m}}G^{p\bar{l}}-R^{l\overline{n}m\overline{p}}\right)\left(\overline{D}_{\bar{l}}\overline{Z}\right)\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(D_nZ\right)D_pZ\right]_{non-susy}\\
&=\left\{\frac{1}{4|Z|^2}\left[2\left(G^{i\bar{i}}D_iZ\overline{D}_{\bar{i}}\overline{Z}\right)^2-R_{l\overline{n}m\overline{p}}\left(\overline{D}^l\overline{Z}\right)\left(\overline{D}^m\overline{Z}\right)\left(D^{\overline{n}}Z\right)D^{\overline{p}}Z\right]\right\}_{non-susy}.
\end{aligned}
\tag{4.4.2.20}
$$

Now, by recalling that in a (commutative) Kähler manifold the completely covariant Riemann–Christoffel tensor $R_{i\bar{j}l\overline{m}}$ is given by (3.1.8) and the SKG constraints may correspondingly be rewritten as in (3.1.9)–(3.1.11), the obtained result may be further elaborated by writing

$$
\begin{aligned}
&\left[G^{i\bar{i}}D_iZ\overline{D}_{\bar{i}}\overline{Z}\right]_{non-susy}\\
&=\left\{\frac{1}{4|Z|^2}\left\{\begin{array}{l}2\left(G^{i\bar{i}}D_iZ\overline{D}_{\bar{i}}\overline{Z}\right)^2\\ -\left[\begin{array}{l}\overline{\partial}_{\overline{n}}\partial_l\overline{\partial}_{\overline{p}}\partial_mK\\ -G^{r\overline{s}}\left(\overline{\partial}_{\overline{s}}\partial_l\partial_mK\right)\left(\partial_r\overline{\partial}_{\overline{n}}\overline{\partial}_{\overline{p}}K\right)\end{array}\right]\left(\overline{D}^l\overline{Z}\right)\left(\overline{D}^m\overline{Z}\right)\left(D^{\overline{n}}Z\right)D^{\overline{p}}Z\end{array}\right\}\right\}_{non-susy}.
\end{aligned}
\tag{4.4.2.21}
$$

Summarizing, in the (regular) SKG of $\mathcal{M}_{z,\overline{z}}$, the following expressions for the (semiclassical, leading order) BH entropy $S_{BH} = \pi V_{BH}$ at the *NON-(BPS-)SUSY* extreme BH attractor(s) are equivalent:

$$
\begin{aligned}
&S_{BH,non-susy}\\
&=\pi\left[|Z|^2+G^{i\bar{i}}D_iZ\overline{D}_{\bar{i}}\overline{Z}\right]_{non-susy}\\
&=\pi\left\{\begin{array}{l}|Z|^2\\ +\frac{1}{4}\frac{G^{i\bar{i}}}{|Z|^2}C_{ijk}\overline{C}_{\bar{i}\bar{j}\overline{n}}G^{j\overline{m}}G^{k\bar{k}}G^{m\bar{j}}G^{n\overline{n}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\bar{k}}\overline{Z}\right)\left(D_mZ\right)D_nZ\end{array}\right\}_{non-susy}
\end{aligned}
$$

$$
= \pi \left\{ \begin{array}{l} |Z|^2 \\ + \frac{1}{4|Z|^2} \left[2 \left(G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z} \right)^2 - R_{l\overline{n}m\overline{p}} \left(\overline{D}^l \overline{Z} \right) \left(\overline{D}^m \overline{Z} \right) \left(D^{\overline{n}} Z \right) D^{\overline{p}} Z \right] \end{array} \right\}_{non-susy}
$$

$$
= \pi \left\{ \begin{array}{l} |Z|^2 + \frac{1}{4|Z|^2} G^{l\bar{l}} G^{m\overline{m}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \left(\overline{D}_{\overline{m}} \overline{Z} \right) \\ \cdot \left[2 \left(D_l Z \right) D_m Z - R_{l\overline{n}m\overline{p}} G^{n\overline{n}} G^{p\overline{p}} \left(D_n Z \right) D_p Z \right] \end{array} \right\}_{non-susy}
$$

$$
= \pi \left\{ \begin{array}{l} |Z|^2 + \frac{1}{4|Z|^2} G^{l\bar{l}} G^{m\overline{m}} \left(\overline{D}_{\bar{l}} \overline{Z} \right) \left(\overline{D}_{\overline{m}} \overline{Z} \right) \\ \cdot \left\{ \begin{array}{l} 2 \left(D_l Z \right) D_m Z \\ - \left[\overline{\partial}_{\overline{n}} \partial_l \overline{\partial}_{\overline{p}} \partial_m K - G^{r\overline{s}} \left(\overline{\partial}_{\overline{s}} \partial_l \partial_m K \right) \left(\partial_r \overline{\partial}_{\overline{n}} \overline{\partial}_{\overline{p}} K \right) \right] \\ \cdot G^{n\overline{n}} G^{p\overline{p}} \left(D_n Z \right) D_p Z \end{array} \right\} \end{array} \right\}_{non-susy} . \tag{4.4.2.22}
$$

It is interesting to analyze the case of only one complex modulus z more in depth (see [72]). In this case $i = 1$, $\bar{i} = \bar{1}$. Let us define the following quantities (assumed to be nonvanishing):

$$
\begin{array}{l} R_{1\bar{1}1\bar{1}} \equiv \mathcal{R} \in \mathbb{R}_0; \\ G_{1\bar{1}} = \partial_z \overline{\partial}_{\bar{z}} K \equiv \mathcal{G} \in \mathbb{R}_0^+; \\ C_{111} \equiv \mathcal{C} \in \mathbb{C}_0; \\ D_1 Z = D_z Z \equiv DZ \in \mathbb{C}_0; \end{array} \tag{4.4.2.23}
$$

notice that the regularity of the SKG of $\mathcal{M}_{z,\bar{z}}$ (in this case $\dim_{\mathbb{C}} \mathcal{M}_{z,\bar{z}} = 1$) implies the strict positivity of $G_{1\bar{1}}$, whose inverse is the unique component of the completely contravariant metric tensor

$$
G^{1\bar{1}} = \left(G_{1\bar{1}} \right)^{-1} = \left(\partial_z \overline{\partial}_{\bar{z}} K \right)^{-1} \equiv \mathcal{G}^{-1}. \tag{4.4.2.24}
$$

Moreover, in this case the defining conditions (4.4.2.1) and (4.4.2.2) read

$$
\left\{ \begin{array}{l} (DZ)_{non-susy} = -\frac{i}{2} \left[\frac{Z}{|Z|^2} \mathcal{C} \left(\partial_z \overline{\partial}_{\bar{z}} K \right)^{-2} \left(\overline{DZ} \right)^2 \right]_{non-susy}; \\ (DZ)_{non-susy} \neq 0. \end{array} \right. \tag{4.4.2.25}
$$

By expliciting DZ, condition (4.4.2.25) may be reformulated as the following differential condition on the Kähler potential $K(z, \bar{z})$:

$$0 \neq \left[\partial_z Z + \tfrac{1}{2}\left(\partial_z K\right) Z\right]_{non-susy}$$
$$= -\tfrac{i}{2}\left[\tfrac{Z}{|Z|^2}\mathcal{C}\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-2}\left(\overline{\partial}_{\overline{z}}\overline{Z} + \tfrac{1}{2}\left(\overline{\partial}_{\overline{z}}K\right)\overline{Z}\right)^2\right]_{non-susy}. \tag{4.4.2.26}$$

Thus, one gets

$$\begin{aligned}
&S_{BH,non-susy}\\
&= \pi\left[|Z|^2 + \mathcal{G}^{-1}|DZ|^2\right]_{non-susy}\\
&= \pi\left\{|Z|^2 + \tfrac{1}{4|Z|^2}\mathcal{G}^{-5}|\mathcal{C}|^2|DZ|^4\right\}_{non-susy}\\
&= \pi\left\{|Z|^2 + \tfrac{1}{4|Z|^2}\left[2\mathcal{G}^{-2}|DZ|^4 - \mathcal{G}^{-4}\mathcal{R}|DZ|^4\right]\right\}_{non-susy}\\
&= \pi\left\{|Z|^2 + \tfrac{1}{4|Z|^2}\mathcal{G}^{-2}|DZ|^4\left[2-\mathcal{G}^{-2}\mathcal{R}\right]\right\}_{non-susy}\\
&= \pi\left\{\begin{array}{l} |Z|^2 + \frac{1}{4|Z|^2}\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-2}|DZ|^4\\ \cdot\left[2-\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-2}\left(\partial_z^2\overline{\partial}_{\overline{z}}^2K - \left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-1}\left|\overline{\partial}_{\overline{z}}\partial_z^2K\right|^2\right)\right]\end{array}\right\}_{non-susy}.
\end{aligned} \tag{4.4.2.27}$$

The second and third lines of such an expression yield that

$$4\left.|Z|^2\right|_{non-susy} = \left(\mathcal{G}^{-4}|\mathcal{C}|^2|DZ|^2\right)_{non-susy}; \tag{4.4.2.28}$$
$$\Updownarrow$$
$$\left.|DZ|^2\right|_{non-susy} = 4\left(\frac{\mathcal{G}^4}{|\mathcal{C}|^2}|Z|^2\right)_{non-susy}. \tag{4.4.2.29}$$

By substituting such a result back into (4.4.2.27), one gets

$$\begin{aligned}
&S_{BH,non-susy}\\
&= \pi\left[|Z|^2\left(1+4\tfrac{\mathcal{G}^3}{|\mathcal{C}|^2}\right)\right]_{non-susy}\\
&= \pi\left\{|Z|^2\left[1+4\tfrac{\mathcal{G}^6}{|\mathcal{C}|^4}\left(2-\mathcal{G}^{-2}\mathcal{R}\right)\right]\right\}_{non-susy}\\
&= \pi\left\{|Z|^2\left\{\begin{array}{l} 1+4\frac{\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^6}{|\mathcal{C}|^4}\\ \cdot\left[2-\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-2}\left(\partial_z^2\overline{\partial}_{\overline{z}}^2K - \left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-1}\left|\overline{\partial}_{\overline{z}}\partial_z^2K\right|^2\right)\right]\end{array}\right\}\right\}_{non-susy}.
\end{aligned} \tag{4.4.2.30}$$

Thus, by considering two (sets of) *NON-(BPS-)SUSY* and $\frac{1}{2}$-*BPS-SUSY preserving* extreme BH attractors with the same $|Z|^2_{cr}(p,q)$ (see (4.4.2.18)), one obtains that

$$\begin{aligned}
0 &\leqslant \tfrac{\Delta(p,q)}{\pi|Z|^2_{cr}(p,q)}\\
&\equiv \tfrac{S_{BH,non-susy}-S_{BH,susy}}{\pi|Z|^2_{cr}(p,q)}\\
&= 4\left(\tfrac{\mathcal{G}^3}{|\mathcal{C}|^2}\right)_{non-susy} = 4\left[\tfrac{\mathcal{G}^6}{|\mathcal{C}|^4}\left(2-\mathcal{G}^{-2}\mathcal{R}\right)\right]_{non-susy}\\
&= 4\left\{\tfrac{\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^6}{|\mathcal{C}|^4}\left[2-\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-2}\left(\partial_z^2\overline{\partial}_{\overline{z}}^2K-\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-1}\left|\overline{\partial}_{\overline{z}}\partial_z^2K\right|^2\right)\right]\right\}_{non-susy}.
\end{aligned}\tag{4.4.2.31}$$

By recalling that we assume $Z\neq 0$ in all $\mathcal{M}_{z,\overline{z}}$, the positivity of $\Delta(p,q)$ implies that the following relations hold in all $\mathcal{M}_{z,\overline{z}}$:

$$\Delta(p,q)\geqslant 0;$$

$$\Updownarrow$$

$$\left.\mathcal{R}\right|_{non-susy}\leqslant 2\left.\mathcal{G}^2\right|_{non-susy};\tag{4.4.2.32}$$

$$\Updownarrow$$

$$\left[\partial_z^2\overline{\partial}_{\overline{z}}^2K-\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^{-1}\left|\overline{\partial}_{\overline{z}}\partial_z^2K\right|^2\right]_{non-susy}\leqslant 2\left.\left(\partial_z\overline{\partial}_{\overline{z}}K\right)^2\right|_{non-susy}.\tag{4.4.2.33}$$

By identifying the attractor points as fixed moduli configurations on the EH of an extremal BH, the differential conditions (4.4.1.1) and (4.4.2.1) and (4.4.2.2) defining the above treated two classes of extreme BH attractors may be considered as the differential form of the so-called attractor equations. They may respectively be rewritten as follows:

$$\forall i=1,\ldots,n_V:\quad
\begin{cases}
\left(D_iZ\right)\left(z_{susy}(p,q),\overline{z}_{susy}(p,q);p,q\right)=0;\\
\Updownarrow\\
\left\{\left[\partial_i+\frac{1}{2}\partial_iK(z,\overline{z})\right]Z(z,\overline{z};p,q)\right\}_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)}=0;\\
\Updownarrow\\
\left\{\left[\partial_i+\frac{1}{2}\partial_iK(z,\overline{z})\right]\left[q_\Lambda L^\Lambda(z,\overline{z})-p^\Lambda M_\Lambda(z,\overline{z})\right]\right\}_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)}=0;
\end{cases}\tag{4.4.2.34}$$

$$\begin{cases} \forall i = 1, \ldots, n_V : \\ \left(\partial_i V_{BH}\right)\left(z_{non-susy}\left(p,q\right), \overline{z}_{non-susy}\left(p,q\right); p,q\right) = 0; \\ \Updownarrow \\ \left[2\overline{Z}D_i Z + iC_{ijk}G^{j\overline{m}}G^{k\overline{k}}\left(\overline{D}_{\overline{m}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\right]_{\left(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q)\right)} = 0. \\ \\ i \in \mathcal{I} \subseteq \{1, ..., n_V\}, \mathcal{I} \neq \emptyset : \\ \left(D_i Z\right)\left(z_{non-susy}\left(p,q\right), \overline{z}_{non-susy}\left(p,q\right); p,q\right) \neq 0. \end{cases} \tag{4.4.2.35}$$

Equations (4.4.2.34) are the differential forms of the $\frac{1}{2}$-BPS extreme BH attractor equations, whereas (4.4.2.35) are an equivalent reformulation of the non-BPS extreme BH attractor equations. When the symplectic, Kähler-covariantly holomorphic sections $L^{\Lambda}(z,\overline{z})$ and $M_{\Lambda}(z,\overline{z})$ appear, clearly the equations strictly pertain to $\mathcal{N}=2$, $d=4$ n_V-fold MESGT.

By inserting as input the BH charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right) \in \Gamma$, the Kähler potential $K(z,\overline{z})$ and the symplectic, Kähler-covariantly holomorphic complex sections L^{Λ} and M_{Λ} of the $U(1)$-bundle over $\mathcal{M}_{z,\overline{z}}$, the differential equations (4.4.2.34) and (4.4.2.35) output (if any) the $\left(p^{\Lambda}, q_{\Lambda}\right)$-dependent extreme BH attractors $\left(z^i_{susy}(p,q), \overline{z}^{\overline{i}}_{susy}(p,q)\right)$ ($\frac{1}{2}$-BPS-SUSY) and $\left(z^i_{non-susy}(p,q), \overline{z}^{\overline{i}}_{non-susy}(p,q)\right)$ (non-BPS-SUSY), respectively. Due to the homogeneity of degree 1 (under a complex rescaling) in the BH charges, the actual independent real degrees of freedom are $2n_V$, and not $2n_V+2$. This is perfectly consistent with the n_V complex moduli configurations fixed by the differential attractor equations (4.4.2.34) and (4.4.2.35) at the EH of the BH.

A completely equivalent formulation of the attractor equations for extreme BHs in $\mathcal{N}=2$, $d=4$ n_V-fold MESGT may be obtained by evaluating at the attractor point(s) – to be found – some identities previously obtained in the context of the SKG of $\mathcal{M}_{z,\overline{z}}$.

In order to determine the most general reformulation of the attractor equations, let us recall the fundamental SKG vector identity (3.2.104), by stressing the moduli dependence of its terms

$$\begin{aligned} & n - i\epsilon\mathcal{M}\left(\mathcal{N}\left(z,\overline{z}\right)\right)n \\ & = -2i\overline{V}\left(z,\overline{z}\right)Z\left(z,\overline{z};p,q\right) - 2iG^{i\overline{i}}\left(z,\overline{z}\right)\left(D_i V\right)\left(z,\overline{z}\right)\left(\overline{D}_{\overline{i}}\overline{Z}\right)\left(z,\overline{z};p,q\right), \end{aligned} \tag{4.4.2.36}$$

where the $(2n_V + 2) \times 1$ symplectic vectors V and n are respectively defined by (3.1.23) and (3.2.35), and as above we renamed $n_m^\Lambda \equiv p^\Lambda$ and $n_\Lambda^e \equiv q_\Lambda$.

Let us now evaluate such an identity at the most general critical points of the effective BH potential function $V_{BH}(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$. In order to do this, we recall the condition of criticality for $V_{BH}(z, \overline{z}; p, q)$ in $\mathcal{M}_{z,\overline{z}}$, which for $Z \neq 0$, is expressed by (4.4.2.6):

$$\overline{D}_{\overline{i}}\overline{Z} = \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{j\overline{j}}G^{k\overline{k}}(D_j Z) D_k Z, \ \forall i \in \{1, \ldots, n_V\}. \tag{4.4.2.37}$$

By substituting the criticality condition (4.4.2.37) back into the $Sp(2n_V+2)$-covariant vector identity (4.4.2.36), one obtains

$$n - i\epsilon\, \mathcal{M}(\mathcal{N})|_{\partial V=0}\, n = \left[-2i\overline{V}Z + \frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}(D_i V)(D_j Z) D_k Z\right]_{\partial V=0}, \tag{4.4.2.38}$$

where "$\partial V = 0$" denotes the evaluation at the critical points of V_{BH} in $\mathcal{M}_{z,\overline{z}}$.

Equation (4.4.2.38) is the most general, $Sp(2n_V+2)$-covariant formulation of the attractor equations for the considered class of extreme BHs in $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT. Once again, the counting of degrees of freedom is consistent: due to the homogeneity of degree 1 (under a complex rescaling) in the BH charges, the actual independent real degrees of freedom are $2n_V$, and not $2n_V + 2$.

Here we shortly note that for critical points of V_{BH} with $Z = 0$ Eqs. (4.4.2.4) and (4.4.2.38) gets simplified respectively as follows:

$$C_{ijk}G^{j\overline{m}}G^{k\overline{k}}\overline{D}_{\overline{m}}\overline{Z}\overline{D}_{\overline{k}}\overline{Z} = 0\,; \tag{4.4.2.39}$$

$$n - i\epsilon\, \mathcal{M}(\mathcal{N})|_{\partial V=0}\, n = -2i\left[G^{i\overline{i}}D_i V\overline{D}_{\overline{i}}\overline{Z}\right]_{\partial V=0}. \tag{4.4.2.40}$$

Even though in what follows we will usually assume $Z \neq 0$, the critical points of V_{BH} with $Z = 0$ have been recently investigated in particular geometric frameworks [209].

By recalling the symplectic-orthogonality relations (3.2.105) and using the criticality condition (4.4.2.37), (4.4.2.38) may also be equivalently rewritten as follows:

$$\begin{cases} \langle V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle_{\partial V=0} = -2\, Z|_{\partial V=0}\,; \\ \langle \overline{V}, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle_{\partial V=0} = 0; \\ \langle D_i V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle_{\partial V=0} = 0; \\ \langle \overline{D}_{\overline{i}} V, n - i\epsilon\mathcal{M}(\mathcal{N})\, n\rangle_{\partial V=0} = -2\left(\overline{D}_{\overline{i}}\overline{Z}\right)_{\partial V=0} \\ = -i\left[\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{j\overline{j}}G^{k\overline{k}}(D_j Z) D_k Z\right]_{\partial V=0}. \end{cases} \tag{4.4.2.41}$$

The real and imaginary parts of (4.4.2.38) respectively yield

$$n = 2\left\{Im\left[Z\overline{V} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)D_iV\right]\right\}_{\partial V=0}$$
$$= -2\left\{Im\left[\overline{Z}V - \frac{i}{2}\frac{Z}{|Z|^2}C_{ijk}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{D}_{\overline{i}}\overline{V}\right]\right\}_{\partial V=0}; \qquad (4.4.2.42)$$

$$\epsilon\,\mathcal{M}\left(\mathcal{N}\right)|_{\partial V=0}\,n| = 2\left\{Re\left[Z\overline{V} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)D_iV\right]\right\}_{\partial V=0}$$
$$= 2\left\{Re\left[\overline{Z}V - \frac{i}{2}\frac{Z}{|Z|^2}C_{ijk}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\overline{D}_{\overline{i}}\overline{V}\right]\right\}_{\partial V=0}, \qquad (4.4.2.43)$$

implying, in turn

$$\left\{Re\left[Z\overline{V} + \tfrac{i}{2}\tfrac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)D_iV\right]\right\}_{\partial V=0}$$
$$= \epsilon\,\mathcal{M}\left(\mathcal{N}\right)|_{\partial V=0}\left\{Im\left[Z\overline{V} + \tfrac{i}{2}\tfrac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)D_iV\right]\right\}_{\partial V=0}. \qquad (4.4.2.44)$$

It is interesting to notice that, as it is evident by looking for instance at (4.4.2.42), at the critical points of V_{BH} the coefficients of $\overline{V}$ and D_iV in the attractor equations have the same holomorphicity in the central charge Z, i.e., they may be expressed only in terms of Z and D_iZ, without considering $\overline{Z}$ and $\overline{D}_{\overline{i}}\overline{Z}$. Such a fact does not happen in a generic point of $\mathcal{M}_{z,\overline{z}}$, as it is easy to realize by looking, e.g., at the identity (3.2.106). As it is evident, the price to be paid in order to obtain the same holomorphicity in Z at the critical points of V_{BH} is the fact that the coefficient of D_iV is not linear in some Kähler-covariant derivative of Z any more, and now it also explicitly depends on the rank-3 tensor $\overline{C}_{\overline{i}\overline{j}\overline{k}}$.

The vector notation of (4.4.2.42) may be explicited as follows:

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix}$$

$$= 2\left\{ Im\left[Z\begin{pmatrix} \overline{L}^{\Lambda} \\ \\ \overline{M}_{\Lambda} \end{pmatrix} + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(D_jZ\right)\left(D_kZ\right)\begin{pmatrix} D_iL^{\Lambda} \\ \\ D_iM_{\Lambda} \end{pmatrix}\right]\right\}_{\partial V=0}$$

$$= -2\left\{ Im\left[\overline{Z}\begin{pmatrix} L^{\Lambda} \\ \\ M_{\Lambda} \end{pmatrix} - \frac{i}{2}\frac{Z}{|Z|^2}C_{ijk}G^{i\overline{i}}G^{j\overline{j}}G^{k\overline{k}}\left(\overline{D}_{\overline{j}}\overline{Z}\right)\left(\overline{D}_{\overline{k}}\overline{Z}\right)\begin{pmatrix} \overline{D}_{\overline{i}}\overline{L}^{\Lambda} \\ \\ \overline{D}_{\overline{i}}\overline{M}_{\Lambda} \end{pmatrix}\right]\right\}_{\partial V=0} . \tag{4.4.2.45}$$

We may now specialize such an expression for the two classes of extreme BH attractors in $\mathcal{N}=2$, $d=4$, n_V-fold MESGT, namely for the $\frac{1}{2}$-*BPS-SUSY* and *non(-BPS)-SUSY* ones.

At the $\frac{1}{2}$-*BPS-SUSY extreme BH attractors* (4.4.2.45) greatly simplifies because $D_iZ=0\ \forall i$, and one finally gets

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix} = 2\left[Im\begin{pmatrix} Z\overline{L}^{\Lambda} \\ \\ Z\overline{M}_{\Lambda} \end{pmatrix}\right]_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)}$$
$$= i\begin{pmatrix} \overline{Z}L^{\Lambda}-Z\overline{L}^{\Lambda} \\ \\ \overline{Z}M_{\Lambda}-Z\overline{M}_{\Lambda} \end{pmatrix}_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)} . \tag{4.4.2.46}$$

This is an equivalent, purely algebraic form of the $\frac{1}{2}$-*BPS-SUSY extreme BH attractor equations* (4.4.2.34). It may be fully explicited by writing

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix} = 2\left[Im\begin{pmatrix} \left[q_{\Sigma}L^{\Sigma}\left(z,\overline{z}\right)-p^{\Sigma}M_{\Sigma}\left(z,\overline{z}\right)\right]\overline{L}^{\Lambda}\left(z,\overline{z}\right) \\ \\ \left[q_{\Sigma}L^{\Sigma}\left(z,\overline{z}\right)-p^{\Sigma}M_{\Sigma}\left(z,\overline{z}\right)\right]\overline{M}_{\Lambda}\left(z,\overline{z}\right) \end{pmatrix}\right]_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)} . \tag{4.4.2.47}$$

By inserting as input the BH charge configuration $\left(p^{\Lambda},q_{\Lambda}\right)\in\Gamma$ and the symplectic, Kähler-covariantly holomorphic complex sections L^{Λ} and M_{Λ} of the $U(1)$-bundle over $\mathcal{M}_{z,\overline{z}}$, (4.4.2.46) and (4.4.2.47) output (if any) the $\left(p^{\Lambda},q_{\Lambda}\right)$-dependent $\frac{1}{2}$-BPS-SUSY extreme BH attractors $\left(z^{i}_{susy}\left(p,q\right),\overline{z}^{\overline{i}}_{susy}\left(p,q\right)\right)$. It is worth pointing out that (4.4.2.46) and (4.4.2.47) are purely algebraic ones, whereas (4.4.2.34) are differential equations, thus, in general, much more complicated to be solved. Consequently, at least in the case of $\frac{1}{2}$-BPS-SUSY extreme BH attractors, the use of the SKG identities allows one to greatly simplify the search of fixed points in the radial evolution of the moduli configurations.

The situation is more complicated at the *non(-BPS)-SUSY extreme BH attractors,* because at such points $D_i Z \neq 0$ for some value(s) of the index i. Equation (4.4.2.45) holds in its fully general form:

$$\begin{pmatrix} p^\Lambda \\ \\ q_\Lambda \end{pmatrix} = 2\left\{ Im \left[\begin{array}{l} Z \begin{pmatrix} \overline{L}^\Lambda \\ \\ \overline{M}_\Lambda \end{pmatrix} \\ + \frac{i}{2}\frac{\overline{Z}}{|Z|^2}\overline{C}_{\overline{i}\overline{j}\overline{k}} G^{i\overline{i}} G^{j\overline{j}} G^{k\overline{k}} \\ \cdot \left(D_j Z\right)\left(D_k Z\right) \begin{pmatrix} D_i L^\Lambda \\ \\ D_i M_\Lambda \end{pmatrix} \end{array} \right] \right\}_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))} . \tag{4.4.2.48}$$

This is an equivalent form of the *non-BPS-SUSY extreme BH attractor equations* (4.4.2.35)*,* which may be fully explicited by writing

$$\begin{pmatrix} p^\Lambda \\ \\ q_\Lambda \end{pmatrix} = 2\left\{ Im \left[\begin{array}{l} \left[q_\Sigma L^\Sigma(z,\overline{z}) - p^\Sigma M_\Sigma(z,\overline{z})\right] \begin{pmatrix} \overline{L}^\Lambda(z,\overline{z}) \\ \\ \overline{M}_\Lambda(z,\overline{z}) \end{pmatrix} \\ + \frac{i}{2}\left[q_\Delta L^\Delta(z,\overline{z}) - p^\Delta M_\Delta(z,\overline{z})\right]^{-1} \\ \cdot \overline{C}_{\overline{i}\overline{j}\overline{k}}(z,\overline{z}) G^{i\overline{i}}(z,\overline{z}) G^{j\overline{j}}(z,\overline{z}) G^{k\overline{k}}(z,\overline{z}) \\ \cdot \left[\partial_j + \frac{1}{2}\partial_j K(z,\overline{z})\right]\left[q_\Xi L^\Xi(z,\overline{z}) - p^\Xi M_\Xi(z,\overline{z})\right] \\ \cdot \left[\partial_k + \frac{1}{2}\partial_k K(z,\overline{z})\right]\left[q_\Theta L^\Theta(z,\overline{z}) - p^\Theta M_\Theta(z,\overline{z})\right] \\ \cdot \begin{pmatrix} \left[\partial_i + \frac{1}{2}\partial_i K(z,\overline{z})\right] L^\Lambda(z,\overline{z}) \\ \\ \left[\partial_i + \frac{1}{2}\partial_i K(z,\overline{z})\right] M_\Lambda(z,\overline{z}) \end{pmatrix} \end{array} \right] \right\}_{(z_{non-susy}(p,q),\overline{z}_{non-susy}(p,q))} . \tag{4.4.2.49}$$

In this case, beside the BH charge configuration $\left(p^\Lambda, q_\Lambda\right) \in \Gamma$ and the symplectic, Kähler-covariantly holomorphic complex sections L^Λ and M_Λ of the $U(1)$-bundle over $\mathcal{M}_{z,\overline{z}}$, the input of (4.4.2.48) and (4.4.2.49) also necessarily includes the Kähler potential $K(z,\overline{z})$ (and consequently the contravariant metric tensor $G^{i\overline{j}}(z,\overline{z})$) and the completely symmetric, Kähler-covariantly holomorphic complex rank-3 tensor $C_{ijk}(z,\overline{z})$. The solutions of (4.4.2.48) and (4.4.2.49) (if any) are the $\left(p^{\Lambda},q_{\Lambda}\right)$-dependent non(-BPS)-SUSY extreme BH attractors $\left(z^{i}_{non-susy}(p,q),\overline{z}^{\overline{i}}_{non-susy}(p,q)\right)$. Differently from the previous supersymmetric case, both (4.4.2.48), (4.4.2.49), and (4.4.2.35) are differential equations, in general hard to be solved.

Concerning the non-BPS, $Z = 0$ critical points of V_{BH} (denoted by $(z_{non-susy,Z=0}(p,q),\overline{z}_{non-susy,Z=0}(p,q))$), here we just mention that the real part of Eq. (4.4.2.40) yields the differential equation

$$\begin{pmatrix} p^{\Lambda} \\ \\ q_{\Lambda} \end{pmatrix} = 2\left\{ Im\left[G^{i\overline{i}}\overline{D}_{\overline{i}}\overline{Z}\begin{pmatrix} D_iL^{\Lambda} \\ \\ D_iM_{\Lambda} \end{pmatrix}\right]_{(z_{non-susy,Z=0}(p,q),\overline{z}_{non-susy,Z=0}(p,q))}\right\}. \tag{4.4.2.50}$$

5

Black Hole Thermodynamics and Geometry

In the previous section we have seen that, in the context of n_V-fold, $\mathcal{N}=2$, $d=4$ MESGT, the fundamental properties of 4-d extreme BHs may be described by the "BH effective potential" function $V(z,\overline{z};p,q)$, and in particular by its critical points in the moduli space $\mathcal{M}_{z,\overline{z}}$, which is endowed with a (regular) special Kähler geometry. New insights can be gained by considering the formalism of the geometric approach to the thermodynamical fluctuation theory. Once again, we will mainly follow Ferrara, Gibbons, and Kallosh [55] (for a complete review, see, e.g., [82]).

5.1 Geometric Approach to Thermodynamical Fluctuation Theory

In the attempt to geometrize the fluctuations in thermodynamical theories, the first metric to be encountered is Weinhold's one. Weinhold suggested to use as a metric the Hessian of the energy M, considered as a function of $n+1$ extensive variables $N^{\mu}=(S,N^{a})$, where S is the entropy and N^{a}, $a=1,\ldots,n$, are conserved numbers (lower Greek indices run $0,1,\ldots,n$ throughout, unless otherwise indicated):

$$M=M\left(S,N^{a}\right)=M\left(N^{\mu}\right). \tag{5.1.1}$$

The (real) extensive variables N^{μ}'s may be considered as coordinates in the (real) $(n+1)$-d extensive thermodynamical configuration space $\mathcal{N}$. Notice that in such a formulation, the volume of an ordinary gas should be included as one of the N^{a}'s. In the case of BHs, the N^{a}'s include conserved charges, angular momenta (in the nonstatic case), and also the values ϕ^{i}_{∞} of the moduli at spatial infinity.

Thus, in conventional thermodynamics the Weinhold metric $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}\left(N\right)\equiv\frac{\partial^{2}M\left(N\right)}{\partial N^{\mu}\partial N^{\nu}}. \tag{5.1.2}$$

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Since in conventional thermodynamics the energy is always the least among the equilibrium configurations with a given entropy S and conserved total numbers N^a's, the Weinhold metric $W_{\mu\nu}$ is always (not necessarily strictly) positive definite.

It should be pointed out that the (geo)metric structure of the $(n+1)$-d extensive thermodynamical configuration space $\mathcal{N}$ and the metric structure given by the covariant Weinhold metric $W_{\mu\nu}$ are, in general, completely different. Usually, in conventional thermodynamics the space $\mathcal{N}$ of the (extensively parameterized) thermodynamical configurations is flat, endowed with a $(n+1)$-d Euclidean metric $\delta_{\mu\nu}$, whereas $W_{\mu\nu}(N)$, describing the (extensively parameterized) thermodynamical fluctuations, is, in general, nontrivially flat (see (5.1.2)).

By differentiating (5.1.1), one gets

$$dM(N) = \frac{\partial M(N)}{\partial S} dS + \frac{\partial M(N)}{\partial N^a} dN^a \equiv T dS + \mu_a dN^a, \tag{5.1.3}$$

where the temperature T and the potentials μ_a's are the intensive variables $\mu_\mu = (T, \mu_a)$ canonically conjugated to the N^μ's:

$$\mu_\mu(N) : \begin{cases} \pi_S \equiv \frac{\partial M(N)}{\partial S} \equiv T(N); \\ \\ (\pi_{N^a})_{Weinhold} \equiv \pi_a \equiv \frac{\partial M(N)}{\partial N^a} \equiv \mu_a(N). \end{cases} \tag{5.1.4}$$

The (real) intensive variables μ_μ's may be considered as coordinates in the (real) $(n+1)$-d intensive thermodynamical configuration space $\widehat{\mathcal{N}}$, which is the intensive counterpart of $\mathcal{N}$, i.e., the intensively parameterized space of the thermodynamical configurations of the physical system being considered. The space $\widehat{\mathcal{N}}$ generally inherits the metric structure of $\mathcal{N}$, and it is thus flat, endowed with a $(n+1)$-d Euclidean (contravariant) metric $\delta^{\mu\nu}$. System (5.1.4), assumed to be determined, solvable and invertible, defines the change of parametrization of the thermodynamical configuration space of the physical system being considered: from extensive to intensive variables, and vice versa.

By Legendre-transforming the energy M w.r.t. the extensive variables N^μ's, one gets the (opposite of the) Gibbs free energy $G(\mu)$

$$\begin{aligned} \mathcal{L}(M) &\equiv \pi_S S + (\pi_{N^a})_{Weinhold} N^a - M \\ &= TS(\mu) + \mu_a N^a(\mu) - M(N(\mu)) = -G(\mu), \end{aligned} \tag{5.1.5}$$

where system (5.1.4) has been inverted in order to give $N^\mu = N^\mu(\mu)$. Clearly, (5.1.5) yields

$$M + \mathcal{L}(M) = M - G = TS + N^a \mu_a = N^\mu \mu_\mu. \tag{5.1.6}$$

By differentiating (5.1.5) and using (5.1.3), one gets

$$\begin{aligned} dG &= -d\left(\mathcal{L}\left(M\right)\right) = dM - SdT - TdS - \mu_a dN^a - N^a d\mu_a \\ &= TdS + \mu_a dN^a - SdT - TdS - \mu_a dN^a - N^a d\mu_a \\ &= -SdT - N^a d\mu_a = -N^\mu d\mu_\mu . \end{aligned} \tag{5.1.7}$$

Thus, the inversion of definition (5.1.2) yields

$$W^{\mu\nu}\left(\mu\right) \equiv \frac{\partial^2 \left(\mathcal{L}\left(M\right)\right)\left(\mu\right)}{\partial\mu_\mu \partial\mu_\nu} = -\frac{\partial^2 G\left(\mu\right)}{\partial\mu_\mu \partial\mu_\nu}, \tag{5.1.8}$$

with

$$W^{\mu\rho}\left(\mu\left(N\right)\right) W_{\rho\nu}\left(N\right) = \delta^\mu_\nu \Longleftrightarrow \frac{\partial^2 G\left(\mu\left(N\right)\right)}{\partial\mu_\mu \partial\mu_\rho}\frac{\partial^2 M\left(N\right)}{\partial N^\rho \partial N^\nu} = -\delta^\mu_\nu,\ \forall N^\sigma, \tag{5.1.9}$$

where $\mu_\mu = \mu_\mu\left(N\right)$ is given by system (5.1.4). Once again, it is worth distinguishing the flat, Euclidean (geo)metric structure of the $(n+1)$-d intensive thermodynamical configuration space $\widehat{\mathcal{N}}$ and the metric structure, in general nontrivially flat, given by the contravariant Weinhold metric $W^{\mu\nu}\left(\mu\right)$, describing the (intensively parameterized) thermodynamical fluctuations, and defined by (5.1.8).

Notice also that the covariant Weinhold metric (5.1.2) and its contravariant inverse (5.1.8) are symmetric, because both $M\left(N\right)$ and $G\left(\mu\right)$, respectively as functions of the N^μ's and μ_μ's, are assumed to satisfy the Schwarz lemma on partial derivatives. Therefore, by recalling definitions (5.1.2) and (5.1.8), the $(n+1)\times(n+1)$ matrix forms of $W_{\mu\nu}$ and $W^{\mu\nu}$ respectively read

$$\begin{aligned} W_{\mu\nu}\left(N\right) &= \begin{pmatrix} \frac{\partial^2 M(N)}{(\partial S)^2} & \frac{\partial^2 M(N)}{\partial N^a \partial S} \\ \frac{\partial^2 M(N)}{\partial S \partial N^a} & \frac{\partial^2 M(N)}{\partial N^b \partial N^a} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial T(N)}{\partial S} & \frac{\partial T(N)}{\partial N^a} \\ \frac{\partial T(N)}{\partial N^a} & \frac{\partial \mu_b(N)}{\partial N^a} \end{pmatrix} = \begin{pmatrix} \frac{\partial T(N)}{\partial S} & \frac{\partial \mu_a(N)}{\partial S} \\ \frac{\partial \mu_a(N)}{\partial S} & \frac{\partial \mu_a(N)}{\partial N^b} \end{pmatrix}; \end{aligned} \tag{5.1.10}$$

$$W^{\mu\nu}\left(\mu\right) = -\begin{pmatrix} \frac{\partial^2 G(\mu)}{(\partial T)^2} & \frac{\partial^2 G(\mu)}{\partial \mu_a \partial T} \\ \frac{\partial^2 G(\mu)}{\partial T \partial \mu_a} & \frac{\partial^2 G(\mu)}{\partial \mu_b \partial \mu_a} \end{pmatrix} = -\begin{pmatrix} \frac{\partial^2 G(\mu)}{(\partial T)^2} & \frac{\partial^2 G(\mu)}{\partial T \partial \mu_a} \\ \frac{\partial^2 G(\mu)}{\partial T \partial \mu_a} & \frac{\partial^2 G(\mu)}{\partial \mu_a \partial \mu_b} \end{pmatrix}. \tag{5.1.11}$$

In the second line of (5.1.10) we used the definition (5.1.4), implying

$$\begin{cases} \frac{\partial T(N)}{\partial N^a} = \frac{\partial \mu_a(N)}{\partial S}; \\ \\ \frac{\partial \mu_b(N)}{\partial N^a} = \frac{\partial \mu_a(N)}{\partial N^b}. \end{cases} \tag{5.1.12}$$

Moreover, the inverting condition (5.1.9) yields

$$\begin{pmatrix} \frac{\partial^2 G}{(\partial T)^2}\left(\mu\left(N\right)\right) & \frac{\partial^2 G\mu}{\partial T \partial \mu_a}\left(\mu\left(N\right)\right) \\ \\ \frac{\partial^2 G}{\partial T \partial \mu_a}\left(\mu\left(N\right)\right) & \frac{\partial^2 G}{\partial \mu_a \partial \mu_b}\left(\mu\left(N\right)\right) \end{pmatrix} \otimes \begin{pmatrix} \frac{\partial^2 M(N)}{(\partial S)^2} & \frac{\partial^2 M(N)}{\partial S \partial N^c} \\ \\ \frac{\partial^2 M(N)}{\partial S \partial N^c} & \frac{\partial^2 M(N)}{\partial N^c \partial N^d} \end{pmatrix} = -\mathbb{I}_{n+1}, \tag{5.1.13}$$

where $\otimes$ denotes the usual row–column matrix product and $\mathbb{I}_{n+1}$ stands for the $(n+1)$-d unit matrix.

Some time after Weinhold, Ruppeiner proposed instead to focus on the entropy S, seen as a function of the extensive charges $Q^\mu = (M, N^a)$,

$$S = S\left(M, N^a\right) = S\left(Q\right). \tag{5.1.14}$$

As before, the (real) extensive variables Q^μ's may be considered as coordinates in the (real) $(n+1)$-d extensive thermodynamical configuration space $\mathcal{V}$. Usually, in conventional thermodynamics such a space is flat, endowed with a $(n+1)$-d Euclidean metric $\delta_{\mu\nu}$.

The intensive variables canonically conjugated to the extensive charges Q^μ's are the $\beta_\mu = \left(\frac{1}{T}, -\frac{1}{T}\mu_a\right)$. This can be shown by using (5.1.3):

$$dM = TdS + \mu_a dN^a$$

$$\Updownarrow$$

$$dS\left(M, N^a\right) = \frac{dM}{T} - \frac{\mu_a}{T} dN^a = \frac{\partial S\left(Q\right)}{\partial M} + \frac{\partial S\left(Q\right)}{\partial N^a} dN^a \tag{5.1.15}$$

$$\Updownarrow$$

$$\beta_\mu\left(Q\right) : \begin{cases} \pi_M \equiv \frac{\partial S(Q)}{\partial M} = \frac{1}{T}\left(Q\right); \\ \left(\pi_{N^a}\right)_{Ruppeiner} \equiv \frac{\partial S(Q)}{\partial N^a} = -\frac{\mu_a}{T}\left(Q\right) \equiv \beta_a\left(Q\right). \end{cases} \tag{5.1.16}$$

Once again, the (real) intensive variables β_μ's may be considered as coordinates in the (real) $(n+1)$-d intensive thermodynamical configuration space $\widehat{\mathcal{V}}$, which is nothing but the intensive counterpart of $\mathcal{V}$, i.e., the intensively parameterized space of the thermodynamical configurations of the physical system being considered (in Ruppeiner's formalism). $\widehat{\mathcal{V}}$ generally inherits the metric structure of $\mathcal{V}$, and it is thus flat, endowed with a $(n+1)$-d Euclidean (contravariant) metric $\delta^{\mu\nu}$. System (5.1.16), assumed to be determined, solvable and invertible, defines the change of parametrization of the thermodynamical configuration space of the physical system being considered: from extensive to intensive variables, and vice versa.

By Legendre-transforming the entropy S w.r.t. the extensive charges Q^{μ}'s, one gets

$$\begin{aligned}\mathcal{L}(S) &\equiv \pi_M M + (\pi_{N^a})_{Ruppeiner} N^a - S \\ &= \frac{M(\beta)}{T} - \frac{1}{T}\mu_a N^a(\beta) - S(Q(\beta)) \\ &= \frac{G(\mu)}{T} \equiv \Gamma(\beta), \end{aligned} \tag{5.1.17}$$

where we used (5.1.5) and system (5.1.16) has been inverted in order to give $Q^{\mu} = Q^{\mu}(\beta)$. $\Gamma(\beta)$ is defined as the Legendre-transform of the entropy $S(Q)$ in Ruppeiner's formalism.

Equation (5.1.17) then implies

$$S + \mathcal{L}(S) = S + \Gamma = \frac{M}{T} - \frac{1}{T}\mu_a N^a = Q^{\mu}\beta_{\mu}\ . \tag{5.1.18}$$

By differentiating (5.1.17) and using (5.1.15), one gets

$$\begin{aligned} d\Gamma = d(\mathcal{L}(S)) &= \frac{dM}{T} - \frac{M}{T^2}dT + \frac{1}{T^2}\mu_a N^a dT - \frac{1}{T}\mu_a dN^a - \frac{1}{T}N^a d\mu_a - dS \\ &= \frac{dM}{T} - \frac{M}{T^2}dT + \frac{1}{T^2}\mu_a N^a dT - \frac{\mu_a}{T}dN^a - \frac{1}{T}N^a d\mu_a - \frac{dM}{T} + \frac{\mu_a}{T}dN^a \\ &= -\frac{M}{T^2}dT + \frac{1}{T^2}\mu_a N^a dT - \frac{1}{T}N^a d\mu_a \\ &= Md\left(\frac{1}{T}\right) + N^a d\left(-\frac{\mu_a}{T}\right) = Q^{\mu}d\beta_{\mu}. \end{aligned} \tag{5.1.19}$$

The Ruppeiner metric is defined as

$$S_{\mu\nu}(Q) \equiv -\frac{\partial^2 S(Q)}{\partial Q^{\mu}\partial Q^{\nu}}. \tag{5.1.20}$$

Since in conventional thermodynamics the entropy is always the biggest among the equilibrium configurations with a given energy M and conserved total numbers N^a's, the Ruppeiner metric $S_{\mu\nu}$ is always (not necessarily strictly) positive definite. Thus, the inversion of definition (5.1.20) yields

$$S^{\mu\nu}(\beta) \equiv -\frac{\partial^2(\mathcal{L}(S))(\beta)}{\partial\beta_{\mu}\partial\beta_{\nu}} = -\frac{\partial^2\Gamma(\beta)}{\partial\beta_{\mu}\partial\beta_{\nu}}, \tag{5.1.21}$$

with

$$S^{\mu\rho}(\beta(Q))\,S_{\rho\nu}(Q) = \delta^{\mu}_{\nu} \Longleftrightarrow \frac{\partial^2\Gamma(\beta(Q))}{\partial\beta_{\mu}\partial\beta_{\rho}}\frac{\partial^2 S(Q)}{\partial Q^{\rho}\partial Q^{\nu}} = \delta^{\mu}_{\nu},\ \forall Q^{\sigma}, \tag{5.1.22}$$

where $\beta_{\mu} = \beta_{\mu}(Q)$ is given by system (5.1.16). Notice that the covariant Ruppeiner metric (5.1.20) and its contravariant inverse (5.1.21) are symmetric, because both $S(Q)$ and $\Gamma(\beta)$, respectively as functions of the Q^{μ}'s and β_{μ}'s, are assumed to satisfy the Schwarz lemma on partial derivatives. Therefore, by recalling definitions (5.1.20) and (5.1.21), the $(n+1)\times(n+1)$ matrix forms of $S_{\mu\nu}$ and $S^{\mu\nu}$ respectively read

$$\begin{aligned} S_{\mu\nu}(Q) &= -\begin{pmatrix} \frac{\partial^2 S(Q)}{(\partial M)^2} & \frac{\partial^2 S(Q)}{\partial N^a \partial M} \\ \\ \frac{\partial^2 S(Q)}{\partial M \partial N^a} & \frac{\partial^2 S(Q)}{\partial N^b \partial N^a} \end{pmatrix} = -\begin{pmatrix} \frac{\partial^2 S(Q)}{(\partial M)^2} & \frac{\partial^2 S(Q)}{\partial M \partial N^a} \\ \\ \frac{\partial^2 S(Q)}{\partial M \partial N^a} & \frac{\partial^2 S(Q)}{\partial N^a \partial N^b} \end{pmatrix} \\ &= -\begin{pmatrix} \frac{\partial\left(\frac{1}{T}(Q)\right)}{\partial M} & \frac{\partial\left(\frac{1}{T}(Q)\right)}{\partial N^a} \\ \\ \frac{\partial\left(\frac{1}{T}(Q)\right)}{\partial N^a} & -\frac{\partial\left(\frac{\mu_b}{T}(Q)\right)}{\partial N^a} \end{pmatrix} = -\begin{pmatrix} \frac{\partial\left(\frac{1}{T}(Q)\right)}{\partial M} & -\frac{\partial\left(\frac{\mu_a}{T}(Q)\right)}{\partial M} \\ \\ -\frac{\partial\left(\frac{\mu_a}{T}(Q)\right)}{\partial M} & -\frac{\partial\left(\frac{\mu_a}{T}(Q)\right)}{\partial N^b} \end{pmatrix}; \end{aligned} \tag{5.1.23}$$

$$\begin{aligned} S^{\mu\nu}(\beta) &= -\begin{pmatrix} \frac{\partial^2 \Gamma(\beta)}{\left(\partial\left(\frac{1}{T}\right)\right)^2} & -\frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{\mu_a}{T}\right)\partial\left(\frac{1}{T}\right)} \\ \\ -\frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{1}{T}\right)\partial\left(\frac{\mu_a}{T}\right)} & \frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{\mu_b}{T}\right)\partial\left(\frac{\mu_a}{T}\right)} \end{pmatrix} \\ &= -\begin{pmatrix} \frac{\partial^2 \Gamma(\beta)}{\left(\partial\left(\frac{1}{T}\right)\right)^2} & -\frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{1}{T}\right)\partial\left(\frac{\mu_a}{T}\right)} \\ \\ -\frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{1}{T}\right)\partial\left(\frac{\mu_a}{T}\right)} & \frac{\partial^2 \Gamma(\beta)}{\partial\left(\frac{\mu_a}{T}\right)\partial\left(\frac{\mu_b}{T}\right)} \end{pmatrix}. \end{aligned} \tag{5.1.24}$$

In the second line of (5.1.23) we used definitions (5.1.16), implying

$$\begin{cases} \frac{\partial\left(\frac{1}{T}(Q)\right)}{\partial N^a} = -\frac{\partial\left(\frac{\mu_a}{T}(Q)\right)}{\partial M}; \\ \\ \frac{\partial\left(\frac{\mu_b}{T}(Q)\right)}{\partial N^a} = \frac{\partial\left(\frac{\mu_a}{T}(Q)\right)}{\partial N^b}. \end{cases} \tag{5.1.25}$$

Moreover, the inverting condition (5.1.22) yields

$$\begin{pmatrix} \frac{\partial^2 \Gamma}{\left(\partial\left(\frac{1}{T}\right)\right)^2}(\beta(Q)) & -\frac{\partial^2 \Gamma}{\partial\left(\frac{1}{T}\right)\partial\left(\frac{\mu_a}{T}\right)}(\beta(Q)) \\ \\ -\frac{\partial^2 \Gamma}{\partial\left(\frac{1}{T}\right)\partial\left(\frac{\mu_a}{T}\right)}(\beta(Q)) & \frac{\partial^2 \Gamma}{\partial\left(\frac{\mu_a}{T}\right)\partial\left(\frac{\mu_b}{T}\right)}(\beta(Q)) \end{pmatrix} \otimes \begin{pmatrix} \frac{\partial^2 S(Q)}{(\partial M)^2} & \frac{\partial^2 S(Q)}{\partial M \partial N^c} \\ \\ \frac{\partial^2 S(Q)}{\partial M \partial N^c} & \frac{\partial^2 S(Q)}{\partial N^c \partial N^d} \end{pmatrix} = \mathbb{I}_{n+1}. \tag{5.1.26}$$

As done for Weinhold's formalism, also for Ruppeiner's formalism it should be pointed out that the (geo)metric structure of the $(n+1)$-d thermodynamical configuration spaces $\mathcal{V}$ (extensively parameterized) and $\widehat{\mathcal{V}}$ (intensively parameterized) and the metric structure given by the Ruppeiner metrics $W_{\mu\nu}$ and $W^{\mu\nu}$ are, in general, completely different. While $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are Euclidean $(n+1)$-d spaces, the Ruppeiner metrics, describing the thermodynamical fluctuations of the physical system being considered, have, in general, a nontrivially flat nature. Notice that in the adopted formalism the covariant thermodynamical metrics are functions of extensive variables, whereas the contravariant thermodynamical metrics are defined on the relevant space of intensive variables.

In order to determine the relation between the Weinhold and the Ruppeiner thermodynamical metrics, let us notice that (5.1.2) and (5.1.3) imply that the infinitesimal square metric interval determined by Weinhold metric in the space $\mathcal{N}$ reads

$$\begin{aligned}(dN)^2 &\equiv W_{\mu\nu}(N)\, dN^\mu dN^\nu = \frac{\partial^2 M(N)}{\partial N^\mu \partial N^\nu} dN^\mu dN^\nu \\ &= d\pi_S \otimes_s dS + d\left(\pi_{N^a}\right)_{Weinhold} \otimes_s dN^a = dT \otimes_s dS + d\mu_a \otimes_s dN^a,\end{aligned} \tag{5.1.27}$$

where $\otimes_s$ denotes the (symmetric) product of the components of the differential forms defined in the relevant thermodynamical configuration space. Analogously, (5.1.20) and (5.1.15) imply that the infinitesimal square metric interval determined by Ruppeiner metric in the space $\mathcal{V}$ reads

$$\begin{aligned}(dQ)^2 &\equiv S_{\mu\nu}(Q)\, dQ^\mu dQ^\nu = -\frac{\partial^2 S(Q)}{\partial Q^\mu \partial Q^\nu} dQ^\mu dQ^\nu \\ &= -d\pi_M \otimes_s dM - d\left(\pi_{N^a}\right)_{Ruppeiner} \otimes_s dN^a \\ &= -d\left(\frac{1}{T}\right) \otimes_s dM - d\left(-\frac{\mu_a}{T}\right) \otimes_s dN^a.\end{aligned} \tag{5.1.28}$$

By using (5.1.15), (5.1.27) may be rewritten as

$$\begin{aligned}(dN)^2 &= dT \otimes_s dS + d\mu_a \otimes_s dN^a \\ &= \frac{dT}{T} \otimes_s dM - \frac{\mu_a}{T} dT \otimes_s dN^a + d\mu_a \otimes_s dN^a \\ &= -T\left[d\left(\frac{1}{T}\right) \otimes_s dM + d\left(-\frac{\mu_a}{T}\right) \otimes_s dN^a\right].\end{aligned} \tag{5.1.29}$$

Thus, (5.1.28) and (5.1.29) yield the simple and perhaps surprising result

$$(dN)^2 = T\,(dQ)^2 \Longleftrightarrow W_{\mu\nu}(N)\, dN^\mu dN^\nu = T S_{\mu\nu}(Q)\, dQ^\mu dQ^\nu, \tag{5.1.30}$$

namely, the Weinhold and Ruppeiner thermodynamical metrics are conformally related, and the conformal factor is the temperature T. Consequently, only the conformal level of the geometrization of fluctuations in (conventional) thermodynamical systems is relevant; this implies that ratios of specific heats should be conformally invariant.

Finally, let us note that the Euclidean, flat nature of the $(n+1)$-d real thermodynamical configuration spaces $\mathcal{N}$, $\widehat{\mathcal{N}}$ (in Weinhold's formalism) and $\mathcal{V}$, $\widehat{\mathcal{V}}$ (in Ruppeiner's formalism), usually assumed as default in conventional thermodynamical systems, may be modified depending on the physical system being considered. As we will see in the next subsection, that this happens in BH thermodynamics, since the (asymptotical) moduli space M_{ϕ} (or its complexification $M_{z,\overline{z}}$) is generally nonflat, and also not necessarily with a global regular (geo)metric structure (even though in the presentation given above we restricted ourselves to the regular "geometric regime").

5.2 Geometrization of Black Hole Thermodynamics

We will now apply the concepts and results exposed above to BH thermodynamics.

5.2.1 Weinhold Black Hole Thermodynamics

Let us start from the Weinhold's geometrizing formalism. In general, the Weinhold metric is defined by (5.1.2) and (5.1.1):

$$\begin{cases} W_{\mu\nu}(N) \equiv \frac{\partial^2 M(N)}{\partial N^{\mu}\partial N^{\nu}}; \\ \\ N^{\mu} = (S, N^a), \ a = 1, \dots, n. \end{cases} \tag{5.2.1.1}$$

Thence, the corresponding definition for BH physics reads

$$\begin{cases} W_{\mu\nu,BH}(N_{BH}) \equiv D_{N^{\mu}_{BH}} D_{N^{\nu}_{BH}} M_{BH}(N_{BH}); \\ \\ N^{\mu}_{BH} = \left(S_{BH} = \frac{1}{4}A_H, p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right), \\ \\ \Lambda = 0, 1, \dots, n_V, \quad a = 1, \dots, m_{\phi_{\infty}}, \quad |\{\mu\}| = 2n_V + m_{\phi_{\infty}} + 3, \end{cases} \tag{5.2.1.2}$$

where M_{BH}, S_{BH}, and A_H respectively are the BH mass, entropy, and horizon area, and the leading-order, (semi)classical BHEA law has been applied. $\left(p^{\Lambda}, q_{\Lambda}\right)$ denotes the $Sp\left(2n_V + 2, \mathbb{Z}\right)$-covariant vector of BH conserved magnetic and electric charges (related to the $\left(U(1)\right)^{n_V+1}$ gauge invariance of the considered theory), whereas ϕ^a_{∞}'s are the spatial-asymptotical values of the $m_{\phi_{\infty}}$ real scalar fields of the theory. Thus, in the case at hand $n = 2n_V + m_{\phi_{\infty}} + 2$. The choice of extensive BH thermodynamical variables N^{μ}_{BH}'s is consistent with the previous treatment; of course, we do not consider

BH angular momenta because we are dealing with static BH metric backgrounds. In the conventional thermodynamics of static (spherically symmetric, asymptotically flat, 4-d) BHs, since the energy is always the least among the equilibrium configurations with a given entropy S_{BH}, conserved magnetic-electric charges $\left(p^{\Lambda}, q_{\Lambda}\right)$, and asymptotical scalar configurations ϕ_{∞}^{a}'s, the BH Weinhold metric $W_{\mu\nu,BH}$ is always (not necessarily strictly) positive definite.

$D_{N_{BH}^{\mu}}$ stands for the covariant derivative in the $(2n_V + m_{\phi_\infty} + 3)$-d (real) extensive BH thermodynamical configuration space $\mathcal{N}_{BH}$, of which the N_{BH}^{μ}'s may be considered as (real) local coordinates. $\mathcal{N}_{BH}$ is assumed to be given by the direct product

$$\mathcal{N}_{BH} \equiv \mathbb{A}^{2n_V+3} \times \mathcal{M}_{\phi_\infty}. \tag{5.2.1.3}$$

$\mathbb{A}^{2n_V+3}$ is the flat $(2n_V + 3)$-d thermodynamical configuration space related to the extensive variables S_{BH}, p^{Λ}, and q_{Λ}. In the electric–magnetic sector of p^{Λ}'s and q_{Λ}'s, such a space is endowed with a symplectic metric structure, given by the $(2n_V + 2)$-d symplectic metric ϵ (see (3.1.24)); at the quantum level such a sector gets discretized in the $(2n_V + 2)$-d BH charge symplectic lattice Γ. $\mathcal{M}_{\phi_\infty}$ instead is the (real parametrization of) the asymptotical moduli space ($dim_{\mathbb{R}}\mathcal{M}_{\phi_\infty} = m_{\phi_\infty}$), generally assumed to be endowed with a Riemannian metric, given by the metric tensor $G_{ab}(\phi_\infty)$. Consequently, when deriving w.r.t. the variables ϕ_{∞}^{a}'s, in order to be consistent with the overall covariance in $\mathcal{N}_{BH}$, one will have to consider covariant – rather than "flat," ordinary – derivatives, determined by the affine structure given by a (linear) connection in $\mathcal{M}_{\phi_\infty}$.

Summarizing, the factorized Ansatz (5.2.1.3) for the BH extensive thermodynamical configuration space $\mathcal{N}_{BH}$ yields the following expression for the covariant differential operator in $\mathcal{N}_{BH}$:

$$D_{N_{BH}^{\mu}} = \left(\partial_{S_{BH}}, \partial_{p^{\Lambda}}, \partial_{q_{\Lambda}}, D_{\phi_{\infty}^{a}}\right); \tag{5.2.1.4}$$

in other words, the only nontrivially ordinary derivatives are those w.r.t. the variables ϕ_{∞}^{a}'s.

Thus, since M_{BH} is a scalar function in $\mathcal{N}_{BH}$ and thus in $\mathcal{M}_{\phi_\infty}$, the Weinhold metric related to the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$ (i.e., the covariant Weinhold metric $W_{\mu\nu,BH}$ restricted to the asymptotical moduli configurations ϕ_{∞}^{a}'s) reads

$$\begin{aligned} W_{ab,BH}\left(N_{BH}\right) &\equiv D_{\phi_{\infty}^{a}} D_{\phi_{\infty}^{b}} M_{BH}\left(N_{BH}\right) = D_{\phi_{\infty}^{a}} \frac{\partial M_{BH}\left(N_{BH}\right)}{\partial \phi_{\infty}^{b}} \\ &= \frac{\partial^2 M_{BH}\left(N_{BH}\right)}{\partial \phi_{\infty}^{a} \partial \phi_{\infty}^{b}} - \Gamma_{ab}{}^{c}\left(\phi_\infty\right) \frac{\partial M_{BH}\left(N_{BH}\right)}{\partial \phi_{\infty}^{c}}, \end{aligned} \tag{5.2.1.5}$$

where $\Gamma_{ab}{}^{c}(\phi_\infty)$ is the (linear) connection determining the affine structure of the m_{ϕ_∞}-d real manifold $\mathcal{M}_{\phi_\infty}$. Since the function M_{BH} is assumed to satisfy the Schwarz lemma in $\mathcal{N}_{BH}$ w.r.t. flat, ordinary derivatives, it is clear that the

Weinhold metric $W_{\mu\nu,BH}$ defined by (5.2.1.2) is symmetric in all $\mathcal{N}_{BH}$, except eventually in the sector "$\mathcal{M}_{\phi_\infty}$,$\mathcal{M}_{\phi_\infty}$" of the ϕ_∞^a's, where the symmetry of $W_{ab,BH}$ strictly depends on the symmetry of the (linear) connection $\Gamma_{ab}{}^{c}(\phi_\infty)$ of $\mathcal{M}_{\phi_\infty}$:

$$\Gamma_{[ab]}{}^{c}(\phi_\infty) = 0 \Longleftrightarrow W_{[ab],BH}(N_{BH}) = 0. \tag{5.2.1.6}$$

Notice that, in general, the affine and metric structures of the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$, respectively, determined by the connection $\Gamma_{ab}{}^{c}(\phi_\infty)$ and the metric tensor $G_{ab}(\phi_\infty)$, are a priori completely independent one from the other. If, as done above, a generic, real Riemannian geometry is assumed for $\mathcal{M}_{\phi_\infty}$, then

$$\Gamma_{ab}{}^{c}(\phi_\infty) = \Gamma_{(ab)}{}^{c}(\phi_\infty) = \left\{{}^{\,c}_{ab}\right\}(\phi_\infty), \tag{5.2.1.7}$$

where $\left\{{}^{\,c}_{ab}\right\}$ are the Christoffel symbols of the second kind of the metric $G_{ab}(\phi_\infty)$

$$\begin{aligned}\Gamma_{ab}{}^{c}(\phi_\infty) &= \left\{{}^{\,c}_{ab}\right\}(\phi_\infty) = \left\{{}^{\;c}_{(ab)}\right\}(\phi_\infty)\\ &= \frac{1}{2}G^{cd}(\phi_\infty)\left[\partial_a G_{db}(\phi_\infty) + \partial_b G_{da}(\phi_\infty) - \partial_d G_{ab}(\phi_\infty)\right].\end{aligned} \tag{5.2.1.8}$$

Consequently, the affine and metric structures of the Riemannian asymptotical moduli space $\mathcal{M}_{\phi_\infty}$ are compatible and strictly related by the "metrization" of the affine parallel-transporting connection expressed by (5.2.1.8). Therefore, in this case $W_{[ab],BH}(N_{BH}) = 0$ and $W_{\mu\nu,BH}$ is symmetric in all $\mathcal{N}_{BH}$.

On the other hand, if $\mathcal{M}_{\phi_\infty}$ is assumed to be endowed with more general (geo)metric structures, such as Einstein–Cartan (commutative) ones or noncommutative ones, in general, no relation exists between the affine and metric structures, and $\Gamma_{[ab]}{}^{c}(\phi_\infty) \neq 0$, generally implying that $\mathcal{M}_{\phi_\infty}$ is a nontrivial, torsionful real m_ϕ-d manifold. While the complexification of the Riemannian structure of $\mathcal{M}_{\phi_\infty}$ may (of course not necessarily) generate a (torsionless) Kähler metric structure (in n_V-fold, $N = 2$, $d = 4$ MESGT this is actually a regular special Kähler one), it is clear that the complexification of more general (geo)metric structures of $\mathcal{M}_{\phi_\infty}$ with nonsymmetric (linear) connection will not generate (torsionless) Kähler metric structures, since the following theorem holds for the complex connection $\Theta_{jk}{}^{i}$ of a Hermitian manifold $\mathcal{H}$ [37]:

$$\Theta_{jk}{}^{i} = \Theta_{kj}{}^{i} \Longleftrightarrow \mathcal{H} \text{ is (torsionless) Kähler.} \tag{5.2.1.9}$$

Thence, by assuming the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$ to be an m_ϕ-d real Riemannian manifold, the BH covariant Weinhold thermodynamical metric $W_{\mu\nu,BH}(N_{BH})$ defined by (5.2.1.2) has the following $(2n_V + m_{\phi_\infty} + 3) \times (2n_V + m_{\phi_\infty} + 3)$ matrix form:

$$
\begin{aligned}
&W_{\mu\nu,BH}\left(N_{BH}\right) \equiv D_{N^{\mu}_{BH}} D_{N^{\nu}_{BH}} M_{BH}\left(N_{BH}\right) \\
&= \begin{pmatrix}
D_{S_{BH}} D_{S_{BH}} M_{BH} & D_{p^{\Lambda}} D_{S_{BH}} M_{BH} & D_{q_{\Lambda}} D_{S_{BH}} M_{BH} & D_{\phi^{a}_{\infty}} D_{S_{BH}} M_{BH} \\
D_{S_{BH}} D_{p^{\Lambda}} M_{BH} & D_{p^{\Sigma}} D_{p^{\Lambda}} M_{BH} & D_{q_{\Sigma}} D_{p^{\Lambda}} M_{BH} & D_{\phi^{a}_{\infty}} D_{p^{\Lambda}} M_{BH} \\
D_{S_{BH}} D_{q_{\Lambda}} M_{BH} & D_{p^{\Sigma}} D_{q_{\Lambda}} M_{BH} & D_{q_{\Sigma}} D_{q_{\Lambda}} M_{BH} & D_{\phi^{a}_{\infty}} D_{q_{\Lambda}} M_{BH} \\
D_{S_{BH}} D_{\phi^{a}_{\infty}} M_{BH} & D_{p^{\Lambda}} D_{\phi^{a}_{\infty}} M_{BH} & D_{q_{\Lambda}} D_{\phi^{a}_{\infty}} M_{BH} & D_{\phi^{b}_{\infty}} D_{\phi^{a}_{\infty}} M_{BH}
\end{pmatrix} \\
&= \begin{pmatrix}
\frac{\partial^2 M_{BH}(N_{BH})}{(\partial S_{BH})^2} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial S_{BH}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial S_{BH}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Sigma}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Sigma}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial p^{\Lambda}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial S_{BH}\partial q_{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Sigma}\partial q_{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Sigma}\partial q_{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial q_{\Lambda}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial S_{BH}\partial \phi^{a}_{\infty}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Lambda}\partial \phi^{a}_{\infty}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Lambda}\partial \phi^{a}_{\infty}} & \begin{array}{l} \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial \phi^{b}_{\infty}} \\ -\left\{ {c \atop ab} \right\}(\phi_{\infty}) \frac{\partial M_{BH}(N_{BH})}{\partial \phi^{c}_{\infty}} \end{array}
\end{pmatrix} \\
&= \begin{pmatrix}
\frac{\partial^2 M_{BH}(N_{BH})}{(\partial S_{BH})^2} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial S_{BH}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial p^{\Sigma}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Sigma}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial p^{\Lambda}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Lambda}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Lambda}\partial p^{\Sigma}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial q_{\Sigma}\partial q_{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial q_{\Lambda}} \\
\frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial S_{BH}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial p^{\Lambda}} & \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial q_{\Lambda}} & \begin{array}{l} \frac{\partial^2 M_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}\partial \phi^{b}_{\infty}} \\ \\ -\left\{ {c \atop ab} \right\}(\phi_{\infty}) \frac{\partial M_{BH}(N_{BH})}{\partial \phi^{c}_{\infty}} \end{array}
\end{pmatrix}.
\end{aligned}
\tag{5.2.1.10}
$$

We may now introduce the BH heat capacity, defined as

$$
\begin{aligned}
C_{BH} &\equiv \left[D_{N^{1}_{BH}} D_{N^{1}_{BH}} M_{BH}\left(N_{BH}\right) \right]_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} \\
&= \left[D_{S_{BH}} D_{S_{BH}} M_{BH}\left(N_{BH}\right) \right]_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} \\
&= \left(\frac{\partial^2 M_{BH}\left(N_{BH}\right)}{\left(\partial S_{BH}\right)^2} \right)_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} \\
&= 16 \left(\frac{\partial^2 M_{BH}\left(N_{BH}\right)}{\left(\partial A_{H}\right)^2} \right)_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} = W_{11,BH}\left(N_{BH}\right),
\end{aligned}
\tag{5.2.1.11}
$$

where $S_{BH} = \frac{1}{4}A_H$ has been defined to be the BH extensive variable N^1_{BH}. In [83, 84], and [85] it was found that C_{BH} changes its sign for nonextreme, dilatonic BHs. Indeed, as explicitly shown in [85] at least for the case $n_V = 1$, during the process of BH "evaporation" the temperature increases, reaches its maximum, and rapidly drops to zero when the BH mass reaches the value of the central charge. The change of sign of C_{BH} happens at nonvanishing temperature, and this implies that the corresponding component $W_{11,BH}(N_{BH})$ of the Weinhold metric also changes its sign. It can be concluded that, in general, for nonextreme BHs the Weinhold metric (and therefore, through (5.1.30), also the Ruppeiner metric) is positive definite, but not everywhere in $\mathcal{N}_{BH}$.

Let us now reconsider the BH mass

$$M_{BH} = M_{BH}(N_{BH}) = M_{BH}\left(S_{BH}, p^\Lambda, q_\Lambda, \phi^a_\infty\right); \tag{5.2.1.12}$$

by differentiating it, we get

$$\begin{aligned} dM_{BH} = {} & \tfrac{\partial M_{BH}(N_{BH})}{\partial S_{BH}} dS_{BH} + \tfrac{\partial M_{BH}(N_{BH})}{\partial p^\Lambda} dp^\Lambda \\ & + \tfrac{\partial M_{BH}(N_{BH})}{\partial q_\Lambda} dq_\Lambda + D_{\phi^a_\infty} M_{BH}(N_{BH})\, d\phi^a_\infty, \end{aligned} \tag{5.2.1.13}$$

where $D_{\phi^a_\infty}$ denotes the Riemann-covariant derivative in $\mathcal{M}_{\phi_\infty}$. By canonically conjugating the extensive variables N^μ_{BH}, we may introduce the intensive variables $(\pi_{BH})_\mu \equiv \left(T_{BH}, \psi^\Lambda, \chi_\Lambda, -\Sigma_a\right)$ as follows:

$$(\pi_{BH})_\mu (N_{BH}) : \begin{cases} \pi_{S_{BH}} \equiv \frac{\partial M_{BH}(N_{BH})}{\partial S_{BH}} \equiv T_{BH}(N_{BH}); \\ \left(\pi^\Lambda_q\right)_{Weinhold} \equiv \frac{\partial M_{BH}(N_{BH})}{\partial q_\Lambda} \equiv \psi^\Lambda(N_{BH}); \\ (\pi_{p^\Lambda})_{Weinhold} \equiv \frac{\partial M_{BH}(N_{BH})}{\partial p^\Lambda} \equiv \chi_\Lambda(N_{BH}); \\ (\pi_{\phi_\infty})_{a,Weinhold} \equiv D_{\phi^a_\infty} M_{BH}(N_{BH}) = \frac{\partial M_{BH}(N_{BH})}{\partial \phi^a_\infty} \\ \qquad \equiv -\Sigma_a(N_{BH}), \end{cases} \tag{5.2.1.14}$$

where T_{BH} is the BH temperature and ψ^Λ and χ_Λ are the BH horizon electric and magnetic potentials, respectively. In the definition of the canonical momenta $(\pi_{\phi_\infty})_{a,Weinhold}$'s, conjugated to the ϕ^a_∞'s in the Weinhold's formalism, we exploited the scalarity of the function M_{BH} in $\mathcal{M}_{\phi_\infty}$, and, in order to introduce the "scalar charges" Σ_a's associated with the scalars ϕ^a_∞'s, we used the result of [59], in which it was shown that (see also (4.1.47))

$$lim_{\tau\to 0^-} \frac{d\phi^a(\tau)}{d\tau} = \left.\frac{d\phi^a(\tau)}{d\tau}\right|_{\tau\to 0^-} \equiv \Sigma^a = -G^{ab}(\phi_\infty)\frac{\partial M_{BH}(N_{BH})}{\partial \phi^b_\infty}. \tag{5.2.1.15}$$

The (real) intensive variables $(\pi_{BH})_\mu$'s may be considered as coordinates in the (real) $(2n_V + m_{\phi_\infty} + 3)$-d intensive thermodynamical configuration space $\widehat{\mathcal{N}}_{BH}$, which is the intensive counterpart of $\mathcal{N}_{BH}$, i.e., the intensively parameterized space of the thermodynamical configurations of the physical system being considered in Weinhold's formalism. System (5.2.1.14), assumed to be determined, solvable, and invertible, defines the change of parametrization of the BH thermodynamical configuration space of the physical system being considered in Weinhold's formalism: from extensive to intensive variables, and vice versa.

Hence, by definitions (5.2.1.14), the total differential of M_{BH} given by (5.2.1.13) may be rewritten as

$$dM_{BH} = T_{BH}dS_{BH} + \chi_\Lambda dp^\Lambda + \psi^\Lambda dq_\Lambda - \Sigma_a d\phi^a_\infty. \tag{5.2.1.16}$$

Also, by using definitions (5.2.1.14), the $(2n_V + m_{\phi_\infty}+ 3)\times(2n_V + m_{\phi_\infty}+ 3)$ matrix expression (5.2.1.10) of the covariant Weinhold metric $W_{\mu\nu,BH}$ may be further elaborated as follows:

$$\begin{aligned}
&W_{\mu\nu,BH}\left(N_{BH}\right) \equiv D_{N^\mu_{BH}} D_{N^\nu_{BH}} M_{BH}\left(N_{BH}\right)\\
&= \begin{pmatrix}
\frac{\partial T_{BH}(N_{BH})}{\partial S_{BH}} & \frac{\partial T_{BH}(N_{BH})}{\partial p^\Lambda} & \frac{\partial T_{BH}(N_{BH})}{\partial q_\Lambda} & \frac{\partial T_{BH}(N_{BH})}{\partial \phi^a_\infty} \\[2ex]
\frac{\partial T_{BH}(N_{BH})}{\partial p^\Lambda} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial p^\Sigma} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial q_\Sigma} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial \phi^a_\infty} \\[2ex]
\frac{\partial T_{BH}(N_{BH})}{\partial q_\Lambda} & \frac{\partial \chi_\Sigma(N_{BH})}{\partial q_\Lambda} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial q_\Sigma} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial \phi^a_\infty} \\[2ex]
\frac{\partial T_{BH}(N_{BH})}{\partial \phi^a_\infty} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial \phi^a_\infty} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial \phi^a_\infty} & \begin{array}{c} -\frac{\partial \Sigma_b(N_{BH})}{\partial \phi^a_\infty} \\ + \left\{{}^{\,c}_{ab}\right\}\left(\phi_\infty\right)\Sigma_c\left(N_{BH}\right) \end{array}
\end{pmatrix}\\
&= \begin{pmatrix}
\frac{\partial T_{BH}(N_{BH})}{\partial S_{BH}} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial S_{BH}} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial S_{BH}} & -\frac{\partial \Sigma_a(N_{BH})}{\partial S_{BH}} \\[2ex]
\frac{\partial \chi_\Lambda(N_{BH})}{\partial S_{BH}} & \frac{\partial \chi_\Lambda(N_{BH})}{\partial p^\Sigma} & \frac{\partial \psi^\Sigma(N_{BH})}{\partial p^\Lambda} & -\frac{\partial \Sigma_a(N_{BH})}{\partial p^\Lambda} \\[2ex]
\frac{\partial \psi^\Lambda(N_{BH})}{\partial S_{BH}} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial p^\Sigma} & \frac{\partial \psi^\Lambda(N_{BH})}{\partial q_\Sigma} & -\frac{\partial \Sigma_a(N_{BH})}{\partial q_\Lambda} \\[2ex]
-\frac{\partial \Sigma_a(N_{BH})}{\partial S_{BH}} & -\frac{\partial \Sigma_a(N_{BH})}{\partial p^\Lambda} & -\frac{\partial \Sigma_a(N_{BH})}{\partial q_\Lambda} & \begin{array}{c} -\frac{\partial \Sigma_b(N_{BH})}{\partial \phi^a_\infty} \\ + \left\{{}^{\,c}_{ab}\right\}\left(\phi_\infty\right)\Sigma_c\left(N_{BH}\right) \end{array}
\end{pmatrix},
\end{aligned} \tag{5.2.1.17}$$

where in the last line we exploited the fact that the scalar function $M_{BH}\left(N_{BH}\right)$ is assumed to satisfy the Schwarz lemma in $\mathcal{N}_{BH}$ w.r.t. flat, ordinary derivatives, which by using definitions (5.2.1.14) implies

$$\begin{cases} \frac{\partial T_{BH}(N_{BH})}{\partial p^{\Lambda}} = \frac{\partial \chi_{\Lambda}(N_{BH})}{\partial S_{BH}}, \\ \frac{\partial T_{BH}(N_{BH})}{\partial q_{\Lambda}} = \frac{\partial \psi^{\Lambda}(N_{BH})}{\partial S_{BH}}, \\ \frac{\partial T_{BH}(N_{BH})}{\partial \phi^{a}_{\infty}} = -\frac{\partial \Sigma_{a}(N_{BH})}{\partial S_{BH}}, \\ \frac{\partial \chi_{\Lambda}(N_{BH})}{\partial q_{\Sigma}} = \frac{\partial \psi^{\Sigma}(N_{BH})}{\partial p^{\Lambda}}, \\ \frac{\partial \chi_{\Lambda}(N_{BH})}{\partial \phi^{a}_{\infty}} = -\frac{\partial \Sigma_{a}(N_{BH})}{\partial p^{\Lambda}}, \\ \frac{\partial \psi^{\Lambda}(N_{BH})}{\partial \phi^{a}_{\infty}} = -\frac{\partial \Sigma_{a}(N_{BH})}{\partial q_{\Lambda}}. \end{cases} \tag{5.2.1.18}$$

Consequently, by using definitions (5.2.1.14), also (5.2.1.11) may be rewritten as

$$\begin{aligned} C_{BH} &= \left(\frac{\partial^2 M_{BH}(N_{BH})}{(\partial S_{BH})^2} \right)_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} \\ &= \left(\frac{\partial T_{BH}(N_{BH})}{\partial S_{BH}} \right)_{(p^{\Lambda}, q_{\Lambda}, \phi^{a}_{\infty})\ fixed} = W_{11,BH}(N_{BH}). \end{aligned} \tag{5.2.1.19}$$

The fact that the asymptotical moduli space $\mathcal{M}_{\phi_{\infty}}$ (here assumed to be a Riemannian m_{ϕ}-d manifold) is not flat complicates, but clearly does not invalidate, the usual thermodynamical formalism involving Legendre transforms. In this case, one should rigorously consider Legendre submanifolds in the "ϕ^{a}_{∞}'s sector," etc., but in what follows we will not deal with such subtleties; nevertheless, such a simplification will not invalidate the exposed results at all.

Thus, by Legendre-transforming the BH energy M_{BH} w.r.t. the Weinhold extensive BH variables N^{μ}_{BH}'s, one gets

$$\begin{aligned} &\mathcal{L}(M_{BH}) \equiv \pi_{S_{BH}} S_{BH} + \left(\pi^{\Lambda}_{q}\right)_{Weinhold} q_{\Lambda} \\ &+ (\pi_{p\Lambda})_{Weinhold}\, p^{\Lambda} + (\pi_{\phi_{\infty}})_{a,Weinhold}\, \phi^{a}_{\infty} - M_{BH} \\ &= T_{BH} S_{BH}(\pi_{BH}) + \psi^{\Lambda} q_{\Lambda}(\pi_{BH}) + \chi_{\Lambda} p^{\Lambda}(\pi_{BH}) \\ &- \Sigma_{a} \phi^{a}_{\infty}(\pi_{BH}) - M_{BH}(N_{BH}(\pi_{BH})) \\ &= -G_{BH}(\pi_{BH}), \end{aligned} \tag{5.2.1.20}$$

where system (5.2.1.14) has been inverted in order to give $N^{\mu}_{BH} = N^{\mu}_{BH}(\pi_{BH})$. $G_{BH}(\pi_{BH})$ is the BH Gibbs free energy function of the BH intensive thermodynamical variables $(\pi_{BH})_{\mu}$'s. Clearly, (5.2.1.20) yields

$$\begin{aligned}M_{BH}+\mathcal{L}\left(M_{BH}\right)&=M_{BH}-G_{BH}=\\&=T_{BH}S_{BH}+\psi^{\Lambda}q_{\Lambda}+\chi_{\Lambda}p^{\Lambda}-\Sigma_{a}\phi_{\infty}^{a}=N_{BH}^{\mu}\left(\pi_{BH}\right)_{\mu}.\end{aligned} \tag{5.2.1.21}$$

By differentiating (5.2.1.20) and using (5.2.1.16), one gets

$$\begin{aligned}&dG_{BH}=-d\left(\mathcal{L}\left(M_{BH}\right)\right)\\&=-T_{BH}dS_{BH}-S_{BH}dT_{BH}-\psi^{\Lambda}dq_{\Lambda}-q_{\Lambda}d\psi^{\Lambda}\\&-\chi_{\Lambda}dp^{\Lambda}-p^{\Lambda}d\chi_{\Lambda}+\Sigma_{a}d\phi_{\infty}^{a}+\phi_{\infty}^{a}d\Sigma_{a}+dM_{BH}\\&=-S_{BH}dT_{BH}-q_{\Lambda}d\psi^{\Lambda}-p^{\Lambda}d\chi_{\Lambda}+\phi_{\infty}^{a}d\Sigma_{a}\\&=-N_{BH}^{\mu}d\left(\pi_{BH}\right)_{\mu}.\end{aligned} \tag{5.2.1.22}$$

The Legendre transform corresponds to the switching from the $(2n_V+m_{\phi_\infty}+3)$-d (real) space of the BH thermodynamical configurations

$$\mathcal{N}_{BH}\equiv\mathbb{A}^{2n_V+3}\times\mathcal{M}_{\phi_\infty}, \tag{5.2.1.23}$$

parameterized by the Weinhold's BH extensive variables $N_{BH}^{\mu}=(S_{BH},p^{\Lambda},q_{\Lambda},\phi_{\infty}^{a})$, to the canonically conjugated space

$$\widehat{\mathcal{N}}_{BH}\equiv\widehat{\mathbb{A}}^{2n_V+3}\times\mathcal{M}_{\Sigma}, \tag{5.2.1.24}$$

parameterized by the Weinhold's BH intensive variables $(\pi_{BH})_{\mu}=(T_{BH},\psi^{\Lambda},\chi_{\Lambda},-\Sigma_a)$ defined by (5.2.1.14). Correspondingly, we obtain that the inversion of definition (5.2.1.2) yields the contravariant BH Weinhold metric

$$\begin{aligned}W_{BH}^{\mu\nu}\left(\pi_{BH}\right)&\equiv D_{\left(\pi_{BH}\right)_{\mu}}D_{\left(\pi_{BH}\right)_{\nu}}\left(\mathcal{L}\left(M_{BH}\right)\right)\left(\pi_{BH}\right)\\&=-D_{\left(\pi_{BH}\right)_{\mu}}D_{\left(\pi_{BH}\right)_{\nu}}G_{BH}\left(\pi_{BH}\right),\end{aligned} \tag{5.2.1.25}$$

with

$$W_{BH}^{\mu\rho}\left(\pi_{BH}\left(N_{BH}\right)\right)W_{\rho\nu,BH}\left(N_{BH}\right)=\delta_{\nu}^{\mu},\ \forall N_{BH}^{\sigma}, \tag{5.2.1.26}$$

where $\left(\pi_{BH}\right)_{\mu}=\left(\pi_{BH}\right)_{\mu}\left(N_{BH}\right)$ is given by system (5.2.1.14).

Particular attention should be paid to distinguish the metric structure of the extensive and intensive BH thermodynamical configuration spaces in Weinhold's formalism, namely $\mathcal{N}_{BH}$ and $\widehat{\mathcal{N}}_{BH}$ (whose factorized structure is respectively given by (5.2.1.23) and (5.2.1.24)), from the Weinhold covariant and contravariant BH thermodynamical metrics, namely $W_{\mu\nu,BH}\left(N_{BH}\right)$ and

$W^{\mu\nu}_{BH}(\pi_{BH})$, defined on such spaces. Indeed, the (geo)metric structure of the $(2n_V + m_{\phi_\infty} + 3)$-d real space $\mathcal{N}_{BH}$ is given by the metric.

$$G_{\mu\nu}(\phi_\infty) = \begin{pmatrix} 1 & & \\ & \epsilon_{n_V+1} & \\ & & G_{ab}(\phi_\infty) \end{pmatrix}, \tag{5.2.1.27}$$

where the unwritten components vanish, and ϵ_{n_V+1} denotes the $(2n_V+2)$-d symplectic metric. $G_{ab}(\phi_\infty)$ is the covariant Riemann metric tensor of the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$. Thus, it is clear that $G_{\mu\nu}(\phi_\infty)$ is completely different from the covariant Weinhold metric $W_{\mu\nu,BH}(N_{BH})$, whose matrix form is given by (5.2.1.10) and (5.2.1.17). Analogously, the (geo)metric structure of the $(2n_V + m_{\phi_\infty} + 3)$-d real space $\widehat{\mathcal{N}}_{BH}$ is given by the metric

$$G^{\mu\nu}(\phi_\infty) = \begin{pmatrix} 1 & & \\ & -\epsilon_{n_V+1} & \\ & & G^{ab}(\phi_\infty) \end{pmatrix}, \tag{5.2.1.28}$$

which is the inverse of $G_{\mu\nu}(\phi_\infty)$ given by (5.2.1.27). $G^{ab}(\phi_\infty)$ is the contravariant Riemann metric tensor of the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$, or equivalently the metric tensor of the canonically conjugated space $\mathcal{M}_\Sigma$ of the "scalar charges" asymptotically associated with the scalars ϕ^a's. Thus, it is clear that $G^{\mu\nu}(\phi_\infty)$ is completely different from the contravariant Weinhold metric $W^{\mu\nu}_{BH}(\pi_{BH})$, which is defined by (5.2.1.25); indeed, through (5.2.1.26), $W^{\mu\nu}_{BH}(\pi_{BH})$ is the inverse of $W_{\mu\nu,BH}(N_{BH})$, and thus its matrix form is given by the inversion of (5.2.1.10) and (5.2.1.17). Clearly, since the scalar function G_{BH} is assumed to satisfy the Schwarz lemma on partial derivatives in the space $\widehat{\mathcal{N}}_{BH}$, in the formulated hypothesis of Riemann structure of $\mathcal{M}_{\phi_\infty}$ (and therefore of $\mathcal{M}_\Sigma$) the contravariant BH Weinhold metric is symmetric in all $\widehat{\mathcal{N}}_{BH}$.

5.2.2 Ruppeiner Black Hole Thermodynamics

Let us now consider the Ruppeiner's geometrizing formalism. In general, the Ruppeiner metric is defined by (5.1.20) and (5.1.14):

$$\begin{cases} S_{\mu\nu}(Q) \equiv -\frac{\partial^2 S(Q)}{\partial Q^\mu \partial Q^\nu}; \\ \\ Q^\mu = (M, N^a), \ a = 1, \dots, n. \end{cases} \tag{5.2.2.1}$$

Thence, the corresponding definition for BH physics reads

$$\begin{cases} S_{\mu\nu,BH}(Q_{BH}) \equiv -D_{Q^\mu_{BH}} D_{Q^\nu_{BH}} S_{BH}(Q_{BH}); \\ \\ Q^\mu_{BH} = \left(M_{BH}, p^\Lambda, q_\Lambda, \phi^a_\infty\right), \\ \\ \Lambda = 0, 1, \dots, n_V, \quad a = 1, \dots, m_\phi, \quad |\{\mu\}| = 2n_V + m_{\phi_\infty} + 3. \end{cases} \tag{5.2.2.2}$$

In conventional thermodynamics of static (spherically symmetric, asymptotically flat, 4-d) BHs, since the entropy is always the biggest among the equilibrium configurations with a given energy M_{BH}, conserved magnetic–electric charges $\left(p^{\Lambda}, q_{\Lambda}\right)$ and asymptotical scalar configurations ϕ_{∞}^{a}'s, the BH Ruppeiner metric $S_{\mu\nu,BH}$ is always (not necessarily strictly) positive definite.

$D_{Q_{BH}^{\mu}}$ stands for the covariant derivative in the $(2n_V + m_{\phi_\infty} + 3)$-d (real) Ruppeiner's extensive BH thermodynamical configuration space $\mathcal{V}_{BH}$, of which the Q_{BH}^{μ}'s may be considered as (real) local coordinates. $\mathcal{V}_{BH}$ is assumed to be given by the direct product

$$\mathcal{V}_{BH} \equiv \mathbb{B}^{2n_V+3} \times \mathcal{M}_{\phi_\infty}. \tag{5.2.2.3}$$

$\mathbb{B}^{2n_V+3}$ here denotes the flat $(2n_V + 3)$-d Ruppeiner's thermodynamical configuration space related to the extensive variables M_{BH}, p^{Λ}, and q_{Λ}. For what concerns its electric–magnetic sector of p^{Λ}'s and q_{Λ}'s, identical considerations to those made in Weinhold's formalism hold here, too.

Summarizing, the factorized Ansatz (5.2.2.3) for the BH Ruppeiner's extensive thermodynamical configuration space $\mathcal{V}_{BH}$ yields the following expression for the covariant differential operator in $\mathcal{V}_{BH}$:

$$D_{Q_{BH}^{\mu}} = \left(\partial_{M_{BH}}, \partial_{p^{\Lambda}}, \partial_{q_{\Lambda}}, D_{\phi_{\infty}^{a}}\right); \tag{5.2.2.4}$$

as above, the only nontrivially ordinary derivatives are those w.r.t. the variables ϕ_{∞}^{a}'s.

Thus, since S_{BH} is a scalar function in $\mathcal{V}_{BH}$ and thus in $\mathcal{M}_{\phi_\infty}$, the Ruppeiner metric related to the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$ (i.e., the covariant Ruppeiner metric $S_{\mu\nu,BH}$ restricted to the asymptotical moduli configurations ϕ_{∞}^{a}'s) reads

$$\begin{aligned} S_{ab,BH}\left(Q_{BH}\right) &\equiv -D_{\phi_{\infty}^{a}} D_{\phi_{\infty}^{b}} S_{BH}\left(Q_{BH}\right) = -D_{\phi_{\infty}^{a}} \frac{\partial S_{BH}\left(Q_{BH}\right)}{\partial \phi_{\infty}^{b}} \\ &= -\frac{\partial^2 S_{BH}\left(Q_{BH}\right)}{\partial \phi_{\infty}^{a} \partial \phi_{\infty}^{b}} + \Gamma_{ab}{}^{c}\left(\phi_\infty\right) \frac{\partial S_{BH}\left(Q_{BH}\right)}{\partial \phi_{\infty}^{c}}, \end{aligned} \tag{5.2.2.5}$$

where $\Gamma_{ab}{}^{c}\left(\phi_\infty\right)$ is the (linear) connection determining the affine structure of the m_ϕ-d real manifold $\mathcal{M}_{\phi_\infty}$. Since the function S_{BH} is assumed to satisfy the Schwarz lemma in $\mathcal{V}_{BH}$ w.r.t. flat, ordinary derivatives, it is clear that the Ruppeiner metric $S_{\mu\nu,BH}$ defined by (5.2.2.2) is symmetric in all $\mathcal{V}_{BH}$, except eventually in the sector "$\mathcal{M}_{\phi_\infty}$,$\mathcal{M}_{\phi_\infty}$" of the ϕ_{∞}^{a}'s, where the symmetry of $S_{ab,BH}$ strictly depends on the symmetry of the connection $\Gamma_{ab}{}^{c}\left(\phi_\infty\right)$ of $\mathcal{M}_{\phi_\infty}$

$$\Gamma_{[ab]}{}^{c}\left(\phi_\infty\right) = 0 \Longleftrightarrow S_{[ab],BH}\left(Q_{BH}\right) = 0. \tag{5.2.2.6}$$

As above, if a real Riemannian geometry is assumed for $\mathcal{M}_{\phi_\infty}$, then the affine and metric structures of the Riemannian moduli space $\mathcal{M}_{\phi_\infty}$ are compatible and strictly related by the metrization of the affine parallel-transporting connection expressed by (5.2.1.8). Therefore, in this case

$$S_{[ab],BH}\left(Q_{BH}\right)=0, \tag{5.2.2.7}$$

and $S_{\mu\nu,BH}$ is symmetric in all $\mathcal{V}_{BH}$.

Thence, by assuming the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$ to be a m_ϕ-d real Riemannian manifold, the BH covariant Ruppeiner thermodynamical metric $S_{\mu\nu,BH}\left(Q_{BH}\right)$ defined by (5.2.2.2) has the following $(2n_V+m_{\phi_\infty}+3)\times(2n_V+m_{\phi_\infty}+3)$ matrix form:

$$S_{\mu\nu,BH}\left(Q_{BH}\right)\equiv -D_{Q^\mu_{BH}}D_{Q^\nu_{BH}}S_{BH}\left(Q_{BH}\right)$$

$$=-\begin{pmatrix} D_{M_{BH}}D_{M_{BH}}S_{BH} & D_{p^\Lambda}D_{M_{BH}}S_{BH} & D_{q_\Lambda}D_{M_{BH}}S_{BH} & D_{\phi^a_\infty}D_{M_{BH}}S_{BH} \\ D_{M_{BH}}D_{p^\Lambda}S_{BH} & D_{p^\Sigma}D_{p^\Lambda}S_{BH} & D_{q_\Sigma}D_{p^\Lambda}S_{BH} & D_{\phi^a_\infty}D_{p^\Lambda}S_{BH} \\ D_{M_{BH}}D_{q_\Lambda}S_{BH} & D_{p^\Sigma}D_{q_\Lambda}S_{BH} & D_{q_\Sigma}D_{q_\Lambda}S_{BH} & D_{\phi^a_\infty}D_{q_\Lambda}S_{BH} \\ D_{M_{BH}}D_{\phi^a_\infty}S_{BH} & D_{p^\Lambda}D_{\phi^a_\infty}S_{BH} & D_{q_\Lambda}D_{\phi^a_\infty}S_{BH} & D_{\phi^b_\infty}D_{\phi^a_\infty}S_{BH} \end{pmatrix}$$

$$=-\begin{pmatrix} \frac{\partial^2 S_{BH}(Q_{BH})}{(\partial M_{BH})^2} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial M_{BH}} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial M_{BH}\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Sigma\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Sigma\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial p^\Lambda} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial M_{BH}\partial q_\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Sigma\partial q_\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Sigma\partial q_\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial q_\Lambda} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial M_{BH}\partial\phi^a_\infty} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Lambda\partial\phi^a_\infty} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Lambda\partial\phi^a_\infty} & \begin{array}{c}\frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial\phi^b_\infty} \\ -\left\{{c \atop ab}\right\}(\phi_\infty)\frac{\partial S_{BH}(Q_{BH})}{\partial\phi^c_\infty}\end{array} \end{pmatrix}$$

$$=-\begin{pmatrix} \frac{\partial^2 S_{BH}(Q_{BH})}{(\partial M_{BH})^2} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial M_{BH}} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial p^\Sigma\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Sigma\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial p^\Lambda} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Lambda\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Lambda\partial p^\Sigma} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial q_\Sigma\partial q_\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial q_\Lambda} \\ \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial M_{BH}} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial p^\Lambda} & \frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial q_\Lambda} & \begin{array}{c}\frac{\partial^2 S_{BH}(Q_{BH})}{\partial\phi^a_\infty\partial\phi^b_\infty} \\ -\left\{{c \atop ab}\right\}(\phi_\infty)\frac{\partial S_{BH}(Q_{BH})}{\partial\phi^c_\infty}\end{array} \end{pmatrix}. \tag{5.2.2.8}$$

Let us now consider the BH entropy in the Ruppeiner's formalism:

$$S_{BH}=S_{BH}\left(Q_{BH}\right)=S_{BH}\left(M_{BH},p^\Lambda,q_\Lambda,\phi^a_\infty\right). \tag{5.2.2.9}$$

Equation (5.2.1.16) implies

$$dS_{BH} = \frac{dM_{BH}}{T_{BH}} - \frac{\chi_\Lambda}{T_{BH}} dp^\Lambda - \frac{\psi^\Lambda}{T_{BH}} dq_\Lambda + \frac{\Sigma_a}{T_{BH}} d\phi_\infty^a. \tag{5.2.2.10}$$

On the other hand, by differentiating (5.2.2.9), we get

$$\begin{aligned} dS_{BH} = & \frac{\partial S_{BH}(Q_{BH})}{\partial M_{BH}} dM_{BH} \\ & + \frac{\partial S_{BH}(Q_{BH})}{\partial p^\Lambda} dp^\Lambda + \frac{\partial S_{BH}(Q_{BH})}{\partial q_\Lambda} dq_\Lambda + D_{\phi_\infty^a} S_{BH}(Q_{BH})\, d\phi_\infty^a, \end{aligned} \tag{5.2.2.11}$$

where $D_{\phi_\infty^a}$ denotes the Riemann-covariant derivative in $\mathcal{M}_{\phi_\infty}$. From the comparison of the two expressions (5.2.2.10) and (5.2.2.11) of the total differential of the BH entropy in Ruppeiner's formalism, we obtain the definitions of the Ruppeiner's intensive variables $(\beta_{BH})_\mu$'s (canonically conjugated to the Ruppeiner's extensive variables Q_{BH}^μ's):

$$(\beta_{BH})_\mu (Q_{BH}) : \begin{cases} \pi_{M_{BH}} \equiv \frac{\partial S_{BH}(Q_{BH})}{\partial M_{BH}} \equiv \frac{1}{T_{BH}} (Q_{BH}); \\ \\ \left(\pi_q^\Lambda\right)_{Ruppeiner} \equiv \frac{\partial S_{BH}(Q_{BH})}{\partial q_\Lambda} \equiv -\frac{\psi^\Lambda}{T_{BH}} (Q_{BH}); \\ \\ \left(\pi_{p\Lambda}\right)_{Ruppeiner} \equiv \frac{\partial S_{BH}(Q_{BH})}{\partial p^\Lambda} \equiv -\frac{\chi_\Lambda}{T_{BH}} (Q_{BH}); \\ \\ \left(\pi_{\phi_\infty}\right)_{a,Ruppeiner} \equiv D_{\phi_\infty^a} S_{BH}(Q_{BH}) = \frac{\partial S_{BH}(Q_{BH})}{\partial \phi_\infty^a} \equiv \frac{\Sigma_a}{T_{BH}} (Q_{BH}). \end{cases} \tag{5.2.2.12}$$

As before, in the definition of the canonical momenta $\left(\pi_{\phi_\infty}\right)_{a,Ruppeiner}$'s, conjugated to the ϕ_∞^a's in the Ruppeiner's formalism, we exploited the scalarity of the function S_{BH} in $\mathcal{M}_{\phi_\infty}$.

The (real) Ruppeiner's intensive variables $(\beta_{BH})_\mu$'s may be considered as coordinates in the (real) $(2n_V + m_{\phi_\infty} + 3)$-d Ruppeiner intensive thermodynamical configuration space $\widehat{\mathcal{V}}_{BH}$, which is nothing but the intensive counterpart of $\mathcal{V}_{BH}$, i.e., the intensively parameterized space of the thermodynamical configurations of the physical system being considered in Ruppeiner's formalism. System (5.2.2.12), assumed to be determined, solvable, and invertible, defines the change of parametrization of the BH thermodynamical configuration space of the physical system being considered in Ruppeiner's formalism: from extensive to intensive variables, and vice versa.

Also, by using definitions (5.2.2.12), the $(2n_V + m_{\phi_\infty} + 3) \times (2n_V + m_{\phi_\infty} + 3)$ matrix expression (5.2.2.8) of the covariant Ruppeiner metric $S_{\mu\nu,BH}$ may be further elaborated as follows:

$$
\begin{aligned}
&S_{\mu\nu,BH}\left(Q_{BH}\right) \equiv -D_{Q^{\mu}_{BH}} D_{Q^{\nu}_{BH}} S_{BH}\left(Q_{BH}\right)\\
&= -\begin{pmatrix}
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & \frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} & \frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} & \frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} \\
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Sigma}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Sigma}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} \\
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} & -\frac{\partial\left(\frac{\chi_{\Sigma}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Sigma}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} \\
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} & \begin{array}{c}\frac{\partial\left(\frac{\Sigma_{b}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} \\ -\left\{{}^{\,c}_{ab}\right\}(\phi_{\infty})\,\frac{\Sigma_{c}}{T_{BH}}\left(Q_{BH}\right)\end{array}
\end{pmatrix}\\
&= -\begin{pmatrix}
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} \\
-\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Sigma}} & -\frac{\partial\left(\frac{\psi^{\Sigma}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} & \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} \\
-\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Sigma}} & -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Sigma}} & \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} \\
\frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}} & \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} & \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} & \begin{array}{c}\frac{\partial\left(\frac{\Sigma_{b}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} \\ -\left\{{}^{\,c}_{ab}\right\}(\phi_{\infty})\,\frac{\Sigma_{c}}{T_{BH}}\left(Q_{BH}\right)\end{array}
\end{pmatrix},
\end{aligned}
\tag{5.2.2.13}
$$

where in the last line we exploited the fact that the scalar function $S_{BH}\left(Q_{BH}\right)$ is assumed to satisfy the Schwarz lemma in $\mathcal{V}_{BH}$ w.r.t. flat, ordinary derivatives, which by using definitions (5.2.2.12) implies

$$
\left\{
\begin{array}{l}
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}} = -\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}}, \\
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}} = -\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}}, \\
\frac{\partial\left(\frac{1}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} = \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial M_{BH}}, \\
\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Sigma}} = \frac{\partial\left(\frac{\psi^{\Sigma}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}}, \\
-\frac{\partial\left(\frac{\chi_{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} = \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial p^{\Lambda}}, \\
-\frac{\partial\left(\frac{\psi^{\Lambda}}{T_{BH}}(Q_{BH})\right)}{\partial \phi^{a}_{\infty}} = \frac{\partial\left(\frac{\Sigma_{a}}{T_{BH}}(Q_{BH})\right)}{\partial q_{\Lambda}}.
\end{array}
\right.
\tag{5.2.2.14}
$$

Now, by Legendre transforming the BH entropy S_{BH} w.r.t. the Ruppeiner extensive BH variables Q^{μ}_{BH}'s, one gets

$$\begin{aligned}
&\mathcal{L}\left(S_{BH}\right) \equiv \pi_{M_{BH}} M_{BH} + \left(\pi_q^{\Lambda}\right)_{Ruppeiner} q_{\Lambda} \\
&+ \left(\pi_{p\Lambda}\right)_{Ruppeiner} p^{\Lambda} + \left(\pi_{\phi_\infty}\right)_{a,Ruppeiner} \phi_\infty^a - S_{BH} \\
&= M_{BH}\left(\beta_{BH}\right) \tfrac{1}{T_{BH}} - q_{\Lambda}\left(\beta_{BH}\right) \tfrac{\psi^{\Lambda}}{T_{BH}} \\
&-p^{\Lambda}\left(\beta_{BH}\right) \tfrac{\chi_{\Lambda}}{T_{BH}} + \phi_\infty^a\left(\beta_{BH}\right) \tfrac{\Sigma_a}{T_{BH}} - S_{BH}\left(Q_{BH}\left(\beta_{BH}\right)\right) \\
&= \tfrac{G_{BH}\left(\pi_{BH}\right)}{T_{BH}} \equiv \Gamma_{BH}\left(\beta_{BH}\right),
\end{aligned} \tag{5.2.2.15}$$

where system (5.2.2.12) has been inverted in order to give $Q_{BH}^{\mu} = Q_{BH}^{\mu}\left(\beta_{BH}\right)$. $\Gamma_{BH}\left(\beta_{BH}\right)$ is defined as the Legendre transform of the BH entropy $S_{BH}\left(Q_{BH}\right)$ in Ruppeiner's formalism.

Clearly, (5.2.2.15) yields

$$\begin{aligned}
S_{BH} + \mathcal{L}\left(S_{BH}\right) &= S_{BH} + \Gamma_{BH} \\
&= M_{BH}\frac{1}{T_{BH}} - q_{\Lambda}\frac{\psi^{\Lambda}}{T_{BH}} - p^{\Lambda}\frac{\chi_{\Lambda}}{T_{BH}} + \phi_\infty^a\frac{\Sigma_a}{T_{BH}} = Q_{BH}^{\mu}\left(\beta_{BH}\right)_{\mu}.
\end{aligned} \tag{5.2.2.16}$$

By differentiating (5.2.2.15) and using (5.2.2.10), one gets

$$\begin{aligned}
&d\Gamma_{BH} = d\left(\mathcal{L}\left(S_{BH}\right)\right) \\
&= \tfrac{1}{T_{BH}} dM_{BH} + M_{BH} d\left(\tfrac{1}{T_{BH}}\right) - \tfrac{\psi^{\Lambda}}{T_{BH}} dq_{\Lambda} + q_{\Lambda} d\left(-\tfrac{\psi^{\Lambda}}{T_{BH}}\right) \\
&- \tfrac{\chi_{\Lambda}}{T_{BH}} dp^{\Lambda} + p^{\Lambda} d\left(-\tfrac{\chi_{\Lambda}}{T_{BH}}\right) + \tfrac{\Sigma_a}{T_{BH}} d\phi_\infty^a + \phi_\infty^a d\left(\tfrac{\Sigma_a}{T_{BH}}\right) - dS_{BH} \\
&= M_{BH} d\left(\tfrac{1}{T_{BH}}\right) + q_{\Lambda} d\left(-\tfrac{\psi^{\Lambda}}{T_{BH}}\right) + p^{\Lambda} d\left(-\tfrac{\chi_{\Lambda}}{T_{BH}}\right) + \phi_\infty^a d\left(\tfrac{\Sigma_a}{T_{BH}}\right) \\
&= Q_{BH}^{\mu} d\left(\beta_{BH}\right)_{\mu}.
\end{aligned} \tag{5.2.2.17}$$

The Legendre transform corresponds to the switching from the $(2n_V + m_{\phi_\infty} + 3)$-d (real) Ruppeiner space of the BH thermodynamical configurations

$$\mathcal{V}_{BH} \equiv \mathbb{B}^{2n_V+3} \times \mathcal{M}_{\phi_\infty}, \tag{5.2.2.18}$$

parameterized by the Ruppeiner's BH extensive variables $Q_{BH}^{\mu} = (M_{BH}, p^{\Lambda}, q_{\Lambda}, \phi_\infty^a)$, to the canonically conjugated space

$$\widehat{\mathcal{V}}_{BH} \equiv \widehat{\mathbb{B}}^{2n_V+3} \times \mathcal{M}_{\frac{\Sigma}{T_{BH}}}, \tag{5.2.2.19}$$

parameterized by the Ruppeiner's BH intensive variables $\left(\beta_{BH}\right)_{\mu} = \left(\frac{1}{T_{BH}}, -\frac{\psi^{\Lambda}}{T_{BH}}, -\frac{\chi_{\Lambda}}{T_{BH}}, \frac{\Sigma_a}{T_{BH}}\right)$ defined by (5.2.2.12). Correspondingly, we obtain that the inversion of definition (5.2.2.2) yields the contravariant BH Ruppeiner metric

$$\begin{aligned}
S_{BH}^{\mu\nu}\left(\beta_{BH}\right) &\equiv -D_{\left(\beta_{BH}\right)_{\mu}} D_{\left(\beta_{BH}\right)_{\nu}} \left(\mathcal{L}\left(S_{BH}\right)\right)\left(\beta_{BH}\right) \\
&= -D_{\left(\beta_{BH}\right)_{\mu}} D_{\left(\beta_{BH}\right)_{\nu}} \Gamma_{BH}\left(\beta_{BH}\right),
\end{aligned} \tag{5.2.2.20}$$

with

$$S_{BH}^{\mu\rho}\left(\beta_{BH}\left(Q_{BH}\right)\right) S_{\rho\nu,BH}\left(Q_{BH}\right) = \delta_{\nu}^{\mu}, \ \forall Q_{BH}^{\sigma}, \tag{5.2.2.21}$$

where $\left(\beta_{BH}\right)_{\mu} = \left(\beta_{BH}\right)_{\mu}\left(Q_{BH}\right)$ is given by system (5.2.2.12).

Once again, particular attention should be paid to distinguish the metric structure of the extensive and intensive BH thermodynamical configuration spaces in Ruppeiner's formalism, namely $\mathcal{V}_{BH}$ and $\widehat{\mathcal{V}}_{BH}$ (whose factorized structure is respectively given by (5.2.2.18) and (5.2.2.19)), from the Ruppeiner covariant and contravariant BH thermodynamical metrics, namely $S_{\mu\nu,BH}\left(Q_{BH}\right)$ and $S^{\mu\nu}_{BH}\left(\beta_{BH}\right)$, defined on such spaces. Indeed, the (geo)metric structure of the $(2n_V + m_{\phi_\infty} + 3)$-d real space $\mathcal{V}_{BH}$ is given by the metric $G_{\mu\nu}\left(\phi_\infty\right)$ specified by (5.2.1.27), which is completely different from the covariant Ruppeiner metric $S_{\mu\nu,BH}\left(Q_{BH}\right)$, whose matrix form is given by (5.2.2.8) and (5.2.2.13). Analogously, the (geo)metric structure of the $(2n_V + m_{\phi_\infty} + 3)$-d real space $\widehat{\mathcal{V}}_{BH}$ is given by the metric $G^{\mu\nu}\left(\phi_\infty\right)$ specified by (5.2.1.28), which is the inverse of (5.2.1.27). In Ruppeiner's formalism the metric $G^{ab}\left(\phi_\infty\right)$ of (5.2.1.28) is the contravariant Riemann metric tensor of the asymptotical moduli space $\mathcal{M}_{\phi_\infty}$, or equivalently the metric tensor of the canonically conjugated space $\mathcal{M}_{\frac{\Sigma}{T_{BH}}}$. As above, it is clear that $G^{\mu\nu}\left(\phi_\infty\right)$ is completely different from the contravariant Ruppeiner metric $S^{\mu\nu}_{BH}\left(\beta_{BH}\right)$, which is defined by (5.2.2.20); through (5.2.2.21), $S^{\mu\nu}_{BH}\left(\beta_{BH}\right)$ is the inverse of $S_{\mu\nu,BH}\left(Q_{BH}\right)$, and thus its matrix form is given by the inversion of (5.2.2.8) and (5.2.2.13). Clearly, since the scalar function Γ_{BH} is assumed to satisfy the Schwarz lemma on partial derivatives in the space $\widehat{\mathcal{V}}_{BH}$, in the formulated hypothesis of Riemann structure of $\mathcal{M}_{\phi_\infty}$ (and therefore of $\mathcal{M}_{\frac{\Sigma}{T_{BH}}}$) the contravariant BH Ruppeiner metric is symmetric in all $\widehat{\mathcal{V}}_{BH}$.

Let us now show that the conformal relation (5.1.30) between the Weinhold and the Ruppeiner thermodynamical metrics continues to hold also for (conventional) BH thermodynamics, where the thermodynamical configuration spaces are no longer Euclidean ones, but instead have been previously assumed to be Riemannian ones, with flat symplectic sectors (see (5.2.1.27) and (5.2.1.28)).

Let us start by noticing that (5.2.1.2), (5.2.1.13), and (5.2.1.14) imply that the infinitesimal square metric interval determined by BH Weinhold metric in the space $\mathcal{N}_{BH}$ reads

$$
\begin{aligned}
&\left(dN_{BH}\right)^2 \equiv W_{\mu\nu,BH}\left(N_{BH}\right) dN^\mu_{BH} dN^\nu_{BH}\\
&= \left[D_{N^\mu_{BH}} D_{N^\nu_{BH}} M_{BH}\left(N_{BH}\right)\right] dN^\mu_{BH} dN^\nu_{BH}\\
&= d\pi_{S_{BH}} \otimes_s dS_{BH} + d\left(\pi_{p\Lambda}\right)_{Weinhold} \otimes_s dp^\Lambda\\
&+ d\left(\pi^\Lambda_q\right)_{Weinhold} \otimes_s dq_\Lambda + D\left(\pi_{\phi_\infty}\right)_{a,Weinhold} \otimes_s d\phi^a_\infty\\
&= dT_{BH} \otimes_s dS_{BH} + d\chi_\Lambda \otimes_s dp^\Lambda + d\psi^\Lambda \otimes_s dq_\Lambda - D\Sigma_a \otimes_s d\phi^a_\infty\\
&= dT_{BH} \otimes_s dS_{BH} + d\chi_\Lambda \otimes_s dp^\Lambda + d\psi^\Lambda \otimes_s dq_\Lambda\\
&- d\Sigma_a \otimes_s d\phi^a_\infty + \left\{{c \atop ab}\right\}\left(\phi_\infty\right) \Sigma_c d\phi^b_\infty \otimes_s d\phi^a_\infty ,
\end{aligned}
\tag{5.2.2.22}
$$

where we recall that $\otimes_s$is the (symmetric) product of the components of the differential forms defined in the relevant BH thermodynamical configuration space. Moreover, D stands for the Riemann-covariant differential in $\mathcal{M}_{\phi_\infty}$ (and $\mathcal{M}_\Sigma$), and (5.2.1.7) and (5.2.1.8) have been used in the last line.

Analogously, (5.2.2.2), (5.2.2.11), and (5.2.2.12) imply that the infinitesimal square metric interval determined by BH Ruppeiner metric in the space $\mathcal{V}_{BH}$ reads

$$\begin{aligned}
&(dQ_{BH})^2 \equiv S_{\mu\nu,BH}(Q_{BH})\, dQ^\mu_{BH} dQ^\nu_{BH} \\
&= -\left[D_{Q^\mu_{BH}} D_{Q^\nu_{BH}} S_{BH}(Q_{BH})\right] dQ^\mu_{BH} dQ^\nu_{BH} \\
&= -d\pi_{M_{BH}} \otimes_s dM_{BH} - d\left(\pi_{p\Lambda}\right)_{Ruppeiner} \otimes_s dp^\Lambda \\
&-d\left(\pi^\Lambda_q\right)_{Ruppeiner} \otimes_s dq_\Lambda - D\left(\pi_{\phi_\infty}\right)_{a,Ruppeiner} \otimes_s d\phi^a_\infty \\
&= -d\left(\tfrac{1}{T_{BH}}\right) \otimes_s dM_{BH} + d\left(\tfrac{\chi_\Lambda}{T_{BH}}\right) \otimes_s dp^\Lambda \\
&+d\left(\tfrac{\psi^\Lambda}{T_{BH}}\right) \otimes_s dq_\Lambda - D\left(\tfrac{\Sigma_a}{T_{BH}}\right) \otimes_s d\phi^a_\infty \\
&= -d\left(\tfrac{1}{T_{BH}}\right) \otimes_s dM_{BH} + d\left(\tfrac{\chi_\Lambda}{T_{BH}}\right) \otimes_s dp^\Lambda + d\left(\tfrac{\psi^\Lambda}{T_{BH}}\right) \otimes_s dq_\Lambda \\
&-d\left(\tfrac{\Sigma_a}{T_{BH}}\right) \otimes_s d\phi^a_\infty + \left\{{}^{\,c}_{ab}\right\}(\phi_\infty)\left(\tfrac{\Sigma_c}{T_{BH}}\right) d\phi^b_\infty \otimes_s d\phi^a_\infty .
\end{aligned} \tag{5.2.2.23}$$

Here D stands for the Riemann-covariant differential in $\mathcal{M}_{\phi_\infty}$ (and $\mathcal{M}_{\frac{\Sigma}{T_{BH}}}$), and once again (5.2.1.7) and (5.2.1.8) have been used in the last line. By using (5.2.2.10), (5.2.2.22) may be rewritten as

$$\begin{aligned}
(dN_{BH})^2 = &\tfrac{dT_{BH}}{T_{BH}} \otimes_s dM_{BH} + d\chi_\Lambda \otimes_s dp^\Lambda - \tfrac{\chi_\Lambda}{T_{BH}} dT_{BH} \otimes_s dp^\Lambda \\
&+ d\psi^\Lambda \otimes_s dq_\Lambda - \tfrac{\psi^\Lambda}{T_{BH}} dT_{BH} \otimes_s dq_\Lambda \\
&- d\Sigma_a \otimes_s d\phi^a_\infty + \tfrac{\Sigma_a}{T_{BH}} dT_{BH} \otimes_s d\phi^a_\infty + \left\{{}^{\,c}_{ab}\right\}(\phi_\infty)\, \Sigma_c d\phi^b_\infty \otimes_s d\phi^a_\infty .
\end{aligned} \tag{5.2.2.24}$$

Thus, (5.2.2.23) and (5.2.2.24) yield

$$\begin{gathered}
(dN_{BH})^2 = T_{BH}\,(dQ_{BH})^2 \\
\Updownarrow \\
W_{\mu\nu,BH}(N_{BH})\, dN^\mu_{BH} dN^\nu_{BH} = T_{BH} S_{\mu\nu,BH}(Q_{BH})\, dQ^\mu_{BH} dQ^\nu_{BH},
\end{gathered} \tag{5.2.2.25}$$

namely the BH Weinhold and Ruppeiner thermodynamical metrics are conformally related by the BH temperature T_{BH}.

5.2.3 $\mathbf{c}^2$-parameterization and $\mathbf{c}^2$-extremization

Now, in order to study the thermodynamics of extreme (4-d, static, spherically symmetric, and asymptotically flat) BHs, it is useful to introduce the so-called $\mathbf{c}^2$-parametrization. By recalling (4.1.9), we have

$$\mathbf{c}^2 \equiv \frac{\kappa_s A_H}{8\pi} = 2S_{BH}T_{BH}. \tag{5.2.3.1}$$

By considering Weinhold's formalism, we thus obtain that the extensive BH thermodynamical variables N^{μ}_{BH}'s get modified as follows:

$$N^{\mu}_{BH} = \left(\frac{\mathbf{c}^2}{2T_{BH}}, p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right). \tag{5.2.3.2}$$

Consequently, (5.2.1.16) now reads

$$dM_{BH} = \frac{\mathbf{c}^2}{2S_{BH}}dS_{BH} + \chi_{\Lambda}dp^{\Lambda} + \psi^{\Lambda}dq_{\Lambda} - \Sigma_a d\phi^a_{\infty}, \tag{5.2.3.3}$$

yielding

$$\pi_{S_{BH}} \equiv \frac{\partial M_{BH}(N_{BH})}{\partial S_{BH}} \equiv T_{BH}(N_{BH}) = \frac{\mathbf{c}^2}{2S_{BH}}, \tag{5.2.3.4}$$

$$\Updownarrow$$

$$M_{BH}(N_{BH}) = \frac{\mathbf{c}^2}{2}ln\left(\frac{S_{BH}}{\mathbf{c}^2}\right) + \widehat{M}_{BH}\left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right) = M_{BH}\left(N_{BH}; \mathbf{c}^2\right); \tag{5.2.3.5}$$

thus, $\mathbf{c}^2$ may be interpreted as a new parameter, determining the lift of the function $M_{BH}(N_{BH})$ to a whole family of $\mathbf{c}^2$-parameterized functions $M_{BH}\left(N_{BH}; \mathbf{c}^2\right)$ by expliciting the dependence on S_{BH}.

Analogously, by using (5.2.1.20), (5.2.3.1), and (5.2.3.4), we get that the BH Gibbs free energy now reads

$$G_{BH} = M_{BH} - \frac{\mathbf{c}^2}{2} - \psi^{\Lambda}q_{\Lambda} - \chi_{\Lambda}p^{\Lambda} + \Sigma_a\phi^a_{\infty}; \tag{5.2.3.6}$$

$$\Downarrow$$

$$\frac{\partial G_{BH}(\pi_{BH})}{\partial T_{BH}} = \frac{\partial M_{BH}(N_{BH})}{\partial S_{BH}}\frac{\partial S_{BH}}{\partial T_{BH}} = -\frac{\mathbf{c}^2}{2T_{BH}}; \tag{5.2.3.7}$$

$$\Updownarrow$$

$$G_{BH}(\pi_{BH}) = -\frac{\mathbf{c}^2}{2}ln(T_{BH}) + \widehat{G}_{BH}\left(\psi^{\Lambda}, \chi_{\Lambda}, -\Sigma_a\right) = G_{BH}\left(\pi_{BH}; \mathbf{c}^2\right); \tag{5.2.3.8}$$

once again, $\mathbf{c}^2$ determines the lift of the function $G_{BH}(\pi_{BH})$ to a whole family of $\mathbf{c}^2$-parameterized functions $G_{BH}\left(\pi_{BH}; \mathbf{c}^2\right)$, by expliciting the dependence on $\pi_{S_{BH}} \equiv T_{BH}$. Consequently, the thermodynamical configuration spaces $\mathcal{N}_{BH}$ and $\widehat{\mathcal{N}}_{BH}$ remain unchanged, but they acquire a $\mathbf{c}^2$-dependent parametrization

$$\mathcal{N}_{BH} \longrightarrow \left\{\mathcal{N}_{BH}\left(\mathbf{c}^2\right)\right\}; \tag{5.2.3.9}$$

$$\widehat{\mathcal{N}}_{BH} \longrightarrow \left\{\widehat{\mathcal{N}}_{BH}\left(\mathbf{c}^2\right)\right\}. \tag{5.2.3.10}$$

By considering definition (5.2.1.11) of BH heat capacity and using Equations (5.2.3.4) and (5.2.1.14), we obtain

$$\begin{aligned} C_{BH} = W_{11,BH}\left(N_{BH}\right) &= \frac{\partial^2 M_{BH}\left(N_{BH}\right)}{\left(\partial S_{BH}\right)^2} = \frac{\partial \pi_{S_{BH}}\left(N_{BH}\right)}{\partial S_{BH}} \\ &= \frac{\partial}{\partial S_{BH}}\left(\frac{\mathbf{c}^2}{2S_{BH}}\right) = -\frac{\mathbf{c}^2}{2S_{BH}^2} \leqslant 0; \end{aligned} \tag{5.2.3.11}$$

analogously (5.2.1.25) and (5.2.3.7) yield

$$\begin{aligned} \frac{\partial^2 G_{BH}\left(\pi_{BH}\right)}{\left(\partial T_{BH}\right)^2} &= \frac{\partial^2 G_{BH}\left(\pi_{BH}\right)}{\left(\partial \pi_{S_{BH}}\right)^2} = -W_{BH}^{11}\left(\pi_{BH}\right) \\ &= \frac{\partial}{\partial T_{BH}}\left(-\frac{\mathbf{c}^2}{2T_{BH}}\right) = \frac{\mathbf{c}^2}{2T_{BH}^2} \geqslant 0, \end{aligned} \tag{5.2.3.12}$$

where we denoted $\pi_{S_{BH}} \equiv T_{BH} \equiv \pi_{BH}^1$.

Let us now consider the effects of the introduction of $\mathbf{c}^2$ in Ruppeiner's geometrizing formalism. The extensive (real) Ruppeiner BH thermodynamical variables Q_{BH}^{μ}'s are not modified, but the first of the definitions of (5.2.2.12) yields

$$\pi_{M_{BH}} \equiv \frac{\partial S_{BH}\left(Q_{BH}\right)}{\partial M_{BH}} \equiv \frac{1}{T_{BH}} = \frac{2}{\mathbf{c}^2} S_{BH}; \tag{5.2.3.13}$$

this is a partial differential equation for $S_{BH}\left(Q_{BH}\right)$, which can be solved by a separation of variables

$$\begin{gathered} \frac{\partial S_{BH}(Q_{BH})}{\partial M_{BH}} = \frac{2}{\mathbf{c}^2} S_{BH} \\ \Updownarrow \\ \frac{\partial S_{BH}(Q_{BH})}{S_{BH}} = \frac{2}{\mathbf{c}^2} \partial M_{BH} \\ \Updownarrow \\ ln\left[\frac{S_{BH}(Q_{BH})}{\mathbf{c}^2}\right] = \frac{2}{\mathbf{c}^2} M_{BH} + \Phi\left(p^{\Lambda}, q_{\Lambda}, \phi_{\infty}^{a}\right) \\ \Updownarrow \\ S_{BH}\left(Q_{BH}\right) = \mathbf{c}^2 exp\left[\Phi\left(p^{\Lambda}, q_{\Lambda}, \phi_{\infty}^{a}\right)\right] exp\left[\frac{2}{\mathbf{c}^2} M_{BH}\right] \\ +\widehat{S}_{BH}\left(p^{\Lambda}, q_{\Lambda}, \phi_{\infty}^{a}\right) = S_{BH}\left(Q_{BH}; \mathbf{c}^2\right). \end{gathered} \tag{5.2.3.14}$$

Thus, the effect of the introduction of the parameter $\mathbf{c}^2$ is the lift of the function $S_{BH}\left(Q_{BH}\right)$ to a whole family of $\mathbf{c}^2$-parameterized functions $S_{BH}\left(Q_{BH}; \mathbf{c}^2\right)$ by expliciting the dependence on M_{BH}.

Analogously, by recalling (5.2.2.15) and using (5.2.3.8), we get

$$\begin{aligned}\mathit{\Gamma}_{BH}\left(\beta_{BH}\right) &= \frac{G_{BH}\left(\pi_{BH}\right)}{T_{BH}} = -\frac{\mathbf{c}^2}{2}\frac{ln\left(T_{BH}\right)}{T_{BH}} + \frac{\widehat{G}_{BH}\left(\psi^{\Lambda},\chi_{\Lambda},-\Sigma_a\right)}{T_{BH}} \\ &= \frac{\mathbf{c}^2}{2}\frac{1}{T_{BH}}ln\left(\frac{1}{T_{BH}}\right) + \frac{\widehat{G}_{BH}\left(\psi^{\Lambda},\chi_{\Lambda},-\Sigma_a\right)}{T_{BH}} \\ &= \frac{\mathbf{c}^2}{2}\frac{1}{T_{BH}}ln\left(\frac{1}{T_{BH}}\right) + \frac{1}{T_{BH}}\widehat{G}'_{BH}\left(-\frac{\psi^{\Lambda}}{T_{BH}},-\frac{\chi_{\Lambda}}{T_{BH}},\frac{\Sigma_a}{T_{BH}}\right) \\ &= \mathit{\Gamma}_{BH}\left(\beta_{BH};\mathbf{c}^2\right),\end{aligned} \tag{5.2.3.15}$$

where we defined

$$\widehat{G}_{BH}\left(\psi^{\Lambda},\chi_{\Lambda},-\Sigma_a\right) \equiv \widehat{G}'_{BH}\left(-\frac{\psi^{\Lambda}}{T_{BH}},-\frac{\chi_{\Lambda}}{T_{BH}},\frac{\Sigma_a}{T_{BH}}\right); \tag{5.2.3.16}$$

once again, $\mathbf{c}^2$ determines the lift of the function $\mathit{\Gamma}_{BH}\left(\beta_{BH}\right)$ to a whole family of $\mathbf{c}^2$-parameterized functions $\mathit{\Gamma}_{BH}\left(\beta_{BH};\mathbf{c}^2\right)$, by expliciting the dependence on $\pi_{M_{BH}} \equiv \frac{1}{T_{BH}}$. Consequently, the Ruppeiner thermodynamical configuration spaces $\mathcal{V}_{BH}$ and $\widehat{\mathcal{V}}_{BH}$ remain unchanged, but they acquire a $\mathbf{c}^2$-dependent parametrization

$$\mathcal{V}_{BH} \longrightarrow \left\{\mathcal{V}_{BH}\left(\mathbf{c}^2\right)\right\}; \tag{5.2.3.17}$$

$$\widehat{\mathcal{V}}_{BH} \longrightarrow \left\{\widehat{\mathcal{V}}_{BH}\left(\mathbf{c}^2\right)\right\}. \tag{5.2.3.18}$$

Moreover, (5.2.2.2), (5.2.2.12), and (5.2.3.13) yield

$$\begin{aligned}-\frac{\partial^2 S_{BH}\left(Q_{BH}\right)}{\left(\partial M_{BH}\right)^2} &= S_{11,BH}\left(Q_{BH}\right) = -\frac{\partial \pi_{M_{BH}}\left(Q_{BH}\right)}{\partial M_{BH}} \\ &= -\frac{\partial}{\partial M_{BH}}\left(\frac{2}{\mathbf{c}^2}S_{BH}\right) = -\frac{2}{\mathbf{c}^2}\frac{\partial S_{BH}\left(Q_{BH}\right)}{\partial M_{BH}} \\ &\quad -\frac{4}{\mathbf{c}^4}S_{BH} \leqslant 0,\end{aligned} \tag{5.2.3.19}$$

where we denoted $M_{BH} \equiv Q^1_{BH}$. Analogously, (5.2.2.20) and (5.2.3.15) imply

$$\begin{aligned}&-\frac{\partial^2 \mathit{\Gamma}_{BH}\left(\beta_{BH}\right)}{\left(\partial\left(\frac{1}{T_{BH}}\right)\right)^2} = -\frac{\partial^2 \mathit{\Gamma}_{BH}\left(\beta_{BH}\right)}{\left(\partial \pi_{M_{BH}}\right)^2} = S^{11}_{BH}\left(\beta_{BH}\right) \\ &= -\frac{\partial^2}{\left(\partial \pi_{M_{BH}}\right)^2}\left[\begin{array}{l}\frac{\mathbf{c}^2}{2}\pi_{M_{BH}} ln\left(\pi_{M_{BH}}\right) \\ +\pi_{M_{BH}}\widehat{G}'_{BH}\left(\left(\pi^{\Lambda}_q\right)_{Ruppeiner},\left(\pi_{p\Lambda}\right)_{Ruppeiner},\left(\pi_{\phi_\infty}\right)_{a,Ruppeiner}\right)\end{array}\right] \\ &= -\frac{\partial}{\partial \pi_{M_{BH}}}\left[\begin{array}{l}\frac{\mathbf{c}^2}{2}ln\left(\pi_{M_{BH}}\right) + \frac{\mathbf{c}^2}{2} \\ +\widehat{G}'_{BH}\left(\left(\pi^{\Lambda}_q\right)_{Ruppeiner},\left(\pi_{p\Lambda}\right)_{Ruppeiner},\left(\pi_{\phi_\infty}\right)_{a,Ruppeiner}\right)\end{array}\right] \\ &= -\frac{\mathbf{c}^2}{2\pi_{M_{BH}}} = -\frac{\mathbf{c}^2}{2}\left(\frac{1}{T_{BH}}\right)^{-1} \leqslant 0,\end{aligned} \tag{5.2.3.20}$$

where we denoted $\pi_{M_{BH}} \equiv \frac{1}{T_{BH}} \equiv \beta^1_{BH}$ and used the notations defined in (5.2.2.12).

It should be noticed that neither (5.2.3.11) and (5.2.3.12) nor (5.2.3.19) and (5.2.3.20) imply that the $\mathbf{c}^2$-parameterized BH Weinhold and Ruppeiner metrics are (not necessarily strictly) negative definite, respectively. We reported (5.2.3.11), (5.2.3.12), (5.2.3.19), and (5.2.3.20) in order to point out that the $\mathbf{c}^2$-parametrization of the relevant BH thermodynamical configuration spaces allows one to explicitly get some functional dependences of the thermodynamical functions, and consequently to explicitly estimate the sign of some components of the BH Weinhold and Ruppeiner metrics (but clearly their – not necessarily strict – positive definiteness does not get changed).

Finally, through the introduction of $\mathbf{c}^2$ (5.2.2), which conformally relates Weinhold's and Ruppeiner's approaches to the geometrization of BH thermodynamics, may be rewritten as

$$\begin{gathered}(dN_{BH})^2 = \frac{\mathbf{c}^2}{2S_{BH}}(dQ_{BH})^2 \\ \Updownarrow \\ W_{\mu\nu,BH}(N_{BH})\, dN^{\mu}_{BH} dN^{\nu}_{BH} = \frac{\mathbf{c}^2}{2S_{BH}} S_{\mu\nu,BH}(Q_{BH})\, dQ^{\mu}_{BH} dQ^{\nu}_{BH}.\end{gathered} \tag{5.2.3.21}$$

Let us now "$\mathbf{c}^2$-extremize" all the previously presented formalism, i.e., let us put

$$\mathbf{c}^2 \equiv \frac{\kappa A_H}{8\pi} = 2S_{BH}T_{BH} = 0. \tag{5.2.3.22}$$

Such a choice corresponds to the 4-d extreme (or extremal) BHs, whose spherically symmetric, static, and asymptotically flat cases have been treated above. The vanishing of $\mathbf{c}^2$ in extreme BHs is usually understood as the vanishing of the surface gravity $\kappa_s = 4\pi T_{BH}$ (and thus of the temperature T_{BH}) with finite (nonzero) horizon area A_H and entropy S_{BH}, which at the (semi)classical, leading order of large BH charges are related by the BHEA law $S_{BH} = \frac{1}{4}A_H$ (see the discussion at the beginning of Sect. 2):

$$\mathbf{c}^2 = 0 \Longrightarrow \begin{cases} \kappa_s = 4\pi T_{BH} = 0; \\ \\ S_{BH} = \frac{1}{4}A_H \neq 0. \end{cases} \tag{5.2.3.23}$$

By considering Weinhold's formalism, we obtain that the (real) Weinhold $\mathbf{c}^2$-extremized extensive BH thermodynamical variables $N^{\mu}_{BH,\mathbf{c}^2=0}$'s are (here and below the lower Greek indices have cardinality $2n_V + m_{\phi_\infty} + 2$)

$$N^{\mu}_{BH,\mathbf{c}^2=0} = \left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right), \tag{5.2.3.24}$$

i.e., S_{BH} does not belong to such a set of variables any more. Indeed, by recalling (5.2.3.3), (5.2.3.4) and (5.2.3.5), the energy of a (static) extreme BH turns out to be independent of S_{BH}:

$$dM_{BH}\big|_{\mathbf{c}^2=0} = \chi_\Lambda dp^\Lambda + \psi^\Lambda dq_\Lambda - \Sigma_a d\phi_\infty^a; \tag{5.2.3.25}$$

$$\Updownarrow$$

$$\left.\frac{\partial M_{BH}\left(N_{BH}\right)}{\partial S_{BH}}\right|_{\mathbf{c}^2=0} = \left.\frac{\mathbf{c}^2}{2S_{BH}}\right|_{\mathbf{c}^2=0} = 0; \tag{5.2.3.26}$$

$$\Updownarrow$$

$$M_{BH}\left(N_{BH};\mathbf{c}^2=0\right) = \widehat{M}_{BH}\left(p^\Lambda, q_\Lambda, \phi_\infty^a\right). \tag{5.2.3.27}$$

Such a result directly implies that (4-d) extreme BHs have vanishing heat capacity; indeed, the $\mathbf{c}^2$-extremization of (5.2.3.11) yields

$$C_{BH}\left(\mathbf{c}^2=0\right) = \left.\frac{\partial^2 M_{BH}\left(N_{BH}\right)}{\left(\partial S_{BH}\right)^2}\right|_{\mathbf{c}^2=0} = 0; \tag{5.2.3.28}$$

analogously, the $\mathbf{c}^2$-extremization of (5.2.3.12) gives

$$\left.\frac{\partial^2 G_{BH}\left(\pi_{BH}\right)}{\left(\partial T_{BH}\right)^2}\right|_{\mathbf{c}^2=0} = 0. \tag{5.2.3.29}$$

Thence, by recalling (5.2.3.6)–(5.2.3.8), we get that the Gibbs free energy of extreme BHs reads

$$G_{BH}\left(\mathbf{c}^2=0\right) = M_{BH}\left(\mathbf{c}^2=0\right) - \psi^\Lambda q_\Lambda - \chi_\Lambda p^\Lambda + \Sigma_a \phi_\infty^a; \tag{5.2.3.30}$$

$$\Downarrow$$

$$\left.\frac{\partial G_{BH}\left(\pi_{BH}\right)}{\partial T_{BH}}\right|_{\mathbf{c}^2=0} = \left(\frac{\partial M_{BH}\left(N_{BH}\right)}{\partial S_{BH}}\frac{\partial S_{BH}}{\partial T_{BH}}\right)_{\mathbf{c}^2=0} = 0; \tag{5.2.3.31}$$

$$\Updownarrow$$

$$G_{BH}\left(\pi_{BH};\mathbf{c}^2=0\right) = \widehat{G}_{BH}\left(\psi^\Lambda, \chi_\Lambda, -\Sigma_a\right). \tag{5.2.3.32}$$

Summarizing, the $\mathbf{c}^2 = 0$ elements of the $\mathbf{c}^2$-parameterized families of (real) $(2n_V + m_{\phi_\infty} + 3)$-d thermodynamical configuration spaces $\left\{\mathcal{N}_{BH}\left(\mathbf{c}^2\right)\right\}$ (extensive) and $\{\widehat{\mathcal{N}}_{BH}\left(\mathbf{c}^2\right)\}$ (intensive) are actually $(2n_V + m_{\phi_\infty} + 2)$-d spaces, since the dependence respectively on S_{BH} and T_{BH} drops out; by recalling (5.2.1.23) and (5.2.1.24), we respectively get

$$\mathcal{N}_{BH}\left(\mathbf{c}^2=0\right) \equiv \mathbb{W}^{2n_V+2} \times \mathcal{M}_{\phi_\infty}, \text{ coords. } N^\mu_{BH,\mathbf{c}^2=0} = \left(p^\Lambda, q_\Lambda, \phi_\infty^a\right); \tag{5.2.3.33}$$

$$\widehat{\mathcal{N}}_{BH}\left(\mathbf{c}^2=0\right) \equiv \widehat{\mathbb{W}}^{2n_V+2} \times \mathcal{M}_\Sigma, \text{ coords. } \left(\pi_{BH,\mathbf{c}^2=0}\right)_\mu = \left(\chi_\Lambda, \psi^\Lambda, -\Sigma_a\right), \tag{5.2.3.34}$$

where $\mathbb{W}^{2n_V+2}$ and $\widehat{\mathbb{W}}^{2n_V+2}$ are $(2n_V+2)$-d symplectic spaces (respectively discretized in the lattices Γ and $\widehat{\Gamma}$ at the quantized level) with the metric ϵ defined in (3.1.24). Consequently, the Weinhold covariant metric for extreme BHs is given by the $\mathbf{c}^2$-extremization of (5.2.1.2):

$$\begin{cases} W_{\mu\nu,extreme\ BH}\left(N_{BH,\mathbf{c}^2=0}\right) \equiv D_{N^{\mu}_{BH,\mathbf{c}^2=0}} D_{N^{\nu}_{BH,\mathbf{c}^2=0}} \widehat{M}_{BH}\left(N_{BH,\mathbf{c}^2=0}\right); \\ N^{\mu}_{BH,\mathbf{c}^2=0} = \left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right), \\ \Lambda = 0, 1, \ldots, n_V, \quad a = 1, \ldots, m_{\phi_\infty}, \quad \left|\{\mu\}\right| = 2n_V + m_{\phi_\infty} + 2. \end{cases} \tag{5.2.3.35}$$

Correspondingly, the Weinhold contravariant metric for extreme BHs is given by the $\mathbf{c}^2$-extremization of (5.2.1.25):

$$\begin{cases} W^{\mu\nu}_{extreme\ BH}\left(\pi_{BH,\mathbf{c}^2=0}\right) \equiv D_{\left(\pi_{BH,\mathbf{c}^2=0}\right)_{\mu}} D_{\left(\pi_{BH,\mathbf{c}^2=0}\right)_{\nu}} \left(\mathcal{L}\left(\widehat{M}_{BH}\right)\right)\left(\pi_{BH,\mathbf{c}^2=0}\right) \\ \qquad = -D_{\left(\pi_{BH,\mathbf{c}^2=0}\right)_{\mu}} D_{\left(\pi_{BH,\mathbf{c}^2=0}\right)_{\nu}} \widehat{G}_{BH}\left(\pi_{BH,\mathbf{c}^2=0}\right); \\ \left(\pi_{BH,\mathbf{c}^2=0}\right)_{\mu} = \left(\chi_{\Lambda}, \psi^{\Lambda}, -\Sigma_a\right), \\ \Lambda = 0, 1, \ldots, n_V, \quad a = 1, \ldots, m_{\phi_\infty}, \quad \left|\{\mu\}\right| = 2n_V + m_{\phi_\infty} + 2. \end{cases} \tag{5.2.3.36}$$

Let us now consider the $\mathbf{c}^2$-extremization of Ruppeiner's geometrizing approach to BH thermodynamics. It can be immediately seen that such a formalism yields a divergent metric for $\mathbf{c}^2 = 0$; indeed, from the $\mathbf{c}^2$-parameterized conformal relation (5.2.3) we get

$$lim_{\mathbf{c}^2 \to 0^+} \left(dQ_{BH}\right)^2 = lim_{\mathbf{c}^2 \to 0^+} \left[\frac{2S_{BH}}{\mathbf{c}^2}\left(dN_{BH}\right)^2\right] = +\infty \tag{5.2.3.37}$$

$$\Updownarrow$$

$$\begin{aligned} & lim_{\mathbf{c}^2 \to 0^+} S_{\mu\nu,BH}\left(Q_{BH}\right) dQ^{\mu}_{BH} dQ^{\nu}_{BH} \\ & = lim_{\mathbf{c}^2 \to 0^+} \left[\tfrac{2S_{BH}}{\mathbf{c}^2} W_{\mu\nu,BH}\left(N_{BH}\right) dN^{\mu}_{BH} dN^{\nu}_{BH}\right] = +\infty, \end{aligned} \tag{5.2.3.38}$$

where we used the previously assumed nonzero finiteness of the entropy of extreme BHs and also the fact that, as seen above, the $\mathbf{c}^2$-extremization of Weinhold's metric (and consequently of the square metric interval defined by such a metric in the space $\mathcal{N}_{BH}$) yields a finite, usually nonvanishing result.

The divergence of $\left(dQ_{BH}\right)^2$ is actually consistent with some results in the literature (see, e.g., [84] and [85]), pointing out that near the extreme $\mathbf{c}^2 \to 0^+$ the BH thermodynamics breaks down, at least in the conventional sense here understood. Nevertheless, by making use of the BHEA law (given by (2.4) and rigorously holding true only in the semi-classical, leading-order limit of large BH charges), we may still define a "renormalized" Ruppeiner metric $S_{\mu\nu,extreme\ BH}$ for extreme BHs

$$\begin{cases} S_{\mu\nu,extreme\ BH}\left(N_{BH,\mathbf{c}^2=0}\right) \equiv -\frac{1}{4} D_{N^{\mu}_{BH,\mathbf{c}^2=0}} D_{N^{\nu}_{BH,\mathbf{c}^2=0}} A_H\left(N_{BH,\mathbf{c}^2=0}\right); \\ \\ N^{\mu}_{BH,\mathbf{c}^2=0} = \left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right), \\ \\ \Lambda = 0, 1, \ldots, n_V, \quad a = 1, \ldots, m_{\phi_{\infty}}, \quad |\{\mu\}| = 2n_V + m_{\phi_{\infty}} + 2. \end{cases} \tag{5.2.3.39}$$

Notice that, for nonzero $\mathbf{c}^2$ the two previously defined sets of extensive variables N^{μ}_{BH}'s (Weinhold's ones, see (5.2.1.2)) and Q^{μ}_{BH}'s (Ruppeiner's ones, see (5.2.2.2)) are different (they actually differ only by their first element, respectively S_{BH} and M_{BH}), but they do coincide for $\mathbf{c}^2 = 0$:

$$N^{\mu}_{BH,\mathbf{c}^2=0} = \left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right) = Q^{\mu}_{BH,\mathbf{c}^2=0}. \tag{5.2.3.40}$$

At the level of corresponding thermodynamical configuration spaces, we may thus complete (5.2.3.33) as follows:

$$\begin{cases} \mathcal{N}_{BH}\left(\mathbf{c}^2 = 0\right) \equiv \mathbb{W}^{2n_V+2} \times \mathcal{M}_{\phi_{\infty}} \equiv \mathcal{V}_{BH}\left(\mathbf{c}^2 = 0\right), \\ \text{coords. } N^{\mu}_{BH,\mathbf{c}^2=0} = Q^{\mu}_{BH,\mathbf{c}^2=0} = \left(p^{\Lambda}, q_{\Lambda}, \phi^a_{\infty}\right), \end{cases} \tag{5.2.3.41}$$

clearly denoting with $\mathcal{V}_{BH}\left(\mathbf{c}^2 = 0\right)$ the thermodynamical configuration space related to the renormalized Ruppeiner's formalism based on $S_{\mu\nu,extreme\ BH}$ of (5.2.3.39).

In Ruppeiner's renormalized formalism for extreme BHs, we may thus define the intensive variables $\Xi_{\mu,R.r.}$'s, canonically conjugated to the $N^{\mu}_{BH,\mathbf{c}^2=0} = Q^{\mu}_{BH,\mathbf{c}^2=0}$, as follows (the subscript "$R.r.$" stands for "Ruppeiner renormalized")

$$\Xi_{\mu,R.r.}\left(N_{BH,\mathbf{c}^2=0}\right): \begin{cases} \left(\pi^{\Lambda}_q\right)_{R.r.} \equiv \frac{1}{4}\frac{\partial A_H\left(N_{BH,\mathbf{c}^2=0}\right)}{\partial q_{\Lambda}}; \\ \\ \left(\pi_{p\Lambda}\right)_{R.r.} \equiv \frac{1}{4}\frac{\partial A_H\left(N_{BH,\mathbf{c}^2=0}\right)}{\partial p^{\Lambda}}; \\ \left(\pi_{\phi_{\infty}}\right)_{a,R.r.} \equiv \frac{1}{4} D_{\phi^a_{\infty}} A_H\left(N_{BH,\mathbf{c}^2=0}\right) = \frac{1}{4}\frac{\partial A_H\left(N_{BH,\mathbf{c}^2=0}\right)}{\partial \phi^a_{\infty}}. \end{cases} \tag{5.2.3.42}$$

By Legendre-transforming $\frac{1}{4}A_H$ w.r.t. the $N^{\mu}_{BH,\mathbf{c}^2=0}$'s, we obtain

$$\begin{aligned} &\mathcal{L}\left(\tfrac{1}{4}A_H\right) \\ &\equiv \left(\pi_{p\Lambda}\right)_{R.r.} p^{\Lambda} + \left(\pi^{\Lambda}_q\right)_{R.r.} q_{\Lambda} + \left(\pi_{\phi_{\infty}}\right)_{a,R.r.} \phi^a_{\infty} - \tfrac{1}{4} A_H\left(N_{BH,\mathbf{c}^2=0}\left(\Xi_{R.r.}\right)\right) \\ &= \tfrac{1}{4}\left(\mathcal{L}\left(A_H\right)\right)\left(\Xi_{R.r.}\right), \end{aligned} \tag{5.2.3.43}$$

where system (5.2.3.42), assumed to be determined, solvable, and invertible, has been inverted in order to give $N_{BH,\mathbf{c}^2=0} = N_{BH,\mathbf{c}^2=0}\left(\Xi_{R.r.}\right)$; the last line of (5.2.3.43) has been obtained by exploiting the linearity of the Legendre transform, holding also in curved backgrounds and in the renormalized extreme formalism being considered.

Clearly, the set of $2n_V + m_{\phi_\infty} + 2$ intensive variables $\Xi_{\mu,R.r.} = \left((\pi_{p\Lambda})_{R.r.}, \left(\pi_q^\Lambda\right)_{R.r.}, (\pi_{\phi_\infty})_{a,R.r.} \right)$ is the renormalized $\mathbf{c}^2 \to 0^+$ limit of the previously defined set of $2n_V + m_{\phi_\infty} + 3$ Ruppeiner intensive variables $\beta_{\mu,BH}$'s (in this case the index μ recovers a cardinality $2n_V + m_{\phi_\infty} + 3$), defined by (5.2.2.12):

$$\left((\pi_{p\Lambda})_{R.r.}, \left(\pi_q^\Lambda\right)_{R.r.}, (\pi_{\phi_\infty})_{a,R.r.} \right) = lim_{\mathbf{c}^2 \to 0^+} \left(\frac{1}{T_{BH}}, -\frac{\chi_\Lambda}{T_{BH}}, -\frac{\psi^\Lambda}{T_{BH}}, \frac{\Sigma_a}{T_{BH}} \right). \tag{5.2.3.44}$$

Since the $\beta_{\mu,BH}$'s may be considered as coordinates in the (real) $(2n_V + m_{\phi_\infty} + 3)$-d BH Ruppeiner intensive thermodynamical configuration space $\widehat{\mathcal{V}}_{BH}$, we may conclude that the variables $\Xi_{\mu,R.r.}$'s can be seen as coordinates of the $(2n_V + m_{\phi_\infty} + 2)$-d real space $\widehat{\mathcal{V}}_{BH,R.r.}$, given by the renormalized $\mathbf{c}^2 \to 0^+$ limit of $\widehat{\mathcal{V}}_{BH}$. Consequently, the Ruppeiner's, renormalized extreme counterpart of (5.2.3.34) may be written as

$$\begin{cases} \widehat{\mathcal{V}}_{BH,R.r.} \equiv lim_{\mathbf{c}^2 \to 0^+} \widehat{\mathcal{V}}_{BH}\left(\mathbf{c}^2\right) = \widehat{\mathbb{W}}_{R.r.}^{2n_V+2} \times \mathcal{M}_{(\pi_{\phi_\infty})_{R.r.}}, \\ \\ \text{coords. } \Xi_{\mu,R.r.} = \left((\pi_{p\Lambda})_{R.r.}, \left(\pi_q^\Lambda\right)_{R.r.}, (\pi_{\phi_\infty})_{a,R.r.} \right), \end{cases} \tag{5.2.3.45}$$

where, in general, $\widehat{\mathbb{W}}_{R.r.}^{2n_V+2}$ is a $(2n_V + 2)$-d symplectic space, and $\mathcal{M}_{(\pi_{\phi_\infty})_{R.r.}}$ is the m_{ϕ_∞}-d real space canonically conjugated (in the renormalized $\mathbf{c}^2 \to 0^+$ limit) to the space $\mathcal{M}_{\phi_\infty}$.

Finally, we may define the renormalized extreme contravariant Ruppeiner metric as

$$\begin{cases} S^{\mu\nu}_{extreme\ BH}\left(\Xi_{\mu,R.r.}\right) \equiv -\frac{1}{4} D_{\Xi_{\mu,R.r.}} D_{\Xi_{\nu,R.r.}} \left(\mathcal{L}\left(A_H\right)\right)\left(\Xi_{R.r.}\right); \\ \\ \Xi_{\mu,R.r.} = \left((\pi_{p\Lambda})_{R.r.}, \left(\pi_q^\Lambda\right)_{R.r.}, (\pi_{\phi_\infty})_{a,R.r.} \right), \\ \\ \Lambda = 0, 1, \ldots, n_V, \quad a = 1, \ldots, m_{\phi_\infty}, \quad |\{\mu\}| = 2n_V + m_{\phi_\infty} + 2, \end{cases} \tag{5.2.3.46}$$

satisfying the condition

$$S^{\mu\rho}_{extreme\ BH}\left(\Xi_{\mu,R.r.}\left(N_{BH,\mathbf{c}^2=0}\right)\right) S_{\rho\nu,extreme\ BH}\left(N_{BH,\mathbf{c}^2=0}\right) = \delta^\mu_\nu, \ \forall N^\mu_{BH,\mathbf{c}^2=0}, \tag{5.2.3.47}$$

where $\Xi_{\mu,R.r.} = \Xi_{\mu,R.r.}\left(N_{BH,\mathbf{c}^2=0}\right)$ is given by system (5.2.3.42).

It should be clearly pointed out that the "reduced" form $S_{ab,extreme\ BH}\left(\phi_\infty\right)$ of the renormalized Ruppeiner metric $S_{\mu\nu,extreme\ BH}$ for extreme BHs (defined by (5.2.3.39)), obtained by fixing the BH charge configuration $\left(p^\Lambda, q_\Lambda\right) \in$

Γ, vanishes. Indeed, as pointed out by (4.2.54) and (4.2.55), by the attractor mechanism the horizon area of (the considered class of 4-d static, spherically symmetric, asymptotically flat) extreme BHs is purely charge-dependent, and, in general, independent of the asymptotical moduli configurations:

$$A_H(p,q) = 4\pi V_{BH}(\phi_H(p,q);p,q) = 4S_{BH}(p,q); \tag{5.2.3.48}$$

$$\Downarrow$$

$$D_{\phi^a_\infty} A_H = \frac{\partial A_H}{\partial \phi^a_\infty} = 0, \quad \forall a \in \{1,\ldots,m_{\phi_\infty}\}; \tag{5.2.3.49}$$

$$\Downarrow$$

$$S_{ab,extreme\ BH} \equiv -\frac{1}{4} D_{\phi^a_\infty} D_{\phi^b_\infty} A_H = 0,\ \forall (a,b) \in \{1,\ldots,m_{\phi_\infty}\}^2. \tag{5.2.3.50}$$

The only case with nontrivially vanishing renormalized, reduced Ruppeiner metric $S_{ab,extreme\ BH}$ is given by the double-extreme BHs, defined by

$$\phi^a(r) = \phi^a_\infty = \phi^a_H(p,q), \quad \forall r \in [r_H,+\infty), \quad \forall a = 1,\ldots,m_\phi, \tag{5.2.3.51}$$

where the $\phi^a_H(p,q)$'s are the (horizon), "attracted," purely charge-dependent values of the moduli. In such a case, we thus have

$$\frac{\partial}{\partial \phi^a_\infty} = \frac{\partial}{\partial \phi^a_H(p,q)}, \tag{5.2.3.52}$$

and thus, by using (5.2.3.48), we get

$$\begin{aligned}
&S_{ab,double-extreme\ BH}(p,q) \equiv -\tfrac{1}{4} D_{\phi^a_\infty} D_{\phi^b_\infty} A_H(p,q) \\
&= -\tfrac{1}{4} D_{\phi^a_H(p,q)} D_{\phi^b_H(p,q)} A_H(p,q) \\
&= -\pi D_{\phi^a_H(p,q)} D_{\phi^b_H(p,q)} V_{BH}(\phi_H(p,q);p,q) \\
&= -\pi D_{\phi^a_H(p,q)} \partial_{\phi^b_H(p,q)} V_{BH}(\phi_H(p,q);p,q) \\
&= -\pi \partial_{\phi^a_H(p,q)} \partial_{\phi^b_H(p,q)} V_{BH}(\phi_H(p,q);p,q) \\
&-\Gamma_{ab}{}^{c}(\phi_H(p,q))\, \partial_{\phi^c_H(p,q)} V_{BH}(\phi_H(p,q);p,q), \\
&\forall (a,b) \in \{1,\ldots,m_\phi\}^2.
\end{aligned} \tag{5.2.3.53}$$

By switching to the complex parametrization $\mathcal{M}_{z,\bar{z}}$ of the constant moduli space of the considered double-extreme BHs (by assuming $m_\phi = 2n_\phi$, $n_\phi \in \mathbb{N}$), and moreover by considering the context of n_V-fold, $N=2$, $d=4$ MESGT (in which $n_\phi = n_V$ and $M_{z,\bar{z}}$ is endowed with a – regular – SKG), we get that

$$S_{\hat{i}\hat{j},double-extreme\ BH}(p,q) = -\pi D_{\hat{i}} D_{\hat{j}} V_{BH}(\phi_H(p,q);p,q) \equiv -\pi H^{V_{BH}}_{\hat{i}\hat{j}}(p,q), \tag{5.2.3.54}$$

where we also used the definition in the second line of (4.4.1.28), and the "hatted" indices are defined as

$$\hat{\imath}, \hat{\jmath} = \overset{\text{holomorphic indices}}{1, \ldots, n_\phi}, \quad \overset{\text{anti-holomorphic indices}}{n_\phi + 1, \ldots, m_\phi} \quad . \tag{5.2.3.55}$$

Consequently, beside the general expression (4.4.1.28) (depending on the $\frac{1}{2}$-BPS or non-BPS nature of the constant, purely charge-dependent "attractor" moduli configurations considered in the moduli space of the 4-d double-extreme BHs), $S_{\hat{\imath}\hat{\jmath},double-extreme\ BH}(p,q)$ will be nonvanishing, but rather given by $-\pi$ times the expressions of $H^{V_{BH}}_{\hat{\imath}\hat{\jmath}}$ given by (4.4.1.29) and (4.4.2.10) and (4.4.2.11), respectively.

In general, for a generic thermodynamic substance or for a generic (4- d) BH, one expects to be able to say very little about the thermodynamical Weinhold and Ruppeiner metrics. Instead, by exploiting the particular features of the (regular) SKG of the (complex parametrization $\mathcal{M}_{z,\overline{z}}$ of the) moduli space of n_V-fold, $\mathcal{N} = 2$, $d = 4$ MESGT, we have been able to obtain rather precise results in the Weinhold's and Ruppeiner's thermodynamical formalism of 4-d (static, spherically symmetric, and asymptotically flat) extreme and double-extreme BHs.

The study of the thermodynamical Weinhold and Ruppeiner metrics related to s-t singularities in SUGRA theories is not a mere academic exercise. As correctly pointed out by Ferrara, Gibbons, and Kallosh in [55], the main motivation for investigating such metrics is the following one.

In an exact quantum theory of gravity, where (presumably) the s-t geometry would not play a preeminent role as it does in classical (and semi-classical) general relativity, one might nevertheless still be able to characterize the s-t singularities, such as the 1-d one we called "black holes," by their thermodynamical properties, of course at a more abstract level w.r.t. the usual, classical one. The Weinhold and Ruppeiner metrics might provide such a generalized thermodynamical framework, in which the full quantum equation of state of the considered systems might be determined.

Thus, in theories based on an underlying geometric structure, such as the previously considered $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT, which is based on the special Kähler–Hodge geometry of the space M_{n_V} of the complex scalars coming from n_V Abelian vector supermultiplets (see Sect. 3), it is not unreasonable to hope that the (regular) metric of the relevant moduli space and the Weinhold (and properly renormalized Ruppeiner) metric continue to be (closely) related in the full quantum regime.

In the previous treatment we dealt only with 4-d, static, spherically symmetric, asymptotically flat extreme (and double-extreme) BHs. Generalizations, corresponding to the removal of such working hypotheses, would be very interesting. Extension of the previous treatment and results to nonstatic, rotating BHs should be immediate, given by the enlargement of the thermodynamical configuration spaces to the angular momenta. Recent investigations [86] have focused on the Ruppeiner metric of Reissner–Nördstrom BHs (in asymptotically flat and asymptotically adS backgrounds), and also of the corresponding rotating counterparts, given by the Kerr BHs.

In general, it would be very interesting to see if the whole treatment of Sects. 4 and 5 may be generalized to include the nonextreme BHs.[1] ($\mathbf{c}^2 \neq 0$), even though recent works seem to point out that for such BHs the attractor mechanism does not hold (see, e.g., [66] and [61]).

[1] For some results on nonextreme BHs along the last years, see, e.g., [87] and [88]

6

$\mathcal{N} > 2$-extended Supergravity, U-duality and the Orbits of Exceptional Lie Groups

In previous sections we considered the $\mathcal{N} = 2$, $d = 4$ Maxwell–Einstein supergravity theory (MESGT) coupled to n_V Abelian vector supermultiplets, and we have seen explicitly how the attractor mechanism works in the moduli space M_{n_V} of such a theory, in relation to the (covariant derivatives of the) central charge of the $\mathcal{N} = 2$ superalgebra. We mainly used a fundamental feature of $\mathcal{N} = 2$, $d = 4$, n_V-fold MESGT, namely the symplectic, special Kähler–Hodge geometry exhibited by M_{n_V}.

At this point, one should ask: *what about $N > 2$-extended SUGRAs, in which more than one central charge operator may arise?*

In this section we will try to answer such a question, in a sketchy, but as exhaustive as possible, way[1]. In order to illustrate the main features of the attractor mechanism in $\mathcal{N} > 2$-extended SUGRAs, we will consider two explicit cases, corresponding to the maximal $\mathcal{N} = 8$, $d = 4$ and 5 SUGRAs. In particular, we will deal with the versions of such theories obtained by a toroidal compactification of string and M theories respectively down to $d = 4$ and 5 dimensions, preserving $\mathcal{N} = 8$ SUSY. In such cases, the U-duality symmetry group of the resulting SUGRAs is given by the noncompact, exceptional Lie groups $E_{7(7)}$ and $E_{6(6)}$, respectively.

A peculiar feature, which distinguishes the $\mathcal{N} > 2$ cases from the $\mathcal{N} = 2$ one, is the possibility to have different degrees of BPS soliton solutions. For example, in $\mathcal{N} = 8$, beside the maximally SUSY-preserving $\frac{1}{2}$-BPS case, also the $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS cases are possible.

We will see that the entropy of the metric backgrounds having a regular horizon geometry (which will turn out to be only the minimally SUSY-

[1] An alternative, complementary approach w.r.t. the one adopted in the present section has been briefly outlined in Sect. 5 of [55]. It investigates the critical points of the BH effective potential function in the moduli space of $\mathcal{N} > 2$-extended ($d = 4$) SUGRAs by using a geometric formalism in which the duality symmetries are manifest [92]

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preserving, i.e., the $\frac{1}{8}$-BPS, ones) may be determined by using only group-theoretical arguments.

Indeed, we will see how the different degrees of SUSY-preservation exhibited by the BPS interpolating metric solutions may be classified in a U-invariant way, using some constraints involving the quartic invariant I_4 of $E_{7(7)}$ and the cubic invariant I_3 of $E_{6(6)}$, respectively for the $\mathcal{N} = 8$, $d = 4$ and 5 SUGRAs. Then, it will turn out that such a U-invariant classification of BPS states may be interpreted in terms of orbits in the fundamental representation spaces of the U-duality groups $E_{7(7)}$ and $E_{6(6)}$, respectively. In such a framework, a deep connection with the classification of the little groups and the orbits of timelike, lightlike, and spacelike vectors in Minkowski space will be established.

We will essentially give a merged, pedagogical exposition of the two papers [15] and [89], to whom we address the interested reader for references and further elucidations (see also [90], and [209–213] for recent advances). Very recent and interesting developments concerning BPS BHs and quantum attractor flows are contained in [91].

6.1 Attractor Mechanism in $\mathcal{N} = 8$, $d = 5$ Supergravity

Let us start by considering the $\mathcal{N} = 8$, $d = 5$ maximal SUGRA obtained by a $\mathcal{N} = 8$ SUSY-preserving toroidal compactification from 11-d M-theory or 10-d superstring theory.

The essential feature is the presence of a (manifestly realized) additional internal, noncompact symmetry, namely the U-duality symmetry. In this case such a symmetry is given by the exceptional Lie group $E_{6(6)}$, whose fundamental representation (fundamental representation) has dimension 27. It may be shown that the 27-d fundamental representation space of $E_{6(6)}$ may be described in terms of the symplectic (and therefore antisymmetric and traceless) tensor representation of the group $USp(8)$, which is, at the same time, the maximal compact subgroup of $E_{6(6)}$ and the $\mathcal{R}$-symmetry (i.e., the automorphism) group of the $\mathcal{N} = 8$, $d = 5$ SUSY algebra. The resulting (asymptotical) moduli space is a real 42-d symmetric space with coset structure $E_{6(6)}/USp(8)$.

Similarly to its maximal compact subgroup $USp(8)$, $E_{6(6)}$ has a twofold role: it is the U-group, classifying the generalized electric–magnetic transformations of the internal d.o.f.s of the theory represented by the (quantized) conserved charges; but, at the same time, it is also the isometry group of the 42-d manifold related to the nonlinear sigma model of the scalars of the gravity supermultiplet (the only possible in the case at hand, due to the maximality of $\mathcal{N} = 8$, $d = 5$ SUGRA).

By the action of $USp(8)$, every element in the fundamental representation space of $E_{6(6)}$ may be put in a skew-diagonal form, called normal form, reading

$$\begin{pmatrix} \varsigma_1 & & & \\ & \varsigma_2 & & \\ & & \varsigma_3 & \\ & & & \varsigma_4 \end{pmatrix} \otimes \epsilon, \quad \varsigma_\mu \in \mathbb{R}, \ \forall \mu = 1,2,3,4, \tag{6.1.1}$$

where $\otimes$ stands for the tensor product, ϵ is the 2-d symplectic metric $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the unwritten components vanish. Therefore, the suitable action of the $\mathcal{R}$-symmetry group of the $\mathcal{N}=8$, $d=5$ SUSY algebra is equivalent to the choice of the so-called normal frame in the 27-d fundamental representation space of $E_{6(6)}$.

On the other hand, the central charge matrix Z_{AB}, determining the central extension of the $\mathcal{N}=8$, $d=5$ SUSY algebra, is in general, given by an 8×8 complex antisymmetric matrix (here and in the following uppercase Latin indices will run from 1 to the number of supercharges $\mathcal{N}=8$). Also in this case, it is always possible, by suitably acting with $USp(8)$, to choose the normal frame.

Stressing the complete functional dependence of Z_{AB} and of its $USp(8)$-transformed version, we thus have

$$\begin{gathered} \text{Generic frame} : Z_{AB}\left(q,\varphi_\infty\right) \\ \downarrow USp(8) \\ \text{Normal frame:} \\ e_{AB}\left(q,\varphi_\infty\right) \equiv \begin{pmatrix} \lambda_1\left(q,\varphi_\infty\right) & & & \\ & \lambda_2\left(q,\varphi_\infty\right) & & \\ & & \lambda_3\left(q,\varphi_\infty\right) & \\ & & & \lambda_4\left(q,\varphi_\infty\right) \end{pmatrix} \otimes \epsilon, \\ \lambda_\mu\left(q,\varphi_\infty\right) \in \mathbb{R}, \ \forall \mu = 1,2,3,4, \end{gathered} \tag{6.1.2}$$

where the q's are the charges, the φ_∞'s are the asymptotical moduli, and the $\lambda_\mu\left(q,\varphi_\infty\right)$'s are usually named the "skew-diagonal" eigenvalues of the central charge matrix Z_{AB}. Beside being "ordinarily" traceless (because symplectic and therefore antisymmetric), e_{AB} is also "skew-diagonally" traceless, because, in general, it holds

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0. \tag{6.1.3}$$

Therefore, it is clear that e_{AB} may be considered as an element of the normal frame of the 27-d fundamental representation space of $E_{6(6)}$, constrained by the "skew-diagonal" tracelessness condition (6.1.3).

Since the U-duality symmetry is nothing but the generalization of the electric–magnetic duality, in general the set of charges (usually denoted with $\left\{q^i\right\}_{i\in I}$) characterizing a given configuration of the considered theory will fit a vector in the fundamental representation of the U-duality symmetry group. Therefore, the cardinality of the set I will be equal to the dimension of such a group. In the case at hand, the set of charges $\{q^i\}_{i=1,\dots,27}$ will fit a vector in the **27** of $E_{6(6)}$, i being a vector index in such a representation.

Thus, it may be stated that the tracelessness condition (6.1.3) should hold $\forall\ q^i \in \mathbf{27}$ of $E_{6(6)}(\mathbb{Z})$ and $\forall \varphi_\infty \in E_{6(6)}/USp(8)$, which is the (asymptotical) moduli space of the theory. $E_{6(6)}(\mathbb{Z})$ is the "discrete version" of the U-duality group, determined by the quantization of the conserved charges $\{q^i\}_{i\in I}$.

Since the lowest order product of fund. reprs. **27**'s of $E_{6(6)}$ containing the singlet in its symmetric part is the cubic one, the lowest order invariant in the fundamental representation **27** of $E_{6(6)}$ is cubic in the q^i's, and will be therefore given by

$$\begin{aligned} I_3(27)(q) &\equiv T_{ijk} q^i q^j q^k \\ &= \frac{1}{8}\left(\lambda_1(q,\varphi_\infty) + \lambda_2(q,\varphi_\infty)\right) \cdot \\ &\quad \cdot \left(\lambda_1(q,\varphi_\infty) + \lambda_3(q,\varphi_\infty)\right)\left(\lambda_2(q,\varphi_\infty) + \lambda_3(q,\varphi_\infty)\right), \end{aligned} \tag{6.1.4}$$

where T_{ijk} is a particular completely symmetric covariant rank-3 tensor in the **27** of $E_{6(6)}$. By defining[2]

$$s_1 \equiv \frac{1}{2}(\lambda_1 + \lambda_2), \ s_2 \equiv \frac{1}{2}(\lambda_1 + \lambda_3), \ \ s_3 \equiv \frac{1}{2}(\lambda_2 + \lambda_3), \tag{6.1.5}$$

we may finally write

$$I_3(27)(q) = s_1(q,\varphi_\infty)\, s_2(q,\varphi_\infty)\, s_3(q,\varphi_\infty). \tag{6.1.6}$$

One of the main features of the cubic invariant $I_3(27)$ of $E_{6(6)}$ is its dependence only on the charges. Indeed, despite the fact that the s's and the λ's are functions of the charges and of the asymptotical moduli, $I_3(27)$ is moduli-independent. As we will see, the property of moduli independence will play a key role in the use of such a purely charge-dependent, U-invariant quantity in order to describe the SUSY-preservation features and the entropy in the physics of extremal BHs.

Also, it may be shown that the sign of $I_3(27)$ is not relevant, because it may change under CPT, which instead is an exact symmetry of the theory.

It is, in general, possible to give a U-invariant classification of the BPS extremal BH solutions in the $\mathcal{N} = 8$, $d = 5$ SUGRA having $E_{6(6)}$ as U-duality symmetry, by using the lowest order (namely cubic) invariant of such an exceptional Lie group.

The relevant steps to perform are

1. Formulating some suitable constraints on the cubic invariant $I_3(27)$ of the fundamental representation **27** of $E_{6(6)}$.

[2] Up to irrelevant renamings, we may choose λ_1, λ_2 and λ_3 as the independent "skew-diagonal" eigenvalues of the central charge matrix Z_{AB}.

2. Interpreting such constraints as the defining equations of the orbits in the 27-d fundamental representation space of $E_{6(6)}$.
   In general, such orbits admit a "coset expression," i.e., a reformulation as a coset of the starting 27-d space. Indeed, they may be considered as particular nonlinear realizations (NLRs) of the U-duality symmetry group $E_{6(6)}$ in its 27-d fundamental representation space. The "stabilizer" in the coset expression (i.e., the group appearing in the denominator) is called the "little group" of the considered orbit.
3. Counting the independent (real) parameters associated with the considered orbit, and thus calculate its dimension, by looking only at the central charge matrix. The key point is to find the so-called little group of the normal form of the properly constrained central charge matrix, which, in general, will be a (proper) subgroup of the previously mentioned "little group of the orbit"[3].

The dimension of the orbit will be given by adding the number of independent real skew-diagonal eigenvalues of the central charge matrix Z_{AB} to the dimension of the coset space $\mathcal{R}/\mathcal{K}$, where:

1. $\mathcal{R}=USp(8)$, which, as already pointed out, is, at the same time, the maximal compact subgroup of $E_{6(6)}$ and the $\mathcal{R}$-symmetry group of the $\mathcal{N}=8$, $d=5$ SUSY algebra,
   and
2. $\mathcal{K}$ stands for the maximal compact subgroup of the previously mentioned little group of the normal form of the properly constrained central charge matrix.

---

[3] Another, alternative procedure may be used in order to calculate the dimensions of the orbits in the considered exceptional Lie group describing the U-duality symmetry of the theory at hand.

Such a procedure is based on the various symmetry groups and the different gradings of $J_{3,split}^{\mathbb{O}}$, i.e., of the exceptional Jordan algebra of 3×3 Hermitian matrices over the split form of the composition algebra of octonions $\mathbb{O}$. Indeed, it may be shown that the cubic invariant I_3 of the fundamental representation **27** of $E_{6(6)}$ can be identified with the cubic norm of $J_{3,split}^{\mathbb{O}}$.

Such an identification is based on the noteworthy result

$$St_0\left(J_{3,split}^{\mathbb{O}}\right)=E_{6(6)},$$

namely, the invariance group of the norm form (called reduced structure group and denoted by St_0) of the exceptional Jordan algebra $J_{3,split}^{\mathbb{O}}$ is nothing but $E_{6(6)}$.

By the way, the procedure mentioned at point 3 allows one to better relate the set of constraints, defining the orbit in the fundamental representation space of $E_{6(6)}$, with the various degrees of SUSY-preservation exhibited by the BPS extremal BH solutions in the considered $\mathcal{N}=8$, $d=5$ SUGRA.

Clearly, $dim\,(USp(8)/\mathcal{K})$ expresses the number of independent (real) angles in the relevant $\mathcal{R}$-symmetry rotation subgroup related to the considered, properly constrained central charge matrix.

As previously pointed out, for $\mathcal{N} = 8$ we have three different BPS degrees of SUSY-preservation, corresponding to $\frac{1}{8}$-BPS (minimally preserving), $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS (maximally preserving) stable interpolating soliton solutions.

Correspondingly, there are, in general, 3 sets of invariant constraints on $I_3(27)$ which allow one to perform an U-invariant classification. In the remaining part of this Subsection we will consider such constraints, expliciting the 3 steps of the algorithm presented above.

I] U-invariant characterization of $\frac{1}{8}$-BPS (minimally SUSY-preserving) solutions.

I.1) “Timelike–spacelike”[4] $E_{6(6)}$-invariant constraint on the cubic norm of **27**

$$\begin{gathered} I_3(27)\,(q) = s_1\,(q,\varphi_\infty)\,s_2\,(q,\varphi_\infty)\,s_3\,(q,\varphi_\infty) \neq 0 \\ \Updownarrow \\ s_1\,(q,\varphi_\infty)\,, s_2\,(q,\varphi_\infty)\,, s_3\,(q,\varphi_\infty) \in \mathbb{R}_0, \\ \forall q^i \in \mathbf{27} \text{ of } E_{6(6)}\,(\mathbb{Z})\,,\ \forall \varphi_\infty \in E_{6(6)}/USp(8), \end{gathered} \tag{6.1.7}$$

which therefore corresponds to three independent skew-diagonal eigenvalues of the central charge matrix Z_{AB}

$$\lambda_1\,(q,\varphi_\infty)\,, \lambda_2\,(q,\varphi_\infty)\,, \lambda_3\,(q,\varphi_\infty) \in \mathbb{R}. \tag{6.1.8}$$

As mentioned above, in $d = 5$ the sign of $I_3(27)\,(q)$ does not matter, because it actually changes under a CPT transformation.

Moreover, in order to calculate the entropy of the $\frac{1}{8}$-BPS (minimally SUSY-preserving) extremal BHs in $d = 5$, the choice of the “timelike” or “spacelike” determination of disequality (6.1.7) is not relevant. This is due to the existence of the following noteworthy relation in the considered framework of the $\mathcal{N} = 8$, $d = 5$ SUGRA with $E_{6(6)}$ as U-group:

$$M^3_{BR}\,(q) = M^3_{ADM}\,(q, \varphi_H\,(q)) = |I_3(27)\,(q)|\,. \tag{6.1.9}$$

This amounts to say that the mass of the “near-horizon” Bertotti–Robinson (BR) geometry (given by the ADM mass of the BPS interpolating metric solution, evaluated at the “attracted,” purely charge-dependent, “horizon” configurations of the moduli) is equal to the third root of the absolute value of the cubic invariant $I_3(27)\,(q)$ of the **27** fundamental representation of $E_{6(6)}$.

[4] We use the nomenclature of the classification of the orbits in the Minkowski space, but obviously in a slightly generalized sense. Nevertheless, as we will see at the end of this subsection, the performed classification of orbits strongly resembles the Minkowskian one

Moreover, we may assume to have the following ordering of the absolute values of the three real independent skew-diagonal eigenvalues of the central charge matrix Z_{AB}:

$$|\lambda_1(q,\varphi_\infty)| \geqslant |\lambda_2(q,\varphi_\infty)| \geqslant |\lambda_3(q,\varphi_\infty)|. \tag{6.1.10}$$

Such an assumption does not imply any loss of generality, up to irrelevant renamings[5] of the λ's.

Since $I_3(27)(q)$ depends only on the charges, it can be equivalently evaluated at any point in the asymptotical moduli space $E_{6(6)}/USp(8)$. In particular, one may choose to evaluate it at $\{\varphi_\infty\} = \{\varphi_H(q)\}$, i.e., at the asymptotical configuration(s) coinciding with the "near-horizon," "attracted" one(s). Consequently, it may be shown that the following result holds true:

$$\begin{aligned} I_3(27)(q) &= s_1(q,\varphi_\infty=\varphi_H(q))\, s_2(q,\varphi_\infty=\varphi_H(q))\, s_3(q,\varphi_\infty=\varphi_H(q)) \\ &= \lambda_1^3(q,\varphi_\infty=\varphi_H(q)) \equiv \lambda_{1,H}^3(q). \end{aligned} \tag{6.1.11}$$

By using relation (6.1.9), one obtains

$$M_{BR}^3(q) = M_{ADM}^3(q,\varphi_h(q)) = |I_3(27)(q)| = |\lambda_{1,H}(q)|^3. \tag{6.1.12}$$

It should be now mentioned that in $d=5$, (3.2.131) may be shown to be generalized as it follows:

$$S_{BH} = \frac{A_H}{4} \sim M_{BR}^{3/2}. \tag{6.1.13}$$

By specializing it to the case at hand and using (6.1.12), one finally gets

$$\begin{aligned} S_{BH} &= \frac{A_H}{4} \sim M_{BR}^{3/2}(q) = M_{ADM}^{3/2}(q,\varphi_H(q)) \\ &= \sqrt{|I_3(27)(q)|} = |\lambda_{1,H}(q)|^{3/2} > 0. \end{aligned} \tag{6.1.14}$$

Thus, the $\frac{1}{8}$-BPS BH solutions described by the "timelike–spacelike" U-invariant constraint (6.1.7) have a physically consistent, purely charge-dependent, strictly positive entropy.

The fundamental relation

$$S_{BH} \sim \sqrt{|I_3(27)(q)|} \tag{6.1.15}$$

[5] A priori, it might happen that such an ordering of the absolute values of the skew-diagonal eigenvalues of Z_{AB} actually changes, depending on the considered value of the formal couple:

$$(q,\varphi_\infty) \in \left(\mathbf{27} \text{ of } E_{6(6)}(\mathbb{Z})\right) \times \left(E_{6(6)}/USp(8)\right).$$

Such a possibility is strictly related to the functional dependence of the λ's on the couple (q,φ_∞).

For further explanations (given in the treatment of the $d=4$ case in next subsection), the reader is addressed to Footnote 8 formally holds for all existing orbits, and it gives a U-invariant expression for the BH entropy in the considered $\mathcal{N} = 8$, $d = 5$ SUGRA with $E_{6(6)}$ as U-duality symmetry.

The attractor mechanism fixes a model-independent criterion for determining the "near-horizon," "attractor" point(s). Indeed, such points are critical points of the highest absolute-valued skew-diagonal eigenvalue of Z_{AB}, seen as a function of the asymptotical moduli, which may continuously and unconstrainedly vary in the asymptotical moduli space $E_{6(6)}/USp(8)$:

$$E_{6(6)}/USp(8) \ni \varphi_H(q) : \left.\frac{\partial |\lambda_1(q, \varphi_\infty)|}{\partial \varphi_\infty}\right|_{\varphi_\infty = \varphi_H(q)} = 0; \tag{6.1.16}$$

such an extremization at the "near-horizon," "attractor" configuration(s) $\varphi_H(q)$ defines the extremum

$$|\lambda_1(q, \varphi_\infty = \varphi_H(q))| \equiv |\lambda_{1,H}(q)| \in \mathbb{R}_0^+. \tag{6.1.17}$$

I.2) The little group related to the corresponding timelike–spacelike orbit in the 27-d fundamental representation space of $E_{6(6)}$ is $F_{4(4)}$, admitting as proper subgroup $O(4,4)$, which is the little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant condition (6.1.7). Finally, the maximal compact subgroup of $O(4,4)$ is $(SU(2))^4$.

Thus, the timelike–spacelike case is characterized by the following chain of group inclusions:

$$(SU(2))^4 \subset O(4,4) \subset F_{4(4)} \subset E_{6(6)}, \tag{6.1.18}$$

and the corresponding coset expression of the timelike–spacelike orbit reads

$$\frac{E_{6(6)}}{F_{4(4)}}. \tag{6.1.19}$$

The general form of the coset expression of an orbit is given by the starting fundamental representation space coset by the little group of the orbit itself (stabilizer of the orbit).

I.3) As pointed out above, in general the dimension of the orbit is given by the number of independent skew-diagonal eigenvalues of the properly constrained central charge matrix Z_{AB}, + the dimension of the maximal compact subgroup of the U-duality symmetry group coset by the maximal compact subgroup of the little group of the skew-diagonal matrix e_{AB}, i.e., of the "normal frame" form of Z_{AB}.

In the case at hand, such a proper subspace of the fundamental representation space of $E_{6(6)}$ reads

$$\frac{USp(8)}{(SU(2))^4}, \tag{6.1.20}$$

with dimension $36 - 4 \cdot 3 = 24$.

Consequently, the dimension of the timelike–spacelike orbit is $3 + 24 = 27$.

The BPS, nonminimally SUSY-preserving solutions in $\mathcal{N} = 8$, $d = 5$ SUGRA (with U-group given by $E_{6(6)}$) will all correspond to different kinds of "lightlike" orbits in the 27-d fundamental representation space of the exceptional Lie group $E_{6(6)}$.

The degenerate, i.e., lightlike, orbits correspond to invariant constraints imposed on the "generalized lightcone" (defined by the vanishing of the lowest order norm form) in the considered fund. repr. space.

The presence of such a kind of orbit is strictly related to the noncompact nature of $E_{6(6)}$, and, since the lowest order norm form is cubic, we obtain two different "degrees of degeneration." The U-invariant condition for the **27** of $E_{6(6)}$ to be a null vector (with respect to the cubic norm) suffices to guarantee the enhancement of the SUSY (i.e., the $\frac{1}{4}$-BPS nature of the corresponding solution). As reported below, the further enhancement of the extremal BPS SUSY-preservation from the intermediate $\frac{1}{4}$-BPS (case II) to the maximal $\frac{1}{2}$-BPS (case III) degree may be obtained only by also requesting the criticality of the orbit (see the defining constraint (6.1)).

II] U-invariant characterization of $\frac{1}{4}$-BPS (intermediate SUSY-preserving) solutions.

II.1) "Lightlike noncritical" $E_{6(6)}$-invariant constraint on the cubic norm of **27** (up to irrelevant renamings of s_i, $i = 1, 2, 3$):

$$
\begin{gathered}
\begin{cases}
I_3(27)\,(q) = s_1\,(q, \varphi_\infty)\, s_2\,(q, \varphi_\infty)\, s_3\,(q, \varphi_\infty) = 0, \\[2ex]
\frac{\partial I_3(27)(q)}{\partial q^i} \neq 0, \forall i = 1, \ldots, 27.
\end{cases} \\
\Updownarrow \\
\begin{cases}
s_1\,(q, \varphi_\infty)\,, s_2\,(q, \varphi_\infty) \in \mathbb{R}_0, \\[2ex]
s_3\,(q, \varphi_\infty) = 0,
\end{cases} \\
\forall q^i \in \mathbf{27} \text{ of } E_{6(6)}\,(\mathbb{Z})\,,\ \forall \varphi_\infty \in E_{6(6)}/USp(8),
\end{gathered}
\tag{6.1.21}
$$

which, in general, therefore corresponds to two independent (in absolute value) eigenvalues out of the three a priori independent skew-diagonal eigenvalues of the central charge matrix Z_{AB}.

II.2) The little group related to the corresponding degenerate lightlike noncritical orbit in the 27-d fundamental representation space of $E_{6(6)}$ is $O(5,4) \otimes_s T_{16}$, where $\otimes_s$ stands for the semidirect group product, and T_{16} is the group of translations corresponding to the spinor representation of $O(5,4)$. In this case the little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant condition (6.1.21), is $O(5,4)$.

Building up a chain of group inclusions similar to the previous case, one finally obtains that the coset expression of the degenerate "lightlike noncritical" orbit reads

$$
\frac{E_{6(6)}}{O(5,4) \otimes_s T_{16}}. \tag{6.1.22}
$$

II.3) Following the approach introduced in the previous case, the dimension of the degenerate lightlike noncritical orbit can be calculated to be $2 + 24 = 26$.

III] **U-invariant characterization of $\frac{1}{2}$-BPS (maximally SUSY-preserving) solutions.**

III.1) "Lightlike critical" $E_{6(6)}$-invariant constraint on the cubic norm of **27** (up to irrelevant renamings of s_i, $i = 1, 2, 3$):

$$\begin{cases} I_3(27)\,(q) = s_1\,(q, \varphi_\infty)\, s_2\,(q, \varphi_\infty)\, s_3\,(q, \varphi_\infty) = 0, \\ \frac{\partial I_3(27)(q)}{\partial q^i} = 0, \forall i = 1, \ldots, 27. \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} s_1\,(q, \varphi_\infty) \in \mathbb{R}_0, \\ s_2\,(q, \varphi_\infty) = 0 = s_3\,(q, \varphi_\infty)\,. \end{cases} \tag{6.1.23}$$

$$\forall q^i \in \mathbf{27} \text{ of } E_{6(6)}\,(\mathbb{Z})\,,\ \forall \varphi_\infty \in E_{6(6)}/USp(8),$$

which, in general, corresponds to only one independent (in absolute value) eigenvalue out of the three a priori independent skew-diagonal eigenvalues of the central charge matrix Z_{AB}.

III.2) The little group related to the corresponding degenerate lightlike critical orbit in the 27-d fundamental representation space of $E_{6(6)}$ is $O(5,5) \otimes_s T_{16}$, where now T_{16} denotes a proper Abelian subgroup corresponding to the spinor representation of $O(5,5)$. In this case the little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant condition (6.1), is $O(5,5)$. It admits $O(5) \otimes O(5) = USp(4) \otimes USp(4)$ as maximal compact subgroup ($\otimes$ now stands for the direct group product).

Thus, the lightlike critical case is characterized by the following chain of group inclusions:

$$O(5) \otimes O(5) = USp(4) \otimes USp(4) \subset O(5,5) \subset O(5,5) \otimes_s T_{16} \subset E_{6(6)}, \tag{6.1.24}$$

and the corresponding coset expression of the degenerate lightlike critical orbit reads

$$\frac{E_{6(6)}}{O(5,5) \otimes_s T_{16}}. \tag{6.1.25}$$

III.3) In this case, the relevant subspace of the 27-d fund. repr. space of $E_{6(6)}$ we have to consider is

$$\frac{USp(8)}{USp(4) \otimes USp(4)}, \tag{6.1.26}$$

with dimension $36 - 2 \cdot 10 = 16$.

Consequently, the dimension of the degenerate lightlike critical orbit is $1 + 16 = 17$.

The orbits described by cases 2 and 3 correspond to the two possible degrees of degeneration of an orbit in the 27-d fundamental representation space of $E_{6(6)}$. Therefore, they all give rise to the unphysical result of zero BH entropy.

Indeed, (6.1.15) formally holds true also in these cases, yielding

$$S_{BH} = \frac{A_H}{4} \sim \sqrt{|I_3(27)\,(q)|} = 0. \tag{6.1.27}$$

This implies that actually the famous Bekenstein–Hawking entropy-area (BHEA) formula may not be consistently applied in cases II and III.

From this point of view, the only physically consistent case is case I. Consequently, the only BPS solution (in $\mathcal{N} = 8$, $d = 5$ SUGRA with $E_{6(6)}$ as U-duality group) having a regular horizon geometry and strictly positive, physically consistent, purely charge-dependent entropy is the extremal $\frac{1}{8}$-BPS (and therefore minimally SUSY- preserving) soliton solution, with entropy given by (6.1.14).

In general, it can be stated that, for the $d = 4$ and 5 cases, the only extremal, singular interpolating soliton metric solutions of the $\mathcal{N}$-extended SUGRAs which have a properly regular horizon geometry and a physically consistent, finite (nonzero) entropy are the $\frac{1}{\mathcal{N}}$-BPS solutions, which preserve the minimal number of supersymmetries (namely 4 in $d = 4$ and 5).

In the previously performed analysis of the orbits, which will be continued in the next subsection for the $d = 4$ case, we used a Minkowskian nomenclature, speaking of timelike, spacelike, and lightlike orbits, but this is obviously meant in a generalized sense, since the lowest order norm forms of $E_{6(6)}$ and $E_{7(7)}$, instead of being quadratic, are respectively cubic and quartic.

Nevertheless, the used nomenclature is well-grounded, because an interesting connection to the Minkowskian case exists. In what follows, we are going to briefly describe such a connection, which hopefully will make the physical meaning of the analysis of Subsects. 6.1 and 6.2 clearer.

Therefore, let us consider the 4-d Minkowski space $\mathcal{M}_4$. As it is well known, a four-vector in such a manifold may be represented by 2×2 matrices in the following way:

$$\mathcal{M}_4 \ni x = x_\mu \sigma^\mu, \tag{6.1.28}$$

where $\sigma^\mu = \left(\mathbb{I}, \sigma^i\right)$, with $\mathbb{I}$ now denoting the 2×2 unit matrix, and σ^i standing for the three Pauli sigma matrices.

Consequently, seen as 2×2 complex matrices, the coordinates x of a 4-d Minkowskian manifold may be considered as elements of the Jordan algebra $J_2^{\mathbb{C}}$ of the Hermitian matrices over $\mathbb{C}$:

$$\mathcal{M}_4 \ni x \to x = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \in J_2^{\mathbb{C}}, \tag{6.1.29}$$

with the symmetric Jordan product $\circ$ [214–220].

$$x \circ y = y \circ x, \qquad (x, y) \in \left(J_2^{\mathbb{C}}\right)^2 \tag{6.1.30}$$

that satisfies the fundamental Jordan identity

$$x \circ \left(y \circ x^2\right) = (x \circ y) \circ x^2, \tag{6.1.31}$$

and preserves Hermiticity.

It can be proved that the automorphism group of $J_2^{\mathbb{C}}$ is $SU(2)$, i.e., the covering group of the rotation group $SO(3)$ in $\mathcal{M}_4$. The analog of the formal couple $\left(J_2^{\mathbb{C}}, SU(2)\right)$ for the previously considered case of the $\mathcal{N} = 8$, $d = 5$ SUGRA (with $E_{6(6)}$ as U-group) is the couple $\left(J^{\mathbb{O}}_{3,split}, F_{4(4)}\right)$, where $J^{\mathbb{O}}_{3,split}$ is the exceptional Jordan algebra over the split form of the composition algebra of the octonions $\mathbb{O}$.

The lowest order norm form in $J_2^{\mathbb{C}}$ is quadratic, and it is denoted by I_2. It corresponds to the ordinary determinant of the Hermitian matrix representation of the elements of the Jordan algebra, and it is immediate to realize from (6.1.29) that it is nothing but the Minkowskian norm of the related 4-vector,

$$\begin{aligned} I_2\,(x) \equiv det\,(x) &= det \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \\ &= \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 = \eta_{\mu\nu} x^\mu x^\nu, \end{aligned} \tag{6.1.32}$$

$\eta_{\mu\nu}$ being the 4-d Minkowski metric.

The reduced structure group for $J_2^{\mathbb{C}}$, i.e., the invariance group of the quadratic norm form I_2 defined in the 2×2 Hermitian matrix representation of such an algebra, is given by $SL(2, \mathbb{C})$, which is the covering group of the homogeneous component of the maximal Killing group of $\mathcal{M}_4$, namely the Lorentz group $SO(3, 1)$:

$$Str_0\left(J_2^{\mathbb{C}}\right) = SL(2, \mathbb{C}). \tag{6.1.33}$$

The analog of the formal triplet

$$\left(J_2^{\mathbb{C}}, I_2, Str_0\left(J_2^{\mathbb{C}}\right) = SL(2, \mathbb{C})\right), \tag{6.1.34}$$

for the $\mathcal{N} = 8$, $d = 5$SUGRA (with $E_{6(6)}$ as U-group) is the triplet

$$\left(J^{\mathbb{O}}_{3,split}, I_3, Str_0\left(J^{\mathbb{O}}_{3,split}\right) = E_{6(6)}\right). \tag{6.1.35}$$

Such a result allows one to connect the cubic norm form I_3 defined in the 3×3 Hermitian matrix representation of $J^{\mathbb{O}}_{3,split}$ with the lowest order, unique, cubic invariant $I_3(27)$ in the fund. repr. **27** of $E_{6(6)}$.

Thus, as the 27-d fundamental representation space of the exceptional, noncompact Lie group $E_{6(6)}$ may be described by $J_{3,split}^{\mathbb{O}}$, in the same way it is possible to realize the 4-d fundamental representation space of the noncompact Lie group $SL(2,\mathbb{C})$ by the elements of $J_2^{\mathbb{C}}$. By using (6.1.29) we finally obtain that the 4-d Minkowski manifold $\mathcal{M}_4$ is nothing but the real realization of the 4-d fundamental representation space of $SL(2,\mathbb{C})$, i.e., of the covering group of the homogeneous part of its (maximal) Killing group of isometries.

Consequently, in $\mathcal{M}_4$ a 4-vector will be characterized by its Minkowskian quadratic norm and by the parameters of the corresponding orbit.

As previously mentioned, in the (fund.) repr. space of a noncompact (Lie) group G, in general, the orbits, defined by suitable constraints on some invariant(s) of the fundamental representation of G, always admit a coset expression as NLRs G/H, where the stabilizer H is called the little group of the orbit.

In the case of Minkowski, that is to say in the case of the noncompact Lie group $SL(2,\mathbb{C})$ and of the related 4-d fundamental representation space, we obtain three independent typologies of orbits:

1. Timelike orbit, defined by $I_2(x) > 0$, with little group $SU(2)$, and thus expressed by the 4-d coset-space $\frac{SL(2,\mathbb{C})}{SU(2)}$ (yielding that the automorphism of $J_2^{\mathbb{C}}$, corresponding to the covering of the 3-d rotations $SO(3)$, is linearly realized).
2. Spacelike orbit, defined by $I_2(x) < 0$, with little group $SU(1,1)$, and thus expressed by the 4-d coset-space $\frac{SL(2,\mathbb{C})}{SU(1,1)}$.
3. Lightlike orbit, corresponding to the Minkowski lightcone, defined by $I_2(x) = 0$, with little group E_2, and thus expressed by the 3-d coset-space $\frac{SL(2,\mathbb{C})}{E_2}$.

As it may be seen from the previous points 1–3, in the case of $E_{6(6)}$ with norm form I_3, the total number of independent orbits is the same as that of the Minkowski case, but, different from this latter one, the timelike and spacelike orbits do coincide, and there are two kinds of degenerate, lightlike orbits. This is due to the fact that the lowest order norm form I_3 is cubic, and not quadratic, as in Minkowski.

As we will see in the next subsection, the classification of the orbits is still richer for the case of the exceptional, noncompact Lie group $E_{7(7)}$, because its lowest order norm is defined by a quartic form.

6.2 Attractor Mechanism in $\mathcal{N} = 8$, $d = 4$ Supergravity

Let us now consider the $\mathcal{N} = 8$, $d = 4$ SUGRA obtained by a $\mathcal{N} = 8$ SUSY-preserving toroidal compactification from the 11-d M-theory or 10-d superstring theory.

Once again, the fundamental feature is the presence of a (manifestly realized) additional internal, noncompact symmetry, corresponding to the U-duality symmetry. In this case such a symmetry is given by the exceptional

Lie group $E_{7(7)}$, whose fundamental representation has dimension 56. It can be shown that the 56-d fundamental representation space of $E_{7(7)}$ may be described in terms of the symplectic (and therefore antisymmetric and traceless) tensor representation of the group $SU(8)$, which is, at the same time, the maximal compact subgroup of $E_{7(7)}$ and the $\mathcal{R}$-symmetry group of the $\mathcal{N} = 8$, $d = 4$ SUSY algebra.

The asymptotical moduli space is a real 70-d symmetric space with coset structure $E_{7(7)}/SU(8)$.

As it is evident, both $\mathcal{N} = 8$, $d = 4$, and $d = 5$ SUGRA theories, considered in this and in the previous subsection, share the same property: the manifold of the nonlinear sigma model related to the scalar fields has the structure G/H, where G is the U-duality symmetry group and H is its (maximal) compact subgroup[6].

$E_{7(7)}$ plays a double role: it encodes the U-duality symmetry properties of the theory but, at the same time, it is also the isometry group of the 70-d manifold related to the nonlinear sigma model of the scalars of the gravity supermultiplet (the only possible in the case at hand, due to the maximality of the considered $\mathcal{N} = 8$, $d = 4$ theory).

By the action of $SU(8)$, every element in the fundamental representation space of $E_{7(7)}$ may be put in a skew-diagonal form, called normal form, reading

$$\begin{pmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \tau_3 & \\ & & & \tau_4 \end{pmatrix} \otimes \epsilon, \quad \begin{cases} \tau_\mu = \upsilon_\mu e^{i\vartheta_\mu} \in \mathbb{C}, \\ \\ \upsilon_\mu, \vartheta_\mu \in \mathbb{R} \end{cases} \quad \forall \mu = 1, 2, 3, 4, \tag{6.2.1}$$

where as before the unwritten components vanish. Thus, the action of the automorphism group of the $\mathcal{N} = 8$, $d = 4$ SUSY algebra may be made equivalent to the choice of the so-called normal frame in the 56-d fundamental representation space of $E_{7(7)}$.

By further using the symmetry subgroup $(U(1))^3 \subset SU(8)$, the relative phases of the τ_μ's can be changed, thus letting just an overall phase $\vartheta \equiv \sum_{\mu=1}^{4} \vartheta_\mu$ and four real positive skew-diagonal eigenvalues υ_μ's. Therefore, we may say that, without loss of generality, the general form of the elements of the 56-d fundamental representation space of $E_{7(7)}$ in the so-called improved normal frame reads

$$\begin{pmatrix} \upsilon_1 & & & \\ & \upsilon_2 & & \\ & & \upsilon_3 & \\ & & & \upsilon_4 \end{pmatrix} e^{i\vartheta} \otimes \epsilon, \quad \upsilon_\mu \in \mathbb{R}^+, \vartheta \in \mathbb{R}, \ \forall \mu = 1, 2, 3, 4. \tag{6.2.2}$$

On the other hand, the central charge matrix Z_{AB}, determining the central extension of the $\mathcal{N} = 8$, $d = 4$ SUSY algebra is, in general, given by an 8×8

[6] Actually, this is a common feature to all maximally extended SUGRAs, which correspond to the low-energy limit of type II superstrings toroidally compactified

complex antisymmetric matrix. Also in this case, it is always possible, by suitably acting with $SU(8)$ and its proper subgroup $(U(1))^3$, to choose the "improved normal frame." Stressing the complete functional dependence of Z_{AB} and of its $SU(8)$-transformed version, we thus have

$$
\begin{gathered}
\text{Generic frame} : Z_{AB}\left(q, \varphi_\infty\right) \\
\downarrow SU(8) \\
\text{"Normal frame":} \\
\widehat{e}_{AB}\left(q, \varphi_\infty\right) \equiv \begin{pmatrix} z_1\left(q, \varphi_\infty\right) & & & \\ & z_2\left(q, \varphi_\infty\right) & & \\ & & z_3\left(q, \varphi_\infty\right) & \\ & & & z_4\left(q, \varphi_\infty\right) \end{pmatrix} \otimes \epsilon, \\
\begin{cases} z_\mu = \rho_\mu e^{i\kappa_\mu} \in \mathbb{C}, \\ \rho_\mu, \kappa_\mu \in \mathbb{R}, \end{cases} \quad \forall \mu = 1, 2, 3, 4, \\
\downarrow (U(1))^3 \subset SU(8) \\
\text{Improved normal frame:} \\
e_{AB}\left(q, \varphi_\infty\right) \equiv \begin{pmatrix} \rho_1\left(q, \varphi_\infty\right) & & & \\ & \rho_2\left(q, \varphi_\infty\right) & & \\ & & \rho_3\left(q, \varphi_\infty\right) & \\ & & & \rho_4\left(q, \varphi_\infty\right) \end{pmatrix} e^{i\kappa} \otimes \epsilon, \\
\begin{cases} \rho_\mu \in \mathbb{R}^+, \forall \mu = 1, 2, 3, 4, \\ \kappa \equiv \sum_{\mu=1}^4 \kappa_\mu \in \mathbb{R}, \end{cases}
\end{gathered}
\tag{6.2.3}
$$

where the q's are the charges and the φ_∞'s are the asymptotical moduli.

As for the $d = 5$ case, the $z_\mu\left(q, \varphi_\infty\right)$'s are usually called the skew-diagonal eigenvalues of the central charge matrix Z_{AB}, whereas the set made by the $\rho_\mu\left(q, \varphi_\infty\right)$'s and the overall phase $\kappa\left(q, \varphi_\infty\right)$ is usually referred to as the set of the improved normal frame parameters. Notice that, differently from the case $d = 5$, now e_{AB}, beside being "ordinarily" traceless, is not also "skew-diagonally" traceless.

Therefore, it is clear that e_{AB} may be considered as an element of the improved normal frame of the 56-d fundamental representation space of $E_{7(7)}$, without any additional constraint (differently from the case $d = 5$, in which the skew-diagonal tracelessness condition (6.1.3) must hold).

From the considerations previously made in the case $d = 5$, it is clear that the set of charges $\left\{q^i\right\}_{i=1,\dots,56}$ will fit a vector in the **56** of $E_{6(6)}$, i being a vector index in such a representation.

Thus, it may be stated that the formal couple $\left(q, \varphi_\infty\right)$ belongs to the direct product of the **56** of $E_{7(7)}\left(\mathbb{Z}\right)$ and of the 70-d moduli space $E_{7(7)}/SU(8)$ of the theory. $E_{7(7)}\left(\mathbb{Z}\right)$ is the "discrete version" of the U-duality group, since the conserved charges q^i's are quantized.

Since the lowest order product of fund. reprs. **56** of $E_{7(7)}$ containing the singlet in its symmetric part is the quartic one, the lowest order invariant in the fundamental representation **56** of $E_{7(7)}$ is quartic in the q^i's, and will be therefore given by

$$I_4(56)\,(q) \equiv T_{ijkl}q^i q^j q^k q^l, \tag{6.2.4}$$

where T_{ijkl} is a particular completely symmetric rank-4 covariant tensor in the **56** of $E_{7(7)}$.

It may be shown that an equivalent expression in the normal frame of the 56-d fundamental representation space of $E_{7(7)}$ reads as follows:

$$\begin{aligned} I_4(56)\,(q) &= \sum_{\mu=1}^{4} |z_\mu\,(q,\varphi_\infty)|^4 - 2\sum_{\mu<\nu=1}^{4} |z_\mu\,(q,\varphi_\infty)|^2\,|z_\nu\,(q,\varphi_\infty)|^2 \\ &\quad +4\left[\begin{array}{l} z_1\,(q,\varphi_\infty)\,z_2\,(q,\varphi_\infty)\,z_3\,(q,\varphi_\infty)\,z_4\,(q,\varphi_\infty) \\ \\ +\overline{z_1}\,(q,\varphi_\infty)\,\overline{z_2}\,(q,\varphi_\infty)\,\overline{z_3}\,(q,\varphi_\infty)\,\overline{z_4}\,(q,\varphi_\infty) \end{array}\right]. \end{aligned}$$

By passing to the improved normal frame, we get

$$\begin{aligned} I_4(56)\,(q) = {} & [\rho_1\,(q,\varphi_\infty) + \rho_2\,(q,\varphi_\infty) + \rho_3\,(q,\varphi_\infty) + \rho_4\,(q,\varphi_\infty)] \\ & \cdot [\rho_1\,(q,\varphi_\infty) + \rho_2\,(q,\varphi_\infty) - \rho_3\,(q,\varphi_\infty) - \rho_4\,(q,\varphi_\infty)] \\ & \cdot [\rho_1\,(q,\varphi_\infty) - \rho_2\,(q,\varphi_\infty) + \rho_3\,(q,\varphi_\infty) - \rho_4\,(q,\varphi_\infty)] \\ & \cdot [\rho_1\,(q,\varphi_\infty) - \rho_2\,(q,\varphi_\infty) - \rho_3\,(q,\varphi_\infty) + \rho_4\,(q,\varphi_\infty)] \\ & +\rho_1\,(q,\varphi_\infty)\,\rho_2\,(q,\varphi_\infty)\,\rho_3\,(q,\varphi_\infty)\,\rho_4\,(q,\varphi_\infty)\,[cos\,(\kappa\,(q,\varphi_\infty)) - 1]\,. \end{aligned} \tag{6.2.5}$$

In the particular case in which $cos\,(\kappa\,(q,\varphi_\infty)) = 1$, by defining

$$\begin{cases} \eta_1 \equiv \rho_1 + \rho_2 + \rho_3 + \rho_4, \\ \\ \eta_2 \equiv \rho_1 + \rho_2 - \rho_3 - \rho_4, \\ \\ \eta_3 \equiv \rho_1 - \rho_2 + \rho_3 - \rho_4, \\ \\ \eta_4 \equiv \rho_1 - \rho_2 - \rho_3 + \rho_4, \end{cases} \tag{6.2.6}$$

we may rewrite a much simpler expression in the improved normal frame

$$I_4(56)\,(q) = \eta_1\,(q,\varphi_\infty)\,\eta_2\,(q,\varphi_\infty)\,\eta_3\,(q,\varphi_\infty)\,\eta_4\,(q,\varphi_\infty)\,. \tag{6.2.7}$$

As for $I_3(27)$ of $E_{6(6)}$, one of the main features of the quartic invariant $I_4(56)$ of $E_{7(7)}$ is its dependence only on the charges. Indeed, despite the fact that the z's, the ρ's, the η's, and κ are functions of the charges and of the moduli, $I_4(56)$ is moduli independent. It is worth mentioning once again that the property of moduli independence is the key feature in order to use such a U- invariant quantity to describe the SUSY-preservation features and the entropy in the physics of extremal BHs.

Differently from the $d=5$ case, it may be shown that in the considered $\mathcal{N}=8$, $d=4$ SUGRA (with $E_{7(7)}$ as U-group) the sign of $I_4(56)$ is important, because it is conserved under CPT, which is an exact symmetry of the theory.

It is, in general, possible to give a U-duality invariant classification of the BPS extremal BH solutions in the $\mathcal{N}=8$, $d=4$ SUGRA (with $E_{7(7)}$ as U-group) by using the lowest order (namely quartic) invariant of such an exceptional Lie group.

Similarly to the $d=5$ case, the relevant steps are

1. Formulating some suitable constraints on the quartic invariant $I_4(56)$ of the fundamental representation **56** of $E_{7(7)}$.
2. Interpreting such constraints as the defining equations of the orbits in the 56-d fundamental representation space of $E_{7(7)}$. Such orbits, in general, admit a coset expression, i.e., a formulation in terms of a suitable coset of the starting 56-d space; indeed, they can be considered as particular NLRs of the U-duality symmetry group $E_{7(7)}$ in its 56-d fundamental representation space. As above, the little groups of the orbits are defined as the stabilizers in the coset expression.
3. Counting the independent (real) parameters associated with the considered orbit, and thus calculating its dimension, by looking only at the central charge matrix. The key point is finding the so-called little group of the normal form of the properly constrained central charge matrix, which, in general, will be a (proper) subgroup of the so-called little group of the orbit[7].

---

[7] Similarly to the $d=5$ case, an alternative procedure may be used in order to calculate the dimensions of the orbits in the considered exceptional Lie group $E_{7(7)}$.

Such a procedure is based on the various symmetry groups and the different gradings of $J_{3,split}^{\mathbb{O}}$, i.e., of the exceptional Jordan algebra of 3×3 Hermitian matrices over the split form of the composition algebra of octonions $\mathbb{O}$, but in a slightly generalized sense w.r.t. to the $d=5$ case.

Indeed, it may be shown that the quartic invariant I_4 of the fundamental representation **56** of $E_{7(7)}$ can be identified with the quartic norm of the so-called *exceptional Freudhental triple system* $\mathfrak{R}$, which offers a realization of the 56-d fund. repr. space of $E_{7(7)}$ in terms of 2×2 "generalized matrices" of the form

$$\mathfrak{R}\ni q\equiv\begin{pmatrix}\alpha & j_1\\ j_2 & \beta\end{pmatrix},$$

where $\alpha,\beta\in\mathbb{R}$ and $j_1,j_2\in J_{3,split}^{\mathbb{O}}$.

With a slight abuse of notation, we may say that such an identification is based on the result

$$St_0\left(\mathfrak{R}\right)=E_{7(7)},$$

As for the $d = 5$ case, the dimension of the orbit will be obtained by adding the number of independent real improved normal frame parameters of the central charge matrix Z_{AB} to the dimension of the coset space $\mathcal{R}/\mathcal{K}$, where

(i) $\mathcal{R} = SU(8)$, which is, at the same time, the maximal compact subgroup of $E_{7(7)}$ and the $\mathcal{R}$-symmetry group of the $\mathcal{N} = 8$, $d = 4$ SUSY algebra, and
(ii) as for the $d = 5$ case, $\mathcal{K}$ is the maximal compact subgroup of the little group of the normal form of the properly constrained central charge matrix.

Clearly, $dim\,(SU(8)/\mathcal{K})$ expresses the number of independent (real) angles in the relevant $\mathcal{R}$-symmetry rotation subgroup related to the considered properly constrained central charge matrix.

As already mentioned, for $\mathcal{N} = 8$, 3 different BPS degrees of SUSY-preservation exist: $\frac{1}{8}$-BPS (minimally preserving), $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS (maximally preserving).

Differently from $d = 5$, in the $d = 4$ case there are more than three sets of invariant constraints on $I_4(56)$; this is clearly due to the richer algebraic structure of set of constraints imposed on a quartic norm form w.r.t. the ones coming from a cubic norm.

In the remaining part of this Subsection we will consider such constraints, expliciting the three steps of the above presented algorithm.

I] U-invariant characterization of $\frac{1}{8}$-BPS (minimally SUSY-preserving) solutions.

I.1) Timelike $E_{7(7)}$-invariant constraint on the quartic norm of **56**:

$$I_4(56)\,(q) > 0. \tag{6.2.8}$$

In $d = 4$ the sign of such a disequality matters, because, as previously reported, $sgn\,(I_4(56)\,(q))$ is a CPT-invariant quantity. The choice of a timelike constraint instead of a spacelike one is due to the following noteworthy relation, which may be rigorously obtained in the considered framework of $\mathcal{N} = 8$, $d = 4$ SUGRA with $E_{7(7)}$ as U-group:

$$M_{BR}^4\,(q) = M_{ADM}^4\,(q, \varphi_H\,(q)) = I_4(56)\,(q)\,. \tag{6.2.9}$$

---

namely, the invariance group of the quartic norm form of the exceptional Freudhental triple system $\Re$ (defined in terms of the trace operator, of the cubic norm form, and of the quadratic adjoint map of $J_{3,split}^{\mathbb{O}}$) is nothing but $E_{7(7)}$.

By the way, it is worth pointing out that the procedure mentioned at point 3 allows one to better relate the set of constraints, defining the orbit in the fundamental representation space of $E_{7(7)}$, with the various degrees of SUSY-preservation exhibited by the extremal BPS BH solutions in the considered $\mathcal{N} = 8$, $d = 4$ SUGRA

This amounts to say that the mass of the near-horizon BR geometry (given by the ADM mass of the BPS stable interpolating soliton metric solution, evaluated at the purely charge-dependent near-horizon moduli configuration) is equal to the fourth root of the quartic invariant $I_4(56)\,(q)$ of the **56** of $E_{7(7)}$. Thus, in order to obtain a real (strictly positive) mass for the BR geometry, it is mandatory to choose a strictly positive $I_4(56)\,(q)$, or, in other words, a timelike defining constraint.

Moreover, nothing forbids to consider the following ordering of the absolute value of the complex skew-diagonal eigenvalues of the central charge matrix Z_{AB}:

$$|z_1\,(q,\varphi_\infty)| \geqslant |z_2\,(q,\varphi_\infty)| \geqslant |z_3\,(q,\varphi_\infty)| \geqslant |z_4\,(q,\varphi_\infty)|\,. \tag{6.2.10}$$

Such a situation does not imply any loss of generality, up to irrelevant renamings[8] of the z's. Thence, it may be shown that the following near-horizon limit holds:

$$\begin{aligned} &z_1\,(q,\varphi_\infty = \varphi_H\,(q)) \equiv z_{1,H}\,(q) \in \mathbb{C}_0, \\ &z_i\,(q,\varphi_\infty = \varphi_H\,(q)) \equiv z_{i,H}\,(q) = 0, \forall i = 1,2,3. \end{aligned} \tag{6.2.11}$$

Consequently, by looking at the expression (6.2.5) of $I_4(56)\,(q)$ in the improved normal frame of the 56-d fundamental representation space of $E_{7(7)}$, it turns out that choosing $cos\,(\kappa\,(q,\varphi_\infty)) = 1$ does not yield any loss of generality.

Therefore, at least in the near horizon limit, the timelike U-invariant constraint characterizing the $\frac{1}{8}$-BPS (minimally SUSY-preserving) solutions may be rewritten as follows:

8 Actually, nothing a priori forbids that, once fixed, such an ordering in absolute value of the skew-diagonal eigenvalues of Z_{AB} may change, depending on the considered value of the formal couple

$$(q,\varphi_\infty) \in \left(\mathbf{56} \text{ of } E_{7(7)}\,(\mathbb{Z})\right) \times \left(E_{7(7)}/SU(8)\right).$$

Such a possibility depends on the functional dependence of the z's on the couple (q,φ_∞).

Thus, in order to make (6.2.10) and (6.2.11) compatible, we should always conjecture that the ordering expressed by (6.2.10) is "stable" in a suitable neighborhood of the discrete attractor point in the (asymptotical) moduli space (remind also that more than one of such attractors might exist).

Otherwise speaking, a minimal hypothesis of consistence for (6.2.10) and (6.2.11) is the stability of the ordering (6.2.10) in

$$\left(\mathbf{56} \text{ of } E_{7(7)}\,(\mathbb{Z})\right) \times I\,(\varphi_H\,(q))\,,$$

where $I\,(\varphi_H\,(q))$ is a suitable neighborhood of the considered attractor point $\varphi_H\,(q)$ in the moduli space $\left(E_{7(7)}/SU(8)\right)$ of the theory.

The fulfilling of such a request, which might appear as an irrelevant subtlety, is instead strictly dependent on the topological properties of the moduli space, considered as an attractor variety (see, e.g., [31]– [33]).

$$I_4(56)\,(q) = \eta_1\,(q, \varphi_\infty)\,\eta_2\,(q, \varphi_\infty)\,\eta_3\,(q, \varphi_\infty)\,\eta_4\,(q, \varphi_\infty) > 0, \tag{6.2.12}$$

at least implying that all η's are not vanishing for every value of the formal couple (q, φ_∞).

Since $I_4(56)\,(q)$ depends only on the charges, it can be equivalently evaluated at any point in the asymptotical moduli space $E_{7(7)}/SU(8)$. In particular, one may choose to evaluate it at $\{\varphi_\infty\} = \{\varphi_H\,(q)\}$, i.e., at the asymptotical configuration(s) coinciding with the near-horizon, attracted one(s). By the limit expressed by (6.2.11), it is therefore possible to conclude that the timelike U-invariant constraint characterizing the $\frac{1}{8} - BPS$ (minimally SUSY-preserving) solutions reads as

$$I_4(56)\,(q) = |z_{1,H}\,(q)|^4 = (\rho_{1,H}\,(q))^4\,. \tag{6.2.13}$$

By using relation (6.2.9), one obtains

$$M^4_{BR}\,(q) = M^4_{ADM}\,(q, \varphi_H\,(q)) = I_4(56)\,(q) = |z_{1,H}\,(q)|^4 = (\rho_{1,H}\,(q))^4\,. \tag{6.2.14}$$

Finally, by recalling (3.2.131), one gets

$$\begin{aligned} S_{BH} &= \frac{A_H}{4} = \pi M^2_{BR}\,(q) = \pi M^2_{ADM}\,(q, \varphi_H\,(q)) \\ &= \pi\sqrt{I_4(56)\,(q)} = \pi\,|z_{1,H}\,(q)|^2 = \pi\,(\rho_{1,H}\,(q))^2 > 0. \end{aligned} \tag{6.2.15}$$

Therefore, the $\frac{1}{8}$-BPS soliton metric solutions described by the timelike U-invariant constraint (6.2.8) have a physically consistent, purely charge-dependent, strictly positive entropy.

The fundamental relation

$$S_{BH} = \pi\sqrt{I_4(56)\,(q)} \tag{6.2.16}$$

formally holds for all existing orbits, and it gives a U-invariant expression for the BH entropy in the considered $\mathcal{N} = 8$, $d = 4$ SUGRA with $E_{7(7)}$ as U-duality symmetry. Indeed, as its $d = 5$ counterpart given by (6.1.15), (6.2.16) relates the dynamical internal symmetries encoded in the U-duality group $E_{7(7)}$ with the near-horizon Bertotti–Robinson BH geometry, and therefore with its entropy.

In this sense, $d = 4$ and $d = 5$ seem to be quite special cases, because for $d \leqslant 3$ and $d \geqslant 6$ no U-invariant expressions for the entropy, i.e., no U-invariant formulations of the BHEA formula, exist at all. This is the reason why the attractor mechanism yields a vanishing (or physically inconsistent constant) result for the entropy of $p\,(< d - 3)$-d extremal (asymptotically flat) "black branes" in $d \leqslant 3$ and $d \geqslant 6$ s-t dimensions.

As mentioned above, the attractor mechanism shows up by determining the near-horizon configuration(s) , which "attract" the dynamics in the moduli space. Indeed, the attractor points are critical points of the highest absolute-valued skew-diagonal eigenvalue of Z_{AB}, seen as a function of the asymptotical moduli, which may continuously and unconstrainedly vary in the asymptotical moduli space $E_{7(7)}/SU(8)$:

$$E_{7(7)}/SU(8) \ni \varphi_H(q) : \left.\frac{\partial |z_1(q,\varphi_\infty)|}{\partial \varphi_\infty}\right|_{\varphi_\infty=\varphi_H(q)} = \left.\frac{\partial \rho_1(q,\varphi_\infty)}{\partial \varphi_\infty}\right|_{\varphi_\infty=\varphi_H(q)} = 0; \tag{6.2.17}$$

such an extremization at the near-horizon, attractor configuration(s) $\varphi_H(q)$ defines the extremum

$$|z_{1,H}(q)| = \rho_{1,H}(q) \in \mathbb{R}_0^+. \tag{6.2.18}$$

As we will see below in point III, there actually exists another $\frac{1}{8}$-BPS extremal solution constrained in a different way, but it gives rise to zero entropy, and therefore it does not yield a sufficiently regular geometry of the EH (notice that such a "degeneration" of the $\frac{1}{8}$-BPS solutions does not happen in the previously treated $d=5$ case).

Similarly to the $d=5$ case, also in $\mathcal{N}=8$, $d=4$ SUGRA (with $E_{7(7)}$ as U-group) only the (timelike constrained) $\frac{1}{8}$-BPS (minimally SUSY-preserving) soliton metric background will have a physically consistent, purely charge-dependent, strictly positive entropy.

I.2) The little group related to the corresponding timelike orbit in the 56-d fundamental representation space of $E_{7(7)}$ is $E_{6(2)}$, admitting as proper subgroup $O(4,4)$, which is the little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant condition (6.2.8). Finally, the maximal compact subgroup of $O(4,4)$ is $(USp(2))^4$.

Thus, the timelike case is characterized by the following chain of group inclusions:

$$(USp(2))^4 \subset O(4,4) \subset E_{6(2)} \subset E_{7(7)}, \tag{6.2.19}$$

and the corresponding coset expression of the timelike orbit reads

$$\frac{E_{6(6)}}{E_{6(2)}}. \tag{6.2.20}$$

I.3) In this case the proper subspace of the 56-d fundamental representation space of $E_{7(7)}$ to be considered is[9]

$$\frac{SU(8)}{(USp(2))^4}, \tag{6.2.21}$$

with dimension $63 - 4 \cdot 3 = 51$.

Consequently, the dimension of the timelike orbit is $5 + 51 = 56$.

[9] We use group-theoretical conventions such that $USp(2) = SU(2)$

II] U-invariant characterization of non-BPS solutions.

II.1) The $E_{7(7)}$-invariant constraint on the cubic norm of **56** is nothing but the spacelike counterpart of constraint (6.2.8)

$$I_4(56)(q) < 0, \tag{6.2.22}$$

at least implying, in the case in which $cos\left(\kappa\left(q, \varphi_\infty\right)\right) = 1$, that all η's are not vanishing for every value of the formal couple

$$(q, \varphi_\infty) \in \left(\mathbf{56} \text{ of } E_{7(7)}\left(\mathbb{Z}\right)\right) \times \left(E_{7(7)}/SU(8)\right). \tag{6.2.23}$$

II.2) The little group related to the corresponding lightlike orbit in the 56-d fundamental representation space of $E_{7(7)}$ is $E_{6(6)}$, i.e., nothing but the U-duality group of the $d = 5$ counterpart of the considered SUGRA theory (see Subsect. 6.1) .

Building up a chain of group inclusions similar to the case I, one finally obtains that the coset expression of the spacelike orbit reads

$$\frac{E_{7(7)}}{E_{6(6)}}. \tag{6.2.24}$$

II.3) Following the approach introduced in the case I, the dimension of the lightlike orbit can be calculated to be still 56.

Now, since the norm form defined in the 56-d fundamental representation space of $E_{7(7)}$ is quartic, one obtains three different degrees of degeneration of lightlike orbits.

It should also be pointed out that, differently from the $d = 5$ case, the U-invariant condition for the **56** of $E_{7(7)}$ to be a null vector (with respect to the quartic norm) is not enough to guarantee the SUSY enhancement (i.e., the $\frac{1}{4}$-BPS nature of the corresponding solution). This can easily be seen in the normal frame (reached by performing a suitable $SU(8)$ transformation), where it may be explicitly shown that for a null vector there are not, in general, coinciding (at least in absolute value) “skew-diagonal eigenvalues.

As reported below, the enhancement of the BPS SUSY-preservation from the minimal $\frac{1}{8}$-BPS to the intermediate $\frac{1}{4}$-BPS degree may be obtained only by also requesting the criticality feature for the orbit (see the defining constraint (6.2.33)).

Thus, beside the $\frac{1}{4}$-BPS “critical” degenerate orbit (case IV), the triplet of degenerate orbits in $\mathcal{N} = 8$, $d = 4$ SUGRA (with U-group given by $E_{7(7)}$) will be completed by a “noncritical” orbit (case III, related to the additional $\frac{1}{8}$-BPS state with vanishing entropy) and by a $\frac{1}{2}$-BPS (i.e., maximally SUSY-preserving) “doubly critical” orbit (case V).

III] U-invariant characterization of $\frac{1}{8}$-BPS (minimally SUSY-preserving) solutions with vanishing entropy.

III.1) Lightlike noncritical $E_{7(7)}$-invariant constraint on the quartic norm of **56**:

$$\begin{cases} I_4(56)\,(q)=0, \\ \frac{\partial I_4(56)(q)}{\partial q^i}\neq 0, \forall i=1,\ldots,56. \end{cases} \tag{6.2.25}$$

The set of lightlike noncritical constraints (6.2.25) implies that there are four independent improved normal frame parameters out of the possible five ones and, (at least) in the case in which $cos\,(\kappa\,(q,\varphi_\infty))=1$, that (up to irrelevant renamings of the η's)

$$\begin{cases} \eta_1\,(q,\varphi_\infty)\,,\eta_2\,(q,\varphi_\infty)\,,\eta_3\,(q,\varphi_\infty)\in\mathbb{R}_0, \\ \eta_4\,(q,\varphi_\infty)=0, \end{cases} \tag{6.2.26}$$

$$\forall q^i\in 56 \text{ of } E_{7(7)}\,(\mathbb{Z})\,,\ \forall\varphi_\infty\in E_{7(7)}/SU(8).$$

III.2) The little group related to the corresponding degenerate lightlike noncritical orbit in the 56-d fundamental representation space of $E_{7(7)}$ is $F_{4(4)}\otimes_s T_{26}$, where T_{26} is a 26-d Abelian subgroup of $E_{7(7)}$.

Building up the relevant chain of group inclusions, one finally obtains that the coset expression of the degenerate lightlike noncritical orbit reads

$$\frac{E_{7(7)}}{F_{4(4)}\otimes_s T_{26}}. \tag{6.2.27}$$

III.3) Following the previously introduced approach, the dimension of the degenerate lightlike noncritical orbit can be calculated to be 55.

IV] U-invariant characterization of $\frac{1}{4}$-BPS (intermediate SUSY-preserving) solutions.

IV.1) Lightlike critical $E_{7(7)}$-invariant constraint on the quartic norm of **56**:

$$\forall i,j=1,\ldots,56 \begin{cases} I_4(56)\,(q)=0, \\ \frac{\partial I_4(56)(q)}{\partial q^i}=0, \\ \left.\frac{\partial^2 I_4(56)(q)}{\partial q^i\partial q^j}\right|_{Adj\left(E_{7(7)}\right)}=\left.T_{ijkl}q^k q^l\right|_{Adj\left(E_{7(7)}\right)}\neq 0. \end{cases} \tag{6.2.28}$$

The symmetric quadratic polynomials in the charges $\left.T_{ijkl}q^k q^l\right|_{Adj\left(E_{7(7)}\right)}$ appearing in the third constraint of the lightlike critical set (6.2.28) correspond to the projection of the Hessian matrix of $I_4(56)\,(q)$ on the adjoint representation **133** of the U-group $E_{7(7)}$.

Actually, the whole set of lightlike critical constraints may be more rigorously formulated in the following way (up to irrelevant renamings of the z's and therefore of the ρ's):

$$\begin{cases} \eta_1 (q, \varphi_\infty), \eta_2 (q, \varphi_\infty) \in \mathbb{R}_0, \\ \\ \eta_3 (q, \varphi_\infty) = 0 = \eta_4 (q, \varphi_\infty). \end{cases} \tag{6.2.29}$$

$$\Updownarrow$$

$$\begin{cases} \kappa (q, \varphi_\infty) = 0, \\ \\ \rho_1 (q, \varphi_\infty) = \rho_2 (q, \varphi_\infty), \\ \\ \rho_3 (q, \varphi_\infty) = \rho_4 (q, \varphi_\infty), \end{cases} \tag{6.2.30}$$

$$\Downarrow$$

$$\begin{cases} |z_1 (q, \varphi_\infty)| = |z_2 (q, \varphi_\infty)|, \\ \\ |z_3 (q, \varphi_\infty)| = |z_4 (q, \varphi_\infty)|. \end{cases} \tag{6.2.31}$$

$$\Updownarrow$$

$$\frac{\partial I_4(56)(z)}{\partial z_\mu} = 0, \forall \mu = 1, 2, 3, 4. \tag{6.2.32}$$

$$\Updownarrow$$

$$\forall i, j = 1, \ldots, 56 \begin{cases} I_4(56)(q) = 0, \\ \\ \frac{\partial I_4(56)(q)}{\partial q^i} = 0, \\ \\ \left.\frac{\partial^2 I_4(56)(q)}{\partial q^i \partial q^j}\right|_{Adj\left(E_{7(7)}\right)} = T_{ijkl} q^k q^l \big|_{Adj\left(E_{7(7)}\right)} \neq 0. \end{cases} \tag{6.2.33}$$

All constraints must hold $\forall q^i \in 56$ of $E_{7(7)}(\mathbb{Z})$ and $\forall \varphi_\infty \in E_{7(7)}/SU(8)$.

In general, when there is no extra overall phase $\kappa (q, \varphi_\infty)$ in the improved normal frame of the 56-d fund. repr. space of $E_{7(7)}$, or better, when there is no extra term $\sim [cos (\kappa (q, \varphi_\infty)) - 1]$ in the explicit expression (6.2.5) of $I_4(56)$ in such a frame, then the related extremal BPS solution is at least $\frac{1}{4}$-BPS, i.e., it preserves at least 8 supersymmetries out of the 32 preserved by the maximally supersymmetric $\mathcal{N} = 8$, $d = 4$ backgrounds.

IV.2) The little group related to the corresponding degenerate lightlike critical orbit in the 56-d fundamental representation space of $E_{7(7)}$ is $O(6, 5) \otimes_s (T_{32} \otimes T_1)$. The little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant conditions (6.2.29)–(6.2.33), is $O(5, 5)$, corresponding to the simple part of $O(6, 5) \otimes_s (T_{32} \otimes T_1)$. Finally, the maximal compact subgroup of $O(5, 5)$ is $(USp(4))^2$.

Thus, the lightlike critical case is characterized by the following chain of group inclusions:

$$(USp(4))^2 \subset O(5, 5) \subset O(6, 5) \otimes_s (T_{32} \otimes T_1) \subset E_{7(7)}, \tag{6.2.34}$$

and the corresponding coset expression of the degenerate lightlike critical orbit reads

$$\frac{E_{6(6)}}{O(6,5)\otimes_s (T_{32}\otimes T_1)}. \tag{6.2.35}$$

IV.3) In the case at hand the proper subspace of the 56-d fund. repr. space of $E_{7(7)}$ to be considered is

$$\frac{SU(8)}{(USp(4))^2}, \tag{6.2.36}$$

with dimension $63-2\cdot 10=43$.

Thus, the dimension of the degenerate lightlike critical orbit is $2+43=45$.

V] U-invariant characterization of $\frac{1}{2}$-BPS (maximally SUSY-preserving) solutions.

V.1) Lightlike doubly critical $E_{7(7)}$-invariant constraint on the quartic norm of **56** (up to irrelevant renamings of the z's and therefore of the ρ's)[10]:

$$\begin{cases} \eta_1(q,\varphi_\infty)\in\mathbb{R}_0, \\ \\ \eta_2(q,\varphi_\infty)=\eta_3(q,\varphi_\infty)=\eta_4(q,\varphi_\infty)=0. \end{cases} \tag{6.2.37}$$

$$\Updownarrow$$

$$\begin{cases} \kappa(q,\varphi_\infty)=0, \\ \\ \rho_1(q,\varphi_\infty)=\rho_2(q,\varphi_\infty)=\rho_3(q,\varphi_\infty)=\rho_4(q,\varphi_\infty), \end{cases} \tag{6.2.38}$$

$$\Downarrow$$

[10] Notice that the "intermediate" situation between cases IV and V, corresponding to three coinciding (in absolute value) skew-diagonal eigenvalues out of four, is missing (or better, it corresponds to non-BPS states).

Actually, such a case should be considered in the classification of the possible extremal BPS states, but it is forbidden by the noncompact U-duality symmetry expressed by $E_{7(7)}(\mathbb{Z})$.

This is due to the remarkable fact that the $E_{7(7)}$-duality gives additional restrictions on the BPS states, other than the ones merely implied by the SUSY algebra.

The analysis of the (double) extremal BHs yields that the quartic invariant $I_4(56)$ is positive definite for BPS states. Such a result implies that

1. configurations preserving $\frac{1}{4}$ of the maximal number of supersymmetries must have eigenvalues equal (in absolute value) in pairs (see constraining condition (6.2.31));
2. the above-mentioned intermediate configurations, which, up to irrelevant renamings of the z's, may be expressed by the condition (holding $\forall q^i\in 56$ of $E_{7(7)}(\mathbb{Z})$ and $\forall\varphi_\infty\in E_{7(7)}/SU(8)$)

$$|z_1(q,\varphi_\infty)|=|z_2(q,\varphi_\infty)|=|z_3(q,\varphi_\infty)|\neq|z_4(q,\varphi_\infty)|,$$

are not BPS.

$$|z_1(q,\varphi_\infty)| = |z_2(q,\varphi_\infty)| = |z_3(q,\varphi_\infty)| = |z_4(q,\varphi_\infty)|. \tag{6.2.39}$$

$$\Updownarrow$$

$$\begin{cases} \frac{\partial I_4(56)(z)}{\partial z_\mu} = 0, \forall \mu = 1,2,3,4, \\ \\ |z_2(q,\varphi_\infty)| = |z_3(q,\varphi_\infty)|. \end{cases} \tag{6.2.40}$$

$$\Updownarrow$$

$$\forall i,j = 1,\dots,56 \begin{cases} I_4(56)(q) = 0, \\ \\ \frac{\partial I_4(56)(q)}{\partial q^i} = 0, \\ \\ \left.\frac{\partial^2 I_4(56)(q)}{\partial q^i \partial q^j}\right|_{Adj(E_{7(7)})} = T_{ijkl} q^k q^l \big|_{Adj(E_{7(7)})} = 0. \end{cases} \tag{6.2.41}$$

All constraints must hold $\forall q^i \in 56$ of $E_{7(7)}(\mathbb{Z})$ and $\forall \varphi_\infty \in E_{7(7)}/SU(8)$.

Performing the decomposition of the U-group $E_{7(7)}$ in its T-symmetry and S-symmetry components, we obtain

$$E_{7(7)} \to SL(2,\mathbb{R}) \otimes O(6,6), \tag{6.2.42}$$

yielding, at the level of fundamental representations

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}). \tag{6.2.43}$$

Consequently, the charges of the system, which transform in the fund. repr. of $E_{7(7)}$, will undergo the following decomposition ($i = 1,\dots,56$):

$$\left\{q^i\right\} \to \left\{v^\alpha_{\widehat{i}}\right\} \cup \left\{s^a\right\}, \tag{6.2.44}$$

where $\alpha = 1,2$, $\widehat{i} = 1,\dots,12$ is a vector index and $a = 1,\dots,32$ is a spinor index.

Concerning the adjoint representations, the example of "$U \to T \otimes S$" duality decomposition expressed by (6.2.42) yields

$$\mathbf{133} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}). \tag{6.2.45}$$

Due to such a decomposition of the representations, the condition of (adjoint) "double-criticality" of the degenerate orbit corresponding to the extremal maximally SUSY-preserving $\frac{1}{2}$-BPS solutions, namely

$$\left.\frac{\partial^2 I_4(56)(q)}{\partial q^i \partial q^j}\right|_{Adj(E_{7(7)})} = T_{ijkl} q^k q^l \big|_{Adj(E_{7(7)})} = 0, \tag{6.2.46}$$

may be decomposed in three distinct conditions, namely

$$\begin{cases} (\mathbf{3},\mathbf{1})\text{-term}: & \frac{\partial^2 I_4(56)}{\partial v_{\widehat{i}}^{\alpha}\partial v_{\widehat{j}}^{\beta}}\eta_{\widehat{i}\widehat{j}} = 0\ , \\ & \\ (\mathbf{1},\mathbf{66})\text{-term}: & \frac{\partial^2 I_4(56)}{\partial s^a \partial s^b}\left(\gamma^{\widehat{i}\widehat{j}}\right)_{ab} + \frac{\partial^2 I_4(56)}{\partial v_{\widehat{i}}^{\alpha}\partial v_{\widehat{j}}^{\beta}}\epsilon_{\alpha\beta} = 0\ , \\ & \\ (\mathbf{2},\mathbf{32})\text{-term}: & \frac{\partial^2 I_4(56)}{\partial s^a \partial v_{\widehat{i}}^{\alpha}}\left(\gamma_{\widehat{i}}\right)_a = 0\ , \end{cases} \tag{6.2.47}$$

where $\eta_{\widehat{i}\widehat{j}}$ is the $(6,6)$-pseudoEuclidean metric, and $\left(\gamma_{\widehat{i}}\right)_a$ stands for the 12 32×32 gamma-matrices in $O(6,6)$, whereas $\left(\gamma^{\widehat{i}\widehat{j}}\right)_{ab}$ denotes their related commutators.

V.2) The little group related to the corresponding degenerate lightlike doubly critical orbit in the 56-d fundamental representation space of $E_{7(7)}$ is $E_{6(6)}\otimes_s T_{27}$. In this case the little group of the skew-diagonal matrix e_{AB}, properly constrained by the invariant conditions (6.2.37)–(6.2.41), is nothing but $E_{6(6)}$, i.e., the U-duality symmetry group of the case $d=5$ treated in Subsect. 6.1.

From the previous treatment, we know that $E_{6(6)}$ admits $USp(8)$ as maximal compact subgroup. Thus, the lightlike doubly critical case is characterized by the following chain of group inclusions:

$$USp(8)\subset E_{6(6)}\subset E_{6(6)}\otimes_s T_{27}\subset E_{7(7)}, \tag{6.2.48}$$

and the corresponding coset expression of the degenerate lightlike doubly critical orbit reads

$$\frac{E_{6(6)}}{E_{6(6)}\otimes_s T_{27}}. \tag{6.2.49}$$

V.3) In this case, the relevant subspace of the 56-d fund. repr. space of $E_{7(7)}$ we have to consider is

$$\frac{SU(8)}{USp(8)}, \tag{6.2.50}$$

with dimension $63-36=27$.

Consequently, the dimension of the degenerate lightlike doubly critical orbit is $1+27=28$.

The orbits described by cases III, IV, and V are all corresponding to different degrees of degenerate, lightlike orbits in the fund. repr. space of $E_{7(7)}$. Therefore, they all give rise to the unphysical result of zero BH entropy. Indeed, by recalling (3.2.131) and specializing it to such cases, one gets

$$S_{BH}=\frac{A_H}{4}=\pi\sqrt{I_4(56)\,(q)}=0. \tag{6.2.51}$$

Thus, the application of the Bekenstein–Hawking entropy-area formula is inconsistent in these cases.

Finally, disregarding the case II corresponding to non-BPS solutions, the remaining case is case I. As previously announced, in the context of $\mathcal{N} = 8$, $d = 4$ SUGRA with $E_{7(7)}$ as U-duality group, it corresponds to the only extremal $\frac{1}{8}$-BPS (and therefore minimally SUSY-preserving) solution with regular horizon geometry and strictly positive, physically consistent, purely charge-dependent entropy (given by (6.2.15)).

7

Microscopic Description. The Calabi–Yau Black Holes

In this section we will briefly consider the issue of the microscopic, statistical interpretation of the BH entropy, in which string theory plays a crucial role. Critical superstring theory lives in 10 dimensions, and usually the 4-d framework is reached by a compactification of the extra dimensions.

The standard compactification scenario is the Kaluza–Klein (KK) one, which leads to effective 4-d field theories. In such a context, the original n-d s-t is locally a product $M_4 \times Y$, where M_4 (with coordinates x^μ, $\mu = 0, 1, 2, 3$) denotes the 4-d s-t and Y (with coordinates y^m, $m = 4, \ldots, n-1$) stands for the (n–4)-d "internal" manifold of extra dimensions. At every point of M_4 corresponds a space Y, with a size compactified to a nondirectly detectable scale. Usually, Y belongs to a certain fixed topological class, parameterized by a set of parameters called moduli, which will appear as fields in the 4-d effective theory. When moving in M_4, the corresponding "internal" space Y may vary in such a way that the trajectory in M_4 will correspond to a trajectory of configurations inside the space of the moduli. In the general case of a nonconstant solution of the 4-d theory, each patch in M_4 corresponds to a nontrivial image in the moduli space of Y.

Such a scenario yields an higher dimensional interpretation of the singular BH solution. Indeed, the 4-d BH metric will vary in nontrivial way over M_4, and correspondingly in Y. When the gravitational fields become stronger and stronger near the center of the BH, the KK scenario loses any predictive power: we can only say that the local product structure $M_4 \times Y$ may break down in such regions, where the 4-d solution may be lifted to a genuine higher dimensional one.

The field-theoretical approach does not allow one to consider the global degrees of freedom of extended objects which may arise in this context, such as strings and branes. The higher dimensional wrapping properties of these objects determine the string description of the 4-d metric singularity represented by the BH. Indeed, strings and branes can wrap themselves around nontrivial cycles in the manifold Y, and such a phenomenon will occur at a corresponding point in M_4, such that a pointlike object will arise in the 4-d effective field theory. This object is nothing but the BH itself.

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Thus, two different approaches to BHs (and, in general, also to spatially extended s-t singularities) exist. In the *macroscopic* approach, which is the one related to GR and SUGRA theories, a BH is described by a singular global metric, and the gravitational fields may vary enormously from spatial infinity to the actual singularity, and also in time. The *microscopic* approach is instead related to string and brane theory, and expresses the pointlike s-t singularity of the BH as the result of a wrapping process taking place in the higher dimensional manifold of extra (compactified) dimensions, previously named Y. In such a context, gravitational fields are not immediately involved, and generally the string description is based on the 4-d flat s-t. Indeed, we will understand the previously introduced notation M_4 as standing for the 4-d Minkowskian s-t.

At this point a crucial question naturally arises: *can the microscopic and macroscopic descriptions be related in some way? Are they equivalent or not?*

Well, of course a priori one would expect that some kind of connection between these two ways of thinking about s-t singularities should exist, since gravitons can consistently be interpreted as closed string states interacting with wrapped branes. Essentially, one is led to a problem of interpolation in the string coupling constant g_s, which is the fundamental parameter governing the interactions. In general, such an interpolation is very hard to perform, but in the special case of supersymmetric extremal BHs precise and consistent predictions can be formulated, and comparisons between the above-mentioned two alternative approaches can be made, yielding new insights about BHs.

In what follows we will sketchily overview the string, microscopic approach to BHs, mainly relying on the nice review by De Wit [93], to whom we address the reader for the original relevant literature. (Here we just mention the seminal paper by Maldacena, Strominger and Witten [221]).

Let us start by recalling the fundamental relation between the massless 4-d fields and the harmonic forms on Y. Roughly speaking, we may say that harmonic forms are in one-to-one correspondence with the cohomology groups $H^p(Y)$, consisting of equivalence classes of closed but not exact forms on Y. In the KK scenario, a generic higherdimensional tensor field $\Phi(x,y)$ will schematically decompose as

$$\Phi(x,y) = \phi^A(x)\omega_A(y) , \tag{7.1}$$

where $\omega_A(y)$ stands for the relevant basis of independent harmonic forms on Y, and therefore for a basis of the relevant p-th cohomology $H^p(Y)$. The cardinality of the set of indices $\{A\}$, corresponding to the number of independent harmonic forms of degree p, is a topological invariant named Betti number, depending only on the topology of Y. By substituting the KK decomposition (7.1) into the higher dimensional action, we obtain interactions of the 4-d fields ϕ^A's with coupling constants given by the so-called intersection numbers

$$C_{ABC\ldots} \propto \int_Y \omega_A \wedge \omega_B \wedge \omega_C \ldots \quad . \tag{7.2}$$

Notice that for a 6-d manifold Y, the ω_A's are harmonic two-forms, and the intersection numbers C_{ABC} are completely symmetric, since the wedge product of two-forms is symmetric.

In general, the p-branes are p-d ($1 \leqslant p \leqslant n-4$) submanifolds of the $(n-4)$-d internal space Y, but they are not themselves the boundary of a $(p+1)$-d submanifold of Y, because otherwise they could collapse to a point. Thus, they can be related to the homological structure of Y. Now, by Poincarè duality, a p-brane is related to an harmonic $(n-4-p)$-form, and viceversa. Therefore, for example the wrapping of a brane may be expressed by decomposing its corresponding $(n-4-p)$-d cycle $\mathcal{P}$ in the relevant basis of $H_{n-4-p}(Y)$:

$$\mathcal{P} = p^A \varsigma_A, \quad p^A \in \mathbb{Z}. \tag{7.3}$$

The p^A's are called winding numbers (along the considered homology basis), and they express the number of times the extended object is wrapped around the cycle $\mathcal{P}$. They are nothing but magnetic charges in the resulting 4-d effective field theory. Since the p^A's are integer (quantized), one has to deal with the integer homology and cohomology of the topological manifold Y. A topologically invariant characterization of the wrapping of the considered brane around the $(n-4-p)$-d cycle $\mathcal{P}$ on Y may be expressed by

$$p \cdot p \cdot p \cdot \ldots \equiv C_{ABC\ldots} p^A p^B p^C \ldots \quad . \tag{7.4}$$

In general, such a (co)homological composition law of the winding numbers by the Poincarè dual "metric" defined by the intersection numbers $C_{ABC\ldots}$ allows one to invariantly express the number of intersections, and therefore the wrapping configuration, of an extended object embedded in a topological manifold.

As an example of a microscopic approach to BHs and BH entropy, we will now briefly describe the BHs in $\mathcal{N}=2$, $d=4$ SUGRA obtained by the compactification of 11-d M-theory on a 7-d manifold $Y = CY_3 \times S^1$, where CY_3 denotes a 3-d complex Calabi–Yau (CY) manifold.

Let us briefly introduce the CY manifolds.

From a mathematical point of view, a CY manifold is a Kähler manifold with vanishing first Chern class. If the complex dimension of the CY manifold is n, then it is called CY n-fold, which we will denote with CY_n.

In 1957 the mathematician Calabi put forward the suggestion that all CY_n's (with the additional property of compactness) could admit a Ricci-flat metric (precisely, one for each Kähler class). In 1977 Yau proved such a conjecture in the so-called Yau's theorem. Consequently, compact CY_n's may alternatively be defined as compact Ricci-flat n-d Kähler manifolds.

In general, a CY_n is characterized by the existence of a globally defined harmonic spinor φ, implying that the related canonical bundle on CY_n is trivial. This can easily be seen by considering a local system of real $2n$ coordinates

$$\left\{x^1, \ldots, x^n, y^1, \ldots, y^n\right\}, \tag{7.5}$$

in CY_n, such that the related complexification is given by the local system of complex n coordinates (which we will call z-system)

$$z^i \equiv x^i + iy^i, \ \forall i = 1, \ldots, n. \tag{7.6}$$

Thence, the harmonic spinor φ may be defined in the z-system as

$$\varphi_z \equiv dz^1 \wedge \ldots \wedge dz^n, \tag{7.7}$$

corresponding to a local section of the canonical bundle on CY_n. Let us now perform a(n invertible) change of coordinates to a w-system, defined in matrix notation as

$$w \equiv Az, \tag{7.8}$$

where $A \in U(n)$. In the w-system the (nonvanishing) harmonic spinor will be

$$\varphi_w = (detA)\, \varphi_z. \tag{7.9}$$

Therefore, in a generic complex (Kähler) n-d manifold the definition of the nonvanishing harmonic spinor is patch-dependent, and therefore local. In CY_n it is instead possible to extend such a definition to a global level, simply because, independently on the chosen patch, $detA = 1$ always. This can rigorously be stated by saying that the holonomy group of CY_n can be always reduced to the unimodular subgroup of $U(n)$, namely to $SU(n)$. This is an alternative defining characterization of the CY_n's, usually given in the context of Riemannian geometry.

Another equivalent definition in such a geometrical framework is that CY_n is an n-d complex calibrated manifold, i.e., it admits a globally defined calibration form τ, algebraically equivalent to

$$Re\left(dz^1 \wedge \ldots \wedge dz^n\right). \tag{7.10}$$

Otherwise speaking, CY_n may be characterized as admitting a global, nowhere vanishing holomorphic $(n, 0)$-form.

It should also be mentioned that the extra assumptions of simple connectedness and/or compactness are often made, especially in the applications.

Let us see some examples.

For $n = 1$, the CY_1's (also called elliptic curves) are nothing but the torii T^2's ($dim_{\mathbb{C}} T^2 = 1$). For such compact manifolds the Ricci-flat metrics are actually flat, and therefore the holonomy group is the trivial group $SU(1)$.

For $n = 2$, the (compact) CY_2's are given by the torii T^4's and the $K3$-manifolds, both with complex dimension two. Actually, the T^4's are sometimes excluded from such a classification, because their holonomy group is still the trivial group $SU(1) \subset SU(2)$, and not $SU(2)$, as the above given definition would require. On the other hand, $K3$-manifolds are an important example of compact complex surfaces. Generally, they are not algebraic, i.e., they cannot be embedded in any projective space as surfaces defined by polynomial equations. Despite this fact, they originally arose out in algebraic geometry, and they are named after three algebraic geometers, Kummer, Kähler, and Kodaira, also alluding to the mountain K2, who was often in the news when the name was given during the 1950s. Compactifications on $K3$-manifolds are the simplest after the toroidal ones, and they preserve $\frac{1}{2}$ of the supersymmetries of the original, uncompactified theory.

For $n \geqslant 3$, the complete classification of all possible CY_3's is still an open problem. One example of (compact) CY_3 is given by the quintic three fold in the complex projective space $\mathbb{CP}^4$.

For what concerns the physical applications of CY manifolds, they are very important in string theory (and M-theory) compactifications. Indeed, superstring theory has critical $d = 10$ and, as previously mentioned, usually the four dimensions of s-t are obtained through a process of compactification, in which the starting 10-d manifold acquires a fibered structure $M_4 \times V_6$, where M_4 is the 4-d s-t (usually assumed to be Minkowski) and V_6 is the internal manifold of the "extra" compactified dimensions ($dim_{\mathbb{R}} V_6 = 6$). The compactifications where V_6 is a compact CY_3 are particularly important, because they leave $\frac{1}{4}$ of the supersymmetries of the original 10-d superstring theory unbroken.

Finally, we mention that the compact CY_n's, and their moduli spaces, have noteworthy properties of symmetry. In particular, the so-called mirror symmetries of the numbers forming the Hodge diamond of a compact CY_n are very interesting, because they have been discovered to be realized by another CY manifold, which is related to the starting one by a duality called "mirror pairing."

Coming back to our treatment, let us mention that M-theory contains a 5-brane (named M5-brane) which can wrap itself around a 4-cycle $\mathcal{P}$ of CY_3. The massless modes captured by the 4-d effective field theory picture correspond to harmonic forms on CY_3, and they are independent of S^1. In particular, the CY two-forms ω^A's give rise to vector gauge fields A^A in the following way:

$$A^A_\mu \sim \omega^{A\nu\rho} T_{\mu\nu\rho} \ , \tag{7.11}$$

where T is the rank-3 tensor gauge field of 11-d M-theory. The CY two-forms ω^A's are Poincarè dual to $6 - 2 = 4$-cycles, and the wrapping configuration of the M5-brane on CY_3 may be encoded by the wrapping numbers p^A's, which, as previously mentioned, appear in the 4-d effective field theory picture as magnetic charges coupled to gauge fields A^A's.

It is also worth recalling that a CY 3-fold CY_3 has a triple intersection number C_{ABC}, which determines the three-point couplings of the 4-d effective field theory. Therefore, the wrapping of the M5-brane around the 4-d cycle $\mathcal{P}$ on CY_3 is invariantly expressed by the cubic form

$$p \cdot p \cdot p \equiv C_{ABC} p^A p^B p^C . \tag{7.12}$$

Moreover, as it pertains to all complex manifolds admitting nontrivial four cycles, CY_3 is also characterized by a nonvanishing second Chern class κ. κ is a four-form whose integral over the four cycles ς_A's (belonging to the considered basis of $H_4(CY_3,\mathbb{Z})$) defines the set of numbers c_{2A}'s:

$$\int_{\varsigma_A} \kappa \equiv c_{2A}, \tag{7.13}$$

or equivalently

$$\int_{CY_3} \kappa \wedge \widehat{\omega}_A \equiv c_{2A}, \tag{7.14}$$

where $\widehat{\omega}_A$ are the two-forms Poincarè dual to the four-cycles ς_A of the homological basis of (7.3). Therefore, by recalling the decomposition (7.3), the so-called instanton number of the four-cycle $\mathcal{P}$ of CY_3 may be defined as the "scalar product" of the vector of the c_{2A}'s and the vector of the winding numbers of $\mathcal{P}$:

$$\int_{\mathcal{P}=p^A\varsigma_A} \kappa = p^A \int_{\varsigma_A} \kappa \equiv c_{2A}p^A; \tag{7.15}$$

equivalently, by Poincarè-dualizing (7.3) and thus by introducing the two-form $\widehat{\mathcal{P}}$ Poincarè dual to the four-cycle $\mathcal{P}$ of CY_3, we get

$$\int_{CY_3} \kappa \wedge \widehat{\mathcal{P}} = \int_{CY_3} \kappa \wedge p^A \widehat{\omega}_A = p^A \int_{CY_3} \kappa \wedge \widehat{\omega}_A \equiv p^A c_{2A}, \tag{7.16}$$

where in the last passage we used (7.14).

Notice that in this framework the cardinality of the set of indices $\{A\}$ is expressed by the second Betti number of CY_3, which expresses the number of independent harmonic forms of $H^2(CY_3)$ coinciding, by Poincarè duality, with the number of independent four cycles of $H_4(CY_3)$.

In addition, the compactification on the circle S^1 gives rise to an extra vector gauge field A^0: it corresponds to the graviphoton associated with S^1, and it is Poincarè dual to a zero-form. By the standard KK mechanism, the S^1-graviphoton A^0 will couple to electric charges q_0 associated with momentum modes on S^1. It should be pointed out that in such a microscopic picture, the total number of vector gauge fields arising in the resulting $\mathcal{N}=2$, $d=4$ SUGRA theory is equal to the second Betti number of CY_3 plus one, and consequently it depends on the topological properties of (a part of) the higher dimensional manifold of the extra, compactified dimensions.

In order to preserve $\mathcal{N}=1$ SUSY in the BH solution of the resulting $\mathcal{N}=2$, $d=4$ SUGRA theory, the four-cycle $\mathcal{P}$ must be holomorphically embedded in the manifold CY_3. Thus, the number of degrees of freedom of the 4-d BH solution (associated with the massless excitations of the wrapped M5-brane characterized by the wrapping numbers p^A's on $\mathcal{P}$) will correspond to a $(1+1)$-d superconformal field theory (SCFT). Due to compactification of the remaining extra spatial dimension on S^1, one obtains a closed string, with left- and right-moving states.

Therefore, the four supersymmetries preserved by the 4-d BH solution will necessarily reside in one of these two sectors. Without loss of generality, if we choose this to be the right one, one finally gets that the considered BH solution (often called a 4-d CY BH) of the $\mathcal{N}=2$, $d=4$ SUGRA, obtained by a $\mathcal{N}=1$ SUSY-preserving $CY_3 \times S^1$ compactification of the 11-d M-theory, is described by a $(0,4)$ sigma model. As it usually happens for SCFTs in $1+1$ dimensions, a central charge will arise, but in the considered case, it splits in a central charge for the right- and the left-moving sectors, respectively denoted by c_R and c_L.

c_R and c_L can be expressed in terms of the wrapping numbers, intersection numbers and second Chern class, and their explicit form respectively reads

$$c_R = C_{ABC}p^A p^B p^C + \frac{1}{2}c_{2A}p^A; \tag{7.17}$$

$$c_L = C_{ABC}p^A p^B p^C + c_{2A}p^A. \tag{7.18}$$

It is important to notice that such expressions hold just in the limit of large p^A's, i.e., in the semiclassical approximation of large (magnetic) charges. Generally, only in the limit of large electric and magnetic charges one can relate the topological properties of the four-cycle $\mathcal{P}$ with the topological features of the CY_3 space.

Now, in order to calculate the entropy of the resulting 4-d CY BH, we can start with a supersymmetric state in the right-moving sector, with a given momentum q_0. The corresponding left-moving states will not preserve any SUSY, and, in general, it will have a certain degeneracy depending on q_0. Therefore, we obtain a tower of states with momentum q_0, built on supersymmetric right-moving states, and including degenerating nonsupersymmetric left-moving states. In the limit of momentum q_0 large compared to the left and right central charges expressed by (7.17) and (7.18), we may use Cardy's formula to calculate the total degeneracy $\mathcal{D}(q_0)$ of such states:

$$\mathcal{D}(q_0) \overset{q_0 \gg c_L}{\approx} exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{|q_0|\, c_L}\right). \tag{7.19}$$

Thence, in the limit of large charges and large momentum q_0, we get the following result for the entropy of the considered 4-d CY BH:

$$\mathcal{S}_{micro}(q,p) = ln\left(\mathcal{D}(q_0)\right) \overset{q_0 \gg c_L}{\approx} \sqrt{\frac{2}{3}}\pi\sqrt{|\widehat{q}_0|\left(C_{ABC}p^A p^B p^C + c_{2A}p^A\right)}. \tag{7.20}$$

The term proportional to the triple CY intersection number C_{ABC} is clearly the leading one, while the term containing the (integrals of the) second Chern class is the subleading one. Due to the interaction between the electric charges q_A's associated with the vector gauge fields A^A with the 2-brane of 11-d M-theory (named M2-brane), the momentum must be shifted in the following way:

$$q_0 \to \widehat{q}_0 = q_0 + \frac{1}{2}C^{AB}q_A q_B, \tag{7.21}$$

where C^{AB} is the inverse of $C_{AB} = C_{ABC}p^C$.

8

Macroscopic Description. Higher Derivative Terms and Black Hole Entropy

In the previous section, we considered the microscopic description of $\mathcal{N}=1$ extremal BHs in $\mathcal{N}=2, d=4$ SUGRA obtained by a compactification of 11-d M-theory on $CY_3 \times S^1$. In this section, we are going to reconsider the SUGRA macroscopic description, in relation to the presence of higher derivative terms.

As mentioned at the beginning of Sect. 3, the field content of the considered $\mathcal{N}=2, d=4$ SUGRA is given by a gravity supermultiplet, containing the graviphoton field A_μ^0 which may couple to charges q_0 and p^0, and n_V Abelian vector supermultiplets, containing n_V Abelian gauge vector fields A_μ^A. We previously named such theories n_V-fold $\mathcal{N}=2, d=4$ MESGTs. Also n_H hypermultiplets could be taken into account, but, as it was previously explained, they do not enter in the attractor mechanism, because they are dynamically decoupled.

It should be pointed out that the microscopic string description is, in general, more restrictive than the macroscopic SUGRA one. Indeed, in the stringy treatment of the previous section we put $p_0=0$, and the cardinality of the set of indices $\{A\}$ was given by the number $h_{21} \equiv dim\left(H^{2,1}\left(CY_3\right)\right)$, thus depending on the topology of the internal manifold of compactification of the extra dimensions. Instead, in the SUGRA treatment considered here, we do not have to necessarily put $p_0=0$, and the cardinality of the set of indices $\{A\}$ is n_V, a priori unbounded. Nevertheless, in order to compare the results with the ones obtained in previous section, we will also put here $p_0=0$.

In general, there is an infinity of $\mathcal{N}=2, d=4$ SUGRA vector multiplet couplings, but they can all be conveniently encoded into a holomorphic function $F(Y)$, which is called the $\mathcal{N}=2, d=4$ SUGRA prepotential. It can be shown (see, e.g., [39], [94], and [95]) that such a function must be homogeneous of second degree, i.e.,[1]

[1] In the following treatment, we will assume that the considered symplectic reference frame in the moduli space is such that a prepotential F exists. For subtleties concerning some particular cases in which F may not exist, we address the reader, e.g., to [38]

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$$F(\lambda Y)=\lambda^2 F(Y),\quad \forall\lambda\in\mathbb{C}, \tag{8.1}$$

implying that

$$Y^\Lambda\frac{\partial F(Y)}{\partial Y^\Lambda}=2F(Y). \tag{8.2}$$

The complex, Kähler gauge-invariant coordinates $\left\{Y^\Lambda\right\}$ are related to the previously introduced central charge Z of the $\mathcal{N}=2,d=4$ superalgebra and to the Kähler potential K and complex coordinates $\left\{X^\Lambda\right\}$ of the Kähler–Hodge manifold M_{n_V} by the following definition ($\Lambda=0,1,\ldots,n_V$):

$$Y^\Lambda\left(z,\overline{z};p,q\right)\equiv\overline{Z}\left(z,\overline{z};p,q\right)L^\Lambda\left(z,\overline{z}\right)=e^{K(z,\overline{z})/2}\overline{Z}\left(z,\overline{z};p,q\right)X^\Lambda\left(z\right); \tag{8.3}$$

consequently, by using the homogeneity properties of the prepotential, we get that

$$\frac{\partial F(Y)}{\partial Y^\Lambda}=e^{K/2}\overline{Z}\frac{\partial F(X)}{\partial X^\Lambda}. \tag{8.4}$$

Moreover, by recalling definition (3.2.53) of the central charge function Z, we may write

$$\begin{aligned}Z&\equiv\langle n,V\rangle=n^T\epsilon V=L^\Lambda n^e_\Lambda-M_\Lambda n^\Lambda_m=e^{K/2}\left[X^\Lambda n^e_\Lambda-F_\Lambda\left(X\right)n^\Lambda_m\right]\\&=e^{K/2}\left[X^\Lambda n^e_\Lambda-\frac{\partial F(X)}{\partial X^\Lambda}n^\Lambda_m\right]=\frac{1}{\overline{Z}}\left[Y^\Lambda n^e_\Lambda-\frac{\partial F(Y)}{\partial Y^\Lambda}n^\Lambda_m\right],\end{aligned} \tag{8.5}$$

such that the manifestly symplectic-invariant expression of $|Z|^2$ reads

$$|Z|^2=Y^\Lambda n^e_\Lambda-\frac{\partial F(Y)}{\partial Y^\Lambda}n^\Lambda_m=\langle n,J\rangle=n^T\epsilon J, \tag{8.6}$$

where we introduced the $Sp\left(2n_V+2\right)$-covariant vector

$$J\equiv\begin{pmatrix}Y^\Lambda\\ \\ \frac{\partial F(Y)}{\partial Y^\Lambda}\end{pmatrix}=e^{K/2}\overline{Z}V, \tag{8.7}$$

which is (n_m,n^e)-dependent due to the presence of the central charge $\overline{Z}$, and it is Kähler gauge-invariant (i.e., has Kähler weights $(0,0)$). It is worth recalling that, from the microscopic perspective, the Kähler–Hodge manifold M_{n_V} is nothing but the space of the complex structure moduli of the CY three-fold CY_3.

Let us now reconsider the algebraic $\frac{1}{2}$-BPS-SUSY extreme BH attractor equations, given by (4.4.2.46), and corresponding to nothing but the evaluation of the identity (3.2.97) of the special Kähler geometry of M_{n_V} at the $\frac{1}{2}$-BPS-SUSY extreme BH attractors

$$\begin{pmatrix} n_m^\Lambda \\ \\ n_\Lambda^e \end{pmatrix} = -2\left[Im\begin{pmatrix} \overline{Z}L^\Lambda \\ \\ \overline{Z}M_\Lambda \end{pmatrix}\right]_{(z_{susy}(p,q),\overline{z}_{susy}(p,q))} . \tag{8.8}$$

By substituting (8.3) and (8.4) in (8.8), one gets[2]

$$\begin{aligned}\begin{pmatrix} n_m^\Lambda \\ n_\Lambda^e \end{pmatrix} &= -2\left[Im\begin{pmatrix} e^{K/2}\overline{Z}X^\Lambda \\ \\ e^{K/2}\overline{Z}\frac{\partial F(X)}{\partial X^\Lambda} \end{pmatrix}\right]_{(z_{susy}(p,q),\overline{z}_{susy}(p,q))} \\ &= -2\left[Im\begin{pmatrix} Y^\Lambda \\ \\ \frac{\partial F(Y)}{\partial Y^\Lambda} \end{pmatrix}\right]_{(z_{susy}(p,q),\overline{z}_{susy}(p,q))} .\end{aligned} \tag{8.9}$$

By renaming as before $n_m^\Lambda \equiv p^\Lambda$ and $n_\Lambda^e \equiv q_\Lambda$, and by denoting $F_\Lambda(Y) \equiv \frac{\partial F(Y)}{\partial Y^\Lambda}$ and $\overline{F}_\Lambda(\overline{Y}) = \overline{F_\Lambda(Y)}$, we finally obtain[3]

$$\begin{cases} ip^\Lambda = \left[\overline{Y}^\Lambda(z,\overline{z};p,q) - Y^\Lambda(z,\overline{z};p,q)\right]_{(z_{susy}(p,q),\overline{z}_{susy}(p,q))} ; \\ \\ iq_\Lambda = \left[\overline{F}_\Lambda(\overline{Y}(z,\overline{z};p,q)) - F_\Lambda(Y(z,\overline{z};p,q))\right]_{(z_{susy}(p,q),\overline{z}_{susy}(p,q))} . \end{cases} \tag{8.10}$$

It should be recalled that, in the previously formulated hypotheses of staticity and spherical symmetry of the extremal BH solution being considered, all quantities will have a s-t dependence exclusively on the radial coordinate r. As previously noticed, in principle the $2n_V+2$ real equations (8.10) yield the horizon values of the n_V+1 complex coordinates Y^Λ in terms of the charges p^Λ and q_Λ of the BH

$$Y^\Lambda\left(z_{susy}(p,q), \overline{z}_{susy}(p,q); p, q\right) \equiv Y^\Lambda_{Hor.}(q,p) \equiv Y^\Lambda(r_H). \tag{8.11}$$

Thus, (8.10) may be rewritten as

$$\begin{cases} ip^\Lambda = \overline{Y}^\Lambda_{Hor.}(q,p) - Y^\Lambda_{Hor.}(q,p); \\ \\ iq_\Lambda = \overline{F}_{\Lambda,Hor.}(q,p) - F_{\Lambda,Hor.}(q,p), \end{cases} \tag{8.12}$$

---

2 With respect to the notation of [93], the r.h.s.'s of (8.9) has a "–" overall. But this is due to the fact that in [93] the central charge Z is defined as the opposite of the central charge (3.2.49) defined in these lectures. Thus, everything is consistent

3 Due to the holomorphicity of the prepotential, it holds that

$$\overline{F}_\Lambda(\overline{Y}) \equiv \overline{(F_\Lambda(Y))} = \frac{\partial\overline{F}(\overline{Y})}{\partial\overline{Y}^\Lambda} \equiv \overline{F}_\Lambda(\overline{Y}).$$

where

$$F_{\Lambda,Hor.}(q,p) \equiv \frac{\partial F}{\partial Y^{\Lambda}}\left(Y\left(z,\overline{z};p,q\right)\right)\Big|_{\left(z_{susy}(p,q),\overline{z}_{susy}(p,q)\right)} = \frac{\partial F\left(Y\right)}{\partial Y^{\Lambda}}\Big|_{Y=Y_{Hor.}(q,p)}. \tag{8.13}$$

It is worth stressing once again that, depending on the features of the input data of system (8.10), i.e., on the complexity of the functional form of the prepotential $F(Y)$ and on the BH charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right) \in \Gamma$, it might not be possible to explicitly write down the solutions.

In the case at hand, the simplest form of the $\mathcal{N}=2, d=4$ SUGRA prepotential compatible with properties (8.1) and (8.2) is (remind that $A, B, C = 1, \ldots, n_V$)

$$F(Y) = -\frac{1}{6}\frac{C_{ABC}Y^{A}Y^{B}Y^{C}}{Y^{0}}, \tag{8.14}$$

where C_{ABC} is the triple CY intersection number defined above, and the prefactor $-\frac{1}{6}$ is put for later convenience. By calculating the area of the EH of the corresponding BH, we obtain

$$A\left(q,p\right) = 4\sqrt{\frac{2}{3}}\pi\sqrt{\left|\widehat{q}_{0}\right| C_{ABC}p^{A}p^{B}p^{C}}. \tag{8.15}$$

By applying the BHEA law, we therefore obtain

$$\mathcal{S}_{macro}\left(q,p\right) = \sqrt{\frac{2}{3}}\pi\sqrt{\left|\widehat{q}_{0}\right| C_{ABC}p^{A}p^{B}p^{C}}, \tag{8.16}$$

corresponding to the leading contribution to the microscopic result expressed by (7.20).

At this point, a simple question naturally comes out: *is it possible to reproduce the subleading terms of $S_{micro}\left(q,p\right)$ in the SUGRA, macroscopic approach?*

The answer is yes, but new contributions from the SUGRA field content must be considered. What one should consider now is the so-called F-term contributions from the vector supermultiplets. Indeed (see, e.g., [96–100]; for a complete review, see [101]), it was noticed that such terms do not vanish in the presence of a BH solution, and they generate higher derivative corrections to the SUGRA action, which are suppressed in the above considered semiclassical limit of large charges, giving rise to the needed subleading terms in the BH entropy.

Without entering in details, here we just say that such corrections may be encoded in a generalized, F-(term)corrected $\mathcal{N}=2, d=4$ SUGRA prepotential, depending on another complex field $\mathcal{W}^{2}$, proportional to the square of the previously introduced graviphoton field strength $T^{-}_{\mu\nu}$, but with another (complex) scaling weight, namely twice that of Y^{Λ}'s:

$$F(Y) \longrightarrow F\left(Y, \mathcal{W}^2\right) : F\left(\lambda Y, \lambda^2 \mathcal{W}^2\right) = \lambda^2 F\left(Y, \mathcal{W}^2\right), \quad \forall \lambda \in \mathbb{C}, \tag{8.17}$$

implying

$$Y^\Lambda \frac{\partial F(Y, \mathcal{W}^2)}{\partial Y^\Lambda} + \mathcal{W} \frac{\partial F(Y, \mathcal{W}^2)}{\partial \mathcal{W}} = 2F(Y, \mathcal{W}^2). \tag{8.18}$$

A nontrivial dependence on $\mathcal{W}^2$ in the holomorphic function $F\left(Y, \mathcal{W}^2\right)$ has important consequences, because it introduces in the resulting SUGRA action terms proportional to the square of the Riemann–Christoffel tensor, giving rise to the so-called R^2-SUGRA[4]. The $\frac{1}{2}$-BPS-SUSY extreme BH attractor equations (8.10) are unchanged; we just have to replace the old prepotential with the new, F-corrected one. The new thing is the fact that the complex field $\mathcal{W}^2$ will have its own independent attractor value: independently on the charges, at the horizon

$$\mathcal{W}^2 = \mathcal{W}^2_{Hor.} = -64. \tag{8.19}$$

Therefore, the F-corrected $\frac{1}{2}$-BPS-SUSY extreme BH attractor equations may be written as follows:

$$\begin{cases} ip^\Lambda = \overline{Y}^\Lambda_{Hor.}(q,p) - Y^\Lambda_{Hor.}(q,p); \\ \\ iq_\Lambda = \overline{F}_\Lambda(\overline{Y}_{Hor.}(q,p), \overline{\mathcal{W}}^2_{Hor.}) - F_\Lambda(Y_{Hor.}(q,p), \mathcal{W}^2_{Hor.}); \\ \\ \mathcal{W}^2 = \mathcal{W}^2_{Hor.} = -64, \end{cases} \tag{8.20}$$

which can be summarized as

$$\begin{cases} ip^\Lambda = \overline{Y}^\Lambda_{Hor.}(q,p) - Y^\Lambda_{Hor.}(q,p); \\ \\ iq_\Lambda = \overline{F}_\Lambda(\overline{Y}_{Hor.}(q,p), -64) - F_\Lambda(Y_{Hor.}(q,p), -64), \end{cases} \tag{8.21}$$

with

$$\overline{F}_\Lambda(\overline{Y}_{Hor.}(q,p), -64) \equiv \left. \frac{\partial \overline{F}(\overline{Y}, \overline{\mathcal{W}}^2_{Hor.})}{\partial \overline{Y}^\Lambda} \right|_{\overline{Y}=\overline{Y}_{Hor.}(q,p), \overline{\mathcal{W}}^2_{Hor.}=-64}. \tag{8.22}$$

Therefore, the F-term corrections, due to the vector supermultiplets and corresponding to higher derivative terms in the resulting $\mathcal{N}=2, d=4$ SUGRA action, break the uniformity of (complex) scaling of the area and of the entropy of the $\frac{1}{2}$-BPS BH solutions, making them not necessarily homogeneous functions of the BH charges.

---

[4] The attractor mechanism in R^2-SUGRA, also in relation to nonsupersymmetric attractors, has been recently investigated in [102]

In order to reproduce the whole microscopic result $\mathcal{S}_{micro}(q,p)$ expressed by (7.20), attempts have been performed with the following F-corrected holomorphic generalized prepotential:

$$F(Y,\mathcal{W}^2) = F(Y) - \frac{c_{2A}}{24\cdot 64}\frac{Y^A}{Y^0}\mathcal{W}^2 = -\frac{1}{6}\frac{C_{ABC}Y^AY^BY^C}{Y^0} - \frac{c_{2A}}{24\cdot 64}\frac{Y^A}{Y^0}\mathcal{W}^2, \tag{8.23}$$

where c_{2A}'s are defined by the (generally) nonvanishing second Chern class of CY_3. Notice that the above introduced $F(Y,\mathcal{W}^2)$ is holomorphic and homogeneous, as it has to be. Consequently, it is possible to calculate the area of the EH, and try to obtain the result of (7.20) by applying the BHEA law. But all efforts have turned to be unsuccessful, and it has been shown [103] that the problem to recover subleading terms in the BH entropy cannot be solved with such an approach.

At this point, in search of other terms which could possibly save the day, one could be lead to think about the contributions of the hypermultiplets. But the vevs of the hypermultiplets are not fixed at the EH of the BH, and they can vary continuously. If BH entropy depended on such vevs, it could not be an intrinsic physical property of the BH, and moreover the number of microstates for fixed charges, which microscopically originates the entropy, could not be an integer. Thus, at least at perturbative level, one would expect that the vector multiplet F-terms encode all (holomorphic) corrections to the BH entropy[5].

The only way out is to abandon the faith on the validity of the BHEA law. This should not surprise, because such a relation between the EH area and the entropy of a BH generally holds true only for gravitational actions of the Einstein–Hilbert type, i.e., linear in the intrinsic Riemann curvature R. For more general, nonlinear gravitational Lagrangian densities, such as the ones arising when F-term contributions are taken into account, the BHEA ceases to hold.

Nevertheless, Wald et al. (see, e.g., [104–107]) elaborated an alternative, generalized definition of BH entropy, based on the existence of a conserved surface charge, which can be used for any Lagrangian invariant under general coordinate transformations. In the general context of a holomorphic, F-corrected, generalized $\mathcal{N}=2, d=4$ prepotential $F(Y,\mathcal{W}^2)$, by considering all constraints imposed by the restoration of the full $\mathcal{N}=2$ SUSY at the EH of the BH, a general formula for $\mathcal{N}=2, d=4$ SUGRA theories has been obtained, reading as follows [97]:

[5] Once again, the story is not so simple. Actually, a nontrivial dependence on the hypermultiplets could arise, but only at a nonperturbative level. Also, due to the so-called holomorphic anomaly, some nonholomorphicity could be included in the functional dependence of the F-term corrected, generalized prepotential. We address the interested reader to, e.g., [98]

$$
\begin{aligned}
&\mathcal{S}_{macro}(q,p)\\
&=\pi\left|Z\right|^2_{Hor.}(q,p)-256\pi\left[Im\left(\tfrac{\partial F(Y,\mathcal{W}^2)}{\partial(\mathcal{W}^2)}\right)\right]_{Y=Y_{Hor.}(q,p),\ \mathcal{W}^2=\mathcal{W}^2_{Hor.}=-64},
\end{aligned}
\tag{8.24}
$$

where $Y^\Lambda=Y^\Lambda_{Hor.}(q,p)$ is the solution of the AEs (8.21).

$\left|Z\right|^2$ may be rewritten in a manifestly symplectic-invariant, $\mathcal{W}^2$-dependent form by noticing that the generalization of (8.6) in the considered context reads

$$
\left|Z\right|^2=Y^\Lambda q_\Lambda-\frac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}p^\Lambda=\langle n,J_{\mathcal{W}}\rangle=n^T\epsilon J_{\mathcal{W}},
\tag{8.25}
$$

where $J_{\mathcal{W}}$ is nothing but the F-term-corrected counterpart of the vector J defined by (8.7):

$$
J_{\mathcal{W}}\equiv\begin{pmatrix} Y^\Lambda\\ \\ \frac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}\end{pmatrix}.
\tag{8.26}
$$

Consequently, by evaluating (8.25) at the horizon, $\frac{1}{2}$-BPS-SUSY attractor point(s), we obtain the purely charge-dependent, manifestly symplectic-invariant expression

$$
\begin{aligned}
\left|Z\right|^2_{Hor.}&=\left(Y^\Lambda q_\Lambda-\frac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}p^\Lambda\right)\Bigg|_{r=r_H}\\
&=\left(Y^\Lambda q_\Lambda-\frac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}p^\Lambda\right)\Bigg|_{Y=Y_{Hor.}(q,p),\ \mathcal{W}^2=\mathcal{W}^2_{Hor.}=-64}\\
&=q_\Lambda Y_{Hor.}(q,p)-p^\Lambda\left.\frac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}\right|_{Y=Y_{Hor.}(q,p),\ \mathcal{W}^2=\mathcal{W}^2_{Hor.}=-64},
\end{aligned}
\tag{8.27}
$$

where clearly the F-corrected $\frac{1}{2}$-BPS-SUSY extreme BH attractor (8.21) have been used to determine the horizon, “attracted,” purely charge-dependent values $Y_{Hor.}(q,p)$ and $\mathcal{W}^2_{Hor.}$ of the moduli Y^Λ's and of the complex field $\mathcal{W}^2$.

By substituting (8.27) in (8), we finally get an explicitly $Sp\left(2n_V+2\right)$-invariant, purely charge-dependent expression for the (macroscopically determined) entropy of a static, spherically symmetric, $\frac{1}{2}$-BPS extremal BH with charge configuration $\left(p^\Lambda,q_\Lambda\right)\in\Gamma$ in n_V-fold $\mathcal{N}=2$, $d=4$ MESGT in the presence of F-term contributions from the vector multiplets

$$
\begin{aligned}
&\mathcal{S}_{macro}(q,p)\\
&=\pi\left[\left(Y^\Lambda q_\Lambda-\tfrac{\partial F(Y,\mathcal{W}^2)}{\partial Y^\Lambda}p^\Lambda\right)-256Im\left(\tfrac{\partial F(Y,\mathcal{W}^2)}{\partial(\mathcal{W}^2)}\right)\right]_{Y=Y_{Hor.}(q,p),\ \mathcal{W}^2=\mathcal{W}^2_{Hor.}=-64}.
\end{aligned}
\tag{8.28}
$$

As indicated, everything is evaluated at $r = r_H$, i.e., on the EH of the BH, and made exclusively charge-dependent by using the AEs (8.21).

Let us analyze (8.28): by recalling (8.27), we see that the first term in round brackets corresponds to the BHEA law, properly corrected by F-term contributions from the Abelian vector supermultiplets, because $|Z|^2_{Hor.}$ is the horizon area in Planck units divided by 4π. The second term, proportional to the derivative of $F(Y, \mathcal{W}^2)$ w.r.t. $\mathcal{W}^2$, represents the deviation from the BHEA law, due to the higher derivative, non- Einstein–Hilbert gravitational terms in the SUGRA action coming from the F-term contributions of the vector multiplets.

By inserting the holomorphic, F-corrected, generalized $\mathcal{N} = 2, d = 4$ prepotential $F(Y, \mathcal{W}^2)$ expressed by (8.23) in the attractor equations (8.21), and by substituting the corresponding solutions $Y^A_{Hor.}(q, p)$ in (8.28), it is then possible to show that the string, microscopic result (7.20) is fully and faithfully recovered.

In order to explicitly show the deviation from the BHEA law due to the F-term contributions, the ratio between one quarter of the EH area and the BH entropy can be calculated as

$$\frac{A(q,p)}{4S_{macro}(q,p)} = \frac{C_{ABC}p^A p^B p^C + \frac{1}{2}c_{2A}p^A}{C_{ABC}p^A p^B p^C + c_{2A}p^A} = \frac{c_R}{c_L}. \tag{8.29}$$

In the last passage, by using (7.17) and (7.18), we interestingly find that such a ratio equals the ratio of the previously introduced central charges c_R and c_L. Therefore, to some extent the asymmetry between the left- and right-moving sectors of the $(0, 4)$ sigma model describing the above-mentioned 4-d. CY BH is responsible for the failure of the BHEA law when going beyond the semiclassical limit of large charges.

9

Further Developments

We devoted these pedagogical, introductory-level lectures to the dynamics governing the (equilibrium) thermodynamics of extremal singular interpolating soliton solutions in $\mathcal{N}$-extended, d-d SUGRAs.

We analyzed the attractor mechanism in various contexts, each with specific relevant features.

They are, respectively:

1. The $\mathcal{N} = 2$, $d = 4$ dilatonic extremal Reissner– Nördstrom BH arising from heterotic string theories.
2. The $\mathcal{N} = 2$, $d = 4$, n_V-fold Maxwell–Einstein Supergravity theory (MESGT), i.e., the $\mathcal{N} = 2$, $d = 4$ SUGRA coupled to n_V Abelian vector supermultiplets; the moduli space of such a theory has a special Kähler–Hodge geometry with additional symplectic structure.
3. The $\mathcal{N} = 8$, $d = 5$ SUGRA, having the exceptional noncompact Lie group $E_{6(6)}$ as U-duality symmetry group.
4. The $\mathcal{N} = 8$, $d = 4$ SUGRA, having the exceptional noncompact Lie group $E_{7(7)}$ as U-duality symmetry group.

The maximal SUGRAs of points 3 and 4 are usually obtained by a $\mathcal{N} = 8$ SUSY-preserving toroidal compactification from 11-d M-theory or 10-d superstring theory.

Now, we would like here to mention that in the last months string theory has been enlighted by a very interesting conjecture (which we will call the *OSV conjecture*) formulated by Ooguri, Strominger, and Vafa in [108]. Such a conjecture suggests an intriguingly interesting connection between BHs and topological string theory, which may be summarized as

$$Z_{BH} = \left|Z_{top}\right|^2 , \tag{9.1}$$

where

1. Z_{BH} is the partition function related to a "mixed," microcanonical/canonical statistical treatment of the magnetic–electric charge configurations of

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a 4-d static, spherically symmetric, asymptotically flat, $\frac{1}{2}$- BPS BH in the $\mathcal{N}=2$, $d=4$, n_V-fold MESGT obtained by a CY compactification of 10-d type II superstrings.
2. Z_{top} is the second-quantized, perturbatively expanded partition function for a gas of topological strings on a CY background three- fold.

In such a context, the attractor mechanism turns out to play a key role, because in the conjecture (9.1) Z_{top} is evaluated at the horizon, attractor point(s) of CY moduli space. Such attractors are determined by the attractor equations, and (up to some subtleties concerning the background dependence) they are purely dependent on the BH charge configuration.

By adopting a perturbative approach based on the expansion parameter $\frac{1}{Q}$, where Q is the so-called graviphoton charge, OSV came to formulate the relation (9.1), in which substantially BHs with large charges get mapped to the weak coupling limit of topological strings. Equation (9.1) immediately suggests an alternative interpretation of (the horizon) Z_{top}. Indeed,

$$Z_{top,Hor.}(\phi_{Hor.}, p), \tag{9.2}$$

could be interpreted as a wavefunction, expressing the quantum amplitude to find a BH solution with charge configuration $\left(p^{\Lambda}, q_{\Lambda}\right)$, where the horizon electric potentials $\phi^{\Lambda}_{Hor.}$ and the electric charges q_{Λ} are related by canonical conjugation. As explained in [108], a BH "mixed" microcanonical/canonical ensemble may be introduced, resulting in a "BH degeneracy function" $\Omega\left(p,q\right)$, which can be seen as a *Wigner function,* related to the "topological wave-function" $Z_{top,Hor.}(\phi_{Hor.}, p)$ and defined on the $(2n_V+2)$-d electric–magnetic charge real lattice Γ (whose corresponding semiclassical, continuum limit may reasonably be chosen to be $\mathbb{R}^{2n_V+2}$).

The seminal OSV paper [108] stimulated lot of work in the last months in different research directions, leading to exciting new developments shedding new light on the interconnections between black holes, string theory, topological string theory, and dynamical systems. In what follows we try to mention a few of such advances, apologizing since the beginning for the incompleteness of our listing.

1. The connection between BPS BHs and topological strings expressed by (9.1) has been explicitly studied in a number of special cases with few charges (see, e.g., [109–112]). Moreover, it should also be mentioned that the matching of the BH entropy with the degeneracy of elementary string states has been recently studied in depth for the particular class of purely electrically charged BHs in heterotic string theory [110,112–120]; such BHs have vanishing entropy in the SUGRA approximation, where only terms linear in the curvature R are considered in the gravitational sector of the action [121–123]).
2. Then, in [124] Verlinde faced the general case with an arbitrary number of charges in the context of type IIB superstrings compactified on a CY

three-fold. He considered a subtle but crucial point: in order to derive the conjecture (9.1), OSV used a holomorphic topological string amplitude F_{top}. But actually F_{top} has, through its higher order terms in the perturbative expansion with parameter g_{top}, an antiholomorphic dependence; this phemonenon is called holomorphic anomaly of the topological string theory, and it is encoded by the so- called holomorphic anomaly equations, obtained from the world-sheet formulation of topological strings in [125].

Thus, the question is: *how to reconcile the OSV conjecture (9.1) (founded on the holomorphicity of* F_{top}*, and thus of* $Z_{top} = exp\,(F_{top})$*) with the holomorphic anomaly?*

The way out to the problem is the wavefunction interpretation of the holomorphic anomaly phenomenon. Such an interpretation was firstly proposed by Witten [126]. Almost simultaneously to the appearance of [125], he interpreted $Z_{top} = exp\,(F_{top})$ as a wavefunction obtained by quantizing the real cohomology $H^3\,(CY_3, \mathbb{R})$, i.e., the space of the three-forms on the considered CY background three-fold, in a complex polarization. From such a viewpoint, the holomorphic anomaly equations express nothing but the behavior of the wavefunction Z_{top} under an infinitesimal change of polarization. Further elaborations along this direction have been made by Dijkgraaf, Verlinde, and Vonk in [127].

By studying the interconnections between the attractor equations and the holomorphic anomaly equations in the framework of the wavefunction interpretation, Verlinde realized that the background dependence expressed by the antiholomorphic dependence of F_{top} is actually due to the choice of complex polarization in the quantization of $H^3\,(CY_3, \mathbb{R})$. Consequently, he performed a change to a real polarization suggested by the attractor equations, in which, as we have seen above, the horizon CY (complex structure and Kähler) moduli are expressed in terms of the BH real (integer, at the quantized level) charges. In such an *"attractor equations-induced" real polarization*, the background dependence of F_{top} disappears, and one ends with a generalized, background-independent generalization of the *OSV conjecture* (9.1).

3. A few months later, Ooguri, Vafa, and Verlinde obtained in [128] an interesting progress toward a realistic formulation of quantum cosmology in string theory, connected also with the *OSV conjecture*. Let us spend a few more words, taken from the introduction of [128].

   Generally, the approach to the compactifications of extra dimensions in string theory is (often implicitly) founded on the assumption that the universe should include a noncompact, 4-d macroscopic s-t. If this is absolutely natural to be conceived in the present cosmological epoch, it is unclear if the primordial universe had actually shared such a property. At its early stage, it is instead much reasonable to conjecture that the universe would have been arising from a microscopic/Planckian size compact space.

   If such a picture had to be implemented within string theory, clearly it would lead to compactify all spatial dimensions. As a consequence, one would not be allowed to freeze the string moduli out. Thus, in order to construct a physical framework where comparisons between different string compactifications could be made on probabilistic grounds, the notion of a wavefunction, defined on the moduli space of string compactifications, would naturally arise. The classical physical behavior, analytically described by the equations of motion, would then correspondingly be given by the peak(s) of the amplitude of such a wavefunction.

As correctly pointed out by Ooguri, Vafa, and Verlinde, this should be the guiding idea in the formulation of a realistic quantum cosmology in the context of string theory. In [128] they present a result on the (partially) SUSY-preserving compactifications, where usually calculations are much simpler to be carried out. They argue that the previously considered topological string partition function Z_{top} should be considered as a "wavefunction of the universe" in the mini-superspace sector of physical superstring theory, analogously to the Hartle–Hawking description [129]. Moreover, such an identification turns out to be exact, because it takes into account all orders of quantum corrections in string perturbation theory.

Consequently, [128] represents an interesting development of the hypothesized wavefunction interpretation of Z_{top}, arosen out from the original *OSV conjecture*, which gets extended to all-loop orders, and related to Hartle–Hawking approach to quantum cosmology.

4. Two months later, in [130] Dijkgraaf, Gopakumar, Ooguri, and Vafa extended such results to the process of creation of baby universes in quantum cosmology, implemented within string theory. Such an extension has been achieved by conjecturing that the holographic description of 4-d BPS-SUSY-preserving BHs would naturally include multicenter solutions, yielding a coherent ensemble of BR geometries $AdS_2 \times S^2$'s. In the particular example treated by the authors, the Euclidean wavefunction of multi-center BHs gets mapped into the "Hartle–Hawking wavefunction" of baby universes.
5. A few days before Dijkgraaf, Gopakumar, Ooguri, and Vafa put their paper [130] on ArXiv, the *OSV conjecture* was extended to the case of open topological strings in a nice paper [131] by Aganagic, Neitzke, and Vafa. Since the *OSV conjecture* formulated in [108] concerns the closed topological string theory, such a generalization seemed quite natural. Clearly, the microscopic, D-brane characterization of the electric and magnetic BH charges in the open case is different from the one exploited in the previously considered closed case. It should be pointed out that, despite the simplicity of its formal expression, schematically reading

$$Z_{BH}^{open} = \left| Z_{top}^{open} \right|^2, \tag{9.3}$$

the groundness of what may be called the *"open generalization of the OSV conjecture"* is less solid than the one of its closed counterpart. Indeed, in the open case the macroscopic SUGRA description at large charges of the BPS states has not been studied. Also, the possible existence of an "open analog" to the attractor mechanism has not been checked for this case, and thence, in general, one does not even have a SUGRA, macroscopic derivation of the (leading term of the) BH entropy. As Aganagic, Neitzke, and Vafa pointed out, further investigation and results in this direction would be precious in order to check, or at least to strengthen, the conjecture (9.3).

Another generalization of the *OSV conjecture* is the one given by Pestun in [132]. He extended the *OSV conjecture* to topological strings on generalized CY manifolds X's, arguing that the classical BH entropy is given by the so-called generalized Hitchin functional, which defines generalized complex structure on X by means of properly defined critical points.

Motivated by the *OSV conjecture*, in [133] Shih and Yin computed the semi-canonical partition function of BPS BHs in $\mathcal{N}=4$ and $\mathcal{N}=8$ string theories, to all orders in perturbation theory. Surprisingly, they found that the BH partition functions have a simple form, and they encode the full topological string amplitudes, confirming the *OSV conjecture*. Nevertheless, Shih and Yin found a slight disagreement between the BH and topological string partition functions also at the perturbative level, and thus they proposed a "minimal" modification of the starting conjecture, in which such differences are understood as a nontrivial measure factor for the topological string (a "minimal redefinition" of the *OSV conjecture* has been proposed also in [134]).

On the other hand, recent agreements with the *OSV conjecture* have been obtained, in the context of the computing of exact degeneracies of BPS BHs on toric CY_3's, by Aganagic, Jafferis, and Saulina in [135].

6. Finally, in [136] Gukov, Saraikin, and Vafa further developed the Hartle–Hawking wavefunction approach to the topological string theory partition function, determining an entropy functional on the moduli space of the (compact or noncompact) CY three-fold of the "internal," "extra" dimensions in compactifications of type IIB 10-d superstrings. Rather interestingly, they found that the maximization of such a functional is related to the appearance of asymptotic freedom in the corresponding effective field theory. Recent advances and observations along this direction have been made in [137] and [72].
7. In [138] Sen, who in the last years devoted various papers to the issue of BH entropy (see, e.g., [112, 116, 119], and [120][1]), developed an "entropy function formalism" which generalizes some results related to the *OSV conjecture* to not necessarily supersymmetric contexts.

   He started by observing that generally all results listed in points 1–6 heavily rely on SUSY. In particular, only a particular class of higher derivative terms, computable by the (perturbative expansion of the) topological string partition function [125,140], and [141] have been considered in the SUGRA

[1] The results of this last paper have been recently generalized to all dimensions in [139].

action (representing the low-energy, effective limit of the related – fundamental – string theory action). The effect of the introduction of such terms may be encoded in the generalized prepotentials (in the case of heterotic BHs, they may also be nonholomorphic: see, e.g., [96–99]).

But, even if the terms encoded by generalized prepotentials are an important class of higher derivative corrections to the effective string theory action, they are by no means the only ones, and it is actually not clear why one should consider only such terms. Indeed, as shown by Sen himself in [116], in type II superstrings some cases exist in which the generalized prepotential is not enough to reproduce the matching between BH entropy and the degeneracy counting in the microscopic description by elementary string states.

In [138] the complete set of higher derivative terms was considered in relation to the near-horizon geometry $AdS_2 \times S^{d-2}$ of a d- d extremal BH, by applying the general formalism elaborated by Wald et al. in [104]– [107]. Without relying on the explicit structure of the higher derivative terms, Sen was able to derive two fundamental results:

1. The Legendre transform of $\frac{S_{BH}(\underline{q},\underline{p})}{2\pi}$ w.r.t. the electric BH charge variables q's is nothing but the integral of the whole Lagrangian density over the $(d-2)$-d sphere S^{d-2} enclosing the considered BH. As in the OSV treatment, the (canonically) conjugated variables to the q's are the horizon, radial electric potentials $\phi_{Hor.}$'s.
2. In the working hypotheses formulated by Sen, the family of the d-d generalizations $AdS_2 \times S^{d-2}$ of the BR background geometries is parameterized by the BH magnetic charges p's, the horizon values of the BH electric potentials $\phi_{Hor.}$'s, the horizon values $u_{Hor.}$'s of the various considered real scalars, and the sizes v_1 and v_2 of AdS_2 and S^{d-2}, respectively.

   In such a framework, an "entropy function" $\mathcal{S}\left(\underline{u}, v_1, v_2, ; \underline{q}, \underline{p}\right)$ is constructed by performing the following steps:
   a. integrating the whole Lagrangian density (related to the $AdS_2 \times S^{d-2}$ family) over S^{d-2}, then
   b. Legendre-transforming the result w.r.t. the horizon, radial, electric potentials $\phi_{Hor.}$'s, and finally
   c. multiplying by 2π.

Thus, the attractor mechanism was shown to work, also in determining the sizes of the d-d BR geometries, because Sen obtained that, for a given BH charge configuration $\left(\underline{p}, \underline{q}\right)$, the horizon values $u_{Hor.}$'s of the scalar fields and the sizes v_1 and v_2 are purely charge-dependent quantities, determined by the extremization of $\mathcal{S}\left(\underline{u}, v_1, v_2, ; \underline{q}, \underline{p}\right)$ w.r.t. $\underline{u}$, v_1, and v_2, respectively

$$\begin{cases} \underline{u}_{Hor.}\left(\underline{q},\underline{p}\right): \left.\frac{\partial \mathcal{S}\left(\underline{u},v_1,v_2,;\underline{q},\underline{p}\right)}{\partial \underline{u}}\right|_{\underline{u}_{Hor.}=\underline{u}_{Hor.}\left(\underline{q},\underline{p}\right)} = \underline{0}; \\ \\ v_1\left(\underline{q},\underline{p}\right): \left.\frac{\partial \mathcal{S}\left(\underline{u},v_1,v_2,;\underline{q},\underline{p}\right)}{\partial v_1}\right|_{v_1=v_1\left(\underline{q},\underline{p}\right)} = 0; \\ \\ v_2\left(\underline{q},\underline{p}\right): \left.\frac{\partial \mathcal{S}\left(\underline{u},v_1,v_2,;\underline{q},\underline{p}\right)}{\partial v_2}\right|_{v_2=v_3\left(\underline{q},\underline{p}\right)} = 0. \end{cases} \tag{9.4}$$

Finally, the BH entropy is nothing but the purely charge-dependent horizon value of $\mathcal{S}\left(\underline{u},v_1,v_2,;\underline{q},\underline{p}\right)$:

$$S_{BH}\left(\underline{q},\underline{p}\right) = \mathcal{S}\left(\underline{u}_{Hor.}\left(\underline{q},\underline{p}\right),v_1\left(\underline{q},\underline{p}\right),v_2\left(\underline{q},\underline{p}\right);\underline{q},\underline{p}\right) \equiv \mathcal{S}_{Hor.}\left(\underline{q},\underline{p}\right). \tag{9.5}$$

Sen's elaboration of the Wald et al.'s higher order derivative Riemannian formalism is based on a set of working assumptions, which are, respectively,
– asymptotical flatness,
– spherical symmetry,
– Abelian gauge fields.
Moreover, Sen's results rely on the assumption that the Lagrangian density can be expressed only in terms of gauge-invariant field strengths, and does not involve explicitly the related gauge fields. Such a condition is clearly violated in presence of Chern–Simons terms. If such terms cannot be removed by switching to dual field variables, the results obtained in [138] still hold true if the additional Chern–Simons terms do not affect, for some reasons, the equations of motion and the entropy of the particular BH solution under consideration.
A progressive, possibly combined generalizing relaxation of such working hypotheses might lead to further generalizations, which hopefully might allow one to gain new interesting insights into the attractor mechanism dynamics in supersymmetric and nonsupersymmetric frameworks.
The results of [138] have been applied in [142] in order to analyze the effects of Gauss–Bonnet terms in the entropy of 4-d heterotic BHs, obtaining interesting results with possible implications on the supersymmetrization procedures of R^2-terms in the SUGRA action (indeed, an approach based on the Gauss–Bonnet combination might be more convenient than the usual one based on the Weyl tensor square term).

8. Recent advances in the study of the nonsupersymmetric attractor mechanism have been obtained by Goldstein, Iizuka, Jena, and Trivedi in [66]. In an approach conceptually similar to Sen's one, they directly studied the equations of motion in theories with gravity, gauge fields, and scalars in static, spherically symmetric BH backgrounds. In such a framework, they were able to determine two sufficient conditions for the attractors mechanism to work, independently on the supersymmetric nature of the system

being considered. Such conditions define the attractor points to be the stable critical points of a suitably defined "effective potential" V_{eff}, which is generally a real positive function only of the BH charges and of the scalar fields[2] (see also Sect. 4).

The simplest case treated in [66], characterized by spherical symmetry, asymptotical flatness, and only magnetic charges in $d = 4$, corresponds to

$$\mathcal{L}_4 \sim R - 2\left(\partial_r \phi^i\right)^2 - f_{ab}(\phi) F^a_{\mu\nu} F^{b\mu\nu}, \tag{9.6}$$

where r is the radial coordinate, R is the s-t Riemann scalar curvature, ϕ^i $(i = 1, \ldots, n_\phi)$ are the (real) scalar fields, and $F^a_{\mu\nu}$ $(a = 1, \ldots, n_A)$ are the field strengths related to the considered n_A Abelian gauge fields. $f_{ab}(\phi)$ denotes the (real, symmetric) $n_A \times n_A$ matrix, determining the dilaton-like couplings between the scalars and the $U(1)$ field strengths. Each Abelian field strength is related to a (integer, quantized) magnetic charge p^a. The definition of V_{eff} corresponding to the action (9.6) reads

$$V_{eff}(\phi; p) \equiv f_{ab}(\phi) p^a p^b, \tag{9.7}$$

and the sufficient conditions for the existence of an attractor mechanism in the space of the configurations of the (purely r-dependent) scalar fields may be written as

$$\begin{cases} I. \quad \left.\frac{\partial V_{eff}(\phi;p)}{\partial \phi^i}\right|_{\phi^j = \phi^j_{Hor.}(p)} = 0 \text{ (criticality)}; \\ \\ II. \quad \left.\frac{\partial^2 V_{eff}(\phi;p)}{\partial \phi^j \partial \phi^i}\right|_{\phi^k = \phi^k_{Hor.}(p)} > 0 \text{ (stability)}, \end{cases} \tag{9.8}$$

where $\phi^i_{Hor.}(p)$ denote the horizon, "attracted," purely (magnetic) charge-dependent values of the scalar fields. Condition I means that the attractor point(s) are critical points for $V_{eff}(\phi; p)$, whereas condition II is a shorthand notation meaning that the Hessian of $V_{eff}(\phi; p)$ at the attractor point(s) must be strictly positive definite.

The generalization to the 4-d case with both electric and magnetic charges is characterized by the presence of "axion"-like couplings in the Lagrangian density, which correspondingly becomes[3]

$$\mathcal{L}_4 \sim R - 2\left(\partial_r \phi^i\right)^2 - f_{ab}(\phi) F^a_{\mu\nu} F^{b\mu\nu} - \frac{1}{2} \widetilde{f}_{ab}(\phi) F^a_{\mu\nu} F^b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}, \tag{9.9}$$

[2] The relevance of V_{eff} in relation to the inner, "attractive" dynamics of BH solutions in the absence of SUSY was firstly remarked in [143]

[3] A comparison of (9.9) with (4.1.1) yields that the analysis performed in the first part of [66] is very similar to the one of [55] (see Subsects. 4.1 and 4.2)

where $\widetilde{f}_{ab}(\phi)$ is the (real, symmetric) $n_A \times n_A$ matrix of the axion-like couplings between the scalars and the $U(1)$ field strengths; in general, it is independent of its dilatonic counterpart $f_{ab}(\phi)$:

$$\frac{1}{2} F^b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho} \equiv \ {}^*F^{b\mu\nu}, \tag{9.10}$$

denotes the usual Hodge-dual gauge field strengths, with $\epsilon^{\mu\nu\lambda\rho}$ being the totally antisymmetric Ricci–Levi–Civita tensor in $d = 4$. In this case the considered n_A Abelian field strengths are related to magnetic charges p^a's and electric charges q_a's. Thence, the V_{eff} related to the Lagrangian density (9.9) may be defined as

$$V_{eff}(\phi; q, p) \equiv f^{ab}(\phi) \left(q_a - \widetilde{f}_{ac}(\phi) p^c\right) \left(q_b - \widetilde{f}_{bd}(\phi) p^d\right) + f_{ab}(\phi) p^a p^b, \tag{9.11}$$

with $f^{ab}(\phi)$ being the inverse[4] of matrix $f_{ab}(\phi)$. Thus, the sufficient conditions for the existence of an attractor mechanism in the space of the configurations of the (purely r-dependent) scalar fields may now be generalized to the following system:

$$\begin{cases} I. \quad \left.\frac{\partial V_{eff}(\phi;q,p)}{\partial \phi^i}\right|_{\phi^j = \phi^j_{Hor.}(q,p)} = 0 \text{ (criticality);} \\ \\ II. \quad \left.\frac{\partial^2 V_{eff}(\phi;q,p)}{\partial \phi^j \partial \phi^i}\right|_{\phi^k = \phi^k_{Hor.}(q,p)} > 0 \text{ (stability),} \end{cases} \tag{9.12}$$

where $\phi^i_{Hor.}(q,p)$ now stand for the horizon, attracted, purely (electric and magnetic) charge-dependent values of the scalar fields.

In all cases the BH entropy is proportional to the horizon, attracted, purely charge-dependent value of the "effective" potential

$$S_{BH}(q,p) \sim V_{eff}(\phi_{Hor.}(q,p); q, p), \tag{9.13}$$

and it is therefore purely charge dependent itself, too.

The results on the attractor dynamics obtained in [66] hold for BH solutions in nonsupersymmetric theories, or also for nonsupersymmetric or partially SUSY-preserving solutions in $\mathcal{N} \geqslant 1$-supersymmetric contexts. Clearly, as explicitly checked in [66], 4-d $\frac{1}{2}$-BPS extremal BHs in $\mathcal{N} = 2$, $d = 4$ SUGRA do satisfy the sufficient conditions (9.12).

Goldstein, Iizuka, Jena, and Trivedi were also able to extend the above $d = 4$ results to generic (higher) d's, and to asymptotically nonflat metric

[4] As correctly pointed out by Goldstein, Iizuka, Jena, and Trivedi, $f_{ab}(\phi)$ is symmetric, and therefore it is always diagonalizable. Its possibly existing zero eigenvalues would correspond to gauge fields with vanishing kinetic terms, which may therefore be omitted from the action (9.9). See also Footnote 1 of Sect. 4

backgrounds, e.g., to asymptotically AdS and dS BHs in d dimensions. While in AdS cases the Breitenlohner–Freedman bound [144] naturally enters the game, in dS cases some additional assumptions were needed in order to take into account the infrared divergences in the far past (or future) of dS spaces[5].

Different, complementary approaches were used in the analysis of [66]: both perturbation theory and numerical simulations were considered. By starting from explicit expressions of V_{eff}, the numerical approach allowed one to go beyond the perturbative regime; in the treated cases, only one basin of attraction was found in the dynamics of radial evolution of the configurations of the n_ϕ scalar fields, even though, at least in supersymmetric frameworks, multiple basins may exist. In some special cases, the equations of motion could be mapped onto an integrable dynamical system of Toda type [147].

It should be clearly stressed that, at least in the asymptotically flat and AdS cases, the whole analysis of [66] is based on the fact that the scalars do not have a potential in the original action, and in particular they are massless. The explicit introduction of a potential for the scalars in the Lagrangian density would destroy their moduli nature. Clearly, $\mathcal{N} \geqslant 1$-supersymmetric theories can be characterized by such an absence of potential, by requiring suitably arranged couplings between scalar and gauge fields. But when SUSY is not there, there is actually no way to naturally avoid a potential for the scalars; in this sense, when considering nonsupersymmetric theories, the results of [66] should more properly be conceived as a "mathematical investigation."

Recently, in [78] Tripathy and Trivedi extended the results of [66] to nonsupersymmetric BH solutions emerging from type II superstrings compactified on a CY three-fold, still obtaining that nonsupersymmetric attractors are related to the minimization of a suitably defined effective potential. In [71] a c-function, to some extent mimicking the central charge function in nonsupersymmetric settings, was found and generalized also to $d > 4$.

Also other extensions of the results of [66] could interestingly be considered, for example the ones related to the relaxation of the working hypotheses of staticity (e.g., extension to rotating BHs) and pointlike nature of the singularity (i.e., generalization to black rings). Recently, the attractor mechanism for spherically symmetric extremal BHs in a theory of general R^2 gravity in $d = 4$, coupled to gauge fields and moduli fields, have been investigated, also in the case of nonsupersymmetric attractors, in [102].

Beside the recent developments concerning the *OSV conjecture*, the advances mentioned above are related to the search for some kind of attractor mechanism when (some of) the hypotheses on which all the previous treatment was founded are removed. Similarly to Sen's previously listed working assumptions, such Ansätze are:

- *asymptotical flatness*,
- *spherical symmetry*,

[5] For (quite) recent reviews on the (classical) features of de Sitter spaces in any dimensions, see, e.g., [145] and [146]

- *staticity*,
- *extremality*,
- *supersymmetric (partially BPS) nature*

of the BH metric solution being considered.

Also, it would be interesting to see what happens for $d \neq 4$ (considerably for $d > 4$)[6] and/or considering spatially extended singular solutions, such as $p\,(> 1)$-black branes.

Let us also mention that very interesting results have been recently obtained about the variational approaches to BH entropy, relating the BH entropy to the exact counting of M/string theory microstates, mostly in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric frameworks (see also [155]). They are mainly achieved by the research group of de Wit et al.; after anticipations by Mohaupt in [156] and some presentations at various conferences (see, e.g., [157] and [158]), the most recent results are given in [159]. For further discussion, the reader is addressed also to [93], [160], and [161].

Finally, also the question of non-Abelian charges (and singular metric backgrounds carrying non-Abelian charges), and the related issue of the non-Abelian generalization of electric/magnetic duality (see, e.g., [163–168]) (and its insertion in the previous treatment, possibly suitably generalized) might be addressed.

Below we give a list (far from being exhaustive) of just some of the possible directions that appear to be a natural extension of the results briefly reported

[6] For what concerns $d = 5$, we mention the results of [148], where Gaiotto, Strominger, and Yin proposed a simple, linear relation between the BPS partition function $Z_{BH}^{d=4}$ of a 4-d BH obtained by CY compactification of 10-d type IIA superstrings (usually named type IIA CY BH) and the BPS partition function $Z_{BH}^{d=5}$ of a 5-d spinning BH obtained by CY compactification of 11-d M-theory. Consequently, by using the *OSV conjecture* (9.1), they were able to directly relate $Z_{BH}^{d=5}$ with the $\mathcal{N} = 2$ topological string partition function Z_{top}. Due to the appearance of $|Z_{top}|^2$ in formula (9.1), the resulting relation between $Z_{BH}^{d=5}$ and Z_{top} is nonlinear, differently from the previously proposed, linear relation between the same quantities at different points of CY moduli space [149], [150]. Thus, combining the results of [148] with the older ones of [149] and [150] might lead to new nontrivial relations between the different values of Z_{top}, obtained by evaluation at different points in the CY moduli space.

The relation proposed in [148] has then been used in [151] and [152], where Shih, Strominger, and Yin, by starting from exactly known 5-d degeneracies, derived weighted BPS dyonic BH degeneracies for 4-d $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theory, respectively. In particular, the result obtained for the $\mathcal{N} = 4$ case perfectly matches the conjecture formulated by Dijkgraaf, Verlinde, and Verlinde in [153].

It should also be mentioned that recently in [154] Guica, Huang, Li and Strominger studied the R^2-corrections to the BH and black ring solutions of 5-d SUGRA. As pointed out by them, the nature of such terms is less clear than that of their 4-d counterparts, mostly due to the relatively limited understanding of F-term contributions in 5-d SUGRA.

above, mostly being related to the removal of some hypotheses made in our treatment.

First of all, it should be pointed out the following remark on the

I) *large charge (semiclassical) limit approach (and its removal) in* $N = 8$, $d = 4$ *and* 5 *SUGRAs.*

In the above treatment we often mentioned the quantization of the conserved BH electric and magnetic charges, related to the topological nontriviality of the backgrounds arising from string theories, i.e., to the intrinsic quantum nature of the basic, fundamental theories of which the considered asymptotically flat SUGRAs constitute a low-energy effective theory.

But we did not mention that the corresponding "discretization" of the U-duality symmetry group, which turns out to be defined on the numeric field of integers $\mathbb{Z}$, leads also to some corrections to the entropy formulae for the BHs, in particular to their U-invariant expressions, which seem to exist only in $d = 4$ and 5, respectively given by (6.2.16) and (6.1.15).

Indeed, it turns out that the cubic and quartic invariant appearing on the r.h.s.'s of such formulae (namely, $I_4(56)$ for the $d = 4$ U-group $E_{7(7)}$ and $I_3(27)$ for its 5-d counterpart $E_{6(6)}$) acquire some quantum corrections, containing the related modular forms. Roughly speaking, such modular forms are some kind of invariants of both the "continuous" and "discrete" versions of the U-groups, but they vanish in the "continuous," classical limit of large values of the charges.

Thus, (6.2.16) and (6.1.15) should not be considered as exact ones, because the reported approach allows one to take into account only the leading term for large charges. Equations (6.2.16) and (6.1.15) should better be seen as semiclassical limits of more general quantum formulae, approximated for large values of the (quantized) conserved charges of the system.

II) *Removal of the hypothesis of asymptotical flatness of metric backgrounds.*

Asymptotically nonflat (maximal) SUGRAs, in general corresponding to (maximal) gauged SUGRAs, do deserve a completely different treatment w.r.t. their asymptotically flat, ungauged counterparts.

For instance, the asymptotically AdS, $\mathcal{N} = 8$, $d = 4$ SUGRA does not have any moduli space, because the moduli are all fixed. Indeed, the $SO(8)$-invariance of the asymptotically AdS, maximally supersymmetric vacuum state "freezes" out all possible moduli of the theory. This happens because the relevant set of scalar fields generally transforms under a nontrivial representation of $SO(8)$, but, at the same time, it must also be $SO(8)$-invariant in the vacuum: the only possibility of simultaneous fulfilling of such requests is to make the scalars constant.

Consequently, at least in $d = 4$, in order to allow for the existence of some moduli space and therefore for some internal nontrivial evolution dynamics of the relevant set of scalars, the interesting idea arises to consider nonmaximal, $\mathcal{N}$-extended gauged SUGRAs, in which therefore the nonmaximality of the localized SUSY would not completely "freeze" the dynamics of the moduli.

In previous sections we treated the ungauged $\mathcal{N} = 2$, $d = 4$ MESGT, i.e., the asymptotically flat, nonmaximal $\mathcal{N} = 2$, $d = 4$ SUGRA coupled to n_V Abelian vector supermultiplets, and possibly to n_H hypermultiplets, too. In such a case we have seen that the scalar fields coming from the hypermultiplets are not fixed at the EH, because the central charge of the local SUSY algebra does not depend on the asymptotical configuration of these fields. Thus, such scalars are completely decoupled from the dynamical behavior of the system, and they do remain moduli of the theory also in the asymptotical near-horizon radial evolution of the considered extremal (spherically symmetric) BH.

Instead, in the nonmaximal gauged SUGRAs *all* the scalars coming from the field contents of the extra matter multiplets coupled to the SUGRA one should be taken into account, including the ones related to the hypermultiplets, which now cannot be decoupled from the dynamics of the system.

Asymptotically AdS backgrounds have been quite extensively considered in the literature, also in their relation with string theories. In a recent work [66] some advances were made in the study of the attractor mechanism in such backgrounds, also in the de Sitter (dS) case. The obtained results are quite general, because they do not rely on SUSY, but nevertheless some other aspects of the attractor mechanism in asymptotically (A)dS background still wait for a detailed examination.

In particular, the moduli space dynamics related to the radial evolution of the scalars of the hypermultiplets coupled to asymptotically nonflat, nonmaximal (spherically symmetric) $\mathcal{N}$-extended, d-d SUGRAs (e.g., to the spherically symmetric, asymptotically AdS, gauged $\mathcal{N} = 2$, $d = 4$ MESGT) has not yet been considered, but its study seemingly appears an interesting direction of development to be pursued.

III) *Removal of the hypotheses of spherically symmetry and /or staticity.*

All the extremal BH solutions considered in our treatment, and in most of the literature, have spherical symmetry. That is why we always considered only the evolution flow in the moduli space which was related to radial dynamics of the relevant set of scalar fields.

The study of nonspherically symmetric singular metric solutions in the context of SUGRAs should naturally lead to the "merging" of the radial and angular dynamics, and consequently to a deeper understanding of the attractor mechanism, possibly involved in both of them.

Also the removal of the hypothesis of staticity (i.e., time- independence) of the considered solutions should shed some new light on interesting aspects. Some spinning (for example, Kerr–Newman-like) BH metrics could be considered, and their possible interpolating soliton nature could be investigated, together with the possibility to obtain higher dimensional spinning extensions of such backgrounds.

IV) *Attractor mechanism in higher dimensions and black rings.*

Reasonably, the removal of the basic hypotheses about the structure of the BH metrics should possibly determine a modification of the attractor mechanism itself, as recent works seem to point out.

Indeed, a deeper, recently gained understanding of the BPS equations in SUGRA [169–171] has led to new examples of general solutions, such as 5-d black rings [172–182]. Also multi-centered BHs in four dimensions have been considered [143, 183–185], and they share some of the features of their ringy 5-d counterparts[7].

For these new classes of singular metrics the entropy turns out to be function not only of the conserved (quantized) charges related to a certain number of (Abelian) gauge symmetries exhibited by the low-energy effective SUGRA theory, but it also depends on the values of the *dipole charges*. These are nonconserved quantities, which may be defined by flux integrals on particular surfaces linked with the ring.

Thus, it could be reasonably conjectured that the near-horizon, attracted configurations of the moduli should in this case *also* depend on the dipole charges.

Rather intriguingly, they actually turn out to *exclusively* depend on the dipole charges [189–191].

Consequently, for black rings the attractor mechanism cannot be related to some kind of extremum principle involving the central charge, because such a quantity depends on the conserved charges, and not on the dipole charges. In [189] Larsen and Kraus formulated a new extremum principle for the attractor mechanism in 5-d black ring solutions, in which a certain function of the dipole charges plays a key role.

A general analysis revealed the existence of two general classes of solutions, whose internal, near-horizon dynamics is governed by the universal attractor mechanism, differently realized in terms of extremum principles for different functions of different charges.

The discriminating, key point is the vanishing or not of certain components of the field strengths (the so-called *dipole field strengths*). The framework corresponding to nonvanishing *dipole field strengths* represents a new arena to generalize the possible realizations of the attractor mechanism.

Finally, for very recent advances in the study of extreme BHs and attractors (also in relation to quantum information), we address the reader to [29, 61, 205, 209, 211, 213, 222–233].

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[7] An interesting line of research on black ring solutions in 5-d SUGRA has recently been pursued by Strominger et al. .

By using M-theory, in [181] Cyrier, Guica, Mateos, and Strominger exploited the microscopic interpretation of the entropy of a recently discovered new black ring solution in 5-d SUGRA.

Moreover, as previously mentioned, in [148] Gaiotto, Strominger, and Yin proposed a simple relation between $Z_{BH}^{d=4}$ and $Z_{BH}^{d=5}$ based on the demonstration that the M-theory lift of a 4-d CY type IIA BH is a 5-d BH spinning at the center of a Taub-NUT-flux geometry. Such a result on M-theory liftings was then further generalized to the case of 4-d multi-BH geometries, which in [186] were shown to correspond to 5-d black rings in a Taub-NUT-flux geometry (see also [187] and [188] for related further developments)

We end these introductory lectures devoted to the attraction mechanism in BHs by saying that it appears to be particularly relevant also in the framework of *flux compactifications* in string theory (see [192] for a nice recent review). In such a context, the attractor mechanism turns out to be encoded in phenomena of moduli stabilization occurring for compactifications in the presence of internal form fluxes and geometrical fluxes *à la Scherk-Schwarz.* In these cases the fluxes originate masses and distort the original geometry, thus restricting the moduli space of solutions.

Despite the considerable number of papers written on the attractor mechanism in the past years, lots of research directions have still to be pursued, paving the way to further developments in the deep comprehension of the inner dynamics of (possibly extended) s-t singularities in SUGRA theories, and hopefully in their fundamental high-energy counterparts, such as superstrings and M-theory.

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