A simple and fast look-up table method to compute the exp(x) and ln(x) functions

Claude Baumann, Director of the Boarding School „Convict Episcopal de Luxembourg[i]“, 5 avenue Marie-Thérèse, m.b. 913, L-2019 Luxembourg[ii] claude.baumann@education.lu

July 29th 2004

Abstract

Normally today's computers shouldn't present any problems to rapidly calculate trigonometric or transcendental functions by any means. Developing better algorithms appears to be more sporty than really practical. However the growing number of embedded applications based on micro-controllers may ask for gain in speed and memory economy, since those applications are fundamentally characterized by limited resources and optimized designs in order to reduce costs. Thus, fast and memory economic algorithms keep playing an important role in this domain.

The present algorithm was developed during the elaboration of *Ultimate ROBOLAB*, an extension to the *ROBOLAB* software[iii] , known as a powerful graphical programming environment for the LEGO RCX. By difference to standard *ROBOLAB* that works with interpreted code, *Ultimate ROBOLAB*[iv] directly generates Assembly code from the graphical code, compiles it to Hitachi H8 byte codes which are then downloaded to the RCX.

The algorithm has three important sections: the first one -universally known- consists in reducing the function f(x)=exp(x) to $f_0(x)=2^x$ - respectively g(x)=ln(x) to $g_0(x)=\log_2(x)$ - the second section concerns the arrangement of the numbers in the particular way to have the CPU only compute $f_0(x)$ or $g_0(x)$ for $x \in [1,2[$ and the third manages a fast computing of $f_0(x)$ and $g_0(x)$ to any desired precision according to a look-up table composed of 22 numbers only for IEEE 754 standard single precision floating point numbers. During the execution of the elementary functions, in any case the CPU has to operate less than 22 products. The average case turns around 11 multiplications.

Introduction

The development of *Ultimate ROBOLAB*, a LabVIEW-based graphical compiler for Hitachi H8/300 code, is a particularly interesting demonstration of the typical constraints to which embedded design with micro-controllers may be submitted. *Ultimate ROBOLAB* is destined as a development tool for advanced users of the well-known LEGO RCX. This module had been invented as the central part of the LEGO Robotics Invention System (RIS), a highly sophisticated toy that, due to its excellent features, has found many applications as an educational tool in schools, high schools and even universities.

The heart of the RCX is a 16MHz clocked Hitachi H8/3292 micro-controller. Besides peripheral devices and 16kB on-chip ROM and 512 bytes RAM, this micro-controller encapsulates an H8/300 CPU that has eight 16-bit registers r0..r7 or sixteen 8-bit registers r0H, r0L, … r7H, r7L (r7 is used as the stack-pointer); 16-bit program counter; 8-bit condition code register; register-register arithmetic and logic operations, of which 8 or 16-bit add/subtract (125ns at 16MHz), $8 \cdot 8$-bit multiplying (875ns), $16 \div 8$-bit division (875ns); concise instruction set (lengths 2 or 4 bytes); 9 different addressing modes. Together with 32k external RAM, LCD-display, buttons, infrared communication module, analog sensor ports and H-bridge output ports, the RCX is an ideal instrument for the exploration of micro-controllers in educational contexts.

In order to attribute most flexibility to the RCX for all kind of robot projects, the programming environment *Ultimate ROBOLAB* required a firmware kernel that should guarantee memory economy and execution speed. Thus each part of the kernel had to be optimized to the limits, so that users could dispose of a maximum of memory for their own code and sufficient reaction speed for reliable robot behaviors and closed loop controls. This was particularly challenging concerning the inclusion of an advanced mathematical kernel that comprised IEEE 754 standard single precision floating-point operations, among which the square root, trigonometric and exponential functions. Scrutinizing the subject with the target to find algorithms that could best balance the requirements rapidly ended in the choice of Heron's algorithm for the square root and CORDIC[v] for the trigonometric functions. However the exponential functions revealed themselves as more difficult.

1. Different methods of calculating y=exp(x) and y=ln(x)

In order to dispose of valuable algorithms for the exponential function and its reciprocal, several existing methods have been evaluated. An alternative CORDIC[vi] algorithm using a hyperbolic *atanh* table instead of the *atan* value list was the first possibility that we tested applying the equations:

$$\ln(x) = 2 \cdot a\tanh\left(\frac{x-1}{x+1}\right)$$

$$\exp(x) = \cosh(x) + \sinh(x)$$

But the implementation of this solution led to an excessively long and complex code, because of a necessary additional algorithm to assure the convergence. Another alternative was an application of Taylor's series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + -... \quad -1 < x \leq 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...$$

While being easily programmable, these series present the disadvantage of converging only very slowly, leading either to unacceptably long computing time or unsatisfying inaccuracy.

Other high-speed alternatives known from FPGA or DSP implementations, and based on large constant tables had to be excluded as well. A remarkable representative of efficient table based algorithms was presented by DEFOUR et al[vii], who succeeded in obtaining high precision function approximation with a multipartite method by performing 2 multiplications and adding 6 terms. The trade-off, and the reason for the disqualification in our application, is that the table size grows exponentially with the precision. For example, in order to get 23 correct bits, corresponding to an error of ½ **ulp** (***units in the last place***) in the case of IEEE 754 standard single precision floating-point data, the exp(x) function needs a table size of 82432 constants. Obviously such a table would explode the RCX memory capacity.

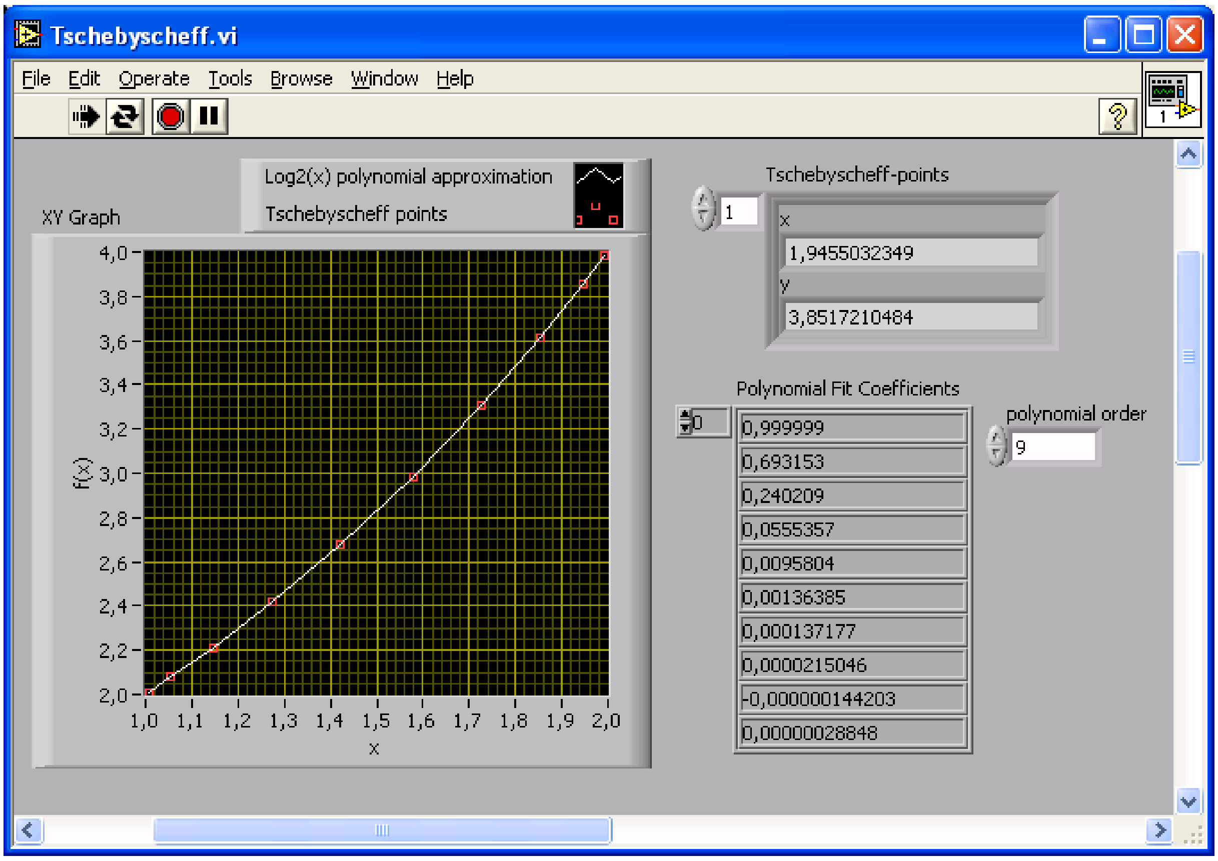
A better approximation choice was the use of a polynomial series based on error minimizing Chebycheff[viii] nodes[ix]. For the purpose of $y=2^x$, $x\in[1,2[$, the applied function to determine 10 Chebycheff-points is:

$$x_i = \frac{1}{2}\cdot\left(3+\cos\left(\frac{i-0.5}{10}\cdot\pi\right)\right), \quad i=1,..,10$$

These nodes serve as reference-points for a 9th order polynomial curve fitting. While applying the Horner Scheme[x] to the polynomial, the number of operations is a total of 9 multiplications and 9 additions per function call, compared to 7·2 + 1=15 multiplications and 9 additions for the straightforward polynomial calculation.

$$\begin{aligned} P(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \\ &= a_0 + x(a_1 + x(a_2 + x(a3 + \ldots + x(a_{n-1} + a_nx)))), \quad \textit{Horner Scheme} \end{aligned}$$

The accuracy that can be obtained with the algorithm is about 3E-7, as can be seen in the result of a LabVIEW simulation (picture 2).



Tscheybscheff-points

1

x
1,9455032349
y
3,8517210484

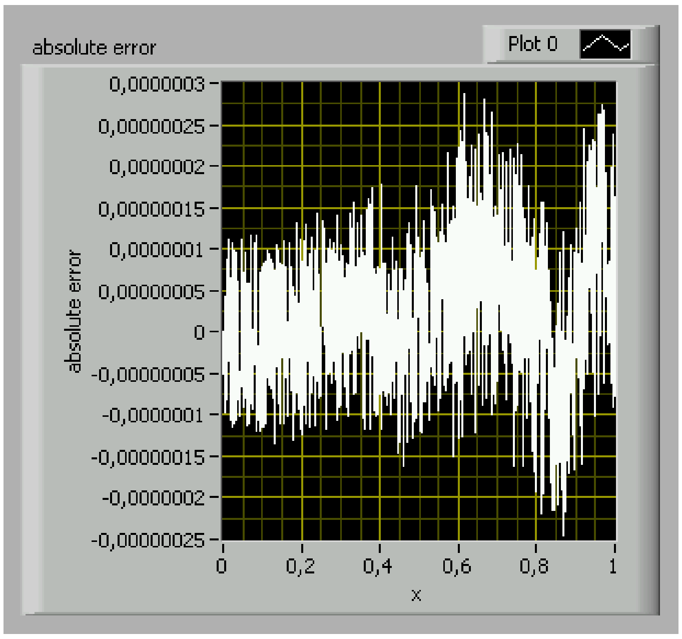
polynomial order
9

Polynomial Fit Coefficients

0

0,999999
0,693153
0,240209
0,0555357
0,0095804
0,00136385
0,000137177
0,0000215046
-0,000000144203
0,00000028848

Picture 1: Operating the polynomial fit with LabVIEW 7.1 on the base of 10 Chebycheff nodes.



Picture 2: Absolute error of the previous polynomial approximation. (mse=0.33)

A similarly efficient polynomial approximation could be set up for the $\log_2(x)$ function in the special case of $x \in [1,2[$. Of course satisfying algorithms must be added to extend the functions to the ranges $]-\infty,+\infty[$ respectively $]0,+\infty[$.

The further investigation of the subject however led us to the algorithm that is described in this paper as a better solution for the purpose, combining memory economy and execution speed in the case of IEEE 754 standard single point precision floating-point representation[xi].

2. Calculation of f(x)=exp(x)

2.1. Simplification to $f_0(x)=2^x$

For any real x we have:

$$e^x = \left(2^{\frac{1}{\ln(2)}} \right)^x$$

$$= 2^{\frac{x}{\ln(2)}}$$

This equation signifies that any x must be multiplied with the constant $c=1/\ln(2)$ before computing the f_0 function.

Convention: we may suppose $x>0$, because of the trivial $e^0 = 1$ and $e^{-x} = 1/e^x$ allowing an easy computation for any x.

2.2. Arranging of $f_0(x)=2^x$

The IEEE standard floating point representation transports three information-parts about the concerned number: the sign, the biased exponent and the fractional part (=mantissa).

Example: single precision representation

```
00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
s  e  e  e  e  e  e  e  e  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m  m
```

where :

- if e=255 and m<>0, then x = NaN ("Not a number")
- if e=255 and m=0, then x = $(-1)^{s} * \infty$
- if 0<e<255 then x = $(-1)^{s} * 2^{(e-127)} * (1.m)$ with (1.m) representing the binary number created by preceding m with a leading 1 and the binary point
- if e=0 and m<>0, then x = $(-1)^{s} * 2^{(-126)} * (0.m)$ representing denormalized values
- if e=0 and m=0 and s=1, then x = -0
- if e=0 and m=0 and s=0, then x = 0

The particular representation of the number 2.25 therefore is:

1. s = 0
2. e = 1 + 127 = 128 = b'10000000'
3. m = 0.125 = b'0010000 00000000 00000000'

Normalized mantissas always describe numbers belonging to the interval [1,2[.

Convention: ***unbiased exponent = q (integer)***

x = $(-1)^{s} * 2^{q} * (1.m)$

Note: it is not possible to represent the number 0 in this form!

$$\forall x > 0,$$
$$2^{x} = 2^{\left(2^{q}\cdot(1.m)\right)} = \left(2^{1.m}\right)^{\left(2^{q}\right)}$$

2.3 Study of $f_1(x = 1.m) = 2^{1.m}$

$$x = 1\cdot 2^{0} + a_1\cdot 2^{-1} + a_2\cdot 2^{-2} + a_3\cdot 2^{-3} + \ldots,\quad a_i \in \{0,1\} = bit - representation\ of\ m$$
$$= 1 + a_1\cdot 0.5 + a_2\cdot 0.25 + a_3\cdot 0.125 + \ldots$$

$$2^{x} = 2^{(1+a_1\cdot 0.5+a_2\cdot 0.25+a_3\cdot 0.125+\ldots)}$$
$$= 2^{1}\cdot 2^{a_1\cdot 0.5}\cdot 2^{a_2\cdot 0.25}\cdot 2^{a_3\cdot 0.125}\cdot\ldots$$
$$= 2\cdot\sqrt{2^{a_1}}\cdot\sqrt[4]{2^{a_2}}\cdot\sqrt[8]{2^{a_3}}\cdot\ldots$$
$$= 2\cdot\prod_{i=1}^{n}\sqrt[2^{i}]{2^{a_i}}$$

(formula 1)

In any case we have: $2^{(x-1)} \in [1,2[$, since $\lim_{i\to\infty}\prod^{i}\sqrt[2^{i}]{2} = 2$ or simpler:

$m \in [0,1[\Rightarrow 2^{m} \in [1,2[$

Thus the result $2^{(x-1)}$ will always be normalized to the IEEE standard.

To fast compute $2^{(x-1)}$, we only need to first set up a look-up table with the constant values of successive square-roots of 2, then operate consecutive multiplying of a selection of those numbers while scanning the mantissa m of x. The table is filled with values that have been previously yielded by any iterative means with a precision <0.5 **ulp**. (For instance in a practical micro-controller application, the values can be produced by the cross-compiler host and stored as constants in the micro-controllers ROM, EEPROM or even as part of the program-code). Due to the limitation of accuracy of the single precision representation, the LUT only needs 22 different values. Thus the last bit won't add any information as can be seen in the following example. (For better comprehension, we added the value 2 to the table at index 0.)

Example:

Compute $y = 2^{1.171875}$

Look-up table (LUT):

index	$2^{\left(2^{-i}\right)}$
0	**2**
1	1.41421353816986084
2	1.18920707702636719
3	**1.09050774574279785**
4	1.0442737340927124
5	**1.02189719676971436**
6	**1.010889176330566**
7	1.00542986392974853
8	1.00271129608154297
9	1.00135469436645508
10	1.00067710876464844
11	1.00033855438232422
12	1.00016927719116211
13	1.00008463859558105
14	1.00004231929779053
15	1.00002110004425049
16	1.00001060962677002
17	1.00000524520874023
18	1.00000262260437012
19	1.00000131130218506
20	1.0000007152557373
21	1.00000035762786865
22	1.00000011920928955
→ *23*	*1,00000011920928955*

(b'0000000 00000000 00000001' is the smallest mantissa that can be represented in IEEE 754 single precision mode. The corresponding normalized number equals 1.0000001192… in decimal notation.)

IEEE mantissa (1.171875) = b'0010110 00000000 00000000'

$2^{1.171875} \quad = \text{LUT}(0) \cdot \text{LUT}(3) \cdot \text{LUT}(5) \cdot \text{LUT}(6)$

= 2 · 1.0905077 · 1.0218971 · 1.0108891
= 2.2530429 (only considering the base-10 significant digits here)

In practice, in order to maintain the result normalized during the whole computation, it is advantageous of operating the multiplication by 2 only at the end of the calculations simply by increasing the exponent.

2.3.1. Accuracy and speed

As a consequence of the theorem postulating that the error η of successive floating-point multiplications may be expressed as:

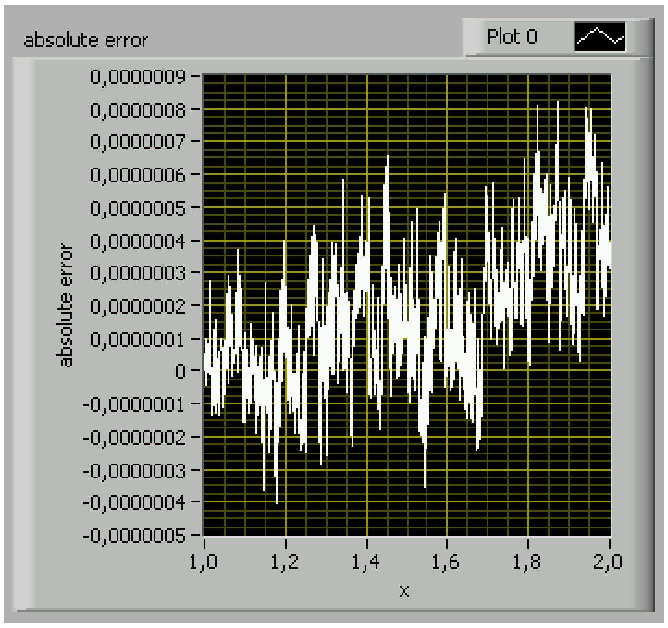
$\frac{|\eta|}{|x_1 \cdot x_2 \cdot \ldots \cdot x_n|} \le \frac{(2n-1)ulp}{1-(2n-1)ulp}$ under the condition[xii] that:

$x_{\min} \le |x_i| \le x_{\max}$ and

$x_{\min} \cdot c_1 \le |x_1 \cdot x_2 \cdot \ldots \cdot x_n| \le x_{\max} \cdot c_2$

$where\ c_1 = \frac{1-(2n-2)ulp}{1-(4n-4)ulp}\ and\ c2 = (1-(2n-2)ulp)$

We can conclude that η won't grow out of an acceptable range. In fact the maximum relative error can be theoretically estimated for n=22 as 5.6E-6 and for n=12 as 2.7E-6. Nonetheless, the LabVIEW simulation gives a better result of about 8E-7 in the worst case. The simulation also reveals a growing error in function of x. (Note that the simulation compared the square-root algorithm with the current LabVIEW (v.7.1) y=2^x function that has been applied to extended precision numbers.)



Picture 3: A LabVIEW simulation reveals that the maximum absolute error may be considered as 8E-7. (mse=2.33)

Compared to the Chebycheff polynomial approximation, there is a non-negligible loss of accuracy. Another trade-off is the fact that the number of multiplications depends on the number of significant bits, whereas the polynomial algorithm keeps the number of operations constant. This certainly is an issue, if higher precision representations are chosen.

The computational worst case would be to operate a product with *m* = b'1111111 11111111 11111111'. This would require 22 multiplications (remember that the last bit may be ignored).

Since the numbers 0 and 1 have the same probability to appear in *m*, the average case only requires 11 products.

If we assume the empirical assertion that a floating-point multiplication needs about 5 times longer than an addition, the algorithm would be operated within the time of 11*5=55 additions, while the polynomial approximation would need 9*5+9=50 additions. Thus the average computing speed may be considered as sensibly equal for both algorithms.

However, a considerable speed gain can be obtained, if the algorithm is programmed at lowest level, where the access to the floating-point representation itself is possible. (This is practically required anyway, since the look-up-table items are selected on the mantissa bits!) One disadvantage of the polynomial algorithm is the fact that the multiplications produce un-normalized results that need to be re-expanded and also aligned for the additions. With the square-root algorithm, such normalizing procedures are superfluous, since it is guaranteed that after each multiplication the result is normalized. Thus, calling a primitive floating-point multiplying function without the normalizing procedure can substantially accelerate the execution.

Another possibility to reduce the computing time may be given, if less significant mantissa-bits are being considered. Indeed, it might be useful in some cases to gain speed instead of accuracy. In this case, the gain, compared to the Chebycheff polynomial approach may be remarkable.

2.4. The influence of the exponent $q \neq 0$

Normally mathematicians would consider the exp(x) and specially the 2^x functions differently than presented in this document, when extending the domain of definition from [1,2[to]-∞,+∞[-or the portion of this interval that corresponds to the single precision representation.

In fact they'd prefer writing:

$$\forall x\, real, \exists r\, relative\, number, \quad 2^x = 2^{r+1.m} = 2^r \cdot 2^{1.m},\ where$$
$$r = floor(x) - 1$$

At first sight, the operation looks like being very easy and fast, since only the f_0 function must be executed and the result of that operation be multiplied by the r^th power of two. The latter operation could be done simply and rapidly by increasing or decreasing the binary exponent by |r|. However there is a hook ! The issue is that the floor-function, representing the truncating of the floating point value, needs quite an impressive amount of execution steps, because it cannot be deduced by simple means from the IEEE 754 floating point representation. Therefore we propose a rather unorthodox approach that will end in a very reduced program code size.

2.4.1. First case : $q > 0$

$$2^x = 2^{\left(2^q \cdot (1.m)\right)}$$
$$= \left(2^{(1.m)}\right)^{\left(2^q\right)}$$
$$= \left(2^{(1.m)}\right)^{2\cdot 2\cdot 2 \ldots}$$
$$= \left(\left(\left(2^{(1.m)}\right)^2\right)^2\right)^{2\ldots}$$

This equation doesn't mean anything else but to compute successive squares of the $2^{1.m}$ value, each one representing a single multiplying.

2^x rapidly reaches the limit of the IEEE floating point representation as can be seen in the following series :

q :	1	2	3	4	5	6	7
$2^{\left(2^q\right)}$:	4	16	256	65536	4294967296	1.8446744E19	3.4028236E38

For example: the largest number that can be represented in single precision is $2^{127} \cdot 1.999999 = 3.4E38$. Thus the largest number x that can be computed to 2^x is

$\log_2(2^{128}) = 128 \Rightarrow q = 7$

Example:

Compute $y = 2^{9.375}$

The IEEE single precision representation of this number gives:

q = 3, 1.m=1.171875

$$y = \left(\left(\left(2^{1.171875}\right)^2\right)^2\right)^2$$
$$= 2.2530429^8$$
$$= 663.981852$$

Note that the relative error grows in a benign way with each multiplication, even if the 2nd condition of the cited theorem in 2.3.1. is no longer fulfilled.

2.4.2. Second case : q<0

$$\begin{aligned}2^x &= 2^{\left(2^q\cdot(1.m)\right)}\\ &= \left(2^{(1.m)}\right)^{\left(2^q\right)}\\ &= \ldots\sqrt{\sqrt{\sqrt{\left(2^{(1.m)}\right)}}}\\ &= \sqrt[2^{(-q)}]{\left(2^{(1.m)}\right)}\end{aligned}$$

Replacing $2^{(1.m)}$ according to *formula 1*, we get:

$$2^x = \sqrt[2^{(-q)}]{2\cdot\sqrt{2^{a_1}}\cdot\sqrt[4]{2^{a_2}}\cdot\sqrt[8]{2^{a_3}}\cdot\ldots}$$

If q = -1, the equation may be rewritten: $2^x = \sqrt{2}\cdot\sqrt[4]{2^{a_1}}\cdot\sqrt[8]{2^{a_2}}\cdot\sqrt[16]{2^{a_3}}\cdot\ldots$

If q = -2, it may be written as: $2^x = \sqrt[4]{2}\cdot\sqrt[8]{2^{a_1}}\cdot\sqrt[16]{2^{a_2}}\cdot\sqrt[32]{2^{a_3}}\cdot\ldots$

a.s.o….

To compute the product for any q, we can operate a simple offset-shift in the look-up table only depending on the value of q.

***Example**:*

Compute y = $2^{0.146484375}$

The IEEE single precision representation of this number gives:

q = -**3** and 1.m=1.171875

y = LUT(0+**3**) · LUT(3+**3**) · LUT(5+**3**) · LUT(6+**3**)
   = 1.0905077 · 1.0108891 · 1.0027112 · 1.0013546
   = 1.1068685

Note that ∀q<-22, 2^x will be considered a 1.

2.5. Practical implementation of the algorithm $y=2^x$

Preliminary notes: For this sample the DELPHI language -former TURBO PASCAL- was chosen because of the better overview concerning stacked *ifs* and *for* loops, than known from C++, even if no such compiler seems to exist for micro-controllers. The bulk of this section is to most clearly present an astute and short implementation of the algorithm that can be easily traduced to any current language. For simplicity, the “NaN”, “infinity” and “denormalized” handlers have been omitted. However they should be added to a real implementation in order to keep everything standardized according to IEEE 754. Obviously the code is set up with respect to the single precision data representation.

```
const MAX_ITERATIONS=22;                        //reduce, if less precision and more
var LUT: array [0..MAX_ITERATIONS] of single;    //computing speed is desired

procedure initialize_LUT; //create LUT
var index:integer;
begin                //values 0 and MAX_ITERATIONS included in the for loop!
  for index:=0 to MAX_ITERATIONS do LUT[index]:=power(2,power(2,(-index)));
end;

procedure expand_SGL(x:single;var exponent:integer; var signum:integer;
                      var mantissa:Cardinal);  //use var type to pass data
var tmp:Cardinal;                    //temporary variable
    pIEEE_754_raw: ^Cardinal;         //pointer to unsigned 32-bit variable
begin
  pIEEE_754_raw:=@x;                 //get address of x
  if (pIEEE_754_raw^ and $80000000)=0 then signum:=0 else signum:=1; //0 :+  1:-
  tmp:=pIEEE_754_raw^ and $7FFFFFFF;  //clear sign bit
  exponent:=(tmp shr 23)-127;         //unbiased exponent
  mantissa:=$80000000 or (tmp shl 8); //set hidden bit and shift rest
end;

function two_exp(x:single):single;
var exponent,signum:integer;
    mantissa,test:Cardinal;
    offset,iterations,index,counter:integer;
    temp:single;
begin
  expand_SGL(x,exponent,signum,mantissa);  //extract data
  //special cases
  if exponent<-MAX_ITERATIONS then         //values below precision
  begin
     result:=1                             //case 2^0.0000... = 1
  end else
  if exponent=-MAX_ITERATIONS then        //no multiplications are needed
     begin                                //since the last item is chosen only
        if signum=0 then result:=LUT[MAX_ITERATIONS] else
                         result:=1/LUT[MAX_ITERATIONS]
     end else
     if exponent>7 then         //out of range
        begin
           if signum=0 then result:=1E99 else      //+infinity
                            result:=+0             //+0
        end else
        begin
          iterations:=MAX_ITERATIONS;          //initialize iterations
          //normal cases
          if exponent<0 then
          begin
            offset:=abs(exponent);             //get LUT offset
            iterations:=iterations+exponent;   //less iterations are required //+= in C++
            index:=offset;                     //start LUT at offset
            temp:=LUT[index]                   //initialize temporary variable
          end else
          begin   //exponent>=0
            index:=0;                          //initialize index
            temp:=1                            //initialize temp for products
          end;

          for counter:=1 to iterations do      //now iterate
          begin
            index:=index+1;                    //increment index //+= operation in C++
            mantissa:=mantissa shl 1;          //left shift mantissa
            test:=mantissa and $80000000;      //get most significant bit
                                  //← in Assembly language could use faster bit-test
            if test<>0 then                    //check msb
            begin
              temp:=temp*LUT[index]; //← could choose normalized multiplying here //*= in C++
            end;
          end;
          //now do the final multiplying, if exponent positive
          if exponent>=0 then result:=2*temp else result:=temp; //← could fast multiply by 2
                                                                //simply increase result's
                                                                //binary exponent
          if signum=1 then result:=1/result;   //negative value, so inverse

          if exponent>0 then
            for counter:=1 to exponent do      //now compute squares
               result:=result*result;          //*= in C++
        end;
end;
```

3. Study of the calculation of g(x)=ln(x)

3.1. Simplification to $g_0(x)=\log_2(x)$

For any strictly positive real x we have:

$$\ln(x) = \log_2(x)\cdot\ln(2)$$

$$\begin{aligned}\log_2(x) &= \log_2(2^q\cdot(1.m))\\ &= \log_2(2^q)+\log_2(1.m)\\ &= q\cdot\log_2(2)+\log_2(1.m)\\ &= q+\log_2(1.m)\end{aligned}$$

3.2. Study of the function $g_1(x=1.m) = \log_2(1.m)$

$$2^{\log_2(x)} = id(x) = x$$

Calculating $\log_2(x)$ may be considered as the inverse operation to doing 2^y with:

$x = 2^y$

Since we have $x\in[1,2[\Rightarrow y=\log_2(x) \in [0,1[$ and:

$$2^y = \sqrt{2^{a_1}}\cdot\sqrt[4]{2^{a_2}}\cdot\sqrt[8]{2^{a_3}}\cdot\ldots,a_i\in\{0,1\}$$

Note: From the IEEE standard point of view, y must be considered as partly “denormalized”, since the leading 1 must be missing before the binary point.

Operating the following iterative tests may yield the set of elements a_i:

$$if\left(t_1=\frac{x}{\sqrt{2}}\right)\geq 1\, then\ x'=t_1;a_1=1\, else\ x'=x;a_1=0$$

$$if\left(t_2=\frac{x'}{\sqrt[4]{2}}\right)\geq 1\, then\ x''=t_2;a_2=1\, else\ x''=x';a_2=0\,....$$

Example:

Compute $y = \log_2(1.12652145)$

$$t_1 = \frac{1.12652145}{1.41421356} < 1 \quad \Rightarrow a_1 = 0$$
$$t_2 = \frac{1.12652145}{1.1892070} < 1 \quad \Rightarrow a_2 = 0$$
$$t_3 = \frac{1.12652145}{1.0905077} = 1.0330247 \geq 1 \quad \Rightarrow a_3 = 1$$
$$t_4 = \frac{1.0330247}{1.0442737} < 1 \quad \Rightarrow a_4 = 0$$
$$t_5 = \frac{1.0330247}{1.0218971} = 1.0108891 \geq 1 \quad \Rightarrow a_5 = 1$$
$$t_3 = \frac{1.0108891}{1.0108891} = 1 \quad (\geq 1) \quad \Rightarrow a_6 = 1$$

$\Rightarrow$ m(y) = 001011 $\Rightarrow$ y = $0 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + 0 \cdot 2^{-4} + 1 \cdot 2^{-5} + 1 \cdot 2^{-6}$
= 0.125 + 0.03125 + 0.015625
= 0.171875

Instead of using divisions, that need a considerable computing time, it might be a good idea to additionally use a variant of the LUT, composed of the inverse values of the initial LUT. This procedure will allow doing only multiplications. The tests then have the following aspect:

$$if \;\; x \geq \sqrt{2} \;\; then \;\; x' = \frac{1}{\sqrt{2}} \cdot x \, ...$$
$$if \;\; x' \geq \sqrt[4]{2} \;\; then \;\; x'' = \frac{1}{\sqrt[4]{2}} \cdot x' \, ...etc$$

Look-up table with the inverse values:

index	$\frac{1}{\sqrt[2^i]{2}}$
1	0,70710676908493042
2	0,840896427631278174
3	0,917004048824310303
4	0,957603275775909424
5	0,978572070598602295
Etc.	

3.3. Note for the practical implementation of the algorithm y=log$_2$(x)

Since the successive values of x and the LUT numbers that are used with the comparisons are always normalized, the comparisons can be fast computed. But this time the multiplications are operated with un-normalized numbers (2nd LUT). However, it is obvious that the multiplications always remain in the interval]0,1[. A low level fast multiplying routine can be set up for these numbers, increasing the computing speed. Nonetheless the log$_2$ algorithm implementation seems trivial and a sample code can be omitted in this document.

Conclusion

This algorithm package may be helpful to implement fast calculations for the exp(x) and the ln(x) functions in the special case of IEEE 754 standard single precision representation to any desired precision in the given range.

---

[i] "Convict" doesn't have the same meaning in French-spoken countries than in Anglo-Saxons. The term is derived from the Latin "convivere" = living together.
[ii] Since 1999 the author teaches advanced LEGO robotics in after-school classes and maintains the related widely known website www.convict.lu/Jeunes/RoboticsIntro.htm
[iii] ROBOLAB has been created by Prof. Chris Rogers and his team at Tufts University Massachusetts in collaboration with the LEGO Company and National Instruments
[iv] Ultimate ROBOLAB has been developed by the author of this paper in collaboration with Prof. Chris Rogers
[v] VOLDER J.E. : *CORDIC Trigonometric Computing Technique*, IRE Transactions on Electronic Computers, EC-8, Sept. 1959
[vi] TURNER P.R.: *Guide to Numerical Analysis* MacMillan UK, 1989 & Guide to Scientific Computing MacMillan, 2000
[vii] DEFOUR D., DE DINECHIN F., MULLER J.M., *A new scheme for table-based evaluation of functions*, Institut National de Recherche en Informatique et en Automatique, ISSN 0249-6399, 2002
[viii] ABRAMOWITZ M., STEGUN I.A., eds, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Ch. 22. NY, 1972
[ix] PLATO R.: *Numerische Mathematik*, GE, 2004, p.10ff
[x] HORNER W.G.: *A new method of solving numerical equations of all orders, by continuous approximation*. In Philosophical Transactions of the Royal Society of London, pp. 308-335, 1819
[xi] IEEE Computer Society (1985), *IEEE Standard for Binary Floating-Point Arithmetic*, IEEE Std 754, 1985
[xii] PLATO R.: *Numerische Mathematik*, GE, 2004, p.394ff